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Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 13

(1) Let A be the matrix $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. The eigenvalues of A are 2 and 8 with geometric multiplicity 2 and 1.

(a) Recalling a theorem from class, explain why you can find an 3×3 orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$. (Don't find Q and D just explain why it is possible.)

(b) Two linearly independent eigenvectors of A corresponding to the eigenvalue 2 are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Note that $\mathbf{v}_1, \mathbf{v}_2$ are not orthogonal. Find two orthonormal eigenvectors $\mathbf{w}_1, \mathbf{w}_2$ of A corresponding to eigenvalue 2.

(c) The vector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 8. What can you say about the orthogonality relationship between $\mathbf{w}_i$'s and $\mathbf{v}_3$?

(d) Now find an 3×3 orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.

(2) We now want to understand this decomposition a little bit better. For that it is important to recall:

Outer Product Rule for computing AB.

Let A be $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix}$$

where each $\mathbf{a}_i$ is a **column vector** and each $\mathbf{b}_i^T$ is a **row vector**. Then

$$AB = \mathbf{a}_1\mathbf{b}_1^T + \cdots + \cdots + \mathbf{a}_n\mathbf{b}_n^T$$

(a) Let $A = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Using the notation above, what are $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{b}_1^T$, $\mathbf{b}_2^T$, and $\mathbf{b}_3^T$?

(b) Compute AB using the outer product rule.

(c) Let $\mathbf{q} \in \mathbb{R}^n$ be a unit vector. Explain why $\mathbf{q}\mathbf{q}^T$ is symmetric. Explain why $\mathbf{q}\mathbf{q}^T$ is the projection matrix of the orthogonal projection onto the span of $\mathbf{q}$. (Hint: What is the formula for a projection matrix?)

(3) Let A be an $n \times n$-matrix such that $A^T = A$. Then there is an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$. Suppose $Q = \begin{bmatrix} \mathbf{q_1} & \cdots & \mathbf{q_n} \end{bmatrix}$ and that the diagonal entries of D are $\lambda_1, \ldots, \lambda_n$.

(a) Using the outer product rule, explain why

$$A = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T.$$

(This is a called the **spectral decomposition** of A.)

(b) Suppose $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Determine the spectral decomposition of A using your results from Problem 1.

(c) The lines spanned by $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are called the **principal axes** of A. A friend tells you the following:

> "Suppose I am given a vector $\mathbf{v}$ and want to compute $A\mathbf{v}$. If A is symmetric, this is easy! Simply project $\mathbf{v}$ onto the each of principal axes, multiply the projects by the corresponding eigenvalues and sum up the n resulting vectors."

Explain why this is true. (Hint: Problems 2(c) and 3(a) should be helpful.)

(d) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Observe that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In the following plot, draw (without making any further computations)

- the principal axes of A (definition in (c)),
- the orthogonal projection of $\mathbf{v}$ onto the principal axes of A, and
- the vector $A\mathbf{v}$.