Student Automorphic Forms Seminar

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1 September 4th: Nir Elber

Today we're talking about the local theory of Tate's thesis.

1.1 A Little Global Theory

In order to not lose perspective in the Fourier analysis that makes up the body of this talk, we discuss a little global theory. The goal of Tate's thesis is to derive analytic properties of L-functions such as the following.

Definition 1. We define the *Riemann ζ-function* as

$$\zeta(s) \coloneqq \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Definition 2. Given a Dirichlet character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, we define the *Dirichlet L-function* by

$$L(s,\chi) \coloneqq \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}.$$

Definition 3. For a number field K, we define the Dedekind ζ-function of a number field K

$$\zeta_K(s) \coloneqq \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1-\operatorname{N}(\mathfrak{p})^{-s}}.$$

For now, "analytic properties" means deriving a meromorphic continuation, which in practice means deriving a functional equation.

Remark 4. There is a common generalization of the above two L-functions called a "Hecke L-function," but streamlined definitions would require a discussion of the adéles, which we are temporarily avoiding.

In fact, Hecke had proven functional equations before Tate, but Tate's arguments modernize better, which is why we will talk about them.

By way of example, we state the functional equation for the Riemann ζ-function $\zeta(s)$.

Theorem 5 (Riemann)**.** Define the completed Riemann ζ-function by

$$\Xi(s) := \pi^{-s/2}\Gamma(s)\zeta(s).$$

Then $\Xi(s)$ admits an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation $\Xi(s) = \Xi(1-s)$.

In some sense, the real goal of Tate's thesis is to explain the presence of the mysterious factor $\pi^{-s/2}\Gamma(s)$ which permits the functional equation $\Xi(s) = \Xi(1-s)$. To understand this, we write

$$\Xi(s) = \pi^{-s/2}\Gamma(s) \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

and the idea is that we should view this product over all places of $\mathbb{Q}$: the factor $\pi^{-s/2}\Gamma(s)$ belongs to the archimedean place of $\mathbb{Q}$!

Idea 6 (Tate)**.** Completed L-functions should be products over all places.

Very roughly, Idea 6 allows us to reduce global functional equations into products of local ones.

1.2 Local Z-integrals

We are interested in proving equations of the type $Z(s) \approx Z(1-s)$ for some suitable function Z. We will emply the following trick: we will show that both $Z(s)$ and $Z(1-s)$ live in the same one-dimensional space of functions and then study the scale factor between the two functions later.

To define our function Z, we take motivation from the definition of

$$\Gamma(s) = \int_{\mathbb{R}^+} e^{-t}t^s\,\frac{dt}{t}.$$

We know that this should correspond to the archimedean place of $\mathbb{Q}$, but we would like to extend this definition to the finite places. As such, we make the following observations.

- $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the archimedean place.
- The function $t \mapsto e^{-t}$ is an additive character $\mathbb{R} \to \mathbb{C}^\times$.
- The function $t \mapsto t^{-s}$ is a multiplicative character $\mathbb{R}^\times \to \mathbb{C}^\times$.
- The measure dt/t is a Haar measure of $\mathbb{R}^+$.

To begin our generalizations, we recall the definition of a local field, which will place $\mathbb{R}$ in the correct context.

Definition 7 (local field)**.** A *local field* is a locally compact nondiscrete topological field. It turns out that local fields of characteristic zero are exactly the finite extensions of $\mathbb{R}$ and $\mathbb{Q}_p$.

Next up, to place dt/t in the correct context, we should define a Haar measure.

Definition 8 (Haar measure)**.** Fix a locally compact topological group G. Then a *left-invariant Haar measure* $d\mu_\ell(g)$ is a Radon measure such that $\mu_\ell(gS) = \mu_\ell(S)$ for each $g \in G$ and measureable set S. In terms of intergrals, this is equivalent to

$$\int_G f(gh)\,dh = \int_G f(h)\,dh$$

for each $g \in G$ and integrable function f. It turns out that the Haar measure is unique up to scalar.

Remark 9. In general, a left-invariant Haar measure need not be right-invariant. However, this is frequently true: for example, if G is abelian or "reductive" (such as $G = \mathrm{GL}_n$), then left-invariant Haar measures are right-invariant.

Example 10. The Lebesgue measure dt is a Haar measure on $\mathbb{R}$. The measure $dt/\left|t\right|$ is a Haar measure on $\mathbb{R}^+$ and $\mathbb{R}^\times$.

Example 11. Fix a prime p. There is a unique Haar measure μ on $\mathbb{Q}_p$ such that $\mu(\mathbb{Z}_p) = 1$. For example, we find that $\mu(a + p\mathbb{Z}_p) = \frac{1}{p}$ for each $a \in \mathbb{Q}_p$.

Remark 12. Local fields turn out be normed, though this is not immediately obvious from the definition. Even though $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$ all have natural norms (extended from $\mathbb{Q}$), here is a hands-free way to obtain this norm from a local field K: choosing a Haar measure dt on K, we define the norm $|a|$ of some $a \in K$ as the scalar such that

$$d(at) = |a|\,dt.$$

It is not at all obvious that $|\cdot|$ is a norm (in particular, why does it satisfy the triangle inequality?), but it is true. As an example, $|\cdot|$ is the square of the Euclidean norm on $\mathbb{C}$.

Example 13. We are now able to say that $dt/\left|t\right|$ is a Haar measure of $K^\times$ for any local field K.

At this point, we may expect that our generalization of $\Gamma(s)$ to a generic local field K to be

$$Z(\psi, \omega) = \int_K \psi(t)\omega(t)\,\frac{dt}{|t|},$$

where $\psi\colon K \to \mathbb{C}^\times$ and $\omega\colon K^\times \to \mathbb{C}^\times$ are characters. However, this is a little too rigid for our purposes.

Notably, by taking linear combinations of additive characters, Fourier analysis explains that understanding Γ very well should permit understanding integrals of the form

$$\int_{\mathbb{R}^+} f(t)t^s\,\frac{dt}{t},$$

where $f\colon \mathbb{R} \to \mathbb{C}$ is some sufficiently nice function. It will help to have the flexibility that this extra f permits.

Definition 14. Fix a local field K. For a nice enough function f and character $\omega\colon K^\times \to \mathbb{C}^\times$, we define the *local Z-integral*

$$Z(f, \omega) := \int_{K^\times} f(t)\omega(t)\,\frac{dt}{|t|}.$$

We will not dwell on this, but perhaps we should explain what is required by "nice enough" function f. The keyword is "Schwartz–Bruhat."

Definition 15 (Schwartz–Bruhat)**.** Fix a local field K.

- If $K \in \{\mathbb{R}, \mathbb{C}\}$, we say $f\colon K \to \mathbb{C}$ is *Schwartz–Bruhat* if and only if it is infinitely differentiable and for all of its derivatives to decay rapidly (namely, $f(t)p(t) \to 0$ as $|t| \to \infty$ for any polynomial p).
- For other K, we say $f\colon K \to \mathbb{C}$ is *Schwartz–Bruhat* if and only if it is locally constant and compactly supported.

We let $S(K)$ denote the vector space of Schwartz–Bruhat functions on K, and we let $S(K)'$ denote its dual (i.e., the vector space of distributions).

Example 16. The function $t \mapsto e^{-t^2}$ is a Schwartz–Bruhat function $\mathbb{C} \to \mathbb{C}$.

Example 17. Fix a prime p. Then the indicator function $1_{\mathbb{Z}_p}$ on $\mathbb{Q}_p$ is Schwartz–Bruhat.

Importantly, the definition of Schwartz–Bruhat will promise that

1.3 Fourier Analysis

We now take a moment to review where we are standing. We were hoping to prove a statement like $Z(s) \approx Z(1-s)$, where $Z(s)$ perhaps has some kind analytic properties. However, we currently have a function $Z(f,\omega)$, so it's not at all obvious how to replace f and ω with "dual" entries or how to make sense of analytic properties. In this subsection, we address both of these concerns; they are both related to character theory.

Going in order of difficulty, it is a little easier to explain how to add analytic structure to Z. After taking norms, we can find some $s \in \mathbb{C}$ such $|\omega| = |\cdot|^s$ so that $\eta \coloneqq \omega\,|\cdot|^{-s}$ outputs to S^1, and

$$\omega = \eta\,|\cdot|^s\,.$$

Thus, we see that we can decompose characters $K^\times \to \mathbb{C}^\times$ into a unitary part η and an "unramified" part $|\cdot|^s$, and the unramified part now has a complex parameter $s \in \mathbb{C}$. Namely, $\omega\,|\cdot|^{-s}$ now outputs to S^1; i.e., this character is unitary. Fix a local field K.

Thus, we can view $Z(f,\eta)$ as having three parameters as $Z(f,\eta,s) \coloneqq Z\left(f,\eta\,|\cdot|^s\right)$, where we now require that η is unitary. Because we already have some notion of smoothness in the parameter $s \in \mathbb{C}$, it remains to understand smoothness in the parameter of unitary character η.

Example 18. In the archimedean case, the parameter η is not so interesting.

- The characters $\mathbb{R}^\times \to S^1$ take the form $t \mapsto t^{-a}\,|t|^s$ where $a \in \{0,1\}$ and $s \in \mathbb{C}$.
- The characters $\mathbb{C}^\times \to S^1$ take the form $z \mapsto z^a\overline{z}^b\,|z|^s$ where $a, b \in \mathbb{Z}$ and $s \in \mathbb{C}$.

We now understand that our functional equation for Z should arise from $Z(f,\eta,s)$. We hope to take $s \mapsto 1-s$, and it seems reasonable (by looking at functional equations for $L(s,\chi)$) to replace η with η^{-1}.

However, we still need to replace f with some dual function. In the archimedean case, we expect this to be the Fourier transform defined by

$$\widehat{f}(s) \coloneqq \int_{\mathbb{R}} f(t)e^{2\pi i s t}\,dt.$$

Such a definition will more or less carry through for arbitrary local fields, but we will have to go through approximately the same dictionary that defined $Z(f,\omega)$ from $\Gamma(s)$. In particular, the functions $t \mapsto e^{2\pi i s t}$ list the characters of $\mathbb{R}$, so the above is the integration of our function f against the list of characters. To this end, we pick up the following theorem.

Theorem 19. Fix a local field K. Then there exists a nontrivial character $\psi\colon K \to S^1$. Any choice of ψ defines a bijection $K \to \widehat{K}$ by sending $a \in K$ to the character $\psi_a(t) \coloneqq \psi(at)$.

Remark 20. For a general locally compact abelian group G, one can give $\widehat{G} := \operatorname{Hom}(G, \mathbb{C})$ a locally compact topology. With this topology in hand, one actually finds that the map $K \to \widehat{K}$ given by $a \mapsto \psi_a$ is an isomorphism of locally compact abelian groups. However, we do not currently have the need to work in this level of generality.

We are now ready to define our Fourier transform.

Definition 21. Fix a local field K, and choose a nontrivial character $\psi\colon K \to S^1$. For $f \in S(K)$, we define the *Fourier transform*

$$\mathcal{F}_\psi f(t) := \int_K f(t)\psi(st)\,dt.$$

Example 22. For $K = \mathbb{R}$, choose $\psi(t) := e^{2\pi it}$. Then the Fourier transform (up to normalization) agrees with the usual one

$$\mathcal{F}_\psi f(t) := \int_{\mathbb{R}} f(t)e^{2\pi ist}\,dt.$$

It turns out that one has the usual properties for the Fourier transform, such as $\mathcal{F}_\psi\mathcal{F}_\psi f(t) = f(-t)$.

We are now ready to state the local functional equation.

Theorem 23. Fix a local field K, and choose a nontrivial character $\psi\colon K \to S^1$. For every $f \in S(K)$, the function $Z(f, \omega, s)$ has a meromorphic continuation to $s \in \mathbb{C}$ (with well-understood poles) and satisfies a functional equation of the form

$$Z(f, \omega, s) = \gamma(\psi, \omega, s, dx)Z\left(\mathcal{F}_\psi f, \omega^{-1}, 1-s\right),$$

where $\gamma(\psi, \omega, s, dx)$ is meromorphic in $s \in \mathbb{C}$ (with well-understood poles).

Tate's original proof of Theorem 23 was more or less by an explicit computation: by some argument interchanging integrals, one can relate $Z(f, \omega, s)$ with $Z(g, \omega, s)$ for separate $f, g \in S(K)$, which allows us to reduce the proof to a single f. Then one chooses some f such that $\mathcal{F}_\psi f = f$ (e.g., one chooses a Gaussian when $K = \mathbb{R}$) and does an explicit computation to check the result.

1.4 Multiplicity One

We are not going to prove Theorem 23 in detail, but we will gesture towards an argument which uses a bit more functional analysis and less explicit computation. The key point is that $Z(-, \omega)\colon S(K) \to \mathbb{C}$ is a distribution with the curious property that

$$\int_{K^\times} f(at)\omega(t)\,\frac{dt}{t} = \omega^{-1}(a)\int_{K^\times} f(t)\omega(t)\,\frac{dt}{|t|}.$$

Thus, we see that $Z(-, \omega)$ is an eigendistribution of sorts. To make this more explicit, we let $K^\times$ act on $S(K)$ by translation: for $a \in K$ and $f \in S(K)$, we define $(r(a)f)(t) := f(at)$. Then $K^\times$ acts on the distributions $S(K)'$ accordingly: for $a \in K$ and $\lambda \in S(K)'$ and $f \in S(K)$, we can define

$$\langle r'(a)\lambda, f\rangle := \left\langle \lambda, r(a)^{-1}f\right\rangle.$$

Tracking everything through, we see that $Z(-, \omega) \in S(K)'$ is an eigendistribution for the action of $K^\times$ on $S(K)'$ with eigenvalue given by the character ω.

Notation 24. Let $S(K)'(\omega)$ denote the space of $K^\times$-eigendistributions with eigenvalue ω.

Now, we see that $Z(-,\omega) \in S(K)'(\omega)$, and one can check that $Z\left(\mathcal{F}_\psi -, \omega^{-1}\right)$ is also in $S(K)'(\omega)$. Thus, a statement like Theorem 23 will follow from the following "representation-theoretic" multiplicity one result.

Theorem 25. Fix a local field K and character $\omega\colon K \to \mathbb{C}^\times$. Then

$$\dim S(K)'(\omega) = 1.$$

Let's explain the sort of inputs that go into proving Theorem 25, but we will not say more.

- One can verify that $Z(-,\omega)$ provides some vector in $S(K)'(\omega)$, so this space is at least nonempty.
- Let $C_c^\infty(K)$ denote the set of compactly supported Schwartz functions on K. Duality provides an exact sequence
$$0 \to S(K)'_0 \to S(K)' \to C_c^\infty(K)' \to 0,$$
where $S(K)'_0$ denotes the distributions supported at 0. Taking eigenspaces, we get an exact sequence
$$0 \to S(K)'_0(\omega) \to S(K)'(\omega) \to C_c^\infty(K)'(\omega).$$
- One can show that $C_c^\infty(K)'(\omega)$ is one-dimensional with basis given by $f \mapsto \int_K f(t)\omega(t)\, dt/\,|t|$.
- Understanding $S(K)'_0$ comes down to some casework. For example, we look at the nonarchimedean case. If ω is trivial, we have the distribution $f \mapsto f(0)$; otherwise, $S(K)'_0(\omega)$ is zero.