Names: ____________________

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Discussion section: **D** ________ **Group number:** ________

MATH 257 - WORKSHEET 9

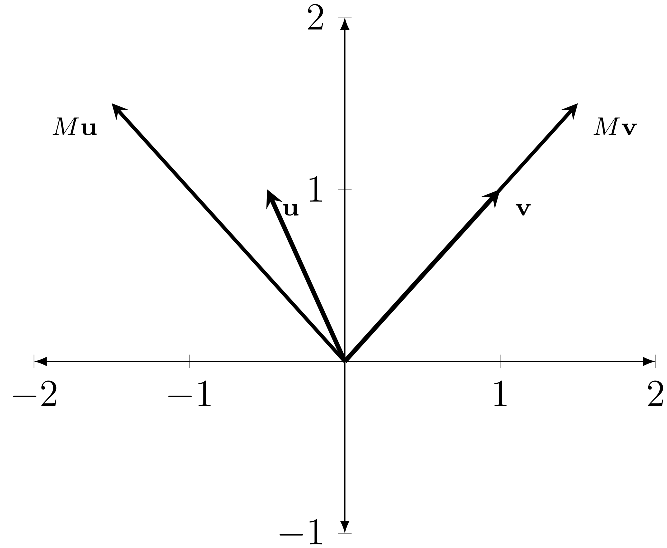
Eigenvectors (and eigenvalues) have countless applications in the real world. But what is an eigenvector? What does that defining equation, $M\mathbf{v} = \lambda\mathbf{v}$, really mean?

In this worksheet, your team will explore two scenarios: one that suggests a conceptual picture of an eigenvector, and another that applies the picture to multiple matrices' eigenvectors.

Once your team has explored the concepts through these scenarios, you will get a chance to practice using them in abstract problems.

(1) **Review**
Discuss the following questions as a group. Write and briefly explain your answers below.

(a) In the plot below, is $\mathbf{v}$ an eigenvector of M? Is $\mathbf{u}$? Use the definition $M\mathbf{v} = \lambda\mathbf{v}$.



Yes, $\mathbf{v}$ is an eigenvector. $M\mathbf{v}$ can be created from $\mathbf{v}$ by multiplying it by a number (about 1.5), which would be λ. No, $\mathbf{u}$ is not an eigenvector because there's no way of scaling $\mathbf{u}$ by a number to get a vector equal to $M\mathbf{u}$.

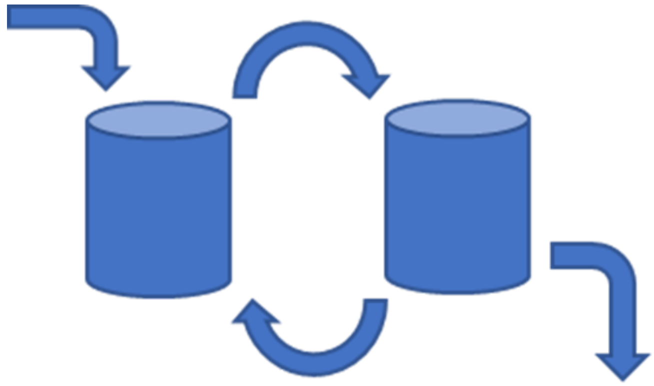
(b) A student makes the following claim:

"Intuitively, the definition of an eigenvector $M\mathbf{v} = \lambda\mathbf{v}$ means that $\mathbf{v}$ stays pointing in the same direction after it gets multiplied by M."

How can you make "stays pointing in the same direction" more precise? To what extent does your group agree or disagree?

"Pointing in the same direction" could mean the vector's image $M\mathbf{v}$ is on the same *half–line* starting at the origin as $\mathbf{v}$ is (which would be false, because when λ is negative the vector flips to the opposite quadrant). Or it could mean the same *line* through the origin, which would be true because $\lambda\mathbf{v}$ must be in the span of $\mathbf{v}$. Another version of "pointing in the same direction" is that the slope stays the same (or more generally all the ratios between the components stay the same), which would be true because λ multiplies both/all components by the same factor.

(2) **Scenario: Brewing beverages**



A distillery makes a fermented beverage in a process that uses two tanks, tank A and tank B. The table below describes how the fermenting liquid is moved between tanks each morning.

Of the initial volume of liquid in tank A,...	Of the initial volume of liquid in tank B,...	
12%	7%	...is mixed into the **other tank**.
—	7%	...is **dispensed from that tank** into bottles.
88%	86%	...is the amount remaining in the **same tank**.
6%	—	...is amount the of **new ingredients added** into **that tank**.

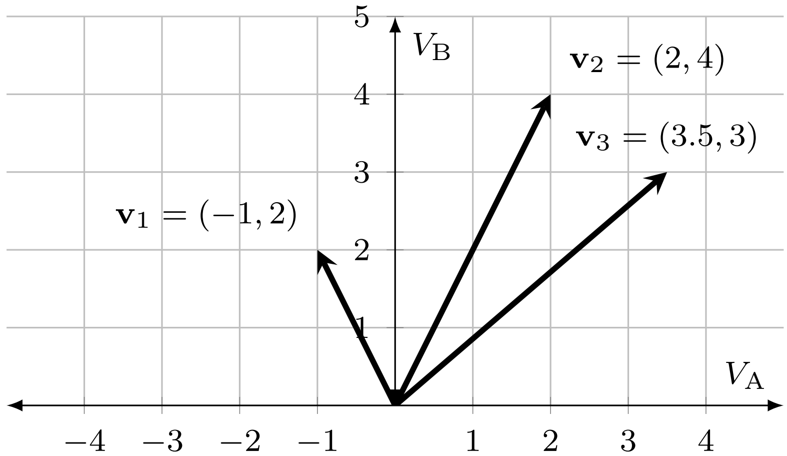
All percentages are calculated **before any liquid is moved**, as a portion of the **initial volume of liquid** (not as a portion of the capacity of the tank).

(a) The volumes of liquid in the two tanks resulting from this mixing, dispensing, and adding can be expressed as the matrix M, below, multiplied by the column vector of initial volumes. As a group, discuss and write out how to derive this matrix from the information above. (Hint: if no liquid were moved, M would be the identity matrix.)

$$\begin{bmatrix} V_{\text{A,after}} \\ V_{\text{B,after}} \end{bmatrix} = M \begin{bmatrix} V_{\text{A,before}} \\ V_{\text{B,before}} \end{bmatrix}, \quad M = \begin{bmatrix} 0.94 & 0.07 \\ 0.12 & 0.86 \end{bmatrix}$$

The diagonal components represent how much volume remains relative to the starting volume in that tank. The 94% comes from the 88% remaining plus the 6% new ingredients based on the tank A volume. The 86% comes directly from the table. The off–diagonal components represent mixing, so 7% from tank B to tank A and 12% from tank A to tank B. The 7% drained from tank B leaves the system entirely, so it's already taken into account in the 86% figure.

(b) For each vector $\mathbf{v}$ on the plot below, calculate and draw $M\mathbf{v}$. Then, discuss whether each $\mathbf{v}$ is an eigenvector of M. If so, state the eigenvalue. If not, explain why not.



$M\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-0.8\\1.6\end{bmatrix}$ eigenvector, eigenvalue $= 0.8$

$M\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}2.16\\3.68\end{bmatrix}$ not an eigenvector

The first component is multiplied by 1.08, but the second component is multiplied by 0.92; you can't factor an eigenvalue out. Equivalently, the new vector points in a slightly different direction.

$M\begin{bmatrix}3.5\\3\end{bmatrix} = \begin{bmatrix}3.5\\3\end{bmatrix}$ eigenvector, eigenvalue $= 1$

(c) The distillery needs to choose what initial volume of liquid to put in each tank. They would like to know whether there are volumes they can keep in the tanks that will **remain the same** after the process. Using the results of part (b), explain why it *is* possible. If tank A contains 700 gallons of liquid, how many gallons of liquid should tank B contain?

The eigenvector with eigenvalue 1 represents the ratio of volumes to put in each tank that will end up at the same values afterward. So if tank A has 700 gallons, tank B needs to have (700 gallons)(3/3.5) = 600 gallons.

Question to consider: If M were chosen randomly, would you expect it to be possible to find tank volumes that end up the same after the process? No. Most matrices don't have 1 as an eigenvalue, let alone for an eigenvector in the first quadrant. Remember, in part (b), the eigenvector with eigenvalue 0.8 was in the second quadrant, meaning one of the tank volumes would have to be negative to create that eigenvalue!

(3) **Reflection**

Discuss the following questions as a group. Write and briefly explain your answers below.

(a) Let's think back to what the student said before:

"Intuitively, the definition of an eigenvector $M\mathbf{v} = \lambda\mathbf{v}$ means that $\mathbf{v}$ stays pointing in the same direction after it gets multiplied by M."

In the brewing beverages scenario, what does "same direction" mean in terms of volumes of liquid in the two tanks?

> The ratio between the volumes of the two tanks stays the same after the process. (For the eigenvector with eigenvalue 1, the actual *values* of the volumes stay the same, but that's not a requirement for it to be an eigenvector.)

(b) Recall that an **eigenspace** is a vector space of eigenvectors with a particular eigenvalue. Geometrically, a one-dimensional eigenspace of a matrix M is a line through the origin, and it **doesn't change slope** when the vectors in it are multiplied by M. Why?

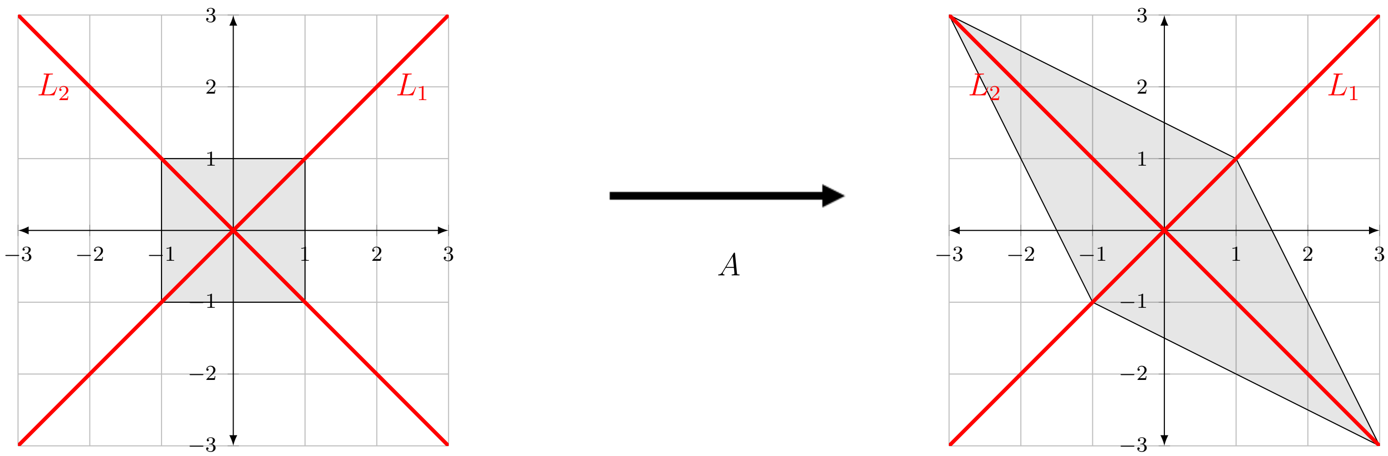
> Given a nonzero element of the eigenspace, you can get the rest by taking the span, which gives you a line through the origin. All of the elements of the eigenspace are eigenvectors by definition (except the zero vector), so when you multiply them by M, they end up pointing in "the same direction," i.e. on the same line through the origin, with the same slope. (The zero vector is not an eigenvector, but it stays put.)

(4) **Scenario: Stretching cloth**

A square piece of elastic cloth is mounted on a frame. Then, the frame is distorted according to multiplication by the accompanying 2×2 matrix.

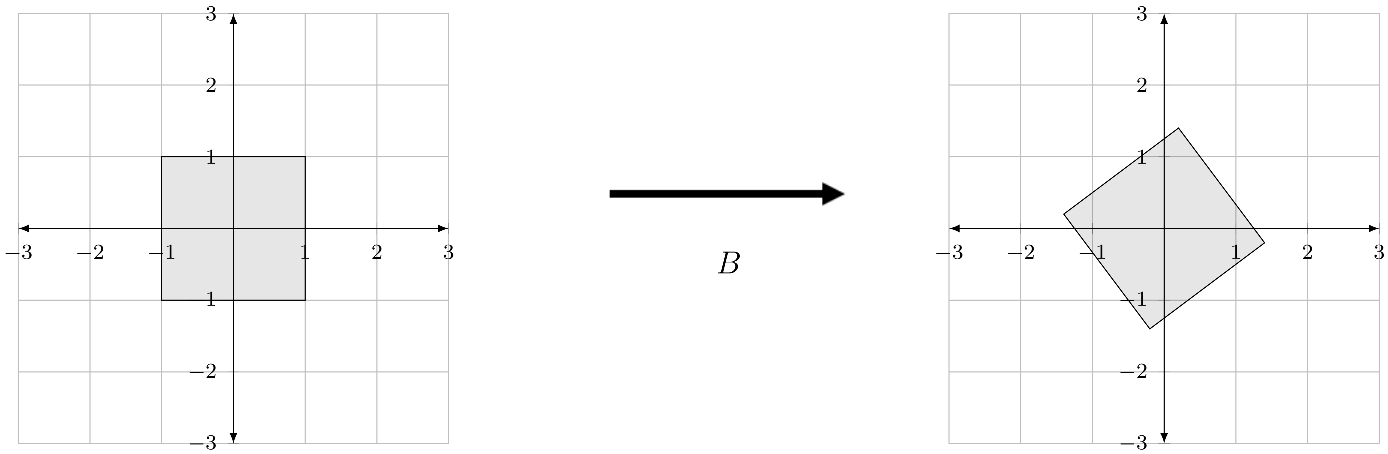
(a) The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ stretches the cloth along one diagonal. A has **two eigenspaces**, one spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the other by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(i) On both plots below, plot both eigenspaces.
(ii) Write a sentence or two below explaining why these are eigenspaces. What are their eigenvalues?



L_1 and L_2 are plotted above. L_1 is an eigenspace because the vectors on it don't move at all; they're on the balance point of the perpendicular stretching. For L_1, the eigenvalue is 1. L_2 is an eigenspace because although the vectors on it move, they end up on the same line, just stretched by a factor of 3. For L_2, the eigenvalue is 3.

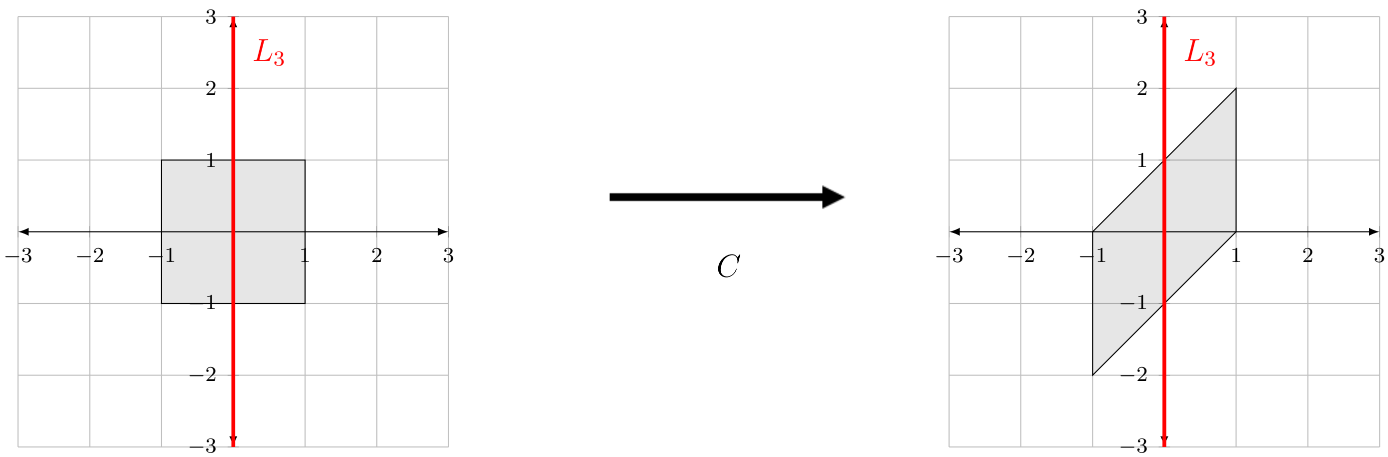
(b) The matrix $B = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ rotates the cloth slightly less than 45° counterclockwise around the origin. B has **no eigenspaces**. Write a sentence or two below explaining why.



Every vector in the plane is rotated, so it's impossible for any vector to end up on the same line through the origin that it started on. The slope of every line through the origin is changed by this transformation.

(c) The matrix $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ shears the cloth in one direction. C has **one eigenspace** spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(i) On both plots below, plot the eigenspace.
(ii) Write a sentence or two below explaining why this is an eigenspace. What is its eigenvalue? Why does C have no other eigenspace?



L_3 is plotted above. This shear changes the x_2 value of every vector proportional to its x_1 value, so all the vectors on the x_2 axis (i.e. with $x_1 = 0$) are left alone. Hence, the eigenvalue is 1. Every other line through the origin gets sheared and changes slope, so there can't be any other eigenspace.

(5) **Abstract problems**

Now that your team has explored the concept of an eigenvector in two applied scenarios, let's try applying it to abstract problems.

Work as a group to solve the following problems.

(a) Let L be a line through the origin in $\mathbb{R}^n$, and A and B be $n \times n$ matrices. True or false? If L is an eigenspace of both A and B, then L is contained in an eigenspace of the product matrix AB.

True. In the product $AB\mathbf{v}_1$ (where $\mathbf{v}_1$ is in L), first B sends $\mathbf{v}_1$ to another vector $\mathbf{v}_2$ in L, and then A sends $\mathbf{v}_2$ to another vector $\mathbf{v}_3$, which is still in L. Hence L is a subset of an eigenspace of AB.
However, L itself may be a proper subset of this eigenspace. For example, take $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B$ so that AB is the identity matrix. Let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis vectors in $\mathbb{R}^2$. Then, $L_1 = \text{span}(\{\mathbf{e}_1\})$ is the 1-eignspace for both A, B but it is only a proper subset of the sole eigenspace $E_1 = \mathbb{R}^2$ of AB. A similar statement holds for $L_2 = \text{span}(\{\mathbf{e}_2\})$.

(b) True or false? If A has an eigenvalue λ, then $\lambda + 1$ is an eigenvalue of $A + I$.

True. Let $\mathbf{v}$ be an eigenvector of A with that eigenvalue λ. Then $(A+I)\mathbf{v} = A\mathbf{v} + I\mathbf{v} = \lambda\mathbf{v} + \mathbf{v} = (\lambda + 1)\mathbf{v}$. This only works because every nonzero vector is an eigenvector of the identity matrix, with eigenvalue 1.