Student Arithmetic Geometry Seminar

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Spring 2024

Contents

1 January 19: Martin Olsson

Today, we will provide an introduction to what this seminar will be about.

1.1 Motivating Paper

We are motivated by Deligne–Mumford's paper "The irreducibility of moduli of curves of given genus." The main theorem is as follows.

Theorem 1 (Deligne–Mumford)**.** Fix a nonnegative integer $g \geq 2$ and a field k. Then the moduli space $\mathcal{M}_g$ of curves of genus g over k is irreducible.

There is quite a bit which goes into this proof.

- Curve theory.
- The formalism of moduli problems.
- The Semistable reduction theorem.
- Jacobian varieties.
- The notion of stable curves and the compactification $\overline{\mathcal{M}}_g$.
- Deformation theory.
- The theory of stacks and coarse moduli spaces.
- An analytic description to get from $\mathbb{C}$.

From these ingredients, there are other directions to go in.

- Finer geometry of $\mathcal{M}_g$.
- Moduli of curves with marked points $\mathcal{M}_{g,n}$.
- Higher-dimensional directions, such as abelian varieties.

Let's go ahead and provide a sketch.

Sketch of Theorem 1. The result is known over $\mathbb{C}$ via some analytic methods. We want to get to other fields. We begin with the following exercise.

Proposition 2. Fix a complete discrete valuation ring $(R, \mathfrak{m})$ with fraction field K and residue field $\kappa := R/\mathfrak{m}$. Fix a smooth proper morphism $f\colon X \to \operatorname{Spec} R$. If $X_{\overline{K}}$ is connected, then $X_{\overline{\kappa}}$ is also connected.

Sketch. One can assume that κ is algebraically closed by some argument. Then the Formal functions theorem tells us that

$$H^0(X, \mathcal{O}_X) = \varprojlim_n H^0(X_n, \mathcal{O}_{X_n}),$$

where $X_n := X \times_V \operatorname{Spec} V/\mathfrak{m}^n$. Now, suppose for the sake of contradiction that $X_\kappa = X_\kappa^1 \sqcup X_\kappa^2$. Then by taking nilpotent thickenings, we see that $X_n = X_n^1 \sqcup X_n^2$, so

$$\varprojlim_n H^0(X_n, \mathcal{O}_{X_n}) = \varprojlim_n H^0(X_n^1, \mathcal{O}_{X_n^1}) \times \varprojlim_n H^0(X_n^2, \mathcal{O}_{X_n^2}).$$

So we are receiving a product of rings $A_1 \times A_2$, which are flat over R, so by viewing the global sections back in $H^0(X, \mathcal{O}_X)$, which should be $\overline{K}$ upon algebraic closure, we will receive our contradiction. ■

Now, the (coarse) moduli space $\mathcal{M}_g$ fails to be either smooth or proper, but some theory of stacks allows us to reduce to this case. Namely, the point is to find some compactification $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$ where $\overline{\mathcal{M}}_g$ is a smooth proper $\mathbb{Z}$-stack with $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ dense in each fiber. As such, we can imagine using Proposition 2 to pull back the result from $\overline{\mathcal{M}}_g$ to $\mathcal{M}_g$. ■

1.2 Thinking about the Moduli Spaces

Let's describe what $\mathcal{M}_g$ is. By the functor of points description, we merely need to describe maps $S \to \mathcal{M}_g$, which we declare to be in natural bijection with genus-g curves $\pi\colon C \to S$; i.e., π is a proper, flat, and smooth morphism whose geometric fibers are genus-g curves.

Example 3. There is a family of curves over a field k given by the equations

$$y^2 = (x - a_1) \cdots (x - a_n),$$

where $a_1, \ldots, a_n$ are allowed to vary. Viewing the $a_\bullet$s as giving a point in affine space, we see that we are (approximately speaking) producing a rational map $\mathbb{A}^n \to \mathcal{M}_g$, where perhaps we need to check that we have a curve of the correct genus.

Now, using a functor of points description, smoothness of $\mathcal{M}_g$ over $\operatorname{Spec}\mathbb{Z}$ is requiring the following (in the sense of formal smoothness): for any surjection $A' \twoheadrightarrow A$ with kernel J such that $J^2 = 0$, any morphism $\operatorname{Spec} A \to \mathcal{M}_g$ induces a unique dashed arrow.

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[d] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

As such, we can unwind the functor of points description of $\mathcal{M}_g$ to prove something like smoothness, which is quite remarkable.

Analogously, we can describe $\overline{\mathcal{M}}_g$ via the functor of points: maps $S \to \overline{\mathcal{M}}_g$ are "stable curves" $\pi\colon C \to S$, which will be a proper flat morphism whose geometric fibers by $\overline{s} \to S$ satisfy the following.

- The geometric fibers are nodal curves, meaning that the completion at any closed point is $\kappa(\overline{s})\llbracket x \rrbracket$ or $\kappa(\overline{s})\llbracket x, y \rrbracket/(xy)$.

- Every rational component (namely, irreducible component whose normalization is $\mathbb{P}^1$) has three distinguished points. (These three points are desirable, for example, to ensure that its automorphism group is trivial.)

Now, we can also check being proper via the functor of points description, using the valuative criterion. Namely, for a complete discrete valuation ring R with fraction field K, we would like to know that any map $\operatorname{Spec} K \to \overline{\mathcal{M}}_g$ induces a unique dashed arrow.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \overline{\mathcal{M}}_g \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "!"] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

And again, we can directly turn this into a geometry problem of curves by tracking through the moduli interpretation, which is the Semistable reduction theorem.