ECE 210/211 - Homework 8: Problems and Solutions

Due: Tuesday, October 17 by 11.59pm

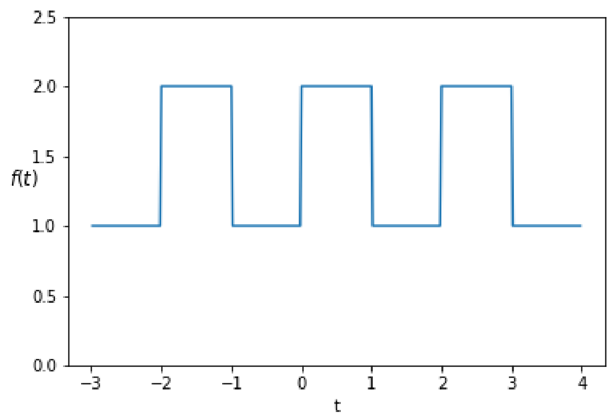
Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
   sign: ________________________.

2. The function $f(t)$ is periodic with period $T = 2$s. Between $t = 0$ and $t = 2$s, the function is described by

$$f(t) = \begin{cases} 2 & 0 < t < 1s \\ 1 & 1 < t < 2s \end{cases}$$

   (a) Plot $f(t)$ between $t = -3s$ and $t = 4s$.
   Solution:



   (b) Determine the exponential Fourier coefficients F_n of $f(t)$ for $n = 0$, $n = \pm 1$, and $n = \pm 2$.
   Solution: $\omega_0 = \frac{2\pi}{T} = \pi$ rad/s.
   By definition,

$$F_n = \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt = \frac{1}{2}\int_0^2 f(t)e^{-jn\pi t}dt = \frac{1}{2}\left(\int_0^1 2e^{-jn\pi t}dt + \int_1^2 e^{-jn\pi t}dt\right)$$

   For $n = 0$,

$$F_0 = \int_0^1 dt + \int_1^2 \frac{1}{2}dt = \frac{3}{2}$$

   For $n \neq 0$,

$$\begin{aligned} F_n &= \frac{2e^{-jn\pi t}}{-2jn\pi}\Big|_0^1 - \frac{e^{-jn\pi t}}{2jn\pi}\Big|_1^2 = \frac{2}{n\pi}\left(\frac{1-e^{-jn\pi}}{2j}\right) + \frac{1}{n\pi}\left(\frac{e^{-jn\pi}-e^{-jn2\pi}}{2j}\right) \\ &= \frac{2}{n\pi}e^{-jn\frac{\pi}{2}}\left(\frac{e^{jn\frac{\pi}{2}}-e^{-jn\frac{\pi}{2}}}{2j}\right) + \frac{e^{-jn\frac{3\pi}{2}}}{n\pi}\left(\frac{e^{jn\frac{\pi}{2}}-e^{-jn\frac{\pi}{2}}}{2j}\right) = \sin\left(\frac{n\pi}{2}\right)\frac{e^{-jn\frac{\pi}{2}}}{n\pi}(2+e^{-jn\pi}) \end{aligned}$$

Notice that

$$\sin(\frac{n\pi}{2}) = \begin{cases} 0 & n \text{ even} \\ \pm 1 & n \text{ odd} \end{cases}$$

Hence,

$$F_1 = \sin\left(\frac{\pi}{2}\right)\frac{e^{-j\frac{\pi}{2}}}{\pi}(2 + e^{-j\pi}) \;=\; -\frac{j}{\pi}$$
$$F_{-1} = -\sin\left(-\frac{\pi}{2}\right)\frac{e^{j\frac{\pi}{2}}}{\pi}(2 + e^{j\pi}) \;=\; \frac{j}{\pi}$$
$$F_2 = F_{-2} = 0$$

(c) Using the results of part (b), determine the compact-form Fourier coefficients c_0, c_1, and c_2.
Solution: $c_n = 2|F_n|$
$\mathbf{c_0} = 2(\frac{3}{2}) = \mathbf{3}, \mathbf{c_1} = \frac{\mathbf{2}}{\pi}, \mathbf{c_2} = \mathbf{0}$

3. For each of the following functions $f(t)$, obtain the Fourier series in exponential, trigonometric, and compact forms.

(a) $f(t) = \sin^4(t)$
Solution:
Trigonometric form:

$$\begin{aligned}\sin^4(t) \;&=\; \left[\sin^2(t)\right]^2 \;=\; \left[\frac{1-\cos(2t)}{2}\right]^2 \;=\; \frac{1}{4}\left[1 - 2\cos(2t) + \cos^2(2t)\right] \\ &=\; \frac{1}{4} - \frac{1}{2}\cos(2t) + \frac{1}{4}\left(\frac{1+\cos(4t)}{2}\right) \;=\; \frac{1}{4} - \frac{1}{2}\cos(2t) + \frac{1}{8} + \frac{1}{8}\cos(4t) \;=\; \frac{3}{8} - \frac{1}{2}\cos(2t) + \frac{1}{8}\cos(4t)\end{aligned}$$

Exponential form: Using the trigonometric form and Euler's formula:

$$f(t) \;=\; \frac{3}{8} - \frac{1}{2}\left(\frac{e^{2t}+e^{-2t}}{2}\right) + \frac{1}{8}\left(\frac{e^{4t}+e^{-4t}}{2}\right) \;=\; \frac{3}{8} - \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} + \frac{1}{16}e^{4t} + \frac{1}{16}e^{-4t}$$

Compact form: Again using the trigonometric form and that fact that we need $c_n > 0$,

$$f(t) \;=\; \frac{3}{8} + \frac{1}{2}\cos(2t+\pi) + \frac{1}{8}\cos(4t)$$

(b) $f(t) = e^t$ for $-\pi \le t < \pi$, with period $T = 2\pi$ s.
Solution: $\omega_0 = \frac{2\pi}{T} = 1$ rad/s.
Exponential form:

$$\begin{aligned}F_n \;&=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t e^{-jnt}dt \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-jn)t}dt \;=\; \frac{1}{2\pi}\frac{e^{(1-jn)t}}{1-jn}\Big|_{-\pi}^{\pi} \;=\; \frac{e^{(1-jn)\pi} - e^{-(1-jn)\pi}}{2\pi(1-jn)} \\ &=\; \frac{e^{\pi}e^{-jn\pi} - e^{-\pi}e^{jn\pi}}{2\pi(1-jn)} \;=\; \frac{e^{\pi}(-1)^n - e^{-\pi}(-1)^n}{2\pi(1-jn)} \;=\; \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1-jn)} \;=\; \frac{(-1)^n\sinh(\pi)}{\pi(1-jn)}\end{aligned}$$

Hence,

$$f(t) \;=\; \sum_{n=-\infty}^{\infty} \frac{(-1)^n\sinh(\pi)}{\pi(1-jn)}e^{jnt} \;=\; \sum_{n=-\infty}^{\infty} \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1-jn)}e^{jnt}$$

Trigonometric form: Using the Fourier series table from the textbook:

$$a_0 \;=\; 2F_0 = \frac{2\sinh(\pi)}{\pi}$$

$$a_n = F_n + F_{-n} = \frac{(-1)^n \sinh(\pi)}{\pi}\left(\frac{1}{1-jn}+\frac{1}{1+jn}\right) = \frac{(-1)^n \sinh(\pi)}{\pi}\left(\frac{1+jn+1-jn}{1+n^2}\right) = \frac{(-1)^n 2\sinh(\pi)}{\pi(1+n^2)}$$

$$\begin{aligned} b_n &= j(F_n - F_{-n}) = j\left[\frac{(-1)^n \sinh(\pi)}{\pi(1-jn)} - \frac{(-1)^n \sinh(\pi)}{\pi(1+jn)}\right] = j\frac{(-1)^n \sinh(\pi)}{\pi}\left(\frac{1}{1-jn}-\frac{1}{1+jn}\right) \\ &= j\frac{(-1)^n \sinh(\pi)}{\pi}\left(\frac{1+jn-(1-jn)}{1+n^2}\right) = j\frac{(-1)^n \sinh(\pi)}{\pi}\left(\frac{j2n}{1+n^2}\right) = \frac{(-1)^{n+1}2n\sinh(\pi)}{\pi(1+n^2)} \end{aligned}$$

Hence,

$$f(t) = \frac{\sinh(\pi)}{\pi}\left(1+\sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}(\cos(nt)-n\sin(nt))\right) = \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos(nt)-n\sin(nt))\right)$$

Compact form: Using the Fourier series table from the textbook:

$$c_n = 2|F_n| = 2\left|\frac{(-1)^n \sinh(\pi)}{\pi(1-jn)}\right| = \frac{2\sinh(\pi)}{\pi\sqrt{1+n^2}}$$

$$\theta_n = \angle F_n = \begin{cases} \tan^{-1}(n) & n \text{ even} \\ \tan^{-1}(n)+\pi & n \text{ odd} \end{cases}$$

Hence,

$$\begin{aligned} f(t) &= \frac{\sinh(\pi)}{\pi}\left[1+2\left(\sum_{n=2,n \text{ even}}^{\infty}\frac{1}{\sqrt{1+n^2}}\cos(nt+\tan^{-1}(n)) + \sum_{n=1,n \text{ odd}}^{\infty}\frac{1}{\sqrt{1+n^2}}\cos(nt+\tan^{-1}(n)+\pi)\right)\right] \\ &= \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos(nt+\tan^{-1}(n)+n\pi)}{\sqrt{1+n^2}}\right) \end{aligned}$$

(c) $f(t) = t^2$ for $-\pi \le t < \pi$, with period $T = 2\pi$ s. Hint: To simplify the problem, make use of the derivative property of Fourier series.
Solution: $\omega_0 = \frac{2\pi}{T} = 1$ rad/s.
Exponential form: Let

$$g(t) = \frac{df(t)}{dt} = 2t$$

Since $f(t)$ is continuous, by the derivative property of the Fourier series

$$G_n = jn\omega_0 F_n = jnF_n \longrightarrow F_{n\neq 0} = \frac{G_n}{jn}$$

Using integration by parts, for $n \neq 0$:

$$\begin{aligned} G_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2te^{-jnt}dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\frac{d}{dt}\left(\frac{e^{-jnt}}{-jn}\right)dt = \frac{1}{\pi}\left(t\frac{e^{-jnt}}{-jn}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-jnt}}{-jn}dt\right) \\ &= \frac{1}{\pi}\left(\frac{-\pi e^{jn\pi}-\pi e^{-jn\pi}}{jn} - \frac{e^{-jnt}}{(-jn)^2}\Big|_{-\pi}^{\pi}\right) = \frac{1}{\pi}\left(\frac{j2\pi\cos(n\pi)}{n} - 0\frac{j2\sin(n\pi)}{n^2}\right) = \frac{j2(-1)^n}{n} \end{aligned}$$

Hence,

$$F_{n\neq 0} = \frac{2(-1)^n}{n^2}$$

$$F_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{2\pi}\frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^3-(-\pi)^3}{6\pi} = \frac{\pi^2}{3}$$

Therefore,

$$f(t) \;=\; \frac{\pi^2}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2(-1)^n}{n^2} e^{jnt}$$

Trigonometric form: Using the exponential form,

$$f(t) \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}e^{jnt} + \frac{2(-1)^{-n}}{(-n)^2}e^{-jnt}\right) \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(e^{jnt}+e^{-jnt}) \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nt)$$

Compact form: Recall that $c_n = 2|F_n|$ and $\theta_n = \angle F_n$, hence

$$c_{n\neq 0} = 2|\frac{2(-1)^n}{n^2}| = \frac{4}{n^2}$$

$$\theta_n = \begin{cases} 0 & n \text{ even} \\ \pi & n \text{ odd} \end{cases}$$

Therefore,

$$f(t) \;=\; \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nt+\theta_n) \;=\; \boldsymbol{\frac{\pi^2}{3}} + \mathbf{4}\left(\sum_{\mathbf{n=2}(\text{even})}^{\infty}\frac{\cos(\mathbf{nt})}{\mathbf{n^2}} + \sum_{\mathbf{n=1}(\text{odd})}^{\infty}\frac{\cos(\mathbf{nt}+\pi)}{\mathbf{n^2}}\right) \;=\; \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos(nt+n\pi)}{n^2}$$

4. Let the function $f(t)$ be the input to an LTI system with frequency response

$$H(\omega) = \begin{cases} 2e^{-j\omega\frac{\pi}{2}} & \omega \in [-3.5, 3.5]\,\frac{\text{rad}}{s} \\ 0 & \text{otherwise} \end{cases}$$

Obtain the steady state response $y_{ss}(t)$ of the system to the following inputs:

(a) $f(t) = \sin^4(t)$ (same as in 3a)
Solution: Recall

$$Y_n \;=\; H(n\omega_0)F_n \;=\; H(2n)F_n$$

Hence,

$$Y_0 \;=\; H(0)F_0 \;=\; 2\left(\frac{3}{8}\right) \;=\; \frac{3}{4}$$

$$Y_{\pm 1} \;=\; H(\pm 2)F_{\pm 1} \;=\; 2e^{-j(\pm 2)\frac{\pi}{2}}\left(-\frac{1}{4}\right) \;=\; -\frac{1}{2}e^{\mp j\pi} \;=\; -\frac{1}{2}(-1) \;=\; \frac{1}{2}$$

$$Y_n \;=\; 0 \text{ otherwise.}$$

Since the input is periodic,

$$y(t) \;=\; y_{ss}(t) \;=\; \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t} \;=\; \frac{3}{4} + \frac{1}{2}(e^{j2t} + e^{-2jt}) \;=\; \frac{3}{4} + \cos(2t)$$

(b) $f(t) = e^t$ for $-\pi \leq t < \pi$, with period $T = 2\pi$ s (same as in 3b)
Solution: Recall

$$Y_n \;=\; H(n\omega_0)F_n \;=\; H(n)F_n$$

Hence,

$$Y_0 = H(0)F_0 = 2\frac{\sinh(\pi)}{\pi}$$

$$Y_1 = H(1)F_1 = 2e^{-j\frac{\pi}{2}}\frac{(-1)\sinh(\pi)}{\pi(1-j)} = \frac{2j\sinh(\pi)}{\pi(1-j)} = \frac{(-1+j)\sinh(\pi)}{\pi}$$

$$Y_{-1} = H(-1)F_{-1} = 2e^{j\frac{\pi}{2}}\frac{(-1)\sinh(\pi)}{\pi(1+j)} = \frac{-2j\sinh(\pi)}{\pi(1+j)} = -\frac{(1+j)\sinh(\pi)}{\pi}$$

$$Y_2 = H(2)F_2 = 2e^{-j2\frac{\pi}{2}}\frac{\sinh(\pi)}{\pi(1-j2)} = \frac{-2(1+j2)\sinh(\pi)}{\pi(1+2^2)} = -\frac{2(1+j2)\sinh(\pi)}{5\pi}$$

$$Y_{-2} = H(-2)F_{-2} = 2e^{j2\frac{\pi}{2}}\frac{\sinh(\pi)}{\pi(1+j2)} = \frac{-2(1-j2)\sinh(\pi)}{\pi(1+2^2)} = \frac{2(-1+j2)\sinh(\pi)}{5\pi}$$

$$Y_3 = H(3)F_3 = 2e^{-j3\frac{\pi}{2}}\frac{(-1)\sinh(\pi)}{\pi(1-j3)} = \frac{-2j(1+j3)\sinh(\pi)}{\pi(1+3^2)} = \frac{(3-j)\sinh(\pi)}{5\pi}$$

$$Y_{-3} = H(-3)F_{-3} = 2e^{j3\frac{\pi}{2}}\frac{(-1)\sinh(\pi)}{\pi(1+j3)} = \frac{2j(1-j3)\sinh(\pi)}{\pi(1+3^2)} = \frac{(3+j)\sinh(\pi)}{5\pi}$$

Therefore,

$$\begin{aligned} y(t) &= y_{ss}(t) = \sum_{n=-\infty}^{\infty} Y_n e^{jnt} \\ &= \frac{\sinh(\pi)}{\pi}\left(2 + (-1+j)e^{jt} - (1+j)e^{-jt} - \frac{2(1+j2)}{5}e^{j2t} + \frac{2(-1+j2)}{5}e^{-j2t} + \frac{(3-j)e^{j3t}}{5} + \frac{(3+j)e^{j3t}}{5}\right), \end{aligned}$$

where $\sinh(\pi) = \frac{e^{\pi}-e^{-\pi}}{2}$.

5. Determine the average power of the following signals:

   (a) $f(t) = \sin^4(t)$ (same as in 3a)
   Solution:

$$P = \sum_{n=-\infty}^{\infty} |F_n|^2 = \left(\frac{3}{8}\right)^2 + 2\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{16}\right)^2 = \frac{35}{128}$$

   (b) $f(t) = e^t$ for $-\pi \le t < \pi$, with period $T = 2\pi$ s (same as in 3b)
   Solution:

$$P = \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^t|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t} dt = \frac{1}{2\pi}\frac{e^{2t}}{2}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{e^{2\pi}-e^{-2\pi}}{2} = \frac{\mathbf{e^{2\pi}-e^{-2\pi}}}{\mathbf{4\pi}} = \frac{\mathbf{1}}{\mathbf{2\pi}}\sinh(\mathbf{2\pi})$$

   Alternative method:

$$\begin{aligned} P &= \sum_{n=-\infty}^{\infty} |F_n|^2 = \sum_{n=-\infty}^{\infty}\left|\frac{(-1)^n\sinh(\pi)}{\pi(1-jn)}\right|^2 = \left(\frac{\sinh(\pi)}{\pi}\right)^2 \sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} = \left(\frac{\sinh(\pi)}{\pi}\right)^2 \pi\coth(\pi) \\ &= \frac{(\sinh(\pi))^2}{\pi}\frac{\cosh(\pi)}{\sinh(\pi)} = \frac{\sinh(\pi)\cosh(\pi)}{\pi} = \frac{1}{2\pi}\sinh(2\pi) = \frac{e^{2\pi}-e^{-2\pi}}{4\pi} \end{aligned}$$

   Using the identities $\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2} = \pi\coth(\pi)$, and $\sinh(\theta+\phi) = \sinh(\theta)\cosh(\phi)+\cosh(\theta)\sinh(\phi)$.

6. This problem will introduce several python programming concepts to reinforce your understanding of Fourier series.

   - You will use an IPython notebook via PrairieLearn with unlimited attempts.
   - Make sure that you complete the PrairieLearn question before this homework's due date to get credit for it.

- You do not need to submit an answer for this question on Gradescope, we will download this problem's grade from PrairieLearn.
- You can access the question by clicking on this link:
  https://us.prairielearn.com/pl/course_instance/139840/assessment/2364401
- If you do not have access to the question, please email Prof. Alvarez.

7. Fill out the following survey before this homework's deadline. We will use these data to improve the course over time.
   - Your grade for this HW problem will not depend on the content of your survey responses, only on the completion of the survey.
   - There are no correct or incorrect answers. Your grade for this HW problem will not depend on the content of your form responses, only on the completion of the form.
   - In order to complete the survey, you will need to view your Mindset Plot, which will be emailed to you on the evening of October 10, so please check your email.
   - The link to the survey is here:
     https://illinois.qualtrics.com/jfe/form/SV_eFFbbygXQlPrwj4
   - You do not need to submit an answer for this question on Gradescope, we will be notified of your submission and will upload the points to Canvas.
   - If you do not have access to the survey, please email Prof. Alvarez.

Exam 2 coming up

- Date: Wednesday, October 18, 7.15-8.30pm.
- Review: Tuesday, October 17, 7-9 p.m., 1013 ECEB.
- Coverage: Exam 2 will cover up to the end of Chapter 6 and homeworks 1-8, with emphasis on post-exam 1 topics.
- We will provide the Fourier series tables (6.1 and 6.3 from the textbook, or 1-2 from the online table handout).
- Room assignments (same as for exam 1). Students with last names beginning with:
  * Aa - Bs will go to room ECEB 2017
  * Bt - Gf will go to room ECEB 3017
  * Gg - Jj will go to room ECEB 1015
  * Jk - Uq will go to room ECEB 1002
  * Ur - Zz will go to room ECEB 1013
- DRES students should have emailed Prof. Alvarez their DRES letters weeks ago and scheduled their exam as indicated by him.

