

# The Local Fundamental Class

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## Abstract

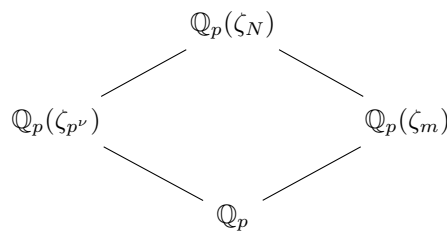
We compute the local fundamental class of the extension  $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$  when  $p$  is an odd prime. This requires making a number of standard group cohomology constructions fully explicit in the process.

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## 1 Set-Up

We will work over  $\mathbb{Q}_p$  as our base field, where  $p$  is an odd prime. Set  $N := p^\nu m$  where  $k$  and  $m$  integers with  $p \nmid m$ . This gives us the following tower of fields.



To help us a little later, we will assume that the extension  $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$  is not totally ramified nor as unramified, for in this case we can understand the extension by viewing it as a cyclic extension. We provide some quick commentary on these extensions.

- The extension  $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$  is unramified of degree  $f := \text{ord}_p(m)$ ; note we are assuming  $1 < f < n$ . Its Galois group is thus generated by the Frobenius element defined by  $\bar{\sigma}_K: \zeta_m \mapsto \zeta_m^p$ .
- The extension  $\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p$  is totally ramified of degree  $\varphi(p^\nu)$ . Its Galois group is thus isomorphic to  $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$ , where the isomorphism takes  $x \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times$  to

$$\sigma_x: \zeta_{p^\nu} \mapsto \zeta_{p^\nu}^{x-1}.$$

The group  $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$  is cyclic, so we will fix a generator  $x$ , which gives us a distinguished generator  $\sigma_x \in \text{Gal}(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p)$ .

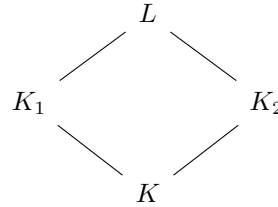
- Because  $\mathbb{Q}_p(\zeta_{p^\nu})$  is totally ramified and  $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$  is unramified, we have that the fields  $\mathbb{Q}_p(\zeta_{p^\nu})$  and  $\mathbb{Q}_p(\zeta_m)$  are linearly disjoint over  $\mathbb{Q}_p$ . As such,  $\mathbb{Q}_p(\zeta_N) = \mathbb{Q}_p(\zeta_{p^\nu})\mathbb{Q}_p(\zeta_m)$  has

$$\begin{aligned}\mathrm{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p(\zeta_{p^\nu})) &\simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) = \langle \bar{\sigma}_K \rangle \\ \mathrm{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p(\zeta_m)) &\simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_x \rangle \\ \mathrm{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p) &\simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \bar{\sigma}_K \rangle \times \langle \sigma_x \rangle.\end{aligned}$$

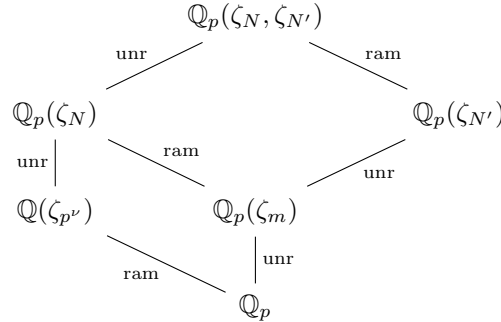
In light of these isomorphisms, we will upgrade  $\bar{\sigma}_K$  to the automorphism of  $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$  sending  $\zeta_m \mapsto \zeta_m^p$  and fixing  $\mathbb{Q}_p(\zeta_{p^\nu})$ ; we do analogously for  $\sigma_x$ . We also acknowledge that our degree is

$$n := [\mathbb{Q}_p(\zeta_N) : \mathbb{Q}_p] = [\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p] \cdot [\mathbb{Q}_p(\zeta_{p^\nu}) : \mathbb{Q}_p] = f\varphi(p^\nu).$$

For brevity, we will also set  $K_1 := \mathbb{Q}_p(\zeta_{p^\nu})$  and  $K_2 := \mathbb{Q}_p(\zeta_m)$ , which makes the fields under  $L$  look like the following.

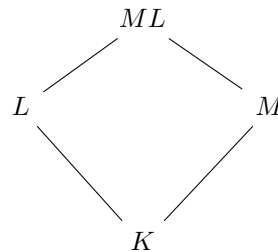


Now, the main idea in the computation is to use an unramified extension of the same degree as  $\mathbb{Q}_p(\zeta_N)$ . As such, we set  $N' := p^n - 1$  so that  $[\mathbb{Q}_p(\zeta_{N'}) : \mathbb{Q}_p]$  is  $\mathrm{ord}_{N'}(p) = n$ . This modifies our diagram of fields as follows.



We have labeled the unramified extensions by “unr” and the totally ramified extensions by “ram.”

For brevity, we set  $K := \mathbb{Q}_p$  and  $L := \mathbb{Q}_p(\zeta_N)$  and  $M := \mathbb{Q}_p(\zeta_{N'})$  so that  $ML = \mathbb{Q}_p(\zeta_N, \zeta_{N'})$ . This abbreviates our diagram into the following.



As before, we provide some comments on the field extensions.

- The extension  $\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p$  is unramified of degree  $n$ . As before, its Galois group is cyclic, generated by  $\sigma_K : \zeta_{N'} \mapsto \zeta_{N'}^p$ . Observe that  $\sigma_K$  restricted to  $\mathbb{Q}_p(\zeta_m)$  is  $\bar{\sigma}_K$ , explaining our notation. In particular,  $\sigma_K$  has order  $n$ , but  $\bar{\sigma}_K$  has order  $f < n$ .

- As before, note that  $\mathbb{Q}_p(\zeta_{p^\nu})$  and  $\mathbb{Q}(\zeta_{N'})$  are linearly disjoint because  $\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p$  is totally ramified while  $\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p$  is unramified. As such, we may say that

$$\begin{aligned}\mathrm{Gal}(ML/M) &\simeq \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_x \rangle \\ \mathrm{Gal}(ML/\mathbb{Q}_p(\zeta_{p^\nu})) &\simeq \mathrm{Gal}(M/K) = \langle \sigma_K \rangle \\ \mathrm{Gal}(ML/K) &\simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_K \rangle \times \langle \sigma_x \rangle.\end{aligned}$$

Again, we will upgrade  $\sigma_K$  and  $\sigma_x$  to their corresponding automorphisms on any subfield of  $ML$ .

- We take a moment to compute

$$\mathrm{Gal}(ML/L) \simeq \{ \sigma_K^{a_1} \sigma_x^{a_2} \in \mathrm{Gal}(ML/K) : \sigma_K^{a_1} \sigma_x^{a_2}|_L = \mathrm{id}_L \}.$$

Because  $L$  is  $\mathbb{Q}_p(\zeta_{p^\nu})\mathbb{Q}_p(\zeta_m)$ , it suffices to fix each of these fields individually. Well, to fix  $\mathbb{Q}_p(\zeta_{p^\nu})$ , we need  $\sigma_x^{a_2}$  to vanish, so we might as well force  $a_2 = 0$ . But to fix  $\mathbb{Q}_p(\zeta_m)$ , we need  $\sigma_K^{a_1}|_{\mathbb{Q}(\zeta_m)} = \bar{\sigma}_k^{a_1}$  to be the identity, so we are actually requiring that  $f \mid a_1$  here. As such,

$$\mathrm{Gal}(ML/L) = \langle \sigma_K^f \rangle.$$

These comments complete the Galois-theoretic portion of the analysis.

## 2 Idea

We will begin by briefly describe the outline for the computation. For a finite extension of finite fields  $L/K$ , let  $u_{L/K} \in H^2(L/K)$  denote the fundamental class.

Now, take variables as in our set-up in [section 1](#). The main idea is to translate what we know about the unramified extension  $M/K$  over to the general extension  $L/K$ . In particular, we are able to compute the fundamental class  $u_{M/K} \in H^2(M/K)$ , so we observe that

$$\mathrm{Inf}_{M/K}^{ML/K} u_{M/K} = [ML : M] u_{M/K} = n \cdot u_{ML/K} = [ML : L] u_{ML/L} = \mathrm{Inf}_{L/K}^{ML/K} u_{L/K}.$$

As such, we will be able to compute  $u_{L/K}$  as long as we are able to invert the inflation map  $\mathrm{Inf} : H^2(L/K) \rightarrow H^2(ML/K)$ . This is not actually very easy to do in general,<sup>1</sup> but we are in luck because this inflation map here comes from the Inflation–Restriction exact sequence

$$0 \rightarrow H^2(L/K) \xrightarrow{\mathrm{Inf}} H^2(ML/K) \xrightarrow{\mathrm{Res}} H^2(ML/L).$$

The argument for the Inflation–Restriction exact sequence is an explicit computation on cocycles (involving some dimension shifting), but it can be tracked backwards to give the desired cocycle.

## 3 Computation

In this section we record the details of the computation.

### 3.1 Group Cohomology

Throughout this section,  $G$  will be a group (usually finite) and  $H \subseteq G$  will be a subgroup (usually normal). We denote  $\mathbb{Z}[G]$  by the group ring and  $I_G \subseteq \mathbb{Z}[G]$  by the augmentation ideal, defined as the kernel of the map  $\varepsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$  which sends  $g \mapsto 1$  for all  $g \in G$ .

We begin by recalling the statement of the Inflation–Restriction exact sequence.

<sup>1</sup> The difficulty comes from the fact that a generic cocycle might be off from an inflated cocycle by some truly hideous coboundary.

**Theorem 1 (Inflation–Restriction).** Let  $G$  be a finite group with normal subgroup  $H \subseteq G$ . Given a  $G$ -module  $A$ , suppose that the  $H^i(H, A) = 0$  for  $1 \leq i < q$  for some index  $q \geq 1$ . Then the sequence

$$0 \rightarrow H^q(G/H, A^H) \xrightarrow{\text{Inf}} H^q(G, A) \xrightarrow{\text{Res}} H^q(H, A)$$

is exact.

*Sketch.* The proof is by induction on  $q$ , via dimension shifting. For  $q = 1$ , we can just directly check this on 1-cocycles. The main point is the exactness at  $H^q(G, A)$ : if  $c \in Z^1(G, A)$  has  $\text{Res}(c) \in B^1(H, A)$ , then find  $a \in A$  with

$$\text{Res}(c)(a) := h \cdot a - a.$$

As such, we define  $f_a \in B^1(G, A)$  by  $f_a(g) := g \cdot a - a$ , which implies that  $c - f_a$  vanishes on  $H$ . It is then possible to stare at the 1-cocycle condition

$$(c - f_a)(gg') = (c - f_a)(g) + g \cdot (c - f_a)(g')$$

to check that  $c - f_a$  only depends on the cosets of  $H$  (e.g., by taking  $g' \in H$ ) and that  $\text{im}(c - f_a) \subseteq A^H$  (e.g., by taking  $g \in H$ ).

For  $q > 1$ , we use dimension shifting via the following lemma.

**Lemma 2 (Dimension shifting).** Let  $G$  be a group with subgroup  $H \subseteq G$ . Given a  $G$ -module  $A$ , all indices  $q \geq 1$  have

$$\delta: H^q(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^{q+1}(H, A).$$

*Sketch.* Recall that we have the short exact sequence of  $\mathbb{Z}[H]$ -modules

$$0 \rightarrow I_G \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0.$$

In fact, this short exact sequence splits over  $\mathbb{Z}$ , so it will still be short exact after applying  $\text{Hom}_{\mathbb{Z}}(-, A)$ , which gives the short exact sequence

$$0 \rightarrow A \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \rightarrow \text{Hom}_{\mathbb{Z}}(I_G, A) \rightarrow 0$$

of  $\mathbb{Z}[H]$ -modules. The result now follows from the long exact sequence of cohomology upon noting that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$  is coinduced and hence acyclic for cohomology. ■

Using the above lemma, we have the following commutative diagram with vertical arrows which are isomorphisms.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^q(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H) & \longrightarrow & H^q(G, \text{Hom}_{\mathbb{Z}}(I_G, A)) & \longrightarrow & H^q(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \longrightarrow & H^{q+1}(G/H, A^H) & \longrightarrow & H^{q+1}(G, A) & \longrightarrow & H^{q+1}(H, A) \end{array}$$

The top row is exact by the inductive hypothesis, so the bottom row is therefore also exact. ■

Our goal is to make the above proof explicit in the case of  $q = 2$ , which is the only reason we sketched the above proofs at all. We begin by making the dimension shifting explicit.

**Lemma 3.** Let  $G$  be a group with subgroup  $H \subseteq G$ , and let  $\{g_\alpha\}_{\alpha \in \lambda}$  be coset representatives for  $H \backslash G$ . Now, given a  $G$ -module  $A$ , the maps

$$\begin{aligned} \delta_H: Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) &\rightarrow Z^2(H, A) \\ c &\mapsto [(h, h') \mapsto h \cdot c(h')(h^{-1} - 1)] \\ [h \mapsto ((h'g_\bullet - 1) \mapsto h' \cdot u((h')^{-1}, h))] &\leftarrow u \end{aligned}$$

are group homomorphisms which descend to the isomorphism  $\bar{\delta}: H^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^2(H, A)$  of Lemma 2. The map  $\delta$  above is surjective, and the reverse map is a section; when  $H = G$ , these are isomorphisms.

*Proof.* We begin by noting that our short exact sequence can be written more explicitly as follows.

$$\begin{aligned} 0 &\longrightarrow A \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \longrightarrow \text{Hom}_{\mathbb{Z}}(I_G, A) \longrightarrow 0 \\ a &\longmapsto (z \mapsto \varepsilon(z)a) \\ f &\longmapsto f|_{I_G} \end{aligned}$$

We now track through the induced boundary morphism  $\delta: H^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \rightarrow H^2(H, Q)$ .

- We begin with  $c \in Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A))$ , which means that we have  $c(h): I_G \rightarrow A$  for each  $h, h' \in H$ , and we satisfy

$$c(hh') = c(h) + h \cdot c(h').$$

Tracking through the action of  $H$  on  $\text{Hom}_{\mathbb{Z}}(I_G, A)$ , this means that

$$c(hh')(g - 1) = c(h)(g - 1) + h \cdot c(h')(h^{-1}g - h^{-1})$$

for any  $g \in G$ .

- To pull  $c$  back to  $C^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ , we need to lift  $c(h): I_G \rightarrow A$  to a  $\tilde{c}(h): \mathbb{Z}[G] \rightarrow A$ . Recalling that we only need to preserve group structure, we simply precompose  $c(h)$  with the map  $\mathbb{Z}[G] \rightarrow I_G$  given by  $z \mapsto z - \varepsilon(z)$ . That is, we define

$$\tilde{c}(h)(z) := c(h)(z - \varepsilon(z)).$$

- We now push  $\tilde{c}$  through  $d: C^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \rightarrow Z^2(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ . This gives

$$(d\tilde{c})(h, h') = g\tilde{c}(h') - \tilde{c}(hh') + \tilde{c}(h)$$

for any  $h, h' \in H$ . Concretely, plugging in some  $z \in \mathbb{Z}[G]$  makes this look like

$$\begin{aligned} (d\tilde{c})(h, h')(z) &= (h\tilde{c}(h'))(z) - \tilde{c}(hh')(z) + \tilde{c}(h)(z) \\ &= h \cdot c(h')(h^{-1}z - \varepsilon(h^{-1}z)) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) \\ &= h \cdot c(h')(h^{-1}z - \varepsilon(z)) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)). \end{aligned}$$

Now, from the 1-cocycle condition on  $c$ , we recall

$$-c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) = -h \cdot (c(h')(h^{-1}z - \varepsilon(z)h^{-1})),$$

so

$$\begin{aligned} (d\tilde{c})(h, h')(z) &= h \cdot c(h')(h^{-1}z - \varepsilon(z)) \\ &= \varepsilon(z) \cdot (h \cdot c(h')(h^{-1} - 1)). \end{aligned}$$

In particular, we see that  $d\tilde{c} \in Z^2(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$  pulls back to  $(h, h') \mapsto h \cdot c(h')(h^{-1} - 1)$  in  $Z^2(H, A)$ . It is not too difficult to check that we have in fact defined a 2-cocycle, but we will not do so because it is not necessary for the proof.

Now, we do know that  $\delta_H$  is a homomorphism abstractly on elements of our cohomology classes by the Snake lemma, but it is also not too hard to see that

$$\delta_H: Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \rightarrow Z^2(H, A)$$

is in fact a homomorphism of groups directly from the construction. In short,

$$\delta_H(c + c')(h, h') = h' \cdot c(h) (h^{-1} - 1) + h' \cdot c'(h) (h^{-1} - 1) = (\delta_H(c) + \delta_H(c'))(h, h')$$

for any  $h, h' \in H$ .

It remains to prove the last sentence. We run the following checks; given  $u \in Z^2(H, A)$ , define  $c_u \in C^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A))$  by

$$c_u(h)(h'g_{\bullet} - 1) = h' \cdot u((h')^{-1}, h).$$

Note that this is enough data to define  $c_u(h): I_G \rightarrow A$  because  $I_G$  is a free  $\mathbb{Z}$ -module generated by  $\{g - 1 : g \in G\}$ .

- We verify that  $c_u$  is a 1-cocycle. This is a matter of force. Pick up  $h, h' \in H$  and  $g_{\bullet}h'' \in G$  and write

$$\begin{aligned} & (hc_u(h'))(h''g_{\bullet} - 1) + c_u(hh')(h''g_{\bullet} - 1) + c_u(h)(h''g_{\bullet} - 1) \\ &= h \cdot c_u(h') (h^{-1}h''g_{\bullet} - h^{-1}) + c_u(hh')(h''g_{\bullet} - 1) + c_u(h)(h''g_{\bullet} - 1) \\ &= h \cdot (h^{-1}h''u((h'')^{-1}h, h') - h^{-1}u(h, h')) + h''u((h'')^{-1}, hh') + h''u((h'')^{-1}, h) \\ &= h''u((h'')^{-1}h, h') - u(h, h') + h''u((h'')^{-1}, hh') + h''u((h'')^{-1}, h). \end{aligned}$$

This is just the 2-cocycle condition for  $u$  upon dividing out by  $h''$ , so we are done.

- For  $u \in Z^2(H, A)$ , we verify that  $\delta_H(c_u) = u$ . Indeed, given  $h, h' \in H$ , we check

$$\begin{aligned} \delta_H(c_u)(h, h') &= h \cdot c_u(h') (h^{-1} - 1) \\ &= h \cdot h^{-1} \cdot u(h, h') \\ &= u(h, h'). \end{aligned}$$

So far we have verified that  $\delta$  has section  $u \mapsto c_u$  and hence must be surjective. Lastly, we take  $H = G$  and show that  $c_{\delta c} = c$  to finish. Indeed, for  $g, g' \in G = H$ , we write

$$\begin{aligned} c_{\delta_H c}(g)(g' - 1) &= g' \cdot (\delta_H c)((g')^{-1}, g) \\ &= g'(g')^{-1} \cdot c(g)(g' - 1) \\ &= c(g)(g' - 1), \end{aligned}$$

which is what we wanted. ■

We also have used dimension shifting to show that  $H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H) \rightarrow H^2(G/H, A^H)$  is an isomorphism, but this requires a little more trickery. To begin, we discuss how to lift from  $\text{Hom}_{\mathbb{Z}}(I_G, A)^H$  to  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$ .

**Lemma 4.** Let  $G$  be a group with subgroup  $H \subseteq G$ . Fix a  $G$ -module  $A$  with  $H^1(H, A) = 0$ . Then, for any  $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ , the function  $h \mapsto h\psi(h^{-1} - 1)$  is a cocycle in  $Z^1(H, A) = B^1(H, A)$ , so we can define a function  $I_{\bullet}: \text{Hom}_{\mathbb{Z}}(I_G, A)^H \rightarrow A$  such that

$$\psi(h - 1) = h \cdot I_{\varphi} - I_{\varphi}$$

for all  $h \in H$ . In fact, given  $\varphi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ , we can construct  $\tilde{\varphi} \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$  by

$$\tilde{\varphi}(z) := \varphi(z - \varepsilon(z)) + \varepsilon(z)I_{\varphi}$$

so that  $\tilde{\varphi}|_{I_G} = \varphi$ .

*Proof.* We will just run the checks directly.

- We start by checking  $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$  give 1-cocycles  $c(h) := \varphi(h - 1)$  in  $Z^1(A, H)$ . To begin, we note that  $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$  simply means that any  $z - \varepsilon(z) \in I_G$  has

$$\psi(z - \varepsilon(z)) = (h\psi)(z - \varepsilon(z)) = h\psi(h^{-1}z - h^{-1}\varepsilon(z))$$

for all  $h \in H$ . In particular, replacing  $h$  with  $h^{-1}$  tells us that

$$h\psi(z - \varepsilon(z)) = \psi(hz - h\varepsilon(z)).$$

Now, we can just compute

$$\begin{aligned} (dc)(h, h') &= hc(h') - c(hh') + c(h) \\ &= hc(h' - 1) - c(hh' - 1) + c(h - 1) \\ &= c(hh' - h) - c(hh' - 1) + c(h - 1), \end{aligned}$$

where in the last equality we used the fact that  $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ . Now,  $(dc)(h, h')$  manifestly vanishes, so we are done.

- Note that  $\tilde{\varphi} \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$  because it is a linear combination of (compositions of) homomorphisms.
- Note that any  $z \in I_G$  has  $\varepsilon(z) = 0$ , so

$$\tilde{\varphi}(z) = \varphi(z - 0) + 0 \cdot I_{\varphi} = \varphi(z),$$

so  $\tilde{\varphi}|_{I_G} = \varphi$ .

- It remains to check that  $\tilde{\varphi}$  is fixed by  $H$ . This requires a little more effort. Recall that  $\varphi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$  means that any  $z - \varepsilon(z) \in I_G$  has

$$h\varphi(z - \varepsilon(z)) = \varphi(hz - h\varepsilon(z))$$

for any  $h \in H$ . Now, we just compute

$$\begin{aligned} (h\tilde{\varphi})(z) &= h\tilde{\varphi}(h^{-1}z) \\ &= h(\varphi(h^{-1}z - \varepsilon(h^{-1}z)) + \varepsilon(h^{-1}z)I_{\varphi}) \\ &= \varphi(z - h\varepsilon(z)) + \varepsilon(z) \cdot hI_{\varphi} \\ &= \varphi(z - h\varepsilon(z)) + \varepsilon(z)\varphi(h - 1) + \varepsilon(z)I_{\varphi} \\ &= \varphi(z - \varepsilon(z)) + \varepsilon(z)I_{\varphi} \\ &= \tilde{\varphi}(z). \end{aligned}$$

The above checks complete the proof. ■

**Remark 5.** For motivation, the  $\tilde{\varphi}$  was constructed by tracking through the following diagram.

$$\begin{array}{ccccccc} \frac{C^0(H, A)}{B^0(H, A)} & \longrightarrow & \frac{C^0(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))}{B^0(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))} & \longrightarrow & \frac{C^0(H, \text{Hom}_{\mathbb{Z}}(I_G, A))}{B^0(H, \text{Hom}_{\mathbb{Z}}(I_G, A))} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & Z^1(H, A) = B^1(H, A) & \longrightarrow & Z^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) & \longrightarrow & Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \end{array}$$

In short, take  $\varphi \in Z^0(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) = \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ , pull it back to  $z \mapsto \varphi(z - \varepsilon(z))$ . Pushing this down to  $Z^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$  and pulling back to  $Z^1(H, A)$  takes us to the 1-cocycle  $h \mapsto h\varphi(h^{-1} - 1)$ . Here we use the  $H^1(H, A) = 0$  condition above and adjust our lift  $z \mapsto \varphi(z - \varepsilon(z))$  accordingly.

And now we can now make our dimension shifting explicit.

**Lemma 6.** Work in the context of Lemma 4 and assume that  $H \subseteq G$  is normal. We track through the isomorphism

$$\delta: H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H) \simeq H^2(G/H, A^H)$$

given by the exact sequence

$$0 \rightarrow A^H \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H \rightarrow \text{Hom}_{\mathbb{Z}}(I_G, A)^H \rightarrow 0.$$

*Proof.* We begin with some  $c \in H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H)$ . To track through the  $\delta$ , we define

$$\tilde{c}(gH) := c(gH)(z - \varepsilon(z)) + I_{c(gH)}\varepsilon(z)$$

to be the lift given in Lemma 4. Now, we are given that  $dc = 0$ , which here means that any  $z \in \mathbb{Z}[G]$  and  $gH, g'H \in G/H$  will have

$$\begin{aligned} 0 &= (dc)(gH, g'H)(z - \varepsilon(z)) \\ 0 &= (gH \cdot c(g'H) - c(gg'H) + c(gH))(z - \varepsilon(z)) \\ 0 &= g \cdot c(g'H) (g^{-1}z - g^{-1}\varepsilon(z)) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)) \\ g \cdot c(g'H) (g^{-1} - 1) \varepsilon(z) &= g \cdot c(g'H) (g^{-1}z - \varepsilon(z)) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)) \\ g \cdot c(g'H) (g^{-1} - 1) \varepsilon(z) &= g \cdot c(g'H) (g^{-1}z - \varepsilon(g^{-1}z)) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)). \end{aligned}$$

We now directly compute that

$$\begin{aligned} (d\tilde{c})(gH, g'H)(z) &= (gH \cdot c(g'H) - c(gg'H) + c(gH))(z) \\ &= g \cdot c(g'H) (g^{-1}z - \varepsilon(g^{-1}z)) + gI_{c(g'H)}\varepsilon(z) \\ &\quad - c(gg'H)(z - \varepsilon(z)) - I_{c(gg'H)}\varepsilon(z) \\ &\quad + c(gH)(z - \varepsilon(z)) + I_{c(gH)}\varepsilon(z) \\ &= (g \cdot c(g'H) (g^{-1} - 1) + g \cdot I_{c(g'H)} - I_{c(gg'H)} + I_{c(gH)}) \varepsilon(z) \end{aligned}$$

As such, we have pulled ourselves back to the 2-cocycle given by

$$u(gH, g'H) := g \cdot c(g'H) (g^{-1} - 1) + g \cdot I_{c(g'H)} - I_{c(gg'H)} + I_{c(gH)}.$$

We quickly note that this is in fact independent of our choice of representative  $g \in gH$ : changing representative of  $g$  to  $gh$  for  $h \in H$  will only affect the terms

$$h \cdot c(g'H) (h^{-1}g^{-1} - 1) + hI_{c(g'H)} = c(g'H) (g^{-1} - h) + c(g'H) (h - 1) + I_{c(g'H)} = c(g'H) (g^{-1} - 1) + I_{c(g'H)},$$

so we are indeed safe. This completes the proof.  $\blacksquare$

We now make Theorem 1 explicit in the case of  $q = 2$ .

**Lemma 7.** Let  $G$  be a group with normal subgroup  $H \subseteq G$ . Fix a  $G$ -module  $A$  with  $H^1(H, A) = 0$ , and define the function  $I_{\bullet}: \text{Hom}_{\mathbb{Z}}(I_G, A)^H \rightarrow A$  of Lemma 4. Given  $c \in Z^2(G, A)$  such that  $\text{Res}_H^G c \in B^2(H, A)$ ; in particular, suppose we have  $b \in \text{Hom}_{\mathbb{Z}}(I_G, A)$  such that all  $h \in H$  have

$$\text{Res}_H^G(\delta^{-1}c)(h) = (db)(h) = h \cdot b - h,$$

where  $\delta^{-1}$  is the inverse isomorphism of Lemma 3. Then we find  $u \in Z^2(G/H, A^H)$  such that

$$[\text{Inf } u] = [c]$$

in  $H^2(G, A)$ .



*Proof.* The main point is that boundary morphisms  $\delta$  commute with Res and Inf. By construction, we have that  $(\text{Res}_H^G \delta^{-1}c) - db = 0$  in  $Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A))$ . Pulling back to  $Z^1(G, \text{Hom}_{\mathbb{Z}}(I_G, A))$ , we note that

$$c' := (\delta^{-1}c - db) \in Z^1(G, \text{Hom}_{\mathbb{Z}}(I_G, A))$$

vanishes on  $H$  by hypothesis. Because  $\delta^{-1}c - db$  is a 1-cocycle, we are able to write

$$c'(gg') = c'(g) + gc'(g').$$

Letting  $g'$  vary over  $H$ , we see that  $\delta^{-1}c - db$  is well-defined on  $G/H$ . On the other hand, for any  $h \in H$  and  $g \in G$ , we note that  $g^{-1}hg \in H$ , so

$$c'(g) = c'(g \cdot g^{-1}hg) = c'(hg) = c'(h) + hc(g),$$

implying that  $c'(g) \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ .

We are now ready to apply [Lemma 6](#), which we use on  $c'$ , thus defining  $u := \delta(c')$ . Explicitly, we have

$$u(gH, g'H) = g \cdot c'(g'H) (g^{-1} - 1) + g \cdot I_{c'(g'H)} - I_{c'(gg'H)} + I_{c'(gH)}.$$

This is explicit enough for our purposes. Observe that  $[\text{Inf } u] = [c]$  because  $[\text{Inf } c'] = [\delta^{-1}c]$ , and  $\delta$  commutes with Inf. ■

### 3.2 Number Theory

Throughout, we will let  $u_{L/K}$  denote a representative of the fundamental class in  $H^2(L/K)$  rather than the actual cohomology class, mostly out of laziness.

We now return to the set-up in [section 1](#) and track through [Lemma 7](#) in our case. For reference, the following is the diagram that we will be chasing around; here  $G := \text{Gal}(ML/K)$  and  $H := \text{Gal}(ML/L)$ .

$$\begin{array}{ccccccc} & & & & H^2(\text{Gal}(M/K), M^\times) & & \\ & & & & \downarrow \text{Inf} & & \\ 0 & \longrightarrow & H^2(\text{Gal}(L/K), L^\times) & \xrightarrow{\text{Inf}} & H^2(G, ML^\times) & \xrightarrow{\text{Res}} & H^2(\text{Gal}(ML/L), ML^\times) \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ 0 & \longrightarrow & H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)^H) & \xrightarrow{\text{Inf}} & H^1(G, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)) & \xrightarrow{\text{Res}} & H^1(H, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)) \end{array}$$

To begin, we know that we can write

$$u_{M/K}(\sigma_K^i, \sigma_K^j) = p^{\lfloor \frac{i+j}{n} \rfloor} = \begin{cases} 1 & i+j < n, \\ p & i+j \geq n. \end{cases}$$

Inflating this down to  $H^2(G, ML^\times)$  gives

$$(\text{Inf } u_{M/K}) (\sigma_K^{a_1} \sigma_x^{a_2}, \sigma_K^{b_1} \sigma_x^{b_2}) = p^{\lfloor \frac{a_1+b_1}{n} \rfloor}.$$

Now, we use [Lemma 2](#) to move down to  $H^1(G, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times))$  as

$$\delta^{-1}(\text{Inf } u_{M/K}) (\sigma_K^{a_1} \sigma_x^{a_1}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) = \sigma_K^{b_1} \sigma_x^{b_2} \cdot (\text{Inf } u_{M/K}) (\sigma_K^{[-b_1]} \sigma_x^{[-b_2]}, \sigma_K^{a_1} \sigma_x^{a_2}) = p^{\lfloor \frac{a_1+[-b_1]}{n} \rfloor},$$

where  $[k]$  denote the integer  $0 \leq [k] < n$  such that  $k \equiv [k] \pmod{n}$ .

Now, we need to show that the restriction to  $H = \langle \sigma_K^f \rangle$  is a coboundary. That is, we need to find  $b \in \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)$  such that

$$\delta^{-1}(\text{Inf } u_{M/K}) (\sigma_K^{fa_1}) = \frac{\sigma_K^{fa_1} \cdot b}{b}.$$

Because  $I_G$  is freely generated by elements of the form  $g - 1$  for  $g \in G$ , it suffices to plug in some arbitrary  $\sigma_K^{b_1} \sigma_x^{b_2} - 1$ , which we see requires

$$\begin{aligned} p^{\lfloor \frac{fa_1 + [-b_1]}{n} \rfloor} &= \frac{(\sigma_K^{fa_1} \cdot b) (\sigma_K^{b_1} \sigma_x^{b_2} - 1)}{b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)} \\ &= \frac{\sigma_K^{fa_1} b (\sigma_K^{b_1 - fa_1} \sigma_x^{b_2} - 1)}{\sigma_K^{fa_1} b (\sigma_K^{-fa_1} - 1) b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)}. \end{aligned}$$

We can see that  $b$  should not depend on  $b_2$ , so we define  $\hat{b}(\sigma_K^a) = b(\sigma_K^a \sigma_x^\bullet - 1)$ ; the above is then equivalent to

$$\begin{aligned} p^{\lfloor \frac{fa_1 + [-b_1]}{n} \rfloor} &= \frac{\sigma_K^{fa_1} \hat{b}(\sigma_K^{b_1 - fa_1})}{\sigma_K^{fa_1} \hat{b}(\sigma_K^{-fa_1}) \hat{b}(\sigma_K^{b_1})} \\ p^{\lfloor \frac{fa_1 + b_1}{n} \rfloor} &= \frac{\hat{b}(\sigma_K^{-b_1 - fa_1})}{\hat{b}(\sigma_K^{-fa_1}) \sigma_K^{-fa_1} \hat{b}(\sigma_K^{-b_1})}, \end{aligned}$$

where we have negated  $b_1$  in the last step. At this point, the right-hand side will look a lot more natural if we set  $\tau := \sigma_K^{-1}$ , which turns this into

$$\frac{\hat{b}(\tau^{fa_1}) \tau^{fa_1} \hat{b}(\tau^{b_1})}{\hat{b}(\tau^{b_1 fa_1})} = (1/p)^{\lfloor \frac{fa_1 + b_1}{n} \rfloor}$$

after taking reciprocals. Thus, we see that  $\hat{b}$  should be counting carries of  $\tau$ s. With this in mind, we note that  $1 - \zeta_{p^\nu} \in L$  is a uniformizer because  $L/\mathbb{Q}_p(\zeta_{p^\nu})$  is an unramified extension. It follows that

$$(1 - \zeta_{p^\nu})^{\varphi(p^\nu)} \in N_{ML/L}(ML^\times).$$

Further,  $(1 - \zeta_{p^\nu})^{\varphi(p^\nu)}$  is only a unit (in  $\mathcal{O}_L^\times$ ) multiplied  $p$ , so in fact  $p$  is a norm from  $ML^\times$  because  $ML/L$  is unramified and so all units in  $\mathcal{O}_L^\times$  are norms from  $ML^\times$ . Thus, we find  $\alpha \in ML^\times$  such that

$$N_{ML/L}(\alpha) = p.$$

The point of doing all of this is so that we can codify our carrying by writing

$$\hat{b}(\tau^a) := \prod_{i=0}^{\lfloor a/f \rfloor - 1} \tau^{if}(\alpha)^{-1}.$$

Tracking out  $\hat{b}$  backwards to  $b$ , our desired  $b \in \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)$  is given by

$$b(\sigma_K^{a_1} \sigma_x^{a_2} - 1) = \prod_{i=0}^{\lfloor [-a_1]/f \rfloor - 1} \sigma_K^{-if}(\alpha)^{-1}.$$

We take a moment to write out  $c := \delta^{-1}(\text{Inf } u_{M/K})/db$ , which looks like

$$\begin{aligned}
 c(\sigma_K^{a_1} \sigma_x^{a_2}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) &= \frac{\delta^{-1}(\text{Inf } u_{M/K})}{db} (\sigma_K^{a_1} \sigma_x^{a_2}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) \\
 &= \frac{\delta^{-1}(\text{Inf } u_{M/K}) (\sigma_K^{a_1} \sigma_x^{a_2}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1)}{(\sigma_K^{a_1} \sigma_x^{a_2} b) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) / b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)} \\
 &= \frac{p^{\lfloor (a_1 + \lceil -b_1 \rceil) / n \rfloor}}{\sigma_K^{a_1} \sigma_x^{a_2} b (\sigma_K^{b_1 - a_1} \sigma_x^{b_2 - a_2} - \sigma_K^{-a_1} \sigma_x^{-a_2}) / b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)} \\
 &= p^{\lfloor (a_1 + \lceil -b_1 \rceil) / n \rfloor} \cdot \hat{b}(\sigma_K^{b_1}) \cdot \sigma_K^{a_1} \sigma_x^{a_2} \left( \frac{\hat{b}(\sigma_K^{-a_1})}{\hat{b}(\sigma_K^{b_1 - a_1})} \right).
 \end{aligned}$$

Before proceeding, we discuss a few special cases.

- Taking  $\sigma_K^{a_1} \sigma_x^{a_2} = \sigma_x$ , we get

$$\begin{aligned}
 c(\sigma_x) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) &= p^{\lfloor (0 + \lceil -b_1 \rceil) / n \rfloor} \cdot \hat{b}(\sigma_K^{b_1}) \cdot \sigma_x \left( \frac{1}{\hat{b}(\sigma_K^{b_1})} \right) \\
 &= \hat{b}(\sigma_K^{b_1}) / \sigma_x \hat{b}(\sigma_K^{b_1}).
 \end{aligned}$$

In particular,  $c(\sigma_x) (\sigma_K^{-1} - 1) = 1$ , provided that  $f > 1$ . Additionally,  $c(\sigma_x) (\sigma_x^{b_2} - 1) = 1$ .

Our general theory says that  $h \mapsto c(\sigma_x)(h - 1)$  is a 1-cocycle in  $Z^1(H, ML^\times)$  (though we could also check this directly), so Hilbert's Theorem 90 promises us a magical element  $I_{c(\sigma_x)} \in ML^\times$  such that

$$\frac{\sigma_K^{fb_1} I_{c(\sigma_x)}}{I_{c(\sigma_x)}} = \frac{\hat{b}(\sigma_K^{fb_1})}{\sigma_x \hat{b}(\sigma_K^{fb_1})}$$

for all  $\sigma_K^{fb_1} \in H$ . This condition will be a little clearer if we write everything in terms of  $\tau := \sigma_K^{-1}$ , which transforms this into

$$\frac{\tau^{fb_1} I_{c(\sigma_x)}}{I_{c(\sigma_x)}} = \frac{\hat{b}(\tau^{-fb_1})}{\sigma_x \hat{b}(\tau^{-fb_1})} = \prod_{i=0}^{b_1-1} \frac{\tau^{if}(\alpha^{-1})}{\sigma_x \tau^{if}(\alpha^{-1})} = \prod_{i=0}^{b_1-1} \frac{\sigma_x \tau^{if}(\alpha)}{\tau^{if}(\alpha)}.$$

Because we are dealing with a cyclic group  $H$ , it is not too hard to see that it suffices merely for  $b_1 = 1$  to hold, so our magical element  $I_{c(\sigma_x)}$  merely requires

$$\boxed{\frac{\sigma_K^{-f} (I_{c(\sigma_x)})}{I_{c(\sigma_x)}} = \frac{\sigma_x(\alpha)}{\alpha}}$$

after inverting  $\tau$  back to  $\sigma_K$ .

- Taking  $\sigma_K^{a_1} \sigma_x^{a_2} = \sigma_K$ , we get

$$c(\sigma_K) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) = p^{\lfloor (1 + \lceil -b_1 \rceil) / n \rfloor} \cdot \hat{b}(\sigma_K^{b_1}) \cdot \sigma_K \left( \frac{\hat{b}(\sigma_K^{-1})}{\hat{b}(\sigma_K^{b_1-1})} \right).$$

In particular,  $\sigma_K^{b_1} \sigma_x^{b_2} = \sigma_K^{-1}$  will give  $c(\sigma_K) (\sigma_K^{-1} - 1) = 1$ . We will also want  $c(\sigma_K) (\sigma_K^{-b_1} - 1)$  for  $0 \leq b_1 < f$ . Using the fact that  $f < n$  and  $f > 1$ , it is not too hard to see that everything will cancel

down to 1 except in the case where  $b_1 = f - 1$ , where we get

$$c(\sigma_K) \left( \sigma_K^{-(f-1)} - 1 \right) = \sigma_K \left( \frac{1}{\hat{b}(\sigma_K^{-f})} \right) = \sigma_K(\alpha).$$

Continuing as before, our general theory says that  $h \mapsto c(\sigma_x)(h - 1)$  is a 1-cocycle in  $Z^1(H, ML^\times)$ , though again we could just check this directly. It follows that Hilbert's Theorem 90 promises us a magical element  $I_{c(\sigma_K)} \in ML^\times$  such that

$$\frac{\sigma_K^{fb_1} I_{c(\sigma_K)}}{I_{c(\sigma_K)}} = p^{\lfloor (1+[-fb_1])/n \rfloor} \cdot \hat{b}(\sigma_K^{fb_1}) \cdot \sigma_K \left( \frac{\hat{b}(\sigma_K^{-1})}{\hat{b}(\sigma_K^{fb_1-1})} \right)$$

for all  $\sigma_K^{fb_1} \in H$ . Using  $f > 1$ , this collapses down to

$$\frac{\sigma_K^{fb_1} I_{c(\sigma_K)}}{I_{c(\sigma_K)}} = \frac{\hat{b}(\sigma_K^{fb_1})}{\sigma_K \hat{b}(\sigma_K^{fb_1-1})}.$$

As before, this condition will be a little clearer if we set  $\tau := \sigma_K^{-1}$ , which turns the condition into

$$\frac{\tau^{fb_1} I_{c(\sigma_K)}}{I_{c(\sigma_K)}} = \frac{\hat{b}(\tau^{fb_1})}{\sigma_K \hat{b}(\tau^{fb_1+1})} = \prod_{i=0}^{b_1-1} \frac{\tau^{if}(\alpha^{-1})}{\sigma_K \tau^{if}(\alpha^{-1})} = \prod_{i=0}^{b_1-1} \frac{\sigma_K \tau^{if}(\alpha)}{\tau^{if}(\alpha)}.$$

(Notably,  $\hat{b}(\tau^{fb_1}) = \hat{b}(\tau^{fb_1+1})$  because  $f > 1$ .) Again, because  $H$  is cyclic generated by  $\tau^f$ , an induction shows that it suffices to check this condition for  $b_1 = 1$ , which means that our magical element  $I_{c(\sigma_K)} \in ML^\times$  is constructed so that

$$\boxed{\frac{\sigma_K^{-f} (I_{c(\sigma_K)})}{I_{c(\sigma_K)}} = \frac{\sigma_K(\alpha)}{\alpha}}$$

where we have again inverted back from  $\tau$  to  $\sigma_K$ .

- We will not actually need a more concrete description of this, but we remark that we can run the same story for any  $g \in G$  through to get an element  $I_{c(g)} \in ML^\times$  such that

$$\frac{\sigma_K^{fb_1} I_{c(g)}}{I_{c(g)}} = \frac{1}{c(g)(\sigma_K^{fb_1} - 1)}$$

for any  $\sigma_K^{fb_1} \in H$ . As usual, this follows from our general theory.

We are now ready to describe the local fundamental class. Piecing what we have so far, we know from [Lemma 7](#) that we can write

$$u_{L/K}(g, g') := gc(g') (g^{-1} - 1) \cdot \frac{g I_{c(g')} \cdot I_{c(g)}}{I_{c(gg')}}.$$

Here are the values that we care about for our specific computation.

- We write

$$\begin{aligned} u_{L/K}(\sigma_K, \sigma_x) &= \sigma_K c(\sigma_x) (\sigma_K^{-1} - 1) \cdot \frac{\sigma_K I_{c(\sigma_x)} \cdot I_{c(\sigma_K)}}{I_{c(\sigma_K \sigma_x)}} \\ &= \frac{\sigma_K I_{c(\sigma_x)} \cdot I_{c(\sigma_K)}}{I_{c(\sigma_K \sigma_x)}}. \end{aligned}$$

- We write

$$\begin{aligned} u_{L/K}(\sigma_x, \sigma_K) &= \sigma_x c(\sigma_K) (\sigma_x^{-1} - 1) \cdot \frac{\sigma_x I_{c(\sigma_K)} \cdot I_{c(\sigma_x)}}{I_{c(\sigma_x \sigma_K)}} \\ &= \frac{\sigma_x I_{c(\sigma_K)} \cdot I_{c(\sigma_x)}}{I_{c(\sigma_x \sigma_K)}}. \end{aligned}$$

- In particular, we know that we can set  $\beta$  in a triple equal to

$$\begin{aligned} \beta &:= \frac{u_{L/K}(\sigma_K, \sigma_x)}{u_{L/K}(\sigma_x, \sigma_K)} \\ &= \frac{\sigma_K I_{c(\sigma_x)} \cdot I_{c(\sigma_K)} / I_{c(\sigma_K \sigma_x)}}{\sigma_x I_{c(\sigma_K)} \cdot I_{c(\sigma_x)} / I_{c(\sigma_x \sigma_K)}} \\ &= \boxed{\beta = \frac{\sigma_K (I_{c(\sigma_x)})}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x (I_{c(\sigma_K)})}}. \end{aligned}$$

As a sanity check, we can hit this  $\beta$  with  $\sigma_K^{-f}$  to show that  $\beta \in (ML)^H = L$ ; namely,  $\sigma_K^{-f} I_{c(\sigma_K)} = \frac{\sigma_K \alpha}{\alpha} \cdot I_{c(\sigma_K)}$  and  $\sigma_K^{-f} I_{c(\sigma_x)} = \frac{\sigma_x \alpha}{\alpha} \cdot I_{c(\sigma_x)}$  by construction, so we can see that everything will appropriately cancel out.

- We will go ahead and compute  $\alpha_1$  and  $\alpha_2$ , for completeness. For  $\alpha_1$ , our element is given by

$$\begin{aligned} \alpha_1 &:= \prod_{i=0}^{f-1} u_{L/K}(\sigma_K^i, \sigma_K) \\ &= \prod_{i=0}^{f-1} \left( \sigma_K^i c(\sigma_K, \sigma_K^{-i} - 1) \cdot \frac{\sigma_K^i I_{c(\sigma_K)} \cdot I_{c(\sigma_K^i)}}{I_{c(\sigma_K^{i+1})}} \right). \end{aligned}$$

Recall from our general theory that  $I_{c(g)}$  only depends on the coset of  $g$  in  $G/H$ , so we see that the product of the quotients  $I_{c(\sigma_K^i)} / I_{c(\sigma_K^{i+1})}$  will cancel out. As for the  $c$  term, we know from our computation that this is 1 until  $i = f - 1$ , which gives  $\sigma_K(\alpha)$ . As such, we collapse down to

$$\boxed{\alpha_1 = \sigma_K^f(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^i (I_{c(\sigma_K)})}.$$

- For  $\alpha_2$ , our element is given by

$$\begin{aligned} \alpha_2 &:= \prod_{i=0}^{\varphi(p^\nu)-1} u_{L/K}(\sigma_x^i, \sigma_x) \\ &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i c(\sigma_x) (\sigma_x^{-i} - 1) \cdot \frac{\sigma_x^i I_{c(\sigma_x)} \cdot I_{c(\sigma_x^i)}}{I_{c(\sigma_x^{i+1})}}. \end{aligned}$$

Recalling that  $\sigma_x$  has order  $\varphi(p^\nu)$ , our quotient term  $I_{c(\sigma_x^i)} / I_{c(\sigma_x^{i+1})}$  will again cancel out. Additionally, the cocycle  $c$  always spits out 1 on these inputs, so we are left with

$$\boxed{\alpha_2 = \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i (I_{c(\sigma_x)})}.$$

We summarize the results above in the following theorem.

**Theorem 8.** Fix everything as in the set-up. Then there exists some  $\alpha \in ML^\times$  such that  $N_{ML/L}(\alpha) = p$  and elements in  $I_{c(\sigma_K)}, I_{c(\sigma_x)} \in ML^\times$  such that

$$\frac{\sigma_K^{-f}(I_{c(\sigma_K)})}{I_{c(\sigma_K)}} = \frac{\sigma_K(\alpha)}{\alpha} \quad \text{and} \quad \frac{\sigma_K^{-f}(I_{c(\sigma_x)})}{I_{c(\sigma_x)}} = \frac{\sigma_x(\alpha)}{\alpha}.$$

Then the triple

$$(\alpha_1, \alpha_2, \beta) := \left( \sigma_K^f(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^i(I_{c(\sigma_K)}), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(I_{c(\sigma_x)}), \quad \frac{\sigma_K(I_{c(\sigma_x)})}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x(I_{c(\sigma_K)})} \right)$$

corresponds to the fundamental class  $u_{L/K} \in H^2(\text{Gal}(L/K), L^\times)$ .

### 3.3 Checks and Consequences

In this section we run some checks and discuss some consequences of [Theorem 8](#).