University of Illinois **Fall 2023**

ECE 210/211 - Homework 9: Problems and Solutions

Due: Tuesday, October 24 by 11.59pm

Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
   sign: ______________________.

2. Consider the signal

$$f(t) = \begin{cases} e^t & -1 \le t \le 1 \\ 0 & \text{else.} \end{cases}$$

   Obtain its Fourier transform, $F(\omega)$, via integration.

   Solution: By definition

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt = \int_{-1}^{1} e^t e^{-j\omega t} dt = \int_{-1}^{1} e^{(1-j\omega)t} dt = \left[\frac{e^{(1-j\omega)t}}{1-j\omega}\right]_{-1}^{1} = \frac{e^{1-j\omega} - e^{-1+j\omega}}{1-j\omega}$$

3. Consider the function

$$G(\omega) = \begin{cases} e^{-\omega} & -1 \le \omega \le 1 \\ 0 & else. \end{cases}$$

   (a) Obtain its inverse Fourier transform, $g(t)$, via integration.

   Solution: By definition

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{-\omega} e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{(-1+jt)\omega} d\omega = \frac{1}{2\pi}\left[\frac{e^{(-1+jt)\omega}}{-1+jt}\right]_{-1}^{1}$$

$$= \frac{1}{2\pi}\frac{\left(e^{-1+jt} - e^{1-jt}\right)}{-1+jt} = \frac{1}{2\pi}\frac{\left(e^{1-jt} - e^{-1+jt}\right)}{1-jt}$$

   (b) If possible, express $g(t)$ in terms of $F(t)$ from the previous problem, where $F(\omega)$ is the Fourier transform from the previous problem. If not possible, indicate why not.

   Solution: From problem 2,

$$F(t) = \frac{e^{1-jt} - e^{-1+jt}}{1-jt}$$

   Now comparing $g(t)$ of problem 3(a) with above expression of $F(t)$,

$$g(t) = \frac{1}{2\pi} F(t)$$

4. It is known that the signal $f(t) = \text{rect}(t)\cos(\alpha t)$, has Fourier transform

$$F(\omega) = \frac{1}{2}\left(\text{sinc}\left(\frac{\omega-\alpha}{2}\right) + \text{sinc}\left(\frac{\omega+\alpha}{2}\right)\right),$$

   where α is a real-valued constant. Use the symmetry property of the Fourier transform to determine $g(t)$, the inverse Fourier transform of the function $G(\omega) = \text{rect}(\omega)\cos(\alpha\omega)$.

Solution:

$$F(t) = \frac{1}{2}\left\{\text{sinc}\left(\frac{t-\alpha}{2}\right) + \text{sinc}\left(\frac{t+\alpha}{2}\right)\right\}$$

$$f(-\omega) = \text{rect}(-\omega)\cos(-\alpha\omega)$$

$f(-\omega)$ is an even function. Hence,

$$f(-\omega) = \text{rect}(\omega)\cos(\alpha\omega) = G(\omega)$$

Using the symmetry property from Table 7.1,

$$F(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi f(-\omega)$$

$$\frac{1}{2}\left\{\text{sinc}\left(\frac{t-\alpha}{2}\right) + \text{sinc}\left(\frac{t+\alpha}{2}\right)\right\} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\text{rect}(\omega)\cos(\alpha\omega)$$

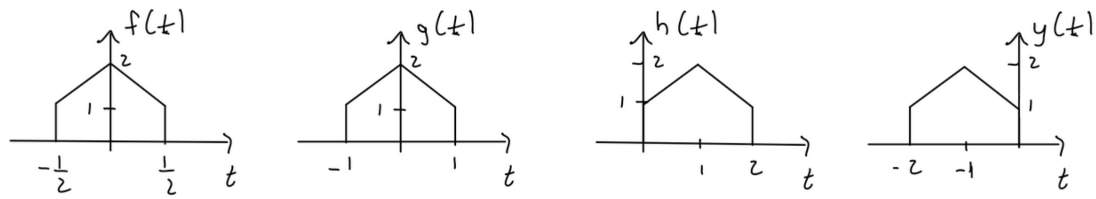
Using amplitude scaling,

$$\frac{1}{4\pi}\left\{\text{sinc}\left(\frac{t-\alpha}{2}\right) + \text{sinc}\left(\frac{t+\alpha}{2}\right)\right\} \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(\omega)\cos(\alpha\omega) = G(\omega)$$

Therefore, inverse Fourier transform of $G(\omega)$ is,

$$g(t) = \frac{1}{4\pi}\left\{\text{sinc}\left(\frac{t-\alpha}{2}\right) + \text{sinc}\left(\frac{t+\alpha}{2}\right)\right\}$$

5. Several time signals are represented in the figure below. All parts of this problem will use properties of Fourier transform.



(a) Determine $F(\omega)$, the Fourier transform of the signal $f(t)$.
Solution: $f(t)$ can be decomposed into sum of unit ractangle and unit triangle as,

$$f(t) = \text{rect}(t) + \Delta(t)$$

Using Fourier transform pair from Table 7.2, $\text{rect}(t) \leftrightarrow \text{sinc}\left(\frac{\omega}{2}\right)$ and $\Delta(t) \leftrightarrow \frac{1}{2}\text{sinc}^2\left(\frac{\omega}{4}\right)$
Therefore,

$$F(\omega) = \text{sinc}\left(\frac{\omega}{2}\right) + \frac{1}{2}\text{sinc}^2\left(\frac{\omega}{4}\right)$$

(b) Express $G(\omega)$, the Fourier transform of the signal $g(t)$ in terms of $F(\omega)$ from part (a).
Solution:

$$g(t) = f\left(\frac{t}{2}\right)$$

Using time-scaling property of Fourier transform from Table 7.1,

$$G(\omega) = \frac{1}{\left|{}^1\!/\!_2\right|}F(2\omega) = 2F(2\omega)$$

(c) Express $H(\omega)$, the Fourier transform of the signal $h(t)$ in terms of $F(\omega)$ from part (a).

Solution:

$$h(t) = g(t-1) = f\left(\frac{t-1}{2}\right)$$

Using time-shift property of Fourier transform from Table 7.1 and the result from part (b),

$$H(\omega) = G(\omega)\, e^{-j\omega} = 2F(2\omega)\, e^{-j\omega}$$

(d) Use the time-scaling property to express $Y(\omega)$, the Fourier transform of the signal $y(t)$ in terms of $H(\omega)$ from part (c).

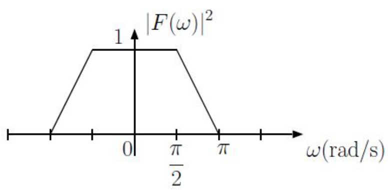
Solution:

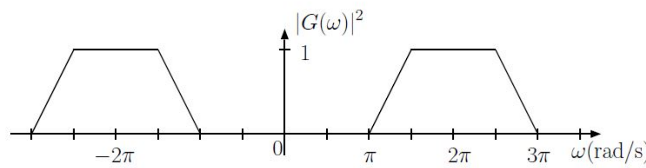
$$y(t) = h(-t) = g(-t-1) = f\left(\frac{-t-1}{2}\right)$$

Using time-scaling property of Fourier transform and the result from part (c),

$$Y(\omega) = \frac{1}{|-1|} H(-\omega) = 2F(-2\omega)\, e^{j\omega}$$

6. Consider the signals $f(t)$ and $g(t)$ with the following energy spectra:





(a) Determine the 3-dB bandwidth of $f(t)$.

Solution: The linear part of the energy spectrum $|F(\omega)|^2$ goes from the maximum value to the minimum value, hence a drop of half the amplitude requires moving half way horizontally, which is a distance $\frac{\pi}{4}$. Therefore

$$\Omega_{3dB} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \text{ rad/s.}$$

(b) Determine the 3-dB bandwidth of $g(t)$.

Solution: Each trapezoid is identical to the one for part (a). Hence, if we consider centering it at $\pm 2\pi$ we would get a corresponding low-pass 3dB bandwidth of $\frac{3\pi}{4}$ rad/s. To get the 3dB bandwidth of the band-pass version, we simply double it and obtain

$$\Omega_{3dB} = 2\left(\frac{3\pi}{4}\right) = \frac{3\pi}{2} \text{ rad/s.}$$

(c) Determine the 95%-bandwidth of $f(t)$.

Solution: First, the energy content is the area of the trapezoid divided by 2π:

$$W = \frac{3\pi/2}{2\pi} = \frac{3}{4}$$

and we can focus just on the positive frequency side, where there is half the energy:

$$W_+ = 3/8.$$

Clearly, the 95% bandwidth must be in the range $\left(\frac{\pi}{2}, \pi\right)$ and the remaining 5% of the nergy will be in the small triangle near $\omega = \pi$. We focus on finding where that 5% of the energy in the positive frequency occurs, based on the equation of the decreasing line

$$|F(\omega)|^2 = -\frac{2}{\pi}\omega + 2$$

and the area of a triangle:

$$\begin{aligned}
\frac{\text{area of triangle}}{2\pi} &= \frac{5}{100} W_+ \\
frac(\pi - \Omega_{95}) \left(-\frac{2}{\pi}\Omega_{95} + 2 \right) 2(2\pi) &= \frac{5}{100} \left(\frac{3}{8} \right)
\end{aligned}$$

Solving the equation we obtain

$$\Omega_{95} = \frac{\pi}{4} \left(4 - \sqrt{\frac{3}{5}} \right) \text{ rad/s.}$$

(d) Determine the 95%-bandwidth of $g(t)$.

Solution: Each trapezoid is identical to the one for part (c). Hence, if we consider centering it at $\pm\, 2\pi$ we would get a corresponding 95% bandiwdth of

$$\frac{\pi}{4} \left(4 - \sqrt{\frac{3}{5}} \right) \text{ rad/s.}$$

To get the 95% bandwidth of the band-pass version, we simply double it and obtain

$$\Omega_{95} = 2 \left(\frac{\pi}{4} \left(4 - \sqrt{\frac{3}{5}} \right) \right) = \frac{\pi}{2} \left(4 - \sqrt{\frac{3}{5}} \right) \text{ rad/s.}$$

