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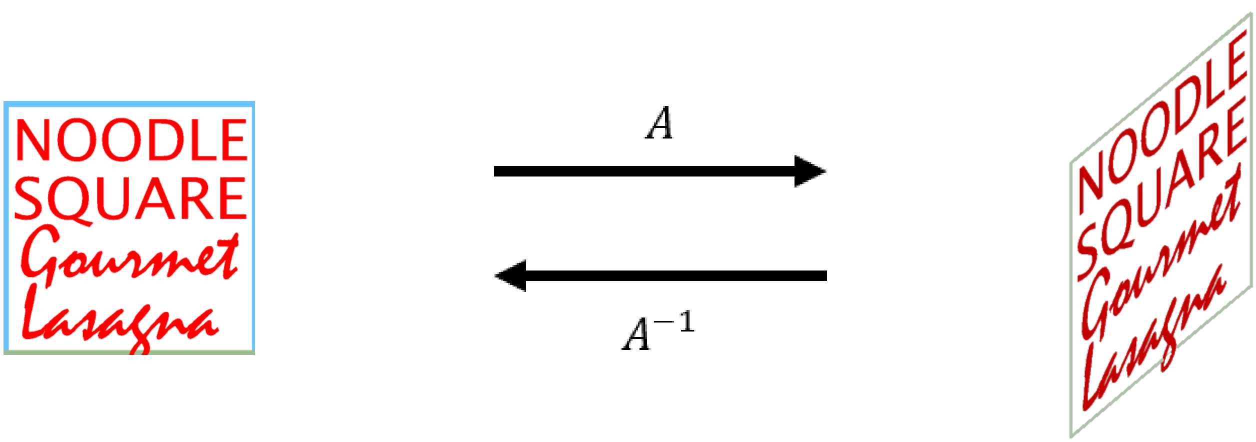
MATH 257 - WORKSHEET 5: CONCEPTUAL PROBLEM SOLVING

One of the challenges of applied linear algebra is modeling the real world in mathematical language. Once a model is created, the challenge becomes to determine the characteristics of that model. What is true in the *modeled* world?

Let's say you've modeled a common engineering problem as a system of linear equations that follows a certain pattern. You might ask: "Can we use the pattern to devise a method for quickly solving the system?" If you can prove that such a method exists, the model becomes easier to use.

Throughout this course, you'll practice this type of challenge by solving conceptual problems that ask you to verify the truth of mathematical statements. **Unlike computational problems, conceptual problems usually don't have an obvious method for solving them. You'll need to creatively use your conceptual understanding.**

Today, your group will practice solving conceptual problems about matrix inverses. Since we won't talk about the real-world meaning of these concepts much today, let's briefly go back to the sign we worked with in week 3 to think about an analogy.



If A is an invertible matrix that turns a square sign into a distorted sign, A^{-1} is the matrix that turns the distorted sign back into a square sign.

(1) Consider the following problem.

> Let A be an $n \times n$ matrix such that A^2 is the zero matrix. Show that the matrix $A - I_n$ is invertible and find its inverse.

(a) Work together as a group to try solving this problem for no more than about 5 minutes. This is a difficult problem, so you might get stuck. The next part will guide you through one mathematician's thought process for solving it.

If your group manages to solve the problem quickly, congrats! You can use the next part to understand your solution better and learn some tools for the next time you get stuck.

(b) *"Maybe we can get some intuition from* 1×1 *matrices."*

Let B be a 1×1 matrix. Work as a group to figure out all the possible values of B such that $B^2 = 0$. For each, determine the inverse of $B - I_1$.

The only 1×1 matrix that squares to zero is $B = [0]$. In this case,

$$B - I_1 = [0] - [1] = [-1].$$

The inverse of $B - I_1$ is just $[-1]$.

(c) *"Did that help?"*

Discuss: From considering the 1×1 matrices in part (b), what insight did you gain into the original problem, if any? Summarize your discussion below in one or two sentences. If you're not sure, go on to part (d) and revisit this question later.

It's hard to get any sense of a pattern from this that can be extended to larger dimensions. The main issue is that there are larger matrices that square to zero without necessarily being zero. Thus when considering $n \times n$ matrices, we have to consider infinitely many different matrices, and not just one. So we have to find a method that works for all of them.

(d) *"There might be more interesting* 2×2 *matrices with the right property."*

Together, invent one example of a nonzero 2×2 matrix C such that $C^2 = 0$. **Determine the inverse of** $C - I_2$ (not the inverse of C!).

One example is $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. $C - I_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$. We can use the formula for the inverse of a 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ here, $M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, to get the inverse $\frac{1}{(-1)(-1)-(0)(1)} \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$. Alternatively, we could use row reduction.

(e) *"Maybe we can use this example to guess the general solution."*

Work together to figure out how to express the **inverse** of $C - I_2$, which you calculated in part (d), as a linear combination of C and I_2.

Since the diagonal elements of C are zero, the term with I_2 is solely responsible for the diagonal elements of the inverse of $C - I_2$. That term must have a coefficient of -1. That leaves the bottom–left element, which is -1 times its value in C. Indeed $-C - I_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$.

(f) *"Let's check if our guess works in general."*

Rewrite your answer to (e) with the variables A and I_n in place of C and I_2 to arrive at a **guess for the inverse of** $A - I_n$ **in general**. Multiply your guess by $A - I_n$ and discuss the result to figure out if your guess satisfies the definition of the inverse.

If there is any doubt or disagreement, take this opportunity to discuss it.

Our guess for the inverse of $A - I_n$ is $-A - I_n$. To check this, we multiply the inverse by the original to see if we get the identity matrix I:

$$(A - I_n)(-A - I_n) = -A^2 - AI_n + I_nA + I_n^2 = 0 - A + A + I_n = I.$$

So it works!

(g) *"The solution doesn't need all our scratch work. Let's present a clean general solution."*

Write out a general solution to the original problem, using variables and equations and avoiding specific examples or vague reasoning.

The matrix $A - I_n$ is invertible, because its inverse is $-A - I_n$. In order to check this, we just have to show that $(A - I_n)(-A - I_n) = I_n$. By the definition of the inverse, this makes $-A - I_n$ the inverse of $A - I_n$. We compute

$$\begin{aligned}(A - I_n)(-A - I_n) &= A^2 - AI_n + I_nA + I_n^2 \\ &= 0 - A + A + I_n \\ &= I_n.\end{aligned}$$

(2) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

(a) Think back to part (1)(a). How did it feel to look at the problem for the first time?

(b) How did it feel to work with an example compared to the general case? What was useful about coming up with examples, and what were the limitations?

(c) The provided thought process involves using both a *computational* approach to the inverse of a written–out matrix and a *abstract* approach to the inverse based on matrix algebra using the definition. How was each approach useful? How necessary was each for your final solution?

(d) In retrospect, can you think of a way to discover the general solution directly, without exploring example matrices?

Summarize your discussion below in 3–4 bullet points.

(a) Maybe intimidating! Maybe it piques some curiosity too. It takes practice to draw on your knowledge and put the puzzle pieces together when you don't yet know what will work and what won't.

(b) Maybe a relief. Maybe confusing—how will this help? Examples can help you get a grasp of a very abstract problem so you can think about it and try to generalize. But some examples don't capture enough of the complexity of the general case to help you generalize. In this case, a 2×2 matrix was just complicated enough.

(c) The computational approach is useful when you have written-out matrices that you can row-reduce. It doesn't help when you try to reason about arbitrary matrices.

(d) One option is factoring. Start with $A^2 = 0$ and manipulate the equation to try to get I_n on the right-hand side. That suggests that we should aim to get the identity $A^2 - I_n = (A - I_n)(A + I_n)$ on the left-hand side. (This identity is the matrix version of the more familiar $a^2 - 1 = (a-1)(a+1)$.) The trick is to start by *subtracting* I_n from both sides and later multiply both sides by -1.

Nice work! By following a mathematician's train of thought, your group has reasoned your way to a solution. The problem below can be approached with a similar process.

(3) Solve the following problem.

> Consider an invertible $n \times n$ matrix $A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}$. Suppose A has the property $A^{-1} = A^T$. What properties must the columns of A have (individually, and in pairs)?

Hint: try using the definition of the inverse and starting with matrices with only a few columns. Coming up with a specific matrix and plotting its columns as vectors might help too.

> Using the definition of the inverse, the property can be rephrased as $A^T A = I_n$. The only 1×1 matrices with this property are $[1]$ and $[-1]$. Those don't help us much because we can't look at pairs of columns, and the transpose doesn't do anything. If try this with a 2×2 matrix, you should find that multiplying A^T by A takes the inner product of the columns: the inner product of each column with itself gives a diagonal element of the product matrix, and the inner product of one column with the other gives the off-diagonal elements of the product matrix. If you find a specific 2×2 matrix with this property and plot its columns, you should notice that the columns are 1 unit long and perpendicular. Generalizing, you can see that in order to get the identity matrix from the matrix multiplication $\begin{bmatrix} — & \mathbf{v}_1^T & — \\ & \vdots & \\ — & \mathbf{v}_n^T & — \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}$, the inner product of each column with itself must equal 1 (i.e. every column has length 1), and the inner product of any two different columns must equal 0 (i.e. every column is orthogonal to every other column).