**Names:** ______________________________

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**Discussion section:** D ________ **Group number:** ________

# MATH 257 - WORKSHEET 4: INVERTIBLE MATRICES

## 1. The basics

**1.** *The matrix* $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ *is reduced to the echelon matrix* $U = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ *using the following row operations (in the given order):*

*(1)* $R_2 \to R_2 + 2R_1$,
*(2)* $R_3 \to R_3 - R_1$,
*(3)* $R_3 \to R_3 + R_2$,
*(4)* $R_4 \to R_4 + R_2$.

*Find a lower-triangular matrix* $L$ *that can appear in a* $LU$*-decomposition* $A = LU$.

**2.** *Find the inverses of the following invertible matrices:*

*(1)* $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$
*(2)* $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

**3.** *Is the sum of two invertible matrices always an invertible matrix? Justify your answer.*

**4.** *If a matrix $A$ is invertible, is the matrix $A^2$ invertible? Justify your answer.*

**5.** *Let $A$ be an invertible $n \times n$-matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^n$. How many solution does $A\mathbf{x} = \mathbf{b}$ have? Explain why!*

**6.** *Let* $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, *and* $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$.

*Note that $A = LU$. Let $A_i$ be the submatrix obtained by taking the first $i$ rows and the first $i$ columns of $A$, for $i = 1, 2, 3$, i.e.,*

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad A_1 = \begin{bmatrix} 2 \end{bmatrix}.$$

*What is an LU-decomposition of $A_i$, for $i = 1, 2, 3$?*

**7.** *Let $A$ be an $n \times n$-matrix such that $A^2$ is the zero matrix. Is $A - I$ invertible? If so, what is the inverse of this matrix?*

**8.** *The objective of this exercise is to study the question: Do all matrices have an LU decomposition?*

(a) *Explain why any triangular matrix (upper or lower triangular) $T$ has an LU decomposition*

(b) *Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Verify that $A$ has no LU decomposition (*Hint: *Suppose that there exists two matrices $L$ and $U$ such that $A = LU$. Eventually you will get a contradiction!)*