Sato–Tate Groups of Generic Superelliptic Curves

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Fall 2024

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman [Shu16]

Chapter 0
Introduction

What we didn't do is make the construction at all usable in practice!
This time we will remedy this.

—Kiran S. Kedlaya, [Ked21]

0.1 Overview

Over the past few decades, there has been a growing interest in understanding how the geometry of a space (such as a smooth projective variety) affects its arithmetic.

One example of these effects arises in the form of the Sato–Tate conjecture, which takes an abelian variety A over $\mathbb{Q}$ and predicts the distribution of the point-counts $\#A(\mathbb{F}_p)$ (suitably interpreted) as the primes p varies. Here, one finds that the "geometric" invariant $\mathrm{End}_{\mathbb{C}}(A)$ essentially determines the desired distribution. We refer to section 3.1 for a more precise discussion, but approximately speaking, the point is that one expects a "motivic monodromy group" to control this distribution, and the motivic monodromy group can be computed either in a geometric situation over $\mathbb{C}$ or understood via such point-counts in an arithmetic situation.

To be slightly more explicit, there are various monodromy groups at play: in the complex analytic situation, there is the Mumford–Tate group $\mathrm{MT}(A)$, and in the ℓ-adic situation, there is the ℓ-adic monodromy group $G_\ell(A)$. There are conjectural relations between these, and these conjectures codify the interplay between geometry and arithmetic; for example, the Mumford–Tate conjecture predicts that $\mathrm{MT}(A)_{\mathbb{Q}_\ell} = G_\ell(A)^\circ$. Ultimately, to understand point-counts, one becomes interested in the groups $G_\ell(A)$, but this group is difficult to compute directly, so it is frequently profittable to compute $\mathrm{MT}(A)$ instead and then use one of the aforementioned conjectures.

In this article, we are interested in the effect of so-called "exceptional" geometry on arithmetic, continuing the work of [GGL24]. The exceptional geometry we are interested in concerns exceptional Hodge classes, which are Hodge classes on A (or a power of A) which are not generated by an endomorphism of A or the polarization of A. The absence of such classes gives control of the geometry of A and hence makes $\mathrm{MT}(A)$ and $G_\ell(A)$ easy to compute. As another application, in the absence of exceptional classes, one knows the Hodge conjecture for all powers of A, so exceptional geometry is in some sense "the enemy" of proving the Hodge conjecture.

0.1.1 Fermat Curves

Roughly speaking, most abelian varieties do not support exceptional classes, so it requires some effort to find abelian varieties with exceptional classes in nature (and then prove and study their existence!). In [GGL24], Gallese, Goodson, and Lombardo are able to control exceptional classes in the Jacobians of the "Fermat" hyperelliptic curves

$$y^2 = x^N + 1$$

as $N \geq 1$ varies over positive integers. Namely, they are able to write down an algorithm which computes the groups MT and G_ℓ for moderately sized N (say, $N \leq 100$), and they are able to prove general results in certain cases (such as N prime). It is still true that some N fail to support exceptional classes, such as when N is a prime, but composite N frequently support exceptional geometry, which must be understood to execute the computation.

The present article can be considered a continuation of the work of [GGL24]. For example, the authrors there remark that their methods should be able to be used to compute MT and G_ℓ for the Jacobians of quotients of the smooth projective Fermat curve

$$X_N \colon X^N + Y^N + Z^N = 0,$$

which includes the hyperelliptic curves $y^2 = x^N + 1$ above. This is carried out in section 4.2; we note that the main theorem is Theorem 4.33, where we provide an explicit description of the Galois action on (absolute) Hodge classes in terms of Galois action on certain explicitly computed periods, but we will not give the statement in the introduction because it is somewhat technical.

> **Remark 0.1.** As an aside, we note that the authors of [GGL24] recourse to more general Fermat hypersurfaces
>
> $$X_0^N + X_1^N + \cdots + X_m^N = 0.$$
>
> in order to understand powers of the Fermat curve X_N. This theory rests on somewhat technical algebraic geometry due to Deligne [Del18, Section 7]. In this article, we rebuild the thoery of [GGL24] while only handling powers of X_N directly, allowing us to avoid Deligne's algebraic geometry. The key point is that a careful analysis of the Künneth isomorphism allows one to gain the same level of control on the Hodge classes of a power of X_N as one would get with embedding in a Fermat hypersurface. This is carried out in section 4.2.1.

Having access to more general quotients allows us to see more geometry. To explain one example, we recall the definition of $G_\ell(A)$. Given an abelian variety A defined over a number field K, one can use the Galois action on the Tate module $V_\ell A$ of A to define a Galois representation

$$\rho_\ell \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V_\ell A).$$

Here, $V_\ell A$ turns out to be a vector space ver $\mathbb{Q}_\ell$ of dimension $2 \dim A$. We then define $G_\ell(A)$ to be the smallest algebraic $\mathbb{Q}_\ell$-subgroup containing the image of ρ_ℓ. The Mumford–Tate conjecture explains that one expects to recover $G_\ell(A)^\circ$ from the complex geometry of A, so it becomes interesting to understand the quotient $G_\ell(A)/G_\ell(A)^\circ$, which we note is finite because $G_\ell(A)$ is an algebraic group. In light of the definition of $G_\ell(A)$, we see that we are interested in the pre-image $\rho_\ell^{-1}(G_\ell(A)^\circ)$; this needs to be a finite-index open subgroup of $\mathrm{Gal}(\overline{K}/K)$, so there is a finite extension K_A^{conn} of K such that $\rho_\ell(\sigma) \in G_\ell(A)^\circ$ if and only if σ fixes K_A^{conn}.

In [GGL24, Theorem 7.1.1], the authors find that their hyperelliptic curves $y^2 = x^N + 1$ all have K_A^{conn} to be a multiquadratic extension of $\mathbb{Q}(\zeta_N)$, and they provide an algorithm to compute it. Further, they find that the prime-power case will always have $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_N)$. One can now ask if one can hope for such control for general quotients of the Fermat curve. Well, [Del18, Theorem 7.15] explains that the extension $K_A^{\mathrm{conn}}/\mathbb{Q}(\zeta_N)$ should always be abelian. However, it turns out that one cannot hope for much more than this.

Example 0.2. Using the techniques of section 4.2, one can show that the Jacobian of the superelliptic curve

$$y^9 = x\left(x^2+1\right),$$

which is a quotient of the Fermat curve $X^{18}+Y^{18}+Z^{18}=0$, has $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_{18}, \sqrt[18]{432})$, which is a degree-18 cyclic extension of $\mathbb{Q}(\zeta_{18})$.

Example 0.3. The Jacobian of the previous example is not simple. At the cost of having slightly higher dimension, one can show something similar for the Jacobian of $y^{11} = x^2\left(x^2+1\right)$, but now this Jacobian is simple.

In section 4.2, we work out the example curve $y^9 = x^3+1$ in detail. Here, one does find exceptional classes, but we still have $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$. In the current draft of this article, we do not work out the above two examples because it would require a somewhat lengthy discussion of algebraicity of products of the Γ-function which has not been included in this first version.

0.1.2 Beyond CM

One aspect of these Fermat curves is that they have so many automorphisms (given by multiplying X or Y by an Nth root of unity) that their Jacobians have complex multiplication. Complex multiplication aides the computation in a few key ways: in this case, $\mathrm{MT}(A)$ and $G_\ell(A)$ are both tori, thus making them much easier to control. For example, the Mumford–Tate conjecture is known in this case, and there exist algorithms to compute $\mathrm{MT}(A)$ from certain combinatorial data attached to A.

As such, to the author's knowledge, the literature does not have an example computation of $G_\ell(A)$ when A does not have complex multiplication and is not fully of Lefschetz type.[1] In this article, we work out such an example. Admittedly, we do not go far from complex multiplication: where complex multiplication would require $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ to contain a CM field of dimension $2\dim A$, we work with certain abelian varieties A such that $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ contains a CM field of dimension $\dim A$. Our limitations are rather technical, and we expect that one can do much better.

As an example difficulty, let's focus on computing $\mathrm{MT}(A)$. Recall that $\mathrm{MT}(A)$ is a connected reductive algebraic group defined over $\mathbb{Q}$, so we can split up its computation into computing the derived subgroup $\mathrm{MT}(A)^{\mathrm{der}}$ and the neutral component $Z(\mathrm{MT}(A))^\circ$ of the torus. In section 2.2, we explain how the current arguments used to understand $\mathrm{MT}(A)$ for A with complex multiplication can be used to compute $Z(\mathrm{MT}(A))^\circ$. To explain this result, we pick up some notation: set $F \coloneqq Z(\mathrm{End}(A))$, and then one can diagonalize the action of F on $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}),\mathbb{C})$ to produce a piece of combinatorial data called the "signature" $\Phi\colon \mathrm{Hom}(F,\mathbb{C})\to\mathbb{Z}_{\geq 0}$; for brevity, we will set $\Sigma_F \coloneqq \mathrm{Hom}(F,\mathbb{C})$. It turns out that one can embed $Z(\mathrm{MT}(A))^\circ$ into the torus $\mathrm{T}_F \coloneqq \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$, and our first main result explains how to recover this subtorus.

Corollary 2.77. Fix an abelian variety A over $\mathbb{C}$ such that $Z(\mathrm{End}(A))$ equals a CM algebra F, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$. Let $\Phi\colon \Sigma_F\to\mathbb{Z}_{\geq 0}$ be the signature defined in Lemma 2.72. Then $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$ has cocharacter group equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$ containing

$$\sum_{\sigma\in\Sigma_F}\Phi(\sigma)\sigma^\vee.$$

Remark 0.4. In fact, a careful reading of the arguments in section 2.2 reveal that we are actually able to compute an explicit power of $Z(\mathrm{MT}(A))$, which technically contains more information. For example, one could provide a sufficient condition for $Z(\mathrm{MT}(A))$ being disconnected.

[1] Roughly speaking, "fully of Lefschetz type" means that all Hodge classes on A can be explained by endomorphisms and the polarization. In type III, it turns out that these classes do imply the existence of an exceptional class, which is the difference between not supporting exceptional cycles and being "fully of Lefschetz type."

It remains to compute $\mathrm{MT}(A)^{\mathrm{der}}$. Under certain simplifying hypotheses given above, we work this out in Proposition 2.120, which we restate below for convenience. Here $\mathrm{L}(A)$ is the Lefschetz group, which is intuitively what $\mathrm{MT}(A)$ would be in the absence of exceptional classes.

Proposition 2.120. Fix a geometrically simple abelian variety A over a number field K. Suppose that $F = Z(\mathrm{End}_{\overline{K}}(A)$ equals a CM field such that $\dim A = \dim F$. Letting Φ be the corresponding signature, we further suppose that $\Phi(\sigma) = 1$ for exactly two $\sigma \in \Sigma_F$. Then we show the Mumford–Tate conjecture holds for A, and

$$\mathrm{MT}(A)^{\mathrm{der}} = \mathrm{L}(A)^{\mathrm{der}}.$$

The argument proving Proposition 2.120 achieves something slightly stronger, but it is technical to state and not required for our application. In short, the idea of the proof is to upgrade the fact that the real Lie groups $\mathrm{SU}(2,0)$ and $\mathrm{SU}(1,1)$ are not isomorphic using the Galois action.

Now that we understand $\mathrm{MT}(A)$, we would like to upgrade this to an understanding of $G_\ell(A)$. After the Mumford–Tate conjecture, we (roughly speaking) need to understand the quotient $G_\ell(A)/G_\ell(A)^\circ$, whch section 2.3.4 explains that this amounts to computing the Galois action on certain "Tate classes." Thus, the trick is to not look at a particular Galois representation ρ_ℓ but instead a family of them. We can engineer everything so that generic members of the family satisfy the properties needed for the rest of the present subsection to go through. Then our last trick is ensure that some special members of the family are quotients of a Fermat curve, where we know the Galois action! In this way, we can "transport" the understanding of the Galois action afforded by the Fermat curves to a generic curve. Here is the toy result we are able to prove.

Theorem 4.39. For given $\lambda \in \mathbb{Q}(\zeta_9) \setminus \{0,1\}$, define A to be the Jacobian of the proper curve $\widetilde{C}$ with affine chart $y^9 = x(x-1)(x-\lambda)$. Suppose that A does not have complex multiplication. Then we show $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, and we compute $\mathrm{ST}(A)$.

0.2 Odds and Ends

In this section, we explain some existential properties of this article.

0.2.1 What Is in This Article

Let's take a moment to explain the layout. In chapter 1, we review all the Hodge theory we will need. Notably, in section 2.2, we explain the algorithm used to compute the neutral component of the Mumford–Tate group. This chapter ends by reviewing cohomology and discussing absolute Hodge classes.

In chapter 2, we review everything we need to know about abelian varieties. The ground of the theory is discussed in section 2.1, and then we move on to more specialized topics. For example, section 2.3 discusses the ℓ-adic representation (and the Tate module in particular), and then we explain how the Galois action on Tate classes is used to compute $G_\ell(A)$ from $G_\ell(A)^\circ$.

Lastly, in chapter 4, we apply the built theory to Fermat curves and their quotients. In particular, we explain how to compute the Galois action on Tate classes by passing to absolute Hodge classes. We then go on to compute the connected monodromy field and $G_\ell(A)$ in a few cases.

0.2.2 What Is Not in This Article

There is a large supply of topics which are not included in the first draft of this article but really should be included in a second. We list them in rough order of importance and provide some of their applications.

1. Algorithms to compute products of Γ.
   (a) We would be able to compute the connected monodromy fields and monodromy groups of the curves $y^9 = x\left(x^2+1\right)$ and $y^{11} = x\left(x^2+1\right)$. An argument with twisting would then allow us to

execute the same computations for $y^9 = x(x-1)(x-\lambda)$ and $y^{11} = x(x-1)(x-\lambda)$ for generic $\lambda \in \mathbb{Q}$. The obstruction here is that the period computations are slightly too large to be done by hand.

(b) We would also be able to compute the connected monodromy field for more general Fermat curves and hypersurfaces. It is expected that the connected monodromy field for the Fermat curves $X^p + Y^p + Z^p = 0$ is $\mathbb{Q}(\zeta_p)$. In the presence of these algorithms, one would be able to place strong upper bounds on the connected monodromy field.

2. A discussion of Tannakian formalism.

   (a) Tannakian formalism provides uniform definitions of our monodromy groups $\mathrm{MT}(A)$ and $G_\ell(A)$.

   (b) We would be able to discuss pure motives and especially abelian motives using the existing discussion of absolute Hodge classes. For example, this would allow us to include the definition of the motivic Galois group and explain its importance to (for example) the Algebraic Sato–Tate conjecture.

3. A discussion of rigid cohomology and Kedlaya's algorithm to compute Frobenius matrices.

   (a) This would allow us to computationally verify Theorem 4.33.

   (b) This would allow us to form p-adic analogues of many parts of our computation, such as Proposition 2.120.

0.2.3 Acknowledgements

The author is indebted to many people for the existence of this article. Most importantly, the author is extremely grateful to his advisor Yunqing Tang for many, many patient and enlightening conversations, for example by suggesting the key ideas that went into the main results. The author would also like to thank Sug Woo Shin and Yiannis Sakellaridis for the opportunity to speak about the Sato–Tate conjecture and helpful conversation to this end. Additionally, the author thanks Hannah Larson for help understanding Hurtwitz spaces. None of this mathematics would have been possible without their constant encouragement.

Most of the research in this article was conducted at the University of California, Berkeley, and the remainder was conducted at Johns Hopkins University. Simply put, this article would not be possible without the extremely welcoming mathematics communities at both places. In particular, the author would like to thank Jad Damaj, Sophie McCormick, and Julie Shields for diverting conversations, and the author would like to thank Sam Goldberg and Justin Wu for productive conversations.

There are also ennumerable people who have helped the author get to where he is today who were not directly involved in the creation of this article. To name just a few, the author thanks his parents Ron Elber and Virginia Yip for believing in his blooming academic carreer. And most importantly, the author is forever in debt to Hui Sun for constant support and companionship. Without her, the author would be without soul.

Chapter 1
A Little Hodge Theory

Once we explicitely know a Mumford-Tate group, we can let it work for us.

—Moonen [Moo, (5.5)]

In this chapter, we define the notion of a Hodge structure as well as some related groups (the Mumford–Tate group and the Hodge group). Our exposition follows Moonen's unpublished notes [Moo; Moo99] and Lombardo's master's thesis [Lom13, Chapter 3]. Throughout, we find motivation from geometry (and in particular the cohomology of complex varieties), but we will review cohomology only later.

1.1 Hodge Structures

Cohomology of a variety frequently comes with some extra structure. On the étale site, we will later get significant utility of the fact that étale cohomology is a Galois representaion. On the analytic site, the corresponding structure is called a "Hodge structure."

1.1.1 Definition and Basic Properties

Here is our defintion.

Definition 1.1 (Hodge structure)**.** A $\mathbb{Q}$-*Hodge structure* is a finite-dimensional vector space $V \in \mathrm{Vec}_{\mathbb{Q}}$ such that $V_{\mathbb{C}}$ admits a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V_{\mathbb{C}}^{p,q}$$

where $V_{\mathbb{C}}^{p,q} = \overline{V_{\mathbb{C}}^{q,p}}$. For fixed $m \in \mathbb{Z}$, if $V_{\mathbb{C}}^{p,q} \neq 0$ unless $p+q=m$, we say that V is *pure of weight* m. We let $\mathrm{HS}_{\mathbb{Q}}$ denote the category of $\mathbb{Q}$-Hodge structures, where a morphism of Hodge structures is a linear map preserving the decomposition over $\mathbb{C}$. In the sequel, it may be helpful to note that one can bring this definition down to $\mathbb{Z}$ as well.

Example 1.2. We give the "Tate twist" $\mathbb{Q}(1) \coloneqq 2\pi i\mathbb{Q}$ a Hodge structure of weight -2 where the only nonzero entry is $\mathbb{Q}(1)^{-1,-1} = \mathbb{Q}(1)$.

Example 1.3. Given a complex projective smooth variety X, the Betti cohomology $\mathrm{H}^n_{\mathrm{B}}(X,\mathbb{Q})$ admits a Hodge structure via the comparison isomorphisms: we find that

$$\mathrm{H}^n_{\mathrm{B}}(X,\mathbb{C}) \simeq \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X),$$

where $\mathrm{H}^{p,q}(X) \coloneqq \mathrm{H}^q(X,\Omega^p_{X/\mathbb{C}})$. This construction is even functorial: a morphism of complex projective smooth varieties $\varphi\colon X \to Y$ induces a morphism of Hodge structures $\varphi^*\colon \mathrm{H}^n_{\mathrm{B}}(Y,\mathbb{Q}) \to \mathrm{H}^n_{\mathrm{B}}(X,\mathbb{Q})$.

Perhaps one would like to check that the category $\mathrm{HS}_{\mathbb{Q}}$ is abelian. The quickest way to do this is to realize $\mathrm{HS}_{\mathbb{Q}}$ as a category of representations of some group. The relevant group is the Deligne torus.

Notation 1.4 (Deligne torus)**.** Let $\mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}}$ denote the Deligne torus. We also let $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ denote the *weight cocharacter* given by $w(r) \coloneqq r \in \mathbb{C}$ on $\mathbb{R}$-points.

Remark 1.5. One can realize $\mathbb{S}$ more concretely as

$$\mathbb{S}(R) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathrm{GL}_2(R) : a^2+b^2 \in R^\times \right\},$$

where R is an $\mathbb{R}$-algebra. Indeed, there is a ring isomorphism from $R \otimes_{\mathbb{R}} \mathbb{C}$ to $\left\{ \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] : a,b \in R \right\}$ by sending $1\otimes 1 \mapsto \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right]$ and $1 \otimes i \mapsto \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$. For example, one can define two characters $z,\overline{z}\colon \mathbb{S}_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ given by $z\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a+bi$ and $\overline{z}\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a-bi$ so that $(z,\overline{z})$ is an isomorphism $\mathbb{S}_{\mathbb{C}} \to \mathbb{G}^2_{m,\mathbb{C}}$. Thus, the character group $X^*(\mathbb{S})$ is a free $\mathbb{Z}$-module of rank 2 with basis $\{z,\overline{z}\}$, and the action of complex conjugation $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ simply swaps z and $\overline{z}$.

Example 1.6. The following cocharacters of $\mathbb{S}$ will be helpful.

- We define the *weight cocharacter* $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ given by $w(r) \coloneqq r \in \mathbb{C}$ on $\mathbb{R}$-points.
- We define the *miniscule cocharacter* $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ given by $\mu(z) \coloneqq (z,1)$ on $\mathbb{C}$-points.

Here is the relevance of $\mathbb{S}$ to Hodge structures.

Lemma 1.7. Fix some $V \in \mathrm{Vec}_{\mathbb{Q}}$. Then a Hodge structure on V has equivalent data to a representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$.

Proof. Remark 1.5 informs us that the character group $X^*(\mathbb{S})$ of group homomorphisms $\mathbb{S} \to \mathbb{G}_m$ is a rank-2 free $\mathbb{Z}$-module generated by $z\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a+bi$ and $\overline{z}\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a-bi$ on $\mathbb{C}$-points.[1] Without too many details, upon passing to the Hopf algebra, one is essentially looking for units in $\mathbb{R}\left[a,b,\left(a^2+b^2\right)^{-1}\right]$, of which there are not many. Note that there is a Galois action by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on these two characters $\{z,\overline{z}\}$, given by swapping them. Let $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denote complex conjugation, for brevity.

Now, a representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ must have $V_{\mathbb{C}}$ decompose into eigenspaces according to the characters $X^*(\mathbb{S})$, so one admits a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\chi \in X^*(\mathbb{S})} V^{\chi}_{\mathbb{C}}.$$

However, one also needs $V^{\iota\chi}_{\mathbb{C}} = \overline{V^{\chi}_{\mathbb{C}}}$ because ι swaps $\{\chi,\iota\chi\}$. By Galois descent, this is enough data to (conversely) define a representation $h\colon \mathbb{S} \to \mathrm{Gal}(V)_{\mathbb{R}}$.

[1] Alternatively, note one has an isomorphism $(\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C})^\times \cong \mathbb{C}^\times \times \mathbb{C}^\times$ by sending $(z,w) \mapsto z\otimes w$. Then these two characters are $(z,w)\mapsto z$ and $(z,w)\mapsto w$.

To relate the previous paragraph to Hodge structures, we recall that $X^*(\mathbb{S})$ is a rank-2 free $\mathbb{Z}$-module, so write $\chi_{p,q} \coloneqq z^{-p}\overline{z}^{-q}$ so that $\iota\chi_{p,q} = \chi_{q,p}$. Setting $V_{\mathbb{C}}^{p,q} \coloneqq V_{\mathbb{C}}^{\chi_{p,q}}$ now explains how to relate the previous paragraph to a Hodge structure, as desired. ■

Remark 1.8. The weight of a Hodge structure on some $V \in \mathrm{HS}_{\mathbb{Q}}$ can be read off of h as follows: note the weight cocharacter $h \circ w$ equals the $(-m)$th power map if and only if the weight is m.

Thus, we see immediately the category $\mathrm{HS}_{\mathbb{Q}}$ is abelian. Additionally, representation theory explains how to take tensor products and duals.

Example 1.9. We see that $V \in \mathrm{HS}_{\mathbb{Q}}$ has $V^\vee$ inherit a Hodge structure by setting $(V^\vee)^{p,q} \coloneqq (V^{-p,-q})^\vee$.

Example 1.10. We are now able to define the Tate twists $\mathbb{Q}(n) \coloneqq \mathbb{Q}(1)^{\otimes n}$, where negative powers indicates taking a dual. In particular, one can check that $\mathbb{Q}(n) \otimes \mathbb{Q}(m) = \mathbb{Q}(n+m)$ for any $n, m \in \mathbb{Z}$.

Notation 1.11. For any Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ and integer $m \in \mathbb{Z}$, we may write

$$V(m) \coloneqq V \otimes \mathbb{Q}(m).$$

We conclude this section by explaining one important application of Hodge structures.

Definition 1.12 (Hodge class)**.** Fix a $\mathbb{Q}$-Hodge structure V. A *Hodge class* of V is an element of $V \cap V^{0,0}$.

Remark 1.13. Looking at the construction in the proof of Lemma 1.7, we see that $v \in V$ is a Hodge class if and only if it is fixed by the corresponding representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$.

Example 1.14. Fix a complex projective smooth variety X of dimension n and some even nonnegative integer $2p \geq 0$. Then one has Hodge classes given by elements of

$$\mathrm{H}_{\mathrm{B}}^{2p}(X, \mathbb{Q}) \cap \mathrm{H}^{p,p}(X)(p).$$

Now, any algebraic subvariety $Z \subseteq X$ of codimension k defines a linear functional on $\mathrm{H}_{\mathrm{dR}}^{2n-2k}(X, \mathbb{C})$ defined by

$$\omega \mapsto \int_Z \omega,$$

which one can check is supported on $\mathrm{H}^{k,k}$. Thus, by Poincaré duality, one finds that Z produces a Hodge cycle in $\mathrm{H}_{\mathrm{B}}^{2k}(X, \mathbb{Q})$.

In light of the above example, one has the following conjecture.

Conjecture 1.15 (Hodge)**.** Fix a complex projective smooth variety X. Then any Hodge class can be written as a linear combination of classes arising from algebraic subvarieties.

Remark 1.16. Here are some remarks on what is known about the Hodge conjecture, though it is admittedly little in this level of generality.

- The Hodge classes in $\mathrm{H}^2_{\mathrm{B}}(X)(1)$ come from algebraic subvarieties.
- The cup product of any two classes arising from algebraic subvarieties continues to be Hodge and arises from algebraic subvarieties.

For example, if one can show that all Hodge classes are cup products of Hodge classes of codimension 1 on a variety X, then one knows the Hodge conjecture for X.

We are not interested in proving (cases of) the Hodge conjecture in this thesis, so we will not say much more.

1.1.2 Polarizations

Here is an important example of a morphism of Hodge structures.

Definition 1.17 (polarization)**.** Fix a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m given by the representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. A *polarization* on V is a morphism $\varphi\colon V \otimes V \to \mathbb{Q}(-m)$ of Hodge structures such that the induced bilinear form on $V_{\mathbb{R}}$ given by

$$\langle v, w\rangle \coloneqq (2\pi i)^m \varphi(h(i)v \otimes w)$$

is symmetric and positive-definite. If V admits a polarization, we may say that V is *polarizable*, and we let $\mathrm{HS}^{\mathrm{pol}}_{\mathbb{Q}} \subseteq \mathrm{HS}_{\mathbb{Q}}$ be the full subcategory of polarizable $\mathbb{Q}$-Hodge structures.

Remark 1.18. The positive-definiteness condition on $\langle\cdot,\cdot\rangle$ implies that φ is non-degenerate. Indeed, one may check non-degeneracy upon base-changing to $\mathbb{R}$ (because this is equivalent to inducing an isomorphism of vector spaces $V \to V^\vee$, which can be checked by fixing some $\mathbb{Q}$-bases and computing a determinant). Then we see that $\langle\cdot,\cdot\rangle$ being non-degenerate implies that

$$\varphi(v \otimes w) = (2\pi i)^{-m}\langle h(-i)v, w\rangle$$

is non-degenerate because $h(-i)\colon V \to V$ is an isomorphism of vector spaces (because $h(-i)^4 = \mathrm{id}_V$).

Remark 1.19. The symmetry condition on $\langle\cdot,\cdot\rangle$ implies a symmetry or alternating condition on φ. Indeed, we compute

$$\begin{aligned}
\varphi(v \otimes w) &= (2\pi i)^{-m}\langle h(-i)v, w\rangle \\
&= (2\pi i)^{-m}\langle w, h(-i)v\rangle \\
&= \varphi(h(i)w \otimes h(-i)v) \\
&= h_{\mathbb{Q}(-m)}(i)\varphi(w \otimes h(-1)v) \\
&= 1\varphi\left(w \otimes (-1)^m w\right) \\
&= (-1)^m \varphi(w \otimes v).
\end{aligned}$$

Thus, φ is symmetric when m is even, and φ is alternating when m is odd.

Let's give some constructions of polarizable Hodge structures.

Example 1.20. It will turn out that $\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ of any abelian variety A (over $\mathbb{C}$) is polarizable, explaining the importance of this notion for our application. Because we are reviewing abelian varieties in chapter 2, we will not say more here.

Example 1.21. If V is polarizable and pure of weight m, then any Hodge substructure $W \subseteq V$ is still polarizable (and pure of weight m). Indeed, one can simply restrict the polarization to W, and all the checks go through. For example, positive-definiteness of $\langle \cdot, \cdot \rangle$ means $\langle v, v \rangle > 0$ for all nonzero $v \in V$, so the same will be true upon restricting to W.

Example 1.22. If V and W are polarizable and pure of weight m, then $V \oplus W$ is also polarizable. Indeed, letting φ and ψ be polarizations on V and W respectively, we see that $(\varphi \oplus \psi)$ defined by

$$(\varphi \oplus \psi)((v, w), (v', w')) \coloneqq \varphi(v, v') + \psi(w, w')$$

succeeds at being a polarization: certainly it is a morphism of Hodge structures to $\mathbb{Q}(-m-n)$, and one can check that the corresponding bilinear form on $V \oplus W$ simply splits into a sum of the forms on V and W and is therefore symmetric and positive-definite.

Example 1.23. If V and W are polarizable and pure of weights m and n respectively, then $V \otimes W$ is also polarizable. Indeed, as in Example 1.22, let φ and ψ be polarizationson V and W respectively, and then we find that $(\varphi \otimes \psi)$ can be defined on pure tensors by

$$(\varphi \otimes \psi)(v \otimes w, v' \otimes w') \coloneqq \varphi(v, v')\psi(w, w').$$

One checks as before that this gives a polarization on $V \otimes W$: we certainly have a morphism of Hodge structures, and the corresponding bilinear form is the product of the bilinear forms on V and W and is therefore symmetric and positive-definite.

Example 1.24. If V is polarizable and pure of weight m with polarization φ, and $W \subseteq V$ is a Hodge substructure (which is polarizable by Example 1.21), then we claim $W^\perp$ (taken with respect to $\langle \cdot, \cdot \rangle$) is also a Hodge substructure and hence polarizable by Example 1.21. Well, for any $w' \in W_{\mathbb{R}}^\perp$ and $z \in \mathbb{S}(\mathbb{R})$, we must check that $h(z)w' \in W_{\mathbb{R}}^\perp$. For this, we note that any $w \in W$ has

$$\begin{aligned} \langle w, h(z)w' \rangle &= (2\pi i)^{-m} \varphi(h(i)w \otimes h(z)w') \\ &= h_{\mathbb{Q}(-m)}(1/z)(2\pi i)^{-m} \varphi(h(i/z)w \otimes w') \\ &= h_{\mathbb{Q}(-m)}(1/z)\langle h(i/z)w, w' \rangle \\ &= 0, \end{aligned}$$

where the last equality holds because $W \subseteq V$ is a Hodge substructure.

Note that one does not expect any Hodge substructure to have a complement, so Example 1.24 is a very important property of polarizations.

1.1.3 The Albert Classification

The presence of a polarization places strong restrictions on the endomorphisms of a Hodge structure. To explain how this works, we begin by reducing to the irreducible case: given a polarizable Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$, we begin by noting that V can be decomposed into irreducible Hodge substructures

$$V = \bigoplus_{i=1}^{N} V_i^{\oplus m_i},$$

where V_i is an irreducible Hodge structure (i.e., an irreducible representation of $\mathbb{S}$) and $m_i \geq 0$ is some nonnegative integer. Then standard results on endomorphisms of representations tell us that

$$\mathrm{End}_{\mathrm{HS}}(V) = \bigoplus_{i=1}^{N} M_{m_i}(\mathrm{End}_{\mathrm{HS}}(V_i)),$$

and Schur's lemma implies that $\mathrm{End}_{\mathrm{HS}}(V_i)$ is a division algebra. The point of the above discussion is that we may reduce our discussion of endomorphisms to irreducible Hodge structures. We remark that polarizability of V implies that irreducible Hodge substructures continue to be polarizable by Example 1.21.

We are thus interested in classifying what algebras may appear as $\mathrm{End}_{\mathrm{HS}}(V)$ for irreducible Hodge structures $V \in \mathrm{HS}_{\mathbb{Q}}$. To this end, we note that $\mathrm{End}_{\mathrm{HS}}(V)$ comes with some extra structure.

Definition 1.25 (Rosati involution)**.** Let φ be a polarization on a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$. The *Rosati involution* is the function $(\cdot)^\dagger \colon \mathrm{End}_{\mathbb{Q}}(V) \to \mathrm{End}_{\mathbb{Q}}(V)$ defined by

$$\varphi(dv \otimes w) = \varphi(v \otimes d^\dagger w)$$

for all $d \in \mathrm{End}_{\mathrm{HS}}(V)$ and $v, w \in V$.

Remark 1.26. In light of Remark 1.18, we see that $d^\dagger$ is simply the adjoint of $d \colon V \to V$ associated to φ viewed as a non-degenerate bilinear pairing. For example, we immediately see that $(\cdot)^\dagger$ induces a well-defined linear operator $\mathrm{End}_{\mathbb{Q}}(V) \to \mathrm{End}_{\mathbb{Q}}(V)$.

Here are the important properties of the Rosati involution.

Lemma 1.27. Fix a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with polarization φ and associated Rosati involution $(\cdot)^\dagger$.

(a) If $d \in \mathrm{End}_{\mathrm{HS}}(V)$, then $d^\dagger \in \mathrm{End}_{\mathrm{HS}}(V)$.

(b) Anti-involution: for any $d, e \in \mathrm{End}_{\mathbb{Q}}(V)$, we have $d^{\dagger\dagger} = d$ and $(de)^\dagger = e^\dagger d^\dagger$.

(c) Positive: for any nonzero $d \in \mathrm{End}_{\mathbb{Q}}(V)$, we have $\operatorname{tr} dd^\dagger > 0$.

Proof. We show the claims in sequence.

(a) This follows because φ is a morphism of Hodge structures. Formally, we would like to check that $d^\dagger$ commutes with the action of $\mathbb{S}$. Let $h \colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be the representation corresponding to the Hodge structure. Well, for any $g \in \mathbb{S}(\mathbb{C})$ and $v, w \in V$, we compute

$$\begin{aligned}
\varphi(v \otimes d^\dagger h(g)w) &= \varphi(dv \otimes h(g)w) \\
&= h_{\mathbb{Q}(-m)}(g)\varphi\left(h(g^{-1})dv \otimes w\right) \\
&\overset{*}{=} h_{\mathbb{Q}(-m)}(g)\varphi\left(dh(g^{-1})v \otimes w\right) \\
&= h_{\mathbb{Q}(-m)}(g)\varphi\left(h(g^{-1})v \otimes d^\dagger w\right) \\
&= \varphi(v \otimes h(g)d^\dagger w)
\end{aligned}$$

where $\overset{*}{=}$ holds because d is a morphism of Hodge structures. The non-degeneracy of φ given in Remark 1.18 now implies that $d^\dagger h(g) = h(g)d^\dagger$, so we are done.

(b) This is a purely formal property of adjoints.

(c) The point is to reduce this to the case where V is a matrix algebra over $\mathbb{R}$ and $(\cdot)^\dagger$ is the transpose. Indeed, this positivity can be checked after a base-change to $\mathbb{R}$. As such, we let $\langle\cdot,\cdot\rangle$ be the symmetric positive-definite bilinear form assocated to φ defined by

$$\langle v, w\rangle := (2\pi i)^{-m}\varphi(h(i)v \otimes w)$$

for any $v, w \in V_{\mathbb{R}}$. We thus see that $(\cdot)^\dagger$ is also the adjoint operator with respect to $\langle\cdot,\cdot\rangle$: we know

$$(2\pi i)^{-m}\langle h(i)dv, w\rangle = (2\pi i)^{-m}\langle h(i)v, d^\dagger w\rangle$$

for any $v, w \in V_{\mathbb{R}}$, which is equivalent to always having $\langle dv, w\rangle = \langle v, d^\dagger w\rangle$. Now, we may fix an orthonomal basis of $V_{\mathbb{R}}$ with respect to $\langle\cdot,\cdot\rangle$ so that $\mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ is identified with $M_n(\mathbb{R}^{\dim V})$ and $(\cdot)^\dagger$ is identified with the transpose. Then $\operatorname{tr} dd^{\intercal}$ is the sum of the squares of the matrix entries of d and is therefore positive when d is nonzero. ■

We are now ready to state the Albert classification, which classifies division algebras over $\mathbb{Q}$ equipped with a positive anti-involution.

Theorem 1.28 (Albert classification)**.** Let D be a division algebra over $\mathbb{Q}$ equipped with a Rosati involution $(\cdot)^\dagger\colon D \to D$. Further, let F be the center of D, and let $F^\dagger$ be the subfield fixed by $(\cdot)^\dagger$. Then D admits exactly one of the following types.

- Type I: D is a totally real number field so that $D = F = F^\dagger$, and $(\cdot)^\dagger$ is the identity.
- Type II: D is a totally indefinite quaternion division algebra over F where $F = F^\dagger$, and $(\cdot)^\dagger$ corresponds to the transpose on $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$.
- Type III: D is a totally definite quaternion division algebra over F where $F = F^\dagger$, and $(\cdot)^\dagger$ corresponds to the canonical involution on $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$ (where $\mathbb{H}$ is the quaternions).
- Type IV: D is a division algebra over F, where F is a totally imaginary quadratic extension of $F^\dagger$, and $(\cdot)^\dagger$ is the complex conjugation automorphism of F. In other words, F is a CM field, and $F^\dagger$ is the maximal totally real subfield.

Proof. This is a rather lengthy computaion. We refer to [Mum74, Section 21, Application I]. ■

1.2 Monodromy Groups

In this section, we define the Mumford–Tate group and the Hodge group.

1.2.1 The Mumford–Tate Group

We are now ready to define the Mumford–Tate group. Intuitively, it is the monodromy group of the associated representation of a Hodge structure.

Definition 1.29 (Mumford–Tate group)**.** For some $V \in \mathrm{HS}_{\mathbb{Q}}$, the *Mumford–Tate group* $\mathrm{MT}(V)$ is the smallest algebraic $\mathbb{Q}$-group containing the image of the corresponding representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$.

Remark 1.30. Because $\mathbb{S}$ is connected, we see that h is also connected. Namely, $\mathrm{MT}(V)^\circ \subseteq \mathrm{MT}(V)$ will be an algebraic $\mathbb{Q}$-group containing the image of h if $\mathrm{MT}(V)$ does too, so equality is forced.

Example 1.31. Suppose that $V \in \mathrm{HS}_{\mathbb{Q}}$ is pure of weight m.

- If $m = 0$, then we claim that $\mathrm{MT}(V) \subseteq \mathrm{SL}(V)$. It is enough to check that h outputs into $\mathrm{SL}(V)$.
- If $m \neq 0$, then we claim that $\mathrm{MT}(V)$ contains $\mathbb{G}_{m,\mathbb{Q}}$. It is enough to check that $\mathrm{MT}(V)_{\mathbb{C}}$ contains $\mathbb{G}_{m,\mathbb{C}}$. Well, for any $z \in \mathbb{C}$ $h(z, \overline{z})$ acts on the component $V^{p,q} \subseteq V_{\mathbb{C}}$ by $z^{-p}z^{-q} = z^{-m}$, so $\mathrm{MT}(V)_{\mathbb{C}}$ must contain the scalar z^{-m} for all $z \in \mathbb{C}$. The conclusion follows.

Because Hodge structures are defined after passing to $\mathbb{C}$, it will be helpful to have a definition of $\mathrm{MT}(V)$ as a monodromy group corresponding to a morphism over $\mathbb{C}$.

Lemma 1.32. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$, and let $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be the corresponding representation. Then $\mathrm{MT}(V)$ is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ such that $\mathrm{MT}(V)_{\mathbb{C}}$ contains the image of $h_{\mathbb{C}} \circ \mu$.

Proof. Let M' be the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ containing $h_{\mathbb{C}} \circ \mu$. We want to show that $M' = M$.

- To show $M' \subseteq \mathrm{MT}(V)$, we must show that $\mathrm{MT}(V)_{\mathbb{C}}$ contains the image of $h_{\mathbb{C}} \circ \mu$. Well, $\mathrm{MT}(V)_{\mathbb{R}}$ contains the image of h, so $\mathrm{MT}(V)_{\mathbb{C}}$ contains the image of $h_{\mathbb{C}}$, which contains the image of $h_{\mathbb{C}} \circ \mu$.
- Showing $\mathrm{MT}(V) \subseteq M'$ is a little harder. We must show that M' contains the image of $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. It is enough to check that M' contains the image of $h_{\mathbb{C}}$ because then we can descend everything to $\mathbb{R}$, and because $\mathbb{C}$ is algebraically closed, we see that $\mathbb{C}$-points are certainly dense enough so that it is enough to chek that $M'(\mathbb{C})$ contains the image $h(\mathbb{S}(\mathbb{C}))$.

  The point is that M' is defined over $\mathbb{Q}$, so $M'_{\mathbb{C}}$ is stable under the action of complex conjugation, which we denote by ι. Similarly, h being defined over $\mathbb{R}$ guarantees that it commutes with complex conjugation. In particular, we already know that M' contains the points of the form $h(z, 1)$ for $(z, 1) \in \mathbb{S}(\mathbb{C})$. Thus, we see that M' also contains the points

  $$\iota(h(z, 1)) = h(\iota(z, 1)) = h(1, z)$$

  because everything is defined over $\mathbb{R}$. (This last equality follows by tracking through the action of ι on $\mathbb{S}(\mathbb{C})$.) We conclude that M' contains $h(z, w)$ for any $(z, w) \in \mathbb{S}(\mathbb{C})$, so we are done. ∎

Roughly speaking, the point of the group $\mathrm{MT}(V)$ is that $\mathrm{MT}(V)$ is an algebraic $\mathbb{Q}$-group remembering everything one wants to know about the Hodge structure. One way to rigorize this is as follows.

Proposition 1.33. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$. Suppose $T \in \mathrm{HS}_{\mathbb{Q}}$ can be written as

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^{\vee})^{\otimes n_i} \right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $W \subseteq T$ is a Hodge substructure if and only if the action of $\mathrm{MT}(V)$ on T stabilizes W.

Proof. For each $W \in \mathrm{HS}_{\mathbb{Q}}$, we let h_W denote the corresponding representation. In the backwards direction, we note that $\mathrm{MT}(V)$ stabilizing W implies that $h(s)$ stabilizes $W_{\mathbb{R}}$ for any s. We can thus view $W_{\mathbb{R}} \subseteq T_{\mathbb{R}}$ as a subrepresentation of $\mathbb{S}$, so taking eigenspaces reveals that W can be given the structure of a Hodge substructure of T.

The converse will have to use the construction of T. Indeed, suppose that $W \subseteq T$ is a Hodge substructure, and let $M \subseteq \mathrm{GL}(V)$ be the smallest algebraic $\mathbb{Q}$-group stabilizing $W \subseteq T$. We would like to show that $\mathrm{MT}(V) \subseteq M$. By definition of $\mathrm{MT}(V)$, it is enough to show that h factors through $M_{\mathbb{R}}$, meaning we must show that $h(s)$ stabilizes W for each $s \in \mathbb{S}$. Well, $h(s)$ will act by characters on the eigenspaces $W^{p,q}_{\mathbb{C}} \subseteq W_{\mathbb{C}}$, so $h(s)$ does indeed stabilize W. ∎

Corollary 1.34. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$. Suppose $T \in \mathrm{HS}_{\mathbb{Q}}$ can be written as

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^\vee)^{\otimes n_i} \right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $t \in T$ is a Hodge class if and only if it is fixed by $\mathrm{MT}(V)$.

Proof. We apply Proposition 1.33 to $\mathbb{Q}(0) \oplus T$. Then we note that $\mathrm{span}_{\mathbb{Q}}\{(1,t)\} \subseteq \mathbb{Q}(0) \oplus T$ is a Hodge substructure if and only if it is preserved by $\mathrm{MT}(V)$. We now tie each of these to the statement.

- On one hand, we see that being a one-dimensional Hodge substructure implies that $(1,t)$ must have bidegree (p,p) for some $p \in \mathbb{Z}$, but we have to live in $(0,0)$ because our 1 lives in $\mathbb{Q}(0)$. Thus, this is equivalent to being a Hodge class.
- On the other hand, being preserved by $\mathrm{MT}(V)$ implies that $\mathrm{MT}(V)$ acts by scalars on $(1,t)$, but $\mathrm{MT}(V)$ acts trivially on $\mathbb{Q}(0)$, so all the relevant scalars must be 1. Thus, this is equivalent to being fixed by $\mathrm{MT}(V)$. ∎

We thus see that understanding the Mumford–Tate group is important from the perspective of the Hodge conjecture (Conjecture 1.15). It will be helpful to note that this characterizes $\mathrm{MT}(V)$ in some cases.

Proposition 1.35. Fix a field k of characteristic 0. Let $H \subseteq \mathrm{GL}_{n,k}$ be a reductive subgroup. Suppose H' is the algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}_{n,k}$ defined by fixing all H-invariants occuring in any tensor representation

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^\vee)^{\otimes n_i} \right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $H = H'$.

Proof. Note $H \subseteq H'$ is automatic, so the main content comes from proving the other inclusion. Proving this would step into the (rather deep) theory of algebraic groups, which we will avoid. Instead, we will mention that the key input is Chevalley's theorem, which asserts that any subgroup H of G is the stabilizer of some line in some representation of G. We refer to [Del18, Proposition 3.1]; see also [Mil17, Theorem 4.27]. ∎

Corollary 1.36. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ such that $\mathrm{MT}(V)$ is reductive. Then $\mathrm{MT}(V)$ is exactly the algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ fixing all Hodge classes.

Proof. Corollary 1.34 explains that the Hodge classes are exactly the vectors fixed by $\mathrm{MT}(V)$, so this follows from Proposition 1.35. ∎

Remark 1.37. Corollary 1.36 is true without a reductivity assumption (see [Del18, Proposition 3.4]), but we will not need this in our applications. (On the other hand, one does not expect Proposition 1.35 to be true without any assumptions on H.) Namely, we will be interested in abelian varieties, whose Hodge structures are polarizable by Example 1.20, and we will shortly see that this implies that $\mathrm{MT}(V)$ is reductive in Lemma 1.44.

1.2.2 The Hodge Group

In computational applications, it will be frequently be easier to compute a smaller monodromy group related to $\mathrm{MT}(V)$.

Definition 1.38 (Hodge group)**.** Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight. Then the *Hodge group* $\mathrm{Hg}(V)$ is the smallest algebraic $\mathbb{Q}$-subgroup $\mathrm{GL}(V)$ containing the image of $h|_{\mathbb{U}}$, where $\mathbb{U} \subseteq \mathbb{S}$ is defined as the kernel of the norm character $z\overline{z}\colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$.

Remark 1.39. Even though z and $\overline{z}$ are only defined as characters $\mathbb{S}_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$, the norm character $z\overline{z}$ is defined as a character $\mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ because it is fixed by complex conjugation. For example, we see that

$$\mathbb{U}(\mathbb{R}) = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus, we see that $\mathbb{U}$ stands for "unit circle." While we're here, we remark that $\mathbb{U}(\mathbb{C}) \subseteq \mathbb{S}(\mathbb{C})$ is identified with the subset $\{(z, 1/z) : z \in \mathbb{C}^\times\}$.

Remark 1.40. The same argument as in Remark 1.30 shows that the connectivity of $\mathbb{U}$ implies the connectivity of $\mathrm{Hg}(V)$.

Intuitively, $\mathrm{Hg}(V)$ removes the scalars that might live in $\mathrm{MT}(V)$ by Example 1.31. These scalars are an obstruction to $\mathrm{MT}(V)$ being a semisimple group, and we will see in Proposition 2.67 that $\mathrm{Hg}(V)$ will thus frequently succeed at being semisimple. Let's rigorize this discusison.

Lemma 1.41. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m, and let $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be the corresponding representation.

(a) We have $\mathrm{Hg}(V) \subseteq \mathrm{SL}(V)$.

(b) Thus,

$$\mathrm{MT}(V) = \begin{cases} \mathrm{Hg}(V) & \text{if } m = 0, \\ \mathbb{G}_{m,\mathbb{Q}} \mathrm{Hg}(V) & \text{if } m \neq 0, \end{cases}$$

where the almost direct product in the second case is given by embedding $\mathbb{G}_{m,\mathbb{Q}} \to \mathrm{GL}(V)$ via scalars.

Proof. We show the claims in sequence.

(a) It is enough to check that $\mathrm{SL}(V)$ contains the image of $h|_{\mathbb{U}}$. In other words, we want to check that $\det h(z) = 1$ for all $z \in \mathbb{U}(\mathbb{R})$. By extending scalars, it is enough to compute the determinant as an operator on $V_{\mathbb{C}}$. For this, we note that $h(z)$ acts on the component $V^{p,q} \subseteq V_{\mathbb{C}}$ by the scalar $z^{-p}\overline{z}^{-q}$, so the determinant of $h(z)$ acting on $V^{p,q} \oplus V^{q,p}$ is

$$\left(z^{-p}\overline{z}^{-q}\right)^{\dim V^{p,q}} \cdot \left(z^{-q}\overline{z}^{-p}\right)^{\dim V^{q,p}} = (z\overline{z})^{-(p+q)\dim V^{p,q}}$$

because $\dim V^{p,q} = \dim V^{q,p}$. This simplifies to $(z\overline{z})^{-\frac{1}{2}m \dim(V^{p,q} \oplus V^{q,p})}$ because V is pure of weight m, so the result follows by summing over all pairs (p, q).[2]

(b) Before doing anything serious, we remark that $\mathbb{G}_{m,\mathbb{Q}} \mathrm{Hg}(V)$ is in fact an almost direct product. Namely, we should check that the intersection $\mathbb{G}_{m,\mathbb{Q}} \cap \mathrm{Hg}(V)$ is finite (even over $\mathbb{C}$). Well, by (a), $\mathrm{Hg}(V) \subseteq \mathrm{SL}(V)$. Thus, it is enough to notice that $\mathbb{G}_{m,\mathbb{Q}} \cap \mathrm{SL}(V)$ is finite because V is finite-dimensional over $\mathbb{C}$: over $\mathbb{C}$,

[2] If m is even, this argument does not work verbatim for the component $(m/2, m/2)$. Instead, one can compute the determinant of $h(z)$ acting on $V^{m/2,m/2}$ directly as $(z\overline{z})^{-\frac{1}{2}m \dim V^{m/2,m/2}}$.

the intersection consisits of scalar matrices $\lambda \operatorname{id}_V$ such that $\lambda^{\dim V} = 1$, so the intersefction is the finite algebraic group $\mu_{\dim V}$.

We now proceed with the argument. Because $\mathbb{U} \subseteq \mathbb{S}$, we of course have $\operatorname{Hg}(V) \subseteq \operatorname{MT}(V)$, and if $m \neq 0$, then Example 1.31 implies that $\mathbb{G}_{m,\mathbb{Q}} \subseteq \operatorname{MT}(V)$ so that $\mathbb{G}_{m,\mathbb{Q}} \operatorname{Hg}(V) \subseteq \operatorname{MT}(V)$. It is therefore enough to check the given equalities after base-changing to $\mathbb{R}$. Namely, using Lemma 1.32, we should check that $\operatorname{Hg}(V)(\mathbb{C})$ contains the image of $h_{\mathbb{C}} \circ \mu$ when $m = 0$, and $\mathbb{C}^\times \operatorname{Hg}(V)(\mathbb{C})$ contains the image of $h_{\mathbb{C}} \circ \mu$ when $m \neq 0$. Well, for any $z \in \mathbb{C}^\times$, we may write $z = re^{i\theta}$ where $r \in \mathbb{R}^+$ and $\theta \in \mathbb{R}$. Then we compute

$$\begin{aligned} h(\mu(z)) &= h(z, 1) \\ &= h\left(re^{i\theta}, 1\right) \\ &= h\left(\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{-i\theta/2}\right) h\left(\sqrt{r}e^{i\theta/2}, \frac{1}{\sqrt{r}e^{i\theta/2}}\right). \end{aligned}$$

Now, $h\left(\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{-i\theta/2}\right)$ is a scalar as computed in Example 1.31, and $\left(\sqrt{r}e^{i\theta/2}, \frac{1}{\sqrt{r}e^{i\theta/2}}\right)$ lives in $\mathbb{U}(\mathbb{C}) = \{(z, w) : zw = 1\}$. Thus, we see that $h(\mu(z))$ is certainly contained in $\mathbb{C}^\times \operatorname{Hg}(V)(\mathbb{C})$, completing the proof in the case $m \neq 0$. In the case where $m = 0$, the scalar $h\left(\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{-i\theta/2}\right)$ is actually the identity, so we see that $h(\mu(z)) \in \operatorname{Hg}(V)(\mathbb{C})$. ■

It is worthwhile to note that there is also a tensor characterization of $\operatorname{Hg}(V)$.

Proposition 1.42. Fix $V \in \operatorname{HS}_{\mathbb{Q}}$ of pure weight. Suppose $T \in \operatorname{HS}_{\mathbb{Q}}$ is of pure weight n and can be written as

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^\vee)^{\otimes n_i}\right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $W \subseteq T$ is a Hodge substructure if and only if the action of $\operatorname{Hg}(V)$ on T stabilizes W.

Proof. Of course, if $W \subseteq T$ is a Hodge substructure, then W is preserved by the action of $\operatorname{MT}(V)$, so W will be preserved by the action of $\operatorname{Hg}(V) \subseteq \operatorname{MT}(V)$.

Conversely, if $\operatorname{Hg}(V)$ stabilizes W, then we would like to show that $W \subseteq T$ is a Hodge substructure, which by Proposition 1.33 is the same as showing that $\operatorname{MT}(V)$ stabilizes W. For this, we use Lemma 1.41, which tells us that $\operatorname{MT}(V) \subseteq \mathbb{G}_{m,\mathbb{Q}} \operatorname{Hg}(V)$. Namely, because $\operatorname{Hg}(V)$ already stabilizes W, it is enough to note that of course the scalars $\mathbb{G}_{m,\mathbb{Q}}$ stabilize the subspace $W \subseteq T$. ■

Corollary 1.43. Fix an irreducible Hodge structure $V \in \operatorname{HS}_{\mathbb{Q}}$ of pure weight. Observe that the inclusion $\operatorname{Hg}(V) \subseteq \operatorname{GL}(V)$ makes V into a representation of $\operatorname{Hg}(V)$. Then V is irreducible as a representation of $\operatorname{Hg}(V)$.

Proof. By Proposition 1.42, a $\operatorname{Hg}(V)$-submodule is a Hodge substructure, but there are no nonzero proper Hodge substructures because V is an irreducible Hodge structure. ■

1.3 Computational Tools

In this section, we provide some discussion which will help the computations used later in this thesis.

1.3.1 Bounding with Known Classes

Here, we use endomorphisms and the polarization to bound the size of $\operatorname{MT}(V)$ and $\operatorname{Hg}(V)$.

Lemma 1.44. Fix a polarizable Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight. Then $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$ are reductive.

Proof. By [Mil17, Corollary 19.18], it is enough to find faithful semisimple representations of $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$. We claim that the inclusions $\mathrm{MT}(V) \subseteq \mathrm{GL}(V)$ and $\mathrm{Hg}(V) \subseteq \mathrm{GL}(V)$ provide this representation: certainly this representation is faithful, and it is faithful because any subrepresentation is a Hodge substructure by Propositions 1.33 and 1.42. ■

Lemma 1.45. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$. Let $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ be the endomorphism algebra of V. Then $\mathrm{MT}(V)$ is an algebraic $\mathbb{Q}$-subgroup of

$$\mathrm{GL}_D(V) \coloneqq \{g \in \mathrm{GL}(V) : g \circ d = d \circ g \text{ for all } d \in D\}.$$

Proof 1. Noting that $\mathrm{GL}_D(V)$ is an algebraic $\mathbb{Q}$-group (it is a subgroup of $\mathrm{GL}(V)$ cut out by the equations given by commuting with a basis of D), it is enough to show that $\mathrm{GL}_D(V)$ contains the image of the representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. Well, by definition D consists of morphisms commuting with the action of $\mathbb{S}$, so the image of h must commute with D. ■

Proof 2. Motivated by Corollary 1.36, one expects to find Hodge classes corresponding to the condition of commuting with D. Well, there is a canonical isomorphism $V \otimes V^\vee \to \mathrm{End}_{\mathbb{Q}}(V)$ of $\mathbb{S}$-representations, so by tracking through how representations of $\mathbb{S}$ correspond to Hodge structures, we see that $f\colon V \to V$ preserves the Hodge structure if and only if it is fixed by $\mathbb{S}$, which is equivalent to the corresponding element $f \in V \otimes V^\vee$ being fixed by $\mathbb{S}$, which is equivalent to f being a Hodge class by Remark 1.13. This completes the proof of the lemma upon comparing with Corollary 1.34. ■

Remark 1.46. Of course, we also have $\mathrm{Hg}(V) \subseteq \mathrm{GL}_D(V)$ because $\mathrm{Hg}(V) \subseteq \mathrm{MT}(V)$.

Lemma 1.47. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with polarization φ. Then $\mathrm{MT}(V)$ is an algebraic $\mathbb{Q}$-subgroup of

$$\mathrm{GSp}(\varphi) \coloneqq \{g \in \mathrm{GL}(V) : \varphi(gv \otimes gw) = \lambda(g)\varphi(v \otimes w) \text{ for fixed } \lambda(g) \in \mathbb{Q}\}.$$

Proof 1. Once again, we note that $\mathrm{GSp}(\varphi)$ is an algebraic $\mathbb{Q}$-group cut out by equations of the form

$$\varphi(gv \otimes gw)\varphi(v' \otimes w') = \varphi(v \otimes w)\varphi(gv' \otimes gw')$$

as $v, w, v', w' \in V$ varies over a basis. Thus, it is enough to check that $\mathrm{GSp}(\varphi)$ contains the image of $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. Well, for any $z \in \mathbb{S}(\mathbb{R})$, we note that

$$\varphi(h(z) \otimes h(z)w) = h_{\mathbb{Q}(-m)}(z)\varphi(v \otimes w)$$

for any $v, w \in V_{\mathbb{R}}$ because φ is a morphism of Hodge structures. ■

Proof 2. Once again, Corollary 1.36 tells us to expect the polarization to produce a Hodge class corresponding to the above equations cutting out $\mathrm{MT}(V)$.

This construction is slightly more involved. We begin by constructing two Hodge classes.

- Note $\varphi\colon V \otimes V \to \mathbb{Q}(-m)$ is a morphism of Hodge structrures, so it is an $\mathbb{S}$-invariant map and hence given by an $\mathbb{S}$-invariant element of $V^\vee \otimes V^\vee(-m)$. Thus, $\varphi \in V^\vee \otimes V^\vee(-m)$ is a Hodge class by Remark 1.13.

- Because φ is non-degenerate, it induces an isomorphism $V(m) \to V^\vee$. Now, $\mathrm{End}_{\mathbb{Q}}(V)$ is canonically isomorphic to $V \otimes V^\vee$, which we now see is isomorphic (via φ) to $V \otimes V(m)$. We let $C \in V \otimes V(m)$ be the image of $\mathrm{id}_V \in \mathrm{End}_{\mathbb{Q}}(V)^{\mathbb{S}}$ in $V \otimes V(m)$, which we note is a Hodge class again by Remark 1.13. (Here, C stands for "Casimir.")

In total, we see that we have produced a Hodge class $C \otimes \varphi$. It remains to show that $g \in \mathrm{GL}(V)$ fixing $C \otimes \varphi$ implies that $g \in \mathrm{GSp}(\varphi)$, which will complete the proof by Corollary 1.34.

Well, suppose $g(C \otimes \varphi) = C \otimes \varphi$. Note $g(C \otimes \varphi) = gC \otimes g\varphi$, which can only equal $C \otimes \varphi \in (V \otimes V) \otimes_{\mathbb{Q}} (V^\vee \otimes V^\vee)$ if there is a scalar $\lambda \in \mathbb{Q}^\times$ such that $gC = \lambda C$ and $g\varphi = \lambda^{-1}\varphi$. This second condition amounts to requiring

$$\varphi\left(g^{-1}v \otimes g^{-1}w\right) = \lambda^{-1}\varphi(v \otimes w)$$

for any $v, w \in V$, which rearranges into $g \in \mathrm{GSp}(\varphi)$. ∎

Remark 1.48. The construction given in the above proof is described in [GGL24, Remark 8.3.4]. They also show the converse claim that any $g \in \mathrm{GSp}(\varphi)$ fixes $C \otimes \varphi$.

To see this, one has to do an explicit computation with C. For this, let $\{v_1, \dots, v_n\}$ be a basis of V, and $\{v_1^*, \dots, v_n^*\}$ be the dual basis of $V(m)$ taken with respect to φ. Then $C = \sum_{i=1}^n v_i \otimes v_i^*$. Similarly, we see that $\{gv_1, \dots, gv_n\}$ is a basis of V with a dual basis $\{(gv_1)^*, \dots, (gv_n)^*\}$ so that $C = \sum_{i=1}^n (gv_i) \otimes (gv_i)^*$. Now, on one hand, if g has multiplier λ, then $g\varphi = \lambda^{-1}\varphi$. On the other hand, $\varphi(gv_i, gv_j^*) = \lambda 1_{i=j}$, so $(gv_i)^* = \lambda^{-1} gv_i^*$, which allows us to compute $gC = \lambda C$. In total, $g(C \otimes \varphi) = C \otimes \varphi$.

Remark 1.49. One can check that $\mathrm{GSp}(\varphi)$ does not depend on the choice of polarization. Roughly speaking, the point is that the choice of a different polarization amounts to some choice of an element in $D^\times$ which we can track through.

In light of the above two lemmas, we pick up the following notation.

Notation 1.50. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ and polarization φ. Then we define

$$\mathrm{GSp}_D(\varphi) \coloneqq \mathrm{GL}_D(V) \cap \mathrm{GSp}(\varphi).$$

By Lemmas 1.45 and 1.47, we see that $\mathrm{MT}(V) \subseteq \mathrm{GSp}_D(\varphi)$.

Remark 1.51. In "most cases," we expect that generic Hodge structures V should have the equality $\mathrm{MT}(V) = \mathrm{GL}_D(V)$, and if V admits a polarization φ, then we expect the equality $\mathrm{MT}(V) = \mathrm{GSp}_D(\varphi)$. To rigorize this intuition, one must discuss Shimura varieties, which we will avoid doing for now.

We can also apply Lemmas 1.45 and 1.47 to bound $\mathrm{Hg}(V)$.

Notation 1.52. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ and polarization φ. Then we define

$$\mathrm{Sp}(\varphi) \coloneqq \{g \in \mathrm{GL}(V) : \varphi(gv \otimes gw) = \varphi(v \otimes w)\},$$

and

$$\mathrm{Sp}_D(\varphi) \coloneqq \mathrm{GL}_D(V) \cap \mathrm{Sp}(\varphi).$$

Remark 1.53. Let's explain why $\mathrm{Hg}(V) \subseteq \mathrm{Sp}_D(\varphi)$. By Lemma 1.45, we see that $\mathrm{Hg}(V) \subseteq \mathrm{MT}(V) \subseteq \mathrm{GL}_D(V)$, so it remains to check that $\mathrm{Hg}(V) \subseteq \mathrm{Sp}(\varphi)$. Proceeding as in Lemma 1.47, it is enough to check that the image of $h|_{\mathbb{U}}$ lives in $\mathrm{Sp}(\varphi)_{\mathbb{R}}$, for which we note that any $z \in \mathbb{U}(\mathbb{R})$ has

$$\varphi(h(z)v \otimes h(z)w) = h_{\mathbb{Q}(-m)}(z)\varphi(v \otimes w),$$

but $h_{\mathbb{Q}(-m)}(z) = |z|^{-2m} \mathrm{id}_{\mathbb{Q}(-m)}$ is the identity because $z \in \mathbb{U}(\mathbb{R})$.

Thus far, our tools have been upper-bounding $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$. Here is a tool which sometimes provides a lower bound.

Lemma 1.54. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight, and let $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ be the endomorphism algebra of V. Then

$$D = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)} = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{Hg}(V)}.$$

Proof. As discussed in the second proof of Lemma 1.45, the Hodge calsses of $\mathrm{End}_{\mathbb{Q}}(V) \cong V \otimes V^\vee$ are exactly the endomorphisms of the Hodge structure, so the first equality follows from Corollary 1.34.

The second equality is purely formal: note that the scalar subgroup $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{GL}(V)$ acts trivially on $V \otimes V^\vee \cong \mathrm{End}_{\mathbb{Q}}(V)$. Thus, we use Lemma 1.41 to compute

$$\begin{aligned}\mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{Hg}(V)} &= \mathrm{End}_{\mathbb{Q}}(V)^{\mathbb{G}_{m,\mathbb{Q}}\,\mathrm{Hg}(V)} \\ &= \mathrm{End}_{\mathbb{Q}}(V)^{\mathbb{G}_{m,\mathbb{Q}}\,\mathrm{MT}(V)} \\ &= \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)},\end{aligned}$$

as required. ∎

Remark 1.55. To understand Lemma 1.54 as providing a lower bound, note that if $\mathrm{MT}(V)$ is "too small," then there will be many invariant elements in $\mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)}$, perhaps exceeding D. On the other hand, the upper bound $\mathrm{MT}(V) \subseteq \mathrm{GL}_D(V)$ corresponds to the inequality $D \subseteq \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(V)}$.

1.3.2 Sums

For later use in computations, it will be helpful to have a few remarks on computing the Mumford–Tate and Hodge groups of a sum. Here the Hodge group really shines: given two Hodge structures $V_1, V_2 \in \mathrm{MT}(V)$ pure of nonzero weight, Lemma 1.41 tells us that $\mathrm{MT}(V_1)$ and $\mathrm{MT}(V_2)$ and $\mathrm{MT}(V_1 \oplus V_2)$ are all equal to some smaller group times scalars. It will turn out to be reasonable to hope that

$$\mathrm{Hg}(V_1 \oplus V_2) \stackrel{?}{=} \mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2),$$

but then the introduction of scalars makes the hope $\mathrm{MT}(V_1 \oplus V_2) \stackrel{?}{=} \mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$ unreasonable!

With this in mind, let's begin to study Hodge groups of sums of Hodge structures.

Lemma 1.56. Fix Hodge structures $V_1, \dots, V_k \in \mathrm{Hg}_{\mathbb{Q}}$ pure of the same weight.

(a) The subgroup $\mathrm{Hg}(V_1 \oplus \cdots \oplus V_k) \subseteq \mathrm{GL}(V_1 \oplus \cdots \oplus V_k)$ is contained in $\mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k) \subseteq \mathrm{GL}(V_1 \oplus \cdots \oplus V_k)$.

(b) For each i, the projection map $\mathrm{pr}_i \colon \mathrm{Hg}(V_1 \oplus \cdots \oplus V_k) \to \mathrm{Hg}(V_i)$ is surjective.

Proof. For each i, let h_i denote the representations of $\mathbb{S}$ corresponding to the Hodge structures V_i, and let $h \coloneqq (h_1, \dots, h_k)$ be the representation $\mathbb{S} \to \mathrm{GL}(V)$ where $V \coloneqq V_1 \oplus \cdots \oplus V_k$. We show the claims in sequence.

(a) We must show that $\mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k)$ contains the image of $h|_{\mathbb{U}}$. Well, for any $z \in \mathbb{U}(\mathbb{R})$ and index i, we see that $h_i(z) \in \mathrm{Hg}(V_i)$, so

$$h(z) = \mathrm{diag}(h_1(z), \ldots, h_k(z))$$

lives in $\mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k)$, as required.

(b) Fix an index i. It is enough to show that smallest algebraic $\mathbb{Q}$-group containing the image of pr_i also contains the image of $h_i|_{\mathbb{U}}$. Well, by definition of h, we see that h_i is equal to the composite

$$\mathbb{S} \xrightarrow{h} \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_k) \xrightarrow{\mathrm{pr}_i} \mathrm{GL}(V_i),$$

from which the claim follows. ∎

Remark 1.57. All the claims in Lemma 1.56 are true if Hg is replaced by MT everywhere. One simply has to replace $\mathbb{U}$ with $\mathbb{S}$ in the proof.

Lemma 1.56 makes $\mathrm{Hg}(V_1 \oplus V_2) \stackrel{?}{=} \mathrm{Hg}(V_1) \times \mathrm{Hg}(V_2)$ appear to be a reasonable expectation. However, we note that we cannot in general expect this to be true: roughly speaking, there may be Hodge cycles on $V_1 \oplus V_2$ which are not seen on just V_1 or V_2. Here is a degenerate example.

Example 1.58. Fix a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight, and let $n \geq 1$ be a positive integer. Letting $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be the corresponding representation, we get another Hodge structure $h^n\colon \mathbb{S} \to \mathrm{GL}\left(V^{\oplus n}\right)$. We claim that the diagonal embedding of $\mathrm{Hg}(V)$ into $\mathrm{GL}(V)^n \subseteq \mathrm{GL}\left(V^{\oplus n}\right)$ induces an isomorphism

$$\mathrm{Hg}(V) \to \mathrm{Hg}\left(V^{\oplus n}\right).$$

On one hand, we note that $\mathrm{Hg}\left(V^{\oplus n}\right)$ lives inside the diagonal embedding of $\mathrm{Hg}(V)$: note $\mathrm{Hg}\left(V^{\oplus n}\right) \subseteq \mathrm{Hg}(V)^n$ by Lemma 1.56, and $\mathrm{Hg}\left(V^{\oplus n}\right)$ must live inside the diagonal embedding of $\mathrm{GL}(V) \subseteq \mathrm{GL}\left(V^{\oplus n}\right)$ becuase all components of $h^n\colon \mathbb{S} \to \mathrm{GL}\left(V^{\oplus n}\right)_{\mathbb{R}}$ are equal. On the other hand, the surjectivity of the projections $\mathrm{Hg}\left(V^{\oplus n}\right) \to \mathrm{Hg}(V)$ from Lemma 1.56 implies that $\mathrm{Hg}\left(V^{\oplus n}\right)$ must equal the diagonal embedding of $\mathrm{Hg}(V)$ (instead of merely being contained in it).

One can upgrade this example as follows.

Lemma 1.59. Fix Hodge structures $V_1, \ldots, V_k \in \mathrm{Hg}_{\mathbb{Q}}$ pure of the same weight, and let $m_1, \ldots, m_k \geq 1$ be positive integers. Then the diagonal embeddings $\Delta_i\colon \mathrm{GL}(V_i) \to \mathrm{GL}\left(V_i^{\oplus m_i}\right)$ induce an isomorphism

$$\mathrm{Hg}(V_1 \oplus \cdots \oplus V_k) \to \mathrm{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right).$$

Proof. We proceed in steps. The proof is a direct generalization of the one given in Example 1.58. For each i, let $h_i\colon \mathbb{S} \to \mathrm{GL}(V_i)_{\mathbb{R}}$ be the representation corresponding to the Hodge structure, and set $h := (h_1^{m_1}, \ldots, h_k^{m_k})$.

1. We claim that $\mathrm{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right)$ lives in the image of $(\Delta_1, \ldots, \Delta_k)$. Indeed, the image is some algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right)$, so we would like to check that this algebraic $\mathbb{Q}$-subgroup contains the image of $h|_{\mathbb{U}}$. Well, for any $z \in \mathbb{U}(\mathbb{R})$, we see that

$$h(z) = (\Delta_1(h_1(z)), \ldots, \Delta_k(h_k(z)))$$

lives in the image of $(\Delta_1, \ldots, \Delta_k)$.

2. For each i, let H_i be the projection of $\mathrm{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right)$ onto one of the V_i components as in Lemma 1.56; the choice of V_i component does not matter by the previous step. By Lemma 1.56, we see that $H_i = \mathrm{Hg}(V_i)$. However, the previous step now requires

$$\mathrm{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right) = \Delta_1(H_1) \times \cdots \times \Delta_k(H_k),$$

so we are done. ∎

Remark 1.60. As usual, this statement continues to be true for MT replacing Hg. One can either see this by applying Lemma 1.41 or by redoing the proof with $\mathbb{S}$ replacing $\mathbb{U}$.

The point of the lemma is that we can reduce our computation of Hodge groups to Hodge structures which are the sum of pairwise non-isomoprhic irreducible Hodge strucutures. Let's make a few remarks about this situation for completeness. Let $V_1, \dots, V_k$ be pairwise non-isomorphic irreducuble Hodge structures which are pure of the same weight, and set $V := V_1 \oplus \cdots \oplus V_k$. Here are some remarks on $\mathrm{Hg}(V_1 \times \cdots \times V_k)$, summarizing everything we have done so far.

- We know that $\mathrm{Hg}(V) \subseteq \mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k)$.
- We know that the projections of $\mathrm{Hg}(V)$ onto each factor $\mathrm{Hg}(V_i)$ are surjective.
- For each i, we may view V_i as a representation of $\mathrm{Hg}(V_i)$ via the inclusion $\mathrm{Hg}(V_i) \subseteq \mathrm{GL}(V_i)$. Then Corollary 1.43 tells us that V_i is an irreducible representation of $\mathrm{Hg}(V_i)$.
- One can also apply Lemma 1.54 to the full space V to see that

$$\begin{aligned}\mathrm{End}_{\mathrm{Hg}(V)}(V) &= \mathrm{End}_{\mathrm{HS}}(V) \\ &= \prod_{i=1}^{k} \mathrm{End}_{\mathrm{HS}}(V_i) \\ &= \prod_{i=1}^{k} \mathrm{End}_{\mathrm{Hg}(V_i)}(V_i).\end{aligned}$$

The following results take the above situation and provides some criteria to have

$$\mathrm{Hg}(V) \stackrel{?}{=} \mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k).$$

Before stating the lemma, we remark that all groups in sight are connected by Remark 1.40, and we already have one inclusion by Lemma 1.56, so it suffices to pass to an algebraic closure and work with Lie algebras instead of the Lie groups. The following lemma is essentially due to Ribet [Rib76, pp. 790–791].

Lemma 1.61 (Ribet)**.** Work over an algebraically closed field of characteristic 0. Let $V_1, \dots, V_k$ be finite-dimensional vector spaces, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)$. For each index i, let $\mathrm{pr}_i \colon (\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)) \to \mathfrak{gl}(V_i)$ be the ith projection, and set $\mathfrak{g}_i := \mathrm{pr}_i(\mathfrak{g})$. Suppose the following.

(i) Each $\mathfrak{g}_i$ is nonzero and simple.

(ii) For each pair (i, j) of distinct indices, the projection map $(\mathrm{pr}_i, \mathrm{pr}_j) \colon \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is surjective.

Then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$.

Proof. We proceed by induction on k. If $k \in \{0, 1\}$, then there is nothing to say. For the induction, we now assume that $k \geq 2$ and proceed in steps.

1. For our set-up, we let J be the kernel of $\mathrm{pr}_k \colon \mathfrak{g} \to \mathfrak{g}_n$. By definition, $J \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ takes the form $I \oplus 0$ for some subspace $I \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$. Formally, one may let I be the set of vectors v such that $(v, 0) \in J$ and argue for the equality $J = I \oplus 0$ because all vectors in J take the form $(v, 0)$.

   The main content of the proof goes into showing that I is actually an ideal. To set ourselves up to prove this claim, let $\mathfrak{n} \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ denote its normalizer. We would like to show that $\mathfrak{n} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$, for which we use the inductive hypothesis.

2. For each pair of distinct indices $i, j < k$, we claim that the projection $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{n} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is surjective. Well, choose $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$, and we need to find an element in $\mathfrak{n}$ with X_i and X_j at the correct coordinates.

   To begin, we note that (ii) yields some $(X_1, \dots, X_k) \in \mathfrak{g}$ such that with the correct $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$ coordinates. We would like to show that $X \coloneqq (X_1, \dots, X_{k-1})$ lives in $\mathfrak{n}$, which will complete this step. Well, select any $Y \coloneqq (Y_1, \dots, Y_{k-1})$ in I, and we see $(Y, 0) \in J$, so

   $$[(X, X_k), (Y, 0)] = ([X, Y], 0)$$

   lives in J too (recall J is an ideal), so we conclude $[X, Y] \in I$. We conclude that X normalizes I, so $X \in \mathfrak{n}$.

3. We take a moment to complete the proof that $I \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ is an ideal. It is enough to check that the normalizer $\mathfrak{n}$ of I in $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ equals all of $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$. For this, we use the inductive hypothesis. The previous step shows that $\mathfrak{g}_i = \mathrm{pr}_i(\mathfrak{n})$ for each i, and we know by (i) that each $\mathfrak{g}_i$ is already nonzero and simple. Lastly, the previous step actually checks condition (ii) for the inductive hypothesis, completing the proof that $\mathfrak{n} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$.

4. We claim $I = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$. Because $I \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ is an ideal of a sum of simple algebras, we know that

   $$I = \bigoplus_{i \in S} \mathfrak{g}_i$$

   for some subset $S \subseteq \{1, \dots, k-1\}$ of indices. Thus, to achieve the equality $I \stackrel{?}{=} \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$, it is enough to check that each projection $\mathrm{pr}_i\colon I \to \mathfrak{g}_{k-1}$ is surjective. Unravelling the definition of I, it is enough to check that each $X_i \in \mathfrak{g}_i$ has some $(X_1, \dots, X_k) \in \mathfrak{g}$ with the correct X_i coordinate and $X_k = 0$. This last claim follows from hypothesis (ii) of $\mathfrak{g}$!

5. We now finish the proof of the lemma. Certainly $\mathfrak{g} \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$, so it is enough to compute dimensions to prove the equality. By the short exact sequence

   $$0 \to J \to \mathfrak{g} \to \mathfrak{g}_n \to 0,$$

   it is enough to show that $\dim J = \dim \mathfrak{g}_1 + \cdots + \dim \mathfrak{g}_{k-1}$. However, this follows from the previous step because $\dim J = \dim I$. ∎

In practice, it is somewhat difficult to check (ii) of Lemma 1.61. Here is an automation.

Lemma 1.62 (Moonen–Zarhin)**.** Work over an algebraically closed field of characteristic 0. Let $V_1, \dots, V_k$ be finite-dimensional vector spaces, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)$. For each index i, let $\mathrm{pr}_i\colon (\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)) \to \mathfrak{gl}(V_i)$ be the ith projection, and set $\mathfrak{g}_i \coloneqq \mathrm{pr}_i(\mathfrak{g})$. Suppose the following.

(i) Each $\mathfrak{g}_i$ is nonzero and simple.

(ii) Fix a simple Lie algebra $\mathfrak{l}$, and define $I(\mathfrak{l}) \coloneqq \{i : \mathfrak{g}_i \cong \mathfrak{l}\}$. If $\#I(\mathfrak{l}) > 1$, we require the following to hold.

- All automorphisms of $\mathfrak{l}$ are inner.
- One can choose isomorphisms $\mathfrak{l} \to \mathfrak{g}_i$ for each $i \in I(\mathfrak{l})$ such that the representations $\mathfrak{l} \to \mathfrak{g}_i \to \mathfrak{gl}(V_i)$ are all isomorphic.
- The diagonal inclusion

  $$\prod_{i \in I(\mathfrak{l})} \mathrm{End}_{\mathfrak{g}_i}(V_i) \to \mathrm{End}_{\mathfrak{g}} \left(\bigoplus_{i \in I(\mathfrak{l})} V_i \right)$$

  is surjective.

Then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$.

Proof. We will show that (ii) in the above lemma implies (ii) of Lemma 1.61, which will complete the proof. We will proceed by contraposition in the following way. Fix a pair (i,j) of distinct indices, and we are interested in the map $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$. Supposing that $(\mathrm{pr}_i, \mathrm{pr}_j)$ fails to be surjective (which is a violation of (ii) of Lemma 1.61), we will show that (ii) cannot be true. In particular, we will assume the first two points of (ii) and show then that the third point of (ii) is false.

Roughly speaking, we are going to use the first two points of (ii) to find an $\mathfrak{h}$ and then produce an endomorphism of $\bigoplus_{i\in I(\mathfrak{h})} V_i$ which does not come from gluing together endomorphisms of the V_is. Having stated the outline, we proceed with the proof in steps.

1. We claim that the image $\mathfrak{h}$ of the map $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is the graph of an isomorphism $\mathfrak{g}_i \to \mathfrak{g}_j$. For this, we use the hypothesis that $(\mathrm{pr}_i, \mathrm{pr}_j)$ fails to be surjective. Well, we claim that the projections $\mathfrak{h} \to \mathfrak{g}_i$ and $\mathfrak{h} \to \mathfrak{g}_j$ are isomorphisms, which implies that $\mathfrak{h}$ is the graph of the composite isomorphism

$$\mathfrak{g}_i \leftarrow \mathfrak{h} \to \mathfrak{g}_j.$$

   By symmetry, it is enough to merely check that $\mathfrak{h} \to \mathfrak{g}_i$ is an isomorphism. On one hand, $\mathfrak{h} \to \mathfrak{g}_i$ is surjective because $\mathrm{pr}_i\colon \mathfrak{g} \to \mathfrak{g}_i$ is surjective by construction of $\mathfrak{g}_i$. On the other hand, the kernel of the projection $\mathfrak{h} \to \mathfrak{g}_i$ will be an ideal of $\mathfrak{h}$ of the form $0 \oplus I$ where $I \subseteq \mathfrak{g}_j$ is some subspace. In fact, becasue the projection $\mathfrak{h} \to \mathfrak{g}_j$ is also surjective, we see that $I \subseteq \mathfrak{g}_j$ must be an ideal, so the simplicity of $\mathfrak{g}_j$ grants two cases.

   - If $I = 0$, then $\mathrm{pr}_i\colon \mathfrak{h} \to \mathfrak{g}_i$ becomes injective and is thus an isomorphism, completing this step.
   - If $I = \mathfrak{g}_j$, then $\mathfrak{h}$ fits into a short exact sequence

$$0 \to (0 \oplus \mathfrak{g}_j) \to \mathfrak{h} \to \mathfrak{g}_i \to 0,$$

     so $\dim \mathfrak{h} = \dim(\mathfrak{g}_i \oplus \mathfrak{g}_j)$, implying the inclusion $\mathfrak{h} \subseteq \mathfrak{g}_i \oplus \mathfrak{g}_j$ is an equality. However, this cannot be the case because we assumed that $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{g} \to \mathfrak{g}_i \to \mathfrak{g}_j$ fails to be surjective!

2. We construct an isomorphism of $\mathfrak{g}$-representations $V_i \to V_j$. For this, we use the first two points of (ii). Let's begin by collecting some data.

   - The previous step informs us that $\mathfrak{g}_i \cong \mathfrak{g}_j$. In fact, because this isomorphism is witnessed by the projections $\mathrm{pr}_i\colon \mathfrak{g} \to \mathfrak{g}_i$ and $\mathrm{pr}_j\colon \mathfrak{g} \to \mathfrak{g}_j$, we see that we are granted an isomorphism $f\colon \mathfrak{g}_i \to \mathfrak{g}_j$ such that $\mathrm{pr}_j = f \circ \mathrm{pr}_i$.
   - We now let $\mathfrak{l}$ be a simple Lie algebra isomorphic to both(!) $\mathfrak{g}_i$ and $\mathfrak{g}_j$. The second point of (ii) grants isomorphisms $f_i\colon \mathfrak{l} \to \mathfrak{g}_i$ and $f_j\colon \mathfrak{l} \to \mathfrak{g}_j$ of Lie algebras and an isomorphism $d\colon V_i \to V_j$ of $\mathfrak{l}$-representations.

   We now construct our isomorphism from d. Because d is only an isomorphism of $\mathfrak{l}$-representations, we are only granted that $(X_1, \dots, X_k) \in \mathfrak{g}$ satisfies $f(X_i) = X_j$ and hence

$$\begin{aligned} d\left((f_i f_j^{-1} f)(X_i) v_i\right) &= d\left(f_i\left(f_j^{-1} f(X_i)\right) v_i\right) \\ &= f_j\left(f_j^{-1} f(X_i)\right) d(v_i) \\ &= X_j d(v_i) \end{aligned}$$

   for all $v_i \in V_i$. We would be done if we could remove the pesky automorphism $f_i f_j^{-1} f\colon \mathfrak{g}_i \to \mathfrak{g}_i$. This is possible because all automorphisms of $\mathfrak{g}_i \cong \mathfrak{l}$ are inner (!), so one may simply "change bases" to remove the inner automorphism. Explicitly, find $a \in \mathrm{GL}(V_i)$ such that $f_i f_j^{-1} f(X) = aXa^{-1}$ for all $X \in \mathfrak{g}_i$, and then we define $e \coloneqq d \circ a$. Then we find that any $v_i \in V_i$ has

$$\begin{aligned} e(X_i v_i) &= d\left(aX_i a^{-1} \cdot av\right) \\ &= d\left((f_i f_j^{-1} f)(X_i) \cdot av\right) \\ &= X_j d(av) \\ &= X_j e(v). \end{aligned}$$

3. We complete the proof. The previous step provides a morphism $e\colon V_i \to V_j$ of $\mathfrak{g}$-representations. We thus note that the composite
$$\bigoplus_{i'\in I(\mathfrak{l})} V_{i'} \twoheadrightarrow V_i \xrightarrow{e} V_j \hookrightarrow \bigoplus_{i'\in I(\mathfrak{l})} V_{i'}$$
is an endomorphism which does not come from the diagonal inclusion of $\prod_{i\in I(\mathfrak{l})} \mathrm{End}_{\mathfrak{g}_i}(V_i)$. This completes the proof by showing that the third point of (ii) fails to hold. ■

Remark 1.63. We should remark on some history. Lemma 1.61 is due to Ribet [Rib76, pp. 790–791], but the given formulation is due to Moonen and Zarhin [MZ95, Lemma 2.14]. In the same lemma, Moonen and Zarhin prove Lemma 1.62, and they seem to be the first to recognize the utility of this lemma for computing Hodge groups. For example, Lombardo includes this result in his master's thesis [Lom13, Lemma 3.3.1] and includes a generalized version in another paper as [Lom16, Lemma 3.7], where it is used to compute Hodge groups of certain products of abelian varieties.

Remark 1.64. Let's explain how Lemma 1.62 is typically applied, which is admittedly somewhat different from the application used in this thesis. In the generic case, one expects (i), for example if $\mathrm{Hg}(V) = \mathrm{Sp}_D(\varphi)^\circ$ for D of Types I–III as in Remark 1.51. In this case, one can also check the first condition of (ii) by a direct computation, the second condition of (ii) has no content, and the third condition of (ii) comes from Lemma 1.54. For more details, we refer to (for example) the applications given in [Lom13; Lom16].

1.3.3 The Lefschetz Group

For motivational reasons, we mention the Lefschetz group $\mathrm{L}(V)$, which contains $\mathrm{Hg}(V)$ but is more controlled. Here is our definition.

Definition 1.65 (Lefschetz group)**.** Fix a polarizable Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight. Then we define
$$\mathrm{L}(V) \coloneqq \mathrm{Sp}_D(\varphi),$$
where $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$, and φ is a polarization.

Thus, Remark 1.53 that $\mathrm{Hg}(V) \subseteq \mathrm{L}(V)$.

Remark 1.66. Let's interpret $\mathrm{L}(V)$ geometrically. Roughly speaking, $\mathrm{L}(V)$ is a form of $\mathrm{Hg}(V)$ which only keeps track of endomorphisms and the polarization instead of keeping track of all Hodge classes. As such, we generically expect $\mathrm{Hg}(V) = \mathrm{L}(V)$ to hold, but we do not expect it to hold always. (Technically, there are generic cases when we do not expect this equality; for example, if V is irreducible of Type III in ths sense of the Albert classificaiotn Theorem 1.28, then $\mathrm{L}(V)$ is not connected, so we cannot have equality.) Furthermore, when $\mathrm{Hg}(V) = \mathrm{L}(V)$, we expect to have strong control on the Hodge classes of V; for example, the Hodge conjecture is known in many such cases [Mur84, Theorem 3.1].

Computationally, one reason why $\mathrm{L}(V)$ is more controlled is that it is much easier to compute. For example, L behaves well in sums.

Lemma 1.67. Fix pairwise non-isomorphic irreducible polarizable Hodge structures $V_1, \dots, V_k$ of the same pure weight, and let $m_1, \dots, m_k \geq 1$ be integers. Then the diagonal embeddings $\Delta_i\colon \mathrm{GL}(V_i) \to \mathrm{GL}\left(V_i^{\oplus m_i}\right)$ induce an isomorphism
$$\mathrm{L}(V_1) \times \cdots \times \mathrm{L}(V_k) \to \mathrm{L}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right).$$

Proof. The main idea is to compute some endomorphism algebras and polarizations. We proceed in steps. Set $V \coloneqq V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$ for brevity.

1. We work with endomorphisms. We may view Hodge structures as $\mathbb{S}$-representations, whereupon we find that
$$\mathrm{End}_{\mathrm{HS}}(V) = \mathrm{End}_{\mathrm{HS}}(V_1)^{m_1 \times m_1} \times \cdots \times \mathrm{End}_{\mathrm{HS}}(V_k)^{m_k \times m_k}.$$
In particular, we see that any f commuting with $\mathrm{End}_{\mathrm{HS}}(V)$ implies that f must preserve each $V_i^{\oplus m_i}$ (because there is a separate algebra $\mathrm{End}_{\mathrm{HS}}\left(V_i^{\oplus m_i}\right)$ for each i). Further, $f|_{V_i^{\oplus m_i}}$ must come from the diagonal embedding $\mathrm{End}_{\mathrm{HS}}(V_i) \to \mathrm{End}_{\mathrm{HS}}\left(V_i^{\oplus m_i}\right)$ because $\mathrm{End}_{\mathrm{HS}}(V_i)^{m_i \times m_i}$ may swap any of the m_i copies of V_i.
We conclude that f commutes with endomorphisms implies that
$$f = (\Delta_1 f_1, \ldots, \Delta_k f_k),$$
where $\Delta_i \colon \mathrm{End}(V_i) \to \mathrm{End}\left(V_i^{\oplus m_i}\right)$ is the diagonal embedding, and each f_i commutes with $\mathrm{End}_{\mathrm{HS}}(V_i)$. Conversely, the computation of $\mathrm{End}_{\mathrm{HS}}(V)$ above allows us to conclude that any f in the above form commutes with $\mathrm{End}_{\mathrm{HS}}(V)$.

2. We work with the polarization. Choose polarizations $\varphi_1, \ldots, \varphi_k$ on $V_1, \ldots, V_k$ (respectively), and we note that these polarizations glue into a polarization φ on V. With this choice of polarization, we see that $f = (\Delta_1 f_1, \ldots, \Delta_k f_k)$ as in the previous step preserves φ if and only if each factor $\Delta_i f_i$ preserves the polarization $\varphi|_{V_i^{\oplus m_i}}$, which is equivalent to f_i preserving the polarization φ_i. In total, we thus see that $f \in \mathrm{L}(V)$ if and only $f_i \in \mathrm{L}(V_i)$ for each i, so we are done. ∎

Lemma 1.67 tells us that we can always reduce the computation of the Lefschetz group to irreducible components. In this way, it now suffices to compute $\mathrm{L}(V)$ by working with V according to the Albert classification (Theorem 1.28). All these computations are recorded in [Mil99, Section 2]. Because we will only be interested in Type IV in the sequel, we will only record the part of this computation we need for completeness.

Lemma 1.68. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight with $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ and polarization φ. Suppose $D = F$ is a CM field. Then
$$\mathrm{L}(V)_{\mathbb{C}} \cong \mathrm{GL}_{[V:F]}(\mathbb{C})^{\frac{1}{2}[F:\mathbb{Q}]}.$$

Proof. We proceed in steps. Let $F^\dagger \subseteq F$ be the maximal totally real subfield, and choose embeddings $\rho_1, \ldots, \rho_{e_0} \colon F^\dagger \hookrightarrow \mathbb{R}$, where $e_0 \coloneqq \frac{1}{2}[F:\mathbb{Q}]$. For each i, we will let σ_i and τ_i be complex conjugate embeddings $F \hookrightarrow \mathbb{C}$ restricting to ρ_i.

1. We begin by explaining the exponent $e_0 = \frac{1}{2}[F:\mathbb{Q}]$. Note V is a free $F^\dagger$-module of rank $[V:F]$, so $V_{\mathbb{R}}$ is a free module over
$$F^\dagger \otimes \mathbb{R} = \prod_{i=1}^{e_0} F^\dagger_{\rho_i},$$
where $F^\dagger_\rho = \mathbb{R}$ refers to the $F^\dagger \otimes \mathbb{R}$ module where F acts by ρ. The above decomposition of $F \otimes \mathbb{R}$ implies a decomposition
$$V_{\mathbb{R}} = V_1 \oplus \cdots \oplus V_{e_0},$$
where each V_i of a vector space over $F^\dagger_{\rho_i}$, all the same dimension.
We now understand the effect of endomorphisms and the polariaztion on our decomposition. Thus, we see that $f \colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ commutes with $F^\dagger \otimes \mathbb{R}$ if and only if f preserves each factor V_i (due to the decomposition of $F^\dagger \otimes \mathbb{R}$) and commute with the action of $F^\dagger_{\rho_i}$ on each V_i. Similarly, we see that the

polarization φ makes the V_is orthogonal: for each $d \in F^\dagger$, we see that any $v_i \in V_i$ and $v_j \in V_j$ has

$$\begin{aligned}\rho_i(d)\varphi(v_i, v_j) &= \varphi(dv_i, v_j)\\ &= \varphi(v_i, \overline{d}v_j)\\ &= \varphi(v_i, dv_j)\\ &= \rho_j(d)\varphi(v_i, v_j),\end{aligned}$$

so $i \neq j$ implies that $\varphi(v_i, v_j) = 0$. Thus, we see that φ must restrict to non-degenerate skew-symmetric bilinear forms on each V_i individually. In total, $f\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ preserves φ if and only if $f|_{V_i}$ preserves $\varphi|_{V_i}$ for each i. In total, we see that

$$\mathrm{L}(V)_{\mathbb{R}} = \mathrm{Sp}_{F\otimes_{\rho_1}\mathbb{R}}(\varphi|_{V_1}) \times \cdots \times \mathrm{Sp}_{F\otimes_{\rho_k}\mathbb{R}}(\varphi|_{V_{e_0}}).$$

2. It remains to show that $\mathrm{Sp}_{F\otimes_{\rho_i}\mathbb{R}}(\varphi|_{V_i})_{\mathbb{C}}$ is isomorphic to $\mathrm{GL}_{[V:F]}(\mathbb{C})$; here, note $[V : F] = [V_i : F^\dagger_{\rho_i}]$. For this, we abstract the situation somewhat: suppose that a vector space V over $\mathbb{R}$ has been equipped with an action by $\mathbb{C} \subseteq \mathrm{End}_{\mathbb{R}}(V)$, and furthermore, φ is a skew-Hermitian form on V. Then we want to show $\mathrm{Sp}_{\mathbb{C}}(\varphi)_{\mathbb{C}} \cong \mathrm{GL}_{[V:\mathbb{R}]}(\mathbb{C})$.

   The trick is that we can keep track of commuting with the action of $\mathbb{C}$ on V by merely commuting with the action of $i \in \mathbb{C}$. Thus, let $J\colon V \to V$ be this map, which satisfies $J^2 = -1$. Now, the action of $J_{\mathbb{C}}$ on $V_{\mathbb{C}}$ must diagonalize into eigenspaces $V_i \oplus V_{-i}$ with eigenvalues i and $-i$ respectively; note that we must have $\dim V_i = \dim V_{-i}$ in order for the characateristic polynomial of J to have real coefficients. The point is that $f \in \mathrm{End}(V_{\mathbb{C}})$ commutes with the action of $\mathbb{C}$ if and only if it commutes with the action of J, which we can see is equivalent to f preserving the decomposition $V_i \oplus V_{-i}$.

   We now study the polarization φ. Note that φ vanishes on $V_{\pm i} \oplus V_{\pm i}$: for any $v, v' \in V_{\pm i}$, we see that

$$\begin{aligned}\pm i\varphi(v, v') &= \varphi(Jv, v')\\ &= \varphi(v, -Jv')\\ &= \mp i\varphi(v, v'),\end{aligned}$$

   from which $\varphi(v, v') = 0$ follows. For example, this implies that any $f \in \mathrm{End}(V_{\mathbb{C}})$ commuting with the J-action will automatically preserve φ on $V_{\pm i} \times V_{\pm i}$. Additionally, we see that φ must restrict to a non-degenerate bilinear form on $V_i \times V_{-i}$.

   We are now ready to claim that restriction defines an isomorphism $\mathrm{Sp}_{\mathbb{C}}(\varphi)_{\mathbb{C}} \to \mathrm{GL}_{\mathbb{C}}(V_i)$. This restriction does actually output to $\mathrm{GL}_{\mathbb{C}}(V_i)$ because $g \in \mathrm{Sp}_{\mathbb{C}}(\varphi)_{\mathbb{C}}$ must preserve the decomposition $V_i \oplus V_{-i}$. To see the injectivity, we note that preserving φ requires

$$\varphi(v, gw) = \varphi\left(g^{-1}v, w\right)$$

   for all $v \in V_i$ and $w \in V_{-i}$; thus, the non-degeneracy of φ implies that $g \in \mathrm{Sp}_{\mathbb{C}}(\varphi)_{\mathbb{C}}$ is uniquely determined by its action on V_i. Conversely, for the surjectivity, we see that we can take any element in $\mathrm{GL}(V_i)$ and use the previous sentence to extend it uniquely to an element of $\mathrm{Sp}_{\mathbb{C}}(\varphi)_{\mathbb{C}}$. ∎

1.4 Absolute Hodge Classes

We now discuss the main application of Hodge structures: cohomology. This will allow us to discuss absolute Hodge classes. Our exposition an abbreviated form [Del18].

1.4.1 Some Cohomology Theories

In this subsection, we will give a lighting introduction to the cohomology theories that we will use. We begin with sheaf cohomology.

Definition 1.69 (sheaf cohomology)**.** Fix a topological space X. Then the category $\mathrm{Ab}(X)$ of abelian sheaves on X has enough injectives. Given a sheaf $\mathcal{F}$ on X, we then may define the *sheaf cohomology* as the abelian groups

$$\mathrm{H}^i(X,\mathcal{F}) \coloneqq \mathrm{R}^i\Gamma(X,\mathcal{F}),$$

where $\Gamma\colon \mathrm{Ab}(X) \to \mathrm{Ab}$ is the global-sections functor. Explicitly, one can compute these cohomology groups by taking the cohomology of an acyclic resolution of $\mathcal{F}$.

This allows us a quick definition of Betti cohomology.

Definition 1.70 (Betti cohomology)**.** Fix a topological space X and a ring R. Then we define the *Betti cohomology* of X with coefficients in R as $\mathrm{H}^i(X,\underline{R})$, where $\underline{R}$ denotes the constant sheaf R.

It will be helpful to a more geometric description of $\mathrm{H}^\bullet_{\mathrm{B}}$.

Definition 1.71 (singular homology, singular cohomology)**.** Fix a topological space X and a ring R. For each $n \geq 0$, we define the n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ as the set of points $(t_0,\dots,t_n) \subseteq [0,1]^{n+1}$ summing to 1. Then we define the complex $S_\bullet(X,R)$ as having entries which are the free R-module with basis given by the maps $\Delta_\bullet \to X$ and boundary morphism given by $\partial\colon S_n(X,R) \to S_{n-1}(X,R)$ given by

$$\partial(\sigma) \coloneqq \sum_{i=0}^{n} (-1)^i \sigma([0,\dots,\widehat{i},\dots,n])$$

for $\sigma\colon \Delta_n \to X$, where $[0,\dots,\widehat{i},\dots,n]$ denotes the $(n-1)$-simplex with vertices $\{0,\dots,\widehat{i},\dots,n\}$. Then we define the *singular homology* $\mathrm{H}^{\mathrm{B}}_i(X,R)$ as the homology of this complex. We now define *singular cohomology* as the cohomology of the dual cocomplex $S^\bullet(X,R)$.

Remark 1.72. The universal coefficient theorem shows that singular homology and cohomology are dual if R is a principal ideal domain, such as $\mathbb{Z}$ or a field.

Our notation suggests that singular cohomology should be Betti cohomology, so we check this.

Theorem 1.73. Fix a topological manifold X. For any field K, there is a canonical isomorphism

$$\mathrm{H}^i(S^\bullet(X,K)) \to \mathrm{H}^i(X,\underline{K}).$$

Proof. The idea is to replace $S^\bullet(X,K)$ with a complex of sheaves $\mathcal{S}^\bullet(X,K)$, and then one finds that this complex is an acyclic resolution of $\underline{K}$. The requirement that X be a topological manifold helps because it allows us to reduce local checks on X to the case of a unit ball. ∎

We now add smoothness to our manifolds, which allows us to define de Rham cohomology.

Definition 1.74 (de Rham cohomology)**.** Fix a smooth manifold X of dimension n. For each $i \geq 0$, we let $\Omega^i_{X_\infty}$ be the sheaf of smooth differential n-forms on X. Then we define *de Rham cohomology* $\mathrm{H}^i_{\mathrm{dR}}(X,\mathbb{R})$ to be the cohomology of the complex

$$0 \to \Omega^0_{X_\infty} \xrightarrow{d} \Omega^2_{X_\infty} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X_\infty} \to 0,$$

where d denotes the de Rham differential.

We once again have a comparison isomorphism.

Theorem 1.75. Fix a smooth manifold X. For each i, there is a functorial perfect paring $\mathrm{H}^{\mathrm{B}}_i(X,\mathbb{R}) \times \mathrm{H}^i_{\mathrm{dR}}(X,\mathbb{R}) \to \mathbb{R}$ given by

$$\langle \sigma, \omega \rangle \coloneqq \int_\sigma \omega$$

for each smooth map $\sigma\colon \Delta^i \to X$.

We next upgrade to complex Kähler manifolds. For example, one can upgrade our de Rham cohomology to use holomorphic differential forms instead of smooth differential forms, and the cohomology does not change. The key benefit of the complex manifold situation is that the de Rham cohomology gains a Hodge structure.

Theorem 1.76. Fix a compact complex Käholer manifold X. For each $n \geq 0$, there is a decomposition

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{pq}(X),$$

where $\mathrm{H}^{pq}(X) \coloneqq \mathrm{H}^p(X,\Omega^q_X)$.

For our last setting, let X be a smooth projective variety over a field K. Here, there are multiple ways to form Betti cohomology.

Notation 1.77. Fix a smooth projective variety over a field K. For any embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, we define Betti cohomology relative to σ as

$$\mathrm{H}^i_\sigma(X,R) \coloneqq \mathrm{H}^i_{\mathrm{B}}(X_\sigma(\mathbb{C}),R)$$

forany ring R. Frequently, we will have fixed once and for all an embedding of K into $\mathbb{C}$, so we may abbreviate $\mathrm{H}^i_\sigma(X,R)$ to just $\mathrm{H}^i_{\mathrm{B}}(X,R)$.

Similarly, one is now able to define de Rham cohomology for X, though we do make a moment to remark that there is a theory of algebraic de Rham cohomology that is able to work in greater generality.

Working with varieties gives access to the last cohomology theory we will need.

Definition 1.78. Fix a smooth projective variety X over a field K. For some étale sheaf $\mathcal{F}$, we are able to define the étale cohomology $\mathrm{H}^i(X,\mathcal{F})$ in the same way as sheaf cohoomology. In particular, for any prime ℓ which is nonzero in K, we define the *ℓ-adic cohomology* by

$$\mathrm{H}^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell) \coloneqq \left(\varprojlim \mathrm{H}^i_{\text{ét}}(X_{\overline{K}},\underline{\mathbb{Z}/\ell^\bullet\mathbb{Z}})\right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Importantly, we note that étale cohomology has the natural action by $\mathrm{Gal}(\overline{K}/K)$. As usual, there is a comparison isomorphism.

Theorem 1.79. Fix a smooth projective variety X over $\mathbb{C}$. Then there is a natural isomorphism

$$\mathrm{H}^i_{\mathrm{B}}(X,\mathbb{Q}_\ell) \to \mathrm{H}^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell).$$

For convenience, we may find it convenient to glue our cohomology theories together.

Notation 1.80. Fix a smooth projective variety X over a field K with an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$. Then we define

$$\mathrm{H}^i_{\mathbb{A}}(X) \coloneqq \mathrm{H}^i_{\mathrm{dR}}(X,\mathbb{R}) \times \left(\varprojlim_n \mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \underline{\mathbb{Z}/n\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \right).$$

We note that there are natural projections π_∞ onto $\mathrm{H}^i_{\mathrm{dR}}(X,\mathbb{R})$ and π_ℓ onto $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell)$.

Remark 1.81. One can realize this as a restricted direct product

$$\mathrm{H}^i_{\mathrm{dR}}(X,\mathbb{R}) \times \prod_\ell \left(\mathrm{H}^i_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell), \mathrm{H}^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_\ell)\right),$$

which provides some motivation for the $\mathbb{A}$ in the notation.

1.4.2 The Definition

To define absolute Hodge classes following [Del18], we must first discuss Tate twists. These change depending on our cohomology theory.

Definition 1.82 (Tate twist)**.** We define our Tate twists as follows.

- If X is a topological manifold, then the Tate twist $\mathbb{Q}_{\mathrm{B}}(1)$ is the $\mathbb{Q}$-vector space $2\pi i\mathbb{Q}$.
- If X is a smooth manifold, then the Tate twist $\mathbb{R}_{\mathrm{dR}}(1)$ is simply $\mathbb{R}$. It has a Hodge structure of pure of weight -2 concentrated in bidegree $(-1,-1)$.
- If X is a smooth projective variety over a field K, then the Tate twist $\mathbb{Q}_\ell(1)$ for any prime ℓ (nonzero in K) is the Galois representation $(\varprojlim \mu_{\ell^\bullet}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Notation 1.83. For any cohomology theory H defined on a space X, we may write

$$\mathrm{H}^i(X)(n) \coloneqq \mathrm{H}^i(X) \otimes \mathrm{T}^{\otimes n},$$

where T denotes the Tate twist, and $i \geq 0$ and $n \in \mathbb{Z}$. If $n \leq 0$, then we take the dual.

As an example application, we want the Hodge classes on a complex Kähler manifold X to be elements of the cohomology group $\mathrm{H}^{2p}_{\mathrm{dR}}(X,\mathbb{C})(p)$ of bidegree $(0,0)$ and satisfying some rationality condition. The definition of an absolute Hodge class comes from trying to be agnostic about the embedding of the base field of X.

Definition 1.84 (absolute Hodge class)**.** Fix a smooth projective variety X over a number field K. An *absolute Hodge class* is an element t of some $\mathrm{H}^{2p}_{\mathbb{A}}(X_{\overline{K}})(p)$ if and only if it satisfies the following properties.

- $\pi_\infty(t)$ lives in the component $(0,0)$ of $\mathrm{H}^{2p}_{\mathrm{dR}}(X,\mathbb{C})$.
- For each embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, the element t is in the image of the embedding $\mathrm{H}^{2p}_{\mathrm{B}}(X,\mathbb{Q})(p)$ into $\mathrm{H}^{2p}_{\mathbb{A}}(X)(p)$.

We denote the collection of these absolute Hodge classes by $C^p_{\mathrm{AH}}(X_{\overline{K}})$ or $C^p_{\mathrm{AH}}(X)$.

Remark 1.85. Deligne [Del18, Section 2] gives a definition for smooth projective varieties defined over a general field of characteristic 0. The above definition makes sense essentially verbatim for any field K of characteristic 0 and finite transcendence degree because then one has access to embeddings into $\mathbb{C}$. For the general case, one must argue that any class with sufficient rationality properties will descend to a field of finite transcendence degree and that the choice of this descent does not matter.

Chapter 2
Abelian Varieties

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions

—Ravi Vakil [Vak23]

In this chapter, we gather together all the results about abelian varieties we need. Many of the results in the earlier sections discussed here can be found in any reasonable text on abelian varieties such as [Mum74; Mil08; EGM]. Results in the later sections are more specialized, and we will provide references when appropriate. Ultimately, our goal is to define ℓ-adic monodromy groups, explain why one might care about them, and indicate how one might compute them.

2.1 Definitions and Constructions

In this section, we set up the theory of abelian varieties rather quickly. We will usually only indicate proofs that work in the complex analytic situation because the general theory usually requires intricate algebraic geometry.

2.1.1 Starting Notions

Let's begin with a definition.

Definition 2.1 (abelian variety)**.** Fix a ground scheme S. An *abelian scheme* A over S is a smooth projective geometrically integral group scheme over S. An *abelian variety* A is an abelian scheme over a field.

Remark 2.2. Throughout, we will work with abelian varieties instead of abelian schemes as much as possible. However, one should be aware that many of the results generalize.

Here, a group variety refers to a group object in the category of varieties over K.

Remark 2.3. With quite a bit of work, one can weaken the hypotheses of being an abelian variety quite significantly. For example, arguments involving group varieties are able to show that being connected and geometrically reduced implies geometrically integral, and it is a theorem that one can replace projectivity with properness. See [SP, Remark 0H2U] for details.

Here are the starting examples.

Example 2.4 (elliptic curves)**.** Any (smooth) cubic equation cuts out a genus-1 curve in $\mathbb{P}^2$. If the curve has points defined over K, this defines an elliptic curve, which can be shown to be an abelian variety. The interesting part comes from defining the group structure. One way to do this is to show that the map $E \to \mathrm{Pic}^0_{E/K}$ given by $x \mapsto [x] - [\infty]$ is an isomorphism of schemes and then give E the group structure induced by $\mathrm{Pic}^0_{E/K}$. (Here, $\mathrm{Pic}^0_{E/K}$ is the moduli space of line bundles over E of degree 0. Smoothness of the curve makes this in bijection with divisors of degree 0.)

Example 2.5. Fix a positive integer $g \geq 0$. If $\Lambda \subseteq \mathbb{C}^g$ is a polarizable sublattice, then $\mathbb{C}^g/\Lambda$ defines an abelian variety over $\mathbb{C}$. Here, polarizable means that there is an alternating map $\varphi\colon \Lambda \times \Lambda \to \mathbb{Z}$ such that the pairing

$$\langle x, y\rangle \coloneqq \psi_{\mathbb{R}}(x, iy)$$

on $\Lambda_{\mathbb{R}}$ is symmetric and positive-definite. (As worked out in [Mil20b, Section I.2], this is equivalent data to a polarization on the Hodge structure $\Lambda = \mathrm{H}_1^{\mathrm{B}}(A, \mathbb{Z})$.) The requirement of polarizability is used to show that the quotient $\mathbb{C}^g/\Lambda$ is actually projective; see [Mum74, Section 3, Theorem].

It is notable that we have not required our abelian varieties A to actually be abelian even though (notably) both examples above are abelian. Indeed, abelian varieties are always abelian groups, which follows from an argument using the Rigidity theorem. We will not give this argument in full because we will not use it, but we state a useful corollary.

Proposition 2.6. Let $\varphi\colon A \to B$ be a smooth map of abelian varieties over a field K. Then φ is the composition of a homomorphism and a translation.

Proof. By composing with a translation, we may assume that $\varphi(0) = 0$. Then one applies the Rigidity theorem to the map $\widetilde{\varphi}\colon A \times A \to B$ defined by

$$\widetilde{\varphi}(a, a') \coloneqq \varphi(a + a') - \varphi(a) - \varphi(a')$$

to find that $\widetilde{\varphi}$ is constantly 0, completing the proof. See [Mil08, Corollary I.1.2] for details. ∎

Corollary 2.7. The group law on an abelian variety A is commutative.

Proof. The inversion map $i\colon A \to A$ on an abelian variety sends the identity to itself, so Proposition 2.6 tells us that i must be a homomorphism. It follows that the group law is commutative. ∎

In particular, we find that morphisms between abelian varieties are rather strutured: we are allowed to basically only ever consider homomorphisms!

It will turn out that considering abelian varieties up to isomorphism is too strong for most purposes, so we introduce the following definition.

Definition 2.8 (isogeny)**.** A morphism $\varphi\colon A \to B$ of abelian varieties over a field K is an *isogeny* if and only if it is a homomorphism satisfying any one of the following equivalent conditions.

(a) φ is surjective with finite kernel.

(b) $\dim A = \dim B$, and φ is surjective.

(c) $\dim A = \dim B$, and φ has finite kernel.

(d) φ is finite, flat, and surjective.

The *degree* of the isogeny is $\#\ker\varphi$ (thought of as a group scheme).

Remark 2.9. Let's briefly indicate why (a)–(d) above are equivalent; see [Mil08, Proposition 7.1] for details. A spreading out argument combined with the homogeneity of abelian varieties implies that

$$\dim B = \dim A + \dim \varphi^{-1}(\{b\})$$

for any b in the image of φ; this gives the equivalence of (a)–(c). Of course (d) implies (a) (one only needs the finiteness and surjectivity); to show (a) implies (d), we note flatness follows by "miracle flatness" because all fibers have equal dimension, and finiteness follows because finite kernel upgrades to quasi-finiteness.

Intuitively, an isogeny is a "squishy isomorphism."

Example 2.10. Any dominant morphism of elliptic curves sending the identity to the identity is an isogeny.

Example 2.11. In the complex analytic setting, an isogeny of two abelian varieties $A = \mathbb{C}^g/\Lambda$ and $B = \mathbb{C}^g/\Lambda'$ amounts (up to change of basis) an inclusion of lattices $\Lambda' \subseteq \Lambda$.

Example 2.12. Fix any abelian variety A. For any nonzero integer n, the multiplication-by-n endomorphism $[n]_A\colon A \to A$ is an isogeny. To see this, note that it is enough to check that $A[n] \coloneqq \ker[n]_A$ is finite. In the complex analytic situation where $A = \mathbb{C}^g/\Lambda$, this follows because $\frac{1}{n}\Lambda/\Lambda$ is finite; in general, one must show that $A[n] \coloneqq \ker[n]_A$ is zero-dimensional, which is somewhat tricky. See [SP, Lemma 0BFG] for details. We remark that one can compute $\deg[n]_A = d^{2\dim A}$, which is again not so hard to see in the complex analytic situation.

Motivated by the complex analytic setting (and the "squishy isomorphism" intuition), one might hope that one can recover weak-ish inverses for isogenies. This turns into an important property of abelian varieties.

Lemma 2.13. Fix an isogeny $\varphi\colon A \to B$ of abelian varieties of degree d. Then there exists an "inverse isogeny" $\beta\colon B \to A$ such that

$$\begin{cases} \alpha \circ \beta = [d]_B, \\ \beta \circ \alpha = [d]_A. \end{cases}$$

Proof. By some theory regrading group scheme quotients, it is enough to check that φ factors through $[d]_A$, which holds because $\ker\varphi$ has order d as a group scheme and thus vanishes under $[d]_A$. ∎

Remark 2.14. As usual, we remark that the above lemma is easier to see in the complex analytic situation, but the key point of trying to factor through $[d]_A$ remains the same.

Lemma 2.13 motivates the following definition, and it codifies our intuition viewing isogenies as squishy isomorphisms.

Definition 2.15 (isogeny category)**.** Fix a field K. We define the *isogeny category* of abelian varieties over K as having objects which are abelian varieties over K, and a morphism $A \to B$ in the isogeny category is an element of $\mathrm{Hom}_K(A, B)_{\mathbb{Q}}$.

We close our discussion of isogenies with one last remark on the size of kernels.

Remark 2.16. If $\varphi\colon X \to Y$ is a finite separable morphism of varieties, then a spreading out argument shows that the number of geometric points in a general fiber of φ equals the degree of φ. Applied to isogenies, the homogeneity of abelian varieties is able to show that the number of geometric points in the fiber of any separable isogeny equals the degree.

Example 2.17. Here is an application of Remark 2.16: if $\mathrm{char}\, K \nmid n$, then one can show that $A[n]$ has $n^{2\dim A}$ geometric points. Again, this is not so hard to see in the complex analytic setting. The hypothesis $\mathrm{char}\, K \nmid n$ is needed to show that $[n]_A$ is separable; in general, the argument is trickier and can (for example) use some intersection theory [Mil08, Theorem I.7.2].

Now that we have a reasonable category, one can ask for decompositions. Here is the relevant result and definition.

Theorem 2.18 (Poincaré reducibility)**.** Fix an abelian subvariety B of an abelian variety A defined over a field K. Then there is another abelian subvariety $B' \subseteq A$ such that the multiplication map induces an isogeny $B \times B' \to A$.

Proof. As usual, we argue only in the complex analytic case. Here write $A = V/\Lambda$ for complex affine space V, and we find that $B = W/(\Lambda \cap W)$ for some subspace $W \subseteq V$. Now, the polarization induces a Hermitian form on V, so we can define $W' \coloneqq W^{\perp}$ so that $B' \coloneqq W'/(\Lambda \cap W')$ will do the trick. For more details, see [Mil20b, Theorem 2.12] for more details. ∎

Definition 2.19 (simple)**.** Fix a field K. An abelian variety A over K is *simple* if and only if it is irreducible in the isogeny category.

Remark 2.20. Theorem 2.18 implies that any abelian variety can be decomposed uniquely into a product of simple abelian varieties, of course up to isogeny and permutation of factors.

2.1.2 The Jacobian

In this thesis, the abelian varieties of interest to us will be Jacobians. There are a few approaches to their definition, which we will not show are equivalent, but we refer to [Mil08, Chapter III] for details. The most direct definition is as a moduli space.

Definition 2.21 (Jacobian)**.** Fix a smooth proper curve C over a field K such that $C(K)$ is nonempty. Then the *Jacobian* $\mathrm{Jac}\, C$ is the group variety $\mathrm{Pic}^0_{C/K}$, where $\mathrm{Pic}^0_{C/K}$ is the moduli space of line bundles on C with degree 0.

Remark 2.22. We will not check that we have defined an abelian variety, nor that we have even defined a scheme. There are interesting questions regarding the representability of moduli spaces, which we are omitting a discussion of. Milne provides a reasonably direct construction in [Mil08, Section III.1], but we should remark that one expects representability to be true in a broader context. In particular, there are formal ways to check (say) properness of $\mathrm{Pic}^0_{C/K}$, from which it does follow that we have defined an abelian variety.

Remark 2.23. One can actually weaken the smoothness assumption on C to merely being "compact type." This is occasionally helpful when dealing with moduli spaces because it allows us to work a little within the boundary of the moduli space of curves.

Remark 2.24. Notably, Example 2.4 tells us that the Jacobian of a curve is E itself.

Note that the assumption $C(K) \neq \varnothing$ allows us to choose some point $\infty \in C(K)$ and then define a map $C(K) \to \operatorname{Jac} C$ by $p \mapsto [p] - [\infty]$. This map turns out to be a regular closed embedding [Mil08, Proposition 2.3]. It is psychologically grounding to see that this map is universal in some sense.

Proposition 2.25. Fix a smooth proper curve C over a field K such that $C(K) \neq \varnothing$. Choose $\infty \in C(K)$, and consider the map $\iota\colon C \to \operatorname{Jac} C$ given by $\iota(p) \coloneqq [p] - [\infty]$. For any abelian variety A over K and smooth map $\varphi\colon C \to A$ such that $\varphi(\infty) = 0$, there exists a unique map $\widetilde{\varphi}\colon \operatorname{Jac} C \to A$ making the following diagram commute.

$$\begin{array}{ccc} C & \xrightarrow{\iota} & \operatorname{Jac} C \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}} \\ & & A \end{array}$$

Proof. We will not need this, so we won't even point in a direction of a proof. We refer to [Mil08, Proposition III.6.1]. ∎

It is worthwhile to provide a complex analytic construction of the Jacobian. Given a curve C, line bundles are in bijection with divisor classes, and divisor classes of degree 0 can all be written in the form $\sum_{i=1}^{k}([P_i]-[Q_i])$ for some points $P_1, Q_1, \dots, P_k, Q_k \in C(\mathbb{C})$. One can take such a divisor and define a linear functional on $\mathrm{H}^1(C, \Omega^1_C)$ by

$$\omega \mapsto \sum_{i=1}^{k} \int_{Q_i}^{P_i} \omega.$$

The construction of this linear functional is not technically well-defined up to divisor class; instead, one can check that changing the divisor class adjusts the linear functional exactly by the choice of a cycle in $\mathrm{H}^{\mathrm{B}}_1(C, \mathbb{Z})$ embedded into $\mathrm{H}^1(C, \Omega^1_C)^\vee$ via the integration pairing. In this one way, one finds that

$$\operatorname{Jac} C(\mathbb{C}) = \frac{\mathrm{H}^1(C, \Omega^1_C)^\vee}{\mathrm{H}^{\mathrm{B}}_1(C, \mathbb{Z})}.$$

In particular, we have realized $\operatorname{Jac} C$ explicitly as a complex affine space modulo some lattice, exactly as in Example 2.5. (One sees that $\operatorname{rank}_{\mathbb{Z}} \mathrm{H}^{\mathrm{B}}_1(C, \mathbb{Z}) = \dim_{\mathbb{R}} \mathrm{H}^1(C, \Omega^1_C)^\vee$ by the Betti-to-de Rham comparison isomorphism.) This construction makes it apparent that

$$\mathrm{H}^{\mathrm{B}}_1(\operatorname{Jac} C(\mathbb{C}), \mathbb{Z}) \cong \mathrm{H}^{\mathrm{B}}_1(C, \mathbb{Z}).$$

This is in fact a general property.

Proposition 2.26. Fix a smooth proper curve C over a field K such that $C(K) \neq \varnothing$. Choose $\infty \in C(K)$, and consider the map $\iota\colon C \to \operatorname{Jac} C$ given by $\iota(p) \coloneqq [p] - [\infty]$. Then the induced map

$$\iota^*\colon \mathrm{H}^1(\operatorname{Jac} C) \to \mathrm{H}^1(C)$$

is an isomorphism, where H is any of the Weil cohomology theories of section 1.4.1.

Proof. The proof requires analyzing each cohomology theory individually. Above we outlined the proof when H is Betti cohomology, and we note that the result follows for de Rham cohomology by the comparison isomorphism. ∎

Corollary 2.27. Fix a smooth proper curve C over a field K such that $C(K) \neq \varnothing$. Then $\dim \operatorname{Jac} C$ equals the genus of the curve C.

Proof. Again, this is easy to see in the complex analytic case from the explicit construction. In general, one can read off the dimension of an abelian variety A from $\dim \mathrm{H}^1(A)$ and then apply Proposition 2.26. ∎

2.1.3 The Dual

Even though we will technically not need it, we take a moment to discuss duality and polarizations of abelian varieties; we do want to understand these notions so that we can make sense of the Weil pairing. Motivated by the utility of the Picard group in defining the Jacobian, we make the following definition.

Definition 2.28 (dual abelian variety)**.** Fix an abelian variety A over a field K. Then we define the *dual abelian variety* $A^\vee$ as the group scheme $\operatorname{Pic}^\circ_{A/K}$ over K.

Remark 2.29. As usual, we will not check that $A^\vee$ is an abelian variety or even a scheme, but it is. (The ingredients that go into these arguments will not be relevant for us.) We refer to [EGM, Chapter 6] for these arguments, in addition to the useful fact that $\dim A = \dim A^\vee$.

Remark 2.30. It is worthwhile to note that, in the complex analytic situation, there already is a notion of a dual abelian variety. If $A = V/\Lambda$ is an abelian variety, then $A^\vee = V^*/\Lambda^*$, where V^* is the vector space of conjugation-semilinear functionals $V^* \to \mathbb{C}$, and Λ^* consists of the functionals which are integral on Λ. It is rather tricky to explain how this definition relates to the one above, so we will not do so and instead refer to [Ros86, Section 4].

It is worth our time to explain why this is called duality. To begin, there is a duality for morphisms.

Lemma 2.31. Fix a homomorphism $f\colon A \to B$ of abelian varieties over a field K. Then there is a dual homomorphism $f^\vee\colon B^\vee \to A^\vee$.

Proof. We define the homomorphism on geometric points. Then a point of $B^\vee(\overline{K})$ is a line bundle $\mathcal{L}$ on $B_{\overline{K}}$, which we can pull back to a line bundle $f^*\mathcal{L}$ on $A_{\overline{K}}$, which is a point of $A^\vee(\overline{K})$. ∎

Lemma 2.32. Fix an abelian variety A over a field K. Then there is a canonical isomorphism $A \to A^{\vee\vee}$.

Proof. We sketch the construction of the map and refer to [EGM, Theorem 7.9] for details. Because $A^\vee$ is a moduli space of line bundles, there is a universal Poincaré line bundle $\mathcal{P}_A$ on $A \times A^\vee$. Unravelling the definition of $A^\vee$, we see that morphisms $S \to A^\vee$ correspond to line bundles on $A \times S$. Turning this around, we thus see that we can view $\mathcal{P}_A$ as a family of line bundles on $A^\vee$ parameterized by A and thus providing a map $A \to A^{\vee\vee}$. This map is the required isomorphism. ■

Most of the utility one achieves from the dual is that it allows us to the complex-analytic notion of a polarization into algebraic geometry. As in Remark 2.30, we view $A = V/\Lambda$ as a complex torus, and the dual abelian variety $A^\vee$ can be realized concretely as some V^*/Λ^*. Now, a polarization of A refers to a polarization of $\Lambda = \mathrm{H}_1^{\mathrm{B}}(A, \mathbb{Z})$, which as mentioned in Example 2.5 has equivalent data to an alternating form $\psi\colon \Lambda \otimes \Lambda \to \mathbb{Z}$ such that the bilinear form

$$\langle x, y\rangle \coloneqq \psi_{\mathbb{R}}(x, iy)$$

on $\Lambda_{\mathbb{R}}$ is symmetric and positive-definite. But now we see that this choice of ψ determines a map $A \to A^\vee$ given by $v \mapsto \psi(v, \cdot)$.

Thus, we would like our polarizations some kind of map $A \to A^\vee$. However, we need to keep track of all the adjectives that ψ had in order to make this an honest definition. For example, perhaps we want to keep track of the constraint that ψ is alternating. To do so, we use cohomology. We will shortly explain in Proposition 2.88 that the cup product provides an isomorphism $\wedge^2 \mathrm{H}^1(A, \mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Z})$, which induces an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}\left(\wedge^2 \Lambda, \mathbb{Z}\right) \cong \mathrm{H}^2(A, \mathbb{Z})$$

upon taking duals. Thus, ψ being an alternating form can be traced backed to it coming from a class in $\mathrm{H}^2(A, \mathbb{Z})$.

Continuing, perhaps we want to keep track of the constaint that $\langle \cdot, \cdot \rangle$ is symmetric. This is equivalent to having $\psi_{\mathbb{R}}(ix, iy) = \psi(x, y)$, which turns out to be equivalent to $\psi_{\mathbb{C}} \in \mathrm{H}^2(A, \mathbb{C})$ living in the $(1,1)$ component. Well, it turns out that the exponential short exact sequence

$$0 \to \mathbb{Z} \overset{2\pi i}{\to} \mathcal{O}_A \overset{\exp}{\to} \mathcal{O}_A^\times \to 0$$

induces a (first Chern class) map $c_1 : \mathrm{H}^1(A, \mathcal{O}_A^\times) \to \mathrm{H}^2(A, \mathbb{Z})$, which is an isomorphism onto the $(1,1)$ component. Thus, the condition that $\langle \cdot, \cdot \rangle$ is symmetric can be traced back to $\psi_{\mathbb{C}}$ coming from a class in $\mathrm{H}^1(A, \mathcal{O}_A^\times)$, which has equivalent data to a line bundle $\mathcal{L}$.

Lastly, it turns out that positive-definiteness of $\langle \cdot, \cdot \rangle$ corresponds to the $\mathcal{L}$ being ample. On the other hand, given a line bundle $\mathcal{L}$ on A, we remark that there already is a natural way to construct a map $A \to A^\vee$ from a line bundle. This gives our definition.

Definition 2.33 (polariaztion)**.** Fix an abelian variety A over a field K. A *polarization* is a morphism $\varphi\colon A \to A^\vee$ such that there is an ample line bundle $\mathcal{L}$ on $A_{\overline{K}}$ giving the equality

$$\varphi(x) = t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

for any $x \in A_{\overline{K}}$. We say that φ is *principal* if and only if it is an isomorphism, and we say that A is a *pricipally polarized*.

Remark 2.34. It turns out that the construction of the above map does correspond to the map $A \to A^\vee$ defined complex-analytically.

Remark 2.35. It turns out that polarizations are isogenies.

Remark 2.36. Here is the sort of thing that one can do with this definition. One may also want to define a Rosati involution on $\mathrm{End}(A)_{\mathbb{Q}}$, analogous to the Rosati involution on polarized Hodge structures. Well, given a (principal) polarization $\varphi\colon A \to A^\vee$, we can define a Rosati involution $(\cdot)^\dagger$ on $\mathrm{End}(A)_{\mathbb{Q}}$ by sending any $f \in \mathrm{End}(A)_{\mathbb{Q}}$ to

$$f^\dagger \coloneqq \varphi^{-1} \circ f^\vee \circ \varphi.$$

If λ is a principal polarization, then this Rosati involution descends to $\mathrm{End}(A)$. One can check that $(\cdot)^\dagger$ continues to be a positive anti-involution, but it is not easy; see for example [EGM, Theorem 12.26]. This allows us to apply the Albert classification Theorem 1.28 to our situation.

Example 2.37. For any smooth proper curve C such that $C(K) \neq \varnothing$, it turns out that the Jacobian $\mathrm{Jac}\, C$ is principally polarized. It is not too hard to describe the line bundle which gives the polarization: let $\iota\colon C \to \mathrm{Jac}(C)$ be an embedding given be one of the points in $C(K)$, and then the line bundle is given by the divisor

$$\underbrace{f(C) + \cdots + f(C)}_{g-1},$$

where g is the genus of C. See [EGM, Theorem 14.23] or [Mil08, Theorem 6.6] for more details.

Analogous to the complex-analytic setting $A = V/\Lambda$, we may still want to be able to define an alternating form on $\Lambda = \mathrm{H}_1^{\mathrm{B}}(A, \mathbb{Z})$. We will achieve a satisfying version of this in Lemma 2.93, but for now, let us point that this is not immediately obvious how to do this because we currently have no analogue for Λ in the general setting. However, we note that the alternating form Λ is able to induce an alternating form on V, and we can access a dense subset of V by taking torsion. Thus, for now, we will aim to provide a pairing

$$A[n](K^{\mathrm{sep}}) \times A[n](K^{\mathrm{sep}}) \to \mathbb{Z}/n\mathbb{Z}$$

for each integer n such that $\mathrm{char}\, K \nmid n$. Unwinding how we took a polarization to a map $A \to A^\vee$, we note that we may as well define the above map using a polarization $\varphi\colon A \to A^\vee$ by instead defining a pairing

$$A[n](K^{\mathrm{sep}}) \times A^\vee[n](K^{\mathrm{sep}}) \to \mathbb{Z}/n\mathbb{Z}$$

and then pre-composing with $A \to A^\vee$. More generally, given an isogeny $f\colon A \to B$, we will be able to show that there is a perfect pairing

$$(\ker f) \times (\ker f^\vee) \to \mathbb{G}_m,$$

upon which we find the desired pairing by taking $f = [n]_A$ and taking K^{sep}-points.

Proposition 2.38 (Weil pairing)**.** Fix an isogeny $f\colon A \to B$ of abelian varieties over K. Then there is a perfect pairing

$$(\ker f) \times (\ker f^\vee) \to \mathbb{G}_m.$$

Proof. We provide an explicit construction of the pairing on K^{sep}-points, but we will not check that it is perfect, for which we refer to [Ton15, Theorem 8.1.3]. Select $x \in (\ker f)(K^{\mathrm{sep}})$ and $y^\vee \in (\ker f^\vee)(K^{\mathrm{sep}})$. The point $y^\vee$ corresponds to a line bundle $\mathcal{L}$ on $B^\vee_{K^{\mathrm{sep}}}$. Being in the kernel of f grants a trivialization $\sigma\colon f^*\mathcal{L} \to \mathcal{O}_{A_{K^{\mathrm{sep}}}}$, which is unique up to a scalar. Now, note that $t_a^* f^*\mathcal{L} = f^* t_{f(a)}^* \mathcal{L} = f^*\mathcal{L}$ because $a \in \ker f$, so there is another trivialization of $f^*\mathcal{L}$ given by $t_a^*\beta\colon \mathcal{L} \to \mathcal{O}_{A_{K^{\mathrm{sep}}}}$. We now define our Weil pairing as $t_a^*\beta \circ \beta^{-1}$, which we realize as an element of $\mathbb{G}_m(K^{\mathrm{sep}})$ by noting that $t_a^*\beta \circ \beta^{-1}$ is an automorphism of $\mathcal{O}_{A_{K^{\mathrm{sep}}}}$ and is therefore a scalar. ∎

Corollary 2.39. Fix an abelian variety A over a field K, and let $\varphi\colon A \to A^\vee$. For each positive integer n, there is a Galois-invariant perfect symplectic pairing

$$e_\varphi\colon A[n](K^{\mathrm{sep}}) \times A[n](K^{\mathrm{sep}}) \to \mu_n.$$

Furthermore, for any positive integer m, the following diagram commutes.

$$\begin{array}{ccc} A[nm](K^{\mathrm{sep}}) & \times & A[nm](K^{\mathrm{sep}}) \xrightarrow{e_\varphi} \mu_{mn} \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \quad \downarrow{\scriptstyle m} \\ A[n](K^{\mathrm{sep}}) & \times & A[n](K^{\mathrm{sep}}) \xrightarrow{e_\varphi} \mu_n \end{array}$$

Proof. We described above how to construct the pairing from the one given in Proposition 2.38 by setting $f = [n]_A$ and then using the polarization φ. The remaining properties of e_φ (such as Galois-invariance) can be checked using the explicit construction given in Proposition 2.38. ∎

2.1.4 Applying Hodge Theory

We now explain the utility of chapter 1 to our application. Here is the main result.

Theorem 2.40 (Riemann)**.** The functor $A \mapsto \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ provides an equivalence of categories between the isogeny category of abelian varieties defined over $\mathbb{C}$ and the category of polarizable $\mathbb{Q}$-Hodge structures V such that $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$.

Proof. Writing $A = \mathbb{C}^g/\Lambda$ for a polarizable lattice Λ, we see that the given functor takes A to $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. It is thus not hard to see that this functor is fully faithful. To see that it is essentially surjective, we begin with any polarizable $\mathbb{Q}$-Hodge structure V and find a polarizable sublattice Λ in order to produce the desired abelian variety A/Λ. Admittedly, most of the work for this theorem was already done in Example 1.20 when we showed that the previous sentence actually gives an abelian variety! ∎

The moral of the story is that we can keep track of abelian varieties A over $\mathbb{C}$ by only keeping track of their Hodge structures $\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. With this in mind, we allow ourselves the following notation.

Notation 2.41. Fix an abelian variety A over $\mathbb{C}$. Then we define the *Mumford–Tate group* of A to be

$$\mathrm{MT}(A) \coloneqq \mathrm{MT}\left(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})\right).$$

We define $\mathrm{Hg}(A)$ and $\mathrm{L}(A)$ similarly.

Here is the main corollary of Theorem 2.40 that we will want.

Corollary 2.42. Fix an abelian variety A over $\mathbb{C}$. Then the natural map

$$\mathrm{End}_{\mathbb{C}}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}_{\mathbb{Q}}\left(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})\right)^{\mathrm{MT}(A)}$$

is an isomorphism.

Proof. By Lemma 1.54, we see that the right-hand side is simply $\mathrm{End}_{\mathrm{HS}}\left(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})\right)$. The result now follows from Theorem 2.40. ∎

As another aside, we go ahead and restate the Albert classification (Theorem 1.28) for our abelian varieties.

Proposition 2.43. Fix a simple abelian variety A of dimension g, defined over a field K of characteristic 0, and set $D \coloneqq \mathrm{End}_K(A)_{\mathbb{Q}}$ and $F \coloneqq Z(D)$. Letting $(\cdot)^\dagger$ be the Rosati involution on D, we also let $F^\dagger$ be the $(\cdot)^\dagger$-invariants of F. Further, set $d \coloneqq \sqrt{[D:F]}$ and $e \coloneqq [F:\mathbb{Q}]$ and $e_0 \coloneqq [F^\dagger:\mathbb{Q}]$. Then we have the following table of restrictions on (g, d, e, e_0).

Type	e	d	Restriction
I	e_0	1	$e \mid g$
II	e_0	2	$2e \mid g$
III	e_0	2	$2e \mid g$
IV	$2e_0$	d	$e_0 d^2 \mid g$

Proof. Recall that D is amenable to the Albert classification as discussed in Remark 2.36. The middle two columns follow from the discussion of the various types; for example, in Type I, we see $d = 1$ because $D = F$, and $e = e_0$ because F is totally real. To receive the dimension restrictions, we note that some descent argument allows us to reduce to the case where $K = \mathbb{C}$, where we receive an inclusion $D \subseteq \mathrm{End}(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q}))$ by Theorem 2.40.[1] This is an inclusion of division $\mathbb{Q}$-algebras, so we see that $\dim_{\mathbb{Q}} D \mid 2g$; this implies

$$d^2 e \mid 2g,$$

which rearranges into the required restrictions. ∎

Remark 2.44. The requirement that $\mathrm{char}\, F = 0$ is necessary in the table; the restrictions are somewhat different (and weaker!) in positive characteristic.

While we're here, we state the main theorem of [Del18] on absolutely Hodge cycles.

Theorem 2.45 (Deligne)**.** Fix an abelian vareity A defined over a number field K. Then all Hodge classes on A are absolutely Hodge.

2.1.5 Complex Multiplication

Even though it is not strictly necessary for our exposition, we take a moment to discuss some theory surrounding complex multiplication. We refer to [Mil20b] throughout for more details. The relevance of this discussion to us mostly arises because we have defined analogous notions in sections 2.2.2 and 2.2.3.

Intuitively, complex multiplication simply means that an abelian variety has many endomorphisms. To explain this properly, we note that the endomorphism algebra of a simple abelian variety A is a division $\mathbb{Q}$-algebra described in Proposition 2.43; if we drop the assumption that A is simple, then it could be a product of matrix algebras of such division $\mathbb{Q}$-algebras. This motivates the following definition to properly account for such matrix algebras.

Definition 2.46 (reduced degree)**.** Write a semisimple algebra D over a field K as a product $D_1 \times \cdots \times D_k$ of simple algebras. Then we define the *reduced degree* as

$$[D:K]_{\mathrm{red}} \coloneqq \sum_{i=1}^{k} \sqrt{[D_i : F_i]} \cdot [D_i : K],$$

where $F_i \coloneqq Z(B_i)$ for each i

[1] It is still possible to get an inclusion like this in general. It requires a discussion of the ℓ-adic representations, which we engage in later.

Remark 2.47. It is not technically obvious that $[D_i : F_i]$ is a square, but this follows from the theory of central simple algebras. Roughly speaking, one can show that $D_i \otimes \overline{D_i} \cong M_n(\overline{D_i})$ for some $n \geq 0$, from which the result follows; see [Mil20a, Corollary IV.2.16].

Remark 2.48. Given an inclusion $B \subseteq \mathrm{End}_K(V)$, one receives a bound

$$[B : K]_{\mathrm{red}} \leq [V : K].$$

Roughly speaking, this follows by breaking up B into simple pieces (which are matrix algebras of division algebras) and then looking for these pieces in $\mathrm{End}_K(V)$. See [Mil20b, Proposition I.1.2]

In light of the previous remark, we are now able to make a definition.

Definition 2.49 (complex multiplication)**.** Fix an abelian variety A over a field K. Then A has *complex multiplication over* K if and only if

$$[\mathrm{End}_K(A)_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}} = 2 \dim A.$$

Namely, we see that $2 \dim A$ is as large as possible by Remark 2.48, by taking V to be H^1 for some Weil cohomology H.[2]

Remark 2.50. The key benefit of the reduced degree is that it is additive: given abelian varieties A and A', we claim

$$[\mathrm{End}(A \oplus A')_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}} \stackrel{?}{=} [\mathrm{End}(A)_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}} + [\mathrm{End}(A')_{\mathbb{Q}} : \mathbb{Q}]_{\mathrm{red}}.$$

Indeed, by breaking everything into simple pieces, we may assume that A and A' are both powers of a simple abelian variety. If they are powers of different simple abelian varieties, then this is a direct computation. Otherwise, they are powers of the same simple abelian variety, in which case all central simple algebras in sight are matrix algebras over the same division algebra, and the result follows by another computation.

Remark 2.51. A computation with Proposition 2.43 shows that a simple abelian variety A has complex multiplication only in Type IV when $d = 1$; i.e., we require $\mathrm{End}_K(A)$ to be a CM field. Combining this with Remark 2.50, we find that an abelian variety A has complex multiplication if and only if each of its factors does.

Remark 2.52. If an abelian variety A with complex multiplication is a sum of non-isomorphic simple abelian varieties, then its endomorphism algebra is simply a product of CM fields. In general, one can show that it is still the case that any abelian variety A with complex multiplication has a CM algebra of degree $2 \dim A$ contained in its endomorphism algebra. However, this requires a little structure theory of semisimple algebras; see [Mil20b, Proposition 3.6].

Complex multiplication places strong constraints on the Mumford–Tate group.

Proposition 2.53. Fix an abelian variety A over $\mathbb{C}$. Then A has complex multiplication if and only if $\mathrm{MT}(A)$ is a torus.

Proof. We show the two implications separately.

[2] Outside the complex-analytic case, it may look like one wants to use the ℓ-adic result Theorem 2.104 over a general field. However, it turns out to be enough to merely achieve the injectivity of the map Theorem 2.104, which is easier.

- In one direction, if A has complex multiplication, then Remark 2.52 grants a CM algebra $E \subseteq \mathrm{End}_{\mathbb{C}}(A)_{\mathbb{Q}}$ with $[E:\mathbb{Q}] = 2\dim A$. Then $\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$ is a free module over E of rank 1, so we see that $\mathrm{GL}_F\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)$ is isomorphic to T_F. We conclude by Lemma 1.45.
- In the other direction, suppose $\mathrm{MT}(A)$ is a torus. Find a maximal torus T containing $\mathrm{MT}(A)$. Then Corollary 2.42 tells us that
$$\mathrm{End}_{\mathbb{C}}(A)_{\mathbb{Q}} = \mathrm{End}_{\mathbb{Q}}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)^{\mathrm{MT}(A)},$$
which then contains $\mathrm{End}_{\mathbb{Q}}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)^{T}$. However, the latter is a commutative semisimple $\mathbb{Q}$-algebra of dimension $2g$: it suffices to check this after base-changing to $\mathbb{C}$, whereupon we may identify T with the diagonal torus, from which the claim follows. This completes the proof. ■

One benefit of complex multiplication is that it lets move difficult geometric questions into combinatorial ones. To see this, we need to define the following combinatorial gadget.

Definition 2.54. Fix an abelian variety A with complex multiplication defined over $\mathbb{C}$, and set $V := \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$. Choose a CM algebra $E \subseteq \mathrm{End}_{\mathbb{C}}(A)_{\mathbb{Q}}$ with $\dim E = 2\dim A$. Then we define the *CM type* $\Phi\colon \Sigma_E \to \mathbb{Z}_{\geq 0}$ of A to be the CM signature (E,Φ) given by
$$V^{1,0} \cong \bigoplus_{\sigma\in\Sigma_F} \mathbb{C}_\sigma^{\Phi(\sigma)}.$$
Note that $\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$ is then a one-dimensional E-vector space, so $\mathrm{im}\,\Phi \subseteq \{0,1\}$, so we can realize Φ as a subset of $\mathrm{Hom}(E,\mathbb{C})$.

Remark 2.55. Note that we are not requiring $E = Z(\mathrm{End}_{\mathbb{C}}(A)_{\mathbb{Q}})$, though this is automatically the case when the simple components of A all have multiplicity 1. Of course, there still is a CM signature coming from the case $E = Z(\mathrm{End}_{\mathbb{C}}(A)_{\mathbb{Q}})$.

Remark 2.56. There is a still a way to recover the CM type even when A is not defined over $\mathbb{C}$. For example, one can note that H^{10} is supposed to be the Lie algebra $\mathrm{Lie}\,A$, so one can instead recover Φ from the E-action on $\mathrm{Lie}\,A$.

Remark 2.57. One can read the simplicity of A off of the CM type (E,Φ). To begin, one needs E to be a field for A to be simple. Now that E is a field, we know that $A \sim B^r$ where B is an abelian variety with complx multiplication; say that it has CM type (E',Φ'). Then the Hodge structure on A is determined by the Hodge structure on B. Tracking this through as in [Lan11, Theorem 3.6] shows that A is simple if and only if any Galois extension $L/\mathbb{Q}$ of E has that
$$\{\sigma \in \mathrm{Gal}(L/\mathbb{Q}) : \Phi\sigma = \Phi\} = \mathrm{Gal}(L/E),$$
where Φ is being suitably thought of as an element of $\mathbb{Z}[\mathrm{Hom}(E,L)]$.

Remark 2.58. It turns out that there is (essentially) exactly one abelian variety with CM type (E,Φ), up to isogeny over the algebraic closure. See [Mil20b, Proposition 3.12].

Remark 2.58 tells us that we are basically allowed to only pay attention to the CM type in the theory of complex multiplication.

2.2 The Center of MT

In this section, we begin with a computational tool to compute $\mathrm{MT}(A)$ for an abelian variety A. This discussion is somewhat involved, so we will spend a full section here.

Let's begin with some motivation. Fix an abelian variety A. In the application of this thesis, we will use Lemma 1.62 to compute $\mathrm{Hg}(A)^{\mathrm{der}}$: note $\mathrm{Hg}(A)^{\mathrm{der}}$ is semisimple and hence its Lie algebra can be written as the sum of simple Lie algebras which may be amenable to the lemma. Because $\mathrm{Hg}(A)$ is reductive by Lemma 1.44, it remains to compute the center $Z(\mathrm{Hg}(A))$; recall $\mathrm{Hg}(A)$ is connected by Remark 1.40, so we may as well compute the connected component $Z(\mathrm{Hg}(A))^\circ$. As usual, the same discussion holds for $\mathrm{MT}(A)$, but we note that $Z(\mathrm{MT}(A))^\circ$ tends to be nontrivial because usually $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{MT}(A)$ by Example 1.31.

In Proposition 2.67, we find that $Z(\mathrm{Hg}(A))^\circ$ is trivial unless A has irreducible factors of Type IV in the sense of the Albert classification (Theorem 1.28). As such, we spend the rest of the section focusing on computations in Type IV. Computations are well-understood when V comes from an abelian variety with complex multiplication, so the main contribution here is that these arguments generalize with only slight modifications.

2.2.1 General Comments

In this subsection, we will mostly work with general polarizable Hodge structures V.

Lemma 2.59. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight, and set $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ with $F \coloneqq Z(D)$. Viewing D as a $\mathbb{Q}$-group, we have

$$Z(\mathrm{Hg}(V)) \subseteq \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F},$$

where $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ embeds into $\mathrm{GL}(V)$ via the D-action on V.

Proof. Here, F is a product of number fields because it is a commutative semisimple $\mathbb{Q}$-algebra. Recall from Lemma 1.54 that

$$D = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{Hg}(V)},$$

which upgrades to an equality of algebraic subgroups of $\mathrm{End}_{\mathbb{Q}}(V)$ because $\mathbb{Q}$-points are dense in these algebraic groups by combining [Mil17, Corollary 17.92] and [Mil17, Definition 12.59]. In particular, we see that $\mathrm{Hg}(V)$ commutes with $D^\times$, so the double centralizer theorem enforces $Z(\mathrm{Hg}(V)) \subseteq D^\times$ even as algebraic groups. However, $Z(\mathrm{Hg}(V))$ now commutes fully with $D^\times$, so in fact $Z(\mathrm{Hg}(V)) \subseteq Z(D)^\times$, which is what we wanted. ∎

Remark 2.60. One also has $Z(\mathrm{MT}(V)) \subseteq \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ because $\mathrm{MT}(V) \subseteq \mathbb{G}_{m,\mathbb{Q}} \mathrm{Hg}(V)$ by Lemma 1.41, and the scalars $\mathbb{G}_{m,\mathbb{Q}}$ already live in $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$.

Lemma 2.59 is that it places the center $Z(\mathrm{Hg}(V))$ in an explicit torus $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$. Subgroups of tori are well-understood by (co)character groups, so this puts us in good shape. This torus will be important enough to have its own notation.

Notation 2.61. Fix a commutative semisimple $\mathbb{Q}$-algebra F (i.e., a product of number fields). Then we define the torus

$$\mathrm{T}_F \coloneqq \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}.$$

Remark 2.62. Writing F as a product of number fields $F_1 \times \cdots \times F_k$, we find

$$\mathrm{T}_F = \mathrm{T}_{F_1} \times \cdots \times \mathrm{T}_{F_k}$$

because $F = F_1 \times \cdots \times F_k$ is an equality of $\mathbb{Q}$-algebras.

Remark 2.63. Let's compute the character group $\mathrm{X}^*(\mathrm{T}_F)$. By Remark 2.62, it's enough to do this computation when F is a field. The choice of a primitive element $\alpha \in F$ with minimal monic polynomial $f(x)$ yields an isomorphism $F \cong \mathbb{Q}[x]/(f(x))$. Upon base-changing to $\overline{\mathbb{Q}}$, we get a ring isomorphism

$$F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \prod_{i=1}^{n} \frac{\overline{\mathbb{Q}}[x]}{(x-\alpha_i)},$$

where $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are the roots of $f(x)$ in $\overline{\mathbb{Q}}$. Each root α_i provides a unique embedding $F \hookrightarrow \overline{\mathbb{Q}}$, so we see that $(\mathrm{T}_F)_{\overline{\mathbb{Q}}} \cong \mathbb{G}_{m,\overline{\mathbb{Q}}}^n$, where the n maps $(\mathrm{T}_F)_{\overline{\mathbb{Q}}} \to \mathbb{G}_{m,\overline{\mathbb{Q}}}$ are given by the embedding $\sigma_i \colon F \hookrightarrow \overline{\mathbb{Q}}$ defined by $\sigma_i(\alpha) \coloneqq \alpha_i$. In total, we find that $\mathrm{X}^*(\mathrm{T}_F)$ is a free $\mathbb{Z}$-module spanned by the embeddings $\Sigma_F \coloneqq \{\sigma_1, \dots, \sigma_n\}$, and it has the natural Galois action. Dually, $\mathrm{X}_*(\mathrm{T}_F)$ has the dual basis $\Sigma_F^\vee = \{\sigma_1^\vee, \dots, \sigma_n^\vee\}$.

In the light of the above remark, we will want the following notation.

Notation 2.64. Given a number field F, we let Σ_F denote the collection of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$. Given a product of number fields $F \coloneqq F_1 \times \cdots \times F_k$, we define $\Sigma_F \coloneqq \Sigma_{F_1} \sqcup \cdots \sqcup \Sigma_{F_k}$.

The point of the above notation is that $\mathrm{X}^*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F]$ as Galois modules.

It is possible to upgrade Lemma 2.59 in the presence of a polarization.

Lemma 2.65. Fix a polarizable $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight, and set $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ with $F \coloneqq Z(D)$. Then

$$Z(\mathrm{Hg}(V)) \subseteq \left\{ g \in \mathrm{T}_F : gg^\dagger = 1 \right\},$$

where $(\cdot)^\dagger$ is the Rosati involution.

Proof. As usual, everything in sight upgrades to algebraic groups. Let φ be a polarization. Fix some $g \in \mathrm{Hg}(V)$; note that Lemma 2.59 implies $g \in \mathrm{T}_F$, so it makes sense to write down $g^\dagger$.

Now, by the non-degeneracy of φ, it is enough to show that

$$\varphi\left(gg^\dagger v \otimes w\right) \stackrel{?}{=} \varphi(v \otimes w)$$

for any $v, w \in V$. Well, the definition of $(\cdot)^\dagger$ tells us that the left-hand side equals $\varphi\left(g^\dagger v \otimes g^\dagger w\right)$, which equals $\varphi(v \otimes w)$ because $\mathrm{Hg}(V) \subseteq \mathrm{Sp}(\varphi)$ by Remark 1.53. ∎

Once again, this torus is important enough to earn its own notation.

Notation 2.66. Fix a commutative semisimple $\mathbb{Q}$-algebra F with involution $(\cdot)^\dagger$. Then we define the torus

$$\mathrm{U}_F \coloneqq \left\{ g \in \mathrm{T}_F : xx^\dagger = 1 \right\}.$$

Here is an application of Lemma 2.65.

Proposition 2.67. Fix polarizable $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight. Suppose that V has no irreducible Hodge substructures with endomorphism algebra of Type IV in the sense of the Albert classification (Theorem 1.28). Then $Z(\mathrm{Hg}(V))$ is finite, and $\mathrm{Hg}(V)$ is semisimple.

Proof. Quickly, recall from Lemma 1.44 that $\mathrm{Hg}(V)$ is reductive, so the finitness of $Z(\mathrm{Hg}(V))$ implies that $Z(\mathrm{Hg}(V))^\circ = 1$ and thus $\mathrm{Hg}(V) = \mathrm{Hg}(V)^{\mathrm{der}}$, making $\mathrm{Hg}(V)$ is semisimple. (See also [Mil17, Proposition 19.10].) As such, we will focus on the first claim.

Set $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ with $F \coloneqq Z(D)$ so that $\mathrm{Hg}(V) \subseteq \mathrm{U}_F$ by Lemma 2.65. It is therefore enough to check that U_F is finite. Well, F is a product of number fields, and upon comparing with Theorem 1.28, we see that avoiding Type IV implies that F is a product of totally real fields. Totally real fields have only two units, so finiteness of U_F follows. ■

Thus, we see that we have pretty good control outside of Type IV factors, so we will spend the rest of this section on Type IV. For some applications outside Type IV, see (for example) [Lom16].

2.2.2 Type IV: The Signature

The arguments in the next two subsections are motivated by the computation of [Yu15, Lemma 4.2] and [Yan94, Proposition 1.1]. For this subsection, A is an abelian variety over $\mathbb{C}$ whose irreducible factors are of Type IV in the sense of the Albert classification (Theorem 1.28). Note that $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ is a Hodge structure concentrated in $V^{0,1}$ and $V^{1,0}$, so we do so.

By assumption, we know that $D \coloneqq \mathrm{End}_{\mathrm{HS}}(V)$ is a division algebra over its center $F \coloneqq Z(D)$, where F is a CM algebra (i.e., a product of CM fields), and the Rosati involution $(\cdot)^\dagger$ restricts to complex conjugation on F. In particular, $F^\dagger$ is the product of the maximal totally real subfields of F.

The basic approach of this subsection is that Lemma 2.59 requires $Z(\mathrm{Hg}(A))^\circ \subseteq \mathrm{T}_F$, and one can compute subtori using the machinery of (co)character groups. In particular, we recall that $\mathrm{X}^*(\Sigma_F) = \mathbb{Z}[\Sigma_F]$ and $\mathrm{X}_*(\Sigma_F) = \mathbb{Z}[\Sigma_F^\vee]$ as Galois modules. We will need a way to work directly with the Hodge structure on V. It will be described by the following piece of combinatorial data. Recall that a CM algebra is a product of CM fields.

Definition 2.68 (signature)**.** Fix a CM algebra F, and recall that Σ_F is the set of homomorphisms $F \hookrightarrow \overline{\mathbb{Q}}$. Then a *signature* is a function $\Phi\colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ such that the sum

$$\Phi(\sigma) + \Phi(\overline{\sigma})$$

is constant with respect to $\sigma \in \Sigma_F$; here, $\overline{\sigma}$ denotes the complex conjugate embedding to σ. We may call the pair (F, Φ) a *CM signature*.

Remark 2.69. One can also view Φ as an element of $\mathbb{Z}[\Sigma_F]$ as

$$\Phi \coloneqq \sum_{\sigma \in \Phi} \Phi(\sigma)\sigma.$$

Remark 2.70. The case that $\Phi(\sigma) + \Phi(\overline{\sigma})$ always equals 1 corresponds to Φ being a CM type.

Remark 2.71. If we expand F as a product of CM fields $F = F_1 \times \cdots \times F_k$, then $\Sigma_F = \Sigma_{F_1} \sqcup \cdots \sqcup \Sigma_{F_k}$. Thus, we see that a signature of F has only a little more data than a signature on each of the $\Sigma_{F_\bullet}$s individually; in particular, one should make sure that $\Phi(\sigma) + \Phi(\overline{\sigma})$ remains equal across the different fields.

The idea is that we can keep track of a signature with a Hodge structure.

Lemma 2.72. Fix an abelian variety A over $\mathbb{C}$ such that $\mathrm{End}(A)$ contains a CM algebra F, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. Then the function $\Phi\colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ defined by

$$V^{1,0} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_\sigma^{\Phi(\sigma)}$$

is a signature, which we will call the induced signature. This is an isomorphism of F-representations, where $\mathbb{C}_\sigma$ is a complex F-representation via the embedding σ.

Proof. In short, the condition that $\Phi(\sigma) + \Phi(\overline{\sigma})$ is constant comes from the condition $V^{0,1} = \overline{V^{1,0}}$. To see this, note that V is a free module over F, so $V_{\mathbb{C}}$ is a free module over $F \otimes \mathbb{C}$ of finite rank. As such, we may set $d \coloneqq [V : F]$ so that $V \cong F^d$ as F-representations, and then we find

$$V_{\mathbb{C}} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_\sigma^d.$$

Now, $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, and because F acts by endomorphisms of Hodge structures, we get a well-defined action of F on $V^{1,0}$ and $V^{0,1}$ individually. In particular, the definition of Φ also grants

$$V^{0,1} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_\sigma^{d-\Phi(\sigma)}$$

as F-representations, so

$$\overline{V^{0,1}} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_{\overline{\sigma}}^{d-\Phi(\sigma)}$$

To complete the proof, we note that $V^{0,1} = \overline{V^{1,0}}$ continues to be true as F-representations, so we must have $\Phi(\sigma) = d - \Phi(\overline{\sigma})$ for all σ. The result follows. ∎

Of course, we cannot expect this signature Φ to remember everything about the Hodge structure. For example, if $\mathrm{End}(A)$ contains a larger CM algebra F' than F, then the signature induced by F' knows more about the Hodge structure than the one induced by F. However, in "generic cases," this signature is expected to suffice. For our purposes, we will take generic to mean that there are no more endomorphisms than the ones promised by F; i.e., this explains why we will assume $Z(\mathrm{End}(A)) = F$ in the sequel.

We now relate our signature to cocharacters of $Z(\mathrm{Hg}(A))^\circ$. For this, it will be helpful to realize $Z(\mathrm{Hg}(A))$ as some kind of monodromy group. The trick is to consider the determinant.

Lemma 2.73. Fix an abelian variety A over $\mathbb{C}$ such that $Z(\mathrm{End}(A))$ equals an algebra F, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. Then $Z(\mathrm{Hg}(A))^\circ$ equals the largest algebraic $\mathbb{Q}$-subgroup of T_F containing the image of $(\det_F \circ h) \colon \mathbb{U} \to (\mathrm{T}_F)_{\mathbb{R}}$.

Proof. The point is that taking the determinant will kill $\mathrm{Hg}(A)^{\mathrm{der}}$ because $\mathrm{Hg}(A) \subseteq \mathrm{GL}_F(V)$. There are two inclusions we must show.

- We show that $Z(\mathrm{Hg}(A))^\circ$ contains the image of $(\det F \circ h|_{\mathbb{U}})$. Well, $\mathrm{Hg}(A)$ contains the image of $h|_{\mathbb{U}}$, so it is enough to show that $Z(\mathrm{Hg}(A))^\circ$ contains the image of $\det_F \colon \mathrm{Hg}(A) \to \mathrm{T}_F$. For this, we recall that $\mathrm{Hg}(A)$ is connected (by Remark 1.40), so

  $$\mathrm{Hg}(A) = Z(\mathrm{Hg}(A))^\circ \, \mathrm{Hg}(A)^{\mathrm{der}}.$$

  Note that $\det_F$ is simply $(\cdot)^{\dim_F V}$ on the torus $Z(\mathrm{Hg}(V))^\circ$, so that piece surjects onto $Z(\mathrm{Hg}(A))^\circ$! Thus, it is enough to check that $\det_F \colon \mathrm{Hg}(A)^{\mathrm{der}} \to \mathrm{T}_F$ is trivial, which is true by the definition of the derived subgroup upon noting that $\det_F$ is a homomorphism with abelian target.
- Suppose that $T \subseteq \mathrm{T}_F$ contains the image of $(\det_F \circ h|_{\mathbb{U}})$. Then we claim that T contains $Z(\mathrm{Hg}(A))^\circ$. Let $H \subseteq \mathrm{GL}_F(V)$ be the pre-image of T under $\det_F \colon \mathrm{GL}_F(A) \to \mathrm{T}_F$. Then H must contain the image of $h|_{\mathbb{U}}$, so it contains $\mathrm{Hg}(A)$ by defintion. In particular, H contains $Z(\mathrm{Hg}(A))^\circ$! Now, T contains $\det_F(H)$, so T contains $\det_F(Z(\mathrm{Hg}(A))^\circ)$, but the previous point check remarked that this simply equals $Z(\mathrm{Hg}(A))^\circ$, so we are done. ∎

Proposition 2.74. Fix an abelian variety A over $\mathbb{C}$ such that $Z(\mathrm{End}(A))$ equals a CM algebra F, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. Let $\Phi \colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the induced signature. Then the induced representation $(\det_F \circ h) \colon \mathbb{U} \to (\mathrm{T}_F)_{\mathbb{R}}$ sends the generator of $\mathrm{X}_*(\mathbb{U})$ to

$$-\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee.$$

Proof. This boils down to computing the map $\det_F \circ h|_{\mathbb{U}}$. We proceed in steps.

1. To set ourselves up, recall that
$$\mathbb{U}_{\mathbb{C}} = \{(z, 1/z) : z \in \mathbb{G}_{m,\mathbb{C}}\},$$
so one has an isomorphism cocharacter $z^\vee \colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{U}_{\mathbb{C}}$ given by $z^\vee \mapsto z \mapsto (z, 1/z)$. Thus, we have left to show that
$$\det_F \circ h_{\mathbb{C}} \circ z^\vee \stackrel{?}{=} -\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee.$$
We may check this equality on geometric points.

2. We describe the map $h_{\mathbb{C}} \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V)_{\mathbb{C}}$. By definition, $h(z, w) \in \mathrm{GL}(V)$ acts by z^{-1} on $V^{1,0}$ and by w^{-1} on $V^{0,1}$. Thus, the definition of Φ grants that $h(z, w)$ diagonalizes. To be more explicit, for each $\sigma \in \Sigma_F$, we define $V_\sigma^{p,q}$ to be the σ-eigenspace for the F-action on $V^{p,q} \subseteq V_{\mathbb{C}}$. Then we see that $h(z, w)$ acts on $V_\sigma^{1,0}$ by the scalar z^{-1} and on $V^{0,1}$ by the scalar w^{-1}.

3. We describe the map $(\det_F \circ h_{\mathbb{C}}) \colon \mathbb{S}_{\mathbb{C}} \to (\mathrm{T}_F)_{\mathbb{C}}$. Realizing geometric points in $(\mathrm{T}_F)_{\mathbb{C}}$ as tuples in $\mathbb{C}^{\Sigma_F}$, we see that $\det_F$ simply takes the determinant of the matrix $h_{\mathbb{C}}(z, w)|_{V_\sigma}$ to the σ-component in $(\mathrm{T}_F)_{\mathbb{C}}$. (One finds this by tracking through the definition of $\det_F$ as a morphism of algebraic groups.) As such, we see that
$$\det h_{\mathbb{C}}(z, w)|_{V_\sigma} = z^{-\Phi(\sigma)} w^{-\Phi(\overline{\sigma})}$$
because Φ is a signature.

4. We complete the proof. The previous step shows that $(\det_F \circ h_{\mathbb{C}} \circ z^\vee)(z)$ goes to the element
$$\left(z^{-\Phi(\sigma)+\Phi(\overline{\sigma})}\right)_{\sigma \in \Sigma(F)} \in \mathbb{C}^{\Sigma_F}.$$
This completes the proof upon noting that the cocharacter $\sigma^\vee \colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{T}_F$ simply maps into the σ-component of $\mathbb{C}^{\Sigma_F}$ on geometric points. ∎

Remark 2.75. Notably, the given element sums to 0, which corresponds to the fact that $\mathrm{Hg}(A) \subseteq \mathrm{SL}(V)$ as seen in Lemma 1.41. Indeed, by diagonalizing the F-action on V, we see that $(\mathrm{T}_F \cap \mathrm{SL}(V))^\circ$ consists of the $g \in \mathrm{T}_F$ such that the product of the elements in g equals 1.

Proposition 2.74 easily translates into a computation of the cocharacter group $\mathrm{X}_*(\mathrm{Hg}(A))^\circ$. In the next few results, saturated simply means that the induced quotient is torsion-free.

Corollary 2.76. Fix an abelian variety A over $\mathbb{C}$ such that $Z(\mathrm{End}(A))$ equals a CM algebra F, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. Let $\Phi \colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the induced signature. Then $Z(\mathrm{Hg}(A))^\circ \subseteq \mathrm{T}_F$ has cocharacter group equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$ containing
$$\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee.$$

Proof. This is immediate from combining Lemma 2.73 and Proposition 2.74 with the equivalence of categories X_* between algebraic tori and Galois modules. See [Mil17, Theorem 12.23] for the proof that X^* is an equivalence, which is similar. ∎

Corollary 2.77. Fix an abelian variety A over $\mathbb{C}$ such that $Z(\mathrm{End}(A))$ equals a CM algebra F, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. Let $\Phi \colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the signature defined in Lemma 2.72. Then $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$ has cocharacter group equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$ containing
$$\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee.$$

Proof. This follows from Corollary 2.76. By Lemma 1.41, it is enough to add in the cocharacter given by the scalars $\mathbb{G}_{m,\mathbb{Q}} \to \mathrm{T}_F$, which is $\sum_{\sigma\in\Sigma_F} \sigma^\vee$. Thus, the fact that Φ is a signature implies that

$$\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee$$

certainly lives in $\mathrm{X}_*(\mathrm{MT}(A)) \subseteq \mathrm{X}_*(\mathrm{T}_F)$.

Conversely, if X is some saturated Galois submodule containing $\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee$, then we would like to show that $\mathrm{X}_*(\mathrm{MT}(A)) \subseteq X$. Well, X is a Galois submodule, so it must contain the complex conjugate element $\sum_{\sigma\in\Sigma_F} \Phi(\overline{\sigma})\sigma^\vee$. On one hand, this then sums with the given element to produce

$$\sum_{\sigma\in\Sigma_F} \sigma^\vee \in X$$

because X is saturated. On the other hand, we can take a difference to see that

$$\sum_{\sigma\in\Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee \in X.$$

We conclude that X contains the cocharacter of the scalars $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{T}_F$ and the cocharacter lattice of $Z(\mathrm{Hg}(A))^\circ \subseteq \mathrm{T}_F$, so we conclude that X must also contain the cocharacter lattice of $Z(\mathrm{MT}(A))^\circ$. ∎

Remark 2.78. One can also prove the above corollary by following the proof of Corollary 2.76. For example, this approach provides a monodromy interpretation of $Z(\mathrm{MT}(A))^\circ$ analogous to Lemma 2.73. Here, one replaces the generator of $\mathrm{X}_*(\mathbb{U})$ with the cocharacter $\mu \in \mathrm{X}_*(\mathbb{S})$, and one finds that $\det_F \circ h_{\mathbb{C}}$ sends μ to $\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee$. One is then able to prove statements analogous to Proposition 2.74 and Corollary 2.76.

Let's pause for a moment with an explanation of how one can use Corollary 2.77 to compute $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$. The approach for $Z(\mathrm{Hg}(A))^\circ$ is similar but only a little more complicated.

We will only compute over a Galois extension $L/\mathbb{Q}$ containing all factors of F. In this case, the F-action on V_L diagonalizes, so one can identify $(\mathrm{T}_F)_L \subseteq \mathrm{GL}(V)_L$ as the diagonal torus for some basis of V_L. In particular, for each $\sigma \in \Sigma_F$, the cocharacter $\sigma^\vee$ corresponds to one of the standard cocharacters for the diagonal torus of $\mathrm{GL}(V)_L$. Now, Corollary 2.77 tells us that $\mathrm{X}_*(Z(\mathrm{MT}(A))^\circ) \subseteq \mathrm{X}_*(\mathrm{T}_F)$ equals the saturation of the sublattice spanned by the vectors

$$g\left(\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee\right) = \sum_{\sigma\in\Sigma_F} \Phi(\sigma)(g\sigma)^\vee,$$

where g varies over $\mathrm{Gal}(L/F)$. By computing a basis of the saturation of this sublattice, we get a family of 1-parameter subgroups of the diagonal torus of $\mathrm{GL}(V)_L$ which together generate $Z(\mathrm{MT}(A))^\circ$. This more or less computes $Z(\mathrm{MT}(A))^\circ$.

2.2.3 Type IV: The Reflex

In the sequel, we will be most interested in equations cutting out $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$. One could imagine proceeding as above to compute $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$ via 1-parameter subgroups and then afterwards finding the desired equations. This is somewhat computationally intensive, so instead we will turn our attention to computing character groups. As in [Yu15, Lemma 4.2], this will require a discussion of the reflex.

Definition 2.79 (reflex signature)**.** Fix CM fields F and F^* and CM signatures (F,Φ) and (F^*,Φ^*). We say that these CM signatures are *reflex* if and only if there is a Galois extension $L/\mathbb{Q}$ containing F and F^* such that each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ has

$$\Phi(\sigma|_F) = \Phi^*\left(\sigma^{-1}|_{F^*}\right).$$

In this situation, we may call (F^*,Φ^*) a *reflex signature* for (F,Φ).

Remark 2.80. We check that (F, Φ) and (F^*, Φ^*) does not depend on the choice of Galois extension L. Indeed, suppose that we have another Galois extension $L'/\mathbb{Q}$ containing F and F^*; let L'' be a Galois extension containing both L and L'. By symmetry, it is enough to check that (F, Φ) are reflex with respect to L if and only if they are reflex with respect to L''. Well, for any $\sigma \in \mathrm{Gal}(L''/\mathbb{Q})$, we see that $\Phi(\sigma|_F) = \Phi^*\left(\sigma^{-1}|_{F^*}\right)$ is equivalent to $\sigma|_L \in \mathrm{Gal}(L/\mathbb{Q})$ satisfying $\Phi(\sigma|_L|_F) = \Phi^*\left(\sigma|_L^{-1}|_{F^*}\right)$, so we are done after remarking that restriction $\mathrm{Gal}(L''/\mathbb{Q}) \to \mathrm{Gal}(L/\mathbb{Q})$ is surjective.

Remark 2.81. We check that reflex signatures certainly exist: one can choose any Galois closure L of F and then define $\Phi^*\colon \mathrm{Gal}(L/\mathbb{Q}) \to \mathbb{Z}_{\geq 0}$ by $\Phi^*(\sigma) \coloneqq \Phi\left(\sigma^{-1}|_L\right)$.

Remark 2.82. In the theory of abelian varieties with complex multiplication, it is customary to make F^* as small as possible, which makes it unique. This is useful for moduli problems. However, this is not our current interest, and we are not requiring that the reflex signature be unique because it will be convenient later to take large extensions.

The point of introducing the reflex is that it provides another monodromy interpretation of $Z(\mathrm{MT}(A))^\circ$. To achieve this, we need the reflex norm.

Definition 2.83 (reflex norm)**.** Fix CM fields F and F^* and reflex CM signatures (F, Φ) and (F^*, Φ^*). Then we define the *reflex norm* as the map $\mathrm{N}_{\Phi^*}\colon F^* \to \overline{\mathbb{Q}}$ by

$$\mathrm{N}_{\Phi^*}(x) \coloneqq \prod_{\sigma \in \Sigma_{F^*}} \sigma(x)^{\Phi^*(\sigma)}.$$

Note that this is a character in $\mathrm{X}^*(\mathrm{T}_{F^*})$.

Technically, this definition does not require us to remember that (F^*, Φ^*) is reflex to (F, Φ), but we will want to know this in the following checks.

Lemma 2.84. Fix CM fields F and F^* and reflex CM signatures (F, Φ) and (F^*, Φ^*).

(a) If (F_1^*, Φ_1^*) is a CM signature restricting to (F^*, Φ^*), then (F, Φ) and (F_1^*, Φ_1^*) are still reflex, and

$$\mathrm{N}_{\Phi_1^*} = \mathrm{N}_{\Phi^*} \circ \mathrm{N}_{F_1^*/F^*}.$$

(b) The image of N_{Φ^*} lands in F.

Proof. Here, "restricting" simply means that F_1^* contains F^* and $\Phi_1^*(\sigma) = \Phi^*(\sigma|_{F^*})$ for all $\sigma \in \Sigma_{F_1^*}$.

(a) That (F, Φ) and (F_1^*, Φ_1^*) are still reflex follows from the definition: choose a Galois extension L containing F and F_1^*, and then each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ has

$$\begin{aligned}\Phi(\sigma|_F) &= \Phi^*\left(\sigma^{-1}|_{F^*}\right) \\ &= \Phi_1^*\left(\sigma^{-1}|_{F_1^*}\right).\end{aligned}$$

To check the equality of reflex norms, we extend each $\sigma \in \Sigma_{F^*}$ to some $\widetilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and then we

directly compute

$$\begin{aligned}\mathrm{N}_{\Phi^*}\left(\mathrm{N}_{F_1^*/F^*}(x)\right) &= \prod_{\sigma\in\Sigma_{F^*}} \sigma\left(\mathrm{N}_{F_1^*/F^*}(x)\right)^{\Phi^*(\sigma)} \\ &= \prod_{\substack{\sigma\in\Sigma_F^* \\ \tau\in\mathrm{Hom}_{F^*}(F_1^*,\overline{\mathbb{Q}})}} \widetilde{\sigma}\tau(x)^{\Phi^*(\sigma)} \\ &= \prod_{\substack{\sigma\in\Sigma_F^* \\ \tau\in\mathrm{Hom}_{F^*}(F_1^*,\overline{\mathbb{Q}})}} \widetilde{\sigma}\tau(x)^{\Phi_1^*(\widetilde{\sigma}\tau)} \\ &= \mathrm{N}_{\Phi_1^*}(x),\end{aligned}$$

where the last step holds by noting that $\widetilde{\sigma}\circ\tau$ parameterizes Σ_{F^*}.

(b) We begin by reducing to the case where $F^*/\mathbb{Q}$ is Galois. Indeed, the previous step tells us that extending F^* merely passes to a norm subgroup of F^*, but norm subgroups are Zariski dense in T_{F^*}, so it suffices to check the result on such norm subgroups. Thus, we may assume that $F^*/\mathbb{Q}$ is Galois, contains F, and thus $\Phi^*(\sigma) = \Phi\left(\sigma^{-1}|_F\right)$. Now, for any $g\in\mathrm{Gal}(F^*/F)$, we see $\Phi^*(\sigma) = \Phi^*\left(g^{-1}\sigma\right)$, so

$$\begin{aligned}g\left(\mathrm{N}_{\Phi^*}(x)\right) &= \prod_{\sigma\in\mathrm{Gal}(F^*/\mathbb{Q})} g\sigma(x)^{\Phi^*(\sigma)} \\ &= \prod_{\sigma\in\mathrm{Gal}(F^*/\mathbb{Q})} \sigma(x)^{\Phi^*\left(g^{-1}\sigma\right)} \\ &= \mathrm{N}_{\Phi^*}(x),\end{aligned}$$

as required. ∎

At long last, we move towards our monodromy interepretation using the reflex. The following argument generalizes [Yu15, Lemma 4.2].

Lemma 2.85. Fix reflex CM signatures (F,Φ) and (F^*,Φ^*). Suppose that F^* contains F and is Galois over $\mathbb{Q}$. For each $g\in\mathrm{Gal}(F^*/\mathbb{Q})$, the reflex norm $\mathrm{N}_{\Phi^*}\colon \mathrm{T}_{F^*}\to\mathrm{T}_F$ sends the cocharacter $g^\vee\in\mathrm{X}_*(\mathrm{T}_{F^*})$ to

$$\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)\left(g^\vee\right) = \sum_{\sigma\in\Sigma_F}\Phi(\sigma)(g\sigma)^\vee.$$

Proof. Notably, N_{Φ^*} outputs to T_F by Lemma 2.84. To begin, we expand

$$\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)\left(g^\vee\right) = \sum_{\sigma\in\Sigma_{F^*}}\Phi^*(\sigma)\mathrm{X}_*(\sigma)(g^\vee).$$

We now check $\mathrm{X}_*(\sigma)(g^\vee) = \left(g\sigma^{-1}\right)^\vee$: for any $\tau\in\mathrm{X}^*(\mathrm{T}_{F^*})$, we compute the perfect pairing

$$\langle\tau,\mathrm{X}_*(\sigma)(g^\vee)\rangle = \langle\tau\sigma,g^\vee\rangle,$$

which is the indicator function for $\tau\sigma = g$ and hence equals $\left\langle\cdot,(g\sigma^{-1})^\vee\right\rangle$. We are now able to write

$$\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)\left(g^\vee\right) = \sum_{\sigma\in\Sigma_{F^*}}\Phi^*(\sigma)\left(g\sigma^{-1}\right)^\vee.$$

Replacing σ with σ^{-1}, we are done upon recalling $\Phi^*\left(\sigma^{-1}\right) = \Phi(\sigma|_F)$ and collecting terms which together restrict to the same embedding of F. ∎

Proposition 2.86. Fix an abelian variety A over $\mathbb{C}$ such that $Z(\mathrm{End}(A))$ equals a CM algebra $F = F_1 \times \cdots \times F_k$, and define $V \coloneqq \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$. Let $\Phi\colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the induced signature, which we decompose as $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_k$ where $(F_\bullet, \Phi_\bullet)$ is a CM signature for all $F_\bullet$. Suppose F^* is a CM field equipped with CM signatures $\Phi_1^*, \ldots, \Phi_k^*$ such that (F_i, Φ_i) and (F^*, Φ_i^*) are reflex for all i. Then $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$ is the image of

$$(\mathrm{N}_{\Phi_1^*}, \ldots, \mathrm{N}_{\Phi_k^*})\colon \mathrm{T}_{F^*} \to \mathrm{T}_F.$$

Proof. Note that norms are surjective on these algebraic tori, so Lemma 2.84 tells us that the image of N_{Φ^*} will not change if we pass to an extension of F^*. As such, we will go ahead and assume that F^* contains F and is Galois over $\mathbb{Q}$.

In light of Corollary 2.77, it is enough to show that the image of $\mathrm{X}_*(\mathrm{N}_{\Phi^*})$ (which we note is already a Galois submodule) has saturation equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F)$ containing $\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee$. This follows from the computation of Lemma 2.85 upon letting g vary over $\mathrm{Gal}(F^*/\mathbb{Q})$. ∎

Let's explain how Proposition 2.86 is applied to compute equations cutting out $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$, where $F = F_1 \times \cdots \times F_k$ is a CM algebra. As before, we will only compute over an extension $L = F^*$ of F which is Galois over $\mathbb{Q}$; let $\Phi_1^*, \ldots, \Phi_k^*$ be the signatures on L making (L, Φ_i^*) and (F_i, Φ_i) reflex for each i. Note, we know that $(\mathrm{T}_F)_L \subseteq \mathrm{GL}(V)_L$ may embed as a diagonal torus.

An equation cutting out $Z(\mathrm{MT}(A))^\circ_L$ in the (subtorus of the) diagonal torus $(\mathrm{T}_F)_L \subseteq \mathrm{GL}(V)_L$ then becomes a character of $(\mathrm{T}_F)_L$ which is trivial on $Z(\mathrm{MT}(A))^\circ$. In other words, these equations are given by the kernel of

$$\mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(Z(\mathrm{MT}(A))^\circ).$$

We now use Proposition 2.86. We know that $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$ is the image of $(\mathrm{N}_{\Phi_1^*}, \ldots, \mathrm{N}_{\Phi_k^*})\colon \mathrm{T}_L \to \mathrm{T}_F$, so the kernel of the above map is the same as the kernel of

$$\mathrm{X}^*\left((\mathrm{N}_{\Phi_1^*}, \ldots, \mathrm{N}_{\Phi_k^*})\right) : \mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(\mathrm{T}_L).$$

To compute this kernel cleanly, note Lemma 2.85 computes $\mathrm{X}_*\left(\mathrm{N}_{\Phi_i^*}\right)$ for each i, so we see $\mathrm{X}^*\left(\mathrm{N}_{\Phi_i^*}\right)$ can be computed as the transpose of the matrix of $\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)$. Attaching these matrices together gives a matrix representation for the above map, and we get our equations by computing the kernel of this matrix.

Remark 2.87. In practice, one can expand $V = V_1 \oplus \cdots \oplus V_k$ into irreducible Hodge substructures and then work with $E \coloneqq F_1 \times \cdots \times F_k$ where $F_i \coloneqq Z(\mathrm{End}_{\mathrm{HS}}(V_i))$ for each i. Technically speaking, F may only embed into E "diagonally" because some $V_\bullet$s may be isomorphic to each other. However, this does not really affect anything we do because we may as well work with the image of $Z(\mathrm{MT}(A))^\circ$ under the inclusion $\mathrm{T}_F \subseteq \mathrm{T}_E$. Working with T_E is more convenient because it can actually be identified with the diagonal torus of $\mathrm{GL}(V)_E$ instead of merely a diagonally embedded subtorus.

2.3 The ℓ-Adic Representation

In this subsection, we now define the ℓ-adic representation and give some of its basic properties.

2.3.1 The Construction

A priori, an abelian variety A gives rise to many ℓ-adic Galois representations via each of its cohomology groups $\mathrm{H}^\bullet_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$. However, it turns out that we are allowed to only care about one of them.

Proposition 2.88. Fix an abelian variety A over a field K, and let H be a Weil cohomology theory. Then the cup product defines an isomorphism between the exterior algebra $\wedge \mathrm{H}^1(A)$ and the cohomology ring $\mathrm{H}^*(A)$.

Proof. In the complex analytic case, we proceed as in [Mil20b, Proposition 2.6]. Write $A = \mathbb{C}^g/\Lambda$ for a lattice Λ. Fixing some index p, we will show that the cup product defines an isomorphism

$$\wedge^p \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Z}) \to \mathrm{H}^p_{\mathrm{B}}(A, \mathbb{Z}).$$

Well, we note that A is homeomorphic to $\left(S^1\right)^{2g}$, so the Künneth formula allows us to reduce the question to S^1, where the result is true by a direct computation. In the general case, one notes that the group structure on A induces a Hopf bialgebra structure on both $\wedge \mathrm{H}^1(A)$ and $\mathrm{H}^*(A)$; then one can appeal to some structure theory to deduce the equality. See [EGM, Corollary 6.13] or more precisely [EGM, Corollary 13.32]. ■

Thus, in ℓ-adic cohomology (where $\operatorname{char} K \nmid \ell$ in K), we see that one can understand all cohomology groups of A by merely understanding $\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell)$. Analogous to the complex analytic case, we will be able to work with the dual "homology group" more concretely.

Let's spend some time giving a more elementary description of $\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell)^\vee$. We refer to [EGM, Corollary 10.38] and the surrounding discussion for more details. We will do this by passing to the fundamental group. In particular, note that there is a Galois-invariant isomorphism

$$\mathrm{H}^1(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell) \cong \operatorname{Hom}\left(\pi_1(A_{K^{\mathrm{sep}}}, a), \mathbb{Z}_\ell\right),$$

where $a \in A(K^{\mathrm{sep}})$ is some basepoint. We will go ahead and choose $a = 0$.

Remark 2.89. Let's take a moment to explain this isomorphism. By taking limits, it is enough to show this isomorphism with $\mathbb{Z}_\ell$ replaced by μ_n where $\operatorname{char} K \nmid n$. Then one knows that $\mathrm{H}^1(A_{K^{\mathrm{sep}}}, \mu_n)$ is in bijection with Galois coverings with Galois group μ_n by using the short exact sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1.$$

This completes the proof upon unravelling the definition of π_1 on the right-hand side.

We now use the fact that A is an abelian variety to compute $\pi_1(A_{K^{\mathrm{sep}}}, 0)$: one can show that any étale covering of A is still an abelian variety and hence is an isogeny onto A (for suitable choice of group law). Thus, Lemma 2.13 promises that the multiplication-by-n maps $[n]_A \colon A \to A$ provide a cofinal sequence of Galois étale coverings of A (at least when $\operatorname{char} K \nmid n$), allowing us to compute that the ℓ-part of $\pi_1(A_{K^{\mathrm{sep}}}, 0)$ equals

$$\varprojlim A\left[\ell^\bullet\right](K^{\mathrm{sep}}).$$

In conclusion, we see that $\mathrm{H}^1(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell)$ is naturally isomorphic to

$$\left(\varprojlim A\left[\ell^\bullet\right](K^{\mathrm{sep}})\right)^\vee$$

as Galois representations. We are now allowed to define the Tate module.

Definition 2.90 (Tate module)**.** Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Then we define the ℓ-adic Tate module as

$$T_\ell A \coloneqq \varprojlim A\left[\ell^\bullet\right](K^{\mathrm{sep}}),$$

and we define the rational ℓ-adic Tate module as $V_\ell A \coloneqq T_\ell A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 2.91. Intuitively, $T_\ell A$ should be thought of as an ℓ-adic stand-in for $\mathrm{H}_1(A)$.

The discussion above suggesets that $T_\ell A$ should be a free $\mathbb{Z}_\ell$-module of rank 2. Let's check this directly. By taking limits, it is enough to show the following.

Lemma 2.92. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. For each $\nu \geq 0$, there is a group isomorphism

$$A\left[\ell^{\nu}\right](K^{\mathrm{sep}}) \cong \mathbb{Z}/\ell^{2\nu \dim A}\mathbb{Z}.$$

Proof. The two groups have the same size by Example 2.17, so the result follows for $\nu \in \{0,1\}$ automatically. For $\nu \geq 2$, we induct using the short exact sequence

$$0 \to A[\ell](K^{\mathrm{sep}}) \to A\left[\ell^{\nu+1}\right](K^{\mathrm{sep}}) \xrightarrow{\ell} A\left[\ell^{\nu}\right](K^{\mathrm{sep}}) \to 0$$

and some cardinality arguments. For example, one can finish by applying the classification of finite abelian groups. ∎

One benefit of a more concrete object is that it is easier to work with directly. For example, we can now find a perfect pairing on $\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell)$.

Lemma 2.93. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Choose a polarization $\varphi\colon A \to A^\vee$. Then the Weil pairing induces a Galois-invariant perfect symplectic pairing

$$e_\varphi\colon \mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell(-1).$$

Proof. By taking duals, it is enough to induce a Galois-invariant perfect symplectic pairing

$$e_\varphi\colon T_\ell A \otimes_{\mathbb{Q}_\ell} T_\ell A \to \mathbb{Z}_\ell(1).$$

This follows by taking a limit of the Weil pairing given in Corollary 2.39. Recall that $\mathbb{Z}_\ell(1)$ is the Galois representation $\varprojlim \mu_{\ell^\bullet}$. ∎

One can also see the Galois action more explicitly: being careful about the Galois action on cohomology and the Tate module, we see that the induced Galois representation

$$\rho_\ell\colon \operatorname{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}(T_\ell A)$$

is simply given by the Galois action on the points in the limit $A\left[\ell^\bullet\right](K^{\mathrm{sep}})$.

2.3.2 The ℓ-Adic Monodromy Group

Now that we have a representation, we may as well define a monodromy group.

Definition 2.94 (ℓ-adic monodromy group)**.** Fix an abelian A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Then the *$\ell$-adic monodromy group* $G_\ell(A)$ is the smallest algebraic $\mathbb{Q}_\ell$-group containing the image of the Galois representation

$$\operatorname{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)\right).$$

Remark 2.95. By taking duals, we see that one produces an isomorphic Galois representation by working with $T_\ell A$ instead. Note that this dual is not very expensive: by using the Weil pairing of Lemma 2.93, we can remove the dual in exchange for a twist, writing

$$\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell) \cong T_\ell A(-1).$$

Remark 2.96. Unlike $\mathrm{MT}(A)$ and $\mathrm{Hg}(A)$, we do not expect $G_\ell(A)$ to be connected in general. However, being an algebraic $\mathbb{Q}_\ell$-group, it will only have finitely many connected components. Thus, we see that the pre-image of $G_\ell(A)^\circ$ in $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is an open subgroup of finite index, so there is a unique minimal field extension K_A^{conn}/K such that $G_\ell(A_{K_A^{\mathrm{conn}}}) = G_\ell(A)^\circ$. Thus, our group becomes connected, only at the cost of a field extension.

The interesting geometric objects arising from Hodge theory were the Hodge classes, which Remark 1.13 explains were exactly the vectors fixed by the group action. Analagously, we pick up the following definition.

Definition 2.97 (Tate class)**.** Fix an abelian A over a field K, and suppose ℓ is a prime such that $\mathrm{char}\, K \nmid \ell$. Then a *Tate class* is a vector of some tensor construction

$$\bigoplus_{i=1}^{k} \mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)^{\otimes n_i} \otimes \mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)^{\vee \otimes m_i}(p_i),$$

where the $n_\bullet$s, $m_\bullet$s, and $p_\bullet$s are some nonnegative integers, fixed by the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$

Remark 2.98. We remark as in Corollary 1.34 that a vector v as above is a Tate class if and only if it is fixed by the indcued action by $G_\ell(A)$. Indeed, the subset of $\mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)\right)$ fixing v is some algebraic $\mathbb{Q}_\ell$-subgroup, so if it contains the image of $\mathrm{Gal}\left(K^{\mathrm{sep}}/K\right)$, then it contains $G_\ell(A)$. We also take a moment to note that Proposition 1.35 explains that one can now cut out $G_\ell(A)$ by requiring it to hold all the Tate classes invariant, as discussed in Corollary 1.36.

Analogous to Conjecture 1.15, one has a Tate class, which we will only state for abelian varieties.

Conjecture 2.99 (Tate)**.** Fix an abelian variety A over a number field K, and fix a prime number ℓ. Then any Tate class can be written as a $\mathbb{Q}_\ell$-linear combination of classes arising from algebraic subvarieties of powers of A.

Remark 2.100. Of course, there are Tate classes and there is a Tate conjecture for more general varieties.

We conclude this section with a few bounds on the ℓ-adic monodromy group, analogous to the discussion for Mumford–Tate groups in section 1.3.1. Let's begin with endomorphisms.

Lemma 2.101. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\mathrm{char}\, K \nmid \ell$. Set $D \coloneqq \mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$G_\ell(A) \subseteq \left\{ g \in \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)\right) : g \circ d = d \circ g \text{ for all } d \in D \right\}.$$

Proof. We proceed as in Lemma 1.45. The right-hand group is an algebraic $\mathbb{Q}_\ell$-group, so it suffices to check that it contains the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Well, for any $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, we see that

$$g \circ d = d \circ g$$

is an equality which holds on the level of endomorphisms of A because d is defined over K (which g fixes). ∎

Lemma 2.102. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\mathrm{char}\, K \nmid \ell$. Choose a polarization $\varphi\colon A \to A^\vee$. Then there is a perfect symplectic pairing e_φ such that

$$G_\ell(A) \subseteq \left\{ g \in \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)\right) : e_\varphi(gv \otimes gw) = \lambda(g) e_\varphi(v \otimes w) \text{ for fixed } \lambda(g) \in \mathbb{Q}_\ell \right\}.$$

Proof. We proceed as in Lemma 1.47. The right-hand group is an algebraic $\mathbb{Q}_\ell$-group, so it suffices to check that it contains the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Well, for any $g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, we see that

$$e_\varphi(gv \otimes gw) = ge_\varphi(v \otimes w)$$

by the Galois-invariance of Lemma 2.93. Now, we note that $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on $\mathbb{Q}_\ell(-1)$ through the cyclotomic character, so the right-hand side equals a scalar $\lambda(g)$ times $e_\varphi(v \otimes w)$, so we are done. ∎

Remark 2.103. There are of course alternate proofs of Lemmas 2.101 and 2.102 by finding Tate classes and then appealing to Remark 2.98. One uses the same classes constructed in the alternate proofs of Lemmas 1.45 and 1.47.

Lastly, we would like to recover the bound of Corollary 2.42 on endomorphisms, sharpening Lemma 2.101. However, the proof is not so easy: the proof of Corollary 2.42 had to translate endomorphisms of the Hodge structure back to endomorphisms of the abelian variety via Theorem 2.40. Recovering the equivalence of Theorem 2.40 is rather difficult: this result is due to Faltings [Fal86, Theorem 3], in his proof of Mordell's conjecture.

Theorem 2.104 (Faltings)**.** Fix an abelian variety A over a number field K, and suppose ℓ is a prime. Then the induced map

$$\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{End}_{\mathrm{Gal}(\overline{K}/K)}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$$

is an isomorphism.

We will definitely not attempt to summarize a proof here, but we will remark that it is not even totally obvious that this map is injective! Speaking from experience, this makes for a reasonable topic for a final term paper in a first course in algebraic geometry.

Remark 2.105. Via the isomorphism

$$\mathrm{End}_{\mathbb{Q}_\ell}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right) \cong \mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell) \otimes \mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)^\vee,$$

we see that Theorem 2.104 can be viewed as asserting that all the Tate classes in the above space arise from endomorphisms of A. This verifies Conjecture 2.99.

Remark 2.106. We have snuck in the hypothesis that K is a number field into the statement of Theorem 2.104. It is also true for finite fields, where it is due to Tate [Tat66]. However, it is not expected to be true in general!

We are now able to provide a satisfying analogue to Lemma 1.54.

Corollary 2.107. Fix an abelian variety A over a number field K, and suppose ℓ is a prime. Then the natural map

$$\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{End}_{G_\ell(A)}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$$

is an isomorphism.

Proof. Remark 2.105 explains that the endomorphisms of A are exactly the Tate classes, so the result follows from the discussion in Remark 2.98. ∎

Remark 2.108. The above corollary allows us to prove the following analogue of Proposition 2.53 (by the same proof!): A has CM defined over a number field K if and only if $G_\ell(A)$ is a torus.

While we're here, we remark on another property of $G_\ell(A)$ due to Faltings.

Theorem 2.109 (Faltings)**.** Fix an abelian variety A over a number field K, and suppose ℓ is a prime. Then $G_\ell(A)$ is reductive.

Proof. By [Mil17, Corollary 19.18], it is enough to find a faithful semisimple representation of $G_\ell(A)$. As in Lemma 1.44, we see that the inclusion

$$G_\ell(A) \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$$

is semisimple by [Fal86, Theorem 3], so we are done. ∎

Remark 2.110. Over finite fields, Tate [Tat66] has proven that the Galois representation $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ is semisimple. Because the Galois group is (topologically) generated by the Frobenius, this amounts to checking that the endomorphism Frob_q has semisimple action.

To finish up our discussion of computational tools for $G_\ell(A)$, we repeat the results Lemmas 1.56 and 1.59 for our new context. Their proofs are exactly the same, replacing $\mathbb{U}$ (or $\mathbb{S}$) with $\mathrm{Gal}(\overline{F}/F)$ and then making the same minimality arguments for our monodromy groups.

Lemma 2.111. Fix abelian varieties $A_1, \dots, A_k$ over a field F.

(a) The subgroup
$$G_\ell(A_1 \times \cdots \times A_k) \subseteq \mathrm{GL}(\mathrm{H}^1_{\text{ét}}((A_1 \times \cdots \times A_k)_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell))$$
is contained in $G_\ell(A_1) \times \cdots \times G_\ell(A_k)$.

(b) For each i, the projection map $\mathrm{pr}_i \colon G_\ell(A_1 \times \cdots \times A_k) \to G_\ell(A_i)$ is surjective.

Lemma 2.112. Fix abelian varieties $A_1, \dots, A_k$ over a field F, and let $m_1, \dots, m_k \geq 1$ be positive integers. Then the diagonal embeddings $\Delta_i \colon \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{i,F^{\mathrm{sep}}}, \mathbb{Q}_\ell)\right) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A^{m_i}_{i,F^{\mathrm{sep}}}, \mathbb{Q}_\ell)\right)$ induce an isomorphism
$$G_\ell(A_1 \times \cdots \times A_k) \to G_\ell\left(A_1^{m_1} \times \cdots \times A_k^{m_k}\right).$$

2.3.3 The Mumford–Tate Conjecture

Over the next two subsections, we will explain some tools used to compute $G_\ell(A)$. In this subsection, we will discuss $G_\ell(A)^\circ$. Suppose that A is defined a number field K.

A motivic perspective would have us hope that all the monodromy groups attached to A are essentially the same. However, as explained in Remark 2.96, we only expect $G_\ell(A)$ to be connected after an extension K. Thus, for example, one can only hope that $\mathrm{MT}(A)$ knows about $G_\ell(A)^\circ$. We may now state the following conjecture.

Conjecture 2.113 (Mumford–Tate)**.** Fix an abelian variety A over a number field K. For all primes ℓ, we have
$$\mathrm{MT}(A)_{\mathbb{Q}_\ell} = G_\ell(A)^\circ$$
as subgroups of $\mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A, \mathbb{Q}_\ell)\right)$. Here, $\mathrm{MT}(A)$ is embedded into this group by the Betti-to-étale comparison isomorphism.

Our work in chapter 1 provides many tools for computing $\mathrm{MT}(A)$, so Conjecture 2.113 would allow us to translate this knowledge into a computation of $G_\ell(A)^\circ$.

Even though Conjecture 2.113 is not fully proven, there is a lot known. Let's review a small amount. For example, both groups are reductive by Lemma 1.44 and Theorem 2.109. Additionally, Theorem 2.104

provides a suitable analogue of Theorem 2.40, telling us that both groups $\mathrm{MT}(A)$ and $G_\ell(A)$ cut out endomorphisms in $\mathrm{End}(A)$.

As a philosophical check, one can show that $G_\ell(A)^\circ$ "contains" the Hodge structure morphism; the following result is due to Sen [Sen73, Theorem 1].

Theorem 2.114 (Sen)**.** Fix an abelian variety A over a number field K. Define the operator Φ as acting by mutltiplication-by-i on each eigenspace

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)_{\mathbb{C}_\ell}(i),$$

where the (i)th eigenspace acts by ith power of the cyclotomic character. Then $\mathrm{Lie}\, G_\ell(A)^\circ$ is the smallest Lie algebra containing Φ.

Continuing, one inclusion of Conjecture 2.113 is known, due to Deligne [Del18, Corollary 6.2].

Theorem 2.115 (Deligne)**.** Fix an abelian variety A over a number field K. For all primes ℓ, we have

$$G_\ell(A)^\circ \subseteq \mathrm{MT}(A)_{\mathbb{Q}_\ell}.$$

In particular, it becomes enough to compare numerical invariants of the two groups (such as rank) to argue for an equality. For example, the following independence result is due to Larsen and Pink [LP95, Theorem 4.3].

Theorem 2.116 (Larsen–Pink)**.** Fix an abelian variety A over a number field K. If $\mathrm{MT}(A)_{\mathbb{Q}_\ell} = G_\ell(A)^\circ$ holds for any prime ℓ, then it holds for all primes ℓ.

One even knows that the centers of the groups coincide, due to Vaisu [Vas07, Theorem 1.3.1].

Theorem 2.117 (Vaisu)**.** Fix an abelian variety A over a number field K. For each prime ℓ, we have

$$Z(\mathrm{MT}(A))^\circ_{\mathbb{Q}_\ell} = Z(G_\ell(A))^\circ.$$

Vaisu [Vas07] has in fact shown quite a bit about the Mumford–Tate conjecture; see in particular [Vas07, Theorem 1.3.4].

Much is known about products, especially products with restricted endomorphism types. For example, Theorem 2.117 immediately implies the Mumford–Tate conjecture for abelian varieties with complex multiplication by combiing with Proposition 2.53 and Remark 2.108.

Remark 2.118. In fact, the Mumford–Tate conjecture for abelian varieties with complex multiplication is much older: it is originally due to Pohlmann [Poh68, Theorem 4], but Ribet in [Rib04] has pointed out that the result is a corollary of results due to Shimura and Tanimaya [ST61], and [Yu15] has recently explicated this argument.Include proof if include CM theorem

Continuing, by combining [Ich91; Lom16], one is able to compute both $\mathrm{MT}(A)$ and $G_\ell(A)^\circ$ for many abelian varieties of Types I–III and control contributions coming from Type IV; this permits a proof of the Mumford–Tate conjecture for products of abelian varieties of dimension at most 3. More generally, the following result is due to Commelin [Com18, Theorem 1.2].

Theorem 2.119 (Commelin)**.** Fix abelian varieties A and B over a number field K. If the Mumford–Tate conjecture holds for both A and B, then it holds for $A \times B$.

To give a taste for how some of these results are proven, we show the following, which follows from [Vas07, Theorem 1.3.4].

Proposition 2.120. Fix a geometrically simple abelian variety A over a number field K. Suppose that $F = Z(\mathrm{End}_{\overline{K}}(A)$ equals a CM field such that $\dim A = \dim F$. Letting Φ be the corresponding signature, we further suppose that $\Phi(\sigma) = 1$ for exactly two $\sigma \in \Sigma_F$. Then we show the Mumford–Tate conjecture holds for A, and

$$\mathrm{MT}(A)^{\mathrm{der}} = \mathrm{L}(A)^{\mathrm{der}}.$$

Proof. For special ℓ, we will actually compute $\mathrm{MT}(A)^{\mathrm{der}}$ and $G_\ell(A)^{\circ,\mathrm{der}}$ "simultaneously" to show that they are equal to the suitable version of $\mathrm{GSp}_F(\varphi)^{\mathrm{der}}$ or $\mathrm{GSp}_F(e_\varphi)^{\mathrm{der}}$. By adding in what we know about the centers from Theorem 2.117 (and the independence of ℓ given in Theorem 2.116), the Mumford–Tate conjecture follows for A. The outline is to base-change to $\mathbb{C}$, where the Lie algebra of $\mathrm{L}(A)^{\mathrm{der}}$ becomes a product of $\mathfrak{sl}_2(\mathbb{C})$s, from which we can appeal to Lemma 1.62.

Before beginning the computation, we set up some notation. In practice, it will be convenient to only write down the computation for $\mathrm{MT}(A)^{\mathrm{der}}$, but we will indicate along the way the changes that need to be made for $G_\ell(A)^{\circ,\mathrm{der}}$. Now, for brevity, set $V := \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ so that $\mathrm{Hg}(A) = \mathrm{Hg}(V)$ and $\mathrm{L}(A) = \mathrm{L}(V)$; we remark that V is a free module over F of rank 2.

Continuing with the set-up, we recall some part of the computation from Lemma 1.68. Fix a polarization φ on V. Then let $\rho_1, \dots, \rho_{e_0}$ be the embeddings of $F_i^\dagger$ into a Galois closure $E^\dagger$, which is the totally real subfield of the Galois closure E of F. Then we admit a decomposition

$$V_{E^\dagger} = V_1 \oplus \cdots \oplus V_{e_0}$$

so that

$$\mathrm{L}(V)_{E^\dagger} = \mathrm{Sp}_{F\otimes_{\rho_1} E^\dagger}(\varphi|_{V_1}) \times \cdots \times \mathrm{Sp}_{F\otimes_{\rho_{e_0}} E^\dagger}(\varphi|_{V_{e_0}}).$$

We now also recall from Lemma 1.68 that each $\mathrm{Sp}_{F\otimes_{\rho_i} E^\dagger}(\varphi|_{V_i})_E$ is isomorphic to $\mathrm{GL}_2(E)$; in particular, this group is connected. In particular, to achieve this decomposition, we diagonalize the induced action of E on V_i and then projects onto one of the eigenspaces.

Now, we would like to show that the inclusion

$$\mathrm{Hg}(V)^{\mathrm{der}}_E \subseteq \mathrm{Sp}_{F\otimes_{\rho_1} E^\dagger}(\varphi|_{V_1})_E \times \cdots \times \mathrm{Sp}_{F\otimes_{\rho_{e_0}} E^\dagger}(\varphi|_{V_{e_0}})_E$$

is an isomorphism, where the last group is embedded in $\mathrm{GL}(V)_E$. All groups involved are connected, so we may check this inclusion on the level of the Lie algebra, so we would like for the inclusion

$$\mathrm{Lie}\,\mathrm{Hg}(V)^{\mathrm{der}}_E \subseteq \mathrm{Sp}_{F\otimes_{\rho_1} E^\dagger}(\varphi|_{V_1})_E \times \cdots \times \mathrm{Sp}_{F\otimes_{\rho_{e_0}} E^\dagger}(\varphi|_{V_{e_0}})_E$$

is surjective. For this, we use Lemma 1.62. Here are our checks; for brevity, set $\mathfrak{hg}(V) := \mathrm{Lie}\,\mathrm{Hg}(V)_E$, and let $\mathfrak{sl}_2(E)_i$ be the factor $\mathrm{Lie}\,\mathrm{Sp}_{F\otimes_{\rho_i} E^\dagger}(\varphi|_{V_i})^{\mathrm{der}}_E$, which we note is isomorphic to $\mathfrak{sl}_2(E)$.

(i) We claim that $\mathfrak{hg}(V)^{\mathrm{der}}$ surjects onto $\mathfrak{sl}_2(E)_i$, which we note is nonzero and simple. Because the $\mathfrak{hg}(V)$ is semisimple, its image in $\mathfrak{sl}_2(E)_i$ continues to be reductive.

Now, reductive subgroups of $\mathfrak{sl}_2(E)$ are either tori of all of $\mathfrak{sl}_2(E)$, so we merely need to check that the image cannot be a torus. If the image in some $\mathfrak{sl}_2(E)_i$ is a torus, then because the Galois action $\mathrm{Gal}(E/\mathbb{Q})$ permutes the decomposition of V into $\{V_i\}_i$ (but will fix $\mathrm{Hg}(V)$), so we see that the image in $\mathfrak{sl}_2(E)_i$ will continue to be a torus for all i. Explicitly, we note that the image of $\mathrm{Hg}(V)$ in $\mathrm{Sp}_{F\otimes_{\rho_i} E^\dagger}(\varphi|_{V_i})_E$ needs to be preserved under $\mathrm{Gal}(E/\mathbb{Q})$, so if the projection is commutative in one factor, then it is commutative in all factors because the $\mathbb{Q}$-points are dense. In particular, $\mathrm{Hg}(V)$ must be a torus, so A has complex multiplication by Proposition 2.53, which is a contradiction to its definition.

(ii) The first point of (ii) is automatic from the construction. The second point follows because all the $\mathfrak{sl}_2(E)_i$s include as the standard representation into $\mathfrak{gl}(V_i)$.

For the last point, we use the Galois action together with the hypothesis on the signature. Arguing as in the proof of Lemma 1.62, it is really enough to check the $(V_i)_E$s are non-isomorphic as $\mathfrak{hg}(V)^{\mathrm{der}}_E$-modules. To make sense of the signature, we choose an embedding $\varepsilon\colon E \to \mathbb{C}$, and then Lemma 2.72

grants a signature Φ_ε from the decomposition of V_ε into $F \otimes_\varepsilon \mathbb{C}$-eigenspaces: explicitly, for each embedding $\sigma \in \mathrm{Hom}(F, E)$, we find

$$\Phi_\varepsilon(\sigma) = \dim(V_\sigma)^{1,0}_\varepsilon,$$

where $(\cdot)^{1,0}$ signifies that we are taking the eigenspace where $i \in \mathbb{C}$ acts by i^{-1}. However, the choice of a different embedding ε will permute the V_σs in sight.

To explain how the signature is now used, we note that if $\{\Phi_\varepsilon(\sigma), \Phi_\varepsilon(\overline{\sigma})\} \neq \{\Phi_\varepsilon(\tau), \Phi_\varepsilon(\overline{\tau})\}$ for two embeddings $\sigma, \tau \in \mathrm{Hom}(F, E)$ where $\rho_i = \sigma|_{F^\dagger}$ and $\rho_j = \tau|_{F^\dagger}$, then we must have $V_i \ncong V_j$ as $\mathfrak{hg}(V)^{\mathrm{der}}_\varepsilon$-modules. Indeed, unwrapping the definition of the signature, we know that the projection of $\mathfrak{hg}(V)_\mathbb{R}$ (where the embedding $E^\dagger \hookrightarrow \mathbb{R}$ is given by the restriction of ε) into $\mathfrak{gl}_4(\mathbb{R})$ is

$$\mathfrak{so}(\Phi_\varepsilon(\sigma), \Phi_\varepsilon(\overline{\sigma})).$$

To see this, note that this a semisimple algebra of the correct rank, so it is enough remark that the image of $\mathfrak{hg}(V)^{\mathrm{der}}$ must land in the above Lie subalgebra by tracking the action of $h(i)$. (One should use Theorem 2.114 in the ℓ-adic computation.) Thus, we are now able to remark that $\mathfrak{so}(\Phi_\varepsilon(\sigma), \Phi_\varepsilon(\overline{\sigma})) \ncong \mathfrak{so}(\Phi_\varepsilon(\tau), \Phi_\varepsilon(\overline{\tau}))$.

To complete the proof, the hypothesis implies there exists exactly one pair $\{\sigma_0, \overline{\sigma}_0\}$ of embeddings $F \hookrightarrow \mathbb{C}$ such that $\Phi_\varepsilon(\sigma_0) = \Phi_\varepsilon(\overline{\sigma}_0) = 1$. Thus, for any two distinct embeddings $\sigma, \tau \in \mathrm{Hom}(F, E)$, we can choose ε so that $\varepsilon\sigma = \sigma_0$ but $\varepsilon\tau \neq \sigma_0$ and apply the previous paragraph. ∎

Remark 2.121. This argument is inspired by [Zar83, Remark 1.9.4], where "changing the embedding" is used similarly to conclude that the Hodge group is large.

2.3.4 Computing ℓ-Adic Monodromy

The previous subsection explains that one expects to be able to compute $G_\ell(A)^\circ = \mathrm{MT}(A)$. We now explain how to use a computation of $G_\ell(A)^\circ$ to compute $G_\ell(A)$ in full. The idea is to use the Galois action on Tate classes. Our exposition follows [GGL24, Sections 8.1–8.2]. We begin with some notation.

Notation 2.122. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\mathrm{char}\, K \nmid \ell$. We will write $V := \mathrm{H}^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_\ell)$. For each $n \geq 0$, we define W_n to be the spce of Tate classes in the nth tensor power, writing

$$W_n := \left(V^{\otimes n} \otimes V^{\vee \otimes n}\right)^{G_\ell(A)^\circ}.$$

We also write $W := \bigoplus_{n \geq 0} W_n$ for brevity.

Remark 2.123. Because A is an abelian variety, one has a polarization $V \otimes V \to \mathbb{Q}_\ell(1)$, so we see that one can replace W_n with

$$\left(V^{\otimes 2n}(n)\right)^{G_\ell(A)^\circ}.$$

Roughly speaking, the point is that the spaces $W_\bullet$ of Tate classes are able to keep track of $G_\ell(A)^\circ$.

Lemma 2.124. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\mathrm{char}\, K \nmid \ell$, and define V and $W_\bullet$ as in Notation 2.122.

(a) If $G \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ fixes W, then $G \subseteq G_\ell(A)^\circ$.

(b) There is a finite-dimensional subspace $W' \subseteq W$ such that $G \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ fixes W' if and only if $G \subseteq G_\ell(A)^\circ$.

Proof. This essentially follows from Proposition 1.35.

(a) Recall $G_\ell(A)^\circ$ is reductive by Theorem 2.109. Thus, by Proposition 1.35, we know that if $G \subseteq \mathrm{GL}(V)$ fixes every $G_\ell(A)^\circ$-invariant in any

$$\bigoplus_{i=1}^{k} \left(V^{\otimes m_i} \otimes V^{\vee \otimes n_i}\right),$$

then $G \subseteq G_\ell(A)^\circ$. However, we claim that all $G_\ell(A)^\circ$-invariants in the above space can be found in W, which will complete the proof. Indeed, by Theorem 2.117, we see that the scalars $\mathbb{G}_{m,\mathbb{Q}_\ell}$ can be found in $G_\ell(A)^\circ$; however, these scalars act by the character $z \mapsto z^{m_i - n_i}$ on $V^{\otimes m_i} \otimes V^{\vee \otimes n_i}$, so any $G_\ell(A)^\circ$-invariant subspace must then have $m_i = n_i$.

(b) The above argument provides countably many equations (in the form of invariant tensors) which cut out $G_\ell(A)^\circ$. However, any algebraic subgroup of $\mathrm{GL}(V)$ will be cut out by finitely many equations, so we can choose W' to be the span of any such subset of finitely many defining equations. ■

Remark 2.125. The proof of (b) in fact gives an effective way to compute the subspace W': simply write down enough tensor elements to cut out $G_\ell(A)^\circ \subseteq \mathrm{GL}(V)$.

We would now like to upgrade from $G_\ell(A)^\circ$ to $G_\ell(A)$.

Lemma 2.126. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\mathrm{char}\, K \nmid \ell$, and define V and $W_\bullet$ as in Notation 2.122. For each $n \geq 0$, the subspace W_n is stabilized by $G_\ell(A)$.

Proof. We already know that $G_\ell(A)^\circ$ acts trivially on W_n, so this will follow purely formally from the fact that $G_\ell(A)^\circ$ is a normal subgroup of $G_\ell(A)$.

We would like to show that each $g \in G_\ell(A)$ stabilizes W_n. Well, W_n exactly consists of the $G_\ell(A)^\circ$-invariants inside $V^{\otimes n} \otimes V^{\vee \otimes n}$, so it suffices to show that gW_n is stabilized by $G_\ell(A)^\circ$. Well, for any $g_0 \in G_\ell(A)^\circ$, we see that

$$g_0 g W_n = g \cdot g^{-1} g_0 g W_n,$$

so we conclude by noting that $g^{-1} g_0 g \in G_\ell(A)^\circ$ because $G_\ell(A)^\circ \subseteq G_\ell(A)$ is a normal subgroup. ■

Combining the above two lemmas, we see that we get a faithful representation

$$G_\ell(A)/G_\ell(A)^\circ \to \mathrm{GL}(W).$$

This faithful representation allows us to compute $G_\ell(A)$: we are looking for elements of $\mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ which produce the automorphisms of W seen in the image of the above faithful representation. Tracking through this sort of reasoning produces our main result.

Proposition 2.127. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\mathrm{char}\, K \nmid \ell$, and define V and $W_\bullet$ as in Notation 2.122. Then $G_\ell(A)$ equals the group

$$\bigcup_{\sigma \in \mathrm{Gal}(\overline{K}/K)} \{g \in \mathrm{GL}(V) : g|_W = \sigma|_W\}.$$

In fact, each set in the union is a connected component of $G_\ell(A)$.

Proof. We begin by noting that $\mathrm{Gal}(\overline{K}/K)$ does in fact preserve W: indeed, one has a composite

$$\mathrm{Gal}(\overline{K}/K) \to G_\ell(A) \to \mathrm{GL}(W),$$

where the first map is well-defined by the definition of $G_\ell(A)$, and the second map is well-defined by summing Lemma 2.126.

Now, we have two inclusions to show.

- Suppose $g \in G_\ell(A)$. Then we must find $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $g|_W = \sigma|_W$. Well, $G_\ell(A)$ is by definition the Zariski closure of the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{GL}(V)$, so the open subset $gG_\ell(A)^\circ$ of $G_\ell(A)$ must contain $\sigma|_V$ for some $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Now, $G_\ell(A)^\circ$ acts trivially on W, so we see that $g|_W = \sigma|_W$.
- Suppose $g \in \mathrm{GL}(V)$ satisfies $g|_W = \sigma|_W$. Then we would like to show that $g \in G_\ell(A)$. The argument in the previous point grants $g_0 \in G_\ell(A)$ such that $g_0|_V = \sigma|_V$, so in particular, $g|_W = g_0|_W$. Thus, gg_0^{-1} acts trivially on W, so $gg_0^{-1} \in G_\ell(A)^\circ$, so it follows that $g \in G_\ell(A)$.

Lastly, it remains to discuss connected components. Well, note that $g, g' \in G_\ell(A)$ live in the same connected component if and only if $g'g^{-1} \in G_\ell(A)$, which is equivalent to $g'g^{-1}$ acting trivially on W, which is equivalent to $gG_\ell(A)^\circ = g'G_\ell(A)^\circ$. ∎

Remark 2.128. A careful reading of the above proof shows that we only needed the following facts about W: it is stable under $G_\ell(A)$, and $g \in \mathrm{GL}(V)$ lives in $G_\ell(A)^\circ$ if and only if it fixes W. Thus, we see that we can replace W with any $G_\ell(A)$-subrepresentation $W' \subseteq W$ which cuts out $G_\ell(A)^\circ$ in the sense of Lemma 2.126. This allows us to make W' quite small (e.g., finite-dimensional).

Remark 2.129. It is worth comparing Proposition 2.127 with the twisted Lefschetz group, defined in [BK15, Definition 5.2]. Roughly speaking, the twisted Lefschetz group is simply the construction of Proposition 2.127 with W replaced by the subspace of W generated by endomorphisms and the polarization; see [GGL24, Remark 8.3.5] for precise discussion of the relation. In this way, one expects the twisted Lefschetz group to equal $G_\ell(A)$ in generic cases, but Remark 2.128 explains that one may need to remember more Hodge classes in exceptional cases.

Proposition 2.127 suggests that one can find representatives of each connected component in $G_\ell(A)$ by looping over all $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and finding some $g \in \mathrm{GL}(V)$ such that $g|_W = \sigma|_W$. This is currently not so computable because $\mathrm{Gal}(\overline{K}/K)$ is an infinite group, and W is an infinite-dimensional vector space. Remark 2.128 explains how to replace W with a finite-dimensional subrepresentation, so it remains to explain how to reduce $\mathrm{Gal}(\overline{K}/K)$ to a finite quotient.

Definition 2.130 (connected monodromy field)**.** Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\mathrm{char}\, K \nmid \ell$. Then we define the *connected monodromy field* K_A^{conn} so that the open subgroup $\mathrm{Gal}(\overline{K}/K_A^{\mathrm{conn}})$ is the pre-image of the connected component $G_\ell(A)^\circ$ in the Galois representation

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right).$$

Remark 2.131. Note that such a field K_A^{conn} exists and is finite over K by Galois theory: note $G_\ell(A)^\circ \subseteq G_\ell(A)$ is a finite-index subgroup (because the quotient is a discrete algebraic group), so the pre-image $U \subseteq \mathrm{Gal}(\overline{K}/K)$ of $G_\ell(A)^\circ$ similarly must be open and finite index and hence closed and finite index.

Thus, we see that the Galois reprentation to $\mathrm{GL}(W)$ factors through the finite group $\mathrm{Gal}(K_A^{\mathrm{conn}}/K)$. In this way, we are able to reduce the computation suggested by Proposition 2.127 from the infinite group $\mathrm{Gal}(\overline{K}/K)$ to the finite quotient $\mathrm{Gal}(K_A^{\mathrm{conn}}/K)$.

Remark 2.132. Let's describe how one might compute K_A^{conn} in practice. By combining the definition of K_A^{conn} with Lemma 2.124, we see that $\mathrm{Gal}(\overline{K}/K_A^{\mathrm{conn}})$ is the kernel of the representation

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(W),$$

so one could imagine computing the open subgroup $\mathrm{Gal}(\overline{K}/K_A^{\mathrm{conn}})$ by computing the above representation. As usual, we remark that Lemma 2.124 allows us to replace W with a finite-dimensional subrepresentation W' "cutting out" $G_\ell(A)^\circ$.

Chapter 3
The Sato–Tate Conjecture

Now that we have a good handle on monodromy groups, we describe one of their applications: the Sato–Tate conjecture. These notions are not central for the results we want to prove, so we will be somewhat sketchy throughout.

3.1 The Statement

In this section, we state the Sato–Tate conjecture, and then we explain how it can be numerically verified in some cases.

3.1.1 The Weil Conjectures

Roughly speaking, the Sato–Tate conjecture is about counting points on an abelian variety A over finite fields $\mathbb{F}_q$ as q varies. In this subsection, we will briefly describe the Weil conjectures because they explain why these point-counts ought to be related to cohomology; these conjectures are now theorems due to Deligne [Del74; Del80].

Theorem 3.1 (Weil conjectures)**.** Fix a smooth projective variety X over a finite field $\mathbb{F}_q$ of dimension n. Consider the formal power series.

$$\zeta_X(T) \coloneqq \exp\left(\sum_{r=1}^{\infty} \#X\left(\mathbb{F}_{q^r}\right) \frac{T^r}{r}\right)$$

(a) Rationality: one can write

$$\zeta_X(T) = \frac{P_1(T)\cdots P_{2n-1}(T)}{P_2(T)\cdots P_{2n}(T)}$$

for polynomials $P_\bullet(T) \in 1 + T\mathbb{Z}[T]$.

(b) Riemann hypothesis: the roots of the polynomial $P_\bullet(T)$ are complex numbers with roots of magnitude $q^{-\bullet/2}$.

It is worth explaining a bit of the proof of these conjectures for abelian varieties. Our exposition is an abbreviated form of the exposition in (say) [Mil08, Chapter II].

Fix an abelian variety A of dimension g over a finite field $\mathbb{F}_q$. The main point is to find a way to compute $\#A(\mathbb{F}_q)$, and then Theorem 3.1 will follow. Viewing $A(\mathbb{F}_q)$ is the set of fixed (geometric) points of the Frobenius endomorphism $\mathrm{Frob}_q\colon A \to A$, one would like to use the Lefschetz fixed point formula to con-

clude. In particular, we should be able to read off the value of $\#A(\mathbb{F}_q)$ from a suitably defined characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$.

To be explicit, one finds that the characteristic polynomial $P(T)$ of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ satisfies

$$P(1) = \#A(\mathbb{F}_q).$$

Thus, by factoring $P(T) = \prod_{i=1}^{2g}(T - \alpha_i)$, one finds that

$$\#A(\mathbb{F}_{q^r}) = \prod_{i=1}^{2g}(1 - \alpha_i^r),$$

which proves the rationality conjecture of Theorem 3.1 after some manipulation. In brief, one finds that $P(T) = P_1(T)$, and in general, the polynomial $P_i(T)$ has roots given by multiplying i of the roots in the set $\{\alpha_1, \dots, \alpha_{2g}\}$ together.

Remark 3.2. Comparing the previous paragaph with the proof of the rationality conjecture from the Lefschetz trace formula

$$\#A(\mathbb{F}_q) = \sum_{i=0}^{2g}(-1)^i \operatorname{tr}\left(\mathrm{Frob}_q \mid \mathrm{H}^i_{\text{ét}}(A_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)\right),$$

one sees that what allows us to recover all the polynomials $P_\bullet(T)$ from merely $P_1(T)$ is that the higher cohomology of A is generated by the cohomology in degree 1 by Proposition 2.88.

It remains to prove the Riemann hypothesis conjecture of Theorem 3.1. This amounts to checking that the roots of $P(T)$ have magnitude $1/\sqrt{p}$, which eventually corresponds to the following fact.

Proposition 3.3. Fix an abelian variety A over a finite field $\mathbb{F}_q$, and consider the induced Frobenius endomorphism Frob_q. Then

$$\mathrm{Frob}_q \circ \mathrm{Frob}_q^\dagger = [q]_A.$$

Proof. Proving this requires more tools than we would like to introduce at this time, so we refer to [Mil08, Lemma III.1.2]. ∎

3.1.2 The Sato–Tate Group

In this section, we will define the Sato–Tate group and state the Sato–Tate conjecture. Our exposition loosely follows [Sut19]. Fix an abelian variety A defined over a number field K, and choose a prime ℓ. We also let $\rho_\ell\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ denote the associated Galois representation.

Intuitively, the Sato–Tate conjecture asserts that the Frobenius elements $\rho_\ell(\mathrm{Frob}_{\mathfrak{p}})$ equidistribute in $G_\ell(A)$ as $\mathfrak{p}$ varies over the maximal ideals of $\mathcal{O}_K$. This conjecture does not make sense verbatim, so we will have to work a bit to write down something formal. Consider the following points.

- To begin, we note that $\mathrm{Frob}_{\mathfrak{p}}$ only makes sense as a conjugacy class, and it only makes sense as a conjugacy class when ρ_ℓ vanishes on the relevant inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$.

  Two remarks are thus in order. First, to vanish on the inertia subgroup, we must exclude a finite set of primes $\mathfrak{p}$ where A has bad reduction. (We are using the Néron–Ogg–Shafarevich criterion [BLR90, Theorem 5].) Second, we will simply regard $\rho_\ell(\mathrm{Frob}_{\mathfrak{p}})$ as a conjugacy class as well. Thus, we really want to say that conjugacy classes equidistribute in a suitable space of conjugacy classes.

- It turns out that $\rho_\ell(\mathrm{Frob}_{\mathfrak{p}})$ is not a totally random element of $G_\ell(A)$. Indeed, by Proposition 3.3, we see that the multiplier of $\mathrm{Frob}_{\mathfrak{p}}$ acting on $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ equals $\mathrm{N}(\mathfrak{p})$. Thus, we would like to rescale $\mathrm{Frob}_{\mathfrak{p}}$ back down by $1/\sqrt{\mathrm{N}(\mathfrak{p})}$.

Once again, this requires two remarks. First, after rescaling, we will be working in the smaller subgroup

$$G_\ell^1(A) \coloneqq G_\ell(A) \cap \mathrm{Sp}(e_\varphi),$$

where φ is a choice of polarization on A. Second, the rescaling cannot happen in $\mathbb{Q}_\ell$ because $\mathbb{Q}_\ell$ does not have enough square roots. As such, we must choose an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, allowing us to consider the elements $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ in the complex Lie group $G_\ell^1(A)_\iota(\mathbb{C})$.[1]

- Another piece of structure to keep track of is that $\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ is semisimple (see Remark 2.110). This means that the subgroup topological generated by $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ (which we now see has all eigenvalues equal to 1 after the normalization in the previous step) will be compact! A standard result in the structure theory of complex Lie groups is that they have maximal compact subgroups unique up to conjugacy, so one can find an element in our conjugacy class $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ in any given maximal compact subgroup of $G_\ell^1(A)_\iota(\mathbb{C})$.

With the above preparations, we are now ready to state the Sato–Tate conjecture.

Definition 3.4 (Sato–Tate group)**.** Fix an abelian variety A defined over a number field K, and choose a prime ℓ and an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Then we define the *Sato–Tate group* $\mathrm{ST}(A)$ to be a maximal compact subgroup of the complex Lie group $G_\ell^1(A)_\iota$, where $G_\ell^1(A)$ is the subset of $G_\ell(A)$ with multiplier equal to 1.

Conjecture 3.5 (Sato–Tate)**.** Fix an abelian variety A defined over a number field K, and choose a prime ℓ and an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. For each nonzero prime ideal $\mathfrak{p}$ of K such that A has good reduction at $\mathfrak{p}$, choose the conjugacy class $x_\mathfrak{p} \in \mathrm{Conj}(\mathrm{ST}(A))$ containing the conjugacy class $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$. Then the conjugacy classes $\{x_\mathfrak{p}\}$ equidistribute with respect to the pushforward of the Haar measure along $\mathrm{ST}(A) \to \mathrm{Conj}(\mathrm{ST}(A))$.

The relevance of the Sato–Tate conjecture for us is that it will let us numerically check that we have the correct ℓ-adic monodromy group; precisely how this is done will be explained in the subsequent subsections.

We will spend the rest of the present subsection making some remarks about Conjecture 3.5.

Remark 3.6. Not much is known about Conjecture 3.5. Roughly speaking, all known proofs prove something akin to modularity for not just the Galois representation attached to A but also its symmetric powers (and maybe more!).

- If A has complex multiplication, then this essentially follows from the Fundamental theorem of complex multiplication.
- For elliptic curves, the state of the art is [Bar+14; Bar+11], where the Sato–Tate conjecture is proven for elliptic curves over totally real and CM fields.
- These potential automorphy techniques were extended to some classes of abelian varieties by Johansson in [Joh17, Theorem 1].

One obnoxious defect of Conjecture 3.5 is that we must make choices regarding ℓ and ι. The choice ι is not so egregious because everything ought to descend to something algebraic, but it is quite unclear that $\mathrm{ST}(A)$ and even $G_\ell^1(A)$ does not depend crucially on ℓ. One expects $G_\ell(A)^\circ$ to not depend on ℓ by the Mumford–Tate conjecture (Conjecture 2.113). The relevant conjecture for the full group $G_\ell(A)$ is the Algebraic Sato–Tate conjecture [BK15, Conjecture 2.1].

[1] Another reason for passing to $\mathbb{C}$ is that groups in $\mathbb{C}$ have access to a good measure theory.

Conjecture 3.7 (Algebraic Sato–Tate)**.** Fix an abelian variety A defined over a number field K. Then there exists an algerbaic subgroup $\mathrm{AST}(A) \subseteq \mathrm{GL}_{2g}(\mathbb{Q})$ such that

$$\mathrm{AST}(A)_{\mathbb{Q}_\ell} = G_\ell^1(A)$$

for all primes ℓ.

This conjecture, being similar in spirit to the Mumford–Tate conjecture, has quite a bit known. For example, Banaszak and Kedlaya have shown this conjecture for products of abelian varieties of dimensions at most 3 [BK15, Theorem 6.11]. Roughly speaking, their proof boils down to the fact that one has $\mathrm{Hg}(A) = \mathrm{L}(A)^\circ$ in these small dimensions, which permits a direct computation of $\mathrm{AST}(A)$ along the lines of Proposition 2.127 (see Remark 2.128).

Remarkably, Farfán and Commelin have shown that the Algebraic Sato–Tate conjecture is implied by the Mumford–Tate conjecture in [CC22].

Theorem 3.8 (Farfán–Commelin)**.** Fix an abelian variety A defined over a number field K. If A satisfies the Mumford–Tate conjecture (Conjecture 2.113) that $G_\ell(A)^\circ = \mathrm{MT}(A)$ for all primes ℓ, then A satisfies the Algebraic Sato–Tate conjecture (Conjecture 3.7) that there exists an algebraic group $\mathrm{AST}(A) \subseteq \mathrm{GL}_{2g}(\mathbb{Q})$ such that $\mathrm{AST}(A)_{\mathbb{Q}_\ell} = G_\ell^1(A)$ for all primes ℓ.

Proof. The proof requires a discussion of Tannakian formalism, so we will not include it. We remark that they actually prove that the Mumford–Tate conjecture is equivalent to a more refined version of the Algebraic Sato–Tate conjecture with $\mathrm{AST}(A)$ equal to the "motivic Galois group" of A.Include proof if include abelian motives ∎

3.1.3 Some Examples

In this subsection, we compute some basic Sato–Tate groups. The general outline is to compute the Hodge or Mumford–Tate groups first, check the Mumford–Tate conjecture to get G_ℓ°, and then compute some Galois action to get G_ℓ. We begin with some elliptic curves.

Example 3.9 (no complex multiplication)**.** Consider the elliptic curve $E\colon y^2 = x^3 + x + 1$ over $\mathbb{Q}$. One can compute that $\mathrm{End}_{\mathbb{C}}(E) = \mathbb{Z}$, so E does not have complex multiplication. Thus, $\mathrm{Hg}(E) \subseteq \mathrm{SL}_{2,\mathbb{Q}}$ needs to be a connected reductive subgroup which is not a torus (see Proposition 2.53); however, the only Lie subalgebras of $\mathfrak{sl}_2(\mathbb{C})$ are either commutative or all of $\mathfrak{sl}_2(\mathbb{C})$, so we conclude that $\mathrm{Hg}(E) = \mathrm{SL}_{2,\mathbb{Q}}$. Thus, $\mathrm{MT}(E) = \mathrm{GL}_{2,\mathbb{Q}}$.

The same computation (with Remark 2.108) allows us to conclude that $G_\ell(E) = \mathrm{GL}_{2,\mathbb{Q}_\ell}$ for all primes ℓ, thus proving the Mumford–Tate conjecture (Conjecture 2.113) in this case. We thus find $G_\ell^1(E) = \mathrm{SL}_{2,\mathbb{Q}_\ell}$, so upon choosing $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we see that $G_\ell^1(E)_\iota = \mathrm{SL}_{2,\mathbb{C}}$, so choosing a maximal compact subgroup finds $\mathrm{ST}(E) = \mathrm{SU}_2$.

Example 3.10 (complex multiplication)**.** Consider the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$. Then we see that $\mathrm{End}_{\mathbb{C}}(E) = \mathbb{Z}[\zeta_3]$, where ζ_3 acts by $(x, y) \mapsto (\zeta_3 x, y)$, so E has complex multiplication. Thus, $\mathrm{Hg}(E) \subseteq \mathrm{SL}_{2,\mathbb{Q}(\zeta_3)}$ is a torus (by Proposition 2.53), but it cannot be trivial (by Corollary 2.42), so we conclude that $\mathrm{Hg}(E)$ is the diagonal torus of $\mathrm{SL}_{2,\mathbb{Q}(\zeta_3)}$.

For primes ℓ which split completely in $\mathbb{Q}(\zeta_3)$, the same computation (with Remark 2.108 and Corollary 2.107) where ℓ splits completely in $\mathbb{Q}_\ell$ reveals $G_\ell(E) = \mathbb{G}_{m,\mathbb{Q}_\ell}^2$ equals the diagonal torus in $\mathrm{GL}_{2,\mathbb{Q}(\zeta_3)}$, proving the Mumford–Tate conjecture (Conjecture 2.113) in this case. We thus find $G_\ell^1(E) \cong \mathbb{G}_{m,\mathbb{Q}_\ell}$, so upon choosing $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we see that $G_\ell^1(E) \cong \mathbb{G}_{m,\mathbb{Q}_\ell}$, so choosing a maximal compact subgroup finds $\mathrm{ST}(E) \cong \mathrm{U}_1$.

Example 3.11 (potential complex multiplication)**.** Consider the elliptic curve $E\colon y^2 = x^3+1$ but now over $\mathbb{Q}$. Example 3.10 computed that $\mathrm{MT}(E) \cong \mathbb{G}_{m,\mathbb{Q}}$ and $G_\ell(E)^\circ = \mathbb{G}_{m,\mathbb{Q}_\ell}$ (for primes $\ell \equiv 1 \pmod 3$). In this case, we see that there are endomorphisms not defined over $\mathbb{Q}$ and hence not fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so $K_E^{\mathrm{conn}} \neq \mathbb{Q}$; instead, these endomorphisms are defined over $K_E^{\mathrm{conn}} = \mathbb{Q}(\zeta_3)$. We thus see that $G_\ell(E) \subseteq \mathrm{GL}_{2,\mathbb{Q}_\ell}$ normalizes its index-2 subgroup $G_\ell(E)^\circ$ (which is the diagonal torus), so $G_\ell(E)$ must be the diagonal torus together with the nontrivial Weyl element in $\mathrm{GL}_{2,\mathbb{Q}_\ell}$, which we write as $\mathbb{G}_{m,\mathbb{Q}_\ell}^2 \rtimes S_2$. We thus find $G_\ell^1(E) \cong \mathbb{G}_{m,\mathbb{Q}_\ell} \rtimes S_2$, so $\mathrm{ST}(E) \cong \mathrm{U}_1 \rtimes S_2$.

Remark 3.12. In the above example, we appealed to the fact that the only elements normalizing the diagonal torus are the Weyl elements, which is a bit ad-hoc and will not work in higher dimensions. Roughly speaking, Proposition 2.127 provides the machine which works in higher dimensions, where we know that the Galois representation now factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$, and we are allowed to replace W with merely $W_1 \oplus W_2$, which can be computed to be generated by the endomorphisms and polarization.

We take a moment to remark that the above examples generalize to work with all elliptic curves, doing casework on having no complex multiplication, complex multiplication, and potential complex multiplication.

We now introduce the main example of the present thesis.

Proposition 3.13. Fix $\lambda \in \mathbb{C} \setminus \{0,1\}$, and define A to be the Jacobian of the normalization of the proper curve C with affine chart $y^9 = x(x-1)(x-\lambda)$. If A does not have complex multiplication, then

$$\begin{cases} \mathrm{MT}(A)_{\mathbb{C}}^{\mathrm{der}} \cong \mathrm{SL}_2(\mathbb{C})^3 \\ Z(\mathrm{MT}(A))_{\mathbb{C}}^{\circ} \cong \mathbb{G}_m^4. \end{cases}$$

We use this to compute $\mathrm{ST}(A_K)$ if $\lambda \in K$ and K contains K_A^{conn}.

Proof. We proceed in steps.

1. To begin, we do some prelimarinary algebraic geometry, along the lines of [Moo10, Section 1]. The curve C comes equipped with a natural map $x\colon C \to \mathbb{P}^1$, with Galois with cyclic Galois group μ_9, where μ_9 acts on C by multiplication of the y-coordinate. As such, a computation with the Riemann–Hurwitz formula reveals that the genus is $g = 7$, so $\dim A = 7$. From here, we can find the differentials

$$\left\{ \frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8} \right\}$$

   are all holomorphic on C, and they are linearly independent, so we see that this is a basis of the space of differentials in $\mathrm{H}^0(C, \Omega^1_{C/\mathbb{C}}) = \mathrm{H}^0(A, \Omega^1_{A/\mathbb{C}})$. We remark that the above is also an eigenbasis for the induced μ_9-action on $\mathrm{H}^0(A, \Omega^1_{A/\mathbb{C}})$.

2. We decompose A into pieces. Note that C projects onto the elliptic curve $C_0\colon y^3 = x(x-1)(x-\lambda)$ via the map $(x,y) \mapsto (x, y^3)$, so C_0 is a factor of A. One can see that the basis of differentials of C_0 is given by dx/y^2, which pulls back to the differential dx/y^6 on A. In this way, we see that the quotient $A_1 \coloneqq A/C_0$ will have $\mathrm{H}^0(A_1, \Omega^1_{A_1/\mathbb{C}})$ have a basis given by

$$\left\{ \frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8} \right\}.$$

   Note that we do not yet know if A_1 is simple!

3. We compute some endomorphism algebras. Note C_0 has $\mu_3 \subseteq \mathrm{Aut}(C_0)$ where ζ_3 acts by multiplication on the y-coordinate, so C_0 has complex multiplication by $F_0 \coloneqq \mathbb{Q}(\zeta_3)$.

We conclude this step by showing that A_1 is simple. This will follow from the fact that A does not have complex multiplication. Note the μ_9-action on A fixes C_0 (we can be seen on the level of the Hodge structure), so it must also fix A_1, so we see $\mathbb{Q}(\zeta_9) \subseteq \mathrm{End}_{\mathbb{C}}(A_1)_{\mathbb{Q}}$. Thus, A_1 contains an isotypic component B^r (where B is simple) such that

$$\mathbb{Q}(\zeta_9) \subseteq \mathrm{End}_{\mathbb{C}}(B^r) = M_r\left(\mathrm{End}_{\mathbb{C}}(\mathrm{H}^1_{\mathrm{B}}(B,\mathbb{C}))\right).$$

As such, we set $D \coloneqq \mathrm{End}_{\mathbb{C}}(B)$ and $F \coloneqq Z(D)$ so that $d \coloneqq \sqrt{[D:F]}$ and $e \coloneqq [F:\mathbb{Q}]$ satisfy $6 \mid rde$ (because $\mathbb{Q}(\zeta_9)$ is contained in a maximal subfield of $M_r(D)$) and $r^2d^2e \leq 2\dim A_1 = 12$. If we had $r^2d^2e = 12$, then A_1 would have complex multiplication, which contradicts the fact that A does not have complex multiplication. Thus, we must instead have $rde = r^2d^2e = 6$, which implies that $r = d = 1$ and so $A_1 = B$ with $\mathrm{End}_{\mathbb{C}}(A_1)$ given exactly by $F_1 \coloneqq \mathbb{Q}(\zeta_9)$.

4. We compute some signatures. We begin with C_0. Letting $\tau_i \in \mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$ be given by $\tau_i(\zeta_3) \coloneqq \zeta_3^i$ for $i \in \{1,2\}$, we see that the signature $\Phi_0\colon \mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) \to \mathbb{Z}_{\geq 0}$ of E_0 is thus given by $\Phi_0(\tau_1) = 1$ and $\Phi_0(\tau_2) = 0$ because the second step provided an (eigen)basis of $\mathrm{H}^{10}(C_0) = \mathrm{H}^0(C_0, \Omega^1_{C_0/\mathbb{C}})$.

   We next consider A_1. The second step provided a basis of $\mathrm{H}^{10}(A_1) = \mathrm{H}^0(A_1, \Omega^1_{A_1/\mathbb{C}})$. As such, we define $\sigma_i \in \mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ to be the automorphism given by $\sigma_i(\zeta_9) \coloneqq \zeta_9^i$ for $i \in \{1,2,4,5,7,8\}$, and we are able to compute that our signature $\Phi_1\colon \mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q}) \to \mathbb{Z}_{\geq 0}$ is given by

$$\Phi(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{7,8\}, \\ 1 & \text{if } i \in \{4,5\}, \\ 2 & \text{if } i \in \{1,2\}. \end{cases}$$

5. We compute $\mathrm{MT}(A)^{\mathrm{der}}$; note that this equals $\mathrm{Hg}(A)^{\mathrm{der}}$ by Lemma 1.41. By Lemma 1.56, we have an inclusion

$$\mathrm{Hg}(A) \to \mathrm{Hg}(C_0) \oplus \mathrm{Hg}(A_1)$$

   which surjects onto each factor. Now, C_0 has complex multiplication, so $\mathrm{Hg}(C_0)$ is a torus by Proposition 2.53, so $\mathrm{Hg}(A)^{\mathrm{der}}$ has trivial projection onto $\mathrm{Hg}(C_0)$. We conclude that the above inclusion upgrades into an isomorphism $\mathrm{Hg}(A)^{\mathrm{der}} \to \mathrm{Hg}(A_1)^{\mathrm{der}}$.

   To compute $\mathrm{Hg}(A_1)^{\mathrm{der}}$, we use Proposition 2.120 to see that this equals $\mathrm{L}(A_1)^{\mathrm{der}}$, so we complete this step by noting that $\mathrm{L}(A_1)^{\mathrm{der}}_{\mathbb{C}} \cong \mathrm{SL}_2(\mathbb{C})^3$ by the computation in Lemma 1.68.

6. We compute $Z(\mathrm{MT}(A))^\circ_{\mathbb{C}}$. We use Proposition 2.86 and in particular the discussion following the proof. Indeed, set $L \coloneqq \mathbb{Q}(\zeta_9)$, which we note is a Galois extension of $\mathbb{Q}$ containing F_0F_1. Then we note that $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$, where $F \coloneqq F_0 \times F_1$ has $(\mathrm{T}_F)_L$ embedded into $\mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A,L)\right)$ as a subtorus of the diagonal torus. Explicitly, we can choose an F-eigenbasis of $\mathrm{H}^1_{\mathrm{B}}(A,L) = \mathrm{H}^1_{\mathrm{B}}(C_0,L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_1,L)$ as

$$\{u_1, u_2, v_1, v_1', v_2, v_2', v_4, v_4', v_5, v_5', v_7, v_7', v_8, v_8'\},$$

   where the subscript partially indicates the F-eigenvalue. (For technical reasons, we will want to know that $\{v_i, v_i'\}$ is a dual basis for $\{v_{9-i}, v_{9-i}'\}$ according to the polarization.) Then we see that $(\mathrm{T}_F)_L \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A,L)\right)$ embeds as

$$\{\mathrm{diag}(\mu_1,\mu_2,\lambda_1,\lambda_1,\lambda_2,\lambda_2,\lambda_4,\lambda_4,\lambda_5,\lambda_5,\lambda_7,\lambda_7,\lambda_8,\lambda_8) : \mu_\bullet, \lambda_\bullet \in \mathbb{G}_{m,L}\}.$$

   The discussion following Proposition 2.86 explains that equations cutting out $Z(\mathrm{MT}(A))^\circ_L \subseteq (\mathrm{T}_F)_L$ can be viewed as elements of the kernel of the map

$$\mathrm{X}^*\left((\mathrm{N}_{\Phi_0^*}, \mathrm{N}_{\Phi_1^*})\right) : \mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(\mathrm{T}_L).$$

Using the established bases for these lattices, we see that our map can be written as the matrix

$$\begin{array}{c|cc|cccccc} & \mu_1 & \mu_2 & \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 & \lambda_7 & \lambda_8 \\ \sigma_1 & 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 \\ \sigma_2 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 \\ \sigma_4 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\ \sigma_5 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 0 \\ \sigma_7 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 1 \\ \sigma_8 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \end{array}.$$

Then one can compute a basis of the kernel of the matrix, which tells us that $Z(\mathrm{MT}(A))^\circ_L \subseteq (\mathrm{T}_F)_L$ is cut out by the equations

$$\begin{aligned} \lambda_1\lambda_8 &= \lambda_2\lambda_7, \\ \lambda_1\lambda_8 &= \lambda_4\lambda_5, \\ \mu_1\mu_2\lambda_7 &= \lambda_5\lambda_8, \\ \lambda_1\lambda_4\lambda_7 &= \lambda_2\lambda_5\lambda_8. \end{aligned}$$

Thus, we see that $Z(\mathrm{MT}(A))^\circ_{\mathbb{C}} \cong \mathbb{G}^4_{m,\mathbb{C}}$ with isomorphism given by the cocharacters $(\mu_1, \lambda_1, \lambda_4, \lambda_8)$.

7. We use the previous steps to compute $G^1_\ell(A)$ when ℓ splits completely in K^{conn}_A. Recall we notably know the Mumford–Tate conjecture that $G_\ell(A)^\circ = \mathrm{MT}(A)_{\mathbb{Q}_\ell}$ by Proposition 2.120. Thus, we choose ℓ to split completely in K^{conn}_A so that $\mathbb{Q}(\zeta_9) \subseteq \mathbb{Q}_\ell$, allowing us to engage in the diagonalization of the previous step. For example, the computation in Lemma 1.68 reveals that the isomorphism between $\mathrm{L}(A)^{\mathrm{der}}$ and SL_2^3 is defined over L (indeed, one merely needs to be able to take L-eigenspaces), so we find that
$$G_\ell(A)^{\mathrm{der}} = \left\{\mathrm{diag}\left(1_2, g_1, g_2, g_4, g_4^{-\intercal}, g_2^{-\intercal}, g_1^{-\intercal}\right) : g_1, g_2, g_3 \in \mathrm{SL}_{2,\mathbb{Q}_\ell}\right\}.$$
Continuing, we add in the equation $\det g = 1$ to the equations cutting out $Z(G_\ell(A_L))^\circ \subseteq (\mathrm{T}_F)_{\mathbb{Q}_\ell}$ given in the previous step. This reveals that $Z\left(G^1_\ell(A_L)\right)^\circ \subseteq (\mathrm{T}_F)_{\mathbb{Q}_\ell}$ is cut out by the equations
$$\begin{aligned} \mu_1\mu_2 &= 1, \\ \lambda_1\lambda_8 &= 1, \\ \lambda_2\lambda_7 &= 1, \\ \lambda_4\lambda_5 &= 1, \\ \lambda_2 &= \lambda_1\lambda_4. \end{aligned}$$
In particular, we see that $Z\left(G^1_\ell(A)\right)^\circ \cong \mathbb{G}^3_{m,\mathbb{Q}_\ell}$ given by the cocharacters $(\mu_1, \lambda_1, \lambda_4)$. In total, we find $G^1_\ell(A) \subseteq \mathrm{GL}_{14,\mathbb{Q}_\ell}$ equals
$$\left\{\mathrm{diag}\left(\mu_1, \mu_1^{-1}, \lambda_1 g_1, \lambda_1\lambda_4 g_2, \lambda_4 g_4, \lambda_4^{-1} g_4^{-\intercal}, \lambda_1^{-1}\lambda_4^{-1} g_2^{-\intercal}, \lambda_1^{-1} g_1^{-\intercal}\right) : \mu_\bullet, \lambda_\bullet \in \mathbb{G}_{m,\mathbb{Q}_\ell}, g_\bullet \in \mathrm{SL}_{2,\mathbb{Q}_\ell}\right\}.$$

8. At last, we compute $\mathrm{ST}(A_K)$ where K contains K^{conn}_A. By Theorem 3.8, we see that $\mathrm{ST}(A)$ does not depend on the choice ℓ, so we may as well choose ℓ to split completely in K^{conn}_A. Then we simply base-change the result of the previous step to $\mathbb{C}$, and then we may take maximal compact subgroups to see ST is
$$\left\{\mathrm{diag}\left(\mu_1, \mu_1^{-1}, \lambda_1 g_1, \lambda_1\lambda_4 g_2, \lambda_4 g_4, \lambda_4^{-1} g_4^{-\intercal}, \lambda_1^{-1}\lambda_4^{-1} g_2^{-\intercal}, \lambda_1^{-1} g_1^{-\intercal}\right) : \mu_\bullet, \lambda_\bullet \in \mathrm{U}_1, g_\bullet \in \mathrm{SU}_2\right\}.$$
(It is not too hard to see that the product of maximal compact subgroups continues to be a maximal compact subgroup.) This completes the computation. ■

Remark 3.14. Note that $\mathrm{MT}(A) \neq \mathrm{L}(A)$ because the centers are different! This continues to be visible in the Sato–Tate group: the first four equations $\mu_1\mu_2 = \lambda_1\lambda_8 = \lambda_2\lambda_7 = \lambda_4\lambda_5 = 1$ can be explained by the polarization (see Lemma 2.65), but the last equation $\lambda_2 = \lambda_1\lambda_4$ corresponds to an exceptional Hodge class not generated by endomorphisms or the polarization.

Remark 3.15. Up to squaring, one can replace the equation $\mu_1\mu_2\lambda_7 = \lambda_5\lambda_8$ with the equation $\lambda_1\lambda_8 = \mu_1^2\mu_2^2$, thus making it clear that it arises from the polarization. Note this squaring is not too much of an issue because we had to take a determinant in Remark 2.78 anyway; in particular, by looking at the end result of the computation, we do see that $\mathrm{MT}(A)$ contains the diagonalizable group cut out by our equations where we have done the replacement with $\lambda_1\lambda_8 = \mu_1^2\mu_2^2$.

The hypothesis that A fails to have CM is necessary, as we will see in the following two examples.

Proposition 3.16. Define A to be the Jacobian of the proper curve C with affine chart $y^9 = x^3 - 1$. Then $\mathrm{MT}(A)_{\mathbb{C}}$ is a torus isomorphic to $\mathbb{G}_{m,\mathbb{C}}^4$. We use this to compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}.

Proof. We proceed in steps, following Proposition 3.13.

1. To begin, we once again note that C has genus 7, so A has dimension 7, and we have a basis of holomorphic differentials given by
$$\left\{\frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8}\right\}.$$
This time around, we see that $\mu_3 \times \mu_9$ acts on C by coordinate-wise multiplication on $(x, y) \in C$.

2. We decompose A into pieces.

   - Note C projects onto $C_0 \colon y^3 = x^3 - 1$ by $(x, y) \mapsto \left(x, y^3\right)$. (This is the quotient of C by $\mu_3 \times 1$.) We see that C_0 is an elliptic curve, and it has complex multiplicatino by μ_3; for example, μ_3 can act by multiplication on y. One can compute that C_0 has a basis of holomorphic differentials given by dx/y^2, which pulls back to the differential dx/y^6 on C.
   - Note C projects onto the proper curve C_1 with affine chart $y^9 = x^3(x-1)$ by $(x, y) \mapsto \left(x^3, xy\right)$, so A has $A_1 \coloneqq \mathrm{Jac}\, C_1$ as a factor.[2] (This is the quotient of C by $\mu_3 \subseteq \mu_3 \times \mu_9$ embedded by $\zeta \mapsto (\zeta, \overline{\zeta})$.) One can compute that C_1 is genus 3 using the Riemann–Hurwitz formula, and then we can compute that it has a basis of holomorphic differentials given by $\left\{x^2\,dx/y^8, x^2\,dx/y^7, x\,dx/y^5\right\}$, which pull back to the differentials $\left\{dx/y^8, x\,dx/y^7, dx/y^5\right\}$ on C (up to a scalar).

     Note that C_1 has an action by μ_9 by multiplying on the y-coordinate, so $\mathbb{Q}(\zeta_9) \subseteq \mathrm{End}_{\mathbb{C}}(A_1)_{\mathbb{Q}}$. However, $\dim A_1 = 3$, so we see that A_1 has complex multiplication. We will check that A_1 is simple shortly.
   - Note C projects onto the proper curve C_2 with affine chart $y^9 = x^6(x-1)$ by $(x, y) \mapsto \left(x^3, x^2y\right)$, so A has $A_2 \coloneqq \mathrm{Jac}\, C_2$ as a factor. (This is the quotient of C by $\mu_3 \subseteq \mu_3 \times \mu_9$ embedded by $\zeta \mapsto (\zeta, \zeta)$.) One can compute that C_2 has genus 3 using the Riemann–Hurwitz formula, and then we can compute that it has a basis of holomorphic differentials given by $\left\{x^5\,dx/y^8, x^4\,dx/y^7, x^2\,dx/y^4\right\}$, which pull back to the differentials $\left\{x\,dx/y^8, dx/y^7, dx/y^4\right\}$ on C (up to a scalar).

     Note that C_2 has an action by μ_9 by multiplying on the y-coordinate, so $\mathbb{Q}(\zeta_9) \subseteq \mathrm{End}_{\mathbb{C}}(A_2)_{\mathbb{Q}}$. However, $\dim A_2 = 3$, so we see that A_2 has complex multiplication. We will check that A_2 is simple shortly.

[2] Technically, we should take normalizations everywhere. We will omit these normalizations.

We spend a moment checking that A is isogenous to $C_0 \times A_1 \times A_2$. The above computations have provided a map $C_0 \times A_1 \times A_2 \to A$, so it is enough to check that this is an isomorphism after base-changing to $\mathbb{C}$. The computations above have shown that this map provides an isomorphism

$$\mathrm{H}^0\left(A, \Omega^1_{A/\mathbb{C}}\right) \to \mathrm{H}^0\left(C_0, \Omega^1_{C_0/\mathbb{C}}\right) \oplus \mathrm{H}^0\left(A_1, \Omega^1_{A_1/\mathbb{C}}\right) \oplus \mathrm{H}^0\left(A_2, \Omega^1_{A_2/\mathbb{C}}\right).$$

(We take a moment to remark that the right-hand side is even a decomposition of $\mathrm{H}^0\left(A, \Omega^1_{A/\mathbb{C}}\right)$ into μ_3-eigenspaces!) This corresponds to an isomorphism on one piece of the Hodge structure, which we note upgrades to an isomorphism of Hodge structures because the relevant Hodge structures are concentrated in $(0,1)$ and $(1,0)$, which are complex conjugates. We conclude that A is isogenous to $C_0 \times A_1 \times A_2$ by Theorem 2.40.

3. We compute some signatures. For our notation, we let $F_0 \coloneqq \mathbb{Q}(\zeta_3)$ have the embeddings $\{\tau_1, \tau_2\}$, where $\tau_\bullet \in \mathrm{Gal}(F_0/\mathbb{Q})$ sends $\zeta_3 \mapsto \zeta_3^\bullet$; similarly, we let $F_1 = F_2 \coloneqq \mathbb{Q}(\zeta_9)$ have the embeddings $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_7, \sigma_8\}$ where $\sigma_\bullet \in \mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ sends $\zeta_9 \mapsto \zeta_9^\bullet$. Here are our signatures.

   - On C_0, we see that H^{10} is spanned by dx/y^2, so with μ_3 acting on y, we get the signature $\Phi_0(\tau_1) = 1$ and $\Phi_0(\tau_2) = 0$.
   - On C_1, we see that H^{10} has basis given by $\left\{x^2\,dx/y^8, x^2\,dx/y^7, x\,dx/y^5\right\}$. Thus, with μ_9 acting on y, we get the signature
   $$\Phi_1(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{5,7,8\}, \\ 1 & \text{if } i \in \{1,2,4\}. \end{cases}$$
   One can check that Φ_1 satisfies the check of Remark 2.57, proving that A_1 is simple.
   - On C_2, we see that H^{10} has basis given by $\left\{x\,dx/y^8, dx/y^7, dx/y^4\right\}$. Thus, with μ_9 acting on y, we get the signature
   $$\Phi_2(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{4,7,8\}, \\ 1 & \text{if } i \in \{1,2,5\}. \end{cases}$$
   One can check that Φ_2 satisfies the check of Remark 2.57, proving that A_1 is simple.

   The above computation allows us to conclude that we have decomposed A into simple abelian varieties with complex multiplication.

4. We compute $\mathrm{MT}(A)_{\mathbb{C}}$. Because A has complex multiplication, we see that $\mathrm{MT}(A)$ is a torus by Proposition 2.53 embedded in T_F, where $F \coloneqq F_0 \times F_1 \times F_2$. As such, we may use Proposition 2.86 and the surrounding discussion following the proof to compute equations cutting out $\mathrm{MT}(A) \subseteq \mathrm{T}_F$. In particular, set $L \coloneqq \mathbb{Q}(\zeta_9)$, which we note is a Galois extension of $\mathbb{Q}$ containing $F_0F_1F_2$. Then we note that $\mathrm{H}^1_{\mathrm{B}}(A, L) = \mathrm{H}^1_{\mathrm{B}}(C_0, L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_1, L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_2, L)$ can be given a basis

   $$\{u_1, u_2, v_1, v_2, v_4, v_5, v_7, v_8, w_1, w_2, w_4, w_5, w_7, w_8\},$$

   where the subscript partially indicates the F-eigenvalue. Then we see that $(\mathrm{T}_F)_L \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A, L)\right)$ embeds as

   $$\{\mathrm{diag}(\mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_7, \lambda_8, \kappa_1, \kappa_2, \kappa_4, \kappa_5, \kappa_7, \kappa_8) : \mu_\bullet, \lambda_\bullet, \kappa_\bullet \in \mathbb{G}_{m,L}\}.$$

   The discussion following Proposition 2.86 explains that equations cutting out $Z(\mathrm{MT}(A))^\circ_L \subseteq (\mathrm{T}_F)_L$ can be viewed as elements of the kernel of the map

   $$\mathrm{X}^*\left((\mathrm{N}_{\Phi_0^*}, \mathrm{N}_{\Phi_1^*}, \mathrm{N}_{\Phi_2^*})\right) : \mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(T_L).$$

Using the established bases for these lattices, we see that our map can be written as the matrix

$$\begin{array}{c|cc|cccccc|cccccc} & \mu_1 & \mu_2 & \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 & \lambda_7 & \lambda_8 & \kappa_1 & \kappa_2 & \kappa_4 & \kappa_5 & \kappa_7 & \kappa_8 \\ \sigma_1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \sigma_2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \sigma_4 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \sigma_5 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \sigma_7 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \sigma_8 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}.$$

Then one can compute a basis of the kernel of the matrix, which tells us that $\mathrm{MT}(A)_L \subseteq (\mathrm{T}_F)_L$ is cut out by the following equations. To begin, it turns out that $(A_1)_L$ and $(A_2)_L$ are isogenous, which we can see from the six equations

$$\begin{aligned} \lambda_1 &= \kappa_5, \\ \lambda_2 &= \kappa_1, \\ \lambda_4 &= \kappa_2, \\ \lambda_5 &= \kappa_7, \\ \lambda_7 &= \kappa_8, \\ \lambda_8 &= \kappa_4. \end{aligned}$$

(Namely, these equations imply an isomorphism of $\mathrm{MT}(A)$-representations $\mathrm{H}^1_{\mathrm{B}}(A_1, L) \cong \mathrm{H}^1_{\mathrm{B}}(A_2, L)$ and hence an isomorphism of Hodge structures, which gives the isogeny by Theorem 2.40.) Then there are the equations given by the polarization (via Lemma 2.65)

$$\begin{aligned} \mu_1\mu_2 &= \kappa_1\kappa_8, \\ \kappa_1\kappa_8 &= \kappa_2\kappa_7, \\ \kappa_1\kappa_8 &= \kappa_4\kappa_5. \end{aligned}$$

Lastly, there is the exceptional equation

$$\mu_1\kappa_7 = \kappa_5\kappa_8.$$

In total, we find that $\mathrm{MT}(A)_L$ is a torus isomorphic to $\mathbb{G}^4_{m,L}$ via the cocharacters $(\kappa_1, \kappa_2, \kappa_4, \kappa_8)$.

5. We use the previous step to compute $G^1_\ell(A_K)$ when ℓ splits completely in $K \coloneqq K_A^{\mathrm{conn}}$. Recall that we know the Mumford–Tate conjecture that $G_\ell(A)^\circ = \mathrm{MT}(A)_{G_\ell}$ by Remark 2.118. Thus, we choose ℓ to split completely in K_A^{conn} so that $L \subseteq \mathbb{Q}_\ell$, allowing us to engage in the diagonalization of the previous step. Now, to compute $G^1_\ell(A_K)$ from $G_\ell(A_K)$, we simply need to add in the equation that the multipler is 1. This reveals that $G^1_\ell(A_{K_A^{\mathrm{conn}}}) \subseteq (\mathrm{T}_F)_{\mathbb{Q}_\ell}$ is cut out by the following equations. As before, we have the six equations

$$\begin{aligned} \lambda_1 &= \kappa_5, \\ \lambda_2 &= \kappa_1, \\ \lambda_4 &= \kappa_2, \\ \lambda_5 &= \kappa_7, \\ \lambda_7 &= \kappa_8, \\ \lambda_8 &= \kappa_4 \end{aligned}$$

given by the isogeny $(A_1)_L \sim (A_2)_L$, and we have the equations given by the polarization

$$\begin{aligned} \mu_1\mu_2 &= 1, \\ \kappa_1\kappa_8 &= 1, \\ \kappa_2\kappa_7 &= 1, \\ \kappa_4\kappa_5 &= 1. \end{aligned}$$

Lastly, there is still the exceptional equation

$$\mu_1\kappa_7 = \kappa_5\kappa_8.$$

In total, we find that $G_\ell^1(A)$ is a torus isomorphic to $\mathbb{G}_{m,L}^3$ via the cocharacters $(\kappa_1, \kappa_2, \kappa_4)$. In total, we see $G_\ell^1(A_K)^\circ \subseteq \mathrm{GL}_{14}$ is

$$\left\{\operatorname{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4}, \frac{\kappa_1\kappa_4}{\kappa_2}, \kappa_4^{-1}, \kappa_1, \kappa_2, \kappa_2^{-1}, \kappa_1^{-1}, \kappa_4, \kappa_1, \kappa_2, \kappa_4, \kappa_4^{-1}, \kappa_2^{-1}, \kappa_1^{-1}\right) : \kappa_\bullet \in \mathbb{G}_{m,\mathbb{Q}_\ell}\right\}.$$

6. At last, we compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}. By Theorem 3.8, we see that ST does not depend on the choice of ℓ, so we may as well choose ℓ to split completely in K_A^{conn}. Then we may simply base-change the result of the previous step to $\mathbb{C}$, and then we may take maximal compact subgroups to see ST is

$$\left\{\operatorname{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4}, \frac{\kappa_1\kappa_4}{\kappa_2}, \kappa_4^{-1}, \kappa_1, \kappa_2, \kappa_2^{-1}, \kappa_1^{-1}, \kappa_4, \kappa_1, \kappa_2, \kappa_4, \kappa_4^{-1}, \kappa_2^{-1}, \kappa_1^{-1}\right) : \kappa_\bullet \in \mathrm{U}_1\right\}.$$

Once again, we remark that the product of maximal compact subgroups continues to be maximal compact. ■

Proposition 3.17. Define A to be the Jacobian of the proper curve C with affine chart $y^9 = x\left(x^2+1\right)$. Then $\mathrm{MT}(A)_{\mathbb{C}}$ is a torus isomorphic to $\mathbb{G}_{m,\mathbb{C}}^4$. We use this to compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}.

Proof. This argument is essentially the same as Proposition 3.16, so we will be a bit briefer.

1. Once again, we see that C has genus 7, so A has dimension 7, and we have a basis of holomorphic differentials given by
$$\left\{\frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8}\right\}.$$
This time around, we see that μ_{18} acts on C by $\zeta_{18}\cdot(x,y) = (-x, -\zeta_9 y)$.

2. We decompose A into pieces.

   - As usual, C_0 projects onto $y^3 = x\left(x^2+1\right)$ by $(x,y) \mapsto \left(x, y^3\right)$. (This is the quotient of C by μ_3.) The Riemann–Hurwitz formula yields that C_0 is an elliptic curve with complex multiplication by μ_3 acting on the y-coordinate. We see that C_0 has a basis of holomorphic differentials given by dx/y^2, which pulls back to dx/y^6 ln C.
   - Now, C projects onto the proper curve C_1 with affine chart $y^9 = x^5(x+1)$ by $(x,y) \mapsto \left(x^2, xy\right)$, so A has $A_1 \coloneqq \operatorname{Jac} C_1$ as a factor. (This is the quotient of C by μ_2.) The Riemann–Hurwitz formula implies that C_1 has genus 3, and then we can compute that it has a basis of holomorphic differentials given by $\left\{x^4\,dx/y^8, x^3\,dx/y^7, x^2\,dx/y^5\right\}$, which pulls back to $\left\{x\,dx/y^8, dx/y^7, dx/y^5\right\}$ on C (up to scalar).

     Note that C_1 has an action by μ_9 acting on the y-coordinate, so $\mathbb{Q}(\zeta_9) \subseteq \mathrm{End}_{\mathbb{C}}(A_1)_{\mathbb{Q}}$. We will check in the next step that A_1 is simple by computing its signature and applying Remark 2.57.

   We can see on the level of differentials that the induced map $C_0 \times A_1 \to A$ is injective, so we let A_2 be the cokernel. In terms of Hodge structures, we can see from the computation that

$$\mathrm{H}_{\mathrm{B}}^1(A,\mathbb{Q}) = \mathrm{H}_{\mathrm{B}}^1(C_0,\mathbb{Q}) \oplus \mathrm{H}_{\mathrm{B}}^1(A_1,\mathbb{Q}) \oplus \mathrm{H}_{\mathrm{B}}^1(A_2,\mathbb{Q})$$

   is a decomposition of μ_{18}-representations because the left two spaces on the right-hand side are stable under the μ_{18}-action. We conclude that $\mathbb{Q}(\zeta_9) \subseteq \mathrm{End}_{\mathbb{C}}(A_2)_{\mathbb{Q}}$ as well.

3. We compute some signatures. As before, we let $F_0 \coloneqq \mathbb{Q}(\zeta_3)$ have $\{\tau_1, \tau_2\} = \mathrm{Gal}(\mathbb{Q}(F_0)/\mathbb{Q})$ where $\tau_\bullet \colon \zeta_3 \mapsto \zeta_3^\bullet$, and we let $F_1 = F_2 \coloneqq \mathbb{Q}(\zeta_9)$ have $\{\sigma_1, \dots, \sigma_8\} = \mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ has $\sigma_\bullet \colon \zeta_9 \mapsto \zeta_9^\bullet$.

   - On C_0, we look at the μ_9-eigenbasis of H^{10} to conclude that our signature has $\Phi_0(\tau_1) = 1$ and $\Phi_0(\tau_2)$.
   - On C_1, we look at the μ_9-eigenbasis of H^{10} to conclude that our signature is

$$\Phi_1(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{5,7,8\}, \\ 1 & \text{if } i \in \{1,2,4\}. \end{cases}$$

     One can check that Φ_1 satisfies the check of Remark 2.57, proving that A_1 is simple.
   - On A_2, we take the remaining differentials from A to find that our signature is

$$\Phi_2(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{4,7,8\}, \\ 1 & \text{if } i \in \{1,2,5\}. \end{cases}$$

     Again, one checks that Φ_2 satisfies the check of Remark 2.57

4. At this point, we recognize that our signatures are the same as in Proposition 3.16 up to swapping Φ_1 and Φ_2. Thus, up to some reordering of letters, the exact same computation goes through. Let's provide the result.

   To be explicit, we give $\mathrm{H}^1_{\mathrm{B}}(A, L) = \mathrm{H}^1_{\mathrm{B}}(C_0, L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_1, L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_2, L)$ a basis

$$\{u_1, u_2, v_1, v_2, v_4, v_5, v_7, v_8, w_1, w_2, w_4, w_5, w_7, w_8\},$$

   where the subscript partially indicates the F-eigenvalue, where $F \coloneqq F_0 \times F_1 \times F_2$. Then we set $L \coloneqq \mathbb{Q}(\zeta_9)$, and we see $(\mathrm{T}_F)_L \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A, L)\right)$ embeds as

$$\left\{\mathrm{diag}(\mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_7, \lambda_8, \kappa_1, \kappa_2, \kappa_4, \kappa_5, \kappa_7, \kappa_8) : \mu_\bullet, \kappa_\bullet, \lambda_\bullet \in \mathbb{G}_{m,L}\right\}.$$

   With this choice of lettering, the equations that end up cutting out $\mathrm{MT}(A)_L \subseteq (\mathrm{T}_F)_L$ are exactly the same, so $\mathrm{MT}(A)_L \cong \mathbb{G}^4_{m,L}$ via the cocharacters $(\kappa_1, \kappa_2, \kappa_4, \kappa_8)$.

   One is now able to compute $G^1_\ell(A)$ in the case where ℓ splits completely in $K \coloneqq K_A^{\mathrm{conn}}$. One finds the exact same equations via the same computation, so we find $G^1_\ell(A_K) \subseteq \mathrm{GL}_{14}$ is given by

$$\left\{\mathrm{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4}, \frac{\kappa_1\kappa_4}{\kappa_2}, \kappa_4^{-1}, \kappa_1, \kappa_2, \kappa_2^{-1}, \kappa_1^{-1}, \kappa_4, \kappa_1, \kappa_2, \kappa_4, \kappa_4^{-1}, \kappa_2^{-1}, \kappa_1^{-1}\right) : \kappa_\bullet \in \mathbb{G}_{m,\mathbb{Q}_\ell}\right\}.$$

   Base-changing to $\mathbb{C}$ and taking a maximal compact subgroup, we find $\mathrm{ST}(A_K)$ is

$$\left\{\mathrm{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4}, \frac{\kappa_1\kappa_4}{\kappa_2}, \kappa_4^{-1}, \kappa_1, \kappa_2, \kappa_2^{-1}, \kappa_1^{-1}, \kappa_4, \kappa_1, \kappa_2, \kappa_4, \kappa_4^{-1}, \kappa_2^{-1}, \kappa_1^{-1}\right) : \kappa_\bullet \in \mathrm{U}_1\right\},$$

   as required. ∎

3.1.4 Moment Statistics

In this subsection, we explain how to numerically verify the Sato–Tate conjecture (Conjecture 3.7). Fix an abelian variety A of dimension g defined over a number field K, and choose a prime ℓ and embedding $\iota \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$; for example, this allows us to define the usual ℓ-adic representation $\rho_\ell \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$.

The main idea is that the map sending $g \in \mathrm{ST}(A)$ to the characteristic polynomial of $g \in \mathrm{GL}_{2g}(\mathbb{C})$ is well-defined up to conjugacy classes, so it defines a (continuous) map $\mathrm{Conj}(\mathrm{ST}(A)) \to \mathbb{C}^{2g+1}$, where $\mathbb{C}^{2g+1}$ simply lists out the coefficients of the characteristic polynomial. In this way, we can push the Haar measure on $\mathrm{ST}(A)$ all the way to $\mathbb{C}^{2g+1}$ to compute what the distribution of the characteristic polynomial will be.

Of course, in practice, it may be difficult to compute the characteristic polynomial of

$$\left[\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right] \in \mathrm{Conj}(\mathrm{ST}(A))$$

for some prime $\mathfrak{p}$ of K such that A has good reduction at $\mathfrak{p}$. For our application, we will only be interested in superelliptic curves, for which this can be computed in SageMath [Aru+19]. To help out the computation a bit more, we make two quick remarks.

Remark 3.18. Let $P(T)$ be the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ acting on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}_\mathfrak{p}}}, \mathbb{Q}_\ell)$. Then we remark that $P(1)$ has a geometric interpretation as $\#A(\mathbb{F}_\mathfrak{p})$.

Remark 3.19. It suffices to only consider primes $\mathfrak{p}$ which are totally split in K because such primes have density 1. This is helpful because primes that split $\mathfrak{p}$ completely have residue fields isomorphic to $\mathbb{F}_p$ where $p \in \mathbb{Z}$ is the prime sitting below $\mathfrak{p}$, so we are frequently able to reduce the computation to something only involving integral coefficients.

As before, let's begin with some elliptic curve examples. Here, we note that the characteristic polynomial of $\frac{1}{\mathrm{N}(\mathfrak{p})}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ will have degree 2, with leading coefficient 1, and the condition on the multiplier (from Proposition 3.3) implies that the constant coefficient is 1. Thus, we see that the only interesting coefficient of the characteristic polynomial is given by the trace.

Lemma 3.20. The map $\mathrm{tr}\colon \mathrm{Conj}(\mathrm{SU}_2) \to [-2,2]$ is a homeomorphism, and the pushforward of the normalized Haar measure of SU_2 onto $\mathrm{Conj}(\mathrm{SU}_2) = [-2,2]$ is given by the semicircle measure $\frac{1}{2\pi}\sqrt{4-t^2}\,dt$.

Proof. We show the claims separately.

1. We show that $\mathrm{tr}\colon \mathrm{Conj}(\mathrm{SU}_2) \to [-2,2]$ is a well-defined homeomorphism. Note that $\mathrm{tr}\colon \mathrm{Conj}(\mathrm{SU}_2) \to \mathbb{C}$ is continuous, and all spaces in sight are compact and Hausdorff, so it is enough to check that tr is a bijection.

   A priori, tr is only defined as a map $\mathrm{tr}\colon \mathrm{Conj}(\mathrm{SU}_2) \to \mathbb{C}$. To begin, we note that any element of SU_2 is diagonalizable by a unitary matrix, and the corresponding diagonal matrix must then look like $\mathrm{diag}(\lambda, \overline{\lambda})$ where $|\lambda|^2 = 1$. By writing $\lambda = e^{i\theta}$, we see that the trace of this element is $2\cos\theta$, so we see that $\mathrm{tr}\colon \mathrm{Conj}(\mathrm{SU}_2) \to [-2,2]$ is a well-defined surjection.

   It remains to check that tr is injective. Because each conjugacy class is represented by a diagonal matrix, it is enough to check that $g_1 \coloneqq \mathrm{diag}(\lambda_1, \overline{\lambda}_1)$ and $g_2 \coloneqq \mathrm{diag}(\lambda_2, \overline{\lambda}_2)$ have $\mathrm{tr}\, g_1 = \mathrm{tr}\, g_2$ only if g_1 and g_2 are conjugate. Well, write $\lambda_\bullet = e^{i\theta_\bullet}$, and then we see that

   $$2\cos\theta_1 = 2\cos\theta_2,$$

   which implies that $\{\pm\theta_1\} = \{\pm\theta_2\}$, so $\{\lambda_1, \overline{\lambda}_1\} = \{\lambda_2, \overline{\lambda}_2\}$. We now do casework: if $\lambda_1 = \lambda_2$, then we see that $g_1 = g_2$ on the nose; otherwise, $\lambda_1 = \overline{\lambda}_2$, and we see that

   $$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \overline{\lambda}_1 \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \begin{bmatrix} \lambda_2 & \\ & \overline{\lambda}_2 \end{bmatrix},$$

   so g_1 is conjugate to g_2.

2. We now compute the required measures. A linear algebra argument with the condition $gg^\dagger = 1_2$ shows that any element of SU_2 can be written uniquely in the form

   $$\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}$$

where $|\alpha|^2 + |\beta|^2 = 1$. In this way, we see that SU_2 is isomorphic to the unit group of the quaternions $\mathbb{H}$, so SU_2 is diffeomorphic to S^3 and inherits a Haar measure by pullback. Explicitly, one finds that SU_2 inherits an action on S^3 by rotations, so the Lebesgue measure on S^3 is invariant under the group. Note that we have yet to normalize the Haar mesure on SU_2.

We would now like to compute the volume of SU_2 with given trace t. Writing $\alpha = a + bi$ and $\beta = c + di$, we see that we are forcing $a = \frac{1}{2}t$, which then requires the remaining coordinates to live in a sphere of radius $\sqrt{1 - \frac{1}{4}t^2}$. Thus, we see that our normalized Haar measure is

$$\frac{\sqrt{1 - \frac{1}{4}t^2}\, dt}{\int_{-2}^{2} \sqrt{1 - \frac{1}{4}t^2}\, dt}.$$

A quick substitution with $t = 2\cos\theta$ in the bottom integral reveals that it equals π, whereupon we find that the desired measure is $\frac{1}{2\pi}\sqrt{4 - t^2}\, dt$ after some rearranging. ■

Remark 3.21. In the sequel, it is occasionally more convenient to identify $\mathrm{Con}(\mathrm{SU}_2)$ with the collection of diagonal matrices $\mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ where $\theta \in [0, \pi)$. Then we see that the trace is $2\cos\theta$, so we produce a measure of $\frac{2}{\pi}\sin^2\theta\, d\theta$ on $[0, \pi)$.

Example 3.22 (no complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + x + 1$ over $\mathbb{Q}$ studied in Example 3.9. Then we recall that $\mathrm{ST}(E) = \mathrm{SU}_2$, so we may use the computation of Lemma 3.20 to see that the Sato–Tate conjecture (Conjecture 3.5) implies that the values

$$\left\{ \mathrm{tr}\, \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p}) \right\}_{\mathfrak{p}\text{ prime}}$$

equidistribute according to the semicircle measure $\frac{1}{2\pi}\sqrt{4 - t^2}\, dt$ on $[-2, 2]$.

Example 3.23 (complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$ studied in Example 3.10. Then we recall that $\mathrm{ST}(E) \cong \mathrm{U}_1$ embedded as $z \mapsto \mathrm{diag}(z, \overline{z})$. We may write U_1 as $\mathrm{U}_1 = \left\{e^{i\theta} : \theta \in [0, 2\pi)\right\}$, so we can equip this group with the normalized Haar measure $\frac{1}{2\pi}\, d\theta$. (The map $e^{i\theta} \mapsto \theta$ is a homeomorphism away from a set of measure 0.) Noting the trace of $\mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ is $2\cos\theta$, we see the Sato–Tate conjecture (Conjecture 3.5) implies that the values

$$\left\{ \mathrm{tr}\, \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p}) \right\}_{\mathfrak{p}\text{ prime}}$$

equidistribute according to the measure $\frac{1}{\pi} \cdot \frac{1}{\sqrt{4 - t^2}}\, dt$ on $[-2, 2]$.

Example 3.24 (potential complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$ studied in Example 3.11. Then we recall that $\mathrm{ST}(E) \cong \mathrm{U}_1 \rtimes S_2$, where $\mathrm{U}_1 \subseteq \mathrm{GL}_{2,\mathbb{C}}$ is embedded as $z \mapsto \mathrm{diag}(z, \overline{z})$, and $S_2 = \{1, w\}$ acts by switching the coordinates. Again, we give $\mathrm{U}_1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ the normalized Haar measure $\frac{1}{2\pi}\,d\theta$, so $\mathrm{U}_1 \rtimes S_2$ gets the normalized Haar measure $\frac{1}{4\pi}\,d\theta$. For $u = \mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right) \in U$, we note that the trace of $(u, 1) \in \mathrm{U}_1 \rtimes S_2$ is simply $2\cos\theta$ while the trace of $(u, w) \in \mathrm{U}_1 \rtimes S_2$ vanishes. Thus, we see the Sato–Tate conjecture (Conjecture 3.5) implies that the values

$$\left\{ \mathrm{tr}\, \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p}) \right\}_{\mathfrak{p}\text{ prime}}$$

equidistribute according to the measure $\frac{1}{2\pi} \cdot \frac{1}{\sqrt{4-t^2}}\,dt + \frac{1}{2}\delta_0\,dt$ on $[-2, 2]$. Here, δ_0 refers to the δ-distribution concentrated at 0.

We now return to the Jacobian of (the normalization of the proper curve with affine chart) $y^9 = x(x-1)(x-\lambda)$. It will be helpful to take products of Haar measures in the sequel. The following result is an easier form of [DE14, Proposition 1.5.6].

Lemma 3.25. Fix a locally compact topological group G. Choose closed subgroups $H, K \subseteq G$ such that $G = HK$ and $K \subseteq C_G(H)$. Letting dh and dg be left Haar measures on H and K, respectively, we find that $dk\,dh$ is a left Haar measure on G.

Proof. We are tasked with showing that the integral

$$\int_H \int_K f(hk)\,dk\,dh$$

is left-invariant for G. It is left-invariant for H with no content, so it suffices to show the same for K. This follows after some manipulation because K commutes with H. ∎

Remark 3.26. In fact, [DE14, Proposition 1.5.6] shows something much stronger: one can replace the strong group-theoretic condition that $K \subseteq C_G(H)$ with merely that K is compact. In fact, a careful reading of the proof there reveals that we may even replace the condition that K is compact with merely having $H \cap K$ compact and $\Delta_G|_K = 1$, where Δ_G is the modular function on G.

Here is our application.

Proposition 3.27. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x(x-1)(x-\lambda)$, where λ lives in a number field. Suppose that A does not have complex multiplication. We compute a Haar measure on $\mathrm{ST}(A_K)$ whenever K contains K_A^{conn}.

Proof. The Sato–Tate computation of Proposition 3.13 (combined with the conjugacy class computation of Lemma 3.20) reveals that an element of $\mathrm{Conj}(\mathrm{ST}(A))$ can be written as

$$\mathrm{diag}\left(\begin{bmatrix} e^{i\alpha_0} & \\ & e^{-i\alpha_0} \end{bmatrix}, \begin{bmatrix} e^{i\alpha_1+i\theta_1} & \\ & e^{i\alpha_1-i\theta_1} \end{bmatrix}, \begin{bmatrix} e^{i\alpha_1+i\alpha_4+i\theta_2} & \\ & e^{i\alpha_1+i\alpha_4-i\theta_2} \end{bmatrix}, \begin{bmatrix} e^{i\alpha_4+i\theta_4} & \\ & e^{i\alpha_4-i\theta_4} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} e^{-i\alpha_4+i\theta_4} & \\ & e^{-i\alpha_4-i\theta_4} \end{bmatrix}, \begin{bmatrix} e^{-i\alpha_1-i\alpha_4+i\theta_2} & \\ & e^{-i\alpha_1-i\alpha_4-i\theta_2} \end{bmatrix}, \begin{bmatrix} e^{-i\alpha_1+i\theta_1} & \\ & e^{-i\alpha_1-i\theta_1} \end{bmatrix} \right)$$

where $\alpha_\bullet \in [0, 2\pi)$ and $\theta_\bullet \in [0, \pi)$. Technically, the map $(\alpha_\bullet, \theta_\bullet)\colon [0, 2\pi)^4 \times [0, \pi)^3 \to \mathrm{Conj}(\mathrm{ST}(A))$ is the finite-to-one because $Z(\mathrm{ST}(A))^\circ \cap \mathrm{ST}(A)^{\mathrm{der}}$ is finite, but this will make no effect on our computations as long as we normalize to have total volume 1 and only integrate against genuine functions on $\mathrm{Conj}(\mathrm{ST}(A))$.

Anyway, we see that the trace is given by

$$2\cos\alpha_0 + 2\cos(\alpha_1+\theta_1) + 2\cos(\alpha_1-\theta_1) + 2\cos(\alpha_1+\alpha_4+\theta_2) + 2\cos(\alpha_1+\alpha_4-\theta_2) + 2\cos(\alpha_4+\theta_4) + 2\cos(\alpha_4-\theta_4).$$

We finish by remarking that Lemma 3.25 gives our Haar measure as

$$\frac{1}{(2\pi)^3}\,d\alpha_0\,d\alpha_1\,d\alpha_4 \cdot \frac{1}{\pi^2}\left(2\sin^2\theta_1 \cdot 2\sin^2\theta_2 \cdot 2\sin^2\theta_4\right) d\theta_1\,d\theta_2\,d\theta_4,$$

which is what we wanted. (Note we used Remark 3.21 for the Haar measure on SU_2.) ■

Proposition 3.28. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x^3 - 1$. Suppose that A does not have complex multiplication. We compute a Haar measure on $\mathrm{ST}(A_K)$ whenever K contains K_A^{conn}.

Proof. The Sato–Tate computation of Proposition 3.16 reveals that an element of $\mathrm{Conj}(\mathrm{ST}(A))$ can be written as

$$\mathrm{diag}\left(e^{i\alpha_2-i\alpha_1-\alpha_4}, e^{i\alpha_1+i\alpha_4-i\alpha_2}, e^{-i\alpha_4}, e^{i\alpha_1}, e^{i\alpha_2}, e^{-\alpha_2}, e^{-\alpha_1}, e^{i\alpha_4}, e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_4}, e^{-i\alpha_4}, e^{-i\alpha_2}, e^{-i\alpha_1}\right)$$

where $\alpha_\bullet \in [0, 2\pi)$. For example, we see that the trace is given by

$$2\cos\cos(\alpha_1-\alpha_2+\alpha_4) + 4\cos\alpha_1 + 4\cos\alpha_2 + 4\cos\alpha_4$$

We finish by remarking that Lemma 3.25 gives our Haar measure as

$$\frac{1}{(2\pi)^3}\,d\alpha_1\,d\alpha_2\,d\alpha_4,$$

which is what we wanted. ■

Remark 3.29. As remarked at the end of the proof of Proposition 3.17, we can run the exact same computation with working the curve given by $y^9 = x\left(x^2+1\right)$ because the resulting Sato–Tate group is the same up to reordering the basis.

Remark 3.30. For the previous examples, there are more interesting coefficients in the characteristic polynomial than merely the trace. Hoever, they are rather lengthy to write down, so we have chosen not to.

It still remains to explain how we numerically verify the Sato–Tate conjecture. The idea is that we can try to compute

$$\mathrm{tr}\,\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$$

for various primes $\mathfrak{p}$ and then compare it with what is expected from

$$\int_{\mathrm{Conj}(\mathrm{ST}(A))} \mathrm{tr}\,g\,dg,$$

where dg refers to the pushforward of the Haar measure from $\mathrm{Conj}(\mathrm{ST}(A))$. One usually expects the above integral to vanish, so one can either look at other coefficients of the characteristic polynomial or at powers of $\mathrm{tr}\,g$. In the sequel, we will compute with only powers of $\mathrm{tr}\,g$ for simplicity, but we do remark that one can typically recover the other coefficients via a combination of Vieta's formulae and Newton's sums.

As usual, let's begin with elliptic curves. Here, explicit formulae are possible.

Example 3.31 (no complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + x + 1$ over $\mathbb{Q}$ studied in Examples 3.9 and 3.22. Fix some integer $m \geq 0$. Using the given Haar measure (from Remark 3.21), we find that one expects the average of $\left\{\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m\right\}_{\mathfrak{p} \text{ prime}}$ to be

$$\int_0^\pi (2\cos\theta)^m \frac{2}{\pi} \sin^2\theta \, d\theta = \begin{cases} \frac{1}{m/2+1}\binom{m}{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where the last equality is verified by expanding $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and $4\sin^2\theta = 2 - e^{2i\theta} - e^{-2i\theta}$.

Example 3.32 (complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$ studied in Examples 3.10 and 3.23. Fix some integer $m \geq 0$. Using the given Haar measure, we find that one expects the average of $\left\{\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m\right\}_{\mathfrak{p} \text{ prime}}$ to be

$$\int_0^{2\pi} (2\cos\theta)^m \frac{1}{2\pi} \, d\theta = \begin{cases} \binom{m}{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where the last equality is verified by expanding $2\cos\theta = e^{i\theta} + e^{-i\theta}$.

Example 3.33 (complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}$ studied in Examples 3.11 and 3.24. Fix some integer $m \geq 0$. Using the given Haar measure, we find that one expects the average of $\left\{\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m\right\}_{\mathfrak{p} \text{ prime}}$ to be

$$\int_0^{2\pi} (2\cos\theta)^m \frac{1}{4\pi} \, d\theta = \begin{cases} \frac{1}{2}\binom{m}{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where the last equality is verified by expanding $2\cos\theta = e^{i\theta} + e^{-i\theta}$.

We now return to $y^9 = x(x-1)(x-\lambda)$. Here, we do not attempt to give explicit formulae, but we list the first few expected values, which were computed using SageMath.

Example 3.34. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x(x-1)(x-10)$. SageMath can verify that A does not have complex multiplication. For $m \in \{0, 1, \ldots, 6\}$, we use Proposition 3.27 to find that we expect the aveage of $\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m$ as $\mathfrak{p}$ varies over primes K (for K containing K_A^{conn}) to be as follows.

m	0	1	2	3	4	5	6
expected	1	0	8	0	186	0	7160
actual	1.0	0.0	7.8	0.2	180	16	6400

Here, the "actual" amounts have been rounded to two significant digits, and they were computed by averaging over primes $p < 216289$ which were $1 \pmod 9$; the condition $p \equiv 1 \pmod 9$ corresponds to splitting completely in $\mathbb{Q}(\zeta_9)$ (see Remark 3.19). These "actual" amounts suggest that $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, a fact which we will verify in the next chapter.

Example 3.35. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x^3 - 1$, where λ lives in a number field. For $m \in \{0, 1, \ldots, 6\}$, we use Proposition 3.28 to find that we expect the aveage of $\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m$ as $\mathfrak{p}$ varies over primes K (for K containing K_A^{conn}) to be as follows.

m	0	1	2	3	4	5	6
expected	1	0	26	0	2118	0	239300
actual	1.0	0.0	25	6.0	2000	890	220000

Here, the "actual" amounts have been rounded to two significant digits, and they were computed by averaging over primes $p < 100000$ which were $1 \pmod 9$; the condition $p \equiv 1 \pmod 9$ corresponds to splitting completely in $\mathbb{Q}(\zeta_9)$ (see Remark 3.19). These "actual" amounts suggest that $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, a fact which we will verify in the next chapter.

Remark 3.36. If one runs the same computation as in the previous example with $y^9 = x\left(x^2 + 1\right)$, one should further restrict primes past $p \equiv 1 \pmod 9$ in order to see the correct moment statistics. This is because now $K_A^{\mathrm{conn}} \neq \mathbb{Q}(\zeta_9)$.

3.2 The Utility of L-Functions

In this section, we will explain how L-functions are used in analytic number theory. Before delving into the main content of this section, we give a rough indication of what an L-function is, though we will wait to explain why we care. One generally expects an L-function to have a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

which converges in some region $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}$, where σ is a real number. (We may call σ the "abscissa" of convergence.) In this situation, one may find that σ to is a pole of $L(s)$ (though not always), but we usually expect $L(s)$ to admit a meromorphic continuation beyond $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}$.

Another important feature is that L-functions frequently come with "Euler products" that look like

$$L(s) = \prod_p L_p(s),$$

where the "Euler factor" $L_p(s)$ is a rational function in p^{-s}. We will be mostly interested in non-vanishing and holomorphy of our L-functions, and these properties tend to be insensitive to adjusting finitely many Euler factors. Thus, we pick up the following notation.

Notation 3.37. Given two infinite products $\prod_p a_p$ and $\prod_p b_p$, we write

$$\prod_p a_p \doteq \prod_p b_p$$

if and only if the two products are equal up to a finite number of nonzero terms.

3.2.1 The Prime Number Theorem

To prove an equidistribution result, one needs to end up proving some natural density results. For a natural density result, one needs to be able to count a total in order to estimate the denominator. Thus, for Conjecture 3.5, we will need to count the number of primes. As such, in this subsection, we will pick up some

tools from analytic number theory, and then we will prove the prime number theorem as an application. Our exposition is very standard; for example, all arguments are results can be found in [Mur08, Chapter 3].

Formally, the prime number theorem states that

$$\sum_{p \le x} 1 \sim \frac{x}{\log x}.$$

Now, even though we are interested in counting primes, it is easier to prove a result of the form

$$\sum_{p \le x} \log p \sim x$$

because the right-hand side is simpler (roughly speaking). Quickly, we give names to our "prime-counting" functions of interest.

Definition 3.38. For $x > 0$, define $\pi(x)$ as the number of primes $p \le x$, and define

$$\psi(x) \coloneqq \sum_{\substack{p \text{ prime}, k > 0 \\ p^k \le x}} \log p.$$

For brevity, we let $\Lambda(n)$ be $\log p$ if n is a power of a prime p and 0 otherwise; then $\psi(x) = \sum_{n \le x} \Lambda(n)$.

It is easier to estimate ψ than π, but their estimates can be shown to be equivalent. To explain this, we use Abel summation.

Proposition 3.39 (Abel summation)**.** Choose a sequence of complex numbers $\{b_n\}_{n \ge 1}$, and set $B(x) \coloneqq \sum_{n \le x} b_n$. For any continuously differentiable $f \colon [0, \infty) \to \mathbb{C}$, we have

$$\sum_{a \le n \le x} b_n f(n) = B(x) f(x) - \int_1^x B(t) f'(t)\, dt.$$

Proof. The main idea is to write $b_n = B(n) - B(n-1)$, so telescoping shows

$$\sum_{n \le x} b_n f(n) = B(\lfloor x \rfloor) f(\lfloor x \rfloor) - \sum_{n < \lfloor x \rfloor} B(n)(f(n+1) - f(n)).$$

Now, $f(n+1) - f(n) = \int_n^{n+1} f'(t)\, dt$, so the sum collapses into the integral

$$\sum_{n \le x} b_n f(n) = B(\lfloor x \rfloor) f(\lfloor x \rfloor) - \int_1^{\lfloor x \rfloor} B(t) f'(t)\, dt.$$

It remains to move from $\lfloor x \rfloor$ to x, for which we note that $B(t) = B(\lfloor x \rfloor)$ for $t \in [\lfloor x \rfloor, x]$, so

$$B(x) f(x) - B(\lfloor x \rfloor) f(\lfloor x \rfloor) = \int_{\lfloor x \rfloor}^x B(t) f'(t)\, dt,$$

thereby completing the proof. ∎

Corollary 3.40. If $\psi(x) \sim x$, then $\pi(x) \sim x / \log x$.

Proof. Given $\psi(x) \sim x$, we begin by claiming $\sum_{p \le x} \log p \sim x$. Indeed,

$$\left| \psi(x) - \sum_{p \le x} \log p \right| = \sum_{\substack{p \text{ prime}, k > 1 \\ p^k \le x}} \log p.$$

We bound this sum unintelligently: it is

$$\sum_{k=2}^{\log_2 x} \sum_{p \le x^{1/k}} \log p \le (\log_2 x)(\sqrt{x} \log x),$$

which is $o(x)$, and the claim follows.

We now show $\pi(x) \sim x/\log x$. This requires Abel summation in the form of Proposition 3.39. Indeed, we see $\pi(x)$ equals

$$\sum_{n \le x} 1_{\text{is prime}}(n) \log n \cdot \frac{1}{\log n} = \frac{1}{\log x} \sum_{p \le x} \log p + \int_2^x \left(\sum_{p \le x} \log p \right) \frac{1}{t(\log t)^2} \, dt.$$

Thus, it remains to show that the integral is $o(x/\log x)$. Well, $\sum_{p \le x} \log p \sim x$, so it is enough to show that the intgral $\int_2^x (\log t)^{-2} \, dt$ is $o(x/\log x)$. Well, for x large, we see that

$$\int_2^{\sqrt{x}} \frac{1}{(\log t)^2} \, dt + \int_{\sqrt{x}}^x \frac{1}{(\log t)^2} \, dt \le \sqrt{x} + \frac{x}{(\log \sqrt{x})^2},$$

which is manifestly $o(x/\log x)$. ∎

Remark 3.41. In fact, one can reverse the application of Proposition 3.39 to show the reverse implication, but we will not need this.

We will spend the rest of our time trying to show that $\psi(x) \sim x$. We will use a weak form of the Weiner–Ikehara theorem to prove this from some analytic properties of the Riemann zeta function. As such, we spend some time working towards the Weiner–Ikehara theorem. Our approach follows [New80] and uses the following Tauberian theorem.

Theorem 3.42 (Newman)**.** Let $f \colon [0, \infty) \to \mathbb{C}$ be a bounded and piecewise continuous function, and let $F(s) \coloneqq \int_{\mathbb{R}^+} f(t) e^{-st} \, dt$ denote the Laplace transform. Suppose that $F(s)$ admits an analytic continuation to the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s \ge 0\}$. Then the integral

$$\int_{\mathbb{R}^+} f(t) \, dt$$

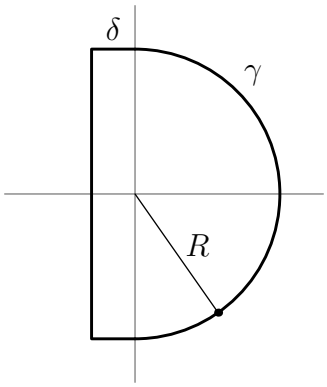
converges and equals $F(0)$.

Proof. In order to estimate with convergent integrals, for any $T > 0$, we define the function $F_T \colon \mathbb{C} \to \mathbb{C}$

$$F_T(s) \coloneqq \int_0^T f(t) e^{-st} \, dt.$$

We quickly remark that F is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ for free because boundedness of f implies that the integral converges in this region; similarly, we note that F_T is automatically entire for any $T > 0$.

Our goal is to show that $F_T(0) \to F(0)$ as $T \to \infty$. We will estimate $|F(0) - F_T(0)|$ via some clever contour integration. Fix some $R > 0$, which will eventually tend to ∞. Then we note that compactness of the interval $\{bi : -R \le b \le R\}$ implies that there is $\delta > 0$ such that the analytic continuation of F extends to an open set containing the box $\{a + bi : a \ge -\delta, -R \le b \le R\}$. We now let γ denote the following contour, oriented counterclockwise.



We also let γ_+ and γ_- denote the parts in the right-half and left-half planes, respectively. Now, the main trick is to note that

$$F(0) - F_T(0) = \frac{1}{2\pi i} \int_\gamma \frac{F(s) - F_T(s)}{s} \cdot e^{sT} \left(1 + \frac{s^2}{R^2}\right) ds$$

by the Cauchy integral formula. (The magic will come from the strange factor $e^{sT}\left(1 + s^2/R^2\right)$.) We now estimate this integral as (in order) $T \to \infty$, $\delta \to 0$, and $R \to \infty$.

- We estimate the integral on γ_+. This can be done directly. On one hand, expanding out the integral reveals

  $$|F(s) - F_T(s)| \leq \|f\|_\infty \cdot \frac{e^{-T \operatorname{Re} s}}{\operatorname{Re} s}.$$

  On the other hand, we note s/R is on the unit circle, so

  $$\left|\frac{e^{st}}{s}\left(1 + \frac{s^2}{R^2}\right)\right| \leq \frac{e^{T \operatorname{Re} s}}{R} \cdot 2 \operatorname{Re}(s/R).$$

  Combining estimates, we bound our integral by

  $$|F(0) - F_T(0)| \leq \frac{\|f\|_\infty}{R},$$

  which vanishes as $R \to \infty$, as required.

- To estimate the integral on γ_-, we split the integral into a sum of integrals of F and F_T separately. In this point, we bound the integral of F_T. Here, F_T is entire, we may replace the contour γ_- with a semicircle of radius R in the left-half plane. Proceeding as in the above point, we note that

  $$|F_T(s)| \leq \|f\|_\infty \cdot \frac{e^{-T \operatorname{Re} s}}{-\operatorname{Re} s}$$

  by expanding out the integral, and then estimating $e^{st}\left(1 + s^2/R^2\right)$ as before yields

  $$\left|\frac{1}{2\pi i} \int_{\gamma_-} \frac{F_T(s)}{s} \cdot e^{sT} \left(1 + \frac{s^2}{R^2}\right) ds\right| \leq \frac{\|f\|_\infty}{R},$$

  which again vanishes as $R \to \infty$.

- It remains to bound the integral of F over γ_-. This will require some care. We will split the estimates into the horizontal and vertical pieces. Throughout, R and δ remain fixed, and we will only send $T \to \infty$; in particular, F is bounded in the region of interest, so we may ignore its contribution.

  – On the horizontal pieces, for $\delta > 0$ small enough, we may still find that our integrand is on the order of $e^{T \operatorname{Re} s} \cdot 2 \operatorname{Re} s$, as in the first point. However, we note that the function $x \mapsto xe^{-x}$ on $\mathbb{R}^+$ achieves its maximum at $(1, 1/e)$, so with $\operatorname{Re} s < 0$, we see that our integrand is bounded by e^{-1}/T. To complete our estimate, we send $T \to \infty$.

– On the vertical piece, for $\delta > 0$ small enough, we note

$$\left| \frac{e^{sT}}{s} \left(1 + \frac{s^2}{R^2}\right) \right| \leq \frac{3e^{-\delta T}}{\delta}.$$

Sending $T \to \infty$ causes this piece to vanish. ■

We are now ready to prove our weakened Weiner–Ikehara theorem. We follow [Vat15, Theorem 2].

Theorem 3.43 (Weiner–Ikehara)**.** Choose a sequence of nonnegative real numbers $\{b_n\}_{n \geq 1}$, and set $L(s) \coloneqq \sum_{n \geq 1} b_n n^{-s}$ and $B(x) \coloneqq \sum_{n \leq x} b_n$. Suppose the following.

(i) The series $L(s)$ converges absolutely for $\operatorname{Re} s > 1$.

(ii) The function $L(s)$ admits a meromorphic continuation to $\operatorname{Re} s = 1$ and has no poles except possibly a simple pole at $s = 1$ with residue c.

(iii) We have $B(x) = O(x)$.

Then $B(x) = cx + o(x)$.

Proof. There are two steps.

1. By Proposition 3.39, we see that
$$L(s) = s \int_1^\infty B(t) t^{-s-1} \, dt$$
holds for $\operatorname{Re} s > 1$. Now, the idea is to apply Theorem 3.42 to the integral
$$\int_0^\infty \frac{B(e^t) - ce^t}{e^t} e^{-st} \, dt = \frac{L(s+1)}{s+1} - \frac{c}{s},$$
where the equality follows from the previous one after the substitutions $s \mapsto s+1$ and $t \mapsto e^t$. Notably, we are allowed to apply Theorem 3.42 because one already knows that the function $e^{-t} B(e^t) - c$ is bounded by (iii), and the right-hand side provides the required analytic continuation. Thus, we are told that
$$\int_0^\infty \frac{B(e^t) - ce^t}{e^t} \, dt = \int_1^\infty \frac{B(t) - ct}{t^2} \, dt$$
converges.

2. We are now ready to conclude. We must show that $B(x)/x \to 1$ as $x \to \infty$. Suppose for the sake of contradiction this is not the case; then either $\limsup_{x \to \infty} B(x)/x > c$ or $\liminf_{x \to \infty} B(x)/x < c$. We handle the case $\limsup_{x \to \infty} B(x)/x > c$ because the other case is similar. In this case, there is $\varepsilon > 0$ and an infinite sequence $\{x_i\}_{i \geq 1}$ tending to infinity such that $B(x_i)/x_i > c(1+\varepsilon)$ for all $i \geq 1$. For any such x_i, we see that
$$\int_{x_i}^{(1+\varepsilon)x_i} \frac{B(t) - ct}{t^2} \, dt \geq \int_{x_i}^{(1+\varepsilon)x_i} \frac{(c+\varepsilon)x_i - ct}{t^2} \, dt.$$
Upon a change of variables, we see this integral equals $\int_1^{1+\varepsilon} (c(1+\varepsilon) - ct) t^{-2} \, dt$, which is some nonzero constant not depending on x_i. Because we can let the x_i tend to infinity, we conclude that the integral $\int_1^\infty (B(t) - ct) t^{-2} \, dt$ cannot converge! This is our required contradiction. ■

Remark 3.44. Note that L is by definition real on the real axis (when the series converges), which implies that the residue c must be real because the residue equals the limit of $sL(s)$ as $s \to 1^+$.

Remark 3.45. The hypothesis (c) in the statement of Theorem 3.43 is not necessary, but one requires a somewhat more technical proof.

We will now apply Theorem 3.43 to show $\psi(x) \sim x$. Because (iii) of Theorem 3.43 is unrelated to the other two conditions, we handle it first. The argument is combinatorial.

Lemma 3.46 (Chebychev)**.** We have $\psi(x) = O(x)$.

Proof. Arguing as in Corollary 3.40, it is enough to show that $\sum_{p \le x} \log p = O(x)$. We proceed in steps.

1. For any $n \ge 0$, we claim that $\sum_{n<p\le 2n} \log p < 2n \log 2$. The idea is to consider $\binom{2n}{n}$. By expanding out its prime factorization, we note that $\binom{2n}{n}$ has each prime factor p in the range $n < p \le 2n$, so $\log \binom{2n}{n} \ge \sum_{n<p\le 2n} \log p$. On the other hand, the binomial theorem requires $\binom{2n}{n} < (1+1)^{2n}$, so $\log \binom{2n}{n} < 2n \log 2$, as required.
2. For any $\nu \ge 0$, we claim that $\sum_{p \le 2^\nu} \log p < 2^{\nu+1} \log 2$. Indeed, this sum is
$$\sum_{k=0}^{\nu-1} \left(\sum_{2^k < p \le 2^{k+1}} \log p \right) \le \sum_{k=0}^{\nu-1} 2^{k+1} \log 2$$
by the previous step, from which the claim follows.
3. We complete the proof. For any $x > 1$, we may find $\nu \ge 0$ such that $2^\nu \le x < 2^{\nu+1}$. Then $\sum_{p \le x} \log p$ is bounded by $2^{\nu+2} \log 2$ by the previous step, but this is in turn bounded by $4x \log 2$, so we are done. ∎

For (i) of Theorem 3.43, we must explain the relevance of the Riemann ζ-function to our argument.

Definition 3.47 (Riemann ζ-function)**.** We define the *Riemann ζ-function* by
$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Lemma 3.48. For s such that $\operatorname{Re} s > 1$, we have $\zeta(s) \ne 0$ and
$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$
This series also converges absolutely for $\operatorname{Re} s > 1$.

Proof. Unique prime factorization produces the Euler product
$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$
so taking logarithms implies
$$\log \zeta(s) = \sum_p -\log\left(1 - p^{-s}\right).$$
Using the Taylor expansion of $\log(1-x)$, we find that
$$\log \zeta(s) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k p^{ks}}.$$

The claimed equality would now follow by taking the derivative with respect to s, but of course, we must know that $\log \zeta(s)$ is an analytic function to be able to do this. Well, we will actually show that the right-hand side is absolutely convergent, which we note then implies $\zeta(s) \neq 0$. To check absolute convergence, we may rearrange our sum, so we sum over k. The $k = 1$ term is bounded by $\zeta(\operatorname{Re} s)$, which is finite. For the remaining terms, we bound our sum in magnitude by

$$\sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{n^k} = \sum_{n=1}^{\infty} \frac{1/n^2}{1 - 1/n}.$$

The summand is $\frac{1}{n(n-1)}$, so the entire sum converges. ■

Thus, the function $L(s)$ arising from trying to show $\psi(x) \sim x$ is simply $\zeta'(s)/\zeta(s)$. It remains to show the required facts about meromorphic continuation for (ii). We begin by showing that ζ continues.

Lemma 3.49. The function $\zeta(s)$ admits a meromorphic continuation to $\operatorname{Re} s > 0$ with no poles except a simple pole at $s = 1$ with residue 1.

Proof. We use Abel summation. By Proposition 3.39, we see that

$$\zeta(s) = s \int_1^{\infty} \lfloor t \rfloor \, t^{-s-1} \, dt.$$

Now, we write $\lfloor t \rfloor = t - \{t\}$ to see

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \{t\} t^{-s-1} \, dt.$$

The listed claims will follow once we show that the reminaing integral $I(s)$ is analytic on $\operatorname{Re} s > 0$. Well, we see

$$|I(s)| \leq \int_1^{\infty} \frac{1}{t^{\operatorname{Re} s + 1}} \, dt = \frac{1}{\operatorname{Re} s},$$

so the integral is always finite, so the integral is analytic because the integrand is.[3] ■

Thus, the check (ii) of Theorem 3.43 amounts to the following non-vanishing result.

Proposition 3.50. If $s \in \mathbb{C}$ has $\operatorname{Re} s = 1$, then $\zeta(s) \neq 0$.

Proof. The following proof is tricky. We proceed in steps, following [Mur08, Section 3.2].

1. For $\sigma > 1$ and $t \in \mathbb{R}$, we claim that

$$\operatorname{Re} \log \zeta(\sigma + it) \stackrel{?}{=} \sum_{n=2}^{\infty} \frac{\Lambda(n) \cos(t \log n)}{n^{\sigma} \log n}.$$

Well, the argument of Lemma 3.48 (equivalently, integrating the statement) shows that

$$\log \zeta(s) = \sum_{p} \sum_{k=1}^{\infty} \frac{1}{k p^{ks}},$$

[3] This point techincally requires some care because one needs to apply some kind of dominated convergence theorem as in [Mat01]. Our proof actually shows that $I(s)$ is analytic on any region $\{s : \operatorname{Re} s > \sigma\}$ for any $\sigma > 0$, from which $I(s)$ being analytic on $\{s : \operatorname{Re} s > 0\}$ follows by taking unions.

where $s = \sigma + it$. This sum absolutely converges (as shown in Lemma 3.48), so we may view it as a sum over prime-powers $n = p^k$, in which case we see that the summand is $\Lambda(n)n^{-s}/\log n$. Thus, we see that

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} \cdot n^{-it}.$$

We conclude by noting that $\operatorname{Re} n^{-it} = \cos(t \log n)$.

2. For $\sigma > 1$ and $t \in \mathbb{R}$, we claim that

$$\left|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)\right| \stackrel{?}{\geq} 1.$$

Well, by taking logarithms, it is enough to show that

$$3 \operatorname{Re} \log \zeta(\sigma) + 4 \operatorname{Re} \log \zeta(\sigma + it) + \operatorname{Re} \log \zeta(\sigma + 2it) \stackrel{?}{\geq} 0.$$

By the previous step, we see that it is enough to check that

$$3 + 4\cos\theta + \cos 2\theta \geq 0$$

for any $\theta \in \mathbb{R}$. This amounts to mimizing the function $4\cos\theta + \cos 2\theta$; taking the derivative reveals that minima will occur when $\sin x = 0$ or $\cos x = 1$, so x is a multiple of π. Thus, we complete this step by noting that the above inequality holds when x is a multiple of π.

3. We conclude the proof. Fix some nonzero real number t, and we would like to show that $\zeta(1 + it) \neq 0$. Well, suppose for the sake of contradiction that $\zeta(1 + it) = 0$. Then

$$\lim_{\sigma \to 1^+} \zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it) = 0$$

because the order of the zero at $\sigma = 0$ is at least $-3 + 4 + 0 > 0$. This contradicts the previous step, so we are done. ∎

We are now ready to prove the Prime number theorem.

Theorem 3.51 (Prime number)**.** We have $\pi(x) \sim x/\log x$.

Proof. It only remains to synthesize the discussion from this subsection. By Corollary 3.40, it is enough to show $\psi(x) \sim x$. For this, we will use Theorem 3.43 applied to the sequence $\{\Lambda(n)\}_{n \geq 1}$, for which Lemma 3.48 explains makes the Dirichlet series equal to $-\zeta'(s)/\zeta(s)$. It remains to check the three conditions in Theorem 3.43.

(i) The absolute convergence of $-\zeta'(s)/\zeta(s)$ follows because $\Lambda(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$, so the series converges absolutely and uniformly on compacts on any region $\{s \in \mathbb{C} : \operatorname{Re} s > \varepsilon\}$ for any $\varepsilon > 0$.

(ii) Because $\zeta(s)$ is nonzero on $\{s : \operatorname{Re} s = 1\}$, we conclude that $-\zeta'(s)/\zeta(s)$ admits a meromorphic continuation to this line. We already know that we are defined everywhere except at $s = 1$, and we see that having ζ have a simple pole with reside 1 at $s = 1$ implies the same for $-\zeta'(s)/\zeta(s)$ by expanding out a Taylor series at $s = 1$.

(iii) Lastly, we see $\psi(x) = O(x)$ by Lemma 3.46. ∎

3.2.2 The Prime Ideal Theorem

In the sequel, we will want to count not just rational primes but also for number fields, so we want to extend Theorem 3.51 to number fields. The method remains the same, though we will not give a complete proof now because showing the required meromorphic continuation is harder. Our exposition loosely follows [RV99, Sections 7.4 and 7.7], which in turn follows [Hei10].

Here are our prime-counting functions.

Definition 3.52. Fix a number field K. For $x > 0$, define $\pi_K(x)$ as the number of prime ideals $\mathfrak{p}$ with $\mathrm{N}\,\mathfrak{p} \le x$. Now, define Λ_K as a function on the ideals of $\mathcal{O}_K$ by

$$\Lambda_K(I) \coloneqq \begin{cases} \log \mathrm{N}\,\mathfrak{p} & \text{if } I = \mathfrak{p}^k \text{ for } k \ge 1, \\ 0 & \text{else,} \end{cases}$$

and we set $\psi_K(x) \coloneqq \sum_{\mathrm{N}(I) \le x} \Lambda_K(I)$.

This time around, the relevant L-function for the Weiner–Ikehara theorem is as follows.

Definition 3.53 (Dedekind zeta function)**.** Fix a number field K. Then we define the *Dedekind ζ-function* as

$$\zeta_K(s) \coloneqq \sum_{I \subseteq \mathcal{O}_K} \frac{1}{\mathrm{N}(I)^s}.$$

Remark 3.54. As in Lemma 3.48, we note that one has an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p} \text{ prime}} \frac{1}{1 - \mathrm{N}\,\mathfrak{p}^{-s}}.$$

It will later be convenient to "twist" our Dedekind zeta function slightly.

Definition 3.55 (Hecke L-function)**.** Fix a number field K and a continuous character $\chi\colon K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$. Factoring $\chi = \prod_v \chi_v$ as a product over places of K, we define the *Hecke L-function* as $L(\chi) \coloneqq \prod_{\mathfrak{p}} (1 - \chi_{\mathfrak{p}}(\mathfrak{p}))^{-1}$, where

$$\chi_{\mathfrak{p}}(\mathfrak{p}) \coloneqq \begin{cases} \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}) & \text{if } \chi_{\mathfrak{p}}|_{\mathcal{O}_{\mathfrak{p}}^\times} = 1, \\ 0 & \text{else,} \end{cases}$$

where $\varpi_{\mathfrak{p}} \in \mathfrak{p}$ is a uniformizer. We may call the former case "unramified" and the latter case "ramified." If χ is a unitary character (i.e., $\operatorname{im} \chi \subseteq S^1$), then we may also write $L(s, \chi) \coloneqq L\left(\chi |\cdot|^s\right)$.

Remark 3.56. By expanding out $(1 - \chi_{\mathfrak{p}}(\mathfrak{p}))^{-1} = \sum_{k=0}^{\infty} \chi_{\mathfrak{p}}(\mathfrak{p})^k$, we see that one can recover a "Dirichlet series" expansion for $L(s, \chi)$ in the form

$$L(s, \chi) = \sum_{I \subseteq \mathcal{O}_K} \frac{\chi(I)}{\mathrm{N}(I)^s}$$

for suitably defined $\chi(I)$ (depending on its prime factorization).

Example 3.57. If χ is the trivial character, then we recover the Dedekind ζ-function.

Example 3.58. Take $K = \mathbb{Q}$. Given a character $\chi\colon (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$, we abuse notation and lift χ to a character $K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ via the composition

$$\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times = \mathbb{R}^+ \times \prod_p \mathbb{Z}_p^\times \twoheadrightarrow \prod_p \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

Upon expanding out the Euler product, we find

$$L(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}.$$

We begin by stating part of the required analytic check.

Lemma 3.59. Fix a number field K and a continuous unitary character $\chi\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$. Then $L(s,\chi)$ converges absolutely and is nonzero for s such that $\operatorname{Re} s > 1$.

Proof. We will instead show that

$$\log L(s,\chi) = \sum_{\mathfrak{p}} -\log\left(1 - \chi_\mathfrak{p}(\mathfrak{p})\,\mathrm{N}\,\mathfrak{p}^{-s}\right)$$

absolutely converges when $\operatorname{Re} s > 1$. (Some formal business involving Euler products can then show that the Dirichlet series described in Remark 3.56 also converges absolutely.) Using the Taylor expansion, our sum is

$$\log L(s,\chi) = \sum_{\mathfrak{p}} \sum_{k\geq 1} \frac{\chi_\mathfrak{p}(\mathfrak{p})^k}{k\,\mathrm{N}\,\mathfrak{p}^{ks}}.$$

Now, to check absolute convergence, we see that may replace $\left|\chi_\mathfrak{p}(\mathfrak{p})^k\right| \in \{0,1\}$ with 1, essentially reducing to the case where $\chi = 1$.

We now find a way to reduce to the case for $K = \mathbb{Q}$, where the result follows from the argument of Lemma 3.48. Well, for each prime $\mathfrak{p}$, we see that $\mathrm{N}\,\mathfrak{p} \geq p$ where p is the prime lying over $\mathfrak{p}$. Further, there are at most $[K:\mathbb{Q}]$ primes of K sitting above p, so we see that

$$|\log L(s,\chi)| \leq [K:\mathbb{Q}] \sum_p \sum_{k\geq 1} \frac{1}{kp^{k\operatorname{Re} s}}.$$

We now have reduced to the situation in Lemma 3.48, so we are done. ∎

Remark 3.60. Term-by-term differentiation shows that the Dirichlet series defining $-L'(s,\chi)/L(s,\chi)$ continues to be absolutely convergent for s satisfying $\operatorname{Re} s > 1$. In particular, we see that

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{I\subseteq\mathcal{O}_K} \frac{\Lambda_K(I)}{\mathrm{N}(I)^s}.$$

This time around, one lacks the integration trick done in Lemma 3.49. The proof is significantly more involved, so we merely state the result we need.

Theorem 3.61 (Hecke)**.** Fix a number field K and a continuous unitary character $\chi\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$. Then $L(s,\chi)$ admits a meromorphic continuation to $\{s : \operatorname{Re} s > 0\}$. Further, $L(s,\chi)$ has no poles except a simple pole when $\chi\,|\cdot|^s$ is trivial on all unramified primes.

Proof. It is possible to prove an analytic continuation to $\{s : \operatorname{Re} s = 1\}$ "combinatorially," essentially by counting ideals of bounded norm; for example, see [ME05, Section 11.2]. However, the best proofs of this result go through Tate's thesis [Tat10]. See also [RV99, Theorem 7-19]. ■

Because it is more within reach (and closer in flavor to the results we are interested in), we will prove the needed non-vanishing result.

Proposition 3.62. Fix a number field K and a continuous unitary character $\chi\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$. If $s \in \mathbb{C}$ has $\operatorname{Re} s = 1$, then $L(s,\chi) \neq 0$.

Proof. Note that $L(s+it,\chi) = L\left(s, \chi|\cdot|^{it}\right)$, so we may twist χ in order to assume that $s = 1$. Now, if χ is trivial on the finite adeles $(\mathbb{A}_K^\infty)^\times$, then Theorem 3.61 explains that there is a pole, so there is nothing to do. We now admit two lengthy cases. There are two lengthy cases.

- Suppose that $\chi_{\mathfrak{p}}^2$ is nontrivial on some unramified prime $\mathfrak{p}$. In this case, we may proceed as in Proposition 3.50: for $\sigma > 1$, an expansion as in Lemma 3.59 finds that

$$\operatorname{Re}\log L(\sigma,\chi) = \sum_{\mathfrak{p}}\sum_{k\geq 1}\frac{\cos(k\theta_{\mathfrak{p}})}{\operatorname{N}\mathfrak{p}^{k\sigma}},$$

  where $\theta_{\mathfrak{p}} \in \mathbb{R}$ is chosen so that $\chi_{\mathfrak{p}}(\mathfrak{p}) = e^{i\theta_{\mathfrak{p}}}$. But now the trigonmetric identity $3 + 4\cos\theta + \cos 2\theta$ proven in Proposition 3.50 verifies that

$$\left|L(\sigma,1)^3 L(\sigma,\chi)^4 L(\sigma,\chi^2)\right| \gg 1,$$

  where the implied constant comes from replacing the Euler product for $L(\sigma,1)$ with one with the correct Euler factors at ramified primes. We now send $\sigma \to 0^+$ and see that having $L(\sigma) = 0$ would force the entire quantity to vanish by pole-counting: we have a zero of order at least $-3+4+0 > 0$, where notably, the hypothesis implies that there is no pole at $L(1,\chi^2)$.

- Now suppose that χ^2 is trivial on all unramified primes. The idea is to consider the Dirichlet series $L(s) \coloneqq \zeta_K(s)L(s,\chi)$, for which one can use the Dirichlet convolution to find equals

$$L(s) = \sum_{I\subseteq\mathcal{O}_K}\left(\sum_{I=AB}\chi(B)\right)\frac{1}{\operatorname{N}(I)^s},$$

  for suitably defined $\chi(I)$. We are going to appeal to some somewhat difficult fact about Dirichlet series. To this end, we want some input from the coefficient $b(I) \coloneqq \sum_{I=AB}\chi(B)$. Multiplicativity reveals that

$$b(I) = \prod_{\mathfrak{p}}\left(1 + \chi(\mathfrak{p}) + \cdots + \chi(\mathfrak{p}^{\nu_{\mathfrak{p}}(I)})\right),$$

  and χ outputs to $\{-1,0,1\}$, so we see that $b(I)$ is a nonnegative integer always. Furthermore, $b(I)$ is nonzero when I is a square (because each factor is nonzero), so we see that $L(s) \geq \zeta_K(2s)$. For example, the pole at $s = 1$ for $\zeta_K(2s)$ then implies that $L(s)$'s abscissa of holomorphy cannot go past $\{s : \operatorname{Re} s = 1/2\}$.

  Now, because $L(s)$ has all nonnegative coefficients, its abscissa of holomorphy agrees with its abscissa of absolute convergence [RV99, Lemmas 7-29]. Thus, if $L(s,\chi)$ has a zero at $s = 1$, then $L(s)$ will succeed at being holomorphic at $s = 1$, so the abscissa of holomorphy for $L(s)$ goes all the way to $\operatorname{Re} s = 0$ by Theorem 3.61. This contradicts the previous paragraph. ■

At long last, we are ready to apply Theorem 3.43.

Theorem 3.63 (Prime ideal)**.** We have

$$\{\mathfrak{p} : \mathrm{N}\,\mathfrak{p} \le x\} \sim \frac{x}{\log x}.$$

Proof. Arguing as in Corollary 3.40, it is enough to check that the function

$$\psi_K(x) \coloneqq \sum_{\substack{\mathfrak{p} \text{ prime}, k \ge 1 \\ \mathrm{N}\,\mathfrak{p}^k \le x}} \log \mathrm{N}\,\mathfrak{p}$$

satisfies $\psi_K(x) \sim x$, for which we use Theorem 3.43. Now, Remark 3.60 explains that $-\zeta_K'/\zeta_K$ is the relevant Dirichlet series. We are now ready to run the checks of Theorem 3.43.

(i) The absolute convergence of $-\zeta'(s)/\zeta(s)$ on $\{s : \mathrm{Re}\, s > 1\}$ follows from Lemma 3.59.

(ii) Note that $\zeta_K(s)$ continues to $\{s : \mathrm{Re}\, s = 1\}$ by Theorem 3.61, and it is nonvanishing by combining Lemma 3.59 with Proposition 3.62. Thus, we achieve the continuation of $-\zeta_K'(s)/\zeta_K(s)$, and we can compute that the residue of its simple pole at $s = 1$ is 1.

(iii) To check $\psi_K(x) = O(x)$, we claim that

$$\sum_{\mathrm{N}(I) \le x} \Lambda_K(I) \overset{?}{\le} \sum_{\substack{p \text{ prime}, k \ge 1 \\ p^k \le x}} [K : \mathbb{Q}] \log p^k.$$

Indeed, it is enough to only consider I of the form $\mathfrak{p}^k$; summing over the primes p below $\mathfrak{p}$, we may upper-bound $\Lambda_K(I)$ by $\log p^k$ and then maximize the number of terms in the sum by noting that there are at most $[K : \mathbb{Q}]$ primes $\mathfrak{p}$ above p and bounding $\mathrm{N}\,\mathfrak{p}^k \le p^k$.

Now, arguing as in Corollary 3.40, one finds that $\psi_K(x) = O(x)$ now follows from $\psi(x) = O(x)$. Roughly speaking, the size of this sum is dominated by the $k = 1$ term (what is left is $O\left(\sqrt{x}(\log x)^2\right)$), and the $k = 1$ term is a constant multiple of $\psi(x)$, so we are done. ∎

3.2.3 Equidistribution

In this subsection, we will prove a few facts about equidistribution, following [Fit15, Section 2] and [Ser98, Appendix to Chapter I]. Although the term already appeals in the statement of our conjecture (Conjecture 3.5), we go ahead and provide a suitable definition. We will assume some measure theory throughout, though we remark that our measures μ will all be Radon on compact Hausdorff spaces X, so they may be thought of as continuous linear functionals on $C(X)$ by the Reisz representation theorem [Fol99, Theorem 7.2], where $C(X)$ denotes the space of complex continuous functions on X.

Definition 3.64 (equidistributed)**.** Fix a compact Hausdorff space X with a probability Radon measure μ (namely, $\mu(X) = 1$). Then a sequence $\{x_n\}_{n \ge 1}$ is *equidistriubted* with respect to μ if and only if any $f \in C(X)$ has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_i) = \int_X f \, d\mu.$$

Remark 3.65. One may want to upgrade this definition from $f \in C(X)$ to $f \in L^1(X)$ or similar, but this is somewhat tricky: functions in $L^1(X)$ are well-defined up to a measure-zero subset, and it is frequently the case that countable subsets of X are measure zero. Concretely, with $X = [0, 1]$, we find that no countable sequence will equidistribute by testing against the function f which indicates this sequence!

The definition has been chosen to be quite strong, but this makes it difficult to check. As such, we pick up the following lemma.

Lemma 3.66. Fix a compact Hausdorff space X with a probability Radon measure μ. The following are equivalent for a sequence $\{x_n\}_{n\geq 1}$.

(i) The sequence $\{x_n\}_{n\geq 1}$ equidistributes.

(ii) Suppose that $F \subseteq C(X)$ is a subset of functions such that linear combinations of functions in F forms a dense subspace of $C(X)$. Then for any $f \in F$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_i) = \int_X f \, d\mu.$$

Proof. Of course (i) implies (ii) because $F \subseteq C(X)$. For the reverse inclusion, let $V \subseteq C(X)$ denote the subset for which the conclusion holds. We know $F \subseteq V$, and we would like to show that $V = L^1(X)$. Certainly V is a subspace, and by the hypothesis of F, we see that V is a dense subspace of $L^1(X)$. Thus, for any $f \in C(X)$, we fix some $\varepsilon > 0$, and we may find $g_\varepsilon \in V$ such that $\|f - g_\varepsilon\|_\infty < \varepsilon$. Then

$$\begin{aligned}\left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_i) - \int_X f \, d\mu \right| &\leq \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_\varepsilon(x_i) - \int_X g_\varepsilon \, d\mu \right| \\ &\quad + \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (f - g_\varepsilon)(x_i) \right| + \left| \int_X (f - g_\varepsilon) \, d\mu \right|.\end{aligned}$$

The rightmost term vanishes because $g \in V$, and the remaining terms are bounded by 2ε, which goes to 0 we send $\varepsilon \to 0^+$. ∎

In the sequel, we will be interested in the case where $X = \mathrm{Conj}(G)$ where G is some compact Hausdorff topological group; here X is given the quotient topology induced by the canonical projection $G \twoheadrightarrow X$. We quickly note that X is certainly compact, and X is Hausdorff because G is normal (and conjugacy classes are closed because they are images of certain continuous maps $G \to G$). We now note that Fourier analysis can detect equidistribution.

Lemma 3.67. Fix a compact Hausdorff topological group G with probability Haar measure μ, and set $X \coloneqq \mathrm{Conj}(X)$. The following are equivalent for a sequence $\{x_n\}_{n\geq 1}$ of X.

(i) The sequence $\{x_n\}_{n\geq 1}$ equidistributes.

(ii) For any nontrivial finite-dimensional complex irreducible continuous representation ρ, one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \operatorname{tr} \rho(x_i) = 0.$$

Proof. Quickly, we note that (ii) has $\operatorname{tr}\rho(x_i)$ well-defined because the character of a representation is well-defined up to conjugacy. Now (i) implies (ii) is immediate because $(\operatorname{tr} \circ \rho)\colon X \to \mathbb{C}$ is a continuous function.

For (ii) implies (i), we use Lemma 3.66. By (ii) above, we see that the conclusion of (ii) in Lemma 3.66 holds for each of the nontrivial irreducible characters $\operatorname{tr} \circ \rho$ of G because

$$\int_G \rho \, d\mu = 0$$

by the nontriviality of ρ. Additionally, we note that the conclusion of (ii) in Lemma 3.66 also holds for the trivial character because then everything in sight is 1. Thus, it remains to check that irreducible characters

of G form a dense subset of $C(X)$. In fact, characters are dense in $L^2(G)$ by (a corollary to) the Peter–Weyl theorem [Fol16, Proposition 5.23], so we are done. ■

Remark 3.68. Of course, one may replace the application of the Peter–Weyl theorem when it is easier to prove. For example, if G is a finite abelian group, then the relevant Fourier analysis is much easier to prove.

Example 3.69. Consider the compact abelian group $G = \mathbb{R}/\mathbb{Z}$ so that $G = X$. We claim that the sequence $\{n\alpha\}_{n\geq 0}$ equidistributes in G for any irrational $\alpha \in \mathbb{R}$.

Quickly, we note that the representations of G are one-dimensional because G is abelian. Further, we claim they all take the form $t \mapsto e^{2\pi imt}$ for $m \in \mathbb{Z}$: indeed, any character of G must lift to a character $\mathbb{R} \to \mathbb{C}^\times$, but it must land in S^1 because G is compact, so our character further lifts to a homomomorphism $\mathbb{R} \to \mathbb{R}$. Continuous homomomorphisms $\mathbb{R} \to \mathbb{R}$ are just scalars, so the claim follows upon ensuring that the induced map $\mathbb{R} \to S^1$ has $\mathbb{Z}$ in its kernel.

To conclude the proof, it is now enough to compute that any nonzero m makes

$$\sum_{n=0}^{N} e^{2\pi imn\alpha} = \frac{e^{2\pi im(N+1)\alpha} - 1}{e^{2\pi im\alpha} - 1},$$

which is $O_m(1)$ and hence $o_m(N)$.

As in the example, we remark that the condition (ii) may also be read as

$$\sum_{n=1}^{N} \operatorname{tr} \rho(x_i) = o(N),$$

so it is the sort of thing that one may hope to prove using the Weiner–Ikehara theorem (Theorem 3.43). We explain the application as follows.

Proposition 3.70 (Serre)**.** Fix a compact Hausdorff topological group G with probability Haar measure μ, and set $X := \operatorname{Conj}(X)$. Further, fix a number field K, and order the set of finite places $\mathfrak{p}$ by norm (breaking ties arbitrarily), and let $\{x_\mathfrak{p}\}_\mathfrak{p}$ be a sequence in X. Now, for each finite-dimensional complex continuous representation ρ of G, define the L-function

$$L(s, \rho) := \prod_{\mathfrak{p}} \frac{1}{\det\left(1 - \rho(x_\mathfrak{p}) \operatorname{N}\mathfrak{p}^{-s}\right)},$$

and suppose that $L(s, \rho)$ admits a non-vanishing holomorphic analytic continuation to the line $\{s : \operatorname{Re} s = 1\}$ for each nontrivial irreducible ρ. Then the sequence $\{x_\mathfrak{p}\}_\mathfrak{p}$ equidistributes in G.

Proof. We apply Lemma 3.67. Using Theorem 3.63 to count the number of prime ideals $\mathfrak{p}$ of norm less than some bound, we see that we need

$$\sum_{\operatorname{N}\mathfrak{p} \leq x} \operatorname{tr} \rho(x_\mathfrak{p}) \stackrel{?}{=} o_\rho\left(\frac{x}{\log x}\right)$$

for all nontrivial irreducible complex representations ρ of G. We go ahead and fix such a representation ρ; set $d := \dim \rho$ for brevity. We proceed in steps.

1. As in Theorem 3.51, the idea is to apply the Weiner–Ikehara theorem to the logarithmic derivative $L(s, \rho)$. The correct "twisted" prime-counting function is a little involved, so we postpone its computation for a moment. Instead, let's go ahead and compute $-L'(s, \rho)/L(s, \rho)$. For each $\mathfrak{p}$, let $\{\lambda_{\mathfrak{p}1}, \dots, \lambda_{\mathfrak{p}d}\}$

denote the eigenvalues of $\rho(x_{\mathfrak{p}})$ (counted with multiplicity), so we see that

$$\log\left(\det\left(1-\rho(x_{\mathfrak{p}})\,\mathrm{N}\,\mathfrak{p}^{-s}\right)\right)=\sum_{i=1}^{d}\log\left(1-\lambda_{\mathfrak{p}i}\,\mathrm{N}\,\mathfrak{p}^{-s}\right),$$

so taking the logarithmic derivative as in Lemma 3.48 of $L(s,\rho)$ yields

$$-\frac{L'(s,\rho)}{L(s,\rho)}=\sum_{\mathfrak{p}}\sum_{k\geq 1}\sum_{i=1}^{d}\frac{\lambda_{\mathfrak{p}i}^{k}\log\mathrm{N}\,\mathfrak{p}}{\mathrm{N}\,\mathfrak{p}^{ks}}.$$

Thus, we see that the correct weights are given by

$$\Lambda_{\rho}(I)\coloneqq\begin{cases}\sum_{i=1}^{d}\lambda_{\mathfrak{p}i}^{k}\log\mathrm{N}\,\mathfrak{p} & \text{if } I=\mathfrak{p}^{k} \text{ and } k\geq 1,\\ 0 & \text{else.}\end{cases}$$

In particular, $-L'(s,\rho)/L(s,\rho)=\sum_{I\subseteq\mathcal{O}_K}\Lambda_{\rho}(I)/\,\mathrm{N}(I)^{s}$; this sum is purely formal, in the sense that one side makes sense as soon as the other does. Do note that $\Lambda_{\rho}(\mathfrak{p})=\operatorname{tr}\rho(x_{\mathfrak{p}})\log\mathrm{N}\,\mathfrak{p}$ for each prime $\mathfrak{p}$. Also, note

2. We now see that arguing as in Corollary 3.40 shows that it will be enough to check that

$$\sum_{\substack{I\subseteq\mathcal{O}_K\\ \mathrm{N}(I)\leq x}}\Lambda_{\rho}(I)\stackrel{?}{=}o_{\rho}(x).$$

This is somewhat involved, so we will provide some detail. Well, we group this sum as

$$\sum_{\substack{\mathfrak{p}\text{ prime},k>1\\ \mathrm{N}\,p^{k}\leq x}}\Lambda_{\rho}\left(\mathfrak{p}^{k}\right).$$

We now have two observations.

- We note we may discard the terms with $k\geq 2$. Because G is compact, the eigenvalues $\lambda_{\mathfrak{p}i}$ are all roots of unity, so the sum $\sum_{i=1}^{d}\lambda_{\mathfrak{p}i}^{k}$ is $O_{\rho}(1)$, so we may ignore its contribution. Now, for each prime $\mathfrak{p}$, we may rudely bound $\Lambda_{\rho}\left(\mathfrak{p}^{k}\right)$ as $\log p^{[K:\mathbb{Q}]k}$, where p is the prime under $\mathfrak{p}$. On the other hand, the number of primes $\mathfrak{p}$ with $\mathrm{N}\,\mathfrak{p}$ can be naïvely bounded by $[K:\mathbb{Q}]$, so we will do so. Thus, we see that our contribution totals to

$$[K:\mathbb{Q}]^{2}\sum_{k=2}^{\log_2 x}\sum_{p\leq x^{1/k}}\log p\leq[K:\mathbb{Q}]^{2}(\log_2 x)(\sqrt{x}\log x)$$

as in Corollary 3.40. We conclude that our hypothesis is equivalent to

$$\sum_{\mathrm{N}\,\mathfrak{p}\leq x}\Lambda_{\rho}(\mathfrak{p})\stackrel{?}{=}o_{\rho}(x).$$

- We now use Abel summation in the form of Proposition 3.39 to see

$$\sum_{\mathrm{N}\,\mathfrak{p}\leq x}\operatorname{tr}\rho(x_{\mathfrak{p}})=\frac{1}{\log x}\sum_{\mathrm{N}\,\mathfrak{p}\leq x}\Lambda_{\rho}(\mathfrak{p})+\int_{2}^{x}\left(\sum_{\mathrm{N}\,\mathfrak{p}\leq x}\Lambda_{\rho}(x)\right)\frac{1}{t(\log t)^{2}}\,dt$$

(Technically, we should stratify the sum over terms of given norm before applying Abel summation.) The left term in the right-hand side is now $o(x/\log x)$ by the hypothesis, and the right term is $o(x/\log x)$ as argued in Corollary 3.40.

3. We are now ready to complete the proof using Theorem 3.43. Here are our checks.

   (i) It is enough to check that
   $$\sum_{\mathrm{N}(I)=n} \Lambda_\rho(I) = O_\rho(n^\varepsilon)$$
   for each $\varepsilon > 0$. We may assume that I is a prime-power $\mathfrak{p}^k$. As in the previous step, we see that the contribution from $\sum_{i=1}^d \lambda_{\mathfrak{p}i}^k \log \mathrm{N}\,\mathfrak{p}$ is $O_\rho(1)$ is $O_\rho(1)$, so it has no effect. We now argue as in Theorem 3.63: the number of I with $I = \mathfrak{p}^k$ is bounded by $[K : \mathbb{Q}]$, and they only contribute $\log \mathrm{N}\,\mathfrak{p} = O(\log n)$. The result follows.

   (ii) This follows immediately from the hypothesis on $L(s, \rho)$.

   (iii) From Theorem 3.63, we already know that
   $$\sum_{\mathrm{N}(I)\le x} \Lambda_K(x) = O(x).$$
   The previous step explains that the contribution $\sum_{i=1}^d \lambda_{\mathfrak{p}i}^k \log \mathrm{N}\,\mathfrak{p}$ is $O_\rho(1)$, so we conclude. ∎

Remark 3.71. Essentially the same proof as in (i) of step 3 above shows that $\log L(s, \rho)$ converges absolutely in the region $\{s : \mathrm{Re}\, s > 1\}$, so $L(s, \rho)$ converges absolutely and is nonzero. Indeed, one merely needs to re-weight the Dirichlet series coefficient $\sum_{\mathrm{N}(I)=n} \Lambda_\rho(I)$ to undo the derivative, effectively removing a $\log n$ factor.

3.2.4 The Chebotarev Density Theorem

We now give a standard application of Proposition 3.70, to the Chebotarev desnity theorem. For any Galois extension L/K of number fields, our goal is to show that the Frobenius conjugacy classes $\mathrm{Frob}_\mathfrak{p}$ equidistribute in $\mathrm{Conj}(\mathrm{Gal}(L/K))$. In light of Proposition 3.70, we see that we are interested in the following L-functions.

Definition 3.72 (Artin L-function)**.** For a number field K, let $\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$ be a finite-dimensional complex representation. Then we define the *Artin L-function*
$$L(s, \rho) \coloneqq \prod_\mathfrak{p} \frac{1}{\det\left(1 - \rho(\mathrm{Frob}_\mathfrak{p})\,\mathrm{N}\,\mathfrak{p}^{-s} | V^{I_\mathfrak{p}}\right)},$$
where $I_\mathfrak{p} \subseteq \mathrm{Gal}(\overline{K}/K)$ denotes the inertia subgroup of $\mathfrak{p}$.

Remark 3.73. Let us explain this factor. Formally, one should fix a prime $\mathfrak{P}$ of $\overline{K}$ living above $\mathfrak{p}$ (alternatively, one could choose a compatible system of primes for every subfield of $\overline{K}$), and then $I_\mathfrak{p}$ and $\mathrm{Frob}_\mathfrak{p}$ mean $I_\mathfrak{P}$ and $\mathrm{Frob}_\mathfrak{P}$, respectively. Let's check that this definition is independent of the choice of $\mathfrak{P}$: any other prime living above $\mathfrak{p}$ looks like $g\mathfrak{P}$ for some $g \in \mathrm{Gal}(\overline{K}/K)$. Then $I_{g\mathfrak{P}} = gI_\mathfrak{P}g^{-1}$ and $\mathrm{Frob}_{g\mathfrak{P}} = g\mathrm{Frob}_\mathfrak{P}g^{-1}$. Thus, we see that $v \mapsto \rho(g)v$ sends $V^{I_\mathfrak{P}} \to V^{I_{g\mathfrak{P}}}$ and sends the action of $\mathrm{Frob}_\mathfrak{P}$ to the action of $\mathrm{Frob}_{g\mathfrak{P}}$. As such, the characteristic polynomials must be equal.

Example 3.74. Taking ρ to be the trivial representation, we find that $L(s, 1)$ and $\zeta_K(s)$ are equal to a finite number of Euler factors. Recall that we may write $L(s, 1) = \zeta_K(s)$.

Before going any further, we state some helpful facts about Artin L-functions.

Lemma 3.75. For a number field K, let $\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$ be a finite-dimensional complex representation. Then $\ker\rho$ is open, and ρ has finite image.

Proof. We have two steps.

1. We show the following "no small subgroups" result: we claim that there is an open neighborhood $U \subseteq \mathrm{GL}(V)$ of the identity which does not contain any nontrivial subgroup. Indeed, recall that $\exp\colon \mathfrak{gl}(V) \to \mathrm{GL}(V)$ is a local diffeomorphism; say that it is a diffeomorphism on some bounded open subset $U_1 \subseteq \mathfrak{gl}(V)$, and then set $U \coloneqq \exp\left(\frac{1}{2}U_1\right)$.

   Now, suppose for the sake of contradiction that U contains a subgroup $H \subseteq \mathrm{GL}(V)$. Then note that any $\exp(x) \in H$ for $x \in \frac{1}{2}U_1$ must have $\exp(2x)$ in H and hence in U, so $2x \in \frac{1}{2}U_1$ as well; this shows that $\frac{1}{2}U_1$ is unbounded, which is a contradiction.

2. We complete the proof. Choose an open neighborhood $U \subseteq \mathrm{GL}(V)$ of the identity as in the previous step. Then $\rho^{-1}(U) \subseteq \mathrm{Gal}(\overline{K}/K)$ is an open subset, but the profinite topology of $\mathrm{Gal}(\overline{K}/K)$ promises that this open subset contains an open subgroup $H \subseteq \mathrm{Gal}(\overline{K}/K)$. Then $\rho(H) \subseteq U$ is a subgroup, which must be trivial, so we conclude that $H \subseteq \ker\rho$. Now, H is a subgroup of finite index, so we conclude the same is true for $\ker\rho$. ■

Lemma 3.76 (additive)**.** For a number field K, let ρ_1 and ρ_2 be finite-dimensional complex representations of $\mathrm{Gal}(\overline{K}/K)$. Then $L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2)$.

Proof. This follows because, for any $g \in G$, the characteristic polynomial of $(\rho_1 \oplus \rho_2)(g)$ is the product of the characteristic polynomials of $\rho_1(g)$ and $\rho_2(g)$. ■

Lemma 3.77 (induction)**.** Fix a finite extension L/K of number fields. Given a finite-dimensional complex representation $\rho\colon \mathrm{Gal}(\overline{K}/L) \to \mathrm{GL}(V)$, we have

$$L(s, \rho) = L\left(s, \mathrm{Ind}_{\mathrm{Gal}(\overline{K}/L)}^{\mathrm{Gal}(\overline{K}/K)} \rho\right).$$

Proof. We follow [Neu99, Proposition VII.10.4(iv)]. Once again, we equate Euler factors over a prime $\mathfrak{p}$ of K. For psychological reasons, we come down to finite extensions. By Lemma 3.75, we may find a finite Galois extension M of K extending L such that $\mathrm{Gal}(M/L)$ is in the kernel of ρ. Then we may replace all instances of $\overline{K}$ with M without changing the value of the L-function; for example, ρ is certainly well-defined throughout.

Now, for brevity, set $G \coloneqq \mathrm{Gal}(M/K)$ and $H \coloneqq \mathrm{Gal}(M/L)$, and let $\widetilde{\rho} \coloneqq \mathrm{Ind}_H^G \rho$ denote the induction; further, set $V_\rho \coloneqq V$ and $V_{\widetilde{\rho}} \coloneqq \mathrm{Ind}_H^G V$ for clarity. We want to show that

$$\frac{1}{\det\left(1 - \widetilde{\rho}(\mathrm{Frob}_\mathfrak{p})\,\mathrm{N}\,\mathfrak{p}^{-s} | V_{\widetilde{\rho}}^{I_\mathfrak{p}}\right)} \overset{?}{=} \prod_{\mathfrak{q}|\mathfrak{p}} \frac{1}{\det\left(1 - \rho(\mathrm{Frob}_\mathfrak{q})\,\mathrm{N}\,\mathfrak{q}^{-s} | V_\rho^{I_\mathfrak{q}}\right)},$$

where $\mathfrak{q}$ varies over primes of L lying over $\mathfrak{p}$. Note that these inertia subgroups can now be brought down to automorphisms of M. We now proceed in steps.

1. We begin with a special case. Suppose that there is a single prime $\mathfrak{P}$ of M above $\mathfrak{p}$, and set $\mathfrak{q} \coloneqq \mathfrak{P} \cap L$. In this case, $G = D_\mathfrak{P}$, and we would like to show that

   $$\det\left(1 - \widetilde{\rho}(\mathrm{Frob}_\mathfrak{P})T | V_{\widetilde{\rho}}^{I_\mathfrak{P}}\right) = \det\left(1 - \rho(\mathrm{Frob}_\mathfrak{P})^{[G:H]} T^{[G:H]} | V_\rho^{I_\mathfrak{P}}\right),$$

   where T is a formal variable replacing $\mathrm{N}\,\mathfrak{p}^{-s}$. (Note that $\mathrm{N}\,\mathfrak{q} = \mathrm{N}\,\mathfrak{p}^{[G:H]}$.) Note that we may as well replace L/K with M/L, effectively allowing us to assume that H is trivial.

For psychological reasons, we explain how to reduce to the case where $I_{\mathfrak{P}}$ is trivial by adjusting the representations. On one hand, we'd like to replace V_ρ with $V_\rho^{H\cap I_{\mathfrak{P}}}$. On the other hand, note that $\left(\operatorname{Ind}_H^G \rho\right)^{I_{\mathfrak{P}}}$ can be descended to $\operatorname{Ind}_{H/(H\cap I_{\mathfrak{P}})}^{G/I_{\mathfrak{P}}} V_\rho^{H\cap I_{\mathfrak{P}}}$: a function $f\colon G\to V$ succeeds at being invariant under $I_{\mathfrak{P}}$ if and only if it descends to a function on $G/I_{\mathfrak{P}}$, and we see that we must restrict outputs to $V^{H\cap I_{\mathfrak{P}}}$ because $h\in H\cap I_{\mathfrak{P}}$ will have $f(x)=f(xh)=\rho(h)f(x)$. Thus, by taking the quotient by $I_{\mathfrak{P}}$ everywhere, we may assume that it is trivial.

Now, $D_{\mathfrak{P}}$ has become cyclic of order $f\coloneqq f(\mathfrak{P}/\mathfrak{p})$ generated by the Frobenius, so we will be able to compute $\operatorname{Ind}_1^G \rho$ relatively easily. Indeed, view $V_{\widetilde{\rho}}$ as the vector space $V^{\oplus f}$ by sending the function f to the f-tuple $\left(f(\operatorname{Frob}_{\mathfrak{P}}^i)\right)_{i=0}^{f-1}$. Then we see that $\operatorname{Frob}_{\mathfrak{P}}$ acts on $V^{\oplus f}$ by

$$(v_0,v_1,\dots,v_{f-2},v_{f-1})=\left(\rho(\operatorname{Frob}_{\mathfrak{P}})^f v_{f-1},v_0,\dots,v_{f-3},v_{f-2}\right).$$

We may now compute the determinant of $1-\widetilde{\rho}(\operatorname{Frob}_{\mathfrak{P}})T$ by commuting the determinant of the matrix

$$\begin{bmatrix} 1 & -T & 0 & \cdots & 0 & 0\\ 0 & 1 & -T & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -T\\ -\rho(\operatorname{Frob}_{\mathfrak{P}})^f T & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which we see is $\det\left(1-\rho(\operatorname{Frob}_{\mathfrak{P}})^f T^f\right)$ after some row-reduction.

2. We now return to the general case for the remainder of the proof. All decomposition, inertial, and Frobenius elements will be taken over $\mathfrak{p}$ unless otherwise specified. We begin by computing the action of $\widetilde{\rho}$. The idea is to use Mackey theory. Indeed, for fixed prime $\mathfrak{P}$ of M above $\mathfrak{p}$, we are only interested in the action of $D_{\mathfrak{P}}$ on $\operatorname{Ind}_H^G \rho$, so we note there is an isomorphism

$$\operatorname{Res}_{D_{\mathfrak{P}}}^G \operatorname{Ind}_H^G \rho \overset{?}{\cong} \bigoplus_{g\in H\backslash G/D_{\mathfrak{P}}} \operatorname{Ind}_{D_{\mathfrak{P}}\cap gHg^{-1}}^{D_{\mathfrak{P}}} \rho_g,$$

where $\rho_g(d)=\rho\left(gdg^{-1}\right)$. Let's quickly explain this. There is a forward map sending a function $f\colon G\to V$ to the tuple of functions $(f_g)_g$ where $f_g\colon D_{\mathfrak{P}}\to V$ is defined by $f_g(x)\coloneqq f(xg)$. There is also a backward map sending the tuple $(f_g)_g$ to the function $f\colon G\to V$ given by $f(hgd)\coloneqq\rho(h)f_g(d)$. These maps are G-invariant and can be checked to be $D_{\mathfrak{P}}$-invariant, so we have our isomorphism.

Thus, we see that

$$\det\left(1-\widetilde{\rho}(\operatorname{Frob}_{\mathfrak{P}})\operatorname{N}\mathfrak{p}^{-s}|V_{\widetilde{\rho}}^{I_{\mathfrak{P}}}\right)=\prod_{g\in H\backslash G/D_{\mathfrak{P}}}\det\left(1-\operatorname{Frob}_{\mathfrak{P}}\operatorname{N}\mathfrak{p}^{-s}|(\operatorname{Ind}_{D_{\mathfrak{P}}\cap gHg^{-1}}^{D_{\mathfrak{P}}}\rho_g)^{I_{\mathfrak{P}}}\right).$$

Undoing conjugation by g, we can rewrite this as

$$\det\left(1-\widetilde{\rho}(\operatorname{Frob}_{\mathfrak{P}})\operatorname{N}\mathfrak{p}^{-s}|V_{\widetilde{\rho}}^{I_{\mathfrak{P}}}\right)=\prod_{g\in H\backslash G/D_{\mathfrak{P}}}\det\left(1-\operatorname{Frob}_{g\mathfrak{P}}\operatorname{N}\mathfrak{p}^{-s}|(\operatorname{Ind}_{D_{g\mathfrak{P}}\cap H}^{D_{g\mathfrak{P}}}\rho)^{I_{g\mathfrak{P}}}\right).$$

3. We translate the product using some group theory. For this, we need to enumerate the primes of L above $\mathfrak{p}$. Note $\operatorname{Gal}(M/K)$ acts transitively on the set of primes of M above $\mathfrak{p}$, so $g\mapsto g\mathfrak{P}$ defines a bijection from $G/D_{\mathfrak{P}}$ to this set of primes. Then restricting to L, we see that $g\mapsto(g\mathfrak{P}\cap L)$ is a surjective map from $G/D_{\mathfrak{P}}$ to the set of primes in L above $\mathfrak{p}$; this map descends to $H\backslash G/D_{\mathfrak{P}}$, where we claim that it actually defines a bijection. Indeed, $(g\mathfrak{P}\cap L)=(g'\mathfrak{P}\cap L)$ implies that $g\mathfrak{P}$ and $g'\mathfrak{P}$ are both primes of M sitting above the same prime of L, so there is $h\in\operatorname{Gal}(M/L)$ such tthat $hg\mathfrak{P}=g'\mathfrak{P}$, which implies $hgD_{\mathfrak{P}}=g'D_{\mathfrak{P}}$.

Thus, we see that

$$\prod_{\mathfrak{q}|\mathfrak{p}} \det\left(1-\rho(\mathrm{Frob}_{\mathfrak{q}})\,\mathrm{N}\,\mathfrak{q}^{-s}|V_\rho^{I_{\mathfrak{q}}}\right) = \prod_{g\in H\backslash G/D_{\mathfrak{P}}} \det\left(1-\rho(\mathrm{Frob}_{g\mathfrak{P}})^{f_g}\,\mathrm{N}\,\mathfrak{p}^{-f_g s}|V_\rho^{I_{g\mathfrak{P}}}\right),$$

where $f_g = f(g\mathfrak{P}/(g\mathfrak{P}\cap L)) = [D_{g\mathfrak{P}} : D_{g\mathfrak{P}}\cap H]$ is the required inertial degree.

4. We are now ready to complete the proof. In light of the previous two steps, we would like to show that any $\mathfrak{P}'$ of M above $\mathfrak{p}$ has

$$\det\left(1-\mathrm{Frob}_{\mathfrak{P}'}T|(\mathrm{Ind}_{D_{\mathfrak{P}'}\cap H}^{D_{\mathfrak{P}'}}\rho)^{I_{\mathfrak{P}'}}\right) = \det\left(1-\rho(\mathrm{Frob}_{\mathfrak{P}'})^{[D_{\mathfrak{P}'}:D_{g\mathfrak{P}'}\cap H]}T^{[D_{\mathfrak{P}'}:D_{g\mathfrak{P}'}\cap H]}|V_\rho^{I_{\mathfrak{P}'}}\right),$$

where T is a formal variable replacing $\mathrm{N}\,\mathfrak{p}^{-s}$. Now, we note that we may define $K' \coloneqq M^{D_{\mathfrak{P}'}}$ and $L' \coloneqq M^{D_{\mathfrak{P}'}\cap H}$, whereupon we see that $\mathfrak{P}'$ is the only prime above of M the prime $\mathfrak{p}' \coloneqq \mathfrak{P}'\cap K'$ in K'. The above equality then follows from the special case in the first step applied to the extension L'/K'. ∎

Remark 3.78. Given subgroups $D\subseteq H\subseteq G$ and a representation ρ of H, the above proof used the fact that

$$\mathrm{Res}_D^G\,\mathrm{Ind}_H^G\,\rho \cong \bigoplus_{\eta\in H\backslash G/D} \mathrm{Ind}_{D\cap\eta^{-1}H\eta}^{D}\,\mathrm{Ind}_{D\cap\eta^{-1}H\eta}^{D}\,\rho_\eta,$$

where $\rho_\eta(d) \coloneqq \rho(\eta d\eta^{-1})$. This fact is remarkably useful.

Example 3.79. Let L/K be a Galois extension of number fields with Galois group G. Then $\mathrm{Ind}_{\mathrm{Gal}(\overline{K}/L)}^{\mathrm{Gal}(\overline{K}/K)} 1$ is the regular representation of G, so by decomposing the regular representation into irreducible representations and using Lemmas 3.76 and 3.77, we find

$$\zeta_L(s) = \prod_{\rho\in\mathrm{IrRep}(G)} L(s,\rho)^{\dim\rho},$$

where $\mathrm{IrRep}(G)$ refers to the set of irreducible representations of G.

In light of Proposition 3.70, we need to show that nontrivial irreducible ρ give $L(s,\rho)$ a non-vanishing holomorphic continuation to the line $\{s : \mathrm{Re}\,s = 1\}$. The rough idea is to use the Brauer induction theorem to reduce to the abelian case, and then the abelian case can be turned over to Hecke L-functions by class field theory.

Thus, we begin with the abelian case. As promised, this is essentially class field theory.

Proposition 3.80. Fix a number field K, and let $\rho\colon \mathrm{Gal}(\overline{K}/K)\to\mathbb{C}^\times$ be a continuous character. Then there is a continuous unitary character $\chi\colon K^\times\backslash\mathbb{A}_K^\times\to\mathbb{C}^\times$ such that

$$L(s,\rho) = L(s,\chi)$$

for s such that $\mathrm{Re}\,s > 1$.

Proof. The main point is that global class field theory in the form of [Mil20a, Theorem 5.3] provides an isomorphism

$$\widehat{K^\times\backslash\mathbb{A}_K^\times} \cong \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}.$$

With this in mind as a guide, we construct the character χ. Because the target of ρ is abelian, we see that ρ factors through $\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{ab}}/K)$. For convenience, we recall from Lemma 3.75 that ρ descends

to a representation of $\mathrm{Gal}(L/K)$ for some minimal finite Galois extension L/K, and once again, we find that $\mathrm{Gal}(L/K)$ is abelian. Thus, we may define χ as the composite

$$K^\times\backslash\mathbb{A}_K^\times \twoheadrightarrow K^\times \mathrm{N}_{L/K}(\mathbb{A}_L^\times)\backslash\mathbb{A}_K^\times \cong \mathrm{Gal}(L/K) \xrightarrow{\rho} \mathbb{C}^\times,$$

where the isomorphism is given by global class field theory [Mil20a, Theorem 5.3]; explicitly, on finite primes $\mathfrak{p}$ of K unramified in L, it is trivial on $\mathcal{O}_\mathfrak{p}^\times \subseteq \mathbb{A}_K^\times$ and sends a uniformizer $\varpi_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ to $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(L/K)$. By construction, χ is a continuous character, and it is unitary because ρ must output to S^1 by the compactness of $\mathrm{Gal}(\overline{K}/K)$.

We now compare the Euler factors of $L(s,\rho)$ and $L(s,\chi)$ at a prime $\mathfrak{p}$ of K. There are two cases.

- Suppose that $\mathfrak{p}$ is a prime unramified in L/K. Then we see that

  $$\det\left(1-\rho(\mathrm{Frob}_\mathfrak{p})\,\mathrm{N}\,\mathfrak{p}^{-s} \mid \mathbb{C}\right) = 1-\chi_\mathfrak{p}(\varpi_\mathfrak{p})\,\mathrm{N}\,\mathfrak{p}^{-s}$$

  by construction of χ (and properties of the global class field theory map), so we are done.

- Suppose that $\mathfrak{p}$ is a prime ramified in L/K. On one hand, ρ is nontrivial on $I_\mathfrak{p} \subseteq \mathrm{Gal}(L/K)$, so $\mathbb{C}^{I_\mathfrak{p}}$ must be zero-dimensional, so the Euler factor of $L(s,\rho)$ is 1. On the other hand, we note that $\mathrm{N}_{L/K}(\mathbb{A}_L^\times)$ does not contain $\mathcal{O}_\mathfrak{p}^\times$ by a computation of norm subgroups, so χ is nontrivial on $\mathcal{O}_\mathfrak{p}^\times$ by tracking through the global class field theory isomorphism, so the Euler factor of $L(s,\chi)$ is also 1. ∎

Remark 3.81. Because a Dirichlet series is uniquely determined by its coefficients, we see that the character χ is uniquely determined by ρ. However, this is not a bijection: the disagreement between the topologies of $K^\times\backslash\mathbb{A}_K^\times$ and $\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$ means that there are many more continuous unitary characters $K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$.

Corollary 3.82. Fix a number field K, and let $\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathbb{C}^\times$ be a nontrivial continuous character. Then $L(s,\rho)$ admits a nonvanishing holomorphic continuation to $\{s : \mathrm{Re}\, s = 1\}$.

Proof. Note $L(s,\rho)$ is already holomorphic and nonvanishing on $\{s : \mathrm{Re}\, s > 0\}$ by Remark 3.71. Now, construct the continuous unitary character $\chi\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ as in Proposition 3.80 so that $L(s,\chi) = L(s,\rho)$. The nonvanishing now follows from Proposition 3.62. Lastly, the continuation follows from Theorem 3.61 as soon as we check that $\chi|\cdot|^{1+it}$ is never trivial on all unramified primes. This follows by the nontriviality of ρ, which requires there to be an unramified prime $\mathfrak{p}$ where $\rho(\mathrm{Frob}_\mathfrak{p}) \neq 1$; this corresponds to the needed fact about χ. ∎

We are now in a position to prove equidistribution of Frobenius elements in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, from which one can prove the general case by a clever reduction argument. However, we will be honest to our discussion of equidistribution and prove nonvanishing holomorphic continuation to $\{s : \mathrm{Re}\, s = 1\}$ for $L(s,\rho)$ for all nontrivial irreducible continuous representations ρ.

The idea is to write ρ as a "linear combination" of inductions of characters. Then the result will follow from the abelian case combined with our properties about L-functions. One almost achieves the full holomorphic nonvanishing as well, but it would be technically possible to see the trivial character in our linear combination, thus possibly introducing a pole or zero.

Of course, it does not a priori make sense to talk about linear combination of representations, so we must pass to their linearization: virtual characters. Thus, we will want to define the Artin L-function of a class function. To motivate, use Lemma 3.75 to descend ρ to some representation $\mathrm{Gal}(L/K) \to \mathbb{C}^\times$. For some $g \in \mathrm{Gal}(L/K)$, we let $\lambda_1,\dots,\lambda_d$ denote the eigenvalues of g (with algebraic multiplicities), so we see that

$$\log\left(\det(1-\rho(g)T)^{-1}\right) = \sum_{i=1}^{d} -\log(1-\lambda_i T).$$

Now, expanding out the Taylor series reveals that

$$\frac{1}{\det(1-\rho(g)T)} = \exp\left(\sum_{k=1}^{\infty} \frac{\operatorname{tr}\rho\left(g^k\right)T^k}{k}\right).$$

We are now ready to make the following definition.

Definition 3.83 (Artin L-function)**.** Fix a Galois extension L/K of number fields with Galois group G. For a class function $\chi\colon G \to \mathbb{C}$, we define the *Artin L-function* as

$$L(s,\chi) \coloneqq \prod_{\mathfrak{p}\text{ unr.}} \exp\left(\sum_{k=1}^{\infty} \frac{\chi(\operatorname{Frob}_{\mathfrak{p}}^k)}{k\,\mathrm{N}\,\mathfrak{p}^{-s}}\right),$$

where the product is taken over primes of K unramified in L.

Example 3.84. The discussion preceding the definition shows that $L(s,\rho) \doteq L(s,\operatorname{tr}\circ\rho)$ for any finite-dimensional complex representation $\rho\colon \operatorname{Gal}(L/K) \to \mathrm{GL}(V)$.

Remark 3.85. A notable defect of this definition is that we have not defined our Euler factors at ramified primes. This will cause us to use some $\doteq$s in the sequel; this is no issue because finitely many Euler factors will not change holomorphy or nonvanishing.

Here are the standard properties of these L-functions, which are carried over from our previous discussion.

Lemma 3.86. Fix a Galois extension M/K of number fields with Galois group G.

(a) If $\chi\colon G \to \mathbb{C}$ is a class function, then $L(s,\chi)$ converges absolutely to a nonvanishing holomorphic function in the region $\{s : \operatorname{Re} s > 1\}$.

(b) Additive: if $\chi_1,\chi_2\colon G \to \mathbb{C}$ are class functions, then $L(s,\chi_1+\chi_2) = L(s,\chi_1)L(s,\chi_2)$.

(c) Inflation: let L/K be a Galois subextension such that $\operatorname{Gal}(M/L) = H$. If $\chi\colon G/H \to \mathbb{C}$ is a class function, then $L(s,\chi) \doteq L(s,\widetilde{\chi})$, where $\widetilde{\chi}\colon G \to \mathbb{C}$ is the induced class function.

Proof. Here, (a) follows as in Remark 3.71 by noting that the series expansion for $\log L(s,\chi)$ absolutely converges to a finite value; notably, G is finite, so χ is bounded, so it does not meaningfully contribute. Continuing, (b) follows by a direct expansion of the Euler product, and (c) follows because the Euler factors are exactly the same for any prime $\mathfrak{p}$ of K unramified in M (and hence also unramified in K). ∎

The suitable analogue of Lemma 3.77 on induction remains true, but we will not need it in the full generality of complex class functions. However, we do need to know how to induct character.

Notation 3.87. Fix a subgroup H of a finite group G. Given a class function $\chi\colon H \to \mathbb{C}$, define the induced class function

$$\operatorname{Ind}_H^G \chi(g) \coloneqq \frac{1}{|H|} \sum_{\substack{\eta\in G \\ \eta g \eta^{-1}\in H}} \chi\left(\eta g \eta^{-1}\right).$$

Lemma 3.88. Fix a subgroup H of a finite group G. Given a finite-dimensional representation $\rho\colon H \to \mathrm{GL}(V)$, then we check that $\operatorname{tr}\circ\operatorname{Ind}_H^G\rho = \operatorname{Ind}_H^G(\operatorname{tr}\circ\rho)$.

Proof. We will use many of the same tricks appearing in Lemma 3.77. Fix some $g \in G$, and we would like to check the result at g. We proceed in steps.

1. We begin with the special case where G is cyclic and generated by g. If $H = G$, there is nothing to do. Otherwise, if $H \neq G$, then $\operatorname{Ind}_H^G(\operatorname{tr}\circ\rho)(g)$ is an empty, so we must show $\operatorname{tr}\operatorname{Ind}_H^G \rho(g)$ vanishes. Well, view elements $\operatorname{Ind}_H^G V$ as sequences of vectors $\{v_{Hg'}\}$ indexed by $H\backslash G$, and then we see that $\operatorname{Ind}_G^H \rho(g)$ acts by a (generalized) permutation matrix which is a sum of nontrivial cycles of length $[G : H]$. Thus, this operator has no trace.

2. We now show the general case. By Remark 3.78, we see that

$$\operatorname{Res}_{\langle g\rangle}^G \operatorname{Res}_H^G \rho \cong \bigoplus_{\eta\in H\backslash G/\langle g\rangle} \operatorname{Ind}_{\langle g\rangle\cap\eta^{-1}H\eta}^{\langle g\rangle} \rho_\eta,$$

   where $\rho_\eta(g') \coloneqq \rho\left(\eta g'\eta^{-1}\right)$. Thus, we see that

$$\operatorname{tr}\operatorname{Ind}_H^G \rho(g) = \sum_{\eta\in H\backslash G/\langle g\rangle} \operatorname{tr}\operatorname{Ind}_{\langle g\rangle\cap\eta^{-1}H\eta}^{\langle g\rangle} \rho_\eta\left(g\right).$$

   Now, by the previous case, we see that terms vanish as long as $g \notin \eta^{-1}H\eta$; on the other hand, if $g \in \eta^{-1}H\eta$, then we get a contribution of $\operatorname{tr}\rho\left(\eta g\eta^{-1}\right)$, so we see

$$\operatorname{tr}\operatorname{Ind}_H^G \rho(g) = \sum_{\substack{\eta\in H\backslash G/\langle g\rangle \\ \eta g\eta^{-1}\in H}} \operatorname{tr}\rho\left(\eta g\eta^{-1}\right).$$

   The result now follows by replacing the sum over $H\backslash G/\langle g\rangle$ with a sum over G. ∎

In order to allow us to stop talking about L-functions as quickly as possible, let's go ahead and explicate the inductive approach to meromorphic continuation via Brauer's theorem. We begin with the following non-standard definition.

Definition 3.89 (Brauer)**.** Fix a finite group G. Then G is *Brauer* if and only if, for any finite-dimensional complex irreducible representation ρ, there is a sequence of pairs $\{(a_i, H_i, \psi_i)\}_{i=1}^n$ where $a_i \in \mathbb{Z}$ and $H_i \subseteq G$ is a subgroup and $\psi_i\colon H_i \to \mathbb{C}^\times$ is a representation such that

$$\operatorname{tr}\circ\rho = \sum_{i=1}^n a_i \operatorname{Ind}_{H_i}^G \psi_i$$

as virtual character. A representation of the form $\operatorname{Ind}_{H_i}^G \psi_i$ is said to be *monomial*.

Lemma 3.90. Fix a Galois extension L/K of number fields with Galois group G. Suppose that G is Brauer. For any finite-dimensional complex representaion ρ of G, the function $L(s,\rho)$ admits a meromorphic continuation to $\{s : \operatorname{Re} s = 1\}$ with no poles or zeroes except possibly a pole or zero at $s = 1$. Further, the order of the pole at $s = 1$ is $\langle \operatorname{tr}\circ\rho, 1\rangle$.

Proof. By the additivity of Lemma 3.76, we may assume that ρ is irreducible. By Example 3.84, it is enough to check the result for $L(s, \operatorname{tr}\circ\rho)$. Because G is Brauer, we receive an expansion $\operatorname{tr}\circ\rho = \sum_{i=1}^n a_i \operatorname{Ind}_{H_i}^G \psi_i$ of $\operatorname{tr}\circ\rho$ into a $\mathbb{Z}$-linear combination of inductions of characters, which implies that

$$L(s, \operatorname{tr}\circ\rho) = \prod_{i=1}^n L\left(s, \operatorname{Ind}_{H_i}^G \psi_i\right)^{a_i}$$

by Lemma 3.86. By Example 3.84, we may now think of each $L\left(s, \operatorname{Ind}_{H_i}^G \psi_i\right)$ as an Artin L-function of a representation (up to finitely many Euler factors), so Lemma 3.77 tells us that this L-function is $L(s, \psi_i)$.

The meromorphic continuation now essentially follows from Corollary 3.82, which tells us each nontrivial ψ_i grants a nonvanishing holomorphic continuation of $L(s,\psi_i)$ to $\{s : \operatorname{Re} s \geq 1\}$. Note the same is true for trivial ψ_i except at the point $s = 1$ where we find a pole in $L(s,\psi_i)$ because this is a Dedekind ζ-function by Example 3.74; see Theorem 3.61 and proposition 3.62. Taking the appropriate product of these contributions proves the statement.

It remains to prove the last sentence. This will require a trick. On one hand, by the discussion in the previous paragraph, we see that the order of the pole is

$$\sum_{\substack{1\leq i\leq n\\ \psi_i = 1_{H_i}}} a_i.$$

On the other hand, we see $\langle \operatorname{tr}\circ\rho, 1\rangle$ equals

$$\sum_{i=1}^{n} a_i \left\langle \operatorname{Ind}_{H_i}^{G} \psi_i, 1_G \right\rangle = \sum_{i=1}^{n} a_i \langle \psi_i, 1_{H_i}\rangle$$

by Frobenius reciprocity. The last sentence now follows. ■

Thus, we will achieve our nonvanishing holomorphi continuation as soon as we check that all finite groups are Brauer; we will complete the nonvanishing later by a careful analysis of $s = 1$.

Our current goal is to prove Brauer's theorem that all finite groups are Brauer; our exposition follows [Ser77, Chapter 10]. We begin by creating a large supply of Brauer groups.

Lemma 3.91. Let G be a finite nilpotent group. Then G is Brauer.

Proof. We induct on $|G|$. For our base case, we note that if G is already abelian (for example, $|G| = 1$), then there is nothing to do because all irreducible representations are already one-dimensional.

Thus, for our induction, we may assume that G is nonabelian, and we fix some complex irreducible representation $\rho\colon G \to \mathrm{GL}(V)$ of G. Because taking induction commutes with taking quotients, we may replace G with $G/\ker\rho$, effectively allowing us to assume that ρ is injective. We will show directly that ρ can be induced from a character, which will complete the proof; we proceed in steps.

1. We claim that there is an abelian normal subgroup $N \subseteq G$ strictly containing $Z(G)$. This follows quickly because G is nilpotent: because G is nonabelian and nilpotent, we see that $G/Z(G)$ is nontrivial and has nontrivial center, so we let $N \subseteq G$ be the pre-image of the center. Then N strictly contains $Z(G)$ and is normal because it is the pre-image of a normal subgroup along a surjective homomorphism.

2. We now decompose $\operatorname{Res}_N^G \rho$ into irreducibles as

$$\operatorname{Res}_N^G \rho = \bigoplus_{\psi\in\operatorname{Hom}(N,\mathbb{C}^\times)} V^\psi,$$

   where $V^\psi \subseteq V$ denotes the ψ-eigenvectors of V. (The sum is over the characters of N; this decomposition exists because N is abelian.) Now, because $N \subseteq G$ is normal, we know that each of the spaces $\rho(g)V^\psi \subseteq V$ continues to be N-invariant and in fact will be N-isotypic. Thus, we see that G acts on the collection $\{V^\psi\}_\psi$, and it must act transitively because the span of the G-orbit of some V^ψ will be a G-subrepresentation of the irreducible representation ρ.

3. We claim that $\operatorname{Res}_N^G \rho$ is not isotypic. This is by the construction of N: this would imply that N acts by scalars on V, thereby implying that $\rho(N)$ commutes with $\rho(G)$, thereby giving $N \subseteq Z(G)$ because ρ is faithful. This contradicts the construction of N as strictly containing $Z(G)$.

4. We now complete the proof. Choose some $\psi_0 \in \operatorname{Hom}(N,\mathbb{C}^\times)$, and let $G_0 \subseteq G$ be the stabilizer of the action given in the second step. Then ρ restricts to a representation $\rho_0\colon G_0 \to \mathrm{GL}(V_0)$ where $V_0 \coloneqq V^{\psi_0}$. Now, we claim that $\rho = \operatorname{Ind}_{G_0}^G \rho_0$, which will complete the proof because G_0 is a strictly smaller nilpotent group than G. Well, for the isomorphism, view $\operatorname{Ind}_{G_0}^G \rho_0$ as $\mathbb{C}[G]\otimes_{\mathbb{C}[G_0]} \rho_0$, and then define the map $\operatorname{Ind}_{G_0}^G \rho_0 \to \rho$ by sending $g\otimes v_0$ to gv_0. ■

Thus, to show that any group G is Brauer, one may simply show that any virtual character for an irreducible complex representation is a $\mathbb{Z}$-linear combination of ones induced from nilpotent subgroups. Now that we are working with virtual characters than one-dimensional ones, we pick up the following notation.

Definition 3.92 (virtual character)**.** Fix a finite group G. Then we let $R(G)$ denote the free $\mathbb{Z}$-module of class functions $G \to \mathbb{C}$ generated by the virtual characters $\operatorname{tr} \circ \rho$ as ρ varies over finite-dimensional complex representations of G. One frequently calls $R(G)$ the *ring of virtual characters*.

Remark 3.93. By taking tensor products of representations, we see that $R(G)$ is a subring of the set of functions $G \to \mathbb{C}$. By induction and restriction of representations, we see that Res^G_H and Ind^G_H induce ring homomorphisms $R(G) \to R(H)$ and $R(H) \to R(G)$, respectively.

Thus, to check that a group G is Brauer, it will be enough to show that the map

$$\operatorname{Ind}\colon \bigoplus_{\substack{H \subseteq G \\ H \text{ Brauer}}} R(H) \to R(G)$$

is surjective, for any element of one of the $R(H)$s can be expanded into a sum of virtual characters induced from linear characters. For example, we will eventually show that one can restrict this direct sum to nilpotent subgroups.

Remark 3.94. While we're here, we remark that one can check the surjectivity of this map after tensoring with any free $\mathbb{Z}$-module because this essentially takes both sides to a finite power. In particular, in the sequel, we will frequently work with $R(G)_{\mathbb{Z}[\zeta_n]}$ where $n = |G|$, which is conveneint because the functions in $R(G)$ output to the ring $\mathbb{Z}[\zeta_n]$. (Indeed, for any $g \in G$ and representation ρ, because $g^n = 1$, the eigenvalues of g are all nth roots of unity, so $\operatorname{tr} \rho(g) \in \mathbb{Z}[\zeta_n]$.)

Most of our work in eventually proving that all finite groups are Brauer will come from a construction of many virtual characters. We begin with a couple preliminary lemmas.

Lemma 3.95. Fix a finite group G of order n, and choose a class function $f\colon G \to \mathbb{Z}[\zeta_n]$. Then nf is in the image of the map

$$\operatorname{Ind}\colon \bigoplus_{\substack{H \subseteq G \\ H \text{ cyclic}}} R(H)_{\mathbb{Z}[\zeta_n]} \to R(G)_{\mathbb{Z}[\zeta_n]}.$$

Proof. The proof has two steps.

1. We show that n is in the image of the given map. For each cyclic subgroup $H \subseteq G$, define $\theta_H\colon H \to \mathbb{Z}$ as $|H|$ times the indicator function of generating H. Then we claim that

$$n \overset{?}{=} \sum_{\substack{H \subseteq G \\ H \text{ cyclic}}} \operatorname{Ind}^G_H \theta_H.$$

   Well, for any $g \in G$, we begin by computing $\operatorname{Ind}^G_H \theta_H(g)$ as

$$\frac{1}{|H|} \sum_{\substack{\eta \in G \\ \eta g \eta^{-1} \in H}} \theta_H\left(\eta g \eta^{-1}\right) = \left|\{\eta \in G : \eta g \eta^{-1} \text{ generates } H\}\right|.$$

   Now, upon summing over all H, we see that each $\eta g \eta^{-1}$ surely generates exactly one cyclic subgroup H, so the claim follows.

2. We complete the proof. By the previous step, we see that nf equals

$$\left(\sum_{\substack{H \subseteq G \\ H \text{ cyclic}}} \operatorname{Ind}_H^G \theta_H \right) f = \sum_{\substack{H \subseteq G \\ H \text{ cyclic}}} \operatorname{Ind}_H^G (\theta_H f|_H),$$

so we will be done as soon as we check that $\theta_H f|_H \in R(H)_{\mathbb{Z}[\zeta_n]}$. Well, because H is cyclic, orthogonality of characters permits to merely check that $\langle \theta_H f|_H, \psi \rangle \in \mathbb{Z}[\zeta_n]$ for any character $\psi \colon H \to \mathbb{C}^\times$; however, this follows by a direct expansion of the inner product because ψ outputs to $\mathbb{Z}[\zeta_n]$ and $f|_H$ outputs to $|H|\,\mathbb{Z}[\zeta_n]$. ■

For the next lemma, we need a piece of notation.

Notation 3.96. Fix a finite group G of order n, and fix a prime p. Choose $g \in G$, whose order we write as $\operatorname{ord}(g) = mp^\nu$ where $p \nmid m$. Then we may find integers x and y such that $xm + yp^\nu = 1$. Now, for any $g \in G$, we define $g_p \coloneqq g^{xm}$ (which has prime-power order) and $g_p' \coloneqq g^{yp^\nu}$ (which has order coprime to p) so that $g = g_p g_p'$.

Lemma 3.97. Fix a finite group G of order n. For any class function $f \colon G \to \mathbb{Z}[\zeta_n]$ in $R(G)_{\mathbb{Z}[\zeta_n]}$, we have

$$f(g) \equiv f(g_p') \pmod{p\mathbb{Z}[\zeta_n]}$$

for any $g \in G$ and prime p.

Proof. Because we are only interested in the values of f on powers of g, we may as well work with $f|_{\langle g \rangle}$. Now, because $\langle g \rangle$, we may write $f|_{\langle g \rangle}$ as $\mathbb{Z}[\zeta_n]$-linear combination of linear characters $\langle g \rangle \to \mathbb{C}^\times$. Notably, the conclusion is $\mathbb{Z}[\zeta_n]$-linear in f, so we may as well assume that $f|_{\langle g \rangle}$ is a linear character $\langle g \rangle \to \mathbb{C}^\times$.

Now, recall that g_p' is g^{yp^ν} where the order of g equals mp^ν for $p \nmid m$ and $x, y \in \mathbb{Z}$ satisfy $xm + yp^\nu = 1$. Thus, $f(g)$ will be an mp^νth root of unity, so it will be enough to check that

$$\zeta \equiv \zeta^{yp^\nu} \pmod{p\zeta[\zeta]},$$

where ζ is a primitive mp^νth root of unity. Well, by the Frobenius automorphism, it is enough to check that sufficiently large pth powers of both sides are equal. For this, note that $p^\nu \equiv (1+y)p^\nu \pmod{mp^\nu}$, so it is enough to take p^νth powers. ■

Now is as good a time as any to begin our main argument.

Theorem 3.98 (Brauer)**.** Let G be a finite group. Then G is Brauer.

Proof. Let n be the order of n. The idea is to show that the map

$$\operatorname{Ind}_{\mathbb{Z}[\zeta_n]} \colon \bigoplus_{\substack{H \subseteq G \\ H \text{ nilpotent}}} R(H)_{\mathbb{Z}[\zeta_n]} \to R(G)_{\mathbb{Z}[\zeta_n]}$$

is surjective, which completes the proof. Namely, one can undo the base-change by $\mathbb{Z}[\zeta_n]$ because $\mathbb{Z}[\zeta_n]$ is a free $\mathbb{Z}$-algebra of finite rank; then one merely notes that any complex irreducible representation ρ of G can be written as a linear combination of inductions $R(H)$ for nilpotent subgroups $H \subseteq G$, but anything in $R(H)$ is induced from a linear character by Lemma 3.91. In fact, we will find that we may merely consider H which are the product of a cyclic group and a p-group.

As always, we proceed in steps.

1. To ground ourselves, we note that it is enough to check that 1 is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$. Indeed, it is then enough to check that the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ is an ideal, for which it is enough to check that $\mathrm{Ind}_H^G R(H) \subseteq R(G)$ is an ideal for any subgroup $H \subseteq G$. Well, for any $f_H \in R(H)$ and $f_G \in R(G)$, we see that $f_G \mathrm{Ind}_H^G f_H = \mathrm{Ind}_H^G(f_G|_H f_H)$.

   Our proof will eventually build a small supply of constant functions in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$, which will produce the constant function 1 by taking suitable linear combinations.

2. We proceed with the key construction. Fix a prime p, and choose $x \in G$ of order coprime to p. Then we claim that there is $f\colon G \to \mathbb{Z}$ in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ such that $f(x) \not\equiv 0 \pmod p$ while $f(y) = 0$ of y has order coprime to p and is not conjugate to x.

   In fact, we will induce f directly from the subgroup $H = \langle x\rangle \times P$, where P is a Sylow p-subgroup of the centralizer $C(x) \subseteq G$. (Note that H is nilpotent because it is the product of nilpotent groups. Also, H is in fact a subgroup because $\langle x\rangle$ has cardinality coprime to p, thus making the induced map $\langle x\rangle \times P \to G$ an injective homomorphism.) We now define $f_H\colon H \to G$ by

$$f_H\left(x^i y\right) \coloneqq \begin{cases} |\langle x\rangle| & \text{if } x^i = x, \\ 0 & \text{else,} \end{cases}$$

   for any $x^i \in \langle x\rangle$ and $y \in P$. We quickly check that $f_H \in R(H)$: note that $f_H = f_H \circ \mathrm{pr}_{\langle x\rangle}$, so it is enough to check that $f_H|_{\langle x\rangle}$, which follows from Lemma 3.95

   It remains to check that $f \coloneqq \mathrm{Ind}_H^G f_H$ satisfies the required conditions. For example, of course f is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ and defines a function $G \to \mathbb{Z}$. Before doing anything else, we remark on the condition $\eta g \eta^{-1} \in H$ for $g, \eta \in G$. Namely, if g has order coprime to p, then $\eta g \eta^{-1} \in H$ continues to have order coprime to p; thus, by writing it out as $x^i y$ for $y \in P$, we find that we must have $\eta g \eta^{-1} \in \langle x\rangle$. In particular, to have $f_H\left(\eta g \eta^{-1}\right) \neq 0$, we must have $\eta g \eta^{-1} = x$ on the nose!

   - Using the previous paragraph, we compute $f(x)$ as

$$\frac{1}{|\langle x\rangle| \cdot |P|} \sum_{\substack{\eta \in G \\ \eta x \eta^{-1} = x}} |\langle x\rangle|.$$

     This sum is now $|C(x)| / |P|$, which is coprime to p because $P \subseteq C(x)$ is a Sylow p-subgroup.
   - Again using the paragraph preceding our checks, we see that any g of order coprime to p must have g conjugate to x in order for the sum $\mathrm{Ind}_H^G f_H(g)$ to have any nonzero terms. We conclude that $f(g) = 0$ when g has order coprime to p but is not conjugate to x.

3. Fix a prime p. Then we claim that there is $f\colon G \to \mathbb{Z}$ in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ such that $p \nmid f(x)$ for all $x \in G$. Quickly, we note that Lemma 3.97 allows us to merely check the conclusion for $x \in G$ of order coprime to p.

   Now, let $X \subseteq G$ be a set of representatives of the conjugacy classes of the elements of G with order coprime to p. Then for each $x \in X$, we construct $f_x\colon G \to \mathbb{Z}$ in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ via the previous step. Then we define

$$f \coloneqq \sum_{x \in X} f_x.$$

   We now check that f works. Certainly f is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$. Further, the construction of the f_xs means that any $y \in G$ of order coprime to p will produce a nonzero contribution $\pmod p$ at exactly one summand (namely, the $x \in X$ conjugate to y).

4. We complete the proof. For each prime p, we factor $n = mp^\nu$ where $p \nmid m$; then we claim that m is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$. By letting p vary over the prime factors of n, this allows us to conclude that 1 is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ by taking $\mathbb{Z}$-linear combinations, thereby completing the proof.

   Now fix a prime p. The previous step provides $f\colon G \to \mathbb{Z}$ such that $f(x) \not\equiv 0 \pmod p$ for all $x \in G$. By replacing f with a suitably large power (which we may do because the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$ is an ideal), we

may achieve that $f \equiv 1 \pmod{p^\nu}$ for all $x \in G$. Then $mf - m$ is a class function $G \to \mathbb{Z}$ with values divisible by n, so Lemma 3.95 tells us it is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$. We are now allowed to conclude m is in the image of $\mathrm{Ind}_{\mathbb{Z}[\zeta_n]}$. ■

At long last, we may prove the Chebotarev density theorem.

Theorem 3.99 (Chebotarev density)**.** Fix a number field K. For each prime $\mathfrak{p}$ of K, choose a prime $\mathfrak{P}$ of $\overline{K}$ above $\mathfrak{p}$, and let $x_\mathfrak{p}$ be the conjugacy class of $\mathrm{Frob}_\mathfrak{P}$ in $\mathrm{Gal}(\overline{K}/K)$. Then the sequence $\{x_\mathfrak{p}\}_\mathfrak{p}$ equidistributes in $\mathrm{Conj}(\mathrm{Gal}(\overline{K}/K))$.

Proof. By Proposition 3.70, it is enough to check that the L-functions $L(s, \rho)$ have nonvanishing holomorphic continuation to $\{s : \mathrm{Re}\, s \geq 1\}$ for each nontrivial complex irreducible representation ρ of $\mathrm{Gal}(\overline{K}/K)$. Well, fix some such ρ. By Lemma 3.75, we can find a finite Galois extension L of K with Galois group G such that ρ descends to $\mathrm{Gal}(L/K)$. Now, G is Brauer by Theorem 3.98, so Lemma 3.90 provides a meromorphic continuation to $\{s : \mathrm{Re}\, s \geq 1\}$ which is holomorphic and nonvanishing for $s \neq 1$. Furthermore, $\langle \mathrm{tr} \circ \rho, 1\rangle = 0$ because ρ is nontrivial and irreducible, so holomorphy and nonvanishing follows. ■

3.3 Some Abelian Varieties

In this section, we begin to prove the Sato–Tate conjecture for abelian varieties, using increasing amounts of results about automorphic forms and automorphy.

3.3.1 The Fundamental Theorem of Complex Multiplication

For this subsection, we will let A be an abelian variety of dimension g defined over a number field L with complex multiplication by an order $\mathcal{O}$ of a CM number field K. We let Φ denote the CM type. Our exposition closely follows [Con04, Section 3]. It is slightly beyond the scope of our current discussion to give a precise statement of the Fundamental theorem of complex multiplication; instead, we will work with the following consequence.

Ultimately, we are interested in computing the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ on the Tate module of A. In order to avoid fixing a prime ℓ, we pick up the following notation.

Notation 3.100. Fix an abelian variety A defined over a field L of characteristic 0. Then we define the adelic Tate modules $\widehat{T}_f(A) \coloneqq \prod_\ell T_\ell(A)$ and $\widehat{V}_f(A) \coloneqq T_f(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 3.101. Note that $\widehat{T}_f A$ is a free $\widehat{\mathbb{Z}}$-module of rank $2g$, and $\widehat{V}_f A$ is a free $\mathbb{A}_{\mathbb{Q},f}$-module of rank $2g$.

Thus, we are interested in the Galois representation

$$\rho_A \colon \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to \mathrm{GL}(\widehat{V}_f A).$$

Suitably interpreted, this Galois representation will turn out to be the reflex norm, up to a root of unity.

Because A has complex multiplication by K defined over L, the image of ρ_A commutes with the action of K on $\widehat{V}_f A$, so ρ_A actually factors through $\mathrm{GL}_{K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}}(V_f A)$. By looking factor-by-factor (on each ℓ), we see that this target is contained in $K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ because K is its own commutator. Thus, ρ_A factors as a Galois representation

$$\rho_A \colon \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to (K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times,$$

where the embedding $K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \hookrightarrow \mathrm{GL}(T_f A)$ is given by the action of K on A. We take a moment to note that this target is $(K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times = (K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},f})^\times = \mathbb{A}_{K,f}^\times$. Anyway, because the target is now abelian, we see that ρ_A factors through

$$\rho_A \colon \mathrm{Gal}(L^{\mathrm{ab}}/L) \to \mathbb{A}_{K,f}^\times.$$

Artin reciprocity provides a canonical map $\mathrm{Art}_L\colon \mathbb{A}_L^\times \to \mathrm{Gal}(L^{\mathrm{ab}}/L)$, so we may as well work with the composite $\overline{\rho}_A$

$$\mathbb{A}_L^\times \to \mathrm{Gal}(L^{\mathrm{ab}}/L) \xrightarrow{\rho_A} \mathbb{A}_{K,f}^\times.$$

By [NSW08, Corollary 8.2.2], we note that Art_L is surjective with kernel containing $L^\times$ and $\mathbb{A}_{L,\infty}^\times \subseteq \mathbb{A}_L^\times$. (To see $\mathbb{A}_{L,\infty}^\times$ is in the kernel, we need to know that L has no real places, which holds because L contains the CM field K^* because K^* is the field of definition for the endomorphisms of A.) For example, this implies that $\overline{\rho}_A$ factors through $\mathbb{A}_L^\times \twoheadrightarrow \mathbb{A}_{L,f}^\times$.

It is this map $\overline{\rho}_A$ which will essentially turn out to be the reflex norm. Here is our statement of the fundamental theorem of complex multiplication, which we will not prove.

Theorem 3.102 (Fundamental)**.** Fix an abelian variety A of dimension $2g$ with complex multiplication by (K,Φ) defined over a number field L. Then there is a continuous homomorphism $\lambda\colon \mathbb{A}_{L,f}^\times \to K^\times$ such that any $s_f \in \mathbb{A}_{L,f}^\times$ has

$$\rho_A(\mathrm{Art}_L\, s_f) = \lambda(s_f)\,\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s_f))^{-1}.$$

Here, λ is continuous where $K^\times$ has been given the discrete topology.

Remark 3.103. Technically, the definition of N_{L/K^*} depends on a choice of reflex K^* inside L, which depends on a choice of embedding $L \hookrightarrow \overline{\mathbb{Q}}$. However, it turns out that the composite $\mathrm{N}_\Phi \circ \mathrm{N}_{L/K^*}$ does not depend on the choice of embedding $L \hookrightarrow \overline{\mathbb{Q}}$. We will not need this, so we will not show it.

Remark 3.104. Theorem 3.102 is frequently cited as merely a corollary of the fundamental theorem, and the fundamental theorem is indeed a more precise statement about the Galois action on A. However, the precise statement of the usual fundamental theorem is rather technical, and we will not need it, so we will be happy merely using Theorem 3.102 in this article.

Let's give a few properties of this mysterious character λ for future use.

Proposition 3.105. Fix an abelian variety A of dimension $2g$ and with complex multiplication by (K,Φ) defined over a number field L. Define the continuous character $\lambda\colon \mathbb{A}_{L,f}^\times \to K^\times$ as in Theorem 3.102.

(a) For $s_f \in \mathbb{A}_{L,f}^\times$, the fractional ideal generated by $\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s_f))$ is $\lambda(s_f)\mathcal{O}_K$.

(b) Choose a prime $\mathfrak{P}$ of L. Then A has good reduction at $\mathfrak{P}$ if and only if λ is trivial on $\mathcal{O}_{\mathfrak{P}}^\times \subseteq \mathbb{A}_{L,f}^\times$.

(c) Choose a prime $\mathfrak{P}$ of L with uniformizer $\varpi_{\mathfrak{P}} \in \mathcal{O}_{\mathfrak{P}}^\times$. Suppose A has good reduction at $\mathfrak{P}$. Then $\lambda(\varpi_{\mathfrak{P}}) \in \mathcal{O}_K$, and it agrees with the $q_{\mathfrak{P}}$-Frobenius endomorphism on the reduction $A_{\kappa(\mathfrak{P})}$ (where $q_{\mathfrak{P}} \coloneqq \#\kappa(\mathfrak{P})$).

Proof. We show these parts one at a time. For (b) and (c), it will help to fix a rational prime ℓ not lying under $\mathfrak{P}$.

(a) For each finite prime $\mathfrak{P}$ of K, we must show $\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s_f))$ and $\lambda(s_f)$ have the same $\mathfrak{P}$-valuations. Equivalently, we would like to check that

$$u(s_f) \coloneqq \lambda(s_f)\,\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s_f))^{-1} \overset{?}{\in} \prod_{\mathfrak{P}} \mathcal{O}_{K,\mathfrak{P}}^\times.$$

Well, by Theorem 3.102, we see that $u(s_f)$ acts on the Tate module $\widehat{V}_f A$ as $\rho_A(\mathrm{Art}_L\, s_f)$, which is notably an automorphism fixing the integral sublattice $\widehat{T}_f A \subseteq \widehat{V}_f A$. We conclude that $u(s_f)$ is a unit at all finite places.

(b) We use the Néron–Ogg–Shafaverich criterion [ST68], which tells us that A has good reduction at $\mathfrak{P}$ if and only if $\rho_A\colon \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to V_\ell A$ is trivial on the inertia subgroup $I_\mathfrak{P}$. The Artin map $\mathrm{Art}_L\colon \mathbb{A}_{L,f}^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is surjective, and the image of $\mathcal{O}_\mathfrak{P}^\times \subseteq \mathbb{A}_{L,f}^\times$ is precisely $I_\mathfrak{P}$, so A has good reduction at $\mathfrak{P}$ if and only if the composite

$$\mathcal{O}_\mathfrak{P}^\times \subseteq \mathbb{A}_{L,f}^\times \twoheadrightarrow \mathrm{Gal}(L^{\mathrm{ab}}/L) \to \mathbb{A}_{K,f}^\times \twoheadrightarrow \mathbb{A}_{K,\ell}^\times$$

is trivial. Well, by Theorem 3.102, we see that this composite is λ multiplied by the reflex norm. The image of the reflex norm on $\mathcal{O}_\mathfrak{P}^\times \subseteq \mathbb{A}_{L,f}^\times$ will land away from $\mathbb{A}_{K,\ell}^\times \subseteq \mathbb{A}_{K,f}^\times$, so it does not affect whether this composite is trivial. Thus, we conclude that the composite is trivial if and only if $\lambda|_{\mathcal{O}_\mathfrak{P}^\times}$ is trivial.

(c) Quickly, observe that $\lambda(\varpi_\mathfrak{P}) \in \mathcal{O}_K^\times$ follows from agreeing with the Frobenius on the reduction. Indeed, agreeing with the Frobenius on the reduction implies that $\lambda(\varpi_\mathfrak{P})$ is the root of the characteristic polynomial of the Frobenius, which is monic with coefficients in $\mathbb{Z}$.

It remains to check agreement with the Frobenius on the reduction. The computation of the composite used in the proof of (b) explains that the Galois action of $\mathrm{Frob}_\mathfrak{P} = \mathrm{Art}_L(\varpi_\mathfrak{P})$ on $V_\ell A$ is given by $\lambda(\varpi_\mathfrak{P})_\ell \in \mathbb{A}_{K,\ell}^\times$. Thus, the action of $\lambda(\varpi_\mathfrak{P})$ on the Tate module $T_\ell A_{\kappa(\mathfrak{P})}$ of the reduction also agrees with the action of the Frobenius, which lifts to an equality of actions on the actual reduction $A_{\kappa(\mathfrak{P})}$ because passing to the Tate module is faithful (see Remark 2.106). ■

Remark 3.106. For (c), one may want to say that $\lambda(\varpi_\mathfrak{P})$ is a characteristic-0 lifting of the Frobenius endomorphism on the reduction. However, if we do not have $\mathcal{O}_K \subseteq \mathrm{End}_L(A)$, then we cannot actually guarantee this lifting.

We now move towards understanding the L-function of A. Proposition 3.105 more or less explains that λ remembers the Frobenius action. However, λ is essentially only a character $\mathbb{A}_{L,f}^\times \to L^\times$, so we desire a way to extend λ to a Hecke character $L^\times\backslash\mathbb{A}_L^\times \to \mathbb{C}^\times$. This means that we would like to extend λ to the infinite places of $\mathbb{A}_L^\times$ in such a way that it is trivial on $L^\times$.

Proposition 3.107. Fix an abelian variety A of dimension $2g$ and with complex multiplication by (K, Φ) defined over a number field L. Define the continuous character $\lambda\colon \mathbb{A}_{L,f}^\times \to K^\times$ as in Theorem 3.102. Further, for each embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, define the character $\chi_\sigma\colon \mathbb{A}_L^\times \to \mathbb{C}^\times$ by

$$\chi_\sigma(s) \coloneqq \lambda(s_f)\,\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(s_\infty))_\sigma^{-1}.$$

Then χ_τ is a Hecke character, and

$$L(s, A) = \prod_{\sigma\colon K\hookrightarrow\mathbb{C}} L(s, \chi_\sigma).$$

3.3.2 Potential Modularity

In this subsection, we explain the proof of the Sato–Tate conjecture for elliptic curves (without complex multiplication) over CM fields.

3.3.3 Some Partial Results

In this subsection, we prove some partial results towards the Sato–Tate conjecture for the curve $y^9 = x(x-1)(x-\lambda)$.

3.4 Vertical Considerations

In this section, we average "in the other direction" to produce some distributions which look like Sato–Tate distributions. To be explicit, the idea is to fix an abelian scheme A over a curve C over a fixed finite field $\mathbb{F}_q$. Then we will be interested in the distribution of $\#A_c(\mathbb{F}_q)$ as $c \in C$ varies. As with the usual Sato–Tate conjecture, this question can be refined into one about equidistribution in a suitable monodromy group.

3.4.1 Katz's Machine

In this subsection, we review Katz's machine to produce vertical Sato–Tate results.

3.4.2 The Cartier–Manin Matrix

Our exposition follows [Sut20], but [Ser03] is the original source. In the sequel, it will be useful to have some partial explicit information about the Frobenius. Let C be a smooth affine curve over a finite field $\mathbb{F}_q$. Later, we will see examples where C is superelliptic, but we may work in more generality for now. Further, we let $K := k(C)$ denote the fraction field of meromorphic functions on C.

We begin by picking up some theory around differentials of curves.

Definition 3.108 (separating element)**.** Fix a perfect field k of characteristic p, and let K/k be a field extension with $\operatorname{trdeg}_k K = 1$. Then a *separating element* is an element $x \in K$ such that the field extension $K/k(x)$ is separable.

Remark 3.109. Separating elements exist. Indeed, by [Sti09, Proposition 3.10.2], such elements $z \in K$ are precisely elements in $K \setminus K^p$.

Now, we let Ω_K denote the module of (Weil) differentials for K. For given separating element $x \in K$, because $\operatorname{trdeg}_k K = 1$, it follows that $[K : K^p]$ has degree p so that K has a basis $\{1, x, \dots, x^{p-1}\}$ as a K^p-vector space. (Indeed, the elements $\{1, x, \dots, x^{p-1}\}$ are K^p-linearly independent.) Thus, any element $\omega \in \Omega_K$ can be written uniquely as

$$\omega = \sum_{i=0}^{p-1} z_i^p x^i \, dx$$

for some $z_0, \dots, z_{p-1} \in K$. We are now ready to define our operator.

Definition 3.110 (Cartier operator)**.** Fix a perfect field k of characteristic p, and let K/k be a field extension with $\operatorname{trdeg}_k K = 1$. Fix a separating element $x \in K$. Then the *Cartier operator* $\mathcal{C} \colon \Omega_K \to \Omega_K$ is defined by

$$\mathcal{C}\left(\sum_{i=0}^{p-1} z_i^p x^i \, dx \right) := z_{p-1} \, dx.$$

Here are some basic properties of the Cartier operator.

Lemma 3.111. Fix a perfect field k of characteristic p, and let K/k be a field extension with $\operatorname{trdeg}_k K = 1$.

(a) Additive: for $\omega_1, \omega_2 \in \Omega_K$, we have $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$.

(b) Frobenius-semilinear: for $\omega \in \Omega_K$ and $z \in K$, we have $\mathcal{C}(z^p \omega) = z\mathcal{C}(\omega)$.

(c) The operator $\mathcal{C} \colon \Omega_K \to \Omega_K$ is surjective.

(d) The kernel of $\mathcal{C}$ is $\{dz : z \in K\}$.

(e) We have $\mathcal{C}(dz/z) = dz/z$ for any $z \in K^\times$.

Proof. This is [Sti09, Exercise 4.12] or [Ser03, Proposition 8]. Throughout, fix a separating element x. Note (a) and (b) follow straight from the definition of $\mathcal{C}$. For (c), we note $\mathcal{C}(z^p x^{p-1}\,dx) = z\,dx$. For (d), we compute that $z = \sum_{i=0}^{p-1} z_i^p x^i$ has $dz = \sum_{i=1}^{p-1} i z_i^p x^{i-1}\,dx$, so the image of $d\colon K \to \Omega_K$ is exactly the Weil differentials $\sum_{i=0}^{p-1} z_i^p x^i\,dx$ with $z_{p-1} = 0$.

It remains to show (e), which is more involved. Note that the given equality is equivalent to asking for $\mathcal{C}(z^{p-1}\,dz) = dz$ by (b), and this latter equality is valid at $z = 0$, so we may let $L \subseteq K$ be the set of $z \in K$ satisfying $\mathcal{C}(z^{p-1}\,dz) = dz$. The proof will be over as soon as we show $L = K$, which we do in steps.

1. To get started, we note $K^p \subseteq L$ simply because d vanishes on K^p. Additionally, $x \in L$ by construction of $\mathcal{C}$. Thus, we will be done if we can show that $K^p \subseteq L$ is closed under addition and multiplication.

2. We directly dispose of closure under multiplication. For $z, w \in L$, we compute

$$\begin{aligned}\mathcal{C}\left((zw)^{p-1}\,d(zw)\right) &= \mathcal{C}\left(z^p w^{p-1}\,dw + z^{p-1}w^p\,dz\right)\\ &= z\mathcal{C}\left(w^{p-1}\,dw\right) + w\mathcal{C}\left(z^{p-1}\,dz\right)\\ &= z\,dw + w\,dz,\end{aligned}$$

   which is $d(zw)$, and so $zw \in L$.

3. Closure under addition will be slightly harder, so we start with an easy case: if $z \in L$, then we show $z + 1 \in L$. Indeed, we compute

$$\begin{aligned}\mathcal{C}\left((z+1)^{p-1}\,d(z+1)\right) &= \sum_{i=0}^{p-1}\binom{p-1}{i}\mathcal{C}\left(z^i\,dz\right)\\ &= \mathcal{C}\left(z^{p-1}\,dz\right) + \sum_{i=0}^{p-2}\frac{1}{i+1}\binom{p-1}{i}\mathcal{C}\left(d\left(z^{i+1}\right)\right)\\ &= dz,\end{aligned}$$

   which is $d(z+1)$, and so $z + 1 \in L$.

4. We now complete the proof. It is enough to show that L is closed under addition by nonzero elements, so we pick up $z, w \in L\setminus\{0\}$, and we would like to check $z+w \in L$. Well, by closure under multplication, it is enough to check that $z/w + 1$ is in L, so the previous step tells us that it is enough to check that $z/w \in L$. By closure under multiplication, it is therefore enough to cehck that $1/w \in L$. This last requirement we can check directly: we compute $d\left(w^{-1}\right)/w^{-1} = -dw/w$, so

$$\begin{aligned}\mathcal{C}\left(d\left(w^{-1}\right)/w^{-1}\right) &= -\mathcal{C}\left(dw/w\right)\\ &= -dw/w,\end{aligned}$$

   which is $d\left(w^{-1}\right)/w^{-1}$, so we are done. ∎

Remark 3.112. The properties of Lemma 3.111 uniquely determine $\mathcal{C}$. (This implies that the operator $\mathcal{C}$ does not depend on the choice of separating element x!) Indeed, suppose $\mathcal{C}$ satisfies the given list of properties, and we will show that $\mathcal{C}\left(\sum_{i=0}^{p-1} z_i^p x^i\,dx\right) = z_{p-1}\,dx$. By (d), we see that $\mathcal{C}\left(\sum_{i=0}^{p-1} z_i^p x^i\,dx\right) = \mathcal{C}\left(z_{p-1}^p x^{p-1}\,dx\right)$, which by (b) is $z_{p-1}\mathcal{C}\left(x^{p-1}\,dx\right)$. Thus, it remains to check $\mathcal{C}\left(x^{p-1}\,dx\right) = dx$, which follows from (e).

Lemma 3.113. Let C be a smooth proper curve defined over a perfect field k of characteristic p, and set $K \coloneqq k(C)$. For each $c \in C$ and $\omega \in \Omega_K$, if $\nu_c(\omega) \geq 0$, then $\nu_c(\mathcal{C}(\omega)) \geq 0$.

Proof. Fix some $c \in C$, and choose a uniformizer t at c. Then $t \in K$ is a separating element (because $t \notin K^p$). Thus, any $\omega \in \Omega_K$ can be expanded (uniquel) as

$$\omega = \sum_{i=0}^{p-1} z_i^p x^i \, dx$$

for some $z_0, \dots, z_{p-1} \in K$. If we assume $\nu_c(\omega) \geq 0$, then we must actually have $\nu_c(z_i) \geq 0$ for each i and in particular $\nu_c(z_{p-1}) \geq 0$, so we conclude that $\nu_c(\mathcal{C}(\omega)) \geq 0$ as well. ∎

To explain why we care about $\mathcal{C}$, we would now like to relate $\mathcal{C}$ to the Frobenius action on $\mathrm{H}^1_{\mathrm{dR}}(C)$. Note that the Frobenius action on $\mathrm{H}^0(C, \Omega_C) \subseteq \mathrm{H}^1_{\mathrm{dR}}(C)$ is simply trivial because the Frobenius action factors through $p \cdot \mathrm{H}^0(C, \Omega_C) = 0$. Thus, the relationship between $\mathcal{C}$ and the Frobenius must not be totally trivial. Instead, the idea is to take the "transpose" and work with $\mathrm{H}^1(C, \mathcal{O}_C) \subseteq \mathrm{H}^1_{\mathrm{dR}}(C)$ instead. To make sense of this, we will need to realize $\mathrm{H}^1(C, \mathcal{O}_C)$ and $\mathrm{H}^0(C, \Omega_C)$ as duals, which is a special case of Serre duality [Har77, Theorem II.7.6]. In our case, this is made explicit by the residue theorem.

Definition 3.114 (residue)**.** Let C be a curve defined over $\mathbb{F}_p$, and set $K \coloneqq k(C)$. For any smooth point $c \in C$, we choose a uniformizer $x \in K$. Then any $\omega \in \Omega_K$ can be written as $\omega = z \, dx$, and we define

$$\mathrm{res}_{c,x}(\omega) \coloneqq a_{-1},$$

where z has x-adic expansion $z = \sum_{n \in \mathbb{Z}} a_n x^n$.

Lemma 3.115. Let C be a curve defined over $\mathbb{F}_p$, and set $K \coloneqq k(C)$. Then $\mathcal{C} \colon \Omega_K \to \Omega_K$ restricts to the (induced) action of the p-power Frobenius on $\mathrm{H}^0(C, \Omega_C)$.

Proof. ∎

Remark 3.116. One can extend this result to work with other finite fields than $\mathbb{F}_p$, in which case we need to take suitable composites of $\mathcal{C}$. The key difficulty is that $\mathcal{C}$ should still relate to the p-power Frobenius, but now one should work with the relative instead of the absolute Frobenius.

3.4.3 The Frobenius

Under some rather restrictive hypotheses, we explain how to use Theorem 3.102 in order to compute the Frobenius. Our exposition follows the approach of [vKW05, Section 4], with two important caveats. First, our application will be slightly simpler. Second, the fundamental theorem (Theorem 3.102) is slightly incorrect in [vKW05, Subsection 4.4] because the described Hecke character $\mathbb{A}_K^\times \to K^\times$ is not continuous at the infinite places. Nevertheless, we are under the impression that the stated results are true, and the arguments are correct with slight modifications separating the continuity at the finite places with the reflex norm at the infinite places (as done in Proposition 3.107).

Here is our starting result, explaining how to compute the Frobenius action.

Proposition 3.117. Let A be an abelian variety with complex multiplication by (K, Φ) defined over L, and let S be a finite set of finite places of L containing the places of A of bad reduction. We further suppose that L is Galois over $\mathbb{Q}$. Then there is a continuous character $\lambda_S \colon \widehat{\mathcal{O}}_{L,S}^\times \to \mu(K)$ with the following property: for any principal prime $\mathfrak{P}$ of L not in S, let $\varpi_\mathfrak{P} \in \mathfrak{P}$ be a generator, and then

$$\lambda_S(\varpi_\mathfrak{P})^{-1} \, \mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(\varpi_\mathfrak{P}))$$

agrees with the $q_\mathfrak{P}$-Frobenis endomorphism on the reduction $A_{\kappa(\mathfrak{P})}$ (where $q_\mathfrak{P} \coloneqq \#\kappa(\mathfrak{P})$).

Proof. By Proposition 3.105, we basically want to compute $\lambda(\iota_{\mathfrak{P}}\varpi_{\mathfrak{P}})$, where $\iota_v\colon L_v^\times \to \mathbb{A}_L^\times$ is the inclusion for any place v. However, we see that our statement is asking us to move the evaluation of λ away from $\mathfrak{P}$, for which we use the Hecke character of Proposition 3.107. Fix an embedding $L \subseteq \mathbb{C}$, and then any embedding $K \hookrightarrow \mathbb{C}$ actually factors through L because L is Galois; we let χ be the Hecke character constructed in Proposition 3.107. Then $\lambda(\iota_{\mathfrak{P}}\varpi_{\mathfrak{P}}) = \chi(\iota_{\mathfrak{P}}\varpi_{\mathfrak{P}})$, and because χ is trivial on $L^\times$, we see that this equals

$$\chi\left(\prod_{v\neq\mathfrak{P}} \iota_v\varpi_{\mathfrak{P}}^{-1}\right) = \prod_{v\nmid\mathfrak{P}\infty} \lambda(\iota_v\varpi_{\mathfrak{P}})^{-1}\cdot \mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(\varpi_{\mathfrak{P}})).$$

Now, by Proposition 3.105, $\lambda(\iota_v\varpi_{\mathfrak{P}}) = 1$ for finite $v \notin S\cup\{\mathfrak{P}\}$, so we see that we may define λ_S as the restriction $\lambda|_{\widehat{\mathcal{O}}_{L,S}^\times}$. However, we must still check that λ_S lands in $\mu(K)$. Well, the kernel of $\lambda\colon \widehat{\mathcal{O}}_{L,S}^\times \to K^\times$ is some open subgroup by continuity of λ, but $\widehat{\mathcal{O}}_{L,S}^\times$ is profinite, so it follows that $\lambda|_{\widehat{\mathcal{O}}_{L,S}^\times}$ factors through a finite quotient and hence lands in a finite subgroup of $K^\times$. ∎

Remark 3.118. It is not difficult to relax the hypothesis that $\mathfrak{P}$ is principal: one simply needs to choose representatives for each ideal class and enlarge S to suit. For simplicity, all our examples will have class number 1, so we will not say more.

Remark 3.119. The construction of λ_S implies that $\lambda_S^{-1}\cdot(\mathrm{N}_\Phi\circ\mathrm{N}_{L/K^*})$ is trivial on $\mathcal{O}_K^\times$. Alternatively, this follows from the statement of Proposition 3.117 because the value $\lambda_S(\varpi_{\mathfrak{P}})^{-1}\,\mathrm{N}_\Phi(\mathrm{N}_{L/K^*}(\varpi_{\mathfrak{P}}))$ should be independent of the choice of generator $\varpi_{\mathfrak{P}}$.

Let's see an example.

Example 3.120. Let A be the elliptic curve over $\mathbb{Q}(\zeta_3)$ given by the affine equation $y^2 = x^3+1$. Define the character $\lambda_S\colon \widehat{\mathcal{O}}_{K,\{2,(1-\zeta_3)\}}^\times \to \mu_6$ by $\lambda_S(x) \equiv x \pmod{2(1-\zeta_3)}$ for each x. Then for any rational prime p such that $p\equiv 1 \pmod 3$, write $p = a^2-ab+b^2$ for $a,b\in\mathbb{Z}$, and

$$\lambda_S(a+b\zeta_3)^{-1}\cdot(a+b\zeta_3)$$

embedded in $\mathrm{End}(A)$ (via $\zeta_3\colon (x,y)\mapsto(\zeta_3 x,y)$) agrees with the action of Frob_p.

Proof. We use Proposition 3.117. Note that A admits an order-3 automorphism given on the affine piece $(x,y)\mapsto(\zeta_3 x,y)$, so A admits complex multiplication by $K := \mathbb{Q}(\zeta_3)$ with $\mathrm{End}_K(A) = \mathbb{Z}[\zeta_3]$. Also, note dx/y provides a basis of $\mathrm{Lie}\,A$, so the CM type of A is given by $\Phi = \{\sigma_1\}$, where $\sigma_i \in \mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$ is defined by $\sigma_i(\zeta_3) := \zeta_3^i$ for each $i\in\{1,2\}$.

Continuing, we see that may take $L := \mathbb{Q}(\zeta_3)$, which is Galois over $\mathbb{Q}$ and has class number 1. By a discriminant computation, we see that A has bad reduction only at $S := \{2,3\}$, so our work will go into computing the character $\lambda_S\colon \widehat{\mathcal{O}}_S^\times \to \mu(K)$. By Remark 3.119, we see that $\lambda_S(\zeta) = \zeta$ for each $\zeta\in\mu(K)$, and this will basically be able to compute λ_S. To be precise, we note the inclusion $\mu_6 \hookrightarrow \widehat{\mathcal{O}}_{K,S}^\times$ is a section of the projection

$$\widehat{\mathcal{O}}_{K,S}^\times = \widehat{\mathcal{O}}_{K,(2)}^\times \times \widehat{\mathcal{O}}_{K,(1-\zeta_3)}^\times \twoheadrightarrow \left(\frac{\widehat{\mathcal{O}}_{K,(2)}}{(2)}\right)^\times \times \left(\frac{\widehat{\mathcal{O}}_{K,(1-\zeta_3)}}{(1-\zeta_3)}\right)^\times = \mathbb{F}_4^\times\times\mathbb{F}_3^\times \cong \mu_3\times\mu_2 = \mu_6.$$

Because $\mu_6 \hookrightarrow \widehat{\mathcal{O}}_{K,S}^\times$ is also a section of λ_S, we suspect that λ_S must equal exactly the above projection. ∎

Remark 3.121. Here is an equivalent statement. For any rational prime p such that $p \equiv 1 \pmod 3$, write $p = (a+b\zeta_3)(a-b\zeta_3)$ where $a, b \in \mathbb{Z}$ and $a+b\zeta_3 \equiv 1 \pmod{2(1-\zeta_3)}$. (Equivalently, $p = a^2 - ab + b^2$ with $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$ and $a + b \equiv 1 \pmod 3$.) Then $(a + b\zeta_3)$ agrees with the p-Frobenius on the reduction $A_{\kappa(a+b\zeta_3)}$. While we're here, we then note that $\#A_{\kappa(a+b\zeta_3)}$ equals

$$(p+1) - \operatorname{tr}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a+b\zeta_3) = (p+1) - (2a-b).$$

This has been verified numerically for $p < 10000$.

Here is a similar computation for a twist, stated as in Remark 3.121.

Example 3.122. Let A be the elliptic curve over $\mathbb{Q}(\zeta_3)$ given by the affine equation $y^3 = x^3 + 1$. Then for any rational prime p such that $p \equiv 1 \pmod 3$, write $p = (a + b\zeta_3)(a + b\overline{\zeta}_3)$ where

$$(a + b\zeta_3) \equiv 1 \pmod{(1+\zeta_3)^2}.$$

Embedding $\mathbb{Z}[\zeta_3] \hookrightarrow \operatorname{End}(A)$ by $\zeta_3 \colon (x, y) \mapsto (x, \zeta_3 y)$, the element $(a + b\overline{\zeta}_3)$ agrees with the action of Frob_p.

Proof. Following Example 3.120, we use Proposition 3.117. We use the notation established in Example 3.120, but we note that the CM type is now $\Phi = \{\sigma_2\}$, and A now only has bad reduction at $S = \{(1-\zeta_3)\}$. Thus, we see that $\lambda_S(\zeta) = \overline{\zeta}$ for all $\zeta \in \mu(K)$, which is able to determine λ_S. Indeed, the inclusion $\mu_6 \hookrightarrow \widehat{\mathcal{O}}_{K,S}^\times$ is a section of the projection

$$\widehat{\mathcal{O}}_{K,(1-\zeta_3)}^\times \twoheadrightarrow \left(\frac{\widehat{\mathcal{O}}_{K,(1-\zeta_3)}}{(1-\zeta_3)^2}\right)^\times,$$

which we can detect because every element of μ_6 maps to a distinct unit in the target. Because $\mu_6 \hookrightarrow \widehat{\mathcal{O}}_{K,S}^\times$ is also a section of $\overline{\lambda}_S$, we conclude that λ_S is the above projection (composed with the isomorphism to μ_6). The result follows. ∎

Remark 3.123. Having $a + b\zeta_3 \equiv 1 \pmod{(1-\zeta_3)^2}$ is equivalent to having 9 dividing

$$((a-1) + b\zeta_3)(1 - \overline{\zeta}_3)^2 = (3a - 3b - 3) + (3a - 3)\zeta_3,$$

or $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$. Thus, we can once again make the statement entirely elementary.

Here is roughly the furthest we can push the method used in the previous two examples.

Corollary 3.124. Let A be an abelian variety with complex multiplication by (K, Φ) defined over L, and let S be a finite set of finite places of L containing the places of A of bad reduction. We further suppose that L is Galois over $\mathbb{Q}$, has class number 1, and that

Chapter 4
The Fermat Curve

Usually mathematicians have to shoot somebody to get this much publicity.

—Thomas R. Nicely

In ths chapter, we will study the Galois representation attached to the projective $\mathbb{Q}$-curve $X_N^1 \subseteq \mathbb{P}^1_{\mathbb{Q}}$ cut out by the equation

$$X_N \colon X^N + Y^N + Z^N = 0,$$

where $N \geq 3$ is some nonnegative integer. For the rest of this chapter, we will fix N and thus denote this curve by $X \subseteq \mathbb{P}^1_{\mathbb{Q}}$.

4.1 Homology and Cohomology

The exposition of this section follows [Ots16, Sections 2 and 3]. We will spend this section setting up some notation and proving basic facts about how these objects relate to each other.

4.1.1 The Group Action

Throughout, it will be helpful to note that the finite alegbraic $\mathbb{Q}$-group

$$G_N \coloneqq \frac{\mu_N \times \mu_N \times \mu_N}{\Delta\mu_N}$$

acts on X_N; here, $\Delta\mu_N \subseteq \mu_N \times \mu_N \times \mu_N$ refers to the diagonally embedded copy of μ_N. As with X_N, we will denote this group by G for the rest of the chapter, and we will let $\zeta \coloneqq \zeta_N$ be a primitive Nth root of unity.

Notably, the action map $G \times X \to X$ is defined over $\mathbb{Q}$ even though $G(\mathbb{Q})$ is trivial. For brevity, we will denote elements of G by $g_{[r:s:t]} \coloneqq [\zeta^r : \zeta^s : \zeta^t]$. We will also have occasion to study the character group $\widehat{G} \coloneqq \widehat{G}_N$, which we identify with

$$\widehat{G}_N = \left\{(a,b,c) \in (\mathbb{Z}/N\mathbb{Z})^3 : a+b+c = 0\right\}.$$

Explicitly, given a triple (a,b,c), we let $\alpha_{(a,b,c)}$ denote the corresponding character, which sends $g_{[r:s:t]} \mapsto \zeta^{ra+bs+tc}$.

In the sequel, we will have many vector spaces induced by X via (co)homology, which therefore have a G-action by functoriality. With this in mind, we make the following definition.

Definition 4.1. Given a $\mathbb{Q}(\zeta)$-vector space H with a G-action, we define

$$\mathrm{H}_\alpha \coloneqq \{v \in \mathrm{H} : g \cdot v = \alpha(g)v\}$$

to be the α-eigenspace for each $\alpha \in \widehat{G}$.

One inconvenience of this definition is that the vector spaces H of interest are frequently defined over $\mathbb{Q}$, but H_α is not.

Thus, we note that some $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\widehat{G}$ as follows: say $\tau(\zeta) = \zeta^u$ for some $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, and then

$$(\tau\alpha)([\zeta^r : \zeta^s : \zeta^t]) = \alpha\left([\zeta^{u^{-1}r}, \zeta^{u^{-1}s} : \zeta^{u^{-1}t}]\right),$$

so we see that $\tau\alpha = u^{-1}\alpha$, where the multiplication $u^{-1}\alpha$ is understood to happen where α is a triple in $(\mathbb{Z}/N\mathbb{Z})^3$. With this in mind, given $\alpha \in \widehat{G}$, we let $[\alpha] \subseteq \widehat{G}$ be the collection of characters of the form $u\alpha$ as $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ varies; for example, $-\alpha \in [\alpha]$. The point of this discussion is that we are able to build a decomposition

$$\mathbb{Q}[G] \cong \prod_{[\alpha] \in G/(\mathbb{Z}/N\mathbb{Z})^\times} \mathbb{Q}([\alpha]),$$

where $\mathbb{Q}([\alpha])$ is the image of the map $\mathbb{Q}[G] \to \mathbb{C}$ given by the characters in $[\alpha]$. We are now ready to make the following definition.

Definition 4.2. Given some $\mathbb{Q}$-vector space H with a G-action, we are now ready to define

$$\mathrm{H}_{[\alpha]} \coloneqq \left\{ v \in \mathrm{H} : v \otimes 1 \in \bigoplus_{\beta \in [\alpha]} (\mathrm{H} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_\alpha \right\}.$$

The discussion of the Galois action of the previous paragraph implies that $\mathrm{H}_{[\alpha]}$ is a generalized eigenspace of the G-action on H. In particular, we find that $\mathrm{H}_{[\alpha]} \otimes \overline{\mathbb{Q}} = \bigoplus_{\beta \in [\alpha]} \mathrm{H}_\beta$, so $\mathrm{H} = \bigoplus_{[\alpha]} \mathrm{H}_{[\alpha]}$.

4.1.2 Differential Forms

In this subsection, we will define a few differential forms. A reasonable reference for this subsection is [Lan11, Section 1.7]. A computation with the Riemann–Hurwitz formula shows that the genus of X is $\frac{(N-1)(N-2)}{2}$, so we know that there are many holomorphic differential forms. On the other hand, we know that the space of differential forms lives in $\mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C})$, which is equipped with a G-action. Anyway, we are now ready to define our differential form.

Definition 4.3. Fix notation as above. For $a \in \mathbb{Z}/N\mathbb{Z}$, let $[a]$ be a representative in $\{0, 1, \ldots, N-1\}$. For any $\alpha_{(a,b,c)} \in \widehat{G}$, we define the differential form

$$\omega_{\alpha_{(a,b,c)}} \coloneqq x^{[a]} y^{[b]-N} \frac{dx}{x}$$

in the affine patch $x^N + y^N + 1 = 0$ of X. In the sequel, we may also denote this differential form by $\omega_{(a,b,c)}$.

Remark 4.4. Because $x^N + y^N + 1 = 0$ implies $x^{N-1}\,dx = -y^{N-1}\,dy$, we also see that

$$\omega_{(a,b,c)} = -x^{[a]-N}y^{[b]}\,\frac{dy}{y}.$$

Further, we can pass to the affine patch $1 + v^N + u^N = 0$ of X by substituting $(x, y) = (1/u, v/u)$, for which we note $d(1/u)/(1/u) = -du/u$ so that

$$\omega_{(a,b,c)} = -u^{N-[a]-[b]}v^{[b]-N}\,\frac{du}{u}.$$

Remark 4.5. Following [Col87, Section VI], we remark that it will be numerically convenient to work with a rational multiple of the $\omega_\bullet$s for some computations in the sequel. Namlely, we define $\nu_\alpha := K(\alpha)\omega_\alpha$ when $\alpha = (a, b, c)$ has nonzero entries, where

$$K(a,b,c) := \begin{cases} \frac{N-[a]-[b]}{N} & \text{if } [a]+[b] > N, \\ 1 & \text{if } [a]+[b] < N. \end{cases}$$

From Remark 4.4, we see that $\omega_{(a,b,c)}$ always succeeds at being meromorphic with poles only at points of the form $[X : Y : 0]$, and it is closed (i.e., has vanishing residues) if and only if $0 \notin \{a, b, c\}$. Further, we see that $\omega_{(a,b,c)}$ succeeds at being holomorphic provided that we also have $[a] + [b] < N$, which we note is equivalent to $[a] + [b] + [c] = N$.

We have now provided $\frac{(N-1)(N-2)}{2}$ holomorphic differentials of X, so we would like to check that we have actually found a basis of $\mathrm{H}^0(X(\mathbb{C}), \Omega^1_{X/\mathbb{C}})$. Well, these differential forms are nonzero by construction,[1] and they are linearly independent because they are all eigenvectors for the G-action.

Lemma 4.6. Fix notation as above. For each $\alpha \in \widehat{G}$, the differential form ω_α is an eigenvector for the G-action with eigenvalue α.

Proof. Say $\alpha = \alpha_{(a,b,c)}$ for some $a, b, c \in \mathbb{Z}/N\mathbb{Z}$. Then for any $g_{[r:s:0]} \in G$, we note

$$\begin{aligned}(g_{[r:s:0]})^*\omega_{(a,b,c)} &= (\zeta^r x)^{[a]}(\zeta^s y)^{[b]-N}\,\frac{d(\zeta^r x)}{(\zeta^r x)} \\ &= \zeta^{ar+bs}\cdot x^{[a]}y^{[b]-N}\,\frac{dx}{x} \\ &= \alpha_{(a,b,c)}(g_{[r:s:0]})\omega_{(a,b,c)}.\end{aligned}$$

The reason to $g_{[r:s:0]}$ in the above computation is that we need the G-action to stay in the affine patch of points of the form $[X : Y : 1]$. ∎

Remark 4.7. Thus, we see that our differential forms must be linearly independent because they are eigenvectors with different eigenvalues. As such, we have constructed eigenbases of $\mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C})$ and $\mathrm{H}^0(X(\mathbb{C}), \Omega^1_{X/\mathbb{C}})$.

While we're here, we compute the Poincaré pairing of our basis of differential forms.

[1] Later, Remark 4.13 will give another way to prove this via periods.

Lemma 4.8. Fix notation as above. Choose $\alpha, \alpha' \in \widehat{G}$ such that $\alpha = (a, b, c)$ and $\alpha' = (a', b', c')$ have nonzero entries. Then the Poincaré pairing

$$P\colon \mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C}) \times \mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C}) \to \mathbb{C}$$

given by $(\omega, \eta) \mapsto \frac{1}{2\pi i} \int_X (\omega \wedge \eta)$ sends $(\omega_\alpha, \omega_{\alpha'})$ to

$$P(\omega_\alpha, \omega_{\alpha'}) = \begin{cases} 0 & \text{if } \alpha \neq -\alpha', \\ (-1)^N \frac{N}{N-[a]-[b]} & \text{if } \alpha = -\alpha'. \end{cases}$$

Proof. We use the Poincaré residue, which implies that

$$P(\omega, \eta) = \sum_{x \in X(\mathbb{C})} \mathrm{Res}_x \left(\eta \int \omega \right),$$

where the sum is over poles, and $\int \omega$ refers to any choice of local primitive for ω in the neighborhood of x. To use this, we note that the computation of Remark 4.4 implies that ω_α and $\omega_{\alpha'}$ can only have poles at the points $[1 : -\zeta^s : 0]$ for some $s \in \mathbb{Z}/N\mathbb{Z}$, and in this neighborhood, we may write

$$\omega_\alpha = -u^{N-[a]-[b]} v^{[b]-N} \frac{du}{u}$$

and similarly for $\omega_{\alpha'}$. In particular, we see that

$$-\frac{1}{N-[a]-[b]} u^{N-[a]-[b]} v^{[b]-N}$$

makes a reasonable primitive for ω_α, so the Poincaré residue yields

$$P(\omega_\alpha, \omega_{\alpha'}) = \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Res}_{(-\zeta^s, 0)} \left(-\frac{1}{N-[a]-[b]} u^{N-[a]-[b]} v^{[b]-N} \cdot -u^{N-[a']-[b']} v^{[b']-N} \frac{du}{u} \right).$$

Now, if $\alpha \neq \alpha'$, then we see that we are computing the residues of some monomial times du/u, but the power of u in the monomial is nonzero, so the residues all vanish. Lastly, we need to discuss what happens with $\alpha = -\alpha'$, where we see

$$\begin{aligned} P(\omega_\alpha, \omega_{-\alpha}) &= \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Res}_{(-\zeta^s, 0)} \left(-\frac{1}{N-[a]-[b]} u^{N-[a]-[b]} v^{[b]-N} \cdot -u^{N-[-a]-[-b]} v^{[-b]-N} \frac{du}{u} \right) \\ &= \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Res}_{(-\zeta^s, 0)} \left(-\frac{1}{N-[a]-[b]} u^{N-[a]-[b]} v^{[b]-N} \cdot u^{[a]+[b]-N} v^{-[b]} \frac{du}{u} \right) \\ &= \frac{1}{N-[a]-[b]} \sum_{s \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Res}_{(-\zeta^s, 0)} \left(v^{-N} \frac{du}{u} \right) \\ &= \frac{1}{N-[a]-[b]} \sum_{s \in \mathbb{Z}/N\mathbb{Z}} (-\zeta^s)^{-N} \\ &= (-1)^N \frac{N}{N-[a]-[b]}, \end{aligned}$$

as desired. ∎

Remark 4.9. Following Remark 4.5, we see that $\alpha \in \widehat{G}$ with nonzero entries will have

$$P(\nu_\alpha, \nu_{-\alpha}) = (-1)^N$$

because exactly one of $K(\alpha)$ or $K(-\alpha)$ will absorb the given rational constant. This is essentially the reason for working with the $\nu_\bullet$s instead of the $\omega_\bullet$s.

4.1.3 Some Group Elements

In this subsection, we define a few elements of $\mathbb{Q}[G]$ which we will then use in the next subsection. We begin with the three elements

$$t := \sum_{g \in G} g, \qquad v := \sum_{s \in \mathbb{Z}/N\mathbb{Z}} g_{[0:s:0]}, \qquad \text{and} \qquad h := \sum_{r \in \mathbb{Z}/N\mathbb{Z}} g_{[r:0:0]}.$$

We take a moment to note that these elements satisfy the relations $tg = gt = t$ for any $g \in G$, and $t = hv = vh$, and $v^2 = Nv$ and $h^2 = Nh$. In the sequel, we will get a lot of mileage out of the idempotent

$$p := \frac{1}{N^2} \sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}).$$

Let's check that p is idempotent.

Lemma 4.10. Fix notation as above.

(a) Then p is idempotent.

(b) For any $r, s \in \mathbb{Z}/N\mathbb{Z}$, we have $(1 - g_{[r:0:0]})(1 - g_{[0:s:0]})p = (1 - g_{[r:0:0]})(1 - g_{[0:s:0]})$.

Proof. Both claims hinge upon the fact that a direct expansion of $(1 - g_{[r:0:0]})(1 - g_{[0:s:0]})$ implies

$$p = \frac{1}{N^2} \left(N^2 - Nh - Nv + t \right).$$

We now show the claims separately.

(a) This is a direct computation: write

$$\begin{aligned} p^2 &= \frac{1}{N^4} \left(N^2 - Nh - Nv + t \right) \left(N^2 - Nh - Nv + t \right) \\ &= \frac{1}{N^4} \left(N^4 + N^2h^2 + N^2v^2 + t^2 - 2N^3h - 2N^3v + 2N^2t + N^2hv - 2Nht - 2Nvt \right) \\ &= \frac{1}{N^4} \left(N^4 + N^3h + N^3v + N^2t - 2N^3h - 2N^3v + 2N^2t + N^2t - 2N^2t - 2N^2t \right) \\ &= \frac{1}{N^4} \left(N^4 - N^3h - N^3v + N^2t \right) \\ &= p. \end{aligned}$$

(b) We will compute as in (a): note $h(1 - g_{[r:0:0]}) = 0$ and $v(1 - g_{[0:s:0]}) = 0$, so

$$\begin{aligned} (1 - g_{[r:0:0]})(1 - g_{[0:s:0]})p &= (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}) \cdot \frac{1}{N^2} \left(N^2 - Nh - Nv + hv \right) \\ &= (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}) \cdot \frac{N^2}{N^2} + 0 + 0 + 0 \\ &= (1 - g_{[r:0:0]})(1 - g_{[0:s:0]}), \end{aligned}$$

as required. ∎

4.1.4 Homology

In this subsection, we will study $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q})$. By the end, we will define a 1-cycle $\gamma := \gamma_N$ such that $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}[G]\cdot[\gamma]$. Morally, this means that we can understand our homology by focusing on this one cycle.

To begin, we need some path in $X(\mathbb{C})$, so we define $\delta\colon [0,1]\to X(\mathbb{C})$ by

$$\delta(t) := \left[t^{1/N} : (1-t)^{1/N} : \zeta_{2N}^{-1}\right].$$

Notably, $\delta(0) = [0:1:\zeta_{2N}^{-1}]$ and $\delta(1) = [1:0:\zeta_{2N}^{-1}]$, so $g = [\zeta^r : \zeta^s : 1]$ has $g_*\delta(0) = [0:\zeta^s:\zeta_{2N}^{-1}]$ and $g_*\delta(1) = [\zeta^r : 0 : \zeta_{2N}^{-1}]$. The point of this computation is that we see

$$(1 - g_{[r:0:0]} - g_{[0:s:0]} + g_{[r:s:0]})_*\delta \in \mathrm{Z}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}).$$

We are now ready to define γ.

Definition 4.11. Fix notation (and in particular δ) as above. Then we define

$$\gamma := \frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} (1-g_{[r:0:0]})(1-g_{[0:s:0]})_*\delta.$$

Note $\gamma = p_*\delta$.

The above computation shows that $\gamma\in \mathrm{Z}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q})$. We will want to know to its periods later. Note that the following result is essentially a special case of [Del18, Lemma 7.12].

Lemma 4.12. Fix notation as above. Suppose $(a,b,c)\in(\mathbb{Z}/N\mathbb{Z})^3$ has no nonzero entries. Then

$$\int_\gamma \omega_{(a,b,c)} = \zeta_{2N}^{[a]+[b]-N}\frac{\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[b]}{N}\right)}{\Gamma\left(\frac{[a]}{N}+\frac{[b]}{N}\right)}.$$

Proof. This is a direct computation. Denote the integral by $P(\gamma,\omega_{(a,b,c)})$. By adjunction, $\int_{p_*\delta}\omega_{(a,b,c)} = \int_\delta p^*\omega_{(a,b,c)}$. This allows us to compute

$$\begin{aligned}
P(\gamma,\omega_{(a,b,c)}) &= \frac{1}{N^2}\int_\delta \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} (1-g_{[r:0:0]})(1-g_{[0:s:0]})^*\omega_{(a,b,c)}\\
&= \frac{1}{N^2}\int_\delta \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right)\omega_{(a,b,c)}\\
&= \left(\frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right)\right)\int_\delta \omega_{(a,b,c)}\\
&= \left(\frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right)\right)\zeta_{2N}^{[a]+[b]-N}\int_0^1 t^{[a]/N}(1-t)^{[b]/N-1}\frac{dt}{t}.
\end{aligned}$$

The last integral (famously) equals the Beta function, and it evaluates to $\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[b]}{N}\right)\Gamma\left(\frac{[a]+[b]}{N}\right)^{-1}$. We take a moment to check that

$$\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right) \overset{?}{=} N^2.$$

Well, $\left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right) = 1-\zeta^{ar}-\zeta^{bs}+\zeta^{ar+bs}$, and because $a,b\neq 0$, we see that summing over r and s causes the terms not equal to 1 to vanish. Thus, we are left with N^2. ∎

Remark 4.13. Because the right-hand side is nonzero, Lemma 4.12 implies that the differential forms $\omega_{(a,b,c)}$ are nonzero.

Remark 4.14. Following Remark 4.5, it will be helpful to also compute $\int_\gamma \nu_{(a,b,c)}$. We claim that

$$\int_\gamma \nu_{(a,b,c)} \stackrel{?}{=} (-1)^{\lfloor([a]+[b])/N\rfloor}\zeta_{2N}^{[a]+[b]-N}\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[b]}{N}\right)\Gamma\left(\frac{[a+b]}{N}\right)^{-1}.$$

We have two cases. If $[a]+[b] < N$, then $\nu_{(a,b,c)} = \omega_{(a,b,c)}$, so this is immediate from Lemma 4.12. Otherwise, if $[a]+[b] > N$, then $\nu_{(a,b,c)} = \frac{N-[a]-[b]}{N}\omega_{(a,b,c)}$, so this follows from Lemma 4.12 as soon as we compute

$$\frac{N-[a]-[b]}{N}\Gamma\left(\frac{[a]+[b]}{N}\right)^{-1} \stackrel{?}{=} -\Gamma\left(\frac{[a+b]}{N}\right)^{-1}.$$

This follows because $\Gamma\left(\frac{[a]+[b]}{N}\right) = \frac{[a]+[b]-N}{N}\Gamma\left(\frac{[a+b]}{N}\right)$.

We are now ready to show that $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}[G]\cdot[\gamma]$.

Lemma 4.15. Fix notation as above. Then $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}[G]\cdot[\gamma]$.

Proof. It is enough to show that $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C}) = \mathbb{C}[G]\cdot[\gamma]$. Note that there is a canonical pairing

$$\begin{array}{ccc}\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C}) \times \mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}),\mathbb{C}) & \to & \mathbb{C} \\ (c,\omega) & & \mapsto \int_c \omega\end{array}$$

which is perfect by the de Rham theorem. We already have a basis $\{\omega_{(a,b,c)}\}_{a,b,c\neq 0}$ of $\mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}),\mathbb{C})$, so we will find a dual basis for $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})$ inside $\mathbb{C}[G]\cdot[\gamma]$. Well, for $g\in G$ and $\alpha\in\widehat{G}$, we see

$$\int_{g^*\gamma}\omega_\alpha = \int_\gamma g^*\omega_\alpha$$

equals $\alpha(g)P(\gamma,\omega_\alpha)$, where $P(\gamma,\omega_\alpha) \coloneqq \int_\gamma \omega_\alpha$ is the (nonzero!) period computed in Lemma 4.12. With this in mind, we define

$$c_\alpha \coloneqq \frac{1}{N^2 P(\gamma,\omega_\alpha)}\sum_{g\in G}\alpha(g)^{-1}g^*[\gamma]$$

for each $\alpha = \alpha_{(a,b,c)}$ with $a,b,c\neq 0$. Then we see that $\int_{c_\alpha}\omega_\beta = 1_{\alpha=\beta}$ by the orthogonality relations, so $\{c_\alpha\}$ is a dual basis of $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})$, and it lives in $\mathbb{C}[G]\cdot[\gamma]$ by its construction. ∎

4.2 Galois Action

We now use the notation set up in the previous section to write out the Galois action on the space of some absolute Hodge cycles attached to X. Roughly speaking, we will be interested in computing ℓ-adic monodromy groups of (quotients of) X, which requires us to have some understanding of the Galois representation

$$\rho\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)\right).$$

In particular, we recall from section 2.3.4 that it will really suffice to be able to compute the Galois action on cetain Tate classes living in

$$\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{\otimes p}\otimes\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{\vee\otimes p} \cong \mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{\otimes 2p}(p),$$

for some nonnegative index $p \geq 0$, which is the main point of this section. In particular, the Künneth theorem tells us that we will be interested in the cohomology group $\mathrm{H}^{2p}_{\text{ét}}(X^{2p}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(p)$.

Roughly speaking, the outline will be to pass to absolute Hodge cycles. Indeed, by the Mumford–Tate conjecture, one is able to correspond Tate classes to Hodge classes, and Hodge classes are known to be absolutely Hodge, and our construction of absolute Hodge cycles makes it clear how they should specialize to a Tate class. In this way, we find that we can attempt to compute Galois action on Tate classes by instead computing Galois action on absolute Hodge classes. This is useful because absolute Hodge cycles have a de Rham component, so we can run our computations on the de Rham component, which is the only place where we can hope to have a basis.

Throughout this section, p is a nonnegative index. We take a moment to note that the action of G on X upgrades into an action of G^{2p} on X^{2p}. Our exposition closely follows [GGL24, Subsection 8.5]. As in section 4.1.1, we will identify $\widehat{G}^{2p}$ with some subset of tuples in $(\mathbb{Z}/N\mathbb{Z})^{6p}$. And for a vector space H defined over $\mathbb{Q}(\zeta)$ (respectively, $\mathbb{Q}$) and character $\alpha \in \widehat{G}^{2p}$, we define H_α (respectively, $\mathrm{H}_{[\alpha]}$) as the corresponding α-eigenspace (respectively, $[\alpha]$-generalized eigenspace). Then given a vector $v \in \mathrm{H}$, we may also write v_α for the component in H_α.

In the sequel, we will find utility out of the following two subsets of $\widehat{G}^{2p}$.

Definition 4.16. Fix notation as above.

- We define the subset $\mathfrak{A}^{2p}$ to be equal to the subset of $\alpha \in \widehat{G}^{2p}$ having nonzero entries as a tuple in $(\mathbb{Z}/N\mathbb{Z})^{6p}$.
- We define the subset $\mathfrak{B}^{2p}$ to be equal to the subset of $\alpha \in \mathfrak{A}^{2p}$ such that $\alpha = (a_1, \ldots, a_{6p})$ satisfies

$$\frac{1}{N}\sum_{i=1}^{6p}[ua_i] = 3p$$

for all $u \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Morally, the characters in $\mathfrak{A}_{2p}$ correspond to basis vectors of $\mathrm{H}^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{\otimes p}$, and the characters in $\mathcal{B}_{2p}$ correspond to Hodge classes (see Proposition 4.23).

4.2.1 Hodge Cycles on X^{2p}

To understand the geometry of X, we will only be interested in tensor powers of $\mathrm{H}^1(X)$ (for a choice of cohomology theory H), which by the Künneth formula embed as

$$\mathrm{H}^1(X)^{\otimes 2p} \subseteq \mathrm{H}^{2p}\left(X^{2p}\right).$$

When H is de Rham cohomology H_{dR}, we thus see we are interested in when the image of an element in $\mathrm{H}^1_{\mathrm{dR}}(X)^{\otimes p}$ succeeds at being a Hodge cycle. Well, note that the action of G on $\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{C})$ extends to an action of G^{2p} on $\mathrm{H}^1_{\mathrm{dR}}\left(X, \mathbb{C}\right)^{\otimes 2p}$. This action diagonalizes with one-dimensional eigenspaces by extending Remark 4.7. We will use properties of the diagonalization to read off when we have an element of bidegree (p,p) in $\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right)$.

Following [Del18, Proposition 7.6], it will be useful to have the following definition.

Definition 4.17 (weight)**.** Given a function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, we define its *weight map* as the function $\langle f\rangle\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ defined by

$$\langle f\rangle(u) \coloneqq \frac{1}{N}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} f(ua)[a]$$

For $p \geq 0$, we note that we may identify $\widehat{G}^{2p}$ with a tuple in $(\mathbb{Z}/m\mathbb{Z})^{2p}$, and then we define the *weight* $\langle\alpha\rangle$ of a character $\alpha \in \widehat{G}^{2p}$ as $\langle 1_\alpha\rangle(1)$, where $1_\alpha\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ is the multiplicity of an element in $\mathbb{Z}/N\mathbb{Z}$ in the tuple α.

Remark 4.18. The point of this definition is as follows: given $\alpha \in \widehat{G}$ with $\alpha = (a, b, c)$ having nonzero entries, we note that ω_α has two possible cases.

- If $[a] + [b] + [c] = N$ so that $\langle \alpha \rangle = 1$, then $\omega_{(a,b,c)}$ is holomorphic so that $\omega_\alpha \in \mathrm{H}^{10}(X)$.
- If $[a] + [b] + [c] = 2N$ so that $\langle \alpha \rangle = 2$, then ω_α is not holomorphic so that $\omega_\alpha \in \mathrm{H}^{01}(X)$.

In all cases, we find $\omega_\alpha \in \mathrm{H}^{2-\langle \alpha \rangle, \langle \alpha \rangle - 1}(X)$.

Remark 4.19. If f is instead a function $f \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{C}$, we may similarly define the weight by the formula

$$\langle f \rangle(u) \coloneqq \sum_{a \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}} f(ua)\langle a \rangle,$$

where $\langle a \rangle$ now refers to the element in $[0, 1)$ in the class of a.

Remark 4.20. Suppose that $\alpha \in \mathfrak{A}^{2p}$ has 1_α of constant weight. Then we claim that $\langle \alpha \rangle = 3p$. Indeed, we must have

$$\frac{1}{N}\sum_{i=1}^{6p}[a_i] = \frac{1}{N}\sum_{i=1}^{6p}[-a_i],$$

but $[-a_i] = N - a_i$ then forces the sum to equal $3p$.

We now upgrade Remark 4.18 to $\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{C})^{\otimes 2p}$.

Notation 4.21. Choose $\alpha \in \widehat{G}^{2p}$ as $\alpha = (\alpha_1, \ldots, \alpha_{2p})$ having nonzero entries. Then we set

$$\omega_\alpha \coloneqq \omega_{\alpha_1} \otimes \cdots \otimes \omega_{\alpha_{2p}}.$$

We define ν_α similarly.

Lemma 4.22. Choose $\alpha \in \widehat{G}^{2p}$ as $\alpha = (\alpha_1, \ldots, \alpha_{2p})$ having nonzero entries (i.e., $\alpha \in \mathfrak{A}_{2p}$). Then ω_α embedded in $\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right)$ is of bidegree $(4p - \langle \alpha \rangle, \langle \alpha \rangle - 2p)$.

Proof. Because the Künneth isomorphism upgrades to an isomorphism of Hodge structures, it is enough to note that $\omega_{\alpha_i} \in \mathrm{H}^{2-\langle \alpha_i \rangle, \langle \alpha_i \rangle - 1}$ (see Remark 4.18) implies ω_α has bidegree

$$\left(4p - \sum_{i=1}^{2p}\langle \alpha_i \rangle, \sum_{i=1}^{2p}\langle \alpha_i \rangle - 2p\right).$$

The lemma follows because weight is additive. ∎

Proposition 4.23. Choose $\alpha \in \mathfrak{A}^{2p}$. Then $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}\right)_{[\alpha]}$ is one-dimensional over $\mathbb{Q}([\alpha])$, and the following are equivalent.

(a) $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}\right)_{[\alpha]}(p)$ consists entirely of Hodge classes.

(b) We have $\langle u\alpha \rangle = 3p$ for all $u \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Proof. Expand $\alpha = (\alpha_1, \dots, \alpha_{2p})$. We begin by embedding

$$\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{C}\right)_{u\alpha}$$

into

$$\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right) = \bigoplus_{\substack{q_1, \dots, q_{2p} \\ q_1 + \cdots + q_{2p} = 2p}} \mathrm{H}^{q_1}_{\mathrm{dR}}(X, \mathbb{C}) \otimes \cdots \otimes \mathrm{H}^{q_{2p}}_{\mathrm{dR}}(X, \mathbb{C}),$$

where this last equality holds by the Künneth isomorphism. Quickly, we reduce to the case where $q_1 = \cdots = q_{2p} = 1$: for each $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, we note that $u\alpha$ has nonzero entries. On the other hand, the G-action on $\mathrm{H}^0(X) = \mathbb{C}$ is always trivial, so we note that if any of the $q_\bullet$s are not equal to 1, then one of them must equal 0, meaning that

$$\left(\mathrm{H}^{q_1}_{\mathrm{dR}}(X, \mathbb{C}) \otimes \cdots \otimes \mathrm{H}^{q_{2p}}_{\mathrm{dR}}(X, \mathbb{C})\right)_{u\alpha} = \mathrm{H}^{q_1}_{\mathrm{dR}}(X, \mathbb{C})_{u\alpha_1} \otimes \cdots \otimes \mathrm{H}^{q_{2p}}_{\mathrm{dR}}(X, \mathbb{C})_{u\alpha_{2p}}$$

is the zero vector space. Thus, we see that

$$\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right)_{[\alpha]} = \bigoplus_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} \left(\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{C})^{\otimes 2p}\right)_{u\alpha}.$$

The comparison isomorphism now implies that $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]}$ has dimension $[\mathbb{Q}([\alpha]) : \mathbb{Q}]$ over $\mathbb{Q}$ and thus one dimension over $\mathbb{Q}([\alpha])$.

It remains to show that (a) and (b) are equivalent. Well, the $\mathbb{Q}$-vector space $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]}(p)$ will consist of Hodge classes if and only if $\left(\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{C})^{\otimes 2p}\right)_{u\alpha}$ is of bidegree (p, p), which is equivalent to $\langle u\alpha \rangle = 3p$ by Lemma 4.22. ∎

4.2.2 An Absolute Hodge Cycle

Thus far, we have access to classes ω_α, and we know how to compute their periods against the Betti cycle γ. We will be able to compute the Galois action on γ because it already comes from a Betti cycle, but we then need to know how to translate this into a Galois action on the ω_α; importantly, note that ω_αs have no obvious Galois action, and indeed, they cannot because they may not even be defined over a number field. To do this, we need a way to put γ and the ω_α on the same footing; following [GGL24, Section 8.5], we use absolute Hodge classes.

For example, the machinery of cohomology tells us how to take γ and then apply some cycle class maps to produce an absolute Hodge class. Let's be more explicit: we may pass the class $\gamma^{2p} \otimes (2\pi i)^{-p}$ through the maps

$$\mathrm{H}^{\mathrm{B}}_{2p}\left(X^{2p}, \mathbb{C}\right)(-p) \cong \mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{C}\right)(p) \subseteq \mathrm{H}^{2p}_{\mathbb{A}}(X)(p),$$

where the last map is the cycle class map. In order to ensure that we output an absolute Hodge cycle, we apply Proposition 4.23: we see that the generalized eigenspace for $[\alpha]$ contains all Hodge classes if and only if $\alpha \in \mathfrak{B}^{2p}$, we simply define $\gamma^{2p}_{[\alpha]} \in \mathrm{H}^{\mathrm{B}}_{2p}\left(X^{2p}, \mathbb{Q}\right)$ to be the projection to the $[\alpha]$-component, and we now know that its image $\gamma^{2p}_{[\alpha],\mathrm{AH}}$ is a Hodge class, hence an absolute Hodge class by Theorem 2.45.

> **Remark 4.24.** We remark that this last paragraph actually argues that the projection
>
> $$C^p_{\mathrm{AH}}(X)_{[\alpha]} \twoheadrightarrow \mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right)(p)_{[\alpha]}$$
>
> is an isomorphism for any $\alpha \in \mathfrak{B}^{2p}$. In particular, both spaces are 1-dimensional vector spaces over $\mathbb{Q}([\alpha])$.

Perhaps we should check that $\gamma^{2p}_{[\alpha],\mathrm{AH}}$ is nonzero. Roughly speaking, we expect this to hold by the period computations of Lemma 4.12.

Proposition 4.25. Choose $\alpha \in \mathfrak{B}^{2p}$. Then

$$\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right) = \sum_{\substack{\beta\in[\alpha]\\ \beta=(a_1,b_1,c_1,\ldots,a_{2p},b_{2p},c_{2p})}} \left((2\pi i)^{-p}\prod_{i=1}^{2p}\frac{N-[a_i]-[b_i]}{N}\int_\gamma \omega_{(-a_i,-b_i,-c_i)}\right)\omega_\beta.$$

Proof. We know that the ω_β form an eigenbasis of $\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p},\mathbb{C}\right)(p)_{[\alpha]}$ by restricting Remark 4.7 to the $[\alpha]$-generalized eigenspace. Thus, we know that $\pi_\infty(\gamma_{[\alpha],\mathrm{AH}})$ is certainly a linear combination of the ω_βs, so we write

$$\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right) = \sum_{\beta\in[\alpha]} z_\beta\omega_\beta,$$

and it remains to compute the coefficients z_β. For this, we use the computation of the Poincaré pairing computation from Lemma 4.8 (iterated $2p$ times), whereupon we see that

$$P\left(\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right),\omega_{-\beta}\right) = z_\beta\prod_{i=1}^{2p}(-1)^N\frac{N}{N-[a_i]-[b_i]},$$

where $\beta=(a_1,b_1,c_1,\ldots,a_{2p},b_{2p},c_{2p})$. Thus, to get the correct answer for z_β, we would like to show that

$$P\left(\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right),\omega_{-\beta}\right) \stackrel{?}{=} (2\pi i)^{-p}\int_\gamma \omega_{-\beta}.$$

(Note that the sign has disappeared because $(-1)^{N\cdot 2p}=1$.) To compute this Poincaré pairing, we would like to remember that $\gamma_{[\alpha],\mathrm{AH}}$ comes from a Betti class. As such, we remark that the composite

$$\mathrm{H}^{\mathrm{B}}_{2p}\left(X^{2p},\mathbb{C}\right)(-p)\cong\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p},\mathbb{C}\right)(p)\subseteq\mathrm{H}^{2p}_{\mathbb{A}}\left(X^{2p}\right)(p)\twoheadrightarrow\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p},\mathbb{C}\right)$$

is just the usual cycle class map from Betti to de Rham cohomology. Thus, we see that the Poincaré pairing with $\gamma^{2p}_{[\alpha],\mathrm{AH}}$ may be computed as the integration pairing

$$P\left(\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right),\omega_{-\beta}\right) = (2\pi i)^{-p}\int_{\gamma^{2p}_{[\alpha]}}\omega_{-\beta}.$$

To complete the proof, we note that we may pass from integrating over $\gamma^{2p}_{[\alpha]}$ to γ^{2p} because the adjunctive property of the integration pairing allows us to pass the projection onto the $[\alpha]$-component to $-\beta$, but $\omega_{-\beta}$ already lives in the $[\alpha]$-generalized eigenspace. ∎

Thus, we see that $\gamma_{[\alpha],\mathrm{AH}}$ is nonzero because we have found nonzero coefficients: the integrals are nonzero by Lemma 4.12. While we're here, we translate this into a statement with $\nu_\bullet$s.

Corollary 4.26. Choose $\alpha \in \mathfrak{B}^{2p}$. Then

$$\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right) = \sum_{\substack{\beta\in[\alpha]\\ \beta=(a_1,b_1,c_1,\ldots,a_{2p},b_{2p},c_{2p})}} \left((2\pi i)^{-p}\prod_{i=1}^{2p}\int_\gamma \nu_{(-a_i,-b_i,-c_i)}\right)\nu_\beta.$$

Proof. The same proof as in Proposition 4.25 applies when combined with Remark 4.9. ∎

Remark 4.27. Following Remark 4.9, it will be computationally helpful to rewrite our formula in terms of the $\nu_\bullet$s because this will make the mysterious rational constant disappear.

In order to hide these integrals for now, we introduce the following notation.

Notation 4.28. For $\alpha \in \mathfrak{B}^{2p}$ such that $\alpha = (\alpha_1, \ldots, \alpha_{2p})$, we define

$$\mathrm{Per}\left(\gamma^{2p}, \nu_\alpha\right) \coloneqq (2\pi i)^{-p} \prod_{i=1}^{2p} \int_\gamma \nu_{\alpha_i}.$$

Note that this number is algebraic by Proposition 4.23 because it is the integral of a differential against an absolute Hodge class. (See the end of the proof of Proposition 4.25.)

Remark 4.29. In order to compute these integrals, we note Remark 4.14 grants the product of the integrals equals

$$\prod_{i=1}^{2p} (-1)^{\lfloor ([a_i]+[b_i])/N \rfloor} \zeta_{2N}^{[a_i]+[b_i]-N} \Gamma\left(\frac{[a_i]}{N}\right) \Gamma\left(\frac{[b_i]}{N}\right) \Gamma\left(\frac{[-c_i]}{N}\right)^{-1}.$$

We quickly note that $\left\lfloor \frac{[a_i]+[b_i]}{N} \right\rfloor \in \{0,1\}$ equals 0 exactly p times and equals 1 exactly p times because $\alpha \in \mathfrak{B}^{2p}$; additionally, $\zeta_{2N}^{-N\cdot 2p} = 1$, so that power vanishes. Thus, our period equals

$$(-2\pi i)^{-p} \prod_{i=1}^{2p} \zeta_{2N}^{[a_i]+[b_i]} \frac{\Gamma\left(\frac{[a_i]}{N}\right) \Gamma\left(\frac{[b_i]}{N}\right)}{\Gamma\left(\frac{[-c_i]}{N}\right)}.$$

We will also want to express the $\nu_\bullet$s in terms of γ.

Corollary 4.30. Choose $\alpha \in \mathfrak{B}^{2p}$. For any $\beta \in [\alpha]$, we have

$$\nu_\beta = \frac{1}{\#G^{2p}(\overline{\mathbb{Q}})\,\mathrm{Per}\left(\gamma^{2p}, \nu_{-\alpha}\right)} \sum_{g \in G^{2p}(\overline{\mathbb{Q}})} \beta(g)^{-1} \cdot \pi_\infty\left(g^* \gamma^{2p}_{[\alpha],\mathrm{AH}}\right).$$

Proof. By the orthogonality of characters applied to Corollary 4.26, we find that

$$\frac{1}{\#G^{2p}(\overline{\mathbb{Q}})} \sum_{g \in G^{2p}(\overline{\mathbb{Q}})} \beta(g)^{-1} \cdot \pi_\infty\left(g^* \gamma^{2p}_{[\alpha],\mathrm{AH}}\right) = \mathrm{Per}\left(\gamma^{2p}, \nu_{-\alpha}\right) \nu_{\beta,\mathrm{AH}},$$

so the result follows. ∎

4.2.3 Computation of the Galois Action

In this subsection, we compute the Galois action on our absolute Hodge cycles. To ground ourselves, we begin by noting that we are expecting a permutation matrix.

Lemma 4.31. Choose $\alpha \in \mathfrak{A}^{2p}$ and a prime ℓ such that $\ell \equiv 1 \pmod N$. Given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\zeta_N) = \zeta_N^u$ for some $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, we find that σ maps

$$\mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\right)_\alpha \to \mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\right)_{u^{-1}\alpha}.$$

Proof. Choose $v \in \mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\right)_\alpha$. Then for any $g \in G^{2p}(\mathbb{Q}_\ell)$, we find that

$$\sigma(g \cdot v) = \sigma(g) \cdot \sigma(v)$$

because the action of G^{2p} is defined over $\mathbb{Q}$ and hence Galois-invariant. Rearranging, we see that

$$\begin{aligned} g \cdot \sigma(v) &= \sigma\left(\sigma^{-1}(g)\right) \cdot \sigma(v) \\ &= \sigma\left(\sigma^{-1}(g) \cdot v\right) \\ &= \sigma\left(\alpha\left(\sigma^{-1}(g)\right) \cdot v\right) \\ &= \alpha\left(\sigma^{-1}(g)\right) \sigma(v), \end{aligned}$$

where the last equality holds because the Galois action is $\mathbb{Q}_\ell$-linear. A direct computation then shows $\alpha\left(\sigma^{-1}(g)\right) = \sigma^{-1}(\alpha(g))$ and then $\sigma^{-1}(\alpha(g)) = \left(u^{-1}\alpha\right)(g)$. ∎

We now move towards the computation of the Galois action on absolute Hodge classes. This requires a warning. Our computation will be able to succeed by using de Rham classes as representatives for absolute Hodge classes. However, de Rham classes have no Galois action: only absolute Hodge classes have Galois action (through the ℓ-adic components). The key to keeping track of the differences between these elements is to keep track of our base-changes. In particular, for any prime ℓ, we may specify an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and note the "comparison" isomorphisms

$$\begin{aligned} \mathrm{H}^{2p}_{\mathrm{dR}}(X, \mathbb{C})(p)_{[\alpha]} &= C^p_{\mathrm{AH}}\left(X^{2p}_{\overline{\mathbb{Q}}}\right)_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{C} \\ &= C^p_{\mathrm{AH}}\left(X^{2p}_{\overline{\mathbb{Q}}}\right)_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \otimes_\iota \mathbb{C} \\ &= \mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\right)(p)_{[\alpha]} \otimes_\iota \mathbb{C}, \end{aligned}$$

where the last isomorphism is given by the Betti-to-étale comparison isomorphism. (We remark that these identifications are all G^{2p}-invariant.) For example, in the sequel, we may write bizzarre things such as

$$\gamma^{2p}_{[\alpha],\mathrm{AH}} \otimes 1 \in C^p_{\mathrm{AH}}\left(X^{2p}_{\overline{\mathbb{Q}}}\right) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \qquad \text{or} \qquad \nu_\alpha \otimes 1 \in \mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{Q}\right) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

and then pretend that these elements live in the same vector space.

As promised in the previous section, we are able to compute the Galois action on γ. Explicitly, this amounts to the following.

Lemma 4.32. Choose $\alpha \in \mathfrak{B}^{2p}$.

(a) There is a function $\lambda\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}([\alpha])^\times$ such that

$$\sigma\left(\gamma^{2p}_{[\alpha],\mathrm{AH}}\right) = \lambda(\sigma)\gamma^{2p}_{[\alpha],\mathrm{AH}}.$$

(b) For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $g \in G^{2p}(\overline{\mathbb{Q}})$, we have

$$\sigma\left(g^*\gamma^{2p}_{[\alpha],\mathrm{AH}}\right) = \lambda(\sigma) \cdot \sigma(g)^*\gamma^{2p}_{[\alpha],\mathrm{AH}}.$$

(c) For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we compute $\iota_\alpha(\lambda(\sigma)) \in \mathbb{Q}(\zeta_{2N})$ as

$$\iota_\alpha(\lambda(\sigma)) = \frac{\sigma\left(\mathrm{Per}\left(\gamma^{2p}, \nu_{-\alpha}\right)\right)}{\mathrm{Per}\left(\gamma^{2p}, \nu_{-\alpha}\right)}.$$

Proof. Here, (a) follows because $C^p_{\mathrm{AH}}\left(X^{2p}\right)_{[\alpha]}$ is a one-dimensional vector space over $\mathbb{Q}([\alpha])$ which is stable under the Galois action (because its Betti component is defined over $\mathbb{Q}$); thus, we see that $\gamma^{2p}_{[\alpha],\mathrm{AH}}$ is a basis vector of this space, so (a) follows. Continuing, (b) follows because the action of G^{2p} on X^{2p} is defined over

$\mathbb{Q}$, implying that

$$\begin{aligned}\sigma\left(g^*\gamma_{[\alpha],\mathrm{AH}}^{2p}\right) &= \sigma(g)^*\sigma\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\right)\\ &= \sigma(g)^*\left(\lambda(\sigma)\gamma_{[\alpha],\mathrm{AH}}^{2p}\right)\\ &= \lambda(\sigma)\cdot\sigma(g)^*\gamma_{[\alpha],\mathrm{AH}}^{2p},\end{aligned}$$

where the last equality holds because $\sigma(g)^*$ is linear.

Lastly, (c) will require a computation. We will work in the de Rham component; the idea is to project onto the α-component. Working in $C_{\mathrm{AH}}^p\left(X^{2p}\right)\otimes_{\mathbb{Q}}\mathbb{C}$, one has the equalities

$$\left(\lambda(\sigma)\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right) = \left(\sigma\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right).$$

We now project onto the α-eigenspace; because the G^{2p}-action is defined over $\mathbb{Q}$, the projection commutes with the Galois action, leaving us with

$$\left(\lambda(\sigma)\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right)_\alpha = \sigma\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right)_\alpha.$$

On one hand, by definition of ι_α, we see that the left-hand side will equal $\iota_\alpha(\lambda(\sigma))\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right)$; then projecting onto the de Rham component leaves us with

$$\pi_\infty\left(\left(\lambda(\sigma)\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right)_\alpha\right) = \lambda(\sigma)\,\mathrm{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)\nu_\alpha$$

by Corollary 4.26. On the other hand, for the right-hand side, we will want to project onto the de Rham component first (which commutes with Galois action by our identifications). To complete the proof, we now run computations in $\mathrm{H}_{\mathrm{dR}}^{2p}\left(X^{2p},\mathbb{C}\right) = \mathrm{H}_{\mathrm{dR}}^{2p}\left(X^{2p},\mathbb{Q}\right)\otimes_{\mathbb{Q}}\mathbb{C}$, for which we use Corollary 4.26 to see

$$\begin{aligned}\pi_\infty\left(\sigma\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right)_\alpha\right) &= \sigma\left(\pi_\infty\left(\gamma_{[\alpha],\mathrm{AH}}^{2p}\otimes 1\right)_\alpha\right)\\ &= \sigma\left(\nu_\alpha\otimes\mathrm{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)\right)\\ &= \sigma\left(\mathrm{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)\right)\nu_\alpha,\end{aligned}$$

where the last equality holds because the Galois action on $\mathrm{H}_{\mathrm{dR}}^{2p}\left(X^{2p},\mathbb{Q}\right)$ is trivial. Comparing the previous two computations completes the proof. ∎

We are now ready for our main theorem.

Theorem 4.33. Choose $\alpha\in\mathfrak{B}^{2p}$. For any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\zeta_N)=\zeta_N^u$ for $u\in(\mathbb{Z}/N\mathbb{Z})^\times$, we have

$$\sigma(\nu_\alpha\otimes 1) = \nu_{u^{-1}\alpha}\otimes\frac{\sigma\left(\mathrm{Per}(\gamma^{2p},\nu_{-u^{-1}\alpha})\right)}{\mathrm{Per}(\gamma^{2p},\nu_{-\alpha})},$$

where this Galois action takes place in $\mathrm{H}_{\mathrm{dR}}^{2p}\left(X^{2p},\mathbb{Q}\right)(p)_{[\alpha]}\otimes_{\mathbb{Q}}\overline{\mathbb{Q}} = C_{\mathrm{AH}}^p\left(X^{2p}\right)_{[\alpha]}\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$.

Proof. We combine the computed Galois action in Lemma 4.32 with the change-of-basis results Corollaries 4.26 and 4.30. To begin, Corollary 4.30 lets us write

$$\begin{aligned}\sigma(\nu_\alpha\otimes 1) &= \sigma\left(\frac{1}{\#G^{2p}(\overline{\mathbb{Q}})}\sum_{g\in G^{2p}(\overline{\mathbb{Q}})} g^*\gamma_{\mathrm{AH}}^{2p}\otimes\frac{1}{\alpha(g)\,\mathrm{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)}\right)\\ &= \frac{1}{\#G^{2p}(\overline{\mathbb{Q}})}\sum_{g\in G^{2p}(\overline{\mathbb{Q}})}\sigma\left(g^*\gamma_{\mathrm{AH}}^{2p}\otimes\frac{1}{\alpha(g)\,\mathrm{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)}\right)\\ &= \frac{1}{\#G^{2p}(\overline{\mathbb{Q}})}\sum_{g\in G^{2p}(\overline{\mathbb{Q}})}\sigma\left(g^*\gamma_{\mathrm{AH}}^{2p}\right)\otimes\frac{1}{\alpha(g)\,\mathrm{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)},\end{aligned}$$

where the last equality takes place in $C^p_{\mathrm{AH}}\left(X^{2p}\right) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ so that the Galois action is happening in the left component. Continuing, Lemma 4.32 tells us that

$$\sigma\left(g^*\gamma^{2p}_{[\alpha],\mathrm{AH}}\right) = \sigma(g)\lambda(\sigma)\cdot\sigma(g)^*\gamma^{2p}_{[\alpha],\mathrm{AH}},$$

so

$$\sigma(\nu_\alpha \otimes 1) = \frac{1}{\#G^{2p}(\overline{\mathbb{Q}})}\sum_{g\in G^{2p}(\overline{\mathbb{Q}})} \lambda(\sigma)\cdot\sigma(g)^*\gamma^{2p}_{[\alpha],\mathrm{AH}} \otimes \frac{1}{\alpha(g)\operatorname{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)}.$$

(We will wait to evaluate $\lambda(\sigma)$ until the end because a trick is required to move it through the tensor product.) We now go back to the basis of $\nu_\bullet$s via Corollary 4.26, writing

$$\sigma(\nu_\alpha \otimes 1) = \frac{1}{\#G^{2p}(\overline{\mathbb{Q}})}\sum_{\substack{g\in G^{2p}(\overline{\mathbb{Q}})\\ \beta\in[\alpha]}} \lambda(\sigma)\cdot\sigma(g)^*\nu_\beta \otimes \frac{\operatorname{Per}(\gamma^{2p},\nu_{-\beta})}{\alpha(g)\operatorname{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)}.$$

Now, $\sigma(g)^*\nu_\beta \otimes 1 = \nu_\beta \otimes \beta(\sigma(g))$, where the equality is now taking place in $\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p},\mathbb{Q}\right)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$. Continuing, we see $\beta(\sigma(g)) = \sigma(\beta(g)) = \beta(g)^u$ because evaluating a character is Galois-invariant. Rearranging the sums, we now see that we can isolate the sum

$$\frac{1}{\#G^{2p}(\overline{\mathbb{Q}})}\sum_{g\in G^{2p}(\overline{\mathbb{Q}})}\frac{(u\beta)(g)}{\alpha(g)},$$

which orthogonality of characters tells us is the indicator for $\beta = u^{-1}\alpha$. Thus, we are left with

$$\sigma(\nu_\alpha\otimes 1) = \lambda(\sigma)\nu_{u^{-1}\alpha}\otimes\frac{\operatorname{Per}(\gamma^{2p},\nu_{-u^{-1}\alpha})}{\operatorname{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)}.$$

It remains to move $\lambda(\sigma)$ through the tensor product. Note that this is not totally trivial because the tensor product only lets us move rational numbers through. Anyway, it is enough to check the required equality in the de Rham component, allowing us to use the proof of Lemma 4.32 to note

$$\lambda(\sigma)\nu_{u^{-1}\alpha}\otimes\operatorname{Per}\left(\gamma^{2p},\nu_{-u^{-1}\alpha}\right) = \nu_{u^{-1}\alpha}\otimes\sigma\left(\operatorname{Per}\left(\gamma^{2p},\nu_{-u^{-1}\alpha}\right)\right),$$

from which the required result follows after some rearranging. ■

Remark 4.34. Because the G^{2p}-action commutes with the Galois action, it is not difficult to directly check that an α-eigenvector should go to a $u^{-1}\alpha$-eigenvector.

Remark 4.35. As a sanity check, it is not hard to see that Theorem 4.33 actually defines a group representation.

Remark 4.36. Following Remark 2.132, one can use Theorem 4.33 allows ons to compute the connected monodromy field K_A^{conn} of the Jacobian A of any quotient C of the Fermat curve X_N. Indeed, Remark 2.132 explains that this is essentially a matter of computing enough the field of definition of enough Tate classes (used to cut out the torus $G^\circ_\ell(A)$). In particular, we already know that $\mathbb{Q}(\zeta_N)\subseteq K_A^{\mathrm{conn}}$ (because of the endomorphisms), and then Theorem 4.33 explains that $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ fixes a Tate cycle ν_α if and only if it fixes the period $\operatorname{Per}\left(\gamma^{2p},\nu_{-\alpha}\right)$.

Let's see an example.

Corollary 4.37. Choose $\alpha := (a,b,c) \in \mathfrak{A}^1$, and set $\alpha' := (a',b',c')$ to be $-\alpha$. Then $(\alpha,\alpha') \in \mathfrak{B}^2$, and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\zeta_N) = \zeta_N^u$ for $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have

$$\sigma(\nu_{(\alpha,\alpha')} \otimes 1) = \nu_{u^{-1}(\alpha,\alpha')} \otimes (-1)^{\langle u^{-1}\alpha\rangle - \langle\alpha\rangle}.$$

In particular, σ fixes $\nu_{(\alpha,\alpha')} \otimes 1$ if and only if $u-1$ is divisible by $N/\gcd(a,b,c,N)$.

Proof. To see that $(\alpha,\alpha') \in \mathfrak{B}^2$, we note that any $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ still has $u\alpha = -u\alpha'$, so $\{\langle u\alpha\rangle, \langle -u\alpha\rangle\} = \{1,2\}$.

Looking at Theorem 4.33, we see the main part of proof will be computing our periods. The main point is that the reflection formula for Γ (recalled later in Proposition 4.40) reassures us that

$$\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[-a]}{N}\right) = \frac{\pi}{\sin\frac{a\pi}{N}}.$$

We now combine this with the computation in Remark 4.29 to achieve

$$\mathrm{Per}\left(\gamma^{2p}, \nu_{-(\alpha,\alpha')}\right) = -(2\pi i)^{-1} \cdot \zeta_{2N}^{[-a]+[-b]+[a]+[b]} \cdot \frac{\pi}{\sin\frac{[a]\pi}{N}} \cdot \frac{\pi}{\sin\frac{[b]\pi}{N}} \cdot \frac{\sin\frac{[c]\pi}{N}}{\pi}.$$

Note that $[a]+[-a] = N$, so the power of ζ_{2N} disappears. Continuing, we expand $\sin z = \frac{1}{2i}\left(z + z^{-1}\right)$, which yields

$$\mathrm{Per}\left(\gamma^{2p}, \nu_{-(\alpha,\alpha')}\right) = -\frac{\left(\zeta_{2N}^c - \zeta_{2N}^{-c}\right)}{\left(\zeta_{2N}^a - \zeta_{2N}^{-a}\right)\left(\zeta_{2N}^b - \zeta_{2N}^{-b}\right)}.$$

Continuing, we factor $\zeta_{2N}^c/\zeta_{2N}^{-a-b} = \zeta_{2N}^{N\langle\alpha\rangle} = (-1)^{\langle\alpha\rangle}$, leaving us with

$$\mathrm{Per}\left(\gamma^{2p}, \nu_{-(\alpha,\alpha')}\right) = -(-1)^{\langle\alpha\rangle} \cdot \frac{\left(1 - \zeta_N^{-c}\right)}{\left(\zeta_N^a - 1\right)\left(\zeta_N^b - 1\right)}.$$

We now plug into Theorem 4.33 to reveal

$$\sigma(\nu_{(\alpha,\alpha')} \otimes 1) = \nu_{u^{-1}(\alpha,\alpha')} \otimes \frac{\sigma\left((-1)^{\langle u^{-1}\alpha\rangle} \cdot \frac{\left(1-\zeta_N^{-u^{-1}c}\right)}{\left(\zeta_N^{u^{-1}a}-1\right)\left(\zeta_N^{u^{-1}b}-1\right)}\right)}{(-1)^{\langle\alpha\rangle} \cdot \frac{\left(1-\zeta_N^{-c}\right)}{\left(\zeta_N^a-1\right)\left(\zeta_N^b-1\right)}},$$

which rearranges into the desired expression because $\sigma\left(\zeta_N^{u^{-1}}\right) = \zeta_N$.

It now remains the last sentence. Well, we see that σ fixes $\nu_{(\alpha,\alpha')}$ if and only if $u^{-1}\alpha = \alpha$, which is equivalent to $u\alpha = \alpha$. By taking $\mathbb{Z}$-linear combinations, it is equivalent to asking for $(u-1)\gcd(a,b,c) \equiv 0 \pmod N$, from which the claim follows. ∎

4.2.4 Some Examples

We begin with the superelliptic curve $C\colon y^9 = x^3 - 1$.

Proposition 4.38. Define A to be the Jacobian of the proper curve C with affine chart $y^9 = x^3 - 1$. Then we show $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, and we compute $\mathrm{ST}(A)$.

Proof. We will freely use the computation executed in Proposition 3.16. Thoughout, $A := \mathrm{Jac}\, C$, and we recall that we have a decomposition $A = C_0 \times A_1 \times A_2$ (over $\mathbb{Q}$) into geometrically simple abelian varieties. We proceed in steps.

1. Even though this is not a Fermat curve, it is a quotient of the Fermat curve X_N with $N \coloneqq 9$: this is witnessed by the quotient map from the affine patch $x^9 + y^9 + 1 = 0$ to C given by $\psi(x, y) \coloneqq \left(-x^3, y\right)$. Thus, we will be able to use the Galois-invariant embedding $\psi \colon \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X_{N,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ to use Theorem 4.33 by restricting to the Galois submodule. To make this explicit, we recall that we have a basis
$$\left\{ \frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8} \right\}$$
of $\mathrm{H}^{10}(C)$, we see that we can pass this basis through ψ^* to see that $\mathrm{H}^{10}(C) \subseteq \mathrm{H}^{10}(X)$ has basis
$$\left\{ \nu_{351}, \nu_{342}, \nu_{333}, \nu_{324}, \nu_{315}, \nu_{621}, \nu_{612} \right\}.$$
Combining with the conjugate differentials yeilds a full basis of $\mathrm{H}^1_{\mathrm{dR}}(C, \mathbb{Q}) \subseteq \mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{Q})$.

2. We now explain how to pass the étale site. By Conjecture 3.7, which is known in this case by Theorem 3.8, we may choose any ℓ, so we choose ℓ so that $\mathbb{Q}_\ell$ contains any algebraic numbers we will need in the sequel (most notably, we want ζ_N and our periods). For each $p \geq 0$, we recall that any $\alpha \in \mathfrak{B}^{2p}$ produces idenitifications
$$\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{C} = C^p_{\mathrm{AH}}\left(X^{2p}\right)_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{C} \hookrightarrow \mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}, \mathbb{Q}_\ell\right)(p)_{[\alpha]} \otimes_\iota \mathbb{C},$$
where $\iota \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ is some fixed embedding. In this way, we see that we are allowed to treat an expression like $\nu_{351} \otimes 1$ as an element of $\mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}, \mathbb{Q}_\ell\right) \otimes_\iota \mathbb{C}$; for carefully chosen ℓ, a Galois descent argument is even able to reassure us that the basis vectors $\nu_\alpha \otimes 1$ produces from the previous step can be found in $\mathrm{H}^{2p}_{\text{ét}}\left(X^{2p}, \mathbb{Q}_\ell\right)(p)_{[\alpha]}$.
Thus, in the notation of Proposition 3.16, we see that ψ^* pulls the basis vectors $\{u_1 \otimes 1, v_1 \otimes 1, v_2 \otimes 1, v_4 \otimes 1, w_1 \otimes 1, w_2 \otimes 1, w_5 \otimes 1\}$ to
$$\{\nu_{333} \otimes 1, \quad \nu_{315} \otimes 1, \nu_{621} \otimes 1, \nu_{342} \otimes 1, \quad \nu_{612} \otimes 1, \nu_{324} \otimes 1, \nu_{351} \otimes 1\},$$
and one can recover ψ^* on the rest of the basis by taking conjugates.

3. We are now ready to begin executing Proposition 2.127; for this, Remark 2.128 informs us that we need to build a space of W' of Tate classes cutting out $G_\ell(A)^\circ \subseteq \mathrm{GL}_{14,\mathbb{Q}_\ell}$. We begin by adding W_1, made up of the endomorphisms, which ensures (for example) that $G_\ell(A)^\circ$ is diagonal. Then Proposition 3.16 computed that we also have the "polarization equations"
$$\begin{aligned} \mu_1\mu_2 &= \kappa_1\kappa_8, \\ \kappa_1\kappa_8 &= \kappa_2\kappa_7, \\ \kappa_1\kappa_8 &= \kappa_4\kappa_5, \end{aligned}$$
and the exceptional equation
$$\mu_1\kappa_7 = \kappa_5\kappa_8.$$
We remark that the polarization equations translate into a Tate class like $\nu_{(\alpha,-\alpha,\beta,-\beta)} \otimes 1$ understood as an element in $\mathrm{H}^4_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, but this Tate class actually already come from a class in W_1 (see Corollary 4.37), so we may safely ignore it. Thus, we only have to translate the exceptional equation into the tensor
$$\nu_{333,675,648,612} \otimes 1 \in \mathrm{H}^4_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$
and its Galois orbit.

4. We claim that $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_N)$. By Remark 2.128, it is enough to know that $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is the largest subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing W'. We already know that our endomorphisms, except the isogeny $(A_1)_{\overline{\mathbb{Q}}} \cong (A_2)_{\overline{\mathbb{Q}}}$, are defined over $\mathbb{Q}(\zeta_N)$ (see also Corollary 4.37 for these equations and the polarization). The isogeny corresponds to equations $\kappa_u = \lambda_{2u}$ for each $u \in (\mathbb{Z}/9\mathbb{Z})^\times$, which means that we would like to check that
$$\mathrm{Per}\left(\gamma^{2p}, \nu_{u(612,378)}\right)$$

is in $\mathbb{Q}(\zeta_N)$. Well, by Remark 4.14, this element is

$$(-2\pi i)^{-1}\zeta_{2N}^{u(6+2+3+7)}\cdot\frac{\Gamma\left(\frac{[6u]}{9}\right)\Gamma\left(\frac{[2u]}{9}\right)}{\Gamma\left(\frac{[8u]}{9}\right)}\cdot\frac{\Gamma\left(\frac{[3u]}{9}\right)\Gamma\left(\frac{[7u]}{9}\right)}{\Gamma\left(\frac{[u]}{9}\right)}.$$

A quick application of the reflection formula as in Corollary 4.37 shows this is in $\mathbb{Q}(\zeta_N)$.

It remains to check that $\sigma\in\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ fixes the Galois orbit of $\nu_{333,675,648,612}\otimes 1$. Well, looking at Theorem 4.33, it is enough to check that σ fixes

$$\mathrm{Per}\left(\gamma^4,\nu_{u(333,675,648,612)}\right)$$

for any $u\in(\mathbb{Z}/N\mathbb{Z})^\times$. Well, by Remark 4.14, we see this equals

$$(-2\pi i)^{-2}\zeta_{2N}^{u(3+3+6+7+6+4+6+1)}\cdot\frac{\Gamma\left(\frac{[3u]}{9}\right)\Gamma\left(\frac{[3u]}{9}\right)}{\Gamma\left(\frac{[6u]}{9}\right)}\cdot\frac{\Gamma\left(\frac{[6u]}{9}\right)\Gamma\left(\frac{[7u]}{9}\right)}{\Gamma\left(\frac{[4u]}{9}\right)}\cdot\frac{\Gamma\left(\frac{[6u]}{9}\right)\Gamma\left(\frac{[4u]}{9}\right)}{\Gamma\left(\frac{[u]}{9}\right)}\cdot\frac{\Gamma\left(\frac{[6u]}{9}\right)\Gamma\left(\frac{[u]}{9}\right)}{\Gamma\left(\frac{[7u]}{9}\right)}.$$

After the dust settles, we are left with

$$(-2\pi i)^{-2}\cdot\Gamma\left(\frac{3}{9}\right)^2\Gamma\left(\frac{6}{9}\right)^2.$$

Now, the reflection formula yields $\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right)=\frac{\pi}{\sin\frac{\pi}{3}}$, so we see that this period lives in $\mathbb{Q}(\zeta_N)$ and hence is fixed by σ; in fact, it is rational!

5. Choose $\sigma\in\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ to satisfy $\sigma(\zeta_N)=\zeta_N^u$. We compute the action of σ on W'. For example, the previous step actually shows that σ fixes the Galois orbit of $\nu_{333,675,648,612}\otimes 1$, so it remains to compute the action on W_1. Note that G acts on the $\mathbb{C}$-vector space, so the action can be diagonalized. Given some character $(\alpha,\beta)\in\mathfrak{A}^2$, we note that $(W_1)_{(\alpha,\beta)}$ is at most one-dimensional spanned by $\nu_{(\alpha,\beta)}\otimes 1$, and this element being a Tate class is equivalent to $\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q})(1)_{[\alpha]}$ has Hodge cycles by the Mumford–Tate conjecture (known in this case by Remark 2.118), which is equivalent to $(\alpha,\beta)\in\mathfrak{B}^2$ by Proposition 4.23. With the aide of a computer, we can enumerate all such (α,β), and we see that they come in two forms.

   - We could have $\alpha=(a,b,c)$ and $\beta=-\alpha$. In this case, Corollary 4.37 explains that

   $$\sigma(\nu_{(\alpha,\beta)}\otimes 1)=\nu_{u^{-1}(\alpha,\beta)}\otimes(-1)^{\langle u^{-1}\alpha\rangle-\langle\alpha\rangle}.$$

   - We could have $\alpha=(a,b,c)$ and $\beta=(-a,-c,-b)$. As in Corollary 4.37, the main point is to compute our periods. Well, by Remark 4.14, we find

   $$\mathrm{Per}\left(\gamma^{2p},\nu_{-(\alpha,\beta)}\right)=-(2\pi i)^{-1}\zeta_{2N}^{[a]+[-a]+[-b]+[c]}\cdot\frac{\Gamma\left(\frac{[-a]}{N}\right)\Gamma\left(\frac{[-b]}{N}\right)}{\Gamma\left(\frac{[c]}{N}\right)}\cdot\frac{\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[c]}{N}\right)}{\Gamma\left(\frac{[-b]}{N}\right)},$$

   which after an appliation of the reflection formula gives

   $$\begin{aligned}\mathrm{Per}\left(\gamma^{2p},\nu_{-(\alpha,\beta)}\right)&=(2\pi i)^{-1}\zeta_{2N}^{[-b]+[c]}\cdot\frac{\pi}{\sin\frac{a\pi}{N}}\\&=\frac{\zeta_{2N}^{[-b]+[c]}}{\zeta_{2N}^{a}-\zeta_{2N}^{-a}}\\&=\frac{\zeta_{2N}^{[-b]+[c]+[a]}}{\zeta_N^a-1}.\end{aligned}$$

It will be convenient to write this entirely in terms of ζ_N, so we note that N being odd forces $\zeta_{2N} = -\zeta_N^{(N+1)/2}$, so this equals $(-1)^{[a]+[-b]+[c]}\zeta_N^{(a-b+c)(N+1)/2}$. The purpose of this rewrite is that all ζ_Ns will go away in the computation

$$\sigma(\nu_{(\alpha,\beta)} \otimes 1) = \nu_{u^{-1}(\alpha,\beta)} \otimes \frac{\sigma\left(\mathrm{Per}\left(\gamma^{2p}, \nu_{-u^{-1}(\alpha,\beta)}\right)\right)}{\mathrm{Per}\left(\gamma^{2p}, \nu_{-(\alpha,\beta)}\right)}$$

because $\sigma\left(\zeta_N^{u^{-1}}\right) = 1$, so we are left with

$$\sigma(\nu_{(\alpha,\beta)} \otimes 1) = \nu_{u^{-1}(\alpha,\beta)} \otimes (-1)^{[u^{-1}a]+[-u^{-1}b]+[u^{-1}c]+[a]+[-b]+[c]}.$$

6. Now choose $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ to satisfy $\sigma(\zeta_N) = \zeta_N^5$, which we note is a generator. We now compute

$$\{g \in \mathrm{GL}_{14,\mathbb{Q}_\ell} : g|_{W'} = \sigma|_{W'}\}.$$

For this, we recall from Proposition 2.127 that we are looking at the component of $G_\ell(A)$ containing the image of σ. In particular, we know that σ is a permutation matrix sending $(\nu_\alpha \otimes 1) \mapsto (\nu_{2\alpha} \otimes 1)$ (up to scalar), so we need g to also be a permutation matrix also sending $(\nu_\alpha \otimes 1) \mapsto (\nu_{2\alpha} \otimes 1)$ (again up to scalar). Well, for each available α, we will compute relations among scalars $\{\lambda_\alpha\}$ defined to satisfy $g(\nu_\alpha \otimes 1) = (\nu_{2\alpha} \otimes \lambda_\alpha)$. Because $G_\ell(A)^\circ$ is a torus of rank 4, we are expecting to be able to write all $\lambda_\bullet$s in terms of four of them.

With this in mind, we use the previous step as follows to produce the required relations. For brevity, let λ be the multiplier of g with respect to the pairing induced by the polarization; this multiplier becomes the action of g on $\mathbb{Q}_\ell(1)$.

- We need g to satisfy
$$g(\nu_{(\alpha,-\alpha)} \otimes 1) = \nu_{2(\alpha,-\alpha)} \otimes (-1)^{\langle 2\alpha\rangle - \langle\alpha\rangle},$$
so $\lambda_\alpha\lambda_{-\alpha} = (-1)^{\langle 2\alpha\rangle - \langle\alpha\rangle}\lambda$.
- For available (a,b,c), we need g to satisfy
$$g(\nu_{(a,b,c,-a,-c,-b)}) = \nu_{(2a,2b,2c,-2a,-2c,-2b)} \otimes (-1)^{[2a]+[-2b]+[2c]+[a]+[-b]+[c]},$$
so $\lambda_{(a,b,c)}\lambda_{(-a,-c,-b)} = \lambda(-1)^{[2a]+[-2b]+[2c]+[a]+[-b]+[c]}$. For convenience, we note that $\pmod 2$ computations have
$$[2a] + [-2b] + [2c] + [a] + [-b] + [c] \equiv [2a] + [2b] + [2c] + [a] + [b] + [c] \equiv \langle 2\alpha\rangle - \langle\alpha\rangle,$$
so we are seeing the same sign as before.
- We need g to fix $\nu_{u(333,675,648,612)}$, so $\lambda_{u(333)}\lambda_{u(675)}\lambda_{u(648)}\lambda_{u(612)} = \lambda^2$.

The above points tell us that we can determine g uniquely by choosing $(\kappa_1, \kappa_2, \kappa_4) = (\lambda_{612}, \lambda_{324}, \lambda_{648})$ and λ. Explicitly, we get the matrix

$$\left[\begin{array}{cc|cc|cc}
 & -\kappa_1\kappa_4/\kappa_2 & & & & \\
\lambda\kappa_2/\kappa_1\kappa_4 & & & & & \\
\hline
 & & & \lambda/\kappa_2 & & \\
 & & \lambda/\kappa_4 & & & \\
 & & & \kappa_1 & & \\
 & & & & \lambda/\kappa_1 & \\
 & & & & & \kappa_4 \\
 & & & -\kappa_2 & & \\
\hline
 & & & & & -\lambda/\kappa_4 \\
 & & \kappa_1 & & & \\
 & & & \kappa_2 & & \\
 & & & & & \lambda/\kappa_2 \\
 & & & & & \lambda/\kappa_1 \\
 & & & & \kappa_4 &
\end{array}\right]$$

as representing g.

Thus, upon enforcing the multiplier to equal 1 and base-changing to $\mathbb{C}$, we see that $\mathrm{ST}(A)$ is generated by $\mathrm{ST}(A)^\circ$ (computed in Proposition 3.16) and the matrix

$$\begin{bmatrix} & -1 & & & & & & \\ 1 & & & & & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & -1 & & & & \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & 1 & & \end{bmatrix}.$$

This completes our computation. ■

We now use the above computation to compute the Sato–Tate group of some generic superelliptic curves.

Theorem 4.39. For given $\lambda \in \mathbb{Q}(\zeta_9) \setminus \{0,1\}$, define A to be the Jacobian of the proper curve $\widetilde{C}$ with affine chart $y^9 = x(x-1)(x-\lambda)$. Suppose that A does not have complex multiplication. Then we show $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, and we compute $\mathrm{ST}(A)$.

Proof. As usual, we proceed in steps. Throughout, we freely use the computation of Proposition 3.13.

0. Quickly, we note that we may pass from $y^9 = x(x-1)(x-\lambda)$ (for $\lambda \notin \{0,1\}$) to $y^9 = \left(x^2+x+1\right)(x-\lambda)$ (for $\lambda \notin \{\zeta_3, \overline{\zeta}_3\}$). Indeed, consider the isomorphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{Q}(\zeta_9)$ by fixing ∞ and sending $0 \mapsto \zeta_3$ and $1 \mapsto \overline{\zeta}_3$. Then the curves $y^9 = x(x-1)(x-\lambda)$ and $y^9 = \left(x^2+x+1\right)(x-f(\lambda))$ are isomorphic by an isomorphism of the "ground" $\mathbb{P}^1$. Thus, the connected monodromy field over $\mathbb{Q}(\zeta_9)$ of both curves must be the same. Because K_A^{conn} for both curves must contain $\mathbb{Q}(\zeta_9)$ anyway (there are endomorphisms whose field of definition is $\mathbb{Q}(\zeta_9)$ already), we see that this movement must be harmless!

1. We lift our situation to an abelian scheme. Let S be $\mathbb{A}^1_{\mathbb{Q}} \setminus \{\zeta_3, \overline{\zeta}_3\}$, and we let $\mathcal{C} \to S$ be the curve cut out by the equation $y^9 = \left(x^2+x+1\right)(x-\lambda)$ as λ varies over S; then we can normalize and complete C to produce a family of smooth projective curves $\widetilde{\mathcal{C}} \to S$. Then $\mathcal{A} \coloneqq \mathrm{Pic}^0 \mathcal{C}/S$ is an abelian scheme over S. In particular, for each $\lambda \in \mathbb{Q} \setminus \{\zeta_3, \overline{\zeta}_3\}$, we can specialize to $\lambda \in S$ to produce $A_\lambda \coloneqq \mathcal{A}_\lambda$ as the Jacobian of the curve $\widetilde{C}_\lambda \coloneqq \widetilde{\mathcal{C}}_\lambda$.

   While we're here, we set up a family of Galois representations. In order to avoid any difficult étale cohomology, we will do this cheaply using the Tate module. For each $n \geq 1$, we have a finite flat group scheme $\mathcal{A}[n] \to S$, so each $\lambda \in S(\mathbb{Q})$ gets a natural Galois-invariant pullback square as follows.

$$\begin{array}{ccc} A_\lambda[n] & \longrightarrow & \mathcal{A}[n] \\ \downarrow & & \downarrow \\ \lambda & \longrightarrow & S \end{array}$$

   Taking limits over n, we get Galois-invariant inclusions $V_\ell A \to V_\ell \mathcal{A}$, where $V_\ell \mathcal{A}$ can be interpreted as a sheaf with stalks given by $V_\ell A$. The moral of the story is that we will be able to use a special point in S in order to compute the Galois action for generic $\lambda \in S$.

2. As before, we will use Proposition 2.127 in order to compute $G_\ell(A_\lambda)$ when A_λ does not have complex multiplication. Thus, Remark 2.128 asks us to find a space W' of Tate classes cutting out $G_\ell(A)^\circ$. We may as well work with $\mathrm{MT}(A)$ by the Mumford–Tate conjecture, which is known in our case by Proposition 2.120. As before, we go ahead and add in W_1 to account for the endomorphisms of A. We also add the class of the polarization to W'. Thus, our Tate classes so far cut out $\mathrm{L}(A)$. The computation

of Proposition 3.13 tells us that $\mathrm{MT}(A)$ is $\mathrm{L}(A)$ cut out by one additional equation from the center, given by

$$\lambda_1\lambda_4\lambda_7 = \lambda_2\lambda_5\lambda_8.$$

Writing $g \in \mathrm{MT}(A) \subseteq \mathrm{L}(A)$ as $\mathrm{diag}(g_0, g_1, g_2, g_4, g_5, g_7, g_8)$ as in Proposition 3.13, we see that the above equation corresponds to the equation

$$\det g_1 g_4 g_7 = \det g_2 g_5 g_8,$$

which we see corresponds to the exceptional Tate class

$$(v_1 \wedge v_1') \otimes (v_4 \wedge v_4') \otimes (v_7 \wedge v_7') \otimes 1 \in \mathrm{H}^6_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(3),$$

where $v \wedge v' = \frac{1}{2}(v \otimes v' - v' \otimes v)$. Explicitly, the computation of Proposition 3.13 tells us that $g \in G_\ell(A)$ acts on $(v_1 \wedge v_1') \otimes (v_4 \wedge v_4') \otimes (v_7 \wedge v_7')$ by some power of the multiplier, which is then cancelled out some by the Tate twist.

3. We claim that $K_{A_\lambda}^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$ for generic λ. Our endomorphisms come from the automorphisms of the curve, which are all defined over $\mathbb{Q}(\zeta_9)$. Additionally, the polarization is certainly defined over $\mathbb{Q}(\zeta_9)$.

   It remains to handle the Galois orbit of the exceptional class given in the previous step. By the discussion at the end of the first step, it is enough to compute the Galois action at a single λ where this Tate class can be found. Well, we take $\lambda = 1$ so that we can appeal to the computations of Proposition 4.38. To explicate our basis, we will take $\{v_1, \dots, v_8\} = \{v_1, \dots, v_8\}$ and $\{v_1', \dots, v_8'\} = \{w_1, \dots, w_8\}$. Unravelling the Tate class, we see that it is a linear combination of the Tate classes given by permuting the triples in the subscript of the Tate class

   $$\nu_{315,612,342,648,378,675}.$$

   (We also need to consider $\nu_{2(315,612,342,648,378,675)}$ for the full Galois orbit, but the computation is essentially the same.) We would like to check that this Tate class is defined over $\mathbb{Q}(\zeta_9)$. Well, by Theorem 4.33, it is enough to check that the period

   $$\mathrm{Per}\left(\gamma^6, \nu_{315,612,342,648,378,675}\right)$$

   lives in $\mathbb{Q}(\zeta_9)$. After expanding the Γs, we are eventually left with some power of ζ_9 multiplied by

   $$\left((2\pi i)^{-1}\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right)\right)^3,$$

   which we see is in $\mathbb{Q}(\zeta_9)$.

4. We compute $G_\ell(A_\lambda)$ for generic λ. Above we computed that the Tate classes cutting out $G_\ell(A_\lambda)$ for generic λ are a strict subset of those needed for $\lambda = 1$, so one finds that $G_\ell(A_1) \subseteq G_\ell(\lambda)$ for generic λ. In particular, Proposition 4.38 tells us that $G_\ell(A_\lambda)$ must contain

   $$\begin{bmatrix} & -1 & & & & & & \\ 1 & & & & & & & \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & & \\ & & & -1 & & & & \\ & & & & 1 & & & \end{bmatrix},$$

   where we have reordered the basis. However, having $K_{A_\lambda}^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$ implies by Proposition 2.127 that $[G_\ell(A_\lambda) : G_\ell(A_\lambda)^\circ] = 9$, and we can see that the group generated by $G_\ell(A)^\circ$ and the above matrix also has $G_\ell(A_\lambda)^\circ$ as an index-9 subgroup. Thus, we conclude that $G_\ell(A_\lambda)$ is generated by $G_\ell(A)^\circ$ (computed in Proposition 3.13) and the above matrix.

5. We conclude that $\mathrm{ST}(A)$ equals is generated by $\mathrm{ST}(A)^\circ$ (computed in Proposition 3.13) and the matrix given in the previous step. This completes the computation. ■

4.3 Calculations of the Periods

Our calculation of the Galois action on absolute Hodge cycles above (Theorem 4.33) found that the main difficulty reduces to a computation of the periods $\mathrm{Per}\left(\gamma^{2p}, \nu_\alpha\right)$. In general, it is not an easy problem to compute the periods of a variety, even an abelian variety with complex multiplication. However, we have already put in a lot of work into being able to do this: Remark 4.29 explains that $\alpha \in \mathfrak{B}^{2p}$ will have

$$\mathrm{Per}\left(\gamma^{2p}, \nu_\alpha\right) = (2\pi i)^{-p} \prod_{i=1}^{2p} \zeta_{2N}^{[a_i]+[b_i]} \frac{\Gamma\left(\frac{[a_i]}{N}\right)\Gamma\left(\frac{[b_i]}{N}\right)}{\Gamma\left(\frac{[-c_i]}{N}\right)}.$$

It remains to compute these ratios, which comes down to being able to do arithmetic with products of Γ-functions. This is the primary goal of this section.

4.3.1 Properties of Γ

To set ourselves up for the remaining subsections, we will now prove all needed properties of the Γ-function

$$\Gamma(s) \coloneqq \int_{\mathbb{R}^+} t^s e^{-t} \frac{dt}{t}$$

from scratch. We will be rather streamlined. Our end goal is to prove the following proposition.

Proposition 4.40. The function $\Gamma(s)$ admits a meromorphic continuation to $\mathbb{C}$ with only simple poles at the nonpositive integers. Further, it satisfies the following properties.

(a) Translation: $\Gamma(s+1) = s\Gamma(s)$.

(b) Reflection: $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$.

(c) Multiplication: for any positive integer d,

$$\Gamma\left(\frac{s}{d}\right)\Gamma\left(\frac{s+1}{d}\right)\cdots\Gamma\left(\frac{s+(d-1)}{d}\right) = (2\pi)^{(d-1)/2} d^{1/2-s}\Gamma(s).$$

Among (a)–(c), only (a) admits a quick proof.

Proof of Proposition 4.40(a). Assuming that the integral form is well-defined, we find that the result holds by integration by parts.

$$\begin{aligned}
\Gamma(s+1) &= \int_{\mathbb{R}^+} t^{s+1} e^{-t} \frac{dt}{t} \\
&= -\int_{t\in\mathbb{R}^+} t^s \, d\left(e^{-t}\right) \\
&= -t^s e^{-t}\Big|_{t=0}^{t=\infty} + s\int_{\mathbb{R}^+} t^s e^{-t}\frac{dt}{t} \\
&= s\Gamma(s),
\end{aligned}$$

as required. ∎

Example 4.41. A direct integral computation shows that $\Gamma(1) = 1$, so we note that we may read the integration by parts above backwards to see that we have shown that the integral defining $\Gamma(n)$ converges and equals $(n-1)!$ for any positive integer n.

Now that we have some idea how to bound the integral defining Γ, we are able to prove the meromorphic continuation.

Proof of meromorphic continuation of Proposition 4.40. We have two steps.

1. We claim that the integral converges absolutely and uniformly on compacts in the region $\{s : \operatorname{Re} s > 0\}$, which will prove that Γ is holomorphic there. Here, we may bound the integral absolutely by
$$\int_{\mathbb{R}^+} \left|t^s e^{-t}\right| \, dt \le \int_0^1 t^{\operatorname{Re} s - 1} \, dt + \int_1^\infty t^{\lceil \operatorname{Re} s - 1 \rceil} e^{-t} \, dt.$$
The left integral equals $\frac{1}{\operatorname{Re} s - 1}$, so it converges absolutely on compacts. The right integral is bounded by $\Gamma(\lceil \operatorname{Re} s - 1 \rceil)$, which we know by Example 4.41 to converge.

2. We complete the meromorphic continuation. The equation $\Gamma(s+1) = s\Gamma(s)$ allows us to inductively holomorphically continue $\Gamma(s)$ to the region $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. This equation written as $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$ also explains that Γ admits a simple pole at $s = 0$, which can then be inductively continued to produce simple poles on the nonpositive integers. ■

Example 4.42. We compute $\Gamma(1/2)$. The proof above shows that the integral converges, so we would like to compute $\int_{\mathbb{R}^+} t^{-1/2} e^{-t} \, dt$. Taking $u = \sqrt{t}$, we see that $2\,du = t^{-1/2}\,dt$, so
$$\Gamma(1/2) = \int_{\mathbb{R}} e^{-u^2} \, du.$$
The technique of squaring the integral and passing to polar coordinates shows that the integral equals $\sqrt{\pi}$.

We now turn to the reflection formula.

Proof of Proposition 4.40(b). We will have to do some work. The following slick argument is taken from David Speyer, who credits Paul Monsky [Spe]. We will show that the function $f(s) := \Gamma(s)\Gamma(1-s)\sin \pi s$ is constant. Note that this will complete the proof because we can compute the constant is π by writing
$$f(1/2) = \Gamma(1/2)^2 \sin \frac{\pi}{2}$$
and using Example 4.42. We now proceed in steps. The idea is that the ambient 1-periodicity of f means that we only have worry about bounds on $f(x+iy)$ as $|y| \to \infty$.

1. We claim that there is a holomorphic function $g \colon \mathbb{C}^\times \to \mathbb{C}$ such that $f(s) = g\left(e^{2\pi i s}\right)$. To begin, note that $\Gamma(s)$ has simple poles at the nonpositive integers, so $\Gamma(1-s)$ has simple poles at the positive integers, so $f(s)$ is entire. Furthermore, we claim that $f(s+1) = f(s)$. By analytic continuation, it is enough to check this away from the real axis. Because the function $\sin \pi s$ satisfies $\sin \pi(s+1) = -\sin \pi s$, it is enough to compute
$$\Gamma(s+1)\Gamma(1-(s+1)) = s\Gamma(s) \cdot \frac{1}{-s}\Gamma(1-s).$$
We now turn towards defining g. The function $s \mapsto e^{2\pi i s}$ is an entire surjection $\mathbb{C}^\times \to \mathbb{C}^\times$ with non-vanishing derivative everywhere, so one can at least locally invert it. Thus, we may use a local inverse suitably composed with f to define g locally. This local definition of g however extends to a definition on all $\mathbb{C}^\times$ because $f(s+1) = f(s)$.

2. We provide some bounds on the function g. We begin with some bounds on Γ: if $x \in [0,1]$ and $|y| \geq 1$, then

$$\begin{aligned}|\Gamma(x+iy)| &= \left|\frac{1}{x+iy}\right| \cdot |\Gamma((x+1)+iy)| \\ &\leq \Gamma(x+1) \\ &\leq \max_{x\in[1,2]} \Gamma(x),\end{aligned}$$

which is absolutely bounded by some constant C. Moving to f, we see

$$|f(x+iy)| \leq C^2 e^{-\pi y}.$$

Lastly, moving to g, we see that $\left|g\left(e^{2\pi i(x+iy)}\right)\right| \leq C^2 e^{-\pi y}$. We evaluate this in two extreme cases: sending $y \to \infty$ tells us that $|g(q)| \leq C^2 |q|^{1/2}$ as $|w| \to \infty$; on the other hand, sending $y \to -\infty$ tells us that $|g(q)| \leq C^2 |q|^{-1/2}$ as $q \to 0$.

3. We complete the proof. Our goal is to show that f is constant, so it is enough to show that g is constant. It is enough to show that $g(s)$ and $g(1/s)$ both extend to holomorphic functions at $s = 0$ because this will imply that g extendeds to a bounded holomorphic function, which is constant.

   It is therefore enough to show the following lemma in complex analysis: suppose $g\colon B(0,1) \setminus \{0\} \to \mathbb{C}$ is a holomorphic function such that $|g(q)| \leq |q|^{-1/2}$ as $q \to 0$. Then we want to show that g extends to a holomorphic function at 0. Well, the function $g_1(q) := qg(q)$ continues to be holomorphic on $B(0,1) \setminus \{0\}$, but now we see that it has a removable singularity at 0 with $qg(q) \to 0$ as $q \to 0$, so g_1 admits a holomorphic continuation to $B(0,1)$ by taking $g_1(q) = 0$. We may now divide out by the zero to define $g(q)$ at $q = 0$. ∎

We now turn to the multiplication formula. This will be harder still. We will require two lemmas.

Lemma 4.43 (Stirling's approximation)**.** As $s \to \infty$, we have

$$\Gamma(s+1) \sim \left(\frac{s}{e}\right)^s \sqrt{2\pi s}.$$

Proof. The following argument is taken from [Con, Section 3]. In order to make the asymptotic terms appear, we set $x := \frac{t-s}{\sqrt{s}}$ so that

$$\begin{aligned}\Gamma(s+1) &= \int_{\mathbb{R}^+} t^s e^{-t}\, dt \\ &= \int_{-\sqrt{s}}^{\infty} (\sqrt{s}x + s)^s e^{-(\sqrt{s}x+s)} \sqrt{s}\, dx \\ &= \left(\frac{s}{e}\right)^s \sqrt{s} \underbrace{\int_{-\sqrt{s}}^{\infty} \left(1 + \frac{x}{\sqrt{s}}\right)^s e^{-\sqrt{s}x}\, dx}_{I(\sqrt{s}) :=}\end{aligned}$$

It remains to check that $I(s) \to \sqrt{2\pi}$ as $s \to \infty$. This will be done using the Dominated convergence theorem. Define $f_s\colon \mathbb{R} \to \mathbb{R}$ by $f_s(x) := \left(1 + \frac{x}{s}\right)^{s^2} e^{-sx}$ so that $I(s) = \int_{\mathbb{R}} f_s(x)\, dx$. (Here, f_s is defined to be 0 on $(-\infty, -s]$.) We have two steps.

1. We claim that $f_s(x) \to e^{-x^2/2}$ as $s \to \infty$. It is enough to check equality after taking $\log$s, so we would like to show that

$$\lim_{s\to\infty} \left(s^2 \log\left(1 + \frac{x}{s}\right) - sx\right) \stackrel{?}{=} -\frac{x^2}{2}.$$

Now, $\log\left(1+\frac{x}{s}\right) = \sum_{k\geq 1} \frac{1}{k}\left(\frac{x}{s}\right)^k$, so the Monotone convergence theorem (used for s large) gives

$$\begin{aligned}\lim_{s\to\infty}\left(s^2\log\left(1+\frac{x}{s}\right)-sx\right) &= \lim_{s\to\infty}\left(s^2\sum_{k\geq 1}\frac{1}{k}\left(\frac{x}{s}\right)^k - sx\right)\\ &= \underbrace{0}_{k=1} - \underbrace{\frac{x^2}{2}}_{k=2} + \sum_{k\geq 3}\lim_{s\to\infty}\frac{1}{k}\left(\frac{x}{s}\right)^k,\end{aligned}$$

which evaluates to $-x^2/2$, as needed.

2. We now apply the Dominated convergence theorem to see that $I(s) \to \int_{\mathbb{R}} e^{-x^2/2}\,dx$, where the integral equals $\sqrt{2\pi}$ as remarked in Example 4.42. In light of the previous step, it remains to find a dominating function for the f_ss. We will do this based on sign.

   - For $x \leq 0$, we claim that $f_s(x) \leq e^{-x^2/2}$. If $s \leq -x$, then $f_s(x) = 0$, so there is nothing to do; otherwise, we take $s > -x$. After taking logarithms, we see that we would like to check that the function
   $$s^2\log\left(1+\frac{x}{s}\right) - sx + \frac{x^2}{2}$$
   is nonpositive for $x \leq 0$. This function vanishes at $x = 0$, so it is enough to check that it is increasing, for which we note its derivative (with respect to x) is
   $$\frac{s^2}{1+\frac{x}{s}}\cdot\frac{1}{s} - s + x = \frac{x^2}{s+x},$$
   which is nonnegative for $s \geq -x$.
   - For $x \geq 0$ (and $s \geq 1$), we claim that $f_s(x) \leq f_1(x)$. After taking logarithms, we see that we would like to show that
   $$(\log(1+x) - x) - \left(s^2\log\left(1+\frac{x}{s}\right) - sx\right)$$
   is nonnegative for $x \geq 0$. This function vanishes at $x = 0$, so it is enough to check that it is increasing, for which we note its derivative (with respect to x) is
   $$\left(\frac{1}{1+x} - 1\right) - \left(\frac{s^2}{1+\frac{x}{s}}\cdot\frac{1}{s} - s\right) = \frac{x^2(s-1)}{(1+x)(s+x)},$$
   which is nonnegative for $s \geq 1$.

   Thus, we see that our dominating function may be taken to be $e^{-x^2/2}$ in the negative region and $f_1(x)$ in the positive region. ■

Lemma 4.44 (Euler form)**.** If $s > 0$, then

$$\Gamma(s) = \lim_{n\to\infty}\frac{n!n^s}{s(s+1)\cdots(s+n)}.$$

Proof. We evaluate the limit directly, using Lemma 4.43. Manipulating directly, we see the limit is

$$\Gamma(s)\lim_{n\to\infty}\frac{\Gamma(n+1)n^s}{\Gamma(s+n+1)}.$$

We now see the desired $\Gamma(s)$ term, so we want to show that the remaining limit equals 0. By Lemma 4.43 and taking logarithms, we see that we would like to show that

$$\lim_{n\to\infty}(x\log x - x + s\log x - (s+x)\log(s+x) + s + x) \stackrel{?}{=} 0.$$

After some simplification, this limit is seen to equal the limit of $s - n\log\left(1+\frac{s}{n}\right)$, which can be evaluated to 0 by expanding out the power series for $\log(1+x)$. ■

Remark 4.45. The right-hand side in fact defines a holomorphic function on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$, so the given equality extends to this region by analytic continuation. This prior claim can be checked by verifying that the right-hand side converges uniformly on compact sets in the region $\{s : \operatorname{Re} s > 0\}$ and also satisfies the equation $\Gamma(s+1) = s\Gamma(s)$.

Proof of Proposition 4.40(c). The following argument is taken from [Var]. By analytic continuation, it is enough to check the identity when s is real and positive. We simply expand out the right-hand side using the Euler form (Lemma 4.44) and Stirling's approximation (Lemma 4.43). To avoid off-by-one errors, we note that

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} \frac{n!n^s}{s(s+1)\cdots(s+n)} \\ &= \lim_{n\to\infty} \frac{n!n^{s-1}}{s(s+1)\cdots(s+n-1)} \\ &= \lim_{n\to\infty} \frac{\sqrt{2\pi}e^{-n}n^{n+s-1/2}}{s(s+1)\cdots(s+n-1)}.\end{aligned}$$

Namely, the denominator now has precisely n terms. Now,

$$\begin{aligned}\prod_{k=0}^{d-1} \Gamma\left(\frac{s+k}{d}\right) &= \lim_{n\to\infty} \prod_{k=0}^{d-1} \frac{\sqrt{2\pi}e^{-n}n^{n+(s+k)/d-1/2}}{\left(\frac{s+k}{d}\right)\left(\frac{s+k+d}{d}\right)\cdots\left(\frac{s+k+(n-1)d}{d}\right)} \\ &= \lim_{n\to\infty} \frac{(\sqrt{2\pi})^d e^{-nd} n^{nd+(ds+(0+\cdots+(d-1)))/d-d/2}}{\frac{s}{d}\left(\frac{s+1}{d}\right)\cdots\left(\frac{s+nd-1}{d}\right)} \\ &= \lim_{n\to\infty} \frac{(\sqrt{2\pi})^d e^{-nd} n^{nd+s-1/2} d^{nd}}{s\,(s+1)\cdots(s+nd-1)}.\end{aligned}$$

We would like the Euler form (Lemma 4.44) for $\Gamma(s)$ to come out of this limit, and this will be done by substituting $nd \to \infty$ into the limit for the coordinate $n \to \infty$. With this in mind, we move the strange factors from the right-hand side of the desired equality in Proposition 4.40 to the left-hand side, writing our limit as

$$(2\pi)^{-(d-1)/2} d^{s-1/2} \prod_{k=0}^{d-1} \Gamma\left(\frac{s+k}{d}\right) = \lim_{n\to\infty} \frac{\sqrt{2\pi}e^{-nd}(nd)^{nd+s-1/2}}{s\,(s+1)\cdots(s+nd-1)},$$

which is indeed the Euler form for $\Gamma(s)$. ∎

4.3.2 Unrefined Algebraicity

It will be worthwhile to give ourselves some language to describe the sorts of products we want to evalaute. A priori, we are basically computing a product which looks like

$$\prod_{i\in\mathbb{Z}} \Gamma\left(\frac{i}{N}\right)^{a_i},$$

where $\{a_i\}_{i\in\mathbb{Z}}$ is a sequence of integers arranged so that the above product is finite. (Namely, i/N should never be in $\mathbb{Z}_{\leq 0}$ if $a_i > 0$, and only finitely many of the a_i should fail to vanish.) However, by using the fact that $\Gamma(s+1) = s\Gamma(s)$, we may slide all factors of the product to $(0,1)$, meaning that we want to compute a product which looks like

$$\prod_{i=1}^{N-1} \Gamma\left(\frac{i}{N}\right)^{f(i/N)},$$

where $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ is some function.

Notation 4.46. For any function $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$, we define

$$\Gamma(f) \coloneqq \prod_{i=1}^{N-1} \Gamma\left(\frac{i}{N}\right)^{f(i/N)}.$$

Dually, for any element $a \in \mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$, we may write $a = \sum_{i=0}^{N-1} a_i \cdot \overline{i/N}$, and we define $\Gamma(a)$ according to the function $i/N \mapsto a_i$. Note that $\Gamma(a)$ does not admit a value if a is nonzero at $0/N$.

Remark 4.47. Because we are only interested in computing the periods $\operatorname{Per}\left(\gamma^{2p}, \nu_\alpha\right)$ where $\alpha \in \mathfrak{B}^{2p}$, we may restrict our view to functions $a\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ such that the weight $\langle a\rangle\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}$ is constant.

Now, Proposition 4.40 gives us two further properties about products of Γs we may use. By suitably translating, we are able to compute products which look like

$$\Gamma\left(\frac{a}{N}\right)\Gamma\left(\frac{N-a}{N}\right) \qquad \text{and} \qquad \Gamma\left(\frac{da}{N}\right)^{-1}\prod_{k=0}^{d-1}\Gamma\left(\frac{a}{N}+\frac{k}{d}\right),$$

$a, b \in \{1, \dots, N\}$, and we require $d \mid N$ and $N \nmid da$ in the second product. Here is some notation to keep track of this.

Notation 4.48. For a positive divisor d of N and $a \in \mathbb{Z}/N\mathbb{Z}$, we define the function $\varepsilon_{d,a}\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ to be the characteristic function of the set

$$\left\{\frac{N-da}{N}\right\} \cup \left\{\frac{a}{N}+\frac{k}{d} : k \in \{0, \dots, d-1\}\right\}.$$

Similarly, for any $a \in (\mathbb{Z}/N\mathbb{Z}) \setminus \{0\}$, we define $s_a\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ to be the characteristic function of $\left\{\frac{a}{N}, \frac{-a}{N}\right\}$.

Remark 4.49. Abusing notation slightly, we may identify $\varepsilon_{d,a}$ and s_a with the corresponding elements in $\mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$.

The moral is that we can compute $\Gamma(\varepsilon_{d,a})$ and $\Gamma(s_a)$, so we would like to see which functions $\frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ can be written as linear combinations of $\varepsilon_{d,a}$s and s_as. Recalling that we are only interested in functions of constant weight, we pick up the following results in this direction.

Lemma 4.50. For any positive divisor $d \mid N$ and $a \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, the weight functions $\langle\varepsilon_{d,a}\rangle$ and $\langle s_a\rangle$ are constant.

Proof. This is [Del79, Example, p. 343]. Note $s_a = \varepsilon_{1,a}$ when $a \neq 0$, so we are reduced to the considering $\varepsilon_{d,a}$s. Before doing any computation, we note that we will write $[q]$ to be the representative in $[0,1)$ of an element $q \in \mathbb{Q}/\mathbb{Z}$. We now proceed in steps.

1. For any $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, we find v such that $uv \equiv 1 \pmod N$ and compute

$$\begin{aligned}\langle\varepsilon_{d,a}\rangle(u) &= \frac{1}{N}\sum_{b\in(\mathbb{Z}/N\mathbb{Z})} 1_{\varepsilon_{d,a}}\left(\frac{ub}{N}\right)[b] \\ &= \left[-\frac{dva}{N}\right] + \sum_{k=0}^{d-1}\left[\frac{va}{N}+\frac{vk}{d}\right] \\ &= \langle\varepsilon_{d,va}\rangle(1).\end{aligned}$$

Thus, we see that we would like to show that $\langle \varepsilon_{d,a} \rangle(1) = \langle \varepsilon_{d,ua} \rangle(1)$ for any $u \in (\mathbb{Z}/N\mathbb{Z})^\times$; for example, there is nothing to show in the case where $a = 0$.

2. Setting $e \coloneqq N/d$, we note that $\varepsilon_{d,a} = \varepsilon_{d,a+e}$ pointwise. Thus, we are reduced to the case where $a \in [0, e)$ by shifting a appropriately.

3. Now, for any $q \in \mathbb{R}/\mathbb{Z} \setminus \frac{1}{d}\mathbb{Z}/\mathbb{Z}$, we define $\varepsilon_{d,q}$ as the indicator of the set $\{[-dq]\} \cup \{q + k/d : k \in \{0, 1 \ldots, d-1\}\}$. We claim that $\langle \varepsilon_{d,q} \rangle(1)$ does not depend on q, which will complete the proof in the cases where $a/N = q$ by the first step. As in the second step, we note that $\varepsilon_{d,q}$ only depends on the class of q in $\mathbb{Q}/\frac{1}{d}\mathbb{Z}$, so we may assume that $q \in (0, 1/d)$. Now, as in the first step, we compute

$$\begin{aligned} \langle \varepsilon_{d,q} \rangle(1) &= [-dq] + \sum_{k=0}^{d-1} \left[q + \frac{k}{d} \right] \\ &\overset{*}{=} (1 - dq) + \sum_{k=0}^{d-1} \left(q + \frac{k}{d} \right) \\ &= 1 + \sum_{k=0}^{d-1} \frac{k}{d}, \end{aligned}$$

which is independent of q. Here, the key equality $\overset{*}{=}$ holds notably because $q \in (0, 1/d)$. ■

Remark 4.51. In fact, the above proof shows that $\langle \varepsilon_{d,a} \rangle$ is $\frac{d+1}{2}$ when $N \nmid da$.

One also has a partial converse.

Proposition 4.52 (Koblitz–Ogus)**.** Let $f \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}$ be a function of constant weight such that $f(0) = 0$. Then f is a $\mathbb{Q}$-linear combination of the functions

$$\{\varepsilon_{d,a} : d \mid N, d \text{ is prime}, N \nmid da\} \sqcup \{s_a : N \mid a\}.$$

Proof. This is [Del79, Proposition, p. 344]. Approximately speaking, the idea is that we want to decompose f into a sum over some cosets, which is a job for Fourier analysis. Before doing anything, we set up some notation. Let E be the given set of $\varepsilon_{d,a}$s. Note that the given statement is one about some functions E in a vector space spanning the full space, which can be checked by extending scalars, so we go ahead and extend scalars to $\mathbb{C}$.

Now, for a given function $f \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{C}$ and a divisor $d \mid N$, we define $f_d \colon (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}$ by $f_d(u) \coloneqq f(u/d)$. For example, because $f(0) = 0$, we see that $f_1 = 0$. Continuing, for each function $f \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{C}$, we define $d(f)$ to be the smallest divisor of N such that $f_{d(f)}$ is nonzero, setting $d(f) = N$ if $f = 0$. Lastly, for convenience, we also define $I_d \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ (for $d \mid N$) to be the subgroup of elements $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $u \equiv 1 \pmod{d}$. Note that there is a short exact sequence

$$1 \to I_d \subseteq (\mathbb{Z}/N\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/d\mathbb{Z})^\times \to 1.$$

We now proceed in steps.

1. The general approach is to induct on $d(f)$. In particular, if $d(f) = 1$, then $f = 0$, so there is nothing to do. Thus, we may fix a divisor $d \mid N$ bigger than 1, and we would like to show that any (fixed) f of constant weight with $d(f) = d$ lives in $\operatorname{span}_{\mathbb{C}} E$, assuming this is true for any f' with $d(f') < d(f)$. As such, our goal is to find $g \in \operatorname{span}_{\mathbb{C}} E$ such that $d(f - g) < d(f)$.

   We need to do something to get ourselves off the ground, so we go ahead and specify some kinds of functions f with $d(f) = d$ for which we are already able to conclude.

(a) Suppose that f_d factors through $(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}$. Then $f_d(-a) = f_d(a)$ for each a, so we may define

$$f' \coloneqq f - \sum_{a \in (\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}} f_d(a) s_{a/d}.$$

By construction, $f'_d = 0$ while $f'_e = f_e$ for any other divisor $e \mid N$, so $d(f') < d(f)$.

(b) Suppose that f_d factors through $\left(\mathbb{Z}/\frac{d}{p}\mathbb{Z}\right)^\times$ for some prime factor $p \mid d$. Let $I_{d/p,d}$ be the kernel of the projection $(\mathbb{Z}/d\mathbb{Z})^\times \twoheadrightarrow \left(\mathbb{Z}/\frac{d}{p}\mathbb{Z}\right)^\times$ so that f_d is invariant under $I_{d/p,d}$. Now, for each $a \in (\mathbb{Z}/d\mathbb{Z})^\times$, we note that

$$\frac{a}{d} I_{d/p,d} = \left(\left\{ \frac{d-pa}{d} \right\} \cup \left\{ \frac{a}{d} + \frac{k}{p} : k \in \{0, \dots, p-1\} \right\} \right) \cap \frac{1}{d}(\mathbb{Z}/d\mathbb{Z})^\times$$

because both sides are simply the elements of the form $\frac{b}{d}$ where $b \in (\mathbb{Z}/d\mathbb{Z})^\times$ has $a \equiv b \pmod{\frac{d}{p}}$. Thus, as in (a), we may subtract out suitable multiples of $\varepsilon_{p,a}$s from f to cause f_d to vanish while not changing f_e for any $e > d$, thereby making $d(f)$ smaller.

In the remaining steps, we will show that any f_d is a linear combination of functions of the type described in (a) and (b), which completes the induction and thus the proof.

2. The aforementioned goal will be achieved via Fourier analysis. Discrete Fourier analysis allows one to write f_d as a linear combination of characters $\chi\colon (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$, writing

$$f = \sum_{\chi\colon (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times} \langle f, \chi_d \rangle \chi.$$

Because we want to show f_d is a linear combination of functions which factor through $(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}$ or $\left(\mathbb{Z}/\frac{d}{p}\mathbb{Z}\right)^\times$, we may as well show that f_d is a linear combination of even and imprimitive characters. Taking the contraposition, we must show $\langle f_d, \chi_d \rangle = 0$ for any odd primitive character $\chi_d\colon (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$.

3. Forget the context of the previous step for a sentence. Continuing with the Fourier analysis, we will show in the next step that any function $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{C}$ and any character $\widetilde{\chi}\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ has

$$\langle \langle f \rangle, \widetilde{\chi} \rangle = \sum_{\substack{d \mid N \\ \widetilde{\chi}|_{I_d} = 1}} -L(0, \chi_d) \, |I_d| \, \langle f_d, \chi_d \rangle, \tag{4.1}$$

where $\chi_d\colon (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ is the character induced from $\widetilde{\chi}$. Let's explain how this completes the proof, returning to the context of the previous step.

We apply (4.1) to our f and some character $\widetilde{\chi}\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ induced from a chosen odd primitive character $\chi_d\colon (\mathbb{Z}/d\mathbb{Z})^\times$; we want to show that $\langle f_d, \chi_d \rangle = 0$. Let's look at both sides of (4.1).

- Because $\langle f \rangle$ is constant and $\widetilde{\chi}$ is nontrivial, the left-hand side $\langle \langle f \rangle, \widetilde{\chi} \rangle$ vanishes.
- On the other hand, the right-hand side sees contributions only from divisors $e \mid N$ for which $I_e \subseteq \ker \widetilde{\chi}$. But then the image of I_e in $(\mathbb{Z}/d\mathbb{Z})^\times$ will be contained in $\ker \chi_d$, which forces $I_e \subseteq I_d$ because $\ker \chi_d$ is trivial (because χ_d is primitive). Thus, our sum only consider divisors $e \mid d$, but because $d(f) = d$, we see that $f_e = 0$ whenever $e < d$. In total, our right-hand side features only the term $-L(0, \chi_d) \, |I_d| \, \langle f_d, \chi_d \rangle$.

The above two points combine to imply $-L(0, \chi_d) \, |I_d| \, \langle f_d, \chi_d \rangle = 0$, so $\langle f_d, \chi_d \rangle = 0$ because χ_d being odd and primitive implies $L(0, \chi_d) \neq 0$. (Namely, $L(0, \chi_d) \neq 0$ by combining the functional equation for this Dirichlet L-function with the non-vanishing result Proposition 3.62.)

4. It remains to check the equality (4.1). This is a direct computation. Expanding everything out, we see

$$\langle\langle f\rangle,\widetilde{\chi}\rangle=\frac{1}{N}\sum_{\substack{u\in(\mathbb{Z}/N\mathbb{Z})^\times\\ a\in\mathbb{Z}/N\mathbb{Z}}}\frac{\langle a\rangle}{N}f\left(\frac{au}{N}\right)\overline{\chi}(u).$$

In order to make f_ds appear, we stratify the sum over a, writing

$$\langle\langle f\rangle,\widetilde{\chi}\rangle=\sum_{d|N}\frac{1}{d}\sum_{\substack{u\in(\mathbb{Z}/N\mathbb{Z})^\times\\ v\in(\mathbb{Z}/d\mathbb{Z})^\times}}\langle v\rangle f_d\left(uv\right)\overline{\chi}(u).$$

Eventually, the sum over v will turn into a term like $\langle f_d,\chi_d\rangle$, so we need to get rid of the sum over u. Let $U_d'\subseteq(\mathbb{Z}/N\mathbb{Z})^\times$ be a set of coset representatives for $(\mathbb{Z}/N\mathbb{Z})^\times/I_d$ so that $(\mathbb{Z}/N\mathbb{Z})^\times=U_d'I_d$. Then the internal sum over u looks like

$$\sum_{\substack{u'\in U_d'\\ u\in I_d}}f_d(uu'v)\overline{\chi}(uu').$$

Note $f_d(uu'v)=f_d(u'v)$, so we may sum $\overline{\chi}$ over just u alone. If $I_d\not\subseteq\ker\chi$, then this sum over u vanishes; otherwise, the sum over u is $|I_d|$, so the total sum is

$$\sum_{u'\in U_d'}f_d(u'v)\overline{\chi}(u')\,|I_d|=\chi_d(v)\,|I_d|\,\langle f_d,\chi_d\rangle.$$

Plugging this back in, we see

$$\langle\langle f\rangle,\widetilde{\chi}\rangle=\sum_{d|N}\left(\frac{1}{d}\sum_{v\in(\mathbb{Z}/d\mathbb{Z})^\times}\langle v\rangle\chi_d(v)\right)|I_d|\,\langle f_d,\chi_d\rangle.$$

The claim now follows by [Was12, Proposition 4.1, Theorem 4.2]. ∎

Corollary 4.53. Let $f\colon\frac{1}{N}\mathbb{Z}/\mathbb{Z}\to\mathbb{Z}$ be a function of constant weight w. Then

$$\pi^{-w}\Gamma(f)\in\overline{\mathbb{Q}}.$$

Proof. By adding or subtracting 1_0s (which have weight 0), we may assume that $f(0)=0$. The hypothesis and conclusion are $\mathbb{Q}$-linear in f (note that fractional powers are permitted in an algebraicity question), so Proposition 4.52 tells us that it is enough to check the result for f being one of the $\varepsilon_{d,a}$s in the statement; recall from Remark 4.51 that $\langle\varepsilon_{d,a}\rangle=\frac{d+1}{2}$.

In fact, for any divisor $d\mid N$ and choice of $a\in\mathbb{Z}/N\mathbb{Z}$ with $N\nmid da$, we claim that $\pi^{-w}\Gamma(\varepsilon_{d,a})\in\overline{\mathbb{Q}}^\times$, where $w=\frac{d+1}{2}$ is the weight. Indeed, by combining the reflection and multiplication formulae (Proposition 4.40), we see that $\Gamma(\varepsilon_{d,a})$ is

$$\Gamma\left(\frac{N-da}{N}\right)\prod_{k=0}^{d-1}\Gamma\left(\frac{a}{N}+\frac{k}{d}\right)\equiv\pi^{(d+1)/2}\pmod{\overline{\mathbb{Q}}^\times},$$

so the result follows. ∎

4.3.3 The Universal Distribution

This section follows [Kub79b]. We are now permitted to make the following definition.

Definition 4.54 (distribution)**.** A *distribution relation* is an element of $\mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$ of the form

$$\overline{a} - \sum_{\substack{b \in \frac{1}{N}\mathbb{Z}/\mathbb{Z} \\ db = a}} \overline{b},$$

where $d \mid N$ is a positive divisor. A *distrubtion* is a function $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to A$ to an abelian group A whose natural extension to $\mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$ vanishes on all distribution relations. A distribution is *odd* if and only if it also satisfies $f(-a/N) = -f(a/N)$ for all a.

Example 4.55 (universal)**.** Let U_N be the abelian group given by taking the quotient of $\mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$ by the subgroup generated by the distribution relations. Then there is a natural inclusion $i\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to U_N$, which we see is a distribution by construction. In fact, we see that every distribution $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to A$ factors uniquely through i, so i is initial in the category of distributions.

Example 4.56. By Proposition 4.40, the function $\frac{1}{\sqrt{2\pi}}\Gamma\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{C}/\overline{\mathbb{Q}}$ is an odd distribution. Namely, this function descends to $\mathbb{Q}/\mathbb{Z}$ by the translation property, it is a distribution by the multiplication formula, and it is odd by the reflection formula.

Example 4.56 explains why we are discussing distributions in this section: products of Γs can be tracked through as satisfying these distribution relations. We also remark that integer-valued functions of constant weight 0 live a new life here.

Lemma 4.57. Let $D_N^- \subseteq \mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$ be the $\mathbb{Z}$-module generated by the distribution relations and the elements $\overline{a} + \overline{-a}$ and $\overline{0}$. After identifying $\mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$ with functions $\frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$, we see D_N^- is generated by the elements $\overline{0}$ and $\varepsilon_{d,a}$ where $d \mid N$ is a divisor and $a \in (\mathbb{Z}/N\mathbb{Z})^\bullet$.

Proof. For nonzero a, note that $\varepsilon_{1,a}$ is simply the generator $\overline{a/N} + \overline{-a/N}$, and $\varepsilon_{d,a}$ produces the distribution relation

$$-\varepsilon_{d,a} + \varepsilon_{1,da} = \overline{\frac{da}{N}} - \sum_{k=0}^{d-1} \overline{\frac{a}{N} + \frac{k}{d}}.$$

Thus, up to adding or subtracting some $\varepsilon_{1,\bullet}$, we see that the distribution relations are in bijection with the $\varepsilon_{d,a}$s, so these elements generate the same subgroup of $\mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right]$. ∎

Remark 4.58. It is not hard to see that one may inductively write $\varepsilon_{d,a}$s as a $\mathbb{Z}$-linear combination of $\varepsilon_{p,a}$s where p is a prime. (For that matter, one can inductively write distribution relations in terms of ones where the divisor $d \mid N$ is prime.) The point is that we really only have to consider $\varepsilon_{p,a}$s (with p prime) and $\varepsilon_{1,a}$s in Lemma 4.57.

The goal of the present subsection is to show the following structure result [Kub79b, Theorem 1.8].

Theorem 4.59 (Kubert)**.** Let $i\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to U_N$ be an initial distribution. Then U_N is a free abelian group of rank $\varphi(N)$.

Proof from Propositions 4.61 and 4.63. We will go ahead and outline the argument, referring forward to results we will prove in the sequel. There are two main steps.

1. In Proposition 4.61, we show that any distribution f has $\langle \operatorname{im} f \rangle$ admitting a generating set of $\varphi(N)$ elements.

2. In Proposition 4.63, we exhibit a distribution r with $\dim_{\mathbb{Q}} \langle \operatorname{im} r \rangle_{\mathbb{Q}} = \varphi(N)$.

Let's quickly explain why these two implications allow us to conclude the proof. By the first step, we see that there is a surjection $\mathbb{Z}^{\varphi(N)} \twoheadrightarrow U_N$ of abelian groups, and we will be done as soon as we know that this map is an isomorphism. Well, because i is an initial distribution, we see that the distribution r factors through i, meaning that there is an induced surjection

$$\mathbb{Z}^{\varphi(N)} \twoheadrightarrow U_N \twoheadrightarrow \langle \operatorname{im} r \rangle.$$

However, this composite must become an isomorphism after tensoring with $\mathbb{Q}$ (for dimension reasons) by the second step, so the composite must in particular be injective. We conclude that the map $\mathbb{Z}^{\varphi(N)} \twoheadrightarrow U_N$ is an isomorphism. ■

It remains to provide the proofs of Propositions 4.61 and 4.63. Before going further, we need some notation.

Notation 4.60. By the Chinese remainder theorem, summation provides an isomorphism

$$\sum_{p \mid N} \frac{1}{p^{\nu_p(N)}} \mathbb{Z}/\mathbb{Z} \to \frac{1}{N}\mathbb{Z}/\mathbb{Z}.$$

For any $s \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ and $p \mid N$, we define $s_p \in \frac{1}{p^{\nu_p(N)}}\mathbb{Z}/\mathbb{Z}$ to be the corresponding p-component. Similarly, if we have $\frac{a}{N} \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, we let $\frac{a_p}{p^{\nu_p(N)}}$ be the p-component.

Because it is faster, we now proceed with Proposition 4.63.

Proposition 4.61. Let $f \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to A$ be a distribution. Then $\langle \operatorname{im} f \rangle$ admits a generating set of $\varphi(N)$ elements.

Proof. This result is [Kub79b, Proposition 1.8], though we follow the isomorphic proof given in [Was12, Proposition 12.10]. The idea is to use the distribution relations to minimize the number of generators. There are two steps.

1. We claim that the collection

$$S_N \coloneqq \left\{ f\left(\frac{a}{N}\right) : a_p = 0 \text{ or } \gcd(a_p, p) = 1 \right\}$$

generates $\langle \operatorname{im} f \rangle$. We proceed by induction on the number of primes factors of N, where the statement has little content if $N = 1$.

Now, for a given N, choose some $a/N \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, and we want to show that $f(a/N) \in \langle S_N \rangle$. Quickly, if $a_p = 0$ for some prime $p \mid N$, then in fact $a/N \in \frac{1}{d}\mathbb{Z}/\mathbb{Z}$ for some divisor $d \mid N$ with strictly fewer prime factors, so $f(a/N) \in \langle S_d \rangle$ by the induction.

Thus, we may assume that $a_p \neq 0$ for all $p \mid N$. In this case, we hope to use a distribution relation to find $f(a/N)$ in $\langle S_N \rangle$. In particular, note that we can write $a = dx$ where $d \mid N$ and $\gcd(x, N) = 1$: indeed, simply write $\frac{a}{N}$ in reduced terms as $\frac{x}{e}$, and then $a = \frac{N}{e} \cdot x$ is a suitable expansion. (In particular, e is the order of a, so $p \mid e$ for all primes e, so $\gcd(x, e) = 1$ implies $\gcd(x, N) = 1$.) Thus, $f(a/N)$ equals

$$f\left(d \cdot \frac{x}{N}\right) = \sum_{k=0}^{d-1} f\left(\frac{x}{N} + \frac{k}{d}\right),$$

and now every term in the right-hand side lives in S_N.

2. We claim that the collection

$$T_N \coloneqq \left\{ f\left(\frac{a}{N}\right) : a_p = 0, \text{ or } a_p \neq 1 \text{ and } \gcd(a_p, p) = 1 \right\}$$

generates $\langle \operatorname{im} f \rangle$. Once again, we proceed by induction on the number of prime factors of N, where the statement has little content if $N = 1$. Note that the previous step tells us that it is enough to check that $S_N \subseteq \langle T_N \rangle$.

As such, we go ahead and pick up some $f(a/N) \in S_N$, and to show that $f(a/N) \in \langle T_N \rangle$. As in the prior step, we note that having $a_p = 0$ for any prime p implies that $f(a/N) \in S_d$ for some $d \mid N$ with fewer prime factors, yielding $f(a/N) \in \langle T_N \rangle$ by the induction. Thus, we may assume that $a_p \neq 0$ for all p.

We will induct on the number $\omega(a/N)$ of primes p such that $a_p = 1$. Of course, if $\omega(a/N) = 0$, then $a/N \in T_N$ already, so there is nothing to do. Otherwise, suppose that our a/N has at least one prime $q \mid N$ with $a_q = 1$. We now use the distribution relations twice: set

$$\frac{b}{M} \coloneqq \sum_{\substack{p \mid N \\ p \neq q}} \frac{a_p}{p^{\nu_p(N)}},$$

and then we note that we have two equalities

$$f\left(q^{\nu_q(N)} \cdot \frac{b}{M}\right) = \sum_{k=0}^{q^{\nu_q(N)}-1} f\left(\frac{b}{M} + \frac{k}{q^{\nu_q(N)}}\right),$$

$$f\left(q^{\nu_q(N)-1} \cdot \frac{b}{M}\right) = \sum_{k=0}^{q^{\nu_q(N)-1}-1} f\left(\frac{b}{M} + \frac{qk}{q^{\nu_q(N)}}\right).$$

Both left-hand sides are in $\langle T_N \rangle$ by the induction on the number of prime factors. Now, subtracting these two equations produces the relation

$$\sum_{k \in (\mathbb{Z}/q^{\nu_q(N)})^\times} f\left(\frac{b}{M} + \frac{k}{q^{\nu_q(N)}}\right) \in \langle T_N \rangle.$$

Note $\frac{a}{N} = \frac{b}{M} + \frac{1}{q^{\nu_q(N)}}$ is the first term in this sum while the other terms in the sum have strictly smaller ω (because the q-component is not equal to 1), so we are done by the induction.

Note that the second step completes the proof because $\#S_N$ equals

$$\prod_{p \mid N} \# \left(\{0\} \cup \left(\mathbb{Z}/p^{\nu_p(N)}\mathbb{Z}\right)^\times \setminus \{1\} \right),$$

which is simply $\#(\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N)$ by the Chinese remainder theorem. ∎

Remark 4.62. The proof of Proposition 4.61 actually gives explicit generators of $\operatorname{im} f$. One can unwind this (and the proof of Theorem 4.59) to give explicit generators of U_N defined in Example 4.55.

We now turn to the construction for Proposition 4.63.

Proposition 4.63. There exists a distribution $r \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to A$ such that $\dim_{\mathbb{Q}} \langle \operatorname{im} r \rangle_{\mathbb{Q}} = \varphi(N)$.

Proof. To understand why this is difficult, we note that we are basically trying to compute the dimension of the vector space $U_{N,\mathbb{Q}}$, where U_N is the rather horrendous abelian group constructed Example 4.55. Technically, Proposition 4.61 tells us what should be a basis, but this vector space has so many relations that

it is difficult to determine if these elements are actually linearly independent. The usual proof (in [Was12, Chapter 12] or [Kub79b, Section 3]) uses cyclotomy theory and some facts about character sums, reducing the task to a non-vanishing of some special value. These topics are moderately tangential to this thesis, so we will not discuss them. Instead, we will follow [Kub79b, Section 4] and provide a direct combinatorial construction.

Our target space will be $A_N \coloneqq \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^\times]$, and we note that $(\mathbb{Z}/N\mathbb{Z})^\times$ has a natural permutation action on A_N. Throughout, ord denotes the additive order of a group element. We require two elements of A_N.

- For $s \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, we define
$$X_N(s) \coloneqq \sum_{\substack{x \in (\mathbb{Z}/N\mathbb{Z})^\times \\ x \cdot N/\operatorname{ord} s \equiv Ns}} \overline{x}.$$
For example, if $\operatorname{ord} s = N$, then $X_N(s) = \{Ns\}$. In general, if $s = a/d$ where $d = \operatorname{ord} s$ so that $\gcd(a,d) = 1$, then the xs take the form $(a + kd)$.

- For prime divisors $p \mid N$, we define
$$Y_N(p) \coloneqq \sum_{\substack{y \in (\mathbb{Z}/N\mathbb{Z})^\times \\ py \equiv 1 \pmod{N/p^{\nu_p(N)}}}} \overline{y}.$$
Notably, the value of $y \in (\mathbb{Z}/N\mathbb{Z})^\times$ only has freedom in the p-component, so $Y_N(p)$ has $\varphi\left(p^{\nu_p(N)}\right)$ elements.

Because $X_N(s)$ and $Y_N(p)$ are basically subsets of $(\mathbb{Z}/N\mathbb{Z})^\times$, we may write $\#X_N(s)$ or $\#Y_N(p)$ to mean the number of their elements. We are now ready to define $r_N \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^\times]$ by

$$r_N(s) \coloneqq \frac{X_N(s)}{\varphi(N)} \prod_{p \mid \operatorname{ord} s} \left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right).$$

It remains to run many checks on r_N. They are all some explicit combinatorial manipulations.

1. For $c \in (\mathbb{Z}/N\mathbb{Z})^\times$, we check that $r_N(cs) = cr_N(s)$. Note that $cX_N(s) = X_N(cs)$ because both contain the x such that $cxN/\operatorname{ord} s \equiv Ns$. It now suffices to check that
$$c\left(\left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right) \cdot X\right) \stackrel{?}{=} \left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right) \cdot cX$$
for any prime divisor $p \mid N$ and $X \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^\times]$. Well, it is enough to check this claim for $X \in (\mathbb{Z}/N\mathbb{Z})^\times$, whereupon doing some rearragnement shows that it is enough to check that $c(Y_N(p)X) = Y_N(p)(cX)$, which is true by definition of the $(\mathbb{Z}/N\mathbb{Z})^\times$-action on A_N.

2. For any divisor $M \mid N$, we claim that the diagram
$$\begin{array}{ccc} \frac{1}{M}\mathbb{Z}/\mathbb{Z} & \xrightarrow{r_M} & \mathbb{Q}[(\mathbb{Z}/M\mathbb{Z})^\times] \\ \cap\mathrel{|} & & \downarrow i \\ \frac{1}{N}\mathbb{Z}/\mathbb{Z} & \xrightarrow{r_N} & \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^\times] \end{array}$$
commutes, where i is given by $i(\overline{y}) = \frac{\varphi(M)}{\varphi(N)} \sum_{x \equiv y \pmod M} \overline{x}$ for any $y \in (\mathbb{Z}/M\mathbb{Z})^\times$. Note that i is injective and $\mathbb{Q}$-linear by construction, but it is not a ring map because it does not map $1 \mapsto 1$. However, the leading constant is chosen to make i multiplicative: for $\overline{y}_1, \overline{y}_2 \in (\mathbb{Z}/M\mathbb{Z})^\times$, we see $i(\overline{y}_1)i(\overline{y}_2)$ equals
$$\left(\frac{\varphi(M)}{\varphi(N)}\right)^2 \sum_{\substack{x_1 \equiv y_1 \pmod M \\ x_2 \equiv y_2 \pmod M}} \overline{x}_1\overline{x}_2 = \left(\frac{\varphi(M)}{\varphi(N)}\right)^2 \sum_{\substack{x \equiv y_1 y_2 \pmod M \\ x' \equiv 1 \pmod M}} \overline{x},$$

where we have substituted $(x, x') = (x_1x_2, x_1/x_2)$. We conclude $i(\overline{y}_1)i(\overline{y}_2) = i(\overline{y}_1\overline{y}_2)$, an equation which extends $\mathbb{Q}$-linearly to all A_N.

The main computation will be to compute $i(r_M(s))$ for $s \in \frac{1}{M}\mathbb{Z}/\mathbb{Z}$. Using the multiplicativity of the previous paragraph, we see

$$i(r_M(s)) = \frac{i(X_M(s))}{\varphi(M)} \prod_{p|\operatorname{ord} s} i\left(1 - \frac{Y_M(p)}{\#Y_M(p)}\right).$$

We see that we have to compute $i(X_M(s))$ and $i(Y_M(p))$.

- Note $\frac{\varphi(N)}{\varphi(M)} i(X_M(s)) = X_N(s)$: some $x \in (\mathbb{Z}/N\mathbb{Z})^\times$ finds itself in $\frac{\varphi(N)}{\varphi(M)} i(X_M(s))$ if and only if $x \cdot M/\operatorname{ord} s \equiv Ms$, which is equivalent to $x \cdot N/\operatorname{ord} s \equiv Ns$.
- We claim $\frac{i(Y_M(p))}{\#Y_M(p)} = i(1)\frac{Y_N(p)}{\#Y_N(p)}$. Note that the reduction map $Y_N(p) \to Y_M(p)$ is surjective: any y with $py \equiv 1 \pmod{M/p^{M/\nu_p(M)}}$ may be lifted to a multiplicative inverse of $p \pmod{N/p^{N/\mu_p(N)}}$. We thus see that the support of $\frac{\varphi(N)}{\varphi(M)} i(1) Y_M(p)$ agrees with the support of $\frac{\varphi(N)}{\varphi(M)} i(Y_M(p))$; however, each element in $\frac{\varphi(N)}{\varphi(M)} i(Y_M(p))$ is overcounted by a factor of $\varphi\left(p^{\nu_p(N)}\right)/\varphi\left(p^{\nu_p(M)}\right)$ because we already had freedom in the p-component. Adjusting for this completes the claim.

We now see

$$i(r_M(s)) = \frac{i(X_M(s))}{\varphi(M)} \prod_{p|\operatorname{ord} s} i(1)\left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right).$$

To get rid of the factor of $i(1)$, we note that $i(X_M(s))i(1) = i(X_M(s))$ by the multiplicativity. Lastly, we may substitute $\frac{i(X_M(s))}{\varphi(M)} = \frac{X_N(s)}{\varphi(N)}$, writing

$$i(r_M(s)) = \frac{X_N(s)}{\varphi(N)} \prod_{p|\operatorname{ord} s} \left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right),$$

which is indeed $r_N(s)$.

3. We claim that r_N is a distribution. Namely, for any divisor $d \mid N$ and $s \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, we must check hat

$$r_N(ds) \stackrel{?}{=} \sum_{k=0}^{d-1} r_N\left(s + \frac{k}{d}\right).$$

We begin with a few reductions. By adjusting s by some k/d, we may assume that $\operatorname{ord} s$ is divisible by d. By inductively applying the distribution relations, we may assume that d is prime. Lastly, because i defined in the previous step is injective, we can pass from r_N to $r_{\operatorname{ord} s}$, allowing us to assume that $\operatorname{ord} s = N$. We now have two cases for the prime divisor d of N.

- Suppose that $d^2 \mid N$. In this case, all primes dividing $\operatorname{ord} s = N$ continue to divide $\operatorname{ord} ds = N/d$. Additionally, $s + \frac{k}{d}$ always has order N, so

$$\sum_{k=0}^{d-1} r_N\left(s + \frac{k}{d}\right) = \left(\frac{1}{\varphi(N)} \sum_{k=0}^{d-1} X_N\left(s + \frac{k}{d}\right)\right) \prod_{p|N}\left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right).$$

Because $s + \frac{k}{d}$ has order N, we see $X_N\left(s + \frac{k}{d}\right) = \overline{N(s + \frac{k}{d})}$. On the other hand, $X_N(ds)$ consists of the x for which $dx \equiv d(Ns)$, which is equivalent to having $x = N(s + \frac{k}{d})$. We conclude

$$\sum_{k=0}^{d-1} r_N\left(s + \frac{k}{d}\right) = \frac{X_N(ds)}{\varphi(N)} \prod_{p|N}\left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right),$$

which is $r_N(ds)$.

- Suppose $d \mid N$ while $d^2 \nmid N$. The same computation essentially goes through except for two caveats: $\operatorname{ord} ds = N/d$ has one fewer prime factor, and $s + \frac{k}{d}$ need not have order N. In particular, the p-component of $s + \frac{k}{d}$ is the same as the same p-component of s, so the order of $s + \frac{k}{d}$ is either N or N/d. Further, and we see that it will be N/d only when $s + \frac{k}{d}$ has d-component equal to 0 for exactly when k; say that $t = s + \frac{k_0}{d}$ is this value of k. Then

$$\sum_{k=0}^{d-1} r_N\left(s + \frac{k}{d}\right) = \frac{1}{\varphi(N)}\left(X_N(t) + \sum_{k=1}^{d-1} X_N\left(t + \frac{k}{d}\right)\left(1 - \frac{Y_N(d)}{\#Y_N(d)}\right)\right) \prod_{p \mid N/d}\left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right).$$

  Comparing this to $r_N(ds) = r_N(dt)$, we see that we have left to show

$$X_N(dt) \stackrel{?}{=} X_N(t) + \sum_{k=1}^{d-1} X_N\left(t + \frac{k}{d}\right)\left(1 - \frac{Y_N(d)}{\#Y_N(d)}\right),$$

  which is equivalent to

$$X_N(dt) + \frac{1}{\#Y_N(d)} \sum_{k=1}^{d-1} X_N\left(t + \frac{k}{d}\right) Y_N(d) \stackrel{?}{=} X_N(t) + \sum_{k=1}^{d-1} X_N\left(t + \frac{k}{d}\right).$$

  We now must compute the various X_Ns.

  - Each $t + \frac{k}{d}$ has order N by construction of t, so $X_N\left(t + \frac{k}{d}\right) = \overline{N(t + \frac{k}{d})}$. As such, multiplying by $Y_N(d)$ will leave us with $x \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $x \equiv \frac{N}{d}(t + \frac{k}{d}) \pmod{N/d}$, which is equivalent to $x \equiv \frac{N}{d} t \pmod{N/d}$; in particular, the sum on the left-hand side counts all these elements $\#Y_N(d) = (d-1)$ times. On the other hand, $X_N(t)$ consists of the x for which $dx \equiv Nt$, which is equivalent to $x \equiv \frac{N}{d} t \pmod{N/d}$, so

$$\frac{1}{\#Y_N(d)} \sum_{k=1}^{d-1} X_N\left(t + \frac{k}{d}\right) = X_N(t).$$

  - Similarly, dt has order N/d, so $X_N(dt)$ consists of the $x \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $dx \equiv Ndt$. Well, this is equivalent to having $x \equiv \overline{N(t + \frac{k}{d})}$, so

$$X_N(dt) = \sum_{k=1}^{d-1} X_N\left(t + \frac{k}{d}\right).$$

  Combining the above two points completes the computation.

4. We begin computing $\langle \operatorname{im} r_N \rangle_{\mathbb{Q}}$. For each prime $p \mid N$, define the fractional ideal

$$U_p \coloneqq X_N\left(\frac{p^{\nu_p(N)}}{N}\right) \mathbb{Z}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right] + \left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right) \mathbb{Z}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right].$$

   We claim that $\langle \operatorname{im} r_N \rangle$ equals $\prod_{p \mid N} U_p$. Because r_N respects the $(\mathbb{Z}/N\mathbb{Z})^\times$-action, it is enough to check that $\langle \operatorname{im} r_N \rangle$ is given by generators of this ideal. Well, a generic generator of $\prod_{p \mid N} U_p$ looks like

$$\prod_{p \nmid M} X_N\left(\frac{p^{\nu_p(N)}}{N}\right) \prod_{p \mid M}\left(1 - \frac{Y_N(p)}{\#Y_N(p)}\right),$$

   where M is some divisor of N; in fact, we may as well assume $\nu_p(M) \in \{0, \nu_p(N)\}$ for all primes p. We claim that the above element is $\varphi(N) r_N(1/M)$; this claim completes this step. To show the claim, we note the right product is already seen in $r_N(1/M)$. It thus remains to show that

$$\prod_{p \nmid M} X_N\left(\frac{p^{\nu_p(N)}}{N}\right) \stackrel{?}{=} X_N\left(\frac{1}{M}\right).$$

Indeed, the left-hand side is made of products $\prod_{p\nmid M} x_p$ where $x_p \cdot p^{\nu_p(N)} \equiv p^{\nu_p(N)} \pmod N$, which is equivalent to a condition on $x_p \equiv 1 \pmod{N/p^{\nu_p(N)}}$. By the Chinese remainder theorem, such products are in bijection with xs such that $x \equiv 1 \pmod{N/M}$, which is $X_N(1/M)$.

5. We claim that $(U_p)_{\mathbb{Q}} = \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^\times]$. Because $(\#Y_N(p) - Y_N(p)) \in U_p$, it is enough to check that $Y_N(p) \in U_p$. In fact, we claim that $Y_N(p)$ is a multiple of $X_N\left(p^{\nu_p(N)}/N\right)$, which will complete the proof. Well, $Y_N(p)$ has x such that $px \equiv 1 \pmod{N/p^{\nu_p(N)}}$, and $X_N\left(p^{\nu_p(N)}/N\right)$ has x such that $x \equiv 1 \pmod{N/p^{\nu_p(N)}}$ as discussed in the previous step. Thus, we see

$$p' X_N\left(\frac{p^{\nu_p(N)}}{N}\right) = Y_N(p),$$

where $p' \in (\mathbb{Z}/N\mathbb{Z})^\times$ is chosen so that $p' \equiv p \pmod{N/p^{\nu_p(N)}}$, and the claim follows.

Thus, we have checked that r_N is a distribution, and the last two steps check that $\langle \operatorname{im} r_N\rangle_{\mathbb{Q}} = A_N$, so $\dim\langle \operatorname{im} r_N\rangle_{\mathbb{Q}} = \varphi(N)$ follows. ∎

4.3.4 Cohomology of the Universal Distribution

Let $i\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to U_N$ be the initial distribution of Example 4.55, and further let U_N^- be the quotient of U_N by the elements $\langle \overline{a} + \overline{-a}\rangle_{a\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}}$. The quotient U_N^- is of interest to us because Γ factors through U_N^- by combining the reflection formula (Proposition 4.40) with Example 4.56.

We are now ready to state the main result of this subsection.

Theorem 4.64. Let $i\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to U_N$ be the initial distribution of Example 4.55, and further let U_N^- be the quotient of U_N by the elements $\langle \overline{a} + \overline{-a}\rangle_{a\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}}$.

(a) The torsion subgroup $U_{N,\mathrm{tors}}^-$ is 2-torsion.

(b) If N is odd or divisible by 4, then $\dim_{\mathbb{F}_2} U_{N,\mathrm{tors}}^- = 2^{\omega(N)-1}$, where $\omega(N)$ is the number of distinct prime factors of N.

Proof from Propositions 4.65 and 4.67. As in the previous subsection, we go ahead and outline the argument, referring forward to results we will prove in the sequel. There are two steps: in Proposition 4.65, we show that $U_{N,\mathrm{tors}}^-$ is isomorphic to the cohomology group $\mathrm{H}^2(\langle\pm1\rangle, U_N)$, thereby proving (a). The dimension computation for this cohomology group is carried out in Proposition 4.67. ∎

We now turn our attention to the proofs of Proposition 4.65 and Proposition 4.67.

Proposition 4.65. Let $i\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to U_N$ be the initial distribution of Example 4.55. Further, let U_N^- be the quotient by the elements $\langle \overline{a} + \overline{-a}\rangle_{a\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}}$. Then the torsion subgroup of U_N^- is isomorphic to

$$\mathrm{H}^2(\langle\pm1\rangle, U_N).$$

Proof. This is an application of Theorem 4.59. We follow [GGL24, Proposition 6.3.3]. Note that the action of $\langle\pm1\rangle \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ on $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ extends to U_N. We will actually show that $U_{N,\mathrm{tors}}^-$ is isomorphic to the Tate cohomology group

$$\mathrm{H}_{\mathrm{T}}^0(\langle\pm1\rangle, U_N) = \frac{U_N^{\langle\pm1\rangle}}{\mathrm{N}_{\langle\pm1\rangle}(U_N)},$$

which is enough because the group cohomology of a cyclic group is 2-periodic. We have two inclusions.

- On one hand, the denominator of $\mathrm{H}_{\mathrm{T}}^0(\langle\pm1\rangle, U_N)$ is basically modding out by the elements $\overline{a} + \overline{-a}$. Thus, we have an inclusion $\mathrm{H}_{\mathrm{T}}^0(\langle\pm1\rangle, U_N) \subseteq U_N^-$, so $\mathrm{H}_{\mathrm{T}}^0(\langle\pm1\rangle, U_N) \subseteq U_{N,\mathrm{tors}}^-$ because Tate cohomology groups are torsion.

- On the other hand, choose some $f \in U_{N,\mathrm{tors}}^-$, and we would like to check that $f \in U_N^{\langle \pm 1 \rangle}$. Well, we are given that there is some $D > 0$ such that Df vanishes in U_N^-, so $Df = (\overline{1} + \overline{-1})g$ (in U_N) for some $g \in U_N$. However, this implies that $(\overline{1} - \overline{-1})Df = 0$ in U_N, which requires $(\overline{1} - \overline{-1})f = 0$ because U_N is torsion-free by Theorem 4.59! We conclude that $f \in U_N^{\langle \pm 1 \rangle}$. ∎

Before proceeding with the long proof of Proposition 4.67, we pick up a group-theoretic lemma.

Lemma 4.66. Fix finite abelian groups G and H. If M is a free $\mathbb{Z}[G \times H]$-module, then M^H and M/M^H are both free $\mathbb{Z}[G]$-modules.

Proof. Because M is a module over $G \times H$, we see that M^H is still a G-module. Quickly, note that M is a sum of $\mathbb{Z}[G \times H]$s, so because taking $(\cdot)^H$ and the quotient are both additive functors, it suffices to check the result for $M = \mathbb{Z}[G \times H]$. We now show that M^H and M/M^H are free independently.

- We show that M^H is a free $\mathbb{Z}[G]$-module. Indeed, some element $\sum_{(g,h)} a_{(g,h)}(g,h)$ is H-invariant if and only if $a_{(g,h)} = a_{(g,h')}$ always, in which case we see that

$$\sum_{(g,h) \in G \times H} a_{(g,h)}(g,h) = \sum_{g \in G} \left(a_{(g,1)}(g,1) \sum_{h \in H} (1,h) \right).$$

  Thus, we see that the map $\mathbb{Z}[G] \to \mathbb{Z}[G \times H]^H$ given by multiplying by $\sum_h (1,h)$ is an isomorphism.

- We show that M/M^H is a free $\mathbb{Z}[G]$-module. Quickly, observe that $\mathbb{Z}[G \times H]$ is free over $\mathbb{Z}[G]$ with a basis given by $\{(1,h)\}_{h \in H}$, so we may apply a linear transformation to see that $\mathbb{Z}[G \times H]$ is free over $\mathbb{Z}[G]$ with basis instead given by

$$\left\{ \sum_h (1,h) \right\} \sqcup \{(1,h)\}_{h \neq 1}.$$

  The first element is a basis of $\mathbb{Z}[G \times H]^H$ over $\mathbb{Z}[G]$ by the previous point, so we see that the quotient is free over $\mathbb{Z}[G]$ with basis given by the remaining entries. ∎

Proposition 4.67 (Kubert)**.** Fix a positive integer N which is odd or divisible by 4, and let $i \colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to U_N$ be the initial distribution of Example 4.55. Then

$$\dim_{\mathbb{F}_2} \mathrm{H}_{\mathrm{T}}^{\bullet}(\langle \pm 1 \rangle, U_N) = 2^{\omega(N)-1},$$

where $\omega(N)$ is the number of distinct prime factors of N.

Proof. Our argument follows [Kub79a, Section 2]. We continue with the set-up of Proposition 4.63, but we drop all the subscript Ns because we will work with fixed N throughout. Thus, we may also set $\nu_p \coloneqq \nu_p(N)$ for each prime p. In particular, by the universal property (and as outlined in Theorem 4.59), we see that U_N is isomorphic to the image of induced map $r \colon \mathbb{Z}\left[\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right] \to \mathbb{Q}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$.

We will need a few other pieces of notation. For bookkeeping reasons, we say that a divisor $M \mid N$ is admissible if and only if $\nu_p(M) \in \{0, \nu_p\}$ for all primes p; roughly speaking, M keeps track of a subset of primes dividing N. For example, for each admissible divisor $M \mid N$, we define

$$U(M) \coloneqq \prod_{p \mid M} U_p,$$

where U_p is the ideal defined at the end of the proof of Proposition 4.63; for example, $U(1) = \mathbb{Z}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$ and $U(N) = \operatorname{im} r$. In short, $U(M)$s will allow us to make certain inductive arguments.

Continuing, for each admissible divisor $M \mid N$, we define the subgroup $C(M) \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ by

$$C(M) \coloneqq \{a \in (\mathbb{Z}/N\mathbb{Z})^\times : a \equiv 1 \pmod{N/M}\}.$$

Thus, $C(M) \cong \prod_{p|M} (\mathbb{Z}/p^{\nu_p}\mathbb{Z})^\times$ is isomorphic to $(\mathbb{Z}/M\mathbb{Z})^\times$. For example, $C(1) = (\mathbb{Z}/N\mathbb{Z})^\times$ and $C(N) = \{1\}$. We also remark that the sum $Y(p)$ is fixed by $C(p^{\nu_p})$ by contruction (in fact, the set admits a transitive action), and a quick expansion of the definitions reveals that $X\left(p^{\nu_p}/N\right) = C(p^{\nu_p})$. As usual, we may identify $C(p^{\nu_p})$ with an element of $\mathbb{Z}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$ given by $\sum_{a \in C(p^{\nu_p})} \overline{a}$.

In the end, we will show that

$$\dim_{\mathbb{F}_2} \mathrm{H}^\bullet_{\mathrm{T}}\left(\langle \pm 1 \rangle, U(M)^{C(N/M)}\right) \stackrel{?}{=} 2^{\omega(M)-1}$$

for any admissible divisor $M \mid N$ bigger than 1, via an induction; taking $M = N$ then produces the desired result. Our proof now proceeds in many steps. We remark that our first few steps are picking up some technica tools used later.

1. Let $\varepsilon_p \in \mathbb{Q}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$ be the idempotent $\frac{1}{\#C(p^{\nu_p})} \sum_{a \in C(p^{\nu_p})} \overline{a}$. Then we claim that $x \in \mathbb{Q}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$ is fixed by $C(p^{\nu_p})$ if and only if $(1-\varepsilon_p)x = 0$. This is some abstract group theory. In one direction, if x is fixed by $C(p^{\nu_p})$, then

$$\frac{1}{\#C(p)} \sum_{a \in C(p^{\nu_p})} ax = \frac{1}{\#C(p^{\nu_p})} \sum_{a \in C(p^{\nu_p})} x$$

is simply x. In the other direction, if $(1-\varepsilon_p)x = 0$, then $x = \varepsilon_p x$; however, $a\varepsilon_p = \varepsilon_p$ for all $a \in C(p^{\nu_p})$ by a rearrangement of the terms in ε_p, so we see that $\varepsilon_p x$ is certainly fixed by $C(p^{\nu_p})$.

2. For an admissible divisor $M \mid N$ and prime $p \mid (N/M)$, we claim that $(1-\varepsilon_p)U\left(Mp^{\nu_p}\right) = (1-\varepsilon_p)U_M$ and

$$U\left(Mp^{\nu_p}\right)^{C(p^{\nu_p})} \stackrel{?}{=} C(p^{\nu_p}) \cdot U(M) + \left(1 - \frac{Y(p)}{\#Y(p)}\right) U(M)^{C(p^{\nu_p})},$$

   where $C(p)$ refers to the element $\sum_{a \in C(p)} \overline{a}$ by abuse of notation.

   For this, we note $U\left(Mp^{\nu_p}\right) = U_p U(M)$ by definition, so

$$U\left(Mp^{\nu_p}\right) = C\left(p^{\nu_p}\right) U(M) + \left(1 - \frac{Y(p)}{\#Y(p)}\right) U(M).$$

   The first and the second claimed equalities are linked by the previous step, which tells us that we are interested in the kernel of $(1-\varepsilon_p)$. As such, let's look at how $(1-\varepsilon_p)$ behaves on each term.

   - Certainly $(1-\varepsilon_p)$ vanishes on $C\left(p^{\nu_p}\right)$, so the left term above lives in the kernel.
   - Similarly, $Y(p)$ is fixed by $C\left(p^{\nu_p}\right)$, so it is in the kernel of $(1-\varepsilon_p)$, from which one sees $(1-\varepsilon_p)\left(1 - \frac{Y(p)}{\#Y(p)}\right) = (1-\varepsilon_p)$. For example, we see that multiplying by $(1-\varepsilon_p)$ kills the coefficient $\left(1 - \frac{Y(p)}{\#Y(p)}\right)$. Additionally, the kernel of $(1-\varepsilon_p)$ will simply be the kernel of $(1-\varepsilon_p)$ acting on $U(M)$.

   Combining these two points completes the proof.

3. Suppose that M and M' are admissible divisors of N such that $MM' \mid N$. Then we claim that $U(M)$ is free as a $C(M')$-module; further, if $MM' \neq N$, we claim that $U(M)$ is free as a $\pm C(M')$-module.

   For this, we induct on M. If $M = 1$, then $U(M)$ is free over all subgroups of $(\mathbb{Z}/N\mathbb{Z})^\times$, so there is nothing to do. Thus, we focus on the inductive step, so suppose that the statement is true for M, and we would like to show it for Mp^{ν_p} for some prime $p \mid (N/M)$. Then the previous step provides short exact sequences as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U(M)^{C(p^{\nu_p})} & \longrightarrow & U(M) & \longrightarrow & (1-\varepsilon_p)U(M) & \longrightarrow & 0 \\
 & & & & & & \| & & \\
0 & \longrightarrow & U(Mp^{\nu_p})^{C(p^{\nu_p})} & \longrightarrow & U(Mp^{\nu_p}) & \longrightarrow & (1-\varepsilon_p)U(Mp^{\nu_p}) & \longrightarrow & 0
\end{array}$$

The main claim is that $U(M)^{C(p^{\nu_p})} = U(Mp^{\nu_p})^{C(p^{\nu_p})}$. Let's quickly explain why this claim will complete this step. Fix an admissible divisor $M' \mid (N/Mp^{\nu_p})$, and then there are two things to show.

- We would like to show that $U(Mp^{\nu_p})$ is free over $C(M')$. By the inductive hypothesis, we know that $U(M)$ is free over $C(M')$ and $C(p^{\nu_p})$ and even over the product of the two groups. Thus, Lemma 4.66 tells us that $U(M)^{C(p^{\nu_p})} = U(Mp^{\nu_p})^{C(p^{\nu_p})}$ and $(1-\varepsilon_p)U(M) = (1-\varepsilon_p)U(Mp^{\nu_p})$ are both free over $C(M')$. Because the right term of the bottom short exact sequence is free, we conclude that the bottom short exact sequence thus splits, forcing $U(Mp^{\nu_p})$ to be a sum of free $C(M')$-modules and hence free.
- Suppose $MM'p^{\nu_p} \neq N$. Then we would like to show that $U(Mp^{\nu_p})$ is free over $\pm C(M')$. In this case, $U(M)$ is free over $\pm C(p^{\nu_p})C(M')$ by the induction, but $\pm C(M') \cap C(p^{\nu_p})$ is trivial: any element a in the intersection has $a \equiv 1 \pmod{N/p^{\nu_p}}$ and $a \equiv \pm 1 \pmod{N/M'}$, but the $+1$ is forced by having $N/(M'p^{\nu_p})$ be bigger than 2 by the hypotheses on N. Thus, $U(M)$ is actually free over $\pm C(M') \times C(p^{\nu_p})$, and now the argument can proceed as in the previous step.

It remains to prove the main claim $U(M)^{C(p^{\nu_p})} = U(Mp^{\nu_p})^{C(p^{\nu_p})}$. Well, the previous step grants

$$U\left(Mp^{\nu_p}\right)^{C(p^{\nu_p})} = C(p^{\nu_p}) \cdot U(M) + \left(1 - \frac{Y(p)}{\#Y(p)}\right) U(M)^{C(p^{\nu_p})}.$$

By the inductive hypothesis, $U(M)$ is free over $C(p^{\nu_p})$, so $U(M)^{C(p^{\nu_p})} = C(p^{\nu_p}) \cdot U(M)$. Thus, it is enough to show that $\left(1 - \frac{Y(p)}{\#Y(p)}\right) U(M)^{C(p^{\nu_p})} \subseteq U(M)^{C(p^{\nu_p})}$. Well, $Y(p)$ is stable under $C(p^{\nu_p})$, so we can express $\left(1 - \frac{Y(p)}{\#Y(p)}\right) \cdot C(p^{\nu_p})$ as $(1-y) \cdot C(p^{\nu_p})$ for some $y \in Y(p)$, and the result follows because $U(M)$ is a fractional ideal for $\mathbb{Z}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$.

4. As a last tool, we show that the induced action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathrm{H}^\bullet\left(\langle \pm 1\rangle, U(M)^{C(N/M)}\right)$ is trivial for any admissible divisor $M \mid N$. In fact, it's enough to check that the action of $C(p^{\nu_p})$ is trivial because these subgroups generate $(\mathbb{Z}/N\mathbb{Z})^\times$. Now, note that $U(M)^{C(N/M)}$ automatically has trivial action by $C(p^{\nu_p})$ if $p \nmid M$, so we now focus on the case $p \mid M$.

   Well, for a given $a \in C(p^{\nu_p})$, we would like to show that the action of a is trivial, for which it is enough to show that the action of $(1-a)$ is zero. Well, we claim that multiplication by $(1-a)$ factors through $\mathrm{H}^\bullet_{\mathrm{T}}\left(\langle \pm 1\rangle, U(M/p^{\nu_p})^{C(N/M)}\right)$, which we note vanishes because $U(M/p^{\nu_p})^{C(N/M)}$ is free over $\langle \pm 1\rangle$ by the previous step!

   Now, to show that multiplication by $(1-a)$ factors as claimed, it is enough by functoriality to show that multiplication by $(1-a)$ on $U(M)$ factors through $U(M/p^{\nu_p})$. Well, we see

$$U(M) = C\left(p^{\nu_p}\right) U(M/p^{\nu_p}) + \left(1 - \frac{Y(p)}{\#Y(p)}\right) U(M/p^{\nu_p}).$$

   Note $(1-a)C\left(p^{\nu_p}\right) = 0$, and $(1-a)\left(1 - \frac{Y(p)}{\#Y(p)}\right) = (1-a)$ because $Y(p)$ is fixed by a. Thus, $(1-a)U(M) \subseteq U(M/p^{\nu_p})$, as needed.

5. If M and Mp^{ν_p} are admissible divisors of N, we claim that there is a short exact sequence

$$0 \to \mathrm{H}^\bullet_{\mathrm{T}}\left(\langle \pm 1\rangle, U(M)^{C(N/M)}\right) \to \mathrm{H}^\bullet_{\mathrm{T}}\left(\langle \pm 1\rangle, U(Mp^{\nu_p})^{C(N/Mp^{\nu_p})}\right) \to \mathrm{H}^{\bullet+1}_{\mathrm{T}}\left(\langle \pm 1\rangle, U(M)^{C(N/M)}\right) \to 0.$$

   Note that we will have to do something nontrivial (beyond immediately applying a long exact sequence) because the middle group has a different invariant subgroup C acting on it. Our extra input will come from the morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U(M)^{C(p^{\nu_p})} & \longrightarrow & U(M) & \xrightarrow{(1-\varepsilon_p)} & (1-\varepsilon_p)U(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (1-\frac{Y(p)}{\#Y(p)})} & & \downarrow{\scriptstyle (1-\frac{Y(p)}{\#Y(p)})} & & \downarrow & & \\
0 & \longrightarrow & U(Mp^{\nu_p})^{C(p^{\nu_p})} & \longrightarrow & U(Mp^{\nu_p}) & \xrightarrow{(1-\varepsilon_p)} & (1-\varepsilon_p)U(Mp^{\nu_p}) & \longrightarrow & 0
\end{array}$$

of exact sequences discussed in the third step; note that the left arrow is well-defined by the second step, and right arrow is then induced by the diagram. Before continuing, we make a few simplifiactions to this diagram.

- In the third step, we showed that $U(M)^{C(p^{\nu_p})} = U(Mp^{\nu_p})^{C(p^{\nu_p})}$.
- Recall that $Y(p)$ is fixed by $C(p^{\nu_p})$, so $(1-\varepsilon_p)\left(1-\frac{Y(p)}{\#Y(p)}\right) = (1-\varepsilon_p)$, thereby implying that the right arrow is simply the identity. We no longer care about the exact content of this right-hand term, so we denote it by $K := (1-\varepsilon_p)U(M) = (1-\varepsilon_p)U(Mp^{\nu_p})$.
- Using the fact that the set $Y(p)$ has a transitive action by $C(p^{\nu_p})$, we see that multiplying an element of $U(M)^{C(p^{\nu_p})}$ by $Y(p)$ is the same as multiplying it by any other element. Thus, we go ahead and fix some element $y_p \in Y(p)$, and we see that the left arrow is simply multiplication by $(1-y_p)$.

Our diagram now looks like the following.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(M)^{C(p^{\nu_p})} & \longrightarrow & U(M) & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (1-y_p)} & & \downarrow{\scriptstyle \left(1-\frac{Y(p)}{\#Y(p)}\right)} & & \| & & \\ 0 & \longrightarrow & U(M)^{C(p^{\nu_p})} & \longrightarrow & U(Mp^{\nu_p}) & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

We now take $C(N/Mp^{\nu_p})$-invariants and $\langle\pm1\rangle$-cohomology to recover the result. Taking $C(N/Mp^{\nu_p})$-invariants keeps the exactness because $U(M)^{C(p^{\nu_p})}$ is free over $C(N/Mp^{\nu_p})$ by using Lemma 4.66 and the result in step 3. Thus, our diagram looks like the following.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(M)^{C(N/M)} & \longrightarrow & U(M)^{C(N/Mp^{\nu_p})} & \longrightarrow & K' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (1-y_p)} & & \downarrow{\scriptstyle \left(1-\frac{Y(p)}{\#Y(p)}\right)} & & \| & & \\ 0 & \longrightarrow & U(M)^{C(N/M)} & \longrightarrow & U(Mp^{\nu_p})^{C(N/Mp^{\nu_p})} & \longrightarrow & K' & \longrightarrow & 0 \end{array}$$

Here, K' is the induced quotient, which we continue to not care about. We now take $\langle\pm1\rangle$-cohomology. For brevity, we will set $\mathrm{H}^\bullet_{\mathrm{T}}(M') := \mathrm{H}^\bullet_{\mathrm{T}}\left(\langle\pm1\rangle, U(M')^{C(N/M')}\right)$ for any admissible divisor $M' \mid N$.

$$\begin{array}{ccccccccccc} \mathrm{H}^{\bullet-1}_{\mathrm{T}}(\langle\pm1\rangle, K') & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{T}}(M) & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{T}}(\langle\pm1\rangle, K') & \longrightarrow & \mathrm{H}^{\bullet+1}_{\mathrm{T}}(M) & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle (1-y_p)} & & \downarrow & & \| & & \downarrow{\scriptstyle (1-y_p)} & & \\ \mathrm{H}^{\bullet-1}_{\mathrm{T}}(\langle\pm1\rangle, K') & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{T}}(M) & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{T}}(Mp^{\nu_p}) & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{T}}(\langle\pm1\rangle, K') & \longrightarrow & \mathrm{H}^{\bullet+1}_{\mathrm{T}}(M) & & \end{array}$$

Here, the 0s arise because $U(M)^{C(N/Mp^{\nu_p})}$ is free over $\langle\pm1\rangle$ by the third step. We now make a few simplications.

- The 0s in the top row imply that $\mathrm{H}^\bullet_{\mathrm{T}}(\langle\pm1\rangle, K') \to \mathrm{H}^{\bullet+1}_{\mathrm{T}}(M)$ is an isomorphism.
- By the previous step, we know that the $(1-y_p)$ arrows are the 0 map. Thus, the commutativity of the diagram implies that the arrows $\mathrm{H}^{\bullet-1}_{\mathrm{T}}(\langle\pm1\rangle, K') \to \mathrm{H}^\bullet_{\mathrm{T}}(M)$ and $\mathrm{H}^\bullet_{\mathrm{T}}(\langle\pm1\rangle, K') \to \mathrm{H}^{\bullet+1}_{\mathrm{T}}(M)$ in the bottom row are both the zero map.

The above two observations turns the bottom row into

$$0 \to \mathrm{H}^\bullet_{\mathrm{T}}(M) \to \mathrm{H}^\bullet_{\mathrm{T}}(Mp^{\nu_p}) \to \mathrm{H}^{\bullet+1}_{\mathrm{T}}(M) \to 0.$$

6. We now complete the proof by induction. We want to compute $\dim_{\mathbb{F}_2} \mathrm{H}^\bullet_{\mathrm{T}}\left(\langle\pm1\rangle, U(M)^{C(N/M)}\right)$ for any admissible divisor $M \mid N$. The previous step grants an "inductive step" that

$$\dim_{\mathbb{F}_2} \mathrm{H}^\bullet_{\mathrm{T}}\left(\langle\pm1\rangle, U(Mp^{\nu_p})^{C(N/Mp^{\nu_p})}\right) = \sum_{i\in\{0,1\}} \dim_{\mathbb{F}_2} \mathrm{H}^i_{\mathrm{T}}\left(\langle\pm1\rangle, U(M)^{C(N/M)}\right)$$

whenever M and Mp^{ν_p} is an admissible divisor of N. For example, we find that

$$\dim_{\mathbb{F}_2} \mathrm{H}_{\mathrm{T}}^{\bullet}\left(\langle\pm1\rangle, U(p^{\nu_p})^{C(N/p^{\nu_p})}\right) = \sum_{i\in\{0,1\}} \dim_{\mathbb{F}_2} \mathrm{H}_{\mathrm{T}}^{i}\left(\langle\pm1\rangle, U(1)^{C(N)}\right)$$

by taking $M = 1$. But now this dimension is independent of the cohomological index, so we inductively see that

$$\dim_{\mathbb{F}_2} \mathrm{H}_{\mathrm{T}}^{\bullet}\left(\langle\pm1\rangle, U(M)^{C(N/M)}\right) = 2^{\omega(M)-1} \sum_{i\in\{0,1\}} \dim_{\mathbb{F}_2} \mathrm{H}_{\mathrm{T}}^{i}\left(\langle\pm1\rangle, U(1)^{C(N)}\right)$$

for any admissible divisor $M \mid N$ such that $M > 1$.

The proof will be over as soon as we check

$$\sum_{i\in\{0,1\}} \dim_{\mathbb{F}_2} \mathrm{H}_{\mathrm{T}}^{i}\left(\langle\pm1\rangle, U(1)^{C(N)}\right) = 1.$$

Well, note $U(1) = \mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times]$, so the $C(N)$-fixed points are given by $C(N)\cdot\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times] = \mathbb{Z}C(N)$. This has trivial action by $\langle\pm1\rangle$, so we are computing the Tate cohomology of the trivial $\langle\pm1\rangle$-module $\mathbb{Z}$. Well, one has

$$\begin{cases} \mathrm{H}_{\mathrm{T}}^{0}(\langle\pm1\rangle, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \\ \mathrm{H}_{\mathrm{T}}^{-1}(\langle\pm1\rangle, \mathbb{Z}) = 0, \end{cases}$$

so we see that the sum of the $\mathbb{F}_2$-dimensions is in fact 1. ∎

Remark 4.68. Choose admissible divisors $M \mid M'$. The fifth step of the argument shows that there is an inclusion $U(M)^{C(N/M)} \subseteq U(M')^{C(N/M')}$ which then induces an inclusion

$$\mathrm{H}_{\mathrm{T}}^{\bullet}\left(\langle\pm1\rangle, U(M)^{C(N/M)}\right) \to \mathrm{H}_{\mathrm{T}}^{\bullet}\left(\langle\pm1\rangle, U(M')^{C(N/M')}\right)$$

on (Tate) cohomology. As seen in the fifth step of the argument, these inclusions explain "half" of the elements of a given $\mathrm{H}_{\mathrm{T}}^{\bullet}\left(U(M')^{C(N/M')}\right)$ by taking $M \mid M'$ to be M'/p^{ν_p} for some prime $p \mid M'$. The "other half" arises from a quotient and is thus harder to describe.

Example 4.69. Let's exhibit a nontrivial element in $\mathrm{H}_{\mathrm{T}}^{0}(\langle\pm1\rangle, U_N)$. Remark 4.68 explains that there is an inclusion $\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times]^{(\mathbb{Z}/N\mathbb{Z})^\times} \subseteq U_N$ which induces an inclusion

$$\mathrm{H}_{\mathrm{T}}^{0}\left(\langle\pm1\rangle, \mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times]^{(\mathbb{Z}/N\mathbb{Z})^\times}\right) \subseteq \mathrm{H}_{\mathrm{T}}^{0}(\langle\pm1\rangle, U_N).$$

Now, $\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times]^{(\mathbb{Z}/N\mathbb{Z})^\times} \subseteq U_N$ is isomorphic to $\mathbb{Z}$ generated by $\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \overline{a}$. Because this module has the trivial $\langle\pm1\rangle$-action, we see that this generating element $\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \overline{a}$ provides a nontrivial class in $\mathrm{H}_{\mathrm{T}}^{0}(\langle\pm1\rangle, U_N)$.

4.3.5 Refined Algebraicity

The previous subsections (and in particular Theorem 4.64) allows us to upgrade Proposition 4.52.

Lemma 4.70. Let $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ be a function of constant weight. Then $2f$ is a $\mathbb{Z}$-linear combination of the functions 1_0 and $\varepsilon_{d,a}$ where $N \nmid da$.

Proof. This is [GGL24, Proposition 6.3.6]. By adding or subtracting 1_0s (which have weight 0), we may assume that $f(0) = 0$. By Proposition 4.52, we know that there is some denominator $D > 0$ such that Df is a $\mathbb{Z}$-linear combination of the functions 1_0 and $\varepsilon_{d,a}$ where $N \nmid da$, and we can see that there are no 1_0s because $f(0) = 0$. Thus, Lemma 4.57 tells us that Df (up to 1_0) vanishes in the group U_N^- described in Theorem 4.64. This group is actually 2-torsion by Theorem 4.64, so we conclude that $2f$ vanishes in U_N^-. Another application of Lemma 4.57 tells us that $2f$ is a $\mathbb{Z}$-linear combination of the $\varepsilon_{d,a}$s. ∎

Remark 4.71. In fact, once we know that $2f$ is a $\mathbb{Z}$-linear combination of 1_0 and the $\varepsilon_{d,a}$s, one can use some linear algebra to explicitly find this linear combination. We take a moment to note that Remark 4.58 tells us that we are allowed to only use $\varepsilon_{1,a}$s and $\varepsilon_{p,a}$s where $p \mid N$ is prime.

Here is the appliation to products of Γ.

Lemma 4.72. Let K_N be the extension of $\mathbb{Q}(\zeta_{2N}, i)$ generated by the elements $\pi^{-w}\Gamma(f)$, where $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ is a function of constant weight w which is a $\mathbb{Z}$-linear combination of the $\varepsilon_{d,a}$s. Then

$$K_N = \mathbb{Q}(i, \zeta_{2N})\left(\{p^{p/N} : \text{prime } p \mid N\}\right).$$

Proof. It is enough to handle f which are equal to some $\varepsilon_{d,a}$. One can inductively write $\varepsilon_{d,a}$ as a sum of $\varepsilon_{1,\bullet}$s and $\varepsilon_{p,\bullet}$s, so we can just handle those. By the reflection formula (Proposition 4.40), $\Gamma(\varepsilon_{1,a})$ is in $\mathbb{Q}(\zeta_{2N}, i)$, so we don't have to worry about these elements.

Continuing, by the multipliation formula (Proposition 4.40), we see

$$\frac{\Gamma(\varepsilon_{p,a})}{\Gamma(\varepsilon_{1,pa})} = (2\pi)^{(p-1)/2} p^{1/2-pa/N}.$$

We now have two cases on the parity of p.

- If p is odd, then these elements show $p^{1/2-pa/N} \in K_N$. However, $p^{1/2} \in \mathbb{Q}(i, \zeta_{2N})$ already, so we are only generating $p^{p/N} \in K_N$.
- Similarly, if $p = 2$, then these elements show $2^{1/2} \cdot 2^{1/2-2/N} \in K_N$. Thus, we are again only generating $2^{2/N} \in K_N$. ∎

Proposition 4.73. Let L_N be the extension of

$$K_N = \mathbb{Q}(i, \zeta_{2N})\left(\{p^{p/N} : \text{prime } p \mid N\}\right)$$

generated by the elements $\pi^{-w}\Gamma(f)$, where $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ is a function of constant weight w. Then the extension L_N/K_N is multiquadratic. If N is odd or divisible by 4, the degree is bounded by

$$\log_2[L_N : K_N] \le 2^{\omega(N)-1} - 1.$$

Proof. We proceed in steps, showing the various claims separately.

1. To check that this extension is multiquadratic, we will actually check that $(\pi^{-w}\Gamma(f))^2$ is in K_N for each f; note that $\pi^{-w}\Gamma(f)$ is already algebraic by Corollary 4.53. Now, by Lemma 4.70, we may write $2f$ as a $\mathbb{Z}$-linear combination of $\varepsilon_{d,a}$s, so $\Gamma(f)^2$ can be written as a product of $\Gamma(\varepsilon_{d,a})$s. But up to a power π, Lemma 4.72 assures us that $\Gamma(\varepsilon_{d,a})$ is in K_N.

2. It remains to bound $[L_N : K_N]$ when N is odd. By the previous step and Kummer theory [Lan02, Theorem VI.8.1], we would like to show that the 2-subgroup $\Gamma_N \subseteq K_N^\times / K_N^{\times 2}$ generated by the elements $(\pi^{-w}\Gamma(f))^2$ has
$$\dim_{\mathbb{F}_2} \Gamma_N \overset{?}{\leq} 2^{\omega(N)-1} - 1.$$
This bound will come from Theorem 4.64. To be formal, let $\varphi\colon \operatorname{Mor}_{\mathrm{cw}}(\frac{1}{N}\mathbb{Z}/\mathbb{Z}, \mathbb{Z}) \to K_N^\times$ be the homomorphism taking functions $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ of constant weight w to $(\pi^{-w}\Gamma(f))^2 \in K_N^\times$. By construction, we see that this homomorphism sends elements of the form $\varepsilon_{d,a}$ to $K_N^{\times 2}$, as discussed in Lemma 4.72. Thus, Lemma 4.57 tells us that φ descends to a homomorphism
$$\overline{\varphi}\colon U_{N,\mathrm{tors}}^- \to \frac{K_N^\times}{K_N^{\times 2}},$$
and Γ_N is the image of this map. Theorem 4.64 explains that the domain $U_{N,\mathrm{tors}}^-$ is already a 2-torsion group and has $\mathbb{F}_2$-dimension bounded by $2^{\omega(N)-1}$, so we will be done if we can lower the dimension any further.

3. We complete the proof by showing that $\overline{\varphi}$ has a nontrivial kernel. Indeed, consider the constant function $f_1 \equiv 1$. We have two checks.

   - On one hand, we claim that $f_1 \in \ker \varphi$. Then
   $$f_1 = 1_0 + 1_{2|N} 1_{1/2} \sum_{a=1}^{\lfloor (N-1)/2 \rfloor} \varepsilon_{1,a},$$
   where 1_0 and $1_{1/2}$ are indicators. Certainly 1_0 and $1_{1/2}$ are in $\ker \overline{\varphi}$ because $\Gamma(1_0) = \Gamma(1_{1/2}) = 1$ (see Example 4.42), and the $\varepsilon_{1,a}$s are in $\ker \overline{\varphi}$ as already noted. We conclude $f_1 \in \ker \varphi$.
   - On the other hand, we claim that f_1 is a nontrivial element of $U_{N,\mathrm{tors}}^-$. This is a little tricky. Under the isomorphism $U_{N,\mathrm{tors}}^- \cong \mathrm{H}_{\mathrm{T}}^0(\langle \pm 1 \rangle, U_N)$ of Proposition 4.65, f_1 corresponds to the (Tate) cohomology class
   $$\sum_{a \in (\mathbb{Z}/N\mathbb{Z})} \overline{a} \in \mathrm{H}_{\mathrm{T}}^0(\langle \pm 1 \rangle, U_N).$$
   However, this element is nontrivial by Example 4.69!

   We conclude that $\ker \overline{\varphi}$ is nontrivial, so $\Gamma_N = \operatorname{im} \overline{\varphi}$ satisfies
   $$\dim_{\mathbb{F}_2} \operatorname{im} \overline{\varphi} < \dim_{\mathbb{F}_2} U_{N,\mathrm{tors}}^-,$$
   so we are done by Theorem 4.64. ∎

Remark 4.74. The first step of the proof has the pleasant consequence of providing an explicit algorithm to compute the algebraic numbers $\pi^{-w}\Gamma(f)$, as discussed in [GGL24, Theorem 6.3.9]. Indeed, it suffices to compute the square $\pi^{-2w}\Gamma(2f)$. Now, Remark 4.71 says that we can use linear algebra to write $2f$ as a $\mathbb{Z}$-linear combination of some $\varepsilon_{d,a}$s, and then we can compute $\Gamma(\varepsilon_{d,a})$ using the reflection and multiplication formulae of Proposition 4.40 (as explained in Corollary 4.53).

Remark 4.75. Whether equality is achieved in Proposition 4.73 is an interesting question. It seems to be true in small examples; see Remark 4.78.

We now apply our theory to periods of the Fermat curve. To begin, we note that periods of Fermat curves can handle fairly general functions of constant weight.

Lemma 4.76. Let $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ be a function of constant weight w such that $f(0) = 0$. Then there is some index $p \geq 0$ and $\alpha \in \mathfrak{B}^{2p}$ and a list $\{a_i\}_{i=1}^n \subseteq (\mathbb{Z}/N\mathbb{Z})$ such that

$$f = 1_\alpha - \sum_{i=1}^n \varepsilon_{1,a_i}.$$

Proof. We will actually show that there is a list $\{a_i\}_{i=1}^n$ such that $f + \sum_{i=1}^n \varepsilon_{1,a_i}$ equals 1_α for some $\alpha \in \mathfrak{B}^{2p}$. In fact, it is enough to get $\alpha \in \mathfrak{A}^{2p}$: we already know that $f + \sum_{i=1}^n \varepsilon_{1,a_i}$ has constant weight by Lemma 4.50, so the weight will correctly be $3p$ as soon as this is some suitably $\alpha \in \mathfrak{A}^{2p}$ by Remark 4.20. As a last reduction, we note that we may assume $\operatorname{im} f \subseteq \mathbb{Z}_{\geq 0}$ by adding in suitable $\varepsilon_{1,\bullet}$s.

We now induct on $\|f\|_1 = \sum_{i=0}^n f(i/N)$. Here are some small cases.

- If $\|f\|_1 = 0$, then $f = 0$, and we can take $p = 0$ and α to be empty.
- It is not possible for f to be supported on a single nonzero entry because such a function cannot have constant weight.
- Suppose $\|f\|_1 = 2$. Because f should not be supported at a single point, we have $f = \overline{a/N} + \overline{b/N}$ for some $a, b \in \mathbb{Z}/N\mathbb{Z}$. We claim that $f = \varepsilon_{1,a}$. Well, f needs to have constant weight, so

  $$[a] + [b] = [-a] + [-b].$$

  Thus, $[a] + [b] = N$, so $b = -a$, as required.

We now proceed with the induction. Suppose that $\|f\|_1 > 2$. Because f is nonzero, f is supported on at least two points, which we name a/N and b/N where $a, b \in (\mathbb{Z}/N\mathbb{Z})$. We have two cases.

- Suppose that $b = -a$. Then $f - \varepsilon_{1,a}$ continues to have nonnegative image and constant weight, but $\|f - \varepsilon_{1,a}\|_1 < \|f\|_1$, so we may apply the inductive hypothesis to $f - \varepsilon_{1,a}$ to conclude the proof.
- Suppose that $b \neq -a$. Then there is a nonzero $c \in (\mathbb{Z}/N\mathbb{Z})$ such that $a + b + c = 0$, and we define $\alpha := (a, b, c)$ to be in $\mathfrak{A}^1$. We now see that

  $$f - 1_\alpha + \varepsilon_{1,c}$$

  has nonnegative image and constant weight, but $\|f - 1_\alpha + \varepsilon_{1,c}\|_1 < \|f\|_1$. We now again conclude by applying the inductive hypothesis. ∎

Theorem 4.77. Let K_A^{conn} be the connected monodromy field of the Jacobian A of the Fermat curve X_N, and define the field

$$K_N = \mathbb{Q}(i, \zeta_{2N})\left(\{p^{p/N} : \text{prime } p \mid N\}\right).$$

(a) We have $K_N \subseteq K_A^{\mathrm{conn}}(i, \zeta_{2N})$.

(b) The extension $K_A^{\mathrm{conn}}(i, \zeta_{2N})/K_N$ is multiquadratic.

(c) If N is odd or divisible by 4, then

$$\log_2[K_A^{\mathrm{conn}}(i, \zeta_{2N}) : K_N] \leq 2^{\omega(N)-1} - 1.$$

Proof. As explaind in Remark 4.36, K_A^{conn} is the extension of $\mathbb{Q}(\zeta_N)$ which contains the periods

$$\operatorname{Per}\left(\gamma^{2p}, \nu_\alpha\right) = (2\pi i)^{-p} \prod_{i=1}^{2p} \zeta_{2N}^{[a_i]+[b_i]} \frac{\Gamma\left(\frac{[a_i]}{N}\right)\Gamma\left(\frac{[b_i]}{N}\right)}{\Gamma\left(\frac{[-c_i]}{N}\right)},$$

where $\alpha \in \mathfrak{B}^{2p}$ varies. By the reflection formula (Proposition 4.40), this period is in $\pi^{-\langle\alpha\rangle}\Gamma(1_\alpha)\mathbb{Q}(i,\zeta_{2N})$. Now, Lemma 4.76 explains that any function $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ of constant weight can be transformed into some 1_α for $\alpha \in \mathfrak{B}^{2p}$ at merely the cost of some 1_0s and $\varepsilon_{1,a}$s, so $K_A^{\mathrm{conn}}(i,\zeta_{2N})$ is actually generated by $\pi^{-w}\Gamma(f)$, where f may now vary over all functions $f\colon \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}$ of some constant weight w. Part (a) now follows from Lemma 4.72, and parts (b) and (c) now follow from Proposition 4.73. ■

Remark 4.78. Let A be the Jacobian of the curve $y^2 = x^N - 1$, which is a quotient of the Fermat curve X_N. In [GGL24, Theorem 7.1.1], it is shown that K_A^{conn} is multiquadratic over merely $\mathbb{Q}(\zeta_N)$ via some algebro-geometric arguments. If N is odd, then Theorem 4.77 shows that

$$\log_2[K_A^{\mathrm{conn}}(i) : \mathbb{Q}(i,\zeta_N)] \le 2^{\omega(N)-1} - 1.$$

In particular, note that the $p^{p/N}$s define odd-degree cyclic extensions of $\mathbb{Q}(i,\zeta_N)$ and hence cannot live in the multiquadratic extension K_A^{conn} of $\mathbb{Q}(\zeta_N)$. The above bound agrees with the table in [GGL24, Example 6.4.10]; in fact, that table suggests that equality may hold without the added is!

Let's see an example computation.

Proposition 4.79. Define A to be the Jacobian of the proper curve C with affine chart $y^9 = x\left(x^2+1\right)$. Then we show $K_A^{\mathrm{conn}} = \mathbb{Q}\left(\zeta_N, 2^{1/3}, 2^{2/9}\cdot 3^{1/6}\right)$.

Proof. This computation follows the one in Proposition 4.38. We will freely use the computation executed in Proposition 3.17. Thoughout, $A \coloneqq \operatorname{Jac} C$, and we recall that we have a decomposition $A = C_0 \times A_1 \times A_2$ (over $\mathbb{Q}$) into geometrically simple abelian varieties. We proceed in steps.

1. Set $N \coloneqq 18$, and we note that there is a quotient map $X_N \to C$ from the affine patch $x^{18}+y^{18}+1 = 0$ to C given by $\psi(x,y) \coloneqq \left(x^9, xy^2\right)$. Thus, we will be able to use the Galois-invariant embedding $\psi\colon \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X_{N,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$ to use Theorem 4.33 by restricting to the Galois submodule. To make this explicit, we recall that we have a basis

$$\left\{\frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8}\right\}$$

of $\mathrm{H}^{10}(C)$, we see that we can pass this basis through ψ^* to see that $\mathrm{H}^{10}(C) \subseteq \mathrm{H}^{10}(X)$ has basis

$$\left\{\nu_{5,10,3}, \nu_{4,8,6}, \nu_{3,6,9}, \nu_{2,4,12}, \nu_{1,2,15}, \nu_{11,4,3}, \nu_{10,2,6}\right\}.$$

Combining with the conjugate differentials yeilds a full basis of $\mathrm{H}^1_{\mathrm{dR}}(C,\mathbb{Q}) \subseteq \mathrm{H}^1_{\mathrm{dR}}(X,\mathbb{Q})$.

2. We pass to the étale site in exactly the same way as in Proposition 4.38. In the notation of Proposition 3.17, we see that ψ^* pulls the basis $\{u_1\otimes 1, v_1\otimes 1, v_2\otimes 1, v_4\otimes 1, w_1\otimes 1, w_2\otimes 1, w_5\otimes 1\}$ to

$$\{\nu_{3,6,9}\otimes 1, \quad \nu_{10,2,6}\otimes 1, \nu_{2,4,12}\otimes 1, \nu_{4,8,6}\otimes 1, \quad \nu_{1,2,15}\otimes 1, \nu_{11,4,3}\otimes 1, \nu_{5,10,3}\otimes 1\}.$$

3. We are now ready to begin executing Proposition 2.127; for this, Remark 2.128 informs us that we need to build a space of W' of Tate classes cutting out $G_\ell(A)^\circ \subseteq \mathrm{GL}_{14,\mathbb{Q}_\ell}$. We begin by adding W_1, made up of the endomorphisms, which ensures (for example) that $G_\ell(A)^\circ$ is diagonal. Then Proposition 3.17 computed that we also have the "polarization equations"

$$\begin{aligned}\mu_1\mu_2 &= \kappa_1\kappa_8,\\ \kappa_1\kappa_8 &= \kappa_2\kappa_7,\\ \kappa_1\kappa_8 &= \kappa_4\kappa_5,\end{aligned}$$

and the exceptional equation

$$\mu_1\kappa_7 = \kappa_5\kappa_8.$$

We remark that the polarization equations translate into a Tate class like $\nu_{(\alpha,-\alpha,\beta,-\beta)} \otimes 1$ understood as an element in $\mathrm{H}^4_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, but this Tate class actually already come from a class in W_1 (see Corollary 4.37), so we may safely ignore it. Thus, we only have to translate the exceptional equation into the tensor

$$\nu_{(3,6,9),(7,14,15),(13,8,15),(1,2,15)} \otimes 1 \in \mathrm{H}^4_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

and its Galois orbit.

4. Arguing as in Remark 4.36, we know that the periods of the Tate classes given in the previous step generate K_A^{conn}, so it remains to compute these periods. We already know that our endomorphisms, except for the isogeny $(A_1)_{\overline{\mathbb{Q}}} \cong (A_2)_{\overline{\mathbb{Q}}}$, are defined over $\mathbb{Q}(\zeta_N)$ (see also Corollary 4.37). We now handle the remaining cycles.

   - The isogeny $A_1 \cong A_2$ corresponds to equations $\kappa_u = \lambda_{2u}$ for each $u \in (\mathbb{Z}/18\mathbb{Z})^\times$, which means that we would like to compute

   $$\operatorname{Per}\left(\gamma^{2p}, \nu_{u(1,2,15),u(16,14,6)}\right).$$

   Well, by Remark 4.14, this element is

   $$(-2\pi i)^{-1}\zeta_{2N}^{u(1+2+16+14)} \cdot \frac{\Gamma\left(\frac{[u]}{18}\right)\Gamma\left(\frac{[2u]}{18}\right)}{\Gamma\left(\frac{[3u]}{18}\right)} \cdot \frac{\Gamma\left(\frac{[16u]}{18}\right)\Gamma\left(\frac{[14u]}{18}\right)}{\Gamma\left(\frac{[12u]}{18}\right)}.$$

   One can check that the term on the left is in $\pi^{-1}\mathbb{Q}(\zeta_N)$, so it remains to handle the product of Γs. We handle the case where $u = 1$ because the others turn out to be essentially Galois conjugates. (Indeed, Theorem 4.33 explains that the remaining us belong to the same Galois orbit.) Using the algorithm suggested in Remark 4.74, one finds that this product equals

   $$\Gamma(-\varepsilon_{1,8} - \varepsilon_{2,3} - \varepsilon_{2,4} - \varepsilon_{2,6} - \varepsilon_{2,7} + \varepsilon_{3,1} + \varepsilon_{3,2} + 2\varepsilon_{3,4}),$$

   which evaluates to

   $$\left(-\zeta_N^5 + \zeta_N^2 + \zeta_N + 1\right) \cdot \left(2^{22} \cdot 3^3\right)^{1/18},$$

   up to a (correct) power of π.

   - It remains to compute

   $$\operatorname{Per}\left(\gamma^4, \nu_{(3,6,9),(7,14,15),(13,8,15),(1,2,15)} \otimes 1\right).$$

   Well, by Remark 4.14, we see this equals

   $$(-2\pi i)^{-2}\zeta_{2N}^{(3+6+7+14+13+8+1+2)} \cdot \frac{\Gamma\left(\frac{3}{18}\right)\Gamma\left(\frac{6}{18}\right)}{\Gamma\left(\frac{9}{18}\right)} \cdot \frac{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{14}{18}\right)}{\Gamma\left(\frac{3}{18}\right)} \cdot \frac{\Gamma\left(\frac{13}{18}\right)\Gamma\left(\frac{8}{18}\right)}{\Gamma\left(\frac{3}{18}\right)} \cdot \frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{2}{18}\right)}{\Gamma\left(\frac{3}{18}\right)}.$$

   As above, the term on the left belongs to $\pi^{-2}\mathbb{Q}(\zeta_N)$, so it remains to handle the product of Γs. Once again using the algorithm suggested in Remark 4.74, one finds that this product equals

   $$\Gamma\left(\varepsilon_{1,7} - \varepsilon_{1,8} + \frac{1}{2}\varepsilon_{1,9} + \varepsilon_{2,5} - \varepsilon_{2,7} + \varepsilon_{2,8} + \varepsilon_{3,1} - \varepsilon_{3,3} + \varepsilon_{3,4} - \varepsilon_{3,5}\right),$$

   which evaluates to $4 \cdot 2^{6/18}$ up to a (correct) power of π.

Altogether, we can combine these two calculations to show $K_A^{\text{conn}} = \mathbb{Q}\left(\zeta_N, 2^{1/3}, 2^{2/9} \cdot 3^{1/6}\right)$. ■

Remark 4.80. Proposition 4.79 provides a reasonably small example of a quotient of a Fermat curve where $\mathbb{Q}(\zeta_N)$, the endomorphism field, and the connected monodromy field all disagree!

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List of Definitions