Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 13

(1) Let A be the matrix $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. The eigenvalues of A are 2 and 8 with geometric multiplicity 2 and 1.

(a) Recalling a theorem from class, explain why you can find an 3×3 orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$. (Don't find Q and D just explain why it is possible.)

Since A is real, symmetric, this conclusion is assured by the Spectral Theorem.

(b) Two linearly independent eigenvectors of A corresponding to the eigenvalue 2 are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Note that $\mathbf{v}_1, \mathbf{v}_2$ are not orthogonal. Find two orthonormal eigenvectors $\mathbf{w}_1, \mathbf{w}_2$ of A corresponding to eigenvalue 2.

Using the Gram-Schimdt process on $\mathbf{v}_1, \mathbf{v}_2$ in this order yields $\mathbf{w}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

(c) The vector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 8. What can you say about the orthogonality relationship between $\mathbf{w}_i$'s and $\mathbf{v}_3$?

The Spectral Theorem assures that distinct eigenspaces are orthogonal. Thus, $\mathbf{v}_3$ is orthgonal to the vectors $\mathbf{w}_i$.

(d) Now find an 3×3 orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.

Let $\mathbf{w}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and define 3×3 matrices:

$$D = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 8 \end{pmatrix} \qquad Q = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{pmatrix}$$

Then, by the Spectral Theorem, $A = QDQ^T$.

(2) We now want to understand this decomposition a little bit better. For that it is important to recall:

Outer Product Rule for computing AB.

Let A be $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_n^T \end{bmatrix}$$

where each $\mathbf{a}_i$ is a **column vector** and each $\mathbf{b}_i^T$ is a **row vector**. Then

$$AB = \mathbf{a}_1\mathbf{b}_1^T + \cdots + \cdots + \mathbf{a}_n\mathbf{b}_n^T$$

(a) Let $A = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Using the notation above, what are $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{b}_1^T$, $\mathbf{b}_2^T$, and $\mathbf{b}_3^T$?

$\mathrm{a}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad \mathbf{a}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mathbf{b}_1^T = \begin{pmatrix} 2 & 0 \end{pmatrix}$ $\quad \mathbf{b}_2^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$ $\quad \mathbf{b}_3^T = \begin{pmatrix} 4 & -7 \end{pmatrix}$

(b) Compute AB using the outer product rule.

$$\begin{aligned} AB &= \begin{pmatrix} 2 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 4 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 4 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 3 \\ 2 & -7 \end{pmatrix} \end{aligned}$$

(c) Let $\mathbf{q} \in \mathbb{R}^n$ be a unit vector. Explain why $\mathbf{q}\mathbf{q}^T$ is symmetric. Explain why $\mathbf{q}\mathbf{q}^T$ is the projection matrix of the orthogonal projection onto the span of $\mathbf{q}$. (Hint: What is the formula for a projection matrix?)

One checks that $(\mathbf{q}\mathbf{q}^T)^T = (\mathbf{q}^T)^T\mathbf{q}^T = \mathbf{q}\mathbf{q}^T$. Recall for nonzero $\mathbf{v}$, orthogonal projection of $\mathbf{w}$ onto the subspace $V = \mathit{span}(\mathbf{v})$ is $\mathit{proj}_V\mathbf{w} = \frac{\mathbf{w}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. For any $\mathbf{w} \in \mathbf{R}^n$, we compute

$$(\mathbf{q}\mathbf{q}^T)\mathbf{w} = \mathbf{q}(\mathbf{q}^T\mathbf{w}) = (\mathbf{q}^T\mathbf{w})\mathbf{q} = \frac{\mathbf{w}\cdot\mathbf{q}}{\|\mathbf{q}\|^2}\mathbf{q}$$

so that $\mathbf{q}\mathbf{q}^T$ is the projection matrix for orthogonal projection onto $\mathit{span}(\mathbf{q})$

(3) Let A be an $n \times n$-matrix such that $A^T = A$. Then there is an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$. Suppose $Q = \begin{bmatrix} \mathbf{q_1} & \cdots & \mathbf{q_n} \end{bmatrix}$ and that the diagonal entries of D are $\lambda_1, \ldots, \lambda_n$.

(a) Using the outer product rule, explain why

$$A = \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \cdots + \lambda_n\mathbf{q}_n\mathbf{q}_n^T.$$

(This is a called the **spectral decomposition** of A.)

$$A = QDQ = \begin{pmatrix} \mathbf{q}_1 & \dots & \mathbf{q}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 & \dots & \mathbf{q}_n \end{pmatrix}^T$$

$$= \begin{pmatrix} \lambda_1 \mathbf{q}_1 & \dots & \lambda_n \mathbf{q}_n \end{pmatrix} \begin{pmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{pmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^T$$

(b) Suppose $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Determine the spectral decomposition of A using your results from Problem 1.

$$A = 2\mathbf{w}_1\mathbf{w}_1^T + 2\mathbf{w}_2\mathbf{w}_2^T + 8\mathbf{w}_3\mathbf{w}_3^T$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} + \frac{8}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} + \frac{8}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) The lines spanned by $\mathbf{q}_1, \dots, \mathbf{q}_n$ are called the **principal axes** of A. A friend tells you the following:

> "Suppose I am given a vector $\mathbf{v}$ and want to compute $A\mathbf{v}$. If A is symmetric, this is easy! Simply project $\mathbf{v}$ onto the each of principal axes, multiply the projects by the corresponding eigenvalues and sum up the n resulting vectors."

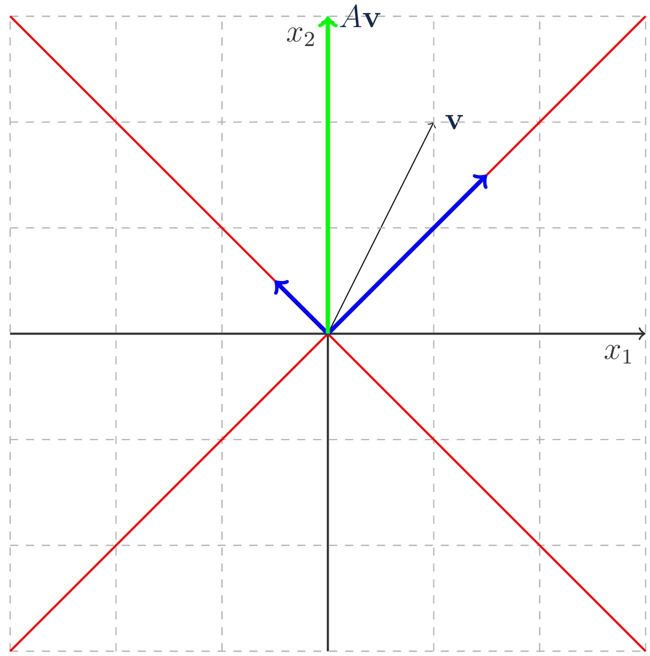
Explain why this is true. (Hint: Problems 2(c) and 3(a) should be helpful.)

Write $Q_i = span(\mathbf{q}_i)$. By the Spectral Decomposition,

$$A\mathbf{v} = \left(\sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^T\right)\mathbf{v} = \sum_{i=1}^{n} \lambda_i (\mathbf{q}_i \mathbf{q}_i^T)\mathbf{v} = \sum_{i=1}^{n} \lambda_i (proj_{Q_i} \mathbf{v})$$

(d) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Observe that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In the following plot, draw (without making any further computations)

- the principal axes of A (definition in (c)),
- the orthogonal projection of $\mathbf{v}$ onto the principal axes of A, and
- the vector $A\mathbf{v}$.



Principal axes (red), projections of $\mathbf{v}$ onto the principal axes (blue), and $A\mathbf{v}$ (green).