Names: ______________________________

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Discussion section: **D** ________ Group number: ________

MATH 257 - WORKSHEET 6: SUBSPACES, SPAN, LINEAR INDEPENDENCE

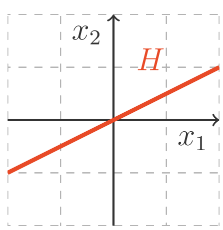
It is easy to picture objects in $\mathbb{R}^2$ or $\mathbb{R}^3$ - like lines, circles, planes or spheres. On the other hand, the human mind has cannot visualize higher-dimensional objects, such as subspaces of $\mathbb{R}^4$, $\mathbb{R}^5$ or $\mathbb{R}^{1242}$. Luckily, there are tools from linear algebra that allow you to describe and understand such object efficiently - even if you can't picture them in your head. In this worksheet we see how this process of using linear algebra to study higher-dimensional spaces works. This plays an absolutely crucial role in the applications of linear algebra to data science.

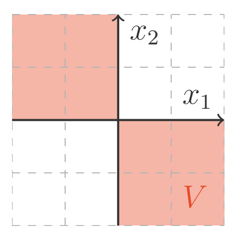
(1) We start by reviewing a few concepts from class.

A non-empty subset H of $\mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if it satisfies the following two conditions:

- If $\mathbf{u}, \mathbf{v} \in H$, then the sum $\mathbf{u} + \mathbf{v} \in H$. ($H$ is *closed under vector addition*).
- If $\mathbf{u} \in H$ and c is a scalar, then $c\mathbf{u} \in H$. (H is *closed under scalar multiplication.*)

(a) As a group, discuss which of the following is a subspace of $\mathbb{R}^2$:





A theorem in class stated that for vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^n$ their span $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$ is a subspace of $\mathbb{R}^n$.

(b) As a group, recall the definition of the span of vectors.

The span $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$ is the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_m$. That is, it is the set of all vectors of the form

$$c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m.$$

(c) Someone on the internet mentioned the following analogy: *"Consider the colors red and blue. Then you can think of set the of all colors you can mix from red and blue as the span of red and blue."* Can you explain this?

If you mix red and blue, then you get purple. Thus purple can create by adding red and blue. Similarly, given to vector $\mathbf{v}_1$ and $\mathbf{v}_2$ can generate their sum $\mathbf{v}_1 + \mathbf{v}_2$. So we can thinking of mixing as taking linear combination. Thus the set of all linear combination corresponds to the set of all colors you can mix from the given colors red and blue.

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).

(d) As a group, discuss why the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are not linearly independent.

Note that

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(2) Ok, let's get started. Consider the following problem.

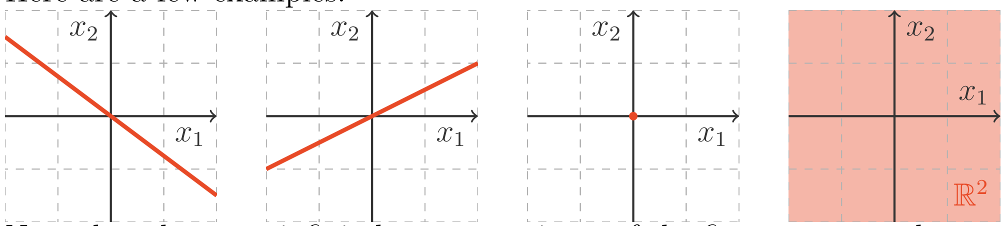
Describe all subspaces of $\mathbb{R}^6$.

(a) Discuss as a group how best to describe subspaces of $\mathbb{R}^6$ geometrically. This is a difficult problem, because it is hard to imagine geometric objects outside of $\mathbb{R}^3$. The next part of this worksheet will show you how linear algebra can help you understand and manipulate such higher dimensional objects. Write down how you think about subspaces of $\mathbb{R}^6$ now, and then move on to the next question.

(b) *"Let's look at $\mathbb{R}^2$. Maybe we can get some intuition from looking at this case."*

Together as a group, find examples of subspaces of $\mathbb{R}^2$. How many different subspaces are there?

Here are a few examples:



Note that there are infinitely many variants of the first two examples.

(c) *"While there are infinitely many subspaces of $\mathbb{R}^2$, maybe we can classify into three distinct geometric classes."*

Work together to figure out how to group subspaces of $\mathbb{R}^2$ into three categories.

Examples you came up with in the previous question should fall into three categories:

- $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$, the subspace of consisting of the zero vectors,
- $\mathbb{R}^2$,
- lines through the origin.

There are no subspace of $\mathbb{R}^2$ that do not fall in one of the three categories. We will see in a little bit why.

(d) *"Ok, we now geometrically understand subspaces of $\mathbb{R}^2$. But can we describe them algebraically?"*

Express each of your examples of subspaces of $\mathbb{R}^2$, as a span of vectors. That is, for each subspace V, find vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that $V = \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

As a group, discuss whether for each example there is a minimum number of vectors you need to span the subspace.

The line in the first example is $\text{span}(\begin{bmatrix} 2 \\ -1.5 \end{bmatrix})$, and the second one is $\text{span}(\begin{bmatrix} 2 \\ 1 \end{bmatrix})$. The third example is a little tricker. The span of no vectors (ie. span()) is just $\{\mathbf{0}\}$. Finally the last example ($\mathbb{R}^2$) is spanned by any two linearly independent vectors of $\mathbb{R}^2$. However, $\mathbb{R}^2$ can not be spanned by a single vector, as the span of single vectors is all of $\mathbb{R}^2$.

So a subspace of $\mathbb{R}^2$ can be spanned by 0 zero (third example), by 1 vector (first and second example) or by 2 linearly independent vectors (fourth example). Note that the minimum number of generators corresponds to the dimension of the object (a point is 0-dimensional, a line is one-dimensional, a plane is two-dimensional).

(e) In the previous problems you figured out how to describe subspaces of $\mathbb{R}^2$. Now let's see how this transfer to $\mathbb{R}^3$. As a group discuss how subspaces of $\mathbb{R}^3$ look like.

Subspace of $\mathbb{R}^2$ can be spanned by 0 zero (a point), by 1 vector (a line), by 2 linearly independent vectors (a plane) or by 3 linearly independent vectors (all of $\mathbb{R}^3$)).
Why do we insist on having linear independent vectors?

(f) Let's come back to subspaces of $\mathbb{R}^6$. Even if you still can picture them in your head, can you describe them in terms of the number of (linearly independent) vectors they are spanned by?

> Every subspace V of $\mathbb{R}^6$ is of the form $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$, where $m \in \{0, 1, 2, 3, 4, 5, 6\}$. If we know that $\mathbf{v}_1, \ldots, \mathbf{v_m}$ are linearly independent, then we can think of V is a m-dimensional object. For example, if $m = 1$, then V is a line (through the origin) in the six dimensional space $\mathbb{R}^6$. If $m = 2$, the V is a plane in $\mathbb{R}^6$. It gets harder to imagine V geometrically when $m > 2$, but we still have a pretty good description of it as the span of m linearly independent vectors.

(3) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

(a) Think back to part (2)(a). How did your understanding of subspace change?
(b) How does the textbook definition relate to how you think of subspaces now?
(c) What role did linear independence play in describing subspaces of $\mathbb{R}^6$?

Summarize your discussion below in 3–4 bullet points.

Nice work! If you still have time, use your better understanding of subspaces and spans to attack the following problem.

(4) Let V be a subspace of $\mathbb{R}^n$.
  (a) Is $V = \text{span}(V)$?
  (b) Is V a linearly independent collection of vectors?
  (c) Can you find linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in V$ such that $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m) = V$?

Nice work! BONUS problems: 1) Suppose $v_1, ..., v_4$ is a list of vectors in $\mathbb{R}^n$ such that *any* 3 of them form a linearly independent set. Must the whole list $v_1, ..., v_4$ be linearly independent?

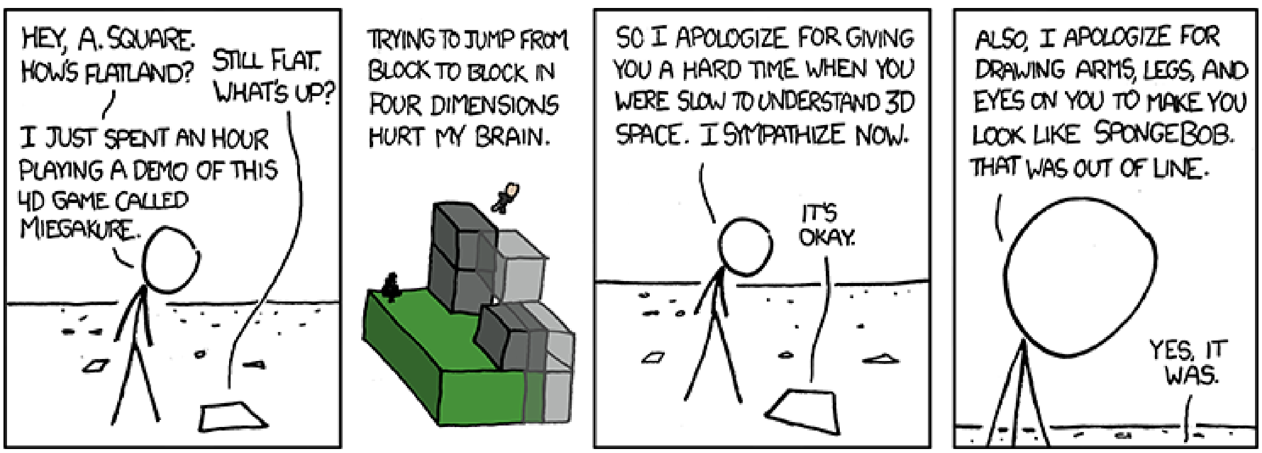
No, a counterexample is given by taking $v_1, .., v_3$ to be any 3 linearly independent vectors in $\mathbb{R}^3$ (e.g. the standard basis vectors) and taking $v_4 = v_1 + v_2 + v_3$.

2) (Advanced) Prove that if U and V are subspaces of $\mathbb{R}^n$ and their union is a subspace, then one is contained in the other.

Suppose they do not contain each other. Then there is a vector $u \in U$ that is not in V, and a vector $v \in V$ that is not in U. Then $u+v$ is neither in U (because if it were $(u+v)+(-u) = v$ would be in U) nor is it in V (analogous reason). So the union cannot be a subspace since it is not closed under addition.

3) (VERY Advanced) Prove that if U, V, W are subspaces of $\mathbb{R}^n$ such that their union is also a subspace, then one of them contains the other two.

This is pretty difficult, do NOT worry about this if you don't want to. First, note that we can assume that the intersection of the three subspaces is just the zero vector, by throwing out any vectors in U that are already in V, etc (since that doesn't change their union). Now by the previous problem we can assume that the union of U and V does not form a subspace. But now if the triple union were a subspace, calling Z, by the zero-intersection property we forced at the beginning, we are saying that Z is a subspace, with two further subspaces U and V in it, which do not contain each other, and their compliment in Z, call it Y, together with the zero vector, is also a subspace of Z. But that cannot be closed under addition, since, by the previous $u+v$ and $u-v$ are both in Y but $(u+v)+(u-v)=2u$ is in U and hence not in Y.



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