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Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 11

In this worksheet we investigate the meaning of diagonalizability via a concrete example. Given square $n \times n$ matrix A we have a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$ via $L(\mathbf{v}) = A\mathbf{v}$. If $\mathcal{E}$ denotes the standard basis on $\mathbb{R}^n$, then $L_{\mathcal{E},\mathcal{E}} = A$. Our program will be to successively exchange one choice of basis for another so that the coordinate matrix for L becomes progressively simpler. We will find that diagonalizable matrices are those yielding the simplest outcomes.
Define matrix

$$A = \frac{1}{4}\begin{pmatrix} 7 & 3 & 1 & 1 \\ 3 & 7 & 1 & 1 \\ -1 & -1 & 1 & 5 \\ -1 & -1 & 5 & 1 \end{pmatrix}$$

and let $\mathbf{v}_i$ be the columns of the matrix

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Define subspaces $X = span\{\mathbf{v}_1, \mathbf{v}_2\}$ and $Y = span\{\mathbf{v}_3, \mathbf{v}_4\}$. Also, define linear transformation $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $L(\mathbf{v}) = A\mathbf{v}$. In particular, $A = L_{\mathcal{E},\mathcal{E}}$.

(1) Explain why $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is an ordered basis for $\mathbb{R}^4$. [Hint: what are the dot products $\mathbf{v}_i \cdot \mathbf{v}_j$?]

(2) Show that $\mathcal{C} = (\mathbf{v}_1, \mathbf{v}_2)$ is a (ordered) basis for X and $\mathcal{D} = (\mathbf{v}_3, \mathbf{v}_4)$ one for Y. Show that $X \cap Y = \{\mathbf{0}\}$.

(3) For each basis vector $\mathbf{v}_i$, write $L(\mathbf{v}_i)$ as a linear combination of the basis vectors in $\mathcal{B}$. Use your answer to show that if $\mathbf{x} \in X$ then $L(\mathbf{x}) \in X$ and that if $\mathbf{y} \in Y$ then $L(\mathbf{y}) \in Y$.

(4) Find $L_{\mathcal{B},\mathcal{B}}$. What is special about its structure? How is its appearance simpler than $A = L_{\mathcal{E},\mathcal{E}}$?

(5) Let $L_X \colon X \to X$ be the restriction of L to X and define L_Y analogously. What are $(L_X)_{\mathcal{C},\mathcal{C}}$ and $(L_Y)_{\mathcal{D},\mathcal{D}}$?

(6) Define $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$ and set $\mathcal{C}' = (\mathbf{w}_1, \mathbf{w}_2)$. Show that $\mathcal{C}'$ is an ordered basis for X.

(7) Define $X_1 = span\{\mathbf{w}_1\}$ and $X_2 = span\{\mathbf{w}_2\}$ subspaces of X. Show that $X_1 \cap X_2 = \{\mathbf{0}\}$. Show that if $\mathbf{x} \in X_1$ then $L(\mathbf{x}) = L_X(\mathbf{x}) \in X_1$ and analogously for $\mathbf{x} \in X_2$. What do we call $\mathbf{w}_1, \mathbf{w}_2$?

(8) What is $(L_X)_{\mathcal{C}',\mathcal{C}'}$? Let $\mathcal{B}' = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \mathbf{v}_4)$. What is $L_{\mathcal{B}',\mathcal{B}'}$?

(9) Verify that $\mathbf{v}_3$ is an eigenvector for A. Show that if $\mathcal{D}' = (\mathbf{v}_3, \mathbf{y})$ is an ordered basis for Y for some $\mathbf{y} \in Y$, then $\mathbf{y}$ cannot be an eigenvector of A. Conclude that there is no choice of basis $\mathcal{D}'$ for Y such that $(L_Y)_{\mathcal{D}',\mathcal{D}'}$ is diagonal.

Summary: In choosing basis $\mathcal{B}$ so that $L_{\mathcal{B},\mathcal{B}}$ is simple, choice of eigenbasis (when such exists) yields the simplest result, namely a diagonal matrix.