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MATH 257 - WORKSHEET 2

Your team has been hired by your city to fill a pair of storefronts in a business district. Each business would attract some number of adult and child customers to the business district per day based on its width. To encourage a thriving but not overcrowded business district, the pair of businesses need to have the right widths so that together they:

(1) Attract two hundred (200) adults to the district every day (not more or less).
(2) Attract fifty (50) children to the district every day (not more or less).

The total width is not fixed. The table below summarizes the five candidate businesses.

Candidate business	**Adult customers attracted per day**	**Children attracted per day**
Hair salon	12 per meter of width	2 per meter of width
Cafe	18 per meter of width	3 per meter of width
Deli	12 per meter of width	3 per meter of width
Shoe store	8 per meter of width	2 per meter of width
Toy store	10 per meter of width	5 per meter of width

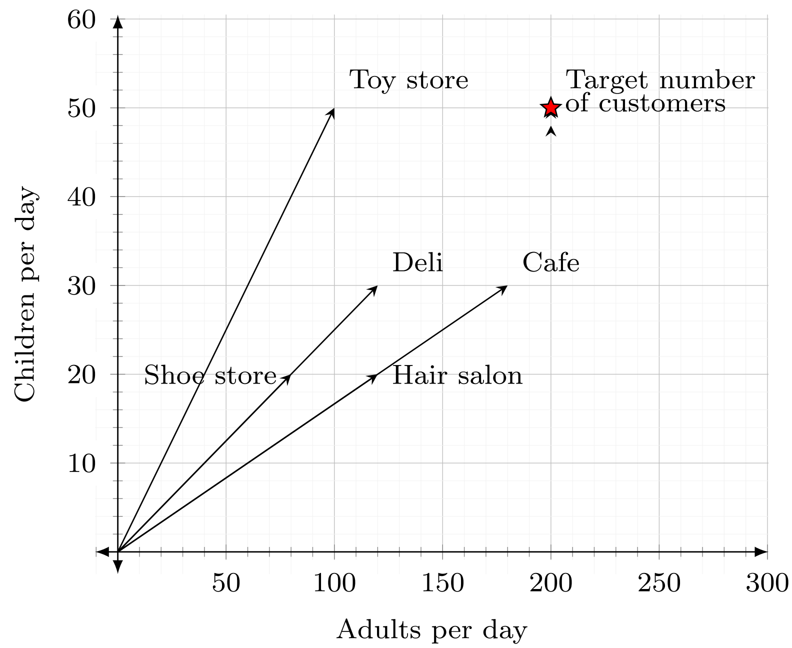
Reminder: For a matrix in echelon form, a **pivot position** is the position of a leading 1. A **free variable** is a variable corresponding to a column with no pivot.
Example:

$$\left[\begin{array}{ccc}1 & 0 & -2\\ 0 & 1 & -3\\ 0 & 0 & 0\end{array}\right]\left[\begin{array}{c}-2\\ -3\\ 0\end{array}\right]$$

Pivot position
x_3 is a free variable

(1) In this problem, your team will gain intuition for pivots and free variables by using a visualization of a linear system.

The candidate businesses are plotted as vectors below. Adjusting the width of a business corresponds to adjusting the length of the corresponding vector while keeping its slope the same. To visualize a successful pairing, imagine picking two businesses and adjusting their widths so that the vectors add together tip-to-tail to reach the star.



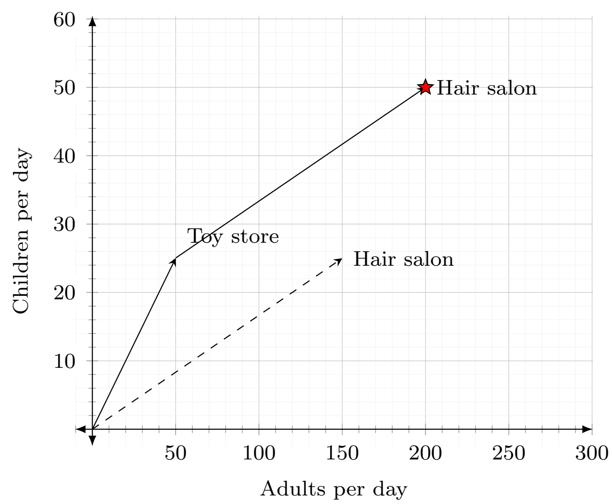
(a) Using the numbers from the table, set up the augmented matrix for the following pairing: **the toy store and the hair salon**. Put the system in echelon form (not RREF) to check the number of pivots and free variables. Discuss: Why should this system have a solution, and why should it be unique? Explain below.

Let's call the toy store's width T and the hair salon's width H. Then to get 200 adult customers, we need $10T + 12H = 200$. Likewise, $5T + 2H = 50$. So the augmented matrix is

$$\left[\begin{array}{cc|c} 10 & 12 & 200 \\ 5 & 2 & 50 \end{array}\right] \xrightarrow{R_2 - 0.5R_1} \left[\begin{array}{cc|c} 10 & 12 & 200 \\ 0 & -4 & -50 \end{array}\right]$$

This system has 2 pivots and no free variable. We can see that continuing row reduction will give us a full diagonal of ones on the left, which means each variable in the system will be assigned to a single value on the right.

(b) Now use the plotted vectors to visualize the pairing of the toy store and the hair salon. Discuss: Why can this system reach the target, and why is the solution unique? Explain below. Draw the solution on the plot as two vectors added tip-to-tail.



The toy store points to one side of the target, and the hair salon points to the other side. So the toy store's width can be picked until the remaining separation between the tip of that vector and the target is parallel to the hair salon vector; then the hair salon can do the rest. That means we can find a solution, and since those two lines only have one intersection point, there's only one way to do it.

(c) Now let's look at some pairings that don't have exactly one solution.

(i) Use the plotted vectors to find a business pairing that has infinitely many solutions. How many pivots and free variables does it have?

Pairing the deli with the shoe store yields infinitely many solutions. You can see that since both vectors point directly at the target, length in one vector can be reduced and the other increased by any amount to get another solution. Writing the pair's augmented matrix:

$$\left[\begin{array}{cc|c} 12 & 8 & 200 \\ 3 & 2 & 50 \end{array}\right] \xrightarrow{R_2 - R_1/4} \left[\begin{array}{cc|c} 12 & 8 & 200 \\ 0 & 0 & 0 \end{array}\right]$$

The last row is all zeros, which does not create a contradiction. The first column is a pivot column, but because the second row is zeroed out, the second column has no pivot—it's a free variable. So this pairing has one pivot and one free variable.

(ii) Use the plotted vectors to find a business pairing that has no solution. How many pivots and free variables does it have?

Pairing the hair salon with the cafe yields no solution. Since the two vectors lie on the same line that does not include the target, there's no way to combine the two of them to reach the target. Writing the pair's augmented matrix:

$$\left[\begin{array}{cc|c} 12 & 18 & 200 \\ 2 & 3 & 50 \end{array}\right] \xrightarrow{R_2 - R_1/6} \left[\begin{array}{cc|c} 12 & 18 & 200 \\ 0 & 0 & 50/3 \end{array}\right]$$

The echelon form is similar to the deli/shoe store pair except that the bottom-right entry is nonzero. As a result, the second row is a contradiction—there's no solution. The number of pivots and free variables is the same, one of each.

(iii) **Reflection.** A pairing with infinitely many solutions and a pairing with no solution have a lot in common. What's similar between them? What's the difference? Think about what happens when the target is moved. Discuss as a group and summarize your discussion below.

In this problem, pairings with infinitely many solutions and pairings with no solution both must have one pivot and one free variable. Equivalently, the two vectors must be collinear. The only difference is whether the target is on that same line (infinitely many solutions) or not on that same line (no solution). In the echelon form, the difference is whether the bottom-right element gets zeroed out (infinitely many solutions) or stays nonzero (no solution).

(d) **Reflection.** Think about this problem from the following perspective:

> *A pivot represents a direction or dimension that the columns of the matrix can "access." A free variable represents a redundant direction/dimension in the columns.*

Discuss the following questions as a group.

- How can you visualize **pivots** using the plotted vectors? What role do they play in determining the number of solutions?
- How can you visualize **free variables**? What role do they play in determining the number of solutions?

Write a couple of bullet points below summarizing your discussion.

- Pivots happen when a column of the system points in a "new direction/dimension"—when it *is not* in the span of the columns to its left. In the plot above, it's when the two vectors have different slopes. Each additional pivot increases the likelihood that the target will be in the span of the columns, meaning that there will be at least one solution.
- Free variables are when a column points in a "redundant direction/dimension"—when it *is* in the span of the columns to its left. In the plot above, it's when the two vectors are collinear. If you imagine two collinear vectors added together tip-to-tail, you can "slide" the contact point along the line, making one longer and the other shorter, while keeping the sum the same. That's the "freedom" of the free variable. As long as the pivots can "access" the solution in the first place, the free variables determine whether there is a unique solution (no free variables) or infinitely many solutions (at least one free variable).

(2) The previous problem developed some intuition for pivots and free variables using small, square systems. In this problem, try **applying that intuition** to larger, "wide" and "tall" matrices.

(a) The city has realized they have more room than they thought: all five businesses will fit in the space. The target numbers of adults and children are the same. How many **pivots** and **free variables** will this system have? How many **solutions**? *(Think about what directions/dimension the columns can "access" and how much redundancy there is.)*

$$\left[\begin{array}{ccccc|c} a & b & c & d & e & f \\ g & h & i & j & k & l \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 12 & 18 & 12 & 8 & 10 & 200 \\ 2 & 3 & 3 & 2 & 5 & 50 \end{array}\right]$$

First, let's think about what's possible based on the shape. There can't be more than one pivot per row, so there will be no more than 2 pivots. You can't get both rows zeroed out, so you'll have at least 1 pivot. So we'll have either 1 pivot and 4 free variables or 2 pivots and 3 free variables.

We know from the last problem that there is at least one pair of columns that spans all of $\mathbb{R}^2$, so there have to be **2 pivots**. That leaves **3 free variables**.

Since the columns span $\mathbb{R}^2$ and the target is in $\mathbb{R}^2$, there must be a solution. Since there is at least one free variable, the number of solutions has to be infinite.

(b) The city now wants your team to think about the entire multi-block business district, which consists of **23 businesses**. The city would like to set targets for the number of adult and child customers per day entering through each of the 15 distinct entrances to the business district, for a total of **30 requirements**. How many pivots and free variables are possible for this system? How many solutions is it **possible** for the system to have, and how many is **most likely**? *(In this part, you don't even have numbers! Use the shape alone to think about what the columns can do and are likely to do.)*

$$\left[\begin{array}{ccc|c} A_{1,1} & \cdots & A_{1,23} & b_1 \\ \vdots & & \vdots & \vdots \\ & \ddots & & \\ \vdots & & \vdots & \vdots \\ A_{30,1} & \cdots & A_{30,23} & b_{30} \end{array}\right]$$

Each column has either a pivot or a free variable. We'll have at least one pivot (unless every column is all zeros), and up to 23 pivots as long as each column points in a "new" direction in the 30-dimensional space, which is most likely because that's a lot of possible new directions. In other worlds, probably each new business will increase customer counts at the 15 locations differently, in a way that's not a linear combination of the other businesses' effects. But redundant columns are possible, each of which would reduce the number of pivots by 1.

With only up to 23 of the 30 dimensions "accessible" by A, it's not likely that the target will lie in the span of the columns—so *probably* there will be no solution. However, it is *possible* for that to happen. In that case, if there are no free variables, there will be a unique solution, and if there is at least one free variable, there will be infinitely many solutions.

(c) **Reflection.** Discuss the following as a group.

- At what stage(s) of problem solving could it be useful to glance at the columns and shape of a large matrix? Why? *(Hint: calculating the RREF of a large matrix is time-consuming!)*
- Have you encountered any applications where this could be useful, or can you think of any now?

Write a couple of bullet points below summarizing your discussion.

- Wide matrices usually have infinitely many solutions (too much freedom), and tall matrices usually have no solution (too many constraints). If you can find that out *before* doing any calculation, it can give you some of the information you might need without needing to calculate the RREF. Alternatively, it might suggest a harder look at the problem setup, or that maybe RREF isn't the best tool for the job.
- Many applications use least squares to find a best-fit line. It's not called a "perfect-fit line"—it's only an *approximate* fit for all the data points. A least squares problem with one dependent variable and one independent variable corresponds to a tall matrix, where there's a row for each data point and only two columns: one for the slope and one for the intercept. If the RREF of that matrix yielded a solution (an exact one), it would correspond to all the data points falling *exactly* on the same line. Instead of RREF, least squares is used to get an approximate solution.

Nice work! Your team helped the city create a thriving business district. Now that you have had some practice working in an applied context with the relationships between a matrix's shape, the form of its RREF, and its solutions, your group can try using what you have learned on abstract problems.

(3) Let A be an $m \times n$-matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$. Suppose that A has m pivots and that $n > m$. Which of the following statements must be true?
(a) The linear system whose augmented matrix is $[A|\mathbf{b}]$ has infinitely many solutions.
(b) The linear system whose augmented matrix is $[A|\mathbf{b}]$ has a unique solution.
(c) The linear system whose augmented matrix is $[A|\mathbf{b}]$ has no solution.
(d) None of the other answers.

Explain why!

Because A has m pivots and is an $m \times n$-matrix, not every column is a pivot columns since $n > m$. Because A has m pivots, every row of A must contain a pivot position. Thus the reduced echelon form $[U|c]$ of $[A|b]$ does not contain a row of the form $[0 \ \dots \ 0 \ |z]$, where $z \neq 0$. Therefore the system $Ax = \mathbf{b}$ has infinitely many solutions.

(4) Let A be a 4×4-matrix and let $\mathbf{b}, \mathbf{c}$ be two vectors in $\mathbb{R}^4$ such that the equation $A\mathbf{x} = \mathbf{b}$ has no solution. What can you say about the number of solutions of the equation $A\mathbf{x} = \mathbf{c}$?

The equation $A\mathbf{x} = \mathbf{c}$ either has no solution or infinitely many solutions.
The equation $A\mathbf{x} = \mathbf{c}$ cannot have a unique solution. Suppose it does. Then the equation cannot have free variables. Thus the echelon form of A has a pivot in the every row. However, then echelon from of the augmented matrix $[A|\mathbf{b}]$ cannot have a row of the form $[0 \ 0 \ 0 \ 0 \mid z]$, where $z \neq 0$. Thus the equation $A\mathbf{x} = \mathbf{b}$ cannot be inconsistent, which contradicts what we were given. Therefore, our supposition that $A\mathbf{x} = \mathbf{c}$ can have a unique solution must be false.

To see that $A\mathbf{x} = \mathbf{c}$ can have infinitely many solutions, consider $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Check that $A\mathbf{x} = \mathbf{c}$ has infinitely many solutions, while $A\mathbf{x} = \mathbf{b}$ has no solution.

You can think about switching from $\mathbf{b}$ to $\mathbf{c}$ as "moving the star". Since the RREF of A has a redundant column (which produced the inconsistency), then $\mathbf{c}$ can either be "on the line" with infinitely many solutions (because of the redundancy in the columns) or "off the line" with no solutions.