Complex Representations of $\mathrm{GL}(2, K)$ for Finite Fields K

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Forward

These are lectures notes of a course that I gave in Tel Aviv University. The aim of these notes is to present the theory of the representations of $\mathrm{GL}(2, K)$ where K is a finite field. However, the presentation of the material has in mind the theory of infinite-dimensional representations of $\mathrm{GL}(2, K)$ for local fields K.

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Introduction

The aim of these notes is to give a description of the complex irreducible representations of the group $G = \mathrm{GL}(2, K)$, where K is a finite field with $q > 2$ elements. In addition, these notes should also serve as a motive for the study of the representation of $\mathrm{GL}(2, K)$, where K is a local field. Therefore, an attempt has been made to reprove theorems by not explicitly using the finiteness of K.

A central role in the description of the representations of G is played by the Borel subgroup consisting of all matrices

$$b = \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \qquad \alpha, \delta \in K^\times, \qquad \beta \in K.$$

If μ_1, μ_2 are characters of $K^\times$, then a character μ of B can be defined by $\mu(b) = \mu_1(\alpha)\mu_2(\delta)$. Let $\hat{\mu} = \mathrm{Ind}_B^G$ be the induced representation. If $\mu_1 = \mu_2$, then $\hat{\mu}$ splits as the direct sum of a one-dimensional representation ρ'_{μ_1,μ_1} which is given by formula $\rho'_{\mu_1,\mu_1}(g) = \mu_1(\deg g)$, and a q-dimensional irreducible representation ρ_{μ_1,μ_1}. There are $q-1$ representations of each kind. If $\mu_1 \neq \mu_2$, then $\hat{\mu} = \rho_{\mu_1,\mu_2}$ is an irreducible representation of dimension $q+1$. There are $\frac{1}{2}(q-1)(q-2)$ representations of this kind. Irreducible representations that are not of the above types are of dimension $q-1$ and are called cuspidal representations. They are however also connected with linear characters in the following way. Let L be the unique quadratic extension of K and let ν be a character of $L^\times$ for which there does not exist a character χ of $K^\times$ such that $\chi(\mathrm{N}_{L/K}\, z) = \nu(z)$ for every $z \in L^\times$. Such a ν is said to be non-decomposable. For each non-decomposable character ν of $L^\times$, we explicitly construct an irreducible representation ρ_ν of G and prove that it is cuspidal. Conversely, w prove that every cuspidal representation of G is of the form ρ_ν for some non-decomposable character ν of $L^\times$. Thus, there are $\frac{1}{2}\left(q^2 - q\right)$ cuspidal representations.

The connection between the irreducible representations of G and the characters of $K^\times$ and $L^\times$ gives rise to a reciprocity law. Let $W(L/K) = L^\times \rtimes \mathrm{Gal}(L/K)$ be the semi-direct product of $L^\times$ by $\mathrm{Gal}(L/K)$. The irreducible representations of $W(L/K)$ (which is called the small Weil group) of dimension ≤ 2. The announced reciprocity law is a natural bijection between the two-dimensional representations of $W(L/K)$ (including the reducible ones) and the irreducible representations of G of dimension > 1.

Next, we attempt to give explicit models for the irreducible representations of G. Let ψ be a non-unit character of K^+. The additive group K^+ can be canonically identified with the subgroup U of G consisting of all the matrices of the form

$$\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \qquad \beta \in K.$$

Therefore, ψ can also be constructed as a character of U. We prove that $\mathrm{Ind}_U^G \psi$ splits into the direct sum of all irreducible representations ρ of G of dimension > 1.; each ρ appears with multiplicity 1. The space V_ρ on which ρ acts can therefore be embedded into $\mathrm{Ind}_U^G V_\psi$. Thus, to each $v \in V_\rho$, there corresponds a function $W_v \colon G \to \mathbb{C}$ such that $W_v(ug) = \psi(u)W'_v(g)$ for every $u \in U$ and $g \in G$. The action of ρ on these functions is given by $W_{\rho(s)v}(g) = W_v(gs)$. The collection of all the W_v is called a Whittaker model for ρ. It has the following property: for all characters ω of $K^\times$, except possible two, there exists complex numbers $\Gamma_\rho(\omega)$

such that

$$\Omega_\rho(\omega) \sum_{x\in K^\times} W_v \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \omega(x) = \sum_{x\in K^\times} W_v \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \omega(x) \tag{0.1}$$

for every $v \in V_\rho$. If ρ is a cuspidal representation, then $\Gamma_\rho(\omega)$ is defined for every ω.

Among the Whittaker functions for ρ, there is a special one, J_ρ, called the Bessel function of ρ, that satisfies

$$J_\rho(gu) = J_\rho(ug) = \psi(u) J_\rho(g) \qquad \text{for} \qquad u \in U, g \in G.$$

Further, $J_\rho(1) = 1$ and $J_\rho(u) = 0$ for $u \in U$ and $u \neq 1$. Substituting this function for W_v in (0.1),

$$\Gamma_\rho(w) = \sum_{x\in K^\times} J_\rho \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \omega(x).$$

This formula is then used to express $\Gamma_\rho(\omega)$ in terms of Gauss sums.

- If $\rho = \rho_{\mu_1,\mu_2}$ is a non-cuspidal representation of G, then

$$\Gamma_\rho(\omega) = \frac{\omega(-1)}{q} G_K\left(\mu_1^{-1}\omega^{-1}, \psi\right) G_K\left(\mu_2^1\omega^{-1}, \psi\right).$$

- If $\rho = \rho_\nu$ is a cuspidal representation, then

$$\Gamma_\rho(\omega) = \frac{\nu(-1)}{q} G_L\left(\nu \circ (\omega \circ \mathrm{N}_{L/K})^{-1}, \psi \circ \mathrm{Tr}_{L/K}\right).$$

The Gauss sum $G_K(\chi, \psi)$ is defined for a character ψ of $K^\times$ by

$$G_K(\chi, \psi) = \sum_{x\in K^\times} \chi(x)\psi(x).$$

In particular, it follows that in every case $|\Gamma_\rho(\omega)| = 1$.

All of these results are finally applied in order to compute the character table for G.

Theme 1

Preliminaries: Represetation Theory; the general linear group

1.1 Linear representations of finite groups

1.2 Induced representations

1.3 The Schur algebra

1.4 The group $\mathrm{GL}(2, K)$

1.5 The conjugacy classes of $\mathrm{GL}(2, K)$

Theme 2
The Representations of $\mathrm{GL}(2, K)$

THEME 3

Γ-FUNCTIONS AND BESSEL FUNCTIONS