Notes on Bump

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Contents

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We plan on covering SS1.1–1.3.

1.1 Dirichlet L-Functions

We begin by defining Dirichlet characters.

Definition 1 (Dirichlet character)**.** Fix a positive integer N. A *Dirichlet character* $\pmod N$ is a character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times$ extended to $\mathbb{Z}$ by declaring $\chi(n) = 0$ whenever $\gcd(n, N) > 1$. If $N \mid M$ where $N < M$, then a Dirichlet character $\chi \pmod N$ *induces* a Dirichlet character $\pmod M$ by the canonical projection $\mathbb{Z}/M\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$. If χ is not induced by any other character, then χ is *primitive*; otherwise, χ is *imprimitive*.

Dirichlet characters $\chi \pmod N$ have two important attached invariants.

Definition 2 (L-function)**.** Fix a Dirichlet character $\chi \pmod N$. Then we define the *Dirichlet L-function* by

$$L(s,\chi) \coloneqq \sum_{n=1}^\infty \frac{\chi(n)}{n^s}.$$

Remark 3. We note that $L(s,\chi)$ converges absolutely for $\operatorname{Re} s > 1$. If χ is not induced by the trivial character, then one sees $L(s,\chi)$ actually converges for $\operatorname{Re} s > 0$ uniformly on compacts. If $\chi = 1$, then $L(s,\chi) = \zeta(s)$, and one can use a summation-by-parts argument to show that $\zeta(s)$ has an integral representation valid for $\operatorname{Re} s > 0$.

Remark 4. The usual argument with unique prime factorization implies $L(s,\chi)$ admits an Euler product

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

Our goal for the time being is to show that $L(s, \chi)$ admits a meromorphic continuation and functional equation. To this end, we introduce the second invariant of a Dirichlet character.

Definition 5 (Gauss sum)**.** Fix a primitive Dirichlet character $\chi \pmod N$. Then we define the *Gauss sum*

$$\tau(\chi) := \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \tau(n)e^{2\pi in/N}.$$

We may like to adjust the character $n \mapsto e^{2\pi in/N}$. To this end, we have the following lemma.

Lemma 6. Fix a primitive Dirichlet character $\chi \pmod N$. Then

$$\sum_{n\in\mathbb{Z}/N\mathbb{Z}} \chi(n)e^{2\pi inm/N} = \overline{\chi}(m)\tau(\chi).$$

Proof. If $\gcd(m, N) = 1$, then this is a matter of rearranging the sum. Otherwise, the right-hand side vanishes by definition of χ, and one shows that the left-hand side vanishes essentially because the "periods" of χ and $n \mapsto e^{2\pi inm/N}$ differ. ■

We will want to know that $\tau(\chi)$ is nonzero. As is common in harmonic analysis, it will be easier to compute the norm.

Lemma 7. Fix a primitive Dirichlet character $\chi \pmod N$. Then $|\tau(\chi)|^2 = N$.

Proof. Some rearranging reveals that

$$\tau(\chi)\overline{\tau(\chi)} = \frac{1}{\varphi(N)} \sum_{m\in\mathbb{Z}/N\mathbb{Z}} \sum_{n_1,n_2\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(n_1)\overline{\chi}(n_2)e^{2\pi i(n_1-n_2)m/N}.$$

(The point is that each m produces the same value.) Summing over m, we see that we only care about terms where $n_1 \equiv n_2 \pmod N$, from which the result follows. ■

Our proof of the functional equation requires the Poisson summation formula. Thus, we introduce a little more harmonic analysis.

Definition 8 (Fourier transform)**.** For a Schwartz function $f : \mathbb{R} \to \mathbb{C}$, we define its *Fourier transform* by

$$\mathcal{F}f(x) := \int_{\mathbb{R}} f(y)e^{2\pi ixy}\,dy.$$

Example 9. For $t \in \mathbb{R}$, define $f_t(x) := e^{-\pi tx^2}$. Then one can compute that $\mathcal{F}f_t = t^{-1/2}f_{1/t}$. Bump includes a proof using contour integration, but of course other proofs exist.

Example 10. For $t \in \mathbb{R}$, define $g_t(x) := xe^{-\pi tx^2}$. Integrating by parts and using Example 9, one finds that $\mathcal{F}g_t = it^{-3/2}g_{1/t}$.

Proposition 11 (Poisson summation)**.** For a Schwarz function $f : \mathbb{R} \to \mathbb{C}$, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n).$$

Proof. The trick is to consider the periodic function

$$F(x) := \sum_{n\in\mathbb{Z}} f(x+n).$$

Because f is Schwartz, F is infinitely differentiable, so it admits a Fourier series. A computation of the Fourier coefficients then reveals that

$$F(x) = \sum_{m\in\mathbb{Z}} \mathcal{F}f(m)e^{2\pi imx},$$

from which the result follows by taking $m=0$. ■

Corollary 12. For a Schwarz function $f\colon \mathbb{R}\to\mathbb{C}$ and primitive Dirichlet character $\chi \pmod N$, we have

$$\sum_{n\in\mathbb{Z}} \chi(n)f(n) = \frac{\tau(\overline{\chi})}{N}\sum_{n\in\mathbb{Z}} \widehat{f}(n/N).$$

Proof. Apply Poisson summation to the function

$$g(x) := \left(\frac{\tau(\overline{\chi})}{N}\sum_{m\in\mathbb{Z}/N\mathbb{Z}} \overline{\chi}(m)e^{2\pi ixm/N}\right) f(x).$$

For example, the left-hand side equals $\sum_{n\in\mathbb{Z}} g(n)$ because the big factor equals $\chi(n)$ when $x=n$ is an integer. ■

We now move towards our proof of the functional equation. Our functional equation for Dirichlet L-functions will be bootstrapped from the functional equation for certain θ-functions.

Proposition 13. Fix a primitive Dirichlet character $\chi \pmod N$. Say $\chi(-1)=(-1)^\varepsilon$, where $\varepsilon\in\{0,1\}$. Define the θ-function

$$\theta_\chi(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}} n^\varepsilon\chi(n)e^{-\pi n^2 t}.$$

Then

$$\theta_\chi(t) = \frac{(-i)^\varepsilon\tau(\chi)}{N^{1+\varepsilon}t^{\varepsilon+1/2}}\theta_{\overline{\chi}}\left(\frac{1}{N^2t}\right).$$

Proof. Doing casework on ε, combine Corollary 12 with Examples 9 and 10. ■

At long last, here is our result.

Theorem 14. Fix a primitive Dirichlet character $\chi \pmod N$. Say $\chi(-1)=(-1)^\varepsilon$, where $\varepsilon\in\{0,1\}$. Then the completed L-function

$$\Lambda(s,\chi) := \pi^{-(s+\varepsilon)/2}\Gamma\left(\frac{s+\varepsilon}{2}\right)L(s,\chi)$$

has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,\chi) = (-i)^\varepsilon\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi}).$$

Proof. It is enough to show the functional equation. A u-substitution proves

$$\int_{\mathbb{R}+} e^{-\pi t n^2} t^{(s+\varepsilon)/2} \frac{dt}{t} = \pi^{-(s+\varepsilon)/2} \Gamma\left(\frac{s+\varepsilon}{2}\right) n^{-s-\varepsilon}.$$

Summing over $n \geq 0$ reveals

$$\Lambda(s, \chi) = \int_{\mathbb{R}+} \theta_\chi(t) t^{(s+\varepsilon)/2} \frac{dt}{t}.$$

Proposition 13 completes the proof. ■

1.2 The Modular Group

The natural action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ descends to an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ by fractional linear transformations. Explicitly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z := \frac{az+b}{cz+d}.$$

We would like some arithmetic input to this action, so we introduce some subgroups.

Definition 15 (congruence subgroup)**.** For a positive integer N, we define $\Gamma(N)$ as the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Explicitly,

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod N \right\}.$$

A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup* if and only if it contains $\Gamma(N)$ for some positive integer N.

We will spend the rest of the section collecting some facts about $\mathrm{SL}_2(\mathbb{Z})$ and its action on $\mathbb{H}$.

Proposition 16. The group $\mathrm{SL}_2(\mathbb{Z})$ acts discontinuously on $\mathbb{H}$.

Proof. For compact subsets $K_1, K_2 \subseteq \mathbb{H}$, we must show that

$$S := \{g \in \mathrm{SL}_2(\mathbb{Z}) : K_2 \cap gK_1 \neq \varnothing\}$$

is a finite set. Well, note that $g := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has

$$\mathrm{Im}\, g(z) = \frac{y}{|cz+d|^2}$$

by a direct computation. Thus, the values of c and d are bounded in S. Because $\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ behaves a lateral shift (to the left or right by $|a|$), we see that there are only finitely many possible values of b. Lastly, a is determined by (b, c, d) because $ad - bc = 1$, so we conclude that S is finite. ■

Proposition 17. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ has a fundamental domain given by

$$F := \left\{ z \in \mathbb{H} : |\mathrm{Re}\, z| < \frac{1}{2}, |z| > 1 \right\}.$$

Namely, any class of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ has a representative in $\overline{F}$, and $z_1, z_2 \in F$ with $g(z_1) = z_2$ must have $z_1 = z_2$ (and $g = \pm I_2$).

Proof. This is essentially a matter of making the previous proof explicit. For the first claim, choose $z \in \mathbb{H}$, and apply $\mathrm{SL}_2(\mathbb{Z})$ until $\mathrm{Im}\, z$ is maximized; then apply elements of the form $\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ until $\mathrm{Re}\, z \in [-1/2, 1/2]$. For the second claim, one does some explicit algebra and casework on z and g. ■

Remark 18. Any finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ can also be given a fundamental domain by taking $\bigcup_{g\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})} gF$, where the union is merely over a set of representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$.

Proposition 19. Fix a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then the quotient $\Gamma\backslash\mathbb{H}$ can be compactified and then given the structure of a compact Riemann surface.

Proof. Define $\mathbb{H}^* := \mathbb{H} \sqcup \mathbb{P}^1_\mathbb{Q}$, where the points of $\mathbb{P}^1_\mathbb{Q}$ are called "cusps." Note that Γ acts on $\mathbb{P}^1_\mathbb{Q}$ separately and with only finitely many orbits (because $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ has finite index). We will explain how $\Gamma\backslash\mathbb{H}^*$ can be given the structure of a compact Riemann surface. Let $\overline{\Gamma} \subseteq \mathrm{PSL}_2(\mathbb{R})$ be the image of Γ; there are three cases for $a \in \mathbb{H}^*$

- If the stabilizer of a in $\overline{\Gamma}$ is trivial, then the discontinuity of our action implies that this is the case in an open neighborhood of a. So we map a to the fundamental domain and take a chart there.
- If the stabilizer of a in $\overline{\Gamma}$ is nontrivial and $a \in \mathbb{H}$, then we use the map $z \mapsto \frac{z-a}{z-\overline{a}}$ to send a to the origin, and it sends everything else to the unit disk. Tracking through how fractional linear transformations behave, we see that the stabilizer must now be a finite collection of rotations about the origin, so we take roots to build our charts.
- If the stabilizer of a in $\overline{\Gamma}$ is nontrivial and $a \in \mathbb{P}^1_\mathbb{Q}$, use $\mathrm{SL}_2(\mathbb{Z})$ to move a to ∞, and a similar argument as the previous point can move everything to the unit disk again. ■

1.3 Modular Forms

Here is our definition.

Definition 20 (modular form)**.** Fix an integer k and finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then a *modular form* of weight k and level Γ is a holomorphic function f on $\mathbb{H}^*$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for any $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$. The vector space of such f is denoted by $M_k(\Gamma)$. If f vanishes on the cusps of $\mathbb{H}^*$, we say that f is a *cusp form*, and

Remark 21. Being holomorphic on $\mathbb{H}^*$ is a somewhat tricky condition. Because $\Gamma\backslash\mathbb{H}^*$ has already been given the structure of a compact Riemann surface, it is enough to show that f has at worst removable singularities, so it is enough to show that f is bounded approaching the cusps in $\mathbb{P}^1_\mathbb{Q}$. More explicitly, if $\Gamma \supseteq \Gamma(N)$, then $q := e^{2\pi i z/N}$ is a local chart around $\infty \in \mathbb{P}^1_\mathbb{Q}$, so we want the Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n$$

to have $a_n = 0$ for $n < 0$.

Remark 22. Suppose k is odd and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then $g = -I_2$ tells us that $f(z) = (-1)^k f(z)$, so $f = 0$.

Remark 23. If $k = 0$, then we are asking for holomorphic functions on $\Gamma\backslash\mathbb{H}^*$, but this is a compact Riemann surface, so our modular forms of weight 0 are constant.

Remark 24. More formally, we see that $M(\Gamma)$ is a graded ring, with grading given by the weight. The point is that the product of modular forms of weights k and ℓ produces a modular form of weight $k+\ell$.

We would like to classify modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 25. Fix a finite-index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then $M_k(\Gamma)$ is finite-dimensional.

Proof. If $M_k(\Gamma)$ only has 0, then we are done. Else, choose a nonzero element f_0. Then division by f_0 sends $f \in M_k(\Gamma)$ to meromorphic functions f/f_0 on $X := \Gamma\backslash\mathbb{H}^*$. Now, this collection of holomorphic functions f/f_0 on X have prescribed poles at the zeroes of f, so an argument with Laurent expansions in local charts around these poles explains that the space of such holomorphic functions on X is finite-dimensional. ■

Thus, $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is relatively small. We now want to show that it is frequently nonempty when k is even (see Remark 22).

Lemma 26. For even $k \geq 4$, define

$$E_k(z) := \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}.$$

Then $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Proof. With $k \geq 4$, one can check that E_k is absolutely convergent, and it is weight k essentially by construction. To check that E_k is holomorphic at ∞, we compute its Fourier expansion. The Fourier transform of $f(u) := (u-\tau)^{-1}$ is

$$\mathcal{F}f(v) = \begin{cases} 2\pi i \operatorname{Res}_{u=t}\left(e^{2\pi i uv}(u-\tau)^{-k}\right) & \text{if } v > 0 \\ 0 & \text{if } v \leq 0, \end{cases} = \begin{cases} \frac{(2\pi i)^k}{(k-1)!} v^{k-1} e^{2\pi i v\tau} & \text{if } v > 0, \\ 0 & \text{if } v \leq 0. \end{cases}$$

Thus, the Poisson summation formula and a little rearrangement tells us that

$$E_k(z) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n)$ is the sum of the $(k-1)$st powers of the divisors of n. ■

Remark 27. A computation of $\zeta(k)$ (for even k) reveals that $G_k(z) := \zeta(k)^{-1}E_k(z)$ has rational coefficients. For example, one can see that $\Delta := G_4^3 - G_6^2$ lives in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

Lemma 28. There exists an element in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ which does not vanish on $\mathbb{H}$.

Proof. We recall the Jacobi triple product formula given by

$$\sum_{n\in\mathbb{Z}} q^{n^2} x^n = \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1+q^{2n-1}x\right)\left(1+q^{2n-1}x^{-1}\right).$$

Substituting $q \mapsto q^{3/2}$ and $x \mapsto -q^{-1/2}$ and rearranging, we see

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \sum_{n\in\mathbb{Z}} \chi(n) q^{n^2/24},$$

where $\chi \pmod{12}$ is the primitive quadratic character. (Explicitly, $\chi(\pm) = 1$ and $\chi(\pm 5) = -1$.) Note $\eta(z) = \theta_\chi(-z/12)$.

We claim that η^{24} is the required function. The infinite product tells us that η does not vanish on $\mathbb{H}$, but η vanishes at $\infty \in \mathbb{H}^*$ (which is $q = 0$). Thus, it remains to show that η^{24} is modular with weight 12. The infinite product explains that η^{24} satisfies the modularity property for $\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right]$, so it remains to check for $\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$. Well, plugging θ_χ into Proposition 13, we see

$$\sqrt{-iz}\eta(z) = \eta\left(-\frac{1}{z}\right),$$

which completes the proof upon raising to the 24th power. ■

Remark 29. The argument of Proposition 25 tells us that $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is actually one-dimensional. Thus, it is spanned by Δ.

And here is our classification result.

Theorem 30. The ring $M(\mathrm{SL}_2(\mathbb{Z}))$ is generated by G_4 and G_6. In particular,

$$\dim M_{12a+2b}(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} a+1 & \text{if } 2b \in \{0,4,6,8,10\}, \\ a & \text{if } 2b = 2. \end{cases}$$

Proof. We abbreviate the group $\mathrm{SL}_2(\mathbb{Z})$ from our notation. Dimension arguments imply that it is enough to show the last computation. The argument of Proposition 25 implies that multiplication by Δ provides an isomorphism $M_k \to S_{k+12}$ for all k; additionally, because we have only one cusp, we see that either $M_k = S_k$ or $\dim M_k = \dim S_k + 1$. Thus, $\dim M_{k+12} = 1 + \dim M_k$ always, so it remains to show the result for $k < 12$.

Examining what we've done so far, it remains to show $\dim M_k = 1$ for even $k \in [4, 10]$ and $\dim M_2 = 0$.

- Take $k \in \{4, 6, 8, 10\}$. To show $\dim M_k = 1$, we will show $\dim S_k = 0$ (and then use E_k to increase dimension). Well, suppose for contradiction that we have a nonzero element $f \in S_k$. On one hand, we see $E_{6(12-k)}(f/\Delta)^6$ is a modular form of weight 0, so it is constant, so we may say $E_{6(12-k)} = \Delta^6/f^6$ by adjusting f by a constant multiple. On the other hand, this means $E_{6(12-k)}$ fails to vanish on $\mathbb{H}$, so $\Delta^{(12-k)/2}/E_{6(12-k)}$ is a modular form of weight 0 with no poles but a zero at the cusp, which is impossible.
- Take $k = 2$. Suppose for contradiction that we have a nonzero element $f \in M_2$. By adjusting f by a constant multiple, the previous tells us we have $fE_4 = E_6$. However, a computation shows $E_4\left(e^{2\pi i/3}\right) = 0$, which would Δ has a zero in $\mathbb{H}$, which we know is false. ■

Our next goal is to make a discussion of L-functions.

Definition 31 (L-function)**.** For $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansion $f(z) = \sum_{n=1}^\infty a_n q^n$, we define

$$L(s, f) := \sum_{n=1}^\infty \frac{a_n}{n^s}.$$

We should probably check that this converges.

Proposition 32. For $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansion $f(z) = \sum_{n=1}^\infty a_n q^n$. Then $|a_n| = O\left(n^{k/2}\right)$.

Proof. A direct computation shows that $\left|f(z)(\operatorname{Im} z)^{k/2}\right|$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant; because f is a cusp form, we see that $\left|f(z)(\operatorname{Im} z)^{k/2}\right|$ is bounded on $\mathbb{H}$ by some constant C. Now, for any $y \in \mathbb{R}$, we see

$$|a_n| e^{-2\pi ny} = \left|\int_{\mathbb{R}/\mathbb{Z}} f(x+iy)e^{-2\pi inx}\,dx\right| \le \int_0^1 |f(x+iy)|\,dx \le Cy^{-k/2}.$$

Choosing $y = 1/n$ completes the proof. ■

Remark 33. In general, we know we can write $f = f_0 + cE_k$ for cusp form f_0, so our computation of the Fourier expansion of E_k reveals that

Thus, $L(s,f)$ converges for $\operatorname{Re} s$ sufficiently large. Here is our functional equation.

Theorem 34. For $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, define

$$\Lambda(s,f) := (2\pi)^{-s}\Gamma(s)L(s,f).$$

Then Λ has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,f) = (-1)^{k/2}\Lambda(k-s,f).$$

Proof. Summing the identity

$$\int_{\mathbb{R}^+} e^{-2\pi ny}y^s\,\frac{dy}{y} = (2\pi)^{-s}\Gamma(s)n^{-s}$$

for $n \ge 1$ shows that

$$\Lambda(s,f) = \int_{\mathbb{R}^+} f(iy)y^s\,\frac{dy}{y}.$$

The result now follows because $f(iy) = (-1)^{k/2}y^{-k} = f(i/y)$ by the modularity of f. ■