

# Sato–Tate Groups of Generic Superelliptic Curves

Nir Elber

Fall 2024

# CONTENTS

---

*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

<b>Contents</b>	<b>2</b>
<b>1 Introduction</b>	<b>4</b>
1.1 Notation . . . . .	4
<b>2 A Little Hodge Theory</b>	<b>5</b>
2.1 Hodge Structures . . . . .	5
2.1.1 Definition and Basic Properties . . . . .	5
2.1.2 Hodge Classes . . . . .	7
2.1.3 Polarizations . . . . .	8
2.1.4 The Albert Classification . . . . .	9
2.2 Monodromy Groups . . . . .	11
2.2.1 The Mumford–Tate Group . . . . .	11
2.2.2 The Hodge Group . . . . .	14
2.3 Computational Tools . . . . .	15
2.3.1 Bounding with Known Classes . . . . .	15
2.3.2 Sums . . . . .	18
2.4 The Center . . . . .	23
2.4.1 Generalities . . . . .	23
2.4.2 Type IV: The Signature . . . . .	25
2.4.3 Type IV: The Reflex . . . . .	29
<b>3 Review of Abelian Varieties</b>	<b>33</b>
<b>4 Absolute Hodge Cycles</b>	<b>34</b>
<b>5 The Sato–Tate Conjecture</b>	<b>35</b>
5.1 Computing Monodromy Groups . . . . .	35

<b>6 The Fermat Curve</b>	<b>36</b>
6.1 Homology and Cohomology . . . . .	36
6.1.1 The Group Action . . . . .	36
6.1.2 Differential Forms . . . . .	37
6.1.3 Some Group Elements . . . . .	39
6.1.4 Homology . . . . .	40
6.2 Galois Action . . . . .	42
6.2.1 Hodge Cycles on $X^{2p}$ . . . . .	42
6.2.2 An Absolute Hodge Cycle . . . . .	43
6.2.3 Computations on de Rham Component . . . . .	43
6.2.4 End of the Computation . . . . .	43
6.3 Fermat Hypersurfaces . . . . .	43
<b>7 Families of Curves</b>	<b>44</b>
<b>Bibliography</b>	<b>45</b>
<b>List of Definitions</b>	<b>47</b>

# CHAPTER 1

## INTRODUCTION

---

### 1.1 Notation

Elements.

- $V$  and  $W$  are vector spaces, frequently  $\mathbb{Q}$ -Hodge structures.
- $\mathbb{Q}(n)$  is the Tate twist.
- $H_{\mathbb{B}}^{\bullet}$  is Betti cohomology,  $H_{\mathrm{dR}}^{\bullet}$  is de Rham cohomology, and  $H_{\mathrm{\acute{e}t}}^{\bullet}$  is étale cohomology.
- $\mathfrak{g}$  and  $\mathfrak{h}$  are Lie algebras.

Groups.

- If  $V$  is a  $\mathbb{Q}$ -Hodge structure, then  $\mathrm{MT}(V)$  and  $\mathrm{Hg}(V)$  are the Mumford–Tate and Hodge groups, respectively.
- $\mathbb{S}$  is the Deligne torus  $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}}$ .
- Given a number field  $F$ , we define the torus  $T_F := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$ .
- Given a CM or totally real number field  $F$ , we define the subtorus  $U_F \subseteq T_F$  by

$$U_F := \{x \in T_F : x\bar{x} = 1\},$$

where  $\bar{x}$  is complex conjugation when  $F$  is CM and the identity when  $F$  is totally real.

- Given an algebraic group  $G$ ,  $G^{\circ}$  denotes the connected component,  $Z(G)$  denotes its center, and  $G^{\mathrm{der}}$  denotes the derived subgroup.

Categories.

- For a field  $F$ ,  $\mathrm{Vec}_F$  is the category of vector spaces over  $F$ .
- $\mathrm{HS}_{\mathbb{Q}}$  is the category of  $\mathbb{Q}$ -Hodge structures.

## CHAPTER 2

# A LITTLE HODGE THEORY

---

*Once we explicitly know a Mumford-Tate group, we can let it work for us.*

—Moonen, [Moo, (5.5)]

In this chapter, we define the notion of a Hodge structure as well as some related groups (the Mumford–Tate group and the Hodge group). Our exposition follows Moonen’s unpublished notes [Moo; Moo99] and Lombardo’s master’s thesis [Lom13, Chapter 3]. Throughout, we find motivation from geometry (and in particular the cohomology of complex varieties), but this discussion is not technically necessary.

## 2.1 Hodge Structures

Cohomology of a variety frequently comes with some extra structure. On the étale site, we will later get significant utility of the fact that étale cohomology is a Galois representation. On the analytic site, the corresponding structure is called a “Hodge structure.”

### 2.1.1 Definition and Basic Properties

Here is our definition.

**Definition 2.1** (Hodge structure). A  $\mathbb{Q}$ -Hodge structure is a finite-dimensional vector space  $V \in \text{Vec}_{\mathbb{Q}}$  such that  $V_{\mathbb{C}}$  admits a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V_{\mathbb{C}}^{p,q}$$

where  $V_{\mathbb{C}}^{p,q} = \overline{V_{\mathbb{C}}^{q,p}}$ . For fixed  $m \in \mathbb{Z}$ , if  $V_{\mathbb{C}}^{p,q} \neq 0$  unless  $p+q = m$ , we say that  $V$  is *pure of weight  $m$* . We let  $\text{HS}_{\mathbb{Q}}$  denote the category of  $\mathbb{Q}$ -Hodge structures, where a morphism of Hodge structures is a linear map preserving the decomposition over  $\mathbb{C}$ . In the sequel, it may be helpful to note that one can bring this definition down to  $\mathbb{Z}$  as well.

**Example 2.2.** We give the “Tate twist”  $\mathbb{Q}(1) := 2\pi i \mathbb{Q}$  a Hodge structure of weight  $-2$  where the only nonzero entry is  $\mathbb{Q}(1)^{-1,-1} = \mathbb{Q}(1)$ .

**Example 2.3.** Given a complex projective smooth variety  $X$ , the Betti cohomology  $H_B^n(X, \mathbb{Q})$  admits a Hodge structure via the comparison isomorphisms: we find that

$$H_B^n(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X),$$

where  $H^{p,q}(X) := H^q(X, \Omega_{X/\mathbb{C}}^p)$ . (For technical reasons, one wants to say that  $H_B^n(X, \mathbb{Q})$  has weight  $-n$ .) This construction is even functorial: a morphism of complex projective smooth varieties  $\varphi: X \rightarrow Y$  induces a morphism of Hodge structures  $\varphi^*: H_B^n(Y, \mathbb{Q}) \rightarrow H_B^n(X, \mathbb{Q})$ .

Perhaps one would like to check that the category  $\text{HS}_{\mathbb{Q}}$  is abelian. The quickest way to do this is to realize  $\text{HS}_{\mathbb{Q}}$  as a category of representations of some group. The relevant group is the Deligne torus.

**Notation 2.4 (Deligne torus).** Let  $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$  denote the Deligne torus. We also let  $w: \mathbb{G}_{m,\mathbb{R}} \rightarrow \mathbb{S}$  denote the *weight cocharacter* given by  $w(r) := r \in \mathbb{C}$  on  $\mathbb{R}$ -points.

**Remark 2.5.** One can realize  $\mathbb{S}$  more concretely as

$$\mathbb{S}(R) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \text{GL}_2(R) : a^2 + b^2 \in R^\times \right\},$$

where  $R$  is an  $\mathbb{R}$ -algebra. Indeed, there is a ring isomorphism from  $R \otimes_{\mathbb{R}} \mathbb{C}$  to  $\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in R \right\}$  by sending  $1 \otimes 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $1 \otimes i \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . For example, one can define two characters  $z, \bar{z}: \mathbb{S}_{\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}$  given by  $z: \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a + bi$  and  $\bar{z}: \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a - bi$  so that  $(z, \bar{z})$  is an isomorphism  $\mathbb{S}_{\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}^2$ . Thus, the character group  $X^*(\mathbb{S})$  is a free  $\mathbb{Z}$ -module of rank 2 with basis  $\{z, \bar{z}\}$ , and the action of complex conjugation  $\iota \in \text{Gal}(\mathbb{C}/\mathbb{R})$  simply swaps  $z$  and  $\bar{z}$ .

**Example 2.6.** The following cocharacters of  $\mathbb{S}$  will be helpful.

- We define the *weight cocharacter*  $w: \mathbb{G}_{m,\mathbb{R}} \rightarrow \mathbb{S}$  given by  $w(r) := r \in \mathbb{C}$  on  $\mathbb{R}$ -points.
- We define  $\mu: \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$  given by  $\mu(z) := (z, 1)$  on  $\mathbb{C}$ -points.

Here is the relevance of  $\mathbb{S}$  to Hodge structures.

**Lemma 2.7.** Fix some  $V \in \text{Vec}_{\mathbb{Q}}$ . Then a Hodge structure on  $V$  has equivalent data to a representation  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ .

*Proof.* Remark 2.5 informs us that the character group  $X^*(\mathbb{S})$  of group homomorphisms  $\mathbb{S} \rightarrow \mathbb{G}_m$  is a rank-2 free  $\mathbb{Z}$ -module generated by  $z: \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a + bi$  and  $\bar{z}: \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a - bi$  on  $\mathbb{C}$ -points.<sup>1</sup> Without too many details, upon passing to the Hopf algebra, one is essentially looking for units in  $\mathbb{R} \left[ a, b, (a^2 + b^2)^{-1} \right]$ , of which there are not many. Note that there is a Galois action by  $\text{Gal}(\mathbb{C}/\mathbb{R})$  on these two characters  $\{z, \bar{z}\}$ , given by swapping them. Let  $\iota \in \text{Gal}(\mathbb{C}/\mathbb{R})$  denote complex conjugation, for brevity.

Now, a representation  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$  must have  $V_{\mathbb{C}}$  decompose into eigenspaces according to the characters  $X^*(\mathbb{S})$ , so one admits a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\chi \in X^*(\mathbb{S})} V_{\mathbb{C}}^{\chi}.$$

<sup>1</sup> Alternatively, note one has an isomorphism  $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^{\times} \cong \mathbb{C}^{\times} \times \mathbb{C}^{\times}$  by sending  $(z, w) \mapsto z \otimes w$ . Then these two characters are  $(z, w) \mapsto z$  and  $(z, w) \mapsto w$ .

However, one also needs  $V_{\mathbb{C}}^{\iota\chi} = \overline{V_{\mathbb{C}}^{\chi}}$  because  $\iota$  swaps  $\{\chi, \iota\chi\}$ . By Galois descent, this is enough data to (conversely) define a representation  $h: \mathbb{S} \rightarrow \text{Gal}(V)_{\mathbb{R}}$ .

To relate the previous paragraph to Hodge structures, we recall that  $X^*(\mathbb{S})$  is a rank-2 free  $\mathbb{Z}$ -module, so write  $\chi_{p,q} := z^{-p}\bar{z}^{-q}$  so that  $\iota\chi_{p,q} = \chi_{q,p}$ . Setting  $V_{\mathbb{C}}^{p,q} := V_{\mathbb{C}}^{\chi_{p,q}}$  now explains how to relate the previous paragraph to a Hodge structure, as desired. ■

**Remark 2.8.** The weight of a Hodge structure on some  $V \in \text{HS}_{\mathbb{Q}}$  can be read off of  $h$  as follows: note the weight cocharacter  $h \circ w$  equals the  $(-m)$ th power map if and only if the weight is  $m$ .

Thus, we see immediately the category  $\text{HS}_{\mathbb{Q}}$  is abelian. Additionally, representation theory explains how to take tensor products and duals.

**Example 2.9.** We see that  $V \in \text{HS}_{\mathbb{Q}}$  has  $V^{\vee}$  inherit a Hodge structure by setting  $(V^{\vee})^{p,q} := (V^{-p,-q})^{\vee}$ .

**Example 2.10.** We are now able to define the Tate twists  $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ , where negative powers indicates taking a dual. In particular, one can check that  $\mathbb{Q}(n) \otimes \mathbb{Q}(m) = \mathbb{Q}(n+m)$  for any  $n, m \in \mathbb{Z}$ .

**Notation 2.11.** For any Hodge structure  $V \in \text{HS}_{\mathbb{Q}}$  and integer  $m \in \mathbb{Z}$ , we may write

$$V(m) := V \otimes \mathbb{Q}(m).$$

### 2.1.2 Hodge Classes

We conclude this section by explaining one important application of Hodge structures.

**Definition 2.12** (Hodge class). Fix a  $\mathbb{Q}$ -Hodge structure  $V$ . A *Hodge class* of  $V$  is an element of  $V \cap V^{0,0}$ .

**Remark 2.13.** Looking at the construction in the proof of Lemma 2.7, we see that  $v \in V$  is a Hodge class if and only if it is fixed by the corresponding representation  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ .

**Example 2.14.** Fix a complex projective smooth variety  $X$  of dimension  $n$  and some even nonnegative integer  $2p \geq 0$ . Then one has Hodge classes given by elements of

$$H_{\mathbb{B}}^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)(p).$$

Now, any algebraic subvariety  $Z \subseteq X$  of codimension  $k$  defines a linear functional on  $H_{\text{dR}}^{2n-2k}(X, \mathbb{C})$  defined by

$$\omega \mapsto \int_Z \omega,$$

which one can check is supported on  $H^{k,k}$ . Thus, by Poincaré duality, one finds that  $Z$  produces a Hodge cycle in  $H_{\mathbb{B}}^{2k}(X, \mathbb{Q})$ .

In light of the above example, one has the following conjecture.

**Conjecture 2.15** (Hodge). Fix a complex projective smooth variety  $X$ . Then any Hodge class can be written as a linear combination of classes arising from algebraic subvarieties.

**Remark 2.16.** Here are some remarks on what is known about the Hodge conjecture, though it is admittedly little in this level of generality.

- The Hodge classes in  $H_B^2(X)(1)$  come from algebraic subvarieties.
- The cup product of any two classes arising from algebraic subvarieties continues to be Hodge and arises from algebraic subvarieties.

For example, if one can show that all Hodge classes are cup products of Hodge classes of codimension 1 on a variety  $X$ , then one knows the Hodge conjecture for  $X$ .

We are not interested in proving (cases of) the Hodge conjecture in this thesis, so we will not say much more.

### 2.1.3 Polarizations

Here is an important example of a morphism of Hodge structures.

**Definition 2.17 (polarization).** Fix a Hodge structure  $V \in \text{HS}_{\mathbb{Q}}$  pure of weight  $m$  given by the representation  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ . A *polarization* on  $V$  is a morphism  $\varphi: V \otimes V \rightarrow \mathbb{Q}(-m)$  of Hodge structures such that the induced bilinear form on  $V_{\mathbb{R}}$  given by

$$\langle v, w \rangle := (2\pi i)^m \varphi(h(i)v \otimes w)$$

is symmetric and positive-definite. If  $V$  admits a polarization, we may say that  $V$  is *polarizable*, and we let  $\text{HS}_{\mathbb{Q}}^{\text{pol}} \subseteq \text{HS}_{\mathbb{Q}}$  be the full subcategory of polarizable  $\mathbb{Q}$ -Hodge structures.

**Remark 2.18.** The positive-definiteness condition on  $\langle \cdot, \cdot \rangle$  implies that  $\varphi$  is non-degenerate. Indeed, one may check non-degeneracy upon base-changing to  $\mathbb{R}$  (because this is equivalent to inducing an isomorphism of vector spaces  $V \rightarrow V^{\vee}$ , which can be checked by fixing some  $\mathbb{Q}$ -bases and computing a determinant). Then we see that  $\langle \cdot, \cdot \rangle$  being non-degenerate implies that

$$\varphi(v \otimes w) = (2\pi i)^{-m} \langle h(-i)v, w \rangle$$

is non-degenerate because  $h(-i): V \rightarrow V$  is an isomorphism of vector spaces (because  $h(-i)^4 = \text{id}_V$ ).

**Remark 2.19.** The symmetry condition on  $\langle \cdot, \cdot \rangle$  implies a symmetry or alternating condition on  $\varphi$ . Indeed, we compute

$$\begin{aligned} \varphi(v \otimes w) &= (2\pi i)^{-m} \langle h(-i)v, w \rangle \\ &= (2\pi i)^{-m} \langle w, h(-i)v \rangle \\ &= \varphi(h(i)w \otimes h(-i)v) \\ &= h_{\mathbb{Q}(-m)}(i) \varphi(w \otimes h(-1)v) \\ &= 1 \varphi(w \otimes (-1)^m w) \\ &= (-1)^m \varphi(w \otimes v). \end{aligned}$$

Thus,  $\varphi$  is symmetric when  $m$  is even, and  $\varphi$  is alternating when  $m$  is odd.

Let's give some constructions of polarizable Hodge structures.

**Example 2.20.** It will turn out that  $H_B^1(A, \mathbb{Q})$  of any abelian variety  $A$  (over  $\mathbb{C}$ ) is polarizable, explaining the importance of this notion for our application. Because we are reviewing abelian varieties in chapter 3, we will not say more here.



**Example 2.21.** If  $V$  is polarizable and pure of weight  $m$ , then any Hodge substructure  $W \subseteq V$  is still polarizable (and pure of weight  $m$ ). Indeed, one can simply restrict the polarization to  $W$ , and all the checks go through. For example, positive-definiteness of  $\langle \cdot, \cdot \rangle$  means  $\langle v, v \rangle > 0$  for all nonzero  $v \in V$ , so the same will be true upon restricting to  $W$ .

**Example 2.22.** If  $V$  and  $W$  are polarizable and pure of weight  $m$ , then  $V \oplus W$  is also polarizable. Indeed, letting  $\varphi$  and  $\psi$  be polarizations on  $V$  and  $W$  respectively, we see that  $(\varphi \oplus \psi)$  defined by

$$(\varphi \oplus \psi)((v, w), (v', w')) := \varphi(v, v') + \psi(w, w')$$

succeeds at being a polarization: certainly it is a morphism of Hodge structures to  $\mathbb{Q}(-m-n)$ , and one can check that the corresponding bilinear form on  $V \oplus W$  simply splits into a sum of the forms on  $V$  and  $W$  and is therefore symmetric and positive-definite.

**Example 2.23.** If  $V$  and  $W$  are polarizable and pure of weights  $m$  and  $n$  respectively, then  $V \otimes W$  is also polarizable. Indeed, as in Example 2.22, let  $\varphi$  and  $\psi$  be polarizations on  $V$  and  $W$  respectively, and then we find that  $(\varphi \otimes \psi)$  can be defined on pure tensors by

$$(\varphi \otimes \psi)(v \otimes w, v' \otimes w') := \varphi(v, v')\psi(w, w').$$

One checks as before that this gives a polarization on  $V \otimes W$ : we certainly have a morphism of Hodge structures, and the corresponding bilinear form is the product of the bilinear forms on  $V$  and  $W$  and is therefore symmetric and positive-definite.

**Example 2.24.** If  $V$  is polarizable and pure of weight  $m$  with polarization  $\varphi$ , and  $W \subseteq V$  is a Hodge substructure (which is polarizable by Example 2.21), then we claim  $W^\perp$  (taken with respect to  $\langle \cdot, \cdot \rangle$ ) is also a Hodge substructure and hence polarizable by Example 2.21. Well, for any  $w' \in W^\perp$  and  $z \in \mathbb{S}(\mathbb{R})$ , we must check that  $h(z)w' \in W^\perp$ . For this, we note that any  $w \in W$  has

$$\begin{aligned} \langle w, h(z)w' \rangle &= (2\pi i)^{-m} \varphi(h(i)w \otimes h(z)w') \\ &= h_{\mathbb{Q}(-m)}(1/z)(2\pi i)^{-m} \varphi(h(i/z)w \otimes w') \\ &= h_{\mathbb{Q}(-m)}(1/z) \langle h(i/z)w, w' \rangle \\ &= 0, \end{aligned}$$

where the last equality holds because  $W \subseteq V$  is a Hodge substructure.

Note that one does not expect any Hodge substructure to have a complement, so Example 2.24 is a very important property of polarizations.

## 2.1.4 The Albert Classification

The presence of a polarization places strong restrictions on the endomorphisms of a Hodge structure. To explain how this works, we begin by reducing to the irreducible case: given a polarizable Hodge structure  $V \in \text{HS}_{\mathbb{Q}}$ , we begin by noting that  $V$  can be decomposed into irreducible Hodge substructures

$$V = \bigoplus_{i=1}^N V_i^{\oplus m_i},$$

where  $V_i$  is an irreducible Hodge structure (i.e., an irreducible representation of  $\mathbb{S}$ ) and  $m_i \geq 0$  is some nonnegative integer. Then standard results on endomorphisms of representations tell us that

$$\mathrm{End}_{\mathrm{HS}}(V) = \bigoplus_{i=1}^N M_{m_i}(\mathrm{End}_{\mathrm{HS}}(V_i)),$$

and Schur's lemma implies that  $\mathrm{End}_{\mathrm{HS}}(V_i)$  is a division algebra. The point of the above discussion is that we may reduce our discussion of endomorphisms to irreducible Hodge structures. We remark that polarizability of  $V$  implies that irreducible Hodge substructures continue to be polarizable by Example 2.21.

We are thus interested in classifying what algebras may appear as  $\mathrm{End}_{\mathrm{HS}}(V)$  for irreducible Hodge structures  $V \in \mathrm{HS}_{\mathbb{Q}}$ . To this end, we note that  $\mathrm{End}_{\mathrm{HS}}(V)$  comes with some extra structure.

**Definition 2.25 (Rosati involution).** Let  $\varphi$  be a polarization on a Hodge structure  $V \in \mathrm{HS}_{\mathbb{Q}}$ . The *Rosati involution* is the function  $(\cdot)^{\dagger}: \mathrm{End}_{\mathbb{Q}}(V) \rightarrow \mathrm{End}_{\mathbb{Q}}(V)$  defined by

$$\varphi(dv \otimes w) = \varphi(v \otimes d^{\dagger}w)$$

for all  $d \in \mathrm{End}_{\mathrm{HS}}(V)$  and  $v, w \in V$ .

**Remark 2.26.** In light of Remark 2.18, we see that  $d^{\dagger}$  is simply the adjoint of  $d: V \rightarrow V$  associated to  $\varphi$  viewed as a non-degenerate bilinear pairing. For example, we immediately see that  $(\cdot)^{\dagger}$  induces a well-defined linear operator  $\mathrm{End}_{\mathbb{Q}}(V) \rightarrow \mathrm{End}_{\mathbb{Q}}(V)$ .

Here are the important properties of the Rosati involution.

**Lemma 2.27.** Fix a Hodge structure  $V \in \mathrm{HS}_{\mathbb{Q}}$  pure of weight  $m$  with polarization  $\varphi$  and associated Rosati involution  $(\cdot)^{\dagger}$ .

- (a) If  $d \in \mathrm{End}_{\mathrm{HS}}(V)$ , then  $d^{\dagger} \in \mathrm{End}_{\mathrm{HS}}(V)$ .
- (b) Anti-involution: for any  $d, e \in \mathrm{End}_{\mathbb{Q}}(V)$ , we have  $d^{\dagger\dagger} = d$  and  $(de)^{\dagger} = e^{\dagger}d^{\dagger}$ .
- (c) Positive: for any nonzero  $d \in \mathrm{End}_{\mathbb{Q}}(V)$ , we have  $\mathrm{tr} \, dd^{\dagger} > 0$ .

*Proof.* We show the claims in sequence.

- (a) This follows because  $\varphi$  is a morphism of Hodge structures. Formally, we would like to check that  $d^{\dagger}$  commutes with the action of  $\mathbb{S}$ . Let  $h: \mathbb{S} \rightarrow \mathrm{GL}(V)_{\mathbb{R}}$  be the representation corresponding to the Hodge structure. Well, for any  $g \in \mathbb{S}(\mathbb{C})$  and  $v, w \in V$ , we compute

$$\begin{aligned} \varphi(v \otimes d^{\dagger}h(g)w) &= \varphi(dv \otimes h(g)w) \\ &= h_{\mathbb{Q}(-m)}(g)\varphi(h(g^{-1})dv \otimes w) \\ &\stackrel{*}{=} h_{\mathbb{Q}(-m)}(g)\varphi(dh(g^{-1})v \otimes w) \\ &= h_{\mathbb{Q}(-m)}(g)\varphi(h(g^{-1})v \otimes d^{\dagger}w) \\ &= \varphi(v \otimes h(g)d^{\dagger}w) \end{aligned}$$

where  $\stackrel{*}{=}$  holds because  $d$  is a morphism of Hodge structures. The non-degeneracy of  $\varphi$  given in Remark 2.18 now implies that  $d^{\dagger}h(g) = h(g)d^{\dagger}$ , so we are done.

- (b) This is a purely formal property of adjoints.

- (c) The point is to reduce this to the case where  $V$  is a matrix algebra over  $\mathbb{R}$  and  $(\cdot)^\dagger$  is the transpose. Indeed, this positivity can be checked after a base-change to  $\mathbb{R}$ . As such, we let  $\langle \cdot, \cdot \rangle$  be the symmetric positive-definite bilinear form associated to  $\varphi$  defined by

$$\langle v, w \rangle := (2\pi i)^{-m} \varphi(h(i)v \otimes w)$$

for any  $v, w \in V_{\mathbb{R}}$ . We thus see that  $(\cdot)^\dagger$  is also the adjoint operator with respect to  $\langle \cdot, \cdot \rangle$ : we know

$$(2\pi i)^{-m} \langle h(i)dv, w \rangle = (2\pi i)^{-m} \langle h(i)v, d^\dagger w \rangle$$

for any  $v, w \in V_{\mathbb{R}}$ , which is equivalent to always having  $\langle dv, w \rangle = \langle v, d^\dagger w \rangle$ . Now, we may fix an orthonormal basis of  $V_{\mathbb{R}}$  with respect to  $\langle \cdot, \cdot \rangle$  so that  $\text{End}_{\mathbb{R}}(V_{\mathbb{R}})$  is identified with  $M_n(\mathbb{R}^{\dim V})$  and  $(\cdot)^\dagger$  is identified with the transpose. Then  $\text{tr } dd^\dagger$  is the sum of the squares of the matrix entries of  $d$  and is therefore positive when  $d$  is nonzero. ■

We are now ready to state the Albert classification, which classifies division algebras over  $\mathbb{Q}$  equipped with a positive anti-involution.

**Theorem 2.28 (Albert classification).** Let  $D$  be a division algebra over  $\mathbb{Q}$  equipped with a Rosati involution  $(\cdot)^\dagger: D \rightarrow D$ . Further, let  $F$  be the center of  $D$ , and let  $F^\dagger$  be the subfield fixed by  $(\cdot)^\dagger$ . Then  $D$  admits exactly one of the following types.

- Type I:  $D$  is a totally real number field so that  $D = F = F^\dagger$ , and  $(\cdot)^\dagger$  is the identity.
- Type II:  $D$  is a totally indefinite quaternion division algebra over  $F$  where  $F = F^\dagger$ , and  $(\cdot)^\dagger$  corresponds to the transpose on  $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ .
- Type III:  $D$  is a totally definite quaternion division algebra over  $F$  where  $F = F^\dagger$ , and  $(\cdot)^\dagger$  corresponds to the canonical involution on  $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$  (where  $\mathbb{H}$  is the quaternions).
- Type IV:  $D$  is a division algebra over  $F$ , where  $F$  is a totally imaginary quadratic extension of  $F^\dagger$ , and  $(\cdot)^\dagger$  is the complex conjugation automorphism of  $F$ .

*Proof.* This is a rather lengthy computation. We refer to [Mum74, Section 21, Application I]. ■

## 2.2 Monodromy Groups

In this section, we define the Mumford–Tate group and the Hodge group.

### 2.2.1 The Mumford–Tate Group

We are now ready to define the Mumford–Tate group. Intuitively, it is the monodromy group of the associated representation of a Hodge structure.

**Definition 2.29 (Mumford–Tate group).** For some  $V \in \text{HS}_{\mathbb{Q}}$ , the *Mumford–Tate group*  $\text{MT}(V)$  is the smallest algebraic  $\mathbb{Q}$ -group containing the image of the corresponding representation  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ .

**Remark 2.30.** Because  $\mathbb{S}$  is connected, we see that  $h$  is also connected. Namely,  $\text{MT}(V)^\circ \subseteq \text{MT}(V)$  will be an algebraic  $\mathbb{Q}$ -group containing the image of  $h$  if  $\text{MT}(V)$  does too, so equality is forced.

**Example 2.31.** Suppose that  $V \in \text{HS}_{\mathbb{Q}}$  is pure of weight  $m$ .

- If  $m = 0$ , then we claim that  $\text{MT}(V) \subseteq \text{SL}(V)$ . It is enough to check that  $h$  outputs into  $\text{SL}(V)$ .
- If  $m \neq 0$ , then we claim that  $\text{MT}(V)$  contains  $\mathbb{G}_{m, \mathbb{Q}}$ . It is enough to check that  $\text{MT}(V)_{\mathbb{C}}$  contains  $\mathbb{G}_{m, \mathbb{C}}$ . Well, for any  $z \in \mathbb{C}$   $h(z, \bar{z})$  acts on the component  $V^{p, q} \subseteq V_{\mathbb{C}}$  by  $z^{-p} \bar{z}^{-q} = z^{-m}$ , so  $\text{MT}(V)_{\mathbb{C}}$  must contain the scalar  $z^{-m}$  for all  $z \in \mathbb{C}$ . The conclusion follows.

Because Hodge structures are defined after passing to  $\mathbb{C}$ , it will be helpful to have a definition of  $\text{MT}(V)$  as a monodromy group corresponding to a morphism over  $\mathbb{C}$ .

**Lemma 2.32.** Fix  $V \in \text{HS}_{\mathbb{Q}}$ , and let  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$  be the corresponding representation. Then  $\text{MT}(V)$  is the smallest algebraic  $\mathbb{Q}$ -subgroup of  $\text{GL}(V)$  such that  $\text{MT}(V)_{\mathbb{C}}$  contains the image of  $h_{\mathbb{C}} \circ \mu$ .

*Proof.* Let  $M'$  be the smallest algebraic  $\mathbb{Q}$ -subgroup of  $\text{GL}(V)$  containing  $h_{\mathbb{C}} \circ \mu$ . We want to show that  $M' = M$ .

- To show  $M' \subseteq \text{MT}(V)$ , we must show that  $\text{MT}(V)_{\mathbb{C}}$  contains the image of  $h_{\mathbb{C}} \circ \mu$ . Well,  $\text{MT}(V)_{\mathbb{R}}$  contains the image of  $h$ , so  $\text{MT}(V)_{\mathbb{C}}$  contains the image of  $h_{\mathbb{C}}$ , which contains the image of  $h_{\mathbb{C}} \circ \mu$ .
- Showing  $\text{MT}(V) \subseteq M'$  is a little harder. We must show that  $M'$  contains the image of  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ . It is enough to check that  $M'$  contains the image of  $h_{\mathbb{C}}$  because then we can descend everything to  $\mathbb{R}$ , and because  $\mathbb{C}$  is algebraically closed, we see that  $\mathbb{C}$ -points are certainly dense enough so that it is enough to check that  $M'(\mathbb{C})$  contains the image  $h(\mathbb{S}(\mathbb{C}))$ .

The point is that  $M'$  is defined over  $\mathbb{Q}$ , so  $M'_{\mathbb{C}}$  is stable under the action of complex conjugation, which we denote by  $\iota$ . Similarly,  $h$  being defined over  $\mathbb{R}$  guarantees that it commutes with complex conjugation. In particular, we already know that  $M'$  contains the points of the form  $h(z, 1)$  for  $(z, 1) \in \mathbb{S}(\mathbb{C})$ . Thus, we see that  $M'$  also contains the points

$$\iota(h(z, 1)) = h(\iota(z, 1)) = h(1, z)$$

because everything is defined over  $\mathbb{R}$ . (This last equality follows by tracking through the action of  $\iota$  on  $\mathbb{S}(\mathbb{C})$ .) We conclude that  $M'$  contains  $h(z, w)$  for any  $(z, w) \in \mathbb{S}(\mathbb{C})$ , so we are done. ■

Roughly speaking, the point of the group  $\text{MT}(V)$  is that  $\text{MT}(V)$  is an algebraic  $\mathbb{Q}$ -group remembering everything one wants to know about the Hodge structure. One way to rigorize this is as follows.

**Proposition 2.33.** Fix  $V \in \text{HS}_{\mathbb{Q}}$ . Suppose  $T \in \text{HS}_{\mathbb{Q}}$  can be written as

$$T = \bigoplus_{i=1}^N (V^{\otimes m_i} \otimes (V^{\vee})^{\otimes n_i}),$$

where  $m_i, n_i \geq 0$  are nonnegative integers. Then  $W \subseteq T$  is a Hodge substructure if and only if the action of  $\text{MT}(V)$  on  $T$  stabilizes  $W$ .

*Proof.* For each  $W \in \text{HS}_{\mathbb{Q}}$ , we let  $h_W$  denote the corresponding representation. In the backwards direction, we note that  $\text{MT}(V)$  stabilizing  $W$  implies that  $h(s)$  stabilizes  $W_{\mathbb{R}}$  for any  $s$ . We can thus view  $W_{\mathbb{R}} \subseteq T_{\mathbb{R}}$  as a subrepresentation of  $\mathbb{S}$ , so taking eigenspaces reveals that  $W$  can be given the structure of a Hodge substructure of  $T$ .

The converse will have to use the construction of  $T$ . Indeed, suppose that  $W \subseteq T$  is a Hodge substructure, and let  $M \subseteq \text{GL}(V)$  be the smallest algebraic  $\mathbb{Q}$ -group stabilizing  $W \subseteq T$ . We would like to show that  $\text{MT}(V) \subseteq M$ . By definition of  $\text{MT}(V)$ , it is enough to show that  $h$  factors through  $M_{\mathbb{R}}$ , meaning we must show that  $h(s)$  stabilizes  $W$  for each  $s \in \mathbb{S}$ . Well,  $h(s)$  will act by characters on the eigenspaces  $W_{\mathbb{C}}^{p, q} \subseteq W_{\mathbb{C}}$ , so  $h(s)$  does indeed stabilize  $W$ . ■

**Corollary 2.34.** Fix  $V \in \text{HS}_{\mathbb{Q}}$ . Suppose  $T \in \text{HS}_{\mathbb{Q}}$  can be written as

$$T = \bigoplus_{i=1}^N (V^{\otimes m_i} \otimes (V^{\vee})^{\otimes n_i}),$$

where  $m_i, n_i \geq 0$  are nonnegative integers. Then  $t \in T$  is a Hodge class if and only if it is fixed by  $\text{MT}(V)$ .

*Proof.* We apply Proposition 2.33 to  $\mathbb{Q}(0) \oplus T$ . Then we note that  $\text{span}_{\mathbb{Q}}\{(1, t)\} \subseteq \mathbb{Q}(0) \oplus T$  is a Hodge substructure if and only if it is preserved by  $\text{MT}(V)$ . We now tie each of these to the statement.

- On one hand, we see that being a one-dimensional Hodge substructure implies that  $(1, t)$  must have bidegree  $(p, p)$  for some  $p \in \mathbb{Z}$ , but we have to live in  $(0, 0)$  because our 1 lives in  $\mathbb{Q}(0)$ . Thus, this is equivalent to being a Hodge class.
- On the other hand, being preserved by  $\text{MT}(V)$  implies that  $\text{MT}(V)$  acts by scalars on  $(1, t)$ , but  $\text{MT}(V)$  acts trivially on  $\mathbb{Q}(0)$ , so all the relevant scalars must be 1. Thus, this is equivalent to being fixed by  $\text{MT}(V)$ . ■

We thus see that understanding the Mumford–Tate group is important from the perspective of the Hodge conjecture (Conjecture 2.15). It will be helpful to note that this characterizes  $\text{MT}(V)$  in some cases.

**Proposition 2.35.** Fix a field  $k$  of characteristic 0. Let  $H \subseteq \text{GL}_{n,k}$  be a reductive subgroup. Suppose  $H'$  is the algebraic  $\mathbb{Q}$ -subgroup of  $\text{GL}_{n,k}$  defined by fixing all  $H$ -invariants occurring in any tensor representation

$$T = \bigoplus_{i=1}^N (V^{\otimes m_i} \otimes (V^{\vee})^{\otimes n_i}),$$

where  $m_i, n_i \geq 0$  are nonnegative integers. Then  $H = H'$ .

*Proof.* Note  $H \subseteq H'$  is automatic, so the main content comes from proving the other inclusion. Proving this would step into the (rather deep) theory of algebraic groups, which we will avoid. Instead, we will mention that the key input is Chevalley’s theorem, which asserts that any subgroup  $H$  of  $G$  is the stabilizer of some line in some representation of  $G$ . We refer to [Del18, Proposition 3.1]; see also [Mil17, Theorem 4.27]. ■

**Corollary 2.36.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  such that  $\text{MT}(V)$  is reductive. Then  $\text{MT}(V)$  is exactly the algebraic  $\mathbb{Q}$ -subgroup of  $\text{GL}(V)$  fixing all Hodge classes.

*Proof.* Corollary 2.34 explains that the Hodge classes are exactly the vectors fixed by  $\text{MT}(V)$ , so this follows from Proposition 2.35. ■

**Remark 2.37.** Corollary 2.36 is true without a reductivity assumption (see [Del18, Proposition 3.4]), but we will not need this in our applications. (On the other hand, one does not expect Proposition 2.35 to be true without any assumptions on  $H$ .) Namely, we will be interested in abelian varieties, whose Hodge structures are polarizable by Example 2.20, and we will shortly see that this implies that  $\text{MT}(V)$  is reductive in Lemma 2.44.

### 2.2.2 The Hodge Group

In computational applications, it will be frequently be easier to compute a smaller monodromy group related to  $\text{MT}(V)$ .

**Definition 2.38 (Hodge group).** Fix  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight. Then the *Hodge group*  $\text{Hg}(V)$  is the smallest algebraic  $\mathbb{Q}$ -subgroup  $\text{GL}(V)$  containing the image of  $h|_{\mathbb{U}}$ , where  $\mathbb{U} \subseteq \mathbb{S}$  is defined as the kernel of the norm character  $z\bar{z}: \mathbb{S} \rightarrow \mathbb{G}_{m,\mathbb{R}}$ .

**Remark 2.39.** Even though  $z$  and  $\bar{z}$  are only defined as characters  $\mathbb{S}_{\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}$ , the norm character  $z\bar{z}$  is defined as a character  $\mathbb{S} \rightarrow \mathbb{G}_{m,\mathbb{R}}$  because it is fixed by complex conjugation. For example, we see that

$$\mathbb{U}(\mathbb{R}) = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus, we see that  $\mathbb{U}$  stands for “unit circle.” While we’re here, we remark that  $\mathbb{U}(\mathbb{C}) \subseteq \mathbb{S}(\mathbb{C})$  is identified with the subset  $\{(z, 1/z) : z \in \mathbb{C}^{\times}\}$ .

**Remark 2.40.** The same argument as in Remark 2.30 shows that the connectivity of  $\mathbb{U}$  implies the connectivity of  $\text{Hg}(V)$ .

Intuitively,  $\text{Hg}(V)$  removes the scalars that might live in  $\text{MT}(V)$  by Example 2.31. These scalars are an obstruction to  $\text{MT}(V)$  being a semisimple group, and we will see in Proposition 2.73 that  $\text{Hg}(V)$  will thus frequently succeed at being semisimple. Let’s rigorize this discussion.

**Lemma 2.41.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  pure of weight  $m$ , and let  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$  be the corresponding representation.

(a) We have  $\text{Hg}(V) \subseteq \text{SL}(V)$ .

(b) Thus,

$$\text{MT}(V) = \begin{cases} \text{Hg}(V) & \text{if } m = 0, \\ \mathbb{G}_{m,\mathbb{Q}} \text{Hg}(V) & \text{if } m \neq 0, \end{cases}$$

where the almost direct product in the second case is given by embedding  $\mathbb{G}_{m,\mathbb{Q}} \rightarrow \text{GL}(V)$  via scalars.

*Proof.* We show the claims in sequence.

(a) It is enough to check that  $\text{SL}(V)$  contains the image of  $h|_{\mathbb{U}}$ . In other words, we want to check that  $\det h(z) = 1$  for all  $z \in \mathbb{U}(\mathbb{R})$ . By extending scalars, it is enough to compute the determinant as an operator on  $V_{\mathbb{C}}$ . For this, we note that  $h(z)$  acts on the component  $V^{p,q} \subseteq V_{\mathbb{C}}$  by the scalar  $z^{-p}\bar{z}^{-q}$ , so the determinant of  $h(z)$  acting on  $V^{p,q} \oplus V^{q,p}$  is

$$(z^{-p}\bar{z}^{-q})^{\dim V^{p,q}} \cdot (z^{-q}\bar{z}^{-p})^{\dim V^{q,p}} = (z\bar{z})^{-(p+q)\dim V^{p,q}}$$

because  $\dim V^{p,q} = \dim V^{q,p}$ . This simplifies to  $(z\bar{z})^{-\frac{1}{2}m \dim(V^{p,q} \oplus V^{q,p})}$  because  $V$  is pure of weight  $m$ , so the result follows by summing over all pairs  $(p, q)$ .<sup>2</sup>

(b) Before doing anything serious, we remark that  $\mathbb{G}_{m,\mathbb{Q}} \text{Hg}(V)$  is in fact an almost direct product. Namely, we should check that the intersection  $\mathbb{G}_{m,\mathbb{Q}} \cap \text{Hg}(V)$  is finite (even over  $\mathbb{C}$ ). Well, by (a),  $\text{Hg}(V) \subseteq \text{SL}(V)$ . Thus, it is enough to notice that  $\mathbb{G}_{m,\mathbb{Q}} \cap \text{SL}(V)$  is finite because  $V$  is finite-dimensional over  $\mathbb{C}$ : over  $\mathbb{C}$ ,

<sup>2</sup> If  $m$  is even, this argument does not work verbatim for the component  $(m/2, m/2)$ . Instead, one can compute the determinant of  $h(z)$  acting on  $V^{m/2, m/2}$  directly as  $(z\bar{z})^{-\frac{1}{2}m \dim V^{m/2, m/2}}$ .

the intersection consists of scalar matrices  $\lambda \text{id}_V$  such that  $\lambda^{\dim V} = 1$ , so the intersection is the finite algebraic group  $\mu_{\dim V}$ .

We now proceed with the argument. Because  $\mathbb{U} \subseteq \mathbb{S}$ , we of course have  $\text{Hg}(V) \subseteq \text{MT}(V)$ , and if  $m \neq 0$ , then Example 2.31 implies that  $\mathbb{G}_{m,\mathbb{Q}} \subseteq \text{MT}(V)$  so that  $\mathbb{G}_{m,\mathbb{Q}} \text{Hg}(V) \subseteq \text{MT}(V)$ . It is therefore enough to check the given equalities after base-changing to  $\mathbb{R}$ . Namely, using Lemma 2.32, we should check that  $\text{Hg}(V)(\mathbb{C})$  contains the image of  $h_{\mathbb{C}} \circ \mu$  when  $m = 0$ , and  $\mathbb{C}^\times \text{Hg}(V)(\mathbb{C})$  contains the image of  $h_{\mathbb{C}} \circ \mu$  when  $m \neq 0$ . Well, for any  $z \in \mathbb{C}^\times$ , we may write  $z = re^{i\theta}$  where  $r \in \mathbb{R}^+$  and  $\theta \in \mathbb{R}$ . Then we compute

$$\begin{aligned} h(\mu(z)) &= h(z, 1) \\ &= h(re^{i\theta}, 1) \\ &= h\left(\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{-i\theta/2}\right) h\left(\sqrt{r}e^{i\theta/2}, \frac{1}{\sqrt{r}e^{i\theta/2}}\right). \end{aligned}$$

Now,  $h\left(\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{-i\theta/2}\right)$  is a scalar as computed in Example 2.31, and  $\left(\sqrt{r}e^{i\theta/2}, \frac{1}{\sqrt{r}e^{i\theta/2}}\right)$  lives in  $\mathbb{U}(\mathbb{C}) = \{(z, w) : zw = 1\}$ . Thus, we see that  $h(\mu(z))$  is certainly contained in  $\mathbb{C}^\times \text{Hg}(V)(\mathbb{C})$ , completing the proof in the case  $m \neq 0$ . In the case where  $m = 0$ , the scalar  $h\left(\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{-i\theta/2}\right)$  is actually the identity, so we see that  $h(\mu(z)) \in \text{Hg}(V)(\mathbb{C})$ . ■

It is worthwhile to note that there is also a tensor characterization of  $\text{Hg}(V)$ .

**Proposition 2.42.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight. Suppose  $T \in \text{HS}_{\mathbb{Q}}$  is of pure weight  $n$  and can be written as

$$T = \bigoplus_{i=1}^N (V^{\otimes m_i} \otimes (V^\vee)^{\otimes n_i}),$$

where  $m_i, n_i \geq 0$  are nonnegative integers. Then  $W \subseteq T$  is a Hodge substructure if and only if the action of  $\text{Hg}(V)$  on  $T$  stabilizes  $W$ .

*Proof.* Of course, if  $W \subseteq T$  is a Hodge substructure, then  $W$  is preserved by the action of  $\text{MT}(V)$ , so  $W$  will be preserved by the action of  $\text{Hg}(V) \subseteq \text{MT}(V)$ .

Conversely, if  $\text{Hg}(V)$  stabilizes  $W$ , then we would like to show that  $W \subseteq T$  is a Hodge substructure, which by Proposition 2.33 is the same as showing that  $\text{MT}(V)$  stabilizes  $W$ . For this, we use Lemma 2.41, which tells us that  $\text{MT}(V) \subseteq \mathbb{G}_{m,\mathbb{Q}} \text{Hg}(V)$ . Namely, because  $\text{Hg}(V)$  already stabilizes  $W$ , it is enough to note that of course the scalars  $\mathbb{G}_{m,\mathbb{Q}}$  stabilize the subspace  $W \subseteq T$ . ■

**Corollary 2.43.** Fix an irreducible Hodge structure  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight. Observe that the inclusion  $\text{Hg}(V) \subseteq \text{GL}(V)$  makes  $V$  into a representation of  $\text{Hg}(V)$ . Then  $V$  is irreducible as a representation of  $\text{Hg}(V)$ .

*Proof.* By Proposition 2.42, a  $\text{Hg}(V)$ -submodule is a Hodge substructure, but there are no nonzero proper Hodge substructures because  $V$  is an irreducible Hodge structure. ■

## 2.3 Computational Tools

In this section, we provide some discussion which will help the computations used later in this thesis.

### 2.3.1 Bounding with Known Classes

Here, we use endomorphisms and the polarization to bound the size of  $\text{MT}(V)$  and  $\text{Hg}(V)$ .

**Lemma 2.44.** Fix a polarizable Hodge structure  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight. Then  $\text{MT}(V)$  and  $\text{Hg}(V)$  are reductive.

*Proof.* By [Mil17, Corollary 19.18], it is enough to find faithful semisimple representations of  $\text{MT}(V)$  and  $\text{Hg}(V)$ . We claim that the inclusions  $\text{MT}(V) \subseteq \text{GL}(V)$  and  $\text{Hg}(V) \subseteq \text{GL}(V)$  provide this representation: certainly this representation is faithful, and it is faithful because any subrepresentation is a Hodge substructure by Propositions 2.33 and 2.42. ■

**Lemma 2.45.** Fix  $V \in \text{HS}_{\mathbb{Q}}$ . Let  $D := \text{End}_{\text{HS}}(V)$  be the endomorphism algebra of  $V$ . Then  $\text{MT}(V)$  is an algebraic  $\mathbb{Q}$ -subgroup of

$$\text{GL}_D(V) := \{g \in \text{GL}(V) : g \circ d = d \circ g \text{ for all } d \in D\}.$$

*Proof 1.* Noting that  $\text{GL}_D(V)$  is an algebraic  $\mathbb{Q}$ -group (it is a subgroup of  $\text{GL}(V)$  cut out by the equations given by commuting with a basis of  $D$ ), it is enough to show that  $\text{GL}_D(V)$  contains the image of the representation  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ . Well, by definition  $D$  consists of morphisms commuting with the action of  $\mathbb{S}$ , so the image of  $h$  must commute with  $D$ . ■

*Proof 2.* Motivated by Corollary 2.36, one expects to find Hodge classes corresponding to the condition of commuting with  $D$ . Well, there is a canonical isomorphism  $V \otimes V^{\vee} \rightarrow \text{End}_{\mathbb{Q}}(V)$  of  $\mathbb{S}$ -representations, so by tracking through how representations of  $\mathbb{S}$  correspond to Hodge structures, we see that  $f: V \rightarrow V$  preserves the Hodge structure if and only if it is fixed by  $\mathbb{S}$ , which is equivalent to the corresponding element  $f \in V \otimes V^{\vee}$  being fixed by  $\mathbb{S}$ , which is equivalent to  $f$  being a Hodge class by Remark 2.13. This completes the proof of the lemma upon comparing with Corollary 2.34. ■

**Remark 2.46.** Of course, we also have  $\text{Hg}(V) \subseteq \text{GL}_D(V)$  because  $\text{Hg}(V) \subseteq \text{MT}(V)$ .

**Lemma 2.47.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  pure of weight  $m$  with polarization  $\varphi$ . Then  $\text{MT}(V)$  is an algebraic  $\mathbb{Q}$ -subgroup of

$$\text{GSp}(\varphi) := \{g \in \text{GL}(V) : \varphi(gv \otimes gw) = \lambda(g)\varphi(v \otimes w) \text{ for fixed } \lambda(g) \in \mathbb{Q}\}.$$

*Proof 1.* Once again, we note that  $\text{GSp}(\varphi)$  is an algebraic  $\mathbb{Q}$ -group cut out by equations of the form

$$\varphi(gv \otimes gw)\varphi(v' \otimes w') = \varphi(v \otimes w)\varphi(gv' \otimes gw')$$

as  $v, w, v', w' \in V$  varies over a basis. Thus, it is enough to check that  $\text{GSp}(\varphi)$  contains the image of  $h: \mathbb{S} \rightarrow \text{GL}(V)_{\mathbb{R}}$ . Well, for any  $z \in \mathbb{S}(\mathbb{R})$ , we note that

$$\varphi(h(z) \otimes h(z)w) = h_{\mathbb{Q}(-m)}(z)\varphi(v \otimes w)$$

for any  $v, w \in V_{\mathbb{R}}$  because  $\varphi$  is a morphism of Hodge structures. ■

*Proof 2.* Once again, Corollary 2.36 tells us to expect the polarization to produce a Hodge class corresponding to the above equations cutting out  $\text{MT}(V)$ .

This construction is slightly more involved. We begin by constructing two Hodge classes.

- Note  $\varphi: V \otimes V \rightarrow \mathbb{Q}(-m)$  is a morphism of Hodge structures, so it is an  $\mathbb{S}$ -invariant map and hence given by an  $\mathbb{S}$ -invariant element of  $V^{\vee} \otimes V^{\vee}(-m)$ . Thus,  $\varphi \in V^{\vee} \otimes V^{\vee}(-m)$  is a Hodge class by Remark 2.13.



- Because  $\varphi$  is non-degenerate, it induces an isomorphism  $V(m) \rightarrow V^\vee$ . Now,  $\text{End}_{\mathbb{Q}}(V)$  is canonically isomorphic to  $V \otimes V^\vee$ , which we now see is isomorphic (via  $\varphi$ ) to  $V \otimes V(m)$ . We let  $C \in V \otimes V(m)$  be the image of  $\text{id}_V \in \text{End}_{\mathbb{Q}}(V)^\mathbb{S}$  in  $V \otimes V(m)$ , which we note is a Hodge class again by Remark 2.13. (Here,  $C$  stands for “Casimir.”)

In total, we see that we have produced a Hodge class  $C \otimes \varphi$ . It remains to show that  $g \in \text{GL}(V)$  fixing  $C \otimes \varphi$  implies that  $g \in \text{GSp}(\varphi)$ , which will complete the proof by Corollary 2.34.

Well, suppose  $g(C \otimes \varphi) = C \otimes \varphi$ . Note  $g(C \otimes \varphi) = gC \otimes g\varphi$ , which can only equal  $C \otimes \varphi \in (V \otimes V) \otimes_{\mathbb{Q}} (V^\vee \otimes V^\vee)$  if there is a scalar  $\lambda \in \mathbb{Q}^\times$  such that  $gC = \lambda C$  and  $g\varphi = \lambda^{-1}\varphi$ . This second condition amounts to requiring

$$\varphi(g^{-1}v \otimes g^{-1}w) = \lambda^{-1}\varphi(v \otimes w)$$

for any  $v, w \in V$ , which rearranges into  $g \in \text{GSp}(\varphi)$ . ■

**Remark 2.48.** The construction given in the above proof is described in [GGL24, Remark 8.3.4]. They also show the converse claim that any  $g \in \text{GSp}(\varphi)$  fixes  $C \otimes \varphi$ .

To see this, one has to do an explicit computation with  $C$ . For this, let  $\{v_1, \dots, v_n\}$  be a basis of  $V$ , and  $\{v_1^*, \dots, v_n^*\}$  be the dual basis of  $V(m)$  taken with respect to  $\varphi$ . Then  $C = \sum_{i=1}^n v_i \otimes v_i^*$ . Similarly, we see that  $\{gv_1, \dots, gv_n\}$  is a basis of  $V$  with a dual basis  $\{(gv_1)^*, \dots, (gv_n)^*\}$  so that  $C = \sum_{i=1}^n (gv_i) \otimes (gv_i)^*$ . Now, on one hand, if  $g$  has multiplier  $\lambda$ , then  $g\varphi = \lambda^{-1}\varphi$ . On the other hand,  $\varphi(gv_i, gv_j^*) = \lambda 1_{i=j}$ , so  $(gv_i)^* = \lambda^{-1}gv_i^*$ , which allows us to compute  $gC = \lambda C$ . In total,  $g(C \otimes \varphi) = C \otimes \varphi$ .

**Remark 2.49.** One can check that  $\text{GSp}(\varphi)$  does not depend on the choice of polarization. Roughly speaking, the point is that the choice of a different polarization amounts to some choice of an element in  $D^\times$  which we can track through.

In light of the above two lemmas, we pick up the following notation.

**Notation 2.50.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  pure of weight  $m$  with  $D := \text{End}_{\text{HS}}(V)$  and polarization  $\varphi$ . Then we define

$$\text{GSp}_D(\varphi) := \text{GL}_D(V) \cap \text{GSp}(\varphi).$$

By Lemmas 2.45 and 2.47, we see that  $\text{MT}(V) \subseteq \text{GSp}_D(\varphi)$ .

**Remark 2.51.** In “most cases,” we expect that generic Hodge structures  $V$  should have the equality  $\text{MT}(V) = \text{GL}_D(V)$ , and if  $V$  admits a polarization  $\varphi$ , then we expect the equality  $\text{MT}(V) = \text{GSp}_D(\varphi)$ . To rigorize this intuition, one must discuss Shimura varieties, which we will avoid doing for now.

We can also apply Lemmas 2.45 and 2.47 to bound  $\text{Hg}(V)$ .

**Notation 2.52.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  pure of weight  $m$  with  $D := \text{End}_{\text{HS}}(V)$  with polarization  $\varphi$ . Then we define

$$\text{Sp}(\varphi) := \{g \in \text{GL}(V) : \varphi(gv \otimes gw) = \varphi(v \otimes w)\},$$

and

$$\text{Sp}_D(\varphi) := \text{GL}_D(V) \cap \text{Sp}(\varphi).$$

**Remark 2.53.** Let's explain why  $\mathrm{Hg}(V) \subseteq \mathrm{Sp}_D(\varphi)$ . By Lemma 2.45, we see that  $\mathrm{Hg}(V) \subseteq \mathrm{MT}(V) \subseteq \mathrm{GL}_D(V)$ , so it remains to check that  $\mathrm{Hg}(V) \subseteq \mathrm{Sp}(\varphi)$ . Proceeding as in Lemma 2.47, it is enough to check that the image of  $h|_{\mathbb{U}}$  lives in  $\mathrm{Sp}(\varphi)_{\mathbb{R}}$ , for which we note that any  $z \in \mathbb{U}(\mathbb{R})$  has

$$\varphi(h(z)v \otimes h(z)w) = h_{\mathbb{Q}(-m)}(z)\varphi(v \otimes w),$$

but  $h_{\mathbb{Q}(-m)}(z) = |z|^{-2m} \mathrm{id}_{\mathbb{Q}(-m)}$  is the identity because  $z \in \mathbb{U}(\mathbb{R})$ .

Thus far, our tools have been upper-bounding  $\mathrm{MT}(V)$  and  $\mathrm{Hg}(V)$ . Here is a tool which sometimes provides a lower bound.

**Lemma 2.54.** Fix  $V \in \mathrm{HS}_{\mathbb{Q}}$ , and let  $D := \mathrm{End}_{\mathrm{HS}}(V)$  be the endomorphism algebra of  $V$ . Then

$$D = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)} = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{Hg}(V)}.$$

*Proof.* As discussed in the second proof of Lemma 2.45, the Hodge classes of  $\mathrm{End}_{\mathbb{Q}}(V) \cong V \otimes V^{\vee}$  are exactly the endomorphisms of the Hodge structure, so the first equality follows from Corollary 2.34.

The second equality is purely formal: note that the scalar subgroup  $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{GL}(V)$  acts trivially on  $V \otimes V^{\vee} \cong \mathrm{End}_{\mathbb{Q}}(V)$ . Thus, we use Lemma 2.41 to compute

$$\begin{aligned} \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{Hg}(V)} &= \mathrm{End}_{\mathbb{Q}}(V)^{\mathbb{G}_{m,\mathbb{Q}} \mathrm{Hg}(V)} \\ &= \mathrm{End}_{\mathbb{Q}}(V)^{\mathbb{G}_{m,\mathbb{Q}} \mathrm{MT}(V)} \\ &= \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)}, \end{aligned}$$

as required. ■

**Remark 2.55.** To understand Lemma 2.54 as providing a lower bound, note that if  $\mathrm{MT}(V)$  is “too small,” then there will be many invariant elements in  $\mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)}$ , perhaps exceeding  $D$ . On the other hand, the upper bound  $\mathrm{MT}(V) \subseteq \mathrm{GL}_D(V)$  corresponds to the inequality  $D \subseteq \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)}$ .

### 2.3.2 Sums

For later use in computations, it will be helpful to have a few remarks on computing the Mumford–Tate and Hodge groups of a sum. Here the Hodge group really shines: given two Hodge structures  $V_1, V_2 \in \mathrm{MT}(V)$  pure of nonzero weight, Lemma 2.41 tells us that  $\mathrm{MT}(V_1)$  and  $\mathrm{MT}(V_2)$  and  $\mathrm{MT}(V_1 \oplus V_2)$  are all equal to some smaller group times scalars. It will turn out to be reasonable to hope that

$$\mathrm{Hg}(V_1 \oplus V_2) \stackrel{?}{=} \mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2),$$

but then the introduction of scalars makes the hope  $\mathrm{MT}(V_1 \oplus V_2) \stackrel{?}{=} \mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$  unreasonable!

With this in mind, let's begin to study Hodge groups of sums of Hodge structures.

**Lemma 2.56.** Fix Hodge structures  $V_1, \dots, V_k \in \mathrm{Hg}_{\mathbb{Q}}$  pure of the same weight.

- (a) The subgroup  $\mathrm{Hg}(V_1 \oplus \dots \oplus V_k) \subseteq \mathrm{GL}(V_1 \oplus \dots \oplus V_k)$  is contained in  $\mathrm{Hg}(V_1) \times \dots \times \mathrm{Hg}(V_k) \subseteq \mathrm{GL}(V_1 \oplus \dots \oplus V_k)$ .
- (b) For each  $i$ , the projection map  $\mathrm{pr}_i: \mathrm{Hg}(V_1 \oplus \dots \oplus V_k) \rightarrow \mathrm{Hg}(V_i)$  is surjective.

*Proof.* For each  $i$ , let  $h_i$  denote the representations of  $\mathbb{S}$  corresponding to the Hodge structures  $V_i$ , and let  $h := (h_1, \dots, h_k)$  be the representation  $\mathbb{S} \rightarrow \mathrm{GL}(V)$  where  $V := V_1 \oplus \dots \oplus V_k$ . We show the claims in sequence.

- (a) We must show that  $\mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k)$  contains the image of  $h|_{\mathbb{U}}$ . Well, for any  $z \in \mathbb{U}(\mathbb{R})$  and index  $i$ , we see that  $h_i(z) \in \mathrm{Hg}(V_i)$ , so

$$h(z) = \mathrm{diag}(h_1(z), \dots, h_k(z))$$

lives in  $\mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k)$ , as required.

- (b) Fix an index  $i$ . It is enough to show that smallest algebraic  $\mathbb{Q}$ -group containing the image of  $\mathrm{pr}_i$  also contains the image of  $h_i|_{\mathbb{U}}$ . Well, by definition of  $h_i$ , we see that  $h_i$  is equal to the composite

$$\mathbb{S} \xrightarrow{h} \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_k) \xrightarrow{\mathrm{pr}_i} \mathrm{GL}(V_i),$$

from which the claim follows. ■

**Remark 2.57.** All the claims in Lemma 2.56 are true if  $\mathrm{Hg}$  is replaced by  $\mathrm{MT}$  everywhere. One simply has to replace  $\mathbb{U}$  with  $\mathbb{S}$  in the proof.

Lemma 2.56 makes  $\mathrm{Hg}(V_1 \oplus V_2) \stackrel{?}{=} \mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2)$  appear to be a reasonable expectation. However, we note that we cannot in general expect this to be true: roughly speaking, there may be Hodge cycles on  $V_1 \oplus V_2$  which are not seen on just  $V_1$  or  $V_2$ . Here is a degenerate example.

**Example 2.58.** Fix a Hodge structure  $V \in \mathrm{HS}_{\mathbb{Q}}$  of pure weight, and let  $n \geq 1$  be a positive integer. Letting  $h: \mathbb{S} \rightarrow \mathrm{GL}(V)_{\mathbb{R}}$  be the corresponding representation, we get another Hodge structure  $h^n: \mathbb{S} \rightarrow \mathrm{GL}(V^{\oplus n})$ . We claim that the diagonal embedding of  $\mathrm{Hg}(V)$  into  $\mathrm{GL}(V)^n \subseteq \mathrm{GL}(V^{\oplus n})$  induces an isomorphism

$$\mathrm{Hg}(V) \rightarrow \mathrm{Hg}(V^{\oplus n}).$$

On one hand, we note that  $\mathrm{Hg}(V^{\oplus n})$  lives inside the diagonal embedding of  $\mathrm{Hg}(V)$ : note  $\mathrm{Hg}(V^{\oplus n}) \subseteq \mathrm{Hg}(V)^n$  by Lemma 2.56, and  $\mathrm{Hg}(V^{\oplus n})$  must live inside the diagonal embedding of  $\mathrm{GL}(V) \subseteq \mathrm{GL}(V^{\oplus n})$  because all components of  $h^n: \mathbb{S} \rightarrow \mathrm{GL}(V^{\oplus n})_{\mathbb{R}}$  are equal. On the other hand, the surjectivity of the projections  $\mathrm{Hg}(V^{\oplus n}) \rightarrow \mathrm{Hg}(V)$  from Lemma 2.56 implies that  $\mathrm{Hg}(V^{\oplus n})$  must equal the diagonal embedding of  $\mathrm{Hg}(V)$  (instead of merely being contained in it).

One can upgrade this example as follows.

**Lemma 2.59.** Fix Hodge structures  $V_1, \dots, V_k \in \mathrm{HS}_{\mathbb{Q}}$  pure of the same weight, and let  $m_1, \dots, m_k \geq 1$  be positive integers. Then the diagonal embeddings  $\Delta_i: \mathrm{GL}(V_i) \rightarrow \mathrm{GL}(V_i^{\oplus m_i})$  induce an isomorphism

$$\mathrm{Hg}(V_1 \oplus \cdots \oplus V_k) \rightarrow \mathrm{Hg}(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}).$$

*Proof.* We proceed in steps. The proof is a direct generalization of the one given in Example 2.58. For each  $i$ , let  $h_i: \mathbb{S} \rightarrow \mathrm{GL}(V_i)_{\mathbb{R}}$  be the representation corresponding to the Hodge structure, and set  $h := (h_1^{m_1}, \dots, h_k^{m_k})$ .

1. We claim that  $\mathrm{Hg}(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k})$  lives in the image of  $(\Delta_1, \dots, \Delta_k)$ . Indeed, the image is some algebraic  $\mathbb{Q}$ -subgroup of  $\mathrm{GL}(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k})$ , so we would like to check that this algebraic  $\mathbb{Q}$ -subgroup contains the image of  $h|_{\mathbb{U}}$ . Well, for any  $z \in \mathbb{U}(\mathbb{R})$ , we see that

$$h(z) = (\Delta_1(h_1(z)), \dots, \Delta_k(h_k(z)))$$

lives in the image of  $(\Delta_1, \dots, \Delta_k)$ .

2. For each  $i$ , let  $H_i$  be the projection of  $\mathrm{Hg}(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k})$  onto one of the  $V_i$  components as in Lemma 2.56; the choice of  $V_i$  component does not matter by the previous step. By Lemma 2.56, we see that  $H_i = \mathrm{Hg}(V_i)$ . However, the previous step requires

$$\mathrm{Hg}(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}) = H_1^{m_1} \times \cdots \times H_k^{m_k},$$

so we are done. ■

**Remark 2.60.** As usual, this statement continues to be true for MT replacing Hg. One can either see this by applying Lemma 2.41 or by redoing the proof with  $\mathbb{S}$  replacing  $\mathbb{U}$ .

The point of the lemma is that we can reduce our computation of Hodge groups to Hodge structures which are the sum of pairwise non-isomorphic irreducible Hodge structures. Let's make a few remarks about this situation for completeness. Let  $V_1, \dots, V_k$  be pairwise non-isomorphic irreducible Hodge structures which are pure of the same weight, and set  $V := V_1 \oplus \dots \oplus V_k$ . Here are some remarks on  $\text{Hg}(V_1 \times \dots \times V_k)$ , summarizing everything we have done so far.

- We know that  $\text{Hg}(V) \subseteq \text{Hg}(V_1) \times \dots \times \text{Hg}(V_k)$ .
- We know that the projections of  $\text{Hg}(V)$  onto each factor  $\text{Hg}(V_i)$  are surjective.
- For each  $i$ , we may view  $V_i$  as a representation of  $\text{Hg}(V_i)$  via the inclusion  $\text{Hg}(V_i) \subseteq \text{GL}(V_i)$ . Then Corollary 2.43 tells us that  $V_i$  is an irreducible representation of  $\text{Hg}(V_i)$ .
- One can also apply Lemma 2.54 to the full space  $V$  to see that

$$\begin{aligned} \text{End}_{\text{Hg}(V)}(V) &= \text{End}_{\text{HS}}(V) \\ &= \prod_{i=1}^k \text{End}_{\text{HS}}(V_i) \\ &= \prod_{i=1}^k \text{End}_{\text{Hg}(V_i)}(V_i). \end{aligned}$$

The following results take the above situation and provides some criteria to have

$$\text{Hg}(V) \stackrel{?}{=} \text{Hg}(V_1) \times \dots \times \text{Hg}(V_k).$$

Before stating the lemma, we remark that all groups in sight are connected by Remark 2.40, and we already have one inclusion by Lemma 2.56, so it suffices to pass to an algebraic closure and work with Lie algebras instead of the Lie groups. The following lemma is essentially due to Ribet [Rib76, pp. 790–791].

**Lemma 2.61 (Ribet).** Work over an algebraically closed field of characteristic 0. Let  $V_1, \dots, V_k$  be finite-dimensional vector spaces, and let  $\mathfrak{g}$  be a Lie subalgebra of  $\mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_k)$ . For each index  $i$ , let  $\text{pr}_i: (\mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_k)) \rightarrow \mathfrak{gl}(V_i)$  be the  $i$ th projection, and set  $\mathfrak{g}_i := \text{pr}_i(\mathfrak{g})$ . Suppose the following.

- (i) Each  $\mathfrak{g}_i$  is nonzero and simple.
- (ii) For each pair  $(i, j)$  of distinct indices, the projection map  $(\text{pr}_i, \text{pr}_j): \mathfrak{g} \rightarrow \mathfrak{g}_i \times \mathfrak{g}_j$  is surjective.

Then  $\mathfrak{g} = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_k$ .

*Proof.* We proceed by induction on  $k$ . If  $k \in \{0, 1\}$ , then there is nothing to say. For the induction, we now assume that  $k \geq 2$  and proceed in steps.

1. For our set-up, we let  $J$  be the kernel of  $\text{pr}_k: \mathfrak{g} \rightarrow \mathfrak{g}_k$ . By definition,  $J \subseteq \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$  takes the form  $I \oplus 0$  for some subspace  $I \subseteq \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$ . Formally, one may let  $I$  be the set of vectors  $v$  such that  $(v, 0) \in J$  and argue for the equality  $J = I \oplus 0$  because all vectors in  $J$  take the form  $(v, 0)$ .

The main content of the proof goes into showing that  $I$  is actually an ideal. To set ourselves up to prove this claim, let  $\mathfrak{n} \subseteq \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$  denote its normalizer. We would like to show that  $\mathfrak{n} = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$ , for which we use the inductive hypothesis.

2. For each pair of distinct indices  $i, j < k$ , we claim that the projection  $(\text{pr}_i, \text{pr}_j): \mathfrak{n} \rightarrow \mathfrak{g}_i \times \mathfrak{g}_j$  is surjective. Well, choose  $X_i \in \mathfrak{g}_i$  and  $X_j \in \mathfrak{g}_j$ , and we need to find an element in  $\mathfrak{n}$  with  $X_i$  and  $X_j$  at the correct coordinates.

To begin, we note that (ii) yields some  $(X_1, \dots, X_k) \in \mathfrak{g}$  such that with the correct  $X_i \in \mathfrak{g}_i$  and  $X_j \in \mathfrak{g}_j$  coordinates. We would like to show that  $X := (X_1, \dots, X_{k-1})$  lives in  $\mathfrak{n}$ , which will complete this step. Well, select any  $Y := (Y_1, \dots, Y_{k-1})$  in  $I$ , and we see  $(Y, 0) \in J$ , so

$$[(X, X_k), (Y, 0)] = ([X, Y], 0)$$

lives in  $J$  too (recall  $J$  is an ideal), so we conclude  $[X, Y] \in I$ . We conclude that  $X$  normalizes  $I$ , so  $X \in \mathfrak{n}$ .

3. We take a moment to complete the proof that  $I \subseteq \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$  is an ideal. It is enough to check that the normalizer  $\mathfrak{n}$  of  $I$  in  $\mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$  equals all of  $\mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$ . For this, we use the inductive hypothesis. The previous step shows that  $\mathfrak{g}_i = \text{pr}_i(\mathfrak{n})$  for each  $i$ , and we know by (i) that each  $\mathfrak{g}_i$  is already nonzero and simple. Lastly, the previous step actually checks condition (ii) for the inductive hypothesis, completing the proof that  $\mathfrak{n} = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$ .
4. We claim  $I = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$ . Because  $I \subseteq \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$  is an ideal of a sum of simple algebras, we know that

$$I = \bigoplus_{i \in S} \mathfrak{g}_i$$

for some subset  $S \subseteq \{1, \dots, k-1\}$  of indices. Thus, to achieve the equality  $I \stackrel{?}{=} \mathfrak{g}_1 \times \dots \times \mathfrak{g}_{k-1}$ , it is enough to check that each projection  $\text{pr}_i: I \rightarrow \mathfrak{g}_{k-1}$  is surjective. Unravelling the definition of  $I$ , it is enough to check that each  $X_i \in \mathfrak{g}_i$  has some  $(X_1, \dots, X_k) \in \mathfrak{g}$  with the correct  $X_i$  coordinate and  $X_k = 0$ . This last claim follows from hypothesis (ii) of  $\mathfrak{g}$ !

5. We now finish the proof of the lemma. Certainly  $\mathfrak{g} \subseteq \mathfrak{g}_1 \times \dots \times \mathfrak{g}_k$ , so it is enough to compute dimensions to prove the equality. By the short exact sequence

$$0 \rightarrow J \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}_n \rightarrow 0,$$

it is enough to show that  $\dim J = \dim \mathfrak{g}_1 + \dots + \dim \mathfrak{g}_{k-1}$ . However, this follows from the previous step because  $\dim J = \dim I$ . ■

In practice, it is somewhat difficult to check (ii) of Lemma 2.61. Here is an automation.

**Lemma 2.62 (Moonen–Zarhin).** Work over an algebraically closed field of characteristic 0. Let  $V_1, \dots, V_k$  be finite-dimensional vector spaces, and let  $\mathfrak{g}$  be a Lie subalgebra of  $\mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_k)$ . For each index  $i$ , let  $\text{pr}_i: (\mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_k)) \rightarrow \mathfrak{gl}(V_i)$  be the  $i$ th projection, and set  $\mathfrak{g}_i := \text{pr}_i(\mathfrak{g})$ . Suppose the following.

- (i) Each  $\mathfrak{g}_i$  is nonzero and simple.
- (ii) Fix a simple Lie algebra  $\mathfrak{l}$ , and define  $I(\mathfrak{l}) := \{i : \mathfrak{g}_i \cong \mathfrak{l}\}$ . If  $\#I(\mathfrak{l}) > 1$ , we require the following to hold.
  - All automorphisms of  $\mathfrak{l}$  are inner.
  - One can choose isomorphisms  $\mathfrak{l} \rightarrow \mathfrak{g}_i$  for each  $i \in I(\mathfrak{l})$  such that the representations  $\mathfrak{l} \rightarrow \mathfrak{g}_i \rightarrow \mathfrak{gl}(V_i)$  are all isomorphic.
  - The diagonal inclusion

$$\prod_{i \in I(\mathfrak{l})} \text{End}_{\mathfrak{g}_i}(V_i) \rightarrow \text{End}_{\mathfrak{g}} \left( \bigoplus_{i \in I(\mathfrak{l})} V_i \right)$$

is surjective.

Then  $\mathfrak{g} = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_k$ .

*Proof.* We will show that (ii) in the above lemma implies (ii) of Lemma 2.61, which will complete the proof. We will proceed by contraposition in the following way. Fix a pair  $(i, j)$  of distinct indices, and we are interested in the map  $(\text{pr}_i, \text{pr}_j): \mathfrak{g} \rightarrow \mathfrak{g}_i \times \mathfrak{g}_j$ . Supposing that  $(\text{pr}_i, \text{pr}_j)$  fails to be surjective (which is a violation of (ii) of Lemma 2.61), we will show that (ii) cannot be true. In particular, we will assume the first two points of (ii) and show then that the third point of (ii) is false.

Roughly speaking, we are going to use the first two points of (ii) to find an  $\mathfrak{h}$  and then produce an endomorphism of  $\bigoplus_{i \in I(\mathfrak{h})} V_i$  which does not come from gluing together endomorphisms of the  $V_i$ s. Having stated the outline, we proceed with the proof in steps.

1. We claim that the image  $\mathfrak{h}$  of the map  $(\text{pr}_i, \text{pr}_j): \mathfrak{g} \rightarrow \mathfrak{g}_i \times \mathfrak{g}_j$  is the graph of an isomorphism  $\mathfrak{g}_i \rightarrow \mathfrak{g}_j$ . For this, we use the hypothesis that  $(\text{pr}_i, \text{pr}_j)$  fails to be surjective. Well, we claim that the projections  $\mathfrak{h} \rightarrow \mathfrak{g}_i$  and  $\mathfrak{h} \rightarrow \mathfrak{g}_j$  are isomorphisms, which implies that  $\mathfrak{h}$  is the graph of the composite isomorphism

$$\mathfrak{g}_i \leftarrow \mathfrak{h} \rightarrow \mathfrak{g}_j.$$

By symmetry, it is enough to merely check that  $\mathfrak{h} \rightarrow \mathfrak{g}_i$  is an isomorphism. On one hand,  $\mathfrak{h} \rightarrow \mathfrak{g}_i$  is surjective because  $\text{pr}_i: \mathfrak{g} \rightarrow \mathfrak{g}_i$  is surjective by construction of  $\mathfrak{g}_i$ . On the other hand, the kernel of the projection  $\mathfrak{h} \rightarrow \mathfrak{g}_i$  will be an ideal of  $\mathfrak{h}$  of the form  $0 \oplus I$  where  $I \subseteq \mathfrak{g}_j$  is some subspace. In fact, because the projection  $\mathfrak{h} \rightarrow \mathfrak{g}_j$  is also surjective, we see that  $I \subseteq \mathfrak{g}_j$  must be an ideal, so the simplicity of  $\mathfrak{g}_j$  grants two cases.

- If  $I = 0$ , then  $\text{pr}_i: \mathfrak{h} \rightarrow \mathfrak{g}_i$  becomes injective and is thus an isomorphism, completing this step.
- If  $I = \mathfrak{g}_j$ , then  $\mathfrak{h}$  fits into a short exact sequence

$$0 \rightarrow (0 \oplus \mathfrak{g}_j) \rightarrow \mathfrak{h} \rightarrow \mathfrak{g}_i \rightarrow 0,$$

so  $\dim \mathfrak{h} = \dim(\mathfrak{g}_i \oplus \mathfrak{g}_j)$ , implying the inclusion  $\mathfrak{h} \subseteq \mathfrak{g}_i \oplus \mathfrak{g}_j$  is an equality. However, this cannot be the case because we assumed that  $(\text{pr}_i, \text{pr}_j): \mathfrak{g} \rightarrow \mathfrak{g}_i \times \mathfrak{g}_j$  fails to be surjective!

2. We construct an isomorphism of  $\mathfrak{g}$ -representations  $V_i \rightarrow V_j$ . For this, we use the first two points of (ii). Let's begin by collecting some data.
  - The previous step informs us that  $\mathfrak{g}_i \cong \mathfrak{g}_j$ . In fact, because this isomorphism is witnessed by the projections  $\text{pr}_i: \mathfrak{g} \rightarrow \mathfrak{g}_i$  and  $\text{pr}_j: \mathfrak{g} \rightarrow \mathfrak{g}_j$ , we see that we are granted an isomorphism  $f: \mathfrak{g}_i \rightarrow \mathfrak{g}_j$  such that  $\text{pr}_j = f \circ \text{pr}_i$ .
  - We now let  $\mathfrak{l}$  be a simple Lie algebra isomorphic to both(!)  $\mathfrak{g}_i$  and  $\mathfrak{g}_j$ . The second point of (ii) grants isomorphisms  $f_i: \mathfrak{l} \rightarrow \mathfrak{g}_i$  and  $f_j: \mathfrak{l} \rightarrow \mathfrak{g}_j$  of Lie algebras and an isomorphism  $d: V_i \rightarrow V_j$  of  $\mathfrak{l}$ -representations.

We now construct our isomorphism from  $d$ . Because  $d$  is only an isomorphism of  $\mathfrak{l}$ -representations, we are only granted that  $(X_1, \dots, X_k) \in \mathfrak{g}$  satisfies  $f(X_i) = X_j$  and hence

$$\begin{aligned} d((f_i f_j^{-1} f)(X_i) v_i) &= d(f_i (f_j^{-1} f(X_i)) v_i) \\ &= f_j(f_j^{-1} f(X_i)) d(v_i) \\ &= X_j d(v_i) \end{aligned}$$

for all  $v_i \in V_i$ . We would be done if we could remove the pesky automorphism  $f_i f_j^{-1} f: \mathfrak{g}_i \rightarrow \mathfrak{g}_i$ . This is possible because all automorphisms of  $\mathfrak{g}_i \cong \mathfrak{l}$  are inner (!), so one may simply “change bases” to remove the inner automorphism. Explicitly, find  $a \in \text{GL}(V_i)$  such that  $f_i f_j^{-1} f(X) = a X a^{-1}$  for all  $X \in \mathfrak{g}_i$ , and then we define  $e := d \circ a$ . Then we find that any  $v_i \in V_i$  has

$$\begin{aligned} e(X_i v_i) &= d(a X_i a^{-1} \cdot a v) \\ &= d((f_i f_j^{-1} f)(X_i) \cdot a v) \\ &= X_j d(a v) \\ &= X_j e(v). \end{aligned}$$

3. We complete the proof. The previous step provides a morphism  $e: V_i \rightarrow V_j$  of  $\mathfrak{g}$ -representations. We thus note that the composite

$$\bigoplus_{k \in I(I)} V_k \rightarrow V_i \xrightarrow{e} V_j \hookrightarrow \bigoplus_{k \in I(I)} V_k$$

is an endomorphism which does not come from the diagonal inclusion of  $\prod_{i \in I(I)} \text{End}_{\mathfrak{g}_i}(V_i)$ . This completes the proof by showing that the third point of (ii) fails to hold. ■

**Remark 2.63.** We should remark on some history. Lemma 2.61 is due to Ribet [Rib76, pp. 790–791], but the given formulation is due to Moonen and Zarhin [MZ95, Lemma 2.14]. In the same lemma, Moonen and Zarhin prove Lemma 2.62, and they seem to be the first to recognize the utility of this lemma for computing Hodge groups. For example, Lombardo includes this result in his master’s thesis [Lom13, Lemma 3.3.1] and includes a generalized version in another paper as [Lom16, Lemma 3.7], where it is used to compute Hodge groups of certain products of abelian varieties.

**Remark 2.64.** Let’s explain how Lemma 2.62 is typically applied, which is admittedly somewhat different from the application used in this thesis. In the generic case, one expects (i), for example if  $\text{Hg}(V) = \text{Sp}_D(\varphi)^\circ$  for  $D$  of types I–III as in Remark 2.51. In this case, one can also check the first condition of (ii) by a direct computation, the second condition of (ii) has no content, and the third condition of (ii) comes from Lemma 2.54. For more details, we refer to (for example) the applications given in [Lom13; Lom16].

## 2.4 The Center

Our last computational tool concerns the center of  $\text{Hg}(V)$ . This discussion is somewhat more involved, so we will spend a full section here.

Let’s begin with some motivation. Fix a Hodge struture  $V \in \text{HS}_{\mathbb{Q}}$ . In the application of this thesis, we will use Lemma 2.62 to compute  $\text{Hg}(V)^{\text{der}}$ : note  $\text{Hg}(V)^{\text{der}}$  is semisimple and hence its Lie algebra can be written as the sum of simple Lie algebras which may be amenable to the lemma. Because  $\text{Hg}(V)$  is reductive by Lemma 2.44, it remains to compute the center  $Z(\text{Hg}(V))$ ; recall  $\text{Hg}(V)$  is connected by Remark 2.40, so we may as well compute the connected component  $Z(\text{Hg}(V))^\circ$ . As usual, the same discussion holds for  $\text{MT}(V)$ , but we note that  $Z(\text{MT}(V))^\circ$  tends to be nontrivial because usually  $\mathbb{G}_{m,\mathbb{Q}} \subseteq \text{MT}(V)$  by Example 2.31.

In Proposition 2.73, we find that  $Z(\text{Hg}(V))^\circ$  is trivial unless  $V$  has irreducible factors of type IV in the sense of the Albert classification (Theorem 2.28). As such, we spend the rest of the section focusing on computations in type IV. Computations are well-understood when  $V$  comes from an abelian variety with complex multiplication, so the main contribution here is that these arguments generalize with only slight modifications.

### 2.4.1 Generalities

The following lemma begins our discussion.

**Lemma 2.65.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight, and set  $D := \text{End}_{\text{HS}}(V)$  with  $F := Z(D)$ . Viewing  $D$  as a  $\mathbb{Q}$ -group, we have

$$Z(\text{Hg}(V)) \subseteq \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F},$$

where  $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$  embeds into  $\text{GL}(V)$  via the  $D$ -action on  $V$ .

*Proof.* Here,  $F$  is a product of number fields because it is a commutative semisimple  $\mathbb{Q}$ -algebra. Recall from Lemma 2.54 that

$$D = \text{End}_{\mathbb{Q}}(V)^{\text{Hg}(V)},$$

which upgrades to an equality of algebraic subgroups of  $\text{End}_{\mathbb{Q}}(V)$  because  $\mathbb{Q}$ -points are dense in these algebraic groups by combining [Mil17, Corollary 17.92] and [Mil17, Definition 12.59]. In particular, we see that  $\text{Hg}(V)$  commutes with  $D^\times$ , so the double centralizer theorem enforces  $Z(\text{Hg}(V)) \subseteq D^\times$  even as algebraic groups. However,  $Z(\text{Hg}(V))$  now commutes fully with  $D^\times$ , so in fact  $Z(\text{Hg}(V)) \subseteq Z(D)^\times$ , which is what we wanted. ■

**Remark 2.66.** One also has  $Z(\text{MT}(V)) \subseteq \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$  because  $\text{MT}(V) \subseteq \mathbb{G}_{m,\mathbb{Q}} \text{Hg}(V)$  by Lemma 2.41, and the scalars  $\mathbb{G}_{m,\mathbb{Q}}$  already live in  $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ .

Lemma 2.65 is that it places the center  $Z(\text{Hg}(V))$  in an explicit torus  $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ . Subgroups of tori are well-understood by (co)character groups, so this puts us in good shape. This torus will be important enough to have its own notation.

**Notation 2.67.** Fix a commutative semisimple  $\mathbb{Q}$ -algebra  $F$  (i.e., a product of number fields). Then we define the torus

$$\mathbf{T}_F := \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}.$$

**Remark 2.68.** Writing  $F$  as a product of number fields  $F_1 \times \cdots \times F_k$ , we find

$$\mathbf{T}_F = \mathbf{T}_{F_1} \times \cdots \times \mathbf{T}_{F_k}$$

because  $F = F_1 \times \cdots \times F_k$  is an equality of  $\mathbb{Q}$ -algebras.

**Remark 2.69.** Let's compute the character group  $X^*(\mathbf{T}_F)$ . By Remark 2.68, it's enough to do this computation when  $F$  is a field. The choice of a primitive element  $\alpha \in F$  with minimal monic polynomial  $f(x)$  yields an isomorphism  $F \cong \mathbb{Q}[x]/(f(x))$ . Upon base-changing to  $\overline{\mathbb{Q}}$ , we get a ring isomorphism

$$F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \prod_{i=1}^n \frac{\overline{\mathbb{Q}}[x]}{(x - \alpha_i)},$$

where  $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$  are the roots of  $f(x)$  in  $\overline{\mathbb{Q}}$ . Each root  $\alpha_i$  provides a unique embedding  $F \hookrightarrow \overline{\mathbb{Q}}$ , so we see that  $(\mathbf{T}_F)_{\overline{\mathbb{Q}}} \cong \mathbb{G}_{m,\overline{\mathbb{Q}}}^n$ , where the  $n$  maps  $(\mathbf{T}_F)_{\overline{\mathbb{Q}}} \rightarrow \mathbb{G}_{m,\overline{\mathbb{Q}}}$  are given by the embedding  $\sigma_i: F \hookrightarrow \overline{\mathbb{Q}}$  defined by  $\sigma_i(\alpha) := \alpha_i$ . In total, we find that  $X^*(\mathbf{T}_F)$  is a free  $\mathbb{Z}$ -module spanned by the embeddings  $\Sigma_F := \{\sigma_1, \dots, \sigma_n\}$ , and it has the natural Galois action. Dually,  $X_*(\mathbf{T}_F)$  has the dual basis  $\Sigma_F^\vee = \{\sigma_1^\vee, \dots, \sigma_n^\vee\}$ .

In the light of the above remark, we will want the following notation.

**Notation 2.70.** Given a number field  $F$ , we let  $\Sigma_F$  denote the collection of embeddings  $F \hookrightarrow \overline{\mathbb{Q}}$ . Given a product of number fields  $F := F_1 \times \cdots \times F_k$ , we define  $\Sigma_F := \Sigma_{F_1} \sqcup \cdots \sqcup \Sigma_{F_k}$ .

The point of the above notation is that  $X^*(\mathbf{T}_F) = \mathbb{Z}[\Sigma_F]$  as Galois modules.

It is possible to upgrade Lemma 2.65 in the presence of a polarization.

**Lemma 2.71.** Fix a polarizable  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight, and set  $D := \text{End}_{\text{HS}}(V)$  with  $F := Z(D)$ . Then

$$Z(\text{Hg}(V)) \subseteq \{g \in \mathbf{T}_F : gg^\dagger = 1\},$$

where  $(\cdot)^\dagger$  is the Rosati involution.



*Proof.* As usual, everything in sight upgrades to algebraic groups. Let  $\varphi$  be a polarization. Fix some  $g \in \mathrm{Hg}(V)$ ; note that Lemma 2.65 implies  $g \in \mathrm{T}_F$ , so it makes sense to write down  $g^\dagger$ .

Now, by the non-degeneracy of  $\varphi$ , it is enough to show that

$$\varphi(gg^\dagger v \otimes w) \stackrel{?}{=} \varphi(v \otimes w)$$

for any  $v, w \in V$ . Well, the definition of  $(\cdot)^\dagger$  tells us that the left-hand side equals  $\varphi(g^\dagger v \otimes g^\dagger w)$ , which equals  $\varphi(v \otimes w)$  because  $\mathrm{Hg}(V) \subseteq \mathrm{Sp}(\varphi)$  by Remark 2.53. ■

Once again, this torus is important enough to earn its own notation.

**Notation 2.72.** Fix a commutative semisimple  $\mathbb{Q}$ -algebra  $F$  with involution  $(\cdot)^\dagger$ . Then we define the torus

$$\mathrm{U}_F := \{g \in \mathrm{T}_F : xx^\dagger = 1\}.$$

Here is an application of Lemma 2.71.

**Proposition 2.73.** Fix polarizable  $V \in \mathrm{HS}_{\mathbb{Q}}$  of pure weight. Suppose that  $V$  has no irreducible Hodge substructures with endomorphism algebra of type IV in the sense of the Albert classification (Theorem 2.28). Then  $Z(\mathrm{Hg}(V))$  is finite, and  $\mathrm{Hg}(V)$  is semisimple.

*Proof.* Quickly, recall from Lemma 2.44 that  $\mathrm{Hg}(V)$  is reductive, so the finiteness of  $Z(\mathrm{Hg}(V))$  implies that  $Z(\mathrm{Hg}(V))^\circ = 1$  and thus  $\mathrm{Hg}(V) = \mathrm{Hg}(V)^{\mathrm{der}}$ , making  $\mathrm{Hg}(V)$  semisimple. (See also [Mil17, Proposition 19.10].) As such, we will focus on the first claim.

Set  $D := \mathrm{End}_{\mathrm{HS}}(V)$  with  $F := Z(D)$  so that  $\mathrm{Hg}(V) \subseteq \mathrm{U}_F$  by Lemma 2.71. It is therefore enough to check that  $\mathrm{U}_F$  is finite. Well,  $F$  is a product of number fields, and upon comparing with Theorem 2.28, we see that avoiding type IV implies that  $F$  is a product of totally real fields. Totally real fields have only two units, so finiteness of  $\mathrm{U}_F$  follows. ■

Thus, we see that we have pretty good control outside of type IV factors, so we will spend the rest of this subsection on type IV. For some applications outside type IV, see (for example) [Lom16].

### 2.4.2 Type IV: The Signature

The arguments in this subsection are motivated by the computation of [Yu15, Lemma 4.2] and [Yan94, Proposition 1.1]. For this subsection,  $V \in \mathrm{HS}_{\mathbb{Q}}$  is a Hodge structure of type IV in the sense of the Albert classification (Theorem 2.28). For brevity, we may say that  $V$  is simply “irreducible of type IV.” In our application, it will also be enough to assume that the Hodge structure of  $V$  is concentrated in  $V^{0,1}$  and  $V^{1,0}$ , so we do so.

By assumption, we know that  $D := \mathrm{End}_{\mathrm{HS}}(V)$  is a division algebra over its center  $F := Z(D)$ , where  $F$  is a CM algebra, and the Rosati involution  $(\cdot)^\dagger$  restricts to complex conjugation on  $F$ . In particular,  $F^\dagger$  is the product of the maximal totally real subfields of  $F$ .

The basic approach of this subsection is that Lemma 2.65 requires  $Z(\mathrm{Hg}(V))^\circ \subseteq \mathrm{T}_F$ , and one can compute subtori using the machinery of (co)character groups. In particular, we recall that  $X^*(\Sigma_F) = \mathbb{Z}[\Sigma_F]$  and  $X_*(\Sigma_F) = \mathbb{Z}[\Sigma_F^\vee]$  as Galois modules. We will need a way to work directly with the Hodge structure on  $V$ . It will be described by the following piece of combinatorial data.

**Definition 2.74 (signature).** Fix a CM algebra  $F$ , and recall that  $\Sigma_F$  is the set of homomorphisms  $F \hookrightarrow \overline{\mathbb{Q}}$ . Then a *signature* is a function  $\Phi: \Sigma_F \rightarrow \mathbb{Z}_{\geq 0}$  such that the sum

$$\Phi(\sigma) + \Phi(\bar{\sigma})$$

is constant with respect to  $\sigma \in \Sigma_F$ ; here,  $\bar{\sigma}$  denotes the complex conjugate embedding to  $\sigma$ . We may call the pair  $(F, \Phi)$  a *CM signature*.

**Remark 2.75.** One can also view  $\Phi$  as an element of  $\mathbb{Z}[\Sigma_F]$  as

$$\Phi := \sum_{\sigma \in \Sigma_F} \Phi(\sigma) \sigma.$$

**Remark 2.76.** The case that  $\Phi(\sigma) + \Phi(\bar{\sigma})$  always equals 1 corresponds to  $\Phi$  being a CM type.

**Remark 2.77.** If we expand  $F$  as a product of CM fields  $F = F_1 \times \cdots \times F_k$ , then  $\Sigma_F = \Sigma_{F_1} \sqcup \cdots \sqcup \Sigma_{F_k}$ . Thus, we see that a signature of  $F$  has only a little more data than a signature on each of the  $\Sigma_{F_i}$ s individually; in particular, one should make sure that  $\Phi(\sigma) + \Phi(\bar{\sigma})$  remains equal across the different fields.

The idea is that we can keep track of a signature with a Hodge structure.

**Lemma 2.78.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  with  $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$  such that  $\text{End}_{\text{HS}}(V)$  contains a CM algebra  $F$ . Then the function  $\Phi: \Sigma_F \rightarrow \mathbb{Z}_{\geq 0}$  defined by

$$V^{1,0} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_{\sigma}^{\Phi(\sigma)}$$

is a signature, which we will call the induced signature. This is an isomorphism of  $F$ -representations, where  $\mathbb{C}_{\sigma}$  is a complex  $F$ -representation via the embedding  $\sigma$ .

*Proof.* In short, the condition that  $\Phi(\sigma) + \Phi(\bar{\sigma})$  is constant comes from the condition  $V^{0,1} = \overline{V^{1,0}}$ . To see this, note that  $V$  is a free module over  $F$ , so  $V_{\mathbb{C}}$  is a free module over  $F \otimes \mathbb{C}$  of finite rank. As such, we may set  $d := [V : F]$  so that  $V \cong F^d$  as  $F$ -representations, and then we find

$$V_{\mathbb{C}} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_{\sigma}^d.$$

Now,  $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ , and because  $F$  acts by endomorphisms of Hodge structures, we get a well-defined action of  $F$  on  $V^{1,0}$  and  $V^{0,1}$  individually. In particular, the definition of  $\Phi$  also grants

$$V^{0,1} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_{\sigma}^{d-\Phi(\sigma)}$$

as  $F$ -representations, so

$$\overline{V^{0,1}} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_{\bar{\sigma}}^{d-\Phi(\sigma)}$$

To complete the proof, we note that  $V^{0,1} = \overline{V^{1,0}}$  continues to be true as  $F$ -representations, so we must have  $\Phi(\sigma) = d - \Phi(\bar{\sigma})$  for all  $\sigma$ . The result follows.  $\blacksquare$

Of course, we cannot expect this signature  $\Phi$  to remember everything about the Hodge structure. For example, if  $\text{End}_{\text{HS}}(V)$  contains a larger CM algebra  $F'$  than  $F$ , then the signature induced by  $F'$  knows more about the Hodge structure than the one induced by  $F$ . However, in “generic cases,” this signature is expected to suffice. For our purposes, we will take generic to mean that there are no more endomorphisms than the ones promised by  $F$ ; i.e., this explains why we will assume  $Z(\text{End}_{\text{HS}}(V)) = F$  in the sequel.

We now relate our signature to cocharacters of  $Z(\text{Hg}(V))^{\circ}$ . For this, it will be helpful to realize  $Z(\text{Hg}(V))$  as some kind of monodromy group. The trick is to consider the determinant.

**Lemma 2.79.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  of pure weight such that  $Z(\text{End}_{\text{HS}}(V))$  equals an algebra  $F$ . Then  $Z(\text{Hg}(V))^{\circ}$  equals the largest algebraic  $\mathbb{Q}$ -subgroup of  $T_F$  containing the image of  $(\det_F \circ h): \mathbb{U} \rightarrow (T_F)_{\mathbb{R}}$ .

*Proof.* The point is that taking the determinant will kill  $\mathrm{Hg}(V)^{\mathrm{der}}$  because  $\mathrm{Hg}(V) \subseteq \mathrm{GL}_F(V)$ . There are two inclusions we must show.

- We show that  $Z(\mathrm{Hg}(V))^\circ$  contains the image of  $(\det F \circ h|_{\mathbb{U}})$ . Well,  $\mathrm{Hg}(V)$  contains the image of  $h|_{\mathbb{U}}$ , so it is enough to show that  $Z(\mathrm{Hg}(V))^\circ$  contains the image of  $\det_F: \mathrm{Hg}(V) \rightarrow T_F$ . For this, we recall that  $\mathrm{Hg}(V)$  is connected (by Remark 2.40), so

$$\mathrm{Hg}(V) = Z(\mathrm{Hg}(V))^\circ \mathrm{Hg}(V)^{\mathrm{der}}.$$

Note that  $\det_F$  is simply  $(\cdot)^{\dim_F V}$  on the torus  $Z(\mathrm{Hg}(V))^\circ$ , so that piece surjects onto  $Z(\mathrm{Hg}(V))^\circ$ . Thus, it is enough to check that  $\det_F: \mathrm{Hg}(V)^{\mathrm{der}} \rightarrow T_F$  is trivial, which is true by the definition of the derived subgroup upon noting that  $\det_F$  is a homomorphism with abelian target.

- Suppose that  $T \subseteq T_F$  contains the image of  $(\det_F \circ h|_{\mathbb{U}})$ . Then we claim that  $T$  contains  $Z(\mathrm{Hg}(V))^\circ$ . Let  $H \subseteq \mathrm{GL}_F(V)$  be the pre-image of  $T$  under  $\det_F: \mathrm{GL}_F(V) \rightarrow T_F$ . Then  $H$  must contain the image of  $h|_{\mathbb{U}}$ , so it contains  $\mathrm{Hg}(V)$  by definition. In particular,  $H$  contains  $Z(\mathrm{Hg}(V))^\circ$ . Now,  $T$  contains  $\det_F(H)$ , so  $T$  contains  $\det_F(Z(\mathrm{Hg}(V))^\circ)$ , but the previous point check remarked that this simply equals  $Z(\mathrm{Hg}(V))^\circ$ , so we are done. ■

**Proposition 2.80.** Fix  $V \in \mathrm{HS}_{\mathbb{Q}}$  with  $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$  such that  $Z(\mathrm{End}_{\mathrm{HS}}(V))$  equals a CM algebra  $F$ . Let  $\Phi: \Sigma_F \rightarrow \mathbb{Z}_{\geq 0}$  be the induced signature. Then the induced representation  $(\det_F \circ h): \mathbb{U} \rightarrow (T_F)_{\mathbb{R}}$  sends the generator of  $X_*(\mathbb{U})$  to

$$- \sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\bar{\sigma})) \sigma^{\vee}.$$

*Proof.* This boils down to computing the map  $\det_F \circ h|_{\mathbb{U}}$ . We proceed in steps.

1. To set ourselves up, recall that

$$\mathbb{U}_{\mathbb{C}} = \{(z, 1/z) : z \in \mathbb{G}_{m, \mathbb{C}}\},$$

so one has an isomorphism cocharacter  $z^{\vee}: \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{U}_{\mathbb{C}}$  given by  $z^{\vee} \mapsto z \mapsto (z, 1/z)$ . Thus, we have left to show that

$$\det_F \circ h_{\mathbb{C}} \circ z^{\vee} \stackrel{?}{=} - \sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\bar{\sigma})) \sigma^{\vee}.$$

We may check this equality on geometric points.

2. We describe the map  $h_{\mathbb{C}}: \mathbb{S}_{\mathbb{C}} \rightarrow \mathrm{GL}(V)_{\mathbb{C}}$ . By definition,  $h(z, w) \in \mathrm{GL}(V)$  acts by  $z^{-1}$  on  $V^{1,0}$  and by  $w^{-1}$  on  $V^{0,1}$ . Thus, the definition of  $\Phi$  grants that  $h(z, w)$  diagonalizes. To be more explicit, for each  $\sigma \in \Sigma_F$ , we define  $V_{\sigma}^{p,q}$  to be the  $\sigma$ -eigenspace for the  $F$ -action on  $V^{p,q} \subseteq V_{\mathbb{C}}$ . Then we see that  $h(z, w)$  acts on  $V_{\sigma}^{1,0}$  by the scalar  $z^{-1}$  and on  $V_{\sigma}^{0,1}$  by the scalar  $w^{-1}$ .
3. We describe the map  $(\det_F \circ h_{\mathbb{C}}): \mathbb{S}_{\mathbb{C}} \rightarrow (T_F)_{\mathbb{C}}$ . Realizing geometric points in  $(T_F)_{\mathbb{C}}$  as tuples in  $\mathbb{C}^{\Sigma_F}$ , we see that  $\det_F$  simply takes the determinant of the matrix  $h_{\mathbb{C}}(z, w)|_{V_{\sigma}}$  to the  $\sigma$ -component in  $(T_F)_{\mathbb{C}}$ . (One finds this by tracking through the definition of  $\det_F$  as a morphism of algebraic groups.) As such, we see that

$$\det h_{\mathbb{C}}(z, w)|_{V_{\sigma}} = z^{-\Phi(\sigma)} w^{-\Phi(\bar{\sigma})}$$

because  $\Phi$  is a signature.

4. We complete the proof. The previous step shows that  $(\det_F \circ h_{\mathbb{C}} \circ z^{\vee})(z)$  goes to the element

$$\left( z^{-\Phi(\sigma) + \Phi(\bar{\sigma})} \right)_{\sigma \in \Sigma(F)} \in \mathbb{C}^{\Sigma_F}.$$

This completes the proof upon noting that the cocharacter  $\sigma^{\vee}: \mathbb{G}_{m, \mathbb{C}} \rightarrow T_F$  simply maps into the  $\sigma$ -component of  $\mathbb{C}^{\Sigma_F}$  on geometric points. ■

**Remark 2.81.** Notably, the given element sums to 0, which corresponds to the fact that  $\mathrm{Hg}(V) \subseteq \mathrm{SL}(V)$  as seen in Lemma 2.41. Indeed, by diagonalizing the  $F$ -action on  $V$ , we see that  $(\mathrm{T}_F \cap \mathrm{SL}(V))^\circ$  consists of the  $g \in \mathrm{T}_F$  such that the product of the elements in  $g$  equals 1.

Proposition 2.80 easily translates into a computation of the cocharacter group  $X_*(\mathrm{Hg}(V))^\circ$ . In the next few results, saturated simply means that the induced quotient is torsion-free.

**Corollary 2.82.** Fix  $V \in \mathrm{HS}_{\mathbb{Q}}$  with  $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$  such that  $Z(\mathrm{End}_{\mathrm{HS}}(V))$  equals a CM algebra  $F$ . Let  $\Phi: \Sigma_F \rightarrow \mathbb{Z}_{\geq 0}$  be the induced signature. Then  $Z(\mathrm{Hg}(V))^\circ \subseteq \mathrm{T}_F$  has cocharacter group equal to the smallest saturated Galois submodule of  $X_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$  containing

$$\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\bar{\sigma}))\sigma^\vee.$$

*Proof.* This is immediate from combining Lemma 2.79 and Proposition 2.80 with the equivalence of categories  $X_*$  between algebraic tori and Galois modules. See [Mil17, Theorem 12.23] for the proof that  $X^*$  is an equivalence, which is similar. ■

**Corollary 2.83.** Fix  $V \in \mathrm{HS}_{\mathbb{Q}}$  with  $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$  such that  $Z(\mathrm{End}_{\mathrm{HS}}(V))$  equals a CM algebra  $F$ . Let  $\Phi: \Sigma_F \rightarrow \mathbb{Z}_{\geq 0}$  be the signature defined in Lemma 2.78. Then  $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$  has cocharacter group equal to the smallest saturated Galois submodule of  $X_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$  containing

$$\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee.$$

*Proof.* This follows from Corollary 2.82. By Lemma 2.41, it is enough to add in the cocharacter given by the scalars  $\mathbb{G}_{m,\mathbb{Q}} \rightarrow \mathrm{T}_F$ , which is  $\sum_{\sigma \in \Sigma_F} \sigma^\vee$ . Thus, the fact that  $\Phi$  is a signature implies that

$$\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee$$

certainly lives in  $X_*(\mathrm{MT}(V)) \subseteq X_*(\mathrm{T}_F)$ .

Conversely, if  $X$  is some saturated Galois submodule containing  $\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee$ , then we would like to show that  $X_*(\mathrm{MT}(V)) \subseteq X$ . Well,  $X$  is a Galois submodule, so it must contain the complex conjugate element  $\sum_{\sigma \in \Sigma_F} \Phi(\bar{\sigma})\sigma^\vee$ . On one hand, this then sums with the given element to produce

$$\sum_{\sigma \in \Sigma_F} \sigma^\vee \in X$$

because  $X$  is saturated. On the other hand, we can take a difference to see that

$$\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\bar{\sigma}))\sigma^\vee \in X.$$

We conclude that  $X$  contains the cocharacter of the scalars  $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{T}_F$  and the cocharacter lattice of  $Z(\mathrm{Hg}(V))^\circ \subseteq \mathrm{T}_F$ , so we conclude that  $X$  must also contain the cocharacter lattice of  $Z(\mathrm{MT}(V))^\circ$ . ■

**Remark 2.84.** One can also prove the above corollary by following the proof of Corollary 2.82. For example, this approach provides a monodromy interpretation of  $Z(\mathrm{MT}(V))^\circ$  analogous to Lemma 2.79. Here, one replaces the generator of  $X_*(\mathbb{U})$  with the cocharacter  $\mu \in X_*(\mathbb{S})$ , and one finds that  $\det_F \circ h_{\mathbb{C}}$  sends  $\mu$  to  $\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee$ . One is then able to prove statements analogous to Proposition 2.80 and Corollary 2.82.

Let's pause for a moment with an explanation of how one can use Corollary 2.83 to compute  $Z(\text{MT}(V))^\circ \subseteq T_F$ . The approach for  $Z(\text{Hg}(V))^\circ$  is similar but only a little more complicated.

We will only compute over a Galois extension  $L/\mathbb{Q}$  containing all factors of  $F$ . In this case, the  $F$ -action on  $V_L$  diagonalizes, so one can identify  $(T_F)_L \subseteq \text{GL}(V)_L$  as the diagonal torus for some basis of  $V_L$ . In particular, for each  $\sigma \in \Sigma_F$ , the cocharacter  $\sigma^\vee$  corresponds to one of the standard cocharacters for the diagonal torus of  $\text{GL}(V)_L$ . Now, Corollary 2.83 tells us that  $X_*(Z(\text{MT}(V))^\circ) \subseteq X_*(T_F)$  equals the saturation of the sublattice spanned by the vectors

$$g \left( \sum_{\sigma \in \Sigma_F} \Phi(\sigma) \sigma^\vee \right) = \sum_{\sigma \in \Sigma_F} \Phi(\sigma) (g\sigma)^\vee,$$

where  $g$  varies over  $\text{Gal}(L/F)$ . By computing a basis of the saturation of this sublattice, we get a family of 1-parameter subgroups of the diagonal torus of  $\text{GL}(V)_L$  which together generate  $Z(\text{MT}(V))^\circ$ . This more or less computes  $Z(\text{MT}(V))^\circ$ .

### 2.4.3 Type IV: The Reflex

In the sequel, we will be most interested in equations cutting out  $Z(\text{MT}(V))^\circ \subseteq T_F$ . One could imagine proceeding as above to compute  $Z(\text{MT}(V))^\circ \subseteq T_F$  via 1-parameter subgroups and then afterwards finding the desired equations. This is somewhat computationally intensive, so instead we will turn our attention to computing character groups. As in [Yu15, Lemma 4.2], this will require a discussion of the reflex.

**Definition 2.85** (reflex signature). Fix CM fields  $F$  and  $F^*$  and CM signatures  $(F, \Phi)$  and  $(F^*, \Phi^*)$ . We say that these CM signatures are *reflex* if and only if there is a Galois extension  $L/\mathbb{Q}$  containing  $F$  and  $F^*$  such that each  $\sigma \in \text{Gal}(L/\mathbb{Q})$  has

$$\Phi(\sigma|_F) = \Phi^*(\sigma^{-1}|_{F^*}).$$

In this situation, we may call  $(F^*, \Phi^*)$  a *reflex signature* for  $(F, \Phi)$ .

**Remark 2.86.** We check that  $(F, \Phi)$  and  $(F^*, \Phi^*)$  does not depend on the choice of Galois extension  $L$ . Indeed, suppose that we have another Galois extension  $L'/\mathbb{Q}$  containing  $F$  and  $F^*$ ; let  $L''$  be a Galois extension containing both  $L$  and  $L'$ . By symmetry, it is enough to check that  $(F, \Phi)$  are reflex with respect to  $L$  if and only if they are reflex with respect to  $L''$ . Well, for any  $\sigma \in \text{Gal}(L''/\mathbb{Q})$ , we see that  $\Phi(\sigma|_F) = \Phi^*(\sigma^{-1}|_{F^*})$  is equivalent to  $\sigma|_L \in \text{Gal}(L/\mathbb{Q})$  satisfying  $\Phi(\sigma|_L|_F) = \Phi^*(\sigma|_L^{-1}|_{F^*})$ , so we are done after remarking that restriction  $\text{Gal}(L''/\mathbb{Q}) \rightarrow \text{Gal}(L/\mathbb{Q})$  is surjective.

**Remark 2.87.** We check that reflex signatures certainly exist: one can choose any Galois closure  $L$  of  $F$  and then define  $\Phi^*: \text{Gal}(L/\mathbb{Q}) \rightarrow \mathbb{Z}_{\geq 0}$  by  $\Phi^*(\sigma) := \Phi(\sigma^{-1}|_L)$ .

**Remark 2.88.** In the theory of abelian varieties with complex multiplication, it is customary to make  $F^*$  as small as possible, which makes it unique. This is useful for moduli problems. However, this is not our current interest, and we are not requiring that the reflex signature be unique because it will be convenient later to take large extensions.

The point of introducing the reflex is that it provides another monodromy interpretation of  $Z(\text{MT}(V))^\circ$ . To achieve this, we need the reflex norm.

**Definition 2.89 (reflex norm).** Fix CM fields  $F$  and  $F^*$  and reflex CM signatures  $(F, \Phi)$  and  $(F^*, \Phi^*)$ . Then we define the *reflex norm* as the map  $N_{\Phi^*} : F^* \rightarrow \overline{\mathbb{Q}}$  by

$$N_{\Phi^*}(x) := \prod_{\sigma \in \Sigma_{F^*}} \sigma(x)^{\Phi^*(\sigma)}.$$

Note that this is a character in  $X^*(T_{F^*})$ .

Technically, this definition does not require us to remember that  $(F^*, \Phi^*)$  is reflex to  $(F, \Phi)$ , but we will want to know this in the following checks.

**Lemma 2.90.** Fix CM fields  $F$  and  $F^*$  and reflex CM signatures  $(F, \Phi)$  and  $(F^*, \Phi^*)$ .

(a) If  $(F_1^*, \Phi_1^*)$  is a CM signature restricting to  $(F^*, \Phi^*)$ , then  $(F, \Phi)$  and  $(F_1^*, \Phi_1^*)$  are still reflex, and

$$N_{\Phi_1^*} = N_{\Phi^*} \circ N_{F_1^*/F^*}.$$

(b) The image of  $N_{\Phi^*}$  lands in  $F$ .

*Proof.* Here, “restricting” simply means that  $F_1^*$  contains  $F^*$  and  $\Phi_1^*(\sigma) = \Phi^*(\sigma|_{F^*})$  for all  $\sigma \in \Sigma_{F_1^*}$ .

(a) That  $(F, \Phi)$  and  $(F_1^*, \Phi_1^*)$  are still reflex follows from the definition: choose a Galois extension  $L$  containing  $F$  and  $F_1^*$ , and then each  $\sigma \in \text{Gal}(L/\mathbb{Q})$  has

$$\begin{aligned} \Phi(\sigma|_F) &= \Phi^*(\sigma^{-1}|_{F^*}) \\ &= \Phi_1^*(\sigma^{-1}|_{F_1^*}). \end{aligned}$$

To check the equality of reflex norms, we extend each  $\sigma \in \Sigma_{F^*}$  to some  $\tilde{\sigma} \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , and then we directly compute

$$\begin{aligned} N_{\Phi^*}(N_{F_1^*/F^*}(x)) &= \prod_{\sigma \in \Sigma_{F^*}} \sigma(N_{F_1^*/F^*}(x))^{\Phi^*(\sigma)} \\ &= \prod_{\substack{\sigma \in \Sigma_{F^*} \\ \tau \in \text{Hom}_{F^*}(F_1^*, \overline{\mathbb{Q}})}} \tilde{\sigma}\tau(x)^{\Phi^*(\sigma)} \\ &= \prod_{\substack{\sigma \in \Sigma_{F^*} \\ \tau \in \text{Hom}_{F^*}(F_1^*, \overline{\mathbb{Q}})}} \tilde{\sigma}\tau(x)^{\Phi_1^*(\tilde{\sigma}\tau)} \\ &= N_{\Phi_1^*}(x), \end{aligned}$$

where the last step holds by noting that  $\tilde{\sigma} \circ \tau$  parameterizes  $\Sigma_{F^*}$ .

(b) We begin by reducing to the case where  $F^*/\mathbb{Q}$  is Galois. Indeed, the previous step tells us that extending  $F^*$  merely passes to a norm subgroup of  $F^*$ , but norm subgroups are Zariski dense in  $T_{F^*}$ , so it suffices to check the result on such norm subgroups. Thus, we may assume that  $F^*/\mathbb{Q}$  is Galois, contains  $F$ , and thus  $\Phi^*(\sigma) = \Phi(\sigma^{-1}|_F)$ . Now, for any  $g \in \text{Gal}(F^*/F)$ , we see  $\Phi^*(\sigma) = \Phi^*(g^{-1}\sigma)$ , so

$$\begin{aligned} g(N_{\Phi^*}(x)) &= \prod_{\sigma \in \text{Gal}(F^*/\mathbb{Q})} g\sigma(x)^{\Phi^*(\sigma)} \\ &= \prod_{\sigma \in \text{Gal}(F^*/\mathbb{Q})} \sigma(x)^{\Phi^*(g^{-1}\sigma)} \\ &= N_{\Phi^*}(x), \end{aligned}$$

as required. ■

At long last, we move towards our monodromy interpretation using the reflex. The following argument generalizes [Yu15, Lemma 4.2].

**Lemma 2.91.** Fix reflex CM signatures  $(F, \Phi)$  and  $(F^*, \Phi^*)$ . Suppose that  $F^*$  contains  $F$  and is Galois over  $\mathbb{Q}$ . For each  $g \in \text{Gal}(F^*/\mathbb{Q})$ , the reflex norm  $N_{\Phi^*}: T_{F^*} \rightarrow T_F$  sends the cocharacter  $g^\vee \in X_*(T_{F^*})$  to

$$X_*(N_{\Phi^*})(g^\vee) = \sum_{\sigma \in \Sigma_F} \Phi(\sigma)(g\sigma)^\vee.$$

*Proof.* Notably,  $N_{\Phi^*}$  outputs to  $T_F$  by Lemma 2.90. To begin, we expand

$$X_*(N_{\Phi^*})(g^\vee) = \sum_{\sigma \in \Sigma_{F^*}} \Phi^*(\sigma)X_*(\sigma)(g^\vee).$$

We now check  $X_*(\sigma)(g^\vee) = (g\sigma^{-1})^\vee$ : for any  $\tau \in X^*(T_{F^*})$ , we compute the perfect pairing

$$\langle \tau, X_*(\sigma)(g^\vee) \rangle = \langle \tau\sigma, g^\vee \rangle,$$

which is the indicator function for  $\tau\sigma = g$  and hence equals  $\langle \cdot, (g\sigma^{-1})^\vee \rangle$ . We are now able to write

$$X_*(N_{\Phi^*})(g^\vee) = \sum_{\sigma \in \Sigma_{F^*}} \Phi^*(\sigma)(g\sigma^{-1})^\vee.$$

Replacing  $\sigma$  with  $\sigma^{-1}$ , we are done upon recalling  $\Phi^*(\sigma^{-1}) = \Phi(\sigma|_F)$  and collecting terms which together restrict to the same embedding of  $F$ . ■

**Proposition 2.92.** Fix  $V \in \text{HS}_{\mathbb{Q}}$  with  $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$  such that  $Z(\text{End}_{\text{HS}}(V))$  equals a CM algebra  $F = F_1 \times \cdots \times F_k$ . Let  $\Phi: \Sigma_F \rightarrow \mathbb{Z}_{\geq 0}$  be the induced signature, which we decompose as  $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_k$  where  $(F_\bullet, \Phi_\bullet)$  is a CM signature for all  $F_\bullet$ . Suppose  $F^*$  is a CM field equipped with CM signatures  $\Phi_1^*, \dots, \Phi_k^*$  such that  $(F_i, \Phi_i)$  and  $(F^*, \Phi_i^*)$  are reflex for all  $i$ . Then  $Z(\text{MT}(V))^\circ \subseteq T_F$  is the image of

$$(N_{\Phi_1^*}, \dots, N_{\Phi_k^*}): T_{F^*} \rightarrow T_F.$$

*Proof.* Note that norms are surjective on these algebraic tori, so Lemma 2.90 tells us that the image of  $N_{\Phi^*}$  will not change if we pass to an extension of  $F^*$ . As such, we will go ahead and assume that  $F^*$  contains  $F$  and is Galois over  $\mathbb{Q}$ .

In light of Corollary 2.83, it is enough to show that the image of  $X_*(N_{\Phi^*})$  (which we note is already a Galois submodule) has saturation equal to the smallest saturated Galois submodule of  $X_*(T_F)$  containing  $\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee$ . This follows from the computation of Lemma 2.91 upon letting  $g$  vary over  $\text{Gal}(F^*/\mathbb{Q})$ . ■

Let's explain how Proposition 2.92 is applied to compute equations cutting out  $Z(\text{MT}(V))^\circ \subseteq T_F$ , where  $F = F_1 \times \cdots \times F_k$  is a CM algebra. As before, we will only compute over an extension  $L = F^*$  of  $F$  which is Galois over  $\mathbb{Q}$ ; let  $\Phi_1^*, \dots, \Phi_k^*$  be the signatures on  $L$  making  $(L, \Phi_i^*)$  and  $(F_i, \Phi_i)$  reflex for each  $i$ . Note, we know that  $(T_F)_L \subseteq \text{GL}(V)_L$  may embed as a diagonal torus.

An equation cutting out  $Z(\text{MT}(V))^\circ_L$  in the (subtorus of the) diagonal torus  $(T_F)_L \subseteq \text{GL}(V)_L$  then becomes a character of  $(T_F)_L$  which is trivial on  $Z(\text{MT}(V))^\circ$ . In other words, these equations are given by the kernel of

$$X^*(T_F) \rightarrow X^*(Z(\text{MT}(V))^\circ).$$

We now use Proposition 2.92. We know that  $Z(\text{MT}(V))^\circ \subseteq T_F$  is the image of  $(N_{\Phi_1^*}, \dots, N_{\Phi_k^*}): T_L \rightarrow T_F$ , so the kernel of the above map is the same as the kernel of

$$X^*((N_{\Phi_1^*}, \dots, N_{\Phi_k^*})) : X^*(T_F) \rightarrow X^*(T_L).$$

To compute this kernel cleanly, note Lemma 2.91 computes  $X_*(N_{\Phi_i^*})$  for each  $i$ , so we see  $X^*(N_{\Phi_i^*})$  can be computed as the transpose of the matrix of  $X_*(N_{\Phi_i^*})$ . Attaching these matrices together gives a matrix representation for the above map, and we get our equations by computing the kernel of this matrix.

**Remark 2.93.** In practice, one can expand  $V = V_1 \oplus \cdots \oplus V_k$  into irreducible Hodge substructures and then work with  $E := F_1 \times \cdots \times F_k$  where  $F_i := Z(\text{End}_{\text{HS}}(V_i))$  for each  $i$ . Technically speaking,  $F$  may only embed into  $E$  “diagonally” because some  $V_\bullet$ s may be isomorphic to each other. However, this does not really affect anything we do because we may as well work with the image of  $Z(\text{MT}(V))^\circ$  under the inclusion  $T_F \subseteq T_E$ . Working with  $T_E$  is more convenient because it can actually be identified with the diagonal torus of  $\text{GL}(V)_E$  instead of merely a diagonally embedded subtorus.



CHAPTER 3

# REVIEW OF ABELIAN VARIETIES

---

CHAPTER 4

# **ABSOLUTE HODGE CYCLES**

---

# THE SATO–TATE CONJECTURE

---

## 5.1 Computing Monodromy Groups

# CHAPTER 6

## THE FERMAT CURVE

---

In this chapter, we will study the Galois representation attached to the projective  $\mathbb{Q}$ -curve  $X_N^1 \subseteq \mathbb{P}_{\mathbb{Q}}^1$  cut out by the equation

$$X_N: X^N + Y^N + Z^N = 0,$$

where  $N \geq 3$  is some nonnegative integer. For the rest of this chapter, we will fix  $N$  and thus denote this curve by  $X \subseteq \mathbb{P}_{\mathbb{Q}}^1$ . It is worthwhile to summarize the basic steps of the computation.

### 6.1 Homology and Cohomology

The exposition of this section follows [Ots16, Sections 2 and 3]. We will spend this section setting up some notation and proving basic facts about how these objects relate to each other.

#### 6.1.1 The Group Action

Throughout, it will be helpful to note that the finite algebraic  $\mathbb{Q}$ -group

$$G_N := \frac{\mu_N \times \mu_N \times \mu_N}{\Delta \mu_N}$$

acts on  $X_N$ ; here,  $\Delta \mu_N \subseteq \mu_N \times \mu_N \times \mu_N$  refers to the diagonally embedded copy of  $\mu_N$ . As with  $X_N$ , we will denote this group by  $G$  for the rest of the chapter, and we will let  $\zeta := \zeta_N$  be a primitive  $N$ th root of unity.

Notably, the action map  $G \times X \rightarrow X$  is defined over  $\mathbb{Q}$  even though  $G(\mathbb{Q})$  is trivial. For brevity, we will denote elements of  $G$  by  $g_{[r:s:t]} := [\zeta^r : \zeta^s : \zeta^t]$ . We will also have occasion to study the character group  $\widehat{G} := \widehat{G}_N$ , which we identify with

$$\widehat{G}_N = \{(a, b, c) \in (\mathbb{Z}/N\mathbb{Z})^3 : a + b + c = 0\}.$$

Explicitly, given a triple  $(a, b, c)$ , we let  $\alpha_{(a,b,c)}$  denote the corresponding character, which sends  $g_{[r:s:t]} \mapsto \zeta^{ra+bs+tc}$ .

In the sequel, we will have many vector spaces induced by  $X$  via (co)homology, which therefore have a  $G$ -action by functoriality. With this in mind, we make the following definition.

**Definition 6.1.** Given a  $\mathbb{Q}(\zeta)$ -vector space  $H$  with a  $G$ -action, we define

$$H_{\alpha} := \{v \in H : g \cdot v = \alpha(g)v\}$$

to be the  $\alpha$ -eigenspace for each  $\alpha \in \widehat{G}$ .

One inconvenience of this definition is that the vector spaces  $H$  of interest are frequently defined over  $\mathbb{Q}$ , but  $H_\alpha$  is not.

Thus, we note that some  $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts on  $\widehat{G}$  as follows: say  $\tau(\zeta) = \zeta^u$  for some  $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ , and then

$$(\tau\alpha)([\zeta^r : \zeta^s : \zeta^t]) = \alpha([\zeta^{u^{-1}r} : \zeta^{u^{-1}s} : \zeta^{u^{-1}t}]),$$

so we see that  $\tau\alpha = u^{-1}\alpha$ , where the multiplication  $u^{-1}\alpha$  is understood to happen where  $\alpha$  is a triple in  $(\mathbb{Z}/N\mathbb{Z})^3$ . With this in mind, given  $\alpha \in \widehat{G}$ , we let  $[\alpha] \subseteq \widehat{G}$  be the collection of characters of the form  $u\alpha$  as  $u \in (\mathbb{Z}/N\mathbb{Z})^\times$  varies; for example,  $-\alpha \in [\alpha]$ . The point of this discussion is that we are able to build a decomposition

$$\mathbb{Q}[G] \cong \prod_{[\alpha] \in G/(\mathbb{Z}/N\mathbb{Z})^\times} \mathbb{Q}([\alpha]),$$

where  $\mathbb{Q}([\alpha])$  is the image of the map  $\mathbb{Q}[G] \rightarrow \mathbb{C}$  given by the characters in  $[\alpha]$ . We are now ready to make the following definition.

**Definition 6.2.** Given some  $\mathbb{Q}$ -vector space  $H$  with a  $G$ -action, we are now ready to define

$$H_{[\alpha]} := \left\{ v \in H : v \otimes 1 \in \bigoplus_{\beta \in [\alpha]} (H \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\alpha} \right\}.$$

The discussion of the Galois action of the previous paragraph implies that  $H_{[\alpha]}$  is a generalized eigenspace of the  $G$ -action on  $H$ . In particular, we find that  $H_{[\alpha]} \otimes \overline{\mathbb{Q}} = \bigoplus_{\beta \in [\alpha]} H_{\beta}$ , so  $H = \bigoplus_{[\alpha]} H_{[\alpha]}$ .

### 6.1.2 Differential Forms

In this subsection, we will define a few differential forms. A reasonable reference for this subsection is [Lan11, Section 1.7]. A computation with the Riemann–Hurwitz formula shows that the genus of  $X$  is  $\frac{(N-1)(N-2)}{2}$ , so we know that there are many holomorphic differential forms. On the other hand, we know that the space of differential forms lives in  $H_{\text{dR}}^1(X(\mathbb{C}), \mathbb{C})$ , which is equipped with a  $G$ -action. Anyway, we are now ready to define our differential form.

**Definition 6.3.** Fix notation as above. For  $a \in \mathbb{Z}/N\mathbb{Z}$ , let  $[a]$  be a representative in  $\{0, 1, \dots, N-1\}$ . For any  $\alpha_{(a,b,c)} \in \widehat{G}$ , we define the differential form

$$\omega_{\alpha_{(a,b,c)}} := x^{[a]} y^{[b]-N} \frac{dx}{x}$$

in the affine patch  $x^N + y^N + 1 = 0$  of  $X$ . In the sequel, we may also denote this differential form by  $\omega_{(a,b,c)}$ .

**Remark 6.4.** Because  $x^N + y^N + 1 = 0$  implies  $x^{N-1} dx = -y^{N-1} dy$ , we also see that

$$\omega_{(a,b,c)} = -x^{[a]-N} y^{[b]} \frac{dy}{y}.$$

Further, we can pass to the affine patch  $1 + v^N + u^N = 0$  of  $X$  by substituting  $(x, y) = (1/u, v/u)$ , for which we note  $d(1/u)/(1/u) = -du/u$  so that

$$\omega_{(a,b,c)} = -u^{N-[a]-[b]} v^{[b]-N} \frac{du}{u}.$$

From Remark 6.4, we see that  $\omega_{(a,b,c)}$  always succeeds at being meromorphic with poles only at points of the form  $[X : Y : 0]$ , and it is closed (i.e., has vanishing residues) if and only if  $0 \notin \{a, b, c\}$ . Further, we see that  $\omega_{(a,b,c)}$  succeeds at being holomorphic provided that we also have  $[a] + [b] < N$ , which we note is equivalent to  $[a] + [b] + [c] = N$ .

We have now provided  $\frac{(N-1)(N-2)}{2}$  holomorphic differentials of  $X$ , so we would like to check that we have actually found a basis of  $H^0(X(\mathbb{C}), \Omega_{X/\mathbb{C}}^1)$ . Well, these differential forms are nonzero by construction,<sup>1</sup> and they are linearly independent because they are all eigenvectors for the  $G$ -action.

**Lemma 6.5.** Fix notation as above. For each  $\alpha \in \widehat{G}$ , the differential form  $\omega_\alpha$  is an eigenvector for the  $G$ -action with eigenvalue  $\alpha$ .

*Proof.* Say  $\alpha = \alpha_{(a,b,c)}$  for some  $a, b, c \in \mathbb{Z}/N\mathbb{Z}$ . Then for any  $g_{[r:s:0]} \in G$ , we note

$$\begin{aligned} (g_{[r:s:0]})^* \omega_{(a,b,c)} &= (\zeta^r x)^{[a]} (\zeta^s y)^{[b]-N} \frac{d(\zeta^r x)}{(\zeta^r x)} \\ &= \zeta^{ar+bs} \cdot x^{[a]} y^{[b]-N} \frac{dx}{x} \\ &= \alpha_{(a,b,c)}(g_{[r:s:0]}) \omega_{(a,b,c)}. \end{aligned}$$

The reason to  $g_{[r:s:0]}$  in the above computation is that we need the  $G$ -action to stay in the affine patch of points of the form  $[X : Y : 1]$ . ■

**Remark 6.6.** Thus, we see that our differential forms must be linearly independent because they are eigenvectors with different eigenvalues. As such, we have constructed eigenbases of  $H_{\text{dR}}^1(X(\mathbb{C}), \mathbb{C})$  and  $H^0(X(\mathbb{C}), \Omega_{X/\mathbb{C}}^1)$ .

While we're here, we compute the Poincaré pairing of our basis of differential forms.

**Lemma 6.7.** Fix notation as above. Choose  $\alpha, \alpha' \in \widehat{G}$  such that  $\alpha = (a, b, c)$  and  $\alpha' = (a', b', c')$  have nonzero entries. Then the Poincaré pairing

$$P: H_{\text{dR}}^1(X(\mathbb{C}), \mathbb{C}) \times H_{\text{dR}}^1(X(\mathbb{C}), \mathbb{C}) \rightarrow \mathbb{C}$$

given by  $(\omega, \eta) \mapsto \frac{1}{2\pi i} \int_X (\omega \wedge \eta)$  sends  $(\omega_\alpha, \omega_{\alpha'})$  to

$$P(\omega_\alpha, \omega_{\alpha'}) = \begin{cases} 0 & \text{if } \alpha \neq -\alpha', \\ (-1)^N \frac{N}{N-[a]-[b]} & \text{if } \alpha = -\alpha'. \end{cases}$$

*Proof.* We use the Poincaré residue, which implies that

$$P(\omega, \eta) = \sum_{x \in X(\mathbb{C})} \text{Res}_x \left( \eta \int \omega \right),$$

where the sum is over poles, and  $\int \omega$  refers to any choice of local primitive for  $\omega$  in the neighborhood of  $x$ . To use this, we note that the computation of Remark 6.4 implies that  $\omega_\alpha$  and  $\omega_{\alpha'}$  can only have poles at the points  $[1 : -\zeta^s : 0]$  for some  $s \in \mathbb{Z}/N\mathbb{Z}$ , and in this neighborhood, we may write

$$\omega_\alpha = -u^{N-[a]-[b]} v^{[b]-N} \frac{du}{u}$$

<sup>1</sup> Later, Remark 6.11 will give another way to prove this via periods.

and similarly for  $\omega_{\alpha'}$ . In particular, we see that

$$-\frac{1}{N - [a] - [b]} u^{N - [a] - [b]} v^{[b] - N}$$

makes a reasonable primitive for  $\omega_{\alpha}$ , so the Poincaré residue yields

$$P(\omega_{\alpha}, \omega_{\alpha'}) = \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \text{Res}_{(-\zeta^s, 0)} \left( -\frac{1}{N - [a] - [b]} u^{N - [a] - [b]} v^{[b] - N} \cdot -u^{N - [a'] - [b']} v^{[b'] - N} \frac{du}{u} \right).$$

Now, if  $\alpha \neq \alpha'$ , then we see that we are computing the residues of some monomial times  $du/u$ , but the power of  $u$  in the monomial is nonzero, so the residues all vanish. Lastly, we need to discuss what happens with  $\alpha = -\alpha'$ , where we see

$$\begin{aligned} P(\omega_{\alpha}, \omega_{-\alpha}) &= \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \text{Res}_{(-\zeta^s, 0)} \left( -\frac{1}{N - [a] - [b]} u^{N - [a] - [b]} v^{[b] - N} \cdot -u^{N - [-a] - [-b]} v^{[-b] - N} \frac{du}{u} \right) \\ &= \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \text{Res}_{(-\zeta^s, 0)} \left( -\frac{1}{N - [a] - [b]} u^{N - [a] - [b]} v^{[b] - N} \cdot u^{[a] + [b] - N} v^{-[b]} \frac{du}{u} \right) \\ &= \frac{1}{N - [a] - [b]} \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \text{Res}_{(-\zeta^s, 0)} \left( v^{-N} \frac{du}{u} \right) \\ &= \frac{1}{N - [a] - [b]} \sum_{s \in \mathbb{Z}/N\mathbb{Z}} (-\zeta^s)^{-N} \\ &= (-1)^N \frac{N}{N - [a] - [b]}, \end{aligned}$$

as desired. ■

### 6.1.3 Some Group Elements

In this subsection, we define a few elements of  $\mathbb{Q}[G]$  which we will then use in the next subsection. We begin with the three elements

$$t := \sum_{g \in G} g, \quad v := \sum_{s \in \mathbb{Z}/N\mathbb{Z}} g_{[0:s:0]}, \quad \text{and} \quad h := \sum_{r \in \mathbb{Z}/N\mathbb{Z}} g_{[r:0:0]}.$$

We take a moment to note that these elements satisfy the relations  $tg = gt = t$  for any  $g \in G$ , and  $t = hv = vh$ , and  $v^2 = Nv$  and  $h^2 = Nh$ . In the sequel, we will get a lot of mileage out of the idempotent

$$p := \frac{1}{N^2} \sum_{r, s \in \mathbb{Z}/N\mathbb{Z}} (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}).$$

Let's check that  $p$  is idempotent.

**Lemma 6.8.** Fix notation as above.

(a) Then  $p$  is idempotent.

(b) For any  $r, s \in \mathbb{Z}/N\mathbb{Z}$ , we have  $(1 - g_{[r:0:0]})(1 - g_{[0:s:0]})p = (1 - g_{[r:0:0]})(1 - g_{[0:s:0]})$ .

*Proof.* Both claims hinge upon the fact that a direct expansion of  $(1 - g_{[r:0:0]})(1 - g_{[0:s:0]})$  implies

$$p = \frac{1}{N^2} (N^2 - Nh - Nv + t).$$

We now show the claims separately.

(a) This is a direct computation: write

$$\begin{aligned}
 p^2 &= \frac{1}{N^4} (N^2 - Nh - Nv + t) (N^2 - Nh - Nv + t) \\
 &= \frac{1}{N^4} (N^4 + N^2 h^2 + N^2 v^2 + t^2 - 2N^3 h - 2N^3 v + 2N^2 t + N^2 hv - 2Nht - 2Nvt) \\
 &= \frac{1}{N^4} (N^4 + N^3 h + N^3 v + N^2 t - 2N^3 h - 2N^3 v + 2N^2 t + N^2 t - 2N^2 t - 2N^2 t) \\
 &= \frac{1}{N^4} (N^4 - N^3 h - N^3 v + N^2 t) \\
 &= p.
 \end{aligned}$$

(b) We will compute as in (a): note  $h(1 - g_{[r:0:0]}) = 0$  and  $v(1 - g_{[0:s:0]}) = 0$ , so

$$\begin{aligned}
 (1 - g_{[r:0:0]})(1 - g_{[0:s:0]})p &= (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}) \cdot \frac{1}{N^2} (N^2 - Nh - Nv + hv) \\
 &= (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}) \cdot \frac{N^2}{N^2} + 0 + 0 + 0 \\
 &= (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}),
 \end{aligned}$$

as required. ■

### 6.1.4 Homology

In this subsection, we will study  $H_1^B(X(\mathbb{C}), \mathbb{Q})$ . By the end, we will define a 1-cycle  $\gamma := \gamma_N$  such that  $H_1^B(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[G] \cdot [\gamma]$ . Morally, this means that we can understand our homology by focusing on this one cycle.

To begin, we need some path in  $X(\mathbb{C})$ , so we define  $\delta: [0, 1] \rightarrow X(\mathbb{C})$  by

$$\delta(t) := [t^{1/N} : (1-t)^{1/N} : \zeta_{2N}^{-1}].$$

Notably,  $\delta(0) = [0 : 1 : \zeta_{2N}^{-1}]$  and  $\delta(1) = [1 : 0 : \zeta_{2N}^{-1}]$ , so  $g = [\zeta^r : \zeta^s : 1]$  has  $g_*\delta(0) = [0 : \zeta^s : \zeta_{2N}^{-1}]$  and  $g_*\delta(1) = [\zeta^r : 0 : \zeta_{2N}^{-1}]$ . The point of this computation is that we see

$$(1 - g_{[r:0:0]} - g_{[0:s:0]} + g_{[r:s:0]})_*\delta \in Z_1^B(X(\mathbb{C}), \mathbb{Q}).$$

We are now ready to define  $\gamma$ .

**Definition 6.9.** Fix notation (and in particular  $\delta$ ) as above. Then we define

$$\gamma := \frac{1}{N^2} \sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - g_{[r:0:0]})(1 - g_{[0:s:0]})_*\delta.$$

Note  $\gamma = p_*\delta$ .

The above computation shows that  $\gamma \in Z_1^B(X(\mathbb{C}), \mathbb{Q})$ . We will want to know its periods later. Note that the following result is essentially a special case of [Del18, Lemma 7.12].

**Lemma 6.10.** Fix notation as above. Suppose  $(a, b, c) \in (\mathbb{Z}/N\mathbb{Z})^3$  has no nonzero entries. Then

$$\int_{\gamma} \omega_{(a,b,c)} = \zeta_{2N}^{[a]/N + [b]/N - 1} \frac{\Gamma\left(\frac{[a]}{N}\right) \Gamma\left(\frac{[b]}{N}\right)}{\Gamma\left(\frac{[a]}{N} + \frac{[b]}{N}\right)}.$$



*Proof.* This is a direct computation. Denote the integral by  $P(\gamma, \omega_{(a,b,c)})$ . By adjunction,  $\int_{p_*\delta} \omega_{(a,b,c)} = \int_{\delta} p^* \omega_{(a,b,c)}$ . This allows us to compute

$$\begin{aligned} P(\gamma, \omega_{(a,b,c)}) &= \frac{1}{N^2} \int_{\delta} \sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - g_{[r:0:0]}) (1 - g_{[0:s:0]})^* \omega_{(a,b,c)} \\ &= \frac{1}{N^2} \int_{\delta} \sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - \zeta^{ar}) (1 - \zeta^{bs}) \omega_{(a,b,c)} \\ &= \left( \frac{1}{N^2} \sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - \zeta^{ar}) (1 - \zeta^{bs}) \right) \int_{\delta} \omega_{(a,b,c)} \\ &= \left( \frac{1}{N^2} \sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - \zeta^{ar}) (1 - \zeta^{bs}) \right) \zeta_{2N}^{[a]/N + [b]/N - 1} \int_0^1 t^{[a]/N} (1-t)^{[b]/N-1} \frac{dt}{t}. \end{aligned}$$

The last integral (famously) equals the Beta function, and it evaluates to  $\Gamma\left(\frac{[a]}{N}\right) \Gamma\left(\frac{[b]}{N}\right) \Gamma\left(\frac{[a]+[b]}{N}\right)^{-1}$ . We take a moment to check that

$$\sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - \zeta^{ar}) (1 - \zeta^{bs}) \stackrel{?}{=} N^2.$$

Well,  $(1 - \zeta^{ar}) (1 - \zeta^{bs}) = 1 - \zeta^{ar} - \zeta^{bs} + \zeta^{ar+bs}$ , and because  $a, b \neq 0$ , we see that summing over  $r$  and  $s$  causes the terms not equal to 0 to vanish. Thus, we are left with  $N^2$ . ■

**Remark 6.11.** Because the right-hand side is nonzero, Lemma 6.10 implies that the differential forms  $\omega_{(a,b,c)}$  are nonzero.

We are now ready to show that  $H_1^B(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[G] \cdot [\gamma]$ .

**Lemma 6.12.** Fix notation as above. Then  $H_1^B(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[G] \cdot [\gamma]$ .

*Proof.* It is enough to show that  $H_1^B(X(\mathbb{C}), \mathbb{C}) = \mathbb{C}[G] \cdot [\gamma]$ . Note that there is a canonical pairing

$$\begin{aligned} H_1^B(X(\mathbb{C}), \mathbb{C}) \times H_{\text{dR}}^1(X(\mathbb{C}), \mathbb{C}) &\rightarrow \mathbb{C} \\ (c, \omega) &\mapsto \int_c \omega \end{aligned}$$

which is perfect by the de Rham theorem. We already have a basis  $\{\omega_{(a,b,c)}\}_{a,b,c \neq 0}$  of  $H_{\text{dR}}^1(X(\mathbb{C}), \mathbb{C})$ , so we will find a dual basis for  $H_1^B(X(\mathbb{C}), \mathbb{C})$  inside  $\mathbb{C}[G] \cdot [\gamma]$ . Well, for  $g \in G$  and  $\alpha \in \hat{G}$ , we see

$$\int_{g^*\gamma} \omega_{\alpha} = \int_{\gamma} g^* \omega_{\alpha}$$

equals  $\alpha(g) P(\gamma, \omega_{\alpha})$ , where  $P(\gamma, \omega_{\alpha}) := \int_{\gamma} \omega_{\alpha}$  is the (nonzero!) period computed in Lemma 6.10. With this in mind, we define

$$c_{\alpha} := \frac{1}{N^2 P(\gamma, \omega_{\alpha})} \sum_{g \in G} \alpha(g)^{-1} g^* [\gamma]$$

for each  $\alpha = \alpha_{(a,b,c)}$  with  $a, b, c \neq 0$ . Then we see that  $\int_{c_{\alpha}} \omega_{\beta} = 1_{\alpha=\beta}$  by the orthogonality relations, so  $\{c_{\alpha}\}$  is a dual basis of  $H_1^B(X(\mathbb{C}), \mathbb{C})$ , and it lives in  $\mathbb{C}[G] \cdot [\gamma]$  by its construction. ■

## 6.2 Galois Action

We now use the notation set up in the previous section to write out the Galois action on the space of some absolute Hodge cycles attached to  $X$ . Throughout this section,  $p$  is a nonnegative index. We take a moment to note that the action of  $G$  on  $X$  upgrades into an action of  $G^{2p}$  on  $X^{2p}$ . Our exposition closely follows [GGL24, Subsection 8.5]. As in section 6.1.1, we will identify  $\widehat{G}^{2p}$  with some subset of tuples in  $(\mathbb{Z}/N\mathbb{Z})^{6p}$ . And for a vector space  $H$  defined over  $\mathbb{Q}(\zeta)$  (respectively,  $\mathbb{Q}$ ) and character  $\alpha \in \widehat{G}^{2p}$ , we define  $H_\alpha$  (respectively,  $H_{[\alpha]}$ ) as the corresponding  $\alpha$ -eigenspace (respectively,  $[\alpha]$ -generalized eigenspace).

### 6.2.1 Hodge Cycles on $X^{2p}$

To understand the geometry of  $X$ , we will only be interested in tensor powers of  $H^1(X)$  (for a choice of cohomology theory  $H$ ), which by the Künneth formula embed as

$$H^1(X)^{\otimes 2p} \subseteq H^{2p}(X^{2p}).$$

When  $H$  is de Rham cohomology  $H_{\text{dR}}$ , we thus see we are interested in when the image of an element in  $H_{\text{dR}}^1(X)^{\otimes p}$  succeeds at being a Hodge cycle. Well, note that the action of  $G$  on  $H_{\text{dR}}^1(X, \mathbb{C})$  extends to an action of  $G^{2p}$  on  $H_{\text{dR}}^1(X, \mathbb{C})^{\otimes 2p}$ . This action diagonalizes with one-dimensional eigenspaces by extending Remark 6.6. We will use properties of the diagonalization to read off when we have an element of bidegree  $(p, p)$  in  $H_{\text{dR}}^{2p}(X^{2p}, \mathbb{C})$ .

Following [Del18, Proposition 7.6], it will be useful to have the following definition.

**Definition 6.13 (weight).** Given a function  $f: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{Z}$ , we define its *weight map* as the function  $\langle f \rangle: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{Q}$  defined by

$$\langle f \rangle := \frac{1}{N} \sum_{a \in \mathbb{Z}/N\mathbb{Z}} f(ua)[a]$$

For  $p \geq 0$ , we note that we may identify  $\widehat{G}^{2p}$  with a tuple in  $(\mathbb{Z}/m\mathbb{Z})^{2p}$ , and then we define the *weight*  $\langle \alpha \rangle$  of a character  $\alpha \in \widehat{G}^{2p}$  as  $\langle 1_\alpha \rangle(1)$ , where  $1_\alpha: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{Z}$  is the multiplicity of an element in  $\mathbb{Z}/N\mathbb{Z}$  in the tuple  $\alpha$ .

**Remark 6.14.** The point of this definition is as follows: given  $\alpha \in \widehat{G}$  with  $\alpha = (a, b, c)$  having nonzero entries, we note that  $\omega_\alpha$  has two possible cases.

- If  $[a] + [b] + [c] = N$  so that  $\langle \alpha \rangle = 1$ , then  $\omega_{(a,b,c)}$  is holomorphic so that  $\omega_\alpha \in H^{10}(X)$ .
- If  $[a] + [b] + [c] = 2N$  so that  $\langle \alpha \rangle = 2$ , then  $\omega_\alpha$  is not holomorphic so that  $\omega_\alpha \in H^{01}(X)$ .

In all cases, we find  $\omega_\alpha \in H^{2-\langle \alpha \rangle, \langle \alpha \rangle-1}(X)$ .

We now upgrade Remark 6.14 to  $H_{\text{dR}}^1(X, \mathbb{C})^{\otimes 2p}$ .

**Lemma 6.15.** Choose  $\alpha \in \widehat{G}^{2p}$  as  $\alpha = (\alpha_1, \dots, \alpha_{2p})$  having nonzero entries. Then

$$\omega_\alpha := \omega_{\alpha_1} \otimes \dots \otimes \omega_{\alpha_{2p}}$$

embedded in  $H_{\text{dR}}^{2p}(X^{2p}, \mathbb{C})$  is of bidegree  $(4p - \langle \alpha \rangle, \langle \alpha \rangle - 2p)$ .

*Proof.* Because the Künneth isomorphism upgrades to an isomorphism of Hodge structures, it is enough to note that  $\omega_{\alpha_i} \in H^{\langle \alpha_i \rangle, \langle \alpha_i \rangle-1}$  (see Remark 6.14) implies  $\omega_\alpha$  has bidegree

$$\left( 4p - \sum_{i=1}^{2p} \langle \alpha_i \rangle, \sum_{i=1}^{2p} \langle \alpha_i \rangle - 2p \right).$$

The proposition follows because weight is additive. ■

**Proposition 6.16.** Choose  $\alpha \in \widehat{G}^{2p}$  as  $\alpha = (\alpha_1, \dots, \alpha_{2p})$  having nonzero entries. Then  $H_B^{2p}(X^{2p})_{[\alpha]}$  is one-dimensional over  $\mathbb{Q}([\alpha])$ , and the following are equivalent.

- (a)  $H_B^{2p}(X^{2p})_{[\alpha]}(p)$  consists entirely of Hodge classes.
- (b) We have  $\langle u\alpha \rangle = 3p$  for all  $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ .

*Proof.* We begin by embedding

$$H_B^{2p}(X^{2p}, \mathbb{Q})_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} H_B^{2p}(X^{2p}, \mathbb{C})_{u\alpha}$$

into

$$H_{dR}^{2p}(X^{2p}, \mathbb{C}) = \bigoplus_{\substack{q_1, \dots, q_{2p} \\ q_1 + \dots + q_{2p} = 2p}} H_{dR}^{q_1}(X, \mathbb{C}) \otimes \dots \otimes H_{dR}^{q_{2p}}(X, \mathbb{C}),$$

where this last equality holds by the Künneth isomorphism. Quickly, we reduce to the case where  $q_1 = \dots = q_{2p} = 1$ : for each  $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ , we note that  $u\alpha$  has nonzero entries. On the other hand, the  $G$ -action on  $H^0(X) = \mathbb{C}$  is always trivial, so we note that if any of the  $q_i$ s are not equal to 1, then one of them must equal 0, meaning that

$$(H_{dR}^{q_1}(X, \mathbb{C}) \otimes \dots \otimes H_{dR}^{q_{2p}}(X, \mathbb{C}))_{u\alpha} = H_{dR}^{q_1}(X, \mathbb{C})_{u\alpha_1} \otimes \dots \otimes H_{dR}^{q_{2p}}(X, \mathbb{C})_{u\alpha_{2p}}$$

is the zero vector space. Thus, we see that

$$H_{dR}^{2p}(X^{2p}, \mathbb{C})_{[\alpha]} = \bigoplus_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} (H_{dR}^1(X, \mathbb{C})^{\otimes 2p})_{u\alpha}.$$

The comparison isomorphism now implies that  $H_B^{2p}(X^{2p}, \mathbb{Q})_{[\alpha]}$  has dimension  $[\mathbb{Q}([\alpha]) : \mathbb{Q}]$  over  $\mathbb{Q}$  and thus one dimension over  $\mathbb{Q}([\alpha])$ .

It remains to show that (a) and (b) are equivalent. Well, the  $\mathbb{Q}$ -vector space  $H_B^{2p}(X^{2p}, \mathbb{Q})_{[\alpha]}(p)$  will consist of Hodge classes if and only if  $(H_{dR}^1(X, \mathbb{C})^{\otimes 2p})_{u\alpha}$  is of bidegree  $(p, p)$ , which is equivalent to  $\langle u\alpha \rangle = 3p$  by Lemma 6.15. ■

## 6.2.2 An Absolute Hodge Cycle

## 6.2.3 Computations on de Rham Component

## 6.2.4 End of the Computation

# 6.3 Fermat Hypersurfaces

We would be remiss without mentioning something about Fermat hypersurfaces. Thus, we will state (but not prove) a few facts about what is known for Fermat hypersurfaces. There is much known here, but the proofs tend to be somewhat harder than what one does with the Fermat curves, which is why we have avoided the theory.

CHAPTER 7

# **FAMILIES OF CURVES**

---

# BIBLIOGRAPHY

---

- [Mum74] David Mumford. *Abelian varieties*. eng. 2nd ed. / with appendices by C.P. Ramanujam and Yuri Manin. Published for the Tata Institute of Fundamental Research, Bombay [by] Oxford University Press, 1974. ISBN: 9780195605280.
- [Rib76] Kenneth A. Ribet. "Galois Action on Division Points of Abelian Varieties with Real Multiplications". In: *American Journal of Mathematics* 98.3 (1976), pp. 751–804. ISSN: 00029327, 10806377. URL: <http://www.jstor.org/stable/2373815> (visited on 11/29/2024).
- [Yan94] H. Yanai. "On Degenerate CM-Types". In: *Journal of Number Theory* 49.3 (1994), pp. 295–303. ISSN: 0022-314X. DOI: <https://doi.org/10.1006/jnth.1994.1095>. URL: <https://www.sciencedirect.com/science/article/pii/S0022314X8471095X>.
- [MZ95] B. J. J. Moonen and Yu. G. Zarhin. "Hodge classes and Tate classes on simple abelian fourfolds". In: *Duke Math. J.* 77.3 (1995), pp. 553–581. ISSN: 0012-7094. DOI: [10.1215/S0012-7094-95-07717-5](https://doi.org/10.1215/S0012-7094-95-07717-5). URL: <https://doi-org.libproxy.berkeley.edu/10.1215/S0012-7094-95-07717-5>.
- [Moo99] Ben Moonen. *Notes on Mumford–Tate Groups*. 1999. URL: <https://www.math.ru.nl/~bmoonen/Lecturenotes/CEBnotesMT.pdf> (visited on 10/18/2024).
- [Lan11] Serge Lang. *Complex Multiplication*. 1st ed. Springer New York, NY, 2011.
- [Lom13] Davide Lombardo. *Mumford–Tate groups and Hodge classes on Abelian varieties of low dimension*. 2013. URL: <https://core.ac.uk/download/pdf/18603931.pdf>.
- [Yu15] Chia-Fu Yu. "A NOTE ON THE MUMFORD–TATE CONJECTURE FOR CM ABELIAN VARIETIES". In: *Taiwanese Journal of Mathematics* 19.4 (2015), pp. 1073–1084. ISSN: 10275487, 22246851. URL: <http://www.jstor.org/stable/taiwjmath.19.4.1073> (visited on 11/30/2024).
- [Lom16] Davide Lombardo. "On the  $\ell$ -adic Galois representations attached to nonsimple abelian varieties". In: *Ann. Inst. Fourier (Grenoble)* 66.3 (2016), pp. 1217–1245. ISSN: 0373-0956. DOI: [10.5802/aif.3035](https://doi.org/10.5802/aif.3035). URL: <https://doi-org.libproxy.berkeley.edu/10.5802/aif.3035>.
- [Ots16] Noriyuki Otsubo. "Homology of the Fermat tower and universal measures for Jacobi sums". In: *Canad. Math. Bull.* 59.3 (2016), pp. 624–640. ISSN: 0008-4395. DOI: [10.4153/CMB-2016-012-0](https://doi.org/10.4153/CMB-2016-012-0). URL: <https://doi.org/10.4153/CMB-2016-012-0>.
- [Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.
- [Mil17] J. S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017. DOI: [10.1017/9781316711736](https://doi.org/10.1017/9781316711736).
- [Del18] P. Deligne. "Hodge Cycles on Abelian Varieties". In: *Hodge Cycles, Motives, and Shimura Varieties*. Vol. 900. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 2018.

- [GGL24] Andrea Gallese, Heidi Goodson, and Davide Lombardo. *Monodromy groups and exceptional Hodge classes*. arXiv:2405.20394. July 2024. DOI: [10 . 48550 / arXiv . 2405 . 20394](https://doi.org/10.48550/arXiv.2405.20394). URL: <http://arxiv.org/abs/2405.20394> (visited on 10/18/2024).
- [Moo] Ben Moonen. *AN INTRODUCTION TO MUMFORD-TATE GROUPS*. en. URL: <https://www.math.ru.nl/~bmoonen/Lecturenotes/MTGps.pdf>.

# LIST OF DEFINITIONS

---

Hodge class, [7](#)  
Hodge group, [14](#)  
Hodge structure, [5](#)

Mumford–Tate group, [11](#)  
polarization, [8](#)

reflex norm, [30](#)  
reflex signature, [29](#)  
Rosati involution, [10](#)

signature, [25](#)  
weight, [42](#)