Special Values

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1 Special Values of Dirichlet L-Functions

This talk was given by Rui. Roughly speaking, the style of these sorts of special values results is that someone observes some equalities, then one works out examples, we make a general conjecture, and eventually it is proven.

1.1 Some Examples

Let's begin by discussing the simplest L-function: the Riemann ζ-function.

Definition 1. The *Riemann ζ-function* ζ is defined by the series

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$.

Example 2. Here is what is known about some small special values. Euler showed that $\zeta(s) = \frac{\pi^2}{6}$, and Apéry showed that $\zeta(3)$ is irrational.

Remark 3. In general, there is a conjecture that the values $\{\pi, \zeta(3), \zeta(5), \ldots\}$ forms an algebrically independent set. Roughly speaking, this is expected by the Grothendieck period conjecture.

Today, we will be happy working in only slightly larger generality, with Dirichlet L-functions.

Definition 4 (Dirichlet character)**.** Fix a positive integer N. Then a *Dirichlet character* $\pmod N$ is a character $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. The Dirichlet character η is *primitive* if and only if it does not factor through $(\mathbb{Z}/D\mathbb{Z})^\times$ for any divisor $D \mid N$. Further, we say that η is even (respectively, odd) if and only if $\eta(-1) = 1$ (respectively, $\eta(-1) = -1$).

Definition 5 (Dirichlet L-function)**.** Given a Dirichlet character $\eta \pmod N$, we define the *Dirichlet L-function* $L(\eta, s)$ by

$$L(\eta, s) := \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s},$$

where implicitly $\eta(n) = 0$ whenever $\gcd(n, N) > 1$.

Example 6. Let $\eta\colon (\mathbb{Z}/2\mathbb{Z})^\times \to \mathbb{C}^\times$ be the nontrivial character. Then $L(\eta, 1) = \frac{\pi}{4}$.

Example 7. Let $\eta\colon (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times$ be the character defined by $\eta(3) = \eta(5) = -1$. This can be proven via a trick. Consider the power series

$$f(x) := x - \frac{1}{3}x - \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots,$$

from which one finds $f'(x) = 1 - x^2 - x^4 + x^6 + \cdots = \frac{1-x^2-x^4+x^6}{1-x^8}$. Then one can integrate $f'(x)$ to get

$$f(x) = \frac{\sqrt{2}}{4} \log \left| \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{x} + 1} \right|.$$

It follows that $L(\eta, 1) = \frac{\sqrt{2}}{2} \log\left(\sqrt{2} + 1\right)$.

The previous example is an example of the class number formula, and right now it looks like a miracle. To give a taste for what is remarkable here, we note that $1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]^\times$ is a fundamental unit. As such, we expect some interesting arithmetic to be going on.

Here is a more general result.

Theorem 8. Suppose $\eta \pmod N$ is a primitive nontrivial Dirichlet character.

(a) If η is even, then for any positive integer m, we have

$$L(\eta, 2m) \equiv \pi^{2m} \pmod{\overline{\mathbb{Q}}^\times}.$$

(b) If η is odd, then for any positive integer m, we have

$$L(\eta, 2m-1) \equiv \pi^{2m-1} \pmod{\overline{\mathbb{Q}}^\times}.$$

The above is an instance of Deligne's conjecture.

For another general result, we note that an even primitive quadratic character $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ has kernel which is an index-2 subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times \cong \operatorname{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, so it corresponds to a quadratic extension F of $\mathbb{Q}$. In fact, the fact that η is even tells us that complex conjugation fixes F, so F is totally real. It then turns out that

$$L(\eta, 1) \equiv \sqrt{\operatorname{disc} \mathcal{O}_F} \cdot \log |u_F| \pmod{\mathbb{Q}^\times},$$

where u_F is a fundamental unit of $\mathcal{O}_F$. This also comes from the class number formula, and it is an instance of Beilinson's conjecture.

1.2 Funtional Equations

As usual, to write down a suitable functional equation for our L-functions, we must add some archimedean factors.

Definition 9 (completed Dirichlet L-function)**.** Let $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Then we define $d \in \{0,1\}$ by $\eta(-1) = (-1)^d$ and then

$$L_\infty(\eta, s) := \pi^{-\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right).$$

Then the *completed Dirichlet L-function* is $\Lambda(\eta,s) := L_\infty(\eta,s)L(\eta,s)$.

Remark 10. Recall that $\Gamma(s)$ is defined by

$$\Gamma(s) := \int_0^\infty e^{-t}t^s\,\frac{dt}{t}$$

for s such that $\operatorname{Re} s > 0$. One also knows that $\Gamma(s)$ admits a meromorphic continuation with understood poles, and it has a functional equation $\Gamma(s+1) = s\Gamma(s)$.

And here is our functional equation.

Theorem 11. Let $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Then $L(\eta,s)$ admits a meromorphic continuation (with poles only at $s \in \{0,1\}$ only when η is trivial) to all $\mathbb{C}$ and satisfies a functional equation

$$\Lambda(\eta,s) = \varepsilon(\eta,s)\Lambda\left(\eta^{-1}, 1-s\right),$$

where $\varepsilon(\eta,s)$ is some appropriately normalized Gauss sum.

We will not prove this today (it is mildly technical). Instead, we will use it to show a partial version of Theorem 8. With that said, we will need to do something in the direction of a meromorphic continuation because we will try to understand negative integer values of $L(\eta,s)$.

By expanding out the series, we see that

$$\Gamma(s)L(\eta,s) = \int_0^\infty \sum_{\substack{n\ge1\\ \gcd(n,N)=1}} \eta(n)e^{-nt}t^s\,\frac{dt}{t} = \int_0^\infty \frac{1}{1-e^{-Nt}}\sum_{n=0}^{N_1}\eta(n)e^{-nt}\,\frac{dt}{t}.$$

One now plugs into the general machine that produces analytic continuation and functional equations.

Lemma 12. Choose a smooth Schwarz function $f\colon \mathbb{R}_{\ge0} \to \mathbb{R}$. Then

$$L(f,s) := \frac{1}{\Gamma(s)}\int_0^\infty f(t)t^s\,\frac{dt}{t}$$

has an analytic continuation to all $\mathbb{C}$ and satisfies $L(f,-n) = (-1)^n f^{(n)}(0)$ for all $n \ge 0$.

Proof. To control singularities, we let $\varphi\colon \mathbb{R}_{\ge0} \to \mathbb{R}$ be a smooth bump function satisfying $\varphi|_{[0,1]} = 1$ and $\varphi_{[2,\infty)} = 0$. Thus, if we expand $f = f_1 + f_2$ with $f_1 = \varphi f$ and $f_2 = (1-\varphi)f$, we see that

$$\int_0^\infty f_2(t)t^s\,\frac{dt}{t} = \int_1^\infty f_2(t)t^s\,\frac{dt}{t},$$

and the rapid decay of f grants this term an analytic continuation to all $\mathbb{C}$, and it even satisfies

$$L(f_2,-n) = \left(\frac{1}{\Gamma(-s)}\int_1^\infty f_2(t)t^s\,\frac{dt}{t}\right)\Bigg|_{s=-n} = 0.$$

Thus, we are allowed to ignore f_2 piece. For the f_1 part, we inductively integrate by parts. For example, our first integration by parts produces

$$L(f,s) = \underbrace{\frac{1}{\Gamma(s)} f_1(t) \frac{t^s}{s}\Big|_0^\infty}_{0} - \frac{1}{s\Gamma(s)} \int_0^\infty f(t)t \cdot t^s \, \frac{dt}{t} = -L(f_1', s+1).$$

Thus, we have moved out s to $s+1$, and we can iteratively produce the needed continuation from the argument above. The result on the special value follows from a computation. ■

One can now use the lemma to see that

$$L(\eta, -n) \in \mathbb{Q}(\eta).$$

Then one can use the functional equation Theorem 11 to prove Theorem 8 after tracking everything through. I apologize, but I chose not to write down the details.