Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 6: SUBSPACES, SPAN, LINEAR INDEPENDENCE

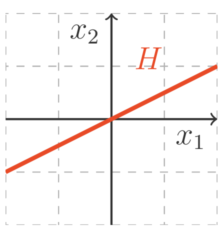
It is easy to picture objects in $\mathbb{R}^2$ or $\mathbb{R}^3$ - like lines, circles, planes or spheres. On the other hand, the human mind has cannot visualize higher-dimensional objects, such as subspaces of $\mathbb{R}^4$, $\mathbb{R}^5$ or $\mathbb{R}^{1242}$. Luckily, there are tools from linear algebra that allow you to describe and understand such object efficiently - even if you can't picture them in your head. In this worksheet we see how this process of using linear algebra to study higher-dimensional spaces works. This plays an absolutely crucial role in the applications of linear algebra to data science.

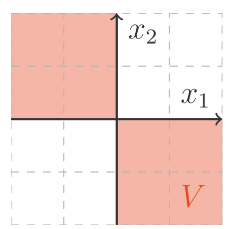
(1) We start by reviewing a few concepts from class.

A non-empty subset H of $\mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if it satisfies the following two conditions:

- If $\mathbf{u}, \mathbf{v} \in H$, then the sum $\mathbf{u} + \mathbf{v} \in H$. ($H$ is *closed under vector addition*).
- If $\mathbf{u} \in H$ and c is a scalar, then $c\mathbf{u} \in H$. (H is *closed under scalar multiplication.*)

(a) As a group, discuss which of the following is a subspace of $\mathbb{R}^2$:





A theorem in class stated that for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ their span $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a subspace of $\mathbb{R}^n$.

(b) As a group, recall the definition of the span of vectors.

(c) Someone on the internet mentioned the following analogy: *"Consider the colors red and blue. Then you can think of set the of all colors you can mix from red and blue as the span of red and blue."* Can you explain this?

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).

(d) As a group, discuss why the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are not linearly independent.

(2) Ok, let's get started. Consider the following problem.

Describe all subspaces of $\mathbb{R}^6$.

(a) Discuss as a group how best to describe subspaces of $\mathbb{R}^6$ geometrically. This is a difficult problem, because it is hard to imagine geometric objects outside of $\mathbb{R}^3$. The next part of this worksheet will show you how linear algebra can help you understand and manipulate such higher dimensional objects. Write down how you think about subspaces of $\mathbb{R}^6$ now, and then move on to the next question.

(b) *"Let's look at* $\mathbb{R}^2$*. Maybe we can get some intuition from looking at this case."*

Together as a group, find examples of subspaces of $\mathbb{R}^2$. How many different subspaces are there?

(c) *"While there are infinitely many subspaces of* $\mathbb{R}^2$*, maybe we can classify into three distinct geometric classes."*

Work together to figure out how to group subspaces of $\mathbb{R}^2$ into three categories.

(d) *"Ok, we now geometrically understand subspaces of* $\mathbb{R}^2$*. But can we describe them algebraically?"*

Express each of your examples of subspaces of $\mathbb{R}^2$, as a span of vectors. That is, for each subspace V, find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

As a group, discuss whether for each example there is a minimum number of vectors you need to span the subspace.

(e) In the previous problems you figured out how to describe subspaces of $\mathbb{R}^2$. Now let's see how this transfer to $\mathbb{R}^3$. As a group discuss how subspaces of $\mathbb{R}^3$ look like.

(f) Let's come back to subspaces of $\mathbb{R}^6$. Even if you still can picture them in your head, can you describe them in terms of the number of (linearly independent) vectors they are spanned by?

(3) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

(a) Think back to part (2)(a). How did your understanding of subspace change?

(b) How does the textbook definition relate to how you think of subspaces now?

(c) What role did linear independence play in describing subspaces of $\mathbb{R}^6$?

Summarize your discussion below in 3–4 bullet points.

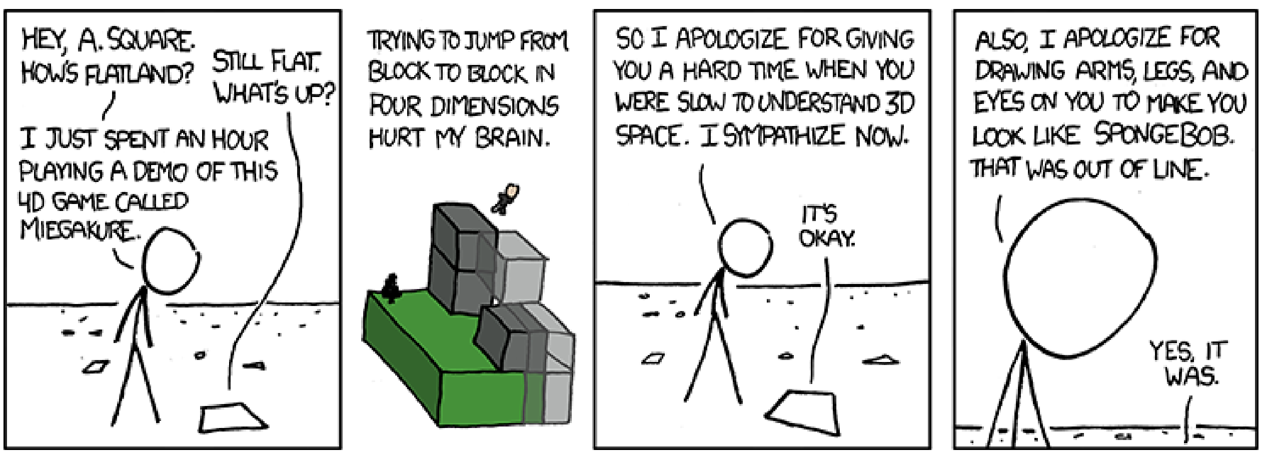
Nice work! If you still have time, use your better understanding of subspaces and spans to attack the following problem.

(4) Let V be a subspace of $\mathbb{R}^n$.
  (a) Is $V = \text{span}(V)$?
  (b) Is V a linearly independent collection of vectors?
  (c) Can you find linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ such that $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = V$?

Nice work! BONUS problems: 1) Suppose $v_1, ..., v_4$ is a list of vectors in $\mathbb{R}^n$ such that *any* 3 of them form a linearly independent set. Must the whole list $v_1, ..., v_4$ be linearly independent?

2) (Advanced) Prove that if U and V are subspaces of $\mathbb{R}^n$ and their union is a subspace, then one is contained in the other.

3) (VERY Advanced) Prove that if U, V, W are subspaces of $\mathbb{R}^n$ such that their union is also a subspace, then one of them contains the other two.



from xkcd.com