Names: ____________________

NetIDs: ____________________

Discussion section: **D** ________ **Group number:** ________

MATH 257 - WORKSHEET 9

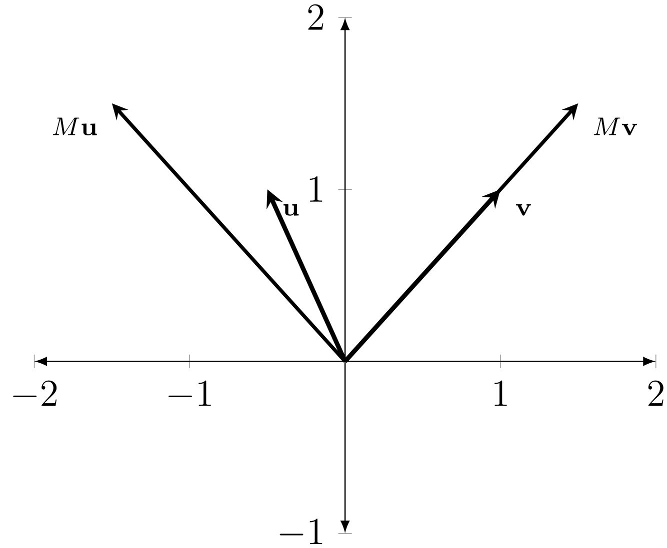
Eigenvectors (and eigenvalues) have countless applications in the real world. But what is an eigenvector? What does that defining equation, $M\mathbf{v} = \lambda\mathbf{v}$, really mean?

In this worksheet, your team will explore two scenarios: one that suggests a conceptual picture of an eigenvector, and another that applies the picture to multiple matrices' eigenvectors.

Once your team has explored the concepts through these scenarios, you will get a chance to practice using them in abstract problems.

(1) **Review**
Discuss the following questions as a group. Write and briefly explain your answers below.

(a) In the plot below, is $\mathbf{v}$ an eigenvector of M? Is $\mathbf{u}$? Use the definition $M\mathbf{v} = \lambda\mathbf{v}$.

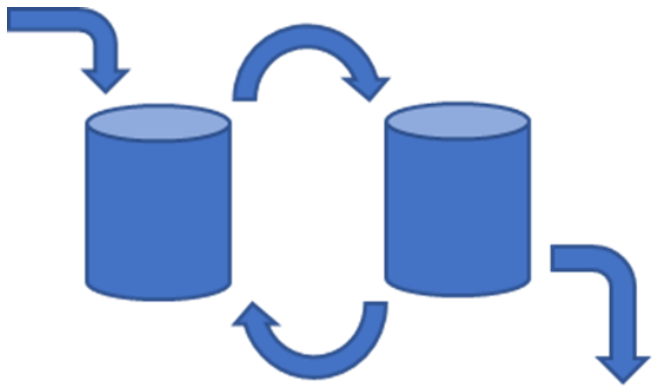


(b) A student makes the following claim:

"Intuitively, the definition of an eigenvector $M\mathbf{v} = \lambda\mathbf{v}$ means that $\mathbf{v}$ stays pointing in the same direction after it gets multiplied by M."

How can you make "stays pointing in the same direction" more precise? To what extent does your group agree or disagree?

(2) **Scenario: Brewing beverages**



A distillery makes a fermented beverage in a process that uses two tanks, tank A and tank B. The table below describes how the fermenting liquid is moved between tanks each morning.

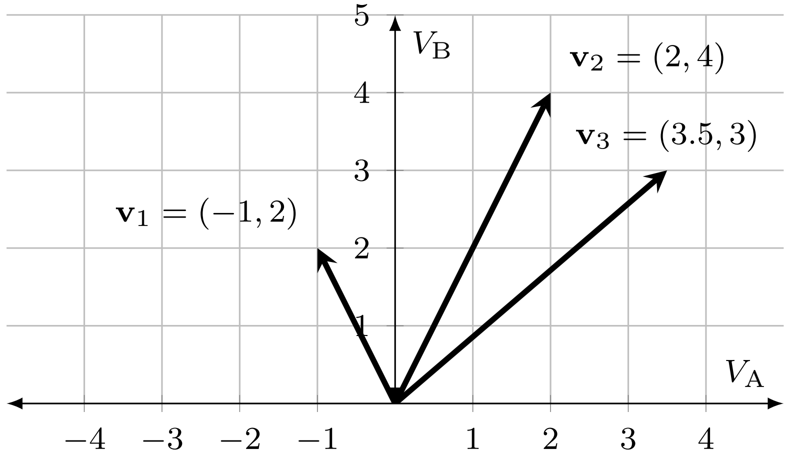
Of the initial volume of liquid in tank A,...	Of the initial volume of liquid in tank B,...	
12%	7%	...is mixed into the **other tank**.
—	7%	...is **dispensed from that tank** into bottles.
88%	86%	...is the amount remaining in the **same tank**.
6%	—	...is amount the of **new ingredients added** into **that tank**.

All percentages are calculated **before any liquid is moved**, as a portion of the **initial volume of liquid** (not as a portion of the capacity of the tank).

(a) The volumes of liquid in the two tanks resulting from this mixing, dispensing, and adding can be expressed as the matrix M, below, multiplied by the column vector of initial volumes. As a group, discuss and write out how to derive this matrix from the information above. (Hint: if no liquid were moved, M would be the identity matrix.)

$$\begin{bmatrix} V_{\text{A,after}} \\ V_{\text{B,after}} \end{bmatrix} = M \begin{bmatrix} V_{\text{A,before}} \\ V_{\text{B,before}} \end{bmatrix}, \quad M = \begin{bmatrix} 0.94 & 0.07 \\ 0.12 & 0.86 \end{bmatrix}$$

(b) For each vector $\mathbf{v}$ on the plot below, calculate and draw $M\mathbf{v}$. Then, discuss whether each $\mathbf{v}$ is an eigenvector of M. If so, state the eigenvalue. If not, explain why not.



(c) The distillery needs to choose what initial volume of liquid to put in each tank. They would like to know whether there are volumes they can keep in the tanks that will **remain the same** after the process. Using the results of part (b), explain why it *is* possible. If tank A contains 700 gallons of liquid, how many gallons of liquid should tank B contain?

(3) **Reflection**

Discuss the following questions as a group. Write and briefly explain your answers below.

(a) Let's think back to what the student said before:

"Intuitively, the definition of an eigenvector $M\mathbf{v} = \lambda\mathbf{v}$ means that $\mathbf{v}$ stays pointing in the same direction after it gets multiplied by M."

In the brewing beverages scenario, what does "same direction" mean in terms of volumes of liquid in the two tanks?

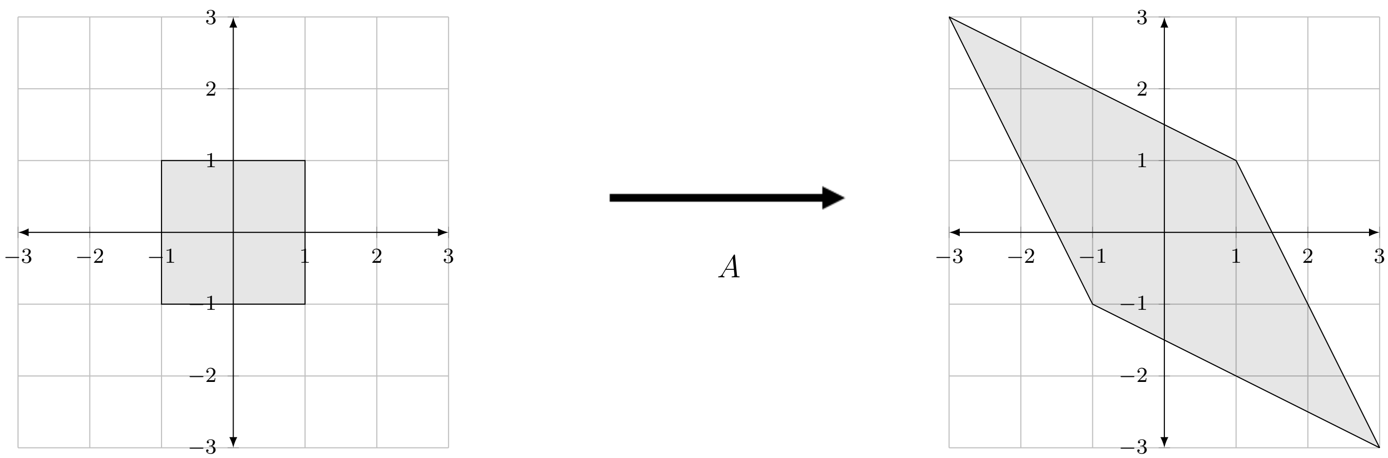
(b) Recall that an **eigenspace** is a vector space of eigenvectors with a particular eigenvalue. Geometrically, a one-dimensional eigenspace of a matrix M is a line through the origin, and it **doesn't change slope** when the vectors in it are multiplied by M. Why?

(4) **Scenario: Stretching cloth**

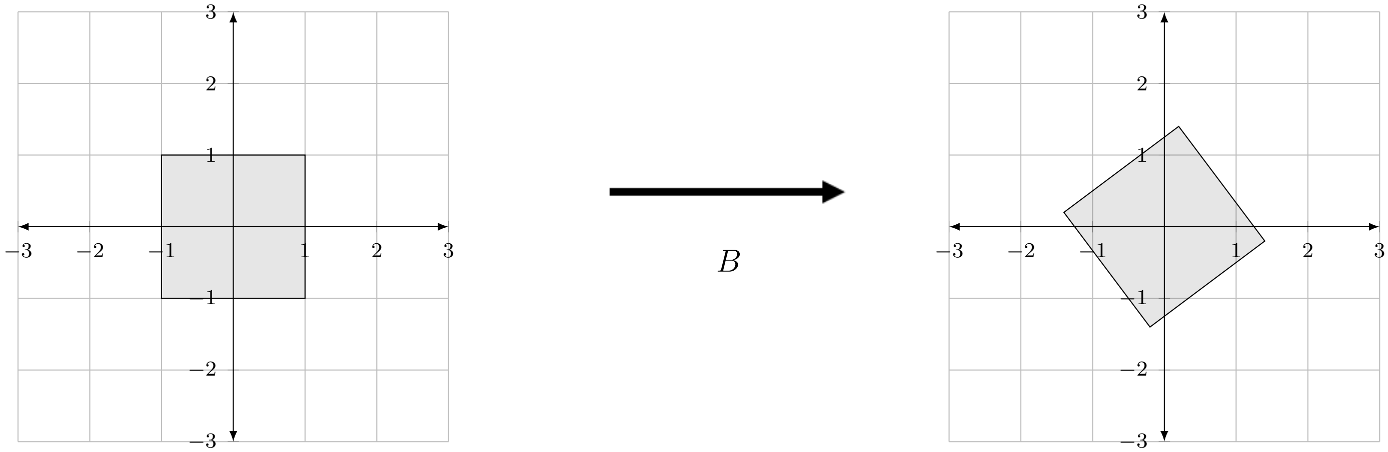
A square piece of elastic cloth is mounted on a frame. Then, the frame is distorted according to multiplication by the accompanying 2×2 matrix.

(a) The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ stretches the cloth along one diagonal. A has **two eigenspaces**, one spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the other by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(i) On both plots below, plot both eigenspaces.
(ii) Write a sentence or two below explaining why these are eigenspaces. What are their eigenvalues?



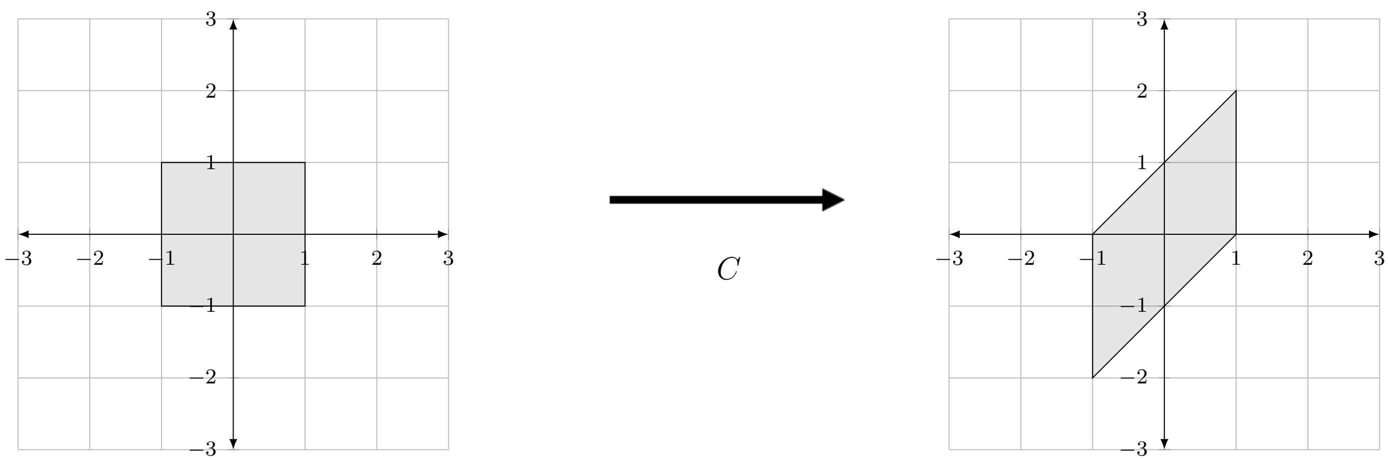
(b) The matrix $B = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ rotates the cloth slightly less than 45° counterclockwise around the origin. B has **no eigenspaces**. Write a sentence or two below explaining why.



(c) The matrix $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ shears the cloth in one direction. C has **one eigenspace** spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(i) On both plots below, plot the eigenspace.

(ii) Write a sentence or two below explaining why this is an eigenspace. What is its eigenvalue? Why does C have no other eigenspace?



(5) **Abstract problems**

Now that your team has explored the concept of an eigenvector in two applied scenarios, let's try applying it to abstract problems.

Work as a group to solve the following problems.

(a) Let L be a line through the origin in $\mathbb{R}^n$, and A and B be $n \times n$ matrices. True or false? If L is an eigenspace of both A and B, then L is contained in an eigenspace of the product matrix AB.

(b) True or false? If A has an eigenvalue λ, then $\lambda + 1$ is an eigenvalue of $A + I$.