University of Illinois **Fall 2023**

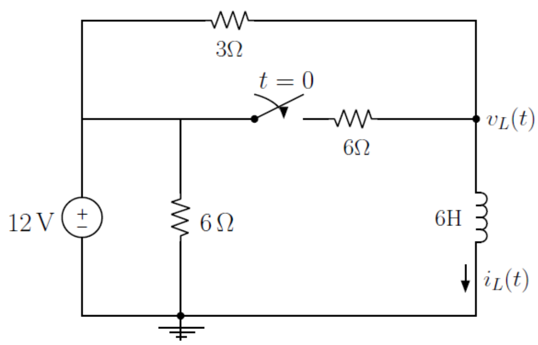
ECE 210/211 - Homework 4: Problems and Solutions

Due: Tuesday, September 19 by 11.59pm

Problems:

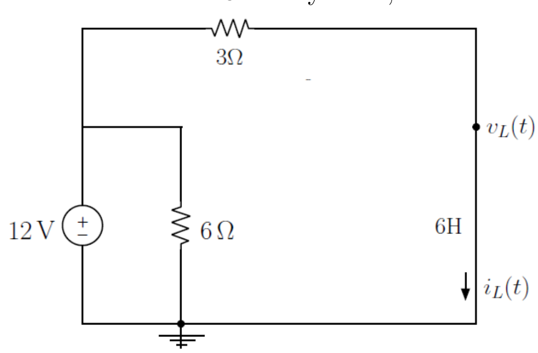
1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
sign: ____________________________.

2. The circuit shown below is in DC steady-state before the switch flips at $t = 0$s.



(a) Determine $v_L(0^-)$ and $i_L(0^-)$.

Solution: In the DC steady-state, before closing the switch, the circuit is as depicted below:



The inductor acts like a short in DC state, hence

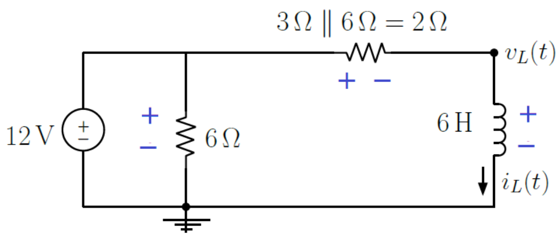
$$v_L(0^-) \;=\; 0\text{V}.$$

Then, by Ohm's Law on the 3Ω resistor, which is in parallel with the 12V source, we have the initial condition

$$i_L(0^-) \;=\; \frac{12}{3} \;=\; 4\,\text{A}.$$

(b) For $t > 0$, determine the zero-state response, $i_{L,ZS}(t)$, the zero-input response, $i_{L,ZI}(t)$ and the full response, $i_L(t)$.

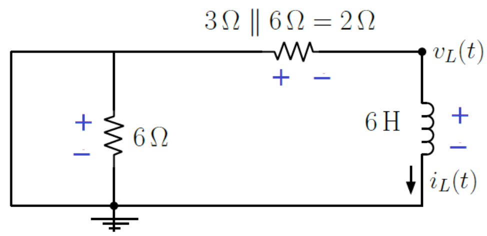
Solution: After closing the switch we have the following circuit



Notice that this is a resistive circuit connected to an inductor. Te general solution for the current in the inductor in an RL circuit with time constant $\tau = L/R_T$ is

$$i_L(t) \;=\; K_1 e^{-t/\tau} \;+\; K_2.$$

In order to obtain R_T, we suppress the independent sources, as observed in the figure below.



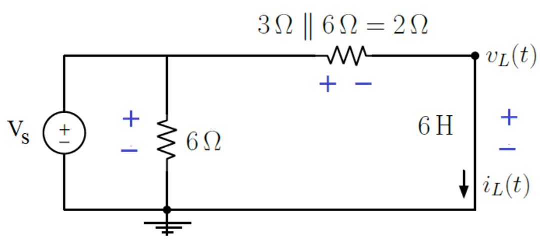
That implies that the 6Ω resistor on the left is shorted because it is in parallel with a short. Hence, from the perspective of the inductor:

$$R_T \;=\; 2\Omega.$$

Therefore,

$$\tau \;=\; L/R_T \;=\; 6/2 \;=\; 3\,\mathrm{s}.$$

To obtain the other constants, we first consider when $t \to \infty$, where the inductor acts like a short. The resulting circuit is:



Therefore, by Ohm's Law on the parallel resistor, which is in parallel with the voltage source V_s:

$$i_L(t \to \infty) \;=\; \frac{V_s}{2} \;=\; K_2.$$

Applying the initial condition, and using the fact that current is continuous in an inductor, we obtain

$$i_L(0^+) \;=\; i_L(0^-) \;=\; K_1 \;+\; K_2 \;=\; K_1 \;+\; \frac{V_s}{2} \;\;\Rightarrow K_1 \;=\; i_L(0^-) \;-\; \frac{V_s}{2}.$$

Therefore, for $t > 0$, $i_L(t)$ is

$$i_L(t) \;=\; \left[\left(i_L(0^-) \;-\; \frac{V_s}{2}\right) e^{-t/3} \;+\; \frac{V_s}{2}\right] \;\mathrm{A}.$$

To obtain the zero-state solution, we simply force the initial state to be zero, that is

$$i_L(0^-) \;=\; 0 \;\mathrm{A},$$

which yields

$$i_{L,ZS}(t) \;=\; \left(0 \;-\; \frac{V_s}{2}\right) e^{-t/3} + \frac{V_s}{2} \;=\; -\frac{V_s}{2} e^{-t/3} + \frac{V_s}{2} \;=\; -\frac{12}{2} e^{-t/3} + \frac{12}{2} \;=\; \left(-6e^{-t/3} \;+\; 6\right) \text{ A.}$$

To obtain the zero-input solution, we simply force the input to be zero, that is

$$V_s = 0 \text{ V},$$

which yields

$$i_{L,ZI}(t) \;=\; \left(i_L(0^-) \;-\; \frac{0}{2}\right) e^{-t/3} \;+\; \frac{0}{2} \;=\; i_L(0^-)e^{-t/3} \;=\; 4e^{-t/3} \text{ A.}$$

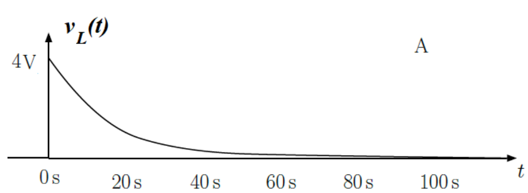
To obtain the full solution we add the two components together:

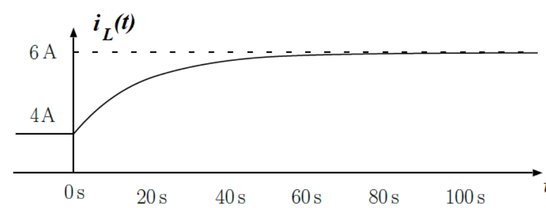
$$i_L(t) \;=\; i_{L,ZS}(t) \;+\; i_{L,ZI}(t) \;=\; 4e^{-t/3} \;-\; 6e^{-t/3} \;+\; 6 \;=\; \left(6 \;-\; 2e^{-t/3}\right) \text{ A.}$$

(c) For $t > 0$, determine $v_L(t)$.
Solution: Use the v-i relation for the inductor

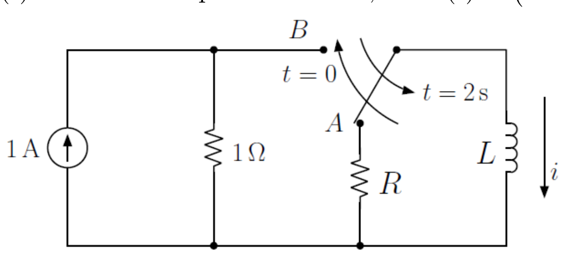
$$v_L \;=\; L\frac{d}{dt}i(t) \;=\; 6 \times \frac{d}{dt}(6 - 2e^{-\frac{t}{3}}) \;=\; 6 \times \left(\frac{2}{3}e^{-\frac{t}{3}}\right) \;=\; 4\,e^{-\frac{t}{3}}\,\text{V}.$$

(d) Sketch $v_L(t)$ and $i_L(t)$ for $t > -1$, clearly labelling important points.
Solution:

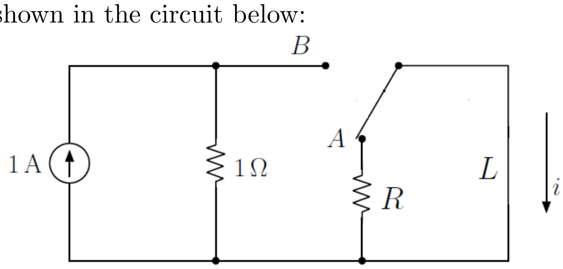




3. In the circuit below, assume the switch has been in position A for a long time. It moves to position B at time $t = 0$s, and back to position A at time $t = 2$s. Determine the values of L and R such that $i(t) = 1 - e^{-1}$ Amperes at $t = 2$s, and $i(t) = \left(1 - e^{-1}\right) e^{-2}$ Amperes at $t = 8$s.



Solution: Before $t = 0$, the circuit is in steady-state and the inductor acts like a short circuit, as shown in the circuit below:



Without any sources there is no current flowing through the inductor, i.e.

$$i(0^-) \;=\; 0\,\text{A},$$

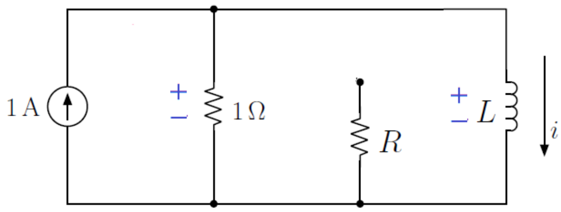
which by continuity of current through the inductor yields

$$i(0^+) \;=\; i(0^-) \;=\; 0\,\text{A}.$$

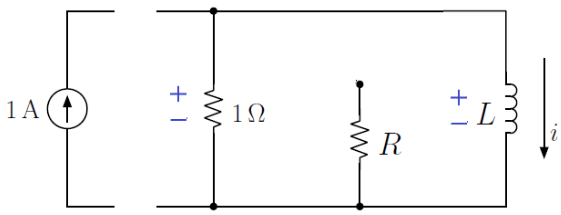
After the switch moves to position B we have an RL circuit with time constant

$$\tau \;=\; L/R_T,$$

as seen in the figure below.



In order to obtain R_T, we suppress the independent sources, as observed in the figure below.



That implies that the 1Ω resistor is R_T from the perspective of the inductor:

$$R_T \;=\; 1\Omega.$$

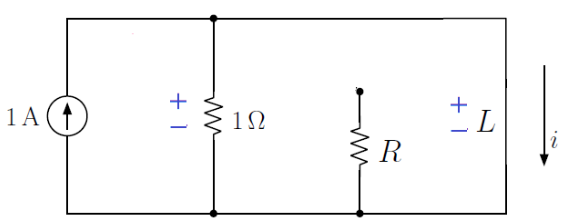
Therefore,

$$\tau \;=\; L/R_T \;=\; L/1 \;=\; L$$

The general equation for the current through the inductor is

$$i(t) \;=\; K_1 e^{-t/\tau} \;+\; K_2 \;=\; K_1 e^{-t/L} \;+\; K_2.$$

If the switch would stay in position B for a long time, the inductor would become a short, as depicted in the figure below:



The current flowing through inductor would be the full current from the source, which means

$$K_2 \;=\; 1\,\text{A}.$$

Applying the initial condition we obtain K_1:

$$i(0^-) \;=\; i(0^+) \;=\; 0 \;=\; K_1 \;+\; K_2 \;=\; K_1 \;+\; 1 \;\Rightarrow\; K_1 \;=\; -1\,\text{A}.$$

Therefore the current through the inductor in the interval $0 < t < 2\,\mathrm{s}$ is

$$i(t) \;=\; 1 \;-\; e^{-t/L}\ \mathrm{A}.$$

When the switch moves back to position A at $t = 2\,\mathrm{s}$, the current through the inductor remains

$$i(2^+) \;=\; i(2^-) \;=\; 1 - e^{-2/L}\,\mathrm{A}.$$

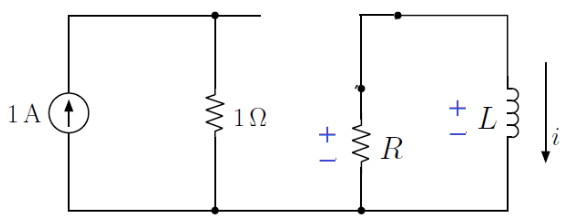
The condition of the problem says that

$$i(2) \;=\; 1 \;-\; e^{-1}\,\mathrm{A}$$

Therefore,

$$i(2) \;=\; 1 \;-\; e^{-2/L} \;=\; 1 \;-\; e^{-1} \;\Rightarrow\; L \;=\; 2\,\mathrm{H}.$$

At time $t = 2\,\mathrm{s}$, the switch moves back to position A, giving another resistive circuit with an inductor, as indicated in the figure below.



Now the general equation for the current after $t > 2$ has a new time constant:

$$\hat{\tau} \;=\; L/\hat{R}_T \;=\; L/R \;=\; 2/R,$$

giving

$$i(t) \;=\; K_3 e^{-(t-2)/\hat{\tau}} \;+\; K_4 \;=\; K_3 e^{-(t-2)R/2} \;+\; K_4.$$

With no source in the new circuit we have

$$i(t \to \infty) = \;0\ \mathrm{A}$$

and consequently

$$K_4 = 0\,\mathrm{A}.$$

Applying the initial condition we can obtain K_3:

$$i(2^+) \;=\; i(2^-) \;=\; 1 \;-\; e^{-1} \;=\; K_3 e^{-(2-2)R/2} \;+\; 0 \;\Rightarrow\; K_3 \;=\; 1 \;-\; e^{-1}.$$

Hence the equation for the current through the inductor for $t > 2\,\mathrm{s}$ is

$$i(t) \;=\; \left(1 - e^{-1}\right) e^{-\frac{(t-2)R}{2}}.$$

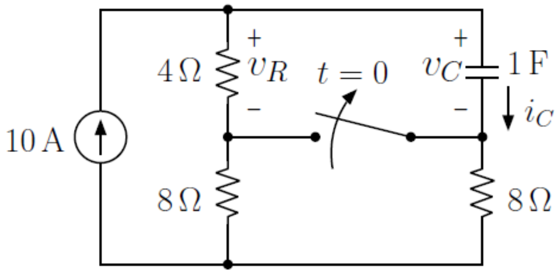
The second condition given in the problem tells us

$$i(8) \;=\; \left(1 - e^{-1}\right) e^{-2}\,\mathrm{A}.$$

Therefore,

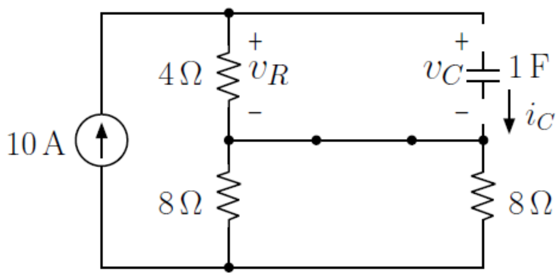
$$i(8) \;=\; \left(1 - e^{-1}\right) e^{-\frac{(8-2)R}{2}} \;=\; \left(1 - e^{-1}\right) e^{-2} \;\Rightarrow\; \frac{(8-2)\,R}{2} \;=\; 2 \;\Rightarrow\; R \;=\; \frac{2}{3}\,\Omega.$$

4. Assume that the switch has been closed for a long time and it opens at $t = 0\,\text{s}$.



(a) Determine $v_C(0^-)$, $v_R(0^-)$, and $i_C(0^-)$.

Solution: In steady-state, before the switch opens, the capacitor acts as an open circuit, resulting in the curcuit below:



There is no current flowing through the (open) fully charged capacitor, so

$$i_C(0^-) = 0 \text{ A}.$$

Since the capacitor is in parallel with the 4Ω before the switch opens,

$$v_R(0^-) \;=\; v_C(0^-).$$

The voltage right before $t = 0$ can be obtained from Ohm's Law on the 4Ω resistor. which gets the full current from the source:

$$v_R(0^-) \;=\; (10)\,(4) \;=\; 40\,\text{V}.$$

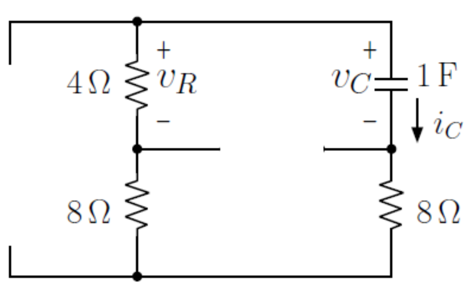
$$v_C(0^-) \;=\; v_R(0^-) \;=\; 40\,\text{V}.$$

(b) For $t > 0$, determine the zero-state response, $v_{C,ZS}(t)$, the zero-input response, $v_{C,ZI}(t)$ and the full response, $v_C(t)$.

Solution: After the switch opens, the general equation for the RC circuit with time constant $\tau = R_T C$ becomes

$$v_C(t) \;=\; K_1 e^{-t/(R_T C)} \;+\; K_2.$$

To obtain R_T, we suppress the current source, as seen in the figure below,



and we notice that the Thevenin resistor that the capacitor sees between its terminals is the series combination of all three resistors:

$$R_T = 4 + 8 + 8 = 20\,\Omega.$$

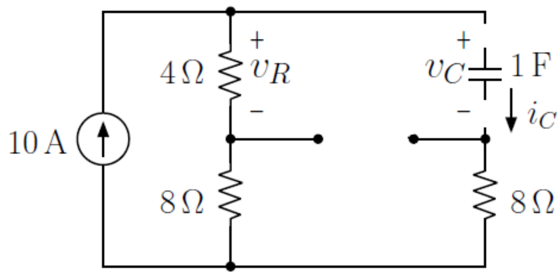
Hence, the time constant

$$\tau = R_T C = 20(1) = 20 \text{ s},$$

and the voltage equation becomes

$$v_C(t) = K_1 e^{-t/20} + K_2.$$

When $t \to \infty$, the capacitor opens again, as depicted in the figure below:



There will be no current flowing through the open capacitor nor the 8Ω resistor below it. The capacitor will be fully charged with the same voltage across the resistors in series $4\,\Omega$ and $8\,\Omega$. Hence, by Ohm's law on the series resistor with the current source I_s:

$$v_C(t \to \infty) = (4+8)\,I_s = 0 + K_2 \Rightarrow K_2 = 12I_s.$$

Finally applying the initial condition and using the fact that voltage in a capacitor is continuous:

$$v_C(0^+) = v_C(0^-) = K_1 + K_2 = K_1 + 12I_s \Rightarrow K_1 = v_C(0^-) - 12I_s.$$

Consequently the equation $v_C(t)$ for $t > 0$ becomes

$$v_C(t) = \big(v_C(0^-) - 12I_s\big)\,e^{-t/20} + 12I_s.$$

To obtain the zero-state solution, we simply force the initial state to be zero, that is

$$v_C(0^-) = 0,$$

which yields

$$\begin{aligned} v_{C,ZS}(t) &= (0 - 12I_s)\,e^{-t/20} + 12I_s = -12I_s e^{-t/20} + 12I_s = -12(10)e^{-t/20} + 12(10) \\ &= -120e^{-t/20} + 120 \text{ V}. \end{aligned}$$

To obtain the zero-input solution, we simply force the input to be zero, that is

$$I_s = 0,$$

which yields

$$v_{C,ZI}(t) = \big(v_C(0^-) - 12(0)\big)\,e^{-t/20} + 12(0) = v_C(0^-)e^{-t/20} = 40e^{-t/20} \text{ V}.$$

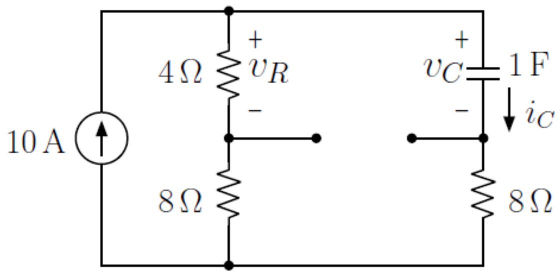
The full solution can be obtained from by adding the two components:

$$v_C(t) = v_{C,ZS}(t) + v_{C,ZI}(t) = 40e^{-t/20} - 120e^{-t/20} + 120 = 120 - 80e^{-t/20}\,\text{V}.$$

(c) For $t > 0$, determine $i_C(t)$.
Solution: We can find $i_C(t)$ using the v-i relation

$$i_C(t) \;=\; C\frac{dv_C}{dt} \;=\; (1)\frac{d}{dt}\left(120 \;-\; 80e^{-t/20}\right) \;=\; 4e^{-t/20}\,\text{A}.$$

(d) For $t > 0$, determine $v_R(t)$.
Solution:



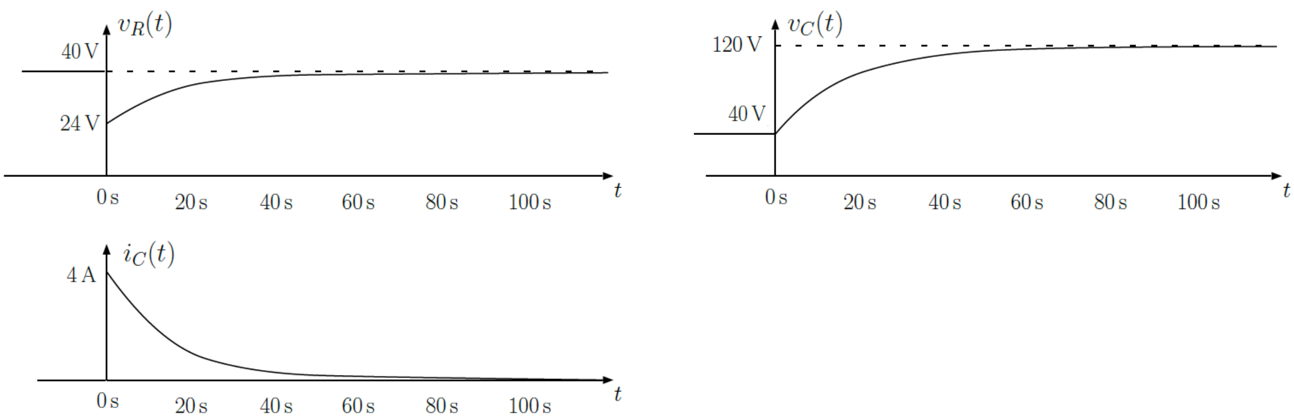
From a KCL in the top node and Ohm's law on the 4Ω resistor we can obtain

$$\frac{v_R(t)}{4} \;+\; i_C(t) \;=\; 10,$$

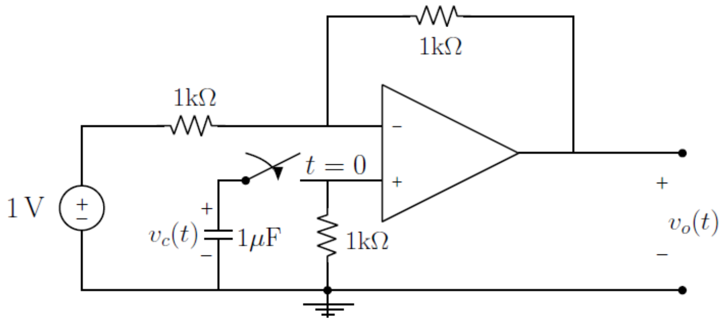
which yields the voltage across the resistor

$$v_R(t) \;=\; 40 \;-\; 4i_C(t) \;=\; 40 \;-\; 16e^{-t/20}\,\text{V}.$$

(e) Sketch $v_R(t), v_C(t)$, and $i_C(t)$ for $t > -1$ clearly labelling important points.
Solution:



5. Assuming linear operation and making use of the ideal op-amp approximations, determine $v_o(t)$ at $t = 2\,\text{ms}$ in the following circuit, given that $v_c(0) = -2\,\text{V}$.



Solution: Assuming ideal op-amp operation there is no current flowing into the op-amp through either terminals (+) nor (-). So we can isolate the RC circuit with time constant

$$\tau \;=\; RC \;=\; (1\text{k}\Omega)(1\mu\text{F}) \;=\; 1\text{ms},$$

which has the general solution:

$$v_c(t) \;=\; K_1 e^{-t/\tau} \;+\; K_2.$$

When $t \to \infty$ all the charge in the capacitor should be gone, therefore

$$v_c(t \to \infty) \;=\; 0 \;=\; K_2.$$

Applying initial condition yields

$$v_c(0^+) \;=\; v_c(0^-) \;=\; -2 \;=\; K_1 \;+\; K_2 \;=\; K_1 \;+\; 0 \;\;\Rightarrow\;\; K_1 \;=\; -2.$$

Therefore, $v_c(t)$ is

$$v_C(t) \;=\; -2e^{-\frac{t}{1\times 10^{-3}}}\,\text{V},$$

for $t > 0$.
Writing a KCL equation in the (-) terminal of the op-amp where

$$v_- \;\approx\; v_+ \;=\; v_c(t),$$

we have

$$\frac{1\text{V} \;-\; v_C(t)}{1\text{k}\Omega} \;=\; \frac{v_C(t) \;-\; v_0(t)}{1\text{k}\Omega},$$

which simplifies to

$$v_0(t) \;=\; 2v_C(t) \;-\; 1 \;=\; \left(-4e^{-\frac{t}{1\times 10^{-3}}} \;-\; 1\right)\,\text{V}.$$

Evaluating at $t = 1ms$,

$$v_0(t = 2\text{ms}) \;=\; \left(-4e^{-2} \;-\; 1\right)\;\text{V} \;\approx\; -1.54\;\text{V}.$$

6. Fill out the following survey before this homework's deadline. We will use these data to improve the course over time.

   - Your grade for this HW problem will not depend on the content of your survey responses, only on the completion of the survey.
   - There are no correct or incorrect answers. Your grade for this HW problem will not depend on the content of your form responses, only on the completion of the form.
   - Course staff will not have access to the data until after the course is over and the data will be anonymized before course staff looks at it.
   - The link to the survey is here:
     https://forms.illinois.edu/sec/609188301

- You do not need to submit an answer for this question on Gradescope, we will be notified of your submission and will upload the points to Canvas. Be sure to enter your netID correctly into the survey so your submission is recorded.
- If you do not have access to the survey, please email Prof. Alvarez.

Exam 1 coming up

- Date: Wednesday, September 20, 7.15-8.30pm.
- Review: Tuesday, September 19, 7-9 p.m., 1013 ECEB.
- Coverage: Exam 1 will cover up to the end of Section 3.4.2 and homeworks 1-4.
- Room assignments. Students with last names beginning with:
  * Aa - Bs will go to room ECEB 2017
  * Bt - Gf will go to room ECEB 3017
  * Gg - Jj will go to room ECEB 1015
  * Jk - Uq will go to room ECEB 1002
  * Ur - Zz will go to room ECEB 1013