University of Illinois **Fall 2023**

ECE 210/211 - Homework 13: Problems and Solutions

Due: Wednesday, November 28 by 11.59pm

Problems:

1. Sign acknowledging you will abide by this course's and the University's Academic Integrity policies or face sanctions for not doing so. These policies include, among others, plagiarism: representing the words or ideas of others as your own; cheating: using unauthorized materials and/or information. The possible sanctions include, among others, reduced letter grades and an F in the course. If your solution upload does not include your signature, your homework will NOT be graded, resulting in a zero.
   sign: ______________________________.

2. For each one of the following signals, $f(t)$, do the following:

   i. Obtain its Laplace transform $\hat{F}(s)$.
   ii. Indicate the poles of $\hat{F}(s)$.
   iii. Indicate the region of convergence, ROC, of $\hat{F}(s)$.

   (a) $f(t) = u(t+1) - u(t-6)$.
   Solution:

   i. Since Laplace transform ignores the signal for $t < 0$, then $u(t+1)$ is equivalent to $u(t)$. Hence
   $$\mathcal{L}\{u(t+1) - u(t-6)\} = \mathcal{L}\{u(t) - u(t-6)\}.$$
   Using the Laplace transform table and time shift property,
   $$\hat{F}(s) = \frac{1}{s} - \frac{1}{s}e^{-6s} = \frac{1 - e^{-6s}}{s}$$
   iii. The pole is at $s = -\infty$ because the exponential will go to infinity. Notice, from L'Hopitals, that there is no pole at $s = 0$.
   iii. The ROC is the region to the right of the right-most pole: $\{s : \sigma = \mathrm{Re}\{s\} > -\infty\}$.

   (b) $f(t) = 2te^{2t-4}u(t+2)$.
   Solution:

   i. Similar to part (a), $u(t+2)$ is equivalent to $u(t)$ when evaluating the Laplace transform. Furthermore, we can rewrite the function as
   $$\mathcal{L}\left\{2te^{2t-4}u(t+2)\right\} = \mathcal{L}\left\{2te^{2t-4}u(t)\right\} = 2e^{-4}\mathcal{L}\left\{te^{2t}u(t)\right\}.$$
   Using the Laplace transform table,
   $$\hat{F}(s) = \frac{2e^{-4}}{(s-2)^2}.$$
   ii. The poles are at $s = 2, 2$ because the denominator is zero there.
   iii. The ROC is the region to the right of the right-most pole: $\{s : \sigma = \mathrm{Re}\{s\} > 2\}$.

   (c) $f(t) = (t-1)e^{-4t} + 3\delta(t)$.
   Solution:

   i. We can rewrite the function as
   $$f(t) = te^{-4t} - e^{-4t} + 3\delta(t).$$
   Using the Laplace transform table,
   $$\hat{F}(s) = \frac{1}{(s+4)^2} - \frac{1}{s+4} + 3 = \frac{1 - (s+4) + 3(s+4)^2}{(s+4)^2} = \frac{1 - s - 4 + 3s^2 + 24s + 48}{(s+4)^2} = \frac{3s^2 + 23s + 44}{(s+4)^2}.$$

ii. The poles are at $s = -4, -4$ because the denominator is zero there.

iii. The ROC is the region to the right of the right-most pole: $\{s : \sigma = \mathrm{Re}\{s\} > -4\}$.

(d) $f(t) = e^{3t}\cos(t)$.

Solution:

i. Using the Laplace transform table,

$$\hat{F}(s) = \frac{s-3}{(s-3)^2+1}.$$

ii. We find poles at $s = 3 \pm j$ because the denominator is zero there.

iii. The ROC is the region to the right of the right-most pole: $\{s : \sigma = \mathrm{Re}\{s\} > 3\}$.

(e) $f(t) = \frac{d}{dt}\left(t^2 u(t)\right)$.

Solution:

i. Taking the derivative, we have

$$f(t) = 2tu(t) + t^2\delta(t) = 2tu(t).$$

Using the Laplace transform table,

$$\hat{F}(s) = \frac{1}{s^2}.$$

ii. The poles are at $s = 0,\ 0$ because the denominator is zero there.

iii. The ROC is the region to the right of the right-most pole$\{s : \sigma = \mathrm{Re}\{s\} > 0\}$.

3. Determine if the LTIC systems with the following transfer functions are BIBO stable and explain why or why not.

(a) $\hat{H}_1(s) = \frac{s+1}{(s+1+j6)(s+1-j6)}$.

Solution: The poles for this system are at $s = -1 \pm j6$, which are on the LHP. Hence, the system is BIBO stable.

(b) $\hat{H}_2(s) = \frac{s^4+4s^3+6s^2+4s+1}{(s+1+j6)(s+1-j6)}$.

Solution: We observe that the system has higher order polynomial in the numerator than in the denominator, so that there is a pole at $s = \infty$, which is not on the LHP. Hence, the system is not BIBO stable.

(c) $\hat{H}_3(s) = 2 + \frac{s}{(s+1)(s-2)}$.

Solution:

$$\hat{H}_3(s) = 2 + \frac{s}{(s+1)(s-2)} = \frac{2(s^2-s-2)+s}{(s+1)(s-2)} = \frac{2s^2-2s-3}{(s+1)(s-2)}.$$

The system has a pole at $s = 2$, which is not on the LHP. Hence, the system is not BIBO stable.

(d) $\hat{H}_4(s) = \frac{1}{s^2+16}$.

Solution: The poles for this system are at $s = \pm j4$, which is not on the LHP. Hence, the system is not BIBO stable.

(e) $\hat{H}_5(s) = \frac{s-2}{s^2-4}$.

Solution: The roots of the denominator polynomial are at $s = \pm 2$, but the pole at $s = 2$ is cancelled by the zero at the same location. The pole is at $s = -2$, which is on the LHP. Hence, the system is BIBO stable.

(f) $\hat{H}_6(s) = \frac{e^s}{s+4}$.

Solution: We observe that the numerator is an exponential function, which grows faster than the first order polynomial at the denominator, such that there is a pole at $s = \infty$, which is not on the LHP. Hence, the system is not BIBO stable.

4. Fill out the following survey before this homework's deadline. We will use these data to improve the course over time.
    - Your grade for this HW problem will not depend on the content of your survey responses, only on the completion of the survey.
    - There are no correct or incorrect answers. Your grade for this HW problem will not depend on the content of your form responses, only on the completion of the form.
    - Course staff will not have access to the data until after the course is over and the data will be anonymized before course staff looks at it.
    - The link to the survey is here:
      https://forms.illinois.edu/sec/306204166
    - You do not need to submit an answer for this question on Gradescope, we will be notified of your submission and will upload the points to Canvas.
    - If you do not have access to the survey, please email Prof. Alvarez.