Student Arithmetic Geometry Seminar

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Spring 2024

Contents

1 January 19: Martin Olsson

Today, we will provide an introduction to what this seminar will be about.

1.1 Motivating Paper

We are motivated by Deligne–Mumford's paper "The irreducibility of moduli of curves of given genus." The main theorem is as follows.

Theorem 1 (Deligne–Mumford)**.** Fix a nonnegative integer $g \geq 2$ and a field k. Then the moduli space $\mathcal{M}_g$ of curves of genus g over k is irreducible.

There is quite a bit which goes into this proof.

- Curve theory.
- The formalism of moduli problems.
- The Semistable reduction theorem.
- Jacobian varieties.
- The notion of stable curves and the compactification $\overline{\mathcal{M}}_g$.
- Deformation theory.
- The theory of stacks and coarse moduli spaces.
- An analytic description to get from $\mathbb{C}$.

From these ingredients, there are other directions to go in.

- Finer geometry of $\mathcal{M}_g$.

- Moduli of curves with marked points $\mathcal{M}_{g,n}$.
- Higher-dimensional directions, such as abelian varieties.

Let's go ahead and provide a sketch.

Sketch of Theorem 1. The result is known over $\mathbb{C}$ via some analytic methods. We want to get to other fields. We begin with the following exercise.

Proposition 2. Fix a complete discrete valuation ring $(R, \mathfrak{m})$ with fraction field K and residue field $\kappa := R/\mathfrak{m}$. Fix a smooth proper morphism $f\colon X \to \operatorname{Spec} R$. If $X_{\overline{K}}$ is connected, then $X_{\overline{\kappa}}$ is also connected.

Sketch. One can assume that κ is algebraically closed by some argument. Then the Formal functions theorem tells us that

$$H^0(X, \mathcal{O}_X) = \varprojlim_n H^0(X_n, \mathcal{O}_{X_n}),$$

where $X_n := X \times_V \operatorname{Spec} V/\mathfrak{m}^n$. Now, suppose for the sake of contradiction that $X_\kappa = X_\kappa^1 \sqcup X_\kappa^2$. Then by taking nilpotent thickenings, we see that $X_n = X_n^1 \sqcup X_n^2$, so

$$\varprojlim_n H^0(X_n, \mathcal{O}_{X_n}) = \varprojlim_n H^0(X_n^1, \mathcal{O}_{X_n^1}) \times \varprojlim_n H^0(X_n^2, \mathcal{O}_{X_n^2}).$$

So we are receiving a product of rings $A_1 \times A_2$, which are flat over R, so by viewing the global sections back in $H^0(X, \mathcal{O}_X)$, which should be $\overline{K}$ upon algebraic closure, we will receive our contradiction. ∎

Now, the (coarse) moduli space $\mathcal{M}_g$ fails to be either smooth or proper, but some theory of stacks allows us to reduce to this case. Namely, the point is to find some compactification $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$ where $\overline{\mathcal{M}}_g$ is a smooth proper $\mathbb{Z}$-stack with $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ dense in each fiber. As such, we can imagine using Proposition 2 to pull back the result from $\overline{\mathcal{M}}_g$ to $\mathcal{M}_g$. ∎

1.2 Thinking about the Moduli Spaces

Let's describe what $\mathcal{M}_g$ is. By the functor of points description, we merely need to describe maps $S \to \mathcal{M}_g$, which we declare to be in natural bijection with genus-g curves $\pi\colon C \to S$; i.e., π is a proper, flat, and smooth morphism whose geometric fibers are genus-g curves.

Example 3. There is a family of curves over a field k given by the equations

$$y^2 = (x - a_1) \cdots (x - a_n),$$

where $a_1, \ldots, a_n$ are allowed to vary. Viewing the $a_\bullet$s as giving a point in affine space, we see that we are (approximately speaking) producing a rational map $\mathbb{A}^n \to \mathcal{M}_g$, where perhaps we need to check that we have a curve of the correct genus.

Now, using a functor of points description, smoothness of $\mathcal{M}_g$ over $\operatorname{Spec}\mathbb{Z}$ is requiring the following (in the sense of formal smoothness): for any surjection $A' \twoheadrightarrow A$ with kernel J such that $J^2 = 0$, any morphism $\operatorname{Spec} A \to \mathcal{M}_g$ induces a unique dashed arrow.

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[d] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

As such, we can unwind the functor of points description of $\mathcal{M}_g$ to prove something like smoothness, which is quite remarkable.

Analogously, we can describe $\overline{\mathcal{M}}_g$ via the functor of points: maps $S \to \overline{\mathcal{M}}_g$ are "stable curves" $\pi\colon C \to S$, which will be a proper flat morphism whose geometric fibers by $\overline{s} \to S$ satisfy the following.

- The geometric fibers are nodal curves, meaning that the completion at any closed point is $\kappa(\overline{s})[\![x]\!]$ or $\kappa(\overline{s})[\![x,y]\!]/(xy)$.
- Every rational component (namely, irreducible component whose normalization is $\mathbb{P}^1$) has three distinguished points. (These three points are desirable, for example, to ensure that its automorphism group is trivial.)

Now, we can also check being proper via the functor of points description, using the valuative criterion. Namely, for a complete discrete valuation ring R with fraction field K, we would like to know that any map $\operatorname{Spec} K \to \overline{\mathcal{M}}_g$ induces a unique dashed arrow.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \overline{\mathcal{M}}_g \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "!"] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

And again, we can directly turn this into a geometry problem of curves by tracking through the moduli interpretation, which is the Semistable reduction theorem.

2 January 26th: Martin Olsson

Today we discuss the setup for our moduli problems.

2.1 The Yoneda Lemma

Here is the statement.

Theorem 4 (Yoneda)**.** Fix a category $\mathcal{C}$, and let $\operatorname{PSh}(\mathcal{C})$ denote the category of presheaves on $\mathcal{C}$.

(a) For $X \in \mathcal{C}$, the functor $h_X \colon A \mapsto \operatorname{Mor}(A, X)$ is a presheaf $h_X \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

(b) There is a natural bijection
$$\operatorname{Mor}_{\operatorname{PSh}(\mathcal{C})}(h_X, F) \to FX.$$

(c) The construction $h_\bullet$ forms a fully faithful embedding $h_\bullet \colon \mathcal{C} \to \operatorname{PSh}(\mathcal{C})$.

Proof. We won't bother to show (a). For (b), the forward map sends $\eta \colon h_X \Rightarrow F$ to $\eta_X(\mathrm{id}_X)$, and one can check that this is a bijection. To show (c), we note that we need to show
$$\operatorname{Mor}_{\mathcal{C}}(X, Y) \simeq \operatorname{Mor}_{\operatorname{Pre}(\mathcal{C})}(h_X, h_Y),$$
but this simply follows by taking $F = h_Y$ in (b). ■

Remark 5. Most of the time, we will take $\mathcal{C} = \operatorname{Sch}(S)$ for a fixed base scheme S.

Anyway, we can now make the following definition.

Definition 6 (representable)**.** Fix a category $\mathcal{C}$. A functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is *representable* if and only if $F \simeq h_X$ for some object $X \in \mathcal{C}$. In the sequel, we may want to fix the isomorphism $F \simeq h_X$, which can be specified by an element $\xi \in FX$.

Here are some examples.

Example 7. Take $\mathcal{C} := \mathrm{Sch}$, and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined by $FY := \Gamma(Y, \mathcal{O}_Y)^n$. We claim F is represented by $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. Indeed, we see

$$\mathrm{Mor}_{\mathrm{Sch}}(Y, \mathbb{A}^n_{\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x_1, \ldots, x_n], \mathcal{O}_Y(Y)) \simeq \mathcal{O}_Y(Y)^n,$$

as desired. To specify this isomorphism, Theorem 4 tells us that it is enough to track through the identity map in $\mathrm{Mor}(\mathbb{A}^n_{\mathbb{Z}}, \mathbb{A}^n_{\mathbb{Z}}) \simeq \Gamma(\mathbb{A}^n_{\mathbb{Z}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{Z}}})^n$, which we can track through is $(x_1, \ldots, x_n)$. Of course, we could choose other isomorphisms, such as determined by the element $(x_n, \ldots, x_1) \in \Gamma(\mathbb{A}^n_{\mathbb{Z}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{Z}}})^n$.

Example 8. Take $\mathcal{C} := \mathrm{Sch}$, and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined by

$$FY := \{(f_\bullet) \in \Gamma(Y, \mathcal{O}_Y)^n : (f_1, \ldots, f_n) = \Gamma(Y, \mathcal{O}_Y)\}.$$

Then F is represented by $\mathbb{A}^n_{\mathbb{Z}} \setminus \{0\}$.

Example 9. Take $\mathcal{C} := \mathrm{Sch}$, and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined by setting FY to the collection of surjections $\mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{L}$ up to isomorphism, where $\mathcal{L}/Y$ is a line bundle. Then F is represented by $\mathbb{P}^n_{\mathbb{Z}}$. The representing element in $F(\mathbb{P}^n_{\mathbb{Z}})$ is given by the surjection $(x_0, \ldots, x_n)\colon \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$.

Remark 10. By identifying a line with the corresponding quotient space, we see that the above interpretation of $\mathbb{P}^n_k$ agrees with the interpretation as "lines in k^{n+1}."

Example 11. Fix a field k and homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree N, and consider $V(f) \subseteq \mathbb{P}^n_k$. A map $Y \to V(f)$ will certainly produce a map $Y \to \mathbb{P}^n_{\mathbb{Z}}$, so our answer should be a subfunctor of Example 9. But then we want to determine which surjections $(s_0, \ldots, s_n)\colon \mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{L}$ (which really consists of $n+1$ global sections of $\mathcal{L}$ globally generating), but then to live in $V(f)$, we request that $f(s_0, \ldots, s_n) = 0$.

Remark 12. Note $\mathbb{A}^n_{\mathbb{Z}} \subseteq \mathbb{P}^n_{\mathbb{Z}}$ (e.g. going to the standard affine open $x_0 \neq 0$). So we expect to have an inclusion $h_{\mathbb{A}^n_{\mathbb{Z}}} \subseteq h_{\mathbb{P}^n_{\mathbb{Z}}}$, and one can track through that it simply corresponds to the provided map $(s_0, \ldots, s_n)\colon \mathcal{O}_Y^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$ having s_0 be an isomorphism. In this case, after identifying $\mathcal{O}_Y$ with $\mathcal{L}$ via s_0, we see that the other sections are indeed providing an element of $\Gamma(Y, \mathcal{O}_Y)^n$.

Example 13 (Hilbert scheme)**.** Fix a flat separated X-scheme S. Then the Hilbert functor assigns a Y-scheme S to flat proper (locally of finite presentation) subschemes $Z \subseteq X \times_S Y$. It turns out that this functor is representable if X is quasi-projective, which is a result due to Grothendieck.

2.2 Functors Not Representable

Representable functors have the tendency to be sheaves. For the Zariski topology, here is the relevant definition.

Definition 14 (Zariski sheaf)**.** A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a *Zariski sheaf* if and only if we have the usual equalizer diagram

$$FY \to \prod_{\alpha \in \kappa} FY_\alpha \leftleftarrows \prod_{\alpha, \beta \in \kappa} F(Y_\alpha \cap Y_\beta),$$

where $\{Y_\alpha\}_{\alpha \in \kappa}$ is a Zariski open cover of Y.

Example 15. If $F \simeq h_X$ is representable, then F is a Zariski sheaf because morphisms glue.

Non-Example 16. Define the functor F as taking a scheme Y and taking the quotient of the set of $(n+1)$-tuples $(s_0, \dots, s_n) \in \Gamma(Y, \mathcal{O}_Y)$ by multiplication by $\Gamma(Y, \mathcal{O}_Y^\times)$. This is not a sheaf, but if we do sheafification, we will recover Example 9. Thus, this functor is not representable.

As another remark, we have the following.

Lemma 17. If X is a scheme, and L/K is a field extension, then the map $X(\operatorname{Spec} K) \to X(\operatorname{Spec} L)$ is injective.

Proof. Indeed, a map $\operatorname{Spec} K \to X$ is simply a point $x \in X$ together with an inclusion $\kappa(x) \to K$, which by a similar description for $\operatorname{Spec} L \to X$ is uniquely determined by that map $\operatorname{Spec} L \to X$. ■

Example 18. Define the functor F by taking a scheme Y and returning elliptic curves over Y (up to isomorphism). But there are distinct elliptic curves over $\mathbb{Q}$ which become isomorphic over $\overline{\mathbb{Q}}$. For example, $y^2 = x^3 + D$ and $y^2 = x^3 - D$ is one such example, which become isomorphic over $\mathbb{Q}(i)$ where we can send $(x, y) \mapsto (-x, iy)$. Thus, F is not representable by Lemma 17.

Remark 19. One can argue similarly for curves of genus $g \geq 2$.