University of Illinois **Fall 2023**

ECE 210/211 - Homework 10: Problems and Solutions

Due: Tuesday, October 31 by 11.59pm

Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
   sign: ________________________.

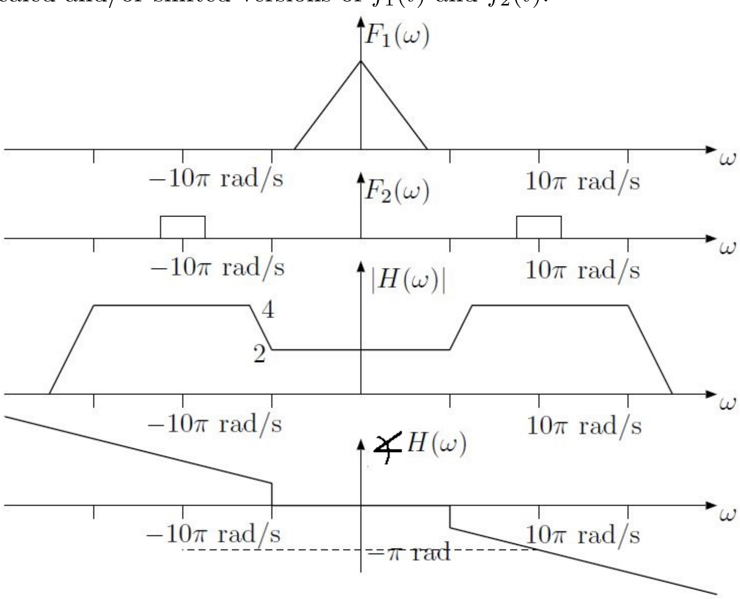
2. Let $f(t) = f_1(t) + f_2(t)$ be such that

$$f_1(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(\omega)$$

and

$$f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_2(\omega).$$

Let $f(t)$ be the input signal of an LTI system with a frequency response $H(\omega)$. The functions $F_1(\omega)$, $F_2(\omega)$, $|H(\omega)|$ and $\angle H(\omega)$ are given graphically below. Express the output of the system, $y(t)$, as a superposition of scaled and/or shifted versions of $f_1(t)$ and $f_2(t)$.



Solution: We are given $f(t) = f_1(t) + f_2(t)$, hence $F(\omega) = F_1(\omega) + F_2(\omega)$. Therefore

$$Y(\omega) = H(\omega)F(\omega) \;=\; H(\omega)F_1(\omega) \;+\; H(\omega)F_2(\omega).$$

In the region where $F_1(\omega) \neq 0$, $H(\omega) = 2$, so

$$Y_1(\omega) \;=\; H(\omega)F_1(\omega) \;=\; 2F_1(\omega) \;\overset{\mathcal{F}}{\longleftrightarrow}\; y_1(t) \;=\; 2f_1(t).$$

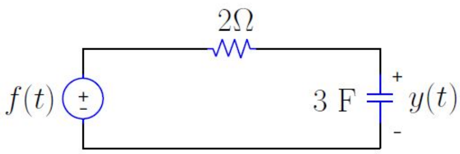
In the region where $F_2(\omega) \neq 0$, $|H(\omega)| = 4$ and the phase angle $\angle H(\omega)$ is linearly decreasing with a slope of $-\frac{1}{10}$, so $H(\omega) \;=\; 4e^{-j\frac{1}{10}\omega}$, and

$$Y_2(\omega) \;-\; H(\omega)F_2(\omega) \;=\; 4e^{-j\frac{1}{10}\omega}F_2(\omega) \;\overset{\mathcal{F}}{\longleftrightarrow}\; y_2(t) \;=\; 4f_2\left(t - \frac{1}{10}\right).$$

Finally, add the two components together to get

$$y(t) \;=\; 2f_1(t) \;+\; 4f_2\left(t - \frac{1}{10}\right).$$

3. Consider the circuit shown below.



   (a) Determine the response $y(t)$ of the circuit to an arbitrary input $f(t)$ and express it in the form of an inverse Fourier transform.

   Solution: Converting to a phasor circuit, the impedance from the capacitor becomes $Z_C = \frac{1}{j3\omega}\ \Omega$. Using voltage division, we have

$$Y(\omega) \;=\; F(\omega)\frac{\frac{1}{3j\omega}}{2+\frac{1}{3j\omega}} \;=\; F(\omega)\frac{1}{6j\omega+1}.$$

   Hence, the output in time becomes

$$y(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)d\omega \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{1}{6j\omega+1}e^{j\omega t}d\omega.$$

   (b) Evaluate your $y(t)$ from part (a) for the case $f(t) = e^{-\frac{t}{6}}u(t)\,\text{V}$.

   Solution: The input $f(t) = e^{-\frac{t}{6}}u(t)$, has Fourier transform $F(\omega) = \frac{1}{\frac{1}{6}+j\omega}$. Plugging into the integral from part (a),

$$Y(\omega) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{6}\frac{1}{(\frac{1}{6}+j\omega)^2}e^{j\omega t}d\omega.$$

   Using the inverse Fourier transform table we obtain

$$y(t) \;=\; \frac{1}{6}te^{-\frac{t}{6}}u(t)\ \text{V}.$$

4. Given that

$$f(t)e^{\pm j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} F(\omega \mp \omega_0),$$

   determine the Fourier transform of $f(t)\sin(\omega_0 t)$.

   Solution:

$$f(t)\sin(\omega_0 t) \;=\; f(t)\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) \;=\; -\frac{j}{2}f(t)e^{j\omega_0 t} \;+\; \frac{j}{2}f(t)e^{-j\omega_0 t}.$$

   Using the frequency-shift property given in the problem statement, we obtain

$$f(t)\sin(\omega_0 t) \;\overset{\mathcal{F}}{\longleftrightarrow}\; -\frac{j}{2}F(\omega-\omega_0) \;+\; \frac{j}{2}F(\omega+\omega_0).$$

5. Given that

$$f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}F(\omega-\omega_o) + \frac{1}{2}F(\omega+\omega_o),$$

   determine the Fourier transform of

$$f\left(t+\frac{\theta}{\omega_o}\right)\cos(\omega_o t+\theta).$$

Solution: We can rewrite

$$f\left(t+\frac{\theta}{\omega_0}\right)\cos(\omega_o t+\theta) \;=\; f\left(t+\frac{\theta}{\omega_o}\right)\cos\left(\omega_o\left(t+\frac{\theta}{\omega_0}\right)\right).$$

Let $g(t) = f(t)\cos(\omega_0 t)$, so

$$g\left(t+\frac{\theta}{\omega_0}\right) \;=\; f\left(t+\frac{\theta}{\omega_o}\right)\cos\left(\omega_o\left(t+\frac{\theta}{\omega_0}\right)\right).$$

We have transformed the problem to get the Fourier transform of $g(t+\frac{\theta}{\omega_0})$. Using time-shifting property,

$$g\left(t+\frac{\theta}{\omega_0}\right) \;\overset{\mathcal{F}}{\longleftrightarrow}\; G(\omega)e^{j\omega\frac{\theta}{\omega_0}}.$$

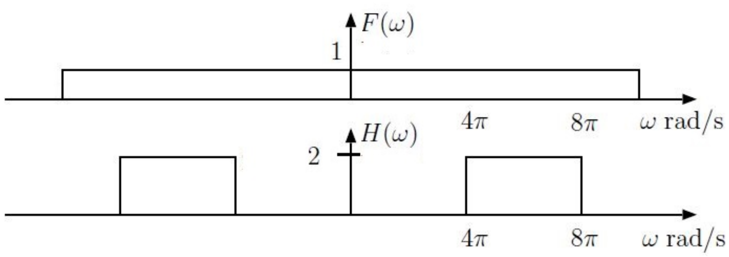
Together with the modulation property

$$g(t) \;=\; f(t)\cos(\omega_0 t) \;\overset{\mathcal{F}}{\longleftrightarrow}\; G(\omega) \;=\; \frac{1}{2}F(\omega-\omega_o) \;+\; \frac{1}{2}F(\omega+\omega_o),$$

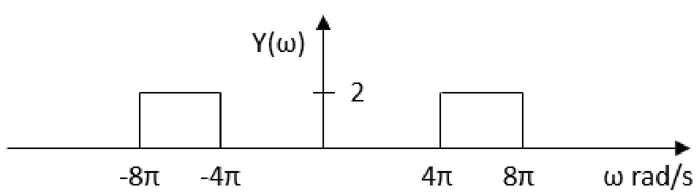
we have

$$f\left(t+\frac{\theta}{\omega_o}\right)\cos(\omega_o t+\theta) \;\overset{\mathcal{F}}{\longleftrightarrow}\; \left(\frac{1}{2}F(\omega-\omega_o)+\frac{1}{2}F(\omega+\omega_o)\right)e^{j\omega\frac{\theta}{\omega_0}}.$$

6. A linear system with frequency response $H(\omega)$ is excited with an input $f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)$. The Fourier transforms $H(\omega)$ and $F(\omega)$ are plotted below:



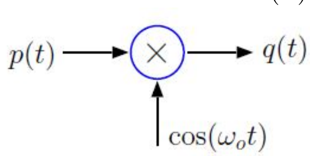
(a) Sketch the Fourier transform of the system output, $Y(\omega)$, and calculate the energy content of $y(t)$, W_y.
Solution: The output of the system is $Y(\omega) = H(\omega)F(\omega)$. Since $F(\omega) = 1$ for all $H(\omega) \neq 1$, we have $Y(\omega) = H(\omega)$.



Using Parseval's theorem, the energy of the signal is

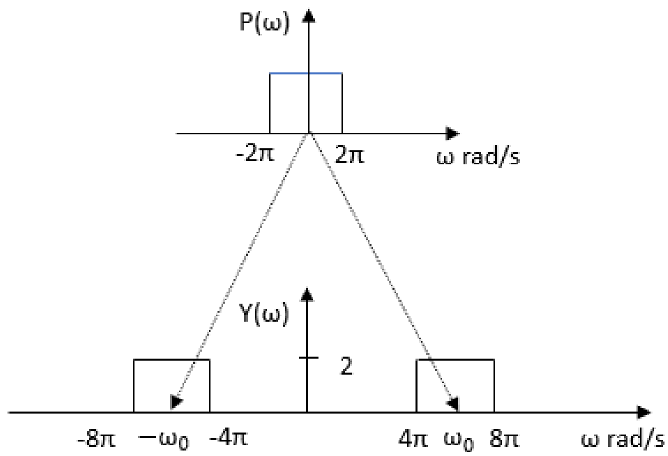
$$W_y \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2 d\omega \;=\; \frac{1}{2\pi}(8\pi-4\pi)(2)^2(2) \;=\; 16.$$

(b) It is observed that output $q(t)$ of the following system equals $y(t)$ determined in part (a). Sketch the Fourier transform $P(\omega)$ and determine ω_o.



Solution: Using $Y(\omega) = Q(\omega)$ and the modulation property

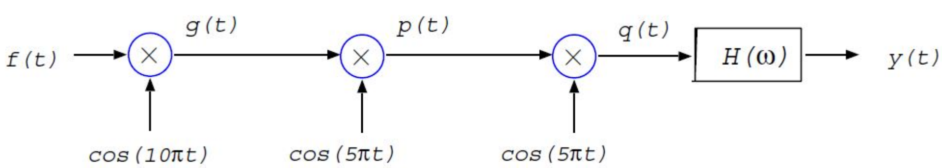
$$Y(\omega) \;=\; \frac{1}{2}P(\omega - \omega_0) \;+\; \frac{1}{2}P(\omega + \omega_0).$$



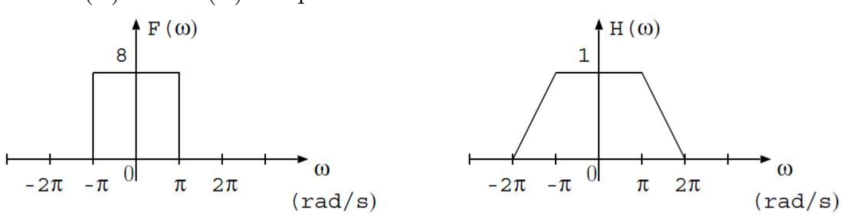
From the graph we can determine that

$$\omega_0 = 6\pi \text{ rad/s}.$$

7. Consider the system below



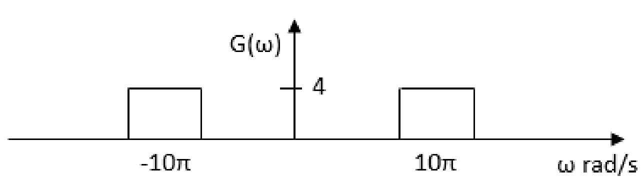
where $F(\omega)$ and $H(\omega)$ are plotted below.



(a) Express $g(t)$ in terms of $f(t)$, express $G(\omega)$ in terms of $F(\omega)$ and sketch $G(\omega)$.

Solution:

$$g(t) \;=\; f(t)\cos(10\pi t) \;\xleftrightarrow{\mathcal{F}}\; G(\omega) \;=\; \frac{1}{2}F(\omega - 10) \;+\; \frac{1}{2}F(\omega + 10)$$

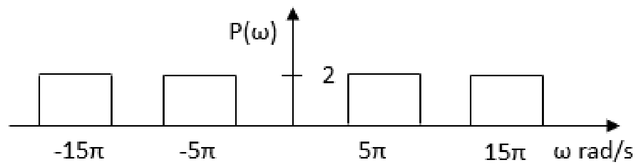


(b) Express $p(t)$ in terms of $f(t)$, express $P(\omega)$ in terms of $F(\omega)$ and sketch $P(\omega)$.

Solution:

$$p(t) \;=\; g(t)\cos(5\pi t) \;=\; f(t)\cos(10\pi t)\cos(5\pi t)$$

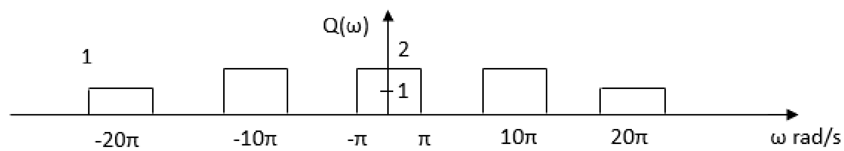
$$\Rightarrow\; P(\omega) \;=\; \frac{1}{2}G(\omega-5) \;+\; \frac{1}{2}G(\omega+5) \;=\; \frac{1}{4}F(\omega-15) \;+\; \frac{1}{4}F(\omega+5) \;+\; \frac{1}{4}F(\omega-5) \;+\; \frac{1}{4}F(\omega+15)$$



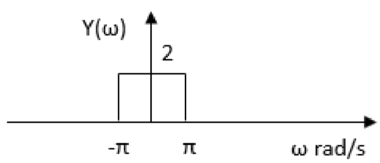
(c) Express $q(t)$ in terms of $f(t)$, express $Q(\omega)$ in terms of $F(\omega)$ and sketch $Q(\omega)$.
Solution:

$$q(t) \;=\; p(t)\cos(5\pi t) \;=\; f(t)\cos(10\pi t)\cos^2(5\pi t)$$

$$\Rightarrow\; Q(\omega) \;=\; \frac{1}{2}P(\omega-5)+\frac{1}{2}P(\omega+5) \;=\; \frac{1}{8}F(\omega-20)+\frac{1}{4}F(\omega)+\frac{1}{4}F(\omega-10)+\frac{1}{4}F(\omega+10)+\frac{1}{8}F(\omega+20)$$



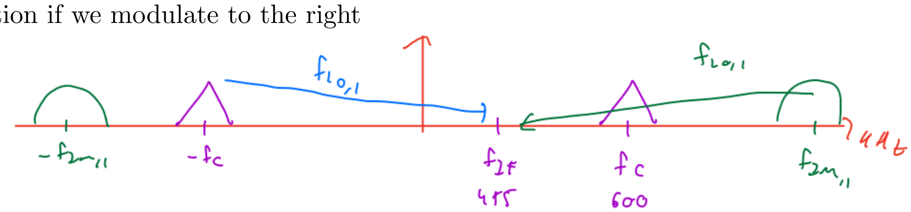
(d) Express $y(t)$ in terms of $f(t)$, express $Y(\omega)$ in terms of $F(\omega)$ and sketch $Y(\omega)$.
Solution:



Clearly from the graph, $Y(\omega) = \frac{1}{4}F(\omega)$ and hence $y(t) = \frac{1}{4}f(t)$.

8. Consider a superheterodyne AM receiver. If we want to listen to the radio station with carrier frequency $fc = 600kHz$, what is the required local oscillator frequency and the resulting image station frequency, assuming that the intermediate frequency is 455kHz?

Solution: There are two possibilities to modulate the signal, left or right. The figure below depicts the situation if we modulate to the right



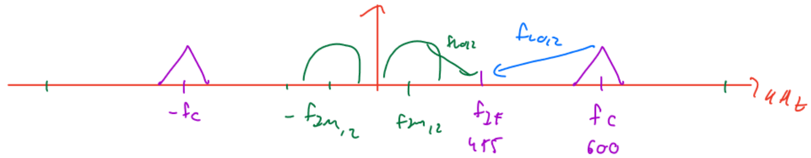
Notice that in this case we need

$$-f_c \;+\; f_{LO} \;= f_{IF} \;\Rightarrow\; f_{LO} \;= f_{IF} \;+\; f_c \;=\; 455+600 \;=\; 1055 \text{ kHz}.$$

On the othe hand,

$$f_{IM} \;-\; f_{LO} \;= f_{IF} \;\Rightarrow\; f_{IM} \;= f_{IF} \;+\; f_{LO} \;=\; 455+1055 \;=\; 1510 \text{ kHz}.$$

The figure below depicts the situation if we modulate to the left



Notice that in this case we need

$$f_c \; - \; f_{LO} \; = f_{IF} \;\Rightarrow\; \; f_{LO} \; = f_c \; - \; f_{IF} \; = \; 600 - 455 \; = \; 145 \text{ kHz}.$$

On the othe hand,

$$f_{IM} \; + \; f_{LO} \; = f_{IF} \;\Rightarrow\; \; f_{IM} \; = f_{IF} \; - \; f_{LO} \; = \; 455 - 145 \; = \; 310 \text{ kHz}.$$

Hence, there are two pairs of possibilities:

$$(f_{LO},\ f_{IM}) \; = \; (1055 \text{ kHz},\ 1510 \text{ kHz})$$

and

$$(f_{LO},\ f_{IM}) \; = \; (145 \text{ kHz},\ 310 \text{ kHz}).$$

