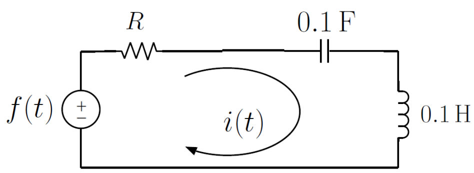
ECE 210/211 - Homework 7: Problems and Solutions

Due: Tuesday, October 10 by 11.59pm

Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
sign: ______________________________.

2. Consider the circuit below, with input $f(t)$ and output $i(t)$.



(a) Determine the frequency response of this circuit, $H(\omega)$ in terms of R.
Solution: The phasor equivalent circuit has an inductance impedance $Z_L = j\omega L = \frac{j\omega}{10}$ Ω and a capacitor impedance $Z_C = \frac{-j}{\omega C} = \frac{-j10}{\omega}$ Ω. Applying KVL yields

$$I\left(R + \frac{j\omega}{10} + \frac{-j10}{\omega}\right) \;=\; F.$$

Therefore, the frequency response of the system is

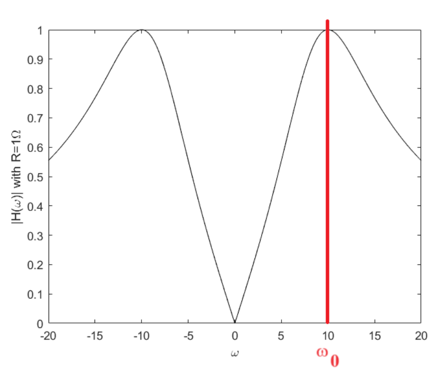
$$H(\omega) \;=\; \frac{I}{F} \;=\; \frac{1}{R + \frac{j\omega}{10} + \frac{-j10}{\omega}} \;=\; \frac{1}{R + j\left(\frac{\omega}{10} - \frac{10}{\omega}\right)}.$$

(b) What is the resonant frequency of this circuit?
Solution: Recall that in an RLC series circuit resonance occurs when

$$\omega_o \;=\; \sqrt{\frac{1}{LC}} \;=\; \sqrt{\frac{1}{0.1 \times 0.1}} \;=\; \sqrt{\frac{1}{0.01}} \;=10\,\text{rad/s}.$$

(c) Let $R = 1$ Ω. Determine $|H(\omega)|$ and plot it on the range $[-20, 20]$ rad/s, labeling the resonant frequency.
Solution:

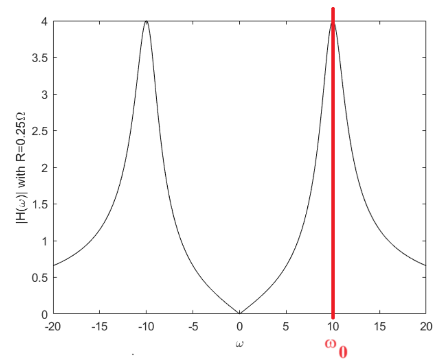
$$|H(\omega)| \;=\; \frac{1}{\sqrt{R^2 + \left(\frac{\omega}{10} - \frac{10}{\omega}\right)^2}} \;=\; \frac{1}{\sqrt{1 + \left(\frac{\omega}{10} - \frac{10}{\omega}\right)^2}}.$$



(d) Repeat part (c) replacing the resistor value with $0.25\,\Omega$.

Solution:

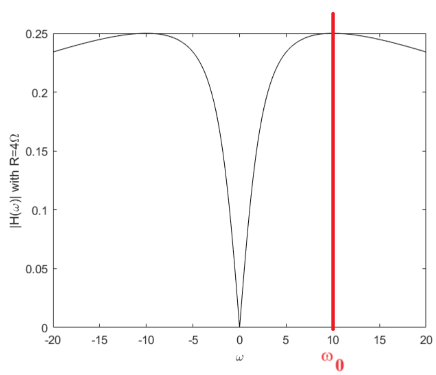
$$|H(\omega)| = \frac{1}{\sqrt{R^2 + \left(\frac{\omega}{10} - \frac{10}{\omega}\right)^2}} = \frac{1}{\sqrt{\frac{1}{16} + \left(\frac{\omega}{10} - \frac{10}{\omega}\right)^2}}.$$



(e) Repeat (b) and (c) replacing the resistor values with $4\,\Omega$.

Solution:

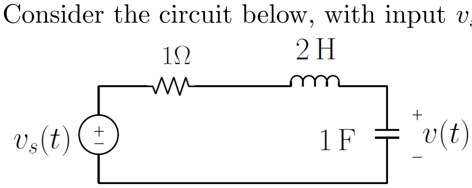
$$|H(\omega)| = \frac{1}{\sqrt{R^2 + \left(\frac{\omega}{10} - \frac{10}{\omega}\right)^2}} = \frac{1}{\sqrt{16 + \left(\frac{\omega}{10} - \frac{10}{\omega}\right)^2}}.$$



(f) How does the resistor value relate to the passband of the filter (e.g., does a larger value for the resistor give a narrower or wider passband)?

Solution: From the results above, we can infer that a lower resistance corresponds to a narrower passband, which is to say that a higher resistance corresponds to a wider passband.

3. Consider the circuit below, with input $v_s(t)$ and output $v(t)$.



(a) Determine its amplitude response $|H(\omega)|$ and plot it in the range $-10 < \omega < 10$.

Solution: The phasor equivalent circuit has an inductance impedance $Z_L = j\omega L = j2\omega$ and a capacitor impedance $Z_C = \frac{1}{j\omega C}\frac{1}{j\omega}$. Applying voltage division,

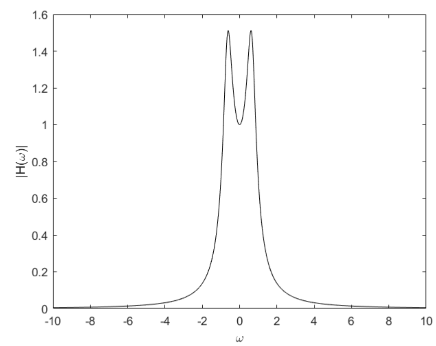
$$V = V_s \frac{\frac{1}{j\omega}}{1 + j\omega 2 + \frac{1}{j\omega}} = \frac{1}{1 - 2\omega^2 + j\omega}.$$

Therefore the frequency response of the system is,

$$H(\omega) = \frac{V}{V_s} = \frac{1}{1-2\omega^2+j\omega}.$$

The magnitude response is

$$|H(\omega)| = \frac{1}{\sqrt{(1-2\omega^2)^2+(\omega)^2)}} = \frac{1}{\sqrt{1-3\omega^2+4\omega^4}}$$



(b) Determine the phase response $\angle H(\omega)$.
Solution: One way to do this is to multiply the numerator and denominator of $H(\omega)$ by the conjugate of the denominator:

$$H(\omega) = \frac{1}{(1-2\omega^2+j\omega)}\left(\frac{1-2\omega^2-j\omega}{1-2\omega^2-j\omega}\right) = \frac{1-2\omega^2-j\omega}{(1-2\omega^2)^2+(\omega)^2}$$

In this case, the location of $H(\omega)$ in the complex plane varies depending on the value of ω. If $1-2\omega^2<0$ then the real part is negative and the angle needs to be adjusted. Hence,

$$\angle H(\omega) = \begin{cases} \arctan\left(\frac{-\omega}{1-2\omega^2}\right) & 0 \le \omega < \frac{1}{\sqrt{2}} \\ -\pi + \arctan\left(\frac{-\omega}{1-2\omega^2}\right) & \frac{1}{\sqrt{2}} \le \omega \\ \arctan\left(\frac{-\omega}{1-2\omega^2}\right) & -\frac{1}{\sqrt{2}} \le \omega < 0 \\ \pi + \arctan\left(\frac{-\omega}{1-2\omega^2}\right) & \omega \le -\frac{1}{\sqrt{2}} \end{cases}$$

4. A linear system with input f(t) and output $y(t)$ is described by the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y(t) = \frac{df}{dt}.$$

(a) Determine the amplitude response $|H(\omega)|$ and plot it versus $\omega for -20<\omega<20$.
Solution: We know that the time-varying signal corresponding to a phasor Y with frequency ω is

$$y(t) = \text{Re}\left\{Ye^{j\omega t}\right\},$$

and its corresponding first and second derivatives can be expressed as

$$\frac{dy(t)}{dt} = \text{Re}\left\{Y\frac{d}{dt}\left(e^{j\omega t}\right)\right\} = \text{Re}\left\{\underbrace{j\omega Y}_{\text{phasor of } \frac{dy}{dt}} e^{j\omega t}\right\}$$

$$\frac{d^2y(t)}{dt^2} = \text{Re}\left\{Y\frac{d^2}{dt^2}\left(e^{j\omega t}\right)\right\} = \text{Re}\left\{\underbrace{(j\omega)^2 Y}_{\text{phasor of } \frac{d^2y}{dt^2}} e^{j\omega t}\right\}$$

Writing the ODE in its phasor equivalent

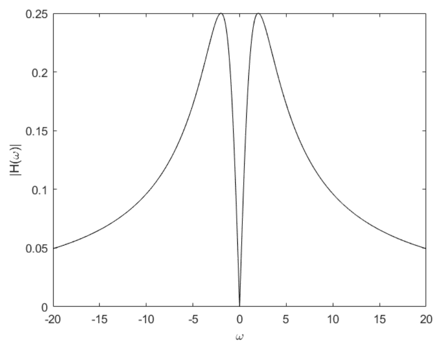
$$(j\omega)^2 Y \;+\; 4(j\omega)Y \;+\; 4Y \;=\; (j\omega)F,$$

yields

$$H(\omega) \;=\; \frac{Y}{F} \;=\; \frac{j\omega}{4-\omega^2+j\omega 4}.$$

From there,

$$|H(\omega)| \;=\; \frac{|\omega|}{\sqrt{(4-\omega^2)^2+(4\omega)^2}} \;=\; \frac{|\omega|}{\sqrt{16-8\omega^2+\omega^4+16\omega}} = \frac{|\omega|}{\sqrt{\omega^4+8\omega^2+16}} \;=\; \frac{|\omega|}{\omega^2+4}.$$



(b) Determine the phase response $\angle H(\omega)$.

Solution: One way to do this is to multiply the numerator and denominator of $H(\omega)$ by the conjugate of the denominator:

$$H(\omega) \;=\; \frac{j\omega}{(4-\omega^2+j\omega 4)}\frac{(4-\omega^2-j\omega 4)}{(4-\omega^2-j\omega 4)} \;=\; \frac{4\omega^2+j\omega\left(4-\omega^2\right)}{16+8\omega^2+\left(\omega^2\right)^2} \;=\; \frac{4\omega^2+j\omega(4-\omega^2)}{(\omega^2+4)^2}.$$

In this case, the real part of $H(\omega)$ is always positive, so no angle adjustment needs to be done. Hence,

$$\angle H(\omega) \;=\; \arctan\left(\frac{\omega\left(4-\omega^2\right)}{4\omega^2}\right) \;=\; \arctan\left(\frac{4-\omega^2}{4\omega}\right).$$

(c) Consider the input $f(t) = 1 + 3\cos(2t) + 2\cos(4\,t)$. Determine the resulting steady-state output $y(t)$.

Solution: By linearity of the system

$$y(t) \;=\; 1H(0) \;+\; 3|H(2)|\cos(2t+\angle H(2)) \;+\; 2|H(4)|\cos(4\,t+\angle H(4)).$$

From part (b), we already know the phase response, hence

$$\angle H(2) \;=\; \arctan\left(\frac{4-2^2}{4(2)}\right) \;=\; 0$$

$$\angle H(4) \;=\; \arctan\left(\frac{4-4^2}{4(4)}\right) \;=\; \arctan\left(\frac{-12}{16}\right) \;=\; \arctan\left(\frac{-3}{4}\right) \;=\; -\arctan\left(\frac{3}{4}\right)$$

The magnitude is also needed:

$$|H(\omega)| \;=\; \frac{|\omega|}{\sqrt{\left(4-\omega^2\right)^2+(4\omega)^2}}$$

$$|H(2)| = \frac{|2|}{\sqrt{(4-2^2)^2+(4(2))^2}} = \frac{2}{\sqrt{64}} = \frac{2}{8} = \frac{1}{4}$$

$$|H(4)| = \frac{|4|}{\sqrt{(4-4^2)^2+(4(4))^2}} = \frac{4}{\sqrt{144+256}} = \frac{4}{\sqrt{400}} = \frac{4}{20} = \frac{1}{5}$$

Hence,

$$\begin{aligned} y(t) &= 1H(0) \;+\; 3|H(2)|\cos(2t+\angle H(2)) \;+\; 2|H(4)|\cos(4\,t+\angle H(4)) \\ &= 1(0) \;+\; 3\left(\frac{1}{4}\right)\cos(2t+0) \;+\; 2\left(\frac{1}{5}\right)|\cos\left(4\,t-\arctan\left(\frac{3}{4}\right)\right) \\ &= \frac{3}{4}\cos(2t)+\frac{2}{5}\cos\left(4\,t-\arctan\left(\frac{3}{4}\right)\right) \end{aligned}$$

5. Given an input

$$f(t) = 1 \;+\; 3e^{-j2t} \;+\; (4+j4)e^{-jt} \;+\; (4-j4)e^{jt} \;+\; 3e^{j2t}$$

and a frequency response

$$H(\omega) = \frac{1+j\omega}{2+j\omega},$$

determine the steady-state response $y(t)$ of the system and express it in terms of real valued signals only.

Solution: Applying the eigenfunction property ($e^{j\omega t}$ is the eigenfunction, and $H(\omega)$ is the eigenvalue) of LTI systems we can write the output as

$$y(t) = 1H(0) \;+\; H(-2)3e^{-j2t} \;+\; H(-1)(4+j4)e^{-jt} \;+\; H(1)(4-j4)e^{jt} \;+\; H(2)3e^{j2t}.$$

We know that $H(\omega)$ is the frequency response of an LTI system. Hence it has the conjugate symmetry property:

$$H(-\omega) = H^*(\omega) = |H(\omega)|\, e^{-j\angle H(\omega)},$$

where

$$|H\omega)| = \sqrt{\frac{1^2+\omega^2}{2^2+\omega^2}} = \sqrt{\frac{1+\omega^2}{4+\omega^2}}$$

and

$$\angle H(\omega) = tan^{-1}\left(\frac{\omega}{1}\right) - tan^{-1}\left(\frac{\omega}{2}\right) = tan^{-1}(\omega) - tan^{-1}(\omega/2)\,.$$

Therefore we can simplify our expression as

$$\begin{aligned} y(t) &= \frac{1}{2}+|H(2)|\,3e^{-j(2t+\angle H(2))} \;+\; |H(1)|\,4\sqrt{2}e^{-j\left(t-\frac{\pi}{4}+\angle H(1)\right)} \;+\; |H(1)|\,4\sqrt{2}e^{j\left(t-\frac{\pi}{4}+\angle H(1)\right)} \\ &\quad+\; |H(2)|\,3e^{j(2t+\angle H(2))} \\ &= \frac{1}{2}+6\,|H(2)|\cos\left(2t+\angle H(2)\right)+8\sqrt{2}\,|H(1)|\cos\left(t-\frac{\pi}{4}+\angle H(1)\right) \\ &= \frac{1}{2}+6\sqrt{\frac{5}{8}}\cos\left(2t+tan^{-1}(2)-tan^{-1}(2/2)\right)+8\sqrt{2}\sqrt{\frac{2}{5}}\cos\left(t-\frac{\pi}{4}+tan^{-1}(1)-tan^{-1}(1/2)\right) \\ &= \frac{1}{2}+\frac{3}{2}\sqrt{10}\cos\left(2t+tan^{-1}(2)-\pi/4\right)+\frac{16}{5}\sqrt{5}\cos\left(t-tan^{-1}(1/2)\right) \end{aligned}$$

6. Consider the function

$$f(t) = 2Re\left\{e^{j\pi t}+2e^{-j3\pi t}\right\} \;-\; 3.$$

Determine its period, T, its fundamental frequency, ω_o, and plot it over at least two periods.

Solution:

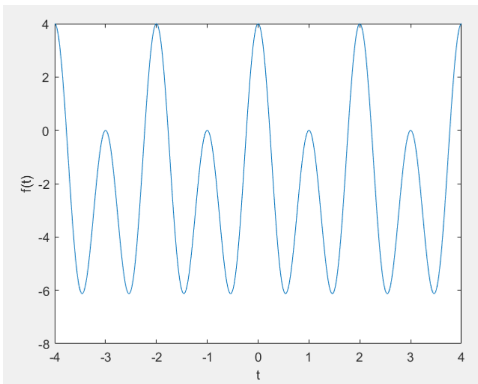
$$f(t) = 2Re\left\{e^{j\pi t}+2e^{-j3\pi t}\right\} \;-\; 3 = 2\cos(\pi t) \;+\; 4\cos(3\pi t) \;-\; 3.$$

The first cosine has frequency $\omega_1 = \pi$ rad/s, the second cosine has frequency $\omega_2 = 3\pi$ rad/s and the constant has frequency zero. The ratio of any pair of those frequencies is an integer divided by an integer, hence the function is periodic. Notice that the largest frequency ω such that ω_1 and ω_2 are integer multiple of it is $\omega = \pi$. Hence,

$$\omega_o \;=\; \pi \text{ rad/s}.$$

Therefore, the period is

$$T \;=\; \frac{2\pi}{\omega_o} \;=\; 2\text{s}.$$



7. For each one of the following functions of t, indicate whether they are periodic or not. If periodic, indicate its period, and if not periodic, indicate why not.

   (a) $e^{\pi t}$

   Solution: The exponential is not complex, hence the function is not periodic due to the exponential increase.

   (b) $e^{j\pi t} \;+\; e^{-2\pi t}$

   Solution: The second exponential is not complex, it is a decaying exponential, so the function as a whole is not periodic.

   (c) $\sin(\pi t) \;+\; \cos\left(\sqrt{2}\pi\, t\right)$

   Solution: This function is not periodic, because the ratio of the two frequencies is not a rational number:

   $$\frac{\pi}{\pi\sqrt{2}} \;=\; \frac{1}{\sqrt{2}} \;\notin\; \mathbb{Q}.$$

   (d) $\sin\left(\frac{\pi}{4}t\right) \;+\; \sin\left(\frac{3\pi}{2}t\right) \;+\; \cos\left(\frac{2\pi}{5}t\right)$

   Solution: It is periodic because all possible ratios of the individual frequencies are rational numbers:

   $$\frac{\pi/4}{3\pi/2} \;=\; \frac{1}{6} \;\in\; \mathbb{Q}, \quad \frac{\pi/4}{2\pi/5} \;=\; \frac{5}{8} \;\in\; \mathbb{Q} \quad \text{and} \quad \frac{3\pi/2}{2\pi/5} \;=\; \frac{15}{4} \;\in\; \mathbb{Q}.$$

   The first term has period

   $$T_1 \;=\; \frac{2\pi}{\omega_1}] = \;\frac{2\pi}{\pi/4} \;=\; 8s,$$

   The second term has period

   $$T_2 \;=\; \frac{2\pi}{\omega_2} \;=\; \frac{2\pi}{3\pi/2} \;=\; \frac{4}{3}s$$

   and the third term has period

   $$T_3 \;=\; \frac{2\pi}{\omega_3} \;=\; \frac{2\pi}{2\pi/5} = 5s.$$

   Their least common multiple is 40, hence

   $$T \;=\; 40s.$$

(e) $\cos\left(n\sqrt{2}t\right)$, where n is a positive integer.

Solution: The function $\cos\left(n\sqrt{2}\,t\right)$ is periodic with period

$$T = \frac{2\pi}{\omega_o} = \frac{2\pi}{n\sqrt{2}} = \frac{\pi}{n}\sqrt{2}\,\mathrm{s}.$$

(f) $\sum_{n=0}^{N} \cos\left(\frac{n\pi}{\sqrt{2}}t\right)$, where N is a finite positive integer.

Solution: The n-th term in the summation has a frequency

$$\omega_n = n\left(\frac{\pi}{\sqrt{2}}\right),$$

hence all possible ratios of the individual frequencies will be rational:

$$\frac{\omega_n}{\omega_m} = \frac{n\pi/\sqrt{2}}{m\pi/\sqrt{2}} = \frac{n}{m} \in \mathbb{Q}.$$

Therefore, the signal is periodic with period

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}s,$$

regardless of the value of N.

(g) $\sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{\sqrt{2}}t\right)$

Solution: From part (f) we know that the signal is periodic with period

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}s,$$

regardless of the value of N, including $N = \infty$.

(h) $\cos(2t)\,|\sin(2t)\,|$

Solution: A similar test as the one to check if a sum of periodic functions is periodic can be applied to a product of them. The first factor of this function has period

$$T_1 = \frac{2\pi}{2} = \pi\ \mathrm{s}.$$

The second factor has period

$$T_2 = \frac{2\pi}{4} = \frac{\pi}{2}\ \mathrm{s}$$

because when the sine term is reflected for the absolute value, it repeats itself twice as fast (you can sketch it to see that).

Hence, the period is their least common multiple,

$$T = \pi\ \mathrm{s}.$$

You can check this by showing that $f(t+\pi) = f(t)$:

$$f(t{+}\pi) = \cos(2(t+\pi))\,|\sin(2(t+\pi))| = \cos(2t+2\pi)\,|\sin(2t+2\pi)| = \cos(2t)\,|\sin(2t)| = f(t).$$