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MATH 257 - WORKSHEET 4: INVERTIBLE MATRICES

1. The basics

1. *The matrix* $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ *is reduced to the echelon matrix* $U = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ *using the following row operations (in the given order):*

(1) $R_2 \to R_2 + 2R_1$,
(2) $R_3 \to R_3 - R_1$,
(3) $R_3 \to R_3 + R_2$,
(4) $R_4 \to R_4 + R_2$.

Find a lower-triangular matrix L that can appear in a LU-decomposition $A = LU$.

Using the algorithm to get the L matrix, using the opposite sign for the replacements and the given order of operations we obtain the lower triangular matrix $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$.

2. *Find the inverses of the following invertible matrices:*

(1) $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(1) In this case, since the matrix is 2×2 there are at least two legitimate ways to do this. The first is directly by the formula for the inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{8 \cdot 4 - 6 \cdot 5} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$$

The other method is to use the Gauss-Jordan algorithm (which is valid for square matrices of any size).

$$\left[\begin{array}{cc|cc} 8 & 6 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{8}R_1} \left[\begin{array}{cc|cc} 1 & 3/4 & 1/8 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 5R_1} \left[\begin{array}{cc|cc} 1 & 3/4 & 1/8 & 0 \\ 0 & 1/4 & -5/8 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to 4R_2} \left[\begin{array}{cc|cc} 1 & 3/4 & 1/8 & 0 \\ 0 & 1 & -5/2 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{3}{4}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -5/2 & 4 \end{array}\right]$$

We conclude once again that $A^{-1} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$.

(2) Here we use the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

We conclude that the inverse is $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

3. *Is the sum of two invertible matrices always an invertible matrix? Justify your answer.*

No. Take for instance $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. They are both invertible, since $A^{-1} = A$ and $B^{-1} = B$, and the sum $A + B$ is 0, that is a non invertible matrix.

4. *If a matrix A is invertible, is the matrix A^2 invertible? Justify your answer.*

Yes. If A is invertible, there is a matrix C such that $AC = CA = I$. Then, if we consider A^2, the matrix $D = C^2$ satisfies that

$$A^2D = AACC = AIC = AC = I.$$

Similarly,

$$DA^2 = CCAA = CIA = CA = I.$$

Thus D is the inverse of A^2, which implies that A^2 is invertible.

5. *Let A be an invertible $n \times n$-matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^n$. How many solution does $A\mathbf{x} = \mathbf{b}$ have? Explain why!*

Only one solution. First of all, $A\mathbf{x} = \mathbf{b}$ has *at least* one solution, $\mathbf{x} = A^{-1}\mathbf{b}$. To see that $A\mathbf{x} = \mathbf{x}$ has *at most* one solution, suppose that $\mathbf{x}_1, \mathbf{x}_2$ are solutions, i.e., $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$. Multiplying through by A^{-1} we see that $\mathbf{x}_1 = \mathbf{x}_2 = A^{-1}\mathbf{b}$, i.e., all solutions *must* be equal to $A^{-1}\mathbf{b}$ and to each other. Hence there is also *at most* one solution.

6. *Let* $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, *and* $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$.
Note that $A = LU$. Let A_i be the submatrix obtained by taking the first i rows and the first i columns of A, for $i = 1, 2, 3$, i.e.,

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad A_1 = \begin{bmatrix} 2 \end{bmatrix}.$$

What is an LU-decomposition of A_i, for $i = 1, 2, 3$?

An LU-decomposition for A_i is L_iU_i where L_i (respectively, U_i) is the matrix introduced by the first i rows and the first i columns of L (respectively U). More specifically,

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A_3} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U_3},$$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 2 & -1 \\ 0 & \frac{3}{2} \end{bmatrix}}_{U_2}, \quad \text{and}$$

$$\underbrace{\begin{bmatrix} 2 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 2 \end{bmatrix}}_{U_1}.$$

7. *Let A be an $n \times n$-matrix such that A^2 is the zero matrix. Is $A - I$ invertible? If so, what is the inverse of this matrix?*

Yes. If we consider $A - I$, by the distributive law, the matrix $D = -A - I$ satisfies that

$$(A - I)D = AD - D = A(-A - I) - (-A - I) = (-A^2 - A) + (A + I) = I.$$

Similarly, we also have

$$D(A - I) = DA - D = (-A - I)A - (-A - I) = (-A^2 - A) + (A + I) = I.$$

Since $(A - I)D = D(A - I) = I$, D is the inverse of $A - I$.

8. *The objective of this exercise is to study the question: Do all matrices have an LU decomposition?*

(a) Explain why any triangular matrix (upper or lower triangular) T has an LU decomposition

First, we observe that the identity *matrix:*

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

is both upper and lower triangular, and has the property that, for any square matrix M, $MI = IM = M$. Now, we have the two following cases for a triangular matrix T:

(1) T is upper triangular. In this case we can write $T = LU$, where $L = I$ and $U = T$.
(2) T is lower triangular. In this case we can write $T = LU$, where $L = T$ and $U = I$.

*(b) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Verify that A has no LU decomposition (*Hint: *Suppose that there exists two matrices L and U such that $A = LU$. Eventually you will get a contradiction!)*

Let's suppose that there are triangular matrices L and U such that $A = LU$. This implies that we have the following equation of matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} l_1 & 0 \\ l_2 & l_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}.$$

By comparing the entries of the matrix multiplication we obtain the following system of equations:

(1) $$0 = l_1 u_1$$
(2) $$1 = l_1 u_2$$
(3) $$1 = l_2 u_1$$
(4) $$1 = l_2 u_2 + l_3 u_3.$$

From Equation (1) we conclude that either l_1 or u_1 must be zero. If $l_1 = 0$, from equation (2) we obtain that $1 = 0$(contradiction). If $u_1 = 0$, from equation (3) we obtain that $1 = 0$(contradiction). This implies that there are no matrices L and U such that $A = LU$.