Johns Hopkins Number Theory Seminar

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1 January 22nd: Friedrich Knop

This is joint work with Zhgoon. For a group G with two subgroups $P, H \subseteq G$, we may be interested in the double coset space $P\backslash G/H$. Today, G will be a connected reductive group over a field K, which may or may not be algebraically closed. Then H and P will be closed subgroups of G.

Example 1. Take $G = \mathrm{GL}_n(\mathbb{C})$ and $H \subseteq \mathrm{O}_n(\mathbb{C})$ and P to be the upper-triangular matrices. Then $P\backslash G/H$ becomes complete flags together with some data of a quadratic form; it turns out to be classified by involutions in S_n, where a σ has attached to it the quadratic form $\sum_i x_i x_{\sigma i}$. For example, the orbit corresponding to $x_1^2 + x_2^2 + x_3^2$ is open.

In general, one can see that having finitely many orbits. For example, there is the following result.

Theorem 2. Fix G a connected reductive group over an algebraically closed field K with Borel subgroup $B \subseteq G$, and let $H \subseteq G$ be a closed subgroup. Then B has an open orbit in G/H if and only if $|B\backslash G/H|$ is finite.

There are examples yielding some level of sharpness for this result. One can ask what is special about the quotient $B\backslash G$, which the following notion helps explain.

Definition 3 (complexity)**.** Fix a group H acting on a variety X over an algebraically closed field K. Then we define the *complexity* $c(X/H)$ as the transcendence degree of $K(X)^H$; note that this is the dimension of X/H if such a quotient makes sense.

Theorem 4 (Vinburg)**.** Let $Y \subseteq X$ be a subvariety with an action by G. Then $c(Y/B) \leq c(X/B)$.

The moral of the story is that we are able to bound open orbits.

We would like to have such theorems over fields K which may not be algebraically closed, but this requires some modifications. For example, one may not have a Borel subgroup B defined over K, so we must work with a minimal parabolic subgroup P. For example, we have the following.

Theorem 5. Work over the field $\mathbb{R}$. Then if P has an open orbit in G/H, then $|P(\mathbb{R})\backslash X(\mathbb{R})|$ is finite, where X refers to the quotient G/H.

However, even this result fails over (say) $\mathbb{Q}$.

Example 6. Consider $G = \mathbb{G}_{m,\mathbb{Q}}$ with $H = \mu_2$. Then $X = G/H$ becomes $\mathbb{G}_m$, but then action is given by the square, so the quotient is the infinite set $\mathbb{Q}^{\times 2}\backslash\mathbb{Q}^{\times}$.

Remark 7. This example suggests that we may be able to salvage the theorem over local fields of characteristic 0.

To fix the result in general, we want to try to work over the algebraic closure.

Theorem 8. Work over a perfect field K. Let $P \subseteq G$ be a minimal parabolic, and let X be a variety with a G-action. Suppose that there is $x \in X(K)$ such that the orbit $Px \subseteq X$ is open. Then the quotient $P(\overline{K})\backslash X(K)$ is finite.

Here, this quotient by $P(\overline{K})$ refers to "geometric" equivalence classes: two points x and x' are identified if and only if one has $p \in P(\overline{K})$ such that $x = px'$. We want the following notion.

Definition 9. Let G act on a variety X. Then X is K*-spherical* if and only if there is a point $x \in X(K)$ such that the orbit by the minimal parabolic is open.

For example, suitably stated (one should assume that $X(K) \subseteq X$ and $Y(K) \subseteq Y$ are dense and that X is normal), one is able to recover the result on complexity. Roughly speaking, the idea is to reduce to the case where G has rank 1. There are two cases for this reduction.

- Previous work explains how to achieve the result when $GY = Y$.
- If GY strictly contains Y, then we pass to a quotient by a parabolic subgroup corresponding to some simple root.

We are now in the rank 1 case. If $K = \overline{K}$, then it turns out that one may merely work with $G = \mathrm{SL}_2$; then one actually directly classify closed subgroups to produce the result. With $K = \mathbb{R}$, a similar idea works, but the casework at the end is harder. However, no such classification is available for general K. Instead, for general K, we develop some structure theory of these sorts of spherical varieties.