Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

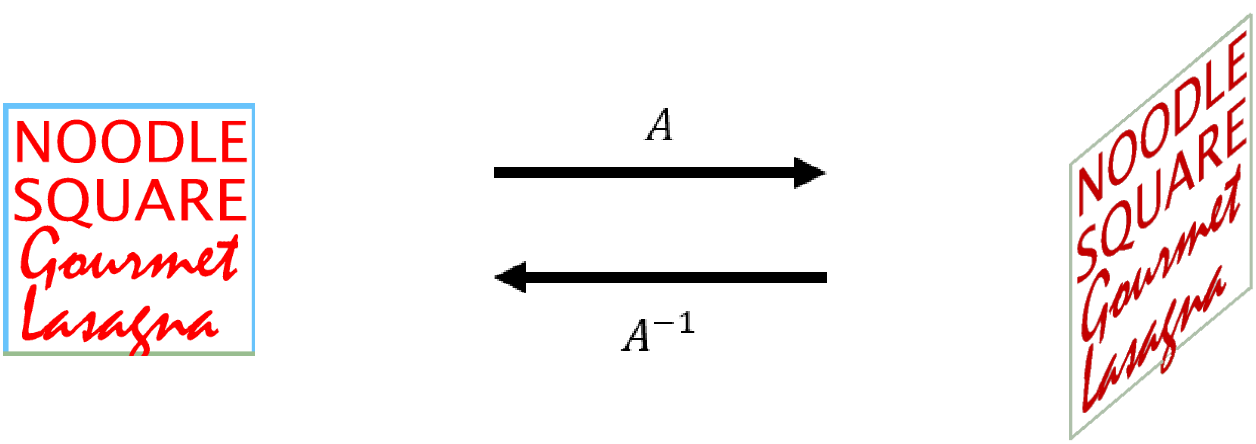
MATH 257 - WORKSHEET 5: CONCEPTUAL PROBLEM SOLVING

One of the challenges of applied linear algebra is modeling the real world in mathematical language. Once a model is created, the challenge becomes to determine the characteristics of that model. What is true in the *modeled* world?

Let's say you've modeled a common engineering problem as a system of linear equations that follows a certain pattern. You might ask: "Can we use the pattern to devise a method for quickly solving the system?" If you can prove that such a method exists, the model becomes easier to use.

Throughout this course, you'll practice this type of challenge by solving conceptual problems that ask you to verify the truth of mathematical statements. **Unlike computational problems, conceptual problems usually don't have an obvious method for solving them. You'll need to creatively use your conceptual understanding.**

Today, your group will practice solving conceptual problems about matrix inverses. Since we won't talk about the real-world meaning of these concepts much today, let's briefly go back to the sign we worked with in week 3 to think about an analogy.



If A is an invertible matrix that turns a square sign into a distorted sign, A^{-1} is the matrix that turns the distorted sign back into a square sign.

(1) Consider the following problem.

> Let A be an $n \times n$ matrix such that A^2 is the zero matrix. Show that the matrix $A - I_n$ is invertible and find its inverse.

(a) Work together as a group to try solving this problem for no more than about 5 minutes. This is a difficult problem, so you might get stuck. The next part will guide you through one mathematician's thought process for solving it.

If your group manages to solve the problem quickly, congrats! You can use the next part to understand your solution better and learn some tools for the next time you get stuck.

(b) *"Maybe we can get some intuition from* 1×1 *matrices."*

Let B be a 1×1 matrix. Work as a group to figure out all the possible values of B such that $B^2 = 0$. For each, determine the inverse of $B - I_1$.

(c) *"Did that help?"*

Discuss: From considering the 1×1 matrices in part (b), what insight did you gain into the original problem, if any? Summarize your discussion below in one or two sentences. If you're not sure, go on to part (d) and revisit this question later.

(d) *"There might be more interesting* 2×2 *matrices with the right property."*

Together, invent one example of a nonzero 2×2 matrix C such that $C^2 = 0$. **Determine the inverse of** $C - I_2$ (not the inverse of C!).

(e) *"Maybe we can use this example to guess the general solution."*

Work together to figure out how to express the **inverse** of $C - I_2$, which you calculated in part (d), as a linear combination of C and I_2.

(f) *"Let's check if our guess works in general."*

Rewrite your answer to (e) with the variables A and I_n in place of C and I_2 to arrive at a **guess for the inverse of $A - I_n$ in general**. Multiply your guess by $A - I_n$ and discuss the result to figure out if your guess satisfies the definition of the inverse.

If there is any doubt or disagreement, take this opportunity to discuss it.

(g) *"The solution doesn't need all our scratch work. Let's present a clean general solution."*

Write out a general solution to the original problem, using variables and equations and avoiding specific examples or vague reasoning.

(2) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

(a) Think back to part (1)(a). How did it feel to look at the problem for the first time?

(b) How did it feel to work with an example compared to the general case? What was useful about coming up with examples, and what were the limitations?

(c) The provided thought process involves using both a *computational* approach to the inverse of a written–out matrix and a *abstract* approach to the inverse based on matrix algebra using the definition. How was each approach useful? How necessary was each for your final solution?

(d) In retrospect, can you think of a way to discover the general solution directly, without exploring example matrices?

Summarize your discussion below in 3–4 bullet points.

Nice work! By following a mathematician's train of thought, your group has reasoned your way to a solution. The problem below can be approached with a similar process.

(3) Solve the following problem.

> Consider an invertible $n \times n$ matrix $A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}$. Suppose A has the property $A^{-1} = A^T$. What properties must the columns of A have (individually, and in pairs)?

Hint: try using the definition of the inverse and starting with matrices with only a few columns. Coming up with a specific matrix and plotting its columns as vectors might help too.