University of Illinois **Fall 2023**

ECE 210/211 - Homework 12: Problems and Solutions

Due: Tuesday, November 14 by 11.59pm

Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
   sign: __________________________.

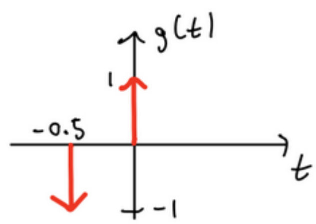
2. Simplify the following expressions involving the impulse, unit step function, rectangular pulse, and/or triangular pulse and sketch the results.

   (a) $g(t) = \cos(2\pi t)\left(\frac{du(t)}{dt} + \delta(t+0.5)\right)$.
   Solution: By the unit-step derivative property of impulse:

$$\cos(2\pi t)\frac{du(t)}{dt} = \cos(2\pi t)\delta(t).$$

   Then, by distributive property of multiplication and the sampling property of impulse:

$$g(t) = \cos(0)\delta(t) + \cos(2\pi(-0.5))\delta(t+0.5) = \delta(t) - \delta(t+0.5).$$



   (b) $a(t) = \int_{-\infty}^{t} \delta(\tau+1)d\tau + \mathrm{rect}\left(\frac{t}{6}\right)\delta(t-2)$.
   Solution: By the definite integral property of impulse:

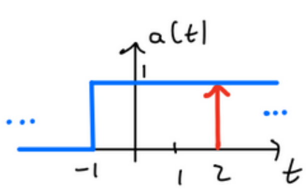
$$\int_{-\infty}^{t} \delta(\tau+1)d\tau = u(t+1).$$

   By the sampling property of impulse:

$$\mathrm{rect}\left(\frac{t}{6}\right)\delta(t-2) = \mathrm{rect}\left(\frac{2}{6}\right)\delta(t-2) = \delta(t-2).$$

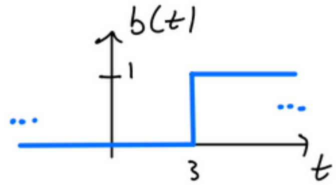
   Hence,

$$a(t) = u(t+1) + \delta(t-2).$$



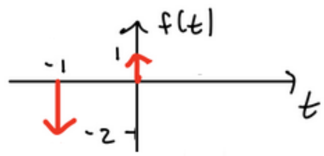
(c) $b(t) = \delta(t-3) * u(t)$.
Solution: By the convolution property of impulse:

$$b(t) \;=\; \delta(t-3) * u(t) = u(t-3).$$



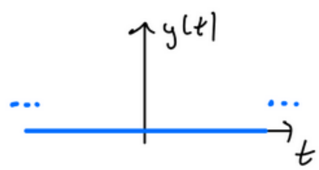
(d) $f(t) = \left(1+t^2\right)\left(\delta(t) - \delta(t+1)\right)$
Solution: By distributive property of multiplication and the sampling property of impulse:

$$f(t) \;=\; (1+0^2)\delta(t) \;-\; (1+(-1)^2)\delta(t+1) \;=\; \delta(t) \;-\; 2\delta(t+1).$$



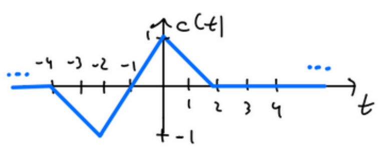
(e) $y(t) = \int_{-1}^{\infty} \left(\tau^2+1\right)\delta(\tau+2)d\tau$
Solution: By the sifting property of impulse and the fact that the impulse is outside the integration region:

$$y(t) \;=\; \int_{-1}^{\infty} \left(\tau^2+1\right)\delta(\tau+2)d\tau = 0.$$



(f) $c(t) = \triangle\left(\frac{t}{4}\right) * \left(\delta(t) - \delta(t+2)\right)$.
Solution: By distributive property of convolution and the convolution property of impulse:

$$c(t) \;=\; \triangle\left(\frac{t}{4}\right) * \delta(t) \;-\; \triangle\left(\frac{t}{4}\right) * \delta(t+2) \;=\; \triangle\left(\frac{t}{4}\right) \;-\; \triangle\left(\frac{t+2}{4}\right).$$



3. For a system with impulse response $h(t)$, the system output $y(t) = h(t) * f(t) = \text{rect}\left(\frac{t-3}{3}\right)$. Determine $h(t)$ if

(a) $f(t) = \text{rect}\left(\frac{t}{3}\right)$.
Solution: The output, $y(t)$, is just a shifted version of $f(t)$, hence

$$y(t) \;=\; \text{rect}\left(\frac{t-3}{3}\right) \;=\; \text{rect}\left(\frac{t}{3}\right) * \delta(t-3).$$

Therefore, the impulse response for this system is

$$h(t) \;=\; \delta(t-3).$$

(b) $f(t) = 2u(t)$.
Solution: The output, $y(t)$, can be written in terms of $f(t)$ as

$$y(t) = \mathrm{rect}\left(\frac{t-3}{3}\right) = u\left(t-\frac{3}{2}\right) - u\left(t-\frac{9}{2}\right) = 2u(t) * \left(\frac{1}{2}\delta\left(t-\frac{3}{2}\right) - \frac{1}{2}\delta\left(t-\frac{9}{2}\right)\right).$$

Therefore, the impulse response for this system is

$$h(t) = \frac{1}{2}\delta\left(t-\frac{3}{2}\right) - \frac{1}{2}\delta\left(t-\frac{9}{2}\right).$$

(c) $f(t) = \mathrm{rect}(t)$.
Solution: The output, $y(t)$, can be written in terms of $f(t)$ as

$$y(t) = \mathrm{rect}\left(\frac{t-3}{3}\right) = \mathrm{rect}(t-2) + \mathrm{rect}(t-3) + \mathrm{rect}(t-4) = \mathrm{rect}(t) * (\delta(t-2) + \delta(t-3) + \delta(t-4)).$$

Therefore, the impulse response for this system is

$$h(t) = \delta(t-2) + \delta(t-3) + \delta(t-4).$$

4. Determine the Fourier transform of the following signals. Simplify the results as much as you can and sketch the result if it is real valued.

(a) $f(t) = 5\cos(5t) + 3\sin(15t)$.
Solution: Using entries #18 and #19 from the Fourier transform table, we have

$$5\cos(5t) \overset{\mathcal{F}}{\longleftrightarrow} 5\pi\left[\delta(\omega-5) + \delta(\omega+5)\right],$$

$$3\sin(15t) \overset{\mathcal{F}}{\longleftrightarrow} 3j\pi\left[\delta(\omega+15) - \delta(\omega-15)\right].$$

Therefore, using the addition property of Fourier transform:

$$F(\omega) = 5\pi\left[\delta(\omega-5) + \delta(\omega+5)\right] + 3j\pi\left[\delta(\omega+15) - \delta(\omega-15)\right].$$

(b) $x(t) = \cos^2(6t)$.
Solution: Using Euler's formula, the function can be written as

$$x(t) = \cos^2(6t) = \frac{1}{4}\left(e^{-j6t} + e^{j6t}\right)^2 = \frac{1}{4}\left(e^{-j12t} + 2 + e^{j12t}\right) = \frac{1}{2} + \frac{1}{2}\cos(12t).$$

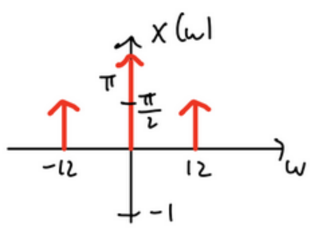
Using entries #18 and #15 from the Fourier transform table, we have

$$\frac{1}{2} \overset{\mathcal{F}}{\longleftrightarrow} \pi\delta(\omega),$$

$$\frac{1}{2}\cos(12t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{2}\left[\delta(\omega-12) + \delta(\omega+12)\right].$$

Therefore, using the addition property of the Fourier transform:

$$X(\omega) = \pi\delta(\omega) + \frac{\pi}{2}\left[\delta(\omega-12) + \delta(\omega+12)\right].$$



(c) $y(t) = e^{-2t}u(t) * \cos(2t)$.

Solution: Using entries #1 and #18 from the Fourier transform table, we have

$$e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j\omega},$$

$$\cos(2t) \overset{\mathcal{F}}{\longleftrightarrow} \pi\left[\delta(\omega-2)+\delta(\omega+2)\right].$$

Therefore, using the time convolution property and the sampling property of the Fourier transform,

$$Y(\omega) = \left(\frac{1}{2+j\omega}\right)\left(\pi\left[\delta(\omega-2)+\delta(\omega+2)\right]\right) = \frac{\pi}{2+j2}\delta(\omega-2) + \frac{\pi}{2-j2}\delta(\omega+2).$$

(d) $z(t) = (1+\cos(3t))\,e^{-t}u(t)$.

Solution: The function $z(t)$ can be written as

$$z(t) = (1+\cos(3t))\,e^{-t}u(t) = e^{-t}u(t) + e^{-t}\cos(3t)\,u(t).$$

Using entries #1 and #12 from the Fourier transform table, we have

$$e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1+j\omega},$$

$$e^{-t}\cos(3t)\,u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1+j\omega}{(1+j\omega)^2+3^2} = \frac{1+j\omega}{(1+j\omega)^2+9}.$$

Therefore, using the addition property of the Fourier transform:

$$Z(\omega) = \frac{1}{1+j\omega} + \frac{1+j\omega}{(1+j\omega)^2+9}.$$

5. Determine the Nyquist sampling frequencies in Hz needed to sample the following analog signals without aliasing error.

(a) Arbitrary signal $f(t)$ with bandwidth 40 kHz.

Solution: The Nyquist sampling frequency is twice of the highest frequency of the signal to avoid the aliasing error. In this case, the sampling frequency should be

$$f_s = 2\cdot 40 = 80 \text{ kHz}.$$

(b) $f(t) = \mathrm{sinc}(40\pi t)$.

Solution: Taking the Fourier transform of the sinc function, we have

$$F(\omega) = \frac{1}{40}\mathrm{rect}\left(\frac{\omega}{80\pi}\right),$$

which is a rectangle function centered at 0 and ranges between $\omega = \pm 40\pi rad/s$, or $f = \pm 20$ Hz. Therefore, the Nyquist sampling frequency should be

$$f_s = 2\cdot 20 = 40 \text{ Hz}.$$

(c) $g(t) = \mathrm{sinc}(100\pi t) + \mathrm{sinc}(40\pi t)\cos(200\pi t)$.

Solution: Using the result from part (b) and the modulation property of Fourier transform, we have

$$G(\omega) = \frac{1}{100}\mathrm{rect}\left(\frac{\omega}{200\pi}\right) + \frac{1}{80}\mathrm{rect}\left(\frac{\omega-200\pi}{80\pi}\right) + \frac{1}{80}\mathrm{rect}\left(\frac{\omega+200\pi}{80\pi}\right).$$

We recognize that the highest frequency of this signal is $\omega = 200\pi + 40\pi = 240\pi$ rad/s, or $f = 120$ Hz. Therefore, the sampling frequency should be

$$f_s = 2\cdot 120 = 240 \text{ Hz}.$$

6. Let $f(t)$ be an arbitrary signal with bandwidth Ω rad/s. Determine the Nyquist sampling frequency ω_s, in rad/s, needed to sample the following analog signals without causing aliasing error.

   (a) $f_1(t) = f(t)\sin(4000\pi t)$.
   Solution: Using the result from HW 11 modulating with a sine function:

   $$f(t)\sin(4000\pi t) \overset{\mathcal{F}}{\longleftrightarrow} j\pi F(\omega + 4000\pi) - j\pi F(\omega - 4000\pi).$$

   The maximum frequency of $F_1(\omega)$ is equal to 4000π plus the bandwidth of $F(\omega)$. Therefore, the Nyquist sampling frequency is

   $$\omega_s \;=\; 2\,(4000\pi \;+\; \Omega) \;=\; 8000\pi \;+\; 2\Omega \text{ rad/s}.$$

   (b) $f_2(t) = f(t) * \sin(4000\pi t)$.
   Solution: From the time convolution property of Fourier transform

   $$\begin{aligned} f(t) * \sin(4000\pi t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad & F(\omega)\left(j\pi\delta(\omega + 4000\pi) \;-\; j\pi\delta(\omega - 4000\pi)\right) \\ & = \; F(-4000\pi)j\pi\delta(\omega + 4000\pi) \;-\; F(4000\pi)j\pi\delta(\omega - 4000\pi). \end{aligned}$$

   If $F(\pm 4000\pi) = 0$ then $F_2(\omega) = 0$ and $\omega_s = 0$ rad/s.
   If $F(\pm 4000\pi) \neq 0$, then $\omega_s = 2(4000\pi) = 8000\pi$ rad/s.
   Hence, the Nyquist sampling frequency is

   $$\omega_s \;=\; \begin{cases} 0 \text{ rad/s} & \text{if } F(\pm 4000\pi) = 0 \\ 8000\pi \text{ rad/s} & \text{else.} \end{cases}$$

   (c) $f_3(t) = f(t) * f(t)$.
   Solution: From the time convolution property of Fourier transform

   $$f(t) * f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)F(\omega).$$

   $F_3(\omega)$ is nonzero for the same frequencies for which $F(\omega)$ is nonzero. Therefore, the Nyquist sampling frequency is

   $$\omega_s \;=\; 2\Omega \text{ rad/s}.$$

   (d) $f_4(t) = f(t)f(t)$.
   Solution: From the frequency convolution property of Fourier transform

   $$f(t)f(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\left(F(\omega) * F(\omega)\right).$$

   The width property of convolution indicates that the width of $F(\omega) * F(\omega)$ is 2Ω. Therefore, the Nyquist sampling frequency is

   $$\omega_s \;=\; 2(2\Omega) \;=\; 4\Omega \text{ rad/s}.$$

   (e) $f_5(t) = f(t-2)$.
   Solution: From the time shift property of Fourier transform

   $$f(t-2) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)e^{-j\omega 2}.$$

   Since $|F_5(\omega)| = |F(\omega)|$, the Nyquist sampling frequency is

   $$\omega_s \;=\; 2\Omega \text{ rad/s}.$$

7. Determine the impulse response, $h(t)$, of the LTI systems having the following *unit-step* responses.

(a) $g(t) = 5u(2t-5)$.
Solution: Recall that the derivative of the unit-step response of an LTI system is $h(t)$. Hence, using the definite integral and scaling properties of the impulse:

$$h(t) \;=\; \frac{d}{dt}g(t) \;=\; 5\delta(2t{-}5)(2) \;=\; 10\delta(2t{-}5) \;=\; 10\delta\left(2\left(t-\frac{5}{2}\right)\right) \;=\; \frac{10}{2}\delta\left(t-\frac{5}{2}\right) \;=\; 5\delta\left(t-\frac{5}{2}\right).$$

(b) $g(t) = t^3u(t)$.
Solution: Recall that the derivative of the unit-step response of an LTI system is $h(t)$. Hence, using the definite integral and sampling properties of the impulse:

$$h(t) \;=\; \frac{d}{dt}g(t) \;=\; 3t^2u(t) \;+\; t^3\delta(t) \;=\; 3t^2u(t) \;+\; 0^3\delta(t) \;=\; 3t^2u(t).$$

(c) $g(t) = (2-e^{-t})u(t-5)$.
Solution: Recall that the derivative of the unit-step response of an LTI system is $h(t)$. Hence, using the definite integral and sampling properties of the impulse:

$$h(t) \;=\; \frac{d}{dt}g(t) \;=\; (2-e^{-t})\delta(t-5) \;+\; u(t-5)(e^{-t}) \;=\; (2-e^{-5})\delta(t-5) \;+\; e^{-t}u(t-5).$$

8. Consider the following zero-state input-output relations for a variety of systems. In each case, determine whether the system is zero-state linear, time invariant, BIBO stable and/or causal.

(a) $y(t) = 5f(t) * u(t)$.
Solution: The system is written as a convolution of $f(t)$ with the impulse response $h\,(t) = 5u\,(t)\,,$ hence the system is LTI.
The impulse response, $h\,(t) = 5u(t)$, is not absolutely integrable, hence this system not BIBO stable.
The impulse response $h\,(t) = 0\;\forall t < 0$, hence this system is causal.

(b) $y(t) = \delta(t-4) * f^2(t)$.
Solution:

$$y(t) \;=\; \delta(t-4) * f^2(t) \;=\; f^2(t-4).$$

The system squares $f\,(t)$ before it is convolved, hence it is not linear.
The system is time invariant due to the time-shift property of convolution and impulse.
The system is BIBO stable because if $|f\,(t)\,| \le C$ then $|f^2\,(t-4)\,| \le C^2$.
The system is causal because the output at time t only depends on the input at time $t-4$, which is not in the future relative to t.

(c) $y(t) = \int_{-\infty}^{t-2} f(\tau)d\tau$.
Solution: The output can be written as

$$y(t) \;=\; \int_{-\infty}^{t-2} f(\tau)d\tau \;=\; f(t) * u(t-2).$$

Since the system is written as a convolution of $f(t)$ with the impulse response $h\,(t) = u\,(t-2)\,,$ the system is LTI.
The system is not BIBO stable because the impulse response, $h\,(t) = u(t-2)$ is not absolutely integrable.
The system is causal because the output is the continuous sum of the input from $-\infty$ to $t-2$, which are not in the future.

(d) $y(t) = f(t-1) + f(t+1)$.
Solution: The system can be written as

$$y\,(t) \;=\; f\,(t) * (\delta\,(t-1) \;+\; \delta\,(t+1))\,,$$

with impulse response

$$h(t) = \delta(t-1) + \delta(t+1).$$

Hence the system is LTI.
The system is BIBO stable because if $|f(t)| \leq C$ then $|f(t-1) + f(t+1)| \leq 2C$.
The system is not causal because the output at time t depends on the future value of the input at time $t+1$.

(e) $y(t) = f(2t)$.
Solution: The system is linear because $y(t)$ is clear a line through the origin as a function of f.
The system is not time invariant because the time is scaled within the function, such that the time shift by t_0 of the input function is the time shift by $t_0/2$ of the output.
The system is BIBO stable because if $|f(t)| \leq C$ then $|f(2t)| \leq C$ too.
The system is not causal because for any time $t > 0$, the output depends on the future input at $2t$.

(f) $y(t) = (t+1)f(t)$.
Solution: The system is linear, because

$$(t+1)(f_1(t) + f_2(t)) = (t+1)f_1(t) + (t+1)f_2(t) = y_1(t) + y_2(t).$$

The system is not time invariant because a time shift at input gives $(t+1)f(t-t_0)$, while the time shift at output gives $(t-t_0+1)f(t-t_0)$.
The system is not BIBO stable because even if the input is bounded, $(t+1)$ is not.
The system is causal because output only depends on the current time from the input's perspective.

9. Fill out the following survey before this homework's deadline. We will use these data to improve the course over time.

   - Your grade for this HW problem will not depend on the content of your survey responses, only on the completion of the survey.
   - There are no correct or incorrect answers. Your grade for this HW problem will not depend on the content of your form responses, only on the completion of the form.
   - Course staff will not have access to the data until after the course is over and the data will be anonymized before course staff looks at it.
   - The link to the survey is here:
     https://forms.illinois.edu/sec/609188301
   - You do not need to submit an answer for this question on Gradescope, we will be notified of your submission and will upload the points to Canvas.
   - If you do not have access to the survey, please email Prof. Alvarez.

Exam 3 coming up

- Date: Wednesday, November 15, 7.15-8.30pm.
- Review: Tuesday, November 14, 7-9 p.m., 1013 ECEB.
- Coverage: Exam 3 will cover up to and including section 10.5 and homeworks 1-12, with emphasis on material after exam 2.
- We will provide the following tables: Fourier series, Fourier transform, convolution, impulse and important signals (6.1, 6.3, 7.1, 7.2, 9.1 and 9.3 from the textbook, or 1-6 from the online table handout)
- Room assignments have **changed**. Students with last names beginning with:
  * Aa - Cv will go to room ECEB 1015

    * Cw - Sv will go to room ECEB 1002
    * Sw - Zz will go to room ECEB 1013

- DRES students should have emailed Prof. Alvarez their DRES letters weeks ago and scheduled their exam as indicated by him.

