Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 2

Your team has been hired by your city to fill a pair of storefronts in a business district. Each business would attract some number of adult and child customers to the business district per day based on its width. To encourage a thriving but not overcrowded business district, the pair of businesses need to have the right widths so that together they:

(1) Attract two hundred (200) adults to the district every day (not more or less).
(2) Attract fifty (50) children to the district every day (not more or less).

The total width is not fixed. The table below summarizes the five candidate businesses.

Candidate business	**Adult customers attracted per day**	**Children attracted per day**
Hair salon	12 per meter of width	2 per meter of width
Cafe	18 per meter of width	3 per meter of width
Deli	12 per meter of width	3 per meter of width
Shoe store	8 per meter of width	2 per meter of width
Toy store	10 per meter of width	5 per meter of width

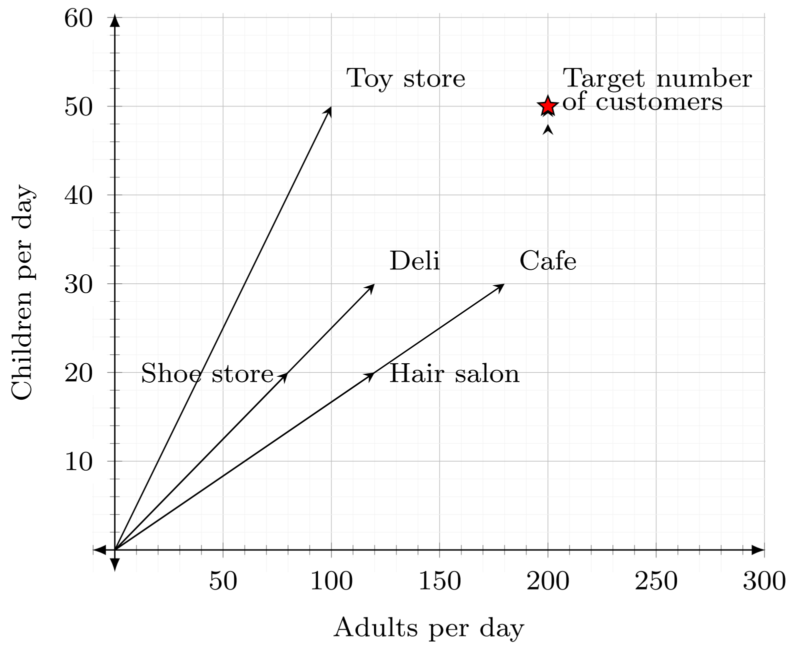
Reminder: For a matrix in echelon form, a **pivot position** is the position of a leading 1. A **free variable** is a variable corresponding to a column with no pivot.
Example:

$$\left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \boxed{0} \end{array}\right] \left[\begin{array}{c} -2 \\ -3 \\ 0 \end{array}\right]$$

Pivot position
x_3 is a free variable

(1) In this problem, your team will gain intuition for pivots and free variables by using a visualization of a linear system.

The candidate businesses are plotted as vectors below. Adjusting the width of a business corresponds to adjusting the length of the corresponding vector while keeping its slope the same. To visualize a successful pairing, imagine picking two businesses and adjusting their widths so that the vectors add together tip-to-tail to reach the star.



(a) Using the numbers from the table, set up the augmented matrix for the following pairing: **the toy store and the hair salon**. Put the system in echelon form (not RREF) to check the number of pivots and free variables. Discuss: Why should this system have a solution, and why should it be unique? Explain below.

(b) Now use the plotted vectors to visualize the pairing of the toy store and the hair salon. Discuss: Why can this system reach the target, and why is the solution unique? Explain below. Draw the solution on the plot as two vectors added tip-to-tail.

(c) Now let's look at some pairings that don't have exactly one solution.

(i) Use the plotted vectors to find a business pairing that has infinitely many solutions. How many pivots and free variables does it have?

(ii) Use the plotted vectors to find a business pairing that has no solution. How many pivots and free variables does it have?

(iii) **Reflection.** A pairing with infinitely many solutions and a pairing with no solution have a lot in common. What's similar between them? What's the difference? Think about what happens when the target is moved. Discuss as a group and summarize your discussion below.

(d) **Reflection.** Think about this problem from the following perspective:

> *A pivot represents a direction or dimension that the columns of the matrix can "access." A free variable represents a redundant direction/dimension in the columns.*

Discuss the following questions as a group.

- How can you visualize **pivots** using the plotted vectors? What role do they play in determining the number of solutions?
- How can you visualize **free variables**? What role do they play in determining the number of solutions?

Write a couple of bullet points below summarizing your discussion.

(2) The previous problem developed some intuition for pivots and free variables using small, square systems. In this problem, try **applying that intuition** to larger, "wide" and "tall" matrices.

(a) The city has realized they have more room than they thought: all five businesses will fit in the space. The target numbers of adults and children are the same. How many **pivots** and **free variables** will this system have? How many **solutions**? *(Think about what directions/dimension the columns can "access" and how much redundancy there is.)*

$$\left[\begin{array}{ccccc|c} a & b & c & d & e & f \\ g & h & i & j & k & l \end{array}\right]$$

(b) The city now wants your team to think about the entire multi-block business district, which consists of **23 businesses**. The city would like to set targets for the number of adult and child customers per day entering through each of the 15 distinct entrances to the business district, for a total of **30 requirements**. How many pivots and free variables are possible for this system? How many solutions is it **possible** for the system to have, and how many is **most likely**? *(In this part, you don't even have numbers! Use the shape alone to think about what the columns can do and are likely to do.)*

$$\left[\begin{array}{ccc|c} A_{1,1} & \cdots & A_{1,23} & b_1 \\ \vdots & & \vdots & \vdots \\ & \ddots & & \\ \vdots & & \vdots & \vdots \\ A_{30,1} & \cdots & A_{30,23} & b_{30} \end{array}\right]$$

(c) **Reflection.** Discuss the following as a group.

- At what stage(s) of problem solving could it be useful to glance at the columns and shape of a large matrix? Why? *(Hint: calculating the RREF of a large matrix is time-consuming!)*
- Have you encountered any applications where this could be useful, or can you think of any now?

Write a couple of bullet points below summarizing your discussion.

Nice work! Your team helped the city create a thriving business district. Now that you have had some practice working in an applied context with the relationships between a matrix's shape, the form of its RREF, and its solutions, your group can try using what you have learned on abstract problems.

(3) Let A be an $m \times n$-matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$. Suppose that A has m pivots and that $n > m$. Which of the following statements must be true?
  (a) The linear system whose augmented matrix is $[A|\mathbf{b}]$ has infinitely many solutions.
  (b) The linear system whose augmented matrix is $[A|\mathbf{b}]$ has a unique solution.
  (c) The linear system whose augmented matrix is $[A|\mathbf{b}]$ has no solution.
  (d) None of the other answers.

Explain why!

(4) Let A be a 4×4-matrix and let $\mathbf{b}, \mathbf{c}$ be two vectors in $\mathbb{R}^4$ such that the equation $A\mathbf{x} = \mathbf{b}$ has no solution. What can you say about the number of solutions of the equation $A\mathbf{x} = \mathbf{c}$?