University of Illinois **Fall 2023**

ECE 210/211 - Homework 11: Problems and Solutions

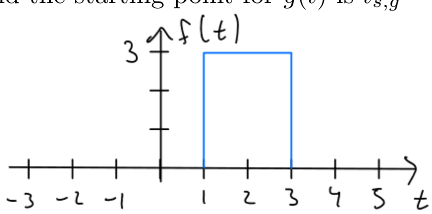
Due: Tuesday, November 7 by 11.59pm

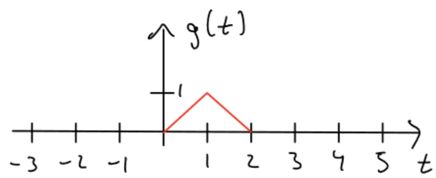
Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
sign: ________________________.

2. Let $f(t) = 3rect(\frac{t-2}{2})$, $h(t) = \triangle(\frac{t-1}{2})$ and $y(t) = f(t) * h(t)$.

   (a) Determine the value of t_I, the first instant in time when $y(t)$ is non-zero.
   Solution: Based on the plots for $f(t)$ and $g(t)$ shown below, the starting point for $f(t)$ is $t_{s,f} = 1$s and the starting point for $g(t)$ is $t_{s,g} = 0$s.

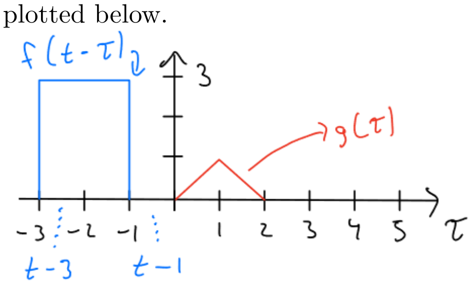




   Therefore, by the start-time property of convolution, $t_I = t_{s,f} + t_{s,g} = 1 + 0 = 1$s.

   (b) Determine the value of t_F, the last time instant in time when $y(t)$ is non-zero.
   Solution: Based on the plots for $f(t)$ and $g(t)$ from part (a), the end point for $f(t)$ is $t_{e,f} = 3$s and the end point for $g(t)$ is $t_{e,g} = 2$s. Therefore, by the end-time property of convolution, $t_F = t_{e,f} + t_{e,g} = 3 + 2 = 5$s.

   (c) Determine the values of $y(0),\ y(1),\ y(2),\ y(3),\ y(4)$.
   Solution: If we choose to flip and shift $f(t)$, the two edges of $f(t-\tau)$ are at $t-3$ and $t-1$, as plotted below.



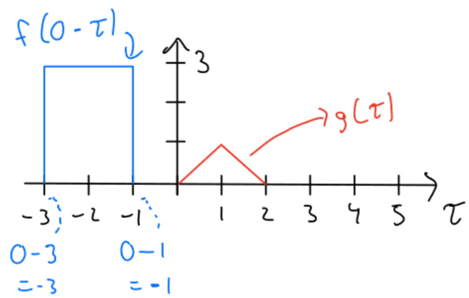
   Recall that convolution is defined as

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau.$$

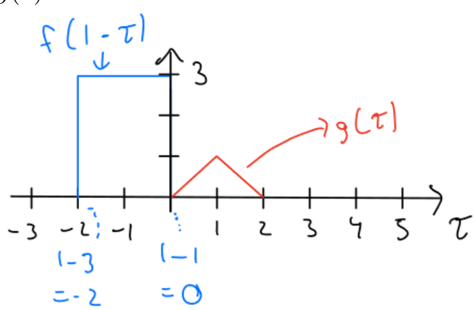
   To obtain $y(0)$:

$$y(0) = \int_{-\infty}^{\infty} f(0-\tau)g(\tau)d\tau.$$

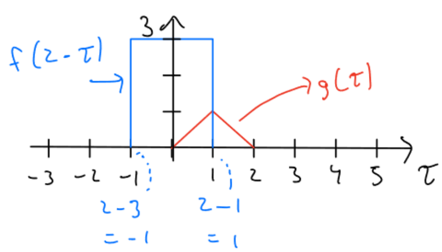
   However, as depicted in the plot below, the product of $f(0-\tau)$ and $g(\tau)$ is always zero, hence $y(0) = 0$.

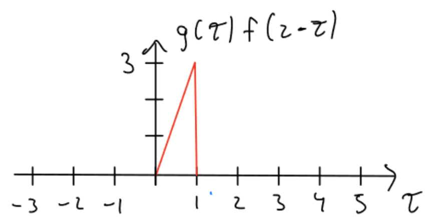


Similarly, as depicted in the plot below, the product of $f(1-\tau)$ and $g(\tau)$ is always zero, hence $y(1) = 0$.



The figure below depicts, $f(2-\tau)$ and $g(\tau)$ on the left hand side, and their product $f(2-\tau)g(\tau)$ on the right hand side.

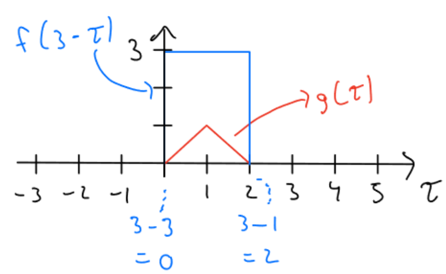


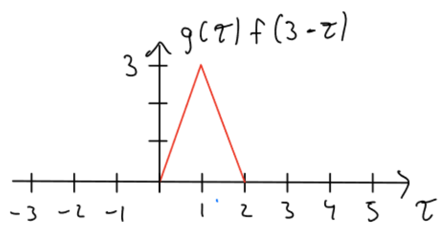


Hence

$$y(2) = \int_{-\infty}^{\infty} f(2-\tau)g(\tau)d\tau = \frac{(1)(3)}{2} = \frac{3}{2}.$$

The figure below depicts, $f(3-\tau)$ and $g(\tau)$ on the left hand side, and their product $f(3-\tau)g(\tau)$ on the right hand side.

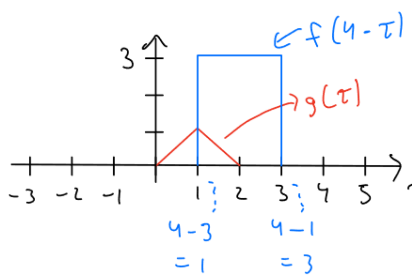


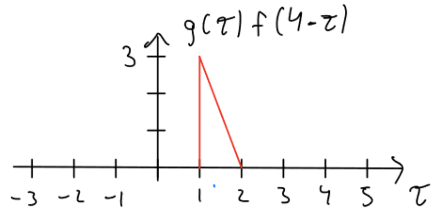


Hence

$$y(3) = \int_{-\infty}^{\infty} f(3-\tau)g(\tau)d\tau = \frac{(2)(3)}{2} = 3.$$

The figure below depicts, $f(4-\tau)$ and $g(\tau)$ on the left hand side, and their product $f(4-\tau)g(\tau)$ on the right hand side.





Hence

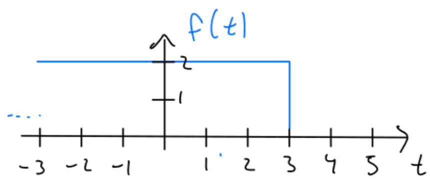
$$y(4) \;=\; \int_{-\infty}^{\infty} f(4-\tau)g(\tau)d\tau \;=\; \frac{(1)(3)}{2} \;=\; \frac{3}{2}.$$

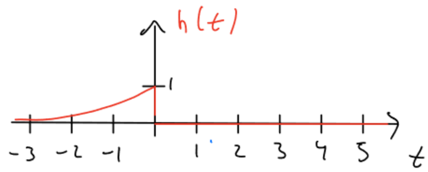
3. Let $f(t) = 2u(3-t)$, $h(t) = e^{\frac{1}{2}t}u(-t)$, and $y(t) = f(t) * h(t)$. Determine $y(t)$ for all $-\infty < t < \infty$.

   Solution: Recall that

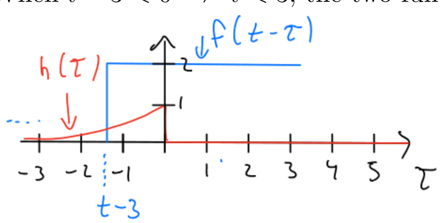
$$y(t) \;=\; \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau.$$

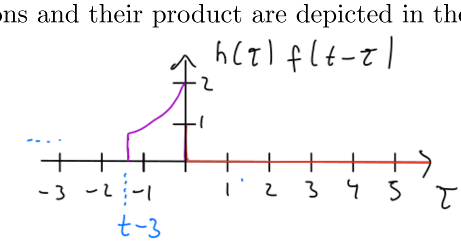
   It is easier to flip and shift $f(t)$ in this case because the values are constant when they are nonzero.





   When $t-3 < 0 \;\Rightarrow\; t < 3$, the two functions and their product are depicted in the plot below:

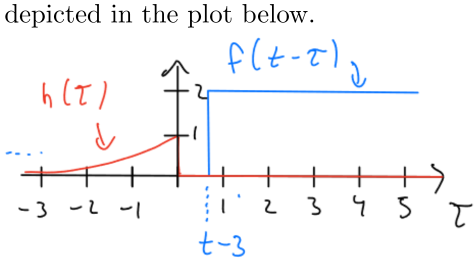




   This yields

$$y(t) \;=\; \int_{t-3}^{0} 2e^{\frac{1}{2}\tau}d\tau \;=\; 4 \;-\; 4e^{\frac{1}{2}(t-3)}.$$

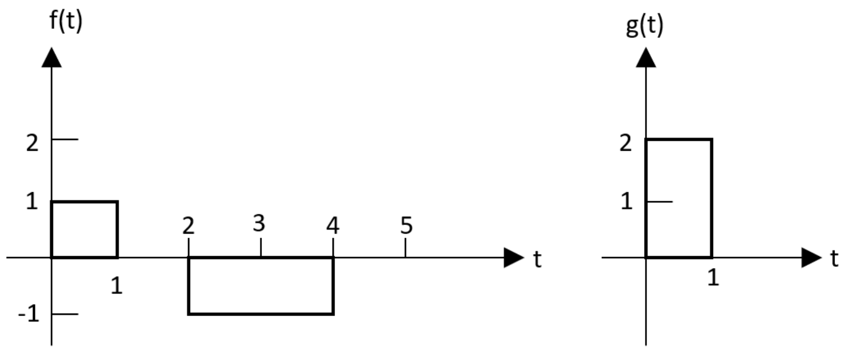
   On the other hand, when $t-3 \geq 0 \;\Rightarrow\; t \geq 3$, we have $y(t) = 0$ because they do not overlap, as depicted in the plot below.



   Hence,

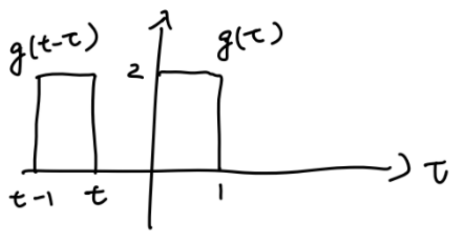
$$y(t) \;=\; \begin{cases} 4(1-e^{\frac{1}{2}(t-3)}) & t < 3 \\ 0 & t \geq 3 \end{cases}$$

4. Consider the signals $f(t)$ and $g(t)$ sketched below.

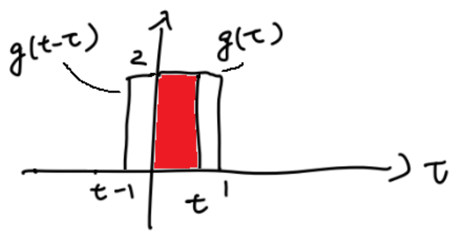


(a) Determine $x(t) = g(t) * g(t)$ by direct integration and sketch the result.

Solution:



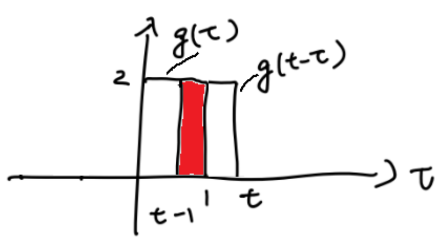
When $t \leq 0$ or $t > 2$, we have $x(t) = 0$ because the two rectangles do not overlap for these times. When $0 < t \leq 1$,



So that

$$x(t) \;=\; \int_0^t (2)(2)d\tau \;=\; 4t.$$

When $1 < t \leq 2$,

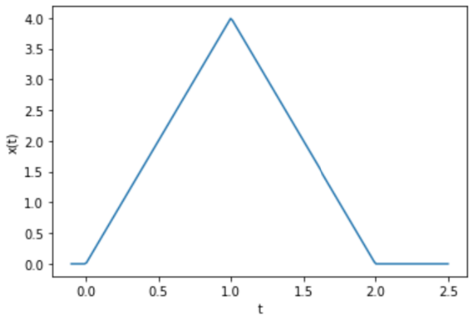


Hence,

$$x(t) \;=\; \int_{t-1}^1 (2)(2)d\tau \;=\; -4t+8.$$

Therefore,

$$x(t) \;=\; \begin{cases} 4t & 0 < t \leq 1 \\ -4t+8 & 1 < t < 2 \\ 0 & else \end{cases} \;=\; 4\triangle\left(\frac{t-1}{2}\right).$$



(b) Express $f(t)$ as a function of $g(t)$.
Solution: Notice that $f(t)$ is composed of several rectangles of the same width as $g(t)$, hence

$$f(t) \;=\; \frac{1}{2}g(t) \;-\; \frac{1}{2}g(t-2) \;-\; \frac{1}{2}g(t-3).$$

(c) Determine $y(t) = f(t) * g(t)$ using appropriate properties of convolution and the result of part (a) and sketch the result.
Solution: Using the result from part (b), along with the distributive and shift properties of convolution:

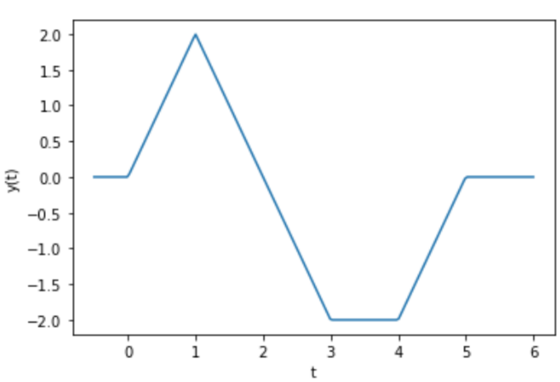
$$\begin{aligned} y(t) &= f(t) * g(t) \;=\; \frac{1}{2}[g(t) - g(t-2) - g(t-3)] * g(t) \\ &= \frac{1}{2}[g(t) * g(t) - g(t-2) * g(t) - g(t-3) * g(t) \;=\; \frac{1}{2}[x(t) - x(t-2) - x(t-3)] \\ &= 2\triangle\left(\frac{t-1}{2}\right) - 2\triangle\left(\frac{t-3}{2}\right) - 2\triangle\left(\frac{t-4}{2}\right). \end{aligned}$$

To sketch it, we can express it as a piece-wise function by adding their individual pieces in the appropriate ranges using the expression

$$2\triangle\left(\frac{t}{2}\right) \;=\; \begin{cases} 2t+2 & -1 < t \le 0, \\ -2t+2 & 0 < t \le 1, \\ 0 & \text{else.} \end{cases}$$

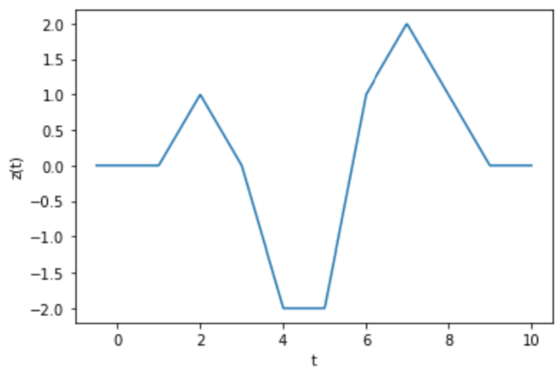
to obtain

$$y(t) = \begin{cases} (2(t-1)+2) - (0) - (0) & 0 \le t \le 1, \\ (-2(t-1)+2) - (0) - (0) & 1 < t \le 2, \\ (0) - (2(t-3)+2) - (0) & 2 < t \le 3, \\ (0) - (-2(t-3)+2) - (2(t-4)+2) & 3 < t \le 4, \\ (0) - (0) - (-2(t-4)+2) & 4 < t \le 5, \\ (0) - (0) - (0) & \text{else.} \end{cases} \;=\; \begin{cases} 2t & 0 \le t \le 1, \\ -2t+4 & 1 < t \le 3, \\ -2 & 3 < t \le 4, \\ 2t-10 & 4 < t \le 5, \\ 0 & \text{else.} \end{cases}$$



(d) Determine $z(t) = f(t) * f(t-1)$ using appropriate properties of convolution and sketch the result.
Solution: Using the results from parts (b) and (c),

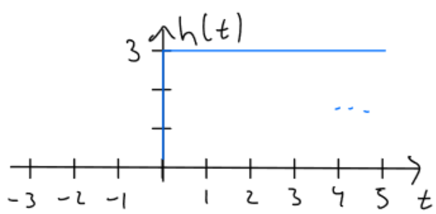
$$\begin{aligned}
z(t) &= f(t) * \frac{1}{2}(g(t-1) - g(t-3) - g(t-4)) = \frac{1}{2}(y(t-1) - y(t-3) - y(t-4)) \\
&= \triangle\left(\frac{t-2}{2}\right) - \triangle\left(\frac{t-4}{2}\right) - \triangle\left(\frac{t-5}{2}\right) - \triangle\left(\frac{t-4}{2}\right) + \triangle\left(\frac{t-6}{2}\right) + \triangle\left(\frac{t-7}{2}\right) \\
&\quad - \triangle\left(\frac{t-5}{2}\right) + \triangle\left(\frac{t-7}{2}\right) + \triangle\left(\frac{t-8}{2}\right) \\
&= \triangle\left(\frac{t-2}{2}\right) - 2\triangle\left(\frac{t-4}{2}\right) - 2\triangle\left(\frac{t-5}{2}\right) + \triangle\left(\frac{t-6}{2}\right) + 2\triangle\left(\frac{t-7}{2}\right) + \triangle\left(\frac{t-8}{2}\right).
\end{aligned}$$

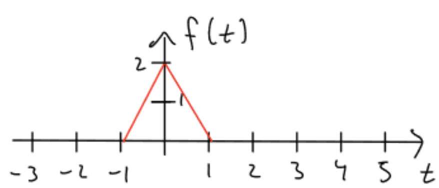


5. Consider the signals $h(t) = 3u(t)$ and $f(t) = 2\triangle(\frac{t}{2})$.

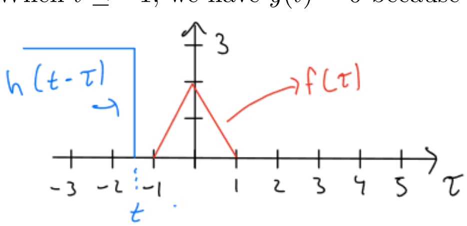
(a) Determine $y(t) = h(t) * f(t)$ and sketch the result.
Solution: Here, we choose to flip and shift $h(t)$. The function $f(t)$ can be written as a piece-wise function:

$$f(t) = \begin{cases} 2 - 2|t| & |t| \leq 1 \\ 0 & \text{else.} \end{cases}.$$

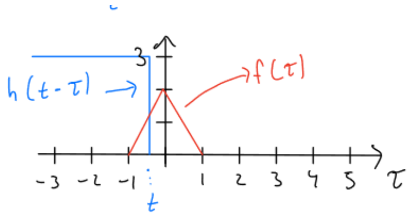


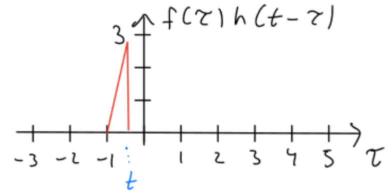


When $t \leq -1$, we have $y(t) = 0$ because there is no overlap, as depicted in the plot below.



When $-1 < t \leq 0$, there is overlap on the first half of the triangle, as depicted in the plot below.

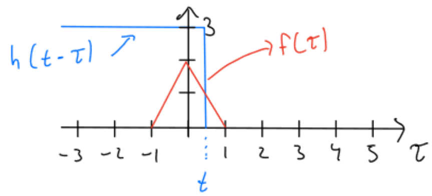


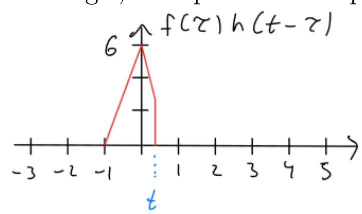


Hence,

$$y(t) \;=\; \int_{-1}^{t} (3)(2\tau + 2)d\tau \;=\; 3t^2 + 6t + 3.$$

When $0 < t \leq 1$, there is overlap with most of the triangle, as depicted in the plot below.

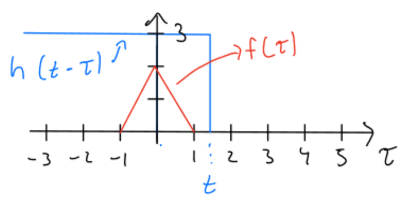


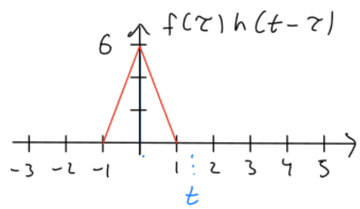


Therefore,

$$y(t) \;=\; \int_{-1}^{0} (3)(2\tau + 2)d\tau \;+\; \int_{0}^{t} (3)(-2\tau + 2)d\tau \;=\; -3t^2 + 6t + 3.$$

Finally, when $t > 1$, the full triangle is integrated, as depicted in the plot below.



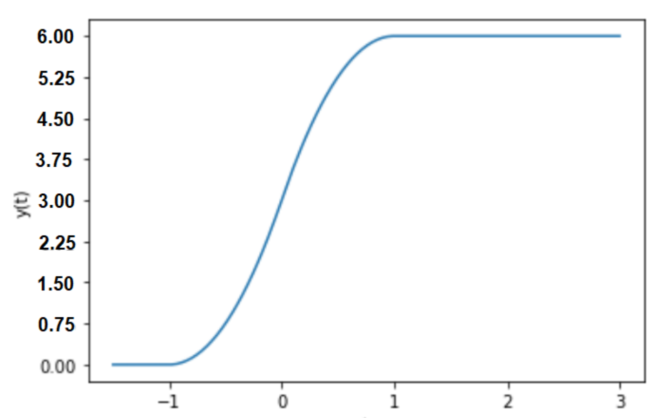


Hence, from the area of the triangle:

$$y(t) \;=\; \frac{(2)(6)}{2} \;=\; 6,$$

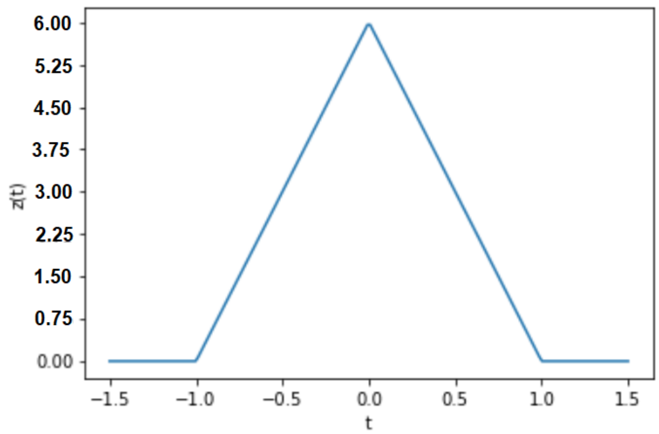
Combining all terms, we obtain

$$y(t) \;=\; \begin{cases} 0 & t \leq -1, \\ 3t^2 + 6t + 3 & -1 < t \leq 0, \\ -3t^2 + 6t + 3 & 0 < t \leq 1, \\ 6 & t > 1. \end{cases}$$

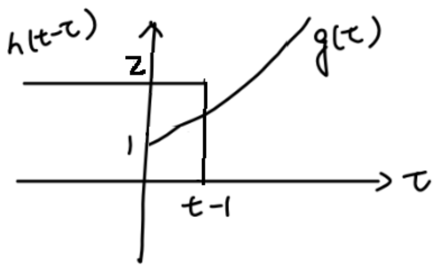


(b) Determine $z(t) = h(t) * \frac{df}{dt}$ using appropriate properties of convolution and sketch the result.
Solution: By the derivative property of convolution,

$$z(t) = \frac{d}{dt}y(t) = \begin{cases} 6t+6 & -1 < t \leq 0 \\ -6t+6 & 0 < t \leq 1 \\ 0 & else \end{cases}$$



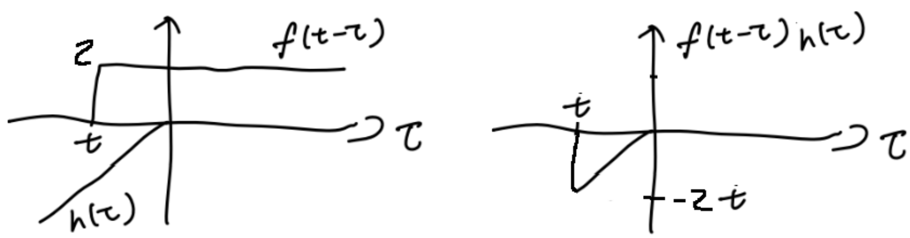
6. Given $f(t) = 2u(t)$, $g(t) = e^{2t}u(t)$ and $q(t) = f(t-1) * g(t)$, determine $q(4)$.
Solution: Let $h(t) = f(t-1) = u(t-1)$ and we choose to flip and shift $h(t)$.



From the plot,

$$q(4) = \int_0^{4-1} 2e^{2\tau}d\tau = e^{2\tau}|_0^3 = e^6 - 1.$$

7. Given $f(t) = 2u(-t)$, $h(t) = tu(-t)$ and $y(t) = f(t) * h(t)$, determine $y(-4)$ and $y(4)$.
Solution: We choose to flip and shift $f(t)$.



$$y(-4) = \int_{-4}^{0} 2\tau d\tau = \tau^2|_{-4}^0 = -16.$$

$$y(4) = 0 \quad \text{because there is no overlap.}$$

