A More Reduced Inventory

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Abstract

We show that the classical reduced inventory for $\mathbb{Z}$ is equivalent to the statement that $\mathbb{Z}$ is initial in Ring.

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0 Review of the Reduced Inventory

In the discussion that follows, all rings have identity, as God intended. We will not require that all rings be commutative.

To review, in first-year number theory, one builds the following reduced inventory which suffices to characterize $\mathbb{Z}$.

Inventory 1 (Reduced inventory, I)**.** We have that $\mathbb{Z}$ is a commutative ring with the following properties.

1. There is a nonempty subset $\mathbb{N} \subseteq \mathbb{Z}$ with the following properties.
   - Closure: the set $\mathbb{N}$ is closed under the operations $+$ and $\times$.
   - Trichotomy: for each integer $n \in \mathbb{Z}$, exactly one of the statements $\{-n \in \mathbb{N}, n = 0, n \in \mathbb{N}\}$ is true. Equivalently, $\mathbb{Z} = -\mathbb{N} \sqcup \{0\} \sqcup \mathbb{N}$.
2. Well-ordering: each nonempty subset $S \subseteq \mathbb{N}$ has a least element.

Observe that well-ordering requires an ordering on $\mathbb{N}$, which one usually defines by saying

$$a < b \iff b - a \in \mathbb{N}$$

for $a, b \in \mathbb{Z}$. This defines an assymetric relation roughly because $0 \notin \mathbb{N}$ by trichotomy; it is transitive because $\mathbb{N}$ is closed under $+$. So indeed, $<$ provides a strict total ordering of $\mathbb{Z}$.

We are interested in showing that the single following statement is sufficient in characterizing the set of integers $\mathbb{Z}$.

Inventory 2 (Reduced inventory, II)**.** We have that $\mathbb{Z}$ is initial in the category Ring.

Formally, our goal is to show the equivalence of Inventory 2 and Inventory 1. Work done during the number theory course can show that Inventory 1 implies that $\mathbb{Z}$ is initial without too much work, so we will be focusing on trying to show Inventory 2 implies Inventory 1. Nevertheless, we will show that Inventory 1 implies Inventory 2 in our work later anyways.

Idea 3. In fact, one can use set thoery to construct a ring $\mathbb{Z}$ which satisfies Inventory 1. It is not difficult to show that Inventory 1 implies Inventory 2, so (for example) $\mathbb{Z}$ is initial in Ring. For the other direction, any ring R which is initial in Ring will be canonically isomorphic to $\mathbb{Z}$, and we can use this isomorphism to show that R satisfies Inventory 1.

I don't really want to write this out, but the construction is pretty standard. I have commented out the rest of the article because its need is obviated by the above argument. I do think some of it is interesting (for example, the proof that $\mathbb{N} = N(\mathbb{Z})$), but it is mostly useless.

There is perhaps room for the possibility of a more category-theoretic argument that Inventory 2 implies Inventory 1, but I am not sure what this would look like, given that Inventory 1 is mostly about subsets.