Student Automorphic Forms Seminar

Nir Elber

Fall 2024

Contents

1 September 4th: Nir Elber

Today we're talking about the local theory of Tate's thesis.

1.1 A Little Global Theory

In order to not lose perspective in the Fourier analysis that makes up the body of this talk, we discuss a little global theory. The goal of Tate's thesis is to derive analytic properties of L-functions such as the following.

Definition 1. We define the *Riemann ζ-function* as

$$\zeta(s) := \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Definition 2. Given a Dirichlet character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, we define the *Dirichlet L-function* by

$$L(s,\chi) := \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}.$$

Definition 3. For a number field K, we define the Dedekind ζ-function of a number field K

$$\zeta_K(s) := \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1-\mathrm{N}(\mathfrak{p})^{-s}}.$$

For now, "analytic properties" means deriving a meromorphic continuation, which in practice means deriving a functional equation.

Remark 4. There is a common generalization of the above two L-functions called a "Hecke L-function," but streamlined definitions would require a discussion of the adeles, which we are temporarily avoiding.

In fact, Hecke had proven functional equations before Tate, but Tate's arguments modernize better, which is why we will talk about them.

By way of example, we state the functional equation for the Riemann ζ-function $\zeta(s)$.

Theorem 5 (Riemann)**.** Define the completed Riemann ζ-function by

$$\Xi(s) := \pi^{-s/2}\Gamma(s)\zeta(s).$$

Then $\Xi(s)$ admits an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation $\Xi(s) = \Xi(1-s)$.

In some sense, the real goal of Tate's thesis is to explain the presence of the mysterious factor $\pi^{-s/2}\Gamma(s)$ which permits the functional equation $\Xi(s) = \Xi(1-s)$. To understand this, we write

$$\Xi(s) = \pi^{-s/2}\Gamma(s) \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

and the idea is that we should view this product over all places of $\mathbb{Q}$: the factor $\pi^{-s/2}\Gamma(s)$ belongs to the archimedean place of $\mathbb{Q}$!

Idea 6 (Tate)**.** Completed L-functions should be products over all places.

Very roughly, Idea 6 allows us to reduce global functional equations into products of local ones.

1.2 Local Z-integrals

We are interested in proving equations of the type $Z(s) \approx Z(1-s)$ for some suitable function Z. We will emply the following trick: we will show that both $Z(s)$ and $Z(1-s)$ live in the same one-dimensional space of functions and then study the scale factor between the two functions later.

To define our function Z, we take motivation from the definition of

$$\Gamma(s) = \int_{\mathbb{R}^+} e^{-t}t^s \, \frac{dt}{t}.$$

We know that this should correspond to the archimedean place of $\mathbb{Q}$, but we would like to extend this definition to the finite places. As such, we make the following observations.

- $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the archimedean place.
- The function $t \mapsto e^{-t}$ is an additive character $\mathbb{R} \to \mathbb{C}^\times$.
- The function $t \mapsto t^{-s}$ is a multiplicative character $\mathbb{R}^\times \to \mathbb{C}^\times$.
- The measure dt/t is a Haar measure of $\mathbb{R}^+$.

To begin our generalizations, we recall the definition of a local field, which will place $\mathbb{R}$ in the correct context.

Definition 7 (local field)**.** A *local field* is a locally compact nondiscrete topological field. (Here, a topological field is one that requires the addition, multiplication, and inversion to all be continuous.) It turns out that local fields of characteristic zero are exactly the finite extensions of $\mathbb{R}$ and $\mathbb{Q}_p$.

Next up, to place dt/t in the correct context, we should define a Haar measure.

Definition 8 (Haar measure)**.** Fix a locally compact topological group G. Then a *left-invariant Haar measure* $d\mu_\ell(g)$ is a Radon measure such that $\mu_\ell(gS) = \mu_\ell(S)$ for each $g \in G$ and measureable set S. In terms of intergrals, this is equivalent to

$$\int_G f(gh)\,dh = \int_G f(h)\,dh$$

for each $g \in G$ and integrable function f. It turns out that the Haar measure is unique up to scalar.

Remark 9. In general, a left-invariant Haar measure need not be right-invariant. However, this is frequently true: for example, if G is abelian or "reductive" (such as $G = \mathrm{GL}_n$), then left-invariant Haar measures are right-invariant.

Example 10. The Lebesgue measure dt is a Haar measure on $\mathbb{R}$. The measure $dt/\,|t|$ is a Haar measure on $\mathbb{R}^+$ and $\mathbb{R}^\times$.

Example 11. Fix a prime p. There is a unique Haar measure μ on $\mathbb{Q}_p$ such that $\mu(\mathbb{Z}_p) = 1$. For example, we find that $\mu(a + p\mathbb{Z}_p) = \frac{1}{p}$ for each $a \in \mathbb{Q}_p$.

Remark 12. Local fields turn out be normed, though this is not immediately obvious from the definition. Even though $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$ all have natural norms (extended from $\mathbb{Q}$), here is a hands-free way to obtain this norm from a local field K: choosing a Haar measure dt on K, we define the norm $|a|$ of some $a \in K$ as the scalar such that

$$d(at) = |a|\,dt.$$

It is not at all obvious that $|\cdot|$ is a norm (in particular, why does it satisfy the triangle inequality?), but it is true. As an example, $|\cdot|$ is the square of the Euclidean norm on $\mathbb{C}$. For $\mathbb{Q}_p$, we find $|p| = 1/p$.

Example 13. We are now able to say that $dt/\,|t|$ is a Haar measure of $K^\times$ for any local field K.

At this point, we may expect that our generalization of $\Gamma(s)$ to a generic local field K to be

$$Z(\psi,\omega) = \int_K \psi(t)\omega(t)\,\frac{dt}{|t|},$$

where $\psi\colon K \to \mathbb{C}^\times$ and $\omega\colon K^\times \to \mathbb{C}^\times$ are characters. However, this is a little too rigid for our purposes.

Notably, by taking linear combinations of additive characters, Fourier analysis explains that understanding Γ very well should permit understanding integrals of the form

$$\int_{\mathbb{R}^+} f(t)t^s\,\frac{dt}{t},$$

where $f\colon \mathbb{R} \to \mathbb{C}$ is some sufficiently nice function. It will help to have the flexibility that this extra f permits.

Definition 14. Fix a local field K. For a nice enough function f and character $\omega\colon K^\times \to \mathbb{C}^\times$, we define the *local Z-integral*

$$Z(f,\omega) := \int_{K^\times} f(t)\omega(t)\,\frac{dt}{|t|}.$$

We will not dwell on this, but perhaps we should explain what is required by "nice enough" function f. The keyword is "Schwartz–Bruhat."

Definition 15 (Schwartz–Bruhat)**.** Fix a local field K.

- If $K \in \{\mathbb{R}, \mathbb{C}\}$, we say $f\colon K \to \mathbb{C}$ is *Schwartz–Bruhat* if and only if it is infinitely differentiable and for all of its derivatives to decay rapidly (namely, $f(t)p(t) \to 0$ as $|t| \to \infty$ for any polynomial p).
- For other K, we say $f\colon K \to \mathbb{C}$ is *Schwartz–Bruhat* if and only if it is locally constant and compactly supported.

We let $S(K)$ denote the vector space of Schwartz–Bruhat functions on K, and we let $S(K)'$ denote its dual (i.e., the vector space of distributions).

Example 16. The function $t \mapsto e^{-t^2}$ is a Schwartz–Bruhat function $\mathbb{C} \to \mathbb{C}$.

Example 17. Fix a prime p. Then the indicator function $1_{\mathbb{Z}_p}$ on $\mathbb{Q}_p$ is Schwartz–Bruhat.

Importantly, the definition of Schwartz–Bruhat will promise that

1.3 Fourier Analysis

We now take a moment to review where we are standing. We were hoping to prove a statement like $Z(s) \approx Z(1-s)$, where $Z(s)$ perhaps has some kind analytic properties. However, we currently have a function $Z(f,\omega)$, so it's not at all obvious how to replace f and ω with "dual" entries or how to make sense of analytic properties. In this subsection, we address both of these concerns; they are both related to character theory.

Going in order of difficulty, it is a little easier to explain how to add analytic structure to Z. After taking norms, we can find some $s \in \mathbb{C}$ such $|\omega| = |\cdot|^s$ so that $\eta := \omega\,|\cdot|^{-s}$ outputs to S^1, and

$$\omega = \eta\,|\cdot|^s\,.$$

Thus, we see that we can decompose characters $K^\times \to \mathbb{C}^\times$ into a unitary part η and an "unramified" part $|\cdot|^s$, and the unramified part now has a complex parameter $s \in \mathbb{C}$. Namely, $\omega\,|\cdot|^{-s}$ now outputs to S^1; i.e., this character is unitary. Fix a local field K.

Thus, we can view $Z(f,\eta)$ as having three parameters as $Z(f,\eta,s) := Z\left(f,\eta\,|\cdot|^s\right)$, where we now require that η is unitary. Because we already have some notion of smoothness in the parameter $s \in \mathbb{C}$, it remains to understand smoothness in the parameter of unitary character η.

Example 18. In the archimedean case, the parameter η is not so interesting.

- The characters $\mathbb{R}^\times \to S^1$ take the form $t \mapsto t^{-a}\,|t|^s$ where $a \in \{0,1\}$ and $s \in \mathbb{C}$.
- The characters $\mathbb{C}^\times \to S^1$ take the form $z \mapsto z^a \overline{z}^b\,|z|^s$ where $a,b \in \mathbb{Z}$ and $s \in \mathbb{C}$.

We now understand that our functional equation for Z should arise from $Z(f,\eta,s)$. We hope to take $s \mapsto 1-s$, and it seems reasonable (by looking at functional equations for $L(s,\chi)$) to replace η with η^{-1}.

However, we still need to replace f with some dual function. In the archimedean case, we expect this to be the Fourier transform defined by

$$\widehat{f}(s) := \int_{\mathbb{R}} f(t)e^{2\pi ist}\,dt.$$

Such a definition will more or less carry through for arbitrary local fields, but we will have to go through approximately the same dictionary that defined $Z(f,\omega)$ from $\Gamma(s)$. In particular, the functions $t \mapsto e^{2\pi ist}$ list the characters of $\mathbb{R}$, so the above is the integration of our function f against the list of characters. To this end, we pick up the following theorem.

Theorem 19. Fix a local field K. Then there exists a nontrivial character $\psi\colon K \to S^1$. Any choice of ψ defines a bijection $K \to \widehat{K}$ by sending $a \in K$ to the character $\psi_a(t) := \psi(at)$.

Remark 20. For a general locally compact abelian group G, one can give $\widehat{G} := \operatorname{Hom}(G, \mathbb{C})$ a locally compact topology. With this topology in hand, one actually finds that the map $K \to \widehat{K}$ given by $a \mapsto \psi_a$ is an isomorphism of locally compact abelian groups. However, we do not currently have the need to work in this level of generality.

We are now ready to define our Fourier transform.

Definition 21. Fix a local field K, and choose a nontrivial character $\psi\colon K \to S^1$. For $f \in S(K)$, we define the *Fourier transform*

$$\mathcal{F}_\psi f(t) := \int_K f(t)\psi(st)\,dt.$$

Example 22. For $K = \mathbb{R}$, choose $\psi(t) := e^{2\pi i t}$. Then the Fourier transform (up to normalization) agrees with the usual one

$$\mathcal{F}_\psi f(t) := \int_{\mathbb{R}} f(t)e^{2\pi i st}\,dt.$$

It turns out that one has the usual properties for the Fourier transform, such as $\mathcal{F}_\psi\mathcal{F}_\psi f(t) = f(-t)$.

We are now ready to state the local functional equation.

Theorem 23. Fix a local field K, and choose a nontrivial character $\psi\colon K \to S^1$. For every $f \in S(K)$, the function $Z(f, \omega, s)$ has a meromorphic continuation to $s \in \mathbb{C}$ (with well-understood poles) and satisfies a functional equation of the form

$$Z(f, \omega, s) = \gamma(\psi, \omega, s, dx)Z\left(\mathcal{F}_\psi f, \omega^{-1}, 1-s\right),$$

where $\gamma(\psi, \omega, s, dx)$ is meromorphic in $s \in \mathbb{C}$ (with well-understood poles).

Tate's original proof of Theorem 23 was more or less by an explicit computation: by some argument interchanging integrals, one can relate $Z(f, \omega, s)$ with $Z(g, \omega, s)$ for separate $f, g \in S(K)$, which allows us to reduce the proof to a single f. Then one chooses some f such that $\mathcal{F}_\psi f = f$ (e.g., one chooses a Gaussian when $K = \mathbb{R}$) and does an explicit computation to check the result.

1.4 Multiplicity One

We are not going to prove Theorem 23 in detail, but we will gesture towards an argument which uses a bit more functional analysis and less explicit computation. The key point is that $Z(-, \omega)\colon S(K) \to \mathbb{C}$ is a distribution with the curious property that

$$\int_{K^\times} f(at)\omega(t)\,\frac{dt}{t} = \omega^{-1}(a)\int_{K^\times} f(t)\omega(t)\,\frac{dt}{|t|}.$$

Thus, we see that $Z(-, \omega)$ is an eigendistribution of sorts. To make this more explicit, we let $K^\times$ act on $S(K)$ by translation: for $a \in K$ and $f \in S(K)$, we define $(r(a)f)(t) := f(at)$. Then $K^\times$ acts on the distributions $S(K)'$ accordingly: for $a \in K$ and $\lambda \in S(K)'$ and $f \in S(K)$, we can define

$$\langle r'(a)\lambda, f\rangle := \left\langle \lambda, r(a)^{-1}f\right\rangle.$$

Tracking everything through, we see that $Z(-, \omega) \in S(K)'$ is an eigendistribution for the action of $K^\times$ on $S(K)'$ with eigenvalue given by the character ω.

Notation 24. Let $S(K)'(\omega)$ denote the space of $K^\times$-eigendistributions with eigenvalue ω.

Now, we see that $Z(-,\omega) \in S(K)'(\omega)$, and one can check that $Z\left(\mathcal{F}_\psi-,\omega^{-1}\right)$ is also in $S(K)'(\omega)$. Thus, a statement like Theorem 23 will follow from the following "representation-theoretic" multiplicity one result.

Theorem 25. Fix a local field K and character $\omega\colon K \to \mathbb{C}^\times$. Then

$$\dim S(K)'(\omega) = 1.$$

Let's explain the sort of inputs that go into proving Theorem 25, but we will not say more.

- One can verify that $Z(-,\omega)$ provides some vector in $S(K)'(\omega)$, so this space is at least nonempty.
- Let $C_c^\infty(K)$ denote the set of compactly supported Schwartz functions on K. Duality provides an exact sequence
$$0 \to S(K)'_0 \to S(K)' \to C_c^\infty(K)' \to 0,$$
where $S(K)'_0$ denotes the distributions supported at 0. Taking eigenspaces, we get an exact sequence
$$0 \to S(K)'_0(\omega) \to S(K)'(\omega) \to C_c^\infty(K)'(\omega).$$
- One can show that $C_c^\infty(K)'(\omega)$ is one-dimensional with basis given by $f \mapsto \int_K f(t)\omega(t)\,dt/\,|t|$.
- Understanding $S(K)'_0$ comes down to some casework. For example, we look at the nonarchimedean case. If ω is trivial, we have the distribution $f \mapsto f(0)$; otherwise, $S(K)'_0(\omega)$ is zero.

2 September 11: Reed Jacobs

Today we are talking about the global theory of Tate's thesis. This will mostly be an excuse to discuss the adeles.

2.1 Adeles

For today, K is a global field; in particular, it is a number field (in characteristic 0) or a finite extension of $\mathbb{F}_p(t)$ (in postive characteristic $p > 0$). Here is our main character for today.

Definition 26 (adele ring)**.** Fix a global field K, and let V_K denote the set of places of K. Then the *adele ring* $\mathbb{A}_K$ is the restricted direct product

$$\mathbb{A}_K \coloneqq \prod_{v\in V_K} (K_v, \mathcal{O}_v).$$

Here, this notation means that $\mathbb{A}_K$ consists of infinite tuples $(a_v)_v \in \prod_{v\in V_K} K_v$ where $a_v \in \mathcal{O}_v$ for all but finitely many places $v \in V_K$.

Remark 27. Even though $\mathcal{O}_v$ may not have a definition for infinite v, we see that this does not matter because there are only finitely many infinite places anyway.

Remark 28. We see that $\mathbb{A}_K$ becomes a ring under the pointwise operations. In fact, it becomes a topological ring when given the subspace topology of the product topology on $\prod_v K_v$. In particular, upon taking the intersection, we see that our basic open sets look like

$$\prod_{v\in V_K} U_v,$$

where $U_v \subseteq K_v$ is open, but $U_v = \mathcal{O}_v$ for all but finitely many $v \in V_K$.

Here are some quick properties of the topology.

Proposition 29. Fix a global field K. Then $\mathbb{A}_K$ is a locally compact topological ring.

Proof. The fact that the ring operations are continuous can be checked pointwise on the level of the product $\prod_v K_v$. Perhaps we should also check that $\mathbb{A}_K$ is Hausdorff, for which we note that this follows from being a subspace of $\prod_v K_v$ again.

To be locally compact, we have to do a little more work. This follows more or less Fix some $x \in \mathbb{A}_K$, and we want some open neighborhood $U \subseteq \mathbb{A}_K$ of x with compact closure. By translation, we may assume that $x = 0$. Then the open subset

$$\prod_{\substack{v \in V_K \\ v \text{ infinite}}} B_v(0,1) \times \prod_{\substack{v \in V_K \\ v \text{ finite}}} \mathcal{O}_v,$$

where $B_v(0,1)$ is the open ball around $0 \in K_v$. This is open by construction, and its closure $\prod_{v \in V_K} \overline{B_v(0,1)}$ is a product of compact sets and hence compact. ■

Remark 30. Do note that $\prod_v K_v$ fails to be locally compact because the open subsets are too big! This is one reason why we want to work with the adeles instead: once has a chance of doing some reasonable topology on $\mathbb{A}_K$.

To misquote Tate, we remark that one can really only extract arithmetic information from $\mathbb{A}_K$ by putting K inside it. Here is this embedding.

Proposition 31. Fix a global field K. Then the embedding $i\colon K \hookrightarrow \mathbb{A}_K$ defined by

$$i\colon a \mapsto (a)_v$$

is discrete and cocompact.

Proof. We will not prove cocompactness; it is equivalent to the finiteness of the class group and Dirichlet's unit theorem. Alternatively, one can use Minkowski theory to show this directly.

Discreteness is a little easier to show. For $a \in K$, we must show that $i(a)$ has an open neighborhood disjoint from the rest of $i(K)$. By translating, we may take $a = 0$. Now, define the open neighborhood of 0 given by

$$U \coloneqq \prod_{\substack{v \in V_K \\ v \text{ infinite}}} B_v(0,1) \times \prod_{\substack{v \in V_K \\ v \text{ finite}}} \mathcal{O}_v.$$

Now, for any $b \in K^\times$, the product formula asserts that $\prod_v |b|_v = 1$, but $\prod_v |b_v|_v < 1$ for any $(b_v)_v \in U$, so $U \cap i(K) = \{0\}$, as required. ■

The above cocompactness tells us that the quotient $\mathbb{A}_K/K$ will be interesting to us; similarly, the group $\mathbb{A}_K^\times/K^\times$ will be interesting to us.

2.2 Some Pontryagin Duals

We have some time, so let's expand the discussion of Remark 20.

Definition 32. Fix a locally compact abelian group G. Then we define the group $G \coloneqq \operatorname{Hom}(G, S^1)$ to be the *Pontryagin dual*, which we turn into a topological group by giving it the compact-open topology. It turns out that $\widehat{G}$ is a locally compact abelian group.

Example 33. Theorem 19 explains that local fields K_v are self-dual.

Example 34. One can check that a character $\mathbb{Z} \to S^1$ has equivalent data to an element of S^1, which essentially proves $\widehat{\mathbb{Z}} \cong S^1$.

Example 35. On the other hand, a continuous homomorphism $S^1 \to S^1$ must be exponentiation by some integer, so $\widehat{S^1} \cong \mathbb{Z}$.

Remark 36. It is a general fact that the Pontryagin dual of $\widehat{G}$ is isomorphic to G. There is at least a map $G \to \operatorname{Hom}\left(\widehat{G}, S^1\right)$ given by $g \mapsto \operatorname{ev}_g$, where $\operatorname{ev}_g \colon \widehat{G} \to S^1$ is given by $\operatorname{ev}_g(\chi) := \chi(g)$.

We even have a nice duality, as in Theorem 19.

Theorem 37. Fix a global field K. Then the locally compact abelian group $\mathbb{A}_K$ is self-dual. More precisely, there exists a nontrivial character $\psi \colon \mathbb{A}_K \to S^1$. Then for any choice of ψ, then the map $\mathbb{A}_K \to \widehat{\mathbb{A}_K}$ given by

$$a \mapsto (\psi_a \colon t \mapsto \psi(at))$$

is an isomorphism of locally compact abelian groups.

Sketch. This follows roughly from Theorem 19, essentially by trying to take a product of the self-duality results for each local field in the restricted direct product. A detailed proof would require understanding the topology of the Pontryagin dual in more detail, so we will avoid it. ■

Here is an application.

Corollary 38. Fix a global field K. Then the Pontryagin dual of K is $\mathbb{A}_K/K$.

Proof. Fix a nontrivial character $\psi \colon \mathbb{A}_K/K \to S^1$ (note the quotient!), and we construct that the composite

$$K \hookrightarrow \mathbb{A}_K \overset{\psi}{\cong} \widehat{\mathbb{A}_K},$$

where the last map is given by Theorem 37. Now, this composite actually outputs to $\widehat{\mathbb{A}_K/K}$: any $a \in K$ produces a character ψ_a satisfying $\psi_a(t) = \psi(at) = 1$ for each $t \in K$, so ψ_a descends to a character of $\mathbb{A}_K/K$. We claim that the above map is an isomorphism, which we do by combining three observations.

- We can check that $\widehat{\mathbb{A}_K/K}$ is a vector space over K.
- Note $\mathbb{A}_K/K$ is compact, so $\widehat{\mathbb{A}_K/K}$ is discrete by some fact of Pontryagin duals.
- Note $\widehat{\mathbb{A}_K/K}/K \subseteq \widehat{\mathbb{A}_K}/K \cong \mathbb{A}_K/K$, and this last space is compcat.

The last two observations imply $\widehat{\mathbb{A}_K/K}$ is finite, but then the first observation requires it to be trivial. ■

2.3 A Poisson Summation Formula

The presence of a duality permits a well-behaved Fourier analysis. For our technicalities, we explain what our Schwartz functions are.

Definition 39 (Schwartz)**.** A *Schwartz–Bruhat* function f on the quotient $\mathbb{A}_K/K$ is a function of the form $\prod_\nu f_\nu$ such that each f_ν is Schwartz, and $f_\nu = 1_{\mathcal{O}_\nu}$ for all but finitely many ν. We also permit finite $\mathbb{C}$-linear combinations of these functions to be Scwhartz–Bruhat.

And here is our corresponding Fourier transform; it is our global analogue of Definition 21.

Definition 40. Fix a global field K, and choose a nontrivial character $\psi\colon \mathbb{A}_K/K \to S^1$. For any Schwartz–Bruhat function f, then we define the *Fourier transform* to be

$$\mathcal{F}_\psi f(y) := \int_{x\in\mathbb{A}_K} f(x)\psi(xy)\,dx.$$

Remark 41. For the discussion which follows shortly, it will be worth our time to write out what the Fourier transform is in general for a locally compact abelian group G. Given a nice enough function $f\colon G \to \mathbb{C}$ (such as Schwartz–Bruhat), the Fourier transform $\widehat{f}$ is a function on $\widehat{G}$ defined by

$$\widehat{f}(\chi) := \int_G f(g)\chi(g)\,dg.$$

For example, the identification of characters of $\mathbb{A}_K$ with $\mathbb{A}_K$ recovers the above definition. One finds that taking the Fourier transform twice almost recovers the original function.

Notably, the Fourier transform depends on a choice of Haar measure. These can be found by taking the product of the Haar measures on the local fields; by scaling, we can get the usual identity $\mathcal{F}_\psi\mathcal{F}_\psi f(x) = f(-x)$.

Riemann's original proof of Theorem 5 has the result more or less follow from some functional equation coming from a Poisson summation formula. As such, we desire a Poisson summation formula in our context as well. The point of a Poisson summation formula is to relate sums for a function f over the discrete cocompact subgroup $\mathbb{Z} \subseteq \mathbb{Q}$ with the same sums of the Fourier transform. Because $K \subseteq \mathbb{A}_K$ is discrete and cocompact, it is natural to have the following statement.

Theorem 42 (Poisson summation)**.** Fix a global field K, and choose a nontrivial character $\psi\colon \mathbb{A}_K/K \to S^1$. For any Schwartz–Bruhat function f, we have

$$\sum_{\gamma\in K} f(\gamma) = \sum_{\gamma\in K} \mathcal{F}_\psi f(\gamma).$$

Proof. Define $F\colon \mathbb{A}_K \to \mathbb{C}$ by

$$F(x) := \sum_{\gamma\in K} f(x+\gamma).$$

Note that F descends to a function on $\mathbb{A}_K/K$. Quickly, note that the convergence of these sums follows from our definition of Schwartz–Bruhat; the convergence is fast enough to ensure that we get a continuous function on $\mathbb{A}_K/K$. We will spend the rest of the proof ignoring convergence issues.

The point is to compute the Fourier transform of F and apply Fourier inversion. However, F has descended to a function on $\mathbb{A}_K/K$, so its Fourier transform should be a function on the Pontryagin dual of $\mathbb{A}_K/K$, which is K by Corollary 38. As such, we let $D \subseteq \mathbb{A}_K$ be a fundamental domain for $\mathbb{A}_K/K$, which we give a volume of 1 to fix our Haar measure, and we compute

$$\begin{aligned}
\mathcal{F}_\psi F(y) &= \int_{\mathbb{A}_K/K} F(x)\psi(xy)\,dx \\
&= \int_D F(x)\psi(xy)\,dx \\
&= \int_D \sum_{\gamma\in K} f(x+\gamma)\psi(xy)\,dx \\
&= \sum_{\gamma\in K} \int_D f(x+\gamma)\psi(xy)\,dx \\
&= \sum_{\gamma\in K} \int_D f(x)\psi((x-\gamma)y)\,dx \\
&\overset{*}{=} \sum_{\gamma\in K} \int_D f(x)\psi(xy)\,dx \\
&= \mathcal{F}_\psi f(y),
\end{aligned}$$

where $\overset{*}{=}$ holds because $\psi|_K = 1$. The general theory of Pontryagin duality allows us to apply a Fourier inversion formula for $\widehat{K} \cong \mathbb{A}_K/K$ to see $F(x)$ equals

$$\sum_{y\in K} \widehat{F}(y)\overline{\psi(xy)} = \sum_{y\in K} \mathcal{F}_\psi f(y)\overline{\psi(xy)}.$$

(Note that this is basically the Fourier transform of the Fourier transform, where we are plugging into the general definition of Remark 41.) Plugging in $x=0$ completes the proof! ■

Remark 43 (Reed)**.** In the case where K is the function field of a curve C over a finite field $\mathbb{F}_q$, one is able to turn Theorem 42 into the Riemann–Roch theorem for C. We will not explain this in detail.

Remark 44. One can basically mimic the proof of Theorem 5 to prove a functional equation for "global Z-integrals" which look like

$$Z(f,\chi) := \int_{\mathbb{A}_K^\times} f(x)\chi(x)\,d^\times x,$$

where $d^\times x$ is some suitably defined Haar measure on $\mathbb{A}_K^\times$. The key analytic input into the proof of Theorem 5 is a Poisson summation formula used to produce the symmetry; analogously, the key input into the global functional equation is the "adelic Poisson summation formula" Theorem 42 used to produce the symmetry.

3 September 18th: Nir Elber

I filled in for Saud Molaib, who has a last-minute cancellation.

4 September 25th: Justin Wu

Today we are discussing how to put a topology on the adeles and related groups.

We begin by fixing some notation. Today, F will be a global field, and $V(F)$ is its set of places. Throughout, $S \subseteq V(K)$ will be a finite subset, usually including the finite set ∞ of infinite places. We define the adele ring as the usual restricted direct product

$$\mathbb{A}_F = \prod_{v \in V(K)} (F_v, \mathcal{O}_v).$$

For our finite set S, we define

$$\begin{aligned}
\mathbb{A}_F^S &:= \prod_{v \notin S} (F_v, \mathcal{O}_v) \\
F_S := \mathbb{A}_{F,S} &:= \prod_{v \in S} F_v \\
\widehat{\mathcal{O}}_F^S &:= \prod_{v \notin S} \mathcal{O}_v.
\end{aligned}$$

For example, $\mathbb{A}_F^{\varnothing} = \mathbb{A}_F$, and $\widehat{\mathcal{O}}_F^{\infty}$ is compact.

Remark 45. There is not a unified consensus on notation. We are agreeing with Getz and Hahn.

4.1 Topology from Topological Rings

For an affine scheme X over F, we would like to give a topology on $X(\mathbb{A}_F)$. It is not totally obvious how to do this. The classical example is that $\mathbb{A}_F$ is a topological ring, and we may want to give a topology on its units $\mathbb{A}_F^\times$, but $\mathbb{A}_F^\times$ will fail to be a topological group if given the subspace topology from $\mathbb{A}_F$: the inversion map $\mathbb{A}_F^\times \to \mathbb{A}_F^\times$ won't be continuous!

The correct way to proceed is to use the closed embedding $j\colon \mathbb{G}_m \to \mathbb{A}^2$ given by $x \mapsto (x, x^{-1})$, which is a closed embedding onto

$$\left\{(x,y) \in \mathbb{A}^2 : xy = 1\right\}.$$

Then we can give $\mathbb{A}_F^\times$ the subspace topology from $\mathbb{A}^2$. In particular, inversion when passed through this embedding j merely permutes the two coordinates, so it succeeds at being continuous, even for a general ring R! Here is the general result.

Theorem 46. Fix a Hausdorff topological ring R. There is a unique way to give a topology on the collection of sets $X(R)$ as X varies over the collection of finite type affine R-schemes satisfying the following.

(a) Functoriality: scheme maps $X \to Y$ produce continuous maps $X(R) \to Y(R)$.

(b) Fiber products: given maps $X \to Z$ and $Y \to Z$, the topology on $(X \times_Z Y)(R)$ is the topology induced by the fiber product $X(R) \times_{Z(R)} \times Y(R)$.

(c) Closed embeddings: given a closed embedding $X \to Y$ of schemes, one gets a closed embedding $X(R) \to Y(R)$ of topological spaces.

(d) Affine space: the topology on $\mathbb{A}^1(R)$ is the topology on R.

Sketch. We show uniqueness, but we won't show existence. For any affine R-scheme X of finite type, we know that there is some $r > 0$ and a closed embedding $X \hookrightarrow \mathbb{A}^r$. Then the topology no $\mathbb{A}^r(R)$ is fixed by the topology on $\mathbb{A}$ and taking fiber products, so the topology on $X(R) \subseteq \mathbb{A}^r(R)$ is fixed by having a closed embedding. ■

Example 47. We see that the group GL_n has a closed embedding into $M_n \times M_n$ via $g \mapsto (g, g^{-1})$. This allows us to give a topology on $\mathrm{GL}_n(R)$ for any topological ring R.

Example 48. For any affine group G of finite type, we can find a closed embedding $G \subseteq \mathrm{GL}_n$ to produce a topology on $G(R) \subseteq \mathrm{GL}_n(R)$. For the adeles, one can check that the topology is given by

$$G(\mathbb{A}_F) = \prod_{v \in V(K)} (G(F_v), G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)).$$

4.2 Hyperspecial Subgroups

We would like to understand the subgroups $G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$ of Example 48 in a way which is ambivalent to the embedding. This is a local object, so we begin by telling a local story.

Definition 49 (unramified)**.** Fix a reductive group G over a nonarchimedean local field F_v. We say that G is *unramified* if and only if G is quasi-split, and there is a finite unramified extension E_w/F_v such that G_{E_w} is split.

Theorem 50. Fix a reductive group G over a nonarchimedean local field F_v. Then G is unramified if and only if there is a smooth model $\mathcal{G}$ of G over $\mathcal{O}_v$ such that $\mathcal{G}_s$ is reductive.

Sketch. We sketch the backwards direction. If we have a model $\mathcal{G}$, then we basically pass to the special fiber to find our Borel subgroup and maximal torus; notably, the maximal torus may only split over a finite extension of the residue field, but then this will only require a finite unramified extension in order to split the torus back in G. ■

The remarkable part of conclusion of the above theorem is not exactly the mere existence of the model but instead having the reductive special fiber. For example, given any embedding $G \subseteq \mathrm{GL}_n$, one can look at the Zariski closure of the composite

$$G \to \mathrm{GL}_{n,F_v} \to \mathrm{GL}_{n,\mathcal{O}_v}$$

to produce a model of G over $\mathcal{O}_v$; in fact, there is a way to smooth out this model as well. However, there is no reason why the special fiber of this thing would be controlled.

This notion of models allows us to discuss subgroups like $G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$.

Definition 51 (hyperspecial)**.** Fix a reductive group G over a nonarchimedean local field F_v. Then a subgroup $H \subseteq G(F_v)$ is *hyperspecial* if and only if there is a smooth model $\mathcal{G}$ of G over $\mathcal{O}_v$ such that $\mathcal{G}$ has reductive special fiber, and $H = \mathcal{G}(\mathcal{O}_v)$.

In fact, we can even give an almost topological description.

Theorem 52. Fix a reductive group G over a nonarchimedean local field F_v.

(a) Every compact subgroup $K \subseteq G(F_v)$ is contained in some maximal compact subgroup.

(b) Every maximal compact subgroup $K \subseteq G(F_v)$ equals $\mathcal{G}(\mathcal{O}_v)$ for some smooth model $\mathcal{G}$ of G.

(c) All maximal compact subgroups are open.

We now transition back to our global theory. The miracle is that we achieve being hyperspecial almost everywhere.

Theorem 53. Fix a reductive group G over a global field F, and find a smooth model $\mathcal{G}$ of G over $\mathcal{O}_F^S$ for some finite subset $S \subseteq V(K)$. Then the subgroup $\mathcal{G}(\mathcal{O}_v)$ is hyperspecial for all but finitely many places $v \notin S$.

The main point is that $\mathcal{G}$ is reductive at the generic fiber, and it turns out that one can spread out the reductivity to a Zariski dense open subset of the special fibers.

Being hyperspecial almost everywhere allows us to recast our topology on $G(\mathbb{A}_F)$.

Proposition 54. Fix an affine scheme X over a global field F with a smooth model $\mathcal{X}$ over $\mathcal{O}_F^S$. Then there is a natural homeomorphism

$$X(\mathbb{A}_F) \to \prod_{v \in V(K)} (X(F_v), \mathcal{X}(\mathcal{O}_v)).$$

Corollary 55. Fix a reductive group G over a global field F. For any compact open subgroup $K \subseteq G(\mathbb{A}_F^S)$ for a finite subset $S \subseteq V(K)$, the projection of K onto $G(F_v)$ is hyperspecial for all but finitely many places $v \notin S$.

Proof. We understand what (compact) open subsets of the restricted direct product look like once given a model, so this follows from the above proposition. ■

4.3 The Adelic Quotient

For this subsection, take F to be a number field. For this seminar, we are essentially interested in the representation theory of $G(\mathbb{A}_F)$ for reductive groups G. To do number theory, we should really embed $G(F) \to G(\mathbb{A}_F)$ via $F \hookrightarrow \mathbb{A}_F$, so perhaps we are really interested in the quotient $G(F)\backslash G(\mathbb{A}_F)$.

Howver, $G(F)\backslash G(\mathbb{A}_F)$ is much too large. For example, it will almost always fail to be compact due to the presence of characteres.

Example 56. Take $G = \mathrm{GL}_n$. Then there is a surective composite

$$\mathrm{GL}_n(\mathbb{A}_F) \stackrel{\det}{\to} \mathrm{GL}_1(\mathbb{A}_F) \stackrel{|\cdot|}{\to} \mathbb{R}^+.$$

However, $\mathrm{GL}_n(F)$ goes to the identity!

Thus, to make $G(F)\backslash G(\mathbb{A}_F)$ smaller, we want to restrict $G(\mathbb{A}_F)$ down to

$$G(\mathbb{A}_F)^1 = \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi),$$

where $X^*(G)$ is the set of characters $G \to \mathbb{G}_m$. Note that $\mathbb{G}_m(F)$ is trivial under $|\cdot|$, so $G(F) \subseteq G(\mathbb{A}_F)^1$.

Another reason the quotient $G(F)\backslash G(\mathbb{A}_F)$ is too large due to the presence of the center. With this in mind, we define $A_G \subseteq G(F_\infty)$ as follows: consider a maximal $\mathbb{Q}$-split torus in $\operatorname{Res}_{F/\mathbb{Q}} Z_G$; then we take the connected component of the identity of the $\mathbb{R}$-points of this thing.

Example 57. We can compute that the maximal $\mathbb{Q}$-split torus of $\operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ is $\mathbb{Q}^\times \subseteq F^\times$. Thus, $A_{\mathbb{G}_m} = \mathbb{R}^+$. More generally,

$$A_{\mathrm{GL}_n} = \{c 1_n : c > 0\} \subseteq \mathrm{GL}_n(F_\infty).$$

We are now ready to define the adelic quotient.

Definition 58 (adelic quotient)**.** Fix a reductive group G over a global field F. Then the *adelic quotient* is the set

$$[G] \coloneqq A_G G(F)\backslash G(\mathbb{A}_F).$$

For a number field F, this is the same as $G(F)\backslash G(\mathbb{A}_F)^1$.

Remark 59 (Tamagawa numbers)**.** It turns out that $G(F)\backslash G(\mathbb{A}_F)^1$ may fail to be compact, but it will at least always have finite measure with respect to the Haar measure on $G(\mathbb{A}_F)^1$.

5 October 2nd: Nir Elber

We had another last-minute cancellation.

6 October 9th: Sam Goldberg

Today we're talking about more on the adeles. We will be giving a topological reason why the adeles are a natural object.

6.1 Characterizing the Adeles

Let's try to state a characterization. Recall that a topological field k requires addition, multiplication, and inversion to be continuous; we also recall a notion of a topologial k-algebra A which requires the scalar multiplication map $k \to A$ to be continuous.

Example 60. Any field k becomes a locally compact topological field by using the discrete topology.

Anyway, here is our theorem.

Theorem 61 (Iwasawa, Levin)**.** Let F be a discrete topological field. Then the following are equivalent.

(a) F is isomorphic to a global field as a field.

(b) There exists a topological F-algebra A_F such that the short exact sequence

$$0 \to F \to A_F \to A_F/F \to 0 \tag{6.1}$$

of topological abelian groups satisfies the following.

- The embedding $F \to A_F$ is a homeomorphism onto its image.
- A_F/F is compact.
- The sequence (6.1) does not split.

In fact, it turns out that A_F is unique up to unique topological F-algebra isomorphism and satisfies $A_F \cong \mathbb{A}_F$.

This is a long theorem statement, so some remarks are in order.

Remark 62. One can think of this statement as characterizing both global fields (algebraically!) and the adele ring A_F. Approximately speaking, we are claiming that the best topology on a global field F is the discrete topology. To understand this claim, note that this is the only way to give a topology to F so that all the completion maps $F \to F_v$ are continuous (for all places v).

Remark 63. This result makes precise the way that $\mathbb{Z} \subseteq \mathbb{R}$ is analogous to $F \subseteq \mathbb{A}_F$: namely, F sits inside $\mathbb{A}_F$ as a discrete cocompact subgroup. In fact, the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

fails to be exact, which provides some motivation for the last condition in (b). Intuitively, we are finding that global fields are precisely the fields which can behave like a lattice, which is why something like Tate's thesis (namely, the Poisson summation formula) can work!

Remark 64. It is worth noting that providing a characterization (like this) provides an explanation for why number fields and function fields over a curve (over a finite field) are similar. One may hope to write topological proofs of all results downstream from characterizations like this, but of course this is hard.

Remark 65. Let's explain why the non-splitting is required. Indeed, if we had a splitting, then we would find that $A_F \cong F \oplus M$ where M is a compact K-algebra. But then it turns out that F acts on M by zero, which is rather uninteresting. Thus, the non-splitting basically asks that A_F is at least a little interesting.

Remark 66. The uniqueness here implies that $\mathbb{A}_F$ has no nontrivial F-algebra automorphisms. In particular, the restricted product decomposition of it is the only way to do this.

Remark 67. The uniqueness of the isomorphism for A_F basically says that there is a unique morphism of short exact sequences. One may hope to upgrade this to a universal property in a category of short exact sequences, but Sam is not sure how to do this.

Let's give an idea for the proof. Going from (a) to (b) has no content by taking $A_F = \mathbb{A}_F$. The key is the structure theory of locally compact Hausdorff abelian groups. If F has characteristic 0, one merely needs to check that $[F : \mathbb{Q}] < \infty$. If F has positive characteristic, then the main point is finding the transcendental element. Anyway, let's provide the structure theorem we need.

Theorem 68 (Levin)**.** Let M be a locally compact Hausdorff $\mathcal{O}_F$-module, where F is some global field (given the discrete topology). Then if M has no "trivial piece," then

$$M \cong R \oplus TM,$$

where R is a finite sum of infinite completions, and TM is topological torsion. In fact, one can further decompose TM into a restricted direct product

$$TM = \prod_{\substack{v \in V(F) \\ v < \infty}} (M^{\mathfrak{p}}, N \cap M^{\mathfrak{p}}),$$

where $M^{\mathfrak{p}}$ is topological $\mathfrak{p}$-primary, and $N \subseteq TM$ is any compact open submodule.

Remark 69. There is another structure theorem which removes some trivial pieces from A_F.

What is amazing here is that these restricted direct products come out of our structure theory of locally compact Hausdorff abelian groups!

7 October 16th: Alex Feiner

Today we are discussing the representation theory of affine algebraic groups.

7.1 The Regular Representation

We begin by setting some notation for today.

- G is a locally compact Hausdorff group.
- If V is a Hausdorff topological vector space, then $\mathrm{End}(V)$ consists of the continuous endomorphisms, and $\mathrm{GL}(V)$ consists of the continuous invertible linear maps.
- If V is also a Hilbert space with inner product $(-,-)$, we let the norm be $\|\cdot\|_2$, and we let $U(V) \subseteq \mathrm{GL}(V)$ be the subgroup of invertible isometries.

In this seminar, we are interested in studying representations of groups G, but when G has some extra structure, we would like to ensure that our representaions remember this structure.

Definition 70 (representation)**.** A *representation* of G on V is a continuous (left) group action $G \times V \to V$.

Remark 71. For each $g \in G$, we thus produce a group homomorphism $\pi \colon G \to \mathrm{GL}(V)$. In the finite-dimensional case, π is continuous, but in general, it may fail to be continuous. (One needs to adjust the topology of $\mathrm{GL}(V)$ in some way.)

Definition 72 (unitary)**.** A representation of G on V is *unitary* if and only if the map $\pi(g) \colon V \to V$ is unitary for all $g \in G$.

As usual, one can define G-equivariant maps between representations (V, π) and (V', π') as linear maps $\varphi \colon V \to W$ such that

$$\varphi(\pi(g)v) = \pi'(g)\varphi(v)$$

for all $g \in G$ and $v \in V$. This will form a category, so one can reasonably talk about isomorphisms of representations and subrepresentations.

Let's describe a general way to build representations.

Proposition 73 (regular action)**.** Suppose that G has a continuous right action on a second-countable locally compact Hausdorff space X. Then X has a Haar measure, and we further assume that the Haar measure dx on X is G-invariant. Then the space $L^2(X)$ of square-integrable functions on X is a Hilbert space (in the usual way), and the action

$$(g\varphi)(x) := \varphi(xg)$$

produces a unitary representation $G \times L^2(X, dx) \to L^2(X)$.

Proof. It is not hard to see that we have a group action, and it is unitary because

$$(g\varphi_1, g\varphi_2) = \int_G \varphi_1(xg)\overline{\varphi_2(xg)}\, dx \overset{*}{=} \int_G \varphi_1(x)\overline{\varphi_2(x)}\, dx = (\varphi_1, \varphi_2),$$

where the key point is that $\overset{*}{=}$ holds because dx is G-invariant. We will not show that the action is continuous, but it is true. ■

Remark 74. Because the action is continuous, we see that the collection of compactly supported functions $C_c(X) \subseteq L^2(X)$ is preserved by the given G-action.

Example 75. Note G acts on itself on the right by right multiplication, so we can take $X = G$ above. Notably, we must take a right Haar measure $d_r g$ on G to make this work.

7.2 The Modular Character

To produce other examples, we recall the following notion.

Definition 76 (unimodular)**.** A locally compact Hausdorff group G is *unimodular* if and only if its left Haar measures are also right Haar measures.

Example 77. Fix a locally compact topological ring R. Then $\mathrm{GL}_n(R)$ has Haar measure

$$\frac{dx_{11}\cdots dx_{nn}}{|\det(x_{ij})|}.$$

This is both a left and right Haar measure, so $\mathrm{GL}_n(R)$ is unimodular.

Example 78. If G is abelian, then of course left and right multiplications are the same, so G is unimodular.

We will explain later how to check that $G(R)$ is unimodular for any reductive group G.

Here is why we care about unimodularity (right now).

Example 79. If G is unimodular, and $H \subseteq G$ is some locally compact Hausdorff subgroup, then $H\backslash G$ gets a right G-action, and it even has a quotient measure which is G-invariant. Thus, we can also consider the representation $G \times L^2(H\backslash G) \to L^2(H\backslash G)$ from Proposition 73.

As such, we would like some ways to detect if a group is unimodular.

Definition 80 (modular character)**.** Fix a locally compact group G. Fixing a right Haar measure d_rg on G, we note that $d_r(gh)$ is still a right Haar measure for any $h \in G$, where $d_r(gh)$ is defined by $d_r(gh)(U) := d_rg(Uh)$. Because Haar measures are unique up to scalar, there exists a scalar $\delta_G(h) \in \mathbb{R}^+$ such that

$$d_r(gh) = \delta_G(h)\, d_rg.$$

Remark 81. It turns out δ_G is a continuous character. Checking the continuity is a corollary of the proof of the existence of the Haar measure, so we will omit it. Checking that it is a character follows from its definition.

Remark 82. Importantly, we see that

$$d_rg(hU) = \delta_G(h) d_rg(U)$$

for any h and U, so G is unimodular if and only if δ_G is the trivial character.

Example 83. If G is compact, then $\operatorname{im}\delta_G$ must be a compact subgroup of $\mathbb{R}^+$. However, the only such subgroup is $\{1\}$, so δ_G is identically 1, so G is unimodular.

7.3 Applications to Affine Groups

We now return to the setting where G is a smooth affine algebraic group defined over a field k. Recall that there is a Lie algebra

$$\mathfrak{g} := \operatorname{Lie} G := \ker\left(G\left(\frac{k\,[t]}{(t^2)}\right) \to G(k)\right),$$

where the morphism is given by $t \mapsto 0$. One can show that this in bijection with derivations at the identity of $\mathfrak{g}$, so $\mathfrak{g}$ is a vector space over k. Namely, one should think of $\mathfrak{g}$ as the tangent space of G at the identity.

For example, the adjoint action $\operatorname{Ad}_g\colon G \to G$ for each $g \in G$ (given by conjugation $h \mapsto ghg^{-1}$) to an action on the Lie algebra $\operatorname{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$, so we get a representation of G in $\mathfrak{g}$. More explicitly, for each k-algebra R, we see that

$$\mathfrak{g}(R) := \ker\left(G\left(\frac{R[t]}{(t^2)} \to G(R)\right)\right)$$

is $\mathfrak{g} \otimes_k R$. Now, $G(R)$ has an action on both the source and the target of our map by embedding $R \hookrightarrow R[t]/\left(t^2\right)$.

Example 84. For $G = \mathrm{GL}_n$, one finds (as expected) that $\mathfrak{g}$ consists of $n \times n$ matrices, and the adjoint action is given by conjugation of matrices.

We now specialize to the case where $k := F$ is a global field, and let v be a place. Now, as in the study of Lie groups, one can use a left G-invariant top differential form $\omega \in \wedge^n \mathfrak{g}^*$ to produce a left Haar measure on $G(F_v)$ via integrating compactly supported functions $f \in C_c^\infty(G(F_v))$ against ω.

The moral of the story is that this adjoint action lets us algebraically compute δ!

Proposition 85. Fix notation as above. For each $g \in G(F_v)$, we have

$$\delta_{G(F_v)}(g) = \left|\det \operatorname{Ad}_g \in \mathrm{GL}(\mathfrak{g}(F_v))\right|.$$

Sketch. The main point is that the modular character can be measured locally, so this will more or less arise from the fact that δ measures how much "conjugation by g" impacts a volume computation, which intuitively is the determinant. ■

Corollary 86. If $G^\circ \subseteq G$ is reductive, then $G(F_v)$ is unimodular.

Proof. It is enough to check that δ vanishes on $G^\circ(F_v)$ (because the quotient $G(F_v)/G^\circ(F_v)$ is finite), so we may assume that G is connected. Then reductive groups split into semisimple parts (which have no nontrivial characters, and hence $\delta = 1$) and abelian (which has $\delta = 1$ for free), so we are done. ■

One can then glue this to see that $G(\mathbb{A}_F)$ is also unimodular!

Corollary 87. Any right Haar measure on $G(\mathbb{A}_F)$ is left invariant for $G(F)$. In fact, $G(\mathbb{A}_F)^1$ is unimodular.

Proof. We want to check that δ vanishes on $G(\mathbb{A}_F)^1 \supseteq G(F)$, for which we use the product formula applied to the formula in Proposition 85. ■

8 October 23: Chris Yao

Today we talk about archimedean representation theory.

8.1 Defining $(\mathfrak{g}, K)$-modules

Today, G will be an affine algebraic group over an archimedean field F, and K will be a compact Lie subgroup of G. We will work with Hilbert spaces throughout, but there is no difficulty in passing to Frechet spaces. Let $\mathfrak{g}$ be the Lie algebra of G. We begin by defining smooth vectors.

Definition 88 (smooth vector). Let (π, V) be a Hilbert space representation of $G(F)$. For $\varphi \in V$ and $X \in \mathfrak{g}$, we define

$$\pi(X)\varphi := \frac{d}{dt}\pi(\exp(tX))\varphi\bigg|_{t=0} = \lim_{t\to 0}\frac{\pi(\exp(tX))\varphi - \varphi}{t},$$

provided the limit exists. (We will abbreviate this to $X\varphi$ whenever possible.) If $X\varphi$ exists for all $X \in \mathfrak{g}$, we say that X is C^1. Recursively, if $X\varphi$ is in C^k for all X, then we say that φ is C^{k+1}. A vector is *smooth* if and only if it is C^∞; we let V_{sm} denote the collection of smooth vectors.

Remark 89. One can check that φ is smooth if and only if the map $g \mapsto \pi(g)\varphi$ is a smooth function $G(F) \to V$.

Remark 90. By interchanging derivatives with actions, we find that the collection of smooth vectors is invariant under $G(F)$.

Remark 91. From the construction, we see that the collection of smooth vectors grants a Lie algebra representation of $\mathfrak{g}$.

The other part of the definition of a $(\mathfrak{g}, K)$-module comes from restriction to a compact subgroup.

Lemma 92. Fix a locally compact Hausdorff group G, and let $K \subseteq G$ be a compact subgroup. For any Hilbert space representation (π, V), there is a K-invariant Hermitian inner product on V.

Proof. Average the inner product over K using the Haar measure to make sense of integration. ∎

The point is that compact groups can unitarize their representations. In particular, taking orthogonal complements allows us to see that any irreducible subrepresentation has a complement, which helps prove the following theorem.

Notation 93. Fix a compact Hausdorff group K. Then we let $\widehat{K}$ denote the collection of irreducible representations of K, which we make unitary.

Theorem 94 (Peter–Weyl). Fix a compact Hausdorf group K.

(a) Any irreducible representation is finite-dimensional.

(b) Any unitary representation decomposes into a Hilbert space direct sum of unitary representations in $\widehat{K}$.

(c) There is an isomorphism

$$L^2(K) \to \widehat{\bigoplus_{\pi\in\widehat{K}}} \operatorname{End} V_\pi.$$

We will not say more about this proof; it is rather hard.

We now move directly towards defining a $(\mathfrak{g}, K)$-module. Well, we would like to temper our representations; one reasonable way to do this is to ask for some kind of finiteness condition, but there are infinitely many elements of $\widehat{K}$ in general

Definition 95. Fix a compact Hausdorff group K. Given a representation (π, V) and $\sigma \in \widehat{K}$, we define

$$V(\sigma) = \{\varphi \in V : \{\pi(g)\varphi : g \in K\} \cong \sigma\}.$$

We call $V(\sigma)$ the σ*-isotypic subspace*, and we say that $v \in V(\sigma)$ has K-type of σ; if $v \in V(\sigma)$ for any σ, we say that v is K*-finite* (because σ is finite-dimensional), and we let V_{fin} denote the collection of these vectors.

Remark 96. Theorem 94 implies that V_{fin} will be dense in V, so there are many of these vectors.

Definition 97 (admissible)**.** A representation (π, V) of a compact Hausdorff group K is *admissible* if and only if $\dim V(\sigma) < \infty$ for all $\sigma \in \widehat{K}$.

Admissible is flexible enough for our purposes, as the following theorem claims.

Theorem 98. If G is reductive, then the unitary representations of $G(F)$ are admissible.

We are now ready to define $(\mathfrak{g}, K)$-modules.

Definition 99 ($(\mathfrak{g}, K)$-module)**.** Fix a compact Lie subgroup K of G and Lie algebra $\mathfrak{g}$ of G. Then a $(\mathfrak{g}, K)$-module is a vector space V equipped with representations $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and $K \to \mathrm{GL}(V)$ satisfying the following.

(a) V is a countable direct sum of finite-dimensional K-invariant subrepresentations.

(b) For all $X \in \mathfrak{k}$ and $\varphi \in V$, the derivative $X\varphi$ exists.

(c) For $k \in K$ and $X \in \mathfrak{g}$, we have

$$\pi(k)\pi(X)\pi(k)^{-1}\varphi = \pi(\mathrm{Ad}_k X)\varphi.$$

The $(\mathfrak{g}, K)$-module is *admissible* if and only if $V(\sigma)$ is finite-dimensional for all $\sigma \in \widehat{K}$.

As usual, one can define morphisms as preserving the relevant $\mathfrak{g}$- and K-actions. This allows us to define $(\mathfrak{g}, K)$-submodules and irreducible $(\mathfrak{g}, K)$-modules.

Example 100. Any Hilbert space representation (π, V) of $G(F)$ makes $V_{\text{fin}} \cap V_{\text{sm}}$ densse in V, and it provides an example of a $(\mathfrak{g}, K)$-module. One shows this by convolution with a smooth function.

Remark 101. In fact, if V is admissible, then $V_{\text{fin}} \subseteq V_{\text{sm}}$.

The underlying $(\mathfrak{g}, K)$-representation remembers quite a bit about the representation theory of G.

Theorem 102. If G is reductive, then an admissible Hilbert space representation (π, V) is irreducible if and only if the underlying $(\mathfrak{g}, K)$-representation is irreducible.

In fact, we can discuss some equivalence.

Definition 103 (infinitesimal equivalence)**.** Two admissible representations of $G(F)$ are *infinitesimally equivalent* if and only if their $(\mathfrak{g}, K)$-modules are isomorphic.

Theorem 104. Two unitary representations of $G(F)$ are equivalent if and only if they are infinitesimally quivalent.

8.2 $(\mathfrak{g}, K)$-modules for $\mathrm{GL}_2(\mathbb{R})$

We now classify $(\mathfrak{g}, K)$-modules for $\mathrm{GL}_2(\mathbb{R})$. Take $G := \mathrm{GL}_2(\mathbb{R})$, and we note that $K := \mathrm{O}_2(\mathbb{R})$ is a maximal compact subgroup with $K^\circ = \mathrm{SO}_2(\mathbb{R})$. We let $B \subseteq G$ be the Borel subgroup of upper-triangular matrices. We now define a few representations.

Definition 105 (principal series)**.** Given a character χ of B, we let the induced representation $\mathrm{Ind}_B^G \chi$ be a *principal series representation*.

Remark 106. Technically, this induction should remember being integrable in some suitable sense, but we will ignore this.

Definition 107 (discrete series)**.** An irreducible unitary representation of G is *square-integrable* if and only if it is isomorphic to an irreducible subrepresentation of the regular representation $L^2(G, \chi)$, where χ is some character. We call these representations the *discrete series*.

The following theorem will be our key tool.

Theorem 108 (subrepresentation)**.** One has the following.

(a) Every irreducible $(\mathfrak{g}, K)$-module arises from an irreducible admissible representation of G.

(b) Every irreducible admissible representation of G is infinitesimally equivalent to a subrepresentation of a principal series of G.

The moral of the story is that we are convinced to be interested in the $(\mathfrak{g}, K)$-modules of G. We do this in steps.

1. We classify representations of K. Well, $K = \mathrm{O}_2(\mathbb{R})$ is an extension of $\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$. Thus, we note that there is a determinant character $\det\colon \mathrm{O}_2(\mathbb{R}) \to \mathbb{C}^\times$, and the only characters of $\mathrm{SO}_2(\mathbb{R})$ are given by $\varepsilon_n(\kappa_\theta) := e^{in\theta}$ where $\kappa_\theta \in \mathrm{SO}_2(\mathbb{R})$ is rotation by θ. One can then show that the only irreducible representations of K are gien by the trivial character, $\det$, and
$$\tau_n := \mathrm{Ind}_{K^\circ}^K \varepsilon_n.$$

2. Next, we note that all characters of B are given by characters of the diagonal because $[B, B]$ contains the upper-triangular matrices of the form $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$. As such, our unitary characters $\chi\colon B \to \mathbb{C}^\times$ look like
$$\chi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = \chi_1(a)\chi_2(b),$$
where χ_1 and χ_2 take the form $x \mapsto \mathrm{sgn}(x)^\varepsilon |x|^s$ where $\varepsilon \in \{0, 1\}$ and s is purely imaginary. (Namely, s being purely imaginary is what gives being unitary.) We will say $\chi = \chi(\varepsilon_1, \varepsilon_2, s_1, s_2)$ where $\varepsilon_\bullet \in \{0, 1\}$ and $s_\bullet \in \mathbb{C}$ is purely imaginary.

3. We now induce from B to G to produce representations $V_\chi^\circ := \mathrm{Ind}_B^G \chi$, and we let V_χ be its completion with respect to the natural inner product. Technically speaking, to make sense of the inner product, one must be careful with the modular character of B, so V_χ° contains the functions $f\colon G \to \mathbb{C}$ satisfying
$$f(bg) = \chi(b)\sqrt{|a/b|} \cdot f(g)$$
for all $g \in G$ and $b \in B$. The point of this is that χ being unitary yields V_χ unitary.

4. From here, one needs to decompose the various V_χ into their K-components and look for what we found.

9 October 30: Jerry Yang

Today we will discuss the representation theory of totally disconnected groups.

9.1 Totally Disconnected Groups

Let's give some motivation for what we are going to discuss today.

- We will not state the local Langlands correspondence, but let us note that the representation theory of totally disconnected groups (in particular, some p-adic groups) plays a key role. For a rough sketch, fix a nonarchimedean local field F_v, and define the Weil–Deligne group W_v to be generated by the Frobenius and the inertia subgroup of $\operatorname{Gal}(F_v^{\text{sep}}/F_v)$. Then the local Langlands correspondence gives a correspondence between n-dimensional representations of W_v and irreducible smooth admissible representations of $\mathrm{GL}_n(F_v)$. It is a primary goal of this talk to define the adjectives "irreducible," "smooth," and "admissible."
- In this seminar, we are interested in representations of $G(\mathbb{A}_F)$ for global fields F. It will turn out (via Flath's theorem) that the representation theory for $G(\mathbb{A}_F)$ will split into representation theory of the localizations.

We begin by discussing totally disconnected groups, which include our p-adic groups.

Definition 109 (totally disconnected)**.** A locally compact Hausdorff space X is *totally disconnected* or *td* if and only if any of the following equivalent conditions are satisfied.

(a) The topology of X has a basis of open compact sets. In particular, this basis is of clopen sets.

(b) Every subspace $A \subseteq X$ with more than one point is disconnected.

(c) The space X is "locally profinite," in that any point $x \in X$ has an open neighborhood which looks like a profinite set.

Example 110. We can show that a locally compact Hausdorff group G is td if and only if it has a neighborhood basis of the identity of open compact subgroups. For example, $G = \mathbb{Q}_p$ has a neighborhood basis of the identity given by the open sets $p^\bullet\mathbb{Z}_p$.

Example 111. More generally, if F is any nonarchimedean local field, then the F-points $G(F)$ of any affine algebraic group scheme G over F is a td group.

The point is that td groups have very small subgroups around the identity. This is in contrast to the theory of real or complex Lie groups, where open neighborhoods of the identity frequently fail to be closed under multiplication. For example, this allows us to show prove the following.

Lemma 112. The kernel of any homomorphism $\varphi\colon G \to H$ from a td group G to a real or complex Lie group H is open.

Remark 113. There is a continuous character on $\mathbb{Q}_p^\times$ given in Tate's thesis which has dense image in S^1. Thus, it is not true that the image is discrete if G fails to be compact.

The moral is that the theory of complex(!) representations of G is allowed to assume that the target groups are given by the discrete topology. With this in mind, we brutally take the following definition.

Definition 114 (smooth)**.** Fix a td group G and a complex vector space V. Then a function $f\colon G \to V$ is *smooth* if and only if it is locally constant. This collection of smooth functions is denoted by $C^\infty(G, V)$, and we let $C_c^\infty(G, V)$ be the subspace of complactly supported functions.

9.2 Representation Theory

We are now ready to define smooth and admissible representations. However, everything should be locally constant for our td groups.

Definition 115 (smooth)**.** Fix a td group G and a complex vector space V. A representation $\pi\colon G \times V \to V$ is *smooth* if and only if the stabilizer of each $v \in V$ is open in G.

Remark 116. One can check that this is equivalent to saying that there is an open compact neighborhood of the identity of G acting trivially on V.

Remark 117. For smoothness, we recall that smoothness for archimedean representations amounted to being able to define a Lie algebra representation. Because we are locally constant, this does not exactly make sense for our context.

Let's see some examples.

Non-Example 118. Choose an isomorphism $\overline{\mathbb{Q}_p} \cong \mathbb{C}$. Then the standard representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ on $\mathbb{C}^n$ is not smooth.

Example 119. Fix a td group G. Then the regular representation of G on $C_c(G, \mathbb{C})$ is smooth. Explicitly, the action is given by

$$(gf)(x) := f(xg).$$

Namely, for a locally constant function f, we see that f is a sum of finitely many indicators of open compact subsets because it is locally constant and compactly supported, and there is a subgroup of G stabilizing these finitely many open compact subgroups. If f fails to be compactly supported, then we may allow f to be a sum of infinitely many disjoint indicators.

And now we define admissibility, analogously to

Definition 120 (admissible)**.** Fix a td group G and a complex vector space V. A representation $\pi\colon G \times V \to V$ is *admissible* if and only if it is smooth, and

$$V^K = \{v \in V : K \text{ stabilizes } v\}$$

is finite-dimensional for each open compact subroup $K \subseteq G$.

This hypothesis is fairly flexible.

Theorem 121. Fix a nonarchimedean local field F, and let G be a reductive group over F.

(a) Any smooth irreducible representation of $G(F)$ is admissible.

(b) Any irreducible unitary representation of $G(F)$ is admissible. More precisely, if V is a Hilbert space representation of $G(F)$, then we can restrict our representation to the "smooth part"

$$V_{\mathrm{sm}} := \bigcup_{\substack{K \subseteq G \\ K \text{ open compact}}} V^K,$$

which turns out to be admissible.

Remark 122. Given any complex representation V, one can define V_{sm} as above, and it turns out that $V_{\mathrm{sm}} \subseteq V$ is a dense subrepresentation, and V_{sm} is irreducible if and only if V is.

Example 123. Any smooth representation of $\mathrm{GL}_1(\mathbb{Q}_p)$ is 1-dimensional, given basically generated by the two characters $x \mapsto |x|^s$ for some $s \in \mathbb{C}$ and $x \mapsto \omega(x)$, where ω is the cyclotomic character.

Example 124 (Steinberg)**.** The representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ is fairly involved, but let's give an interesting example. Let St consist of the locally constant functions $\mathbb{P}^1_{\mathbb{Q}_p} \to \mathbb{C}$ modulo the constnat functions. Then $\mathrm{GL}_2(\mathbb{Q}_p)$ has a natural action on $\mathbb{P}^1_{\mathbb{Q}_p}$, so $\mathrm{GL}_2(\mathbb{Q}_p)$ gets an action on St.

9.3 Hecke Algebras

It turns out that one can view representations of G as modules over a Hecke algebra. We begin by considering modules over $C_c^\infty(G, \mathbb{C})$.

Definition 125 (non-degenerate)**.** Fix a td group G. A $C_c^\infty(G, \mathbb{C})$-module M is *non-degenerate* if and only if any element of M can be written as a linear combination

$$a_1 m_1 + \cdots + a_r m_r$$

where $a_1, \ldots, a_r \in C_c^\infty(G, \mathbb{C})$ and $m_1, \ldots, m_r \in M$.

Observe that this definition would have no content if $C_c^\infty(G, \mathbb{C})$ has a unit, but this frequently fails to be the case. In general, there is some approximate identity given by taking an ascending limit of functions

$$e_K := \frac{1}{\mu(K)} 1_K$$

over all compact open subgroups $K \subseteq G$.

The point is that non-degenerate modules turn out to be in bijection with representations. We now define the Hecke algebra.

Definition 126 (unramified Hecke algebra)**.** Fix a td group G and open compact subgroup K. Then the *unramified Hecke algebra* $C_c^\infty(G/K)$ consists of the functions which are K-bi-invariant. It has ring multiplication given by convolution.

The point is that we can recover K-invariants of a representation V by $e_k \cdot V = V^K$.

Let's end by explaining the use of the Hecke algebra.

Theorem 127. Fix a nonarchimedean local field F_v, and let G be a reductive group over F_v with open compact subgroup $K \subseteq G$. Let $\mathcal{G}$ be a model, producing a hyperspecial subgroup $\mathcal{G}(\mathcal{O}_v)$. For any smooth representation V, if V^K is nonzero, then $\dim V^K = 1$.

10 November 6: Sam Mayo

Today we are talking about the Satake isomorphism. For today, F is a nonarchimedean local field with ring of integers $\mathcal{O}$, and $\pi \in \mathcal{O}$ is a uniformizer. We let q denote the size of the residue field. Lastly, we set $G := \mathrm{GL}_n$, and we let T be the maximal split torus of diagonal matrices, and our Weyl group is $W := S_n$ (which acts on T by permuting the diagonal entries).

10.1 Spherical Representations

We are interested in smooth representations of $G(F)$. We begin with a special subclass.

Definition 128 (spherical, unramified)**.** Fix a split reductive group G over $\mathcal{O}$. A smooth irreducible representation V of G is *spherical* or *unramified* if and only if there is some $v \in V$ fixed by $G(\mathcal{O})$.

The term "spherical" is supposed to be analogous to what happens in the archimedean case, where maximal compact subgroups (which $G(\mathcal{O})$ is supposed to be analogous to) look like spheres.

We also have a notion of Hecke algebra.

Definition 129 (spherical Hecke algebra)**.** Fix a split reductive group G over $\mathcal{O}$. The *spherical Hecke algebra* for G, denoted $\mathcal{H}^0(G)$ consists of the functions $G(F) \to \mathbb{C}$ which are locally constant, compactly supported, and $G(\mathcal{O})$-invariant on the left and right.

The hypotheses of $\mathcal{H}^0(G)$ work together to give it the structure of an algebra with multiplication given by the convolution operation

$$(f * g)(x) := \int_G f(z)g\left(z^{-1}x\right)\,dz.$$

Here is the point.

Theorem 130. Fix a split reductive group G over $\mathcal{O}$. There is a bijection between irreducible spherical representations V of G and irreducible representations of $\mathcal{H}^0(G)$.

In the forward direction, one takes irreducible spherical representations V to the subspace of $G(\mathcal{O})$-invariants $V^{G(\mathcal{O})}$. Notably, $V^{G(\mathcal{O})}$ becomes a representation of $\mathcal{H}^0(G)$ by the action

$$f \cdot v := \int_G f(g)(gv)\,dg.$$

In fact, one can upgrade this construction to an equivalence of categories. The point is that one should think about $\mathcal{H}^0(G)$ as the group ring for spherical representations.

10.2 The Cartan Decomposition

Let's see what our Hecke algebra looks like for T.

Example 131. Spherical representations of T (which is abelian) are equivalent to representations of the quotient $T(F)/T(\mathcal{O})$ because having one $T(\mathcal{O})$-fixed vector means that everything is fixed by $T(\mathcal{O})$. However, T is abelian, so we are really looking at representations of the double quotient

$$T(\mathcal{O})\backslash T(F)/T(\mathcal{O}).$$

Let's try to understand this quotient. By splitting $T = \mathbb{G}_m^r$, we see that $T(F)/T(\mathcal{O})$ is isomorphic to $(F^\times/\mathcal{O}^\times)^r = \mathbb{Z}^r$, where the second isomorphism is given by taking the norm. To avoid having to take a splitting, one should work with the full coweight lattice $X_*(T)$ of homomorphisms $\mathbb{G}_m \to T$, and then we have a short exact sequence

$$1 \to T(\mathcal{O}) \to T(F) \to X_*(T) \to 1,$$

which is independent of any choices. As such, one finds that $\mathcal{H}^0(T) = \mathbb{C}[X_*(T)]$.

For $G = \mathrm{GL}_n$, we need the following linear algebra result.

Theorem 132 (Cartan decomposition)**.** One has

$$\mathrm{GL}_n(F) = \bigsqcup_{\substack{k_1 \le \cdots \le k_n \\ k_1,\ldots,k_n \in \mathbb{Z}^n}} G(\mathcal{O}) \begin{bmatrix} \pi^{k_1} & & \\ & \ddots & \\ & & \pi^{k_n} \end{bmatrix} G(\mathcal{O}).$$

This is simply the Smith normal form for $\mathcal{O}$; in particular, one can hope to do some explicit linear algebra to do the decomposition. In general, one has the following result, given by translating the notion of these increasing sequeneces into weights.

Theorem 133 (Cartan decomposition)**.** Fix a split reductive group G over $\mathcal{O}$. Then

$$G(F) = \bigsqcup_{\substack{\lambda \in X_*(T) \\ \lambda \text{ dominant}}} G(\mathcal{O})\lambda(\pi)G(\mathcal{O}).$$

The point is that $\mathcal{H}^0(G)$ has a basis given by indicators on these double cosets $G(\mathcal{O})\lambda(\pi)G(\mathcal{O})$. In fact, by computing the commutators explicitly, one can see that $\mathcal{H}^0(G)$ is commutative.

Corollary 134. Fix a split reductive group G over $\mathcal{O}$. Then $\mathcal{H}^0(G)$ is commutative.

Proof. We write out the argument for $G = \mathrm{GL}_n$. The point is that we have an anti-involution $(\cdot)^\intercal$ on G (this is the transpose), so it extends to an anti-involution on the spherical Hecke algebra $\mathcal{H}^0(\mathrm{GL}_n)$ via $f^\intercal(x) := f(x^\intercal)$. Namely, we see that

$$(fg)^\intercal = g^\intercal f^\intercal$$

by a direct computation. However, by computing $(\cdot)^\intercal$ on the basis, we see that it is the identity, so we conclude that $\mathcal{H}^0(G)$ is commutative! For general G, one can use a similar involution called the Chevalley involution. ■

Corollary 135. Fix a split reductive group G over $\mathcal{O}$. The irreducible representations of $\mathcal{H}^0(G)$ are one-dimensional.

Proof. This is immediate from $\mathcal{H}^0(G)$ being commutative. ■

Corollary 136. Fix a split reductive group G over $\mathcal{O}$ with maximal split torus T and Weyl group W. There is an isomorphism $\mathcal{H}^0(G) \to \mathcal{H}^0(T)^W$ of vector spaces.

Proof. We know that $\mathcal{H}^0(G)$ has basis indexed by dominant coweights in $X_*(T)$, which can then be summed over W to produce something in $\mathcal{H}^0(T)^W$. ∎

10.3 The Satake Transform

Corollary 136 is not trying to produce an isomorphism of algebras, which is false. To fix this, we will instead try to build a map $\mathcal{H}^0(G) \to \mathcal{H}^0(T)$ by restriction to T. This is a little funny to do, for the following somewhat vague reason: the torus T should really be thought of as a quotient B/N where $B \subseteq G$ is some Borel subgroup, and N is its unipotent radical. Borel subgroups are somewhat more natural to look at from the perspective of representation theory.

We are now ready to define the Satake transform. Given $f \in \mathcal{H}^0(G)$, we want to restrict along the diagram

$$G \supseteq B \twoheadrightarrow B/N \cong T.$$

The inclusion $G \supseteq B$ should correspond to an actual restriction, but along $B \twoheadrightarrow B/N$ one should be integrating along fibers. As such, we define

$$Sf(t) := \delta_B(t)^{1/2} \int_N f(tn)\, dn,$$

where δ_B is the modular character. (The modular character is needed to make sure that our integrals are suitably invariant.)

Theorem 137 (Satake)**.** Fix a split reductive group G over $\mathcal{O}$. Let B be a Borel subgroup with unipotent radical N, and set $T := B/N$ to be a maximal split torus. Then $S\colon \mathcal{H}^0(G) \to \mathcal{H}^0(T)^W$ is an isomorphism.

This is fairly difficult to show, though one can of course write everything out in sufficient detail for (say) $G = \mathrm{SL}_2$. Roughly speaking, one sends the basis of indicator functions to the restriction with some correction terms coming from smaller coweights.

10.4 A Little Langlands

To end our discussion, we say something about the local Langlands correspondence. The classical theory of Lie algebras and Lie groups provides a bijection between finite-dimensional irreducible representations of a complex Lie group $\check{G}(\mathbb{C})$ with dominant weights (which are in bijection with W-orbits in $X^*(T)$).

One can translate this into an isomorphism of the representation ring of $\check{G}(\mathbb{C})$ with $\mathbb{C}[X^*(T)]^W$, which looks suspiciously similar to what we have done above in the nonarchimedean setting. However, the caveat is that the rest of our discussion has been working with $X_*(T)$ instead of $X^*(T)$. As such, one needs a notion of duality: given our nonarchimedean group G, one knows that there is a dual group ${}^LG$ whose torus $\check{T}$ canonically has $X_*(T) \cong X^*(\check{T})$. Then Langlands duality should boil down to the observation

$$\mathcal{H}^0(G) \overset{S}{\cong} \mathcal{H}^0(T)^W \cong \mathbb{C}[X_*(T)]^W \cong \mathbb{C}[X^*(\check{T})]^W \cong \operatorname{Rep} \check{G}(\mathbb{C}).$$

Thus, we have related nonarchimedean representation theory to complex representation theory! Explicitly, we see that irreducible smooth spherical represntations of V are in bijection with irreducible modules of $\mathcal{H}^0(G)$, which are just one-dimensional and hence in bijection with maps $\mathcal{H}^0(G) \to \mathbb{C}$. The above chain then puts this in bijection with maps $\operatorname{Rep} \check{G}(\mathbb{C}) \to \mathbb{C}$, which classical representation theory argues is in bijection with semisimple conjugacy classes (given by some character theory).

Remark 138. One can construct an inverse for this construction given by principal series!

11 November 13: Sam Goldberg

Today we have a sidequest talk about automorphic representations and quantum chaos.

11.1 Classical Chaos

Our motivation comes from physics. We have a Riemannian manifold (M, g), which we will assume to be complete and a surface for ease. Then there is a "geodesic flow" G on T^*M. Namely, for any position-velocity pair $(x, v) \in T^*M$, one has

$$G^t(x, v) = (\gamma(t), \gamma'(t)),$$

where γ is a geodesic starting with (x, v). Some computation shows that the unit cotangent (fiber) bundle UT^*M is invariant under G.

More specifically, today we will work Riemannian surfaces with constant curvature. Up to coverings, there are three of these: $\mathbb{R}^2$ (which has 0 curvature), S^2 (which has positive curvature), and $\mathbb{H}^2$ (which has negative curvature).

Remark 139. Hyperbolic space kinda looks like a pringle chip. Sam does not want to draw a pringle chip.

More precisely, any complete connected surface M with constant curvature has one of $\mathbb{R}^2$, S^2, or $\mathbb{H}^2$ as a universal cover. In fact, we can see that the quotient map can be realized as a group quotient by some discrete subgroup of isometries.

Example 140. The torus T is a surface with constant curvature. It has universal cover $\mathbb{R}^2 \twoheadrightarrow T$, and we can see we can realize T as $\mathbb{Z}^2\backslash\mathbb{R}^2$.

As such, we will be interested in dynamics on $\Gamma\backslash\mathbb{H}^2$.

Theorem 141. Fix a connected hyperbolic surface M of finite volume. Then the geodesic flow on UT^*M is ergodic. For example, this implies that the geodesic flow applied to a point $(x, v) \in UT^*M$ equidistributes over UT^*M.

11.2 Quantum Chaos

Theorem 141 is "classical chaos." We would like a formulation in quantum mechanics. To do this mathematically, we begin with a C^*-algebra $\mathcal{A}$, which is a subalgebra of some bounded (i.e., continuous) operators on a Hilbert space, complete with respect to the norm and closed under the adjoint.

Example 142. Bounded functions on a Hilbert space itself is a C^*-algebra.

Example 143. Complex continuous functions on a locally compact Hausdorff space X make a C^*-algebra; adjoint becomes complex conjugation.

An observable in quantum mechanics amounts to functionals on our C^*-algebra $\mathcal{A}$, and a state should be a positive linear functional sending the identity of $\mathcal{A}$ to 1, from which we get a probability by evaluating at the point. We further recover a notion of time evolution as automorphisms of $\mathcal{A}$.

Concretely, revisiting our classical mechanics, the C^*-algebra will (roughly) be $L^2(M)$, where the measure is given by the Riemannian metric, and its states and observables are given by bounded continuous functions. And we have a time evolution given by some operator $e^{-it\Delta}$, where Δ is some generalization of the Laplacian to Riemannian manifolds. We would now like a notion of "quantum ergodicity" in this set-up.[1] Well, here is a theorem.

[1] This is predicted by the "Bohr correspondence principle."

Theorem 144. Fix a compact hyperbolic surface $\Gamma\backslash\mathbb{H}$. Then the action of Δ on $L^2(\Gamma\backslash\Delta)$ as above has an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ with discrete spectrum $\lambda_\bullet \to \infty$. Then we have a notion of equidistribution in the limit: for a bump function $\psi \in C^\infty(M)$, we have

$$\lim_{j\to\infty} \langle \psi\varphi_j, \varphi_j\rangle = \int_M \psi.$$

This was originally proven under the assumption that we can remove a natural density 0 subset of "exceptional" eigenfunctions. To explain why this is hard, there is a geodesic on $\mathrm{SL}_2(\mathbb{Z})$ which more or less fails to equidistribute. Here is some background, where much is known for arithmetic subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.

- Conditional under GRH, this was proven in 2002.
- In the compact case, this was proven in 2006.
- In the non-compact case, this was pveon in 2010.

11.3 Automorphic Forms

Automorphic forms will come from studying $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We are hoping to understand eigenfunctions on this space. Number theory allows us to bring in the adeles: there is a homeomorphism

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong Z(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}))\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K,$$

where $K \subseteq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is some maximal compact subgroup.

In higher dimensions, we recall $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ is locally compact, so it admits a Haar measure, so we get a space $L^2(\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}))$. Being careful about a quotient, we even get

$$L^2\big(Z(\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}))\,\mathrm{GL}_n(\mathbb{Q})\backslash\,\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})\big).$$

For us, our automorphic representations will be seen as irreducible unitary subrepresentations of the above space. For brevity, set $[G] := Z(\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}))\,\mathrm{GL}_n(\mathbb{Q})\backslash\,\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. One finds that $L^2([G])$ decompses into the following pieces.

- There are cuspidal and residual parts, which are completely reducible.
- Then there is a continuous spectrum which (for example) includes Eisenstein series.

To recover "classical" modular forms, we note that we have an action by $\mathrm{GL}_2(\mathbb{R})$, so we get a $(\mathfrak{g}, K)$-module where $\mathfrak{g} = \mathrm{gl}_2(\mathbb{R})_\mathbb{C}$ and $K = \mathrm{SO}_2(\mathbb{R})$. We can further decompose this via the diagonalizable actions of $Z(U\mathfrak{g})$ and K and the Hecke algebra, which eventually produces a classical automorphic form. Anyway, it will turn out that all these pieces will be able to produce an eigenbasis for our Laplacian Δ.

Remark 145. This decomposition was much earlier known to Selberg in the case of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. This is notable because it allows us to prove something like Parseval's theorem for the non-compact space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

The point is that one is able to use these sorts of decompositions (and in particular our Parseval's identity) to prove Theorem 144 in our case.

12 November 20: Sai Sanjeev Balakrishnan

Today we define an automorphic form.

12.1 Our Definitions

For today, F is a number field; there are definitions for general global fields, but there are a few caveats. Here is our definition.

Definition 146 (automorphic form)**.** Fix an affine algeraic group G over F. A function $\varphi\colon G(\mathbb{A}_F) \to \mathbb{C}$ is an *automorphic form* if and only if it satisfies the following conditions.

(a) φ is smooth.

(b) φ had moderate growth.

(c) φ is $G(F)$-invariant.

(d) φ is K-finite, where $K \subseteq G(\mathbb{A}_F)$ is some subgroup factoring into $K_\infty \subseteq G(F_\infty)$ as a maximal compact subgroup, and $K^\infty \subseteq G(\mathbb{A}_F^\infty)$ is compact open.

(e) φ is $Z(\mathfrak{g})$-finite.

Remark 147. If it turns out that $\int_[N]\varphi(ng)\,dg = 0$ for any unipotent radical N of a parabolic subgroup P of G, then the automorphic form is called cuspidal. Here, $[N] = N(F)\backslash N(\mathbb{A}_F)$.

Remark 148. It turns out that the choice of K does not matter. The point is that K-finiteness is equivalent to being K'-invariant for some other subgroup K'. Thus, it is enough to note that the infinite part K_∞ is unique up to conjugacy, and K^∞ is well-understood as a subgroup of the finite part of the adeles (for example, frequently being hyperspecial).

We will go through the various properties in an example: we will take $G = \mathrm{GL}_2$ and $F = \mathbb{Q}$. Consider a modular form $f \in S_k(N)$ for some weight k and level N. To relate this to a function on $\mathrm{GL}_2(\mathbb{A}_F)$, we must do a little work. Define the subgroup

$$\Gamma_0(N) \coloneqq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \pmod N \right\}$$

if $\mathrm{GL}_2(\mathbb{Z})$. Then one can define $K_0(N)_p$ as the corresponding subgroup of $\mathbb{Z}_p$, and we define $K_0(N) \subseteq \mathrm{GL}_2(\mathbb{A}_F)$ as the product over all ps.

Now, by strong approximation, one has an isoorphism

$$\mathbb{H}/\Gamma_0(N) \cong \mathrm{O}_2(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})/\Gamma_0(N) \cong \Gamma_0(N)\,\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

As such, we will define φ_f on $\mathrm{GL}_2(\mathbb{A}_F)$ as $f(g_\infty i)j(g_\infty i)^{-k}$, where g_∞ is some matrix in $\mathrm{GL}_2(\mathbb{R})$, and j is the automorphy factor seen in modular forms.

Let's now go through our definitions to check on φ_f.

(a) To check that φ is smooth, one remembers that we have a decomposition

$$C^\infty(G(\mathbb{A}_F)) = C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty)).$$

As such, we may view φ as a function on pairs (g_∞, g^∞), where g_∞ is the infinite part, and g^∞ is the finite part. Smoothness requires that $\varphi(g_\infty, \cdot)$ (as a function in the finite part g^∞) to be locally constant and that $\varphi(\cdot, g^\infty)$ (as a function in the infinite part g_∞) to be smooth in the usual analytic sense.

One checks this for φ_f directly from its definition given above: the level provides locally constant in the finite part, and the definition is of course smooth in the infinite part.

(b) To check that φ has moderate growth, we give an embedding $\iota\colon G \to \mathrm{GL}_n$ for some n, which then gives an embedding into SL_{2n} via $g \mapsto \left(h, h^{-1}\right)$. Thus, for $g \in G(F_v)$, we can define $\|g\|_v$ as the supremum norm of the element in $\mathrm{SL}_{2n}(F_v)$, and we define the norm $\|g\|$ for $g \in G(\mathbb{A}_F)$ by taking the product of the local norms, which we note is finite because the product will only have finitely many terms not equal to 1 by the integrality of the adeles. Moderate growth now asserts that

$$|\varphi(g)| \le C\,\|g\|^R$$

for some C and R. (One can check that this does not depend on the choice of embedding.)

For φ_f, one must recall the bound on modular forms given by

$$\left|y^{k/2}f(x+iy)\right| < C$$

for some $C > 0$.

(c) Left invariance by $G(F)$ is self-explanatory. One checks this for φ_f by construction because we lifted the function on $\mathrm{GL}_2(\mathbb{R})^+$ to $\mathrm{GL}_2(\mathbb{A}_F)$ by having a quotient on the left by $\mathrm{GL}_2(\mathbb{Q})$.

(d) Being K-finite is equivalent to asserting that the functions $k\varphi \in G(\mathbb{A}_F)$ defined by $(k\varphi)(g) \coloneqq \varphi(gk)$ spans a finite-dimensional vector space over $\mathbb{C}$.

In our example, our function φ_f is invariant on $K_\infty = \mathrm{SO}_2(\mathbb{R})$ by its construction, and we have taken a quotient by $\mathrm{GL}_2(\mathbb{Q})$, so we will be fully invariant by $\mathrm{O}_2(\mathbb{R})$. Also, it is invariant on $K_0(N)$ again by construction of the modular form and the automorphy factor. The K-finiteness follows.

(e) For $Z\mathfrak{g}$-finiteness, we begin by definining $\mathfrak{g}_v$ as the Lie algebra of $\operatorname{Res}_{F_v/\mathbb{R}} G_{F_v}$ for each infinite place v, and then we set $\mathfrak{g}_\infty \coloneqq \prod_{v\mid\infty}\mathfrak{g}_v$. Then

$$Z\mathfrak{g} \coloneqq Z(U(\mathfrak{g}_\infty)_\mathbb{C}),$$

and we are asking for finiteness exactly as in (d): the action of $Z(U\mathfrak{g}_\mathbb{C})$ on φ spans a finite-dimensional vector space over $\mathbb{C}$. Let's recall what this aciton is: note that we have an action of $X \in \mathfrak{g}$ on φ in the usual way by

$$(X\varphi)(g) \coloneqq \lim_{t\to 0}\frac{(\exp(tX)\varphi - \varphi)(g)}{t}.$$

It turns out that K-finiteness promises that $X\varphi$ exists, and in fact one check that it is an automorphic form.

In our example, we have $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$, and then we want to compute $Z(\mathfrak{gl}_2(\mathbb{C}))$, which turns out to be the $\mathbb{C}$-algebra generated by 1_2 and the Casimir element $C \coloneqq H^2 + 2RL + 2LR$ (in $U\mathfrak{g}$!), where $H \coloneqq -i\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$ and $L \coloneqq \frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right]$ and $R \coloneqq \frac{1}{2}\left[\begin{smallmatrix} i & i \\ i & -1 \end{smallmatrix}\right]$. One can explicitly write down the action of these elements on f, which turns into some differential operators. In particular, it is enough to note that

$$C \cdot \varphi_f = \frac{k}{2}\left(1 - \frac{k}{2}\right)\varphi_f.$$

In particular, we must use the holomorphicity of f in order to do some of these computations, and the weight needs to appear somewhere in life in order to get the above.

(f) Let's check that a cuspidal modular form produces a cuspidal automorphic form. It is enough to check this result for parabolic subgroups containing a fixed minimal parabolic subgroup because parabolic subgroups always contain a conjugate of the given fixed minimal parabolic subgroup.

In our example, let's suppose for simplicity that we are in level 1 so that we only have one cusp. In this case, we may take our minimal parabolic subgroup to be the Borel subgroup of upper-triangular matrices, which we note is also already maximal. Thus, we only have to work with the unipotent radical of matrices of the form $\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]$. As such, we are interested in computing

$$\begin{aligned}\int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})}\varphi_f(ng)\,dn &= \int_0^1\int_{\widehat{\mathbb{Z}}}\varphi_f\left(\left(\begin{bmatrix}1 & t \\ & 1\end{bmatrix}n'\right)g\right)\,dn'\,dt \\ &= \int_0^1 f\left(\begin{bmatrix}1 & t \\ & 1\end{bmatrix}g_\infty i\right)j(g_\infty,-i)^{-k}\,dt,\end{aligned}$$

where j is the automorphy factor.

With the time we have left, let's define an automorphic representation.

Definition 149. Recall that there is an action of $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_F^\infty)$ action on the space of automorphic forms $\mathcal{A}$. An *automorphic representation* is an irreducible admissible module over $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_F^\infty)$ which is isomorphic to a subquotient of $\mathcal{A}$.

13 December 4: Andrew Cheng Jr

Today we are talking about Eisenstein series.

13.1 Parabolic Subgroups

Throughout today, G is a connected reductive group over a field K. In the sequel, K is typically a number field, but we will not ask for this for now.

Definition 150 (parabolic)**.** Fix a connected reductive group G over a field K, and let $B_{\overline{K}} \subseteq G_{\overline{K}}$ be a Borel subgroup. An algebraic subgroup $P \subseteq G$ is *parabolic* if and only if $P_{\overline{K}}$ contains $B_{\overline{K}}$.

Notably, parabolic subgroups are not required to be reductive. However, a Levi decomposition recovers reductivity.

Definition 151 (Levi subgroup)**.** Fix a parabolic subgroup P of a connected reductive group G over a field K. Let $U \subseteq P$ be the unipotent radical. Then there is a reductive subgroup $M \subseteq P$ such that there is an almost direct product

$$P = MU.$$

We call M the *Levi subgroup*. Also, we may write A_M for the maximal split torus in M.

We will not show that M exists. Lastly, we will want the following terminology.

Definition 152 (cuspidal)**.** Fix a connected reductive group G over a field K. For a parabolic subgroup $P \subseteq G$, we will call $P(K) \subseteq G(K)$ a *cuspidal subgroup*. We may then call the pair $(P(K), A_M)$ a *cuspidal pair* of $G(K)$. We may say that two cuspidal pairs $(P(K), A_M)$ and $(P'(K), A_{M'})$ are *associate* if and only if A_M and $A_{M'}$ are conjugate in $G(K)$.

Remark 153. The relevance of these definitions for us is that an automorphic form φ is cuspidal if and only if we have

$$\int_U \varphi(xu)\, du = 0$$

for all $x \in G(K)$, where U varies over the unipotent radicals of the parabolic subgroups of G.

Let's see an example. Take $G = \mathrm{SL}_2(\mathbb{R})$. Then one has an Iwasawa decomposition $G = KAU$ with $K = \mathrm{SO}_2(\mathbb{R})$, and

$$A := \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\},$$

and

$$U := \left\{ \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} : u \in \mathbb{R} \right\}.$$

Namely, we may write any element of G uniquely as $r(\theta)e(t)u(x)$, where $r(\theta)$ is rotation by θ, and $e(t)$ and $u(x)$ are defined as above.

13.2 Eisenstein Series

Let's begin with a quick-and-dirty definition of the Eisenstein series. Fix a discrete arithmetically defined subgroup Γ of $G(K)$, and choose a cuspidal pair $(P(K), A_M)$.

Example 154. Continuing with $G = \mathrm{SL}_2$, one can take $K = \mathbb{Q}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Our Eisenstein series is

$$E(g, \varphi, \Lambda) \coloneqq \sum_{\gamma \in \Gamma / \Gamma \cap P(k)} \varphi(\gamma g) \exp\big(\Lambda(H(\gamma g)) + \rho(H(\gamma g))\big).$$

Here, Λ is a linear functional on $\operatorname{Lie} A_M$, $H\colon G(\mathbb{A}_F) \to A_M$ is some function, and ρ is the half sum of the positive roots appearing in P. The definition of H is somewhat technical, so we are omitting it.

Example 155. Take $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In this case, H sends $r(\theta)e(t)u(x)$ to t, and our Eisenstein series collapses into

$$\frac{1}{2} \sum_{\gamma \in \Gamma / (\Gamma \cap U(K))} e^{(\lambda - 1)H(x\gamma)}.$$

Writing out x and γ out as 2×2 matrices, one finds that this equals

$$\frac{1}{2} \sum_{\gamma \in \Gamma / (U \cap \Gamma)} |(x\gamma)_1|^{\lambda - 1},$$

where $(g)_1 = \left(a^2 + c^2\right)^{1/2}$ when $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The punchline is that this is the usual Eisenstein series. For example, there is a way to give this a meromorphic continuation, and there is a functional equation.

Here is a quick application, which tells us at least that our definition knows a little about group theory.

Proposition 156. Fix cuspidal pairs $(P(K), A_M)$ and $(P'(K), A_{M'})$. Then they are associate if and only if there are (g, φ, Λ) as above such that

$$\int_{U'/(\Gamma \cap U')} E(ug, \varphi, \Lambda)\, du \neq 0.$$

Let's close with another applications: the Eisenstein series produces the continuous spectrum.

Theorem 157. Let G be a connected reductive group over a number field K, and suppose that there all split tori in G are trivial. Then $[G] = G(K) \backslash G(\mathbb{A}_K)$. Note that $G(\mathbb{A}_K)$ acts on $L^2([G])$ be right translation. Then spectral theory providesa decomposition

$$L^2([G]) = L^2_{\text{disc}}([G]) \oplus L^2_{\text{cont}}([G]).$$

In this case, $L^2_{\text{cont}}([G])$ contains integrals parameterized by Eisenstein series, and $L^2_{\text{disc}}([G])$ is conjecturally spanned by a cuspidal part and residues of some Eisenstein series.