Names: ____

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Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 10

Eigenvectors and eigenvalues are arguably the most important concepts you learn in this class. As we have seen in the applications, matrices and matrix multiplication are everywhere. Eigenvectors and eigenvalues are tools that allow us to better understand a given matrix and how it acts on vectors. Here we learn how.

(1) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Determine the eigenvalues of A.

(b) Find a basis of each eigenspace of A. Describe the eigenspaces geometrically.

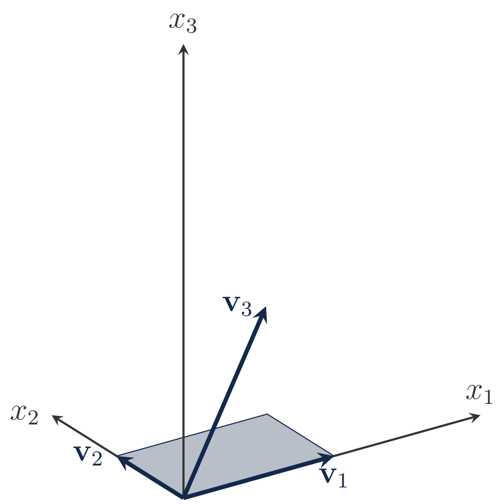
(c) Find three eigenvectors of A that are linearly independent. (Do they form a basis of $\mathbb{R}^3$?)

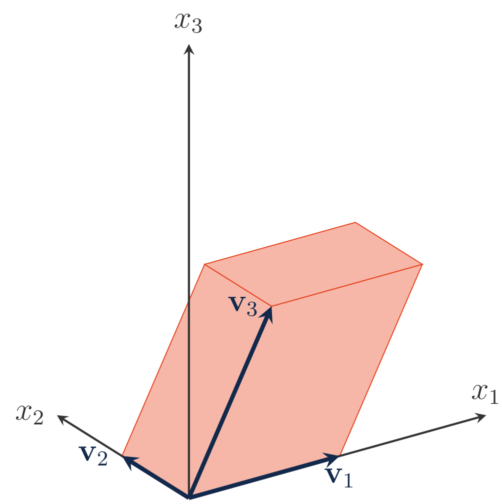
Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the three linearly independent eigenvectors you found in part (c), with $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to the eigenvalue 1.

(d) Let $k \in \mathbb{N}$. What is $A^k\mathbf{v}_1$? What is $A^k\mathbf{v}_2$? What is $A^k\mathbf{v}_3$?

(e) Suppose $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Express $A\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. (Hint: Recall that $A\mathbf{v} = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3$).

(f) Eigenspaces tell you a lot about how the matrix A acts on vectors. Look at the blue rectangle and the orange box below. As a group, discuss how multiplying by A changes the rectangle and the box.





(2) Let B be a 3×3 matrix.

(a) What can you say about the determinant of B if 0 is an eigenvalue of B?

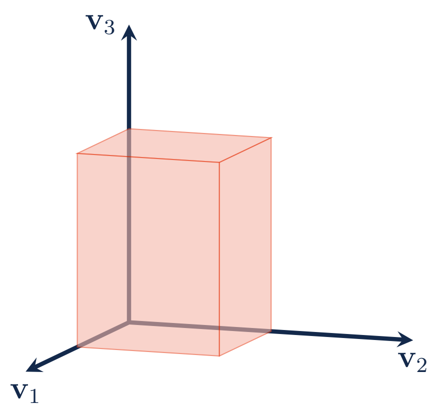
Now assume that B has an eigenvalue 0 corresponding to an eigenspace of dimension 1.

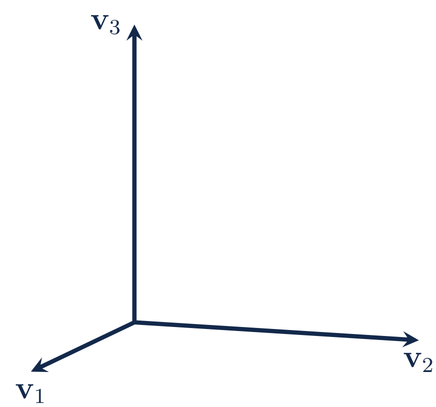
(b) What is $\dim \operatorname{Nul}(B)$? Explain. Can you describe $\operatorname{Col}(B)$ geometrically?

Suppose that $\mathbf{v}_1$ is an eigenvector of B with eigenvalue 0, $\mathbf{v}_2$ is an eigenvector of B with eigenvalue 1.5, $\mathbf{v}_3$ is an eigenvector of B with eigenvalue 0.5.

(c) Without doing any computations, find a basis of $\mathrm{Nul}(B)$ and a basis of $\mathrm{Col}(B)$.

(d) Consider the following box. Discuss how this changes when multiplied by B. Draw the resulting shape into the second picture.





(3) For each of the following matrices, find the eigenvalues of matrix, and for each eigenvalue λ, find a basis of the corresponding eigenspace.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Use your eigenvalue and eigenvector calculations of the above problem as a guide to answer the following questions about a matrix.

(a) At most how many eigenvalues can a 2×2 matrix have? Explain.

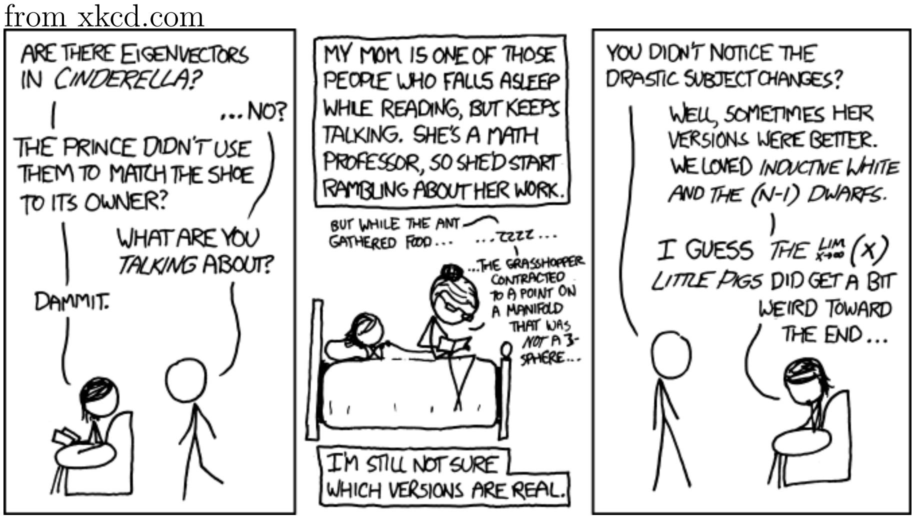
(b) How many linearly independent eigenvectors can be found for a 2×2 matrix? Is it possible to have a matrix without 2 linearly independent eigenvectors? Explain.

(4) Great work! If you still have time, consider the following two challenging questions about how the transpose interact with eigenvalues and eigenvectors. Let A be an $n \times n$-matrix.

(a) Explain why A and A^T have the same eigenvalues. (Hint: Compare the characteristic polynomials of the two matrices.)

(b) (Challenge question) Suppose that $A^T = A$. Let λ_1, λ_2 be such that $\lambda_1 \neq \lambda_2$. Let $\mathbf{x}, \mathbf{y}$ be such that $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$. Check that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other.

from xkcd.com



.. but eigenvalues appear in Avengers: Endgame...