Names: ______________________________

NetIDs: ______________________________

Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 10

Eigenvectors and eigenvalues are arguably the most important concepts you learn in this class. As we have seen in the applications, matrices and matrix multiplication are everywhere. Eigenvectors and eigenvalues are tools that allow us to better understand a given matrix and how it acts on vectors. Here we learn how.

(1) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Determine the eigenvalues of A.

Since A is triangular, the eigenvalues are the main diagonal entries $\lambda = 1, 1, 2$.

(b) Find a basis of each eigenspace of A. Describe the eigenspaces geometrically.

$$E_1 = \text{Nul}(1I - A) = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad E_2 = \text{Nul}(2I - A) = span \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

The displayed vectors are bases for their respective eigenspaces. The 1-eigenspace is the xy-plane and the 2-eigenspace the line through the origin along the direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c) Find three eigenvectors of A that are linearly independent. (Do they form a basis of $\mathbb{R}^3$?)

Since $\mathbf{0} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ implies $x_i = 0$ for all i, these vectors are linearly independent. Any three linearly independent vectors in 3-dimensional vector space necessarily form a basis for the vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the three linearly independent eigenvectors you found in part (c), with $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to the eigenvalue 1.

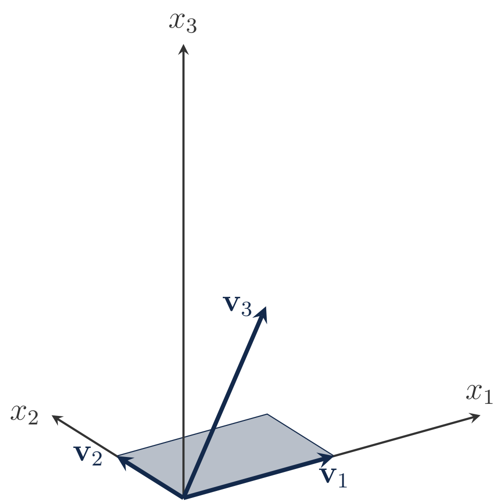
(d) Let $k \in \mathbb{N}$. What is $A^k\mathbf{v}_1$? What is $A^k\mathbf{v}_2$? What is $A^k\mathbf{v}_3$?

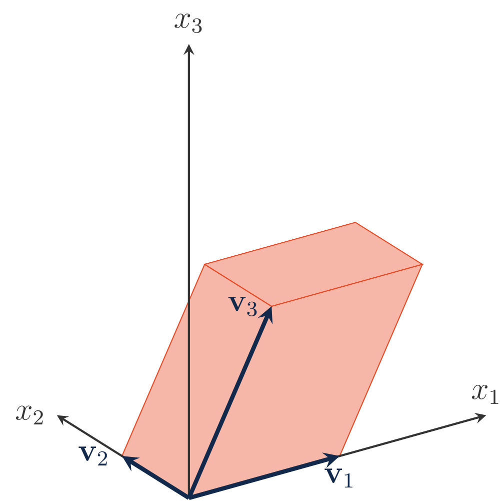
Let $\lambda_1 = 1 = \lambda_2$ and $\lambda_3 = 2$. Then, $A^k\mathbf{v}_1 = A^{k-1}(\lambda_1\mathbf{v}_1) = \lambda_1^k\mathbf{v}_1 = \mathbf{v}_1$, Similarly, $A^k\mathbf{v}_2 = \mathbf{v}_2$ and $A^k\mathbf{v}_3 = 2^k\mathbf{v}_3$.

(e) Suppose $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Express $A\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. (Hint: Recall that $A\mathbf{v} = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3$).

$A\mathbf{v} = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 2c_3\mathbf{v}_3$

(f) Eigenspaces tell you a lot about how the matrix A acts on vectors. Look at the blue rectangle and the orange box below. As a group, discuss how multiplying by A changes the rectangle and the box.





(2) Let B be a 3×3 matrix.

(a) What can you say about the determinant of B if 0 is an eigenvalue of B?

Since $\det(B)$ is the product of the eigenvalues of B, we have $\det(B) = 0$. Alternatively, 0 being an eigenvalue implies $\mathrm{Nul}(B)$ is non-trivial so that B is not invertible. Hence, $\det(B) = 0$.

Now assume that B has an eigenvalue 0 corresponding to an eigenspace of dimension 1.

(b) What is $\dim \mathrm{Nul}(B)$? Explain. Can you describe $\mathrm{Col}(B)$ geometrically?

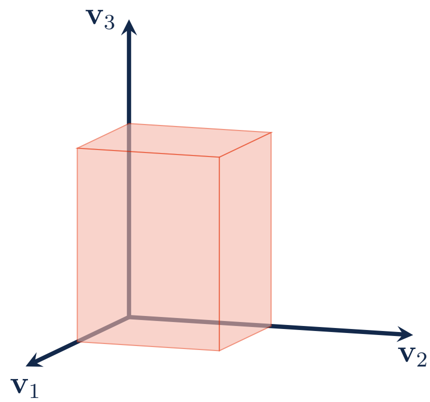
If 0 is an eigenvalue of B, then $E_0 = \mathrm{Nul}(B)$ whence $\dim(E_0) = 1$. By the Rank-Nullity Theorem, $\dim(\mathrm{Col}(B)) = 3 - \dim(\mathrm{Nul}(B)) = 2$ so that $\mathrm{Col}(B)$ is geometrically a plane through the origin in $\mathbb{R}^3$.

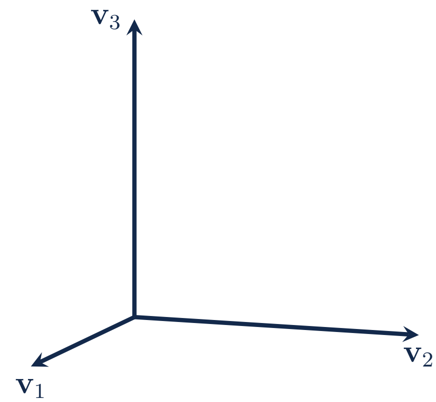
Suppose that $\mathbf{v}_1$ is an eigenvector of B with eigenvalue 0, $\mathbf{v}_2$ is an eigenvector of B with eigenvalue 1.5, $\mathbf{v}_3$ is an eigenvector of B with eigenvalue 0.5.

(c) Without doing any computations, find a basis of $\mathrm{Nul}(B)$ and a basis of $\mathrm{Col}(B)$.

A collection of eigenvectors, each drawn from a different eigenspace, is a linearly independent set. Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an eigenbasis for B. A basis for E_0 is $\mathbf{v}_1$. Since $\mathbf{v}_2, \mathbf{v}_3$ span $\mathrm{Col}(B)$ (see e.g. part (e) of the previous problem), they are a basis for $\mathrm{Col}(B)$.

(d) Consider the following box. Discuss how this changes when multiplied by B. Draw the resulting shape into the second picture.





The eigenspace of eigenvalue 0 is the null space because it's the set of vectors that get sent to the zero vector. So the dimension of the null space, the nullity, is 1. By the rank–nullity theorem, the rank plus the nullity have to add to 3, the number of columns, so the rank must be 2, which by definition is the dimension of the column space. Hence, the column space is a plane.

Geometrically, if you imagine M acting on a sphere centered at the origin, the vectors in L have to get collapsed to the origin. Because the other vectors in the sphere are linear combinations of a vector in L and a vector perpendicular to L, the part in L will go to zero, "flettening" the sphere into circle (which could then get rotated, stretched, sheared, etc.). The plane containing the resulting ellipse is the column space. The one-dimensional eigenspace of eigenvalue 0 "flattens" one dimension that M would otherwise be able to access through its columns.

(3) For each of the following matrices, find the eigenvalues of matrix, and for each eigenvalue λ, find a basis of the corresponding eigenspace.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

For the matrix on the left:

$$E_5 = span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \qquad E_0 = span\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$$

with the given vectors bases for their respective eigenspaces. For the matrix on the right, there is only one eigenspace:

$$E_1 = span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

with the given vector a basis for E_1.

Use your eigenvalue and eigenvector calculations of the above problem as a guide to answer the following questions about a matrix.

(a) At most how many eigenvalues can a 2×2 matrix have? Explain.

Eigenvalues are the roots of the characteristic polynomial. For a $n \times n$ matrix, the characteristic polynomial has degree n and hence at most n roots (counting multiplicity).

(b) How many linearly independent eigenvectors can be found for a 2×2 matrix? Is it possible to have a matrix without 2 linearly independent eigenvectors? Explain.

A $n \times n$ matrix can have at most n linearly independent eigenvectors since $\dim(\mathbb{R}^n) = n$. In this case, the n eigenvectors form an eigenbasis. However, as the above matrix on the right illustrates, sometimes there is no eigenbasis.

(4) Great work! If you still have time, consider the following two challenging questions about how the transpose interact with eigenvalues and eigenvectors. Let A be an $n \times n$-matrix.

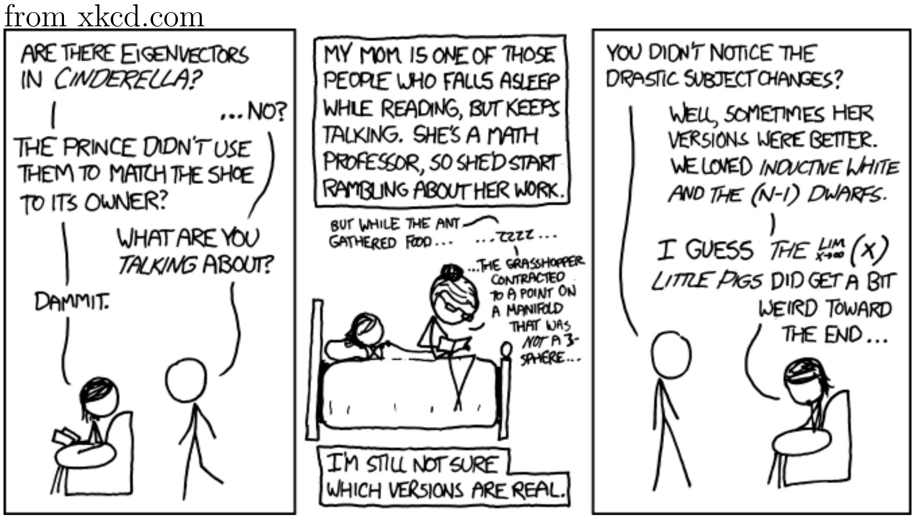
(a) Explain why A and A^T have the same eigenvalues. (Hint: Compare the characteristic polynomials of the two matrices.)

They have identical characteristic polynomials since any square matrix and its transpose have the same determinant: $\det(\lambda I - A^T) = \det((\lambda I - A)^T) = \det(\lambda I - A)$.

(b) (Challenge question) Suppose that $A^T = A$. Let λ_1, λ_2 be such that $\lambda_1 \neq \lambda_2$. Let $\mathbf{x}, \mathbf{y}$ be such that $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$. Check that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other.

Recall we may write $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$. Then, $\lambda_1(\mathbf{x} \cdot \mathbf{y}) = (A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T A \mathbf{y} = \lambda_2(\mathbf{x} \cdot \mathbf{y})$. Since $\lambda_1 \neq \lambda_2$, we conclude $\mathbf{x} \cdot \mathbf{y} = 0$.

from xkcd.com



.. but eigenvalues appear in Avengers: Endgame...