University of Illinois **Fall 2023**

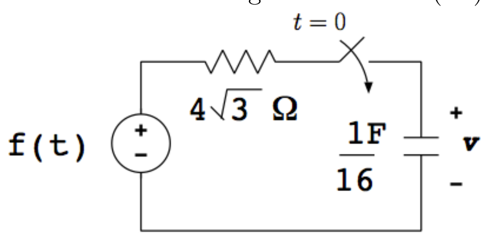
ECE 210/211 - Homework 5: Problems and Solutions

Due: Tuesday, September 26 by 11.59pm

Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
   sign: ______________________.

2. Consider the following circuit with $v(0^-) = 1\,V$ and let $f(t) = 2e^{-t/\sqrt{3}}V$.



For $t > 0$, obtain the following:

(a) the zero-state voltage across the capacitor's terminals, $v_{\text{ZS}}(t)$,
Solution: First we are going to find the ODE that governs the circuit. Applying KVL gives

$$2e^{-t/\sqrt{3}} \;=\; v \;+\; v_R \;=\; v \;+\; R\left(C\frac{dv}{dt}\right),$$

which yields the following ODE:

$$2e^{-t/\sqrt{3}} \;=\; v \;+\; \frac{4\sqrt{3}}{16}\,\frac{dv}{dt}.$$

To solve this ODE we start from the particular solution

$$v_p(t) \;=\; Be^{-t/\sqrt{3}}.$$

Taking the derivative yields

$$\frac{dv_p(t)}{dt} \;=\; \frac{-1}{\sqrt{3}}Be^{-t/\sqrt{3}}.$$

Substituting $v_p(t)$ and $\frac{dv_p(t)}{dt}$ into the ODE, we obtain

$$2e^{-t/\sqrt{3}} \;=\; Be^{-t/\sqrt{3}} \;+\; \frac{4\sqrt{3}}{16}\left(\frac{-1}{\sqrt{3}}Be^{-t/\sqrt{3}}\right),$$

yielding

$$B = \frac{8}{3}.$$

Therefore, the particular solution is

$$v_p(t) \;=\; \frac{8}{3}e^{-t/\sqrt{3}}\,V.$$

The homogeneous solution for the RC circuit has the form

$$v_h(t) \;=\; K_1e^{-t/\tau},$$

with

$$\tau \;=\; RC \;=\; \frac{4\sqrt{3}}{16} \;=\; \frac{\sqrt{3}}{4}\,\text{s}.$$

Adding the homogeneous and the particular solutions yields the general solution

$$v(t) \;=\; v_h(t) \;+\; v_p(t) \;=\; K_1 e^{-t/\tau} \;+\; \frac{8}{3}e^{-t/\sqrt{3}} \;=\; K_1 e^{-t4/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}}\,\text{V}.$$

From the general solution, at time $t = 0$, we can calculate the solution that satisfies $v(0^+) = v(0^-)$:

$$v(0^-) \;=\; K_1 \;+\; \frac{8}{3} \;\;\Rightarrow\;\; K_1 \;=\; v(0^-) \;-\; \frac{8}{3}.$$

This gives the full solution

$$v(t) \;=\; \left(v(0^-) \;-\; \frac{8}{3}\right) e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}}\,\text{V}.$$

To obtain the zero-state response, we simply set $v(0^-) = 0$:

$$v_{\text{ZS}}(t) \;=\; -\frac{8}{3}e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}}\,\text{V}.$$

(b) the zero-input voltage across the capacitor's terminals, $v_{\text{ZI}}(t)$,
Solution: The zero-input response can be obtained from

$$\begin{aligned} v_{ZI}(t) \;&=\; v(t) \;-\; v_{ZS}(t) \;=\; \left(v(0^-) \;-\; \frac{8}{3}\right) e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}} \;-\; \left(-\frac{8}{3}e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}}\right) \\ &=\; v(0^-)e^{-\frac{4t}{\sqrt{3}}} \;=\; e^{-\frac{4t}{\sqrt{3}}}\,\text{V}. \end{aligned}$$

(c) the transient voltage across the capacitor's terminals, $v_{\text{tr}}(t)$,
Solution: From the total response we notice that the transient response is composed of the terms that go to zero as $t \to \infty$:

$$v_{\text{tr}}(t) \;=\; e^{-\frac{4t}{\sqrt{3}}} \;-\; \frac{8}{3}e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}} \;=\; -\frac{5}{3}e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}}\,\text{V}$$

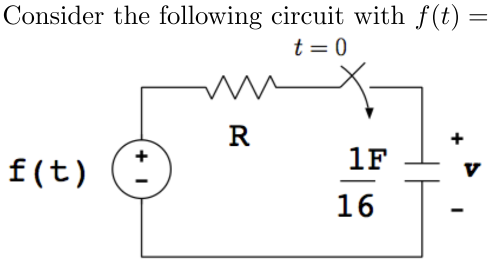
(d) the steady state voltage across the capacitor's terminals, $v_{\text{ss}}(t)$,
Solution: There are no components left after the transient response goes to zero, hence

$$v_{\text{ss}}(t) = 0\,\text{V}.$$

(e) the total voltage across the capacitor's terminals, $v(t)$.
Solution: The total voltage across the capacitor is the sum of the zero-input and the zero-state responses. Also, it is the sum of transient response and steady state response:

$$v(t) \;=\; v_{\text{ZI}}(t) \;+\; v_{\text{ZS}}(t) \;=\; v_{tr}(t) \;+\; v_{\text{ss}}(t) \;=\; -\frac{5}{3}e^{-4t/\sqrt{3}} \;+\; \frac{8}{3}e^{-t/\sqrt{3}}\,\text{V}.$$

3. Consider the following circuit with $f(t) = 2\cos(\omega t)$ V and $v\,(0^-) = v_0$ V.



It is known that for $t > 0$,

$$v(t) \;=\; Be^{-\frac{4t}{\sqrt{3}}} \;+\; \frac{1}{2}\cos(4t) \;+\; \frac{\sqrt{3}}{2}\sin(4t)\;\;\text{V}$$

.

(a) Write the ODE that governs this system for $t > 0$ in terms of $R,\ v(t)$ and ω.
Solution: To write the ODE that governs this system for $t > 0$, we apply KVL which gives

$$f(t) \ = \ v \ + \ v_R \ = \ v \ + \ R\left(C\frac{dv}{dt}\right).$$

This results in the following ODE:

$$2\cos(\omega t) \ = \ v(t) \ + \ \frac{R}{16}\left(\frac{dv(t)}{dt}\right).$$

(b) Determine the value of R.
Solution: Given

$$v(t) \ = \ Be^{-\frac{4t}{\sqrt{3}}} \ + \ \frac{1}{2}\cos(4t) \ + \ \frac{\sqrt{3}}{2}\sin(4t) \ = \ Be^{-\frac{t}{\tau}} \ + \ \frac{1}{2}\cos(4t) \ + \ \frac{\sqrt{3}}{2}\sin(4t),$$

we have the time constant

$$\tau \ = \ \frac{\sqrt{3}}{4} \ = \ RC \ = \ \frac{R}{16}.$$

Therefore, we have

$$R \ = \ 4\sqrt{3}\ \Omega.$$

Alternative: Take derivative of $v(t)$, we get

$$\frac{dv(t)}{dt} \ = \ \frac{-4B}{\sqrt{3}}e^{-\frac{4t}{\sqrt{3}}} \ - \ 2\sin(4t) \ + \ 2\sqrt{3}\cos(4t).$$

Substitute the values of v and dv/dt into simplify the ODE from part(a),

$$2\cos(\omega t) \ = \ \left(1+\frac{-R}{4\sqrt{3}}\right)Be^{-\frac{4t}{\sqrt{3}}} \ + \ \left(\frac{1}{2}+\frac{R\sqrt{3}}{8}\right)\cos(4t) \ + \ \left(\frac{\sqrt{3}}{2}-\frac{R}{8}\right)\sin(4t).$$

This identity imposes the constraint

$$1+\frac{-R}{4\sqrt{3}} \ = \ 0 \ \Rightarrow \ R \ = \ 4\sqrt{3}\ \Omega.$$

(c) What is the value of ω?
Solution: In order to have the ODE in part (a) to be valid, we need

$$\omega \ = \ 4\text{rad/s}$$

so that both input and output have the same frequency.
This can be verified by substituting $R = 4\sqrt{3}\ \Omega$ in the ODE of part (b).

(d) If $v(t)$ is the zero-state response, what is the value of the constant B ?
Solution: The zero-state response is defined as the solution that satisfies

$$v(0^+) \ = \ v(0^-) \ = \ 0.$$

If so, we have

$$v(0) \ = \ B \ + \ \frac{1}{2} \ = \ 0 \ \Rightarrow \ B \ = \ -\frac{1}{2}.$$

(e) Identify $v_{tr}(t)$, the transient component of $v(t)$.
Solution: The zero-input response is the exponential decaying term

$$v_{tr}(t) \ = \ \left(v_0 \ - \ \frac{1}{2}\right)e^{\frac{-4t}{\sqrt{3}}}.$$

(f) Identify $v_{ss}(t)$, the steady-state component of $v(t)$.
Solution: The steady state of $v(t)$ is defined as the value when $t \to \infty$.
Since the term $Ae^{-\frac{4t}{\sqrt{3}}}$ vanishes as $t \to \infty$, and the remainder is the steady state response, we have:

$$v_{\text{ss}}(t) \;=\; \frac{1}{2}\cos(4t) + \frac{\sqrt{3}}{2}\sin(4t)\,.$$

(g) What is the steady-state phasor V ?
Solution: In order to find the phasor V, we modify the expression as

$$v_{\text{ss}}(t) \;=\; \frac{1}{2}\cos(4t) \;+\; \frac{\sqrt{3}}{2}\cos\left(4t - \frac{\pi}{2}\right).$$

In the phasor form, we have

$$V \;=\; \frac{1}{2} \;+\; \frac{\sqrt{3}}{2}e^{-j\frac{\pi}{2}} \;=\; \frac{1}{2} \;-\; \frac{\sqrt{3}}{2}j \;=\; e^{j\left(-\arctan\left(\sqrt{3}\right)\right)} \;=\; e^{-j\frac{\pi}{3}}.$$

4. The different parts of this problem can be solved independently.

(a) Show that $\frac{e^{j4t}+e^{-j4t}}{2} \;=\; \sin(4t+\frac{\pi}{2})$.
Solution:

$$\begin{aligned}\frac{e^{j4t}+e^{-j4t}}{2} \;&=\; \frac{1}{2}\left(\cos(4t) + j\,\sin(4t) + \cos(-4t) + j\,\sin(-4t)\right)\\ &=\; \frac{1}{2}\left(\cos(4t) + j\,\sin(4t) + \cos(4t) - j\,\sin(4t)\right) \;=\; \frac{1}{2}\left(2\cos(4t)\right) \;=\; \cos(4t)\,,\end{aligned}$$

However, we know that $\cos(x) = \sin(x+\pi/2)$, for all real x. Therefore

$$\frac{e^{j4t} \;+\; e^{-j4t}}{2} \;=\; \sin\left(4t + \frac{\pi}{2}\right).$$

(b) Express $\frac{e^{-j2t}-e^{j2t}}{j}$ in terms of a sine function.
Solution:

$$\begin{aligned}\frac{e^{-j2t}-e^{j2t}}{j} \;&=\; \frac{1}{j}\left(\cos(-2t) + j\,\sin(-2t) - \cos(2t) - j\,\sin(2t)\right)\\ &=\; \frac{1}{j}\left(\cos(2t) - j\,\sin(2t) - \cos(2t) - j\,\sin(2t)\right) \;=\; \frac{1}{j}\left(-2j\,\sin(2t)\right) \;=\; -2\sin(2t)\end{aligned}$$

(c) Express $\text{Re}\{2e^{j\frac{\pi}{3}}e^{-j5t}\}$ in terms of a cosine function.
Solution:

$$\text{Re}\{2e^{j\frac{\pi}{3}}e^{-j5t}\} \;=\; \text{Re}\{2e^{j\left(-5t+\frac{\pi}{3}\right)}\} \;=\; \text{Re}\{2\left(\cos\left(-5t+\frac{\pi}{3}\right) \;+\; j\sin\left(-5t+\frac{\pi}{3}\right)\right)\} \;=\; 2\cos\left(-5t+\frac{\pi}{3}\right).$$

(d) Determine the phasor F of $f(t) \;=\; -2\sin(2t-\frac{\pi}{3})$ and express it in both polar and rectangular coordinates.
Solution:

$$f(t) \;=\; -2\sin(2t-\frac{\pi}{3}) \;=\; 2\cos\left(2t+\frac{\pi}{2}-\frac{\pi}{3}\right) \;=\; 2\cos\left(2t+\frac{\pi}{6}\right).$$

The magnitude of the cosine signal is

$$|F| \;=\; 2.$$

The phase shift is

$$\angle F \;=\; \frac{\pi}{6}\,\text{rad}.$$

Therefore the phasor is:

$$F \;=\; 2e^{j\frac{\pi}{6}} \;=\; \sqrt{3} \;+\; j.$$

(e) Determine the phasor F of $f(t) = \cos(3t - \frac{\pi}{2})$. Express F in both polar and rectangular coordinates.

Solution:

$$f(t) \;=\; \cos(3t - \frac{\pi}{2}).$$

The magnitude of the cosine signal is

$$|F| \;=\; 1.$$

The phase shift is

$$\angle F \;=\; -\frac{\pi}{2}\,\text{rad}.$$

Therefore the phasor is:

$$F \;=\; 1e^{j\frac{-\pi}{2}} \;=\; -j.$$

(f) Express the phasor $F = 2 + j2$ in terms of a cosine function $f(t)$ having frequency $\omega = 3\,\frac{\text{rad}}{\text{s}}$.

Solution:

$$F \;=\; 2 \;+\; j2 \;=\; \sqrt{2^2 + 2^2}e^{j\arctan\left(\frac{2}{2}\right)} = 2\sqrt{2}e^{j\frac{\pi}{4}}.$$

Therefore,

$$f(t) \;=\; 2\sqrt{2}\cos\left(3t + \frac{\pi}{4}\right).$$

(g) Express the phasor $F = 3e^{-j\frac{\pi}{3}}$ in terms of a cosine function $f(t)$ having frequency $\omega = 3\,\frac{\text{rad}}{\text{s}}$.

Solution:

$$|F| \;=\; 3,\; \angle F \;=\; -\frac{\pi}{3}\,\text{rad}.$$

Therefore,

$$f(t) \;=\; 3\cos\left(3t - \frac{\pi}{3}\right).$$

5. Use the phasor method to express the following signals in terms of a single cosine function:

(a) $f(t) = -3\sin(3t) - 3\cos(3t)$.

Solution:

$$f(t) \;=\; -3\sin(3t) \;-\; 3\cos(3t) \;=\; 3\cos\left(3t + \frac{\pi}{2}\right) \;-\; 3\cos\left(3t\right).$$

In phasor form

$$F \;=\; 3e^{j\frac{\pi}{2}} \;-\; 3 \;=\; -3 \;+\; j3 \;=\; 3\sqrt{2}e^{j\left(\frac{3\pi}{4}\right)},$$

which yields the time-domain form:

$$f(t) \;=\; 3\sqrt{2}\cos\left(3t + \frac{3\pi}{4}\right).$$

(b) $g(t) = 2\left[\sin(2t) + \sin(2t + \pi/2)\right]$.

Solution:

$$g(t) \;=\; 2\left[\sin(2t) \;+\; \sin(2t + \pi/2)\right] \;=\; 2\cos\left(2t - \frac{\pi}{2}\right) \;+\; 2\cos\left(2t\right).$$

In phasor form

$$F \;=\; 2e^{-j\frac{\pi}{2}} \;+\; 2 \;=\; -j2 \;+\; 2 \;=\; 2\sqrt{2}e^{-j\frac{\pi}{4}},$$

which yields

$$f(t) \;=\; 2\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right).$$

6. Fill out the following survey. We will use these data to improve the course over time.

- NOTE: this survey will be released once Exam 1 grades are released, The deadline to submit the HW might be extended, depending on when the exam grades are released. We will send out an email notifying you that the survey is released.

- Your grade for this HW problem will not depend on the content of your survey responses, only on the completion of the survey.
- There are no correct or incorrect answers. Your grade for this HW problem will not depend on the content of your form responses, only on the completion of the form.
- The link to the survey is here:
  https://forms.illinois.edu/sec/1654712151
- You do not need to submit an answer for this question on Gradescope, we will be notified of your submission and will upload the points to Canvas. Be sure to enter your netID correctly into the survey so your submission is recorded.
- If you do not have access to the survey, please email Prof. Alvarez.

