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Discussion section: D ________ **Group number:** ________

MATH 257 - WORKSHEET 8

It is likely that the majority of your math classes in recent years were focused on studying functions and their properties. Why do you think people are so interested in studying them? In this subject we introduce another important class of functions that share the name with our whole subject *linear*. First let's recall the definition:

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if for all $\mathbf{v}, \mathbf{w} \in V$ and all $a, b \in \mathbb{R}$ we have that $T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$.

Can you give any examples of them (be sure to specify the "input" and the "output" vector spaces)? What do you think is so special about such functions (transformations)?

Consider the following functions $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$

$$f_1(x) = 2x + 3, \quad f_2 = -4x, \quad f_3(x) = x^2, \quad \text{and} \quad f_4(x) = \frac{1}{x}.$$

Are any of them linear transformations? Check the definition.

1. *For each of the following functions, determine if they are linear transformations.*

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$

This is not a linear transformation. Scalar multiplication is violated, for example:

$$(-1)T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq T\left((-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $\det : M_{2\times 2} \to \mathbb{R}$ such that $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$, where $M_{2\times 2}$ is the vector space of all 2×2 matrices.

This is not a linear transformation. Scalar multiplication is violated, for example:

$$\det\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 4 \neq 2\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$$

(c) $\operatorname{tr} : M_{2\times 2} \to \mathbb{R}$ such that $\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$, where $M_{2\times 2}$ is the vector space of all 2×2 matrices.

This is a linear transformation. For any $x_1, x_2 \in \mathbb{R}$ and two $M_{23\times 2}$ matrices

$$\begin{aligned}
\operatorname{tr}\left(x_1\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1\end{bmatrix} + x_2\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2\end{bmatrix}\right)&\\
&= \operatorname{tr}\left(\begin{bmatrix} x_1a_1 + x_2a_2 & x_1b_1 + x_2b_2 \\ x_1c_1 + x_2c_2 & x_1d_1 + x_2d_2\end{bmatrix}\right)\\
&= x_1a_1 + x_2a_2 + x_1d_1 + x_2d_2\\
&= x_1(a_1 + d_1) + x_2(a_2 + d_2)\\
&= x_1 \operatorname{tr}\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1\end{bmatrix}\right) + x_2 \operatorname{tr}\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2\end{bmatrix}\right)
\end{aligned}$$

(d) $T : M_{3\times 3} \to M_{3\times 3}$ such that $T\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i\end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & d & e\end{bmatrix}$ where $M_{3\times 3}$ is the vector space of all 3×3 matrices.

This is a linear transformation. For any $x_1, x_2 \in \mathbb{R}$ and two $M_{3\times 3}$ matrices

$$\begin{aligned}
T\left(x_1\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1\end{bmatrix} + x_2\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2\end{bmatrix}\right)&\\
&= T\left(\begin{bmatrix} x_1a_1 + x_2a_2 & x_1b_1 + x_2b_2 & x_1c_1 + x_2c_2 \\ x_1d_1 + x_2d_2 & x_1e_1 + x_2e_2 & x_1f_1 + x_2f_2 \\ x_1g_1 + x_2g_2 & x_1h_1 + x_2h_2 & x_1i_1 + x_2i_2\end{bmatrix}\right)\\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_1a_1 + x_2a_2 & x_1b_1 + x_2b_2 \\ 0 & x_1d_1 + x_2d_2 & x_1e_1 + x_2e_2\end{bmatrix}\\
&= x_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & d_1 & e_1\end{bmatrix} + x_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & b_2 \\ 0 & d_2 & e_2\end{bmatrix}\\
&= x_1 T\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1\end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2\end{bmatrix}\right)
\end{aligned}$$

(e) **Reflection.** Take a short break from solving problems and discuss with your team what you have learned.
- How did you approach these questions?
- Would you suggest any particular thought process for approaching such problems in the future?

One can approach these problems by checking the definition directly. If if does not seem to work, perhaps, there is a counterexample. One can also gain an intuition about a counterexample depending on what step exactly did not go through originally.

2. *True or False? Justify your answers.*

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T(\mathbf{v}_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Notice that $T(\mathbf{v}_1), T(\mathbf{v}_2)$ are linearly independent. Is it true that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent?

True, since if $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = 0$ then $0 = x_1T(\mathbf{v}_1) + x_2T(\mathbf{v}_2) = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, hence $x_1 = x_2 = 0$.

(b) Let $T : V \to W$ be a linear transformation and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in V. If $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are also linearly independent.

True, since if $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n = 0$ then $x_1T(\mathbf{v}_1) + x_2T(\mathbf{v}_2) + \ldots + x_nT(\mathbf{v}_n) = 0$, but $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are linearly independent so all x_is are equal to 0.

(c) Let $T : V \to W$ be a linear transformation and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in V. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, then $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are also linearly independent.

False, consider $T : \mathbb{R}^3 \to \mathbb{R}$ such that $T(\mathbf{v}) = 0$.

(d) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V and let $T : V \to W$. Do the images $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ span W?

This is false. To see this map the vectors $\mathbf{v}_i$ to zero. The images $T\mathbf{v}_i$ with $i = 1, 2, \cdots, n$ will not span W.

(e) **Reflection.** Take a moment to reflect and state in words what you observed for each claim.

- How does applying a linear transformation interact with linear independence of vectors?
- Can you suggest a sufficient condition for a LT to take a spanning set to a spanning set?

Recall from lectures that if $T : V \to W$ is a linear transformation $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of V and W respectively then one can find a matrix $T_{\mathcal{B}_2\mathcal{B}_1}$ such that $T(\mathbf{v}) = \mathbf{w}$ if and only if $T_{\mathcal{B}_2\mathcal{B}_1}\mathbf{v}_{\mathcal{B}_1} = \mathbf{w}_{\mathcal{B}_2}$. In other words, if we fix bases in the vector spaces of interest then **any** linear transformation can be represented as a multiplication by a matrix.

3. *Suppose* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *rotates the plane by* $90°$ *degrees counterclockwise about the origin.*

(a) Is T a linear transformation?

Yes, see last slide of Module 25.

(b) Let $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ find $T_{\mathcal{E}\mathcal{E}}$.

$$T_{\mathcal{E}\mathcal{E}} = \left[T(\mathbf{e}_1)_{\mathcal{E}} \, T(\mathbf{e}_2)_{\mathcal{E}} \right] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c) Using $T_{\mathcal{E}\mathcal{E}}$ find the result of the transformation applied to $\begin{bmatrix} 15 \\ 10 \end{bmatrix}$.

$$T_{\mathcal{E}\mathcal{E}} \begin{bmatrix} 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 15 \\ 10 \end{bmatrix} = \begin{bmatrix} -10 \\ 15 \end{bmatrix}$$

It turns out that if a transformation is linear one can deduce quite a lot of information about it without knowing its explicit formula. For example, in the next exercise we find its explicit formula (that is given by a coordinate matrix) from knowing its value on just two inputs!

4. *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with*

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Consider the basis $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$ and the basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Determine $T_{\mathcal{B}_2,\mathcal{B}_1}$.

(In these solutions, we use colors for scalars to help the reader keep track of them)

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}5\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{5}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$
$$= \underbrace{\frac{5}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{B}_2}.$$

and

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(-\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\right) = -\frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix}5\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-\frac{5}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$
$$= \underbrace{-\frac{5}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{B}_2}.$$

Then

$$T_{\mathcal{B}_2\mathcal{B}_1} = \begin{bmatrix}\frac{5}{2} & -\frac{5}{2}\\\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}.$$

(b) Is $T(\mathbf{v}) = T_{\mathcal{B}_2\mathcal{B}_1}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$?

Yes, in this case $T(\mathbf{x}) = T_{\mathcal{B}_2,\mathcal{B}_1}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ (since both $\mathcal{B}_1$ and $\mathcal{B}_2$ are standard basis). For example,

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}\frac{5}{2} & -\frac{5}{2}\\\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\0\\1\end{bmatrix}.$$

(c) Consider the basis $\mathcal{C}_1 := \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$ and the basis $\mathcal{C}_2 = \left\{\begin{bmatrix}5\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^3$. Determine $T_{\mathcal{C}_2,\mathcal{C}_1}$.

For (ii),

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}5\\0\\1\end{bmatrix}=\underbrace{1\begin{bmatrix}5\\0\\1\end{bmatrix}+0\begin{bmatrix}0\\1\\0\end{bmatrix}+0\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{C}_2},$$

and

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix}=\underbrace{0\begin{bmatrix}5\\0\\1\end{bmatrix}+1\begin{bmatrix}0\\1\\0\end{bmatrix}+0\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{lin. combo. of vectors from } \mathcal{C}_2}.$$

Then

$$T_{\mathcal{C}_2\mathcal{C}_1}=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}.$$

(d) Is $T(\mathbf{v})=T_{\mathcal{C}_2,\mathcal{C}_1}\mathbf{v}$ for all $\mathbf{v}\in\mathbb{R}^2$?

No, $T(\mathbf{x})\neq B\mathbf{x}$ for some $\mathbf{x}\in\mathbb{R}^2$. For example,

$$\begin{bmatrix}5\\0\\1\end{bmatrix}=T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)\neq\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

(e) Is $T(\mathbf{v})_{\mathcal{C}_2}=T_{\mathcal{C}_2,\mathcal{C}_1}\mathbf{v}_{\mathcal{C}_1}$ for all $\mathbf{v}\in\mathbb{R}^2$?

Yes. See Theorem 51 in Module 27.

(f) **Reflection.** Great job! Thinking back on this problem and they way you have solved discuss with your team the following questions:

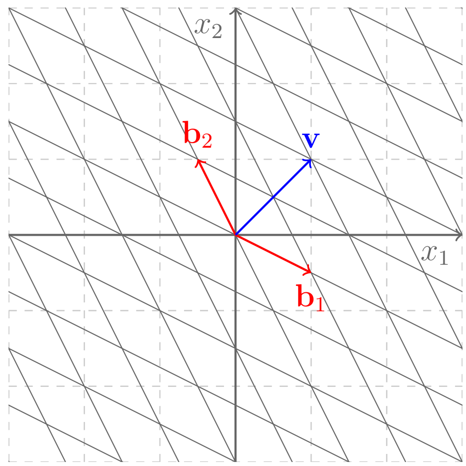
- It is rather remarkable that we learned so much about this transformation from a seemingly small amount of information. Why do you think we could do that?
- Was there anything special about vectors $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ that we used?
- Give examples of the input vectors that would not have been as useful.

We used the fact that the collection $\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ to be able to represent any other vector uniquely as their linear combination. Suppose instead we were given the fact that $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix}$, of course, this would not have been enough information to recover the whole matrix for the linear transformation. We would have known only the first column of the corresponding matrix!

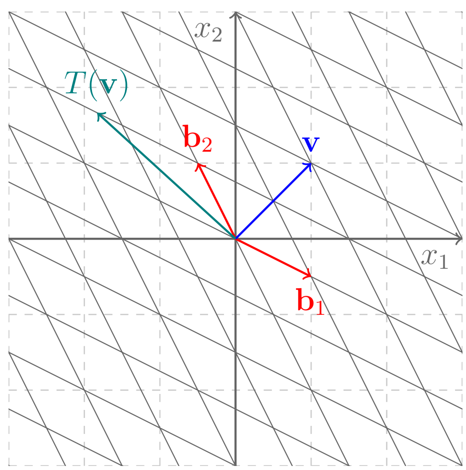
In **Worksheet 3** we discussed how applying a 2×2 matrix corresponds to a distortion of a picture. Since every linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ is associated with a matrix a similar visual representation should be applicable here.

5. *Consider the following basis* $\mathcal{B} := \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}\right)$ *of* $\mathbb{R}^2$. *Let* $\mathbf{v} = 2\mathbf{b}_1 + 2\mathbf{b}_2$.

(a) Draw $\mathbf{b_1}, \mathbf{b_2}$ *and* $\mathbf{v}$ *in the graph below.*



(b) Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $T_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} -2/3 & 0 \\ 0 & 1/2 \end{bmatrix}$. *Draw* $T(\mathbf{v})$.



Nice Work! Your team has thoroughly explored the concept linear transformations and their properties. If you are interested in non trivial application of this theory to quantum mechanics I advise you check out the optional addendum on Moodle about Heisenberg's uncertainty principle!