Algebra and Number Theory Day

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1 Laura DeMarco: From Manin–Mumford to Dynamical Rigidity

I missed the first talk because my flight into Baltimore was delayed. Let's formulate our conjecture. It is supposed to contain as special cases a "uniform" Manin–Mumford conjecture bounding torsion points on Jacobians and dynamical rigidity on $\mathbb{P}^1$.

Conjecture 1. Let V be a smooth quasiprojective algebraic variety over $\mathbb{C}$, and let $F\colon V \times \mathbb{P}^N \to V \times \mathbb{P}^N$ be a family of endomorphisms of $\mathbb{P}^N$ given by

$$F(v, z) \coloneqq (v, f_v(z)).$$

Then there exists an invariant $(1,1)$ current $\hat{T}_f$ given by pulling back the Fubini–Study metric on $\mathbb{P}^N$ iterated along the currents of F as $\lim \frac{1}{d^n}(F^n)^*\omega$. Given a family of subvarieties $\mathcal{X} \subseteq V \times \mathbb{P}^N$ (over V), the following are equivalent.

(a) The F-preperiodic points in $\mathcal{X}$ are Zariski desnse in $\mathcal{X}$.

(b) The current $\hat{T}_F^{\wedge r(F,\mathcal{X})}|_{\mathcal{X}}$ is nonzero, where $r(F, \mathcal{X})$ is the smallest relative dimension (over V) of some family of invariant subvarieties $\mathcal{X} \subseteq \mathcal{Y} \subseteq V \times \mathbb{P}^N$.

Remark 2. The current $\hat{T}_f$ is desirable because it vanishes along the (pre)periodic subvarieties

$$\{(v, z) : F^n(v, z) = (v, z)\}$$

as $n \geq 0$ varies. This more or less explains why we have defined a reasonable gluing of the currents T_f.

Remark 3. In the typical case, one has $r(F, \mathcal{X}) = N$. The reason we need an $r(F, \mathcal{X})$ is that sometimes we may find our $\mathcal{X}$ trapped in a smaller dimension.

Remark 4. Here are some notable special cases.

- One can allow V to be a point.
- Our $\mathcal{X}$ can live inside a family of abelian varieties.
- We can allow F to be a constant family of maps.

Example 5. In the case where V is a point and X is contained in an abelian variety A and F is multiplication by 2, then we recover the Manin–Mumford conjecture. Here, F-preperiodic points are torsion points, which Manin–Mumford asserts is equivalent to being a translate of a subgroup. But now (b) of the conjecture implies $\dim \mathcal{X} \geq r(F, \mathcal{X})$, from which one can conclude because invariant subvarieties are translates of subgroups!

Example 6. In the case where $\mathcal{X}$ is contained a family of abelian varieties $\mathcal{A}$, and F restricts to a family of endomorphisms on $\mathcal{A}$, then this is known as the relative Manin–Mumford conjecture and due to Gao and Habegger. This generalizes the Manin–Mumford conjecture.

Remark 7. It is known that (b) implies (a). For the general case, this is hard. For the abelian varieties setting, this is relatively easy.

Example 8. Suppose $r(F, \mathcal{X}) = N$, which is the generic case. If $\dim \mathcal{X} \geq N$, one expects that $\mathcal{X}$ should have lots of intersections with the (pre)periodic subvarieties of $V \times \mathbb{P}^N$, which have codimension N. This is not quite good enough because $\mathcal{X}$ could "follow along" some of these subvarieties. The condition that $\hat{T}_F^{\wedge r(F,\mathcal{X})}|_{\mathcal{X}} \neq 0$ roughly promises that $\mathcal{X}$ intersects transversally.

Remark 9. Let's say something about how this relates to number theory. Consider the case where V is a point so that we are looking at a single map $f \colon \mathbb{P}^N \to \mathbb{P}^N$ of degree d. Suppose that everything in sight is defined over $\overline{\mathbb{Q}}$. Call and Silverman defiend a canonical height of $\hat{h}_f \colon \mathbb{P}^N(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ as follows: starting with a standard (logarithmic) Weil height h, we define

$$\hat{h}_f(x) \coloneqq \lim_{n \to \infty} \frac{1}{d^n} h\left(f^{\circ n}(x)\right),$$

which is comparable to the original height h. In fact, one finds that $\hat{h}_f(x) = 0$ if and only if x is preperiodic. This defintion generalizes the Néron–Tate (canonical) height. One can relate this to part (b) of the conjecture because $\hat{h}_f$ will decompose locally, and the archimedean component relates to the current $\hat{T}_f$.

2 Sasha Petrov: Characteristic classes of p-adic local systems

Let's begin by telling a story in differential geometry. Consider a topological space M which is locally contractible, such as a manifold. For example, one could be interested in the cohomology and the homotopy groups. There is a notable property that $\operatorname{Hom}(\pi_1(M), \mathbb{Z}) = H^1(M, \mathbb{Z})$, but in general there is not so much more one can do to relate these invariants.

Nonetheless, perhaps we are interested in studying $\pi_1(M)$, which we will do by studying its representations $\rho \colon \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$, which are found to be local systems $\mathbb{L}_\rho$ on M.

Definition 10 (local system)**.** A *local system* $\mathbb{L}$ on M is a sheaf of $\mathbb{C}$-vector spaces on M (of fixed rank), which is locally isomorphic to the constant sheaf.

The idea is that loops in M produce automorphisms between trivializing charts of a local system. In order to bring cohomology back into our story, we introduce characteristic classes.

Example 11. A complex vector bundle E on M has Chern classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$. In the case that E is a line bundle, only $c_1(E)$ is interesting. To define this map, we note that there is a short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_M \to \mathcal{O}_M^\times \to 0,$$

so it is enough to note that a line bundle amounts to an element of $H^1(M, \mathcal{O}_M^\times)$ (because a line bundle is a $\mathbb{G}_m$-torsor), and $c_1(E)$ is then the induced element in $H^2(M, \mathbb{Z})$.

We would now like to understand the characteristic class of a local system $\mathbb{L}$, which we do by consider the vector bundle $\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_M$. The idea is that $\mathbb{L}$ is made of some transition maps, and we can use the exact same transition maps to define a vector bundle.

Example 12. Suppose $M = BG$ for some finite group G, meaning that $\pi_1(M) = G$ but the higher homotopy groups vanish; one finds that $H^{2i}(M, \mathbb{Z}) = H^{2i}(G, \mathbb{Z})$. Then a complex representation ρ of $\pi_1(M)$ amounts to a complex representation ρ of G, and it turns out that the characteristic classes $c_\bullet(\mathbb{L}_\rho \otimes \mathcal{O}_M)$ (together with the rank) form a complete collection of invariants.

Example 13. Suppose M is a finite CW-complex. Then it turns out that $c_i(\mathbb{L} \otimes \mathcal{O}_M) \in H^{2i}(M, \mathbb{Z})$ is torsion for any local system $\mathbb{L}$ and any $i > 0$. Here's one way to see this in the case that M is a manifold: it is enough to check that $c_i(\mathbb{L} \otimes \mathcal{O}_M)$ vanishes in $H^{2i}(M, \mathbb{C})$, but this vector bundle has a flat connection induced by the local system, and one can use analytic theory of Chern–Weil theory to show that our class vanishes in $H^{2i}(M, \mathbb{C})$.

This last example is a bit disheartening, but it suggests that we should actually be looking at the induced class in $H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$, and it turns out that there is a canonical way to choose a "Cherns–Simons" class $\hat{c}_i(\mathbb{L}) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$ going down to $c_i(\mathbb{L} \otimes \mathcal{O}_M)$.

Example 14. Given a representation $\rho\colon \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$, one finds that $\hat{c}_1(\mathbb{L}_\rho) \in H^1(M, \mathbb{C}/\mathbb{Z})$ is simply the determinant class in $H^1(M, \mathbb{C}^\times)$, where we identify $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^\times$ via the exponential map.

Let's describe these classes $\hat{c}$ more explicitly. Given a representation $\rho\colon \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$, the point is that one finds a class $\hat{c}_i \in H^{2i-1}(\mathrm{GL}_n(\mathbb{C}), \mathbb{C}/\mathbb{Z})$ (where $\mathrm{GL}_n(\mathbb{C})$ is given the discrete topology!) such that the composite

$$H^{2i-1}(\mathrm{GL}_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \xrightarrow{\rho} H^{2i-1}(\pi_1(M), \mathbb{C}/\mathbb{Z}) \to H^{2i-1}(M, \mathbb{C}/\mathbb{Z})$$

which produces $\hat{c}_i(\mathbb{L}_\rho)$ for each ρ.

Example 15. Let $M = \mathbb{H}^3/\Gamma$ be an orientable compact hyperbolic manifold. Then $\Gamma = \pi_1(M)$, and the embedding $\Gamma \subseteq \mathrm{Isom}(\mathbb{H}^3)$ induces a canonical representation $\rho\colon \pi_1(M) \to \mathrm{PGL}_2(\mathbb{C})$. Then it turns out that $\hat{c}_2(\mathbb{L}_\rho) \in H^3(M, \mathbb{C}/\mathbb{Z})$ is actually just outputting a number in $\mathbb{C}/\mathbb{Z}$. The imaginary part remembers the volume

In classical algebraic geometry, one has the following.

Theorem 16 (Reznikov)**.** Fix a smooth projective algebraic variety X over $\mathbb{C}$. Then all local systems $\mathbb{L}$ over $X(\mathbb{C})$ make $\hat{c}_i(\mathbb{L}) \in H^{2i-1}(\mathbb{C}/\mathbb{Z})$ is torsio for all $i > 1$.