Notes on Bump

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1 June 7th

We plan on covering §1.1–1.3.

1.1 Dirichlet L-Functions

We begin by defining Dirichlet characters.

Definition 1 (Dirichlet character)**.** Fix a positive integer N. A *Dirichlet character* $\pmod N$ is a character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times$ extended to $\mathbb{Z}$ by declaring $\chi(n) = 0$ whenever $\gcd(n, N) > 1$. If $N \mid M$ where $N < M$, then a Dirichlet character $\chi \pmod N$ *induces* a Dirichlet character $\pmod M$ by the canonical projection $\mathbb{Z}/M\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$. If χ is not induced by any other character, then χ is *primitive*; otherwise, χ is *imprimitive*.

Dirichlet characters $\chi \pmod N$ have two important attached invariants.

Definition 2 (L-function)**.** Fix a Dirichlet character $\chi \pmod N$. Then we define the *Dirichlet L-function* by

$$L(s, \chi) := \sum_{n=1}^\infty \frac{\chi(n)}{n^s}.$$

Remark 3. We note that $L(s, \chi)$ converges absolutely for $\operatorname{Re} s > 1$. If χ is not induced by the trivial character, then one sees $L(s, \chi)$ actually converges for $\operatorname{Re} s > 0$ uniformly on compacts. If $\chi = 1$, then $L(s, \chi) = \zeta(s)$, and one can use a summation-by-parts argument to show that $\zeta(s)$ has an integral representation valid for $\operatorname{Re} s > 0$.

Remark 4. The usual argument with unique prime factorization implies $L(s,\chi)$ admits an Euler product

$$L(s,\chi)=\prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

Our goal for the time being is to show that $L(s,\chi)$ admits a meromorphic continuation and functional equation. To this end, we introduce the second invariant of a Dirichlet character.

Definition 5 (Gauss sum)**.** Fix a primitive Dirichlet character $\chi \pmod N$. Then we define the *Gauss sum*

$$\tau(\chi):=\sum_{n\in\mathbb{Z}/N\mathbb{Z}}\tau(n)e^{2\pi in/N}.$$

We may like to adjust the character $n\mapsto e^{2\pi in/N}$. To this end, we have the following lemma.

Lemma 6. Fix a primitive Dirichlet character $\chi \pmod N$. Then

$$\sum_{n\in\mathbb{Z}/N\mathbb{Z}}\chi(n)e^{2\pi inm/N}=\overline{\chi}(m)\tau(\chi).$$

Proof. If $\gcd(m,N)=1$, then this is a matter of rearranging the sum. Otherwise, the right-hand side vanishes by definition of χ, and one shows that the left-hand side vanishes essentially because the "periods" of χ and $n\mapsto e^{2\pi inm/N}$ differ. ■

We will want to know that $\tau(\chi)$ is nonzero. As is common in harmonic analysis, it will be easier to compute the norm.

Lemma 7. Fix a primitive Dirichlet character $\chi \pmod N$. Then $|\tau(\chi)|^2=N$.

Proof. Some rearranging reveals that

$$\tau(\chi)\overline{\tau(\chi)}=\frac{1}{\varphi(N)}\sum_{m\in\mathbb{Z}/N\mathbb{Z}}\sum_{n_1,n_2\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi(n_1)\overline{\chi}(n_2)e^{2\pi i(n_1-n_2)m/N}.$$

(The point is that each m produces the same value.) Summing over m, we see that we only care about terms where $n_1\equiv n_2 \pmod N$, from which the result follows. ■

Our proof of the functional equation requires the Poisson summation formula. Thus, we introduce a little more harmonic analysis.

Definition 8 (Fourier transform)**.** For a Schwartz function $f\colon\mathbb{R}\to\mathbb{C}$, we define its *Fourier transform* by

$$\mathcal{F}f(x):=\int_{\mathbb{R}}f(y)e^{2\pi ixy}\,dy.$$

Example 9. For $t\in\mathbb{R}$, define $f_t(x):=e^{-\pi tx^2}$. Then one can compute that $\mathcal{F}f_t=t^{-1/2}f_{1/t}$. Bump includes a proof using contour integration, but of course other proofs exist.

Example 10. For $t\in\mathbb{R}$, define $g_t(x):=xe^{-\pi tx^2}$. Integrating by parts and using Example 9, one finds that $\mathcal{F}g_t=it^{-3/2}g_{1/t}$.

Proposition 11 (Poisson summation)**.** For a Schwarz function $f\colon \mathbb{R}\to\mathbb{C}$, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n).$$

Proof. The trick is to consider the periodic function

$$F(x) \coloneqq \sum_{n\in\mathbb{Z}} f(x+n).$$

Because f is Schwartz, F is infinitely differentiable, so it admits a Fourier series. A computation of the Fourier coefficients then reveals that

$$F(x) = \sum_{m\in\mathbb{Z}} \mathcal{F}f(m)e^{2\pi i m x},$$

from which the result follows by taking $m=0$. ■

Corollary 12. For a Schwarz function $f\colon \mathbb{R}\to\mathbb{C}$ and primitive Dirichlet character $\chi \pmod N$, we have

$$\sum_{n\in\mathbb{Z}} \chi(n)f(n) = \frac{\tau(\overline{\chi})}{N}\sum_{n\in\mathbb{Z}} \widehat{f}(n/N).$$

Proof. Apply Poisson summation to the function

$$g(x) \coloneqq \left(\frac{\tau(\overline{\chi})}{N}\sum_{m\in\mathbb{Z}/N\mathbb{Z}} \overline{\chi}(m)e^{2\pi i x m/N}\right) f(x).$$

For example, the left-hand side equals $\sum_{n\in\mathbb{Z}} g(n)$ because the big factor equals $\chi(n)$ when $x=n$ is an integer. ■

We now move towards our proof of the functional equation. Our functional equation for Dirichlet L-functions will be bootstrapped from the functional equation for certain θ-functions.

Proposition 13. Fix a primitive Dirichlet character $\chi \pmod N$. Say $\chi(-1)=(-1)^\varepsilon$, where $\varepsilon\in\{0,1\}$. Define the θ-function

$$\theta_\chi(t) \coloneqq \frac{1}{2}\sum_{n\in\mathbb{Z}} n^\varepsilon\chi(n)e^{-\pi n^2 t}.$$

Then

$$\theta_\chi(t) = \frac{(-i)^\varepsilon\tau(\chi)}{N^{1+\varepsilon}t^{\varepsilon+1/2}}\theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right).$$

Proof. Doing casework on ε, combine Corollary 12 with Examples 9 and 10. ■

At long last, here is our result.

Theorem 14. Fix a primitive Dirichlet character $\chi \pmod N$. Say $\chi(-1)=(-1)^\varepsilon$, where $\varepsilon\in\{0,1\}$. Then the completed L-function

$$\Lambda(s,\chi) \coloneqq \pi^{-(s+\varepsilon)/2}\Gamma\left(\frac{s+\varepsilon}{2}\right)L(s,\chi)$$

has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,\chi) = (-i)^\varepsilon\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi}).$$

Proof. It is enough to show the functional equation. A u-substitution proves

$$\int_{\mathbb{R}+} e^{-\pi t n^2} t^{(s+\varepsilon)/2} \frac{dt}{t} = \pi^{-(s+\varepsilon)/2}\Gamma\left(\frac{s+\varepsilon}{2}\right) n^{-s-\varepsilon}.$$

Summing over $n \geq 0$ reveals

$$\Lambda(s,\chi) = \int_{\mathbb{R}+} \theta_\chi(t) t^{(s+\varepsilon)/2} \frac{dt}{t}.$$

Proposition 13 completes the proof. ■

1.2 The Modular Group

The natural action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ descends to an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by fractional linear transformations. Explicitly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z := \frac{az+b}{cz+d}.$$

We would like some arithmetic input to this action, so we introduce some subgroups.

Definition 15 (congruence subgroup)**.** For a positive integer N, we define $\Gamma(N)$ as the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Explicitly,

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod N \right\}.$$

A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup* if and only if it contains $\Gamma(N)$ for some positive integer N.

We will spend the rest of the section collecting some facts about $\mathrm{SL}_2(\mathbb{Z})$ and its action on $\mathbb{H}$.

Proposition 16. The group $\mathrm{SL}_2(\mathbb{Z})$ acts discontinuously on $\mathbb{H}$.

Proof. For compact subsets $K_1, K_2 \subseteq \mathbb{H}$, we must show that

$$S := \{g \in \mathrm{SL}_2(\mathbb{Z}) : K_2 \cap gK_1 \neq \varnothing\}$$

is a finite set. Well, note that $g := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has

$$\operatorname{Im} g(z) = \frac{y}{|cz+d|^2}$$

by a direct computation. Thus, the values of c and d are bounded in S. Because $\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ behaves a lateral shift (to the left or right by $|a|$), we see that there are only finitely many possible values of b. Lastly, a is determined by (b,c,d) because $ad - bc = 1$, so we conclude that S is finite. ■

Proposition 17. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ has a fundamental domain given by

$$F := \left\{ z \in \mathbb{H} : |\operatorname{Re} z| < \frac{1}{2}, |z| > 1 \right\}.$$

Namely, any class of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ has a representative in $\overline{F}$, and $z_1, z_2 \in F$ with $g(z_1) = z_2$ must have $z_1 = z_2$ (and $g = \pm I_2$).

Proof. This is essentially a matter of making the previous proof explicit. For the first claim, choose $z \in \mathbb{H}$, and apply $\mathrm{SL}_2(\mathbb{Z})$ until $\operatorname{Im} z$ is maximized; then apply elements of the form $\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ until $\operatorname{Re} z \in [-1/2, 1/2]$. For the second claim, one does some explicit algebra and casework on z and g. ■

Remark 18. Any finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ can also be given a fundamental domain by taking $\bigcup_{g\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})} gF$, where the union is merely over a set of representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$.

Proposition 19. Fix a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then the quotient $\Gamma\backslash\mathbb{H}$ can be compactified and then given the structure of a compact Riemann surface.

Proof. Define $\mathbb{H}^* := \mathbb{H} \sqcup \mathbb{P}^1_{\mathbb{Q}}$, where the points of $\mathbb{P}^1_{\mathbb{Q}}$ are called "cusps." Note that Γ acts on $\mathbb{P}^1_{\mathbb{Q}}$ separately and with only finitely many orbits (because $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ has finite index). We will explain how $\Gamma\backslash\mathbb{H}^*$ can be given the structure of a compact Riemann surface. Let $\overline{\Gamma} \subseteq \mathrm{PSL}_2(\mathbb{R})$ be the image of Γ; there are three cases for $a \in \mathbb{H}^*$

- If the stabilizer of a in $\overline{\Gamma}$ is trivial, then the discontinuity of our action implies that this is the case in an open neighborhood of a. So we map a to the fundamental domain and take a chart there.
- If the stabilizer of a in $\overline{\Gamma}$ is nontrivial and $a \in \mathbb{H}$, then we use the map $z \mapsto \frac{z-a}{z-\overline{a}}$ to send a to the origin, and it sends everything else to the unit disk. Tracking through how fractional linear transformations behave, we see that the stabilizer must now be a finite collection of rotations about the origin, so we take roots to build our charts.
- If the stabilizer of a in $\overline{\Gamma}$ is nontrivial and $a \in \mathbb{P}^1_{\mathbb{Q}}$, use $\mathrm{SL}_2(\mathbb{Z})$ to move a to ∞, and a similar argument as the previous point can move everything to the unit disk again. ■

1.3 Modular Forms

Here is our definition.

Definition 20 (modular form)**.** Fix an integer k and finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then a *modular form* of weight k and level Γ is a holomorphic function f on $\mathbb{H}^*$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for any $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$. The vector space of such f is denoted by $M_k(\Gamma)$. If f vanishes on the cusps of $\mathbb{H}^*$, we say that f is a *cusp form*, and

Remark 21. Being holomorphic on $\mathbb{H}^*$ is a somewhat tricky condition. Because $\Gamma\backslash\mathbb{H}^*$ has already been given the structure of a compact Riemann surface, it is enough to show that f has at worst removable singularities, so it is enough to show that f is bounded approaching the cusps in $\mathbb{P}^1_{\mathbb{Q}}$. More explicitly, if $\Gamma \supseteq \Gamma(N)$, then $q := e^{2\pi i z/N}$ is a local chart around $\infty \in \mathbb{P}^1_{\mathbb{Q}}$, so we want the Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n$$

to have $a_n = 0$ for $n < 0$.

Remark 22. Suppose k is odd and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then $g = -I_2$ tells us that $f(z) = (-1)^k f(z)$, so $f = 0$.

Remark 23. If $k = 0$, then we are asking for holomorphic functions on $\Gamma\backslash\mathbb{H}^*$, but this is a compact Riemann surface, so our modular forms of weight 0 are constant.

Remark 24. More formally, we see that $M(\Gamma)$ is a graded ring, with grading given by the weight. The point is that the product of modular forms of weights k and ℓ produces a modular form of weight $k+\ell$.

We would like to classify modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 25. Fix a finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then $M_k(\Gamma)$ is finite-dimensional.

Proof. If $M_k(\Gamma)$ only has 0, then we are done. Else, choose a nonzero element f_0. Then division by f_0 sends $f \in M_k(\Gamma)$ to meromorphic functions f/f_0 on $X \coloneqq \Gamma\backslash\mathbb{H}^*$. Now, this collection of holomorphic functions f/f_0 on X have prescribed poles at the zeroes of f, so an argument with Laurent expansions in local charts around these poles explains that the space of such holomorphic functions on X is finite-dimensional. ■

Thus, $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is relatively small. We now want to show that it is frequently nonempty when k is even (see Remark 22).

Lemma 26. For even $k \geq 4$, define

$$E_k(z) \coloneqq \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}.$$

Then $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. With $k \geq 4$, one can check that E_k is absolutely convergent, and it is weight k essentially by construction. To check that E_k is holomorphic at ∞, we compute its Fourier expansion. The Fourier transform of $f(u) \coloneqq (u-\tau)^{-1}$ is

$$\mathcal{F}f(v) = \begin{cases} 2\pi i \operatorname{Res}_{u=t}\left(e^{2\pi i u v}(u-\tau)^{-k}\right) & \text{if } v > 0 \\ 0 & \text{if } v \leq 0, \end{cases} = \begin{cases} \frac{(2\pi i)^k}{(k-1)!} v^{k-1} e^{2\pi i v \tau} & \text{if } v > 0, \\ 0 & \text{if } v \leq 0. \end{cases}$$

Thus, the Poisson summation formula and a little rearrangement tells us that

$$E_k(z) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n)$ is the sum of the $(k-1)$st powers of the divisors of n. ■

Remark 27. A computation of $\zeta(k)$ (for even k) reveals that $G_k(z) \coloneqq \zeta(k)^{-1}E_k(z)$ has rational coefficients. For example, one can see that $\Delta \coloneqq G_4^3 - G_6^2$ lives in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

Lemma 28. There exists an element in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ which does not vanish on $\mathbb{H}$.

Proof. We recall the Jacobi triple product formula given by

$$\sum_{n\in\mathbb{Z}} q^{n^2} x^n = \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1+q^{2n-1}x\right)\left(1+q^{2n-1}x^{-1}\right).$$

Substituting $q \mapsto q^{3/2}$ and $x \mapsto -q^{-1/2}$ and rearranging, we see

$$\eta(z) \coloneqq q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \sum_{n\in\mathbb{Z}} \chi(n) q^{n^2/24},$$

where $\chi \pmod{12}$ is the primitive quadratic character. (Explicitly, $\chi(\pm) = 1$ and $\chi(\pm 5) = -1$.) Note $\eta(z) = \theta_\chi(-z/12)$.

We claim that η^{24} is the required function. The infinite product tells us that η does not vanish on $\mathbb{H}$, but η vanishes at $\infty \in \mathbb{H}^*$ (which is $q = 0$). Thus, it remains to show that η^{24} is modular with weight 12. The infinite product explains that η^{24} satisfies the modularity property for $\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right]$, so it remains to check for $\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$. Well, plugging θ_χ into Proposition 13, we see

$$\sqrt{-iz}\eta(z) = \eta\left(-\frac{1}{z}\right),$$

which completes the proof upon raising to the 24th power. ■

Remark 29. The argument of Proposition 25 tells us that $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is actually one-dimensional. Thus, it is spanned by Δ.

And here is our classification result.

Theorem 30. The ring $M(\mathrm{SL}_2(\mathbb{Z}))$ is generated by G_4 and G_6. In particular,

$$\dim M_{12a+2b}(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} a+1 & \text{if } 2b \in \{0,4,6,8,10\}, \\ a & \text{if } 2b = 2. \end{cases}$$

Proof. We abbreviate the group $\mathrm{SL}_2(\mathbb{Z})$ from our notation. Dimension arguments imply that it is enough to show the last computation. The argument of Proposition 25 implies that multiplication by Δ provides an isomorphism $M_k \to S_{k+12}$ for all k; additionally, because we have only one cusp, we see that either $M_k = S_k$ or $\dim M_k = \dim S_k + 1$. Thus, $\dim M_{k+12} = 1 + \dim M_k$ always, so it remains to show the result for $k < 12$.

Examining what we've done so far, it remains to show $\dim M_k = 1$ for even $k \in [4, 10]$ and $\dim M_2 = 0$.

- Take $k \in \{4, 6, 8, 10\}$. To show $\dim M_k = 1$, we will show $\dim S_k = 0$ (and then use E_k to increase dimension). Well, suppose for contradiction that we have a nonzero element $f \in S_k$. On one hand, we see $E_{6(12-k)}(f/\Delta)^6$ is a modular form of weight 0, so it is constant, so we may say $E_{6(12-k)} = \Delta^6/f^6$ by adjusting f by a constant multiple. On the other hand, this means $E_{6(12-k)}$ fails to vanish on $\mathbb{H}$, so $\Delta^{(12-k)/2}/E_{6(12-k)}$ is a modular form of weight 0 with no poles but a zero at the cusp, which is impossible.
- Take $k = 2$. Suppose for contradiction that we have a nonzero element $f \in M_2$. By adjusting f by a constant multiple, the previous tells us we have $fE_4 = E_6$. However, a computation shows $E_4\left(e^{2\pi i/3}\right) = 0$, which would Δ has a zero in $\mathbb{H}$, which we know is false. ■

Our next goal is to make a discussion of L-functions.

Definition 31 (L-function)**.** For $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansion $f(z) = \sum_{n=1}^\infty a_n q^n$, we define

$$L(s, f) := \sum_{n=1}^\infty \frac{a_n}{n^s}.$$

We should probably check that this converges.

Proposition 32. For $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansion $f(z) = \sum_{n=1}^\infty a_n q^n$. Then $|a_n| = O\left(n^{k/2}\right)$.

Proof. A direct computation shows that $\left|f(z)(\operatorname{Im} z)^{k/2}\right|$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant; because f is a cusp form, we see that $\left|f(z)(\operatorname{Im} z)^{k/2}\right|$ is bounded on $\mathbb{H}$ by some constant C. Now, for any $y \in \mathbb{R}$, we see

$$|a_n| e^{-2\pi n y} = \left| \int_{\mathbb{R}/\mathbb{Z}} f(x+iy) e^{-2\pi i n x} \, dx \right| \le \int_0^1 |f(x+iy)| \, dx \le C y^{-k/2}.$$

Choosing $y = 1/n$ completes the proof. ■

Remark 33. In general, we know we can write $f = f_0 + cE_k$ for cusp form f_0, so our computation of the Fourier expansion of E_k reveals that

Thus, $L(s,f)$ converges for $\operatorname{Re} s$ sufficiently large. Here is our functional equation.

Theorem 34. For $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, define

$$\Lambda(s,f) := (2\pi)^{-s}\Gamma(s)L(s,f).$$

Then Λ has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,f) = (-1)^{k/2}\Lambda(k-s,f).$$

Proof. Summing the identity

$$\int_{\mathbb{R}^+} e^{-2\pi n y} y^s \, \frac{dy}{y} = (2\pi)^{-s}\Gamma(s) n^{-s}$$

for $n \ge 1$ shows that

$$\Lambda(s,f) = \int_{\mathbb{R}^+} f(iy) y^s \, \frac{dy}{y}.$$

The result now follows because $f(iy) = (-1)^{k/2} y^{-k} = f(i/y)$ by the modularity of f. ■

2 June 14th

Today we plan on covering §1.4–1.6.

2.1 Hecke Operators

Following Bump, we begin by discussing Hecke operators for $\mathrm{SL}_2(\mathbb{Z})$ and then will discuss Hecke operators for different level in remarks later.

Notation 35. Fix a positive integer k. Given holomorphic $f\colon \mathbb{H} \to \mathbb{C}$ and $\gamma := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{R})$ with $\deg \gamma > 0$, we define $f|_\gamma \colon \mathbb{H} \to \mathbb{C}$ by

$$(f|_\gamma)(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma \cdot z).$$

One can check that this creates a right action of $\mathrm{GL}_2(\mathbb{R})^+$ on holomorphic functions $\mathbb{H} \to \mathbb{C}$.

One would like to know that this action sends modular forms to modular forms, but this has the nasty side effect of adjusting level. Nonetheless, one can check a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ continues to make the conjugate $\gamma^{-1}\Gamma\gamma \cap \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup for any $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$.[1] From here, one can indeed check that f being a modular form for a congruence subgroup Γ of weight k makes $f|_\gamma$ continue to be a modular form for the congruence subgroup $\gamma^{-1}\Gamma\gamma \cap \mathrm{SL}_2(\mathbb{Z})$ of weight k.

We will spend most of the rest of this subsection in level 1, so we set $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ until stated otherwise. Our construction of Hecke operators will rest on certain double coset computations, which we now carry out.

[1] The corresponding level depends on α and Γ.

Lemma 36. Fix $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$. Then
$$\left|\frac{\Gamma\gamma\Gamma}{\Gamma}\right| = \left|\frac{\Gamma}{\gamma^{-1}\Gamma\gamma \cap \Gamma}\right|,$$
which is finite.

Proof. Note that the right-hand side is in fact finite because $\gamma^{-1}\Gamma\gamma \cap \Gamma$ is a congruence subgroup, so it suffices to compute
$$\frac{\Gamma\gamma\Gamma}{\Gamma} \cong \frac{\Gamma\gamma\Gamma\gamma^{-1}}{\Gamma} \cong \frac{\gamma\Gamma\gamma^{-1}}{\Gamma \cap \gamma\Gamma\gamma^{-1}} \cong \frac{\Gamma}{\gamma^{-1}\Gamma\gamma \cap \Gamma},$$
as required. ■

Definition 37 (Hecke operator)**.** Fix a modular form f of weight k and level $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For each $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, we define the *Hecke operator*
$$f|T_\alpha := \sum_{\Gamma\gamma \subseteq \Gamma\alpha\Gamma} f|_\gamma.$$
Note T_α only depends on $\Gamma\alpha\gamma$, so we let $\mathcal{R}$ denote the free abelian group generated by $\{T_\alpha\}_{\Gamma\alpha\Gamma}$.

Modularity of f implies that the choice of representatives γ for $\Gamma\gamma \subseteq \Gamma\alpha\Gamma$ does not remember. In fact, for any $\gamma' \in \Gamma$, we see that applying $f|T_\alpha|_{\gamma'}$ merely rearranges right cosets in the sum and thus just equals $f|_{\gamma'}|T_\alpha$, meaning $f|T_\alpha$ will continue to be a modular form of weight k.

As suggested by the letter $\mathcal{R}$, we would like to define a ring structure. Unsurprisingly, this will be by composition. A direct computation reveals that
$$f|T_\alpha|T_\beta = \sum_{\sigma \in \Gamma\backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma} m(\alpha,\beta,\sigma) f|T_\sigma,$$
where
$$m(\alpha,\beta,\sigma) := \#\{(\Gamma\alpha',\Gamma\beta') : \sigma \in \Gamma\alpha'\beta'\}.$$
(Importantly, one must check that $m(\alpha,\beta,\sigma)$ only depends on $\Gamma\sigma\Gamma$, for example.) Thus, we may extrapolate a definition of $T_\alpha \cdot T_\beta$ from the right-hand side above. A direct computation shows that this multiplication is associative, so we get a (a priori non-commutative) ring; the identity is T_{I_2}. (This is not immediate from composition being commutative, sadly.)

We would like to show that $\mathcal{R}$ is commutative. Approximately speaking $\mathcal{R}$ is the convolution algebra on $\Gamma\backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma$, so this will be done by providing an anti-involution on the level of these double cosets. As such, we want to understand these double cosets more.

Lemma 38. We have
$$\Gamma\backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma = \left\{\Gamma\begin{bmatrix} d_1 & \\ & d_2\end{bmatrix}\Gamma : d_1, d_2 \in \mathbb{Q}, \frac{d_1}{d_2} \in \mathbb{Z}^+\right\}.$$

Proof. For $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, we need to show that $\Gamma\alpha\Gamma$ has a unique representative in the required form. Existence follows by putting (some positive integer multiple of) α into Smith normal form. Uniqueness follows by a direct computation of what elements of $\Gamma\left[\begin{smallmatrix} d_1 & \\ & d_2\end{smallmatrix}\right]\Gamma$ look like. ■

Proposition 39. The Hecke algebra $\mathcal{R}$ is commutative.

Proof. We must show that $m(\alpha,\beta;\sigma) = m(\beta,\alpha;\sigma)$. It is enough to show that $m(\alpha,\beta;\sigma) = m(\beta,\alpha;\sigma^\intercal)$ because $T_\sigma = T_{\sigma^\intercal}$ by selecting the representative σ to be diagonal (via the above lemma).

It will be useful to have some explicit representatives for Lemma 36. Begin with any choice of representatives $\{\alpha_i\}$, and then replace a given α_i with an element of $\Gamma\alpha_i \cap \alpha_i^\intercal\Gamma$ so that

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i = \bigcup_i \alpha_i\Gamma.$$

We similarly set $\{\beta_j\}$ and $\{\sigma_k\}$ for representatives for Lemma 36. Then

$$\{(i,j) : \sigma \in \Gamma\alpha_i\beta_j\Gamma\} = \sum_k m(\alpha,\beta;\sigma_k) = \left|\frac{\Gamma\sigma\Gamma}{\Gamma}\right| m(\alpha,\beta;\sigma).$$

We now take $\alpha \mapsto \alpha^\intercal$ and $\beta \mapsto \beta^\intercal$ on the left-hand side to see that $m(\alpha,\beta;\sigma) = m(\beta,\alpha;\sigma^\intercal)$. ■

Notation 40. Now that our Hecke operators are commutative, we will choose to write T_α on the left, writing $T_\alpha f$ for $f|T_\alpha$.

We next show that these Hecke operators are self-adjoint. This requires an inner product.

Definition 41 (Petersson inner product)**.** Fix cusp forms f and g of weight k and some level $\Gamma(N)$. Then we define

$$\langle f,g\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]}\int_{\Gamma(N)\backslash\mathbb{H}} f(z)\overline{g(z)}\, y^k \frac{dx\,dy}{y^2}.$$

One can check that f and g being modular forms implies that the integral is well-defined. Being a cusp form implies that the integral converges (in particular, it will vanish rapidly approaching any cusp of the compact space $\Gamma(N)\backslash\mathbb{H}^*$).

Proposition 42. Each operator $T_\alpha \in \mathcal{R}$ is self-adjoint with respect to the Petersson inner product.

Proof. Fix cusp forms f and g and some $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$. Light rearrangement verifies $\langle f|_\alpha, g\rangle = \langle f, g|_{\alpha^{-1}}\rangle$. For example, this implies that the computed inner product only depends on the ambient double coset $\Gamma\alpha\Gamma$. As such, we compute

$$\langle T_\alpha f, g\rangle = \left|\frac{\Gamma\alpha\Gamma}{\Gamma}\right| \langle f|_\alpha, g\rangle,$$

and we can now move the α over to g and rearrange everything back into $\langle f, T_\alpha g\rangle$ after a little work. ■

Thus, $\mathcal{R}$ becomes a commutative family of self-adjoint operators acting on the finite-dimensional vector space $S_k(\Gamma)$, so the operators in $\mathcal{R}$ are simultaneously diagonalizable by a basis of "Hecke eigenforms."

Our last goal is to show that $L(f,s)$ for Hecke eigenforms f admit Euler products. For this, we will use a special subset of Hecke operators.

Notation 43. For positive integer n, we define

$$T_n := \sum_{\substack{d_1d_2=n \\ d_2|d_1}} T_{\mathrm{diag}(d_1,d_2)}.$$

Letting $\Delta_n \subseteq \mathbb{Z}^{2\times 2}$ be the subset with determinant n, the proof of Lemma 38 implies

$$T_n f = \sum_{\Gamma\delta\subseteq\Delta_n} f|_\delta.$$

Let's compute some representatives.

Lemma 44. For positive integer n, we have

$$\Delta_n = \bigsqcup_{\substack{a,d>0 \\ ad=n \\ 0\le b<d}} \Gamma \begin{bmatrix} a & b \\ & d \end{bmatrix}.$$

Proof. The backward inclusion is clear. The union being disjoint is a direct computation. Lastly, the forward inclusion follows by picking up some element of Δ_n and doing row-reduction to adjust the bottom-left entry. ■

We now compute the behavior of T_n.

Lemma 45. For a cusp form f of weight k and level Γ with Fourier expansion $f = \sum_{m\ge1} a_m q^m$, we have

$$T_n f = \sum_{m\ge1} \left(\sum_{\substack{ad=n \\ a\mid m}} \left(\frac{a}{d}\right)^{k/2} d a_{md/a} \right) q^m.$$

Proof. Direct expansion with the above lemma shows

$$T_n f(z) = \sum_{\substack{ad=n \\ 0\le b<d}} \sum_{m\ge1} \left(\frac{a}{d}\right)^k a_m e^{2\pi i m(az/d)} e^{2\pi i m(b/d)}.$$

Now, we sum over b and rearrange the sum into the desired result. ■

Lemma 46. Fix a nonzero cusp Hecke eigenform f of weight k where the operator T_n has eigenvalue $n^{1-k/2}\lambda_n$ for some function λ. Give f the Fourier expansion $f = \sum_{m\ge1} a_m q^m$.

(a) $a_1 \ne 1$.

(b) If $a_1 = 1$, then $\lambda_m = a_m$ for all $m \ge 1$.

(c) If $a_1 = 1$, then the function $a_\bullet$ is multiplicative.

Proof. Using the previous lemma, we see that

$$n^{1-k/2}\lambda_n a_m = \sum_{\substack{ad=n \\ a\mid m}} \left(\frac{a}{d}\right)^{k/2} d a_{md/a}. \tag{2.1}$$

If $\gcd(m,n) = 1$, then the sum must have $(a,d) = (1,n)$, so the sum collapses to $\lambda_n a_m = a_{mn}$. The result follows. ■

And here is our result.

Theorem 47. Fix a nonzero cusp Hecke eigenform f of weight k. Give f the Fourier expansion $f = \sum_{m\ge1} a_m q^m$, scaled so that $a_1 = 1$. Then

$$L(s,f) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$

Proof. The previous lemma yields

$$L(s,f)=\prod_p\left(\sum_{\nu=0}^{\infty}A\left(p^\nu\right)p^{-\nu s}\right),$$

so we want to compute this infinite sum. Well, (2.1) provides the two-term recurrence

$$a_{p^{\nu+1}}-a_pa_{p^\nu}+p^{k-1}a_{p^{\nu-1}}=0,$$

from which we can evaluate the sum. ■

Let us conclude by saying a little about Hecke operators attached to congruence subgroups $\Gamma(n)$. We require two important congruence subgroups.

Definition 48. For positive integer N, we define

$$\Gamma_0(N):=\left\{\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{SL}_2(\mathbb{Z}):c\equiv 0\pmod N\right\},$$
$$\Gamma_1(N):=\left\{\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{SL}_2(\mathbb{Z}):a,d\equiv 1\pmod N, c\equiv 0\pmod N\right\}.$$

Note $\Gamma(N)\subseteq\Gamma_1(N)\subseteq\Gamma_0(N)$.

It will be helpful to be able to twist a modular form by a Dirichlet character.

Definition 49. For a weight k and positive integer N, we define

$$M_k(\Gamma_0(N),\chi):=\left\{f\in M_k(\Gamma_0(N)):f|_\gamma=\chi(d)f\text{ for }\gamma=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\Gamma_0(N)\right\}.$$

In the sequel, we abbreviate $\chi(d)$ to $\chi(\gamma)$. We define $S_k(\Gamma_0(N),\chi)$ analogously.

Now, one can show that our characters produce an orthogonal decomposition

$$S_k(\Gamma_1(N))=\bigoplus_{\chi\pmod N}S_k(\Gamma_0(N),\chi).$$

As such, it suffices to define Hecke operators on the spaces $M_k(\Gamma_0(N),\chi)$ and reassemble later. One does this essentially by doing double coset computations with $\Gamma_0(N)\backslash\mathrm{GL}_2(\mathbb{Z}_N)^+/\mathrm{GL}_2(\mathbb{Z})$, where $\mathbb{Z}_N$ refers to the localization. The arguments of the above theory more or less goes through.

2.2 Twisting

We are going to discuss a few converse theorems. The proofs are rather technical, so (as usual) we will not include them in any nontrivial detail.

The moral of the story is that we produced a functional equation for our L-function by taking the Mellin transform of a functional equation of a modular form. Morally, one should be able to take the functional equation for the L-function and then take the inverse Mellin transform to recover the functional equation of a modular form. In particular, an L-function satisfying a suitable functional equation will then be forced to arise from a modular form!

Following Bump, we will sketch two results of this type.

Theorem 50. Fix a nonnegative integer k and a sequence $\{a_m\}_{m\geq 1}$ be a sequence of complex numbers of polynomial growth, and define

$$L(s) := \sum_{n\geq 1} \frac{a_n}{n^s}.$$

Assume the following.

(a) Analytic continuation: $\Lambda(s) := (2\pi)^{-s}\Gamma(s)L(s,f)$ has an analytic continuation to all $s \in \mathbb{C}$.

(b) Bounded: $\Lambda(s,f)$ is bounded in vertical strips $\{s : \sigma_1 \leq \operatorname{Re} s \leq \sigma_2\}$.

(c) Functional equation: we have $\Lambda(s) = (-1)^{k/2}\Lambda(k-s)$.

Then $f := \sum_{m\geq 1} a_m q^m$ lives in $S_k(\mathrm{SL}_2(\mathbb{Z}))$.

Remark 51. By controlling the pole produced by a modular form which is not a cusp form, one can state a similar result valid for modular forms.

As outlined above, we want two lemmas.

Lemma 52. For continuous $\varphi\colon \mathbb{R}^+ \to \mathbb{C}$, we note that the Mellin transform

$$\mathcal{M}f(s) := \int_{\mathbb{R}^+} \varphi(y)y^s \, \frac{dy}{y}$$

is converges absolutely on some vertical strip $\{s : \sigma_1 \leq \operatorname{Re} s \leq \sigma_2\}$. Then for σ in this strip, we see

$$\varphi(y) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma} \varphi(s)y^{-s}\, ds.$$

Proof. Bump proves this by relating the Mellin transform to the Fourier transform (which can be done via the isomorphism $\exp\colon \mathbb{R} \to \mathbb{R}^+$ of topological groups) and then appealing Fourier inversion. One can also prove this in the same way as the Fourier inversion formula. ■

Lemma 53 (Phragmén–Lindelöf)**.** Fix a function f holomorphic on some strip $\{s : \sigma_1 \leq \operatorname{Re} s \leq \sigma_2, \operatorname{Im} s > c\}$ and satisfying a growth condition $f(\sigma+it) = O\left(e^{t^\alpha}\right)$ (as $t \to \infty$) for some real α. Then if $f(\sigma+it) = O\left(t^M\right)$ (as $t \to \infty$) for $\sigma \in \{\sigma_1, \sigma_2\}$, then the same bound holds uniformly for $\sigma \in [\sigma_1, \sigma_2]$.

Proof. By replacing f with $f(s)/s^M$, we may assume that $M = 0$. Without loss of generality, we may take t large so that the desired strip occupies a small sector of $\mathbb{C}$. By shifting and dividing up $[\sigma_1, \sigma_2]$, we may assume that $\sigma_2 > 0$ is small and $\sigma_1 = -\sigma_2$. The point is that our arguments are close to $\pi/2$, so we choose $m \equiv 2 \pmod 4$ of moderate size so that $m \arg s \approx \pi$ for desired s. Now, for small $\varepsilon > 0$, we consider

$$g_\varepsilon(s) := f(s)e^{\varepsilon s^m}.$$

One can show that $g_\varepsilon(s)$ is bounded on a rectangle determined by the constraints of f, so one receives a bound on f by taking $\varepsilon \to 0^+$. ■

We now prove Theorem 50.

Proof of Theorem 50. Define f as in the conclusion, and we want to show that $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. (The polynomial growth condition on $a_\bullet$ is included so that $L(s)$ converges for $\operatorname{Re} s$ large.) Note f has a Fourier expansion, so $f(z+1) = f(z)$ already; it is thus sufficient to check the functional equation for $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. By analytic continuation, we may check the functional equation for $iy \in i\mathbb{R}^+$. Now, we recall that

$$\int_{\mathbb{R}^+} f(iy)y^s \frac{dy}{y} = \Lambda(s),$$

so Mellin inversion (Lemma 52) yields

$$f(iy) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=\sigma} \Lambda(s)y^{-s}\,ds.$$

Now, we use the functional equation to replace s with $k-s$. Note Λ exhibits rapid decay for $\operatorname{Re} s$ very large and very small, so Lemma 53 tells us that we exhibit this rapid decay uniformly on any vertical strip $\{s : \sigma_1 \leq \operatorname{Re} s \leq \sigma_2\}$. The point is that we don't have to worry about convergence issues, so we send $k - s \mapsto -s$, from the modularity of f follows. ■

Remark 54. In fact, if $L(s)$ further admits an Euler product of the form

$$L(s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}},$$

then f is a Hecke eigenform. The point is that the Euler product implies a particular recursion among the Fourier coefficients, from which one can use Lemma 45 to show that we have a Hecke operator with the expected Hecke eigenvalues.

Our next converse theorem, due to Weil, requires us to twist our modular forms by Dirichlet characters. In particular, we will deduce our converse theorem from twisted functional equations.

Notation 55. For $f \in S_k(\Gamma_0(N), \psi)$ with Fourier expansion $f = \sum_{m\geq 1} a_m q^m$ and Dirichlet character $\chi \pmod D$, we define

$$\begin{aligned} f_\chi(z) &:= \sum_{n=1}^\infty \chi(n)a_n q^n, \\ L(s,f,\chi) &:= \sum_{n=1}^\infty \frac{\chi(n)a_n}{n^s}, \\ \Lambda(s,f,\chi) &:= (2\pi)^{-s}\Gamma(s)L(s,f,\chi). \end{aligned}$$

Technically, one does not require f to be a modular form.

Here is our functional equation.

Proposition 56. Fix $f, g \in S_k(\Gamma_0(N))$ and primitive Dirichlet equation $\chi \pmod D$ such that $\gcd(N, D) = 1$. If $f = g|_{w_N}$ where $w_N := \left[\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right]$, then

$$\Lambda(s,f,\chi) = i^k \chi(N)\psi(D)\frac{\tau(\chi)^2}{D}\left(D^2N\right)^{-s+k/2}\cdot\Lambda(k-s,g,\overline{\chi}).$$

Proof. Note that w_N normalizes $\Gamma_0(N)$, so our hypothesis at least makes sense. A discrete Fourier transform shows

$$f_\chi = \frac{\chi(-1)\tau(\chi)}{D}\sum_{m\in(\mathbb{Z}/D\mathbb{Z})^\times} \overline{\chi}(m) f|_{\left[\begin{smallmatrix} D & m \\ & D\end{smallmatrix}\right]}.$$

For example, one can use some rearrangement to shows that this implies

$$\begin{aligned} f_\chi|_{\left[\begin{smallmatrix} & -1 \\ D^2N & \end{smallmatrix}\right]} &= \chi(N)\frac{\tau(\chi)}{D} \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} \chi(r) g|_{\left[\begin{smallmatrix} D & -r \\ -Nm & s \end{smallmatrix}\right]\left[\begin{smallmatrix} D & r \\ & D \end{smallmatrix}\right]} \\ &= \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \cdot g_{\overline{\chi}}. \end{aligned}$$

Now, in the usual way, plug in iy into this equation and apply the Mellin transform to conclude. ■

One would like a converse theorem from these functional equations.

Theorem 57 (Weil). Fix a positive integer N and Dirichlet character $\psi \pmod N$. Further, fix sequences of complex $\{a_m\}$ and $\{b_m\}$ exhibiting polynomial growth, and define the functions $f := \sum_m a_m q^m$ and $g := \sum_m b_m q^m$ so that we can define $L(s, f, \chi)$ and so on as usual. Lastly, fix a finite set of primes S (including the prime divisors of N), and we assume the following for all Dirichlet characters χ with conductor D or a prime not in S.

(a) Analytic continuation: $\Lambda(s, f, \chi)$ and $\Lambda(s, g, \overline{\chi})$ has an analytic continuation to all $s \in \mathbb{C}$.

(b) Bounded: $\Lambda(s, f, \chi)$ and $\Lambda(s, g, \overline{\chi})$ are bounded in vertical strips.

(c) Functional equation: we have

$$\Lambda(s, f, \chi) = i^k \chi(N)\psi(D)\frac{\tau(\chi)^2}{D}\left(D^2N\right)^{-s+k/2}\Lambda(s, g, \overline{\chi}).$$

Then $f \in S_k(\Gamma_0(N), \psi)$.

Proof. Several pages of manipulation of 2×2 matrices. The primary difficulty is that $\Gamma_0(N)$ may potentially have lots of generators, so the same proof technique will not work verbatim. Nonetheless, redoing the proof of Theorem 50 does imply

$$f_\chi|_{\left[\begin{smallmatrix} & -1 \\ D^2N & \end{smallmatrix}\right]} = \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \cdot g_{\overline{\chi}}.$$

One now does a lengthy computation to bootstrap this into the required result. ■