

# Seminars

Nir Elber

Spring 2026

## Abstract

This semester, I will just record all seminars I go to in an uncategorized manner. I will try to record the date, the speaker, and which seminar it was to maintain some semblance of organization.

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## 1 Multiplicity Formulae for Spherical Varieties

This talk was given by Toan Pham at Johns Hopkins University for the student automorphic representations seminar.

### 1.1 The Theorems

Fix a homogeneous spherical variety  $X = G/H$  over a local field  $F$ . Also, choose a character  $\chi: H(F) \rightarrow \mathbb{C}^\times$  and an irreducible representation  $\pi$  of  $G(F)$ . Then the local relative Langlands program is interested in the multiplicity

$$m(\pi, \chi) := \dim \text{Hom}_{H(F)}(\pi, \chi).$$

Note that

$$m(\pi, \chi) = \dim \text{Hom}_{G(F)}(\pi, C^\infty(X(F), \chi)).$$

Here are some starting examples.

**Example 1 (Whittaker).** An interesting case is when  $H \subseteq G$  is a unipotent subgroup. For example, one can take the unipotent radical  $U$  of the Borel subgroup of  $\text{GL}_2$ . Then we can take  $\chi$  to be lifted from any additive character  $F \rightarrow \mathbb{C}^\times$ . One can generalize this example to work from any  $\mathfrak{sl}_2$ -triple, and it produces Whittaker models of  $\pi$  in  $C^\infty(X(F), \chi)$ .

**Example 2 (Gan–Gross–Prasad).** Fix a quadratic extension  $E/F$ . Then one can take  $X = \text{SO}_n \backslash (\text{SO}_n \times \text{SO}_{n+1})$  or  $X = \text{U}_n \backslash (\text{U}_n \times \text{U}_{n+1})$ . In general, there are more examples arising from so-called “GGP triples”  $(G, H, \chi)$ .

**Remark 3.** It turns out that  $m(\pi, \chi) \leq 1$  in Examples 1 and 2, but this is not always true.

Nonetheless, it becomes interesting to discover when  $m(\pi, \chi) \geq 1$ . Here are some answers.

**Theorem 4.** Fix everything as in Examples 1 and 2 and choose a Langlands parameter  $\varphi: W_F \rightarrow {}^L G$ . Then the Langlands packet  $\Pi_\varphi$  contains exactly one  $\pi$  for which  $m(\pi, \chi) \neq 0$ .

**Theorem 5.** Fix everything as in Example 2, and choose a Langlands parameter  $\varphi: W_F \rightarrow {}^L G$ . Then  $\pi \in \Pi_\varphi$  has  $m(\pi, \chi) \neq 0$  if and only if it maps to a specified unitary character.

Our method will be the so-called “local trace formulae,” developed originally by Waldspurger.

## 1.2 Local Trace Formulae

For the rest of the talk, we work in the context of Example 2. The type of result we are trying to prove rewrites  $m(\pi, \chi)$  in terms of a sum of orbital integrals.

**Example 6.** For finite groups  $G$ , we can use the character  $\Theta_\pi$  of  $\pi$  to see that

$$m(\pi, \chi) = \sum_{\text{conj. } [x] \subseteq H} \frac{1}{\#C_H(x)} \cdot \Theta_\pi(x) \overline{\chi}(h).$$

This right-hand side can be viewed as some twisted sum of sizes of conjugacy classes.

Our method to prove such formulae will rest on local trace formulae, which basically amount to finding two ways of expressing the trace of some  $f \in C_c^\infty(G)$  acting on  $L^p(H(F) \backslash G(F), \chi)$ . On one hand, we may write

$$\begin{aligned} (Rf \cdot \varphi)(x) &= \int_{G(F)} f(g) \varphi(xg) dg \\ &= \int_{H(F) \backslash G(F)} K_f(x, y) \varphi(y) dy, \end{aligned}$$

where  $K_f(x, y) = \int_{H(F)} f(x^{-1}hy) \chi(h) dh$ . We morally then expect the trace of  $Rf$  to be the sum of  $K_f$  along the diagonal, as one finds with finite groups.

**Proposition 7.** Fix everything as above. Say that  $f \in C_c^\infty(G(F))$  is strongly cuspidal if and only if

$$\int_{U(F)} f(um) du = 0$$

for any parabolic  $P$  with Levi decomposition  $P = MU$ . If  $f$  is strongly cuspidal, then the trace of  $Rf$  converges absolutely.

The local trace formula now amounts to expressing  $J(f)$  either via a spectral or a geometric expansion.

- The spectral expansion is

$$J_{\text{spec}}(f) := \int_{\mathfrak{X}(G)} D(\pi) \widehat{\Theta}_f(\pi) m(\pi) d\pi.$$

Here,  $\mathfrak{X}(G)$  consists of the space of tempered representations (i.e., found in  $L^2(G)$ ) arising from parabolic induction of elliptic representations. Then  $\Theta_f$  is a weighted orbital integral of  $f$ , and  $\widehat{\Theta}_f$  is its Fourier transform. The  $D(\pi)$  is unknown, but we have been reassured that it is not important.

The idea to show that  $J(f) = J_{\text{spec}}(f)$  is to use the Plancheral formula

$$\langle f_1, f_2 \rangle = \int_{\widehat{G}} J_\pi(f_1 \otimes f_2) d_X \pi,$$

where  $J_\pi$  is the natural composite

$$C_c^\infty(X \times X) \rightarrow \pi \otimes \tilde{\pi} \rightarrow \mathbb{C}.$$

It turns out that  $J_\pi$  is non-vanishing if and only if  $m(\pi, \chi) \neq 0$ . Yiannis claims that the forward direction is easy.

- The geometric expansion associates a quasicharacter  $\theta_f: G_{\text{reg}}(F) \rightarrow \mathbb{C}$  defined on the regular semi-simple locus, and then we have

$$m_{\text{geom}}(\theta) := \lim_{s \rightarrow 0} \int_{\Gamma(G, H)} D^G(x)^{1/2} c_\theta(x) \Delta^{s-1/2} dx.$$

Here,  $\Gamma(G, H)$  is the space of semisimple conjugacy classes of  $G(F)$  which are represented by an element in  $H(F)$ , and  $c_\theta$  is some extension of  $\theta$  to the semisimple locus  $G_{\text{ss}}(F)$ . Our geometric expansion is then  $J_{\text{geom}}(f) := m_{\text{geom}}(\Theta_f)$ .

It now turns out that

$$J_{\text{spec}}(f) = J(f) = J_{\text{geom}}(f),$$

though both of these equalities are theorems. Though it requires an argument, this turns out to be equivalent to a multiplicity formula  $m_{\text{geom}}(\pi) = m(\pi)$ .

**Conjecture 8.** One has  $m_{\text{geom}}(\pi) = m(\pi)$  for any spherical  $X$ .