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MATH 257 - WORKSHEET 11

In this worksheet we investigate the meaning of diagonalizability via a concrete example. Given square $n \times n$ matrix A we have a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$ via $L(\mathbf{v}) = A\mathbf{v}$. If $\mathcal{E}$ denotes the standard basis on $\mathbb{R}^n$, then $L_{\mathcal{E},\mathcal{E}} = A$. Our program will be to successively exchange one choice of basis for another so that the coordinate matrix for L becomes progressively simpler. We will find that diagonalizable matrices are those yielding the simplest outcomes.
Define matrix

$$A = \frac{1}{4}\begin{pmatrix} 7 & 3 & 1 & 1 \\ 3 & 7 & 1 & 1 \\ -1 & -1 & 1 & 5 \\ -1 & -1 & 5 & 1 \end{pmatrix}$$

and let $\mathbf{v}_i$ be the columns of the matrix

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Define subspaces $X = \mathit{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $Y = \mathit{span}\{\mathbf{v}_3, \mathbf{v}_4\}$. Also, define linear transformation $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $L(\mathbf{v}) = A\mathbf{v}$. In particular, $A = L_{\mathcal{E},\mathcal{E}}$.

(1) Explain why $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is an ordered basis for $\mathbb{R}^4$. [Hint: what are the dot products $\mathbf{v}_i \cdot \mathbf{v}_j$?]

The collection is orthogonal and does not contain the zero vector. Hence, it is linearly independent, and since $\dim(\mathbb{R}^4) = 4$ it is a basis for $\mathbb{R}^4$.

(2) Show that $\mathcal{C} = (\mathbf{v}_1, \mathbf{v}_2)$ is a (ordered) basis for X and $\mathcal{D} = (\mathbf{v}_3, \mathbf{v}_4)$ one for Y. Show that $X \cap Y = \{\mathbf{0}\}$.

Since $\mathcal{B}$ is linearly independent, so are $\mathcal{C}$ and $\mathcal{D}$. Thus, $\mathcal{C}, \mathcal{D}$ are bases for X, Y respectively. Suppose $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_3\mathbf{v}_3 + a_4\mathbf{v}_4 \in X \cap Y$. Then, $\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 - a_3\mathbf{v}_3 - a_4\mathbf{v}_4$ so that by linear independence, we conclude all the $a_i = 0$ whence $X \cap Y = \{\mathbf{0}\}$.

(3) For each basis vector $\mathbf{v}_i$, write $L(\mathbf{v}_i)$ as a linear combination of the basis vectors in $\mathcal{B}$. Use your answer to show that if $\mathbf{x} \in X$ then $L(\mathbf{x}) \in X$ and that if $\mathbf{y} \in Y$ then $L(\mathbf{y}) \in Y$.

We compute that

$$L(\mathbf{v}_1) = -\mathbf{v}_2 \qquad L(\mathbf{v}_2) = -\mathbf{v}_1 \qquad L(\mathbf{v}_3) = 2\mathbf{v}_3 \qquad L(\mathbf{v}_4) = \mathbf{v}_3 + 2\mathbf{v}_4$$

Thus, for any $\mathbf{x} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \in X$ we have $L(\mathbf{x}) = -r_1\mathbf{v}_2 - r_2\mathbf{v}_1 \in X$. Similarly, for any $\mathbf{y} = r_3\mathbf{v}_3 + r_4\mathbf{v}_4 \in X$ we have $L(\mathbf{x}) = 2r_3\mathbf{v}_3 + r_4(\mathbf{v}_3 + 2\mathbf{v}_4) \in Y$.

(4) Find $L_{\mathcal{B},\mathcal{B}}$. What is special about its structure? How is its appearance simpler than $A = L_{\mathcal{E},\mathcal{E}}$?

Using the previous work, we have $L_{\mathcal{B},\mathcal{B}} = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$ a "block-diagonal" matrix much simpler than the original $A = L_{\mathcal{E},\mathcal{E}}$. Now, we can focus attention on subspaces X, Y separately in attempt to further simplify the two 2×2 blocks along the diagonal of $L_{\mathcal{B},\mathcal{B}}$.

(5) Let $L_X : X \to X$ be the restriction of L to X and define L_Y analogously. What are $(L_X)_{\mathcal{C},\mathcal{C}}$ and $(L_Y)_{\mathcal{D},\mathcal{D}}$?

These are precisely the blocks along the diagonal of $L_{\mathcal{B},\mathcal{B}}$, namely

$$(L_X)_{\mathcal{C},\mathcal{C}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad (L_Y)_{\mathcal{D},\mathcal{D}} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

(6) Define $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$ and set $\mathcal{C}' = (\mathbf{w}_1, \mathbf{w}_2)$. Show that $\mathcal{C}'$ is an ordered basis for X.

There are many ways to do this. One way: note that $\mathbf{v}_1 = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2$ and that $\mathbf{v}_2 = \frac{1}{2}\mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2$. Thus, $\mathbf{w}_1, \mathbf{w}_2$ are a spanning set for 2-dimensional space X, hence a basis for X.

(7) Define $X_1 = \mathit{span}\{\mathbf{w}_1\}$ and $X_2 = \mathit{span}\{\mathbf{w}_2\}$ subspaces of X. Show that $X_1 \cap X_2 = \{\mathbf{0}\}$. Show that if $\mathbf{x} \in X_1$ then $L(\mathbf{x}) = L_X(\mathbf{x}) \in X_1$ and analogously for $\mathbf{x} \in X_2$. What do we call $\mathbf{w}_1, \mathbf{w}_2$?

We have $X_1 \cap X_2 = \{\mathbf{0}\}$ since $\mathbf{w}_1, \mathbf{w}_2$ are linearly independent. Since $L(\mathbf{w}_1) = L_X(\mathbf{w}_1) = -\mathbf{w}_1$ and $L(\mathbf{w}_2) = L_X(\mathbf{w}_2) = \mathbf{w}_2$ the claims follow. Note that we have shown $\mathbf{w}_1$ is a (-1)-eigenvector of A and $\mathbf{w}_2$ is a 1-eigenvector of A.

(8) What is $(L_X)_{\mathcal{C}',\mathcal{C}'}$? Let $\mathcal{B}' = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \mathbf{v}_4)$. What is $L_{\mathcal{B}',\mathcal{B}'}$?

From previous work we have $(L_X)_{\mathcal{C}',\mathcal{C}'} = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right]$ a further simplification. This optimal simplification arises from choosing eigenvectors for our basis in X. Then,

$$L_{\mathcal{B}',\mathcal{B}'} = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

(9) Verify that $\mathbf{v}_3$ is an eigenvector for A. Show that if $\mathcal{D}' = (\mathbf{v}_3, \mathbf{y})$ is an ordered basis for Y for some $\mathbf{y} \in Y$, then $\mathbf{y}$ cannot be an eigenvector of A. Conclude that there is no choice of basis $\mathcal{D}'$ for Y such that $(L_Y)_{\mathcal{D}',\mathcal{D}'}$ is diagonal.

From previous work, $2\mathbf{v}_3 = L(\mathbf{v}_3) = A\mathbf{v}_3$ so that $\mathbf{v}_3$ is a 2-eigenvector for A. Write $\mathbf{y} = r\mathbf{v}_3 + s\mathbf{v}_4$. Since $\mathcal{D}'$ is a basis, $s \neq 0$. Then,

$$A\mathbf{y} = L(\mathbf{y}) = L_Y(r\mathbf{v}_3 + s\mathbf{v}_4) = r(2\mathbf{v}_3) + s(\mathbf{v}_3 + 2\mathbf{v}_4) = 2\mathbf{y} + s\mathbf{v}_3$$

so that $\mathbf{y}$ is not an eigenvector. Thus, there is no choice of basis $\mathcal{D}'$ for Y such that $(L_Y)_{\mathcal{D}',\mathcal{D}'}$ is diagonal.

Summary: In choosing basis $\mathcal{B}$ so that $L_{\mathcal{B},\mathcal{B}}$ is simple, choice of eigenbasis (when such exists) yields the simplest result, namely a diagonal matrix.