Johns Hopkins Number Theory Seminar

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1 January 22nd: Friedrich Knop

This is joint work with Zhgoon. For a group G with two subgroups $P, H \subseteq G$, we may be interested in the double coset space $P\backslash G/H$. Today, G will be a connected reductive group over a field K, which may or may not be algebraically closed. Then H and P will be closed subgroups of G.

Example 1. Take $G = \mathrm{GL}_n(\mathbb{C})$ and $H \subseteq \mathrm{O}_n(\mathbb{C})$ and P to be the upper-triangular matrices. Then $P\backslash G/H$ becomes complete flags together with some data of a quadratic form; it turns out to be classified by involutions in S_n, where a σ has attached to it the quadratic form $\sum_i x_i x_{\sigma i}$. For example, the orbit corresponding to $x_1^2 + x_2^2 + x_3^2$ is open.

In general, one can see that having finitely many orbits. For example, there is the following result.

Theorem 2. Fix G a connected reductive group over an algebraically closed field K with Borel subgroup $B \subseteq G$, and let $H \subseteq G$ be a closed subgroup. Then B has an open orbit in G/H if and only if $|B\backslash G/H|$ is finite.

There are examples yielding some level of sharpness for this result. One can ask what is special about the quotient $B\backslash G$, which the following notion helps explain.

Definition 3 (complexity)**.** Fix a group H acting on a variety X over an algebraically closed field K. Then we define the *complexity* $c(X/H)$ as the transcendence degree of $K(X)^H$; note that this is the dimension of X/H if such a quotient makes sense.

Theorem 4 (Vinburg)**.** Let $Y \subseteq X$ be a subvariety with an action by G. Then $c(Y/B) \leq c(X/B)$.

The moral of the story is that we are able to bound open orbits.

We would like to have such theorems over fields K which may not be algebraically closed, but this requires some modifications. For example, one may not have a Borel subgroup B defined over K, so we must work with a minimal parabolic subgroup P. For example, we have the following.

Theorem 5. Work over the field $\mathbb{R}$. Then if P has an open orbit in G/H, then $|P(\mathbb{R})\backslash X(\mathbb{R})|$ is finite, where X refers to the quotient G/H.

However, even this result fails over (say) $\mathbb{Q}$.

Example 6. Consider $G = \mathbb{G}_{m,\mathbb{Q}}$ with $H = \mu_2$. Then $X = G/H$ becomes $\mathbb{G}_m$, but then action is given by the square, so the quotient is the infinite set $\mathbb{Q}^{\times 2}\backslash\mathbb{Q}^\times$.

Remark 7. This example suggests that we may be able to salvage the theorem over local fields of characteristic 0.

To fix the result in general, we want to try to work over the algebraic closure.

Theorem 8. Work over a perfect field K. Let $P \subseteq G$ be a minimal parabolic, and let X be a variety with a G-action. Suppose that there is $x \in X(K)$ such that the orbit $Px \subseteq X$ is open. Then the quotient $P(\overline{K})\backslash X(K)$ is finite.

Here, this quotient by $P(\overline{K})$ refers to "geometric" equivalence classes: two points x and x' are identified if and only if one has $p \in P(\overline{K})$ such that $x = px'$. We want the following notion.

Definition 9. Let G act on a variety X. Then X is K-*spherical* if and only if there is a point $x \in X(K)$ such that the orbit by the minimal parabolic is open.

For example, suitably stated (one should assume that $X(K) \subseteq X$ and $Y(K) \subseteq Y$ are dense and that X is normal), one is able to recover the result on complexity. Roughly speaking, the idea is to reduce to the case where G has rank 1. There are two cases for this reduction.

- Previous work explains how to achieve the result when $GY = Y$.
- If GY strictly contains Y, then we pass to a quotient by a parabolic subgroup corresponding to some simple root.

We are now in the rank 1 case. If $K = \overline{K}$, then it turns out that one may merely work with $G = \mathrm{SL}_2$; then one actually directly classify closed subgroups to produce the result. With $K = \mathbb{R}$, a similar idea works, but the casework at the end is harder. However, no such classification is available for general K. Instead, for general K, we develop some structure theory of these sorts of spherical varieties.

2 January 24: Milton Lin

Today, we are talking about period sheaves in mixed characteristics. The references are to relative Langlands duality and the geometrization of the local Langlands conjectures. In relative Langlands duality, they stated a duality of pairs of spaces (X, G) and $(\check{X}, \check{G})$. Today, we will be interested in the "Iwasawa–Tate" case, which is the pair $(\mathbb{A}^1, \mathbb{G}_m)$, where the dual is itself. In particular, we will study the period duality.

Today, Λ will be one of the coefficient rings $\{\mathbb{F}_\ell, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell\}$, and E is a p-adic field. Here is our main reesult.

Proposition 10. There is a map of v-stacks $\pi\colon \mathrm{Bun}_G^X \to \mathrm{Bun}_G$.

Intuitively, a v-stack is some kind of geometric object.

Definition 11. We call $\mathcal{P}_X := \pi_! \Lambda$ the *period sheaf.*

Remark 12. These objects all exist in the equal characteristic case. Roughly speaking, $\mathcal{P}_X$ categorifies period functionals.

Remark 13. Conjecturally, one can go down to local systems and define a map $\mathrm{Loc}_{\check{G}}^{\check{X}} \to \mathrm{Loc}_{\check{G}}$. This allows us to define an L-sheaf by $\mathcal{L}_{\check{X}} := \pi_* \omega_{\widehat{\check{X}}}$, which is supposed to categorify L-functions. One expects $\mathcal{L}_{\check{X}}$ and $\mathcal{P}_X$ to correspond to each other in the case of $\mathbb{G}_m$, where the result is a known case of the local geometric Langlands conjecture due to Zou.

We would like to recover the correspondence between $\mathcal{P}_X$ and $\mathcal{L}_{\check{X}}$ in our mixed characteristic setting.

Let's review some background on Bun_G before continuing. We let $\mathrm{Pftd}_{\mathbb{F}_q}$ be the category of perfectoid spaces over $\mathbb{F}_q$. These spaces are glued together from "affines" $\mathrm{Spa}(R, R^+)$ where (R, R^+) is a perfectoid ring: R is some topological ring, and $R^+ \subseteq R$ is a subring of the bounded elements (and equality will hold in our cases of interest).

Example 14. The prototypical examples in positive characteristic look like $\mathbb{F}_p((t))$ embedded in a completion of $\mathbb{F}_p((t^{1/p^\infty}))$.

Example 15. The prototypical examples in mixed characteristic look like $\mathbb{Q}_p$ embedded in a completion of $\mathbb{Q}_p(\mu_{p^\infty})$.

Definition 16 (Fague–Fontaine curve)**.** Fix a perfectoid ring (R, R^+). Set $E := \mathbb{Q}_p$ and $S := \mathrm{Spa}(R, R^+)$. Then we define

$$\mathbb{D}_S := \mathrm{Spa}\left(W(R^+) \otimes_{W(\mathbb{F}_p)} \mathcal{O}_E\right).$$

Morally, we have base-changed a disk to our desired coefficients. We also define the punctured disk $\mathbb{D}_S^\times$ as removing the vanishing set of $\pi[\varpi]$, where $\pi \in E$ is a uniformizer and ϖ is a topologically nilpotent unit. (Here, $[\varpi]$ refers to the Teichmuler lift from R to $W(R)$.) Lastly, we define the Fague–Fontaine curve

$$\Sigma_{S,E} := \mathbb{D}_S^\times / \varphi^{\mathbb{Z}},$$

where φ is the Frobenius on $W(R^+)$.

Remark 17. Morally, the point of $\mathbb{D}_S^\times$ is that we are trying to replicate some object like $\mathrm{Spec}\,\mathbb{Z} \times_{\mathbb{F}_1} \mathrm{Spec}\,\mathbb{Z}$: the point is that we want a second parameter in (R, R^+) to keep track of the distance between our points in the mixed characteristic situation.

Definition 18. We define Bun_G as the functor $\mathrm{Pftd}_{\mathbb{F}_q}$ to anima (here, anima is approximately speaking topological spaces) sending a perfectoid space S to G-torsors on $\Sigma_{S,E}$. This is a v-stack, meaning that it satisfies some kind of descent for the v-topology.

Example 19. For $G = \mathbb{G}_m$, there is an explicit description in terms of quotients $[*/E^\times]$. This allows a computation of $D(\mathrm{Bun}_{\mathbb{G}_m}, \Lambda)$.

3 January 31st: Milton Lin

Let's begin with some motivation today. Fix a reductive group G defined over a number field F. An automorphic form on G is some sort of smooth function f on the automorphic quotient $[G] = G(F)\backslash G(\mathbb{A}_F)$. Given a G-variety X, one finds many interesting period functionals Θ_X, and then the canonical pairings $\langle \Theta_X(\varphi), f\rangle$.

Let's be a bit more explicit about where this functional Θ_X arises from. Roughly speaking, one defines a Schwartz space $\mathcal{S}(X(\mathbb{A}_F))$ and then

$$\Theta_X(\varphi) \coloneqq \sum_{\gamma\in X(F)} \gamma\varphi,$$

where $\varphi \in \mathcal{S}(X(\mathbb{A}_F))$. The point is that choosing Θ_X, φ, and f carefully recovers some special values.

Let's be a bit more explicit. Throughout, $G = \mathbb{G}_m$ and $X = \mathbb{A}^1$.

Example 20. We work over a number field F. Then we know how to define $\mathcal{S}(\mathbb{A}_F)$ as

$$\mathcal{S}(\mathbb{A}_F) = \bigoplus_{v\in V(F)} (\mathcal{S}(F_v), 1_{\mathcal{O}_v}),$$

where this refers to a restricted tensor product. Our special vector $\varphi \in \mathcal{S}(\mathbb{A}_F)$ will be the self-dual one from Tate's thesis: define φ_v by

$$\varphi_v \coloneqq \begin{cases} 1_{\mathcal{O}_v} & \text{if } v < \infty, \\ e^{-\pi|x|^2} & \text{if } v \mid \infty. \end{cases}$$

Next, we note that automorphic forms on $\mathbb{G}_m$ are simply Hecke characters χ, and it turns out that $\langle \Theta_X(\varphi), \chi\rangle$ recovers the special value $L(\chi, 0)$ of the Hecke L-function $L(s, \chi)$.

Example 21. We work over a function field $F = \mathbb{F}_q(\Sigma)$, where Σ is a smooth projective curve over a finite field $\mathbb{F}_q$. Then all places $v \in V(F)$ are finite already, so we should define φ as $\prod_v 1_{X(\mathcal{O}_v)}$. But now we note that φ is notably $\mathcal{O}_v^\times$-invariant for each place v, so $\Theta_X(\varphi)$ will descend to the double quotient

$$G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_F),$$

which has a geometric meaning as $\mathrm{Bun}_G(\mathbb{F}_q)$. In particular, it turns out that $\Theta_X(\varphi)$ is realized as some trace of a constant sheaf on a moduli space $\mathrm{Bun}_G^X(\mathbb{F}_q)$ of G-bundles together with a section from the associated X-bundle. In fact, one can compute that this sends a G-bundle $\mathcal{L}$ (which is a line bundle!) to $\#H^0(\Sigma, \mathcal{L})$.

Let's now return to the setting from last week. Let E be a nonarchimedean local field, and let Λ be the coefficient ring $\mathbb{F}_\ell$. Last time we described a stratification

$$\mathrm{Bun}_G = \bigsqcup_{d\in\mathbb{Z}} [\mathrm{pt}/E^\times],$$

and we label each piece by Bun_G^d. It is interesting, from the perspective of these sheaves, to understand $\mathcal{P}_X \coloneqq \pi_! \underline{\Lambda}_X$. For example, last time we explained

$$\mathcal{P}_{X,d} = \begin{cases} \Lambda_{\text{norm}}[-2d] & \text{if } d > 0, \\ \Lambda_c(E) & \text{if } d = 0, \\ \Lambda_{\text{triv}} & \text{if } d < 0, \end{cases}$$

where these are all elements of the derived category $D^{\mathrm{sm}}(E^\times, \Lambda)$ of smooth representations of $E^\times$ over Λ. Remarkably, we see that $d = 0$ recovers our functions on E, which is seen in Tate's thesis.

This study of periods is all "motivic" in some sense. On the spectral/automorphic side, one has a dual version $\mathrm{Bun}_G^X \to \mathrm{Bun}_G$ named

$$\pi\colon \mathrm{Loc}_{\check{G}}^{\check{X}} \to \mathrm{Loc}_{\check{G}},$$

which are intended to categorify L-functions; however, these objects are not available in mixed characteristic.

Let's begin by discussing $\mathrm{Loc}_{\check{G}}$, which is available. Intuitively, we are looking for "local systems with G-action," which amounts to a group homomorphism from a fundamental group to G. One usually begins by defining a moduli stack of 1-cocycles named $\mathcal{Z}^1(\omega_E, \check{G})$, and then $\mathrm{Loc}_{\check{G}}$ is the quotient of this by the adjoint action from $\check{G}$. So we now want to define $\mathcal{Z}^1(\omega_E, \check{G})$, which as a functor of points takes a commutative $\mathbb{Z}_\ell$-algebra A to

$$\mathrm{Hom}_{\mathrm{cts}}(W_E, \check{G}(A)),$$

where W_E is the Weil group. Here, W_E is to be understood as a stand-in for the fundamental group of the Fargues–Fontaine curve.

Example 22. With $G = \mathbb{G}_m$, the fact that G is abelian yields

$$\mathcal{Z}^1(W, \check{G})/\check{G} = \mathrm{Hom}_{\mathrm{cts}}(W_E, \check{G}) \times B\check{G}.$$

We now expand our coefficients to $\Lambda = \overline{\mathbb{Q}}_\ell$. It turns out that

$$D(\mathrm{Bun}_G, \Lambda) \cong \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}),$$

which is some sort of categorical version of the local Langlands correspondence. The left-hand side is understood to decompose geometrically, and the right-hand side (due to the $B\check{G}$) is seen to decompose similarly. Namely, one computes both sides as

$$\prod_{d \in \mathbb{Z}} D^{\mathrm{sm}}(E^\times, \Lambda) \cong \prod_{d \in \mathbb{Z}} \mathrm{QCoh}(\mathrm{Hom}_{\mathrm{cts}}(W_E, \check{G})),$$

and the two decompositions roughly align (up to a sign). In short, one uses the Hecke again to reduce to $d = 0$. Roughly speaking, this comes from class field theory, though to be formal, one should reduce everything to a "finite level" by taking quotients by a chosen compact open subgroup $K \subseteq E^\times$, which is legal because our representations are smooth.

The moral of the story is that we can move our period sheaf $\mathcal{P}_X$ from the left to the right. In particular, the proof of the previous paragraph even provides an explicit formula how to do this. For example, on the 0th component, we are interested in what quasicoherent sheaf corresponds to $\mathrm{ind}_1^{E^\times} \Lambda = \Lambda_c(E^\times)$. In particular, we find that we get

$$\varinjlim_{K \subseteq E^\times} \Lambda_c(E^\times)_K,$$

where $(-)_K$ denotes the coinvariants. Eventually one computes that we get $\Lambda[E^\times/K]$ at finite level, which becomes the structure sheaf of $\mathrm{Hom}(W_E, \check{G})$ after the colimit. In total, one finds that

$$\mathrm{Loc}^d_{\check{G}} = \begin{cases} i_{\mathrm{cyc}*}\Lambda[2d] & \text{if } d > 0, \\ ? & \text{if } d = 0, \\ i_{\mathrm{triv}*}\Lambda & \text{if } d < 0. \end{cases}$$

To guess the last entry $?$, one uses the short exact sequence

$$0 \to \Lambda_c(E^\times) \to \Lambda_c(E) \to \Lambda \to 0,$$

where the right-hand map is given by evaluation at 0, so one can dualize the short exact sequence.

4 February 5: Ryan Chen

Today we are talking about near-center derivatives and arithmetic 1-cycles. Recall that one can normalize the weight-2 Eisenstein series to have a Fourier expansion

$$E_2^*(z) = \frac{1}{8\pi y} + \frac{\zeta(-1)}{2} + \sum_{t>0} \sigma_1(t) q^t,$$

where $q := e^{2\pi i z}$ as usual. We claim that all terms except for the $\frac{1}{8\pi y}$ admit geometric meanings.

- The value $\zeta(-1)/2$ is the volume of the modular curve $Y(1)$, where the measure is given by the push-forward of the Hodge module Ω^1 along the universal elliptic curve $\pi\colon \mathcal{E} \twoheadrightarrow Y(1)$. Explicitly, one finds that this volume form is $-\frac{1}{4\pi^2}\,dx \wedge dy$.
- The values $\sigma_1(t)$ are degrees of some Hecke correspondences $\mathrm{Hk}(t)$ over $Y(1)$. Imprecisely, $\mathrm{Hk}(t)$ parameterizes degree-t isogenies.

Let's do this for other Shimura varieties. Let's define SL_2-Eisenstein series. Here, $F/\mathbb{Q}$ is an imaginary quadratic field with discriminant Δ, and we assume 2 splits for technical reasons. Then our SL_2 is given the real form $\mathrm{SU}(1,1)$, and it acts on the upper-half plane $\mathcal{H}$ as usual. Of course, there are higher-dimensional versions of this. Let $P \subseteq \mathrm{SU}(1,1)$ be the Siegel parabolic of upper-triangular matrices, and for even positive weight, we have

$$E_n(z,s) := \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in P(\mathbb{Z})\backslash \mathrm{SU}(1,1)(\mathbb{Z})} \frac{(\det y)^s}{\det(cz+d)^n \left|\det(cz+d)\right|^{2s}}.$$

Sometimes, one shifts s by $s_0 = \frac{1}{2}(n-m)$ to make the functional equation look nicer.

It again turns out that $E_2^*(z,s)$ can be written out as a constant plus some Fourier coefficients of the form $E_2^*(y,s)(t)q^t$. Roughly speaking, this Euler product $E_2^*(y,s)$ reaks into archimedean and nonarchimedean parts

$$W_{t,\infty}^*(y,s)\prod_p W_{t,p}^*(s).$$

The archimedean part is something coming from hypergeometric functions, and the nonarchimedean part basically multiplies to something involving divisor functions as $|t|^{s+1/2}\,\sigma_{-2s}(|t|)$ (after shifting s).

We are now ready to state a corollary of the main theorem.

Corollary 23. One has

$$\frac{1}{2}\frac{d}{ds}\bigg|_{s=1/2} \prod_p W_{t,p}^*(s) = \sum_{\substack{\varphi\colon E\to E_0 \\ \deg\varphi = t}} (h_{\mathrm{Fal}}(E) - h_{\mathrm{Fal}}(E_0)),$$

where E_0 is a fixed elliptic curve with CM by $\mathcal{O}_F$ for some imaginary quadratic field $F/\mathbb{Q}$ in which 2 is split, and h_{Fal} is the Faltings height.

Notably, the right-hand expression depends on F, but the left-hand side does not! We remark that one can give geometric meaning to the entire expression $E_2^*(y,s)(t)$ as coming from some cycle.

Remark 24. Let's recall something about the Faltings height. Let $\widetilde{E}$ be an elliptic scheme over $\mathcal{O}_K$; we are only interested in the CM case, so we may as well assume that $\widetilde{E}$ has good reduction everywhere. We also assume that $\Gamma(\widetilde{E},\Omega^1)$ is free over $\mathcal{O}_K$ gnerated by some α. Then

$$h_{\mathrm{Fal}}(E) = -\frac{1}{2}\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma\colon K\hookrightarrow\mathbb{C}} \log\left|\frac{1}{2\pi}\int_{E_\sigma(\mathbb{C})} \alpha\wedge\overline{\alpha}\right|.$$

Remark 25. This complex volume on the right-hand side can also be seen as a volume of the arithmetic curve $\widetilde{E}$ over $Y(1)_{\mathcal{O}_K}$. We remark that one can approximate this "arithmetic volume" by intersecting $\widetilde{E}$ with Hecke translates of a given curve. Some version of equidistribution of Hecke orbits is then able to let us compute a volume! This is a key idea in the proof.

Let's move to a higher-dimensional case. Simply upgrade the group $\mathrm{SU}(1,1)$ to $\mathrm{SU}(m,m)$ everywhere, and we are able to define our Eisenstein series. Then it turns out that one has a Fourier expansion

$$E_n(z,s) = \sum_{T \in \mathrm{Herm}_m(F)} E_n(y,s)(T) \cdot q^T,$$

where $\mathrm{Herm}_m(F)$ refers to the Hermitian $m \times m$ matrices with entries in F. We let $\widetilde{E}(z,s)$ denote some suitable normalization, mostly multiplying in some πs, a discriminant, some gamma factors, and some L-function of a quadratic character.

Our main theorem is some version of the Siegel–Weil formula. Here is the classical result, which tells us that special values of Eisenstein series know about counting lattice points.

Theorem 26 (Seigel–Weil)**.** Choose an integer n divisible by 4. Then with $m = 1$,

$$\widetilde{E}(y,s)(T) = \sum_{\Lambda} \frac{\#\{x \in \Lambda : (x,x) = T\}}{\# \operatorname{Aut} \Lambda},$$

where Λ varies over Hermitian $\mathcal{O}_F$-lattices which are positive-definite, self-dual (with respect to the trace pairing over $\mathbb{Z}$), and rank n.

Remark 27. There is also a straightforwardly stated generalization to higher m. One basically replaces $x \in \Lambda$ with a tuple $\underline{x} \in \Lambda^m$ and replaces the norm condition with a condition on the Gram matrix $(\underline{x}, \underline{x})$.

Remark 28. The set of vector spaces $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is a Shimura variety of unitary type.

To make this result more geometric, we replace all these lattices by abelian varieties. We pick up a few definitions.

Definition 29 (Hermitian abelian scheme)**.** A *Hermitian abelian scheme* over a base S is a triple (A, ι, λ), where A is an abelian scheme over S, $\iota\colon \mathcal{O}_F \to \operatorname{End} A$ is an embedding, $\lambda\colon A \to A^\vee$ is a polarization, along with some compatibility conditions. For example, complex conjugation should correspond to the Rosati involution.

Definition 30 (Kottwitz signature)**.** Fix a Hermitian abelian scheme (A, ι, λ) over S, where S is a scheme over $\mathcal{O}_F$. Then the *Kottwitz signature* is a pair $(n-r, r)$ of integers such that any $\alpha \in \mathcal{O}_F$ has its characteristic polynomial of $\iota(\alpha)$ acting on $\operatorname{Lie} A$ taking the form

$$(X-a)^{n-r}(X-\sigma a)^r.$$

Now, to state our more geometric Siegel–Weil formula, we count points in a lattice $\mathcal{L} = \operatorname{Hom}(E_0, A)$, where E_0 is a fixed CM elliptic curve, which we go ahead and equip with its canonical principal polarization. Roughly speaking, the norm (x,x) comes from a "Rosati involution" (arising from the ambient polarizations) and is defined as $x^\intercal \circ x$. Lastly, the self-duality condition corresponds to having $\ker \lambda$ equaling the kernel of the different ideal acting on A, and it is included for some technical conditions.

At the end of this story, we have defined a subfunctor $\mathcal{Z}$ of $\mathcal{L}^m$, and it is called the Kudla–Rapoport special cycle. This is some 0-dimensional cycle.

Remark 31. Prior work computed an arithmetic degree of $\mathcal{Z}(T)$ as

$$\frac{\#\operatorname{Cl}\mathcal{O}_F}{\#\mathcal{O}_F^\times}\frac{d}{ds}\bigg|_{s=0}E(y,s)(T).$$

The main theorem roughly tells the same story for an analogously defined 1-cycle, proving a case of the Kudla's program.

5 February 19: Stefan Dawydiak

This talk was titled "Affine and asymptotic Hecke algebras." I don't expect to be able to follow much. Today, we are talking about p-adic representation theory. It is well-known that they are controlled by affine Hecke algebra, but today we would like to extend this to asymptotic Hecke algebras. Philosophically, the affine Hecke algebra is important because it has two presentations, each of which know something about one side of the local Langlands correspondence. The asymptotic Hecke algebra will have a similar role.

Let's begin with affine Hecke algebras. This requires us to define Coxeter groups, which tell the story of finite Weyl groups.

Definition 32 (Coxeter)**.** A *finite Coxeter group* W is one which is generated by some reflections $s_i \in S$ with explicit relations which look like $(s_i s_j)^\bullet = 1$.

Definition 33 (affine Hecke algebra)**.** The *affine Hecke algebra* $\mathcal{H}_{\text{fin}}$ is a q-deformation of $\mathbb{Z}[W]$, spanned by some letters $\{T_w : w \in W\}$ satisfying the relations

$$(T_s + 1)(T_s - q) = 0$$

for each $s \in S$.

Example 34. At $q = 1$, we recover $\mathbb{Z}[W]$.

Remark 35. There is a Kazhdan–Lusztig basis which provides a ring J_{fin} (over $\mathbb{Z}$) such that there is an embedding

$$\mathcal{H}_{\text{fin}} \hookrightarrow J_{\text{fin}}[q^{\pm 1}],$$

which is an isomorphism as soon as one localizes by some polynomial $P_W(q) = \sum_{w\in W} q^{\ell(w)}$.

Next we move to affine Weyl groups.

Definition 36 (affine Weyl)**.** An *affine Weyl group* $\widetilde{W}$ takes the form $W_{\text{fin}} \ltimes X_*$ where $(X^*, X_*, R, R^\vee)$, where $(X^*, X_*, R, R^\vee)$ is the root datum of a split connected reductive group G over a p-adic field F.

Remark 37. There is a decomposition

$$G(F) = \bigsqcup_{w\in\widetilde{W}} IwI,$$

where $I \subseteq G(F)$ is the Iwahori subgroup. For example, with $G = \mathrm{GL}_2(\mathbb{Q}_p)$, we find that I consists of the matrices which are $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \pmod p$.

Definition 38. Given an affine Weyl group $\widetilde{W}$, there is an *affine Hecke algebra* $\mathcal{H}_{\mathrm{aff}}$ spanned by some T_w for $w \in \widetilde{W}$ satisfying some relations.

Remark 39. It now turns out that $\mathcal{H}_{\mathrm{aff}}$ is isomorphic to $C_c(I \backslash G(F)/I)$, whose modules tell us something about smooth admissible representations of $G(F)$.

Remark 40. On the other hand, we see that $\mathcal{H}_{\mathrm{aff}}$ can be seen as some kind of equivariant K-theory over a Steinberg variety. For example, this allows us to say something about what the modules of $\mathcal{H}_{\mathrm{aff}}$ are.

Comparing the above two remarks allows us to prove some subcases of the local Langlands correspondence.

Remark 41. We note that there is once again some "Kazhdan–Lusztig" basis which produces a ring J_{aff} over $\mathbb{Z}$ such that one admits an embedding $\mathcal{H}_{\mathrm{aff}} \hookrightarrow J_{\mathrm{aff}}\left[q^{\pm 1}\right]$. This thing becomes an isomorphism after taking some completion on both sides.

Let's indicate the main theorem. As before, G is a connected split reductive group over a p-adic field F. Let $P \subseteq G$ be a varying parabolic subgroup, and let $P = MU$ be the Levi decomposition. Then we fix some square integrable representation σ of the Levi M, and we let ν vary over the unramified characters of M; note that the collection $X_{\mathrm{unr}}(M)$ of unramified characters on M forms a variety. We now note two constructions coming from $\pi = \mathrm{ind}_P^G(\sigma \otimes \nu)$.

- For $f \in C_c(I \backslash G/I)$, one can produce a function $\pi(f)$ on $X_{\mathrm{unr}}(M)$ defined by

$$\pi(f)v := \int_G f(g)\pi(g)v\, dg.$$

  It turns out that this construction will produce infinite tuples η for each choice (P, σ), each η of which varies algebraically in ν.

- The same construction sends Schwartz functins in $\mathcal{C}(I \backslash G/I)$ to endomorphisms which vary smoothly in ν.

It is the proposal of Braverman–Kazhdan that one can fit the ring J_{aff} into the above picture as going to the families $\eta_{(P,\sigma)}$ where $\eta_{(P,\sigma)}$ varies rationally in ν with no poles when one has some positivity condition on ν. This last sentence is our main theorem.