Seminars

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Abstract

This semester, I will just record all seminars I go to in an uncategorized manner. I will try to record the date, the speaker, and which seminar it was to maintain some semblance of organization.

Contents

1 January 30: Multiplicity Formulae for Spherical Varieties

This talk was given by Toan Pham at Johns Hopkins University for the student automorphic representations seminar.

1.1 The Theorems

Fix a homogeneous spherical variety $X = G/H$ over a local field F. Also, choose a character $\chi\colon H(F) \to \mathbb{C}^\times$ and an irreducible representation π of $G(F)$. Then the local relative Langlands program is interested in the multiplicity

$$m(\pi, \chi) := \dim \operatorname{Hom}_{H(F)}(\pi, \chi).$$

Note that

$$m(\pi, \chi) = \dim \operatorname{Hom}_{G(F)}(\pi, C^\infty(X(F), \chi)).$$

Here are some starting examples.

Example 1 (Whittaker)**.** An interesting case is when $H \subseteq G$ is a unipotent subgroup. For example, one can take the unipotent radical U of the Borel subgroup of GL_2. Then we can take χ to be lifted from any additive character $F \to \mathbb{C}^\times$. One can generalize this example to work from any $\mathfrak{sl}_2$-triple, and it produces Whittaker models of π in $C^\infty(X(F), \chi)$.

Example 2 (Gan–Gross–Prasad)**.** Fix a quadratic extension E/F. Then one can take $X = \mathrm{SO}_n \backslash (\mathrm{SO}_n \times \mathrm{SO}_{n+1})$ or $X = \mathrm{U}_n \backslash (\mathrm{U}_n \times \mathrm{U}_{n+1})$. In general, there are more examples arising from so-called "GGP triples" (G, H, χ).

Remark 3. It turns out that $m(\pi, \chi) \le 1$ in Examples 1 and 2, but this is not always true.

Nonetheless, it becomes interesting to discover when $m(\pi, \chi) \ge 1$. Here are some answers.

Theorem 4. Fix everything as in Examples 1 and 2 and choose a Langlands parameter $\varphi\colon W_F \to {}^L G$. Then the Langlands packet Π_φ contains exactly one π for which $m(\pi, \chi) \neq 0$.

Theorem 5. Fix everything as in Example 2, and choose a Langlands parameter $\varphi\colon W_F \to {}^L G$. Then $\pi \in \Pi_\varphi$ has $m(\pi, \chi) \neq 0$ if and only if it maps to a speified unitary character.

Our method will be the so-called "local trace formulae," developed originally by Waldspurger.

1.2 Local Trace Formulae

For the rest of the talk, we work in the context of Example 2. The type of result we are trying to prove rewrites $m(\pi, \chi)$ in terms of a sum of orbital integrals.

Example 6. For finite groups G, we can use the character Θ_π of π to see that

$$m(\pi, \chi) = \sum_{\text{conj. } [x] \subseteq H} \frac{1}{\#C_H(x)} \cdot \Theta_\pi(x) \overline{\chi}(h).$$

This right-hand side can be viewed as some twisted sum of sizes of conjugacy classes.

Our method to prove such formulae will rest on local trace formulae, which basically amount to finding two ways of expressing the trace of some $f \in C_c^\infty(G)$ acting on $L^p(H(F)\backslash G(F), \chi)$. On one hand, we may write

$$\begin{aligned}(Rf \cdot \varphi)(x) &= \int_{G(F)} f(g)\varphi(xg)\, dg \\ &= \int_{H(F)\backslash G(F)} K_f(x,y)\varphi(y)\, dy,\end{aligned}$$

where $K_f(x,y) = \int_{H(F)} f\left(x^{-1}hy\right)\chi(h)\, dh$. We morally then expect the trace of Rf to be the sum of K_f along the diagonal, as one finds with finite groups.

Proposition 7. Fix everything as above. Say that $f \in C_c^\infty(G(F))$ is strongly cuspidal if and only if

$$\int_{U(F)} f(um)\, du = 0$$

for any parabolic P with Levi decomposition $P = MU$. If f is strongly cuspidal, then the trace of Rf converges absolutely.

The local trace formula now amounts to expressing $J(f)$ either via a spectral or a geometric expansion.

- The spectral expansion is
$$J_{\text{spec}}(f) \coloneqq \int_{\mathfrak{X}(G)} D(\pi)\widehat{\Theta}_f(\pi)m(\pi)\, d\pi.$$
Here, $\mathfrak{X}(G)$ consists of the space of tempered representations (i.e., found in $L^2(G)$) arising from parabolic induction of elliptic representations. Then Θ_f is a weighted orbital ingeral of f, and $\widehat{\Theta}_f$ is its Fourier transform. The $D(\pi)$ is unknown, but we have been reassured that it is not important.
The idea to show that $J(f) = J_{\text{spec}}(f)$ is to use the Plancheral formula
$$\langle f_1, f_2\rangle = \int_{\widehat{G}} J_\pi(f_1 \otimes f_2)\, d_X\pi,$$
where J_π is the natural composite
$$C_c^\infty(X \times X) \to \pi \otimes \widetilde{\pi} \to \mathbb{C}.$$
It turns out that J_π is non-vanishing if and only if $m(\pi,\chi) \neq 0$. Yiannis claims that the forward direction is easy.
- The geometric expansion associates a quasicharacter $\theta_f \colon G_{\text{reg}}(F) \to \mathbb{C}$ defined on the regular semisimple locus, and then we have
$$m_{\text{geom}}(\theta) \coloneqq \lim_{s\to 0} \int_{\Gamma(G,H)} D^G(x)^{1/2}c_\theta(x)\Delta^{s-1/2}\, dx.$$
Here, $\Gamma(G,H)$ is the space of semisimple conjugacy classes of $G(F)$ which are represented by an element in $H(F)$, and c_θ is some extension of θ to the semisimple locus $G_{\text{ss}}(F)$. Our geometric expansion is then $J_{\text{geom}}(f) \coloneqq m_{\text{geom}}(\Theta_f)$.

It now turns out that

$$J_{\text{spec}}(f) = J(f) = J_{\text{geom}}(f),$$

though both of these equalities are theorems. Though it requires an argument, this turns out to be equivalent to a multiplicity formula $m_{\text{geom}}(\pi) = m(\pi)$.

Conjecture 8. One has $m_{\text{geom}}(\pi) = m(\pi)$ for any spherical X.

2 January 31: Finiteness Theorems in Arithmetic Statistics

This talk was given by Fatemehzahra Janbazi for the JHUnior number theory days.

2.1 The Birch–Merriman Theorems

For today, K will be a number field, and S is a finite set of places, including the infinite ones.

Notation 9. For a ring R, we define $U_n(R)$ to be the set of binary forms f which are homogeneous of degree n.

Note that GL_2 naturally acts on U_n.

Definition 10. There is also a discriminant Δ on U_n. We say that $f \in \mathrm{U}_n(\mathbb{Q}_p)$ has *good reduction* if and only if $\Delta(f) \in \mathcal{O}_p^\times$.

Remark 11. The discriminant satisfies

$$\Delta(\gamma \cdot f) = \det \gamma^{n(n-1)} \Delta(f).$$

We then have the following theorems.

Theorem 12 (Birch–Merriman)**.** The set $U_n(\mathcal{O}_{K,S})$ with good reduction outside S breaks into infinitely many $\mathrm{GL}_2(\mathcal{O}_{K,S})$-orbits.

Theorem 13. For any nonzero $D \in \mathcal{O}_K$, the set of forms in $U_n(\mathcal{O}_K)$ with disciminant D has finitely many equivalence classes for the action of $\mathrm{GL}_2(\mathcal{O}_K)$.

We are interested in analogous results for ternary forms.

Definition 14. For a ring R, we define $V_n(R)$ to be the set of homogeneous ternary forms of degree n. There is also a discriminant Δ of ternary forms, and we say that f has *good reduction* at p if and only if $\Delta(f) \in \mathcal{O}_p^\times$.

Remark 15. As usual, there is a natural GL_3-action on V_n, and it satisfies

$$\Delta(\gamma \cdot f) = \det \gamma^{n(n-1)} \Delta(f).$$

Here are our main theorems.

Theorem 16. The set of elements in $V_n(\mathcal{O}_{K,S})$ with good reduction outside S breaks up into finitely many $\mathrm{GL}_3(\mathcal{O}_{K,S})$-orbits.

Theorem 17. For any nonzero $D \in \mathcal{O}_K$, the set of forms in $V_n(\mathcal{O}_K)$ with discrimimant D has finitely many equivalence classes for the action of $\mathrm{GL}_3(\mathcal{O}_K)$.

Here is another analogue.

Definition 18. For a ring R, we define

$$V_{2,2}(R) := \frac{R[x_1,x_2,x_3]_2 \otimes_R R[z_1,z_2,z_3]_2}{(\sum_i x_i z_i) \cap R[x_1,x_2,x_3]_2 \otimes_R R[z_1,z_2,z_3]_2}.$$

Remark 19. There is now a natural action of GL_3 on $V_{2,2}$ given by $(\gamma \cdot f)(x,z) := f(x\gamma, z\delta\gamma)$.

Definition 20. A $K3$ surface S is *generic* if and only if it is smooth and cover $\mathbb{P}^2$ via a natural projection from a flag variety. Some $f \in V_{2,2}$ has *good reduction* if and only if it products a generic $K3$ surface.

Theorem 21. The set of elements in $V_{2,2}(\mathcal{O}_{K,S})$ with good reduction outside S and Picard rank 2 breaks up into finitely many orbits for the action of $\mathrm{GL}_2(\mathcal{O}_{K,S})$.

Remark 22. Let's sketch the idea of Theorem 16. Let $T_n(K,S)$ be the set of f with discriminant in $\mathcal{O}_{K,S}^\times$. Each such f cuts out a curve C_f which has good reduction outside S, but Faltings's theorem implies that there are only finitely many such curves, which can be enumerated by some invariants $I_1(f),\ldots,I_k(f)$. One can now chase around to prove the theorem.

Here is an application.

Example 23. Let $W_n(R)$ be the set of pairs (F,G) of elements in V_n. This has a natural action of $\mathrm{GL}_2 \times \mathrm{GL}_3$ given by

$$(\gamma_2,\gamma_3)\cdot(F,G) = (\gamma_3 F, \gamma_3 G)\gamma_2^\intercal.$$

One can prove a similar result about finiteness of elements in $W_n(\mathcal{O}_K)$ with fixed discriminant. The proof uses both the finitenesses of U_n and V_n.

2.2 Geometric Reformulation

We will want more notation.

Notation 24. Let $U_n(K,S)$ be the set of elements in $U_n(\mathcal{O}_{K,S})$ and $\Delta(f) \in \mathcal{O}_{K,S}^\times$ and which split over K. We also let $\Omega_n(\mathbb{P}^1;S)$ to be the set of n points with good reduction outside S.

The point of introducing these sets is that there is a natural map $U_n(K,S) \to \Omega_n(\mathbb{P}^1;S)$, preserving the GL_2-action, and finiteness theorems can be moved from one side to the other.

Thus, we see that we may be interested in analogous questions for curves other than $\mathbb{P}^1$.

Theorem 25. Fix a smooth projective curve C over K with good redution at S. One can define $\Omega_n(C;S)$ in the same way. For any $n \geq 1$, the set of classes in $\Omega_n(C;S)$ up to the action by $\mathrm{Aut}_{\mathcal{O}_{K,S}}(\mathcal{C})$ is finite.

Example 26. For $C = \mathbb{P}^1$ and $n = 1$, finiteness is equivalent to finiteness of the class group of $\mathcal{O}_{K,S}$.

Example 27. If $C = \mathbb{P}^1$ and $n \geq 3$, we recover the Birch–Merriman theorems. Alternatively, one can move some points to $\{0,1,\infty\}$, and then the remaining points are seen to be finite by the unit equation.

Example 28. If C has genus 1, then we have an elliptic curve; moving some point to the origin, we recover Siegel's theorem.

Example 29. Lastly, if C has genus 2, then at $n = 1$, we recover Faltings's theorem.

The proof of Theorem 25 more or less upgrades the above examples by doing some Galois cohomology.

3 January 31: Infinitely Many Supersingular Primes

This talk as given by Fangu Chen at Johns Hopkins University for the JHUnior number theory days.

3.1 Expectations for Supersingular Primes

Here are our definitions for elliptic curves.

Definition 30 (ordinary, supersingular)**.** Fix an elliptic curve E over a finite field of characteristic p. Then E is *ordinary* if and only if $E[p] \cong \mathbb{Z}/p\mathbb{Z}$; it is *supersingular* if and only if $E[p] = 0$.

Remark 31. There are equivalent formulations based on the Frobenius action or a calculation of the endomorphism ring.

Remark 32. It turns out that every elliptic curve is ordinary or supersingular.

Question 33. Fix an elliptic curve E over $\mathbb{Q}$. For how many primes p are there with $E_{\mathbb{F}_p}$ admit supersingular reduction?

Here is a heuristic argument: one knows $|\operatorname{tr}\operatorname{Frob}_p| \leq 2\sqrt{p}$, so one can expect that p is supersingular with probability $\sim 1/\sqrt{p}$.

Conjecture 34 (Lang–Trotter)**.** Fix an elliptic curve E over $\mathbb{Q}$ without complex multiplication. Then the number of primes $p < X$ for which $E_{\mathbb{F}_p}$ has supersingular reduction is asymptotic to $c_E\sqrt{x}/\log x$ for some constant c_E.

For example, Serre has proved that the set of supersingular primes has density 0; Elkies has proved that there are infinitely many.

We are interested in analogous questions for abelian varieties.

Remark 35. In the case of complex multiplication, it is not so hard to study supersingularity via the Newton polygon and the Shimura–Taniyama formula.

One expects that the non-ordinary abelian varieties (measured via the Newton polygon) should have density zero. Recent work has shown that the supersingular primes has density zero. We are interested in showing infinitude.

3.2 The Main Theorem

For our talk, we fix a totally real number field F as our base and a quaternion algebra B over F which is split at exactly one real palce. Then let V be the subset of B with reduced trace equal to zero, and let Q_F be the reduced norm of V.

We will require all these to admit F-multiplication. In the case $[F:\mathbb{Q}]=3$, Mumford has defined some abelian fourfolds with endomorphism algebra $\mathbb{Z}$ but admitting some extra Hodge classes in $\mathrm{H}^4(A;\mathbb{Q})$. There is a Kuga–Satake construction which takes $K3$ surfaces to abelian varieties; it turns out that A^{64} is isogenous to an abelian variety coming from a $K3$ surface.

Remark 36. We have recently computed some of these fourfolds as coming from some explicit Jacobians.

In the case that F has narrow class number 1, the Shimura curve parameterizing Mumford's abelian fourfolds admits a canonical model Sh.

Theorem 37. Assume that Sh is isomorphic to $\mathbb{P}^1_F$ as well as some technical conditions. If A is in Sh and has field of moduli F, then A admits infinitely many primes of supersingular reduction.

Remark 38. The extra conditions are as follows.

- For F: we want 2 to be inert in F and $F(\sqrt{-\varepsilon})$ to have class number 1 for all $\varepsilon \in \mathcal{O}_F^\times$ which is negative at exactly one real place F. This condition simplifies calculations at the archimedean place; some case-by-case analysis could relax this condition.
- For B: there are also some ramification conditions.

There are many examples of such F; for example, there are cubic fields.

Remark 39. Such extensions of Elkies's theorem have a long history. For example, there has been a lot of work on abelian surfaces with some quaternionic multiplication. Notably, all previous work has been done with Shimura curves of PEL type, isomorphic to $\mathbb{P}^1$. This work is notable because the underlying Shimura curve merely has Hodge type.

3.3 Proof Idea

Let's review the proof idea for elliptic curves, due to Elkies.

Theorem 40 (Elkies). Fix an elliptic curve E over $\mathbb{Q}$. Then E admits supersingular reduction at infinitely many primes p.

Proof. Suppose E has finitely many supersingular primes in a set S.

1. We start with some CM points. Fix CM elliptic curves $\{E_1,\ldots,E_h\}$ with $\mathbb{Z}[\sqrt{-\ell}] \subseteq \mathrm{End}_{\overline{\mathbb{Q}}}(E_\bullet)$, where $\ell \equiv -1 \pmod 4$ is prime. Then we set

$$P_\ell(x) = \prod_i (x - j(E_i)),$$

and it turns out to be in $\mathbb{Z}[x]$. We would like to consttruct ℓ for which $\nu_p(P_\ell(j(E))) > 0$ (i.e., E has supersinglar reduction) and $\left(\frac{-\ell}{p}\right) \neq 1$ (i.e., $E_i \pmod p$ has supersingular reduction). Accordingly, let N be the numerator of $P_\ell(j(E))$, and then we have two cases.

- We may want $\ell \mid N$, in which case we choose $p = \ell$.
- We may want $\left(\frac{-\ell}{N}\right) = -1$, in which case we can still find some p.
- Lastly, we may want $p \notin S$, given from $\left(\frac{-\ell}{q}\right) =_1$ for all $q \in S$.

We will achieve these many congruence conditions by some explicit analysis at finite and real places.

2. Now, at finite places, we may want $\left(\frac{N}{\ell}\right) = +1$. The point is that CM liftings are paired, so one finds that $\deg P_\ell(x)$ is even. It follows that $P_\ell(x)$ is a square $\pmod{\ell}$.
3. At the real place, one shows that the roots of P_ℓ are paired, and then we get a non-square at N.

The proof is now completed upon some local calculations with quadratic reciprocity. ■

Our proof uses the following inputs.

1. CM points: these are parameterized by some conjugacy classes of embeddings of CM extensions into B. We then construct $P_\lambda(x)$ and do a quadratic reciprocity argument as before.
2. Finite places: we use something about integral models of Shimura varieties and do pairings as before.
3. Real places: we use something about Hecke equidistributino.

4 January 31: p-adic Higher Green's Functions

This talk was given by Hazem Hassan for JHUnior number theory days.

4.1 Higher Green's Functions

We begin with a story of complex multiplication. Let $\mathcal{H}$ be the complex upper-half plane, and let $j\colon \mathcal{H} \to \mathbb{C}$ is the j-invariant.

Definition 41. A *CM point* τ is a point in $\mathcal{H}$ such that $\mathbb{Q}(\tau)$ is an imaginary quadratic field.

Theorem 42. Fix a CM point $\tau \in \mathcal{H}$. Then $\mathbb{Q}(\tau, j(\tau))$ is an abelian extension of $\mathbb{Q}(\tau)$.

Remark 43. In fact, the theory of complex multiplication allows us to constuct all abelian extensions of $\mathbb{Q}(\tau)$ by working only a little harder.

We now introduce some modular forms.

Example 44. Here are some modular forms of weight 0.

- The function $j - j(\tau)$ is a holomorphic modular form of weight 0. Here, "holomorphic" is on $\mathcal{H}$, not $\mathbb{P}^1_{\mathbb{C}}$.
- The function $\log(j - j(\tau))$ is a multivalued holomorphic modular form of weight 0.
- The function $\log|j - j(\tau)|$ is a real analyti cfunction of weight 0.
- One can also take a logarithmic derivative $d\log(j - j(\tau))$, which is still holomorphic but now of weight 2.

There are also higher-weight versions of some of these.

Example 45. Fix some even $k \geq 4$, and set $n := k - 2$. Then

$$f_{k,\tau}(z) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \left(\frac{\gamma\tau - \gamma\overline{\tau}}{(z - \gamma\tau)(z - \gamma\overline{\tau})} \right)^{k/2}$$

is a meromorphic modular form of weight k with (simple) poles on $\mathrm{SL}_2(\mathbb{Z}) \cdot \tau$. We cannot evaluate this at our CM points because of the poles, so we would like a primitive.

Remark 46. We want a notion of differentiation to say something about $d\log$. However, differentiation is not totally SL_2-invariant: if f has weight $-n$, then one finds

$$d^{n+1}(f|_{-n}\gamma) = \left(d^{n+1}f\right)|_{-k},$$

where $\cdot|_\bullet$ is the twisted action associated with a modular form.

There is now a weight $-n$ modular form $\widetilde{f}_{k,\tau}(z)$ which is a multivalued modular form of weight $-n$; it is constructed via some repeated integration process. We may now define higher Green's functions.

Definition 47 (higher Green's function)**.** We define

$$G_{k/2}(\tau,\sigma) \coloneqq \operatorname{Re}\delta_{-n}^{n/2}\widetilde{f}_{k,\tau}(\sigma),$$

where $\delta_\ell \coloneqq \frac{d}{dz} + \frac{\ell}{z-\overline{z}}$.

Theorem 48 (Gross–Zagier conjecture)**.** Fix a normalized weakly holomorphic modular form f of weight $-n$ with Laurent expansion $f = \sum_m c(m)q^m$. Then for CM points τ and σ,

$$\sum_{m>0} c(-m)m^{n/2}(T_m \cdot G_{k/2})(\tau,\sigma) \in \overline{\mathbb{Q}}.$$

These numbers turn out to be intersections of Heegner cycles, which is why we may expect them to be algebraic (due to the standard conjectures). This is now known to Bruinier–Li–Yang, using regularized θ-lifts.

4.2 Real Multiplication

We now try to tell a similar story for real multiplication, retelling the story of extensions of real quadratic fields via some geometric picture.

- The analogue of $\mathcal{H}$ is the p-adic upper half-plane $\mathcal{H}_p \coloneqq \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$.
- The analogue of $\mathrm{SL}_2(\mathbb{Z})$ is $\Gamma \coloneqq \mathrm{SL}_2(\mathbb{Z}[1/p])$.
- Modular forms come from some sections $\mathrm{H}^0(\mathrm{SL}_2(\mathbb{Z}); \mathcal{M}_k)$ of a line bundle. It turns out that the right analogue is $\mathrm{H}^1(\Gamma; \mathcal{M}_k)$, where we go up in a degree to see some modular symbols.
- The Heegner points admit some archimedean intersection numbers $\log|j(\tau) - j(\sigma)|$. In the p-adic upper half-plane, one has some Stark–Heegner points, but they only conjecturally lie in the real quadratic field. There is something known about algebraicity.
- Heegner cycles have some interesting higher Green's functions. We are interested in putting something on the real multiplication side called "Stark–Heegner cycles."

We cannot define Stark–Heegner cycles, but we can say something about their conjectural intersections, which should be p-adic higher Green's functions.

Theorem 49. Fix an RM point $\tau \in \mathcal{H}_p$ and some integer $k \geq 2$. Then there is a unique rigid meromorphic cocycle (up to analytic cocycles and scaling) $J_{k,\tau} \in \mathrm{H}^1(\Gamma; \mathcal{M}_k)$ with poles supported on $\Gamma \cdot \tau$. Conversely, such cocycles span the "parabolic cohomology."

Remark 50. In fact, the poles of $J_{k,\tau}$ have the same shape as the poles of the earlier defiend $f_{k,\tau}$.

Example 51. By linearity, one can define $J_{k,D}$ for any divisor D with real multiplication. If it has "strong degree zero," then we find

$$J_{k,D}(\gamma)(z) = \sum_{\substack{b\in\Gamma \\ \tau\in|D|}} i\left([r,\gamma r],[b\overline{\tau},b\tau]\right)\left(\frac{b\tau-b\overline{\tau}}{(z-b\tau)(z-b\overline{\tau})}\right)^{k/2}.$$

Here, $i\left([r,\gamma r],[b\overline{\tau},b\tau]\right)$ is a signed intersection number of some geodesics (in $\mathcal{H}$!), where $r\in\mathbb{Q}$ is arbitrary. Notably, these intersection numbers are necessary to make the sum converge.

One now again takes some repeated integration, and we can state a Gross–Zagier conjecture. In fact, one can conjecture something about the prime factorizations of the special values, coming from some intersection numbers on Shimura curves.

5 January 31: Geometric and Arithmetic Siegel–Weil Formulae

This talk was given by Yu Luo at Johns Hopkins University for the JHUnior number theory days.

5.1 The Geometric Formula

For motivation, we may be interested in how an elliptic curve can map into an abelian variety. To geometrize, we let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g.

Definition 52 (Noether–Lefschetz cycle)**.** For each m, we define the functor $\mathrm{NL}(m)$ as living over $\mathcal{A}_1\times\mathcal{A}_g$ and parameterizing tuples (E,A,λ,h) such that $h\colon E\to A$ is some map, and λ is the polarizatino of A, and $\deg h^*\mathcal{L}_\lambda = m$.

Example 53. It turns out that $\mathrm{NL}(1)\cong\mathcal{A}_1\times\mathcal{A}_{g-1}$.

These cycles are interesting but difficult. We will be interested in similar questions with a little additional structure, coming from unitary Shimura varieties.

For today, F is an imaginary quadratic field, and we let V be a Hermitian spcae over F with signature $(n-1,1)$ (where $n\coloneqq\dim V$). We also fix some level structure $K\subseteq\mathrm{U}(V)(\mathbb{A}_{\mathbb{Q}}^\infty)$.

Definition 54 (unitary Shimura variety)**.** Fix everything as above. We define $\mathcal{M}_K$ to parameterize the data

$$(E,\iota_E,A,\iota,\lambda,\eta),$$

where (E,ι_E) is an elliptic curve with CM by F, (A,λ) is a principally polarized abelian variety, ι is an F-structure on A so that $\operatorname{Lie} A$ has signature $(n-1,1)$, and η is K-conjugacy class of a morphism $\widehat{V}(E)\to\widehat{V}(A)$.

Example 55. Choose a line $L\subseteq V$, and let K be the stabilizer of a lattice $\widehat{L}$. Then $\mathcal{M}_K(\mathbb{C})$ consists of copies of symmetric spaces which look like $\operatorname{Aut}(L)\backslash D$, where $D\subseteq\mathbb{P}(V_{\mathbb{R}})$ consists of isotropic lines in $V_{\mathbb{R}}$.

Remark 56. There is a pairing structure on $\operatorname{Hom}_F(E, A)$ by sending a pair (x, y) to the composite

$$E \xrightarrow{x} A \xrightarrow{\lambda} A^\vee \xrightarrow{y^\vee} E^\vee \cong E.$$

In particular, this maps to $\operatorname{End}(E) \cong F$, so we receive a pairing analogous to the Rosati involution.

Having given our analogue of $\mathcal{A}_g$, we need to give an analogue of our special cycles.

Definition 57. Fix a positive definite Hermitian $r \times r$ matrix T with coefficients in F, and choose some $\mu \in K \backslash V(\mathbb{A}_f)^r$. Then we let $Z(T, \mu)$ parameterize triples (E, A, u) where $u \in \operatorname{Hom}_F(E, A)^r$ satisfies $(u, u) = T$ (and some level structure dictated by μ).

It turns out that $Z(T, \mu)$ lives in $\operatorname{CH}^r(\mathcal{M}_K)$, so we may take some linear combinations.

Notation 58. For any $\varphi_f \in \mathcal{S}(V(\mathbb{A}_f)^r)^K$, we may define

$$Z(T, \varphi_f) := \sum_\mu \varphi_f(\mu)[Z(T, \mu)].$$

By definition of S, this is a finite sum.

Example 59. We continue from Example 55. For $x \in L$, we define $D(x)$ to be the lines perpendicular to x. Then $Z(t, 1_{\widehat{L}})$ on complex points is given by the projection

$$\operatorname{Aut}(L) \Bigg\backslash \bigsqcup_{(x,x)=t} D(x) \to \operatorname{Aut}(L) \backslash D.$$

The geometric Siegel–Weil formula arises when $\operatorname{rank} T = r - 1$.

Theorem 60. Fix a positive definite Hermitian $r \times r$ matrix T of rank $r - 1$. Then

$$\deg Z(T, \varphi_f) \sim E_T(\tau, \varphi),$$

where $\sim$ means we are up to some explicit constant (coming from the Shimura variety), φ is an automorphic form constructed with finite part φ_f, and E_T refers to the Tth Fourier coefficient for the Eisenstein series $E(\tau, \varphi)$ associated to φ.

5.2 The Arithmetic Formula

For the arithmetic Siegel–Weil formulae, we need to pass to some integral models. Accordingly, let S be a finite set of primes containing 2 and the ramified primes of the extension $F/\mathbb{Q}$. For simplicity, we will also assume that the lattice L of V is self-dual away from S, and we assume that $K^S = \operatorname{Stab}(\widehat{L}^S)$. In this situation, $\mathcal{M}_K$ admits an integral model over $\mathcal{O}_F\left[S^{-1}\right]$. If we derive, we also receive the special cycles $Z^{\mathbb{L}}(T, \varphi_f)$.

In this case, we will be intersted in the case where $\operatorname{rank} T = r$, which produces a cycle of codimension 1 because $\mathcal{M}_K \to \operatorname{Spec} \mathcal{O}_F\left[S^{-1}\right]$ has relative dimension r.

Notation 61. We let $\operatorname{Diff}(T, V)$ be the set of nonsplit primes p for which $V_p \ncong V(T)_p$, where $V(T)$ is the space with basis $\{e_1, \ldots, e_n\}$ satisfies $(e_i, e_j) = T_{ij}$.

Theorem 62 (Kudla–Rapoport)**.** The special cycle $Z(T, \varphi_f)$ is nonempty only if $\operatorname{Diff}(T, V)$ is a single prime $p \notin S$. In this case, the special cycle is supersingular over $\mathbb{F}_{p^2}$.

Theorem 63. Suppose $\operatorname{Diff}(T,V)=\{p\}$. Then

$$\log\left(p^2 \deg Z^{\mathbb{L}}(T,\varphi_f)\right) \sim E'_T(\tau,\varphi).$$

One can view this as a natural generalization of the Gross–Zagier formula to unitary Shimura varieties. The moral is that we are relating an intersection number to a modular form, which can then be related to a special value. Here are some ingredients of the proof.

1. It turns out that
$$Z(\tau,\varphi)=\sum_{m>0} Z(m,\varphi)q^m$$
is a modular form. One can then take an arihetmic intersection with another special cycle to produce a genuine modular form. This produces a modular form whose coefficients are given by the left-hand side.

2. On the other hand, we can use the right-hand side to produce a modular form. It turns out that the coefficients are the same at all but finitely many coefficients, but the difference is a modular form, so full equality follows.

6 January 31: Higher Siegel–Weil Formulae

This talk was given by Mikayel Mkrtchyan at the Johns Hopkins University for the JHUnior number theory days.

6.1 Shtukas

We are interested in function field analogues of the geometric and arithmetic Siegel–Weil formulae. The analogue of (unitary) Shimura varieties are given by shtukas. For our setup, we fix some étale double cover $X' \to X$ of smooth proper curves over $k := \mathbb{F}_q$; let σ be the nontrivial automorphism.

Definition 64. A U_n-*bundle* is a pair $(\mathcal{F},h)$ of a rank n bundle $\mathcal{F}$ on X' and an isomorphism $h\colon \mathcal{F}\to\sigma^*\mathcal{F}^\vee$. A *Hecke modification* is a diagram $x\colon \mathcal{F}_0\to\mathcal{F}_1$, where $\mathcal{F}_0$ and $\mathcal{F}_1$ are U_n-bundles, $x\in X'$ is a point, and we are euipped with an isomorphism $\mathcal{F}_0|_{X'\setminus\{x,\sigma x\}}\cong\mathcal{F}_1|_{X'\setminus\{x,\sigma x\}}$. We also require there to be a natural short exact sequence

$$0\to\mathcal{F}_0\to\mathcal{F}_1\to\delta_x\to 0,$$

where δ_x is the skyscraper.

Example 65. If $X'\to X$ is split, then a U_n-bundle is a vector bundle of rank n on X.

Definition 66 (shtuka)**.** We define $\mathrm{Sht}^r_{\mathrm{U}_n}$ to be the moduli stack parameterizing diagrams

$$\mathcal{F}_0\to\mathcal{F}_1\to\cdots\to\mathcal{F}_r\cong\mathcal{F}_0^\tau,$$

where each $\mathcal{F}_\bullet$ is a U_n-bundle, each arrow is a Hecke modification, and $\mathcal{F}_0^\tau$ is the Frobenius pullback.

Here are some grounding remarks about shtukas.

Remark 67. There is a "leg morphism" $\mathrm{Sht}^r_{\mathrm{U}_n}\to(X')^r$ recording the points where the Hecke modifications occur. This is analogous to the structure morphism for a Shimura variety. It turns out that this map is smooth of relative dimensino $r(n-1)$. Thus, $\dim\mathrm{Sht}^r_{\mathrm{U}_n}=rn$.

Example 68. One has that $\mathrm{Sht}^0_{\mathrm{U}_n} = \mathrm{Bun}_{\mathrm{U}_n} k$, so functions on this space are basically unramified automorphic forms.

Remark 69. For $n > 1$, the space $\mathrm{Sht}^r_{\mathrm{U}_n}$ is not quasicompact.

Remark 70. Taking the fibers over the various Hecke modifications, we produce "tautological" line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on $\mathrm{Sht}^r_{\mathrm{U}_n}$.

Remark 71. It turns out that $\mathrm{Sht}^r_{\mathrm{U}_n}$ vanishes for odd r, but it is not too hard to define a suitable analogue.

We will also want some special cycles. The following definition is due to Feng–Yun–Zhang.

Definition 72. Fix a pair $(\mathcal{E}, a)$, where $\mathcal{E}$ is a vector bundle of rank m on X', and $a \colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is some Hermitian morphism. (This is an analogue of a lattice.) Then $(\mathcal{E}, a)$ is *non-degenerate* if and only if a is injective. We let $\mathcal{A}(m)$ be the moduli stack of non-degenerate pairs.

Definition 73. Fix a pair $(\mathcal{E}, a)$ as before. Then we define the special cycle $Z^r_{\mathcal{E}}(a)$ on $\mathrm{Sht}^r_{\mathrm{U}_n}$ to parameterize diagrams

$$\begin{tikzcd}
 & & \mathcal{E} \arrow[dll, "t_0"'] \arrow[dl, "t_1"] \arrow[dr, "t_r"'] \arrow[drr, "t_0^\tau"] & & \\
\mathcal{F}_0 \arrow[r] \arrow[drr] & \mathcal{F}_1 \arrow[r] \arrow[dr] & \cdots \arrow[r] & \mathcal{F}_r \arrow[r] \arrow[dl] & \mathcal{F}_0^\tau \arrow[dll] \\
 & & \sigma^*\mathcal{E}^\vee & &
\end{tikzcd}$$

where the total composite is a.

Remark 74. To see that this is an analogue, imagine that $\mathcal{E}$ is a CM elliptic curve, so we are asking for maps to abelian varieties which preserve some structure.

Example 75. If a is an isomorphism, then taking orthogonal complements shows that $\mathcal{E}$ embeds into the shtuka, so $Z^r_{\mathcal{E}}(a) \cong \mathrm{Sht}^r_{\mathrm{U}_{n-m}}$. Even if a is merely generically an isomorphism, then such an isomorphism holds generically.

Example 76. If $a = 0$, then there is a natural map $\mathrm{Sht}^r_{\mathrm{U}_n} \subseteq Z^r_{\mathcal{E}}[0]$ simply by setting the $t_\bullet$s to vanish.

Remark 77. One expects that $\dim Z^r_{\mathcal{E}}(a) = r(n-m)$, though the dimension could be much larger. For example, when a is an isomorphism, equality holds, and when $a = 0$, equality does not hold. Feng–Yun–Zhang have defined virtual fundamental classes $[Z^r_{\mathcal{E}}(a)]$ in $\mathrm{CH}_{r(n-m)} Z^r_{\mathcal{E}}(a)$.

6.2 The Formulae

We will need to define some Eisenstein series.

Definition 78. There is a Siegel Eisenstein series $\mathrm{Eis}(g,s)_m$ for $\mathrm{U}(m,m)$ such that each pair $(\mathcal{E},a)$ (of rank m) has a Fourier coefficient $\mathrm{Eis}_{(\mathcal{E},a)}(s)_m$ with a rather technical definition. Our Eisenstein series are normalized so that the center of the functional equation is at $s=0$.

Remark 79. The Eisenstein series turn out to have explicit combinatorial expansions. For example, if $\operatorname{coker} a$ has length two, then $\mathrm{Eis}_{(\mathcal{E},a)} = q^s \pm q^{-2}$, where the $+$ sign is used if we are split over a split point $x \in X$.

Theorem 80 (Feng–Yun–Zhang)**.** Fix everything as above, and take $m=n$ and a non-degenerate pair $(\mathcal{E},a)$. Then for all $r \geq 0$, we have

$$\deg\,[Z^r_{\mathcal{E}}(a)] \sim \partial^r_{s=0}\,\mathrm{Eis}_{(\mathcal{E},a)}(s)_m.$$

Here, $\partial^r_{s=0}$ means that we take r derivatives and then evalaute at 0.

Theorem 81. Fix everything as above, and take $m=n-1$ and a non-degenerate pair $(\mathcal{E},a)$. Then for every even $r \geq 0$, we have

$$\deg\,[Z^r_{\mathcal{E}}(a)] \cap [\mathcal{L}_1] \cap \cdots \cap [\mathcal{L}_r] \sim \partial^r_{s=0}\left(q^{s\deg\omega_X}\,\mathrm{Eis}_{(\mathcal{E},a)}(s-1/2)_{n-1}L(\eta,2s)\right).$$

Remark 82. There is an analogous statement for $r \geq 0$.

Remark 83. There is some analogous thing known for number fields when $r=1$.

Let's say something about the proofs. The main difficulty in the proof is calculating on the geometric side. One can view the function

$$(\mathcal{E},a) \mapsto \deg\,[Z^r_{\mathcal{E}}(a)]$$

as a k-valued function on the moduli stack $\mathcal{A}(n)$, so we hope that it arises from a sheaf. (The same holds for the $m=n-1$ case.)

Definition 84 (Hitchin)**.** Let $\mathcal{M}(m,n)$ be the moduli stack of morphism composites $\mathcal{E} \to \mathcal{F} \cong \sigma^*\mathcal{F}^\vee$ so that the composite

$$\mathcal{E} \to \mathcal{F} \cong \sigma^*\mathcal{F}^\vee \to \sigma^*\mathcal{E}^\vee$$

is injective.

There is a natural projection $\mathcal{M}(m,n) \to \mathcal{A}(m)$, and it turns out that there is a self-correspondence

$$\mathcal{M}(m,n) \leftarrow \mathrm{Hk}^1_{\mathcal{M}}(m,n) \to \mathcal{M}(m,n).$$

Thus, we can do cohomology to this correspondence, from which a Lefschetz trace formula for correspondences lets us compute $\deg\,[Z^r_{\mathcal{E}}(a)]$ as the trace of Frobenius acting via this correspondence. (Again, something similar happens for $m=n-1$.) We now discuss $m=n$ and $m=n-1$ separately.

- Now, if $m=n$, there is a pullback square

$$\begin{tikzcd} \mathcal{M}(n,n) \arrow[r] \arrow[d] & \mathrm{Lg} \arrow[d] \\ \mathcal{A}(n) \arrow[r] & \mathrm{Herm} \end{tikzcd}$$

  where the right arrow can be controlled étale-locally. Here, Herm is the moduli space of Hermitian torsion sheaves, and Lg parameterizes Langrangians. This allows one to control the fibers of $\mathcal{M}(n,n) \to \mathcal{A}(n)$.

- The case of $m = n-1$ is more difficult. Instead, one factors $\mathcal{M}(n-1,n) \to \mathcal{A}(n-1)$ through some $\mathcal{A}(n-1) \times \mathrm{Bun}_{\mathrm{U}_1}$, and this factored morpism is a $(\mathbb{P}^1)^d$-fibration. It also turns out that this factored map has full support, allowing us to control the fibers.

7 January 31: Hypergeometric Motives with CM

This talk was given by Esme Rosen at Johns Hopkins University for the JHUnior number theory days.

7.1 Hypergeometric Motives

Our modular forms are classical ones. As usual, they have L-functions, which have analytic continuation, and they frequently have Euler products, and so on. We will be considering normalized Hecke eigenforms f, which also have associated Galois representations ρ_f, which are known to come from elliptic curves.

Definition 85 (hypergeometric data)**.** Fix $z \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0,1,\infty\}$ and two lists of rational numbers denoted $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ with $b_0 = 1$. Set M to be their least common multiple. Then there is a hypergeometric motive defined over $\mathbb{Q}(\zeta_M)$ related to classical hypergeometric functions

$$ {}_{n+1}F_n \begin{bmatrix} a_0 & \cdots & a_n & z \\ b_0 & \cdots & b_n & \end{bmatrix} = \sum \frac{(a_0)! \cdots (a_n)!}{(b_0)! \cdots (b_n)!} z^{\bullet}. $$

The associated hypergeometric functions give rise to the periods of the motives.

Example 86. The data $((1/2,1/2),(1,1))$ gives rise to the Legendre family of elliptic curves $y^2 = x(1-x)(1-zx)$.

Example 87. The data $((1/2,1/2,1/2),(1,1,1))$ gives rise to a family of $K3$ surfaces, which are known to be modular at the CM points. These are symmetric powers of elliptic curves. One expects these to be associated to weight 3 modular forms.

Example 88. Let's construct the hypergeometric motives for ${}_2F_1$s; the general case is not too much harder. Start with the normalization X_{HD} of the curve

$$ Y^M = X^i(1-X)^j(1-zX)^k $$

for some $i,j,k < M$ with $i+j+k \nmid M$. There is a standard γ whose period is some quotient of Γs along with a ${}_2F_1$ value. By diagonalizing the μ_M action on X_{HD}, we can decompose the motive X_{HD}. (For example, this certainly produces an absolute Hodge cycle.)

Remark 89. There are finite field versions of our hypergeometric series, basically obtained by replacing Γs with some Gauss sums. Katz has shown that there are Galois representations whose traces are the correct hypergeometric sums over finite fields.

Classical work in this de Rham realization has found that certain periods of modular forms (which are essential special values of the L-function) can be expressed as certain hypergeometric series. There is also work in the étale realization, expressing certain traces of Galois representations.

7.2 Some Theorems

Here is a main theorem.

Theorem 90. There is explicit modularity for some hypergeometric motives of the type

$$((1/2, 1/2, r), (1, 1, s)),$$

where (r, s) are some explicit pairs of rational numbers.

For another application, we consider Fermat motives. Fermat motives have complex multiplication, which means that their Galois representations are induced from Hecke characters. The periods of motives with CM by an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ are known by the Chowla–Selberg formula.

Remark 91. Fermat motives arise in our story because they are ${}_2F_1(1)$ motives, which notably appear in the cohomology of Fermat curves. The associated Hecke characters are some Jacobi sums; we then take induction to get the associated modular form.

Theorem 92. For many of these hypergeometric motives, one can descend the Galois representations from $\mathbb{Q}(\zeta_M)$ to $\mathbb{Q}(\sqrt{-D})$ for some explicit D.

In fact, the theory of complex multiplication now allows us to compute some special values.

8 January 31: Finiteness of Heights of Motives

This talk was given by Alice Lin at Johns Hopkins University for the JHUnior number theory days.

8.1 Our Statement

We are going to be interested in heights today. Here is one such example.

Theorem 93 (Faltings)**.** Fix $g \geq 1$, a number field F, and a finite set S of finite places. Then there are finitely many isomorphism classes of g-dimensional abelian varieties A over F with good reduction outside S.

Sketch. The proof has two steps.

1. Show that there are only finitely many isogeny classes.
2. Show that each isogeny class has only finitely many isomorphism classes. This is done basically by understanding how the Faltings height changes along isogenies. ■

The second step has been better understood recently.

Theorem 94 (Kisin–Mocz)**.** Assume the Mumford–Tate conjecture for an abelian variety A over a number field F. Then for any $cf > 0$, the set of abelian varieties B over $\overline{\mathbb{Q}}$ isogenous to A (over $\overline{\mathbb{Q}}$) with Faltings height at most c.

Remark 95. Unlike in Faltings's situation, the set of abelian varieties B (with unbounded Faltings height) is infinite.

Remark 96. Recall that the Mumford–Tate conjecture asserts that the Mumford–Tate group agrees with the (connected component of the) ℓ-adic monodromy group. Here, the Mumford–Tate group is the Tannakian algebraic group of $\mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}); \mathbb{Q})$ in the category of Hodge structures, and the ℓ-adic monodromy group is the Zariski closure of the Galois representation. The Mumford–Tate conjecture is wide open in the general case, but a lot is known.

Here is our result, with definitions to follow.

Theorem 97. Let M be a Koshikawa motive defied over F, and assume the following.

- The adelic Mumford–Tate conjecture holds for M.
- All Hodge classes are absolutely Hodge and compatible with étale–de Rham comparison maps.

Then for any $c > 0$, the set of motives M' equipped with an isogeny $M' \to M$ with height at most c has only finitely many possible heights.

Remark 98. The second hypothesis is true for abelian varieties, due to Deligne and Blasius.

Remark 99. If $\mathrm{MT}(M)$ acts irreducibly, then we can achieve only finitely many isomorphism classes.

Corollary 100. Fix a point x on a Shimura variety $\mathrm{Sh}(G, X)(F)$. Suppose that the Galois representation

$$\rho_x \colon \operatorname{Gal}(\overline{F}/F) \to G(\mathbb{A}_f)$$

has open image (so that the adelic Mumford–Tate conjecture holds). Then for any $c > 0$, there are only finitely many $\overline{F}$-points in the Hecke orbit with height at most c.

Notably, this result is known due to Kisin–Mocz in the Hodge type case.

8.2 Explanation of the Theorem

Let's start by defining our motives.

Definition 101 (Koshikawa motive)**.** A *Koshikawa (pure) motive* M over a number field F with weight w is (roughly) a system of cohomological realizations, equipped with a $\widehat{\mathbb{Z}}$-integral Galois stable structure on the adelic étale realization. More precisely, we take

- A system of cohomological realizations: all the Betti cohomology realizations for each embedding $F \hookrightarrow \mathbb{C}$, the de Rham cohomology, all the ℓ-adic cohomology realizations, and
- A lattice: such as the $\widehat{\mathbb{Z}}$-coefficients étale cohomology.

Definition 102 (isogeny)**.** An isogeny $M' \to M$ of Koshikawa motives is a Galois-stable inclusion of the underlying integral lattices.

Remark 103. This is analogous to the fact that an isogeny of abelian varieties amounts to the data of an embedding of the (adelic) Tate modules.

We also need to define our height.

Definition 104 (Kato–Koshikawa)**.** Fix a Koshikawa motive M, and set

$$V := \bigotimes_{r\in\mathbb{Z}} \det_F \left(\operatorname{gr}^r M_{\mathrm{dR}}\right)^{\otimes r}.$$

Then integral p-adic Hodge theory provides an $\mathcal{O}_{F_v}$-lattice in $V \otimes_F F_v$ arising from the $\widehat{\mathbb{Z}}$-lattice; these assemble into an $\mathcal{O}_F$-lattice $\mathcal{L} \subseteq V$. Then we define the height

$$h(M) = \frac{1}{[F:\mathbb{Q}]}\left(\log \#\left(\frac{\mathcal{L}}{\mathcal{O}_F\alpha}\right) - \sum_{\sigma\colon F\hookrightarrow\mathbb{C}} \log \|\alpha\|_\sigma\right).$$

Example 105. When $M = \mathrm{H}^1(A)$, we find $V = \det \mathrm{H}^0(A;\Omega^1) = \mathrm{H}^0(A;\Omega^g_{A/F})$. The integral p-adic Hodge theory can be replaced with the Néron model $\mathcal{A}$ over $\mathcal{O}_F$, allowing us to take $\mathcal{L} = \mathrm{H}^0(\mathcal{A};\Omega^g_{\mathcal{A}/\mathcal{O}_F})$. The height is then exactly the Faltings height.

Roughly speaking, the proof boils down to an isogeny formula for how our height changes in isogenies. The Mumford–Tate conjecture (and other inputs) are used in order to move the integral p-adic Hodge theory out into more controlled cohomology theories.