Seminars

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Abstract

This semester, I will just record all seminars I go to in an uncategorized manner. I will try to record the date, the speaker, and which seminar it was to maintain some semblance of organization.

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1   September 3: Arthur Representations and the Unitary Dual

This talk was given by David Vogan at MIT for the Lie groups seminar.

1.1   The Orbit Method

Given a group G, we would like to understand its unitary representations, which more or less amounts to understanding the unitary dual of G. There is a long history of trying to solve this problem for various classes of groups G, and we will focus on reductive groups, roughly speaking because it is suitable for inductive arguments. In particular, we will focus on real reductive groups.

    Our story begins with the ideas of Bertram Kostant. This is the method of coadjoint orbits, which explains where one should look for representations of Lie groups. The point is that unitary representations provide operators on a Hilbert space, as do quantum mechanical systems, so one might want to undo the quantization. It turns out then that a unitary representation un-quantizes to a Hamiltonian G-space, which is a symplectic manifold with G-action (and a moment map).

> **Idea 1.** Irreducible representations of G correspond to a quantization of covers of coadjoint orbits.

One must make precise what a quantization is, which is not known in general.

    Anyway, the point is that want to construct our unitary representations geometrically (which amounts to quantizing coadjoint operators), and then we want to show that we have all them. For real reductive groups, there are not so many kinds of coadjoint orbits.

- Hyperbolic (diagonalizable with real eigenvalues): quantization is real parabolic induction. Here, parabolic induction takes unitary representations to unitary representations.

- Elliptic (diagonalizable with non-real eigenvalues): quantization is cohomological parabolic induction. Here, we take unitary representations (of some Levi subgroups) to unitary representations under some positivity requirement on the codomain.

- Nilpotent: quantization is not totally understood, though something partial was suggested by Arthur and worked out completely by other authors.

In total, here is how one can construct unitary representations using the orbit method.

1. Choose a "θ-stable" Levi subgroup $L_\theta \subseteq L_\mathbb{R}$, meaning that it is the centralizer of a compact torus.

2. Fix some unipotent representation π_θ of L_θ.

3. Twist π_θ by a positive unitary character λ of L_θ.

4. Take cohomological induction of $\pi_\theta(\lambda)$ to $\pi_\mathbb{R}(\lambda)$ on $L_\mathbb{R}$.

5. Twist further by some unitary character ν of $L_\mathbb{R}$.

6. Take real induction up to G.

There is even an explicit way to compute the infinitsemial character at the end, and it is explicit when π_θ is a "special" unipotent representation.

> **Remark 2.** One may want the infinitesimal character to be integral, but it is rarely so.

> **Example 3.** We work some of this out for $\mathrm{Sp}_{2n}(\mathbb{R})$.

Proof. Decompose $n = n_r + \cdots + n_1 + n_0$ into nonnegative integers, and we get a Levi subgroup which looks like
$$L_\mathbb{R} = \mathrm{GL}(n_m, \mathbb{R}) \times \cdots \times \mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{Sp}(2n_0, \mathbb{R}).$$
Having lots of GLs is fairly typical for Levi subgroups, which one can see by suitably taking subgraphs of the Dynkin diagram. One then choose a compact torus suitably and writes down some $L_\theta \subseteq L_\mathbb{R}$. We then see that our orbit method asks for many unipotent representations of groups of smaller rank (mostly GLs or Us) and do some inductions and twisting by controlled characters. ∎

> **Remark 4.** The orbit method cannot provide a complete list of our representations, roughly speaking because of the hyperbolic step. The point is that we only allowed twisting by unitary characters, but sometimes you can twist by a non-unitary character and still end up with a unitary representation at the end due to frequent coincidences.

1.2   Arthur's Conjecture

For the rest of the talk, we will be interested in the following conjecture.

> **Conjecture 5.** Suppose G is a real reductive algebraic group and π is a unitary representation of G having integral infinitesimal character. Then π is an Arthur representation, meaning that we get it from the orbit method.

Much is known: there are no known counterexamples for classical G, nor are there any for G_2 or E_6. However, there are a few counterexamples for some exceptional groups: it fails for two representations of split F_4, at most six representations for split E_7, and at most twenty-seven representations for split E_8. (The "at most" phrase is present here because the definition of Arthur representation is difficult to calculate.)

2   Sepember 8: The Affine Chabauty Method

This talk was given by Marius Leonhardt at Boston University for the number theory seminar.

2.1   The Statement

We are interested in finding the integral solutions to curves, such as hyperelliptic curves. Fix a smooth projective curve X over $\mathbb{Q}$; after removing some finite number of "cusps" $D \subseteq X$ (which are just some closed points), we can form the affine curve Y; today, we will study the S-integral points on Y for some finite set S of primes.

> **Remark 6.** Technically, in order to make sense S-integral points, one should choose a regular model $\mathcal{X}$ of X over $\mathbb{Z}_S$, choose a closure $\mathcal{D}$ of $D \subseteq X$ in $\mathcal{X}$, and set $\mathcal{Y} := \mathcal{X} \setminus \mathcal{D}$. Then, we can speak coherently about S-integral points on $\mathcal{Y}$.

We will want a few more invariants. Choose a basepoint $P_0 \in Y(\mathbb{Q})$, and set g to be the genus of X, r to be the rank of the Jacobian, and n to be the number of geometric points in D. Note we can write $n = n_1 + 2n_2$ for the totally real and (classes) of complex points.

We are interested in making the following classical result effective.

> **Theorem 7** (Siegel). If Y is hyperbolic, then $\mathcal{Y}(\mathbb{Z}_S)$ is finite.

Here is our main result.

> **Theorem 8.** Fix everything as above.
>
> (a) If
> $$r + \#S + n_1 + n_2 - \#D < g + n - 1,$$
> then $\mathcal{Y}(\mathbb{Z}_S)$ is contained in a finite computable subset of $\mathcal{Y}(\mathbb{Z}_p)$.
>
> (b) If we also have $p > 2g + n$, then $\#\mathcal{Y}(\mathbb{Z}_S)$ is bounded above by $\#\mathcal{Y}(\mathbb{F}_p) + 2g + n - 2$ multiplied by the number of reduction types.

> **Remark 9.** This result even works if the genus is 0 or 1!

> **Remark 10.** The reduction type is a piece of combinatorial data $\Sigma = (\Sigma_\ell)_\ell$ partitioning $\mathcal{Y}(\mathbb{Z}_S)$; namely, we choose the component or cusp of $\mathcal{X}_{\mathbb{F}_\ell}$ where the point reduces.

Before stating our modifications, let's recall the usual Chabauty method. Consider the following diagram.

$$
\begin{array}{ccc}
X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) \\
{\scriptstyle \mathrm{AJ}}\downarrow & & {\scriptstyle \mathrm{AJ}}\downarrow \qquad \searrow {\scriptstyle \int} \\
\mathrm{Jac}\, X(\mathbb{Q}) & \hookrightarrow & \mathrm{Jac}\, X(\mathbb{Q}_p) \xrightarrow[\log]{} \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee
\end{array}
$$

The Jacobian has rank r, and the last cohomology group has dimension g, so $r < g$ provides a nontrivial differential ω for which $X(\mathbb{Q})$ is contained in the set of p-adic points A with $\int_{P_0}^A \omega = 0$. This last condition is computable, which is the point of the method.

2.2   How to Fix Chabauty

Let's say something about the proof of Theorem 8.

- Using logarithmic (instead of holomorphic) differentials, meaning that we allow some simple poles at D, then
$$\dim \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D)) = g + n - 1,$$
which exhibits the right-hand side of the inequality in (a).

- One should use generalized Jacobians to build a Jacobian J_Y of Y. For example, one definition is given by $J_Y(K)$ to be the divisors of Y_K of degree zero modulo principal divisors $\mathrm{div}\, f$ where $f \in K(X)^\times$ has value 1 on D. Notably, J_Y is a semiabelian variety, so it fits into a diagram
$$0 \to T_D \to J_Y \to J \to 0,$$
where T_D is some torus. This means that $J_Y(\mathbb{Q})$ has little chance of being finitely generated, so it is not enough to just write out the same Chabauty diagram as before.

- Instead, we look at $\mathbb{Z}_S$-points. Build the Chabauty diagram as follows.

$$
\begin{array}{ccc}
\mathcal{Y}(\mathbb{Z}_S) & \longrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
Y(\mathbb{Q}) & \longrightarrow & Y(\mathbb{Q}_p) \\
\downarrow & & \downarrow \qquad \searrow^{\int} \\
J_Y(\mathbb{Q}) & \longrightarrow J_Y(\mathbb{Q}_p) & \longrightarrow \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^{\vee}
\end{array}
$$

We need to show that the image of $\mathcal{Y}(\mathbb{Z}_S)$ in $J_Y(\mathbb{Q})$ is contained in a (small) finitely generated subgroup; in fact, we can get this rank down to

$$ r + \#S + n_1 + n_2 - \#D, $$

from which (a) follows. From here, (b) follows by arguing as in the Chabauty–Coleman method.

- It remains to prove the bound on the rank, which is done using arithmetic intersection theory. Inspired by something with p-adic heights, we may take mod-ℓ intersections of points in $J_Y(\mathbb{Q})$ with reductions of the cusps, and one can compute that $\mathcal{Y}(\mathbb{Z}_S)$ has controlled image.

3   September 9th: Singularities of Secant Varieties

The (pre-)talk was given by Debaditya Raychaudhury at the Harvard–MIT algebraic geometry seminar, and it takes up the first two subsections; it was titled "The Hodge Filtration on Local Cohomology." The rest is from the main talk.

3.1   Properties of Local Cohomology

Fix a subvariety Z of a smooth variety X over $\mathbb{C}$. Then one has local cohomology sheaves $\mathcal{H}_Z^{\bullet}(-)$, which are understod as the derived functors of $\mathrm{H}_Z^0(-)$, which is the sheaf of sections supported on Z.

> **Example 11.** Locally, we can make everything affine, so say $X = \operatorname{Spec} R$ and $Z = \operatorname{Spec} R/I$. Then $H_I^0(R)$ consists of the $r \in R$ for which $I^k r = 0$ for some k large enough.

We would like to compute these sheaves.

One way is geometric. With $U := X \setminus Z$, we let $j: U \hookrightarrow X$ be the inclusion. Then there is an exact sequence

$$ 0 \to \mathcal{O}_X \to j_* \mathcal{O}_U \to \mathcal{H}^1(\mathcal{O}_X) \to 0 $$

and $\mathrm{R}^{q-1} j_* \mathcal{O}_U \cong \mathcal{H}_Z^q(\mathcal{O}_X)$ for $q \geq 2$.

> **Example 12.** Continuing with the affine situation, suppose further that $I = (f)$. Then we get an exact sequence
>
> $$ 0 \to R \to R_f \to \mathrm{H}_{(f)}^1(R) \to 0, $$
>
> so we can compute $\mathrm{H}_{(f)}^1(R)$ via some Čech complex. In general, with $I = (f_1, \ldots, f_s)$, one builds the Čech complex
>
> $$ 0 \to R \to \bigoplus_{i_1} R_{f_{i_1}} \to \bigoplus_{i_1 \leq i_2} R_{f_{i_1} f_{i_2}} \to \cdots $$
>
> to compute the cohomology $\mathrm{H}_I^q(R)$; here, the differential maps are the usual ones for the Čech complex. For example, if $q > s$, then $\mathrm{H}_I^q(R) = 0$ automatically!

Thus, we see that $\mathcal{H}_Z^q(\mathcal{O}_X)$ vanishes for q not so large: it vanishes as soon as q is larger than the number of local defining equations. We are now allowed to make the following definition.

Definition 13. We define the *local cohomological dimension* $\mathrm{lcd}(X, Z)$ to be the maximum q such that $\mathcal{H}^q_Z(\mathcal{O}_X)$ is nonzero.

Remark 14. It turns out that the minimal q such that $\mathcal{H}^q_Z(\mathcal{O}_X)$ is equal to the codimension of $Z \subseteq X$, which can be seen by a computation on affines.

Remark 15. As discussed above, if $Z \subseteq X$ is a local complete intersection, then $\mathrm{lcd}(Z, X)$ will equal the codimension.

Thus, we see that the local cohomological dimension does a reasonable job keeping track of singularities.

Notation 16. We define $\mathrm{lcdef}(Z) := \mathrm{lcd}(X, Z) - \mathrm{codim}_X(Z)$

Remark 17. This quantity does not depend on Z! One way to see this is to show that $\mathrm{lcdef}(Z)$ is the maximum j for which

$$\mathcal{H}^j(\mathbb{Q}^H_Z[\dim Z]) \neq 0,$$

which is more manifestly independent of X. Here, $\mathbb{Q}^H_X$ is the Hodge module.

3.2 The Hodge Filtration

It turns out that $\mathcal{H}^q_Z(\mathcal{O}_X)$ has the structure of aa filtered $\mathcal{D}_X$-module, where $\mathcal{D}_X$ is defined as the (dual) differential ring, given on affines U by

$$\mathcal{D}_X(U) := \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_X(U)\partial^\alpha,$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$; here ∂_i is dual to dx_i. Approximately speaking, this arises because $\mathcal{H}^q_Z(\mathcal{O}_X)$ is the underling $\mathcal{D}$-module for the filtered module $\mathcal{H}^q(i_*i^!\mathbb{Q}^H_X[z])$, where $i\colon Z \hookrightarrow X$ is the inclusion.

We would like to compute the Hodge filtration $F_\bullet \mathcal{H}^q_Z(\mathcal{O}_X)$. Suppose we have a "resolution of singularities" $f\colon Y \to X$ fitting into a pullback square

$$\begin{array}{ccc} Y \setminus E & \longrightarrow & Y \\ \| & & \downarrow f \\ U & \longrightarrow & X \end{array}$$

where $Z := f^{-1}(Z)$ is some simple normal crossings divisor. Namely, one uses the complex

$$0 \to f^*\mathcal{D}_X \to \Omega^1_Y(\log E) \otimes f^*\mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes f^*\mathcal{D}_X \to 0.$$

Roughly speaking, one hits this with F_{k-n} and computes the image of some cohomology of this complex.

Here is the sort of thing that one can prove.

Theorem 18. Suppose $q \geq 1$ and $\mathcal{H}^k_Z(\mathcal{O}_X) = 0$ for $k > q$. Then $F_\bullet \mathcal{H}^q_Z(\mathcal{O}_X)$ is generated at level ℓ if and only if $\mathrm{R}^{q-1+i}f_*\Omega^{n-i}_Y(\log E) = 0$.

We end this talk by defining one more invariant.

Definition 19. We say that $c(Z) \geq k$ if and only if $F_p\mathcal{H}^q_Z(\mathcal{O}_X) = 0$ for all $p \leq k$ and $q > \mathrm{codim}_X(Z)$.

Roughly speaking, this is expected to "cohomologically" generalize local complete intersections.

Theorem 20. One has $c(Z) \geq k$ if and only if $\operatorname{depth} \Omega_Z^p \geq \dim Z - p$ for all $p \leq k$.

Theorem 21. One has that $\dim Z - \operatorname{lcdef}(Z)$ is the minimal value of $\operatorname{depth} \Omega_Z^k + k$.

3.3 The Secant Variety

Given an ample line bundle $\mathcal{L}$ on a smooth variety X of dimension n, we get an embedding $X \hookrightarrow \mathrm{H}^0(\mathcal{L})$. Then we define the secant variety Σ as the closure of the line spanned by any pair $x_1, x_2 \in X$.

Remark 22. With $\dim X = n$, we expect $\dim \Sigma = 2n + 1$: each point has n degrees of freedom, and then the line adds another degree of freedom.

Example 23. If X is $\mathbb{P}^1$ embedded in $\mathbb{P}^2$ (via $\mathcal{O}(2)$), then $\Sigma = \mathbb{P}^2$, which is smaller than expected.

Example 24. If X is $\mathbb{P}^2$ embedded in $\mathbb{P}^3$ (via $\mathcal{O}(3)$), then $\dim \Sigma = 3$.

The moral is that Σ only achieves the expected dimension for sufficiently positive line bundles.

Proposition 25. Suppose that $\mathcal{L}$ is 3-very ample (which we won't define), then $\dim \Sigma = 2n + 1$, and the singular locus of Σ is precisely X unless Σ is the whole space.

Thus, we will focus on our secant varieties arising from 3-very ample line bundles.

With more positivity, we get more.

Theorem 26 (Ullery–Chou–Song)**.** If $\mathcal{L} = \mathcal{K}_X + (2n + 2)\mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ is very ample, and $\mathcal{B}$ is nef, then Σ is normal, has DB singularities, is Cohen–Macaulay (CM), weakly rational, and thus has rational singularities if and only if $\mathrm{H}^i(\mathcal{O}_X) = 0$ for all $i > 0$.

To improve this result, one needs to know something about $\mathbb{Q}_\Sigma^H[2n + 1]$, where the H adds emphasizes that it is a Hodge module.

3.4 Mixed Hodge Modules

Let's quickly say something about mixed Hodge modules. Suppose Z is a smooth variety.

Definition 27. A *Hodge module M* is the data of a left $\mathcal{D}_Z$-module M along with a Hodge filtration $F_\bullet M$ on M, satisfying some conditions, together with a $\mathbb{Q}$-perverse sheaf K and isomorphism $\alpha \colon K \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{DR}_Z(M)$, where $\mathrm{DR}_Z(M)$ is the de Rham complex of M.

Here, the de Rham complex is M tensored with

$$\mathcal{O}_Z \to \Omega_Z^1 \to \cdots \to \omega_Z.$$

We remark that once we take graded pieces with respect to F, we get an object in $D^b(\operatorname{Coh}(Z))$.

Remark 28. There is a derived category $D^b(\mathrm{MHM}(Z))$, even when Z fails to be singular, and it admits the six functors. Thus, we may define

$$\mathbb{Q}_Z^H := p^* \mathbb{Q}_{\mathrm{pt}}^H,$$

where $p \colon Z \to \mathrm{pt}$ is the constant map.

3.5 Main Result

For our application, we need a log resolution of X. Let $X^{[2]}$ be the Hilbert scheme of lines on X. Then there is a tautological $\Phi \subseteq X^{[2]} \times X$ of pairs (ξ, x) where $x \in \xi$. (Equivalently, Φ is the blow up of $X \times X$ along the diagonal.) Let $\theta \colon \Phi \to X^{[2]}$ and $q \colon \Phi \to X$ be the projections, so we may define $\mathcal{E} \coloneqq \theta_* q^* \mathcal{L}$. It turns out that $\mathrm{H}^0(\mathcal{E}) = \mathrm{H}^0(\mathcal{L})$, and it turns out that $\mathbb{P}\mathcal{E}$ projects onto Σ. We are now able to draw the following diagram.

$$
\begin{array}{ccccc}
\mathrm{Bl}_{\{x\}} X & \hookrightarrow & \Phi & \hookrightarrow & \mathbb{P}\mathcal{E} \\
\downarrow & & \downarrow q & & \downarrow t \\
\{x\} & \hookrightarrow & X & \hookrightarrow & \Sigma & \hookrightarrow & \mathbb{P}\mathrm{H}^0(\mathcal{L})
\end{array}
$$

The squares are all pullback squares.

Here is our main theorem. Recall that there is a complex $\underline{\Omega}^p_Z$ given by

$$
\mathrm{gr}_{-p} \mathrm{DR}(\mathbb{Q}^H_Z[\dim Z])[p - \dim Z].
$$

Theorem 29. Suppose $\mathcal{L}$ is sufficiently positive.

(a) $\underline{\Omega}^p_\Sigma \cong \Omega^{[p]}_Z$ for all $0 \le p \le k$ is equivalent to $\mathrm{H}^i(\mathcal{O}_X)$ for all $1 \le p \le k$. In particular, this implies that $\underline{\Omega}^p_\Sigma$ is a sheaf.

(b) $\mathcal{E}xt^i(\underline{\Omega}^{2n}_\Sigma, \omega^\bullet_\Sigma[-2n-2]) = 0$ for all $i \ge 1$ if and only if $X \cong \mathbb{P}^1$.

(c) If $\mathcal{L}$ is 3-very ample, and $\Sigma \ne \mathbb{P}^n$, then

$$
\mathrm{lcdef}(\Sigma) = \begin{cases}
n - 1 & \text{if } n \ge 2 \text{ and } \mathrm{H}^1(\mathcal{O}_X) \ne 0, \\
n - 2 & \text{if } n \ge 2 \text{ and } \mathrm{H}^1(\mathcal{O}_X) = 0, \\
0 & \text{if } n = 1.
\end{cases}
$$

The speaker then did some intricate calculation to prove a partial result of (c), that $\mathrm{lcdef}(\Sigma) \le n - 1$.

Remark 30. There are other theorems with more calculations of these invariants of secant varieties. For example, under the hypothesis of (c) above, the defect

$$
\sigma(Z) \coloneqq \dim_\mathbb{Q} \frac{\mathrm{WeilDiv}_\mathbb{Q}(Z)}{\mathrm{CarDiv}_\mathbb{Q}(Z)}
$$

is finite if and only if $\mathrm{H}^1(\mathcal{O}_X) = 0$, and it is zero if and only if X is $\mathbb{P}^1$.

4 September 10: Unipotent Representations: Changing q to $-q$

This talk was given by George Lusztig at MIT for the Lie groups seminar.

4.1 A Special Basis

For today, G is a split, connected reductive group over $k = \mathbb{F}_q$. Each Weyl element $w \in W$ produces a subvariety X_w of the flag variety $\mathcal{B} \times \mathcal{B}$ (where $\mathcal{B}$ is made of all the Borel subgroups) consisting of the pairs (B, FB) for $w \in \mathcal{O}_w$. It turns out that a unipotent representation of $G(k)$ appears in the cohomology of X_w if and only if it appears in the Euler characteristic.

These unipotent representations have been classified as follows: as the Weyl group W has its irreducible representations in canonical bijection with the conjugacy classes, the unipotent representations also have

a classification according to these conjugacy classes. To explain how this bijection works, we give each c a finite group Γ_c, then the unipotent representations are parameterized by pairs (g, ρ) (up to conjugacy) where $g \in \Gamma_c$ and ρ is an irreducible representation of $Z(g)$. We let $M(\Gamma_c)$ be this collection of pairs. For a pair m, we may write ξ_m for the corresponding unipotent representation.

Here is the sort of thing that we are able to prove.

> **Theorem 31.** There is an ordered basis of $\mathbb{C}[M(\Gamma_c)]$ where all the matrices relating the elements are upper-triangular with 1s on the diagonal and otherwise positive entries; in fact, these entries are natural always except for a single c in the case of E_8, in which case the entry is in $\mathbb{Z}[\zeta_5]$.

> **Remark 32.** It turns out that there is a canonical pairing on $\mathbb{C}[M(\Gamma_c)]$ (which is akin to a nonabelian Fourier transform), and the basis of the theorem makes this pairing be represented by a matrix with nonnegative rational entries always except for one exception.

4.2   Negating Dimensions

It turns out that the dimensions of the unipotent representations ξ_m are polynomials in q, and we will write $D(\xi_m)$ for this dimension. We will assume that the long Weyl element is central. By tabulating these dimensions, one can show that there is an involution $(\cdot)'$ on the unipotent representations so that

$$D(\xi')(q) = \pm D(\xi)(-q).$$

However, this involution is not uniquely determined by this property.

Let's try to exhibit this involution. It turns out that $M(\Gamma_c)$ can be viewed as the set of irreducible objects in the tensor category of Γ_c-equivariant vector bundles. Then one can do something categorical.

5   September 11: Statements of the Weil Conjectures

This talk was given by Ari Krishna and Sophie Zhu at MIT for the STAGE seminar.

5.1   Some History

For today, X will be a smooth proper variety over a finite field $\mathbb{F}_q$. Let's give a statement of the Weil conjectures in the spirit of counting points.

Conjecture 33 (Weil)**.** Fix a finite field $\mathbb{F}_q$.

(a) Fix a scheme X of finite type over a field $\mathbb{F}_q$. Then there are algebraic integers $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_s\}$ such that

$$\#X(\mathbb{F}_{q^n}) = (\alpha_1^n + \cdots + \alpha_r^n) - (\beta_1^n + \cdots + \beta_r^n)$$

for all $n \geq 0$.

(b) Rationality: suppose further that X is proper of equidimension d. Then we can arrange these algebraic integers as

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d}(-1)^i\left(\sum_{j=0}^{b_i}\alpha_{ij}^n\right).$$

(c) Poincaré duality: with X proper, the multi-sets $\{\alpha_{2d-i,j} : 1 \leq j \leq b_i\}$ and $\{q^d/\alpha_{ij} : 1 \leq j \leq b_i\}$ agree.

(d) Riemann hypothesis: with X proper, $|\alpha_{ij}| = q^{i/2}$ for all i and j.

(e) Betti numbers: with X proper, if X admits an integral model $\mathcal{X}$ over some subring $R \subseteq \mathbb{C}$, then b_i is the ith Betti number of $\mathcal{X}(\mathbb{C})$.

The history of these conjectures is long and fraught.

- In the 1930s, Artin, Hasse, and Schmidt proved everything but the Riemann hypothesis for curves, and they proved the Riemann hypothesis for curves of genus at most 1.

- In 1948, Weil proved the Weil conjectures for curves of any genus. This arose by combining two observations: first, counting $\#X(\mathbb{F}_{q^n})$ should equal the number of fixed points of F^n, and second, these counts could be understood in terms of intersection theory with the graph of the Frobenius.

- In 1949, Weil proved the Riemann hypothesis for other varieties, namely certain Fermat varieties. At this point, the conjectures were finally stated.

- In the 1950s, Grothendieck and many others developed the theory of étale cohomology. By rather formal arguments, this proves everyting but the Riemann hypothesis.

- In 1974, Deligne finishes his first proof of the Weil conjectures.

- In 1980, Deligne strengthens his proof of the Weil conjectures.

5.2 ζ-Functions

The Weil conjectures admit an important reformulation in terms of ζ-functions. Let's begin with the classical ζ-function.

Definition 34. The *Riemann ζ-function $\zeta(s)$* is defined as the analytic continuation of the series

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The Riemann ζ-function admits the following properties.

- Euler product: one can write $\zeta(s)$ as a product

$$\zeta(s) = \prod_{\text{prime } p}\frac{1}{1 - p^{-s}}.$$

- Continuation: there is a meromorphic continuation to the plane, and it has only a simple pole at $s = 1$.

- Functional equation: upon completing the ζ-function as

$$\xi(s) := (s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}+1\right)\zeta(s),$$

  we have the functional equation $\xi(s) = \xi(1-s)$.

- Riemann hypothesis: it is expected that the only zeroes of ζ occur at the negative integer integers and along $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$.

This generalizes as follows.

Definition 35. Fix a scheme X of finite type over $\mathbb{Z}$. Then we define the *arithmetic ζ-function* $\zeta_X(s)$ as

$$\zeta_X(s) := \prod_{\text{closed } \mathfrak{p} \in X} \frac{1}{1 - \#\kappa(\mathfrak{p})^{-s}}.$$

Example 36. The Euler product implies that $\zeta(s) = \zeta_{\operatorname{Spec} \mathbb{Z}}(s)$.

In order to relate this to point-counts, we produce the following definition.

Definition 37. Fix a scheme X of finite type over $\mathbb{F}_q$. Then we define

$$Z_X(T) := \exp\left(\sum_{n \geq 1} \#X(\mathbb{F}_q)\frac{T^n}{n}\right).$$

Remark 38. A direct calculation shows that $Z_X(q^{-s}) = \zeta_X(s)$.

We are now able to rewrite the Weil conjectures.

Conjecture 39 (Weil)**.** Fix a finite field $\mathbb{F}_q$.

(a) Fix a scheme X of finite type over $\mathbb{F}_q$. There are algebraic integers $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_s\}$ such that

$$Z_X(T)\frac{(1 - \beta_1 T)\cdots(1 - \beta_s T)}{(1 - \alpha_1 T)\cdots(1 - \alpha_r T)}$$

for some algebraic integers $\alpha_\bullet$s and $\beta_\bullet$s.

(b) Rationality: let X be a smooth proper variety over $\mathbb{F}_q$ of equidimension d. Then Z_X admits a factorization as

$$\frac{P_1(T)\cdots P_{2d-1}(T)}{P_0(T)\cdots P_{2d}(T)},$$

where $P_i \in 1 + T\mathbb{Z}[T]$ for each T.

(c) Functional equation: with X proper, we have $Z_X(1/q^d T) = \pm q^{d\chi/2} T^\chi Z_X(T)$, where $\pm$ is some sign, and χ is the Euler characteristic.

(d) Riemann hypothesis: with X proper, we have $|\alpha_{ij}| = q^{i/2}$ for all i.

(e) Betti numbers: with X proper, if X admits an integral model $\mathcal{X}$ over some subring $R \subseteq \mathbb{C}$, then $\deg P_i$ is the ith Betti number of $\mathcal{X}(\mathbb{C})$.

These statements are shown to be equivalent by expanding out the definition of Z_X and taking logarithms.

5.3   Proof for Curves

We prove many of the Weil conjectures for curves. By keeping track of completions, we may as well assume that X is smooth and proper. Let's start with rationality.

> **Proposition 40.** Fix a smooth proper curve X over $\mathbb{F}_q$. Then $Z_X(T)$ is a rational function of T.

Proof. The point is to write $Z_X(T)$ out in terms of divisors, which will allow us to use Riemann–Roch. Recall $Z_X(T)$ is the product

$$Z_X(T) = \prod_{\text{closed } p \in X} \left(1 - T^{\deg p}\right)^{-1},$$

which then expands out into the sum

$$Z_X(T) = \sum_{\substack{D \in \text{Div } X \\ D \geq 0}} T^{\deg D},$$

where $D \geq 0$ means that D is effective. There are now two cases: if $\deg D \leq 2g - 2$, we will handle this separately. Otherwise, when $\deg D \geq 2g - 2$, then Riemann–Roch implies that the number of effective divisors with this degree is $\left(q^{d-g+1} - 1\right) / (q - 1)$. (Namely, Riemann–Roch allows one to compute the dimension of the space of effective divisors with given degree; this is a finite vector space over $\mathbb{F}_q$, so we can now compute its size!) This finishes the proof upon rewriting this out as a geometric series. ∎

> **Remark 41.** By inputting more effort, one can use this proof to prove the functional equation. If one is careful, then one can achieve an expansion
>
> $$Z_X(T) = \frac{f(T)}{(1 - T)(1 - qT)}$$
>
> for some polynomial $f(T)$ of degree $2g$ with integral coefficients. Note that this includes the Betti numbers conjecture!

We now turn to the Riemann hypothesis, which of course is the hard part. This will depend on the following size bound.

> **Theorem 42** (Hasse–Weil)**.** Fix a smooth proper curve C over a finite field $\mathbb{F}_q$. Then
>
> $$|\#C(\mathbb{F}_q) - (q + 1)| \leq 2g\sqrt{q}.$$

Let's explain why this produces the Riemann hypothesis. Because we already have an expression

$$Z_X(T) = \frac{f(T)}{(1 - T)(1 - qT)},$$

we may factor $f(T) = \prod_{i=1}^{2g}(1 - \alpha_i T)$, and we note that we are trying to show $|\alpha_i| = \sqrt{q}$ for all i. By the functional equation, it is enough to show merely that $|\alpha_i| \leq \sqrt{q}$ for all i. Now, by definition of $Z_X(T)$, we see that

$$\sum_{n \geq 1} \#C(\mathbb{F}_{q^n})T^n = \frac{d}{dT} \log Z_X(T),$$

which can be computed directly to be

$$\sum_{n \geq 1} \#C(\mathbb{F}_{q^n})T^n = \sum_{i=1}^{2g} \left(\frac{-\alpha_i}{1 - \alpha_i T} + \frac{1}{1 - T} + \frac{1}{1 - qT} \right),$$

which after expanding out the geometric series becomes

$$\sum_{n \geq 1} \#C(\mathbb{F}_{q^n}) T^n = \sum_{n \geq 1} \left(q^n + 1 - \sum_{i=1}^{2g} \alpha_i^n \right).$$

Thus, the Hasse–Weil bound shows that

$$\left| \sum_{i=1}^{2g} \alpha_i^n \right| \leq 2g \sqrt{q^n}.$$

Now, if $|\alpha_i| > \sqrt{q}$ for any i, then we can send $i \to \infty$ to achieve a contradiction because the left-hand side is too large.

5.4   Intersection Theory on a Surface

We will want to know something about intersection theory on a surface. We're in a talk, so we're allowed to just state the result we want.

> **Theorem 43.** Fix a smooth projective surface X over an algebraically closed field k. Then there is a unique integral symmetric bilinear pairing $(\cdot, \cdot)$ on $\mathrm{Div}\, X$ such that any two transverse curves $C, C' \subseteq X$ have
> $$(C, C') = \#(C \cap C').$$

There are many ways to the pairing (C, C'). The most geometric is to show that one can always wiggle one of the curves to make the intersection transverse.

For our bound, we need the following geometric input.

> **Theorem 44** (Hodge index)**.** Fix a smooth projective surface X over an algebraically closed field k. Further, fix an ample line bundle H in $\mathrm{Div}\, X$. If we are given a divisor D on X which is not linearly equivalent to 0 while $D \cdot H = 0$, then $D \cdot D < 0$.

Proof. We will prove this in steps.

1. Suppose instead that $D \cdot H > 0$ and $D^2 > 0$. Then we claim mD is linearly equivalent to an effective divisor for sufficiently large m. Well, beacuse $D \cdot H > 0$, $(K_X - mD) \cdot H < 0$ for m sufficiently large, so $K_X - mD$ cannot be effective. Thus, $\mathrm{H}^0(K_X - mD) = 0$, so $\mathrm{H}^2(mD) = 0$ by Serre duality. However, by Riemann–Roch for surfaces, one has

   $$h^0(mD) = h^1(mD) + \frac{1}{2} mD \cdot (mD - K_X) + \chi(\mathcal{O}_X),$$

   which becomes positive for m large enough.

2. Now, suppose for the sake of contradiction that $D^2 > 0$. Then we can take $H' := D + nH$ to be ample for n large enough, from which we find $D \cdot H' = D^2 > 0$, so the lemma implies that mD is effective for m large enough, which contradicts having $D \cdot H = 0$.

3. Lastly, suppose for the sake of contradiction that $D^2 = 0$. Because $D \cdot H = 0$, we can find an effective divisor E such that $D \cdot E \neq 0$ while $E \cdot H = 0$. Now, consider $D' := nD + E$. One can calculate $(D')^2 > 0$ while $D' \cdot H = 0$, so we reduce to the previous step. ∎

To apply this, we will want to understand ample divisors.

> **Theorem 45.** Fix a divisor D on X. Then D is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C on X.

Here is how this is applied.

> **Theorem 46.** Let $X = C \times C'$ where C and C' are smooth projective curves. Set $\ell := C \times \mathrm{pt}$ and $m := \{\mathrm{pt}\} \times C'$. Then for any divisor D, we have
>
> $$D^2 \leq 2(D \cdot \ell)(D \cdot m).$$

Proof. As a lemma, we claim that if H is ample, then

$$\left(D^2\right) \cdot \left(H^2\right) \leq (D \cdot H)^2.$$

For this, one uses the Hodge index theorem on $E := \left(H^2\right)D - (H \cdot D)H$, from which one can calculate $E^2 < 0$. Thus, as long as $D \neq 0$, we get $\left(D^2\right)\left(H^2\right) - (D \cdot H)^2 < 0$; in all cases, we get the inequality.

Now, by Theorem 45, the divisor $H := \ell + m$ is ample. Applying the above argument with D' defined as

$$D' = \left(H^2\right)\left(E^2\right)D - \left(E^2\right)(D \cdot H)H - \left(H^2\right)(D \cdot E)E$$

where $E := \ell - m$. ∎

We are now ready to prove the Hasse–Weil bound. We will do intersection theory on the surface $X := C \times C$. Let $\Delta \subseteq X$ be the diagonal, and let $\Gamma \subseteq X$ be the graph. Then $\#C(\mathbb{F}_q) = (\Delta \cdot \Gamma)$, which is what we want to bound. Here are our steps.

1. We claim $\Delta^2 = (2 - 2g)$. By the adjunction formula (note $\Delta \cong C$), we see

$$2g - 2 = \Delta^2 + \Delta \cdot K_X.$$

   However, one can expand out K_X as $\mathrm{pr}_1^* C + \mathrm{pr}_2^* C$, which each have intersection number $2g - 2$ with Δ by using the adjunction formula, so the result follows.

2. We claim $\Gamma^2 = q(2 - 2g)$. By the adjunction formula (note $\Gamma \cong C$), we see

$$2g - 2 = \Gamma^2 + \Gamma \cdot K_X.$$

   After doing the same expansion of K_X, one calculates that $\Gamma \cdot \mathrm{pr}_1^* K_C = q(2g-2)$ and $\Gamma \cdot \mathrm{pr}_2^* K_C = 2g - 2$ by using the adjunction formula.

3. We now apply Theorem 46 to $X = C \times C$. Take large integers r and s, and set $D := r\Gamma + s\Delta$. Then $D \cdot \ell = rq + s$ and $D \cdot m = r + s$. From Theorem 46, one calculates that

$$|N - (q+1)| \leq g\left(\frac{rg}{s} + \frac{s}{r}\right),$$

   so the result follows by sending $\frac{r}{s} \to \frac{1}{\sqrt{q}}$.

6 September 15: Rational Points on $X_0(N)^*$ for N Non-Squarefree

This talk was given by Sachi Haschimoto at Boston University of the Boston University number theory seminar. We are discussing joint work with Timo Keller and Samuel Le Fourn.

6.1 Overview

The motivation for our results arises from the following theorem of Mazur.

> **Theorem 47.** Fix a prime p, and let E be an elliptic curve over $\mathbb{Q}$ with (potential) complex multiplication. If E admits a rational isogeny of degree p, then
>
> $$p \in \{2, 3, 5, 7, 13, 37\}.$$

This is proved by classifying the points on certain modular curves.

Definition 48. Fix a positive integer N. Then there is an affine curve $Y_0(N)$ defined over $\mathbb{Q}$ such that $Y_0(N)(K)$ is in bijection with the $\overline{K}$-isomorphism classes of pairs (E, C_N), where E is an elliptic curve over K, and $C_N \subseteq E(K)$ is a cyclic subgroup of order N. We let $X_0(N)$ be the completion, and we set $J_0(N) := \operatorname{Jac} X_0(N)$.

Thus, we see that Theorem 47 is equivalent to saying that $Y_0(p)(\mathbb{Q})$ only has CM points except when p is among the listed exceptions: simply take any such elliptic curve E with rational isogeny φ of degree p, and we can produce the pair $(E, \ker \varphi) \in Y_0(p)(\mathbb{Q})$. Equivalently, we may say that $X_0(p)(\mathbb{Q})$ only has CM points or cusps.

Definition 49 (trivial)**.** A modular curve $X_0(N)$ is *trivial* if and only if $X_0(N)(\mathbb{Q})$ only has cusps and CM points.

Remark 50. The exceptional p have $X_0(p) \cong \mathbb{P}^1_{\mathbb{Q}}$ (and therefore must have many points) except for $p = 37$. When $p = 37$, it turns out that $X_0(37)$ is genus 2 and hence hyperelliptic, and one finds that the hyperelliptic involution produces the non-CM, non-cuspidal points.

For this talk, we are interested in certain quotients of $X_0(N)$ rather than $X_0(N)$ itself.

Definition 51 (Atkin–Lehner involution)**.** Fix a positive integer N and a divisor $Q \mid N$ with $\gcd(Q, N/Q) = 1$. Then we define the *Atkin–Lehner involution* $w_Q \colon X_0(N) \to X_0(N)$ by

$$w_Q((E, C_N)) := \left(\frac{E}{C_N[Q]}, \frac{C_N + E[Q]}{C_N[Q]} \right)$$

One can check that this definition is smooth and therefore extends from $Y_0(N)$ to $X_0(N)$. One can also check that $w_Q^2 = \operatorname{id}$.

Remark 52. We see that w_Q also extends to $J_0(N)$ by pullback, and it still has $w_Q^2 = \operatorname{id}$. To relate to modular forms, we note that $S_2(\Gamma_0(N))$ is the cotangent space of $0 \in J_0(N)$, from which it follows that w_Q also acts on $S_2(\Gamma_0(N))$. But $S_2(\Gamma_0(N))$ is just some finite-dimensional complex vector space, so we end up with some linear algebra.

Notation 53. We define $X_0(N)^*$ as the quotient of $X_0(N)$ by all the Atkin–Lehner involutions.

Remark 54. It turns out that rational points of $X_0(N)^*$ (minus cusps) correspond to elliptic curves which are isogenous to all their Galois conjugates, where the isogenies have degree dividing by N.

Here is what is currently known, which is due to many people.

Theorem 55. Fix a positive integer N. If N is a prime-power, and the genus of $X_0(N)^*$ is positive, then all rational points of $X_0(N)^*(\mathbb{Q})$ are trivial, except when $N = 5^3$, in which case there is only one interesting point.

Remark 56. The hardest part of the theorem covers the cases $N \in \{125, 169\}$, where the rank of the Jacobian equals the genus, so one has to work harder to make something like the Chabauty–Coleman method work.

Here is what is expected

> **Conjecture 57** (Elkies). For N large enough, all rational points on $X_0(N)^*(\mathbb{Q})$ are either CM points or cusps.

6.2 Formal Immersion Method

We will be interested in the following morphisms.

> **Definition 58** (formal immersion). Fix locally Noetherian schemes X and Y. A map $f \colon X \to Y$ is a *formal immersion* at some point $x \in X$ if and only if it induces an isomorphism of residue fields and is injective on tangent spaces.

Here is how these are used.

> **Theorem 59** (Mazur). Fix a morphism $f \colon X_0(N) \to A$ of schemes over $\mathbb{Z}$, where A is an abelian scheme over $\mathbb{Z}$ of rank 0. Suppose that f is a formal immersion at ∞, where $f(\infty) = 0$. Then any $(E, C_N) \in X_0(N)(\mathbb{Q})$ has E with potentially good reduction at all primes $p > 2$.

The point is that we have upgraded rationality of E to some integrality. Indeed, an equivalent statement is that $j(E) \in \mathbb{Z}[1/2]$.

Sketch. Suppose not. Then E becomes a cusp at some prime p, so by using Atkin–Lehner involutions, one can move E to become ∞ at p. Now, $f((E, C_N)) \in A(\mathbb{Q})$ is torsion because $A(\mathbb{Q})$ is torsion, so because torsion reduces $\pmod p$ injectively, we see that $f((E, C_N)) \equiv f(\infty) \equiv 0 \pmod p$ and therefore $f(x) = f(\infty) = 0$. But then our tangent spaces will fail to be injective. ∎

> **Remark 60.** If N is squarefree, then the simple factors of $J_0(N)^*$ over $\mathbb{Q}$ are all expected to have positive rank (by the Birch–Swinnerton-Dyer conjecture), so one does not expect Theorem 59 to be particularly helpful.

Here is what we are able to prove.

> **Theorem 61.** Fix $N \geq 1$ which is not 99 or 147 but is not squarefree and not a prime-power. For any $P \in Y_0(N)^*(\mathbb{Q})$, choose a lift (E, C_N) to $Y_0(N)(K)$ for some number field K. Then
> $$(8 \cdot 3 \cdot 25 \cdot 49 \cdot 31)^N \, j(E) \in \mathcal{O}_K.$$
> In fact, if $p^2 \mid N$ for some $p \notin \{2, 3, 5, 7, 13\}$, then $j(E) \in \mathcal{O}_K$.

> **Remark 62.** The $N \in \{99, 147\}$ cases have positive rank, so we do not have hope for an integrality statement.

> **Remark 63.** The exponent N is more or less sharp. The 31 is never expected to appear.

Our method proof, following Mazur, uses the following.

> **Theorem 64** (Mazur). If $p \notin \{2, 3, 5, 7, 13\}$, then $J_0(p)^- := J_0(p)/(1 + w_p)$ has a rank zero quotient.

The exceptions arise because $\dim J_0(p) = 0$, so there is no way to find such a quotient. These rank zero abelian varieties are good enough for our purposes most of the time, but we have also proven the following.

> **Theorem 65.** Fix distinct primes p and q with $p \in \{2, 3, 5, 7, 13\}$ and $q > 23$. Then $J_0(pq)^{-p,+q} :=$
> $J_0(pq)/(1 + w_p, 1 - w_q)$ has a rank 0 quotient.

This is proven using some analytic techniques showing that some L-functions vanish. Now, the point of the proof of the main theorem is to do some reduction of levels to construct formal immersions to abelian varieties of rank 0.

7   September 16: Higher Siegel–Weil Formula for Unitary Groups

This talk was given by Mikayel Mkrtchyan at MIT for the MIT number theory seminar.

7.1   The Kudla Program

Let's start by recalling the classical formulation of the Siegel–Weil formula. An approximate statement is that an Eisenstein series $E(g, s_0)$ is approximately equal to a period $\int_{[H]} \theta(g, h)\, dh$ for some θ-function θ. This was proved for a dual pair (G, H) of reductive groups over a global field F in the case where $(G, H) = (\mathrm{U}(2n), \mathrm{U}(n))$.

The Kudla program geometrizes this into a statement that $E(g, s_0)$ should be equal to the degree of some 0-cycle on a Shimura variety. There is also an arithmetic Siegel–Weil formula, which describes the derivative of the Eisenstein series as the degree of a 0-cycle on the integral model(!) of a Shimura variety.

For this talk, we will be working over function fields. As such, we go ahead and fix a morphism $X' \to X$ of smooth curves over a finite field $k := \mathbb{F}_q$, and we will assume that $X' \to X$ is an étale double cover, where $\sigma \colon X' \to X'$ is the automorphism. We also set $F := k(X)$ and $F' := k(X')$.

With the above motivation in mind, we are now ready to set some notation. Let's describe the automorphic side of our Siegel–Weil formula.

> **Definition 66** (Siegel–Eisenstein series)**.** Fix the group $G := \mathrm{U}(n, n)$ over a global field, and let $P \subseteq G$ be a parabolic subgroup. Consider the induction
>
> $$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \left(\chi \, |\mathrm{det}|^{s+n/2} \right),$$
>
> where χ is some auxiliary character, and $s \in \mathbb{C}$. To produce an automorphic form, we may choose some $\varphi(g, s)$ in this induction satisfying $\varphi(1, s) = 1$, and then we define our Eisenstein series as
>
> $$E(g, s) := \sum_{\gamma \in P(F) \backslash G(F)} \varphi(\gamma g, s).$$
>
> We may normalize this to $\widetilde{E}(g, s)$ via some silent process.

> **Remark 67.** The Eisenstein series $E(g, s)$ admits a Fourier decomposition, indexed by Hermitian $n \times n$ matrices $T \in \mathrm{Herm}_n(F)$. These can alternatively be parameterized by pairs $(\mathcal{E}, a)$ where $\mathcal{E}$ is a vector bundle of rank n, and $a \colon \mathcal{E} \to \sigma^* \mathcal{E}^\vee$ is some map.

We now turn to the geometric side. Instead of Shimura varieties, we are able to use shtukas.

> **Definition 68.** A $\mathrm{U}(n)$-*bundle* on X' is a pair $(\mathcal{F}, h)$ of a rank-n vector bundle on X' and some isomorphism $h \colon \mathcal{F} \to \sigma^* \mathcal{F}$. We let $\mathrm{Bun}_{\mathrm{U}(n)}$ denote the relevant moduli space.

Definition 69. We define the *Hecke modification* $\mathrm{HK}^1_{\mathrm{U}(n)}$ as the collection of tuples $(x, \mathcal{F}_0, \mathcal{F}_1, f_{1/2}, \iota_0, \iota_1)$ where $x \in X'$, $\mathcal{F}_0$ and $\mathcal{F}_1$ are $\mathrm{U}(n)$-bundles, and $\iota_0 \colon \mathcal{F}_{1/2} \to \mathcal{F}_0$ and $\iota_1 \colon \mathcal{F}_{1/2} \to \mathcal{F}_1$ are some maps respeecting the unitary structure.

Remark 70. It turns out that the canonical map $\mathrm{HK}^1_{\mathrm{U}(n)} \to X' \times \mathrm{Bun}_{\mathrm{U}(n)}$ onto the first two coordinates is a $\mathbb{P}^{n-1}$-bundle.

Definition 71. We define $\mathrm{Sht}^r_{\mathrm{U}(n)}$ as the moduli space of chains of r Hecke modifications

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_r,$$

where $\mathcal{F}_r$ is isomorphic to the Frobenius twist of $\mathcal{F}_0$. This is a Deligne–Mumford stack.

Remark 72. It turns out that $\dim \mathrm{Sht}^r_{\mathrm{U}(n)} = rn$ for even r and vanishes for odd r. (One should modify the definition slightly to get interesting statements when r is odd.)

With our analogue of Shimura varieties in hand, we are able to define our special cycles.

Definition 73. Fix a pair $(\mathcal{E}, a)$ of a vector bundle $\mathcal{E}$ of rank m and a map $a \colon \mathcal{E} \to \sigma^* \mathcal{E}^\vee$. Then we define the special cycle $Z^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{\mathrm{U}(n)}$ to parameterize the data of an r-shtuka

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_r$$

equipped with commuting maps $\mathcal{E} \to \mathcal{F}_0$ which commute with our Frobenius twists and pullbacks by σ.

Remark 74. The map $Z^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{\mathrm{U}(n)}$ is finite, so we are more or less producing a cycle.

Remark 75. If a in the pair $(\mathcal{E}, a)$ is an isomorphism, then $\mathcal{Z}^r_{\mathcal{E}}(a) \cong \mathrm{Sht}^r_{\mathrm{U}(n-\mathrm{rank}\,\mathcal{E})}$.

Remark 76. The expected dimension of $Z^r_{\mathcal{E}}(a)$ is $r(n - \mathrm{rank}\,\mathcal{E})$.

We are now raedy to state our main theorem.

Theorem 77. Fix the following data.

- A pair $(\widetilde{\mathcal{E}}, \widetilde{a})$, where $\mathrm{rank}\,\widetilde{E} = n$ with $\mathrm{rank}\,\mathrm{im}\,\widetilde{a} = n - 1$. Then $Z^r_{\widetilde{\mathcal{E}}}(\widetilde{a})$ is proper, and the degree of the corresponding cycle equals

$$\partial^r_{s=0} \left(\widetilde{E}_{(\widetilde{E}, \widetilde{a})(s)} \right)$$

up to some explicit constant.

8   September 18th: The Étale Site

This talk was given by Yutong Chen for the STAGE seminar at MIT.

8.1   Étale Morphisms

We will be interested in étale morphisms today. Intuitively, they are supposed to be the algebro-geometric version of a covering space in topology. Here is the easiest definition.

Definition 78 (étale)**.** A morphism $f\colon X \to S$ of schemes is *étale* if and only if it is locally of finite presentation, flat, and unramified.

While locally of finite presentation and flatness are fairly common notions, we should define what it means for a morphism to be unramified. We will define this in steps.

Definition 79 (unramified)**.** Fix an extension $A \subseteq B$ of discrete valuation rings with uniformizers π_A and π_B, respectively. Then $A \subseteq B$ is *unramified* if and only if $(\pi_B) = \pi_A \cdot B$ and the extension of residue fields is separable.

Definition 80 (unramified)**.** Fix a map $f\colon A \subseteq B$ of local rings. Then f is *unramified* if and only if $f(\mathfrak{m}_A) = \mathfrak{m}_B$ and the field extension

$$A/\mathfrak{m}_A \to B/\mathfrak{m}_B$$

is separable.

Definition 81 (unramified)**.** Fix a morphism $f\colon X \to S$ of schemes. Then f is *unramified* if and only if the local maps

$$\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$$

are unramified for all $x \in X$.

Example 82. Open and closed immersions are unramified.

Non-Example 83. Consider the squaring map $\mathbb{A}_k^1 \to \mathbb{A}_k^1$ given by the ring map $k[t] \to k[t^2]$ defined by $t \mapsto t^2$. Then this map is not ramified at 0. Indeed, this map is locally given by

$$k[t^2]_{(t^2)} \to k[t]_{(t)},$$

but the maximal ideal fails to go to the maximal ideal.

There are many ways to think about étale morphisms.

Definition 84 (étale)**.** A morphism $f\colon X \to S$ is *étale* if and only if it is smooth of relative dimension 0.

Here is one version of smoothness which is fairly hands-on.

Definition 85 (smooth)**.** Fix a morphism $f\colon X \to S$. Given $x \in X$, we say that f is *smooth* at x if and only if the morphism locally looks like

$$\operatorname{Spec} \frac{A[t_1, \ldots, t_n]}{(g_{r+1}, \ldots, g_n)} \to \operatorname{Spec} A$$

and the corresponding Jacobian matrix has full rank $n - r$. We may also say that f is smooth of *relative dimension r* in this situation.

Of course, there are also many ways to define smoothness. Here is another useful criterion.

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> **Proposition 86.** Fix a flat morphism $f\colon X \to S$ of irreducible varieties over a field k, and set $r := \dim X - \dim S$. Then f is smooth of relative dimension r if and only if $\Omega_{X/S}$ is locally free of rank r.

Here are a few more ways to work with the yoga of étale morphisms.

> **Proposition 87.** Fix a ring A, an extension $B = A[t]/(p)$ where $p \in A[t]$ is monic, and a localization $C = B\left[q^{-1}\right]$ for some q. If $p'(t) \in C^\times$, then the natural map $\operatorname{Spec} C \to \operatorname{Spec} A$.

We will not prove this (all of these proofs are horribly annoying), but we will content ourselves with an example.

> **Example 88.** Fix $A := k[x]$ and $B := k[x,y]/\left(y^2 - x(x-1)(x+1)\right)$. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is basically the projection from an elliptic curve to the affine line, so we expect to have some ramification at $(0,0)$, $(1,0)$, and $(-1,0)$. Accordingly, if we localize out by $x^3 - x$, then we see that the map $\operatorname{Spec} C \to \operatorname{Spec} A$ is successfully étale, which can be checked because the derivative of $p(y) = y^2 - \left(x^3 - x\right)$ is in $C^\times$.

> **Remark 89.** It turns out that all étale morphisms can locally be factored like Proposition 87.

> **Proposition 90.** Fix a smooth morphism $f\colon X \to S$ of relative dimension r at a point $x \in X$. Further, fix some local functions $g_1, \ldots, g_r \in \mathcal{O}_{X,x}$. Then the following are equivalent.
>
> (i) The elements $dg_1, \ldots, dg_r$ form a local basis for $\Omega_{X/S} \otimes k(x)$.
>
> (ii) The elements $g_1, \ldots, g_r$ extend to an open neighborhood U of x such that $(g_1, \ldots, g_r)\colon U \to \mathbb{A}_S^r$ is étale.

> **Remark 91.** Property (i) is relatively easy to satisfy, so we know that such functions surely exist.

> **Remark 92.** The point of (ii) is that f now factors as
>
> $$X \supseteq U \to \mathbb{A}_S^r \to S,$$
>
> where the map $U \to \mathbb{A}_S^r$ is étale. Thus, smooth morphisms are "just" projections up to an étale map.

8.2 The Fundamental Group

Continuing with our intuition that étale morphisms are covering spaces, we now try to define a fundamental group. It is difficult to make sense of paths in algebraic geometry, so instead we will use covering spaces. Here is the construction that we will try to generalize.

Example 93. For a nice topological space X (e.g., a manifold) with a basepoint $x \in X$, then there is a natural "fiber" functor
$$\mathrm{Fib}_x\colon \mathrm{Cover}(X) \to \mathrm{Set}$$
from the category of covering spaces of X to sets given by sending $p\colon Y \to X$ to the fiber $p^{-1}(\{x\})$. By a path-lifting argument, one shows that
$$\pi_1(X, x) = \mathrm{Aut}(\mathrm{Fib}_x).$$
(In particular, path-lifting desribes an action of $\pi_1(X, x)$ on all fibers in a compatible way.) We remark that this allows us to upgrade the fiber functor into an equivalence
$$\mathrm{Fib}_x\colon \mathrm{Cover}(X) \to \mathrm{Set}(\pi_1(X, x)).$$

Remark 94. Topology is aided by the existence of a universal cover. For example, one has a universal cover of S^1 given by $\mathbb{R} \twoheadrightarrow S^1$, but this covering space fails to be finite; similarly, the universal cover of $\mathbb{C}^\times$ is the exponential map $\exp\colon \mathbb{C} \twoheadrightarrow \mathbb{C}^\times$, which is not algebraic. Algebra is going to have some trouble producing coverings which are not finite (or algebraic), so we will have to content ourselves with some finite quotients.

Accordingly, we find that we are contenting ourselves to work with finite covering spaces, which amounts to working with finite étale covers.

Definition 95 (étale fundamental group). Fix a scheme X and a geometric point $\overline{x} \hookrightarrow X$, and consider the corresponding category $\mathrm{Fin\acute{E}t}(X)$ of finite étale covers of X. Then we define the *étale fundamental group* $\pi_1(X, \overline{x})$ to be the automorphism group of the fiber functor
$$\mathrm{Fib}_x\colon \mathrm{Fin\acute{E}t}(X) \to \mathrm{Set}$$
given by sending the cover $p\colon Y \to X$ to the covering to the fiber $Y \times_p \overline{x}$.

Remark 96. As in the topological case, one finds that Fib_x upgrades to an equivalence
$$\mathrm{Fib}_x\colon \mathrm{Fin\acute{E}t}(X) \to \mathrm{Set}(\pi_1(X, \overline{x})).$$

As a sanity check, we note the following comparison theorem.

Theorem 97. Fix an irreducible variety X over $\mathbb{C}$. Then $Y \mapsto Y(\mathbb{C})$ upgrades to an equivalence of categories between the finite étale covers of X and the finite covers of $X(\mathbb{C})$.

Example 98. Consider $X = \mathbb{C}\left[x, x^{-1}\right]$ so that $X(\mathbb{C}) = \mathbb{C}^\times$. Then we see that $\pi_1^{\mathrm{\acute{e}t}}(X, \overline{1})$ will be $\widehat{\mathbb{Z}}$ because it is the colimit of the automorphism groups of the finite covers of $\mathbb{C}^\times$.

But now that we can do algebraic geometry, we can add in some arithmetic information.

Example 99. Consider the point $X = \operatorname{Spec} k$ and an algebraic closure $\overline{x} = \operatorname{Spec} \overline{k}$. Then a finite étale cover $Y \to X$ will be a finite disjoint union of points. To describe our category, we are allowed to work with just the connected covers of X, which amounts to making Y a point, so we may write $Y = \operatorname{Spec} L$. In order for the map $Y \to X$ to be an étale cover, it is equivalent to ask for the induced field extension $k \subseteq L$ to be finite and separable. The fiber of such an L is given by

$$(Y \times \overline{x})(\overline{k}) = \operatorname{Spec}(L \otimes \overline{k})(\overline{k}) = \operatorname{Hom}_k(L, \overline{k}).$$

Thus, $\operatorname{FÉt}(X)$ amounts to the category of finite separable extensions of k, and it is not hard to see that the automorphism group is simply $\operatorname{Gal}(\overline{k}/k)$.

8.3   Grothendieck Topologies

The point of a Grothendieck topology is to recognize that what makes a topology important is not its open sets but instead the notion of covers. Thus, to specify a Grothendieck topology, we will try to specify the covers and make do with that.

Definition 100 (Grothendieck topology)**.** Fix a category $\mathcal{C}$ closed under finite products. A *Grothendieck topology* on $\mathcal{C}$ is a collection of families $\mathcal{T}$ of the form $\{f_i \colon U_i \to U\}_i$ and satisfying the following.

(a) Isomorphisms: the family $\mathcal{T}$ contains all isomorphisms.

(b) Refinement: given a covering $\{U_i \to U\}_i$ in $\mathcal{T}$ and some coverings $\{V_{ij} \to U_i\}_j$, then the composite $\{V_{ij} \to U_i \to U\}_{i,j}$ continues to be in $\mathcal{T}$.

(c) Pullback: given a covering $\{U_i \to U\}_i$ in $\mathcal{T}$ and some object V with a map $V \to U$, then the pullback $\{U_i \times_U V \to V\}_i$ is in $\mathcal{T}$.

In this situation, the pair $(\mathcal{C}, \mathcal{T})$ is a site.

Here is the motivating example.

Example 101 (Zariski site)**.** If X is a topological space, then we can let $\mathcal{C}$ be the category of open sets in X with morphisms given by inclusion. We can endow $\mathcal{C}$ with the structure of a Grothendieck topology by letting the covers simply be the open covers. If X is a scheme, then this site is called the Zariski site.

Here is the site for today.

Definition 102 (small étale site)**.** Fix a scheme X, and consider the category $\operatorname{Ét}(X)$ of all étale covers of X. Then we endow $\operatorname{Ét}(X)$ with the structure of a Grothendieck topology by saying that a collection of morphisms $\{U_i \to U\}_i$ is a covering if and only if $\bigsqcup_i U_i \to U$ is surjective. This is called the *(small) étale site* and is denoted $X_{\text{ét}}$.

Remark 103. It turns out that a morphism of étale covers of X is automatically étale. This can be proven using the usual techniques of cancellation.

Remark 104. By replacing the word étale with other adjectives, we also have an fppf site and fpqc site. We note that the Zariski site has the same definition where étale is replaced with open embeddings.

As usual, once we have an object, we want some morphisms.

Definition 105 (continuous). A *continuous* map $F\colon (\mathcal{C}', \mathcal{T}') \to (\mathcal{C}, \mathcal{T})$ is the data of a functor $F\colon \mathcal{C} \to \mathcal{C}'$ satisfying the following.

(a) For any covering $\{U_i \to U\}_i$ in $\mathcal{T}$, we require that $\{FU_i \to FU\}_i$ to be in $\mathcal{T}'$.

(b) Given a covering $\{U_i \to U\}_i$ in $\mathcal{T}$ and a map $V \to U$, then we require that $F(V \times_U U_i) \to FV \times_{FU} FU_i$ to be an isomorphism.

Remark 106. If $f\colon X' \to X$ is a continuous map of topological spaces, then taking the pre-image induces a functor of the categories of open sets, and one can see directly that taking the pre-image produces a continuous map of the Grothendieck topologies.

Remark 107. For any scheme X, there is a continuous map between the étale site

$$X_{\mathrm{fpqc}} \to X_{\mathrm{fppf}} \to X_{\mathrm{\acute{e}t}} \to X_{\mathrm{Zar}}.$$

The point of having a notion of topology is that it lets us do sheaf theory.

Definition 108. Fix a Grothendieck topology on a category $\mathcal{C}$. Then a presheaf $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is a *sheaf* if and only if the usual exact sequence

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact for all covers $\{U_i \to U\}_i$.

Example 109. A sheaf on the Zariski site is the usual notion of sheaf in scheme theory.

Remark 110. Because open embeddings are already étale, fppf, and fpqc, we see that a sheaf on any of these sites must be a Zariksi sheaf as well.

Remark 111. Because the sites we care about are closed under arbitrary coproduct, it is enough to check it on coverings which look like $U' \to U$, though of course one cannot require either U' or U to be connected.

We have yet to construct any sheaves! Here is the usual way to do so.

Definition 112. Fix a scheme X. For any Zariski quasicoherent sheaf $\mathcal{F}$ on X, we define the étale presheaf $\mathcal{F}^{\mathrm{\acute{e}t}}$ on $X_{\mathrm{\acute{e}t}}$ by sending the cover $p\colon U \to X$ to

$$\mathcal{F}_{\mathrm{\acute{e}t}}(U) := \mathrm{Hom}(p^*\mathcal{O}_X, p^*\mathcal{F}).$$

Remark 113. It turns out that this construction produces a sheaf. Something similar works for the fppf sites and fpqc sites. Let's explain this for the fpqc site. Indeed, fix a fpqc morphism $p\colon S' \to S$, so we set $S' \coloneqq S' \times_S S'$ with projection $q\colon S'' \to S$, and we need to check that the usual sequence

$$\mathcal{F}_{\mathrm{fpqc}}(S) \to \mathcal{F}_{\mathrm{fpqc}}(S') \to \mathcal{F}_{\mathrm{fpqc}}(S'')$$

is exact. Accordingly, we see that we may as well replace $\mathcal{F}$ with the pullback to S (so that $X = S$), and we have left to check that

$$\mathrm{Hom}(\mathcal{O}_S, \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}_{S'}, p^*\mathcal{F}) \to \mathrm{Hom}(\mathcal{O}_{S''}, q^*\mathcal{F})$$

is exact. Exactness now follows from some notion of descent.

The last remark we should make about sheaves on a site is that we can do sheafification.

Definition 114 (sheafification). Fix a site $\mathcal{C}$. Then there is a left adjoint to the forgetful functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$, which we call sheafification.

9 September 22nd: Uniform Boundedness over Function Fields

This talk was given by Jit Wu Yap at Boston University for the Boston University number theory seminar.

9.1 The Main Theorems

For today, we will work over a function field $K \coloneqq \mathbb{C}(B)$, where B is a smooth projective curve over $\mathbb{C}$. We are interested in abelian varieties with semistable reduction.

Definition 115 (semistable reduction). Fix an abelian variety A over K. We say that A has *semistable reduction* if and only if there is a semiabelian scheme G over B such that $G_K \cong A$.

Here are two theorems.

Theorem 116. Fix an integer g. Then there is an integer N only depending on g and the genus of B such that all g-dimensional abelian varieties A over K of semistable reduction has

$$\mathrm{ord}(x) \leq N$$

for all $x \in A(K)_{\mathrm{tors}}$.

Remark 117. Intuitively, this is a result on the boundedness of torsion, analogous to Maur's theorem for $g = 1$ over $K = \mathbb{Q}$. There is a long history of such results. In the 1990s, results were achieved for $g = 1$ over a number field. For $g \geq 2$, Silverberg showed the result when A has complex multiplication. Cadoret–Tamagawa's results were used as an input to Bakkar–Tsiermann showing this for $g \geq 2$ when A has real multiplication in 2018.

Theorem 118. Fix an integer g. Then there is a positive constant c depending only on g and the genus of B such that all g-dimensional abelian varieties A of semistable reduction, then the Néron–Tate height of any $x \in A(K)$ for which $\overline{\mathbb{Z}x} = A$ is bounded below by $c h_{\mathrm{Fal}}(A)$.

The Faltings height and the Weil height machine work just fine for function fields over $\mathbb{C}$. For example, one can define the Faltings height as the usual Weil height of the point $A \in \mathcal{A}^*_{g,3}$ against some canonically defined ample line bundle.

> **Remark 119.** This is referred to as a "Lang–Silverman" result because it was conjectured by them. In the case of $g = 1$, this was shown by Hindry–Silverman in 1988.

> **Remark 120.** The methods are largely arithmetic. The fact that we are working over $\mathbb{C}$ is used only once in the proof: we use Faltings's Arakelov inequality, which asserts that there is a positive constant c depending only on $g = \dim A$ for which
>
> $$h_{\mathrm{Fal}}(A) \le c\left(|S| + g(B) + 1\right),$$
>
> where S is the set of places of bad reduction of A. Accordingly, if this inequality is true over number fields K (which is known as a higher-dimensional Szpiro conjecture), then Theorems 116 and 118 hold. However, Szpiro's conjecture is known to be hard: just in $g = 1$, it is known to imply the abc conjecture.

9.2   Some Ideas

Here is the main idea for the results.

1. If $x \in A(K)$, then the Arakelov inequality is able to place constraints on $x \in A(K_v)$ for many places v.

2. However, K-points of small Néron–Tate height will equidistribute. Here is a formal statement for elliptic curves: given ascending collections $F_n \subseteq A(\overline{K})$ of Galois-invariant points, then it is known that

$$\frac{1}{\#F_n} \sum_{x \in F_n} \delta_x \to \mu,$$

for some suitably defined Haar measure μ. Thus, we cannot expeect to have too many points in $A(K)$ of small Néron–Tate height.

Here are a couple of tools.

1. Over $\mathbb{C}$, there is a notion of "transfinite diameter" of a sequence of points $\{x_1, \ldots, x_n\}$ defined as

$$\frac{1}{n^2} \sum_{i \ne j} -\log|x_i - x_j|.$$

This turns out to measure how close a given set of points are to a large-degree hypersurface.

2. It turns out that abelian varieties over local fields satisfies "degeneration by ultrafilters." Roughly speaking, given a countable collection $\{K_n\}_n$ of complete, algebraically closed field, nonarchimedean fields, and g-dimensional principally polarized abelian varieties A_n over K_n, then there is a "limit" A_ω over some complete, algebraically closed, nonarchimedean field K_ω for any ultrafilter ω on $\mathbb{N}$. Roughly speaking, one fixes a compactification $\overline{X}$ of $X := \mathcal{A}_{g,3}$. Then define $\lambda_{\partial X}(A)$ to be $-\log$ of the distance of A to $\partial \overline{X}$. Then we may define

$$A^\varepsilon := \left\{ (x_n)_n \in \prod_{n=1}^{\infty} K_n : |x_n|^{1/\lambda_{\partial X}(A_n)} \text{ is bounded} \right\}.$$

It now turns out that there is a morphism $\operatorname{Spec} A^\varepsilon \to \mathcal{A}_{g,3}$ making the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A^\varepsilon & \longrightarrow & \mathcal{A}_{g,3} \\
{\scriptstyle \mathrm{pr}_n} \downarrow & \nearrow & \\
\operatorname{Spec} K_n & &
\end{array}
$$

commute. Of course, one can certainly induce a map to $\overline{X}$, so the difficulty is showing that we do not end up in the boundary in the limit.

10   September 23: Unlikely Intersections

This talk was given by Xinyu Fang as a pre-talk for the Harvard–MIT algebraic geometry seminar.

10.1   The Ax–Schanuel Theorem

Today, we are going to discuss the following result, which is a version of unlikely intersections for $\exp$. Define $\pi \colon \mathbb{C}^n \to (\mathbb{C}^\times)^n$ by

$$\pi(z_1, \ldots, z_n) := (\exp(2\pi i z_1), \ldots, \exp(2\pi i z_n)).$$

We are interested in when π sends algebraic subvarieties to algebraic subvarieties.

> **Definition 121** (bialgebraic). A subvariety $L \subseteq \mathbb{C}^n$ is *bialgebraic* if and only if $\pi(L)$ continues to be algebraic.

> **Example 122.** If $L \subseteq \mathbb{C}^n$ is a linear subspace cut out by some (rational) equations of the form
>
> $$\sum_i a_i z_i = c,$$
>
> then $\pi(L)$ continues to be an algebraic subvariety now cut out by
>
> $$\prod_i w_i^{a_i} = 1,$$
>
> where w_i is $\exp(2\pi i z_i)$.

> **Theorem 123.** Every bialgebraic subvariety over $\mathbb{C}$ is a linear subspace.

Here is a slightly easier corollary.

> **Theorem 124** (Ax–Lindermann–Weierstrass). Let $V \subseteq (\mathbb{C}^\times)^n$ be an algebraic subvariety. Then any maximal algebraic subvariety $W \subseteq \pi^{-1}(V)$ is bialgebraic.

Here is our theorem.

> **Theorem 125** (weak Ax–Schanuel). Fix algebraic subvarities $V_1 \subseteq \mathbb{C}^n$ and $V_2 \subseteq (\mathbb{C}^\times)^n$ such that the analytic component U of the intersection $V_1 \cap V_2$ admits unexpected codimension, meaning
>
> $$\operatorname{codim} U < \operatorname{codim} V_1 + \operatorname{codim} V_2.$$
>
> Then U is contained in a proper bialgebraic subvariety.

Intuitively, we are saying that having large intersection is explained by having large linear subspaces.

   We will be interested in applications to Shimura varieties. These are some fancy quotients $\Gamma \backslash \Omega$ where Ω is some complex manifold, and $\Gamma \subseteq \Omega$ is a discrete subgroup. Having such a quotient gives us some uniformization map $\pi \colon \Omega \twoheadrightarrow Y$.

> **Example 126** (Modular curve). The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{H}$, and the quotient is named $Y(1)$.

Example 127 (PEL type)**.** The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on

$$\mathbb{H}_g := \{M \in M_g(\mathbb{C}) : Z^\mathsf{T} = Z \text{ and } \operatorname{im} Z > 0\}$$

via similar fractional linear transformations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot Z := (AZ + B)(CZ + D)^{-1}.$$

The quotient is $\mathcal{A}_g$, which turns out to be a moduli space of principally polarized g-dimensional abelian varieties.

Here is our version of the Ax–Schanuel theorem.

Theorem 128. Let $\pi \colon \Omega \to Y$ be the uniformization map of a Shimura variety Y. Fix algebraic sub-varieties $V_1 \subseteq \Omega$ and $V_2 \subseteq Y$ such that the analytic component U of the intersection $V_1 \cap V_2$ admits unexpected codimension, meaning

$$\operatorname{codim}_\Omega U < \operatorname{codim}_\Omega V_1 + \operatorname{codim}_Y V_2.$$

Then U is contained in a proper weakly special subvariety of Ω.

Here, weakly special means a translate of a special subvareity, where a special subvariety is roughly speaking one which is itself a Shimura variety.

Example 129. The special subvarieties of $Y(1)^n$ are given by $Y(1)^m \times \{0\}$ up to rearranging the coordinates.

11   September 30th: Homological Sieve and Manin's Conjecture

This talk was given by Sho Tanimoto.

11.1   Manin's Conjecture

Define height functions $H \colon \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{>0}$ in the usual way; for a projective variety, one gets a height function

Definition 130 (Fano)**.** A smooth projetive variety X is *Fano* if and only if $-K_X$ is an ample divisor. A two-dimensional Fano variety is a *del Pezzo* variety.

Example 131. Cubic hypersurfaces in $\mathbb{P}^n$ for $n \geq 3$ are Fano.

Example 132. Given a del Pezzo surface S over an algebraically closed field, set $e := (-K_S)^2$. If $e = 9$, then $S \cong \mathbb{P}^2$; if $e = 3$, then S is a smooth cubic surface in $\mathbb{P}^3$.

Conjecture 133 (Manin)**.** Fix a smooth variety X over a number field k such that $X(k)$ is not then. Then there is a thin set $Z \subseteq X(k)$ such that

$$N(X(k) \setminus Z, -K_X, T) \sim cT(\log T)^{\rho - 1},$$

where c is some explicit constant and ρ is the Picard rank.

Here, a thin subset is a finite union of images of maps $f_i \colon Y_i \to X$ which are generically finite but not bira-tional. For example, we are allowed to remove some affie spaces.

> **Remark 134.** There is a lot of history here. For example, del Pezzo surfaces in large degree are toric, so the conjecture is known. In smaller degrees, some is known but nothing in degree smaller than 4.

We are interested in working over global function fields. By valuative criteria, we find that we are basically interested in studying the space of morphisms $\mathbb{P}^1 \to X$, which can in turn by classified by nef α.

12   October 1st: L-Functions and the Homological Theta Lift

This talk was given by Jialiang Zou at MIT for the Lie groups seminar.

12.1   Basics on Theta Lifting

Fix a p-adic field F. We set $G_m \coloneqq \mathrm{GL}_m(F)$ and $H_n \coloneqq \mathrm{GL}_n(F)$, and we let $S(M_{mn})$ be the Schwarz functions on the matrices. We now let ω_{mn}, frequentiyl denoted ω, be the representation of $G_m \times H_n$ acting on $S(M_{mn})$ in the usual way, with some twists by characters to make the representation unitary. For example, Tate's thesis takes $m = n = 1$.

Let π be an irreducible representation of H_n, and we let $\omega[\pi]$ be the maximal π-isotypic quotient of ω, which lets us write

$$\omega[\pi] = \pi \boxtimes \Theta(\pi),$$

where $\Theta(\pi)$ is now some representation of G_m, which is known as the big theta lift of π. This construction is almost functorial: there is a functor $\Theta^\vee \colon \mathrm{Rep}(H_n) \to \mathrm{Rep}(G_m)$, which is also $\mathrm{Hom}_{H_n}(\omega, \pi)$ (where we are silently taking the smooth vectors). Here is some of what is known about this theta lift.

> **Theorem 135** (Minguez)**.** If $\Theta(\pi) \neq 0$, then it admits a unique irreducible quotient $\theta(\pi)$, and
>
> $$\dim_{\mathbb{C}} \mathrm{Hom}_{G_m \times H_n}(\omega, \pi \boxtimes \theta(\pi)) = 1.$$
>
> Furthermore, when $m \geq n$, if π is a Langlands quotient of some product $\tau_1 \times \cdots \times \tau_r$ of essentially discrete series representations, then $\theta(\pi)$ is some explicit Langlands quotient of the contragradients $\tau_\bullet^\vee$.

> **Example 136.** If $m = n$, then $\theta(\pi) = \pi^\vee$.

12.2   Two Approaches

We are interested in the structure of Θ, which we continue to think about as $\mathrm{Hom}_{H_n}(\omega, -)$. Algebraically or homologically, we have the following.

> **Theorem 137** (Adams–Prasad–Savin)**.** For all $i \geq 0$, the representation $\mathrm{Ext}^i_{H_n}(\omega, \pi)$ is of finite length, and it vanishes for $i \geq \mathrm{rank}\, H_n$. Furthermore, the virtual "Euler characteristic" representation
>
> $$\sum_i (-1)^i \mathrm{Ext}^i_{H_n}(\omega, \pi)$$
>
> is 0 if $m < n$ and is $\pi \times 1_{m-n}$ if $m \geq n$.

> **Remark 138.** One can stratify M_{mn} by the rank of the matrices, building subrepresentations J_k of the rank-k matrices. It turns out that the Euler characteristic is additive for this stratification.

Remark 139. When $m \geq n$, one expects that $\operatorname{Ext}^i_{H_n}(\omega, \pi) = 0$ for all $i \geq 1$, which will let us compute $\Theta(\pi)$ as the dual of the Euler characteristic.

Let's describe an analytic approach due to Fang–Sun–Xue. Working in the case $m = n$, the stratification has an open dense subrepresentation $\omega^\circ := S(\operatorname{GL}_n(F))$, also known as $\pi \boxtimes \pi^\vee$ where π is the regular representation. Then we have a short exact sequence

$$0 \to \omega^\circ \to \omega \to \omega/\omega^\circ \to 0,$$

and the long exact sequence produces a map

$$\operatorname{Hom}_{G_n \times H_m}(\omega, \pi \boxtimes \pi^\vee) \to \operatorname{Hom}_{G_n \times H_m}(\omega^\circ, \pi \boxtimes \pi^\vee),$$

and the problem turns into wanting to lift some eigendistributions on ω° to ω.

Idea 140. Obstructions to lifting is given by poles of L-functions.

In particular, we have a Godemont–Jacquet L-function defined by

$$Z(s, \varphi, f) := \int_{\operatorname{GL}_n(F)} \varphi(g) f(g) \, |\deg g|^{s+(n-1)/2} \, dg,$$

where φ is a Schwartz function and f is some matrix coefficient. The greatest common divisor of these L-functions is $L(s, \pi)$, so its poles can be seen to measure Z's ability to extend.

Theorem 141. If $L(s, \pi)$ is holomorphic at $s = 1/2$, then

$$\operatorname{Hom}_{G_n \times H_n}(\omega/\omega^\circ, \pi^\vee \boxtimes \pi) = 0$$

and

$$\operatorname{Hom}_{H_n}(\omega/\omega^\circ, \pi) = 0.$$

This more or less follows from the exact sequence and Howe duality. The point is that the zeta integral provides an explicit lift.

Corollary 142. If $L(s, \pi)$ or $L(s, \pi^\vee)$ is holomorphic at $s = 1/2$, then

$$\Theta(\pi) = \pi^\vee.$$

Idea. Let's argue in the case $L(s, \pi)$. The point is that the map

$$\operatorname{Hom}_{H_n}(\omega, \pi) \to \operatorname{Hom}_{H_n}(\omega^\circ, \pi)$$

is an isomorphism, and the second one is $\pi^\vee$.   ∎

12.3   Relating the Approaches

The moral is that having $\operatorname{Ext}^i_{H_n}(\omega, \pi) = 0$ for $i \geq 1$ or $L(s, \pi)$ holomorphic at $s = 1/2$ will imply $\Theta(\pi) = \pi^\vee$. One may hope that there is some relation between the L-functions and the homological lifting. This motivates the following result.

Theorem 143. Assume $m \geq n$ and π is an irreducible representation of H_n. If either $L(s, \pi)$ or $L(s, \pi^\vee)$ are holomorphic at $s = \frac{1+m-n}{2}$, then $\operatorname{Ext}^i_{H_n}(\omega, \pi) = 0$ for $i \geq 1$, and

$$\Theta(\pi) = \begin{cases} 1_{m-n} \times \pi^\vee & \text{if } L(s, \pi) \text{ is holomorphic at } s = (1+m-n)/2, \\ \pi^\vee \times 1_{m-n} & \text{if } L(s, \pi^\vee) \text{ is holomorphic at } s = (1+m-n)/2. \end{cases}$$

Both claims are proved by a more or less explicit calculation. For example, one can write π as a Langlands quotient and then just calculate. Then $L(s, \pi)$ being holomorphic at a given point can be seen on the level of this decomposition, and Ext vanishing can be seen via a calculation with the long exact sequence, comparing π with a Jacquet lift.

13 October 2: The Lefschetz Trace Formula

This talk was given by Arav Karighattam for the STAGE seminar at MIT.

13.1 The Tools

We are going to use ℓ-adic cohomology to prove all the Riemann conjectures, with the excpetion of the Riemann hypothesis. We willl recall the statements as we get to them.

Let's recall our Chow groups.

> **Definition 144** (Chow group)**.** Fix a variety X over a field k of equidimension d. Then we define the *Chow group* $\mathrm{CH}^\bullet(X)$ as the graded ring, where $\mathrm{CH}^i(X)$ contains the codimension-i cycles (up to rational equivalence). The product $[A] \cdot [B]$ is given by the intersection $[A \cup B]$, which makes sense when A and B are generically transverse, meaning that a generic point x in $A \cap B$ has $T_x A + T_x B = T_x X$.

As usual, we will not bother to show that this product makes sense, which requires some notion of the Moving lemma or a different approach.

Our main tool will be ℓ-adic cohomology.

> **Definition 145** (ℓ-adic cohomology)**.** Fix a variety X over a field k. For a prime ℓ distinct from $\mathrm{char}\, k$, we define ℓ-*adic cohomology* as
> $$\mathrm{H}_\ell^i(X) := \left(\lim \mathrm{H}^i(X_{k^{\mathrm{sep}}}; \underline{\mathbb{Z}/\ell^\bullet\mathbb{Z}}) \right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$
>
> This cohomology groups assemble into a graded commutative ring $\mathrm{H}_\ell^\bullet(X)$, where the product is given by the cup product.

> **Remark 146.** As usual, the cup product can be defined on the level of Čech cocycles.

It turns out that $\mathrm{H}_\ell^\bullet$ assembles into a Weil cohomology theory with coefficients in $\mathbb{Q}_\ell$. Let's quickly review what we are given.

- The cohomology groups $\mathrm{H}_\ell^\bullet$ are supported in degrees $[0, 2 \dim X]$.

- There is a Künneth formula
  $$\mathrm{H}_\ell^\bullet(X \times Y) \cong \mathrm{H}_\ell^\bullet(X) \otimes \mathrm{H}_\ell^\bullet(Y)$$
  induced by the projections.

- If X has equidimension d, then the cup product produces a perfect pairing
  $$\mathrm{H}_\ell^i(X) \times \mathrm{H}_\ell^i(X)(d) \to \mathrm{H}_\ell^{2d}(X)(d) \to \mathbb{Q}_\ell,$$
  where the last map is a trace map.

- There is a cycle class map $\mathrm{cl}_X \colon \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)(i)$.

These data are subject to many compatibilities.

13.2 The Lefschetz Trace Formula

Here is our theorem.

Theorem 147. Fix a regular endomorphism $\varphi \colon X \to X$ of a smooth projective variety X of equidimension d over a field k. Then
$$(\Gamma_\varphi \cdot \Delta_X) = \sum_{r=0}^{2d} (-1)^r \operatorname{tr}\left(\varphi^*; \mathrm{H}_\ell^r(X)\right).$$

Here, Γ_φ is the graph of φ, Δ_X is the diagonal, so $(\Gamma_\varphi \cdot \Delta_X)$ should be thought of as the number of fixed points of φ (counted with the correct multiplicities).

Proof. This is purely formal from the construction of a Weil cohomology theory. It turns out that the intersection number $(\Gamma_\varphi \cdot \Delta_X)$ agrees with the scalar
$$\operatorname{cl}_{X \times X}(\Gamma_\varphi \cdot \Delta) \in \mathrm{H}^{4d}(X \times X)(2d),$$

where the target is identified with $\mathbb{Q}_\ell$ via Poincaré duality. By a coherence property, we see that we want to evaluate $\operatorname{cl}_{X \times X}(\Gamma_\varphi) \cup \operatorname{cl}_{X \times X}(\Delta)$.

Let's explain how to compute $\operatorname{cl}_{X \times X}(\Gamma_\varphi)$, and then one can compute $\operatorname{cl}_{X \times X}(\Delta)$ by setting $\varphi = \operatorname{id}_X$. Well, for each degree r, fix a basis $\{e_{i,r}\}$ of $\mathrm{H}_\ell^r(X)$, which then has a dual basis $\{f_{i,2d-r}\}$ of $\mathrm{H}_\ell^{2d-r}(X)(d)$. We will take $e_{i,r} \cup f_{2d-i,r} = 1$ as our sign convention. The Künneth formula explains that $\mathrm{H}^\bullet(X \times X)$ can be identified with $\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X)$, so we get to write
$$\operatorname{cl}_{X \times X}(\Gamma_\varphi) = \sum_{i,r} a_{i,r} \boxtimes f_{i,2d-r}$$

for some coefficients $a_{i,r}$ which we would like to solve for. To do so, we note that
$$\operatorname{cl}_{X \times X}(\Gamma_\varphi) \cup (1 \boxtimes e_{j,r}) = a_{j,r} \boxtimes e_{2d},$$

where there graded commutative signs cancel out after expanding out $\boxtimes$ as a cup product. Thus,
$$\operatorname{pr}_{1*}\left(\operatorname{cl}_{X \times X}(\Gamma_\varphi) \cup (1 \boxtimes e_{j,r})\right) = a_{j,r}.$$

We can now collapse the left-hand side. Note $\Gamma_\varphi = (\operatorname{id}_X, \varphi)_* 1_X$, so we can rewrite this as
$$a_{j,r} = \operatorname{pr}_{1*}\left((\operatorname{id}_X, \varphi)_* 1_X \cup \operatorname{pr}_2^* e_{j,r}\right).$$

By the projection formula, this collapses to $\varphi^* e_{j,r}$, so
$$\operatorname{cl}_{X \times X}(\Gamma_\varphi) = \sum_{i,r} \varphi^* e_{j,r} \boxtimes f_{i,2d-r}.$$

Plugging in $\varphi = \operatorname{id}_X$, we see similarly that
$$\operatorname{cl}_{X \times X}(\Delta_X) = \sum_{i,r} e_{j,r} \boxtimes f_{i,2d-r}.$$

This is also
$$\operatorname{cl}_{X \times X}(\Delta_X) = \sum_{i,r} (-1)^r f_{i,2d-r} \boxtimes e_{j,r},$$

so $\operatorname{cl}_{X \times X}(\Gamma_\varphi) \cup \operatorname{cl}_{X \times X}(\Delta)$ is
$$\sum_{j,r} (-1)^r \left(\varphi^* e_{j,r} \cup f_{j,2d-r}\right) \boxtimes e_{2d}$$

even after keeping track of signs in the graded commutativity. Now, the sum over j of the piece in parentheses is exactly the trace of φ^* acting on a given basis of $\mathrm{H}_\ell^i(X)$, so we conclude. ∎

13.3 Some Weil Conjectures

Here is the main input to our proofs.

> **Proposition 148.** Fix a smooth projective variety X of equidimension d over $\mathbb{F}_q$. Then
> $$|X(\mathbb{F}_q)| = \sum_{r=0}^{2d} (-1)^r \operatorname{tr}\left(\operatorname{Frob}_q^*; \mathrm{H}_\ell^r(X)\right).$$

Proof. Let φ be the Frobenius. By Theorem 147, we only need to show that $(\Gamma_\varphi \cdot \Delta_X)$ is in fact $|X(\mathbb{F}_q)|$. Certainly the fixed points of the Frobenius acting on $X(\overline{\mathbb{F}}_q)$ is precisely $X(\mathbb{F}_q)$, so it remains to see that our intersection is actually transverse. This is true because all tangent spaces of Γ_φ are horizontal (the derivative of x^q vanishes in $\mathbb{F}_q$) while tangent spaces of Δ_X are diagonal. ∎

> **Corollary 149.** Fix a smooth projective variety X of equidimension d over $\mathbb{F}_q$. Then
> $$Z(X,T) = \exp\left(\sum_{n=1}^\infty X(\mathbb{F}_{q^n})\frac{T^n}{n}\right) = \prod_{r=0}^{2d} \det\left(1 - \operatorname{Frob}_q^* T; \mathrm{H}^i(X)\right)^{(-1)^{i+1}}.$$

Proof. We start with a linear algebraic fact. In general, if $\varphi\colon V \to V$ is a linear opeator over a field k, then we claim
$$-\log\det(1 - \varphi T; V) \overset{?}{=} \sum_{n=1}^\infty \operatorname{tr}(\varphi^n; V)\frac{T^n}{n}.$$

Combining this linear algebraic fact with Proposition 148 completes the argument.

To see the claim, note that both sides of the identities are additive in the pair (V, φ), and the case of scalars c acting on a one-dimensional space amounts to the Taylor expansion of $\log$ as $-\log(1 - \lambda T) = \sum_{n\geq 1} \lambda^n T^n / n$. To complete the proof, we note that the identity is insensitive to changing the base field, so we may base-change to the algebraic closure, diagonalize φ, and reduce to the case of scalars. ∎

We will now have achieved rationality as soon as we can show that the polynomials
$$P_i(T) := \det\left(1 - \operatorname{Frob}_q^* T; \mathrm{H}^i(X)\right)$$

live in $1 + T\mathbb{Z}[T]$. Certainly it is in $1 + T\mathbb{Q}_\ell[T]$, and the alternating product of these polynomials is Z and therefore is in $1 + T\mathbb{Z}[[T]]$, so one can make an argument that the P_is must be in $1 + T\mathbb{Z}[T]$.

It remains to prove the functional equation. This follows from Poincaré duality. Indeed, $\operatorname{Frob}_* \operatorname{Frob}^* = q^d$ on cohomology, so

$$Z\left(X, q^{-d}T^{-1}\right) = \prod_{r=0}^{2d} \det\left(1 - q^{-d}\operatorname{Frob}^* T^{-1}; \mathrm{H}^i(X)\right)^{(-1)^{r+1}}$$
$$= \prod_{r=0}^{2d} \det\left(1 - q^{-d}\operatorname{Frob}_* T^{-1}; \mathrm{H}^{2d-i}(X)(d)\right)^{(-1)^{r+1}}.$$

This can be unwound into the functional equation.

14 October 6: The Visibility of Elements of the Tate–Shafaverich Group

This talk was given by Jerson Caro for the Boston Unviersity number theory seminar. It is joint work with Barinder Banwait and Shiva Chidambaram.

14.1   Visualization

Fix an elliptic curve E over $\mathbb{Q}$, and choose an integer $n \geq 2$. Taking Galois cohomology of the Kummer exact sequence

$$0 \to E[n] \to E \xrightarrow{n} E \to 0$$

produces the short exact sequence

$$0 \to \frac{E(\mathbb{Q})}{nE(\mathbb{Q})} \to \mathrm{H}^1(\mathbb{Q}; E[n]) \to \mathrm{H}^1(\mathbb{Q}; E)[n] \to 0.$$

Comparing with the localizations, we produce a short exact sequence

$$0 \to \frac{E(\mathbb{Q})}{nE(\mathbb{Q})} \to \mathrm{Sel}_n(E/\mathbb{Q}) \to \text{Ш}(E/\mathbb{Q})[n] \to 0,$$

where $\mathrm{Sel}_n(E/\mathbb{Q})$ is the kernel of the map $\mathrm{H}^1(\mathbb{Q}; E[n]) \to \prod_v \mathrm{H}^1(\mathbb{Q}_v; E)[n]$, and $\text{Ш}(E/\mathbb{Q})$ is the kernel of the map $\mathrm{H}^1(\mathbb{Q}; E) \to \prod_v \mathrm{H}^1(\mathbb{Q}_v; E)$.
  One can give $\mathrm{H}^1(\mathbb{Q}; E)$ a geometric realization as torsors.

> **Definition 150** (torsor). A *torsor* of E is a genus 1 curve C equipped with a simply transitive action of E. An isomorphism of torsors is an isomorphism of the underlying curves commuting with the action of E.

> **Remark 151.** It turns out that torsors are parameterized by $\mathrm{H}^1(\mathbb{Q}; E)$. In short, a torsor C is a twist of E: any geometric point $p_0 \in C(\overline{\mathbb{Q}})$ defines an isomorphism $E \to C$ given by $P \mapsto P + p_0$. Twists are parameterized by $\mathrm{H}^1(\mathbb{Q}; \mathrm{Aut}\, E)$, so it is a matter of checking that the torsors correspond to 1-cocycles of $E \subseteq \mathrm{Aut}\, E$.

For example, we see that $\text{Ш}(E/\mathbb{Q})$ contains "everywhere locally trivial" torsors.
  By considering the addition structure on torsors, one can show the following.

> **Theorem 152** (Cassels). If C is an E-torsor representing an element of $\text{Ш}(E/\mathbb{Q})[n]$, then C admits a divisor over $\mathbb{Q}$ of degree n.

Here is another way to visualize, due to Mazur: fix an embedding $\varphi \colon E \to A$, where A is an abelian variety. Given a torsor C, if the corresponding class ξ vanishes under the pushforward $\varphi_* \colon \mathrm{H}^1(E; \mathbb{Q}) \to \mathrm{H}^1(A; \mathbb{Q})$, then C embeds into A. Indeed, the short exact sequence

$$0 \to E \to A \to B \to 0$$

of group schemes produces the exact sequence

$$A(\mathbb{Q}) \to B(\mathbb{Q}) \to \mathrm{H}^1(E; \mathbb{Q}) \to \mathrm{H}^1(\mathbb{Q}; A).$$

We are given that ξ vanishes in $\mathrm{H}^1(\mathbb{Q} : A)$, so it comes from some $\beta \in B(\mathbb{Q})$, and one can check that the fiber in A is isomorphic to C. This motivates the following definition.

> **Definition 153.** Fix an abelian variety A containing E. Then we define the *visibility* as
>
> $$\mathrm{Vis}_A\, \mathrm{H}^1(\mathbb{Q}; E) := \ker\left(\mathrm{H}^1(\mathbb{Q}; E) \to \mathrm{H}^1(\mathbb{Q}; A)\right).$$
>
> Similarly, we define $\mathrm{Vis}_A\, \text{Ш}(E/\mathbb{Q})$ to be $\mathrm{Vis}_A\, \mathrm{H}^1(\mathbb{Q}; E) \cap \text{Ш}(E/\mathbb{Q})$.

Remark 154. Note $\mathrm{Vis}_A \mathrm{H}^1(\mathbb{Q}; E)$ is finite because it fits into the short exact sequence

$$0 \to \frac{A(\mathbb{Q})}{E(\mathbb{Q})} \to B(\mathbb{Q}) \to \mathrm{Vis}_A \mathrm{H}^1(\mathbb{Q}; E) \to 0,$$

and one can compare ranks of the two groups on the left.

It turns out that every element ξ of $\mathrm{III}(E/\mathbb{Q})$ can be visualized. Here are two constructions.

- Let C be the corresponding torsor, and choose a field K with $C(K) \neq \varnothing$. (One can show that K should take degree n when ξ has order n.) Then $A = \mathrm{Res}_{K/\mathbb{Q}} E$ will work.

- It turns out that $\xi \in \mathrm{III}(E/\mathbb{Q})[n]$ produces an Azumaya algebra $\mathcal{A}$ of rank n^2 over $\mathbb{Q}(E)$. Upon choosing some field $L \subseteq \mathcal{A}$ of degree n over $\mathbb{Q}(E)$, one can take A to be the Jacobian of the curve corresponding to L.

14.2   Minimal Visualization

We are interested in the "smallest" abelian variety visualizing an element $\xi \in \mathrm{III}(E/\mathbb{Q})$.

Definition 155 (visibility dimension)**.** For $\xi \in \mathrm{III}(E/\mathbb{Q})$, we define the *visibility dimension* as the minimal dimension of an abelian variety visualizing ξ.

Here is some of what is known.

- Mazur showed that elements in $\mathrm{III}(E/\mathbb{Q})[3]$ have visibility dimension 2, and Bruin–Dahmen gave explicit constructions by Jacobians of genus 2 curves.

- Kenke showed that elements in $\mathrm{III}(E/\mathbb{Q})[2]$ have visibility dimension 2, and Bruin gave an explicit construction by Jacobians of genus 2 curves.

- Fisher gave examples of elements in $\mathrm{III}(E/\mathbb{Q})[6]$ and $\mathrm{III}(E/\mathbb{Q})[7]$ that are not visualized in abelian surfaces. He also showed that elements of $\mathrm{III}(E/\mathbb{Q})[4]$ cannot be visualized in principally polarized abelian varieties, such as Jacobians.

For abelian surfaces, here is one way to make progress.

Definition 156 (congruent)**.** Two elliptic curves E and F are n-*congruent* if and only if there is an isomorphism $E[n] \cong F[n]$ of Galois modules.

Proposition 157. Given $\xi \in \mathrm{III}(E/\mathbb{Q})[n]$, if there is an abelian surface A visualizing ξ, then there is a curve F which is n-congruent to E for which

$$A = \frac{E \times F}{\Delta},$$

where Δ is the diagonal embedding of the n-torsion.

The idea is that visualization provides a short exact sequence

$$0 \to E \to A \to F \to 0$$

of abelian varieties, so there is an isogeny $E \times F \to A$. It is then a matter of studying the kernel.

Another way to make sense of minimality is categorically.

> **Definition 158** (visibility category)**.** We define the *m-visibility category* $\mathcal{V}(E/\mathbb{Q}, m)$ whose objects are abelian varieties visualizing every element in $\text{III}(E/\mathbb{Q})[m]$, and the morphisms are given by inclusions (commuting with the embedding from the torsor).

> **Remark 159.** It turns out that minimal objects in $\mathcal{V}(E/\mathbb{Q}, m)$ can have different dimensions; examples have been found as early as $m = 5$.

The main result is that $n \in \{2, 3\}$ have canonical minimal elements.

15   October 7: Near Coincidences and Nilpotent Division Fields

This talk was given by Harris Daniels for the MIT number theory seminar.

15.1   Constructibility

Our story begins historically.

> **Theorem 160** (Gauss–Wantzel)**.** A regular n-gon is constructible if and only if $\varphi(n)$ is a power of 2.

In other words, n is a power of 2 times a semiprime product of Fermat primes (which are one more than a power of 2). The key input is the following.

> **Theorem 161.** A complex number $\alpha \in \mathbb{C}$ is constructible if and only if α belongs to a number field K for which $\text{Gal}(K/\mathbb{Q})$ is a 2-group.

Thus, we are asking when $\mathbb{Q}(\zeta_n)$ has Galois group which is a 2-group.
    Here is our first main result.

> **Theorem 162.** Let E be the elliptic curve $y^2 = x^3 - x$ and $n \geq 2$. Then $\mathbb{Q}(E[n])$ is constructible if and only if $\varphi(n)$ is a power of 2.

Proof. We show the directions independently.

- If $\mathbb{Q}(E[n])$ is constructible, then the Weil pairing forces $\mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(E[n])$, which then forces ζ_n to be constructible.

- If $\varphi(n)$ is constructible, then n is the product of a power of 2 and some Fermat primes, so we can do casework. In this case, one can explicitly show that $\mathbb{Q}\left(E[2^{\nu+1}]\right)/\mathbb{Q}(E[2^\nu])$ is a power of 2. Additionally, when p is a Fermat prime, one finds that the image of

$$\text{Gal}(\mathbb{Q}(E[p], i)/\mathbb{Q}(i)) \to \text{GL}_2(\mathbb{F}_p)$$

  is diagonal by the theory of complex multiplication, and this diagonal subgroup has size equal to a power of 2 because p is a Fermat prime. ∎

Our motivation comes from the following result.

> **Theorem 163.** Fix an elliptic curve E. If $\mathbb{Q}(E[n])/\mathbb{Q}$ is abelian, then $n \leq 6$.

We are interested in generalizing such a result where abelian is replaced by nilpotent. One way to access this condition is that nilpotent groups are the products of their Sylow subgroups, which is an instance of Theorem 162.

15.2   Nilpotency

We are going to rely on the following "standard" conjecture.

Conjecture 164. If $p > 11$, then one has a reasonable control over rational points on modular curves of the non-split Cartan subgroup.

Theorem 165. If E does not have complex multiplication with $\mathbb{Q}(E[n]/\mathbb{Q})$ is nilpotent, then either $n < 25$ or is a power of 2.

In the base case $E[p]$, one appeals to some classification results about subgroups of PGL_2.

In the $E[p^2]$ case, one finds that $\mathbb{Q}\left(E[p^2]\right) = \mathbb{Q}(E[p], \zeta_{p^2})$, which is not expected to happen. Here is the sort of result one knows about these things.

Theorem 166. Fix an elliptic curve E over $\mathbb{Q}$, and let p be a prime and $n \in \mathbb{Z}^+$. If $\mathbb{Q}\left(E[p^{n+1}]\right) = \mathbb{Q}(E[p^n])$, then $(p, n) = (2, 1)$.

One shows this basically by comparing when p-power roots of unity can be found in $E[p^\bullet]$. Of course, in our application, we should be allowed to add roots of unity.

Theorem 167. If $\mathbb{Q}\left(E[p^{n+1}]\right) = \mathbb{Q}(E[p^n], \zeta_{p^{n+1}})$, then $p \in \{2, 3\}$ and $n = 1$.

This seems to also come down to some group theory.

16   October 8: A new approach to Hecke Correspondences

This talk was given by Keerthi Madapusi for the Harvard number theory seminar.

16.1   Rapoport–Zink Spaces

For today, G is a reductive group over $\mathbb{Z}_p$, and $\mu \colon \mathbb{G}_{m,\mathcal{O}} \to G_{\mathcal{O}}$ is miniscule cocharacter defined over some unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$. We also let $\check{\mathbb{Z}}_p$ be the maximal unramified ring of integers over $\mathbb{Z}_p$, whose fraction field is $\check{\mathbb{Q}}_p$.

Definition 168 (Deligne–Lusztig variety)**.** Fix some $b \in G(\check{\mathbb{Q}}_p)$. Then we define the *affine Deligne–Lusztig variety* as

$$X_\mu b(\overline{\mathbb{F}}_p) := \left\{ g \in G(\check{\mathbb{Q}}_p)/G(\check{\mathbb{Z}}_p) : g^{-1} b \sigma(g) \in G(\check{\mathbb{Z}}_p)\mu(p)G(\check{\mathbb{Z}}_p) \right\}.$$

Here is our main theorem.

Theorem 169. There is a formally smooth formal scheme $RZ_b^{G,\mu}$ over $\check{\mathbb{Z}}_p$ such that

$$RZ_b^{G,\mu}(\overline{\mathbb{F}}_p) \cong X_{\sigma(\mu)}b(\overline{\mathbb{F}}_p).$$

Remark 170. This is special to the reductive case: if G fails to be reductive, formal smoothness can fail.

Remark 171. There are global applications to integral models of Shimura varieties.

The RZ stands for Rapoport–Zink. Accordingly, let's give some historical remarks.

Example 172. With $G = \mathrm{GL}_n$, one can let μ_d be the cocharacter

$$z \mapsto \mathrm{diag}(\underbrace{z, \ldots, z}_{d}, \underbrace{1, \ldots, 1}_{n-d}).$$

It turns out that $X_\mu b(\overline{\mathbb{F}}_p) \neq \varnothing$ if and only if there is a p-divisible group $\mathcal{H}$ over $\overline{\mathbb{F}}_p$ of dimension d and height n such that the Dieudonné module $\mathbb{D}(\mathcal{H})[1/p]$—which is some n-dimensional vector space over $\check{\mathbb{Q}}_p$ equipped with a σ-linear automorphism F—has $F = b\sigma$. When nonempty, we define $RZ_r^{G,\mu}$ to be the formal moduli space of pairs $(\mathcal{G}, \xi)$ where $\mathcal{G}$ is a p-divisible group of height n, and $\xi \colon \mathcal{G} \to \mathcal{H}$ is a quasi-isogeny.

Remark 173. Rapoport and Zink show that $RZ_r^{G,\mu}$ is a formal scheme. (Note that it is much easier to show that it is an ind-formal scheme.) One can use the above case to handle G of "EL type," meaning that (G, μ) embeds into some (GL_n, μ_d), and G is cut out by some endomorphisms.

Remark 174. Wansu Kim and Howard–Pappas handled more general G, still requiring some embedding $(G, \mu) \hookrightarrow (\mathrm{GL}_n, \mu_d)$.

Remark 175. Later, Scholze–Weinstein constructed the diamond associated to $RZ_b^{G,\mu}$ and the shtukas with one leg, which more or less explain as a formal model of our moduli space on perfectoid inputs.

16.2 p-Divisible Groups

The main "philosophical" input is a linear algebraic characterization of p-divisible groups over rather general bases.

Theorem 176. Fix a p-complete ring R. Then there is a canonical equivalence and p-divisible groups over R and vector bundles over R^{syn} with Hodge–Tate weights in $\{0,1\}$. Here, R^{syn} is the "syntonification" of R.

This makes the syntonification appear as a natural construction. Namely, one expects $RZ_b^{G,\mu}$ to be moduli of certain isogenies to given objects which look like these vector bundles on R^{syn}.

Definition 177. We define $\mathrm{BT}^{G,\mu}$ to be the functor taking R to the G-bundles on R^{syn} bounded by μ.

Theorem 178. The moduli functor $\mathrm{BT}^{G,\mu}$ is a smooth formal Artin stack over $\mathcal{O}$.

17   October 9: Constructible Sheaves, Base Change, and L-functions

This talk was given by Mikayel Mkrtchyan for the STAGE seminar at MIT.

17.1   Constructible Sheaves

For today, all schemes will be finite type and separated over a field. Unless otherwise stated, X_0 and $\mathcal{F}_0$ will be objects defined over a finite field $\mathbb{F}_q$, and X and $\mathcal{F}$ will be their base-changes to the algebraic closure $\overline{\mathbb{F}}_q$. This convention also holds for other similar letters. While we're here, we fix our characteristic to be $p > 0$, and we choose a prime $\ell \neq p$. We work with the étale topology throughout.

Let's begin by stating some foundational results.

Theorem 179. Fix a smooth proper morphism $f\colon X \to Y$ of qcqs schemes. For any ℓ-adic local system $\mathcal{L}$, the pushforward $\mathrm{R}^i f_* \mathcal{L}$ is a local system on Y.

Remark 180. The topological intuition is that a proper submersion $f\colon X \to Y$ of real manifolds is a locally trivial fibration, which is known as Ehresmann's lemma.

We may be interested in upgrading this, removing properness or smoothness. One no longer expects to get local systems from higher pushforwards: instead, we get constructible sheaves.

Definition 181. Fix a sheaf $\mathcal{F}$ on X with finite stalks coprime to p. Then $\mathcal{F}$ is *constructible* if and only if there is a Zariski locally closed disjoint union $X = \bigsqcup_i X_i$ such that $\mathcal{F}|_{X_i}$ is a local system for each i.

Example 182. If $i\colon Z \to X$ is a closed subset, then $i_* \underline{\mathbb{Z}/\ell\mathbb{Z}}$ is constructible.

Remark 183. We can also choose a stratification $\bigsqcup_i X_i$ to be some étale locally closed disjoint union. The point is that one can check if a sheaf is a local system after étale base change.

Here are some indications that we have given a good definition.

Theorem 184 (finitude)**.** Fix a morphism $f\colon X \to Y$ of qcqs schemes. If $\mathcal{F}$ is constructible on X, then $\mathrm{R}^i f_* \mathcal{F}$ is constructible.

Theorem 185. Fix an open subset $j\colon U \to X$, and set $i\colon Z \to X$ to be the complement. Then the data of a constructible sheaf on X is equivalent to the data of a triple $(\mathcal{F}_Z, \mathcal{F}_U, f)$, where $\mathcal{F}_Z$ is a constructible sheaf on Z, and $\mathcal{F}_U$ is a constructible sheaf on U, and f is a map $\mathcal{F}_Z \to i^* j_* \mathcal{F}_U$.

Sketch. It is not hard to build the triple from $\mathcal{F}$ by restricting to Z and U. Given a triple $(\mathcal{F}_Z, \mathcal{F}_U, f)$, surely we know what the stalks are, and f tells us how to glue. ∎

Example 186. Fix a smooth curve X over $\mathbb{C}$, and choose a finite subset $Z \subseteq X$ of "cusps," and let $U \coloneqq X \setminus Z$ be its kernel. Then a constructible sheaf on X has equivalent data to a local system $\mathcal{L}$ on U (which is equivalent to the data of a representation $\pi_1(U) \to \mathrm{GL}(L)$ for some given vector space L), a finite group V_z at each $z \in Z$, and the last map $f\colon \mathcal{F}_Z \to i^* j_* \mathcal{F}_U$ amounts to a map $V_z \to L^{I_z}$. To see this last map, we see that $i^* j_* \mathcal{L}$ is the colimit of $\mathcal{L}(U \setminus z)$ where U is an open neighborhood of z, but these sections turn out to be given by L^{I_z} via the Riemann–Hilbert correspondence. In particular, the maps $V_z \to L^{I_z}$ vanishing corresponds to adding skyscraper sheaves.

Example 187. Similarly, let X_0 be a smooth curve over $\mathbb{F}_q$, let $U_0 \subseteq X_0$ be a nonempty open subset, and set $Z_0 \coloneqq X_0 \setminus U_0$. Then the data of a constructible sheaf on X_0 is equivalent to the data of a local system $\mathcal{L}$ on U_0 (which is the data of a representation of $\pi_1^{\text{ét}}(U)$ on some L), a representation of $\mathrm{Gal}(\overline{k(z)}/k(z))$ on some V_z for each $z \in Z_0$, and a map $V_z \to L^{I_z}$ for each z. Here, I_z is the inertia subgroup which is the kernel of $\pi_1^{\text{ét}}(U, z) \to \mathrm{Gal}(\overline{k(z)}/k(z))$.

We are now allowed to make the following central definition.

Definition 188 (ℓ-adic sheaf)**.** An ℓ-*adic sheaf* is a compatible system of constructible sheaves of constructible sheaves of $(\mathbb{Z}/\ell^\bullet\mathbb{Z})$-modules. Here, the compatibility requires that the locally closed stratifications stabilize for higher powers of ℓ. We think about this as by taking inverse limits over the compatible systems.

Remark 189. The stalks of an ℓ-adic sheaf are $\mathbb{Z}_\ell$-modules. We will frequently (and silently) make these $\mathbb{Q}_\ell$-vector spaces.

17.2   Proper Base Change

Suppose we have a commutative square as follows.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Given an ℓ-adic sheaf $\mathcal{F}$ on X, then there is a base-change morphism

$$g^* f_* \mathcal{F} \to f'_* (g')^* \mathcal{F}$$

defned by using various adjunctions: note

$$\begin{aligned} \mathrm{Hom}(g^* f_* \mathcal{F}, f'_* (g')^* \mathcal{F}) &= \mathrm{Hom}(f_* \mathcal{F}, g_* f'_* (g')^* \mathcal{F}) \\ &= \mathrm{Hom}(f_* \mathcal{F}, f_* g'_* (g')^* \mathcal{F}) \\ &\supseteq f_* \mathrm{Hom}(\mathcal{F}, g_* (g')^* \mathcal{F}), \end{aligned}$$

and the last set has a canonical adjunction map. Using something about δ-functors, one can upgrade our given map to a base change map

$$g^* \mathrm{R}^i f_* \mathcal{F} \to \mathrm{R}^i f'_* (g')^* \mathcal{F}.$$

These maps are in general not isomorphisms.

Example 190. Consider the pullback square

$$\begin{array}{ccc} \varnothing & \xrightarrow{g'} & \mathbb{G}_m \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ 0 & \xrightarrow{g} & \mathbb{A}^1 \end{array}$$

of intersections. Then one can compute that $g^* f_* \underline{\mathbb{Q}_\ell} = \underline{\mathbb{Q}_\ell}$, but $f'_* (g')^* \underline{\mathbb{Q}_\ell}$ vanishes.

However, in good situations, the result holds.

Theorem 191 (Proper base change)**.** If the square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a pullback with f proper, then the map $g^* \mathrm{R}^i f_* \mathcal{F} \to \mathrm{R}^i f'_* g^* \mathcal{F}$ is an isomorphism.

One application is that we can make sense of cohomology with compact supports.

Definition 192. Fix a Zariski open embedding $j\colon U \to X$ and a sheaf $\mathcal{F}$ on U. Then there is a functor $j_!\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ which is left adjoint to j^*. It is given by the sheafification of the presheaf sending some étale open $V \to X$ to
$$\begin{cases} f(V) & \text{if } V \text{ factors through } U, \\ 0 & \text{else.} \end{cases}$$

Remark 193. One can see that $j_!$ is exact: indeed, its stalks are identity in U and vanish outside U.

Remark 194. For a sheaf $\mathcal{F}$ on X, we let $j\colon U \to X$ be an open embedding and let $i\colon Z \to X$ be the complement. Then we have a short exact sequence
$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$
One can check exactness on stalks.

Definition 195. Fix a morphism $f\colon X \to Y$ of separated schemes over a field k. Then we define the functor $f_!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ as follows. By a theorem of Nagata, f factors through some compactification $j\colon X \to \overline{X}$ where the induced map $\overline{f}\colon \overline{X} \to Y$ is proper. Then we define
$$\mathrm{R}^i f_! \mathcal{F} := \mathrm{R}^i \overline{f}_* (j_! \mathcal{F}).$$
If X is separated over a field k, then we can define $\mathrm{H}_c^i(X; \mathcal{F})$ as $\mathrm{R}^i p_! \mathcal{F}$ where $p\colon X \to \mathrm{Spec}\, k$ is the structure morphism.

Remark 196. One can use Theorem 191 (and the Leray spectral sequence) to show that this definition is independent of j.

Remark 197. The functor $\mathrm{R}^i f_!$ sends constructible sheaves to constructible sheaves.

Remark 198. Given any Cartesian square
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
then the base change maps $g^* \mathrm{R} f_! \to \mathrm{R}^i f_!'(g')^*$ will always be an isomorphism.

Remark 199. The functors $\mathrm{R}^i f_!$ are not the derived functors of $f_!$.

17.3 L-**Functions**

We now settle into our conventions, where X_0 is a variety over $k := \mathbb{F}_q$, and $\mathcal{F}_0$ is a sheaf on X_0. Then one finds that there are Galois actions on $\mathrm{H}^i(X; \mathcal{F})$ and $\mathrm{H}_c^i(X; \mathcal{F})$ as follows: for $\sigma \in \mathrm{Gal}(\overline{k}/k)$, then we have a morphism $(\sigma \times \mathrm{id}_{X_0})\colon X \to X$, so we induce a map
$$\mathrm{H}^i(X; \mathcal{F}) \to \mathrm{H}^i(X; (\sigma \times \mathrm{id}_{X_0})^* \mathcal{F}) = \mathrm{H}^i(X; \mathcal{F}),$$
where the last identification holds because $\mathcal{F}$ started its life over k.

Notation 200. We let $\mathrm{Frob}_{\mathrm{arith}} \in \mathrm{Gal}(\overline{k}/k)$ be the arithmetic Frobenius $x \mapsto x^{|k|}$, and we let Frob be the geometric Frobenius.

The geometric Frobenius is convenient because it provides the correct morphism on points.

For calculations later, it will be helpful to have L-functions of general sheaves.

Definition 201. Fix a constructible ℓ-adic sheaf $\mathcal{F}_0$ on X_0 over a finite field $k := \mathbb{F}_q$. Then we define the zeta function
$$Z(X_0; \mathcal{F}_0, T) := \prod_{\text{closed } x \in X_0} \frac{1}{\det\left(1 - \mathrm{Frob}T^{\deg x}; \mathcal{F}_x\right)}.$$

Remark 202. A short calculation shows that
$$Z(X_0, \mathcal{F}_0; T) = \exp\left(\sum_{m \geq 1} \left(\sum_{x \in X_0(\mathbb{F}_{q^m})} \mathrm{tr}(\mathrm{Frob}_x; \mathcal{F}_x)\right) \frac{T^m}{m}\right).$$
Here, $\mathrm{Frob}_x \in \mathrm{Gal}(\overline{k(x)}/k(x))$ is the geometric Frobenius, and it acts on the stalk of $\mathcal{F}_x$ by viewing the stalk as the pullback to the point.

Example 203. One can check that $\mathcal{F}_0 = \underline{\mathbb{Z}_\ell}$ recovers the usual zeta function.

One still has a rationality result.

Theorem 204 (Grothendieck–Lefschetz trace formula). Fix a constructible ℓ-adic sheaf $\mathcal{F}_0$ on X_0 over a finite field $k := \mathbb{F}_q$. Then
$$Z(X_0; \mathcal{F}_0, T) = \prod_{i=0}^{2\dim X} \det\left(1 - \mathrm{Frob}T; \mathrm{H}_c^i(X; \mathcal{F})\right)^{(-1)^{i+1}}.$$

Remark 205. One can view this as a "global expression" for the "locally defined" Euler product.

The result becomes more memorable if we pass through the sheaf-function dictionary.

Definition 206. Fix a constructible ℓ-adic sheaf $\mathcal{F}_0$ on X_0 over a finite field $k := \mathbb{F}_q$. Then we define the function $G_{\mathcal{F}_0} \colon X_0(\mathbb{F}_{q^m}) \to \overline{\mathbb{Q}}_\ell$ by
$$G_{\mathcal{F}_0}(x) := \mathrm{tr}(\mathrm{Frob}_x; \mathcal{F}_x).$$

Remark 207. Note $G \colon \mathrm{Sh}(X) \to \mathrm{Fun}(X(\mathbb{F}_{q^m}), \mathbb{Q}_\ell)$ factors through the Grothendieck group $K_0(\mathrm{Sh}(X))$.

This gives a relative version of the trace formula.

Theorem 208 (Grothendieck–Lefschetz trace formula)**.** Fix a morphism $f_0\colon X_0 \to Y_0$ and a sheaf $\mathcal{F}_0$. Then the diagram

$$
\begin{array}{ccc}
K_0(\mathrm{Sh}(X_0)) & \xrightarrow{\;G\;} & \mathrm{Fun}(X_0(\mathbb{F}_{q^m}), \mathbb{Q}_\ell) \\
{\scriptstyle \mathrm{R}f_!}\big\downarrow & & \big\downarrow{\scriptstyle \smallint} \\
K_0(\mathrm{Sh}(Y_0)) & \xrightarrow{\;G\;} & \mathrm{Fun}(Y_0(\mathbb{F}_{q^m}), \mathbb{Q}_\ell)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F} & \longmapsto & g \\
\big\downarrow & & \big\downarrow \\
\sum_i (-1)^i \mathrm{R}^i f_! \mathcal{F} & \longmapsto & y \mapsto \sum_{g(x)=y} g(x)
\end{array}
$$

commutes.

18   October 10: The p-Adic Abel–Jacobi Map

This talk was given by Yashi for the Automorphic student seminar at Johns Hopkins University.

18.1   Construction of the Map

Fix an unramified extension F of $\mathbb{Q}_p$. Choose a variety X_r over F admitting a smooth model $\mathcal{X}_r$ over $\mathcal{O}_F$. Then the Abel–Jacobi map will be a map

$$
\mathrm{AJ}_F^{\mathrm{\acute{e}t}}\colon\ \mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(F) \to \mathrm{H}^{2r+1}\left(F, \varepsilon_{X_r}\mathrm{H}^1_{\mathrm{\acute{e}t}}(\overline{X}_r; \mathbb{Q}_p(r+1))\right).
$$

Let's explain some of this notation. In the application, X_r will be the Kuga–Sato variety $W_r \times A^r$, where A is some elliptic curve with complex multiplication by an order R, and W_r is the desingularization of $\mathcal{E}^r$, where $\mathcal{E}$ is the universal elliptic curve sitting over a curve $C := Y_1(N)$. As such $\dim X_r = 2r+1$. Then $\varepsilon_{X_r} = \varepsilon_{W_r}\varepsilon_{A_r}$, where $\varepsilon_{W_r} \in \mathbb{Q}[\mathrm{Aut}\,W_r]$ and $\varepsilon_{A_r} \in \mathbb{Q}[\mathrm{Aut}\,A^r]$. Roughly speaking, they ensure that we land in Sym^r part of our cohomology. Slightly more precisely, ε_{W_r} is the product of the averaging operators

$$
\frac{1}{N^r}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^r} t_a \qquad \text{and} \qquad \frac{1}{2^r r!}\sum_{\theta\in\{\pm 1\}^r \rtimes S_r} \mathrm{sgn}(\theta)\theta,
$$

and ε_{A^r} is the same as the right averaging operator. Roughly speaking, we are projecting onto the sign component.

We are going to care about the image of our map on some cycles of the form $\Delta_\varphi = \mathcal{E}_{X_r}\gamma_\varphi$, where γ_φ is the r-fold graph of some endomorphism φ on a chosen $(A', \varphi) \in C$. Denote this point by p. (Sorry!) Let's now define our map. By construction, there is a projection $\pi_r\colon X_r \to C$. Given $p \in C$, we define $X_p \subseteq X_r$ as the fiber and X_r^b as the complement. The Gysin sequence of $X_r = X_p \sqcup X_r^b$ reads as

$$
\mathrm{H}^{2r-1}(\overline{X}_p; \mathbb{Q}_p)(r) \to \mathrm{H}^{2r+1}(\overline{X}_r; \mathbb{Q}_p)(r+1) \to \mathrm{H}^{2r+1}(\overline{X}_r^b; \mathbb{Q}_p) \to \mathrm{H}^{2r}(\overline{X}_p; \mathbb{Q}_p)(r)_0 \to 0
$$

after appropriate twisting, and the last term is a kernel. Applying ε_{X_r} turns this into

$$
0 \to \varepsilon_{X_r}\mathrm{H}^{2r+1}(\overline{X}; \mathbb{Q}_p)(r+1) \to \varepsilon_{X_r}\mathrm{H}^{2r+1}(\overline{X}_r^b; \mathbb{Q}_p)(r+1) \to \varepsilon_{X_r}\mathrm{H}^{2r}(\overline{X}_p; \mathbb{Q}_p)(r) \to 0.
$$

Now, we will send our $\Delta_\varphi \in \mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(F)$ to the extension V_Δ fitting into the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varepsilon_X\mathrm{H}^{2r+1}(\overline{X}_r; \mathbb{Q}_p)(r+1) & \longrightarrow & V_\Delta & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\
& & \big\| & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \varepsilon_X\mathrm{H}^{2r+1}(\overline{X}_r; \mathbb{Q}_p)(r+1) & \longrightarrow & \varepsilon_X\mathrm{H}^{2r+1}(\overline{X}_r^b; \mathbb{Q}_p)(r+1) & \longrightarrow & \varepsilon_X\mathrm{H}^{2r}(\overline{X}_p; \mathbb{Q}_p)(r) & \longrightarrow & 0
\end{array}
$$

where the right map sends 1 to the cycle class $\mathrm{cl}_p([\Delta_\varphi]) \in \varepsilon_X\mathrm{H}^{2r}(\overline{X}_p; \mathbb{Q}_p)(r)$.

Remark 209. In general, there is a general definition of AJ as the edge map of a certain Leray spectral sequence.

18.2   Calculation of the Map

It turns out that AJ actually lands in

$$\mathrm{Ext}^1_{\mathrm{cris}}\left(F, \varepsilon_{X_r}\mathrm{H}^{2r+1}(\overline{X}_r; \mathbb{Q}_p(r+1))\right),$$

where the subscript means that we are computing Ext in the abelian category of crystalline representations. Indeed, everything in the construction has been some étale cohomology group of a scheme smooth over $\mathcal{O}_F$, which produces a crystalline representation. Then we can pass through $D_{\mathrm{cris}}(V) := (V \otimes B_{\mathrm{cris}})^{G_F}$, which brings us to the category of filtered Frobenius modules. Crystalline cohomology becomes de Rham cohomology, which allows us to calculate the right-hand side becomes cusp forms.

19   October 14: A Slope Conjecture for Hypergeometric Motives

This talk was given by David Roberts at MIT for the MIT number theory seminar.

19.1   Slopes

Fix positive integers $a \leq b$, and define $M := a + b$. Then we define the polynomial

$$f(a, b; t; x) := a^a m^m - t(mx - b)^a.$$

The discriminant in the variable x is $\pm a^\bullet b^\bullet t^{m-1}(t-1)$, so we can think about this as having singularities at $t \in \{0, 1, \infty\}$; let c be the 2-adic valuation of the discriminant. (The monodromy of this polynomial and in particular the behavior of these singularities determine this polynomial f.) There is a way to change variables to make this an actual trinomial, but we will not bother. As t varies, we may define

$$K(a, b)_t := \frac{\mathbb{Q}_p[x]}{(f(a, b; t; x))}.$$

This is a separable algebra over $\mathbb{Q}_p$.

> **Example 210.** One finds
> $$K(1, 15)_t = \frac{\mathbb{Q}_2[x]}{(x^{16} - 16x + 15t)}.$$
> As $t \to 0$, this looks like $\mathbb{Q}_2[x]/\left(x^{15} - 16t\right)x$. As $t \to \infty$, this looks like $\mathbb{Q}_2[x]/\left(x^{16} + 15t\right)$. (These can be seen using some Newton polygon arguments.) For values of $|t|_2$ in between, there are complicated patterns.

> **Remark 211.** There is an upper numbering filtration of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. This induces a filtration on the p-adic algebra $K(a, b)_t$, which turns into some sum of slopes producing c. Slope 0 means a part with no ramification, slope 1 means a part with tame ramification, and higher slope means a part with wild ramification. Then c is the sum of all the slopes, so we define $\hat{c}$ to be the sum of the slopes where we replaced slopes which are 0s to slopes which are 1s.

It turns out that there is a lot of structure contained in the image of the $(\nu_2(t), c)$s, which has reently been proved for $\nu_2(t)$ odd. Our main result is a refined conjecture to the slopes.

> **Conjecture 212.** There is a piecewise-linear function σ_i such that the augmented slope $\hat{s}_i$ is bounded above by σ_i, and equality holds when $p \nmid \nu_p(t)$.

20   October 15: Intersection Number of Rankin–Selberg Cycles

This talk was given by Zeyu Wang at MIT for the MIT Lie groups seminar.

20.1   Classial Rankin–Selberg

For today, G is a split reductive group over $\mathbb{F}_q$, and C is a smooth projective geometrically connected curve. We let $G^\vee$ be the Langlands dual group (over $\overline{\mathbb{Q}}$), and we choose a prime ℓ distinct from the characteristic p of $\mathbb{F}_q$. Lastly, we let Bun_G denote the stack of G-bundles on C. We may stratify Bun_G as

$$\mathrm{Bun}_G = \bigsqcup_{d \in \pi_1(G)} \mathrm{Bun}_G^d,$$

and we let $\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ be the collection of $\overline{\mathbb{Q}}$-valued functions with finite support on each $\mathrm{Bun}_G^d(\mathbb{F}_q)$. The usual bundles argument shows that

$$\mathrm{Bun}_G(\mathbb{F}_q) = \prod_{\text{closed } x \in C} G(\mathcal{O}_x) \Big\backslash \prod_{\text{closed } x \in C} G(K_x) \Big/ G(K(C)).$$

Note $\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ has an action by the Hecke algebra

$$\mathcal{H}_G := \bigotimes_x{}' \mathrm{Fun}_c(G(\mathcal{O}_x) \backslash G(K_x) / G(\mathcal{O}_x)).$$

We are interested in decomposing the action of the Hecke algebra on these functions.

> **Definition 213.** Let σ be a $G^\vee$-local system on C. Say $f \in \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ is an eigenfunction with eigenvalue σ if and only if
> $$h \cdot f = h(\sigma(\mathrm{Frob}_x)) \cdot f$$
> for all h in the Hecke algebra.

For our setting, we will look at $G := \mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_{n-1}(\mathbb{F}_q)$. Then our test function f can take the form $f_n \otimes f_{n-1}$, and it should have eigenvalue $\sigma = (\sigma_n, \sigma_{n-1})$, and we may as well assume that σ_n and σ_{n-1} are irreducible local systems. Now, given such a σ, we have an L-function

$$L(s, \sigma) := \det\left(1 - q^{-s}\mathrm{Frob}; \mathrm{H}^1(C_{\overline{\mathbb{F}}_q}; \sigma)\right),$$

which is some polynomial in q. A little linear algebra can also identify this with $\mathrm{tr}\left(\mathrm{Frob}; \wedge^\bullet \mathrm{H}^1(\sigma)(s)\right)$, where the s at the end is a Tate twist.

The "Rankin–Selberg" in the title means that we will be interested in periods against the subgroup $H := \mathrm{GL}_{n-1}$, embedded in G as $h \mapsto (\mathrm{diag}(h, 1), h)$. Here is the classical story.

> **Theorem 214** (Jacquet–Piatetski-Shapiro–Shalika)**.** The integral
> $$I(f) := \int_{\mathrm{Bun}_H(\mathbb{F}_q)} f$$
> is a finite sum, and it equals the special value $L(1/2, \sigma)$ up to some explicit normalization factors. Furthermore, if f' has eigenvalue σ^* (which is the dual), then
> $$\frac{I(f)I(f')}{\langle f, f' \rangle} = q^{\dim \mathrm{Bun}_H} \cdot \frac{\widetilde{L}(1/2, \sigma \oplus \sigma^*)}{\mathrm{Res}_{s=1} \widetilde{L}(s, \mathrm{Ad}\, \sigma_n) \cdot \mathrm{Res}_{s=1} \widetilde{L}(s, \mathrm{Ad}\, \sigma_{n-1})},$$
> where $\langle -, - \rangle$ is some canonical inner product, and $\widetilde{L}$ is the norrmalized L-function.

To geometrize, we replace functions with sheaves. For example, an eigenfunction $f \in \mathrm{Bun}_G(\mathbb{F}_q)$ become eigensheaves $\mathbb{L}_\sigma$. Taking integrals $\int_{\mathrm{Bun}_H(\mathbb{F}_q)} f$ corresponds to taking compactly supported cohomology after pullback $\Gamma_c(\pi^*\mathbb{L}_\sigma)$. Then the L-function should correspond to $\wedge^\bullet \mathrm{H}^1(\sigma)(1/2)$ under the sheaf-function correspondence.

> **Theorem 215** (Lysenko)**.** Up to a Whittaker functional, there is an isomorphism
> $$\Gamma_c(\pi^*\mathbb{L}_\sigma) \cong \wedge^\bullet \mathrm{H}^1(\sigma)(1/2).$$

> **Remark 216.** One can recover the original identification of I with $L(1/2, \sigma)$ by taking trace of the Frobenius action.

20.2   Higher Rankin–Selberg

We now pass to shtukas. We pass from $\mathrm{Bun}_G(\mathbb{F}_q)$ to the moduli stack $\mathrm{Sht}_{G,\mu}$ of shtukas over C^r, where μ is some dominant coweight of G. Roughly speaking, $\mathrm{Sht}_{G,\mu}$ is a sequeence of r total G-bundles $\{\mathcal{E}_1, \dots, \mathcal{E}_r\}$ along with some rational maps $\mathcal{E}_i \to \mathcal{E}_{i+1}$ with at most a pole at some $c_i \in C$ and bounded by the μ_i. (There should also be some morphism $\mathrm{Frob}^*\mathcal{E}_r \to \mathcal{E}_1$.) Note that setting $r = 0$ recovers $\mathrm{Bun}_G(\mathbb{F}_q)$ from $\mathrm{Sht}_{G,\mu}$. The replacement of $\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$ is $\mathrm{H}_c^*(\mathrm{Sht}_{G,\mu})$, which turns out to still carry an action by the Hecke algebra.

Consider the coweights $\mu_+ = \mathrm{diag}(1, 0, \dots, 0)$ and $\mu_- = \mathrm{diag}(0, \dots, 0, -1)$, and define μ as some sequence $(\mu_{\varepsilon_1}, \dots, \mu_{\varepsilon_r})$ where $\mu_i \in \{\pm 1\}$ for each i. Then there is a map

$$\pi\colon \mathrm{Sht}_{H,\mu_\varepsilon}^d \to \mathrm{Sht}_{G,\mu_\varepsilon}^d$$

given by $(\mathcal{E}_i) \mapsto (\mathcal{E}_i \oplus 0, \mathcal{E}_i)$. (Here, the d denotes the degree of $\mathcal{E}_r$.) It also turns out that $\mathrm{Sht}_{G,\mu}^d$ contains a special cycle Z_μ^d which is simply the pushforward of $\mathrm{Sht}_{H,\mu}^d$.

> **Theorem 217.** There is a pairing $\langle -, - \rangle$ which relates $\langle Z_{r,\sigma}, Z_{r,\sigma} \rangle$ to a special value.

The proof proceeds by taking trace of Hecke composed with Frobenius action.

21   October 16: Katz's Proof of the Riemann Hypothesis for Curves

This talk was given by Jane Shi at MIT for the STAGE seminar.

21.1   Review

Katz's idea is to "spread out" the Riemann hypothesis over a family of curves. More precisely, suppose that we already have the Riemann hypothesis for a given curve C_0 over $\mathbb{F}_q$. Then given a family of curves $f\colon X \to U$ where one fiber is C_0, then we will be able to prove the Riemann hypothesis for the full family.

Let's begin by reviewing some ℓ-adic cohomology. Fix a scheme X of finite type over $\mathbb{F}_q$. Then we have a zeta function

$$Z(X, T) \coloneqq \exp\left(\sum_{n \geq 1} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right).$$

We already know rationality, which tells us that if X is smooth proper of equidimension d, then it can be written as an alternating product. Here is our goal.

Theorem 218. Fix a smooth projective irreducible curve X_0 over $\mathbb{F}_q$ of genus g. Then

$$Z(X,T) = \frac{P_1(T)}{(1-T)(1-qT)},$$

where $P_1(T)$ is a polynomial of degree $2g$, where all roots have absolute value $q^{1/2}$.

Remark 219. Previously, we showed that we can take $P_1(T)$ to be

$$\det\left(1 - T\mathrm{Frob}; \mathrm{H}^1(X_{\overline{\mathbb{F}_q}}; \mathbb{Q}_\ell)\right).$$

Let's be more precise about our Frobenius polynomials.

- There is an arithmetic Frobenius σ_q, which is the generator of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

- The inverse of the arithmetic Frobenius σ_q is the geometric Frobenius F_q.

- For our variety X defined over $\mathbb{F}_q$, the base-change $X_{\overline{\mathbb{F}}_q}$ admits a Frobenius morphism acting on the $\mathrm{Spec}\,\overline{\mathbb{F}}_q$ factor.

- Given a closed point $x\colon \mathrm{Spec}\,k \to X$ of a variety X over k, there is an induced map

$$\pi_1(\mathrm{Spec}\,k) \to \pi_1(X)$$

of étale fundamental groups. Accordingly, if $k = \mathbb{F}_q$, then the left-hand group is $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, so we can choose a (geometric) Frobenius element generating it (well-defined up to conjugacy). We will call this element Frob_x.

Remark 220. Given an ℓ-adic local system $\mathcal{F}$, we produce a representation

$$\pi_1(X) \to \mathrm{GL}(\mathcal{F}_{\overline{x}})$$

of the fiber. The target is some finite-dimensional $\mathbb{Q}_\ell$-vector space, so we are allowed to consider the action of our Frobneius Frob_x on this vector space.

Remark 221. In the sequel, we will have reason to work with non-constant coefficients. Namely, we will be working with a family $f\colon X \to U$ of curves, so the sheaf $\mathrm{R}^i f_* \mathbb{Q}_\ell$ will be interesting to us. In particular, by the Proper base change theorem, we know that

$$\left(\mathrm{R}^i f_* \mathbb{Q}_\ell\right)_s = \mathrm{H}^i(X_{s,\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$$

for any point $s \hookrightarrow X$.

Let's apply the previous remark. For our family $f\colon X \to U$, we see that

$$L(\mathrm{R}^i f_* \mathbb{Q}_\ell; T) = \prod_{\text{closed } p \in U} \det\left(1 - T^{\deg p}\mathrm{Frob}_p; (\mathrm{R}^i f_* \mathbb{Q}_\ell)_s\right)^{-1}$$

$$= \prod_{\text{closed } p \in U} \det\left(1 - T^{\deg p}F_q; \mathrm{H}^i_{\text{ét}}(X_{p,\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right)^{-1}.$$

By the trace formula (and Poincaré duality), if U is an affine curve, one can alternatively write

$$L(\mathcal{F}; T) = \frac{\det\left(1 - TF_q; \mathrm{H}^1_c(U; \mathcal{F})\right)}{\det\left(1 - TF_q; \mathrm{H}^2_c(U; \mathcal{F})\right)},$$

but by Poincaré duality, this last demoninator is simply

$$\det\left(1 - qTF_q; \mathcal{F}_{\pi_1^{\mathrm{geo}}}\right),$$

where we are silently keeping track of some Tate twist (which turns the TF_q into qTF_q). The subscript $(-)_{\pi_1^{\mathrm{geo}}}$ refers to co-invariants.

21.2   Redution to a Single Curve

For our reduction, we will require the notion of purity. Throughout, the base U is a smooth affine geometrically connected curve.

Definition 222. Fix an embedding $\iota\colon \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. For an ℓ-adic local system $\mathcal{F}$ on U, we say that $\mathcal{F}$ is ι-*pure of weight* w if and only if

$$|\alpha| = \mathrm{N}\, p^{w/2}$$

for all closd points $p \in U$ and all eigenvalues α of the action of Frob_p on $\mathcal{F}_p$. We say that $\mathcal{F}$ is ι-*real* if and only if the characteristic polynomial

$$\iota \det\left(1 - T\mathrm{Frob}_p; \mathcal{F}_p\right)$$

has real coefficients for all closed points $p \in U$.

Remark 223. For a family $f\colon X \to U$ of curves, our local system $\mathcal{F} = \mathrm{R}^i f_* \mathbb{Q}_\ell$ is ι-real (for any ι) by Lefschetz trace formula arguments.

Here is our main result.

Theorem 224. Fix an ι-real ℓ-adic local system $\mathcal{F}$ on a smooth affine geometrically connected curve U. Suppose that every eigenvalue β of F_q acting on $\left(\mathcal{F}^{\otimes 2k}\right)_{\pi_1^{\mathrm{geo}}}$ (for every p and k) has $|\beta| \le 1$. Then for every closed point p, every eigenvalue α of Frob_p acting on $\mathcal{F}_p$ has eigenvalue $|\alpha| \le 1$.

Proof. We compare Euler factors of the L-function L_{2k} of $\mathcal{F}^{\otimes 2k}$ as k gets large. The Euler factor "at p" is

$$L_{p,2k} := \det\left(1 - T^{\deg p}\mathrm{Frob}_p; \mathcal{F}^{\otimes 2k}\right)^{-1} = \exp\left(\sum_{n \ge 1} \mathrm{tr}\left(\mathrm{Frob}_p^n; \mathcal{F}\right)^{2k} \frac{T^{n \deg p}}{n}\right).$$

By looking at the roots and using the fact that $\mathcal{F}$ is ι-real, we see that this Euler factor has nonnegative real coefficients, meaning it lives in $1 + T\mathbb{R}_{\ge 0}[[T]]$. Thus, by multiplying our Euler factors together, we see that the power series $L_{p,2k}$ is bounded above by L_{2k} term-wise.

Now, the radius of convergence of the full L-function L can be read off of a calculation with $\mathcal{F}_{\pi_1^{\mathrm{geo}}}$, which is where we may apply the hypothesis. In particular, the radius of convergence is bounded above by 1, so we find that the eigenvalues of Frob_p acting on $\mathcal{F}^{\otimes 2k}$ has absolute value bounded above by $q^{\deg p}$. Sending $k \to \infty$ completes the proof. ■

Corollary 225. Fix an ι-real ℓ-adic local system $\mathcal{F}$ on a smooth affine geometrically connected curve U. If there is a closed point p on U suc hthat the eigenvalues of Frob_p on $\mathcal{F}_p$ have eigenvalue bounded by 1, then th same is true for all eigenvalues.

Sketch. The key idea is that F_q^d acts by Frob_p on $\left(\mathcal{F}^{\otimes 2k}\right)_{\pi_1^{\mathrm{geo}}}$. (Roughly speaking, Frobenius can be transported to different points because the action will only differ by something in π_1^{geo}.) The result then follows from the theorem. ■

In light of the corollary, it remains to prove the Riemann hypothesis for a well-behaved curve and living in some rather general families.

> **Lemma 226.** Fix a genus $g \geq 1$, and choose two smooth projective geometrically connected curves C_0 and C_1 over $\mathbb{F}_q$. After a field extension, there exists a smooth affine geometically connected curve U and a family $f \colon C \to U$ so that C_0 and C_1 are some fibers of f.

Sketch. For genus 1, choose some $N \geq 4$, and one can use the moduli space $Y(N)$ of a pair of an elliptic curve along with a point of order N. For large N, one finds that U is a curve, and we can use the universal elliptic curve on $Y(N)$.

Now, for genus $g > 1$, we recall (from Deligne and Mumford) that there is a moduli space H_g° classifying genus g quasiprojectve smooth geometrically connected curves. We can get the required U by choosing a generic curve which goes through every $\mathbb{F}_q$-point. ∎

Let's now explain how the Riemann hypothesis gets transferred between curves.

Proof of Theorem 218 from a single curve. Suppose we have the Riemann hypothesis for a single curve C_0 of genus g, and we want to move it to any other curve C_1 of genus g. Then we get an upper bound on the eigenvalues of the Frobenius on the full family U, which comes down to the required upper bound $|\alpha| < q^{1/2}$ of eigenvalues α for C_1. To complete the proof, we know that $\alpha \mapsto q/\alpha$ should be an involution of the roots by the functional equation, so the equality is forced! ∎

21.3   Computations on a Family of Curves

We will compute with the Fermat curves. Choose some d coprime to q, and we define the smooth projective Fermat curve

$$F_d \colon X^d + Y^d = Z^d.$$

The Weil conjectures in this case are due to Weil.

> **Theorem 227** (Weil)**.** The Riemann hypothesis holds for F_d if $\gcd(d, q) = 1$.

Sketch. This is an explicit calculation with some character sums. Given two characters $\chi_1, \chi_2 \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$, we define their Jacobi sum as

$$J(\chi_1, \chi_2) \coloneqq \sum_{a \in \mathbb{F}_q} \chi_1(a)\chi_2(1 - a).$$

If both χ_1 and χ_2 are nontrivial, then one can calculate $|J(\chi_1, \chi_2)|^2 = \sqrt{q}$.

Now, to calculate $\#F_d(\mathbb{F}_{q^n})$, this amounts to calculating the solutions to $x^m + y^m = 1$ and adding some points at infinity. Counting solutions to $x^m + y^m = 1$ turns into a sum over Jacobi sums, which can then be compared to

$$\#F_d(\mathbb{F}_{q^n}) = 1 + q^n - \sum_\alpha \alpha^n,$$

where the sum is over the eigenvalues α of the Frobenius. ∎

Now, the genus of F_d is $\binom{d-1}{2}$, so we have many remaining genera, for which we will take quotients.

> **Remark 228.** Note that a non-constant map $C \to C'$ will have the Riemann hypothesis transfer from C to C'. Indeed, the Frobenius eigenvalues of C' are a subset of the Frobenius eigenvalues of C. For example, one can see this by splitting $\operatorname{Jac} C = \operatorname{Jac} C' \oplus B$ for some other abelian variety B. Then, upon taking Tate modules, we find that the Frobenius eigenvalues of C are precisely the Frobenius eigenvalues of C' along with the Frobenius eigenvalues of B.

Lemma 229. For any genus $g \geq 1$, there is a Fermat curve F_d (with $\gcd(d, q) = 1$) and a curve C of genus g along with a quotient map $F_d \to C$.

Proof. If $p \neq 2$, then use the hyperelliptic curves $y^2 = x^d + 1$, where we choose $d \in \{2g + 1, 2g + 2\}$ to avoid p. If $p = 2$, we can use the curves $y^2 - y + x^{2g+1}$. A short calculation shows that these quotients recover all genera. ∎

21.4   Persistence of Purity

We close the seminar by stating the following result.

Theorem 230. Fix an ι-real ℓ-adic local system $\mathcal{F}$ on some smooth affine geometrically connected curve U. Suppose that there is a closed point p for which every Frobenius eigenvalue of Frob_p acting on $\mathcal{F}_p$ has absolute value equal to 1. Then this is true for all closed points on $\mathcal{F}$.

We will say more about this result next week. It upgrades Theorem 224.

22   October 21: The Arithmetic of Thin Orbits

This talk was given by Katherine Strange at Boston College for the BC–MIT number theory seminar.

22.1   Quadratic Families

We are working with continued fractions today. We remark that one can encode these in terms of Möbius transformations, viewing $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{P}^1(\mathbb{Q})$. Here is the sort of thing we are interested in.

Conjecture 231 (Zaremba)**.** Fix the alphabet $A := \{1, 2, 3, 4, 5\}$, and consider the group

$$\Gamma_A := \left\langle \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

Then

$$Z_A := \{a \in \mathbb{Z} : \text{there is } p \in \mathbb{N} \text{ such that } (p, q) \in \Gamma_A \cdot (0, 1)\}$$

is all $\mathbb{Z}$.

As another sort of dynamics question, given three curvatures (c_1, c_2, c_3), there are exactly two curvatures c_4 and c_4' whose corresponding circle can be tangent to the other three. Thus, there is a canonical involution on the quadruple (c_1, c_2, c_3, c_4) replacing c_4 with c_4'. This is some subgroup of $\mathrm{O}_Q(\mathbb{Z})$ for some quadratic form Q.

Definition 232. A subgroup A of an algebraic group is *thin* if and only if it is infinite index and Zariski dense.

These questions can be seen visually. For example, the Appolonian circles can be placed into a "superpacking."

Remark 233. The "obvious" local-to-global conjecture is false: there are packings missing quadratic or quartic families of curvatures. The point is to compute the Kronecker symbol on the tangencies of the circles, so you can see that a given Kronecker symbol will be constant on the total packing.

Theorem 234. The thin semigroup

$$\Phi_1 := \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \right\rangle$$

has $\Phi_1 \cdot (2, 3)$ missing denominatorics which are squares.

23   October 21: Multiple Polylogarithms

This talk was given by Daniiel Rudenko at Boston College for the BC−MIT number theory seminar.

23.1   Algebraic K-Theory

We are interested in the Goncharov program.

Definition 235 (algebraic K-theory). Fix a field F. Then we define the *algebraic K-groups* $\{K_n(F)\}_{n \geq 0}$ as the nth homotopy group of

$$B\Omega \bigsqcup_{n \geq 1} \mathrm{BGL}_n(F),$$

where $B\Omega$ refers to some group completion operation from algebraic topology.

Example 236. One has $K_0(F) = \mathbb{Z}$ (related to the dimension of a vector space), and $K_1(F) = F^\times$ (related to the determinant).

Example 237. In characteristic not 2, one can calculate $K_2(F) \cong \wedge^2 F^\times / \langle x \otimes (1 - x) \rangle$, which is known as "Milnor K-theory." For example, the Hilbert symbol factors through $K_2(F)$.

We may be interested in calculating $K_n(F)$ for $n \geq 0$. There is also a description for $n = 3$, which we will give later. Here is something of interest to number theorists.

Theorem 238 (Borel). Fix a number field F. Then $K_{2n}(F) = 0$, and $\mathrm{rank}\, K_{2n-1}(F)$ is the number of complex embeddings of K if $2 \mid n$, and it is the number of real embeddings plus the numer of complex embeddings if $2 \nmid n$.

Remark 239. Borel also showed that there is a regulator $\mathrm{reg} \colon K_{2n-1}(F) \to \mathbb{R}^{d_n}$, embedding as a lattice, and its covolume turns out to be a mysterious rational number times an easy transcendental number times $\zeta_F(n)$.

Theorem 240 (Suslin). Fix a field F. Then there is a natural grading on $K_n(F)_{\mathbb{Q}}$ given by

$$K_n^{(p)}(F) := \frac{\wedge^n F^\times}{\langle x \otimes (1 - x) \otimes \cdots \rangle}.$$

Then there is an exact sequence

$$0 \to K_3^{(2)}(F) \to B_2(F) \to \wedge^2 F^\times \to K_2^{(2)} F \to 0,$$

where $B_2(F)$ is the quotient of $\mathbb{Z}[F^\times]$ by the kernel of the map sending $[x] \mapsto (x \wedge (1 - x))$.

Remark 241. It turns out that the regulator on $K_3^{(2)}(\mathbb{C})$ factors through $B_2(\mathbb{C})$. The corresponding regulator $B_2(\mathbb{C}) \to \mathbb{R}$ turns out to be the dilogarithm.

This suggests that we might be able to "compute" K-theory by putting it into some kind of resolution and then be able to work with elements satisfying some properties.

23.2 Mixed Tate Motives

Zagier has some conjectures for number fields, based on some calculations. Beilinson and Deligne have explained why these conjectures might be true. Our story begins with motives.

Conjecture 242 (Beilinson–Deligne). Fix a field F. Then there is an abelian category $\mathrm{MM}(F)$ of mixed Tate motives with the following properties.

(a) There are Tate twists $\mathbb{Q}(n)$ for $n \in \mathbb{Z}$.

(b) One has $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-i}^{(i)}(F)$.

Remark 243. This conjecture is known for number fields F, coming from the (known) triangulated category of Voevodsky motives.

As usual, motives should be thought of as something like a usual cohomology theory for varieties. For example, this category is Tannakian, so there is a certain Lie (co)algebra $\mathcal{L}(F)$ parameterizing the representations. It then turns out tht $K_{2n-i}^{(i)}(F)$ is $\mathrm{Extt}_{\mathrm{MM}}^i(\mathbb{Q}(0), \mathbb{Q}(n))$, which can then be computed as the ith cohomology group of the complex

$$\mathcal{L} \to \wedge^2 \mathcal{L} \to \wedge^3 \to \mathcal{L} \to \cdots,$$

thereby realizing algebraic K-theory in the complex. The Goncharov program now hopes to be able to compute the regulator using generalizations of the dilogarithm, which he named $\mathrm{Li}_{n_1,\ldots,n_k} \in \mathcal{L}_n(F)$ (where $n_1 + \cdots + n_k = n$) as some periods.

Conjecture 244 (Universality). The functions $\mathrm{Li}_{n_1,\ldots,n_k}$ span $\mathcal{L}_n(F)$.

Intuitively, this tells us that very general periods can be found as linear combinations of these generalized polylogarithms. Here is our main result.

Theorem 245. Fix a number field F. Then Conjecture 244 holds, and there are explicit relations.

Example 246. One can calcualte that $\mathcal{L}_3(F) = B_3(F)$.

23.3 Algebraic Topology

Our approach is homotopical in nature. One can view

$$\underline{\mathrm{GL}} := \bigsqcup_{n \geq 1} B\,\mathrm{GL}_n(F)$$

as an $\mathbb{E}_\infty$-algebra. It is known that there is $\mathbb{E}_\infty$-homology, so one can ask to try to compute the homology $\mathrm{H}_{n,d}(\underline{\mathrm{GL}})$.

Theorem 247 (Kupers–Galatino–Randal-Williams). One has $\mathrm{H}_{n,d} = 0$ for $d \leq 2n - 2$ and $n \geq 1$.

> **Conjecture 248.** Fix a field F. Then
> $$\bigoplus \mathrm{H}_{n,2d-1}(\underline{\mathrm{GL}})$$
> is the Lie coalgebra $\mathcal{L}$ of the mixed Tate motives.

Our main theorem then follows from the following.

> **Theorem 249.** Conjecture 248 holds for number fields F.

> **Remark 250.** The method involved uses the Steinberg module of a vector space. In particular, one uses that it satisfies a certain "Koszul" property. Then one shows that
> $$\mathrm{H}_{n,2n-1}(\underline{\mathrm{GL}}) = \mathrm{H}_1(\mathrm{GL}_n, (\mathrm{St}_n \otimes \mathrm{St}_n)/\sim),$$
> and one finds generators which look like polylogarithms. All the main theorems follow.

24   October 22: Canonical Representations of Surface Groups

This talk was given by Aaron Landesman at Harvard for the Dynamics, Geometry and Moduli Spaces seminar. This is joint work with Josh Lam and Daniel Litt.

24.1   Canonical Representations

Today, we are interested in $\Sigma_{g,n}$, which is a genus-g surface with n punctures. For example, we may be interested in the fundamental group, which we can understand throug representations $\rho\colon \pi_1(\Sigma) \to \mathrm{GL}_r(\mathbb{C})$. We may also be interested in the mapping class group $\mathrm{Mod}_{g,n} := \mathrm{Homeo}^+(\Sigma)$, which has an action on representations $\pi_1(\Sigma) \to \mathrm{GL}_r(\mathbb{C})$ (even considered up to conjugation).

> **Definition 251.** A *canonical representaion* $\rho\colon \pi_1(\Sigma) \to \mathrm{GL}_r(\mathbb{C})$ is one whose conjugacy class has finite orbit under the action by $\mathrm{Mod}_{g,n}$.

These representations turn out to be quite important, but there are less interesting constructions.

> **Example 252.** If $\rho\colon \pi_1(\Sigma) \to \mathrm{GL}_r(\mathbb{C})$ has finite image H, then ρ is canonical: $\pi_1(\Sigma)$ is finitely generated, so there are only finitely many maps $\pi_1(\Sigma) \to H$ anyway.

We will be interested in canonical representations with infinite image.

> **Remark 253.** From the perspective of algebraic geometry, it turns out that $\mathrm{Mod}_{g,n} = \pi_1(\mathcal{M}_{g,n})$. Loosely speaking, a canonical representation on $\Sigma_{g,n}$ is then a local system on the universal curve over $\mathcal{M}_{g,n}$, but we have to take some pullback to a dominant cover of $\mathcal{M}_{g,n}$. For example, this perspecitve explains how $\pi_1(\mathcal{M}_{g,n})$ acts on $\pi_1(\Sigma)$: there is a fibration $\Sigma \to \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$, which more or less induces the exact sequence
> $$1 \to \pi_1(\Sigma) \to \pi_1(\mathcal{M}_{g,n+1}) \to \pi_1(\mathcal{M}_{g,n}) \to 1.$$

> **Example 254.** If $(g, r) = (0, 1)$, then all canonical representations have finite image.

> **Example 255.** If $(g, r, n) = (0, 2, 3)$, then $\mathrm{Mod}_{0,3} = S_3$ (the automorphisms of $\mathbb{P}^1$ preserving the three punctures can only permute the three points), which is notably finite. Thus, every representation is canonical.

Example 256 (Tykhyy)**.** If $(g, r, n) = (0, 2, 4)$, then canonical representations roughly correspond to algebraic solutions to some differential equation. These have all been classified, but it took something like fifty to one hundred papers. Here, $\pi_1(\Sigma_{0,4}) = \pi_1(\mathbb{C} \setminus \{0, 1, 2\})$ is the free group on three elements, so a representation to $\mathrm{GL}_2(\mathbb{C})$ is given by three matrices. Then the action by $\mathrm{Mod}_{0,4}$ is more or less given by the braid group on three generators, generated by the maps

$$\sigma_1 \colon (A, B, C) \mapsto \left(B, B^{-1}AB, C\right) \qquad \text{and} \qquad \sigma_2 \colon (A, B, C) \mapsto \left(A, C, C^{-1}BC\right).$$

Thus, the canonical representations turn into some explicit linear algebra.

Example 257. Consider $(g, n) = (0, 4)$. We will use the fact that every representation with $n = 3$ is canonical. In particular, put three of our punctures at $\{0, 1, \infty\}$, and we can specify a local system on $\Sigma_{0,4}$ as having arbitrary monodromy A at 0, monodromy $M = \mathrm{diag}(1, -1)$ at 1, and the forced monodromy B at ∞. Now, we have a double cover $\Sigma_{0,4} \to \Sigma 0, 3$ sending 0 and 1 to 0, ramified at the point over $1 \in \Sigma_{0,3}$, and sending λ and ∞ to ∞. Then one can check that the corresponding composite is canonical. In particular, one calculates that the mondromy at the point of ramification of the double cover is $M^2 = 1$. Letting A vary, we receive a 2-dimensional family of canonical representations. There are other infinite-dimensional families are similar.

Conjecture 258 (Tykhyy)**.** Fix $(g, r) = (0, 2)$. If ρ is a canonical reprsentation with Zariski dense image, and no loop around a puncture has scalar monodromy, then $n \leq 6$ and ρ is explicitly classified.

24.2   Higher Classification Results

Here is our main theorem.

Theorem 259 (Lam−L−Litt)**.** If ρ has at least one loop around a puncture with infinite image, then Conjecture 258 is true.

The complementary case is also known.

Theorem 260 (Bronstrin−Moret)**.** If ρ has all loops around a puncture with finite image, then Conjecture 258 is true.

Let's begin discussing $g > 0$.

Example 261 (Kodaira−Parshings trick)**.** If r is large with respect to g (e.g., $r \geq 6^{2g}$ for $g \geq 2$), then there are lots of these canonical representations. For example, suppose we want to construct a canonical representation of $\pi_1(\Sigma_{g,1}) \to \mathrm{GL}_r(\mathbb{C})$. We need to choose something without choices, so let $D_p \to \Sigma$ be the triple cover branched only at $p \in \Sigma$; note that there are only finitely many of these because they correspond to representations $\pi_1(\Sigma \setminus \{p\}) \to S_3$. Then $p \mapsto \prod_{D_p} \mathrm{H}^1(D_p; \mathbb{C})$ provides us with a canonical representation, and one can check that the image is infinite. (Roughly speaking, the image is infinite because moving in a loop with p must change the covering D_p because there are only finitely many maps between curves of genus at least 2.)

Example 262. If $r = 2$ and ≥ 2, it turns out that all canonical representations have finite image.

Theorem 263 (L−Litt)**.** If $r < \sqrt{g+1}$, then any canonical representation has finite image.

Remark 264. It is conjectured that irreducible canonical representations have r growing exponentially with g.

Corollary 265. Fix $r < \sqrt{g}$. Consider the action on tuples $(A_1, A_2, \ldots, A_{2g})$ of $r \times r$-matrices, where we may permute the entries of the tuple, invert the first entry, or replace A_1 by $A_1 A_2$. If the tuple $(A_1, \ldots, A_{2g})$ has finite orbit up to conjugation, then they generate a finite group.

Proof. This follows by considering the action of the outer automorphism group of the free group (which contains the mapping class group). ∎

25 October 23: The Riemann Hypothesis for Hypersurfaces

This talk was given by Leonid Gorodetskii at MIT for the STAGE seminar.

25.1 Spreading Out for Hypersurfaces

Today, we are giving Katz's proof of the Riemann hypothesis for hypersurfaces of $\mathbb{P}^{n+1}$. The general idea is to spread out from a single example using moduli spaces. Let $f \colon X \to U$ be a smooth proper family over $\mathbb{F}_q$. Then $\mathrm{R}^i f_* \mathbb{Q}_\ell$ defines some local system on U, and one has

$$\det\left(1 - T\mathrm{Frob}_u; \mathrm{R}^i f_* \mathbb{Q}_\ell\right) = \det\left(1 - T\mathrm{Frob}_{q^{\deg u}}; \mathrm{H}^i(X_u; \mathbb{Q}_\ell)\right),$$

where u is a closed point of U. We called this polynomial $P_i(X_u, T)$. With U a curve, we already know that P_0 and P_2 live in $1 + \mathbb{Z}[T]$, so the known rationality results imply that $P_1(X_u, T)$ is in $1 + \mathbb{Z}[T]$ as well.

Corollary 266. Fix everything as above. Then $\mathrm{R}^i f_* \mathbb{Q}_\ell$ is real.

Lemma 267. Fix everything as above. Then the following are equivalent.

(i) The Riemann hypothesis holds for X_u.

(ii) $\mathrm{H}^i(X_u; \mathbb{Q}_\ell)$ is pure of weight i.

(iii) $\mathrm{R}^i f_* \mathbb{Q}_\ell$ is pure of weight i at u.

Proof. Unwind the definitions of purity. ∎

Our key spreading our result is the following.

Theorem 268 (Persistence of purity)**.** Fix a smooth affine geometrically connected curve U over $\mathbb{F}_q$, and let $\mathcal{F}$ be a real ℓ-adic local system on U. Then the following are equivalent.

(i) The ℓ-adic local system $\mathcal{F}$ is pure of weight $w \in \frac{1}{2}\mathbb{Z}$ at some point u.

(ii) The ℓ-adic local system $\mathcal{F}$ is pure of weight $w \in \frac{1}{2}\mathbb{Z}$ at all points u.

Sketch. By twisting, we may assume that $w = 0$. More precisely, we use the Tate twists $\mathbb{Q}_\ell(i)$, which are the ℓ-adic local systems in which the action by Frob_q in the étale fundamental group is given by q^{-i}. For example, we see that $\mathbb{Q}_\ell(i)$ is pure of weight $-2i$.[1] The result now follows from Theorem 224 and some careful duality. ∎

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[1] Technically, we may need some half-integer twists, which requires us to consider field extensions of $\mathbb{Q}_\ell$. This causes no problems as soon as we suitably generalize our definition of ℓ-adic system.

In order to work with hypersurfaces, we need to know something about their cohomology. Here is the cohomology of projective space.

> **Theorem 269.** Fix some $n \geq 0$. Then
> $$\mathrm{H}^i(\mathbb{P}^n; \mathbb{Q}_\ell) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \mathbb{Q}_\ell(-i/2) & \text{if } i \text{ is even.} \end{cases}$$
> In particular, the Riemann hypothesis holds.

Sketch. Use the Gysin sequence on the decomposition $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$, and induct on n. Note that the statement has no content for $n = 0$, and we also already know it for $n = 1$. Approximately speaking, the given even-dimensional classes arise because the whole cohomology ring is generated by the hyperplane class in the image of the cycle class map $\mathrm{CH}^1(\mathbb{P}^n) \to \mathrm{H}^2(\mathbb{P}^n; \mathbb{Q}_\ell)(1)$. ∎

> **Remark 270.** To check that this makes sense, we note that it gives $Z(\mathbb{P}^n; T)$ is
> $$\prod_{i=0}^{n} \frac{1}{\det\left(1 - T\mathrm{Frob}_q; \mathbb{Q}_\ell(-i)\right)} = \prod_{i=0}^{n} \frac{1}{1 - q^i T},$$
> which can then be expanded by hand.

Now, by the weak Lefschetz theorem and Poincaré duality, we are able to say something about the cohomology of hypersurfaces as well.

> **Theorem 271.** Fix a smooth hypersurface $X \subseteq \mathbb{P}^{n+1}$.
> (a) For $i \in \{0, 1, \ldots, 2n\} \setminus \{n\}$, we have $\mathrm{H}^i(X; \mathbb{Q}_\ell) = \mathrm{H}^i\left(\mathbb{P}^{n+1}; \mathbb{Q}_\ell\right)$.
> (b) For $i = n$, we have $\mathrm{H}^n(X; \mathbb{Q}_\ell) \supseteq \mathrm{H}^n\left(\mathbb{P}^{n+1}; \mathbb{Q}_\ell\right)$.

Proof. Omitted. ∎

The hard Lefschetz theorem (which we do not currently know for étale cohomology at our point in the theory) motivates the following defintion.

> **Definition 272.** Fix a hypersurface $X \subseteq \mathbb{P}^{n+1}$. Then
> $$\mathrm{Prim}^n(X) = \begin{cases} \mathrm{H}^n(X; \mathbb{Q}_\ell) & \text{if } n \text{ is odd,} \\ \mathrm{H}^n(X; \mathbb{Q}_\ell)/\mathcal{L}^{n/2} & \text{if } n \text{ is even,} \end{cases}$$
> where $\mathcal{L}$ is some class coming from a hyperplane class.

> **Lemma 273.** Fix a hypersurface $X \subseteq \mathbb{P}^{n+1}$ over $\mathbb{F}_q$. Then the Riemann hypothesis holds for X if and only if $\mathrm{Prim}^n(X)$ is pure of weight n.

Proof. Define
$$P(T) := \det\left(1 - T\mathrm{Frob}_q; \mathrm{Prim}^n(X)\right).$$
Tracking through the definitions, we see that $Z(X; T)$ is $P(T)Z(\mathbb{P}^n; T)$ if n is odd and is $P(T)^{-1}Z(\mathbb{P}^n; T)$ if n is even. The result now follows because we already have the Riemann hypothesis for $\mathbb{P}^n$. ∎

Thus, here is our spreading out result.

Theorem 274. Suppose there is a smooth hypersurface $X_0 \subseteq \mathbb{P}^{n+1}$ of degree d over $\mathbb{F}_p$ satisfying the Riemann hypothesis. Then the Riemann hypothesis holds for all smooth hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree d over $\mathbb{F}_q$ for any power q of p.

Proof. Quickly, note that we may immediately upgrade the Riemann hypothesis from X_0 to $X_0 \otimes \mathbb{F}_q$ by computing some eigenvalues. Thus, we may assume that X_0 is defined over $\mathbb{F}_q$.

Now, choose another smooth hypersurface $X_1 \subseteq \mathbb{P}^{n+1}$ of degree d over $\mathbb{F}_q$. Smoothness allows us to say that X_0 is cut out by a single equation F_0, and X_1 is smooth over F_1. Then the equation

$$tF_1(x) + (1 - t)F_0(X)$$

defines a family of hypersurfaces in $\mathbb{P}^{n+1}$ over $\mathbb{A}^1$. After removing finitely many points $t \in \mathbb{A}^1$, we may assume that this is a smooth family of smooth hypersurfaces over some affine curve $U \subseteq \mathbb{A}^1$. The result now follows from Theorem 268 applied to $\mathcal{F} \coloneqq \mathrm{R}^i f_* \overline{\mathbb{Q}}_\ell(n/2)$. (The algebraic closure is desired here in order to take a half-twist.) ∎

25.2   A Single Example

We are now reduced to proving the Riemann hypothesis for a single hypersurface.

Lemma 275. Fix a smooth hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d over $\mathbb{F}_p$. Then the Riemann hypothesis is true for X if and only if

$$\#X(\mathbb{F}_q) = \#\mathbb{P}^n(\mathbb{F}_q) + O_X\left(q^{n/2}\right)$$

for all powers q of p.

Proof. By Lemma 273, we only have to check the eigenvalues in the primitive part, and by Poincaré duality, we are allowed to only check that the eigenvalues α satisfy $|\alpha| \leq q^{n/2}$. Because we can expand out the size of $\#X(\mathbb{F}_q)$ as sum of some powers of the eigenvalues, it is enough to consider the sum of all these eigenvalues (otherwise we get some domination), and the result follows. ∎

Let's now begin with our calculation. Here are the hypersurfaces used by Katz.

- If $p \nmid d$, then we can take the Fermat hypersurface cut out by the equation $\sum_{i=1}^{n+2} x_i^d = 0$. This calculation is due to Weil.

- If $p \mid d$ and $d \geq 3$, then we can use the Gabber hypersurface $x_1^d + \sum_{i=1}^{n+1} x_i x_{i+1}^{d-1} = 0$.

- If $p = d = 2$ and n is odd, then it turns out that $\mathrm{Prim}^n(X) = 0$, so there is nothing to check.

- Lastly, if $p = d = 2$ and $n = 2m$ is even, then we can use $\sum_{i=1}^{m+1} x_i x_{m+1+i} = 0$.

We will only do the calculation in the Fermat case because the other ones are more of the same. We proceed in steps. Set $N \coloneqq n + 2$ for brevity.

1. Because $X \subseteq \mathbb{P}^{n+1}$, it should be cut out by a single polynomial $F(x) = 0$. Accordingly, let $X^{\mathrm{aff}} \subseteq \mathbb{A}^N$ be cut out by this same polynomial F, and we see that

$$\#X^{\mathrm{aff}}(\mathbb{F}_q) = 1 + (q - 1)\#X(\mathbb{F}_q).$$

Thus, it is now enough to show that

$$\#X^{\mathrm{aff}}(\mathbb{F}_q) = q^{n+1} + O_{d,n}\left(q^{(n+2)/2}\right).$$

2. Define $V^*(\mathbb{F}_q) \subseteq X^{\mathrm{aff}}(\mathbb{F}_q) \cap \mathbb{G}_m(\mathbb{F}_q)^N$ to have the nonzero solutions. We claim that it is enough to achieve

$$\#V^*(\mathbb{F}_q) \overset{?}{=} \frac{1}{q}(q-1)^N + O_{d,n}\left(q^{N/2}\right).$$

This is a matter of stratifying X^{aff}. Indeed, for a subset $S \subseteq \{1, 2, \ldots, N\}$, let V_S^* be the solutions in $X^{\mathrm{aff}}(\mathbb{F}_q)$ whose nonzero entries are exactly in S. Then by choosing what our nonzero entries should be, we calculate

$$\#X^{\mathrm{aff}}(\mathbb{F}_q) = \sum_{S \subseteq \{1,\ldots,N\}} \#V_S^*(\mathbb{F}_q)$$

$$\overset{*}{=} \sum_{S \subseteq \{1,\ldots,N\}} \left(\frac{1}{q}(q-1)^{\#S} + O_d\left(q^{\#S/2}\right)\right)$$

$$= \frac{1}{q} \sum_{S \subseteq \{1,\ldots,N\}} (q-1)^{\#S} + O\left(q^{N/2}\right),$$

where the last error term holds because the smaller error terms get smaller exponentially (even though there are an exponential number of them). Notably, we have applied the hypothesis at $\overset{*}{=}$. The result now follows by noticing that the last sum collapses to $((q-1)+1)^N$ by the binomial theorem.

Before continuing with the calculation, we recall some facts about characters.

Definition 276 (character). Fix a finite abelian group G. Then a *character* is a homomorphism $G \to \mathbb{C}^\times$. We let $G^\vee$ denote the group of characters.

Remark 277. Using the classification of finite abelian gruops, one can check that G and $G^\vee$ have the same size. There is also a non-canonical isomorphism between G and $G^\vee$.

Remark 278. The restriction maps induce a natural isomorphism $(G \times H)^\vee \to G^\vee \times H^\vee$. The inverse is given by sending the pair (χ_G, χ_H) to the character $\chi_G \chi_H$.

Remark 279. Note that there are the dual identities

$$\frac{1}{\#G} \sum_{g \in G} \chi(g) = \begin{cases} 1 & \text{if } \chi = 1, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \frac{1}{\#G} \sum_{\chi \in G^\vee} \chi(g) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{else.} \end{cases}$$

Example 280. Because $\mathbb{F}_q^\times$ is cyclic, it is fairly easy to write down its multiplicative characters.

Example 281. For any $a \in \mathbb{F}_q$, there is an additive character $\psi_a \colon \mathbb{F}_q \to \mathbb{C}$ given by

$$\psi_a(t) := \exp\left(\frac{2\pi i}{p} \cdot \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(ax)\right).$$

One can check that these characters are distinct, so these give all the characters.

Our bounds will come from knowledge of Gauss sums.

Definition 282 (Gauss sum). Fix an additive character ψ_a and a multiplicative character χ of $\mathbb{F}_q$. Then we define

$$g(\chi; \psi_a) := \sum_{t \in \mathbb{F}_q^\times} \chi(t)\psi_a(t).$$

Remark 283. These are in some sense analogous to the function

$$\Gamma(z) = \int_{\mathbb{R}^+} t^z e^{-t} \frac{dt}{t}.$$

Roughly speaking, we are integrading an additive and multiplicative character together over a multiplicative group.

Remark 284. Provided $a \neq 0$, one can calculate

$$|g(\chi; \psi_a)|^2 = \begin{cases} q & \text{if } \chi \neq 1, \\ 1 & \text{if } \chi = 1. \end{cases}$$

This sort of fact is used in many proofs of quadratic reciprocity, where one frequently takes χ to be the Legendre symbol (and receives a "quadratic Gauss sum.")

We now continue with our calculation.

3. To ease our calcuation, we let $\varphi \colon \mathbb{G}_m^N \to \mathbb{G}_m^N$ be the dth power map, and we let $\sigma \colon \mathbb{G}_m^N \to \mathbb{A}^1$ be the summing (i.e., trace) map. As such, V^* is the zero locus of $\sigma \circ \varphi$. Accordingly, note that we have an exact sequence

$$1 \to \ker \varphi \to \mathbb{G}_m^N \xrightarrow{\varphi} \mathbb{G}_m^N \operatorname{coker} \varphi \to 1,$$

so for example we see that $\# \ker \varphi(\mathbb{F}_q)$ is $\# \mu_d^N(\mathbb{F}_q) \leq d^N$ does not depend on q. We now see that

$$\# V^*(\mathbb{F}_q) = \# \ker \varphi(\mathbb{F}_q) \cdot \# \left\{ t \in \mathbb{G}_m^N(\mathbb{F}_q) : t \in \operatorname{im} \varphi(\mathbb{F}_q) \text{ and } \sigma(t) = 0 \right\}.$$

4. We use our character theory. By the orthogonality relations, we know that

$$1_{\sum t_i = 0} = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi_a(t_1 + \cdots + t_N) \qquad \text{and} \qquad 1_{t \in \operatorname{im} \varphi} = \frac{1}{\# \operatorname{coker} \varphi(\mathbb{F}_q)} \sum_{\chi \in \operatorname{coker} \varphi(\mathbb{F}_q)^\vee} \chi(t).$$

Thus, by cancelling out the kernel and cokernel, we see that

$$\# V^*(\mathbb{F}_q) = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \sum_{\chi \in \operatorname{coker} \varphi(\mathbb{F}_q)^\vee} \sum_{t \in \mathbb{G}_m^N(\mathbb{F}_q)} \chi(t) \psi_a(t_1 + \cdots + t_n).$$

For example, we see that the $a = 0$ succeeds at being nonzero only when χ is trivial, where we receive $\frac{1}{q}(q-1)^N$.

5. It remains to handle the values $a \neq 0$, for which we use Gauss sums. Note that a character χ on $\operatorname{coker} \varphi$ can be lifted to $\mathbb{G}_m^N$ and therefore can be factored into N characters $\chi_1 \cdots \chi_N$. Thus, we may factor

$$\sum_{t \in \mathbb{G}_m^N(\mathbb{F}_q)} \chi(t) \psi_a(t) = \prod_{i=1}^N \underbrace{\sum_{t_i \in \mathbb{F}_q^\times} \chi_i(t_i) \psi_a(t_i)}_{g(\chi_i; \psi_a)},$$

so we see that the entire product has absolute value bounded by $q^{N/2}$ because $|g(\chi; \psi_a)| \leq q^{N/2}$.

6. We conclude. Plugging in the previous two steps yields

$$\left| \# V^*(\mathbb{F}_q) - \frac{1}{q}(q-1)^N \right| \leq \frac{1}{q} \underbrace{(q-1)}_{a} \cdot \underbrace{\# \operatorname{coker} \varphi(\mathbb{F}_q)}_{\chi} \cdot q^{N/2}.$$

The term $\frac{1}{q}(q-1)$ dies, and the size of $\operatorname{coker} \varphi(\mathbb{F}_q)$ is the size of $\ker \varphi(\mathbb{F}_q)$ is bounded independently of q. The total error term comes out to $q^{N/2}$, so we are done!

26   October 30: Deligne's Proof of Weil I and the Main Lemma

This talk was given by Mohit Hulse at MIT for the STAGE seminar. The term "main lemma" is due to Milne; Deligne only names a consequence as "the fundamental estimate."

26.1   The Étale Fundamental Group

For today, we fix a connected scheme X of finite type over a field k.

> **Definition 285.** Fix a scheme X of finite type over a field k. For a geometric point $\overline{x} \hookrightarrow X$, we define the fiber functor $\omega_{\overline{x}} \colon \mathrm{F\acute{E}t}(X) \to \mathrm{Sets}$ by
> $$\omega_x(Y) := Y_{\overline{x}}.$$
> We define $\pi_1^{\text{ét}}(X, \overline{x})$ as the automorphism group of ω_x.

> **Remark 286.** Once $\pi_1^{\text{ét}}(X, \overline{x})$ has been defined, we may upgrade $\omega_{\overline{x}}$ to a functor
> $$\omega_x \colon \mathrm{F\acute{E}t}(X) \to \mathrm{Sets}(\pi_1^{\text{ét}}(X, \overline{x})),$$
> and this latter functor turns out to be an equivalence.

The theory of the étale fundamental group proves the following "pro-representatbility" result.

> **Theorem 287.** Fix a scheme X of finite type over a field k, and choose a geometric point $\overline{x} \hookrightarrow X$. There is a cofiltered sequence $\{X_i\}$ of finite étale covers of X such that
> $$\omega_{\overline{x}} = \operatorname*{colim}_i \mathrm{Hom}_X(X_i, -).$$
> In fact, one can choose the covers $X_i \to X$ to be Galois.

> **Corollary 288.** Fix a scheme X of finite type over a field k, and choose a geometric point $\overline{x} \hookrightarrow X$. There is a cofiltered sequence $\{X_i\}$ of finite étale covers of X such that
> $$\pi_1(X, \overline{x}) = \lim_i \mathrm{Aut}_X(X_i).$$

> **Example 289.** If X is the point $\mathrm{Spec}\, k$, then one can choose the X_i to be finite Galois extensions of k, so we find that $\pi_1^{\text{ét}}(X, \overline{x}) = \mathrm{Gal}(k^{\text{sep}}/k)$.

> **Example 290.** If X is a smooth projective variety over $\mathbb{C}$, then we see that $\pi_1^{\text{ét}}(X, \overline{x})$ is the profinite completion of $\pi_1(X, x)$. For example, $X = \mathbb{G}_m$ admits covers $\mathbb{G}_m \to \mathbb{G}_m$ by $x \mapsto x^n$ (where $\mathrm{char}\, k \nmid n$), allowing us to compute $\pi_1(X, \overline{x}) = \widehat{\mathbb{Z}}$ when $\mathrm{char}\, k = 0$. This sort of process works for general Riemann surfaces because the finite étale covers of a Riemann surface all come from varieties.

The reason we care about the étale fundamental group is that it will allow us to understand local systems.

> **Notation 291.** Fix a scheme X of finite type over a field k. Then $\mathrm{Loc}(X_{\text{ét}}, \mathrm{FinSet})$ consists of the locally constant étale sheaves on X valued in finite sets. In other words, there is some étale covering $\{U_i\}$ of X so that the sheaf is constant when restricted to any of the given U_i.

Theorem 292. Fix a scheme X of finite type over a field k, and choose a geometric point $\overline{x} \hookrightarrow X$. Then the fiber functor

$$\mathrm{Loc}(X_{\text{ét}}, \mathrm{FinSet}) \to \mathrm{FinSet}\left(\pi_1^{\text{ét}}(X, \overline{x})\right)$$

given by $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is an equivalence, and any sheaf on the left-hand side is representable.

Sketch. The point is to choose a cover trivializing the sheaf, and then one can prove representability over X explicitly by finding descent datum. ∎

Of course, we would like a way to extend this to ℓ-adic sheaves, which we do as follows.

Notation 293. Fix a scheme X of finite type over a field k. Then $\mathrm{Loc}(X_{\text{ét}}, \mathbb{Q}_\ell)$ consists of the locally constant ℓ-adic étale sheaves on X, meaning that they are valued in finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

Theorem 294. Fix a scheme X of finite type over a field k, and choose a geometric point $\overline{x} \hookrightarrow X$. Then the fiber functor

$$\mathrm{Loc}(X_{\text{ét}}, \mathbb{Q}_\ell) \to \mathrm{Rep}_{\mathbb{Q}_\ell}\left(\pi_1^{\text{ét}}(X, \overline{x})\right)$$

given by $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is an equivalence.

Proof. Unwind to the previous theorem. ∎

Remark 295. If the corresponding representation on the right-hand side does not factor through a finite quotient of $\pi_1^{\text{ét}}(X, \overline{x})$, then one does not expect to be able to find a single cover trivializing the entire ℓ-adic local system.

Of course, in this seminar, we are interested in computing cohomology, so we pick up a few resutls to do so.

Lemma 296. Fix a scheme X of finite type over a field k, and choose a geometric point $\overline{x} \hookrightarrow X$. Then for any locally constant ℓ-adic sheaf $\mathcal{F}$, we have

$$\mathrm{H}^0(X; \mathcal{F}) \cong (\mathcal{F}_{\overline{x}})^{\pi_1^{\text{ét}}(X, \overline{x})}.$$

Sketch. There is nothing to do if $\mathcal{F}$ is constant. If $\mathcal{F}$ is non-constant, then we can locally pass to a Galois étale cover $Y \to X$ where it is constant, and then we can compute $\mathrm{H}^0(X; \mathcal{F})$ via Čech cohomology to prove the result. ∎

26.2   The Main Lemma

Recall the following definition.

Notation 297. Fix a scheme X of finite type over $\mathbb{F}_q$, and let $\mathcal{F}$ be an ℓ-adic local system. Then we define

$$Z(\mathcal{F}_0; T) := \prod_{\text{closed } x \in X} \frac{1}{\det\left(1 - F_{x_0} T^{\deg x_0}; \mathcal{F}_0\right)}.$$

Earlier, we proved the following formula.

Theorem 298 (Lefschetz trace formula)**.** Fix a scheme X of finite type over $k := \mathbb{F}_q$, and let $\mathcal{F}$ be an ℓ-adic local system. Then

$$Z(\mathcal{F}_0; T) = \prod_{i \geq 0} \det \left(1 - \mathrm{Frob}_q T; \mathrm{H}_c^i(X_{\overline{k}}; \mathcal{F}) \right)^{(-1)^{i+1}}.$$

Example 299. If X is an affine curve, then $\mathrm{H}_c^0(X_{\overline{k}}; -)$ vanishes, so we only have to worry about $\mathrm{H}_c^1(X_{\overline{k}}; -)$ and $\mathrm{H}_c^2(X; -)$. By Poincaré duality, we can recover $\mathrm{H}_c^2(X_{\overline{k}}; \mathcal{F})$ as $\mathrm{H}^0(X_{\overline{k}}; \mathcal{F})$ and compute via Lemma 296.

For the main lemma, we recover the following definition.

Definition 300 (weight)**.** Fix an ℓ-adic local system $\mathcal{F}$ on a scheme X of finite type over $\mathbb{F}_q$. Then $\mathcal{F}$ is of *weight* β if and only if, for each closd point $x \in X$, the eigenvalues of the Frobenius F_x acting on $\mathcal{F}_{\overline{x}}$ are algebraic numbers all of whose Galois conjugates have absolute value $q^{\beta/2}$.

Theorem 301 (Main lemma)**.** Fix an ℓ-adic local system $\mathcal{F}$ on an affine curve U of finite type over $\mathbb{F}_q$. Choose an integer β, and assume the following.

(a) Symplectic: there is a perfect alternating pairing $\psi \colon \mathcal{F} \otimes \mathcal{F} \to \mathbb{Q}_\ell(-\beta)$.

(b) Big monodromy: the image of $\pi_1(U_{\overline{\mathbb{F}}_q}, \overline{u})$ in $\mathrm{GL}(\mathcal{F}_{\overline{u}})$ is open in $\mathrm{Sp}(\mathcal{F}_{\overline{u}}; \psi)$ in the ℓ-adic topology for some geometric point $\overline{u}$.

(c) Rationality: the characteristic polynomials of $F_{\overline{u}}$ acting on $\mathcal{F}_{\overline{u}}$ are rational for all geometric points.

Then $\mathcal{F}$ has weight β.

Example 302. Fix a family $\pi \colon Y \to U$ of smooth projective hypersurfaces in $\mathbb{P}^{d+1}$, where d is odd. Then we will apply Theorem 301 with $\mathcal{F} := \mathrm{R}^d \pi_* \mathbb{Q}_\ell$.

(a) Poincaré duality provides a pairing $\mathrm{R}^d \pi_* \mathbb{Q}_\ell \times \mathrm{R}^d \pi_* \mathbb{Q}_\ell \to \mathrm{R}^{2d} \pi_* \mathbb{Q}_\ell = \mathbb{Q}_\ell(-d)$. This is symplectic because d is odd.

(b) Big monodromy turns out to be hard to check (and of course, it is not always true: one can take a constant family).

(c) Rationality follows from known cases of the Weil conjectures: the cohomology of the fiber of $\mathrm{R}^d \pi_* \mathbb{Q}_\ell$ can be computed via proper base change to be $\mathrm{H}^d(Y_u; \mathbb{Q}_\ell)$, which is known to be rational because the full zeta function is rational (and this is the only interesting cohomology group!).

The point is that having big monodromy proves the Riemann hypothesis.

Proof of Theorem 301. We proceed in steps.

1. We can use hypotheses (a) and (b) to produce an isomorphism

$$\mathrm{H}_c^2(U_{\overline{k}}, \mathcal{F}^{\otimes 2k}) \to \mathbb{Q}_\ell(-k\beta - 1)^{\oplus N}$$

for some integer N. Indeed, we chain together the isomorphisms

$$\mathrm{H}_c^2(U_{\overline{k}}; \mathcal{F}^{\otimes 2k}) = \mathrm{H}^0(U_{\overline{k}}; \mathcal{F}^{\vee \otimes 2k})^{\vee}(-1)$$
$$= \left(\mathcal{F}_{\overline{u}}^{\vee \otimes 2k} \right)^{\pi_1^{\text{ét}}(U; \overline{u}), \vee}(-1),$$

64

where we have used Poinaré duality in the first line. We may identify $\mathcal{F}$ with its dual by (a), so we may ignore the dual. By the big monodromy result, we will reduce ourselves to understanding

$$\mathrm{Hom}_{\mathrm{Sp}(\mathcal{F}_u)}(\mathcal{F}^{\otimes 2k}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(k\beta)^{\oplus N},$$

for some N, which completes the proof after tracking through the various twists. (The isomorphism here follows from some representation theory: more or less, one can explicitly construct functionals on $\mathcal{F}_u^{\otimes 2k}$ to be of the form $x_1 \otimes \cdots \otimes x_{2k} \mapsto \psi(x_1, x_2) \cdots \psi(x_{2k-1}, x_{2k})$.) Thus, we see that the main point is to check that π_1-invariants are G-invariants, where $G := \mathrm{Sp}(\mathcal{F}_u)$. Let H be the Zariski closure of the image of π_1 in G; then π_1-invariants are H-invariants because fixing a vector is an algebraic equation. However, $H \subseteq G$ contains an ℓ-adic open subgroup, so $\dim H = \dim G$, so $H = G$ follows because G is connected.

2. We apply Rankin's trick. Our two versions of $Z(\mathcal{F}; T)$ show that

$$\prod_{\substack{\text{closed } u \hookrightarrow U}} \frac{1}{\det\left(1 - F_u T^{\deg u}; \mathcal{F}^{\otimes 2k}\right)} = \frac{\det\left(1 - \mathrm{Frob}_q T; \mathrm{H}_c^1(U; \mathcal{F}^{\otimes 2k})\right)}{\left(1 - q^{k\beta+1}\right)^N},$$

where the denominator in the right-hand side is computed from the first step. Now, the left-hand side is assumed to live in $\mathbb{Q}((t))$ (which we can see in fact needs to have positive coefficients), so the right-hand side also lives in $\mathbb{Q}((t))$. Thus, the radius of convergence of the left-hand side is at most the radius of convergence of each factor in the product. However, the radius of convergence of the right-hand side is just $1/q^{k\beta+1}$, so we conclude that the eigenvalues α of the determinants on the left-hand side must have

$$\frac{1}{|\alpha|^{2k/\deg u}} \geq \frac{1}{q^{k\beta+1}}.$$

Sending $k \to \infty$ (which is Rankin's trick!) implies that $|\alpha| \leq q_{u_0}^{\beta/2}$. Because $\mathcal{F}_u$ is symplectic, we can run the same argument for the dual, so $\mathcal{F}_u$ also has the eigenvalue q^β/α. The result follows from comparing the two resulting inequalities. ∎

> **Remark 303.** The above proof basically features no algebraic geometry.

26.3   Applications

We now use Theorem 301 to derive some estimates.

> **Lemma 304.** Embed an affine curve U over $\mathbb{F}_q$ as $j: U \hookrightarrow \mathbb{P}^1$, and let S be the complement. Further, fix an ℓ-adic local system $\mathcal{F}$ on U. Then Poincaré duality induces a perfect pairing
>
> $$\mathrm{H}^1\left(\mathbb{P}^1; j_*\mathcal{F}^\vee\right) \otimes \mathrm{H}^1\left(\mathbb{P}^1; j_*\mathcal{F}\right) \to \mathbb{Q}_\ell(-1).$$

Proof. There is a pairing already for $\mathrm{R}j_*\mathcal{F}^\vee$ and $j_!\mathcal{F}$, but the differences between $j_!\mathcal{F}$ and $j_*\mathcal{F}$ turns out to be cancelled out. ∎

> **Corollary 305.** Embed an affine curve U over $\mathbb{F}_q$ as $j: U \hookrightarrow \mathbb{P}^1$, and let S be the complement.
>
> (a) Let α be an eigenvalue of $\mathrm{H}^1\left(\mathbb{P}^1; j_!\mathcal{F}\right)$. Then
>
> $$|\alpha| \leq q^{\frac{\beta+1}{2} + \frac{1}{2}}.$$
>
> (b) Let α be an eigenvalue of $\mathrm{H}^1\left(\mathbb{P}^1; j_*\mathcal{F}\right)$. Then
>
> $$q^{\frac{\beta+1}{2} - \frac{1}{2}} \leq |\alpha| \leq q^{\frac{\beta+1}{2} + \frac{1}{2}}.$$

Proof. For (a), we start by noting $\mathrm{H}^1(\mathbb{P}^1; j_!\mathcal{F}) = \mathrm{H}^1(U, \mathcal{F})$. The same sort of calculation as in the first step of Theorem 301 shows that our (inverse) zeta function is

$$\prod_{\substack{\text{closed } u \in U}} \det\left(1 - F_{u_0}T^{\deg u}; \mathcal{F}_{\overline{u}}\right) = \frac{1}{\det\left(1 - \mathrm{Frob}_q T; \mathrm{H}_c^1(U; \mathcal{F})\right)}.$$

Now, the left-hand product coverges absolutely if and only if the sum of the individual eignevalues converges absolutely. The moral is that

$$\sum_{\alpha} q^{\beta \deg u/2} |t|^{\deg u} < \infty,$$

where the sum is taken over all eigenvalues α of all closed points $u \in U$. Now, one can bound the number of closed points of $U \subseteq \mathbb{A}^1$ and bound the geometric series to achieve (a).

For (b), one uses the exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to i_*i^*j_*\mathcal{F} \to 0,$$

where $i\colon S \hookrightarrow \mathbb{P}^1$ is the inclusion. The right-hand term is supported on a finite set, so its H^1 vanishes, so we achieve a surjection

$$\mathrm{H}^1(\mathbb{P}^1; j_!\mathcal{F}) \twoheadrightarrow \mathrm{H}^1(\mathbb{P}^1; j_*\mathcal{F}).$$

We now get use the bound in (a), and the other bound follows from duality. ∎

27   November 3: An Overivew of Lawrence−Venkatesh

This talk was given by Frank for the Kisin seminar at Harvard.

27.1   How to Prove Mordell

The main application of the present method is another proof of the following theorem.

> **Theorem 306** (Faltings, Lawrence−Venkatesh)**.** Let C be a smooth projective curve over a number field K. If the genus of C is at least 2, then $C(K)$ is finite.

The original proof by Faltings first shows the following.

> **Theorem 307** (Tate's conjecture)**.** Fix a number field K. Then the Tate module functor T_ℓ from abelian varieties over K to Galois representations of K is fully faithful.

This then implies the following.

> **Theorem 308** (Shafaverich conjecture)**.** Fix a number field K and a fixed finite set S of places. Then there are only finitely many isomorphism classes of abelian varieties of dimension g and with good reduction outside S.

> **Remark 309.** By Torelli's theorem, one can descend from isomorphism classes of abelian varieties to isomorphism classes of curves.

Proof that Theorem 308 implies Theorem 306. Choose some étale cover $\varphi\colon C' \to C$ of degree more than 2. Each $x \in C(\overline{K})$ then induces the following diagram.

$$\begin{array}{ccc}
C_x & \longrightarrow & C' \xrightarrow{\ \varphi\ } C \\
\downarrow & \lrcorner & \downarrow \\
J' & \xrightarrow{[2]} & J'
\end{array}$$

Here, J' is a generalized Jacobian where we only quotient by rational functions with divisor supported in $\varphi^{-1}(\{x\})$; this J' turns out to be an extension of the Jacobian by a torus. (The vertical arrows are rational, not defined over the fiber $\varphi^{-1}(\{x\})$.) For degree reasons, the cover $C_x \to C$ is only ramified at x.

Now, if $x \in C(K)$, then Riemann–Hurwitz controls the genus of C_x, and we can control the field of definition of C_x by the choice of φ. Additionally, we can control where C_x has good reduction: it may only have bad reduction where C and C' have bad reduction, at 2, or where φ fails to be étale. Thus, there are only finitely many isomorphism classes of such C_x by Theorem 308, and the result follows because there are not many maps $C_x \to C$ (by de Franchis's theorem). ∎

> **Remark 310.** This argument is not due to Faltings: it was already understood in the function field case. This is known as Parshin's trick.

Morally, we are moving around the diagram

$$
\begin{array}{ccc}
C(K) & \longrightarrow & \{C_x \text{ with controlled reduction}\} \\
\downarrow & & \downarrow \\
C(\overline{K}) & \longrightarrow & \{C_x \to C \text{ with controlled genus}\}
\end{array}
$$

where the point is that the vertical arrows are inclusions, and the bottom arrow has finite fibers by de Franchis's theorem. Note that we only had to apply Shafaverich's conjecture to the family of curves of the form $\{C_x\}$, so one might expect to be able to prove Mordell's conjecture using something weaker about families $\mathcal{X} \to C'$ for a given cover $C' \to C$.

27.2   The Period Map

We are now prepared to say what Lawrence–Venkatesh is about. Given a family of varieties, we would like to understand the cohomology of the fibers, where "cohomology" can refer to any of étale, de Rham, cystalline, and so on. We are going to understand such cohomology via a p-adic period map.

Throughout, we fix a number field K and a finite set S of places, and we let $\mathcal{O}_{K,S}$ be the ring of S-integers. Now, fix a smooth projective map $Y \to X$ of smooth varieties over K which spreads out to $\mathcal{Y} \to \mathcal{X}$ over $\mathcal{O}_{K,S}$. We will furthermore suppose that the relative de Rham $\mathcal{H}^q$ and the pushforwards $\mathrm{R}^\bullet \pi_* \Omega^p_{\mathcal{Y}/\mathcal{X}}$ are all locally free on X (which can be achieved by possibly enlarging S). In this situation, thre is a Gauss–Manin connection

$$
\nabla \colon \mathcal{H}^q \to \mathcal{H}^q \otimes \Omega^1_{\mathcal{X}/\mathcal{O}_{K,S}}.
$$

We now fix an archimedean place $\iota \colon K \hookrightarrow \mathbb{C}$ and a place v above an odd prime p (avoiding S) so that v is unramified.

Furthermore, we fix some $x_0 \in \mathcal{X}(\mathcal{O}_{K,S})$, so it has a reduction $\overline{x}_0 \in \mathcal{X}(\mathbb{F}_v)$. Accordingly, we may choose a system of parameters $\{z_1, \ldots, z_m\}$ for $\mathcal{O}_{\mathcal{X}, \overline{x}_0}$. Upon choosing a basis $\{e_i\}$ of $\mathcal{H}^q$, we may expand out the connection as

$$
\nabla(e_i) = \sum_j A_{ji} e_j,
$$

where the coefficients A_{ji} are integral. It follows that we can solve the system of differential equations

$$
\nabla\left(\sum_i f_i e_i \right) = 0
$$

where the $f_\bullet$s live in $K[[z_1, \ldots, z_m]]$. By the aforementioned integrality, the $f_\bullet$s acquire a positive radius of convergence.

Now, if $x \equiv x_0 \pmod v$, one gets an isomorphism

$$
\mathrm{H}^q_{\mathrm{dR}}(\mathcal{Y}_{x_0}/K_v) \to \mathrm{H}^q_{\mathrm{dR}}(\mathcal{Y}_x/K_v),
$$

and this isomorphism is compatible with the comparison isomorphism from de Rham to crystalline cohomology. (The above map arises by plugging in the $f_\bullet$s, which vanish under ∇.) Thus, we can give de Rham cohomology a Frobenius action compatible with the Gauss−Manin connection. On the other hand, the Hodge filtration on de Rham cohomology will vary as x varies over $x_0 \pmod{v}$.

The moral is that we have a map

$$\Phi_v \{x \in X(K_v) : x \equiv x_0 \pmod{v}\} \to \mathcal{L}_v,$$

where $\mathcal{L}_v$ consists of triples $(\mathrm{H}^q_{\mathrm{dR}}(\mathcal{Y}_x/K_v), F_{\mathrm{Hodge}}, \varphi)$, where F_{Hodge} is the Hodge filtration, and φ is the Frobenius action. By the Gauss−Manin connection, we see that we should view $\mathcal{L}_v$ as parameterizing triples of a vector space, a filtration, and a Frobenius.

> **Remark 311.** Everything present also works complex analytically. This produces a map
>
> $$\Phi_{\mathbb{C}} \colon \widetilde{X} \to \mathcal{L}_{\mathbb{C}},$$
>
> where $\widetilde{X}$ is the universal cover, and $\mathcal{L}_{\mathbb{C}}$ parameterizes possible Hodge filtrations on the de Rham cohomology of the fiber.

> **Lemma 312.** The dimension of the Zariski closure of the image of Φ_v is at least the dimension of $\Gamma \cdot h_0$, where h_0 is the image of (some lift of) x_0 under $\Phi_{\mathbb{C}}$, and Γ is the Zariski closure of the image of $\pi_1(X, x_0)$ in $\mathrm{H}^q_{\mathrm{B}}(Y)$.

The moral is that complex analytic control of monodromy provides some v-adic control of monodromy.

27.3   Introducing Galois Representations

The space $\mathcal{L}_v$ turns out to parameterize Galois representations. Let's explain how to pass from crystalline cohomology to Galois representations.

> **Theorem 313.** Fix a smooth proper variety Z over a p-adic field E with good reduction. Then there is an isomorphism
>
> $$\left(\mathrm{H}^*_{\mathrm{ét}}(Z_{\overline{E}}; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\right)^{\mathrm{Gal}(\overline{E}/E)} \cong \mathrm{H}^n_{\mathrm{crys}}(\mathcal{Z}/\mathcal{O}_v)[1/p].$$
>
> In the sequel, we will denote the functor on the left-hand side by D_{crys}.

> **Definition 314.** When Z has good reduction, we may say that the Galois representation
>
> $$\rho \colon \mathrm{Gal}(\overline{E}/E) \to \mathrm{Aut}\left(\mathrm{H}^q_{\mathrm{ét}}(Z_{\overline{E}}; \mathbb{Q}_p)\right)$$
>
> is crystalline, meaning that D_{crys} preserves dimension.

> **Theorem 315** (Fontaine). The functor D_{crys} provides a fully faithful functor from crystalline representations to filtered Frobenius modules.

> **Example 316.** It turns out that D_{crys} sends $\mathrm{H}^q_{\mathrm{ét}}(\mathcal{Y}_x/\mathbb{Q}_p)$ to exactly the triple $\Phi_v(x)$. Furthermore, the above theorem tells us that Φ_v is injective.

Everything so far is local. Here is our global input.

Proposition 317. Fix notation as above. Suppose

$$\dim Z(\phi_v^{[K_v:\mathbb{Q}_p]}) < \dim_{\mathbb{C}} \Gamma \cdot h_0'.$$

Then

$$\{x \in \mathcal{X}(\mathcal{O}_{K,S}) : x \equiv x_0 \pmod{v}, \rho_x \text{ is semisimple}\}$$

lies in a proper K_v-analytic subvariety of the residue disk of $X(K_v)$.

Proof. The smallness will arise from the following finiteness result.

Lemma 318 (Faltings)**.** Fix integers $w, d \geq 0$. There are only finitely many semisimple representations $\rho \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(\mathbb{Q}_p)$ (up to conjugation) satisfying the following.

(a) ρ is unramified outside S.

(b) For all $\mathfrak{p}$ in S, the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$ has all roots algebraic and with complex absolute value $\#\kappa(\mathfrak{p})^{w/2}$.

(c) For all $\mathfrak{p}$ in S, the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$ has integer coefficients.

The proof of this lemma is technically elementary, but it is quite tricky.

Note ρ_x satisfies all three of the above conditions, so we may fix a choice of ρ for $\rho_x \cong \rho$. Now, because D_{crys} is fully faithful, we know that ρ_x and ρ agreeing on $\mathrm{Gal}(\overline{K}/K)$ is equivalent to $\Phi_v(x)$ being equivalent to $D_{\mathrm{crys}}(\rho)$. It follows that $\Phi_v(x)$ lives in some orbit of $Z(\phi_v)$ in $\mathcal{L}_v$.

The proof then follows by comparing dimensions: one finds that the dimension of the $Z(\phi_v)$-orbit is strictly less than the dimension of the Zariski closure of $\Phi_v(\Omega_v)$. It follows that $\Phi_v(\Omega_v)$ cannot be contained in any orbit by $Z(\phi_v)$. This unwinds into the desired statement. ∎

As before, one can view the above argument as chasing around the following diagram

$$\begin{array}{ccc} \mathcal{X}(\mathcal{O}_{K,S}) & \longrightarrow & \{Z(\phi_v) \text{ orbits}\} \\ \downarrow & & \downarrow \\ \Omega_v & \longrightarrow & \{\text{filtrations on } \mathrm{H}^q\} \end{array}$$

with vertical inclusions, and the bottom inclusion has positive codimension.

Here are some problems with applying Proposition 317.

- One may have wanted being in an analytic subvariety to an algebraic subvariety. For example, if the base is in high dimensions, then one may want to show finiteness by showing Zariski non-density in order to inductively apply Proposition 317. Luckily, if the codimension is high enough, one can achieve Zariski non-density by some results of Bakker and Tsimermann.

- One only gains control of x where ρ_x is semisimple. To fix this, it frequently suffices to look at a locus where two representations have the same semisimplification.

- It is difficult to verify the big monodromy hypothesis inductively to achieve finiteness, as described in the first point.

Remark 319. Professor Kisin thinks that the proof of big monodromy in Lawrence−Venkatesh is "sus."

Next time, we (excluding me) will discus the following theorem.

Theorem 320 (Lawrence−Venkatesh)**.** The Shafaverich conjecture holds for hypersurfaces of abelian varieties of dimension at least 4.

What is surprising here is that Lawrence−Venkatesh require a big monodromy result.

28 November 4: The Gross–Siebert Program

This talk was given by Yonghwan at MIT as a pre-talk for the Harvard–MIT algebraic geometry seminar.

28.1 Mirror Symmetry

Mirror symmetry is interested in the following result.

> **Theorem 321** (Closed string mirror symmetry). Fix a smooth Calabi–Yau variety X, and let $X^\vee$ be the mirror Calabi–Yau. Then there is a mirror map
>
> $$\left(\mathrm{H}^*(X^\vee; \wedge^\bullet T_{X^\vee}), \nabla^{\mathrm{GM}}\right) \cong (\mathrm{QH}^*(X), \nabla^q).$$
>
> Further, there is a J-function whih is a canonical flat section of the quantum D-module, and there is an I-funtion which is a canonical flat sectino of the Gauss–Manin system.

Here, $\mathrm{QH}^*(X)$ is the quantum cohomology. Its underlying vector space is $\mathrm{H}^*(X; \wedge)$, where $\wedge$ is the Norkiov ring $\mathbb{Q}[[q^A]]$, where the basis is taken over A in some cohomology classes. The Gromov–Witten invariants then fit into this picture as the coefficients of the multiplication.

> **Example 322.** The mirror of an elliptic curve E is itself. In this case, the quantum cohomology is just regular cohomology.

> **Example 323.** The mirror of a K3 surface X is itself. In this case, the quantum cohomology is again regular cohomology.

There is also a story of mirror symmetry beyond Calabi–Yau surfaces. For Fano surfaces in general, the mirror is a pair of a quasiprojective variety $X^\vee$ and a regular function W on $X^\vee$.

> **Example 324.** For $X = \mathbb{P}^n$, the mirror turns out to be $(\mathbb{C}^\times)^n$ and W is
>
> $$W(z_1, \ldots, z_n) := z_1 + \cdots + z_n + \frac{u^{-1}}{z_1 \cdots z_n}.$$

The statement of mirror symmetry in this case also says something about quantum cohomology of X in terms of some total cohomology of $X^\vee$ of a deformed de Rham cohomology. The Gross–Siebert program aims to compute some mirror varieties in algebraic geometry.

29 November 4: Automorphism Groups of Riemann Surfaces

This talk was given by Jennifer Paulhus at MIT for the MIT number theory seminar.

29.1 Riemann Surfaces

Let X be a Riemann surface of genus g. Then $G := \mathrm{Aut}\, X$ is finite, and X/G is another Riemann surface, say of genus h. The map $X \twoheadrightarrow X/G$ has degree $d := \#G$, so we expect fibers have to size d, but there may be ramification. This covering has some monodromy group; let $\{m_1, \ldots, m_r\}$ be the orders of the monodromy. We are interested in what sort of tuples $[h; m_1, \ldots, m_r]$ can occur.

> **Remark 325.** Everything in the story descends to discuss curves over number fields.

There is already a classification of the possible groups G which can appear.

Theorem 326 (Riemann existence)**.** A finite group G acts on some Riemann surface X of genus $g > 1$ if and only if there are elements $\{a_1, b_1, \ldots, a_h, b_h\} \cup \{c_1, \ldots, c_r\}$ generating G and for which

$$\prod_{i=1}^{r} [a_i, b_i] \cdot \prod_{i=1}^{r} c_i = 1$$

and the Riemann–Hurwitz formula is satisfied with h being the genus of X/G.

Remark 327. One can see that the $c_\bullet$s are the monodromy elements.

Remark 328. Here is one way to view this theorem: X has universal cover given by the upper-half plane $\mathcal{H}$, so X/G becomes the orbit space of some Fuchsian group.

Here is the sort of question we may be interested in. Given a group G, we can look for the possible signatures $[h; m_1, \ldots, m_r]$, and then afterward we have to look for which signatures can be realized.

Example 329. The signature $[0; 3, 3, 9]$ satisfies the Riemann–Hurwitz formula for the cyclic group C_9, but it never happens.

Example 330. The signature $[0, 2, \ldots, 2]$ satisfies the Riemann–Hurwitz formula for $\mathrm{SL}(2, q)$, but there is only one element of order 2 in this group.

Definition 331. A group G acts with *almost all possible signatures* if and only if there are only finitely many signatures satisfying the Riemann–Hurwitz formula whose signature cannot be realized in a Riemann surface X.

Example 332. If the commutator subgroup $[G : G]$ fails to contain an element of all possible orders, then the signatures $[h; n_i]$ for any $h > 0$ is satisfies Riemann–Hurwitz, but it cannot be realized.

Example 333. If G can be generated by elements of order n_i for any possible order n_i, then we can again find infinitely many signatures.

It turns out that the conditions given in the two examples are sharp.

Example 334. It turns out that any finite nonabelian simple group acts with almost all possible signatures. For example, the work of many group theorists showed that every element of G is a commutator.

One may be interested in obtaining the finite list of missed signatures for finite nonabelian simple groups. This turns out to be difficult. For signature $[h; m_1, \ldots, m_r]$ for $h \geq 2$, all signatures are doable; one input is that any finite nonabelian simple group is generated by two elements. It turns out that $h = 0$ is very hard, so let's determine what we can do for $h = 1$

Theorem 335. For $n > 6$, every potential signature $[h; m_1, \ldots, m_r]$ for $h \geq 1$ is an actual signature for the group A_n.

30   November 5: Local Constancy of Automorphic Sheaves

This talk was given by Joakim Faergeman for the Lie Groups seminar at MIT.

30.1   Geometric Langlands

Fix a smooth projective curve X over $\mathbb{C}$, and let G be a complex connected reductive group. Here is one version of the geometric Langlands correspondence.

> **Theorem 336** (de Rham geometric Langlands)**.** Let $\mathrm{Bun}_G(X)$ be the moduli stack of G-bundles on X, and consider the (derived) category $\mathcal{D}(\mathrm{Bun}_G(X))$ of D-modules; this is the automorphic side. Further, let $\mathrm{LS}_{\check{G}}(X)$ be the moduli stack of $\check{G}$-bundles on X with a connection; this is the spectral side. Then
> $$\mathcal{D}(\mathrm{Bun}_G(X)) \cong \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

As usual, there are technical problems here; for example, the right-hand side turns out to be too small.

Ben-Zvi and Nadler have introduced a Betti version of geometric Langlands. The goal is to replace vector bundles with connections with topological $\check{G}$-local systems; this is then merely the data of a group homomorphism $\pi_1(X) \to \check{G}$.

> **Remark 337.** One can make morphisms $\pi_1(X) \to \check{G}$ more explicit: $\pi_1(X)$ can be presented with generators $\{a_1, b_1, \dots, a_g, b_g\}$ (where g is the genus of X) satisfying $\prod_i [a_i, b_i] = 1$.

But now one must modify the automorphic side.

> **Definition 338.** Fix a smoooth algebraic stack $\mathcal{Y}$ over $\mathbb{C}$, and let $\mathrm{Shv}(\mathcal{Y})$ be the category of complex sheaves on $\mathcal{Y}(\mathbb{C})$. For some closed conical subset $\Lambda \subseteq T^*\mathcal{Y}$, we let $\mathrm{Shv}_\Lambda(\mathcal{Y})$ be the collectio of sheaves whose singular support is contained in Λ, where the singular support measures where (and in which direction!) $\mathcal{F}$ fails to be locally constant.

> **Example 339.** With $\Lambda = 0$, we see that $\mathrm{Shv}_\Lambda(\mathcal{Y})$ has all topological local systems on $\mathcal{Y}$.

> **Example 340.** We will take $\mathcal{Y} = \mathrm{Bun}_G(X)$. Then $T^* \mathrm{Bun}_G(X)$ consists of pairs $(\mathcal{P}_G, \varphi)$ of a G-bundle $\mathcal{P}_G$ and a "Higgs field" φ; a Higgs field is just some global section of $\mathcal{Y}^*_{\mathcal{P}_G} \otimes \Omega^1_X$. Then we will take Λ to be the nilpotent cone $\mathrm{Nilp}(X)$ of X inside $T^*\mathrm{Bun}_G(X)$, meaning that the Higgs field φ is nilpotent.

> **Theorem 341** (Betti geometric Langlands)**.** Let $\mathrm{LS}^{\mathrm{top}}_{\check{G}}$ be the moduli of group maps $\pi_1(X) \to \check{G}$ up to $\check{G}$-conjugation. Then
> $$\mathrm{Shv}_{\mathrm{Nilp}(X)}(\mathrm{Bun}_G(X)) \cong \mathrm{QCoh}\left(\mathrm{LS}^{\mathrm{top}}_{\check{G}}(X)\right).$$

30.2   The Main Result

Note that the right-hand side in Theorem 341 only depends on the homtopy type of X (i.e., the genus), so the left-hand side should be the same. This is our main theorem today.

> **Theorem 342.** The category $\mathrm{Shv}_{\mathrm{Nilp}(X)}(\mathrm{Bun}_G(X))$ depends only on the topological space of X.

Of course, we are interested in proving this without using the proof of Betti geometric Langlands.

Example 343. Consider $G = \mathrm{SL}_2$. Then $\mathrm{Bun}_G(X)$ consists of rank-2 vector bundles $\mathcal{E}$ for which $\det \mathcal{E} \cong \mathcal{O}_X$. Now, the nilpotent cone of X stratifies according to degree into some subspaces $\mathrm{Nilp}^d(X)$, which consists of short exact sequences
$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^\vee \to 0$$
where φ is some nonzero function $\mathcal{L}^\vee \to \mathcal{L} \otimes \Omega_X^1$. Thus, the geometry of $\mathrm{Nilp}(X)$ will depend on $\Gamma\left(X, \mathcal{L}^2 \otimes \Omega_X^1\right)$. Riemann–Roch explains what this vector spaces is when $\deg \mathcal{L} < 0$ or $\deg \mathcal{L} > 2g - 2$, but in general, Brill–Noether theory explains that $\mathrm{Nilp}(X)$ will depend on the complex structure of X.

Here is the setup for the argument: we will work with a family $X \to S$ of smooth plane curves. Then the stack $\mathrm{Bun}_G(X/S)$ lies over S with fiber $\mathrm{Bun}_G(X_s)$ over $s \in S$. It is a result of Nadler and Yun that one can find some $\mathrm{Nilp}^U \subseteq T^* \mathrm{Bun}_G(X/S)$ whose fiber over $s \in S$ is $\mathrm{Nilp}(X_s)$, and then our goal becomes to show that the projection maps

$$\mathrm{Shv}_{\mathrm{Nilp}^U}\left(\mathrm{Bun}_G(X/S)\right) \to \mathrm{Shv}_{\mathrm{Nilp}(X_s)}\left(\mathrm{Bun}_G(X_s)\right)$$

is an equivalence when the base S is contractible. A key in the proof is to have an irreducible component $\mathrm{Nilp}^{\mathrm{Kos}}$ which "controls" $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G(X))$ which behaves well in families.

31   November 6: Lefschetz Principles

This talk was given by Jack Miller at MIT for the STAGE seminar.

31.1   Fibrations, Diffeomorphically

We are (still) trying to show the Riemann hypothesis part of the Weil conjectures. As usual, the veracity of the Riemann hypothesis is insensitive to base-change of the base finite field. After some reductions, it turns out that it will be enough to handle the middle-dimension cohomology of a smooth projective even-dimensional varieties. As such, we will let $n + 1$ be the dimension of some varieties to be chosen later, where $n = 2m + 1$ is odd.

As in our previous cases of the Weil conjectures, we are on the hunt for many fibrations $X \to \mathbb{P}^1$. Such fibrations are the algebro-geometric version of finding some smooth submersions $M \to \mathbb{R}$. In the real analytic case, this amounts to the following result.

Theorem 344 (Ehresmann)**.** If $f\colon M \to N$ is a smooth submersion of closed manifolds, then f is a locally trivial fibration.

Corollary 345. Fix a closed manifold M. Every smooth map $M \to \mathbb{R}$ admits a critical point.

Remark 346. Of course, is not hard to prove the corollary directly: a smooth map $M \to \mathbb{R}$ must admit a maximum, which is critical.

Corollary 347. Fix a closed manifold M. If $M \to S^1$ is a submersion, then M is diffeomorphic to a mapping torus.

Thus, admitting a submersion to a curve places strong requirements on M.

Here is the algebro-geometric version of this.

Theorem 348 (Ehresmann)**.** If $f\colon Y \to S$ is smooth proper, and $\mathcal{F}$ is a locally constant constructible étale sheaf on Y, then the higher pushforwards $\mathrm{R}^p f_* \mathcal{F}$ are locally constant constructible.

However, the real analytic version tells us that we are going to need to allow some singularities for our fibrations. Let's explain the sort of singularities one finds in the diffeomorphic setting.

> **Theorem 349** (Morse's lemma)**.** Fix a d-dimensional closed manifold M. Then there is an open subset $U \subseteq \mathrm{Fun}(M, \mathbb{R})$ (using the C^2 topology) satisfying the following for each function in U.
>
> - There are finitely many critical points, and at most one in each fiber.
>
> - The critical points p are non-degenerate, and each bad fiber $f^{-1}(\{p\})$ looks like a conic of the form
>
> $$0 = x_1^2 + \cdots + x_i^2 - x_{i+1}^2 - \cdots - x_d^2.$$
>
> In other words, the Hessian is non-degenerate.

> **Remark 350.** Picard and Lefschetz proved a variant of this for complex manifolds.

31.2   Lefschetz Pencils

Let's now try to find our fibrations in the algebro-geometric setting.

> **Definition 351.** Fix a projective variety X, and let $\mathcal{L}$ be a very ample line bundle on X inducing $X \hookrightarrow \mathbb{P}\Gamma(X, \mathcal{L})$. Then a *pencil D* of $(X, \mathcal{L})$ is a 1-parameter line of hyperplanes of $\mathbb{P}\Gamma(X, \mathcal{L})^\vee$; in other words, D is a map $\mathbb{P}^1 \to \mathbb{P}\Gamma(X, \mathcal{L})^\vee$.

> **Definition 352.** Fix a projective variety X, and let $\mathcal{L}$ be a very ample line bundle on X. The *axis* of a pencil $D \colon \mathbb{P}^1 \to \mathbb{P}\Gamma(X, \mathcal{L})^\vee$ is
>
> $$A(D) := \bigcap_t D_t.$$

> **Remark 353.** In fact, $A = D_0 \cap D_\infty$

> **Definition 354.** Fix a projective variety X, and let $\mathcal{L}$ be a very ample line bundle on X. Then a pencil $D \colon \mathbb{P}^1 \to \mathbb{P}\Gamma(X, \mathcal{L})^\vee$ is *Lefschetz* if and only if it satisfies the following.
>
> (a) Transverse: A intersects D transversely.
>
> (b) Smoothness: the intersectios $X \cap D_t$ are nonsingular for all but finitely many t. Let $S \subseteq \mathbb{P}^1$ be the singular locus.
>
> (c) For each $t \in S$, the singularities P in X_t are single ordinary double points, meaning that
>
> $$\widehat{\mathcal{O}}_{X,p} = \frac{k[[x_1, \ldots, x_d]]}{Q(x_1, \ldots, x_{d+1})},$$
>
> where $d = \dim X$ and Q is a non-degenerate quadratic.

This definition is a little involved, so we should make sure that such things exist.

> **Theorem 355.** Fix a projective variety X over an algebraically closed field k of characteristic 0. For any very ample sheaf $\mathcal{L}$, there is a Lefschetz pencil.

> **Remark 356.** If $\operatorname{char} k > 0$, then there is a Lefschetz pencil for some tensor power of $\mathcal{L}$.

Sketch. Suppose $X \neq \mathbb{P}\Gamma(X, \mathcal{L})$, and we go ahead and assume that X is geometrically connected. As usual, set $d \coloneqq \dim X$ and $N \coloneqq \dim \Gamma(X, \mathcal{L})$. We are going to use incidence correspondence

$$\Phi \coloneqq \{(x, H) : x \in H \text{ and } T_x X \subseteq H\}.$$

To explain this, note that the "enemy" is basically where a hyperplane is tangent to X; otherwise, the intersection is automatically transverse! The condition $T_x X \subseteq H$ is equivalent to the tangency because H has codimension 1. Note that Φ is a correspondence fitting in the diagram

$$
\begin{array}{ccc}
 & \Phi & \\
{\scriptstyle \operatorname{pr}_1} \swarrow & & \searrow {\scriptstyle \operatorname{pr}_2} \\
X & & \mathbb{P}\Gamma(X, \mathcal{L})^\vee
\end{array}
$$

so we let $X^\vee$ denote the image of X in $\mathbb{P}\Gamma(X, \mathcal{L})^\vee$. (We call $X^\vee$ the dual variety, though it notably also depends on $\mathcal{L}$.) Here are some properties of this correspondence.

- The projection $\operatorname{pr}_1 \colon \Phi \to X$ is a projective bundle. Indeed, at each $x \in X$, the fiber above x simply consists of the hyperplanes containing $T_x X \subseteq T_x \mathbb{P}\Gamma(X, \mathcal{L})$. As such, each fiber is isomorphic to $\mathbb{P}^{N-d-1}$, which we can see by working out some equations of hyperplanes. It follows that Φ is irreducible, projective, and has dimension $N - 1$.

- The projection $\operatorname{pr}_2 \colon \Phi \to X^\vee$ has fibers consisting of the singular locus in $H \cap X$. Indeed, the fiber above H is exactly the pairs (x, H) where H is tangent to X at x. Approximately speaking, the ramification behavior of pr_2 controls the singularities we see.

Let's now sketch the rest of the proof. Bertini's theorem lets us find a single valid H to start our Lefschetz pencil, so we just need to show that we can continue it to a pencil. Well, one simply controls the singularities of X (they have positive codimension at least 2), and then one argues that lines exist generically. ∎

Blowing up produces fibrations.

> **Definition 357** (Lefschetz fibration)**.** Fix a Lefschetz pencil D for $(X, \mathcal{L})$. Then the *Lefschetz fibration* is the incidence correspondence X^* sitting in the following diagram.
>
> $$
> \begin{array}{ccc}
> & X^* & \\
> \swarrow & & \searrow \\
> X & & D
> \end{array}
> $$

> **Remark 358.** It turns out that X^* is the blow up of X along $X \cap A(D)$.

> **Remark 359.** It also turns out that the map $X^* \to \mathbb{P}^1$ is proper, flat, and it admits a section.

31.3   Symplectic Monodromy

Let's explain how we will get to use the Main lemma. Fix a nice variety $X \subseteq \mathbb{P}^N$ over $\mathbb{F}_q$, which we return to being n-dimensional, where $n = 2m + 1$. Then one can find a Lefschetz pencil over some extension of $\mathbb{F}_q$: first find it over the algebraic closure, and then descend everything.

Remark 360. It turns out that Lefschetz pencils also exist without doing the extensions. This is due to Poonen, Nguyen, and Gunther.

Now, let $S \subseteq \mathbb{P}^1$ be the collection of singular values, and we let $U \subseteq \mathbb{P}^1 \setminus S$ denote the complement. It turns out that the associated map $\pi \colon X^* \to \mathbb{P}^1$ is now smooth and proper over U, so the higher pushforward $\mathcal{V} := \mathrm{R}^n \pi_* \mathbb{Q}_\ell$ is an ℓ-adic local system over U by the Proper base change theorem.

Thus, we are granted some representation

$$\pi_1(U) \to \mathrm{GL}(V),$$

where V is some fiber of $\mathcal{V}$. The tameness of the singularities of $X^* \to \mathbb{P}^1$ turns out to provide tameness of the representation. To be more precise, we say something about inertia.

Definition 361. For some ramified $s \in S$, let $\mathbb{D}_s$ be the formal disk $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathbb{P}^1, s}$ so that the puncture $\mathbb{D}_s^\circ$ is its fraction field. Then the *inertia subgroup* is the image of the map

$$\pi_1(\mathbb{D}^\circ) \to \pi_1(U).$$

Remark 362. The map is only defined up to conjugation because it has suppressed moving some base-points around.

Example 363. We work over $\mathbb{C}$ for psychological reasons. Consider the Legendre family $\mathcal{E} \to \mathbb{P}^1$ of elliptic curves given by
$$\mathcal{E}_\lambda \colon Y^2 Z = X(X - Z)(X - \lambda Z).$$
This admits singularities at $\{0, 1, \infty\}$, where we have nodal singularities. (There is someting variable change one has to do to produce a definition at $\lambda = \infty$.) Thus, we see that this is a Lefschetz pencil! So we set $U := \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and we get a representation

$$\pi_1(U) \to \mathrm{GL}_2\left(\mathrm{H}^1(E_\eta; \mathbb{Q}_\ell)\right),$$

where η is the generic point. It turns out that the monodromy loops γ_0 and γ_1 around 0 and 1 go to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. There is apparently some geometric argument for this.

Remark 364. In general, for such tame representations, it turns out that each element of the monodromy fixes a codimension 1 subspace, and it modifies the remaining one-dimensional subspace in a controlled way; for example, it turns out that we should output a matrix of determinant 1. This is (definitionally) a transvection. Eventually, one can hope to use these transvections to prove a big monodromy result.

Remark 365. In the specialization
$$\mathrm{H}_1(\mathcal{E}_\eta; \mathbb{Q}_\ell) \to \mathrm{H}_1(\mathcal{E}_s; \mathbb{Q}_\ell),$$
there is a cyclic which vanishes. Appropriately, this may be called a vanishing cycle.

Thus, we see that we will be interested in some specializations. For example, letting $j \colon U \hookrightarrow \mathbb{P}^1$ denote the inclusion, we may be interested in when the canonical map $\mathcal{F} \to j_* j^* \mathcal{F}$ is an isomorphism. It turns out that this is equivalent to the injectivity of the cospecialization map

$$\mathcal{F}_s \to \mathcal{F}_\eta$$

and has image in $\mathcal{F}_\eta^{I_s^{\mathrm{tame}}}$.

Definition 366 (vanishing cycle). Fix everything as above and some $s \in S$. Then we define the space of *vanishing cycles* to be the kernel of

$$E_s := (\mathrm{H}^n(X_{\overline{\eta}}; \mathbb{Q}_\ell(n))^\vee \to \mathrm{H}^n(X_s; \mathbb{Q}_\ell(n))^\vee),$$

where this is the dual of the cospecialization map.

Remark 367. It turns out that E_s is one-dimensional and hence isomorphic to some Tate twist $\mathbb{Q}_\ell(m)$, so we may choose a generator $\delta_s(-m) \in E_s(-m)$.

Remark 368. It turns out that there is an exact sequence

$$0 \to \mathrm{H}^n(X_s; \mathbb{Q}_\ell) \to \mathrm{H}^n(X_\eta; \mathbb{Q}_\ell) \xrightarrow{\delta_s} \mathbb{Q}_\ell(m - n) \to 0.$$

The point of all this is that we are able to compute some transvections.

Theorem 369 (Picard–Lefschetz). Fix everything as above, and choose $\sigma_s \in I_s$. For each $x \in \mathrm{H}^n(X_\eta; \mathbb{Q}_\ell)$, we have
$$\sigma_s(x) = x \pm t(\sigma_s)(x \cup \delta_s)\delta_s.$$
Here, $t \colon I_s \to \mathbb{Z}_\ell(1)$ is the winding number; alternatively, it is the natural projection once I_s is identified with $\widehat{\mathbb{Z}}(1)^{(p)}$, where the superscript means we are taking prime-to-p roots of unity.

Remark 370. One can verify that the Tate twists work out.

Remark 371. The sign $\pm$ is $(-1)^{(n+1)(n+2)/2}$. In particular, it only depends on $n \pmod 4$.

Next time, we will show that these transvections to prove a big monodromy result.

32 November 7: Metaplectic Whittaker Functions and Quantum Groups

This talk was given by Valentin Buciumas at Johns Hopkins University for the Number theory seminar.

32.1 The Statements

For today, F is a nonarchimedean local field, $\mathcal{O} \subseteq F$ is the ring of integers, π is a uniformizer, and q is the cardinality of the residue field. Let G be a split reductive group over F, and let K be a maximal compact subgroup.

Theorem 372 (Satake–Langlands). The spherical Hecke algebra $\mathcal{H}(G, K) := C_c^\infty(K \backslash G / K)$ is isomorphic to $\mathbb{C}[\Lambda^\vee]^W$, which is isomorphic to $K_0(\mathrm{Rep}\, G^\vee)$.

The point is that unramified smooth representations are indexed by semisimple conjugacy classes of $G^\vee$. Here is a generalization to spherical varieties.

Theorem 373 (Sakellaridis). Let X be a spherical variety $H \backslash G$ satisfying some technical conditions. Then the Hecke algebra $\mathcal{H}(G, K)$ acts on $C_c^\infty(X/K)$, which is isomorphic to some $\mathbb{C}[\delta_X^{1/2} \Lambda_X^\vee]^{W_X}$, which is isomorphic to some $\mathrm{Rep}\, G_X^\vee$.

Example 374. With $X = (\mathrm{GL}_n \times \mathrm{GL}_n)\backslash\mathrm{GL}_{2n}$ has $G_X = \mathrm{Sp}_{2n}$.

Example 375. With $X = \mathrm{Sp}_{2n}\backslash\mathrm{GL}_{2n}$ has $G_X^\vee = \mathrm{GL}_n$.

We are interested in generalizations to quantum groups. Suppose we want to pass from G to a metaplectic cover $\widetilde{G}$. It is known that there is an isomorphism

$$\mathcal{H}(\widetilde{G}, K) \cong \mathrm{Rep}\, G_Q^\vee$$

for some reductive group G_Q. It is then conjectured that one can descend this to

$$\left(\mathrm{Ind}_U^{\widetilde{G}} \psi\right)^K = \mathcal{W}_\psi(\widetilde{G}, K) \cong \mathrm{Rep}\, U_\xi \mathfrak{g}^\vee.$$

Now, it remains to make sense of these terms and explain why one might conjecture this. There has been a lot of progress on these conjectures in the last few years.

32.2   Metaplectic Groups

Here is our definition.

Definition 376. Fix a positive integer n, and choose symmetric bilinear form B on $\Lambda^\vee$ which is W-invariant (equivalently, one can use Q). Suppose that $q \equiv 1 \pmod{n}$ so that there is an embedding $\mu_n \subseteq F$. Then there is a covering

$$1 \to \mu_n \to \widetilde{G} \to G \to 1$$

which is known as the metaplectic group. In brief, for torus $T \subseteq G$, we have the commutation relation $[a^\lambda, b^\mu] = (a, b)^{B(\lambda, \mu)}$, where $(-, -)$ is the Hilbert symbol on F.

Remark 377. If G is simple and simply connected, then there is essentially a unique primitive n-cover $\widetilde{G}$. In fact, there is further a splitting of U and K to $\widetilde{G}$.

In the sequel, we will assume the existence of splittings $U \to \widetilde{G}$ and $K \to \widetilde{G}$.

Notation 378. We define the Hecke algebra $\mathcal{H}(\widetilde{G}, K)$ to consist of smooth compactly supported functions f on $\widetilde{G}$ for which

$$f(\varepsilon k_1 g k_2) = \varepsilon f(g)$$

for any root of unity ε.

Notation 379. We define the Whittaker space as follows. Let $(\Lambda^\vee, \Phi^\vee, \Lambda, \Phi)$ be the root datum of G. Set

$$\widetilde{\Lambda}^\vee := \{\lambda \in \Lambda^\vee : B(\lambda, \mu) \equiv 0 \pmod{n} \text{ for } \mu \in \Lambda^\vee\}$$

and $\widetilde{\alpha}_i = n_i \alpha_i$ where n_i is the smallest such that $n_i W(\alpha_i^\vee) \equiv 0 \pmod{n}$. Define $\widetilde{\Lambda}$ and $\widetilde{\alpha}$ analogously, and we let $G_{(Q,n)}$

Example 380. Suppose G is simply connected, and choose Q so that $Q(\alpha_i) = 1$ for each simple root α_i. Then

$$G_{(Q,n)}^\vee = \begin{cases} G^\vee & \text{if } n \text{ is odd,} \\ G & \text{if } n \text{ is even.} \end{cases}$$

One can see this explicitly in some cases; for example, type B and C may swap as long roots become short ones.

> **Theorem 381.** There are isomorphisms
> $$\mathcal{H}(\widetilde{G}, K) \cong \mathbb{C}[\widetilde{\Lambda}^\vee]^W \cong K_0 \operatorname{Rep} G^\vee_{(Q,n)}.$$

It is an observation of Lurie that the correct object to put on the other side of $\mathcal{W}_\psi(\widetilde{G}, K)$ is a quantum group.

32.3 Quantum Groups

By introducing some qs to the Serre relations, one can quantize $U\mathfrak{g}$ to $U_q\mathfrak{g}$, which produces a quantum group. Roughly speaking, the representation theory of $U\mathfrak{g}$ is an abelian symmetric monoidal category, but $U_q\mathfrak{g}$ is an abelian braided monoidal category.

Once we have q in hand, there are a few ways to specialize ξ. The standard plugging in yields $U_\xi^{\mathrm{KC}}\mathfrak{g}$. If we take some divided powers (replacing e_i with $e_i/[k]_q!$ but none for the f_is), then we get $U_\xi^{\mathrm{Lus}}\mathfrak{g}$. It turns out that the representation theory of $U_\xi^{\mathrm{Lus}}\mathfrak{g}$ looks like the representation theory of $G(k)$ in characteristic p when k has characteristic p.

For example, $\operatorname{Rep} U_\xi^{\mathrm{Lus}}\mathfrak{g}$ is category with some highest weight theory, and one can do Kazhdan–Lusztig combinatorics. Let's explain how to build some representations. Start with the representation theory of $G(\mathbb{C})$, and we can build some irreducible representations V_λ. Then V_λ is in fact defined integrally, so one can extend it to $U_q\mathfrak{g}$ and then specialize it to $U_\xi^{\mathrm{Lus}}\mathfrak{g}$. The representation won't be irreducible after the specialization, but there is still some combinatorics with filtrations that one can do.

Returning to our metaplectic groups, there is quantum Frobenius

$$q\mathrm{Fr}\colon \operatorname{Rep} U\mathfrak{g}^\vee_{(Q,n)} \to \operatorname{Rep} U_\xi\mathfrak{g}^\vee,$$

which we will not discuss. Let's take stock.

- Note $\mathcal{H}(\widetilde{G}, K)$ has a canonical basis h_K given by the $\mu_n KgK$. Similarly, $\mathcal{W}_\psi(\widetilde{G}, K)$ has a canonical basis given p-adically.

- Note $K_0 \operatorname{Rep} G^\vee_{(Q,n)}$ has a canonical basis given by the $[V_\lambda]$s. Similarly, $K_q \operatorname{Rep} U_\xi\mathfrak{g}^\vee$ admits a basis from the λs described above.

Our main theorem says that there is a way to go between the various bases listed in the "similarly" sentences. There is some precise formula involving the quantum Frobenius, some Ext-groups, and there are some strange Gauss sums.

33 November 12: Counting Solutions to Generalized Fermat Equations

This talk was given by Santiago Arango-Piñerosat at Harvard for the Harvard number theory seminar.

33.1 The Statement

To start off, supposed that we are interested in counting rational numbers Q such that the numerator is an ath power, the numerator of $a - 1$ is a bth power, and the denominator is a cth power.

To make this precise, we consider Q as a point in $\mathbb{P}^1(\mathbb{Q})$, which is represented by a coprime pair $(s, t) \in \mathbb{Z}^2$ up to sign. The numerator is s, the denominator is t, and the numerator of $Q - 1$ is $s - t$; let $\Omega(a, b, c)$ be the set of projective points of interest. The height $\operatorname{ht} Q$ is simple $|s| + |t|$, and we will be interested in computing

$$\frac{N(\Omega(a, b, c); h)}{N(\mathbb{P}^1(\mathbb{Q}); h)},$$

where the h means we are counting up to h. Assuming inddependence, we expect this fraction to be

$$\frac{h^{1+1/a}}{h^2} + \frac{h^{1+1/b}}{h^2} + \frac{h^{1+1/c}}{h^2} \sim \frac{h^\chi}{h^2},$$

where

$$\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1.$$

Notably, if $\chi < 0$, one may expect to have finitely many solutions; when $\chi > 0$, we expect $N(\Omega(a, b, c); h)$ to be proportional to h^χ. Here is our result.

Theorem 382. Suppose that $\chi > 0$. Then there is an explicit constant κ for which $N(\Omega(a, b, c); h)$ is on the order of κh^χ.

Let's describe some previous work. Thus far, we may be interested in counting solutions to $x^a \pm y^b \pm z^c$, but we may more generally be interested in counting integral solutions to

$$F(x, y, z) = Ax^a + By^b + Cz^c.$$

Let $\Omega(F)$ be the corresponding set, and let U be the scheme cut out by this equation (over $\mathbb{Z}$).

Theorem 383 (Darmon−Granville)**.** If $\chi < 0$, then $\Omega(F)$ is finite.

Theorem 384 (Beukers)**.** The set $\Omega(F)$ can be parameterized by finitely many polynomials. Thus, if $\chi > 0$, then there are infinitely many solutions.

Theorem 385 (Ponnen−Schaefer−Stoll)**.** There is an explicit classificatin of the solutions to $x^2 + y^3 + z^7 = 0$.

In the example $x^2 + y^3 + z^7 = 0$, one finds that $\mathbb{G}_m$ acts on U by $\lambda \cdot (x, y, z) = (\lambda^{21}x, \lambda^{14}y, \lambda^6 z)$. This is not quite a $\mathbb{G}_m$-torsor, but one can still take a quotient stack and study it that way.

To generalize this, consider the subgroup $H \subseteq \mathbb{G}_m^3$ cut out by the equatino $\lambda_0^a = \lambda_1^b = \lambda_\infty^c$. Then away from a finite set of primes S, one has

$$[U/H]_R \cong \mathbb{P}^1(a, b, c)_R,$$

where $\mathbb{P}^1(a, b, c)$ is some explicit stack which is just $\mathbb{P}^1$ except it has μ_a-action at 0, μ_b-action at 1, and μ_c-action on c. Then the method of Poonen−Schaever−Stoll is able to prove our main result.

Theorem 386. Fix F as above for which $\chi > 0$ and $U(\mathbb{Z}) \neq 0$. Then there is an explicitly computable constant $\kappa(F) > 0$ such that

$$N(\Omega(F); h) = \kappa(F)h^\chi + O_{\varepsilon, F}\left(h^{\chi/2 + \varepsilon}\right).$$

The proof is done in three steps.

1. Covering: one can find $\varphi \colon \mathbb{P}^1_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q}$ which is geometrically Galois with Galois group givnen by an explicit triangle group, Beyli of signature (a, b, c) (meaning that it is ramified only at $\{0, 1, \infty\}$ with the given ramifications). For example, for $(2, 3, 3)$, one can use modular curves.

2. Twisting: one can spread φ out to integral model over some $R = \mathbb{Z}_S$. It then turns out that $\mathbb{P}^1(a, b, c)_R$ is $\left[\mathbb{P}^1_R / \operatorname{Aut} \Phi\right]$, which will let us present the points explicitly. In particular, $\Omega(F)$ embeds in

$$\bigsqcup_{\tau \in \mathrm{H}^1(\mathbb{Q}, \operatorname{Aut} \Phi_{\overline{\mathbb{Q}}})} \varphi_\tau(\mathbb{P}^1_\tau(\mathbb{Q})),$$

where φ_τ is some twist.

3. Sieving: one can explicitly compute with H^1 in order to enumerate the disjoint union. In particular, the problem reduces to counting the images of some rational maps, which reduces to the following.

Lemma 387. Let $\varphi\colon \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be a geometrically Galos map of degree d, then there exists an explicitly computable constant $\kappa(\varphi) > 0$ such that

$$N(\varphi(\mathbb{P}^1(\mathbb{Q})); h) = \kappa(\varphi)h^{2/d} + O_{\varphi,\varepsilon}(h^{1/d+\varepsilon}).$$

Sketch. This is basically done by counting points in some bounded region in the target, and then one uses the geometrically Galois hypothesis in order to count fibers. ∎

34   November 12: The Hodge Conjecture

This talk was given by Pierre Deligne at Harvard for the Millenium Prize Problems lectures.

34.1   The Conjecture

The talk began by introducing complex projective varieties, in steps. We started with the real line, completed to $\mathbb{P}^1(\mathbb{R})$, added variables to get to $\mathbb{P}^n(\mathbb{R})$, and then we passed to complex numbers. Then we introduced (Betti) homology via cycles. Cohomology was introduced using the de Rham complex.

Hodge showed that there is an action by S^1 on the de Rham cohomology, which gives rise to the Hodge filtration. In particular, being in bidegree (p, p) corresponds. Hodge conjectured that the algebraic cycles is the intersection of $\mathrm{H}^i_{\mathrm{B}}(X; \mathbb{Z})$ and being in bidegree (p, p). This turns out to be wrong, but it is still conjectured to be true up to tensoring by $\mathbb{Q}$.

Here are some difficulties.

- Proving the conjecture requires constructing cycles. For example, cutting by hyperplanes can only prove the conjecture in codimension 1. Alternatively, one can try to decompose a given reducible cycle into maybe more interesting irreducible ones. For example, the projective surface $wz = xy$ contains the curve
$$\left(s^3, s^2t, st^2, t^3\right).$$
The decomposition here turns out to be complicated.

- Here is one way to disprove: given a Hodge cycle, one can look at the corresponding class in a different cohomology theory (via comparison theorems) such as étale, and we can ask if the corresponding cycle satisfies various criteria that algebraic cycles satisfy such as the Tate conjecture. However, this turns out to not work in most situations, and it is difficult to construct Hodge cycles anyway.

In short, one cares because the Hodge conjecture would produce a well-founded theory of motives. Note that Yves Andre has built a good theory of motives in characteristic zero by passing to motivated cycles, but there is some hope to do this in positive characteristic using abelian varieties.

35   November 13: The Riemann Hypothesis

This talk was given by Xinyu Zhou at MIT for the STAGE seminar.

35.1   Reduction to the Blowup

Today, we will use the technique of Lefschetz pencils in order to prove the Riemann hypothesis part of the Weil conjectures. In short, we would like to show that the action of the Frobenius on $\mathrm{H}^r(X; \mathbb{Q}_\ell)$ has eigenvalues α with magnitude $q^{r/2}$. Note that we are allowed to extend the base field to prove this result.

By embedding X diagonally into $X \times X$ and using the Künneth formula to expand out the cohomology in middle dimension d, we see that it is enough to prove the Riemann hypothesis for $\mathrm{H}^d(X \times X; \mathbb{Q}_\ell)$. Thus,

we may assume that $\dim X = 2m + 2$, and we will prove the result for $\mathrm{H}^{m+1}(X; \mathbb{Q}_\ell)$. Additionally, by taking powers using a tensor-power trick, it is enough to merely check that each Frobenius eigenvalue α satisfies

$$q^{n/2} < |\alpha| < q^{n/2+1},$$

were $n + 1 = \dim X$.

We begin with a rather abstract result on weights.

> **Definition 388.** We say that an operator F on a vector space V satisfies W_n if and only if all eigenvalues α of F have
> $$q^{n/2} < |\alpha| < q^{n/2+1}.$$

> **Lemma 389.** Fix an operator F on a vector space V.
>
> (a) If V satisfies W_n, and $W \subseteq V$ is some F-stable subspace, then both W and V/W satisfy W_n.
>
> (b) If there is an F-stable filtration
> $$V \supseteq V_1 \supseteq \cdots,$$
> and each V_i/V_{i+1} satisfies W_n, then V satisfies W_n.

Proof. This is a linear algebra exercise. Namely, (a) follows by suitably upper-triangularizing F using W, and (b) follows similarly. ∎

We now set up some notation around Lefschetz pencils. Fix a very ample line bundle $\mathcal{L}$ on some X inducing an embedding $X \hookrightarrow \mathbb{P}\Gamma(X, \mathcal{L})$, and we know that there is a Lefschetz pencil $D \colon \mathbb{P}^1 \to \mathbb{P}\Gamma(X, \mathcal{L})^\vee$. The axis will be denoted $A = D_0 \cap D_\infty$, which is smooth, and we let X^* be the blow up of X along the axis. Thus, there is a surjection $\varphi \colon X^* \to X$ providing the blow-up, and there is a projection $\pi \colon X^* \to \mathbb{P}^1$ for which the fiber over $t \in \mathbb{P}^1$ is $X_t := X \cap D_t$.

> **Lemma 390.** Fix everything as above. It suffices to prove the Riemann hypothesis for X^*.

Proof. Let $N_{X/(A \cap X)}$ be the normal bundle of $A \cap X$ in X. It is a property of the Lefschetz pencil that

$$\varphi^{-1}(A \cap X) = \mathbb{P}N_{X/(A \cap X)}.$$

For example, with $A \cap X$ of codimension in 2, we see that the normal bundle has rank 2.

Thus, some theory of Chern classes provides a decomposition

$$\mathrm{H}^*\left(\varphi^{-1}(A \cap X); \mathbb{Q}_\ell\right) = \mathrm{H}^*(A \cap X; \mathbb{Q}_\ell) \oplus \mathrm{H}^{*-2}(A \cap X; \mathbb{Q}_\ell)(-1).$$

In short, one can choose a class $\xi \in \mathrm{H}^2(\mathbb{P}_X^n; \mathbb{Q}_\ell)(1)$ corresponding to the line bundle $\mathcal{O}(1)$, so we get a Lefschetz decomposition

$$\mathrm{H}^*(\mathbb{P}_X^n; \mathbb{Q}_\ell) = \bigoplus_{i=0}^{n} \mathrm{H}^{*-2i}(X; \mathbb{Q}_\ell)(-i)\xi^i.$$

Indeed, this decomposition can be proven by reducing to the case where X is affine, and then it follows by careful calculations of the cohomology of projective space (as one does for fields). Something similar holds for $\mathbb{P}\mathcal{E}$ even when $\mathcal{E}$ is no longer a trivial bundle, which is the requested decomposition.

For example, taking degree 0 shows that there is an isomorphism $\mathbb{Q}_\ell \to \varphi_*\mathbb{Q}_\ell$. Additionally, one finds that $\mathrm{R}^\bullet\varphi_*\mathbb{Q}_\ell$ is supported on $A \cap X$ in higher degrees (because φ is an isomorphism away from $A \cap X$, so the fibers produce some trivial cohomology by proper base change). Further, the higher direct images $\mathrm{R}^\bullet\varphi_*\mathbb{Q}_\ell$ vanishes outside degrees 0 and 2 (because the cohomology of the fiber at some $x \in A \cap X$ is cohomology of $\mathbb{P}^1$, which is supported in degrees 0 and 2). Thus, the Leray spectral sequence

$$E_2^{pq} = \mathrm{H}^p(X; \mathrm{R}^q\varphi_*\mathbb{Q}_\ell) \Rightarrow \mathrm{H}^{p+q}(X^*; \mathbb{Q}_\ell)$$

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degenerates,[2] so we get a decomposition

$$\mathrm{H}^*(X^*; \mathbb{Q}_\ell) = \mathrm{H}^*(X; \mathbb{Q}_\ell) \oplus \mathrm{H}^{*-2}(A \cap X; \mathbb{Q}_\ell)(-1)$$

from the spectral sequence. Thus, we obtain a splitting $\mathrm{H}^*(X; \mathbb{Q}_\ell) \subseteq \mathrm{H}^*(X^*; \mathbb{Q}_\ell)$, and the result follows by Lemma 389. ∎

35.2   The Three Groups

We are now reduced to X^*, which we may understand by understanding the projection $\pi\colon X^* \to \mathbb{P}^1$. Once again, we have a spectral sequence

$$E_2^{pq} = \mathrm{H}^p\left(\mathbb{P}^1; \mathrm{R}^q \pi_* \mathbb{Q}_\ell\right) \Rightarrow \mathrm{H}^{p+q}(X^*; \mathbb{Q}_\ell).$$

Note that $\mathbb{P}^1$ has cohomology supported in degrees $\{0, 1, 2\}$, so we only have $p \in \{0, 1, 2\}$. Additionally, we are only interested in $\mathrm{H}^{n+1}(X^*; \mathbb{Q}_\ell)$, so we see that we are only interested in the groups

$$\mathrm{H}^0\left(\mathbb{P}^1; \mathrm{R}^{n+1} \pi_* \mathbb{Q}_\ell\right), \quad \mathrm{H}^1\left(\mathbb{P}^1; \mathrm{R}^n \pi_* \mathbb{Q}_\ell\right), \quad \text{and} \quad \mathrm{H}^2\left(\mathbb{P}^1; \mathrm{R}^{n-1} \pi_* \mathbb{Q}_\ell\right).$$

We would like to show all three groups satisfy W_n, so W_n is satisfied by any subquotient by Lemma 389, and the result will follow for $\mathrm{H}^{n+1}(X^*; \mathbb{Q}_\ell)$.

> **Remark 391.** Weil II tells us that each of the sheaves $\mathrm{R}^\bullet \pi_* \mathbb{Q}_\ell$ have an expected weight. Thus, we see that the rest of the proof amounts to proving some special cases of Weil II. This explains why it is tricky: Weil II is genuinely hard!

Each of the these groups will be handled separately. Let's quickly handle the left and right groups.

- To handle $\mathrm{H}^2\left(\mathbb{P}^1; \mathrm{R}^{n-1} \pi_* \mathbb{Q}_\ell\right)$, we recall that our vanishing cycles live in $\mathrm{H}^n(X_t; \mathbb{Q}_\ell) = (\mathrm{R}^n \pi_* \mathbb{Q}_\ell)_t$, where $t \in \mathbb{P}^1$, so we receive a specialization map

$$\mathrm{H}^*(X_t; \mathbb{Q}_\ell) \to \mathrm{H}^*(X_\eta^*; \mathbb{Q}_\ell)$$

  which is an isomorphism. This amounts to saying that all the fibers of $\mathrm{R}^{n-1} \pi_* \mathbb{Q}_\ell$ is constant on $\mathbb{P}^1$, so we can compute its cohomology as

$$\mathrm{H}^2\left(\mathbb{P}^1; \mathrm{R}^{n-1} \pi_* \mathbb{Q}_\ell\right) = \left(\mathrm{R}^{n-1} \pi_* \mathbb{Q}_\ell\right)_t(-1) = \mathrm{H}^{n-1}(X_t; \mathbb{Q}_\ell)(-1).$$

  Now, let $Y \subseteq X_t$ be some smooth hyperplane section so that $X_t \setminus Y$ is affine, so $\mathrm{H}^{n-1}(X_t \setminus Y; \mathbb{Q}_\ell) = \mathrm{H}^{n+1}(X_t \setminus Y; \mathbb{Q}_\ell) = 0$ by Poincaré duality, so excision tells us that we have an injection

$$\mathrm{H}_c^{n-1}(X_t \setminus Y; \mathbb{Q}_\ell) \to \mathrm{H}^{n-1}(X_t; \mathbb{Q}_\ell) \to \mathrm{H}^{n-1}(Y; \mathbb{Q}_\ell).$$

  We are thus reduced to proving the statement to Y, which has smaller dimension than X, so we may induct down.

- We omit details for the calculation for $\mathrm{H}^0\left(\mathbb{P}^1; \mathrm{R}^{n+1} \pi_* \mathbb{Q}_\ell\right)$ because it is done with similar tricks. In short, one again finds that $\mathrm{R}^{n+1} \pi_* \mathbb{Q}_\ell$ is constant, and then the Gysin sequence provides a surjection

$$\mathrm{H}^{n-1}(Y; \mathbb{Q}_\ell)(-1) \to \mathrm{H}^{n+1}(X_t; \mathbb{Q}_\ell),$$

  so we are done by an induction.

We now move on to the (hardest) middle cohomology group $\mathrm{H}^1\left(\mathbb{P}^1; \mathrm{R}^n \pi_* \mathbb{Q}_\ell\right)$. Let $S \subseteq \mathbb{P}^1$ be the locus of singular fibers, and let U be the complement of S, and we distinguish some basepoint $u \in U$. For brevity, set $V := (\mathrm{R}^n \pi_* \mathbb{Q}_\ell)_u$, and we let $E \subseteq V$ be the vanishing cycles. The cup product provides a symplectic pairing $V \times V \to \mathbb{Q}_\ell(-n)$, so we may let $E^\perp \subseteq V$ be the orthogonal complement.

---

[2] It seems nontrivial to show that $d_3 = 0$, but it is true.

> **Remark 392.** The Hard Lefschetz theorem would imply that $E \cap E^\perp = 0$. However, the first proof of the Hard Lefschetz theorem was strictly harder than the proof of the Riemann hypothesis.

Without knowing that $E \cap E^\perp$ is trivial, we can still consider the filtration

$$V \supseteq E \supseteq E \cap E^\perp \supseteq 0.$$

To start off, we note that any generator σ_s in the inertia subgroup at s of $\pi_1(U)$ has

$$\sigma_s(x) = x \pm t(\sigma_s)(x \cup \delta_s)\delta_s,$$

so one can check that $\pi_1(U)$ acts trivially on V/E and $E \cap E^\perp$. As such, we may extend the above filtration into

$$\mathcal{V} \supseteq \mathcal{E} \supseteq \mathcal{E} \cap \mathcal{E}^\perp = 0$$

of sheaves on U, where $\mathcal{V} = \mathrm{R}^n \pi_* \mathbb{Q}_\ell|_U$. The aforementioned trivial action by π_1 implies that $\mathcal{V}/\mathcal{E}$ and $\mathcal{E} \cap \mathcal{E}^\perp$ are both constant sheaves. Pushing forward along $j \colon U \to \mathbb{P}^1$, we ge a filtration

$$\mathrm{R}^n \pi_* \mathbb{Q}_\ell \supseteq j_* \mathcal{E} \supseteq j_*(\mathcal{E} \cap \mathcal{E}^\perp) = 0.$$

The main lemma is applied to the non-constant quotient.

> **Lemma 393.** For each $x \in U$, the action of F_x on $\mathcal{E}/(\mathcal{E} \cap E^\perp)$ is rational. In other words, the characteristic polynomial has rational coefficients.

Proof. Omitted. ∎

From here, one applies the Main lemma to $E/(E \cap E^\perp)$. For example, the cup-product gives us our symplectic pairing, and it is a theorem of Kazhdan–Margulis that the monodromy group has open image, so we complete.

It remains to handle the constant sheaves. There are two cases: either $E/(E \cap E^\perp) \neq 0$ or $E \subseteq E^\perp$. We will focus on the first case because it is easier. It turns out that this means that there are no vanishing cycles in $E \cap E^\perp$ because having any vanishing cycles in $E \cap E^\perp$ implies that all of them are in there by the Picard–Lefschetz formula. Observe that there is an exact sequence

$$0 \to j_* \mathcal{E} \to j_* \mathcal{V} \to j_*(\mathcal{V}/\mathcal{E}) \to 0$$

of sheaves on $\mathbb{P}^1$. Observe $j_*(\mathcal{V}/\mathcal{E})$ is constant, so the map $\mathrm{H}^1\left(\mathbb{P}^1; j_*\mathcal{E}\right) \to \mathrm{H}^1\left(\mathbb{P}^1; \mathrm{R}^n \pi_* \mathbb{Q}_\ell\right)$, is surjective, so we are reduced to handling $\mathrm{H}^1\left(\mathbb{P}^1; j_*\mathcal{E}\right)$. For this, we note that we have a short exact sequence

$$0 \to j_*(\mathcal{E} \cap \mathcal{E}^\perp) \to j_* \mathcal{E} \to j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp) \to 0.$$

Again, the left sheaf is constant, so we receive an injection

$$\mathrm{H}^1\left(\mathbb{P}^1; j_*\mathcal{E}\right) \to \mathrm{H}^1\left(\mathbb{P}^1; j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp)\right).$$

Thus, we reduce to the quotient covered by the Main lemma.

36   November 17: The p-adic Simpson Correspondence

This talk was given by Oakley at Harvard for the Mark Kisin seminar.

For today, C is a complete, algebraically closed nonarchimedean field over $\mathbb{Q}_p$, we let B_{dR}^+/t^2 be the usual ring, which is a non-split extension

$$0 \to C(1) \to B_{\mathrm{dR}}^+/t^2 \to C \to 0.$$

We alo let X/C be a smooth proper rigid space, and we let $\widetilde{\Omega}_X^1$ be $\Omega_{X/C}^1(-1)$.

36.1   Complex Nonabelian Hodge Theory

Let's start by recalling the story for complex manifolds. Let (X, ω) be a compact Kähler manifold so that there is a Hodge decomposition

$$\mathrm{H}^n(X; \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^p(X; \Omega_X^q).$$

We are interested in generalizing this to non-constant local systems.

Definition 394 (Higgs bundle)**.** Fix a compact Kähler manifold X. Then a *Higgs bundle* (E, θ) on X is a holomorphic vector bundle E on X together with an $\mathcal{O}_X$-linear map $\theta \colon X \to E \otimes \Omega_X^1$ satisfying $\theta \wedge \theta = 0$.

Remark 395. One can reinterpret θ as a map $(\Omega_X^1)^\vee \to \mathrm{End}(E)$. Then commutativity condition amounts to requiring that θ factors through Sym.

Theorem 396 (Simpson, Corlette, Dondaldson)**.** Fix a smooth projective variety X over $\mathbb{C}$, and let $[\omega]$ be a hyperplane class (coming from a projective embedding). Then there is an equivalence between the category of local systems $\mathrm{Loc}_{\mathbb{C}}(X)$ and the category of semistable Higgs bundles (E, θ) for which $c_i(E)_\theta = 0$ for all Chern classes c_i.

We will ignore semistability and the Chern classes for now because we will not mention such things in the p-adic situation.

Remark 397. If $\mathcal{L}$ is some local system with associated Higgs bundle (E, θ), then $\mathrm{H}_{\mathrm{B}}^n(X; \mathcal{L})$ is isomorphic to the hypercohomoloy

$$\mathbb{H}^n\left(X; [E \xrightarrow{\theta} X \otimes \Omega_X^1 \xrightarrow{\theta} X \otimes \Omega_X^2 \to \cdots]\right).$$

36.2 p-adic Nonabelian Hodge Theory

For our main result, we need a site.

Definition 398. Fix a rigid analytic space X. Then the *proétale site* consists of cofiltered systems $\{U_i\} \to X$ where each $U_i \to X$ is étale and the internal maps are étale surjective for sufficiently large indices.

Roughly speaking, the coverings are given by jointly surjective maps, but there are some set-theoretic issues.

Remark 399. There is a morphism $j \colon X_{\mathrm{proét}} \to X_{\mathrm{ét}}$ of sites.

Example 400. There is a sheaf $\mathcal{O}_X$ on the proétale site, which comes from the étale site.

Example 401. Similarly, there is a completed sheaf $\widehat{\mathcal{O}}_X$ which is

$$\left(\lim_n j^{-1}\mathcal{O}_X^+/p^n\right)[1/p].$$

For example, this sheaf takes a perfectoid $\mathrm{Spa}(R, R^+) \to X$ to R.

Example 402. There is a sheaf $\mathbb{B}_{\mathrm{dR}}^+$ which takes perfectoid $\mathrm{Spa}(R, R^+) \to X$ to $A_{\mathrm{int}}(R^+)[1/p]^\wedge_{\ker \theta}$.

Let's begin by explaining Hodge theory over C.

Theorem 403 (Faltings, Scholze, Conrad–Gabber). Fix a smooth proper rigid analytic space X/C admits a filtration $\mathrm{Fil}^\bullet_{\mathrm{HT}} \mathrm{H}^n_{\text{ét}}(X; C)$ for which

$$\mathrm{gr}^p_{\mathrm{HT}} \mathrm{H}^n_{\text{ét}}(X; C) = \mathrm{H}^{n-p}(X; \widetilde{\Omega}^p_X).$$

Furthermore, this filtration splits after choosing a flat lift of X to B_{dR}^+/t^2.

Let's see an example.

Definition 404 (Higgs–Tate torsor). Suppose we have a lift $\mathbb{X}$ of X to B_{dR}^+/t^2; observe that $\mathbb{X}$ and X have the same topologies. Notate λ as the mapping $X_{\mathrm{proét}} \to X_{\text{ét}}$. We define $\mathcal{L}_{\mathbb{X}} \subseteq \mathrm{Hom}\left(\lambda^{-1}\mathcal{O}_{\mathbb{X}}, \mathbb{B}_{\mathrm{dR}}^+/t^2\right)$ to be given by diagrams

$$
\begin{array}{ccc}
\lambda^{-1}\mathcal{O}_{\mathbb{X}} & \dashrightarrow & \mathbb{B}_{\mathrm{dR}}^+/t^2 \\
\downarrow & & \downarrow \\
\lambda^{-1}\mathcal{O}_X & \longrightarrow & \widehat{\mathcal{O}}_X
\end{array}
$$

of sheaves of algebras over $\mathbb{B}_{\mathrm{dR}}^+/t^2$. (There is some deformation-theoretic argument which explains why this is a torsor.)

Example 405. Suppose X is a curve, and let $\mathbb{X}$ be a lift of X to B_{dR}^+/t^2. Then we are interested in defining a splitting for

$$0 \to \mathrm{H}^1(X; \mathcal{O}_X) \to \mathrm{H}^1_{\text{ét}}(X; C) \to \mathrm{H}^0(X; \widetilde{\Omega}^1_X) \to 0.$$

By some comparison theorems, the middle cohomology group is isomorphic to $\mathrm{H}^1_{\mathrm{proét}}(X; \widetilde{\mathcal{O}})$, so it is enough to define a splitting $\mathrm{H}^0(X; \widetilde{\Omega}^1_X) \to \mathrm{H}^1_{\mathrm{proét}}(X; \widehat{\mathcal{O}})$. For this, we send some $\theta \in \mathrm{H}^0$ to the fiber product $\widehat{\mathcal{O}}_X \times_\theta \mathcal{L}_{\mathbb{X}}$.

Here is our p-adic nonabelian Hodge theory, which is the p-adic Simpson correspondence.

Theorem 406 (Faltings–Hever). Let X be a smooth proper rigid space over C, and choose a lift $\mathbb{X}$ to B_{dR}^+/t^2. Then we admit an exponential $\exp \colon C \to (1 + \mathfrak{m}_C)$ and a $\otimes$-equivalence

$$\mathrm{Vect}(X_{\mathrm{proét}}; \widehat{\mathcal{O}}) \to \mathrm{Higgs}(X),$$

where $\mathrm{Higgs}(X)$ is a(n étale) Higgs bundle (defined as earlier). This equivalence is functorial in $\mathbb{X}$, and

$$\mathbb{H}^n\left(X; [E \xrightarrow{\theta} X \otimes \Omega^1_X \xrightarrow{\theta} X \otimes \Omega^2_X \to \cdots]\right).$$

Of course, the Simpson correspondence is about local systems, so we should explain how we get local systems.

Theorem 407 (Faltings, Scholze). There is a fully faithful $\otimes$-functor $\mathrm{Loc}_C(X_{\mathrm{proét}}) \to \mathrm{Vec}(X_{\mathrm{proét}}; \widehat{\mathcal{O}})$, compatible with cohomology.

The moral is that our p-adic Simpson correspondence depends on two choices: we must choose a lift $\mathbb{X}$ and an exponential map $\exp$.

36.3   The Proof for Line Bundles

We will spend the rest of the talk explaining the proof of the p-adic Simpson correspondence. Let's start with line bundles. To start, we define a Picard sheaf.

Definition 408. Fix a rigid analytic space X. Then we let $\mathrm{Rig}^{\mathrm{sm}}_{C,\text{ét}}$ to be the category of smooth rigid spaces over C, equipped with the étale topology. Then we define the *étale Picard sheaf* $\mathrm{Pic}_{X,\text{ét}}$ by

$$S \mapsto \frac{\mathrm{H}^1_{\text{ét}}(X \times S; \mathcal{O}^\times)}{\mathrm{H}^1_{\text{ét}}(S; \mathcal{O}^\times)}.$$

There is an analogously defined $\mathrm{Pic}_{X,\text{proét}}$ which uses $\widehat{\mathcal{O}}$ instead of $\mathcal{O}$.

Proposition 409. There is a short exact sequence

$$0 \to \mathrm{Pic}_{X,\text{ét}} \to \mathrm{Pic}_{X,\text{proét}} \to \mathrm{H}^0(X; \widetilde{\Omega}^1_X) \otimes \mathbb{G}_a \to 0.$$

Proof. Left-exactness comes from the Leray spectral sequence applied to the composite of sites $X_{\text{proét}} \xrightarrow{\nu} X_{\text{ét}} \to *$ applied to the sheaf $\widehat{\mathcal{O}}^\times$. (One needs to compute that $\mathrm{R}^1\nu_*\widehat{\mathcal{O}}^\times = \mathrm{R}^1\nu_*\widehat{\mathcal{O}} = \widetilde{\Omega}^1_X$, where the left equality is by using the exponential, and the right equality is due to Scholze.) Right exactness is harder, and I did not follow it. ∎

Remark 410. On Lie algebras, we get an exact sequence

$$0 \to \mathrm{H}^1(X; \mathcal{O}) \to \mathrm{H}^1_{\text{proét}}(X; \widehat{\mathcal{O}}) \to \mathrm{H}^0(X; \widetilde{\Omega}^1_X) \to 0.$$

We are now in the following situation.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}_{X,\text{ét}}(C) & \longrightarrow & \mathrm{Pic}_{X,\text{proét}}(C) & \longrightarrow & \mathrm{H}^0(X; \widetilde{\Omega}^1_X) & \longrightarrow & 0 \\
& & & & \big\uparrow & & \big\| & & \\
0 & \longrightarrow & \mathrm{H}^1_{\text{ét}}(X; \mathcal{O}) & \longrightarrow & \mathrm{H}^1_{\text{proét}}(X; \widehat{\mathcal{O}}) & \longrightarrow & \mathrm{H}^0(X; \widetilde{\Omega}^1_X) & \longrightarrow & 0
\end{array}
$$

The bottom-right arrow already admits a splitting $s_{\mathbb{X}}$ (as discussed earlier), and our goal is to show that top-right arrow admits a splitting. Thus, it is enough to exhibit the middle vertical arrow.

Theorem 411 (Warner)**.** The sheaf $\mathrm{Pic}_{X,\text{ét}}$ is representable.

It follows that $\mathrm{Pic}_{X,\text{proét}}$ is representable by the exact sequence above. Thus, we can view the top row is an exact sequence of rigid groups, and the map to the Lie algebra admits a splitting known as the exponential!

One can now compute the functor taking Higgs bundles to line bundles as given by sending the Higgs bundle (E, θ) to $\nu^*E \otimes_{\widehat{\mathcal{O}}} \mathcal{L}_\theta$. Here, $\mathcal{L}_\theta$ is $\exp(s_{\mathbb{X}}(\theta))$, where $\exp$ is the exponential map described in the previous paragraph. There is also a description of the inverse functor given by sending V to $\nu_*(V \otimes_{\widehat{\mathcal{O}}} \mathcal{L}^{-1}_\theta)$, where θ_V is the Higgs bundle corresponding to V, and it is not hard to check that these are inverse isomorphisms.

Remark 412. Once we have a splitting of

$$0 \to \mathrm{Pic}_{X,\text{ét}}(C) \to \mathrm{Pic}_{X,\text{proét}}(C) \to \mathrm{H}^0(X; \widetilde{\Omega}^1_X) \otimes \mathbb{G}_a \to 0$$

on the right, we know that there should be a splitting on the left as well. The functors defined above tell us that this splitting takes V to $\mathcal{L}_{\theta_V}$.

36.4   Gesturing towards Higher Rank

We now give some ideas of how to generalize this story to higher rank.

Notation 413. Fix a Higgs bundle (E, θ) on X, and let B denote the image of $\theta\colon \mathrm{Sym}(\widetilde{\Omega}^1_X)^\vee \to \mathrm{End}(E)$. Then B is a coherent $\mathcal{O}_X$-algebra, so we set $X' := \mathrm{Spa}\, B$ and gain a finite map

$$\pi\colon X' \to X.$$

Remark 414. If (E, θ) admits an action by $\pi_* \mathcal{O}_{X'}$, then E becomes a coherent sheaf on X'.

Remark 415. If π is étale, then E will lift to a line bundle on X'. Thus, we are able to apply the line bundle case to produce a p-adic Simpson correspondence. However, when π is not étale, then π_* does not preserve being vector bundles on the proétale site. In fact, π_* does not even preserve coherence!

We would like to upgrade the argument in the previous remark to work even if π fails to be étale. The solution is to write the same proof but base everything on X. For example, one defines a more general Picard functor for a coherent algebra B on X by

$$\mathrm{Pic}_{B/X, \text{ét}}\colon S \mapsto \frac{\mathrm{H}^1_{\text{ét}}X \times S; B^\times)}{\mathrm{H}^1_{\text{ét}}(S; \mathrm{pr}_{2*} B^\times)}.$$

All the arguments now upgrade: there is a short exact sequence of sheaves as before, and then we want to induce some splitting from group sheaves to their Lie algebras, which is done by choosing an exponential. However, this time the Picard functor is not known to be representable, so one has to work harder to define the exponential maps (e.g., by rigidifying all line bundles). There is a description of the functor $\mathrm{Higgs}(X) \to \mathrm{Vect}(X_{\text{proét}}; \widehat{\mathcal{O}})$ as before. It turns out that describing the inverse functor is harder. Roughly speaking, the problem is the construction of a B when given a vector bundle, but there is a way to do this.

37   November 18: p-adic Hyperboliciy of Shimura Varieties

This talk was given by Ananth Shankar at MIT for the BC–MIT number theory semianr. This is joint work with Ben Bakker, Oswal, and Yao.

37.1   The Complex Story

Let Y be a complex modular curve. Then the connected components of Y all look like $\mathbb{H}/\Gamma$, where Γ is some arithmetic subgroup.

Theorem 416 (Picard)**.** Suppose that $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ is torsion-free. Then every holomorphic function $f\colon D^\times \to Y$ extends to $D \to Y^{\mathrm{BB}}$, where $(-)^{\mathrm{BB}}$ denotes the Bailey–Borel compactification.

Proof. Given a correspondence

$$\begin{array}{ccc} & \widetilde{Y} & \\ \swarrow & & \searrow \\ Y & & Y' \end{array}$$

where each arrow is finite étale, one can pass the theorem from Y to Y'. Using correspondences like this, we may pass to $\Gamma = \Gamma_2$. Then $Y_{\Gamma_2} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Picard's theorem then tells us that $f\colon D^\times \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ has at most poles at 0, so we may extend to D. ∎

We are looking for generalizations to Shimura varieties S. The fact we will need is that Shimura varieties have connected components which look like X/Γ, where X is a Hermitian symmetric domain, and Γ is some congruence subgroup.

Theorem 417 (Borel)**.** If Γ is torsion-free, then every holomorphic map $(D^\times)^a D^b \to S$ extends to $D^a \times D^b \to S^{\mathrm{BB}}$.

Corollary 418. Let V be a complex algebraic variety. Then every holomorphic map $V \to S$ is algebraic.

Proof. Use GAGA. ∎

Remark 419. Both properties fail when S is replaced by an arbitrary variety. Roughly speaking, the problem with extension is the existence of essential singularities. For example, we can consider the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

We are thus motivated to study the two following properties.

Definition 420. A variety S satisfies *extension* if and only if any map $(D^\times)^a \times D^b \to S$ extends to a map $D^a \times D^b \to \overline{S}$, for some completion $\overline{S}$. If S is algebraic, it then satisfies *algebraicity* if and only if any map $V \to S$ from an algebraic variety is in fact algebraic.

37.2   The p-adic Statements

Work of many people (Shimura, Deligne, Borovoi, and Milne) allows one to descend a Shimura variety to be defined over a canonical number field. Thus, we can ask such extension questions over p-adic completions. Here is a main theorem.

Theorem 421. Let S be a Shimura variety of abelian type defined over E. For any prime $\mathfrak{p}$ of $\mathcal{O}_E$ and complete discrete valuation field K, any rigid analytic map $f\colon (D^\times)^a \times D^b \to S$ defined over K extends to a map $f\colon D^a \times D^b \to S^{\mathrm{BB}}$.

The idea is to reduce to $\mathcal{A}_g$. For example, this grants us access to integral models, and certain interesting local systems. There is also an algebraicity corollary, as before.

Corollary 422. Let S be a compact Shimura variety of Hodge type, which we embed into $\mathcal{A}_{g,6}$. Then the extension and algebraicity properties hold for the universal abelian scheme over S.

However, here is our main theorem for today.

Theorem 423. Let S be a nonabelian Shimura variety defined over E. Alternatively, we may let S be a geometric perid image. For sufficiently large primes p (relative to S), choose a prime $\mathfrak{p}$ of E above p, and fix some complete discretely valued field K containing $E_\mathfrak{p}$. Then any map $S \to D^{\times a} \times D^b \to S$ defined over K extends to a map $D^a \times D^b \to S$, provided that $\mathrm{im}\, f$ has nonempty intersection with the good reduction locus.

Observe that we did not have to compactify S for this result!

Remark 424. The good reduction locus of S means what we expect it means if S were (say) $\mathcal{A}_g$. For example, if S is compact, then the good reduction locus is everything.

Remark 425. The large prime p hypothesis is used many times in the proof. For example, integral models are only known to exist for large p.

There is once again an algebraicity corollary.

37.3   Sketch of Proofs

Let's sketch the argument for $\mathcal{A}_g$. We implicitly add enough level structure to avoid stacky issues and so on. Here are the steps.

1. One can show that any map $f \colon D^\times \to \mathcal{A}_g$ either has image contained in the good reduction locus or contained in the bad reduction locus. The point is to use the Néron–Ogg–Shafaverich criterion to test reduction ℓ-adically.

2. The various reduction strata are uniformized by Rapoport–Zink spaces. In the event that f has image contained in the good reduction locus, then one can show that it is in the image of some uniformization $j \colon \mathrm{RZ} \to \mathcal{A}_g$. (This uses integral models.)

3. Furthermore, it turns out that f lifts to $\widetilde{f} \colon D^\times \to \mathrm{RZ}$. (This also uses integral models.)

4. Taking de Rham cohomology of the universal abelian scheme, one gets a vector bundle with flat connection (V, D), which we can then pull back to $j^*(V, D)$ on $D^\times$ on RZ. It turns out that it admits a trivialization on RZ.

5. It turns out that $f^*(V, D)$ being trivial extends to D. A p-adic Riemann hypothesis then shows that $f^* T_p$ extends to 0, so we get something crystalline at 0.

6. Lastly, we use period maps to extend the map $D^\times \to \mathrm{RZ}$ and the map from D to the flag variety (given by the previous step) to a map $D \to \mathrm{RZ}$.

Most of this argument does not work for exceptional type Shimura varieties. However, it is known that one can still get a crystalline local system $\mathbb{V}_p$ on $S_{E_\mathfrak{p}}$, which then extends to a log Fontaine–Lafaille module on the integral model of the toroidal compactification. One then gets objects on the de Rham site, the crystalline site, and the ℓ-adic sites. After proceeding as in step 1 above, one proves the following general result.

Theorem 426. Let $\mathcal{S}$ be a smooth scheme over the Witt vectors W, and let $\mathcal{S}^{\mathrm{rig}}$ denote the rigid geometric fiber. Further, suppose that $\mathbb{L}$ is some crystalline local system. Assume that the associated Fontaine–Lafaille module $\mathbb{V}_{\mathrm{FL}}$ satisfies one of the following.

- The Kodaira–Spencer morphism is everywhere immersive.

- Some bundle associated to $\mathbb{V}_{\mathrm{FL}}$ is ample.

Then one has an extension property: every $f \colon D^\times \to \mathcal{S}^{\mathrm{rig}}$ extends to D.

The talk concluded with a sketch of this, but I am already lost anyway.

38   November 18: Mirror Symmetry and the Breuil–Mezard Conjecture

This talk was given by Tony Feng at MIT for the BC–MIT number theory seminar.

38.1   Modular Local Langlands

We are going to spend most of our time talking about what hte Breuil–Mezard conjecture is. Ahistorically, this is related to the p-adic local Langlands correspondence. Let G be some reductive group over $\mathbb{Q}_p$, and set $k := \overline{\mathbb{F}}_p$.

Roughly speaking, we are supposed to provide a bijection between irreducible representations of $G(\mathbb{Q}_p)$ over k and Galois representations $\mathrm{Gal}_{\mathbb{Q}_p} \to \check{G}(K)$. This was understood for tori by class field theory, and work in the early 2000s extended it to $\mathrm{GL}_2(\mathbb{Q}_p)$.

> **Remark 427.** Tony had a "hot take" that maybe this correspondence does not exist for other groups. For example, we tried to make it work for $\mathrm{GL}_2(\mathbb{Q}_{p^2})$, but we have not succeeded.

However, things look better if we work categorically.

> **Conjecture 428.** There is an emedding of derived categories
> $$D^b\left(\mathrm{Rep}_K G(\mathbb{Q}_p)\right) \hookrightarrow D^b\left(\mathrm{Coh}\,\mathcal{X}_K^{\check{G}}\right).$$
> Here, $\mathcal{X}_K^{\check{G}}$ is the moduli space of Galois representations $\rho\colon \mathrm{Gal}_{\mathbb{Q}_p} \to \check{G}(K)$. It is called the Emerton–Gee stack.

> **Example 429.** For $G = \mathrm{SL}_2$, the stack $\mathcal{X}$ looks like some union of stacky curves, possibly with nilpotent thickenings. Each stacky curve have two distinguished points, and generically the curve looks like
> $$\begin{bmatrix} \eta \chi_{\mathrm{cyclo}}^{\lambda+1} & * \\ & 1 \end{bmatrix},$$
> where λ is fixed, and η varies over some unramified characters. The two distinguished points are given by choices of η where the representation is irreducible or when it is split. The split and irreducible

However, one does not expect this equivalence to preserve t-structures. There is still no cases of the conjecture proven beyond the G listed previously.

38.2   The Breuil–Mezard Conjecture

Now, remembering that everything should be derived, we expect a functor $\mathrm{Rep}_K G(\mathbb{Q}_p) \to \mathrm{Coh}\,\mathcal{X}_K^{\check{G}}$, so we expect to have a composite
$$\mathrm{Rep}_K G(\mathbb{Z}_p) \to \mathrm{Rep}_K G(\mathbb{Q}_p) \to \mathrm{Coh}\,\mathcal{X}_K^{\check{G}}.$$
(The left map is given by co-induction.) De-categorifying, there is a homomorphism
$$K_0 \mathrm{Rep}_K G(\mathbb{Z}_p) \to K_0 \mathrm{Coh}\,\mathcal{X}_K^{\check{G}}.$$

Taking K_0 allows us to stop worrying about things being derived.

Observe that $K_0 \mathrm{Rep}_K G(\mathbb{Z}_p) = K_0 \mathrm{Rep}_K G(\mathbb{F}_p)$ is some free abelian group on the irreducible representations, and one can describe these representations using some kind of highest weight theory $(\mathrm{mod}\ p)$. Thus, the existence of such a map has no content. Instead, we want to impose some compatibility with a characteristic-0 story. In other words, we want some kind of commutativity of a diagram

$$
\begin{array}{ccc}
K_0 \mathrm{Rep}_{\overline{\mathbb{Q}}_p} G(\mathbb{Z}_p) & \longrightarrow & K_0 \mathcal{X}_{\mathbb{Q}_p}^{\check{G}} \\
\downarrow & & \downarrow \\
K_0 \mathrm{Rep}_K G(\mathbb{Z}_p) & \dashrightarrow & K_0 \mathcal{X}_K^{\check{G}}
\end{array}
$$

where the vertical arrows are $(\mathrm{mod}\ p)$. It turns out to be too hard to work with the top map, and it is too hard to construct the bottom map anyway, so we observe that there is a map $K_0 \mathcal{X}_{\mathbb{Q}_p}^{\check{G}} \to \mathrm{H}_{\mathrm{top}}(\mathcal{X}_{\mathbb{Q}_p}^{\check{G}})$, and then

we will ask for there to be a commutative diagram as follows.

$$\begin{array}{ccc} K_0 \operatorname{Rep}_{\overline{\mathbb{Q}}_p} G(\mathbb{Q}_p) & \longrightarrow & \operatorname{H}_{\mathrm{top}}(\mathcal{X}_{\mathbb{Q}_p}^{\check{G}}) \\ \downarrow & & \downarrow \\ K_0 \operatorname{Rep}_K G(\mathbb{Q}_p) & \dashrightarrow & \operatorname{H}_{\mathrm{top}}(\mathcal{X}_K^{\check{G}}) \end{array}$$

The top map will now give us constraints and be computable: it is informed by the classical local Langlands conjectures, p-adic Hodge–Tate theory, and automorphy lifting theorems.

Let's explain how the local Langlands correspondence may help us.

- On the Galois side, we have invariants given by Hodge–Tate weights $\lambda \in X_*(\check{T})^+$ and an inertia type $\tau \colon I_{\mathbb{Q}_p} \to \check{G}(\overline{\mathbb{Q}}_p)$.

- On the automorphic side, there are representations $W(\lambda) \in \operatorname{Rep}_{\overline{\mathbb{Q}}_p} G(\mathbb{Z}_p)$ and $\operatorname{LLC}(\tau) \in \operatorname{Rep}_{\overline{\mathbb{Q}}_p} G(\mathbb{Z}_p)$. (These are analogous to fixing the weight and the level.)

For example, it turns out that $W(\lambda) \otimes \operatorname{LLC}(\tau)$ should go to the connected component $[\mathcal{X}_{\lambda,\tau}] \in \operatorname{H}_{\mathrm{top}}(\mathcal{X}_{\mathbb{Q}_p}^{\check{G}})$. We are now ready to state our conjecture.

> **Conjecture 430** (Breuil–Mezard)**.** There is a map $K_0 \operatorname{Rep}_K G(\mathbb{F}_p) \to \operatorname{H}_{\mathrm{top}}(\mathcal{X}_K^{\check{G}})$ such that all λ and τ have
>
> $$\lim_{p \to 0} W(\lambda) \otimes \operatorname{LLC}(\tau) \mapsto \lim_{p \to 0} [\mathcal{X}_{\lambda,\tau}].$$
>
> Here, the left-hand limit means we are taking $\pmod p$, and the right-hand limit means that it is flat limit over all p.

> **Remark 431.** As a sanity check, we note that there are finitely many "variables": roughly speaking, we merely have to choose some Breuil–Mezard cycle $\mathcal{Z}_\sigma$ for each irreducible $\sigma \in \operatorname{Rep}_K G(\mathbb{F}_p)$. However, there are infinitely many equations given by the constraints, so one can show that there is at most one map.

> **Remark 432.** This is a confusing statement because it is simultaneously a decategorified version of the local Langlands conjecture, and it is a local version of global reciprocity. In particular, this conjecture turns out to be closely related to $R = \mathbb{T}$ theorems.

> **Example 433.** For $G = \operatorname{SL}_2$, irreducible representations of $G(\overline{\mathbb{F}}_p)$ turn out to all be reductions from characteristic 0.
>
> - One finds that $\mathcal{Z}_\lambda$ should be $\lim_{p \to 0} \mathcal{X}_{\lambda,\mathrm{triv}}$.
>
> - Alternatively, if we choose level $\Gamma_0(p)$, then we have a decomposition of the reduced representation $\operatorname{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}$, which corresponds to $\mathcal{X}_{\lambda,\tau}$ now splitting into two stacky curves.
>
> It turns out that the two stacky curves above intersect, which corresponds to the existence of some congruence $\pmod p$ between a modular form in level 1 and a modular form in level p. One can also read the ε conjecture off of these cycles.

> **Remark 434.** Let's give some history.
>
> - The p-adic local Langlands conjecture provides a proof for $G = \operatorname{GL}_2(\mathbb{Q}_p)$.
>
> - For $\lambda = (1, 0)$, one can do it for $G = \operatorname{GL}_2(K)$ by automorphy lifting methods.

Here are our main theorems.

> **Theorem 435** (Feng–Le Hung)**.** The conjecture is true for G unramified and generic λ and τ.

> **Theorem 436** (Feng–Le Hung–Lin)**.** The conjecture is true for general G and generic λ and τ.

The method of the first theorem comes from mirror symmetry. The second theorem still uses mirror symmetry but also a new input.

38.3   Mirror Symmetry

There is an analogy between $\mathbb{Q}_p$ and surfaces Σ.

- $\mathbb{Q}_p$ has an Emerton–Gee stack $\mathcal{X}$, and Σ has a character variety S. Both sides have some symplectic structure on H^1.

- The embedding $\mathcal{X}_{\lambda,\tau} \hookrightarrow \mathcal{X}^{\check{G}}$ looks like Lagrangian in the same way that the character variety S has Lagrangians.

Homological mirror symmetry morally allows one to index Lagrangians in a symplectic variety by some dual object (e.g., coherent sheaves), which explains why it may be helpful in our situation.

Even though homological mirror symmetry is mostly unknown, some known cases can help us. Relevant for us is that there is a Lagrangian skeleton (given by the affine springer fiber) embedding into some Hitchin moduli. On one side, affine springer fibers look like unions of $\mathbb{P}^1$s, so one can imagine matching them up with $\mathcal{X}_{\lambda,\tau}$s. On the other side, there is some map $\mathrm{Rep}_K G(\mathbb{F}_p) \to \mathrm{Coh}^{\mathsf{T}}(G/B)$, so one can use homological mirror symmetry to construct the required map.

Let's say a little about the ramified case because it is funny. Let G be ramiified U_{2n} so that $\mathcal{G}^{\mathrm{red}}(\mathbb{F}_p) = \mathrm{Sp}_{2n}(\mathbb{F}_p)$. Thus, we are interested in producing a map

$$K_0 \operatorname{Rep}_K \mathrm{Sp}_{2n}(\mathbb{F}_p) \xrightarrow{?} \mathrm{H}_{\mathrm{top}}(\mathcal{X}^{\mathrm{GL}_{2n}}).$$

We already have a map $K_0 \operatorname{Rep}_K \mathrm{Sp}_{2n}(\mathbb{F}_p) \to \mathrm{H}_{\mathrm{top}}(\mathcal{X}^{\mathrm{SO}_{2n+1}})$, so we are on the hunt for a map

$$\mathrm{H}_{\mathrm{top}}(\mathcal{X}^{\mathrm{SO}_{2n+1}}) \xrightarrow{?} \mathrm{H}_{\mathrm{top}}(\mathcal{X}^{\mathrm{GL}_{2n}}).$$

This is visibly (but not actually) related to twisted endoscopy, and indeed, it takes some of the same inputs as the Fundamental lemma. They called this "spectral Langlands functoriality" because it is a functoriality happening on the Galois side (whereas usual Langlands functoriality here would be happening on the automorphic side).

39   November 20: The Statement of Weil II

This talk was given by Kenta Suzuki at MIT for the STAGE seminar.

39.1   The Statement

We have spent the last many lectures showing the following.

> **Theorem 437** (Deligne)**.** Fix a smooth projective variety X over $\mathbb{F}_q$. Then each eigenvalue α of the Frobenius Frob_q acting on $\mathrm{H}^i(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ has absolute value $q^{i/2}$.

Observe that $\mathrm{H}^i(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ is $\mathrm{R}^i f_* \mathbb{Q}_\ell$, where $f\colon X \to \operatorname{Spec} \mathbb{F}_q$ is the structure morphism. Thus, we may want to find a relative analogue. Already, in the last lecture, we say that $\mathrm{H}^i(\mathbb{P}^1; \mathcal{L})$ had some controlled weights, where $\mathcal{L}$ was some local system.

Let's recall the notion of weight.

Definition 438 (weight). Fix a scheem X over $\mathbb{F}_q$ of finite type. Choose a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X and an iso-morphism $\tau \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$.

(a) Then $\mathcal{F}$ is τ-*pure of weight* w if and only if all eigenvalues α of the Frobenius acting on $\mathcal{F}_x$ have magnitude $|\tau(\alpha)| = q^{(\deg x)(w/2)}$ for all $x \in X(\overline{\mathbb{F}}_q)$.

(b) Then $\mathcal{F}$ is τ-*mixed* if and only if it admits a filtration by τ-pure.

Similarly, pure means τ-pure for all τ, and mixed means it admits a filtration of pure sheaves.

Remark 439. A priori, it seems that being τ-mixed for all τ does not imply being mixed. It is unclear if these conditions are in fact equivalent.

Example 440. The Tate twist $\overline{\mathbb{Q}}_\ell(1)$ is pure of weight -2. This sign convention appears because the geometric Frobenius is the inverse of $(-)^q$, and $(-)^q$ acts by q on $\lim \mu_{\ell^\bullet}$.

Here is our statement.

Theorem 441 (Weil II). Fix a morphism $f \colon X \to Y$ of schemes of finite type over $\mathbb{F}_q$. If $\mathcal{F}$ is a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X which is τ-mixed with weights at most n, then for any $i \geq 0$, the sheaf $\mathrm{R}^i f_! \mathcal{F}$ is τ-mixed with weights at most $n + i$.

Remark 442. Here, $f_! \mathcal{F}$ is the sheaf whose sections on an open subset U are those sections $s \in \mathcal{F}(f^{-1}U)$ with proper support. Equivalently, one can choose a compactification $j \colon X \hookrightarrow X'$ so that $f' \colon X' \to Y$ is proper, and then $f_! = f'_* \circ j_!$.

As a corollary, let's recover the Riemann hypothesis.

Example 443. If we take $Y = \operatorname{Spec} \mathbb{F}_q$, then $Rf_! \mathcal{F} = \mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}; \mathcal{F})$, which we see is mixed of weights at most $n + i$.

Example 444. If X is smooth and equidimensional of dimension d, then Poincaré duality provides a perfect pairing
$$\mathrm{H}^i(X_{\overline{\mathbb{F}}_q}; \mathcal{F}) \times \mathrm{H}^{2d-i}(X_{\overline{\mathbb{F}}_q}; \mathcal{F}^\vee) \to \mathbb{Q}_\ell(-d).$$
Taking Y to be $\operatorname{Spec} \mathbb{F}_q$ again, we find that if $\mathcal{F}$ has weights at least n, then dualizing shows that $\mathcal{F}^\vee$ has weights at most $-n$, so $\mathrm{H}_c^{2d-i}(X_{\overline{\mathbb{F}}_q}; \mathcal{F}^\vee)$ has weights at most $(2d - i) - n$, so the perfect pairing shows that $\mathrm{H}^i(X_{\overline{\mathbb{F}}_q}; \mathcal{F})$ has weights at least $n + i$.

Example 445. If X is smooth, proper, and equidimensional, and $\mathcal{F}$ is pure of weight n, then $\mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}; \mathcal{F}) = \mathrm{H}^i(X_{\overline{\mathbb{F}}_q}; \mathcal{F})$ is pure of weight n by combining the two corollaries. Notably, $\mathcal{F} = \overline{\mathbb{Q}}_\ell$ recovers the Riemann hypothesis.

39.2 Reductions

We will spend the rest of the talk giving a very sketchy outline of the proof of Weil II. Let's start with some functoriality properties.

Lemma 446. Fix a morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{F}_q$.

   (a) If $\mathcal{F}$ is pure of weight n on Y, then $f^*\mathcal{F}$ is pure of weight n on X.

   (b) Suppose f is finite. If $\mathcal{F}$ is pure of weight n on X, then $f_*\mathcal{F}$ is pure of weight n on Y.

   (c) Weight is preserved by base-change.

Proof. For (a), simply note that $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$ for each geometric point x. For (b), simply note that

$$(f_*\mathcal{F})_y = \bigoplus_{y \in f^{-1}(\{x\})} \mathcal{F}_x.$$

Lastly, for (c), one applies the argument of (b) to the morphism $X_{\mathbb{F}_{q'}} \to X_{\mathbb{F}_q}$ for any extension $\mathbb{F}_q \subseteq \mathbb{F}_{q'}$. ∎

Lemma 447. Fix a pure or mixed sheaf $\mathcal{F}$ on X. Then the same is true for any subquotient of $\mathcal{F}$.

Proof. There is nothing to say for the pure case. In the mixed case, we separate the statement into two steps.

- For any subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, one can intersect the pure filtration of $\mathcal{F}$ with $\mathcal{F}'$.
- For any quotient $\mathcal{F} \twoheadrightarrow \mathcal{F}''$, one can project the pure filtration of $\mathcal{F}$ onto $\mathcal{F}''$. ∎

Lemma 448. Given an exact sequence
$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$
of sheaves, if $\mathcal{F}'$ and $\mathcal{F}''$ are mixed, then $\mathcal{F}$ is sheaf.

Proof. By replacing $\mathcal{F}'$ with its image in $\mathcal{F}$ and $\mathcal{F}''$ with the image of $\mathcal{F}$, we may pass to a short exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$
Now, simply glue together the pure filtrations for $\mathcal{F}'$ and $\mathcal{F}''$ to build a pure filtration of $\mathcal{F}$. ∎

We now do some devissage for Theorem 441.

1. Weights are preserved by field extensions, so we may extend $\mathbb{F}_q$ at will.

2. Note that there is nothing to do if f is quasi-finite because the fibers $(f_!\mathcal{F})_x$ are give by $\bigoplus_{y \in f^{-1}(\{x\})} \mathcal{F}_y$.

3. Given a short exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
of sheaves on X, if we have the theorem for $\mathcal{F}'$ and $\mathcal{F}''$, then we have it for $\mathcal{F}$. Indeed, we simply have to note that the long exact sequence provides us with an exact sequence
$$\mathrm{R}^i f_! \mathcal{F}' \to \mathrm{R}^i \mathcal{F} \to \mathrm{R}^i \mathcal{F}'',$$
so the left and right being mixed (with the correct weights) implies the same for the middle.

4. Suppose we have an open subset $j\colon U \subseteq X$ with complement $i\colon Z \hookrightarrow X$. Then we claim that having the theorem for both $\mathcal{F}|_U$ and $\mathcal{F}|_Z$ yields the theorem for $\mathcal{F}$. To see this, one simply uses the short exact sequence
$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_! i^* \mathcal{F} \to 0,$$
so we use the previous reductino for $j^*\mathcal{F} = \mathcal{F}|_U$ and $i^*\mathcal{F} = \mathcal{F}|_Z$.

5. We may replace X and Y with their reduced subschemes because they have the same étale sites.

6. Because $\mathcal{F}$ is constructible, we see that we may replace it by some subquotients in order to assume that it is a local system.

7. Given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, if we have the theorem for both f and g, then we have it for $g \circ f$. To see this, one uses the Grothendieck spectral sequence

$$E_2^{pq} = \mathrm{R}^p g_!(\mathrm{R}^q f_! \mathcal{F}) \Rightarrow \mathrm{R}^{p+q}(gf)_! \mathcal{F}.$$

The point is that the E_2 page is τ-mixed with weights at most $n + p + q$, and then the E_∞ page is simply subquotients of these, so they also have weights at most $n + p + q$. But now the E_∞ page filters $\mathrm{R}^{p+q}(gf)_! \mathcal{F}$, so we conclude that this sheaf is still τ-mixed with weights at most $n + p + q$.

8. By Noetherian induction, one can use the previous reduction to assume that $f\colon X \to Y$ is smooth of relative dimension 1 (by fibering by curves). By some Noether normalization and other things, one can further reduce to the case that $f\colon X \to Y$ is a smooth, affine, surjective morphism whose fibers are geometrically connected irreducible smooth curves.

39.3 Sketch

To continue, we should recall the definition of a real sheaf.

> **Definition 449.** A sheaf $\mathcal{F}$ is τ-*real* if and only if each x has
>
> $$\tau \det(1 - \mathrm{Frob}T; \mathcal{F}_x) \in \mathbb{R}[t]$$
>
> for all points x.

This is fairly restrictive, but we will be able to use it.

> **Lemma 450.** Fix a local system $\mathcal{G}$ which is τ-pure of weight w. Then $\mathcal{G}$ is the direct summand of a τ-real, τ-pure local system of weight w.

Proof. Consider $\mathcal{G}^\vee = \underline{\mathrm{Hom}}(\mathcal{G}, \overline{\mathbb{Q}}_\ell)$, and choose some $b := \tau^{-1}\left(q^{2w}\right)$. Then we may define a local system $\mathcal{L}_b$ as being the Galois representation where Frob_b acts by b. Then one can take

$$\mathcal{F} := (\mathcal{G}^\vee \otimes \mathcal{L}_b) \oplus \mathcal{G}.$$

Indeed, for each eigenvalue α of $\mathcal{G}$, we get an additional eigenvalue $\overline{\alpha} = q^{4w}/\alpha$ from the left summand, so the fact that $\mathcal{F}$ is real follows. $\blacksquare$

> **Lemma 451.** Let $\mathcal{F}$ be a local system on X. If $\mathcal{F}$ is τ-real, then $\mathcal{F}$ is τ-mixed.

Proof. One uses the Rankin–Selberg method. In other words, we obtain information about $\mathcal{F}$ from $\mathcal{F}^{\otimes k}$ for large k. For sanity, we will only do this in the case that Y is a smooth affine geometrically irreducible curve and that X is a point. Observe that we may reduce to the case that $\mathcal{F}$ is irreducible, so we want to show that $\mathcal{F}$ is τ-pure.

We will need the following fact: for any real $A \in M_n(\mathbb{R})$, one can check that $\det\left(1 - (A \otimes \overline{A})T\right)^{-1}$ is a power series (in T) with nonnegative coefficients. Thus, we find that

$$\frac{1}{\tau \det\left(1 - \mathrm{Frob}_q T; \mathcal{F}_x^{\otimes 2n}\right)}$$

has nonnegative real coefficients for any positive n. On the other hand, the Lefschetz fixed point theorem tells us that

$$\prod_x \frac{1}{\tau \det \left(1 - \mathrm{Frob}_q T; \mathcal{F}_x^{\otimes 2n}\right)} = \frac{\tau \det \left(1 - \mathrm{Frob}_q T; \mathrm{H}_c^1(\mathcal{F}^{\otimes 2n})\right)}{\tau \det \left(1 - \mathrm{Frob}_q T; \mathrm{H}_c^2(\mathcal{F}^{\otimes 2n})\right)}.$$

Now, the right-hand side absolutely converges for $|t| \geq q^{1/(n\beta+1)}$, where β is an appropriately defined maximal weight. Thus, we get the same convergence for the left-hand side, which lets us bound eigenvalues on the left-hand side because each power series has nonnegative coefficients. (This is basically the same argument as Theorem 224.) ∎

40   November 24: Local Systems and the Simpson Correspondence

This talk was given by Oakley at Havard for the Mark Kisin seminar.

40.1   Higgs Bundles from Local Systems

For today, C is an algebraically closed complete nonarchimedean p-adic field. We let X be a smooth proper connected rigid space, and we needed to choose a flat lift $\mathbb{X}$ of X to $\mathbb{B}_{\mathrm{dR}}^+/t^2$ and an expoenential $\exp \colon C \to (1 + \mathfrak{m}_C)$. Last time, we exhibited some $\otimes$-equivalence

$$s \colon \mathrm{Vect}(X_{\mathrm{proét}}; \widehat{\mathcal{O}}) \to \mathrm{Higgs}(X).$$

We would like to use this to say something about local systems. Last time, we noted that there is a fully faithful embedding $\mathrm{Loc}_C(X_{\mathrm{proét}})$ into the vector bundles, so today we will be interested in finding the essential image.

Example 452. Consider $X = \mathbb{P}^1$. Then there is some étale cover $\mathcal{M} \to \mathbb{P}^1_C$ (where $\mathcal{M}$ is the moduli space of elliptic curves), and its Tate module produces some local system $\mathbb{L}$. It turns out that this local system is not semistable. The point is that the Higgs bundle (E, θ) turns out to have $E = \mathcal{O}(1) \oplus \mathcal{O}(-1)$, bwhere we have a map $\theta \colon E \to \mathcal{O}(-1)$ because $\mathcal{O}(-1) = \mathcal{O}(1) \otimes \widetilde{\Omega}^1 = E \otimes \widetilde{\Omega}^1$. However, there is a disagreement in the slopes.

Remark 453. This is basically a minimal computable example: semistability is automatic in rank 1, and examples are difficult to compute in the non-proper situation.

Thus, we see that we are looking at too many local systems, so we restrict our category to

$$\mathrm{Rep}_C \pi_1(X; \overline{x}) \subseteq \mathrm{Loc}_C(X_{\mathrm{proét}}).$$

Note that these more or less coincide in the complex analytic setting, but they are different in the rigid analytic setting because the left-hand side is using the étale fundamental group, which requires our local system to trivialize on a finite étale cover.

Corollary 454. There are equivalences

$$\mathrm{Rep}_C^{\mathrm{uni}} \pi_1(X; \overline{x}) \cong \mathrm{Loc}_C^{\mathrm{uni}} X_{\mathrm{proét}} \cong \mathrm{Vect}^{\mathrm{uni}}(X_{\mathrm{proét}}; \widehat{\mathcal{O}}) \cong \mathrm{Higgs}^{\mathrm{uni}}(X),$$

where the unipotent decoration means that we are looking at extensions of the trivial object.

Proof. There are functors from left to right, and they all preserve H^1. ∎

Anyway, here is our answer to the essential image question.

Lemma 455. The inclusion
$$\mathrm{Rep}_C \pi_1(X; \overline{x}) \hookrightarrow \mathrm{Vect}(X_{\mathrm{proét}}; \widehat{\mathcal{O}})$$
has essential image given by the vector bundles which are trivialized on a profinite étale cover.

Proof. Let's just give the two functors. Given a representation ρ, one produces the natural vector bundle $\mathbb{L}_\rho \otimes \widehat{\mathcal{O}}$. Conversely, given a vector bundle V, we can choose the profinite étale cover $\widetilde{X}$ and take the representation $V(\widetilde{X})$. The proof that these are inverse equivalences is a matter of tracking through descent data. ∎

Theorem 456. Fix a smooth projective curve X. Then the essential image of
$$\mathrm{Rep}_C \pi_1(X; \overline{x}) \hookrightarrow \mathrm{Higgs}(X)$$
factors through the semistable Higgs bundle (E, θ) for which $c_i(E)_{\mathbb{Q}} = 0$.

Proof. Recall that there is a short exact sequence
$$0 \to \mathrm{Pic}_{X,\mathrm{ét}} \to \mathrm{Pic}_{X,\mathrm{proét}} \to \mathrm{H}^0(X; \widetilde{\Omega}^1) \otimes \mathbb{G}_a \to 0,$$
which on π_0 gives identifications $\pi_0 \mathrm{Pic}_{X,\mathrm{ét}} = \mathrm{Pic}_{X,\mathrm{proét}} = \mathbb{Z}$ given by taking the degree.

We now claim that $\deg V = \deg s(V)$ and further that $(N, \theta_N) \subseteq (E, \theta)$ implies that there is a pro-étale vector bundle $W \subseteq V$ for which $s(W) = (N, \theta_N)$. This latter claim follows from the theorem we have. We will not show this claim.

The point is that it now suffices to show that (E, θ) is semistable of degree 0 if and only if $V = s^{-1}(E, \theta)$ is semistable of degree 0. For this, one does two cases. If $X = \mathbb{P}^1$, then all vector bundles are semistable of degree 0 anyway. Otherwise, X has some genus $g \geq 1$, and there is a Riemann–Roch argument which concludes. ∎

Remark 457. There is an extension beyond curves. The point is that checking semistability with vanishing ratinoal Chern class can be checked after pulling back to every curve. There is some technical input that we need any curve $Y \to X$ for which we have our flat lifts $Y \to \mathbb{Y}$ and $X \to \mathbb{X}$, then we need a lifted map $\mathbb{Y} \to \mathbb{X}$. This lifting is doable when $\mathbb{X}$ is smooth.

As such, we have the following conjecture.

Conjecture 458. Let $X/\mathbb{C}_p$ be smooth projective. Then there is an equivalence
$$\mathrm{Rep}_{\mathbb{C}_p} \pi_1(X; \overline{x}) \to \{\text{semistable Higgs bundles } (E, \theta) \text{ with } c_i(E)_{\mathbb{Q}} = 0\}.$$

Remark 459. The conjecture is false if we change the ground field $\mathbb{C}_p$ to any larger field. Approximately speaking, the problem occurs because the residue field of any larger C is no longer a union of finite fields.

Here are some known cases.

Theorem 460 (Heuer)**.** Let X/C be smooth proper. Then there is an equivalence between one-dimensional continuous representations C-representations of $\pi_1(X; \overline{x})$ and the category of profinite étale-locally trivial Higgs line bundles (L, θ) on $X_{\mathrm{ét}}$.

Remark 461. In fact, this latter category was shown to be equivalent to the category of Higgs line bundles (L, θ) for which L is "topologically torsion," meaning that there is a map $L^{n!} \to 1$ for some n. In the case of $C = \mathbb{C}_p$, it is then shown later that being topologically torsion is equivalent to $c_1(L)_{\mathbb{Q}} = 0$, but this is not the case when C is larger!

Theorem 462 (Heuer, …)**.** Let A/C be an abelian variety. Then there is an equivalence between the category $\mathrm{Rep}_C \pi_1(A, 0)$ and the category of profinite étale locally trivially Higgs bundles on $X_{\text{ét}}$.

Remark 463. This second category is then shown to be equivalent to the category of Higgs bundles (E, θ) which decompose into a sum of tensor products of unipotent representations and line bundles. So if $C = \mathbb{C}_p$, then one can show that this latter condition is equivalent to having vanishing rational Chern class again.

40.2   Parallel Transport

We will end the talk by discussing the p-adic parallel transport of Deninger and Werner. Our goal is to actually compute the Simpson correspondence. We will only be able to do this for $C = \mathbb{C}_p$.

Definition 464. Fix a proper scheme X over $\overline{\mathbb{F}}_p$. Then a vector bundle E on X is *strongly semistable of degree*) if and only if all the pullbacks $\mathrm{Frob}_p^{k*} E$ are semistable of degree 0.

Definition 465. Fix a formal scheme $\mathcal{X}$ over $\mathrm{Spf} \, \mathcal{O}_{\mathbb{C}_p}$. Then a vector bundle $\mathcal{E}$ on $\mathcal{X}$ is *Deninger–Werner* if and only if $\mathcal{E}_{\overline{\mathbb{F}}_p}$ is strongly semistable of degree 0.

Definition 466. Fix a proper rigid space X over $\mathbb{C}_p$ and a vector bundle E on X. Then E is *Deninger–Werner* if and only if there is a formal neighborhood $\mathcal{X}$ of X and a lift $\mathcal{E}$ of E for which $\mathcal{E}$ is Deninger–Werner.

Theorem 467 (Deninger–Werner, …)**.** Fix a smooth projective curve X_0 over $\overline{\mathbb{Q}}_p$, and set $X := (X_0)_{\mathbb{C}_p}$. Further, fix a Deninger–Werner Higgs bundles (E, θ). Then $s^{-1}(E, \theta)$ is some profinite étale locally trivial vector bundle.

Sketch. We will only discuss the $\theta = 0$ case. Here, one first shows that $\mathcal{E}/p$ is trivialized after a finite étale cover (up to some Frobenius twist). Then one needs to show this for all powers $\mathcal{E}/p^n$ are trivialized by finite étale covers by some cohomological argument using the exact sequence

$$0 \to M_n(\mathfrak{p}\mathcal{O}_X^+/\mathfrak{p}^2\mathcal{O}_X^+) \to \mathrm{GL}_n(\mathcal{O}_X^+/\mathfrak{p}^2) \to \mathrm{GL}_n(\mathcal{O}_X^+/\mathfrak{p}) \to 1.$$

The point is that all elements in $\mathrm{H}^1(M_n(\mathfrak{p}/\mathcal{O}_X^+/\mathfrak{p}^2\mathcal{O}_X^+))$ can pass from proétale cohomology to étale cohomology using a comparison theorem. Lastly, once we know that the $\mathcal{E}/p^n$s trivialize on finite étale covers, it follows that $\mathcal{E}$ trivializes on a profinite étale cover. ∎

Thus, we have explained how to produce a representation from a Deninger–Werner Higgs bundles. However, our condition on the Higgs bundle (E, θ) has only used E, so it cannot tell a complete story. It turns out that it is enough to merely be "potentially" Deninger–Werner, meaning that our vector bundle is Deninger–Werner after taking a finite cover.

> **Theorem 468** (Xu)**.** Fix a smooth projective curve X_0 over $\overline{\mathbb{Q}}_p$, and set $X \coloneqq (X_0)_{\mathbb{C}_p}$. Then there is an equivalence of categories between $\mathrm{Rep}_{\mathbb{C}_p} \pi_1(X; \overline{x})$ and potentially Deninger–Werner Higgs bundles.

> **Remark 469.** These arguments crucially use the fact that we are over $\mathbb{C}_p$. The fact that we start over $\overline{\mathbb{Q}}_p$ seems not so necessary.

41   November 25: Arithmetic of Fourier Coefficients

This talk was given by Naomi Sweeting at MIT for the number theory seminar.

41.1   Quaternionic Modular Forms

We are going to talk about G_2 eventually, but let's start with GL_2, where the story is Waldspurger's formula.

> **Theorem 470.** Fix a modular form f of weight $2k$ and trivial central character. Then we consider is a Shimura lift φ of weight $k + 1/2$ (on the group $\widetilde{\mathrm{SL}}_2$); let its Fourier expansion be $\varphi = \sum_n a_n q^n$. Then for squarefree D, we have
> $$|a_D|^2 = L(1/2, f \otimes \chi_D).$$

What is interesting here is that a single object φ remembers all quadratic twists of a single modular form.

We are interested in an analogous story where $\widetilde{\mathrm{SL}}_2$ is replaced with G_2. As before, we choose a modular form f, and it admits a lift $\varphi = \mathrm{GG}(f)$, which is a "quaternionic" modular form (on the group G_2). (Such an object is predicted by Langlands functoriality.) Let's say something about quaternionic modular forms.

> **Definition 471.** Fix a reductive group G over $\mathbb{Q}$. Then we define $\mathcal{A}(G) \coloneqq L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}}))$, and an *automorphic representation* is an irreducible $G(\mathbb{A}_{\mathbb{Q}})$-submodule of $\mathcal{A}(G)$.

> **Remark 472.** It turns out that any automorphic representation π admits a tensor decomposition $\pi = \bigotimes_v \pi_v$.

> **Remark 473.** Some of these components are more important than others; for example, π_∞ predicts "arithmeticity" (e.g., whether we expect to find a Galois representation, integrality properties, etc.). The holomorphic discrete series are the best-behaved in this sense. Note that holomorphic discrete series do not always exist: one needs $G(\mathbb{R})/K$ to admit a G-invariant complex structure (where $K \subseteq G(\mathbb{R})$ is a maximal compact). This is notably equivalent to admitting a Hermitian symmetric domain, and it is related to the (non)existence of Shimura varieties.

Notably, G_2 does not admit holomorphic discrete series, but there is still a distinguished family of "quaternionic" discrete series $\{\pi_k\}_{k \geq 1}$. We are now allowed to give the following definition, which is due to Pollack.

> **Definition 474.** A function $\varphi \colon G_2(\mathbb{Q}) \backslash G_2(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ is a *quaternionic modular form* of weight k if and only if φ generates the lowest K-type of π_k under the $G_2(\mathbb{R})$-action. Here, $K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\}$ is the maximal compact subgroup of $G_2(\mathbb{R})$; the lowest K-type is then trivial on one of the $\mathrm{SU}(2)$s and is k-dimensional on the other.

Remark 475 (Gan–Gross–Savin)**.** A quaternionic modular form φ (of appropriate level) admit a Fourier expansion whose coefficients $c_A(\varphi)$ are indexed by totally real cubic rings A. Here, a "totally real cubic ring" A is a ring which is free of rank 3 over $\mathbb{Z}$ and satisfying $A \otimes_\mathbb{Z} \mathbb{R} = \mathbb{R}^3$.

Theorem 476 (Pollack)**.** There is basis of quaternionic modular forms with Fourier coefficients in $\mathbb{Q}^{\mathrm{ab}}$.

This is an amazing result: the analogous result for modular forms uses a moduli description of modular curves to tease out some integrality. Anyway, the proof of the above theorem uses an exceptional theta lift from F_4. Integrality properties are also unknown.

We are now ready to state our desired conjecture.

Conjecture 477 (Gross)**.** Fix a modular form f of level 1, and suppose that we have found a suitable lift $\varphi = \mathrm{GG}(f)$. Then

$$|c_A(\varphi)|^2 = L(1/2, f \otimes V_E) \operatorname{disc}(A)^{k-1/2},$$

where A is a maximal order in some field $E/\mathbb{Q}$, and V_E is defined via the decomposition $\operatorname{Ind}_E^\mathbb{Q} 1 = V_E \oplus 1$.

Here is our main result.

Theorem 478 (Bakic–Horeax–Li-Huerta–S)**.** If f is a CM form and $L(1/2, f) \neq 0$, then a variant of Conjecture 477 holds.

Note that we are only proving a variant because the level of a CM form is always exceeding 1.

41.2   Packet Structure

For our result, we choose f coming from a Hecke character χ over a quadratic imaginary K. Then Arthur's conjecture predicts that f_χ only lifts to a packet of automorphic representations. Here, a packet consists of some family of automorphic representations whose local components are the same for all but finitely components. Note that this is a rather subtle issue: frequently, just changing a single place very rarely will actually produce another automorphic representation.

Now, for each place v, we define π_v^+ to be the local representation of $G_2(\mathbb{Q}_v)$ arising from local functoriality. Then we will be able to modify our lifts at some finite set of places S consisting of the places v which are non-split in K and satisfying $\chi_v^2 \neq 1$; here, there is an additional possible representation π_v^-, understood somewhat explicitly. Here is what Arthur's conjecture says in this situation

Conjecture 479 (Arthur)**.** Fix notation as above. Then the lift $\mathcal{A}_{\mathrm{GG}}(f_\chi) \subseteq \mathcal{A}(G_2)$ is expected to be

$$\mathcal{A}_{\mathrm{GG}}(f_\chi) = \bigoplus_\varepsilon \pi^\varepsilon,$$

where ε is some tuple of signs $S \to \{\pm\}$ for which $\prod_v \varepsilon_v = \varepsilon(1/2, \chi)\varepsilon(1/2, \chi^3)$. Here, $\pi^\varepsilon = \bigotimes_{v \notin S} \pi_v^+ \otimes \bigotimes \pi_v^{\varepsilon_v}$.

Remark 480. This is known by previous work (with the same authors) when $L(1/2, f_\chi) \neq 0$ and K is unramified at 2.

Remark 481. Here are a few remarks about the local behavior.

- If π_∞^- is the quaternionic discrete series, then π_∞^+ is not. Thus, we will need to take $\varepsilon_\infty = -1$ to have Fourier coefficients.

- For $\varphi \in \pi^\varepsilon$, then $c_A(\varphi) = 0$ unless $\varepsilon_\ell = \varepsilon(A \otimes \mathbb{Q}_\ell, \chi_\ell)$ for all $\ell \in S$.

Here is a more precise version of our main result.

Theorem 482. Assume that $L(1/2, f_\chi) \neq 0$, and choose $\varepsilon = (\varepsilon_v)_{v \in S}$ for which $\varepsilon_\infty = -1$ and $\prod_v \varepsilon_v = \varepsilon(1/2, \chi^3)$. Then there is a quaternionic modular form $\varphi \in \pi^\varepsilon$ satisfying the following: for all $A = \mathcal{O}_E$ for totally real étale algebras $E/\mathbb{Q}$, we have

$$|c_A(\varphi)| = \begin{cases} L(1/2, f \otimes \chi_E) \operatorname{disc}(A)^{k-1/2} & \text{if } \varepsilon_\ell = \varepsilon(A \otimes \mathbb{Q}_\ell, \chi_\ell), \\ 0 & \text{otherwise.} \end{cases}$$

41.3   Proof Sketch

The construction of our lifts will come from exceptional theta correspondences. This is basically a correspondence Θ for automorphic representations from one group G to another group H; such a thing exists only when (G, H) forms a dual reductive pair, meaning that there is a map $H \times G \to \widetilde{G}$ and some θ functions, and these θ-functions are then used as integral kernels.

Example 483. Classically, one takes $\widetilde{G} = \widetilde{\mathrm{Sp}}_{2n}$ and $H \times G = \mathrm{Sp} \times \mathrm{O}$ or $\mathrm{U} \times \mathrm{U}$.

Example 484. In the exceptional case, one has finitely many options in the split $\widetilde{G}$ case.

- There is $G_2 \times \mathrm{PGL}_3 \to E_6$.

- There is $G_2 \times \mathrm{PGSp}_6 \to E_7$.

- Lastly, there is $G_2 \times F_4 \to E_8$.

Note that the classical lifting embeds $\mathrm{PGL}_2 \to \mathrm{PGSp}_6$, and then one takes a Θ-lift to G_2. This is a complicated lifting process.

Here is the crux of our calculation.

Idea 485. For this talk, we will use $G_2 \times \mathrm{PU}_3 \to E_6'$, where E_6' is some quasi-split twist.

In particular, starting with f_χ arising from a Θ-lift along $(\mathrm{U}(1), \mathrm{PU}(3))$, and then another Θ-lift brings us to G_2. This is easy enough to do computations with, which is why we are able to prove the theorem.

For example, let's explain how we will approach our Fourier coefficients, following Gan–Gross–Savin. As usual, choose a parabolic P, which we give the Levi decomposition $P = MN$; here, $M = \mathrm{GL}_2$. Given some $\varphi \in \mathcal{A}(G_2)$, we define the Fourier coefficient $c_\lambda(\varphi)$ as

$$c_\lambda(\varphi) = \int_{[N]} \varphi(n) \overline{\psi_\lambda(n)} \, dn.$$

Notably, characters ψ_λ of N turn out to by $\lambda \in \mathbb{Q}^4$. In order to recover the story with cubic rings, one adds some level structure: notably, if φ is invariant under $P(\widehat{\mathbb{Z}})$, then $c_\lambda(\varphi)$ is supported on $\mathbb{Z}^4$, and it depends only on the ring A_λ, where A_λ is some appropriate ring. Lastly, for archimedean reasons, $\varphi_\lambda(\varphi) =)$ vanishes unless A_λ is totally real.

Idea 486. Periods for $\Theta(\pi)$ will frequently reduce to periods for π.

The reasoning here is that Θ arises from integral kernels, so periods (which are integrals) can frequently be unwound to the other group. In particular, for φ as constructed using Θ-lifting as above from f_χ, one finds that

$$c_\lambda(\varphi) \sim \int_{[T_E]} \rho(t) \, dt,$$

where $E = A_\lambda \otimes_\mathbb{Z} \mathbb{Q}$, and T_E is a chosen torus in PU_3 from E, and ρ is the lift from χ to $\mathrm{PU}(3)$. By the doubling method, one finds that

$$\left| \int_{[T_E]} \rho(t)\, dt \right|^2 \sim L(1/2, f_\chi \otimes V_E) L(1/2, f_\chi) \Delta_E^{1/2}.$$

The various $\sim$s turn into some local integrals $I_v(\varphi_v, \lambda)$, and evaluating enough of them completes the proof of the theorem. We remark a few points about the calculation.

- The nonarchimedean integrals at $v = \ell$ away from N is only checked to be 1 when $A \otimes \mathbb{Z}_\ell$ is maximal, which explains our maximality condition in the theorem.

- The achimedean integrals use explicit Whittaker models.

- The nonarchimedean integrals at $v = \ell$ dividing N are only computed at specified test vectors φ_ℓ.

42   December 1: Pan's Proof of a Classicality Result

This talk was given by Hang at Harvard for the Mark Kisin seminar.

42.1   The Fontaine–Mazur Conjecture

For today, we fix a prime p. For a field F, we let G_F be the absolute Galois group of F. We also let $B \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$ be the upper-triangular Borel subgroup, and we let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$, and $\mathfrak{h}$ be the usual Lie subalgebras for $\mathfrak{g} = \mathfrak{gl}(2)$.

Our story starts with the Fontaine–Mazur conjecture.

Conjecture 487. Let $\rho \colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous irreducible representation, and assume that the following conditions are satisfied for some nonnegative integer k.

(a) Unramified: $\rho|_{G_{\mathbb{Q}_\ell}}$ is unramified for almost all ℓ.

(b) Odd: $\det \rho(c) = -1$, where c is the conjugacy class of complex conjugation.

(c) Weights: $\rho|_{G_{\mathbb{Q}_p}}$ is de Rham of Hodge–Tate weights $(0, k)$.

Then ρ arises from an eigenform of weight k.

Remark 488. A lot is known about this conjecture, due to many people.

Here is an approach due to Emerton.

1. Given conditions (a) and (b), show that ρ comes from a p-adic modular form.

2. Then prove a classicality result to show that ρ comes from an actual modular form when given (c). Emerton shows this using some local-global compatibility result.

Today, we are going to try to redo the second step without having to appeal to the Langlands conjectures. As such, let's try to be precise about what was proven.

Definition 489 (completed cohomology). Fix a level $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$. Then we define the *modular curve* to be
$$Y_K := \mathrm{GL}_2(\mathbb{Q}) \backslash (\mathcal{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)) / K.$$
Its natural compactification is denoted X_K. We then define the *completed cohomology* at p to be
$$\widetilde{\mathrm{H}}^i(K^p; \mathbb{Q}_p) := \left(\lim_n \operatorname*{colim}_{K_p} \mathrm{H}^i(Y_{K^p K_p}(\mathbb{C}); \mathbb{Z}/p^n\mathbb{Z}) \right) \left[\frac{1}{p} \right],$$
and $\widetilde{\mathrm{H}}^i(K^p; \mathbb{C}_p) := \widetilde{\mathrm{H}}^i(K^p; \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$. When no confusion is possible, $\widetilde{\mathrm{H}}^1(\mathbb{C}_p) := \operatorname*{colim}_{K^p} \widetilde{\mathrm{H}}^1(K^p; \mathbb{C}_p)$.

Remark 490. If we took the limit over n before the colimit over K_p, then we would get the cohomology of Y_{K^p} with coefficients in $\mathbb{Z}_p$, which sees less than the given definition.

Remark 491. The cohomology group $\widetilde{\mathrm{H}}^i(K^p; \mathbb{C}_p)$ admits actions by $\mathrm{GL}_2(\mathbb{Q}_p)$, the Galois group, and the Hecke algebra (of level K^p).

Here is our classicality result, due to many people.

Theorem 492. Fix an irreducible continuous representation $\rho \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$, where E is a finite extension of $\mathbb{Q}_p$. Suppose that
$$\mathrm{Hom}_{G_{\mathbb{Q}}} \left(\rho, \widetilde{\mathrm{H}}^1(K^p, \mathbb{Q}_p) \otimes E \right) \neq 0.$$

(a) If $\rho|_{G_{\mathbb{Q}_p}}$ is Hodge–Tate of weights $(0,0)$, then ρ arises from an eigenform of weight 1.

(b) If $\rho|_{G_{\mathbb{Q}_p}}$ is Hodge–Tate of weights $(0,k)$, then ρ arises from an eigenform of weight $k+1$.

Remark 493. The first case is separated because it is largely due to different people, and it is a bit easier because the underlying method (inputting p-adic Hodge theory into the Taylor–Wiles method) is easier.

Remark 494. It is potentially surprising that we have found eigenforms of high weight in the cohomology of modular curves. What is going on here is that we are working with completed cohomology, which allows us to work in p-adic families, and it turns out that modular forms of high weight can be p-adically approximated by modular forms of weight 2.

We will explain a new proof of this theorem.

42.2   Hodge–Tate Period Maps

Our story begins with the following result of Scholze, allowing us to work at infinite level.

Theorem 495 (Scholze)**.** Fix some K^p, and let $\mathcal{X}_{K^p K_p}$ be the adic space associated to $X_{K^p K_p, \mathbb{C}_p}$.

(a) Then there is a unique perfectoid space $\mathcal{X}_{K^p}$ over $\mathbb{C}_p$ for which

$$\mathcal{X}_{K^p} = \varprojlim_{K_p} \mathcal{X}_{K^p K_p}.$$

(b) There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant "Hodge–Tate" period map

$$\pi_{\mathrm{HT}} \colon \mathcal{X}_{K^p} \to \mathrm{Flag}\left(\mathbb{Q}_p^2\right).$$

Further, π_{HT} is affine and Hecke-equivariant.

Remark 496. We will say nothing about the proof, but we will recall the definition of π_{HT}. For any elliptic curve E, there is a Hodge–Tate filtration

$$0 \to \mathrm{Lie}\, E^\vee(\mathbb{C}_p) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p E, \mathbb{C}_p) \to (\mathrm{Lie}\, E)^*(\mathbb{C}_p)(-1) \to 0.$$

On the modular curve, this then idnuces a filtration on $\mathcal{X}_{K^p}$ given by

$$0 \to \omega_{K^p}^{-1} \to V \otimes \mathcal{O}_{\mathcal{X}_{K^p}} \to \omega_{K^p}(-1) \to 0,$$

whre $V = \mathbb{Q}_p^2$, and ω_{K^p} is the pullback of the Hodge bundle to $\mathcal{X}_{K^p}$. The position of $\omega_{K^p}^{-1}$ in $V \otimes \mathcal{O}_{\mathcal{X}_{K^p}}$ at a given point in $\mathcal{X}_{K^p}$ defines π_{HT}.

The use $\mathcal{X}_{K^p}$ for our purposes is the following interpretation of completed cohomology.

Theorem 497 (Scholze)**.** There are isomorphisms

$$\widetilde{\mathrm{H}}^i(K^p; \mathbb{C}_p) \cong \mathrm{H}^i(\mathcal{X}_{K^p}; \mathcal{O}_{\mathcal{X}_{K^p}}) \cong \mathrm{H}^i\left(\mathrm{Flag}\, \mathbb{Q}_p^2; \pi_{\mathrm{HT}} \mathcal{O}_{\mathcal{X}_{K^p}}\right)$$

preserving all the various actions.

Remark 498. The left map arises from some almost purity result, apparently. The right map arises from some spectral sequence, which immediately degenerates because π_{HT} is affine.

We are almost ready to state our main result, but we need a little notation.

Notation 499. Let $\mathcal{B}$ be the flag variety $\mathbb{P}^1 = B \backslash \mathrm{GL}_2$. For $x \in \mathcal{B}(\mathbb{C}_p)$, we set $B_x := x^{-1} B x$, and we define $\mathfrak{b}_x$ and $\mathfrak{n}_x$ accordingly. We then define

$$\mathfrak{g}^\circ := \mathcal{O}_{\mathcal{B}} \otimes \mathfrak{g},$$

and we define $\mathfrak{b}^\circ \subseteq \mathfrak{g}^\circ$ and $\backslash^\circ$ accordingly.

Our first main result computes completed cohomology on locally analytic vectors.

Theorem 500 (Pan)**.** For an admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation V over an adic space, we let V^{la} denote the locally analytic vectors for $\mathrm{GL}_2(\mathbb{Q}_p)$.

(a) We have $\mathrm{H}^i(\mathbb{P}^1; \mathcal{O}_{K^p}^{\mathrm{la}}) \cong \widetilde{\mathrm{H}}^i(K^p; \mathbb{C}_p)^{\mathrm{la}}$.

(b) The algebra $\mathfrak{n}^\circ$ annihilates $\mathcal{O}_{K^p}^{\mathrm{la}}$.

Our second main result introduces an action.

Theorem 501 (Pan)**.** There is an action θ_h of $\mathfrak{h} = \mathrm{H}^0(\mathfrak{b}^\circ/\mathfrak{n}^\circ)$ acting on $\mathcal{O}_{K^p}^{\mathrm{la}}$ satisfying the following properties.

(a) The action θ_h extends to an action by $\mathrm{Sym}(\mathfrak{sl}_2 \cap \mathfrak{h})$, which then agrees with the action by $Z(U\mathfrak{sl}_2) \subseteq U\mathfrak{g}$ under the Harish-Chandra isomorphism.

(b) For scalar $z \in \mathfrak{g}$, $\theta_h|_z$ agrees with the expected action of z.

(c) Sen operator: the operator $\theta_h(\mathrm{diag}(0,1))$ is the Sen operator on $\widetilde{\mathrm{H}}^1(K^p; \mathbb{C}_p)^{\mathrm{la}}$. (Technically, it merely agrees with the Sen operator on finite-dimensionsal Galois-invariant subspaces.)

(d) Define the character $\mu_k \colon \mathfrak{b} \to \mathbb{C}_p$ given by $\mu_k\left(\left[\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right]\right) = kd$. Then the μ_k-isotypic component of $\widetilde{\mathrm{H}}^1(\mathbb{C}_p)^{\mathrm{la}}$ admits a Hodge−Tate decomposition

$$\widetilde{\mathrm{H}}^1(\mathbb{C}_p)_k^{\mathrm{la}} = N \oplus N',$$

where θ_h acts by $(k, 0)$ on N and by $(1, k-1)$ on N'.

Remark 502. There is also a description (via a filtration) of N and N'.

It is not too hard to prove Theorem 492(a) from here, but we will not explain the proof. Instead, we will content ourselves with giving some taste for the proof of the above main theorem. There are two main ingredients: relative Sen theory, and an explicit description of $\mathcal{O}_{K^p}^{\mathrm{la}}$. With these two ingredients, the theorem then becomes a calculation.

43 December 2: Unlikely Intersections with Orthogonality

This talk was given by Gal Binyamini at Boston College for the BC−MIT number theory seminar.

43.1 The Pila−Wilkie Theorem

Our story begins with some logic.

Definition 503 (o-minimal)**.** A structure is some nonempty family

$$\mathcal{S} \subseteq \bigsqcup_n \mathcal{P}(\mathbb{R}^n)$$

which is closed under unions, intersections, complements, linear projections, products, and containing algebraic hypersurfaces. The structure $\mathcal{S}$ is o-*minimal* if and only if each $A \in \mathcal{S} \cap \mathcal{P}(\mathbb{R})$ is a finite union of points and intervals.

Remark 504. There are many variations of this definition: one can add more closure properties (e.g., Minkowski sum), one can merely be closed under certain graphs, etc.

Remark 505. Being o-minimal places strong constraints on $\mathcal{S}$ because we can consider projections of some $A \in \mathcal{S}$ to $\mathbb{R}$.

Example 506. The structure $\mathbb{R}_{\mathrm{alg}}$ constists of the semi-algebraic subsets. It is o-minimal.

Example 507. The structure $\mathbb{R}_{\mathrm{an}}$ is the smallest structure consisting of $\mathbb{R}_{\mathrm{alg}}$ along with the graphs of analytic functions $[0,1]^n \to \mathbb{R}$. It is also o-minimal. Similarly, we define $\mathbb{R}_{\mathrm{an,exp}}$ to also be closed under the graph of the exponential $\exp\colon \mathbb{R} \to \mathbb{R}$.

Typically, one works with $\mathbb{R}_{\mathrm{an,exp}}$, but today we will have occasion to work with some larger o-minimality.
   The relevance to number theory is as follows.

Notation 508. Fix a subset $X \subseteq \mathbb{R}^n$, we define

$$X(\mathbb{Q}, H) := \{x \in X \cap \mathbb{Q}^n : H(x) \le H\}.$$

Notation 509. Fix a subset $X \subseteq \mathbb{R}^n$. Then set X^{alg} to be the union of all positive-dimensional connected semi-algebraic subsets of X. The complement $X \setminus X^{\mathrm{alg}}$ is X^{tr}.

Theorem 510 (Pila–Wilkie)**.** Suppose $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structure. For any $\varepsilon > 0$, one has

$$\#X^{\mathrm{tr}}(\mathbb{Q}, H) \lesssim_{X,\varepsilon} H^\varepsilon.$$

Remark 511. One cannot in general improve H^ε.

Remark 512. One passes to X^{tr} because semialgebraic subsets can have lots of rational points.

43.2   Unlikely Intersections

The Pila–Wilkie theorem is typically applied via "unlikely intersections." Here is one such application.

Theorem 513. Fix an algebraic subset $V \subseteq (\mathbb{C}^\times)^n$. If V contains no cosets of positive-dimensional subgroups, then $V \cap (\mathbb{C}^\times)^n_{\mathrm{tors}}$ is finite.

This result has many proofs. Here is one due to Pila–Zannier, and it generalizes nicely.

Proof. We apply Theorem 510. For technical reasons, assume that V is defined over $\mathbb{Q}$. Consider the map $\pi\colon [0,1]^n \to (\mathbb{C}^\times)^n$ defined by

$$\pi(x_1, \ldots, x_n) := (\exp(2\pi i x_1), \ldots, \exp(2\pi i x_n)).$$

This will be our "transcendental input" to Theorem 510; observe that it covers the torsion of $(\mathbb{C}^\times)^n$. Define $X := \pi^{-1}(V)$, which we note is definable in $\mathbb{R}_{\mathrm{an}}$; observe that $V[n]$ can be found in $X(\mathbb{Q}, H)$ for some H (which is a polynomial in n).
   It now turns out that $X^{\mathrm{alg}} = \varnothing$ by the Ax–Schanuel theorem; this uses the hypothesis that V contains no cosets of positive-dimensional subgroups. Now, Theorem 510 implies that

$$\#V[n] \lesssim_{V,\varepsilon} N^\varepsilon.$$

We now need to upgrade this into finiteness, which is done via Galois theory. To do this, if $p \in V[n]$, then one can show that there is some c for which

$$\# \left(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot p \right) \ge N^c.$$

Combining these two bounds implies that $V[n]$ vanishes for sufficiently divisible n, and the result follows. ∎

The sort of claim being proven above is an "unlikely intersection" problem: one has some special points and some special subvarieties, and we claim that any controlled subset avoiding the special subvarieties may only have finitely many special points. For example, this is the same sort of result occurs for Shimura varieties.

43.3   Orthogonal Polynomials

Today, we will be interested in orthogonal polynomials.

Definition 514 (orthogonal). Fix a weight function w on an interval I. A family of polynomials $\{P_n\}_{n\in\mathbb{N}}$ with rational coefficients is w-*orthogonal* if and only if $\deg P_n = n$ for each n, $P_n(1) = 1$, and

$$\int_I P_n(x)P_m(x)w(x)\,dx = 0$$

for all $n \neq m$.

Example 515 (Legendre). The polynomials $\{P_n\}$ are defined with $I = [-1, 1]$ and $w \equiv 1$.

Example 516 (Laguere). The polynomials $\{L_n\}$ are defined with $I = \mathbb{R}^+$ and $w(x) = \exp(-x)$.

Example 517 (Hermite). The polynomials $\{H_n\}$ are defined with $I = \mathbb{R}$ and $w(x) = \exp\left(-x^2\right)$.

Question 518 (Stieltjes). Consider the family of Legendre orthogonal polynomials $\{P_n\}_{n\in\mathbb{N}}$.

(a) Is it possible for P_n and P_m to share roots other than $x = 0$?

(b) Is P_n irreducible over $\mathbb{Q}$ for all n?

Remark 519. Irreducibility is known for the families $\{L_n\}$ and $\{H_n\}$.

Let's explain why this is an instance of unlikely intersections. For concreteness, we work with the polynomials $\{L_n\}$.

Theorem 520. Say that a point $\ell \in \mathbb{R}^n$ is special if and only if each of its coordinates is a root of L_n for some n, and say that a subset $V \subseteq \mathbb{R}^n$ is special if and only if cut out by the equations $x_i = x_j$ or $x_i = \ell$ for a special point ℓ. Then an algebraic variety avoiding all special subvarieties admits only finitely many special points.

The proof turns out to be essentially the same, but the Galois-theoretic input is more complicated.

Remark 521. The Galois theory could use as input that the Galois group of L_n is S_n, which provides a different proof.

One can do something for Legendre polynomials.

Theorem 522. Fix terminology as Theorem 520 but working with the Legendre polynomials $\{P_n\}$. Then there is some N for which

$$\#\{x \neq 0 : P_n(x) = P_m(x) = 0 \text{ for some } n \neq m\} \leq \log(n + m)^N$$

for any n and m.

The difficulty between $P_\bullet$s and $L_\bullet$s is having weaker Galois-theoretic input.

43.4   Orthogonal Subsets

Our next example will come from harmonic analysis.

> **Definition 523** (orthogonal)**.** Consider the disk $\mathbb{D}$. A subset $\Lambda \subseteq \mathbb{R}^2$ is *orthogonal for* $\mathbb{D}$ if and only if
> $$\int_{\mathbb{D}} e^{2\pi i (\lambda_1 - \lambda_2)\cdot v}\, dv = 0$$
> for all distinct $\lambda_1, \lambda_2 \in \Lambda$.

> **Conjecture 524** (Fuglede)**.** Consider the disk $\mathbb{D}$. Then any orthogonal subset Λ has $\#\Lambda \leq 4$.

To motivate this conjecture (and relate it to unlikely intersections), we introduce Bessel functions.

> **Definition 525** (Bessel function)**.** The Bessel function J_1 is the normalized slow-growing solution to the differential equation
> $$x^2 y'' + xy' + \left(x^2 - 1\right) y = 0.$$
> It turns out that J_1 is defined by some explicit integrals.

> **Remark 526.** There are also Bessel functions J_κ for some κ, where we replace the differential equation with
> $$x^2 y'' + xy' + \left(x^2 - \kappa\right) y = 0.$$

> **Remark 527.** It turns out that the Fourier transform of $1_{\mathbb{D}}$ is $J_1(2\pi \|\lambda\|)/\|\lambda\|$.

> **Remark 528.** It turns out that
> $$\int_{\mathbb{D}} e^{2\pi i (\lambda_1 - \lambda_2)\cdot v}\, dv = 0$$
> is equivalent to $\pi \|\lambda_1 - \lambda_2\|$ being a root of J_1. This indicates how we might later relate this to unlikely intersections.

> **Notation 529.** We define $V_{4,2} \subseteq \mathbb{R}^6$ to consist of the six-tuple of distances between four ordered points in $\mathbb{R}^2$. Note that $V_{4,2}$ is an algebraic subset.

> **Remark 530.** It now turns out that Conjecture 524 is equivalent to having no J_1-special points in $V_{4,2}$. This explains why the conjecture may be true: one does not expect to have any algebraic relations between these roots.

> **Remark 531.** It is known that orthogonal subsets Λ for the disk are finite. In fact, it is known that
> $$\#\Lambda \cap [-R, R]^2 \lesssim_\varepsilon R^{\varepsilon + 3/5},$$
> building on work of many people.

Theorem 532. Let $V \subseteq \mathbb{C}^n$ be an algebraic subset with no Bessel special subvarieties. Then the number of special points in $V \cap [-R, R]^n$ is bounded by $c(V, \varepsilon) R^\varepsilon$ for any $\varepsilon > 0$.

Remark 533. If n is odd, then one can replace R^ε by $\log R$. It is expected that one can still do this for n even, but it is more technical.

Let's end the talk by explaining how the previously given orthogonality results relate to unlikely intersections. Let's work with J_1 for concreteness.

Theorem 534. There is an o-minimal structure $\mathbb{R}_G$ (which is not $\mathbb{R}_{\mathrm{an,exp}}$) for which there is a definable map $f_B \colon \mathbb{R} \to \mathbb{R}$ such that f_B sends integers to the zeroes of J_1.

Idea. The point is that
$$J_1(x) = e^{ix} G_1(x) + e^{-ix} G_2(x),$$
where G_1 and G_2 admit some asymptotic expansions $\sum_k a_k x^{-k}$. Then we can define the graph of f_B as having those pairs (t, x) such that $0 \leq x - 2\pi t \leq 2\pi$ and
$$e^{i(x-2\pi t)} G_1(x) + e^{-(x-2\pi t)} G_2(x) = 0.$$

Notably, whenever t is an integer, we find that $J_1(x) = 0$. ∎

From here, one can argue as before using Theorem 510, pulling back V along f_B and hoping that there are no algebraic subsets. There is a similar approach which works for the Legendre polynomials, using different defining polynomials.

44   December 2: Unlikely Intersections with Orthogonality

This talk was given by Jonathan Pila at Boston College for the BC–MIT number theory seminar.

44.1   The Ax–Schanuel Theorem

The following conjecture comes from transcendental number theory.

Conjecture 535 (Schanuel)**.** Fix complex numbers $x_1, \ldots, x_n$ which are $\mathbb{Q}$-linearly independent. Then
$$\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n.$$

Example 536. With $n = 1$, this is known due to Hermite–Lindermann. It translates into the statement that $\exp(x_1)$ is trascendental when x_1 is algebraic.

Example 537. With $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$, this is known to Lindermann–Weierstrass. It translates into the statement that the $\exp(x_\bullet)$s are trascendentally independent. In particular, we see that the lower bound n is sharp.

For $n \geq 2$ and $x_\bullet$s general, very little is known.

There is a known analogue for functions.

Theorem 538 (Ax−Schanuel)**.** Let $x_1, \ldots, x_n$ be holomorphic functions on $\mathbb{D}$. Suppose that the functions $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly independent modulo constants. Then

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n + 1.$$

Example 539. The $n + 1$ is sharp. For example, one can take x_1 to be the identity function.

Remark 540. One can pass to higher-dimensional disks $\mathbb{D}$, in which case $n + 1$ gets replaced with $n + \dim \mathbb{D}$. Again, this is sharp by letting the $x_\bullet$s be coordinate functions on $\mathbb{D}$.

Remark 541. The most natural setting for this theorem works in differential fields.

Example 542. To relate this to geometry, one can imagine $(x_1, \ldots, x_n) \colon \mathbb{D} \to \mathbb{C}^n$ parameterizing some algebraic curve. In this case, $\operatorname{trdeg}_{\mathbb{C}}(x_1, \ldots, x_n) = 1$, so it follows that

$$\operatorname{trdeg}_{\mathbb{C}}(\exp(x_1), \ldots, \exp(x_n)) \geq n.$$

Example 543. Let's explain the application to Theorem 513. Suppose that we have an algebraic subvariety $C \subseteq \pi^{-1}(V)$ of positive dimension, we may as well have $\dim C = 1$. But then the exponentials of the various parameterizing functions $(x_1, \ldots, x_n)$ for C lie in V, so their transcendence degree will be small. It follows by contraposition that some $\mathbb{Q}$-linear combination of the $x_\bullet$s is a constant in $\mathbb{C}$, which translates into V containing a coset of a positive-dimensional subtorus of $(\mathbb{C}^\times)^n$!

44.2   Ax−Schanuel Input to Orthogonal Subsets

Suppose we want to prove Theorem 532. We are interested in finding Bessel-special points. Set $H_1(x) := e^{ix}G_1(x)$ and $H_2(x) := e^{-ix}G_2(x)$ so that $J_1 = H_1 + H_2$. Then define the subset $Z \subseteq \mathbb{R}^n$ to consist of those $(t_1, \ldots, t_n)$ for which there is some $(x_1, \ldots, x_n) \in V \cap (\mathbb{R}^+)^n$ satisfying the bounds $0 \leq x_j - 2\pi t_j < 2\pi$ for all j and

$$e^{-2\pi i t_j} H(x_j) + e^{2\pi i t_j} H_2(x_j) = 0$$

for all j. In particular, when t_j is an integer, it follows that x_j is an integer. Thus, the special points in V arise from points in $Z \cap \mathbb{Z}^n$, and the bounds $0 \leq x_j - 2\pi t_j < 2\pi$ turns out to put these into bijection (due to some separation of the roots of J_1). Importantly, Z is definable in $\mathbb{R}_G$.

   We would now like some input like Ax−Schanuel, so suppose that we ahve found some semialgebraic curve T in Z; parameterize T by $(t_1, \ldots, t_n)$. Passing to the complexification, we see that $T_{\mathbb{C}}$ is contained in some $\widetilde{Z}$, defined as follows: one has $(t_1, \ldots, t_n) \in \mathbb{C}^n$ lives in $\widetilde{Z}$ if and only if one can find $(x_1, \ldots, x_n) \in V$ satisfying

$$e^{-2\pi i t_j} H(x_j) + e^{2\pi i t_j} H_2(x_j) = 0$$

for all j. One can now calculate that the transcendence degree of the t_js, their exponentials, the x_js, the $(H_1 \circ x_j)$s and the $(H_2 \circ x_j)$s have some very small transcedence degree.

   We are now approximately ready to state our main theorem.

Notation 544. There are two distinguished linearly independent solutions z_1 and z_2 to Bessel's differential equation

$$x^2 z'' + xz' + \left(x^2 - \kappa\right) z = 0.$$

We now set

$$W_\kappa := \begin{bmatrix} z_1 & z_2 \\ z_1' & z_2' \end{bmatrix}.$$

Remark 545. It turns out that $\det W(x) = c/x$ for some constant c, but this is the only relation: when $\kappa \notin \frac{1}{2} + \mathbb{Z}$,

$$\operatorname{trdeg}_{\mathbb{C}(x)} \mathbb{C}(x, W) = 3.$$

Here, adjoining W means that we are adjoining all its coordinates.

Theorem 546. Fix everything as above. Choose $\kappa_1, \ldots, \kappa_n \in \mathbb{C} \backslash (\frac{1}{2} + \mathbb{Z})$. Let $\{x_1, \ldots, x_n\}$ be non-constant functions in $s \in \mathbb{D}$, and set $W_i := W_{\kappa_i}(x_i)$. Further, choose more non-constant functions $\{t_1, \ldots, t_\ell\}$. Suppose the following conditions are not satisfied.

(a) The t_js are $\mathbb{Q}$-linearly independent modulo constants.

(b) For some $j \neq k$, we have $x_j = \pm x_k$ and $\kappa_j \pm \kappa_k \in \mathbb{Z}$.

Then

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(t_\bullet, \exp(t_\bullet), x_\bullet, W_\bullet) \geq \ell + 3n + 1.$$

Once again, the relevance of this result for us will be its contraposition, and it allows us to identify the algebraic part, explaining the Ax–Schanuel input to Theorem 532. The key inputs to the proof of the above theorem consist of the "modern" technology of how to produce Ax–Schanuel for differentially closed fields as well as some understanding of the differential Galois group at play.

45   December 3: Cohomology of Banach–Colmez Spaces

The pre-talk and talk were given by Xinyu Zhou as a pre-talk for the Harvard number theory seminar.

45.1   The Étale Site

We are interested in adic spaces today.

Definition 547 (finite étale). A map $\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$ is *finite étale* if and only if B is finite étale over A and B^+ is the integral closure of A^+ in B. A map $f \colon X \to Y$ of analytic adic spaces is *finite étale* if and only if it locally has this form.

Definition 548 (étale). A map $f \colon X \to Y$ is *étale* if and only if each $x \in X$ has an open neighborhood U around x and V around $f(x)$ such that f factors as an open embedding follows by a finite étale map.

Remark 549. For schemes, this definition would be for separated étale morphisms, which is the same as some étale morphism of the underlying schemes are assumed to be separated.

Definition 550. Fix an analytic adic space X. Then the *étale site* consists of those surjective coverings whose maps are étale.

We are interested in the cohomology of local systems on analytic adic spaces. This is expected to be difficult because the étale site is more complicated (even locally) as compared to schemes.

Example 551. Fix complete algebraic closure C of $\mathbb{Q}_p$, let $\mathcal{O}_C$ be the ring of integers and $\mathfrak{m}_C$ be the maximal ideal. Then the unit disk $\mathbb{D} = \mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C\langle T\rangle)$ has

$$\mathrm{H}^1(\mathbb{D}; \mathbb{F}_p) \cong \mathrm{H}^1(\mathbb{D}; \mu_p) \cong \frac{(1 + \mathfrak{m}_C T\langle T\rangle)}{p},$$

where the last equality follows from Kummer theory. Notably, this surjects onto $\mathfrak{m}_C/p\mathfrak{m}_C$, so this dimension is infinite and grows with the cardinality of C.

This problem will be solved by using perfectoid spaces.

1. To start, we will want our rigid spaces X to be covered locally by perfectoid spaces: namely, for an affine $U \subseteq X$, we want to be able to find an affine S covering U by an inverse limit of étale morphisms (which is a pro-étale morphism).

2. We will want the cohomology of affinoid perfect spaces Y to be controlled. More precisely, up to pro-étale covering, one can achieve $\mathrm{H}^i(Y; \mathbb{F}_p) = 0$ for $i > 1$. In short, by tilting, one reduces to characteristic 0, and then one can use Artin–Schreier.

45.2   The Proétale Site

Having seen the use of pro-étale morphisms, we define our site.

Definition 552. Fix an analytic adic space X. Then the *proétale site* $X_{\mathrm{proét}}$ has its objects which are pro-systems $\{U_i\}$ whose internal maps $U_i \to U_j$ are finite étale for large i and j. A covering is said to be *proétale* if and only if it is further surjective.

Remark 553. More intuitively, one can define a proétale site on the category of perfectoid spaces over $\mathbb{F}_p$ where the coverings literally have their morphisms as limits of étale morphisms. Showing this requires showing that rigid analytic spaces are sheaves on this site.

Remark 554. There is a morphism $X_{\mathrm{proét}} \to X_{\text{ét}}$ of sites.

Remark 555. For $\mathbb{F}_p$-local systems, it turns out that the cohomology on the étale and proétale sites agree.

Now that we have our site, we can define some completed structure sheaves.

Definition 556. Fix an analyti adic space X. Then we define the structure sheaf $\mathcal{O}^+/p^n$ as having

$$\mathcal{O}^+/p^n(U) := R^+/p^n$$

for any affinoid perfectoid $U = \mathrm{Spa}(R, R^+)$. We then define $\widehat{\mathcal{O}}^+ := \lim \mathcal{O}^+/p^\bullet$ and $\widehat{\mathcal{O}} := \widehat{\mathcal{O}}^+[1/p]$.

Remark 557. Fix some $\mathbb{F}_p$-local system $\mathbb{L}$. Then there is an Artin–Schreier sequence

$$0 \to \mathbb{L} \to \mathbb{L} \otimes \mathcal{O}^+/p \to \mathbb{L} \otimes \mathcal{O}^+/p \to 0.$$

> **Theorem 558** (Primitive comparison)**.** Let X be a proper rigid analytic space. THen
>
> $$\mathrm{H}^i(X; \mathbb{L} \otimes \mathcal{O}^+/p) \cong \mathrm{H}^i(X; \mathbb{L}) \otimes \mathcal{O}_C/p.$$
>
> (This is an almost isomorphism.)

The moral is that we can reduce to understanding cohomology for $\mathcal{O}^+/p$-local systems. With more work, Scholze showed the following.

> **Theorem 559** (Scholze)**.** FIx a proper rigid analytic space X. Then $\mathrm{H}^i_{\mathrm{ét}}(X; \mathbb{L})$ is finite for any $\mathbb{F}_p$-local system $\mathbb{L}$.

> **Remark 560.** There is also a Poincaré duality, but it is hard. One way to do this is to construct a 6-functor formalism for $\mathcal{O}^+/p$-cohomology and use a version of the Primitive comparison theorem.

45.3   Banach−Colmez Spaces

There are a number of conventions which I do not understand. For example, all maps of v-stacks are assumed to be !-able, whatever that means. Today, we will give the category Perf of perfectoid spacaecs over $\mathbb{F}_p$ the proétale topology.

> **Definition 561** (compact)**.** A collection $\{f_\bullet \colon X_\bullet \to X\}$ of morphisms is *compact* if and only if any quasi-compact $U \subseteq X$ admits $f_1, \ldots, f_n$ for which there are quasicompact $U_i \subseteq X_i$ with
>
> $$U = f_1(U_1) \cup \cdots f_n(U_n).$$
>
> A collection in $\mathrm{Perf}_{\mathrm{proét}}$ is a covering if and only if the maps are proétale and the collection is compact.

> **Example 562.** There is a tilted structure sheaf $\mathcal{O}^\flat \colon \mathrm{Spa}(R, R^+) \mapsto R$.

> **Definition 563.** A *diamond* is a sheaf of the form X/R where X is perfectoid and R is a perfectoid equivalence.

> **Example 564.** Rigid analytic spaces and perfectoid spaces are diamonds.

> **Example 565.** If a group G acts on a space X, then X/G is a diamond.

Here is a distinguished perfectoid space.

> **Definition 566** (Fargue−Fontaine curve)**.** Fix a perfectoid space S, and let φ_S be the absolute Frobenius. Then we define
> $$X_S := (S \times \mathrm{Spd}\,\mathbb{Q}_p)/\varphi_S^{\mathbb{Z}}$$
> to be the *Fargue−Fontaine curve* relative to S.

> **Example 567.** There is a natural functor Φ which takes isocrystals over $\mathbb{F}_p$ to $\mathrm{Bun}\,X_S$. We define $\mathcal{O}(\lambda) := \Phi(D_\lambda)$, where D_λ is the isocrystal of slope λ. It turns out that if we write $\lambda = d/r$ in reduced form, then $\mathrm{rank}\,\mathcal{O}(\lambda) = r$ and $\deg\mathcal{O}(\lambda) = d$.

Here are our objects of interest today.

Definition 568. Fix a positive λ. Then the *negative Banach–Colmez space* $\mathrm{BC}(-\lambda)$ sends a perfectoid S to

$$\mathrm{H}^1(X_S; \mathcal{O}(-\lambda)) = \mathrm{Ext}^1_{X_S}(\mathcal{O}(\lambda), \mathcal{O}).$$

This is a proétale sheaf. There is an open subsheaf $\mathrm{BC}(-\lambda)^*$ classifying the semistable extensions.

Example 569. If $\lambda = -1$, then we see that $\mathrm{BC}(-1)$ will classify extensions

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

It turns out that this is either split (with $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$) or semistable (with $\mathcal{E} = \mathcal{O}(1/2)$).

Remark 570. There is a strange phemenon that cohomology of proétale $\mathbb{Q}_p$-local systems on a proper rigid space over $\mathbb{C}_p$ can be infinite-dimensional over $\mathbb{Q}_p$. The reason turns out to be that the cohomology comes from BCs, and one can recover finiteness from there even when the $\mathbb{Q}_p$-dimension is infinite.

45.4   Cohomological Calculations

Here is our main result.

Theorem 571. Fix positive λ.

(a) The cohomology $\mathrm{H}^i(\mathrm{BC}(-\lambda); \mathcal{O}^\flat)$ is concentrated in degree $i = 0$, where it is $\mathbb{F}_p$.

(b) The cohomology $\mathrm{H}^i(\mathrm{BC}(-1/r); \mathcal{O}^\flat)$ is concentrated in degrees $i \in \{0, r+1\}$, where is $\mathbb{F}_p$ (up to a Tate twist).

In short, one starts by noting that $\mathrm{BC}(-\lambda)$ is not a diamond, but one can calculate that $\mathrm{BC}(-1) \times \mathrm{Spd}\, C$ is some reasonable quotient of $\mathbb{A}^1_{\mathbb{C}_p}$ by the translation action of $\mathbb{Q}_p$. (Something similar is known for $\mathrm{BC}(-1/r)$.) Thus, it more or less remains to extrat out cohomology of $\mathrm{BC}(-1)$, so we need some nice properties of our cohomology.

One can make the site more wild without losing anything.

Definition 572 (v-topology)**.** A collection in Perf_v is a v-*covering* if and only if it satisfies compactness.

Remark 573. It turns out that $\mathrm{H}^*_v(-; \mathcal{O}^\flat) \simeq \mathrm{H}^*_{\mathrm{proét}}(-; \mathcal{O}^\flat)$, but it tends to be easier to calculate in the v-topology because the local structure is better-behaved.

Theorem 574. There is a 6-functor formalism on the (derived) category of (solid) quasicoherent $\mathcal{O}^\flat$-modules.

(a) If f is proper, then f is cohomologically proper (i.e., $f_* = f_!$).

(b) If f is smooth, then f is cohomologically smooth (i.e., $f^! = f^*[2 \dim X]$).

Remark 575. The necessity of the condensed mathematics here more or less comes from the desire to work with the strange quotient $\mathbb{A}^1_{\mathbb{C}_p}/\mathbb{Q}_p$.

Our second main result extracts out the properties of this 6-functor formalism that we will need.

Theorem 576. We work over $\mathbb{C}_p$ for simplicity.

(a) Künneth isomorphism: for rigid X and Y, one has

$$\mathrm{R\Gamma}_c(X \times Y; \mathcal{O}^\flat) \simeq \mathrm{R\Gamma}_c(X; \mathcal{O}^\flat) \otimes \mathrm{R\Gamma}_c(Y; \mathcal{O}^\flat).$$

(b) Drinfeld's lemma: let $Z \to \mathrm{Spd}\,\mathbb{F}_p$ be a v-stack. Then there is an isomorpshim

$$\mathrm{R\Gamma}(Z \times_{\mathrm{Spd}\,\mathbb{F}_p} [\mathrm{Spd}\,C/\varphi_C]; \mathcal{O}^\flat) \simeq \mathrm{R\Gamma}(Z; \mathcal{O}^\flat).$$

One now does a long calculation.

46   December 8: The Goncharov Program

This talk was given by Ismael Sierra Del Rio at MIT for the algebraic topology seminar.

46.1   Algebraic K-Theory

We are interested in studying the K-theory of an infinite field. In particular, we will want to compute the groups $K_*(F)_\mathbb{Q}$.

Example 577. One has that $K_0(F)$ is $K_0(\mathrm{Vec}F)$, which is $\mathbb{Z}$.

Example 578. One can calculate $K_1(F)_\mathbb{Q} = F_\mathbb{Q}^\times$. This object admits the following somewhat bizarre presentation: it is the quotient

$$\frac{\mathbb{Q}[F^\times]}{([a \cdot b] - [a] \cdot [b])}.$$

Example 579. Matsumoto computed

$$K_2(F)_\mathbb{Q} = \frac{\lambda_\mathbb{Q}^2 F_\mathbb{Q}^\times}{([a] \wedge [1-a])}.$$

Example 580. One can split $K_3(F)_\mathbb{Q}$ into the sum of a Milnor part $K_3^\mathrm{M}(F)_\mathbb{Q}$ and an indecomposable part $K_3^\mathrm{ind}(F)_\mathbb{Q}$.

- The Milnor part is $\wedge^3 F_\mathbb{Q}^\times / ([a] \wedge [1-a] \wedge [b])$.

- Suslin computed that the indecomposable part is the kernel of a map

$$\frac{\mathbb{Q}\langle F^\times \rangle}{\left(\{a\} + \{b\} + \left\{\frac{1-a}{1-ab}\right\} + \left\{\frac{1-b}{1-ab}\right\} + \{1 - ab\} \right)} \to \wedge^2 F_\mathbb{Q}^\times,$$

   where the map sends $\{a\} \mapsto [a] \wedge [1-a]$.

Though the presentation of K_3 looks bizarre, it is part of a general feature.

Theorem 581. One has a weight decomposition

$$K_d(F)_{\mathbb{Q}} = \bigoplus_{m=1}^{d} K_d^{(m)}(F)_{\mathbb{Q}},$$

where the Adams operator ψ^{ℓ} acts on the mth component by ℓ^m. Furthermore, $K_d^{(1)} = 0$ for $d \geq 2$.

Here, the Adams operations are induced by taking exterior powers of vector spaces.

Remark 582. The Milnor K-theory group $K_d^{\mathrm{M}}(F)_{\mathbb{Q}}$ is the top weight piece.

46.2 Goncharov Program

We start with the following conjecture.

Conjecture 583 (Beilinson–Deligne)**.** For any field F, there is a Tannakian category of mixed Tate motives $\mathrm{MTM}(F)$ generated by the Tate motive $\mathbb{Q}(-1)$. Furthermore,

$$K_{2m-i}^{(m)}(F)_{\mathbb{Q}} = \mathrm{Ext}_{\mathrm{MTM}(F)}^{i}(\mathbb{Q}(-m), \mathbb{Q}(0)).$$

Remark 584. Being Tannakian implies that one expects to be able to extract a "motivic" graded Lie coalgebra $L_{\bullet}(F)$ so that $\mathrm{MTM}(F)$ consists of representations of $L_{\bullet}(F)$. The isomorphism to the representation category allows one to realize $\mathrm{Ext}_{\mathrm{MTM}(F)}^{i}(\mathbb{Q}(-m), \mathbb{Q}(0))$ as the mth graded piece of $\mathrm{H}_i(L_{\bullet}(F))$.

Remark 585. When F is a number field, the existence of this category is known. One basically passes to the category of Voevodsky motives and then finds the mixed Tate motives living inside.

Let's make this more explicit. The homology groups $\mathrm{H}_i(L_{\bullet}(F))$ come from taking homology of the complexes

$$m = 3 \qquad L_3 \longrightarrow L_1 \otimes L_2 \longrightarrow \wedge^3 L_2$$

$$m = 2 \qquad L_2 \longrightarrow \wedge^2 L_1$$

$$m = 1 \qquad L_1$$

$$i = 1 \qquad\qquad i = 2 \qquad\qquad i = 3$$

and then these homology groups recover the weight pieces of algebraic K-theory.

However, this description is not so explicit because $L_{\bullet}$ was conjured from Tannakian reconstruction. The Goncharov program aims to provide explicit constructions for these L_i. For example, L_3 is conjectured to be $\mathbb{Q}\langle F^{\times}\rangle$ modulo some 22-term relation. Let's explain wher such relations come from.

Conjecture 586 (Goncharov)**.** The group $L_m(F)$ is the quotient

$$\frac{\mathbb{Q}\langle \mathrm{Li}_{m_1,\ldots,m_k}(a_1,\ldots,a_k) : a_1,\ldots,a_k \in F, \sum_i m_i = m \rangle}{(\text{relations})},$$

and the relations can be made explicit.

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Remark 587. The functions $\mathrm{Li}_{m_1,\ldots,m_k}$ are some explicit periods of some motives, and the relations arise from the expected relations of these periods.

Remark 588. Goncharov also gave a formula for the co-algebra bracket of $L_\bullet$.

Here is our main result.

Theorem 589. If F is a number field, then $L_m(F)$ exists, it is spanned by the multiple polylogarithm functions, and all the relations are known.

Corollary 590. There is an explicit presentation of $K_d(F)_{\mathbb{Q}}$.

Remark 591. In fact, the ranks of $K_d(F)_{\mathbb{Q}}$ were already known, due to Borel.

Corollary 592. The universality conjecture follows for number fields: all periods coming from mixed Tate motives arise from polylogarithms.

Example 593. The special values $\zeta_F(m)$ are periods of mixed Tate motives.

46.3   The Homology of GL_n

We are interested in computing the homology of F. To F, we associate the $\mathbb{E}_\infty$-algebra

$$R(F) := \bigsqcup_{m \geq 1} \mathrm{BGL}_m(F).$$

We are only interested in rational homology, so we are allowed to tensor with $\mathbb{Q}$ to get $R(F)_{\mathbb{Q}}$, which lives in the stable ∞-category of $\mathrm{Mod}_{\mathbb{Q}}$. Separating out the gradings allows us to define

$$\mathrm{H}_{m,d}(R(F)_{\mathbb{Q}}) := \mathrm{H}_d(\mathrm{BGL}_m(F); \mathbb{Q}).$$

Algebriac K-theory arises from the homotopy groups of $R(F)$, so it now makes sense that we will be interested in the groups $\mathrm{H}_{m,d}(R(F)_{\mathbb{Q}})$.

Here is one way to approach the calculation: there is a left adjoint to the inclusion from trivial $\mathbb{E}_\infty$-algebras to all $\mathbb{E}_\infty$-algebras, and it is named $\mathrm{B}^{\mathrm{com}}$. Here is the sort of thing which was already known.

Theorem 594. One can calculate

$$\mathrm{H}_{m,d}(B^{\mathrm{com}} R(F)_{\mathbb{Q}}) = \mathrm{H}_{d-2m+2}(\mathrm{GL}_m(F); \mathrm{St}_m^\infty(F)_{\mathbb{Q}}),$$

where $\mathrm{St}_m^\infty(F)_{\mathbb{Q}}$ is some module related to the Steinberg module.

Corollary 595. The homology vanishes in the range $d < 2m - 2$.

Example 596. For $m \geq 2$, one can calculate

$$\mathrm{H}_0(\mathrm{GL}_m(F); \mathrm{St}_m^\infty(F)_{\mathbb{Q}}) = 0.$$

Now, it turns out that there is a spectral sequence which computes $K_\bullet(F)_\mathbb{Q}$ from $\mathrm{H}_{m,d}(B^{\mathrm{conn}}R(F)_\mathbb{Q})$. But there is more structure here: $\Sigma B^{\mathrm{conn}}R(F)_\mathbb{Q}$ is a Lie coalgebra! It is now reasonable to expect that we can relate this story to mixed Tate motives.

Notation 597. The *Goncharov* Lie coalgebra is

$$\mathcal{G}_\bullet(F) := \bigoplus_{m \geq 1} \mathrm{H}_1(\mathrm{GL}_m(F); \mathrm{St}_m^\infty(F)_\mathbb{Q}).$$

Conjecture 598. There is an isomorphism

$$L_\bullet(F) \cong \mathcal{G}_\bullet(F).$$

Theorem 599. Fix an infinite field F.

  (a) The coalgebra $\mathcal{G}_m(F)$ is spanned by symbols coming from multiple polylogarithms.

  (b) All relations are known, and they are already known.

  (c) The cobracket on $\mathcal{G}$ is the expected object.

  (d) One can compute $\mathcal{G}_3(F)$ is the expected $L_3(F)$.

Theorem 600. Fix a number field F. Then there is a canonical map

$$\mathcal{G}_\bullet(F) \to L_\bullet(F)$$

which is an isomorphism of Lie coalgebras. This map sends the multiple polylogarithm symbols to each other.

Idea. Construction of the map is a matter of matching relations on both sides. Note that $L_\bullet(F)$ has homology concentrated in degree 1, so it turns out to be free. On the other hand, the groups $\mathrm{H}_d(\mathrm{BGL}_m(F)_\mathbb{Q})$ are understood due to Borel–Yang (both abstractly and with generating classes), so one can calculate $\mathcal{G}_\bullet(F)$ explicitly as well. One then works harder, I guess. ∎

47   December 12: Heegner Cycles and p-Adic L-functions

This talk was given by Rui at Johns Hopkins University for the automorphic forms learning seminar.

47.1   The Main Theorem

Throughout today, p is a large prime (not dividing 2, $(2r - 1)!$, nor $N\varphi(N)$), χ is some Dirichlet character of conductor coprime to N, f is a newform of weight $2r$ and level N, N is a product of primes split in an imaginary quadratic field K, and $p = \mathfrak{p}\overline{\mathfrak{p}}$ is also split in K.

Theorem 601. Suppose that f is ordinary at p. If $L(f, \chi, r) \neq 0$, then

$$\dim_F \mathrm{Sel}(V_{f,\chi}/K) = 0,$$

where $V_{f,\chi}$ is the associated Galois representation.

> **Remark 602.** Here, $V_{f,\chi} = V_f(r) \otimes \chi$, where V_f is the Galois representation associated to f.

The first ingredient to the proof is a p-adic Gross–Zagier formula, which we lift from Bertolini–Darmon–Prasad.

> **Theorem 603.** Fix a newform $f \in S_{2r}(\Gamma_0(N))$, and asume the strong Heegner condition that N is a product of primes split in K. Additionally, choose an elliptic curve A with complex multiplication by $\mathcal{O}_K$, from which one can construct some generalized Heegner cycles $z_{f,\chi} \in \mathrm{H}^1_f(K; T \otimes \chi)$, where $T \subseteq V$ is some Galois-stable $\mathbb{Z}_p$-lattice.
>
> Now, fix an anticyclotomic character ψ of infinity type $(r, -r)$ and conductor $c_0\mathcal{O}_K$, and choose some ϕ which is the p-adic avatar (via $\mathbb{C} \cong \mathbb{C}_p$) of an anticyclotomic character of infinity type $(r+j, -j-r)$, where $-r < j < r$ and p-power conductor. Then
>
> $$\frac{\mathcal{L}(f)(\hat{\phi}^{-1})}{\Omega_p^{-2j}} \sim \langle \log \mathrm{loc}_{\mathfrak{p}}\, z_{f,\chi}, c \rangle,$$
>
> where c is some explicit class, and $\log$ is the Bloch–Kato logarithm.

Let's now sketch the idea.

1. The p-adic Gross–Zagier formula explains that the infinity types with $j \geq r$ have a p-adic L-function which interpolate L-values.

2. On the other hand, Waldspurger's formula explains that the region $-r \leq j \leq r$ has a p-adic L-function which interpolates pairings from the generalized Heegner cycles.

3. It turns out that it is the same L-function in both points, so one is able to conclude that the p-adic L-function interpolates some kind of extension of the Heegner cycles beyond their region of definition, to $j \geq r$.

4. These Heegner cycles in their region of definition are unramified, but when extended, they become ramified, and the previous step explains that there is an L-value which predicts if the ramification is actually nontrivial.

5. The presence of such ramified classes force the Selmer group to vanish.

48   December 12: Higher Algebraic Geometry

This talk was given by Germán Stefanich at Johns Hopkins University as a colloqium talk. This is joint work with Peter Scholze, David Ben-Zvi, and David Nadler.

48.1   Categorification of Algebraic Geometry

Let's start by reviewing classical algebraic geometry. Fix an algebraically closed field k, and then algebraic geometry is interested in the solution sets to some polynomials in affine space. These make the affine algebraic varieties, and they come equipped with a sheaf of k-algebras of regular functions.

> **Theorem 604** (Hilbert). Fix an algebraically closed field k. There is an equivalence of categories between the category of affine k-varieties and integral k-algebras of finite type.

It was realized that one can relax working with k-algebras.

> **Theorem 605** (Grothendieck). There is an equivalence of categories between affine schemes and commutative rings.

Of course, in algebraic geometry, one is not just interested in affine schemes but rather schemes, which are obtained by carefully gluing affine schemes.

> **Example 606.** The projective line $\mathbb{P}^1$ admits a covering by two affine lines $\mathbb{A}^1$, gluing along $\mathbb{A}^1 \setminus \{0\}$.

For our talk, instead of gluing to produce more geometric objects, we will generalize the notion of commutative ring.

> **Remark 607.** Commutative rings came up in our story because our functions formed sheaves of rings. Thus, to work with more general rings, we need to work with more general functions.

> **Example 608.** Even though the functions on $\mathbb{P}^1$ are all constants (and hence not very interesting), the category of vector bundles on $\mathbb{P}^1$ is much more interesting (though also well-understood). Unsururprisingly, it will actually be more convenient to work with quasicoherent sheaves (for categorical reasons).

The above example fits into a more general framework of "categorification."

- Numbers can be categorified into vector spaces. One recovers the number by taking the dimension.
- Functions can be categorified into quasicoherent sheaves. One recovers the function by taking sections.
- One can add functions, which becomes sums of sheaves.
- Additionally, one can multiply, which become tensor products of sheaves.

Under the above framework, we see that the structure sheaf $\mathcal{O}_X$ can be categorified into $\mathrm{QCoh}(X)$.

> **Remark 609.** In the affine case, this basically corresponds to taking a commutative ring A to $\mathrm{Mod}(A) = \mathrm{QCoh}(\mathrm{Spec}\, A)$.

> **Theorem 610.** There is a fully faithful functor from commutative rings to symmetric monoidal categories which sends a ring A to $\mathrm{Mod}(A)$.

One may hope to upgrade the algebraic side to make this into an equivalence.

48.2   Higher Categorification

It turns out that QCoh is not enough to distinguish geometry if we move to stacks.

> **Example 611.** Consider the stack $\mathrm{B}^2\mathbb{G}_m$, which classifies Azumaya algebras. Then $\mathcal{O}\left(\mathrm{B}^2\mathbb{G}_m\right)$ only has constants, and $\mathrm{QCoh}\left(\mathrm{B}^2\mathbb{G}_m\right)$ only has vector spaces. However, there is a 2-category $2\mathrm{QCoh}\left(\mathrm{B}^2\mathbb{G}_m\right)$ (of quasicoherent sheaves of categories) is interesting.

> **Remark 612.** Working with symmetric monoidal 2-categories is genuinely more interesting.

This example generalizes to some $\mathrm{B}^n\mathbb{G}_m$, so we see that we are going to need symmetric monoidal (i.e., "stable") ∞-categories.

> **Definition 613** (categorical ring). A *categorical ring* is a compatible sequence
> $$(A_0, A_1, A_2, \ldots),$$
> where A_n is a symmetric monoidal n-category.

The compatibility refers to some projection functor which takes symmetric monoidal n-categories to symmetric monoidal $(n-1)$-categories.

Remark 614. Secretly, all these objects may want to be ∞-categories.

So here is our main definition.

Definition 615 (gestalten)**.** The category of *shapes* is the opposite category of the category of categorical rings.

Remark 616. Affine schemes do not hav a good theory of gluing: this is the purpose of schemes. However, the category of shapes turns out to have a good theory of gluings. For example, the category of shapes includes the category of quasicoherent quasiseparated schemes.

Remark 617. Intuitively, a shape consists of the data of a categorical ring $(A_0, A_1, A_2, \ldots)$, where A_0 is the functions on the shape, A_1 is the sheaves on the shape, and so on.

It remains to motivate the need for the notion of a shape.

Example 618. There is a moduli "shape" of (one-dimensional) topological quantum field theories. Here, a (one-dimensional) topological quantum field theory (over a field k) is a symmetric monoidal functor $\mathrm{Cob} \to \mathrm{Vec}(k)$, where Cob is the 1-category of cobordisms. It turns out that our shape is given by the shape corresponding to Cob. For example, one can show that π_0 of the moduli shape of invertible one-dimensional topological quantum field theories is just $\mathbb{Z}/2\mathbb{Z}$; approximately speaking, this corresponds to the theory of "superlinear algebra." Notably, working in higher dimensions predicts the existence of certain higher superlinear algebras.

For our next example, we note that the category of shapes admits a six-functor formalism. Six-functor formalisms frequently admit an associated cohomology theory: $\mathcal{D}$-modules yields de Rham cohomology, étale sheaves yields étale cohomology, and Betti sheaves yields singular cohomology.

Theorem 619. For every six-functor formalism Shv and variety X, there is a canonical shape X_{Shv} for which $\mathrm{QCoh}(X_{\mathrm{Shv}}) = \mathrm{Shv}(X)$.

This explains the bizarre fact that six-functor formalisms frequently look like quasicoherent sheaves on some stack.

Example 620. There is a six-functor formalism SH of (Voevodsky or Ayoub) motivic sheaves. Then one can view $[\mathrm{Spec}\,\mathbb{Z}]_{\mathrm{SH}}$ as the moduli shape of cohomology theories. It also turns out to parameterize ring stacks.

Example 621. There is a six-functor formalism IndCoh, which is like QCoh but knows about singularities. If X is smooth, one expects that turns out that $2\mathrm{QCoh}(X_{\mathrm{IndCoh}})$ is an invariant of T^*X. One can prove this when $X = V$ is a finite-dimensional vector space.