Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 7

The goal of this worksheet is to study the abstract concepts of bases and dimensions of vector spaces. Focusing on null spaces and column spaces, you will develop an intuition for these mathematical objects by trying to understand how they arise in practical applications.

(1) We start by reviewing a few concepts from class.

Let V be a vector space. A sequence of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ in V is a **basis** of V if

- $V = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and
- $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ are linearly independent.

(a) As a group, discuss why $\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ is a basis of $\mathbb{R}^3$.

> Just check the defining properties of a basis. Actually, since we already know that $\dim(\mathbb{R}^3) = 3$ it suffices to check just one of the two properties, as explained in the lectures.

(b) Every vector space V with $\dim V \geq 1$ has infinitely many different bases. Draw your basis from part (a) and use this to propose infinitely many other bases for $\mathbb{R}^3$.

(c) Take the three vectors in (a) and make them the columns of a 3×3 matrix A in the given order. Let matrix B be the result of doing an elementary row operation to A. Explain why the columns of B are a basis for $\mathbb{R}^3$. Do you see how to create infinitely many bases for $\mathbb{R}^3$?

> Row equivalent matrices have the same nullspace. Hence, $\operatorname{Nul}(B) = \operatorname{Nul}(A)$ remains trivial so that the columns of B are linearly independent and hence a basis for $\mathbb{R}^3$.

(d) Let A be an $m \times n$ matrix. We want to review some of the things you learnt about the dimensions of $\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$.

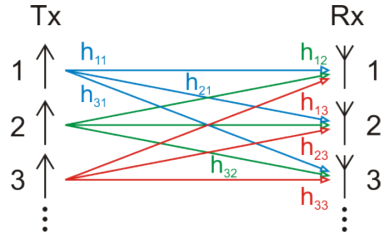
(I) Suppose the columns of A are linearly independent. What are the dimensions of $\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$?

> $\dim(\operatorname{Col}(A)) = n$ and $\dim(\operatorname{Nul}(A)) = 0$.

(II) Suppose $\dim \operatorname{Nul}(A) = \{\mathbf{0}\}$. Discuss as a group why the columns of A have to be linearly independent?

Write $A = \begin{pmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{pmatrix}$ where $\mathbf{a}_i$ are the columns of A. Then, $\mathbf{0} = \sum_{i=1}^{n} x_i \mathbf{a}_i = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ so that the columns of A are linearly independent iff $\mathrm{Nul}(A)$ is trivial.

(2) In multiple-input and multiple-output (short: MIMO) systems, a transmitter sends multiple streams by multiple transmit antennas.



Suppose there are n transmitters and m receivers. This can be modeled using linear algebra:

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}}_{\text{received vector } \mathbf{y}} = \underbrace{\begin{bmatrix} h_{1,1} & \dots & h_{1,n} \\ \vdots & \ddots & \vdots \\ h_{m,1} & \dots & h_{m,n} \end{bmatrix}}_{\text{channel matrix } H} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{transmitted vector } \mathbf{x}} .$$

Here the vector $\mathbf{x}$ describes what the transmitters are sending out. So each of the n transmitter sends out a single number (x_1 is what transmitter 1 sends out, x_2 is what transmitter 2 sends out, and so on). The vector $\mathbf{y}$ is the vector describing what is received. So each receiver receives a single number (y_1 is what receiver 1 receives, y_2 is what receiver 2 receives, and so on). The channel matrix H tells you how the signal $\mathbf{x}$ is transformed into the vector $\mathbf{y}$.

(a) If the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is transmitted, what vector is received?

The received vector is $\begin{pmatrix} h_{1,1} + h_{1,2} \\ \vdots \\ h_{m,1} + h_{m,2} \end{pmatrix}$.

(b) In terms of what is received by the m receivers, what does the first column of H represent? What does the i-th column represent?

The first column represents what the receivers receive from the first transmitter. Note that if the sender transmits the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$, then the received vector $\mathbf{y}$ will be the first column of H. Similarly, the i-th column represents what the receivers receive from the i-th transmitter.

(c) In terms of signals, what is $\mathrm{Col}(H)$?

The set of signals that can be received. All the possible signals that can be received, are precisely the linear combinations of all the columns of H.

(d) If the signal $\mathbf{x}$ belongs to the nullspace of H, what signal $\mathbf{y}$ will be received?

The receivers will receive no signal, since $H\mathbf{x} = \mathbf{0}$.

(e) In a well designed system, what do you want $\mathrm{Nul}(H)$ to be? What does that tell you about the columns of H?

We want that $\mathrm{Nul}(H) = \{\mathbf{0}\}$. This guarantees that the receiver receive no signal if and only if no signal is transmitted. We know that if $\dim \mathrm{Nul}(H)$, then the columns of H have to be linearly independent.

(f) If you add one more transmitters, how does H change and how does the column space of H change? How do you have to choose the new transmitter in order to be able receive more messages?

Each new transmitter adds a new column to H. The column space of H (and hence the set of message we can receive) only increase if the new column was not already in the column space.

(g) Suppose your colleague built a system of transmitters that is not well designed (ie. $\mathrm{Nul}(H) \neq \{\mathbf{0}\}$). Your boss tells you to fix this error by simply tearing down some of the transmitters. Can you do so without changing what kind of message can be received? If so, how do you determine the transmitters you have to tear down. (Hint: how do you find a basis of $\mathrm{Col}(H)$?)

Determine the columns of H indexed by free variables. Destroy the transmitters corresponding to these columns. This guarantees that the column space does not change.

(h) Oh no, a tornado destroyed the n transmitters! Your company built n new transmitters and tells you that the new channel matrix is H'. You are told that the column space of H is equal to the column space of H'. How can you check that this claim is true?

Check that every column of H is a linear combination of the columns of H' and vice versa.