The Chinese Remainder Theorem

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Abstract

In this short note, we introduce the statement and give some example applications of the Chinese remainder theorem. We assume some familiarity with modular arithmetic. The reader is strongly encouraged to work out the exercises.

Contents

1 Introduction

The Chinese remainder theorem codifies the intuition that taking $\pmod{m_1}$ and $\pmod{m_2}$ are pretty independent operations. In particular, because taking $\pmod{m_1}$ and $\pmod{m_2}$ are essentially independent, we should be able to find integers x satisfying the conditions

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}, \end{cases}$$

for any integers a_1 and a_2. This is not literally true, but it is almost true. The objective of this note is to explain how to solve such systems of equations. We will motivate and prove the following theorem.

Theorem 1 (Chinese remainder theorem)**.** Choose two pairs of integers (a_1, m_1) and (a_2, m_2) such that m_1 and m_2 are positive. Suppose that $\gcd(m_1, m_2) = 1$. Then there is exactly one class $x \pmod{m_1 m_2}$ such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}. \end{cases}$$

While reasonably intuitive, there is still something to explain about this statement. Notably, it is not obvious what the condition $\gcd(m_1, m_2) = 1$ does, but we will find that it is quite necessary for both the existence and the uniqueness of x.

Let's take a moment to explain the layout of this note. We split this theorem into two pieces: the existence of x and its uniqueness $\pmod{m_1 m_2}$. In section 2, we handle the existence, and in section 3, we handle the uniqueness. Lastly, in section 4, we answer some questions the reader may be left with after reading the rest of the article.

2 Existence

In this section, we show that systems of equations which look like

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}, \end{cases}$$

frequently have solutions.

2.1 A Starting Example

Let's give a taste for our constructions work.

Example 2. Show that there is an integer x such that $x \equiv 1 \pmod 3$ and $x \equiv 2 \pmod 5$.

Proof. Here is one way to approach this problem: we can search among the integers which are $2 \pmod 5$ until we find one which is $1 \pmod 3$. To this end, we make the following table.

n	0	1	2	3	4	5	6	7	$\cdots$
$5n+2$	2	7	12	17	22	27	32	37	$\cdots$
$5n+2 \pmod 3$	2	1	0	2	1	0	2	1	$\cdots$

Thus, we see that we may take $x = 7$ or $x = 22$ or $x = 37$. ■

Exercise 3. Show that there is an integer x such that $x \equiv 1 \pmod 3$ and $x \equiv 1 \pmod 5$.

Exercise 4. Show that there is an integer x such that $x \equiv 2 \pmod 3$ and $x \equiv 0 \pmod 5$.

Let's make a general argument.

Proposition 5. Choose any two integers a_1 and a_2. Then there is an integer x such that

$$x \equiv \begin{cases} a_1 & \pmod 3, \\ a_2 & \pmod 5. \end{cases}$$

Proof. We argue as in Example 2. We would like to take $x = a_2 + 5k$ for some integer k because such x automatically satisfy $x \equiv a_2 \pmod 5$. Thus, we are asking if there exists an integer k such that

$$a_2 + 5k \overset{?}{\equiv} a_1 \pmod 3.$$

We would like to "solve" for k. This is equivalent to asking for $2k \equiv (a_1 - a_2) \pmod 3$. Multiplying both sides by 2, this is equivalent to asking for $k \equiv 2(a_1 - a_2) \pmod 3$. Taking $k = 2(a_1 - a_2)$ will work. ■

Remark 6. For the sake of completeness, we note that we may unwind this argument to see that we are taking $x = a_2 + 5k = -9a_2 + 10a_1$. One can check directly that $x \equiv a_1 \pmod 3$ and $x \equiv a_2 \pmod 5$.

2.2 Some Trouble

However, there is some trouble in paradise.

Example 7. Show that there is no integer x such that $x \equiv 1 \pmod 4$ and $x \equiv 2 \pmod 6$.

Proof. Suppose that such an integer x exists.

- On one hand, $x = 1 + 4k$ for some integer k, so $x \equiv 1 \pmod 2$.
- On the other hand, $x = 2 + 6\ell$ for some integer ℓ, so $x \equiv 0 \pmod 2$.

However, there is no integer x which is both $0 \pmod 2$ and $1 \pmod 2$! ■

Exercise 8. Show that there is an integer x such that $x \equiv 1 \pmod 4$ and $x \equiv 3 \pmod 6$.

Exercise 9. Show that there is no integer x such that $x \equiv 11 \pmod{15}$ and $x \equiv 12 \pmod{25}$.

Ultimately, the obstruction is "common divisors."

Proposition 10. Choose two pairs of integers (a_1, m_1) and (a_2, m_2) such that m_1 and m_2 are positive. Suppose that there is a common divisor d of m_1 and m_2. If there is an integer x such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}, \end{cases}$$

then $a_1 \equiv a_2 \pmod d$.

Proof. The idea is to reduce everything $\pmod d$. First, we lift everything to integers. We may write $x = a_1 + k_1 m_1$ and $x = a_2 + k_2 m_2$ for some integers k_1 and k_2. Second, we reduce $\pmod d$: because d divides m_1 and m_2, we see that

$$a_1 + k_1 m_1 \equiv a_1 \pmod d,$$

and similarly $a_2 + k_2 m_2 \equiv a_2 \pmod d$. We conclude that

$$a_1 \equiv x \equiv a_2 \pmod d,$$

which is the desired statement. ■

Remark 11. Let's explain the relevance of Proposition 10. The point of is that the presence of common divisors d means that we are not allowed to choose a_1 and a_2 randomly, lest solutions fail to exist! Example 7 is one instance of this: 2 is a common divisor of both 4 and 6, but $1 \not\equiv 0 \pmod 2$.

2.3 The Theorem

However, in the absence of common divisors, this system is solvable.

Theorem 12 (Chinese remainder theorem, existence)**.** Choose two pairs of integers (a_1, m_1) and (a_2, m_2) such that m_1 and m_2 are positive. Suppose that $\gcd(m_1, m_2) = 1$. Then there is an integer x such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}. \end{cases}$$

Proof. We proceed as in Proposition 5. We would like to take $x = a_2 + km_2$ for some integer k because such x automatically satisfy $x \equiv a_2 \pmod{m_2}$. Thus, we are asking if there exists an integer k such that

$$a_2 + km_2 \overset{?}{\equiv} a_1 \pmod{m_1}.$$

We would like to "solve" for k. To this end, we note that this is equivalnt to

$$km_2 \overset{?}{\equiv} (a_1 - a_2) \pmod{m_1}.$$

Now, because $\gcd(m_1, m_2) = 1$, there exists an integer m_2' such that $m_2 m_2' \equiv 1 \pmod{m_1}$.[1] This allows us to fully isolate k, making the above equivalent to

$$k \overset{?}{\equiv} m_2'(a_1 - a_2) \pmod{m_1},$$

which is certainly possible. ■

Let's unwind the construction in this proof, following Remark 6.

Exercise 13. Choose two pairs of integers (a_1, m_1) and (a_2, m_2) such that m_1 and m_2 are positive. Suppose that there are integers m_1' and m_2' such that $m_1' m_1 + m_2' m_2 = 1$. Show that $x := m_2' m_2 a_1 + m_1' m_1 a_2$ satisfies

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}. \end{cases}$$

Exercise 14. Find integers m_1' and m_2' such that $3m_1' + 5m_2' = 1$. Use these integers to find an integer x such that $x \equiv 1 \pmod{3}$ and $x \equiv 2 \pmod{5}$.

Remark 15. Exercise 13 tells us something important: not only is the system of equations solvable in many cases, but it is not difficult to go out and find the solutions. Indeed, it is basically as hard as making Bezout's lemma effective, for which one can use the Extended Euclidean algorithm.

3 Uniqueness

We would now like to examine how unique a solution x to a system

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}, \end{cases}$$

can be.

[1] By Bezout's lemma, we can write $m_1' m_1 + m_2' m_2 = 1$ for some integers m_1' and m_2', and then m_2' is the desired integer.

3.1 Infinitely Many Integers

If we merely ask for integer solutions, we are going to find infinitely many. Let's see this.

Example 16. Show that there are infinitely many integers x such that $x \equiv 1 \pmod{3}$ and $x \equiv 2 \pmod{5}$.

Proof. In Example 2, we showed that $x = 7$ or $x = 22$ or $x = 37$ suffice. Noting that $22 = 7 + 15$ and $37 = 22 + 15$, we are motivated to guess that $x := 7 + 15k$ will work for any integer k. Indeed, it does: $7 + 15k \equiv 1 \pmod{3}$, and $7 + 15k \equiv 2 \pmod{5}$. ■

Exercise 17. Show that there are infinitely many integers x such that $x \equiv 1 \pmod{3}$ and $x \equiv 1 \pmod{5}$.

Exercise 18. Show that there are infinitely many integers x such that $x \equiv 1 \pmod{4}$ and $x \equiv 2 \pmod{7}$.

In general, one has the following.

Proposition 19. Choose two pairs of integers (a_1, m_1) and (a_2, m_2) such that m_1 and m_2 are positive. Suppose that there exists at least one integer x_0 such that

$$x_0 \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}. \end{cases}$$

Then there are infinitely many integers x such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}. \end{cases}$$

Proof. We proceed constructively. The idea is that $m_1m_2 \equiv 0 \pmod{m_1}$ and $m_1m_2 \equiv 0 \pmod{m_2}$. Thus, for any integer k, we claim that $x := x_0 + km_1m_2$ solves the system of equation. Indeed,

$$\begin{aligned} x &= x_0 + km_1m_2 \\ &\equiv x_0 \\ &\equiv a_1 \pmod{m_1}. \end{aligned}$$

A similar argument shows that $x \equiv a_2 \pmod{m_2}$. Letting k vary over all integers produces the desired infinitely many solutions. ■

3.2 Uniqueness of Classes

Thus, it is too much to hope for our solutions x to be unique because we can always add m_1m_2 to a given solution to produce a new one. This suggests that we should be looking at solutions $\pmod{m_1m_2}$! Let's see some examples.

Example 20. Suppose that an integer x satisfies $x \equiv 0 \pmod{3}$ and $x \equiv 0 \pmod{5}$. Then $x \equiv 0 \pmod{15}$.

Proof. In short, because 3 and 5 are coprime, we see that $3 \mid x$ and $5 \mid x$ implies $15 \mid x$.

Here is an argument in more words; this will work in more generality, as we will see later in Theorem 28. Because $3 \mid x$, we can write $x = 3k$. Now, $3k \equiv 0 \pmod 5$. We claim that $k \equiv 0 \pmod 5$, which will then imply $15 \mid x$ and hence complete the proof. Well, multiplying both sides of the equation

$$3k \equiv 0 \pmod 5$$

by 2 yields

$$k \equiv 0 \pmod 5.$$

Thus, we may write $k = 5\ell$ for some integer ℓ, which gives $x = 15\ell$, as desired. ■

Example 21. Suppose that an integer x satisfies $x \equiv 1 \pmod 3$ and $x \equiv 2 \pmod 5$. Then $x \equiv 7 \pmod{15}$.

Proof. We use Example 20. Indeed, note that

$$\begin{aligned} x - 7 &\equiv 1 - 1 \\ &\equiv 0 \pmod 3, \end{aligned}$$

and

$$\begin{aligned} x - 7 &\equiv 2 - 2 \\ &\equiv 0 \pmod 5. \end{aligned}$$

We conclude that $x - 7 \equiv 0 \pmod{15}$ by applying Example 20 to $x - 7$, so $x \equiv 7 \pmod{15}$. ■

Exercise 22. Suppose that an integer x satisfies $x \equiv 1 \pmod 3$ and $x \equiv 1 \pmod{15}$. Show that $x \equiv 1 \pmod{15}$.

Exercise 23. Suppose that there are two integers x_1 and x_2 such that $x_1 \equiv x_2 \equiv 2 \pmod 3$ and $x_1 \equiv x_2 \equiv 0 \pmod 5$. Show that $x_1 \equiv x_2 \pmod{15}$.

These arguments can be generalized.

Proposition 24. Suppose that there are two integers x_1 and x_2 such that $x_1 \equiv x_2 \pmod 3$ and $x_1 \equiv x_2 \pmod 5$. Then

$$x_1 \equiv x_2 \pmod{15}.$$

Proof. We proceed as in Example 21. Note that $x_1 - x_2 \equiv 0 \pmod 3$ and $x_1 - x_2 \equiv 0 \pmod 5$, so applying Example 20 to the integer $x := x_1 - x_2$ implies that $x_1 - x_2 \equiv 0 \pmod{15}$. Thus, $x_1 \equiv x_2 \pmod{15}$. ■

3.3 Some Trouble, Again

Once again, there is some trouble if we permit common divisors. As before, let's start by looking at how this looks at 0.

Example 25. Show that there is an integer x such that $x \equiv 0 \pmod 4$ and $x \equiv 0 \pmod 6$ but $x \not\equiv 0 \pmod{24}$.

Proof. We are asking for x to be divisible by both 4 and 6, which amounts to looking for common multiples of 4 and 6. Thus, we may as well start with the least common multiple $\operatorname{lcm}(4,6)$, which is $x := 12$. Indeed, $12 \not\equiv 0 \pmod{24}$. ■

Note 12 presents a problem for such systems in general.

Exercise 26. Fix integers a_1 and a_2 for which there is an integer x such that $x \equiv a_1 \pmod 4$ and $x \equiv a_2 \pmod 6$. Then show that

$$x + 12 \equiv \begin{cases} a_1 & \pmod 4, \\ a_2 & \pmod 6. \end{cases}$$

However, this is essentially the only obstruction.

Proposition 27. Fix integers a_1 and a_2 for which there is an integer x such that $x \equiv a_1 \pmod 4$ and $x \equiv a_2 \pmod 6$. Suppose that x_1 and x_2 are both integers satisfying the system

$$x \equiv \begin{cases} a_1 & \pmod 4, \\ a_2 & \pmod 6. \end{cases}$$

Then $x_1 \equiv x_2 \pmod{12}$.

Proof. By hypothesis, we know that $x_1 \equiv x_2 \pmod 4$ and $x_1 \equiv x_2 \pmod 6$. Then $x_1 - x_2$ is divisible by 4 and by 6, so it is also divisible by 12. Thus, $x_1 \equiv x_2 \pmod{12}$. ■

We refer to Theorem 31 for a more general statement.

3.4 The Theorem

We are now ready to prove our uniqueness result upon adding the condition $\gcd(m_1, m_2) = 1$.

Theorem 28 (Chinese remainder theorem, uniqueness)**.** Suppose that two positive integers m_1 and m_2 satisfy $\gcd(m_1, m_2) = 1$. If two integers x_1 and x_2 satisfy

$$\begin{cases} x_1 \equiv x_2 & \pmod{m_1}, \\ x_1 \equiv x_2 & \pmod{m_2}, \end{cases}$$

then $x_1 \equiv x_2 \pmod{m_1 m_2}$.

Proof. Our proof follows Proposition 24. Take $x := x_1 - x_2$, which we know satisfies $x \equiv 0 \pmod{m_1}$ and $x \equiv 0 \pmod{m_2}$, and we would like to show that $x \equiv 0 \pmod{m_1 m_2}$. Ultimately, this will hold because $\gcd(m_1, m_2) = 1$.

Because $x \equiv 0 \pmod{m_1}$, we may write $x = m_1 k$ for some integer k. We claim that $k \equiv 0 \pmod{m_2}$, which will quickly complete the argument. Well, we may find an integer m_1' such that $m_1 m_1' \equiv 1 \pmod{m_2}$. (Footnote 1 explains why m_1' exists.) Thus, we can multiply both sides of the equation

$$m_1 k \equiv 0 \pmod{m_2}$$

by m_1' to give

$$k \equiv 0 \pmod{m_2},$$

completing the proof of the claim. To complete the proof, we write $k = m_2 \ell$ for some integer ℓ, which yields $x = m_1 m_2 \ell$ and thus $x \equiv 0 \pmod{m_1 m_2}$. ■

Because it is not totally obvious, we explain how our two theorems Theorem 12 (on existence) and Theorem 28 (on uniqueness) together imply Theorem 1.

Proof of Theorem 1 from Theorems 12 and 28. Let's quickly recall the goal. Assuming $\gcd(m_1, m_2) = 1$, we are hunting for a unique class $x \pmod{m_1m_2}$ such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ a_2 & \pmod{m_2}, \end{cases}$$

where a_1 and a_2 are some given integers. We will handle existence and uniqueness separately. The existence of the class $x \pmod{m_1m_2}$ follows from the existence of the integer x in Theorem 12.

It remains to show that the class $x \pmod{m_1m_2}$ is unique. Well, suppose that there are two classes $x_1 \pmod{m_1m_2}$ and $x_2 \pmod{m_1m_2}$ solving our system. Then

$$x_1 \equiv a_1 \equiv x_2 \pmod{m_1},$$

and similarly $x_1 \equiv x_2 \pmod{m_2}$, so we conclude that $x_1 \equiv x_2 \pmod{m_1m_2}$ by Theorem 28. ■

4 Supplements

In this last section, we mention some asides which aide in the understanding of the Chinese remainder theorem. Because we not need this section as much, we will be briefer.

4.1 More Equations

One can use induction to solve systems with more equations.

Theorem 29. Choose pairs of integers $(a_1, m_1), \ldots, (a_k, m_k)$ such that the integers $m_1, \ldots, m_k$ are positive. Suppose $\gcd(m_i, m_j) = 1$ for any indices i or j. Then there is exactly one class $x \pmod{m_1 \cdots m_k}$ such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ & \vdots \\ a_k & \pmod{m_k}. \end{cases}$$

Proof. We induct on k. If $k = 1$, there is not much to do: we are hunting for a class $x \pmod{m_1}$ such that $x \equiv a_1 \pmod{m_1}$, so we are forced to take $x \equiv a_1 \pmod{m_1}$. If $k = 2$, we appeal to Theorem 1.

For our induction, we assume that the statement holds at k, and we want to show it for $k + 1$. To be explicit, we are given suitable pairs of integers $(a_1, m_1), \ldots, (a_{k+1}, m_{k+1})$, and we want to find a unique class $x \pmod{m_1 \cdots m_{k+1}}$ such that

$$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ & \vdots \\ a_{k+1} & \pmod{m_{k+1}}. \end{cases}$$

We will handle the existence and uniqueness parts of the proof separately.

- Existence: we must find an integer x such that

  $$x \equiv \begin{cases} a_1 & \pmod{m_1}, \\ & \vdots \\ a_{k+1} & \pmod{m_{k+1}}. \end{cases}$$

  By the inductive hypothesis, there exists y such that

  $$y \equiv \begin{cases} a_1 & \pmod{m_1}, \\ & \vdots \\ a_k & \pmod{m_k}. \end{cases}$$

Now, by Theorem 1 (to solve a system of two equations), we are granted an integer x such that

$$x \equiv \begin{cases} y \pmod{m_1 \cdots m_k}, \\ a_{k+1} \pmod{m_{k+1}}. \end{cases}$$

(Why is $\gcd(m_1 \cdots m_k, m_{k+1}) = 1$?) It remains to check that this x works. Well, $x \equiv a_{k+1} \pmod{m_{k+1}}$ by construction, for any for i in $\{1, \ldots, k\}$, we see that

$$x \equiv y \equiv a_i \pmod{m_i}$$

by construction of y. Thus, our constructed x works.

- Uniqueness: suppose that we have two integers x_1 and x_2 solving the system of equations. We would like to show that $x_1 \equiv x_2 \pmod{m_1 \cdots m_{k+1}}$. For this, we group the equations: we do know that

  $$x_1 \equiv x_2 \pmod{a_i}$$

  for each $i \in \{1, \ldots, k\}$, so the inductive hypothesis implies that $x_1 \equiv x_2 \pmod{m_1 \cdots m_k}$. Furthermore, we know that

  $$\begin{cases} x_1 \equiv x_2 \pmod{m_1 \cdots m_k}, \\ x_1 \equiv x_2 \pmod{m_{k+1}}, \end{cases}$$

  so Theorem 1 (for the uniqueness of two equations) gives $x_1 \equiv x_2 \pmod{m_1 \cdots m_{k+1}}$. ■

Exercise 30. Find all integers x such that

$$x \equiv \begin{cases} 1 \pmod{2}, \\ 2 \pmod{3}, \\ 3 \pmod{5}. \end{cases}$$

4.2 Common Divisors

It is possible to relax the condition $\gcd(m_1, m_2) = 1$ in Theorem 1, but it requires some care.

Theorem 31. Choose pairs of integers (a_1, m_1) and (a_1, m_2) such that the integers m_1 and m_2 are positive. Set $d := \gcd(m_1, m_2)$ and $M := \operatorname{lcm}(m_1, m_2)$.

(a) Non-existence: if $a_1 \not\equiv a_2 \pmod{d}$, then there does not exist an integer x satisfying $x \equiv a_1 \pmod{m_1}$ and $x \equiv a_2 \pmod{m_2}$.

(b) Existence: if $a_1 \equiv a_2 \pmod{d}$, then there exists exactly one class $x \pmod{M}$ satisfying $x \equiv a_1 \pmod{m_1}$ and $x \equiv a_2 \pmod{m_2}$.

Proof. With a bit of squinting, one can see that the statement (a) is the equivalent to the statement of Proposition 10.

It remains to handle (b), so we assume that $a_1 \equiv a_2 \pmod{d}$. As usual, we deal with the existence and uniqueness separately. In general, the approach is to reduce to the coprime case by dividing out by d (note $\gcd(m_1/d, m_2/d) = 1$), where we can apply Theorem 1.

- Existence: we would like to find an integer x such that $x \equiv a_1 \pmod{m_1}$ and $x \equiv a_2 \pmod{m_2}$. By subtracting a_1 from both equations, it is enough to find an integer y such that $y \equiv 0 \pmod{m_1}$ and $y \equiv b_2 \pmod{m_2}$, where $b_2 := a_2 - a_1$. Notably, $b_2 \equiv 0 \pmod{d}$ by hypothesis.

Now, because $\gcd(m_1/d, m_2/d) = 1$, Theorem 1 promises an integer y' such that

$$y' \equiv \begin{cases} 0 \pmod{m_1/d}, \\ b_2/d \pmod{m_2/d}. \end{cases}$$

Thus, $y := dy'$ can be checked to satisfy $y \equiv 0 \pmod{m_1}$ and $y \equiv b_2 \pmod{m_2}$. Taking $x := y + a_1$ completes the proof.

- Uniqueness: suppose that x_1 and x_2 are two integers such that $x_1 \equiv x_2 \equiv a_1 \pmod{m_1}$ and $x_1 \equiv x_2 \equiv a_2 \pmod{m_2}$. Then we would like to check that $x_1 \equiv x_2 \pmod{M}$. Consider $x := x_1 - x_2$ so that we would like to show that $x \equiv 0 \pmod{M}$ when we are given that

$$x \equiv \begin{cases} 0 \pmod{m_1}, \\ 0 \pmod{m_2}. \end{cases}$$

Well, $m_1 \mid x$ and $m_2 \mid x$ implies that d divides x, and x/d is divisible by both m_1/d and m_2/d. However, $\gcd(m_1/d, m_2/d) = 1$, so we are allowed to apply Theorem 1 to see that x/d is divisible by m_1m_2/d^2. We conclude that

$$x \equiv 0 \pmod{m_1m_2/d},$$

so we are done because $\operatorname{lcm}(m_1, m_2) = m_1m_2/\gcd(m_1, m_2)$. ■

Exercise 32. Find all integers x such that

$$x \equiv \begin{cases} 3 \pmod{10}, \\ 8 \pmod{15}. \end{cases}$$

Remark 33. It is possible to formulate and prove a theorm which simultaneously generalizes Theorems 29 and 31. The interested reader is encouraged to attempt this exercise.