Names: ____

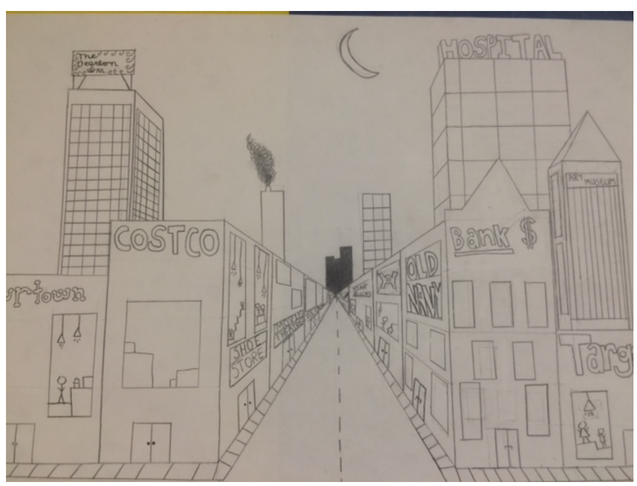
NetIDs: ____

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 3: MATRIX MULTIPLICATION

Today, your team will try to figure out: can **matrix multiplication** model the distortions of seeing a three-dimensional sign from an angle?

Original to distorted



Distorted sign viewed from the street

(1) **Matrix multiplication is simpler than it looks!** Let's see why.

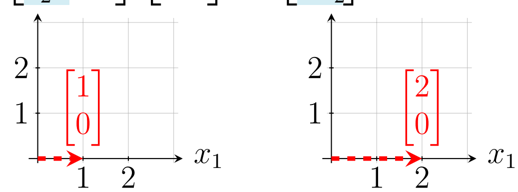
Let $\mathbf{e}_j$ represent the unit vector in the x_j direction. It's all zeros except the jth element, which is 1. We'll just talk about $\mathbb{R}^2$ to start, so $\mathbf{e}_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{e}_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

We'll also need this arbitrarily chosen matrix M:

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

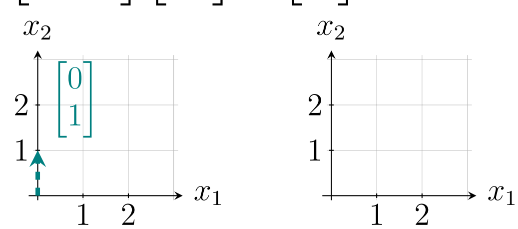
(a) Multiplying M by $\mathbf{e}_1$ gives the *first* column of M. **Why?** As a group, discuss how matrix multiplication works to come up with a brief explanation, and **write** it below.

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



(b) Similarly, multiplying M by $\mathbf{e}_2$ gives the *second* column of M. **Without doing any calculations, write in** $M\mathbf{e}_2$ below and **plot** it below right. As above, discuss and **explain** why this works.

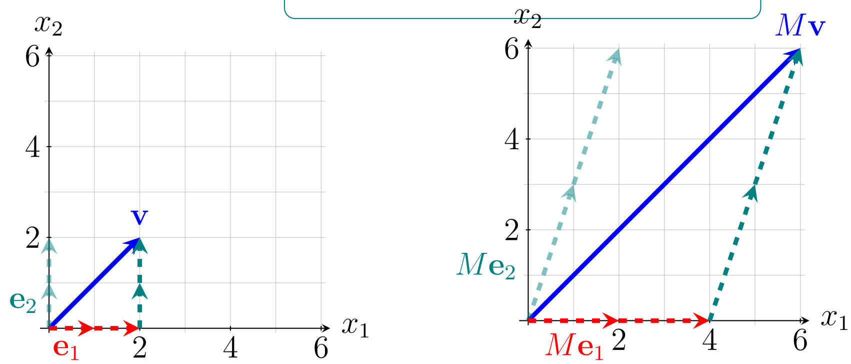
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$



(c) Consider the arbitrarily chosen vector $\mathbf{v} = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$, which is a a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. As it turns out, $M\mathbf{v}$ can be formed from a linear combination of $M\mathbf{e}_1$ and $M\mathbf{e}_2$ with *the same coefficients*.

Compute $M\mathbf{v}$ directly to confirm that this is true. As a group, discuss the matrix multiplication rule so you can **write a brief explanation** of why this trick works.

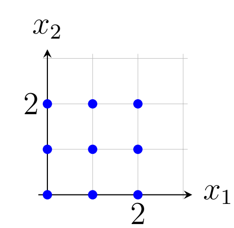
$$\mathbf{v} = 2\,\mathbf{e}_1 + 2\,\mathbf{e}_2 \qquad\qquad M\mathbf{v} = 2\,M\mathbf{e}_1 + 2\,M\mathbf{e}_2$$



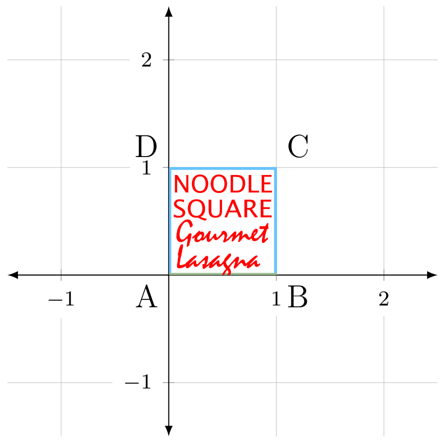
(d) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

- How can you generalize the unit vector shortcut to $m \times n$ matrices?
- Given any vector $\mathbf{u} \in \mathbb{R}^2$, how can you quickly plot $M\mathbf{u}$ using what you learned above?
- Given a square grid G in $\mathbb{R}^2$ (see example below right), how can you quickly plot the transformed grid points $\{M\mathbf{g} : \mathbf{g} \in G\}$?

Summarize your discussion below in 3–4 bullet points.

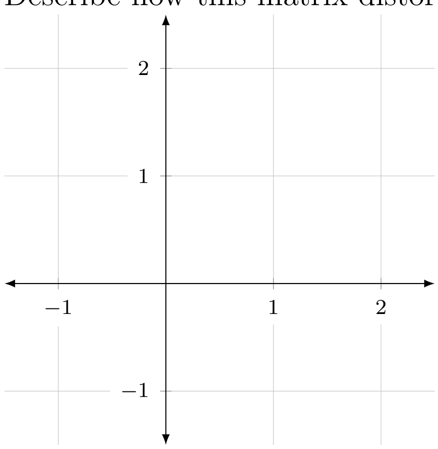


(2) Now that we've explored a single matrix in depth, let's characterize a few different matrices. **Using as little calculation as possible**, figure out what the unit square sign $ABCD$ plotted below will look like when transformed by each matrix. Draw the result on the adjacent empty plot. Keep track of each corner's final location by labeling it with the corresponding letter.



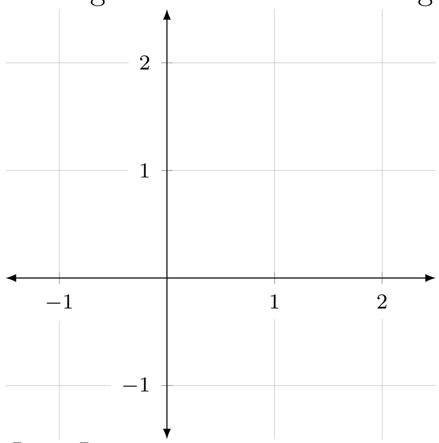
(a) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, a = 2$

Describe how this matrix distorts the sign. What does a represent?



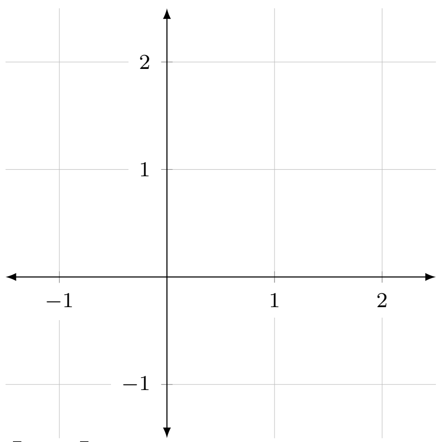
(b) $\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$, $b = -0.5$

Describe how this matrix distorts the sign. What does b represent? How does the negative value affect the figure?



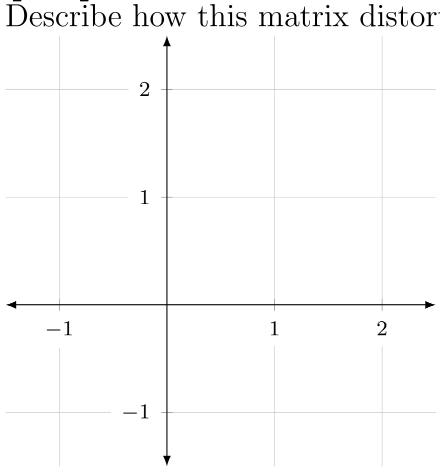
(c) $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, $s = 0.5$

Describe how this matrix distorts the sign. What does s represent?

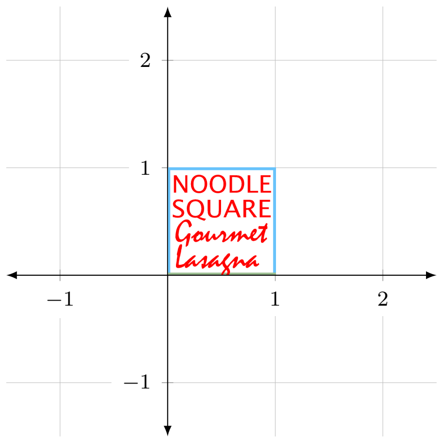


(d) $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$, $t = -0.5$

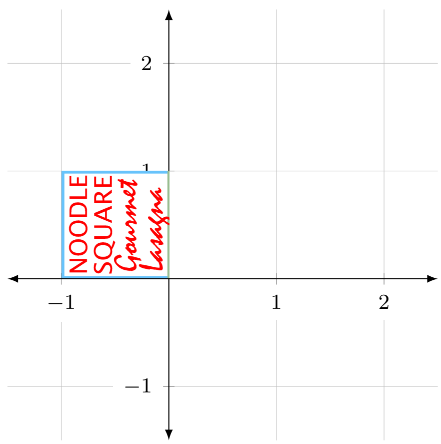
Describe how this matrix distorts the sign. What does t represent?



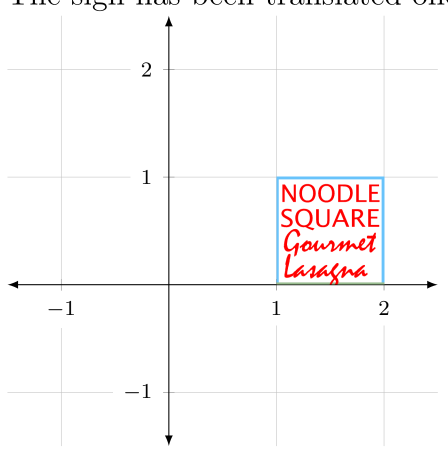
(3) The sign below, a unit square, has been distorted in four different ways. For each, discuss as a group: **can matrix multiplication create** this distortion? If yes, **state** the appropriate matrix. If no, **explain why not.**



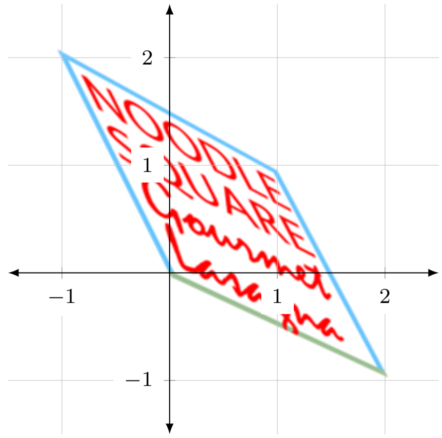
(a) The sign has been rotated by $\pi/2$ radians counterclockwise around the origin.



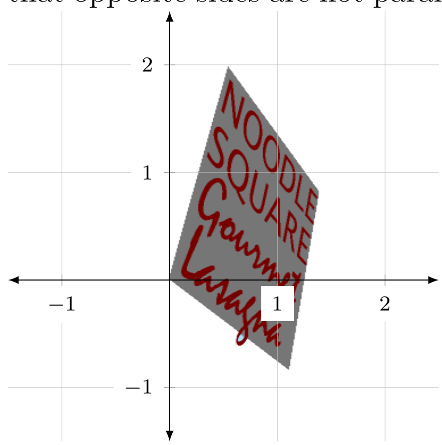
(b) The sign has been translated one unit to the right.



(c) The sign has been stretched along a diagonal.



(d) The sign has been drawn in perspective, as if it were seen from an angle. Note that opposite sides are not parallel because they converge to a “vanishing point”.



(4) **Reflection**. Take a few minutes to pause the problem solving and discuss what you have learned from it.

- Is it possible to use matrix multiplication to create the distortions of a sign seen three–dimensionally from an angle?
- What can matrix multiplication do? What can't it do?
- This problem only considered matrix multiplication from $\mathbb{R}^2$ to $\mathbb{R}^2$. How can you generalize what you learned to higher dimensions?

Summarize your discussion below in 3–4 bullet points.

Nice work! Your team has explored and determined whether matrix multiplication can model the distortion of looking at a three-dimensional sign from an angle.
Now that you have developed a graphical picture of matrix multiplication in an applied context, practice using it on the following abstract problems.

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Using a graphical picture of what left–multiplying by A does, determine $A^n\mathbf{v}$ for any natural number n. Then determine just the matrix, A^n.

(6) Suppose you are told there is a mystery 2×2-matrix A such that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Can you determine A just from this information?

(b) Is always enough to know $A\mathbf{v}$ and $A\mathbf{w}$ for two distinct vectors $\mathbf{v}$ and $\mathbf{w}$ in order to determine A?