Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 1

Today you and your group will review some topics from previous math classes: sets, systems of equations, plotting in three dimensions, and operations on functions.

What is set builder notation?
Set builder notation expresses a set, a collection of mathematical objects (in this course, usually numbers or vectors). One kind of set builder notation, called set comprehension, consists of two or three parts:

$$\{\textbf{template} : \textbf{constraints}\}$$
$$(\textbf{domains} \text{ on either side})$$

- **Template**: What the set's elements look like.
- **Domains**: Try assigning all the values from these sets to the variables...
- **Constraints**: ...but only keep the elements that satisfy these constraints.

Example 1:

$$\{(x, y) \in \mathbb{R}^2 : y = x\}$$

- The part before the ":" says to try all vectors in $\mathbb{R}^2$ (**domain**), calling the first component x and the second component y (**template**).
- The part after the ":" says to include only the elements where those two components are equal (**constraint**).
- The result is the 45-degree diagonal line from the third quadrant, through the origin, to the first quadrant.

Example 2:

$$\{(-2, -2) + a(1, 1) : a \in \mathbb{R}\}$$

- The part before the ":" says to start with the vector $(-2, -2)$, then add the vector $(1, 1)$ to it scaled by the variable a (**template**).
- The part after the ":" says to plug every real number (**domain**) into a and include the resulting vector in the set (there is no **constraint**).
- The resulting set is the same as in Example 1!

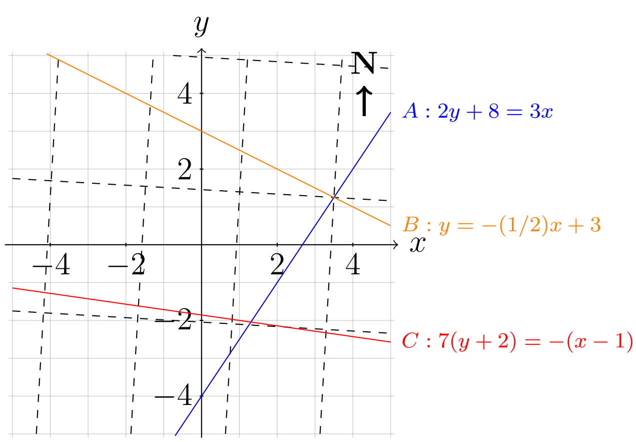
Your team is trying to recover city map data after a data center fire. Planners' notes that survived the fire describe proposed subway routes and electrical lines in set builder notation, but in different formats.

Subway routes		Electrical lines	
Subway route A	$\{(x,y) \in \mathbb{R}^2 : 2y+8=3x\}$	Electrical line J	$\{(2,2)-a(2,-1) : a \in \mathbb{R}\}$
Subway route B	$\{(x,y) \in \mathbb{R}^2 : y=-(1/2)x+3\}$	Electrical line K	$\{(2.5,-1)+a(-2,4) : a \in \mathbb{R}\}$
Subway route C	$\{(x,y) \in \mathbb{R}^2 : 7(y+2)=-(x-1)\}$	Electrical line L	$\{(2,-1)-a(-4,-6) : a \in \mathbb{R}\}$

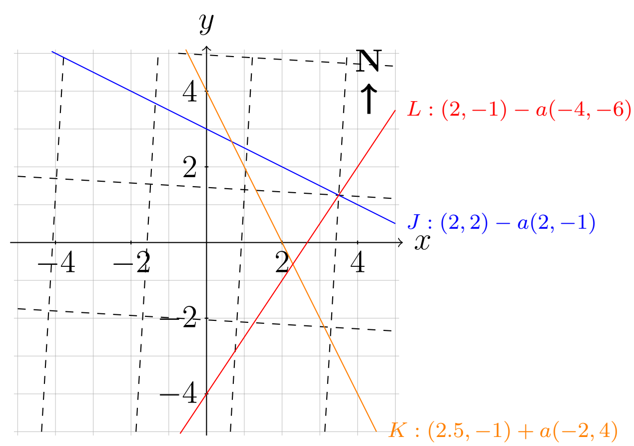
(1) Your team is told that **two of the three subway routes and two of the three electrical lines** were chosen for construction, but you're not sure which two out of each set of three. What you do know is that the subways operate on electric power, so **the electrical lines need to be built on top of the subway routes**. As a group, **plot and label** the remaining subway routes and electrical lines on the corresponding street maps below. Then, use the plots to discuss and decide which **TWO PAIRS** of a subway route with an electrical line should be built.

Two pairs of subway route with an electrical line to be built are **Subway route A** with **Electrical line L** and **Subway route B** with **Electrical line J**. You can infer this from matching the plots below. You can also check when sets are the same or not by thinking about the algebra geometrically. For example, look carefully at the format of the electrical lines. The vector multiplied by a tells you how far to go in the x and y directions to get to another point on the line, so you can use it to calculate the slope of the line. The slope can be compared easily against the format of the subway routes.

Subway routes



Electrical lines



Note: Dashed lines already plotted above represent the street grid.

(2) A subway station needs to be built where the two subway routes intersect.

(a) As a group, set up and solve a system of equations to determine the coordinates of the intersection point.

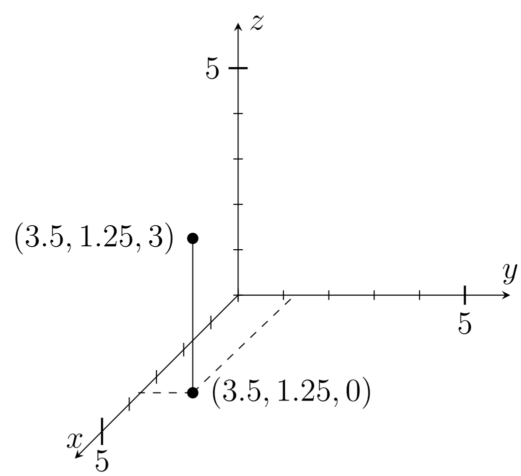
The intersection point of the lines is the point where both lines' equations are satisfied at the same time. So we can write and solve this system of equations:

$$y = -0.5x + 3$$
$$y = 1.5x - 4$$

The solution is $(\mathbf{3.5}, \mathbf{1.25})$.

(b) The subway will be 3 units under street level. Using the x–y plane as the subway level, plot the coordinates of the planned street-level station on the plot below.

Drawing auxiliary lines can help you visualize the three-dimensional space and get the point in the right place. You might want to start by finding the point's "shadow" in the x–y plane: draw a dashed line starting at (3.5, 0, 0) (on the x axis) that's parallel to the y axis, and another dashed line starting at (0, 1.25, 0) (on the y axis) that's parallel to the x axis (drawn diagonally on the page!). Their intersection point is (3.5, 1.25, 0), the point's "shadow".



(c) An elevator shaft needs to be built to connect the street-level station to the subways. Work together to come up with **two different ways** of writing the set, in set builder notation, representing the vertical line containing the elevator shaft.

There are many ways to write the line containing the elevator shaft in set builder notation. Here are two inspired by the formats of the subway routes and electrical lines:

$$\{(3.5, 1.25, 0) + a(0, 0, 1) : a \in \mathbb{R}\}$$
$$\{(x, y, z) \in \mathbb{R}^3 : (x, y) = (3.5, 1.25)\}$$

(3) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

(a) Another kind of set builder notation lists the elements separated by commas, like this: $\{(1,1),(2,2),(3,-1)\}$. What is useful about set comprehensions compared to listing the elements? What is challenging?

(b) What are some strategies you can use to determine if two sets written using set comprehensions are the same? Can you do it without physically plotting them? Summarize your discussion below in 3–4 bullet points.

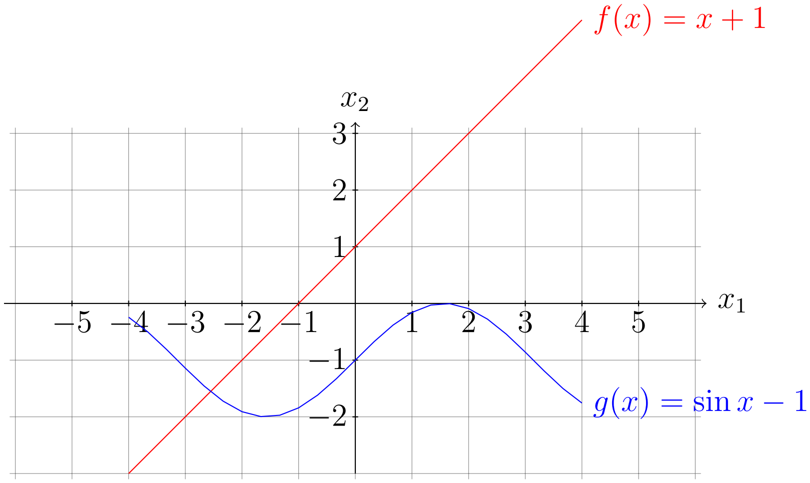
Listing the elements is easier, and fine as long as you don't have too many—no need to write a complicated expression if you already know the only 3 points that satisfy it. But a line contains infinitely many points! Set comprehensions are a quick way to tell you about all those points at once.
There are many possible strategies for thinking about set comprehensions geometrically. In addition to checking slopes, you can check special points to see if both sets contain them. Two points determine a line, so if you know your sets are lines and you find two points that they share, you know they're the same line.

Nice work! Your team has helped the city recover its subway project plans.

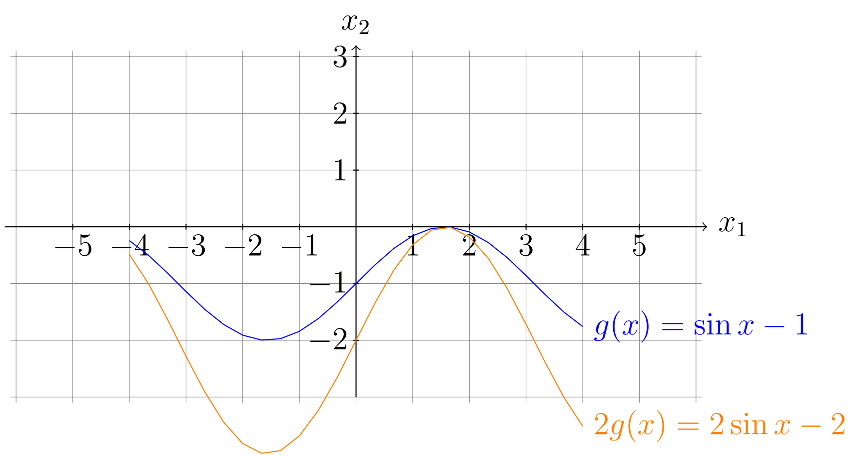
The previous parts use an applied context so that you have real-world analogies for the concepts you're learning. Now let's take the context away and try some abstract problems. Your team will now review working with functions.

In the problems below, let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) := x+1$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) := \sin(x) - 1$.

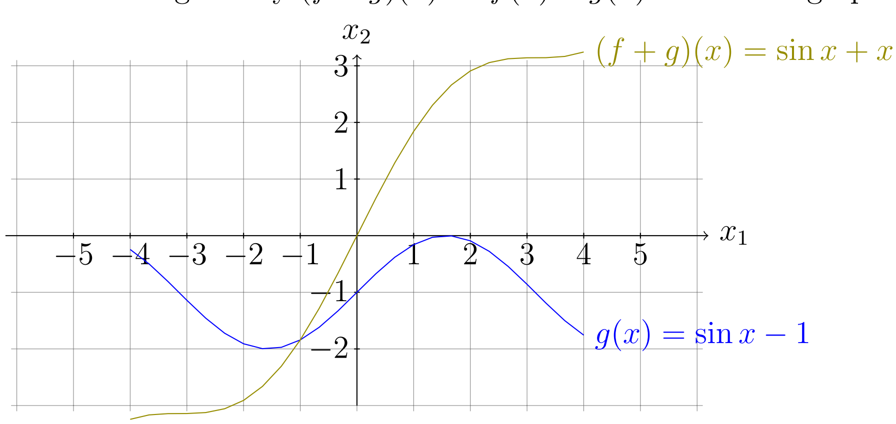
(4) Draw the graphs of f and g.



(5) Let $2g$ be the function given by $(2g)(x) := 2g(x)$. Draw the graph of g and $2g$.



(6) Let $f + g$ be the function given by $(f + g)(x) := f(x) + g(x)$. Draw the graph of $f + g$.



(7) Let $f \circ g$ be the function given by $(f \circ g)(x) := f(g(x))$, and let $g \circ f$ be the function given $(g \circ f)(x) := g(f(x))$. Is $f \circ g = g \circ f$? Explain.

$$\begin{aligned} f \circ g &= f(g(x)) \\ &= (g(x)) + 1 \\ &= (\sin(x) - 1) + 1 \\ &= \sin(x) \\ g \circ f &= g(f(x)) \\ &= \sin(f(x)) - 1 \\ &= \sin(x+1) - 1 \end{aligned}$$

No, $f \circ g \neq g \circ f$, because $(f \circ g)(x) = \sin x$, but $(g \circ f)(x) = \sin(x+1) - 1$.