**University of Illinois** **Fall 2023**

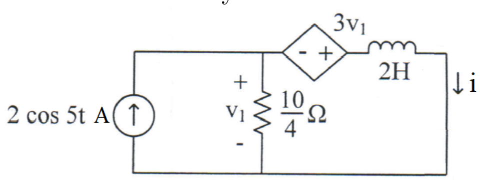
# ECE 210/211 - Homework 6: Problems and Solutions

**Due:** Tuesday, October 3 by 11.59pm

**Problems:**

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
sign: ______________________________.

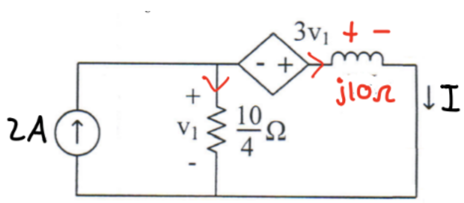
2. Consider the steady-state RL circuit below.




(a) Determine the average absorbed power at each element.

**Solution:** In order to calculate absorbed power in all elements, we need to obtain the equivalent transformed phasor circuit.
The corresponding impedance is:

$$Z_L \;=\; j\omega L \;=\; j(5)(2) \;=\; j10\ \Omega.$$




Then, calculate the phasors $V_1$ and $I$.
Using a KCL at the top-left node:

$$2 \;=\; \frac{V_1}{10/4} \;+\; I \quad\Rightarrow\quad I \;=\; 2 \;-\; \frac{2V_1}{5}.$$

Then, a KVL on the right-side loop:

$$-V_1 \;-\; 3V_1 \;+\; j10I \;=\; 0$$

$$\Rightarrow\quad V_1 \;=\; j\frac{10}{4}I \;=\; j\frac{5}{2}\left(2-\frac{2V_1}{5}\right) \quad\Rightarrow\quad V_1 \;=\; \frac{j5}{1+j} \;=\; \frac{5e^{j\pi/2}}{\sqrt{1^2+1^2}e^{j\pi/4}} \;=\; \frac{5}{\sqrt{2}}e^{j\pi/4}\ \text{V}.$$

Hence,

$$I \;=\; \frac{2}{1+j} \;=\; \frac{2}{\sqrt{1^2+1^2}e^{j\pi/4}} \;=\; \sqrt{2}\,e^{-j\pi/4}\text{A}.$$

Now, we can calculate the absorbed power at each element using SRS:

$$P_{2A} \;=\; \frac{1}{2}\text{Re}\left\{V_{2A}I_{2A}^{*}\right\} \;=\; \frac{1}{2}\text{Re}\left\{V_1(-2)^{*}\right\} \;=\; \frac{1}{2}\text{Re}\left\{\left(\frac{5}{\sqrt{2}}e^{j\pi/4}\right)(-2)\right\} \;=\; -\frac{5}{\sqrt{2}}\cos(\pi/4) \;=\; -\frac{5}{2}\ \text{W}.$$

$$P_{2H} \;=\; 0 \text{ W because inductors always have zero absorbed power with phasors.}$$

$$\begin{aligned} P_{3v_1} &= \frac{1}{2}\mathrm{Re}\left\{V_{3v_1} I^*_{3v_1}\right\} \;=\; \frac{1}{2}\mathrm{Re}\left\{3V_1(-I)^*\right\} \;=\; \frac{1}{2}\mathrm{Re}\left\{3\left(\frac{5}{\sqrt{2}}e^{j\pi/4}\right)\left(-\sqrt{2}\;e^{-j\pi/4}\right)^*\right\} \\ &= \frac{15}{2}\mathrm{Re}\left\{e^{j\pi/2}\right\} = \frac{15}{2}\cos(\pi/2) = 0 \text{ W} \end{aligned}$$

$$P_{10/4\Omega} \;=\; \frac{|V|^2}{2R} \;=\; \frac{|V_1^2|}{2(10/4)} \;=\; \frac{\left(\frac{5}{\sqrt{2}}\right)^2|}{5} \;=\; \frac{25}{2(5)} = \frac{5}{2} \text{ W.}$$

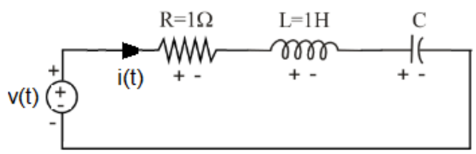
(b) Determine the voltage $v_1(t)$.
**Solution:** From part (a):

$$V_1 \;=\; \frac{5}{\sqrt{2}}e^{j\pi/4} \text{ V} \quad\Rightarrow\quad v_1(t) \;=\; \frac{5}{\sqrt{2}}\cos(5t+\pi/4) \;=\; \frac{5\sqrt{2}}{2}\cos(5t+\pi/4) \text{ V.}$$

(c) Determine the current $i(t)$.
**Solution:** From part (a):

$$I \;=\; \sqrt{2}\;e^{-j\pi/4} \text{ A} \quad\Rightarrow\quad i(t) \;=\; \sqrt{2}\cos(5t-\pi/4) \text{ A.}$$

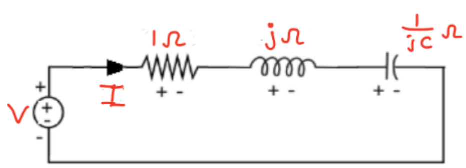
3. The steady-state circuit below is fed by a cosinusoidal voltage source $v(t) = 2\cos\left(t+\frac{\pi}{4}\right)$ $V$.




(a) Determine the phasor current, $I$, in terms of the capacitance, $C$.
**Solution:** In order to calculate the phasor current, we need to obtain the equivalent transformed phasor circuit. The corresponding impedances are:

$$Z_L \;=\; j\omega L \;=\; j(1)(1) \;=\; j\ \Omega.$$

$$Z_C \;=\; \frac{1}{j\omega C} \;=\; \frac{1}{j(1)C} \;=\; \frac{1}{jC} \;=\; \frac{-j}{C}\ \Omega.$$




Using Ohm's law on the series impedance:

$$I \;=\; \frac{V}{1+j-\frac{j}{C}} \;=\; \frac{V}{1+j\left(1-\frac{1}{C}\right)} \;=\; \frac{V\left(1-j\left(1-\frac{1}{C}\right)\right)}{1^2+\left(1-\frac{1}{C}\right)^2} \;=\; \frac{V\left(1-j\left(1-\frac{1}{C}\right)\right)}{1+\left(1-\frac{1}{C}\right)^2} \text{ A.}$$

(b) If $v_L(t) = -v_C(t)$, what is the value of the capacitance, $C$, and what is the steady-state voltage $v_R(t)$ ?
**Solution:** Given $v_L(t) = -v_C(t)$ implies that $v_R(t) = v(t)$ by KVL, hence

$$v_R(t) \;=\; 2\cos\left(t+\frac{\pi}{4}\right) \text{ V.}$$

Also, since the current is the same on the inductor and the capacitor,

$$V_L \;=\; -V_C \quad\Rightarrow\quad Z_L I \;=\; -Z_C I \quad\Rightarrow\quad Z_L \;=\; -Z_C \quad\Rightarrow\quad j = \frac{-j}{C} \quad\Rightarrow\quad C = 1 \text{ F.}$$

(c) If the current leads (peaks earlier than) the voltage at the source, $v(t)$, by $\frac{\pi}{4}$rad, what is the value of the capacitance, $C$?

**Solution:** The current leading the voltage at the source by $\frac{\pi}{4}$rad means

$$I = Vke^{j\pi/4},$$

for some positive real-valued constant $k$.

$$I = \frac{V\left(1 - j\left(1 - \frac{1}{C}\right)\right)}{1 + \left(1 - \frac{1}{C}\right)^2} = Vke^{j\pi/4}$$

$$\Rightarrow \quad \angle\left(1 - j\left(1 - \frac{1}{C}\right)\right) = \frac{\pi}{4} \quad \Rightarrow \quad -\left(1 - \frac{1}{C}\right) = 1 \quad \Rightarrow \quad C = \frac{1}{2}\text{ F}.$$

(d) Let $C = \frac{1}{2}F$.

i. Determine the voltage at each element, $v_R(t)$, $v_L(t)$ and $v_C(t)$.

**Solution:** From part (a):

$$\begin{aligned} I &= \frac{V\left(1 - j\left(1 - \frac{1}{C}\right)\right)}{1 + \left(1 - \frac{1}{C}\right)^2} = \frac{V\left(1 - j\left(1 - \frac{1}{1/2}\right)\right)}{1 + \left(1 - \frac{1}{1/2}\right)^2} = \frac{V(1+j)}{1+1} = \frac{V\sqrt{1^1 + 1^2}e^{j\pi/4}}{2} \\ &= \frac{2e^{j\pi/4}\sqrt{1^1 + 1^2}e^{j\pi/4}}{2} = \sqrt{2}e^{j\pi/2}\text{ A}. \end{aligned}$$

Hence,

$$V_R = RI = (1)\left(\sqrt{2}e^{j\pi/2}\right) = \sqrt{2}e^{j\pi/2} \quad \Rightarrow \quad v_R(t) = \sqrt{2}\cos(t + \pi/2)\text{ V}.$$

$$V_L = Z_L I = (j)\left(\sqrt{2}e^{j\pi/2}\right) = \sqrt{2}e^{j\pi} \quad \Rightarrow \quad v_L(t) = \sqrt{2}\cos(t + \pi) = -\sqrt{2}\cos(t)\text{ V}.$$

$$V_C = Z_C I = \left(\frac{-j}{C}\right)\left(\sqrt{2}e^{j\pi/2}\right) = (-j2)\left(\sqrt{2}e^{j\pi/2}\right) = 2\sqrt{2}e^{j0} \quad \Rightarrow \quad v_C(t) = 2\sqrt{2}\cos(t)\text{ V}.$$

ii. Calculate the average absorbed power at each element.

**Solution:** We can calculate the absorbed power at each element using SRS:

$$\begin{aligned} P_{v(t)} &= \frac{1}{2}\text{Re}\left\{V_{v(t)}I^*_{v(t)}\right\} = \frac{1}{2}\text{Re}\left\{V(-I)^*\right\} = \frac{1}{2}\text{Re}\left\{V\left(-\frac{V(1+j)}{2}\right)^*\right\} \\ &= \frac{-|V|^2}{2}\text{Re}\{1 - j\} = \frac{-|V|^2}{2} = \frac{-\left(2^2\right)}{2} = -1\text{ W}. \end{aligned}$$

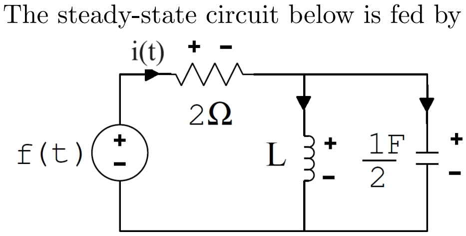
$$P_{1H} = 0\text{ W because inductors always have zero absorbed power with phasors.}$$

$$P_C = 0\text{ W because capacitors always have zero absorbed power with phasors.}$$

By energy conservation,

$$P_{1\Omega} = 1\text{ W}$$

4. The steady-state circuit below is fed by a cosinusoidal voltage source $f(t) = 3\sin\left(2t + \frac{\pi}{6}\right)$V.

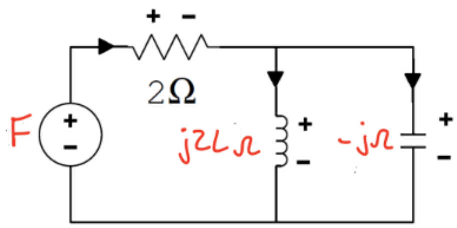


(a) Determine the phasor current, $I$, in terms of the inductance, $L$.

**Solution:** In order to calculate the phasor current, we need to obtain the equivalent transformed phasor circuit. The corresponding impedances are:

$$Z_L = j\omega L = j(2)(L) = j2L\ \Omega.$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j(2)(1/2)} = \frac{1}{j} = -j\ \Omega.$$




In order to obtain the phasor current, let's start with a parallel impedance

$$Z_p = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{(j2L)(-j)}{j2L + (-j)} = \frac{2L}{j(2L-1)} = \frac{j2L}{1-2L}\ \Omega.$$

The current through the $2\Omega$ resistor can be obtained using Ohm's law on the series impedance of resistor and $Z_p$:

$$I_{2\Omega} = \frac{F}{2+Z_p} = \frac{F}{2+\frac{j2L}{1-2L}} = \frac{F(1-2L)}{2-4L+j2L}\ \text{A}.$$

(b) If $i_L(t) = -i_C(t)$, what is the value of the inductance, $L$, and what is the steady-state current $i_R(t)$ ?

**Solution:** Given $i_L(t) = -i_C(t)$, then, by KCL $i_R(t) = 0$ A.
Also, since the voltage is the same on the inductor and the capacitor,

$$I_L = -I_C \quad \Rightarrow \quad \frac{V_L}{Z_L} = -\frac{V_L}{Z_C} \quad \Rightarrow \quad Z_C = -Z_L \quad \Rightarrow \quad -j = -(j2L) \quad \Rightarrow \quad L = \frac{1}{2}\ \text{H}.$$

(c) If the voltage at the source, $f(t)$, lags (peaks later than) the voltage at the inductor, $v_L(t)$, by $\frac{\pi}{4}$ rad, what is the value of the inductance, $L$?

**Solution:** The voltage at the source lagging the voltage at the inductor by $\frac{\pi}{4}$ rad means that

$$F = V_L k e^{-j\pi/4}$$

for some positive real-valued constant $k$.

$$\begin{aligned} F &= (Z_p I) k e^{-j\pi/4} = k\left(\frac{j2L}{1-2L}\right)\left(\frac{F(1-2L)}{2-4L+j2L}\right)e^{-j\pi/4} = Fk\left(\frac{j2L}{2-4L+j2L}\right)e^{-j\pi/4} \\ &= Fk\left(\frac{j2L}{2-4L+j2L}\right)\left(\frac{2-4L-j2L}{2-4L-j2L}\right)e^{+j\pi/4} = Fk\left(\frac{4L^2+j2L(2-4L)}{(2-4L)^2+(2L)^2}\right)e^{+j\pi/4} \end{aligned}$$

$$\Rightarrow \frac{\pi}{4} = \angle\left(\frac{4L^2+j2L(2-4L)}{(2-4L)^2+(2L)^2}\right) = \angle\left(4L^2+j2L(2-4L)\right)$$

$$\Rightarrow 4L^2 = 2L(2-4L) \quad \Rightarrow \quad 2L = 2-4L \quad \Rightarrow 4L^2 = 2L(2-4L) \quad \Rightarrow \quad L = \frac{1}{3}\ \text{H}.$$

(d) Let $L = 1$H.

i. Determine the voltage at each element, $v_R(t),\ v_L(t)$ and $v_C(t)$.
**Solution:** By voltage division,

$$\begin{aligned}
V_{2\Omega} &= F\frac{2}{2+Z_p} = F\frac{2}{2+\frac{j2L}{1-2L}} = 3e^{j(\pi/6-\pi/2)}\frac{2}{2+\frac{j2(1)}{1-2(1)}} = 3e^{-j\pi/3}\frac{2}{2-j2} \\
&= \frac{3}{\sqrt{1^2+(-1)^2}}e^{-j\pi/3+j\pi/4} = \frac{3}{\sqrt{2}}e^{-j\pi/12}\text{ V.}
\end{aligned}$$

$$\Rightarrow \quad v_R(t) = \frac{3}{\sqrt{2}}\cos(2t-\pi/12)\text{ V.}$$

Similarly, due to the inductor and capacitor being in parallel:

$$\begin{aligned}
V_L = V_C &= F\frac{Z_p}{2+Z_p} = F\frac{\frac{j2L}{1-2L}}{2+\frac{j2L}{1-2L}} = 3e^{j(\pi/6-\pi/2)}\frac{\frac{j2(1)}{1-2(1)}}{2+\frac{j2(1)}{1-2(1)}} = 3e^{-j\pi/3}\frac{j2}{-2+j2} \\
&= \frac{3}{\sqrt{(-1)^2+1^2}}e^{-j\pi/3+j\pi/2-j3\pi/4} = \frac{3}{\sqrt{2}}e^{-j7\pi/12}\text{ V.}
\end{aligned}$$

$$\Rightarrow \quad v_L(t) = v_C(t) = \frac{3}{\sqrt{2}}\cos(2t-7\pi/12)\text{ V.}$$

ii. Determine the average absorbed power at each element.
**Solution:** The average absorbed powers can be obtained as:

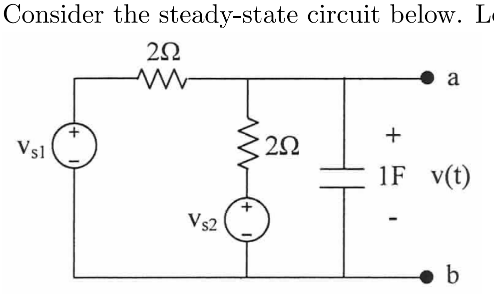
$$\begin{aligned}
P_L &= 0\text{ W because inductors always have zero absorbed power with phasors.} \\
P_C &= 0\text{ W because capacitors always have zero absorbed power with phasors.}
\end{aligned}$$

$$P_{2\Omega} = \frac{|V|^2}{2R} = \frac{\left(\frac{3}{\sqrt{2}}\right)^2}{2(2)} = \frac{9}{8}\text{ W.}$$

By energy conservation

$$P_{f(t)} = -P_{2\Omega} = -\frac{9}{8}\text{ W.}$$

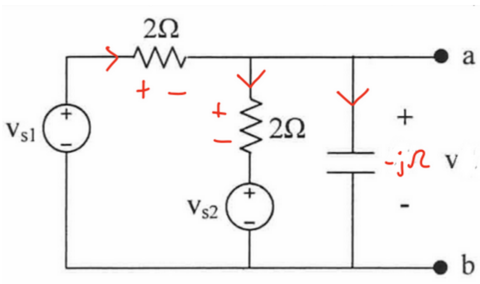
5. Consider the steady-state circuit below. Let $V_{s1} = \cos(t)$ and $V_{s2} = \sin(t)$.




(a) Determine the phasor Thevenin voltage, $V_T$.
**Solution:** In order to obtain the phasor Thevenin voltage, we need to obtain the equivalent transformed phasor circuit. The corresponding impedance is:

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j(1)(1)} = \frac{1}{j} = -j\ \Omega.$$




Using superposition, shorting $V_{s_1}$, the capacitor and resistor are in parallel, hence

$$Z_p = \frac{2(Z_C)}{2+Z_C} = \frac{2(-j)}{2+(-j)} = \frac{-2j}{2-j}\ \Omega.$$

Then, by voltage division, the contribution from $V_{s_2}$ is

$$\begin{aligned} V_{T,2} &= V_{s_2}\left(\frac{Z_P}{2+Z_p}\right) = V_{s_2}\left(\frac{\frac{-2j}{2-j}}{2+\frac{-2j}{2-j}}\right) = V_{s_2}\left(\frac{-2j}{4-j4}\right) = \frac{V_{s_2}}{2}\left(\frac{-j}{1-j}\right) \\ &= \frac{-j}{2}\left(\frac{-j}{1-j}\right) = \frac{-1}{2(1-j)}. \end{aligned}$$

Similarly, the contribution from $V_{s_1}$ is

$$\begin{aligned} V_{T,1} &= V_{s_1}\left(\frac{Z_P}{2+Z_p}\right) = V_{s_1}\left(\frac{\frac{-2j}{2-j}}{2+\frac{-2j}{2-j}}\right) = V_{s_1}\left(\frac{-2j}{4-j4}\right) = \frac{V_{s_1}}{2}\left(\frac{-j}{1-j}\right) \\ &= \frac{1}{2}\left(\frac{-j}{1-j}\right) = \frac{-j}{2(1-j)}. \end{aligned}$$

The total voltage becomes

$$V_T = V_{T,1}+V_{T,2} = \frac{-j}{2(1-j)} + \frac{-1}{2(1-j)} = \frac{-(1+j)}{2(1-j)} = \frac{-\sqrt{1^2+1^2}e^{j\pi/4}}{2\sqrt{1^2+(-1)^2}e^{-j\pi/4}} = \frac{-1}{2}e^{j\pi/2} = \frac{-j}{2}\ \text{V}.$$

(b) Determine the Thevenin impedance, $Z_T$.

**Solution:** To obtain $Z_T$, do source suppression and obtain the resulting parallel impedance

$$\frac{1}{Z_T} = \frac{1}{2}+\frac{1}{2}+\frac{1}{-j} = \frac{-j-j+2}{-j2} = \frac{2-2j}{-j2} = \frac{1-j}{-j} \Rightarrow Z_T = \frac{-j}{1-j} = \frac{-j(1+j)}{1^2+1^2} = \frac{-j+1}{2}\ \Omega.$$

(c) Determine the matched load, $Z_L$ and the available power.

**Solution:** To obtain the available power, we need

$$P_a = \frac{|V_T|^2}{8R_T} = \frac{0^2+(-1/2)^2}{8(1/2)} = \frac{1}{16}\ \text{W}.$$

The corresponding matched load is

$$Z_L = Z_T^* = \left(\frac{-j+1}{2}\right)^* = \frac{j+1}{2}\ \Omega.$$

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**Quiz 2 is coming up:**

- The second quiz begins on Thursday, October 5. You can already register for the quiz through PrairieTest.
- We strongly recommend you sign up for the first day, in case you get sick. If you sign up for the last couple of days and you get sick, you might not be granted an extension.