

The Local Fundamental Class

Nir Elber

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Abstract

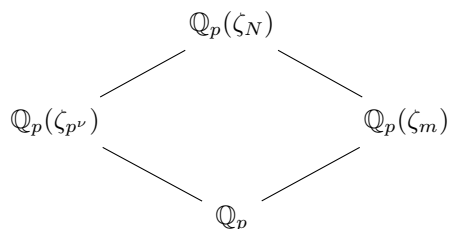
We compute the local fundamental class of the extension $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ when p is an odd prime. This requires making a number of standard group cohomology constructions fully explicit in the process.

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1 Set-Up

We will work over \mathbb{Q}_p as our base field, where p is an odd prime. Set $N := p^\nu m$ where k and m integers with $p \nmid m$. This gives us the following tower of fields.



To help us a little later, we will assume that the extension $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ is not totally ramified nor as unramified, for in this case we can understand the extension by viewing it as a cyclic extension. We provide some quick commentary on these extensions.

- The extension $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$ is unramified of degree $f := \text{ord}_p(m)$; note we are assuming $1 < f < n$. Its Galois group is thus generated by the Frobenius element defined by $\bar{\sigma}_K: \zeta_m \mapsto \zeta_m^p$.
- The extension $\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p$ is totally ramified of degree $\varphi(p^\nu)$. Its Galois group is thus isomorphic to $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$, where the isomorphism takes $x \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times$ to

$$\sigma_x: \zeta_{p^\nu} \mapsto \zeta_{p^\nu}^{x^{-1}}.$$

The group $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$ is cyclic, so we will fix a generator x , which gives us a distinguished generator $\sigma_x \in \text{Gal}(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p)$.

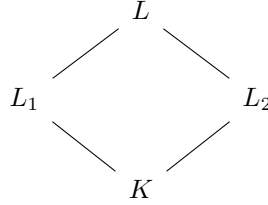
- Because $\mathbb{Q}_p(\zeta_{p^\nu})$ is totally ramified and $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$ is unramified, we have that the fields $\mathbb{Q}_p(\zeta_{p^\nu})$ and $\mathbb{Q}_p(\zeta_m)$ are linearly disjoint over \mathbb{Q}_p . As such, $\mathbb{Q}_p(\zeta_N) = \mathbb{Q}_p(\zeta_{p^\nu})\mathbb{Q}_p(\zeta_m)$ has

$$\begin{aligned}\text{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p(\zeta_{p^\nu})) &\simeq \text{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) = \langle \bar{\sigma}_K \rangle \\ \text{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p(\zeta_m)) &\simeq \text{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_x \rangle \\ \text{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p) &\simeq \text{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) \times \text{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \bar{\sigma}_K \rangle \times \langle \sigma_x \rangle.\end{aligned}$$

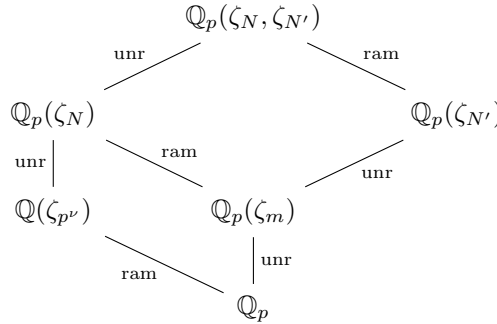
In light of these isomorphisms, we will upgrade $\bar{\sigma}_K$ to the automorphism of $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ sending $\zeta_m \mapsto \zeta_m^p$ and fixing $\mathbb{Q}_p(\zeta_{p^\nu})$; we do analogously for σ_x . We also acknowledge that our degree is

$$n := [\mathbb{Q}_p(\zeta_N) : \mathbb{Q}_p] = [\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p] \cdot [\mathbb{Q}_p(\zeta_{p^\nu}) : \mathbb{Q}_p] = f\varphi(p^\nu).$$

For brevity, we will also set $L_1 := \mathbb{Q}_p(\zeta_{p^\nu})$ and $L_2 := \mathbb{Q}_p(\zeta_m)$, which makes the fields under L look like the following.

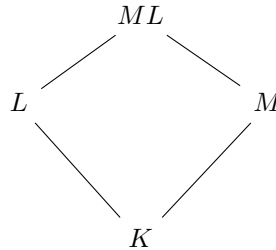


Now, the main idea in the computation is to use an unramified extension of the same degree as $\mathbb{Q}_p(\zeta_N)$. As such, we set $N' := p^n - 1$ so that $[\mathbb{Q}_p(\zeta_{N'}) : \mathbb{Q}_p]$ is $\text{ord}_{N'}(p) = n$. This modifies our diagram of fields as follows.



We have labeled the unramified extensions by “unr” and the totally ramified extensions by “ram.”

For brevity, we set $K := \mathbb{Q}_p$ and $L := \mathbb{Q}_p(\zeta_N)$ and $M := \mathbb{Q}_p(\zeta_{N'})$ so that $ML = \mathbb{Q}_p(\zeta_N, \zeta_{N'})$. This abbreviates our diagram into the following.



As before, we provide some comments on the field extensions.

- The extension $\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p$ is unramified of degree n . As before, its Galois group is cyclic, generated by $\sigma_K : \zeta_{N'} \mapsto \zeta_{N'}^p$. Observe that σ_K restricted to $\mathbb{Q}_p(\zeta_m)$ is $\bar{\sigma}_K$, explaining our notation. In particular, σ_K has order n , but $\bar{\sigma}_K$ has order $f < n$.

- As before, note that $\mathbb{Q}_p(\zeta_{p^\nu})$ and $\mathbb{Q}(\zeta_{N'})$ are linearly disjoint because $\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p$ is totally ramified while $\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p$ is unramified. As such, we may say that

$$\begin{aligned}\mathrm{Gal}(ML/M) &\simeq \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_x \rangle \\ \mathrm{Gal}(ML/\mathbb{Q}_p(\zeta_{p^\nu})) &\simeq \mathrm{Gal}(M/K) = \langle \sigma_K \rangle \\ \mathrm{Gal}(ML/K) &\simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_K \rangle \times \langle \sigma_x \rangle.\end{aligned}$$

Again, we will upgrade σ_K and σ_x to their corresponding automorphisms on any subfield of ML .

- We take a moment to compute

$$\mathrm{Gal}(ML/L) \simeq \{ \sigma_K^{a_1} \sigma_x^{a_2} \in \mathrm{Gal}(ML/K) : \sigma_K^{a_1} \sigma_x^{a_2}|_L = \mathrm{id}_L \}.$$

Because L is $\mathbb{Q}_p(\zeta_{p^\nu})\mathbb{Q}_p(\zeta_m)$, it suffices to fix each of these fields individually. Well, to fix $\mathbb{Q}_p(\zeta_{p^\nu})$, we need $\sigma_x^{a_2}$ to vanish, so we might as well force $a_2 = 0$. But to fix $\mathbb{Q}_p(\zeta_m)$, we need $\sigma_K^{a_1}|_{\mathbb{Q}(\zeta_m)} = \bar{\sigma}_k^{a_1}$ to be the identity, so we are actually requiring that $f \mid a_1$ here. As such,

$$\mathrm{Gal}(ML/L) = \langle \sigma_K^f \rangle.$$

These comments complete the Galois-theoretic portion of the analysis.

2 Idea

We will begin by briefly describe the outline for the computation. For a finite extension of finite fields L/K , let $u_{L/K} \in H^2(L/K)$ denote the fundamental class.

Now, take variables as in our set-up in [section 1](#). The main idea is to translate what we know about the unramified extension M/K over to the general extension L/K . In particular, we are able to compute the fundamental class $u_{M/K} \in H^2(M/K)$, so we observe that

$$\mathrm{Inf}_{M/K}^{ML/K} u_{M/K} = [ML : M] u_{M/K} = n \cdot u_{ML/K} = [ML : L] u_{ML/L} = \mathrm{Inf}_{L/K}^{ML/K} u_{L/K}.$$

As such, we will be able to compute $u_{L/K}$ as long as we are able to invert the inflation map $\mathrm{Inf} : H^2(L/K) \rightarrow H^2(ML/K)$. This is not actually very easy to do in general,¹ but we are in luck because this inflation map here comes from the Inflation–Restriction exact sequence

$$0 \rightarrow H^2(L/K) \xrightarrow{\mathrm{Inf}} H^2(ML/K) \xrightarrow{\mathrm{Res}} H^2(ML/L).$$

The argument for the Inflation–Restriction exact sequence is an explicit computation on cocycles (involving some dimension shifting), but it can be tracked backwards to give the desired cocycle.

3 Computation

In this section we record the details of the computation.

3.1 Group Cohomology

Throughout this section, G will be a group (usually finite) and $H \subseteq G$ will be a subgroup (usually normal). We denote $\mathbb{Z}[G]$ by the group ring and $I_G \subseteq \mathbb{Z}[G]$ by the augmentation ideal, defined as the kernel of the map $\varepsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$ which sends $g \mapsto 1$ for all $g \in G$.

We begin by recalling the statement of the Inflation–Restriction exact sequence.

¹ The difficulty comes from the fact that a generic cocycle might be off from an inflated cocycle by some truly hideous coboundary.

Theorem 1 (Inflation–Restriction). Let G be a finite group with normal subgroup $H \subseteq G$. Given a G -module A , suppose that the $H^i(H, A) = 0$ for $1 \leq i < q$ for some index $q \geq 1$. Then the sequence

$$0 \rightarrow H^q(G/H, A^H) \xrightarrow{\text{Inf}} H^q(G, A) \xrightarrow{\text{Res}} H^q(H, A)$$

is exact.

Sketch. The proof is by induction on q , via dimension shifting. For $q = 1$, we can just directly check this on 1-cocycles. The main point is the exactness at $H^q(G, A)$: if $c \in Z^1(G, A)$ has $\text{Res}(c) \in B^1(H, A)$, then find $a \in A$ with

$$\text{Res}(c)(a) := h \cdot a - a.$$

As such, we define $f_a \in B^1(G, A)$ by $f_a(g) := g \cdot a - a$, which implies that $c - f_a$ vanishes on H . It is then possible to stare at the 1-cocycle condition

$$(c - f_a)(gg') = (c - f_a)(g) + g \cdot (c - f_a)(g')$$

to check that $c - f_a$ only depends on the cosets of H (e.g., by taking $g' \in H$) and that $\text{im}(c - f_a) \subseteq A^H$ (e.g., by taking $g \in H$).

For $q > 1$, we use dimension shifting via the following lemma.

Lemma 2 (Dimension shifting). Let G be a group with subgroup $H \subseteq G$. Given a G -module A , all indices $q \geq 1$ have

$$\delta: H^q(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^{q+1}(H, A).$$

Sketch. Recall that we have the short exact sequence of $\mathbb{Z}[H]$ -modules

$$0 \rightarrow I_G \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0.$$

In fact, this short exact sequence splits over \mathbb{Z} , so it will still be short exact after applying $\text{Hom}_{\mathbb{Z}}(-, A)$, which gives the short exact sequence

$$0 \rightarrow A \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \rightarrow \text{Hom}_{\mathbb{Z}}(I_G, A) \rightarrow 0$$

of $\mathbb{Z}[H]$ -modules. The result now follows from the long exact sequence of cohomology upon noting that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ is coinduced and hence acyclic for cohomology. ■

Using the above lemma, we have the following commutative diagram with vertical arrows which are isomorphisms.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^q(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H) & \longrightarrow & H^q(G, \text{Hom}_{\mathbb{Z}}(I_G, A)) & \longrightarrow & H^q(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \longrightarrow & H^{q+1}(G/H, A^H) & \longrightarrow & H^{q+1}(G, A) & \longrightarrow & H^{q+1}(H, A) \end{array}$$

The top row is exact by the inductive hypothesis, so the bottom row is therefore also exact. ■

Our goal is to make the above proof explicit in the case of $q = 2$, which is the only reason we sketched the above proofs at all. We begin by making the dimension shifting explicit.

Lemma 3. Let G be a group with subgroup $H \subseteq G$, and let $\{g_\alpha\}_{\alpha \in \lambda}$ be coset representatives for $H \backslash G$. Now, given a G -module A , the maps

$$\begin{aligned} \delta_H: Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) &\rightarrow Z^2(H, A) \\ c &\mapsto [(h, h') \mapsto h \cdot c(h')(h^{-1} - 1)] \\ [h \mapsto ((h'g_\bullet - 1) \mapsto h' \cdot u((h')^{-1}, h))] &\mapsto u \end{aligned}$$

are group homomorphisms which descend to the isomorphism $\bar{\delta}: H^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^2(H, A)$ of Lemma 2. The map δ above is surjective, and the reverse map is a section; when $H = G$, these are isomorphisms.

Proof. We begin by noting that our short exact sequence can be written more explicitly as follows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(I_G, A) \longrightarrow 0 \\ & & a & \longmapsto & (z \mapsto \varepsilon(z)a) & & \\ & & & & f & \longmapsto & f|_{I_G} \end{array}$$

We now track through the induced boundary morphism $\delta: H^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \rightarrow H^2(H, Q)$.

- We begin with $c \in Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A))$, which means that we have $c(h): I_G \rightarrow A$ for each $h, h' \in H$, and we satisfy

$$c(hh') = c(h) + h \cdot c(h').$$

Tracking through the action of H on $\text{Hom}_{\mathbb{Z}}(I_G, A)$, this means that

$$c(hh')(g - 1) = c(h)(g - 1) + h \cdot c(h')(h^{-1}g - h^{-1})$$

for any $g \in G$.

- To pull c back to $C^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$, we need to lift $c(h): I_G \rightarrow A$ to a $\tilde{c}(h): \mathbb{Z}[G] \rightarrow A$. Recalling that we only need to preserve group structure, we simply precompose $c(h)$ with the map $\mathbb{Z}[G] \rightarrow I_G$ given by $z \mapsto z - \varepsilon(z)$. That is, we define

$$\tilde{c}(h)(z) := c(h)(z - \varepsilon(z)).$$

- We now push \tilde{c} through $d: C^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \rightarrow Z^2(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$. This gives

$$(d\tilde{c})(h, h') = g\tilde{c}(h') - \tilde{c}(hh') + \tilde{c}(h)$$

for any $h, h' \in H$. Concretely, plugging in some $z \in \mathbb{Z}[G]$ makes this look like

$$\begin{aligned} (d\tilde{c})(h, h')(z) &= (h\tilde{c}(h'))(z) - \tilde{c}(hh')(z) + \tilde{c}(h)(z) \\ &= h \cdot c(h')(h^{-1}z - \varepsilon(h^{-1}z)) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) \\ &= h \cdot c(h')(h^{-1}z - \varepsilon(z)) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)). \end{aligned}$$

Now, from the 1-cocycle condition on c , we recall

$$-c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) = -h \cdot (c(h')(h^{-1}z - \varepsilon(z)h^{-1})),$$

so

$$\begin{aligned} (d\tilde{c})(h, h')(z) &= h \cdot c(h')(h^{-1}z - \varepsilon(z)) \\ &= \varepsilon(z) \cdot (h \cdot c(h')(h^{-1} - 1)). \end{aligned}$$

In particular, we see that $d\tilde{c} \in Z^2(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ pulls back to $(h, h') \mapsto h \cdot c(h')(h^{-1} - 1)$ in $Z^2(H, A)$. It is not too difficult to check that we have in fact defined a 2-cocycle, but we will not do so because it is not necessary for the proof.

Now, we do know that δ_H is a homomorphism abstractly on elements of our cohomology classes by the Snake lemma, but it is also not too hard to see that

$$\delta_H: Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \rightarrow Z^2(H, A)$$

is in fact a homomorphism of groups directly from the construction. In short,

$$\delta_H(c + c')(h, h') = h' \cdot c(h) (h^{-1} - 1) + h' \cdot c'(h) (h^{-1} - 1) = (\delta_H(c) + \delta_H(c'))(h, h')$$

for any $h, h' \in H$.

It remains to prove the last sentence. We run the following checks; given $u \in Z^2(H, A)$, define $c_u \in C^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A))$ by

$$c_u(h)(h'g_{\bullet} - 1) = h' \cdot u((h')^{-1}, h).$$

Note that this is enough data to define $c_u(h): I_G \rightarrow A$ because I_G is a free \mathbb{Z} -module generated by $\{g - 1 : g \in G\}$.

- We verify that c_u is a 1-cocycle. This is a matter of force. Pick up $h, h' \in H$ and $g_{\bullet}h'' \in G$ and write

$$\begin{aligned} & (hc_u(h'))(h''g_{\bullet} - 1) + c_u(hh')(h''g_{\bullet} - 1) + c_u(h)(h''g_{\bullet} - 1) \\ &= h \cdot c_u(h') (h^{-1}h''g_{\bullet} - h^{-1}) + c_u(hh')(h''g_{\bullet} - 1) + c_u(h)(h''g_{\bullet} - 1) \\ &= h \cdot (h^{-1}h''u((h'')^{-1}h, h') - h^{-1}u(h, h')) + h''u((h'')^{-1}, hh') + h''u((h'')^{-1}, h) \\ &= h''u((h'')^{-1}h, h') - u(h, h') + h''u((h'')^{-1}, hh') + h''u((h'')^{-1}, h). \end{aligned}$$

This is just the 2-cocycle condition for u upon dividing out by h'' , so we are done.

- For $u \in Z^2(H, A)$, we verify that $\delta_H(c_u) = u$. Indeed, given $h, h' \in H$, we check

$$\begin{aligned} \delta_H(c_u)(h, h') &= h \cdot c_u(h') (h^{-1} - 1) \\ &= h \cdot h^{-1} \cdot u(h, h') \\ &= u(h, h'). \end{aligned}$$

So far we have verified that δ has section $u \mapsto c_u$ and hence must be surjective. Lastly, we take $H = G$ and show that $c_{\delta c} = c$ to finish. Indeed, for $g, g' \in G = H$, we write

$$\begin{aligned} c_{\delta_H c}(g)(g' - 1) &= g' \cdot (\delta_H c)((g')^{-1}, g) \\ &= g'(g')^{-1} \cdot c(g)(g' - 1) \\ &= c(g)(g' - 1), \end{aligned}$$

which is what we wanted. ■

We also have used dimension shifting to show that $H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H) \rightarrow H^2(G/H, A^H)$ is an isomorphism, but this requires a little more trickery. To begin, we discuss how to lift from $\text{Hom}_{\mathbb{Z}}(I_G, A)^H$ to $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$.

Lemma 4. Let G be a group with subgroup $H \subseteq G$. Fix a G -module A with $H^1(H, A) = 0$. Then, for any $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$, the function $h \mapsto h\psi(h^{-1} - 1)$ is a cocycle in $Z^1(H, A) = B^1(H, A)$, so we can define a function $\eta_{\bullet}: \text{Hom}_{\mathbb{Z}}(I_G, A)^H \rightarrow A$ such that

$$\psi(h - 1) = h \cdot \eta_{\varphi} - \eta_{\varphi}$$

for all $h \in H$. In fact, given $\varphi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$, we can construct $\tilde{\varphi} \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$ by

$$\tilde{\varphi}(z) := \varphi(z - \varepsilon(z)) + \varepsilon(z)\eta_{\varphi}$$

so that $\tilde{\varphi}|_{I_G} = \varphi$.

Proof. We will just run the checks directly.

- We start by checking $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ give 1-cocycles $c(h) := \varphi(h - 1)$ in $Z^1(A, H)$. To begin, we note that $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ simply means that any $z - \varepsilon(z) \in I_G$ has

$$\psi(z - \varepsilon(z)) = (h\psi)(z - \varepsilon(z)) = h\psi(h^{-1}z - h^{-1}\varepsilon(z))$$

for all $h \in H$. In particular, replacing h with h^{-1} tells us that

$$h\psi(z - \varepsilon(z)) = \psi(hz - h\varepsilon(z)).$$

Now, we can just compute

$$\begin{aligned} (dc)(h, h') &= hc(h') - c(hh') + c(h) \\ &= hc(h' - 1) - c(hh' - 1) + c(h - 1) \\ &= c(hh' - h) - c(hh' - 1) + c(h - 1), \end{aligned}$$

where in the last equality we used the fact that $\psi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$. Now, $(dc)(h, h')$ manifestly vanishes, so we are done.

- Note that $\tilde{\varphi} \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ because it is a linear combination of (compositions of) homomorphisms.
- Note that any $z \in I_G$ has $\varepsilon(z) = 0$, so

$$\tilde{\varphi}(z) = \varphi(z - 0) + 0 \cdot \eta_{\varphi} = \varphi(z),$$

so $\tilde{\varphi}|_{I_G} = \varphi$.

- It remains to check that $\tilde{\varphi}$ is fixed by H . This requires a little more effort. Recall that $\varphi \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$ means that any $z - \varepsilon(z) \in I_G$ has

$$h\varphi(z - \varepsilon(z)) = \varphi(hz - h\varepsilon(z))$$

for any $h \in H$. Now, we just compute

$$\begin{aligned} (h\tilde{\varphi})(z) &= h\tilde{\varphi}(h^{-1}z) \\ &= h(\varphi(h^{-1}z - \varepsilon(h^{-1}z)) + \varepsilon(h^{-1}z)\eta_{\varphi}) \\ &= \varphi(z - h\varepsilon(z)) + \varepsilon(z) \cdot h\eta_{\varphi} \\ &= \varphi(z - h\varepsilon(z)) + \varepsilon(z)\varphi(h - 1) + \varepsilon(z)\eta_{\varphi} \\ &= \varphi(z - \varepsilon(z)) + \varepsilon(z)\eta_{\varphi} \\ &= \tilde{\varphi}(z). \end{aligned}$$

The above checks complete the proof. ■

Remark 5. For motivation, the $\tilde{\varphi}$ was constructed by tracking through the following diagram.

$$\begin{array}{ccccccc} \frac{C^0(H, A)}{B^0(H, A)} & \longrightarrow & \frac{C^0(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))}{B^0(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))} & \longrightarrow & \frac{C^0(H, \text{Hom}_{\mathbb{Z}}(I_G, A))}{B^0(H, \text{Hom}_{\mathbb{Z}}(I_G, A))} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & Z^1(H, A) = B^1(H, A) & \longrightarrow & Z^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) & \longrightarrow & Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) \end{array}$$

In short, take $\varphi \in Z^0(H, \text{Hom}_{\mathbb{Z}}(I_G, A)) = \text{Hom}_{\mathbb{Z}}(I_G, A)^H$, pull it back to $z \mapsto \varphi(z - \varepsilon(z))$. Pushing this down to $Z^1(H, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ and pulling back to $Z^1(H, A)$ takes us to the 1-cocycle $h \mapsto h\varphi(h^{-1} - 1)$. Here we use the $H^1(H, A) = 0$ condition above and adjust our lift $z \mapsto \varphi(z - \varepsilon(z))$ accordingly.

And now we can now make our dimension shifting explicit.

Lemma 6. Work in the context of [Lemma 4](#) and assume that $H \subseteq G$ is normal. We track through the isomorphism

$$\delta: H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H) \simeq H^2(G/H, A^H)$$

given by the exact sequence

$$0 \rightarrow A^H \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H \rightarrow \text{Hom}_{\mathbb{Z}}(I_G, A)^H \rightarrow 0.$$

Proof. We begin with some $c \in H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, A)^H)$. To track through the δ , we define

$$\tilde{c}(gH) := c(gH)(z - \varepsilon(z)) + \eta_{c(gH)}\varepsilon(z)$$

to be the lift given in [Lemma 4](#). Now, we are given that $dc = 0$, which here means that any $z \in \mathbb{Z}[G]$ and $gH, g'H \in G/H$ will have

$$\begin{aligned} 0 &= (dc)(gH, g'H)(z - \varepsilon(z)) \\ 0 &= (gH \cdot c(g'H) - c(gg'H) + c(gH))(z - \varepsilon(z)) \\ 0 &= g \cdot c(g'H) (g^{-1}z - g^{-1}\varepsilon(z)) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)) \\ g \cdot c(g'H) (g^{-1} - 1) \varepsilon(z) &= g \cdot c(g'H) (g^{-1}z - \varepsilon(z)) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)) \\ g \cdot c(g'H) (g^{-1} - 1) \varepsilon(z) &= g \cdot c(g'H) (g^{-1}z - \varepsilon(g^{-1}z)) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)). \end{aligned}$$

We now directly compute that

$$\begin{aligned} (d\tilde{c})(gH, g'H)(z) &= (gH \cdot c(g'H) - c(gg'H) + c(gH))(z) \\ &= g \cdot c(g'H) (g^{-1}z - \varepsilon(g^{-1}z)) + g\eta_{c(g'H)}\varepsilon(z) \\ &\quad - c(gg'H)(z - \varepsilon(z)) - \eta_{c(gg'H)}\varepsilon(z) \\ &\quad + c(gH)(z - \varepsilon(z)) + \eta_{c(gH)}\varepsilon(z) \\ &= (g \cdot c(g'H) (g^{-1} - 1) + g \cdot \eta_{c(g'H)} - \eta_{c(gg'H)} + \eta_{c(gH)}) \varepsilon(z) \end{aligned}$$

As such, we have pulled ourselves back to the 2-cocycle given by

$$u(gH, g'H) := g \cdot c(g'H) (g^{-1} - 1) + g \cdot \eta_{c(g'H)} - \eta_{c(gg'H)} + \eta_{c(gH)}.$$

We quickly note that this is in fact independent of our choice of representative $g \in gH$: changing representative of g to gh for $h \in H$ will only affect the terms

$$h \cdot c(g'H) (h^{-1}g^{-1} - 1) + h\eta_{c(g'H)} = c(g'H) (g^{-1} - h) + c(g'H) (h - 1) + \eta_{c(g'H)} = c(g'H) (g^{-1} - 1) + \eta_{c(g'H)},$$

so we are indeed safe. This completes the proof. ■

We now make [Theorem 1](#) explicit in the case of $q = 2$.

Lemma 7. Let G be a group with normal subgroup $H \subseteq G$. Fix a G -module A with $H^1(H, A) = 0$, and define the function $\eta_{\bullet}: \text{Hom}_{\mathbb{Z}}(I_G, A)^H \rightarrow A$ of [Lemma 4](#). Given $c \in Z^2(G, A)$ such that $\text{Res}_H^G c \in B^2(H, A)$; in particular, suppose we have $b \in \text{Hom}_{\mathbb{Z}}(I_G, A)$ such that all $h \in H$ have

$$\text{Res}_H^G(\delta^{-1}c)(h) = (db)(h) = h \cdot b - h,$$

where δ^{-1} is the inverse isomorphism of [Lemma 3](#). Then we find $u \in Z^2(G/H, A^H)$ such that

$$[\text{Inf } u] = [c]$$

in $H^2(G, A)$.

Proof. The main point is that boundary morphisms δ commute with Res and Inf. By construction, we have that $(\text{Res}_H^G \delta^{-1}c) - db = 0$ in $Z^1(H, \text{Hom}_{\mathbb{Z}}(I_G, A))$. Pulling back to $Z^1(G, \text{Hom}_{\mathbb{Z}}(I_G, A))$, we note that

$$c' := (\delta^{-1}c - db) \in Z^1(G, \text{Hom}_{\mathbb{Z}}(I_G, A))$$

vanishes on H by hypothesis. Because $\delta^{-1}c - db$ is a 1-cocycle, we are able to write

$$c'(gg') = c'(g) + gc'(g').$$

Letting g' vary over H , we see that $\delta^{-1}c - db$ is well-defined on G/H . On the other hand, for any $h \in H$ and $g \in G$, we note that $g^{-1}hg \in H$, so

$$c'(g) = c'(g \cdot g^{-1}hg) = c'(hg) = c'(h) + hc(g),$$

implying that $c'(g) \in \text{Hom}_{\mathbb{Z}}(I_G, A)^H$.

We are now ready to apply [Lemma 6](#), which we use on c' , thus defining $u := \delta(c')$. Explicitly, we have

$$u(gH, g'H) = g \cdot c'(g'H) (g^{-1} - 1) + g \cdot \eta_{c'(g'H)} - \eta_{c'(gg'H)} + \eta_{c'(gH)}.$$

This is explicit enough for our purposes. Observe that $[\text{Inf } u] = [c]$ because $[\text{Inf } c'] = [\delta^{-1}c]$, and δ commutes with Inf. ■

3.2 Number Theory

Throughout, we will let $u_{L/K}$ denote a representative of the fundamental class in $H^2(L/K)$ rather than the actual cohomology class, mostly out of laziness.

We now return to the set-up in [section 1](#) and track through [Lemma 7](#) in our case. For reference, the following is the diagram that we will be chasing around; here $G := \text{Gal}(ML/K)$ and $H := \text{Gal}(ML/L)$.

$$\begin{array}{ccccccc} & & & & H^2(\text{Gal}(M/K), M^\times) & & \\ & & & & \downarrow \text{Inf} & & \\ 0 & \longrightarrow & H^2(\text{Gal}(L/K), L^\times) & \xrightarrow{\text{Inf}} & H^2(G, ML^\times) & \xrightarrow{\text{Res}} & H^2(\text{Gal}(ML/L), ML^\times) \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ 0 & \longrightarrow & H^1(G/H, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)^H) & \xrightarrow{\text{Inf}} & H^1(G, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)) & \xrightarrow{\text{Res}} & H^1(H, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)) \end{array}$$

To begin, we know that we can write

$$u_{M/K}(\sigma_K^i, \sigma_K^j) = p^{\lfloor \frac{i+j}{n} \rfloor} = \begin{cases} 1 & i+j < n, \\ p & i+j \geq n. \end{cases}$$

Inflating this down to $H^2(G, ML^\times)$ gives

$$(\text{Inf } u_{M/K}) (\sigma_K^{a_1} \sigma_x^{a_2}, \sigma_K^{b_1} \sigma_x^{b_2}) = p^{\lfloor \frac{a_1+b_1}{n} \rfloor}.$$

Now, we use [Lemma 2](#) to move down to $H^1(G, \text{Hom}_{\mathbb{Z}}(I_G, ML^\times))$ as

$$\delta^{-1}(\text{Inf } u_{M/K}) (\sigma_K^{a_1} \sigma_x^{a_1}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) = \sigma_K^{b_1} \sigma_x^{b_2} \cdot (\text{Inf } u_{M/K}) (\sigma_K^{[-b_1]} \sigma_x^{[-b_2]}, \sigma_K^{a_1} \sigma_x^{a_2}) = p^{\lfloor \frac{a_1+[-b_1]}{n} \rfloor},$$

where $[k]$ denote the integer $0 \leq [k] < n$ such that $k \equiv [k] \pmod{n}$.

Now, we need to show that the restriction to $H = \langle \sigma_K^f \rangle$ is a coboundary. That is, we need to find $b \in \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)$ such that

$$\delta^{-1}(\text{Inf } u_{M/K}) (\sigma_K^{fa_1}) = \frac{\sigma_K^{fa_1} \cdot b}{b}.$$

Because I_G is freely generated by elements of the form $g - 1$ for $g \in G$, it suffices to plug in some arbitrary $\sigma_K^{b_1} \sigma_x^{b_2} - 1$, which we see requires

$$\begin{aligned} p^{\lfloor \frac{fa_1 + [-b_1]}{n} \rfloor} &= \frac{(\sigma_K^{fa_1} \cdot b) (\sigma_K^{b_1} \sigma_x^{b_2} - 1)}{b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)} \\ &= \frac{\sigma_K^{fa_1} b (\sigma_K^{b_1 - fa_1} \sigma_x^{b_2} - 1)}{\sigma_K^{fa_1} b (\sigma_K^{-fa_1} - 1) b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)}. \end{aligned}$$

We can see that b should not depend on b_2 , so we define $\hat{b}(\sigma_K^a) = b(\sigma_K^a \sigma_x^\bullet - 1)$; the above is then equivalent to

$$\begin{aligned} p^{\lfloor \frac{fa_1 + [-b_1]}{n} \rfloor} &= \frac{\sigma_K^{fa_1} \hat{b}(\sigma_K^{b_1 - fa_1})}{\sigma_K^{fa_1} \hat{b}(\sigma_K^{-fa_1}) \hat{b}(\sigma_K^{b_1})} \\ p^{\lfloor \frac{fa_1 + b_1}{n} \rfloor} &= \frac{\hat{b}(\sigma_K^{-b_1 - fa_1})}{\hat{b}(\sigma_K^{-fa_1}) \sigma_K^{-fa_1} \hat{b}(\sigma_K^{-b_1})}, \end{aligned}$$

where we have negated b_1 in the last step. At this point, the right-hand side will look a lot more natural if we set $\tau := \sigma_K^{-1}$, which turns this into

$$\frac{\hat{b}(\tau^{fa_1}) \tau^{fa_1} \hat{b}(\tau^{b_1})}{\hat{b}(\tau^{b_1 fa_1})} = (1/p)^{\lfloor \frac{fa_1 + b_1}{n} \rfloor}$$

after taking reciprocals. Thus, we see that \hat{b} should be counting carries of τ s. With this in mind, we note that $1 - \zeta_{p^\nu} \in L$ is a uniformizer because $L/\mathbb{Q}_p(\zeta_{p^\nu})$ is an unramified extension. It follows that

$$(1 - \zeta_{p^\nu})^{\varphi(p^\nu)} \in N_{ML/L}(ML^\times).$$

Further, $(1 - \zeta_{p^\nu})^{\varphi(p^\nu)}$ is only a unit (in \mathcal{O}_L^\times) multiplied p , so in fact p is a norm from ML^\times because ML/L is unramified and so all units in \mathcal{O}_L^\times are norms from ML^\times . Thus, we find $\alpha \in ML^\times$ such that

$$N_{ML/L}(\alpha) = p.$$

The point of doing all of this is so that we can codify our carrying by writing

$$\hat{b}(\tau^a) := \prod_{i=0}^{\lfloor a/f \rfloor - 1} \tau^{if}(\alpha)^{-1}.$$

Tracking out \hat{b} backwards to b , our desired $b \in \text{Hom}_{\mathbb{Z}}(I_G, ML^\times)$ is given by

$$b(\sigma_K^{a_1} \sigma_x^{a_2} - 1) = \prod_{i=0}^{\lfloor [-a_1]/f \rfloor - 1} \sigma_K^{-if}(\alpha)^{-1}.$$

We take a moment to write out $c := \delta^{-1}(\text{Inf } u_{M/K})/db$, which looks like

$$\begin{aligned}
c(\sigma_K^{a_1} \sigma_x^{a_2}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) &= \frac{\delta^{-1}(\text{Inf } u_{M/K})}{db} (\sigma_K^{a_1} \sigma_x^{a_2}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) \\
&= \frac{\delta^{-1}(\text{Inf } u_{M/K}) (\sigma_K^{a_1} \sigma_x^{a_2}) (\sigma_K^{b_1} \sigma_x^{b_2} - 1)}{(\sigma_K^{a_1} \sigma_x^{a_2} b) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) / b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)} \\
&= \frac{p^{\lfloor (a_1 + [-b_1])/n \rfloor}}{\sigma_K^{a_1} \sigma_x^{a_2} b (\sigma_K^{b_1 - a_1} \sigma_x^{b_2 - a_2} - \sigma_K^{-a_1} \sigma_x^{-a_2}) / b (\sigma_K^{b_1} \sigma_x^{b_2} - 1)} \\
&= p^{\lfloor (a_1 + [-b_1])/n \rfloor} \cdot \hat{b}(\sigma_K^{b_1}) \cdot \sigma_K^{a_1} \sigma_x^{a_2} \left(\frac{\hat{b}(\sigma_K^{-a_1})}{\hat{b}(\sigma_K^{b_1 - a_1})} \right).
\end{aligned}$$

Before proceeding, we discuss a few special cases.

- Taking $\sigma_K^{a_1} \sigma_x^{a_2} = \sigma_x$, we get

$$\begin{aligned}
c(\sigma_x) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) &= p^{\lfloor (0 + [-b_1])/n \rfloor} \cdot \hat{b}(\sigma_K^{b_1}) \cdot \sigma_x \left(\frac{1}{\hat{b}(\sigma_K^{b_1})} \right) \\
&= \hat{b}(\sigma_K^{b_1}) / \sigma_x \hat{b}(\sigma_K^{b_1}).
\end{aligned}$$

In particular, $c(\sigma_x) (\sigma_K^{-1} - 1) = 1$, provided that $f > 1$. Additionally, $c(\sigma_x) (\sigma_x^{b_2} - 1) = 1$.

Our general theory says that $h \mapsto c(\sigma_x)(h - 1)$ is a 1-cocycle in $Z^1(H, ML^\times)$ (though we could also check this directly), so Hilbert's Theorem 90 promises us a magical element $\eta_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{\sigma_K^{fb_1} \eta_{c(\sigma_x)}}{\eta_{c(\sigma_x)}} = \frac{\hat{b}(\sigma_K^{fb_1})}{\sigma_x \hat{b}(\sigma_K^{fb_1})}$$

for all $\sigma_K^{fb_1} \in H$. This condition will be a little clearer if we write everything in terms of $\tau := \sigma_K^{-1}$, which transforms this into

$$\frac{\tau^{fb_1} \eta_{c(\sigma_x)}}{\eta_{c(\sigma_x)}} = \frac{\hat{b}(\tau^{-fb_1})}{\sigma_x \hat{b}(\tau^{-fb_1})} = \prod_{i=0}^{b_1-1} \frac{\tau^{if}(\alpha^{-1})}{\sigma_x \tau^{if}(\alpha^{-1})} = \prod_{i=0}^{b_1-1} \frac{\sigma_x \tau^{if}(\alpha)}{\tau^{if}(\alpha)}.$$

Because we are dealing with a cyclic group H , it is not too hard to see that it suffices merely for $b_1 = 1$ to hold, so our magical element $\eta_{c(\sigma_x)}$ merely requires

$$\boxed{\frac{\sigma_K^{-f}(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} = \frac{\sigma_x(\alpha)}{\alpha}}$$

after inverting τ back to σ_K .

- Taking $\sigma_K^{a_1} \sigma_x^{a_2} = \sigma_K$, we get

$$c(\sigma_K) (\sigma_K^{b_1} \sigma_x^{b_2} - 1) = p^{\lfloor (1 + [-b_1])/n \rfloor} \cdot \hat{b}(\sigma_K^{b_1}) \cdot \sigma_K \left(\frac{\hat{b}(\sigma_K^{-1})}{\hat{b}(\sigma_K^{b_1-1})} \right).$$

In particular, $\sigma_K^{b_1} \sigma_x^{b_2} = \sigma_K^{-1}$ will give $c(\sigma_K) (\sigma_K^{-1} - 1) = 1$. We will also want $c(\sigma_K) (\sigma_K^{-b_1} - 1)$ for $0 \leq b_1 < f$. Using the fact that $f < n$ and $f > 1$, it is not too hard to see that everything will cancel

down to 1 except in the case where $b_1 = f - 1$, where we get

$$c(\sigma_K) \left(\sigma_K^{-(f-1)} - 1 \right) = \sigma_K \left(\frac{1}{\hat{b}(\sigma_K^{-f})} \right) = \sigma_K(\alpha).$$

Continuing as before, our general theory says that $h \mapsto c(\sigma_x)(h - 1)$ is a 1-cocycle in $Z^1(H, ML^\times)$, though again we could just check this directly. It follows that Hilbert's Theorem 90 promises us a magical element $\eta_{c(\sigma_K)} \in ML^\times$ such that

$$\frac{\sigma_K^{fb_1} \eta_{c(\sigma_K)}}{\eta_{c(\sigma_K)}} = p^{\lfloor (1+[-fb_1])/n \rfloor} \cdot \hat{b}(\sigma_K^{fb_1}) \cdot \sigma_K \left(\frac{\hat{b}(\sigma_K^{-1})}{\hat{b}(\sigma_K^{fb_1-1})} \right)$$

for all $\sigma_K^{fb_1} \in H$. Using $f > 1$, this collapses down to

$$\frac{\sigma_K^{fb_1} \eta_{c(\sigma_K)}}{\eta_{c(\sigma_K)}} = \frac{\hat{b}(\sigma_K^{fb_1})}{\sigma_K \hat{b}(\sigma_K^{fb_1-1})}.$$

As before, this condition will be a little clearer if we set $\tau := \sigma_K^{-1}$, which turns the condition into

$$\frac{\tau^{fb_1} \eta_{c(\sigma_K)}}{\eta_{c(\sigma_K)}} = \frac{\hat{b}(\tau^{fb_1})}{\sigma_K \hat{b}(\tau^{fb_1+1})} = \prod_{i=0}^{b_1-1} \frac{\tau^{if}(\alpha^{-1})}{\sigma_K \tau^{if}(\alpha^{-1})} = \prod_{i=0}^{b_1-1} \frac{\sigma_K \tau^{if}(\alpha)}{\tau^{if}(\alpha)}.$$

(Notably, $\hat{b}(\tau^{fb_1}) = \hat{b}(\tau^{fb_1+1})$ because $f > 1$.) Again, because H is cyclic generated by τ^f , an induction shows that it suffices to check this condition for $b_1 = 1$, which means that our magical element $\eta_{c(\sigma_K)} \in ML^\times$ is constructed so that

$$\boxed{\frac{\sigma_K^{-f}(\eta_{c(\sigma_K)})}{\eta_{c(\sigma_K)}} = \frac{\sigma_K(\alpha)}{\alpha}}$$

where we have again inverted back from τ to σ_K .

- We will not actually need a more concrete description of this, but we remark that we can run the same story for any $g \in G$ through to get an element $\eta_{c(g)} \in ML^\times$ such that

$$\frac{\sigma_K^{fb_1} \eta_{c(g)}}{\eta_{c(g)}} = \frac{1}{c(g)(\sigma_K^{fb_1} - 1)}$$

for any $\sigma_K^{fb_1} \in H$. As usual, this follows from our general theory.

We are now ready to describe the local fundamental class. Piecing what we have so far, we know from [Lemma 7](#) that we can write

$$u_{L/K}(g, g') := gc(g') (g^{-1} - 1) \cdot \frac{g\eta_{c(g')} \cdot \eta_{c(g)}}{\eta_{c(gg')}}.$$

Here are the values that we care about for our specific computation.

- We write

$$\begin{aligned} u_{L/K}(\sigma_K, \sigma_x) &= \sigma_K c(\sigma_x) (\sigma_K^{-1} - 1) \cdot \frac{\sigma_K \eta_{c(\sigma_x)} \cdot \eta_{c(\sigma_K)}}{\eta_{c(\sigma_K \sigma_x)}} \\ &= \frac{\sigma_K \eta_{c(\sigma_x)} \cdot \eta_{c(\sigma_K)}}{\eta_{c(\sigma_K \sigma_x)}}. \end{aligned}$$

- We write

$$\begin{aligned} u_{L/K}(\sigma_x, \sigma_K) &= \sigma_x c(\sigma_K) (\sigma_x^{-1} - 1) \cdot \frac{\sigma_x \eta_{c(\sigma_K)} \cdot \eta_{c(\sigma_x)}}{\eta_{c(\sigma_x \sigma_K)}} \\ &= \frac{\sigma_x \eta_{c(\sigma_K)} \cdot \eta_{c(\sigma_x)}}{\eta_{c(\sigma_x \sigma_K)}}. \end{aligned}$$

- In particular, we know that we can set β in a triple equal to

$$\begin{aligned} \beta &:= \frac{u_{L/K}(\sigma_K, \sigma_x)}{u_{L/K}(\sigma_x, \sigma_K)} \\ &= \frac{\sigma_K \eta_{c(\sigma_x)} \cdot \eta_{c(\sigma_K)} / \eta_{c(\sigma_K \sigma_x)}}{\sigma_x \eta_{c(\sigma_K)} \cdot \eta_{c(\sigma_x)} / \eta_{c(\sigma_x \sigma_K)}} \\ \beta &= \frac{\sigma_K (\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x (\eta_{c(\sigma_K)})}. \end{aligned}$$

As a sanity check, we can hit this β with σ_K^{-f} to show that $\beta \in (ML)^H = L$; namely, $\sigma_K^{-f} \eta_{c(\sigma_K)} = \frac{\sigma_K \alpha}{\alpha} \cdot \eta_{c(\sigma_K)}$ and $\sigma_K^{-f} \eta_{c(\sigma_x)} = \frac{\sigma_x \alpha}{\alpha} \cdot \eta_{c(\sigma_x)}$ by construction, so we can see that everything will appropriately cancel out.

- We will go ahead and compute α_1 and α_2 , for completeness. For α_1 , our element is given by

$$\begin{aligned} \alpha_1 &:= \prod_{i=0}^{f-1} u_{L/K}(\sigma_K^i, \sigma_K) \\ &= \prod_{i=0}^{f-1} \left(\sigma_K^i c(\sigma_K, \sigma_K^{-i} - 1) \cdot \frac{\sigma_K^i \eta_{c(\sigma_K)} \cdot \eta_{c(\sigma_K^i)}}{\eta_{c(\sigma_K^{i+1})}} \right). \end{aligned}$$

Recall from our general theory that $\eta_{c(g)}$ only depends on the coset of g in G/H , so we see that the product of the quotients $\eta_{c(\sigma_K^i)} / \eta_{c(\sigma_K^{i+1})}$ will cancel out. As for the c term, we know from our computation that this is 1 until $i = f - 1$, which gives $\sigma_K(\alpha)$. As such, we collapse down to

$$\alpha_1 = \sigma_K^f(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^i (\eta_{c(\sigma_K)}).$$

- For α_2 , our element is given by

$$\begin{aligned} \alpha_2 &:= \prod_{i=0}^{\varphi(p^\nu)-1} u_{L/K}(\sigma_x^i, \sigma_x) \\ &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i c(\sigma_x, \sigma_x^{-i} - 1) \cdot \frac{\sigma_x^i \eta_{c(\sigma_x)} \cdot \eta_{c(\sigma_x^i)}}{\eta_{c(\sigma_x^{i+1})}}. \end{aligned}$$

Recalling that σ_x has order $\varphi(p^\nu)$, our quotient term $\eta_{c(\sigma_x^i)} / \eta_{c(\sigma_x^{i+1})}$ will again cancel out. Additionally, the cocycle c always spits out 1 on these inputs, so we are left with

$$\alpha_2 = \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i (\eta_{c(\sigma_x)}).$$

We summarize the results above in the following theorem.

Theorem 8. Fix everything as in the set-up. Then there exists some $\alpha \in ML^\times$ such that $N_{ML/L}(\alpha) = p$ and elements in $\eta_{c(\sigma_K)}, \eta_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{\sigma_K^{-f}(\eta_{c(\sigma_K)})}{\eta_{c(\sigma_K)}} = \frac{\sigma_K(\alpha)}{\alpha} \quad \text{and} \quad \frac{\sigma_K^{-f}(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} = \frac{\sigma_x(\alpha)}{\alpha}.$$

Then the triple

$$(\alpha_1, \alpha_2, \beta) := \left(\sigma_K^f(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta_{c(\sigma_x)}), \quad \frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\text{Gal}(L/K), L^\times)$.

We remark that we can replace α with $\sigma_K^f(\alpha)$ (which still has norm p) while keeping all other variables the same; this gives us the following slightly prettier presentation. Note that we have multiplied the equations for η_\bullet by σ_K^f on both sides.

Corollary 9. Fix everything as in the set-up. Then there exists some $\alpha \in ML^\times$ such that $N_{ML/L}(\alpha) = p$ and elements in $\eta_{c(\sigma_K)}, \eta_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{\eta_{c(\sigma_K)}}{\sigma_K^f(\eta_{c(\sigma_K)})} = \frac{\sigma_K(\alpha)}{\alpha} \quad \text{and} \quad \frac{\eta_{c(\sigma_x)}}{\sigma_K^f(\eta_{c(\sigma_x)})} = \frac{\sigma_x(\alpha)}{\alpha}.$$

Then the triple

$$(\alpha_1, \alpha_2, \beta) := \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta_{c(\sigma_x)}), \quad \frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\text{Gal}(L/K), L^\times)$.

3.3 Checks

In this section we run some checks and discuss some consequences of [Theorem 8](#), in the form of [Corollary 9](#). For these results, we recall that we set $L := \mathbb{Q}_p(\zeta_N)$ and $L_1 := \mathbb{Q}_p(\zeta_{p^\nu})$ and $L_2 := \mathbb{Q}_p(\zeta_m)$ so that $\bar{\sigma}_K = \sigma_K|_{L_1}$ generates $\text{Gal}(L/L_1)$ and σ_x generates $\text{Gal}(L/L_2)$.

In the discussion which follows, we will make repeated use of the fact that (using notation of [Corollary 9](#))

$$\sigma_K^f(\eta_{c(\sigma_K)}) = \frac{\alpha}{\sigma_K(\alpha)} \cdot \eta_{c(\sigma_K)} \quad \text{and} \quad \sigma_K^f(\eta_{c(\sigma_x)}) = \frac{\alpha}{\sigma_x(\alpha)} \cdot \eta_{c(\sigma_x)}.$$

And here are our checks; we start by showing that our elements are in the right field.

Lemma 10. Fix a triple $(\alpha_1, \alpha_2, \beta)$ as in [Corollary 9](#). Then the following are true.

- (a) $\alpha_1 \in L_1^\times$.
- (b) $\alpha_2 \in L_2^\times$.
- (c) $\beta \in L^\times$.

Proof. We run the checks one at a time.

(a) It suffices to show that α_1 is fixed by $\text{Gal}(M/L_1) = \langle \sigma_K \rangle$. As such, we simply compute

$$\begin{aligned}
 \sigma_K(\alpha_1) &= \sigma_K \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i (\eta_{c(\sigma_K)}) \right) \\
 &= \sigma_K(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^{i+1} (\eta_{c(\sigma_K)}) \\
 &= \sigma_K(\alpha) \cdot \sigma_K^f (\eta_{c(\sigma_K)}) \prod_{i=1}^{f-1} \sigma_K^{i+1} (\eta_{c(\sigma_K)}) \\
 &= \alpha \cdot \eta_{c(\sigma_K)} \prod_{i=1}^{f-1} \sigma_K^{i+1} (\eta_{c(\sigma_K)}) \\
 &= \prod_{i=0}^{f-1} \sigma_K^{i+1} (\eta_{c(\sigma_K)}) \\
 &= \alpha_1.
 \end{aligned}$$

(b) It suffices to show that α_2 is fixed by $\text{Gal}(M/L_2) = \langle \sigma_K^f, \sigma_x \rangle$. On one hand,

$$\begin{aligned}
 \sigma_K^f(\alpha_2) &= \sigma_K^f \left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i (\eta_{c(\sigma_x)}) \right) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i \left(\sigma_K^f \eta_{c(\sigma_x)} \right) \\
 &= \left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i \left(\frac{\alpha}{\sigma_x(\alpha)} \right) \right) \cdot \left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i (\eta_{c(\sigma_x)}) \right) \\
 &= \left(\prod_{i=0}^{\varphi(p^\nu)-1} \frac{\sigma_x^i(\alpha)}{\sigma_x^{i+1}(\alpha)} \right) \cdot \alpha_2 \\
 &= \alpha_2,
 \end{aligned}$$

where the product telescopes because σ_x has order $\varphi(p^\nu)$.

On the other hand,

$$\begin{aligned}
 \sigma_x(\alpha_2) &= \sigma_x \left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i (\eta_{c(\sigma_x)}) \right) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^{i+1} (\eta_{c(\sigma_x)}) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i (\eta_{c(\sigma_x)}),
 \end{aligned}$$

where we have again used the fact that σ_x has order $\varphi(p^\nu)$. This last product is α_2 , so we are done.

(c) It suffices to show that β is fixed by $\text{Gal}(M/L) = \langle \sigma_K^f \rangle$. Applying force, we see

$$\begin{aligned}
 \sigma_K^f(\beta) &= \sigma_K^f \left(\frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \right) \\
 &= \frac{\sigma_K(\sigma_K^f \eta_{c(\sigma_x)})}{\sigma_K^f \eta_{c(\sigma_x)}} \cdot \frac{\sigma_K^f \eta_{c(\sigma_K)}}{\sigma_x(\sigma_K^f \eta_{c(\sigma_K)})} \\
 &= \frac{\sigma_K(\alpha/\sigma_x \alpha) \cdot \sigma_K(\eta_{c(\sigma_x)})}{(\alpha/\sigma_x \alpha) \cdot \eta_{c(\sigma_x)}} \cdot \frac{(\alpha/\sigma_K \alpha) \cdot \eta_{c(\sigma_K)}}{\sigma_x(\alpha/\sigma_K \alpha) \cdot \sigma_x(\eta_{c(\sigma_K)})} \\
 &= \frac{\sigma_K \alpha}{\sigma_K \sigma_x \alpha} \cdot \frac{\sigma_x \alpha}{\alpha} \cdot \frac{\alpha}{\sigma_K \alpha} \cdot \frac{\sigma_x \sigma_K \alpha}{\sigma_x \alpha} \cdot \frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \\
 &= \beta.
 \end{aligned}$$

The above checks complete the proof. ■

Next we show the relations.

Lemma 11. Fix a triple $(\alpha_1, \alpha_2, \beta)$ as in [Corollary 9](#). Then the following are true.

- (a) $N_{L/L_1}(\beta) = \alpha_1/\sigma_x \alpha_1$.
- (b) $N_{L/L_2}(\beta^{-1}) = \alpha_2/\bar{\sigma}_K \alpha_2$.

Proof. We go one at a time.

(a) Note $\text{Gal}(L/L_1) = \langle \bar{\sigma}_K \rangle$. In particular, $\bar{\sigma}_K$ has order f , so we can just compute out

$$\begin{aligned}
 N_{L/L_1}(\beta) &= \prod_{i=0}^{f-1} \sigma_K^i(\beta) \\
 &= \prod_{i=0}^{f-1} \sigma_K^i \left(\frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \right) \\
 &= \prod_{i=0}^{f-1} \frac{\sigma_K^{i+1}(\eta_{c(\sigma_x)})}{\sigma_K^i(\eta_{c(\sigma_x)})} \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}) / \sigma_x \left(\prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}) \right) \\
 &= \frac{\sigma_K^f(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}) / \sigma_x \left(\prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}) \right) \\
 &= \frac{\alpha}{\sigma_x \alpha} \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}) / \sigma_x \left(\prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}) \right) \\
 &= \alpha_1/\sigma_x \alpha.
 \end{aligned}$$

(b) Note $\text{Gal}(L/L_2) = \langle \sigma_x \rangle$, so we compute

$$\begin{aligned}
 N_{L/L_2}(\beta) &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\beta) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i \left(\frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \right) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \frac{\sigma_x^i(\eta_{c(\sigma_K)})}{\sigma_x^{i+1}(\eta_{c(\sigma_K)})} \cdot \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_K(\eta_{c(\sigma_x)}) \Big/ \prod_{i=0}^{\varphi(p^\nu)-1} \eta_{c(\sigma_x)} \\
 &= \sigma_K \left(\prod_{i=0}^{\varphi(p^\nu)-1} \eta_{c(\sigma_x)} \right) \Big/ \prod_{i=0}^{\varphi(p^\nu)-1} \eta_{c(\sigma_x)} \\
 &= \sigma_K \alpha_2 / \alpha_2.
 \end{aligned}$$

Taking the reciprocal finishes; in particular, $\bar{\sigma}_K \alpha_2 = \sigma_K \alpha_2$ is a legal expression because $\alpha_2 \in L^\times$.

The above checks complete the proof. ■

3.4 Consequences

With some checks out of the way, here are some actual consequences. To begin, we state Hilbert's Theorem 90.

Lemma 12. Suppose that L/K is a (finite) cyclic extension of fields such that $\Gamma := \text{Gal}(L/K)$ is generated by $\sigma \in \Gamma$. Given some $\alpha \in L^\times$ such that $N(\alpha) = 1$, there exists $\beta_0 \in L^\times$ such that $\alpha = \beta_0 / \sigma \beta_0$. In fact, this β_0 is unique "up to a multiple in K^\times " in the sense that

$$\{\beta \in L^\times : \alpha = \beta / \sigma \beta\} = \{x \beta_0 : x \in K^\times\}.$$

Proof. That such a β_0 exists follows directly from Hilbert's Theorem 90. For the last sentence, of course any $\beta := x \beta_0 \in L^\times$ with $x \in K^\times$ will have

$$\frac{\beta}{\sigma \beta} = \frac{\beta_0}{\sigma \beta_0} = \alpha.$$

In the other direction, if $\beta \in L^\times$ has $\beta / \sigma \beta = \alpha$, then

$$\sigma(\beta / \beta_0) = (\sigma \beta) / (\sigma \beta_0) = \beta / \beta_0,$$

so $\beta / \beta_0 \in K^\times$ and $\beta = (\beta / \beta_0) \cdot \beta_0$. ■

And here are some quick consequences of this.

Corollary 13. Fix everything as in the set-up, and fix $\alpha \in ML^\times$ such that $N_{ML/L}(\alpha) = p$. Choosing some $\sigma \in \{\sigma_K, \sigma_x\}$, the elements η_σ satisfying

$$\frac{\eta_\sigma}{\sigma_K^f(\eta_\sigma)} = \frac{\sigma(\alpha)}{\alpha}$$

are unique up to a multiple in L^\times , in the sense of [Lemma 12](#).

Proof. Note that $\text{Gal}(ML/L) = \langle \sigma_K^f \rangle$ is cyclic generated by σ_K^f and $N_{ML/L}(\sigma \alpha / \alpha) = p/p = 1$, so we may simply apply [Lemma 12](#) directly to get the result. ■

We might be worried that our choice α is affecting the set of $\eta_{c(\sigma_K)}$ or $\eta_{c(\sigma_x)}$, but in fact they are not, more or less.

Corollary 14. Fix everything as in the set-up, and choose $\sigma \in \{\sigma_K, \sigma_x\}$. Given $\alpha \in ML^\times$ such that $N_{ML/L}(\alpha) = p$, define

$$S_\alpha := \left\{ \eta_\sigma \in ML^\times : \frac{\eta_\sigma}{\sigma_K^f(\eta_\sigma)} = \frac{\sigma(\alpha)}{\alpha} \right\}.$$

Then the set S_α is “unique up to a multiple in ML^\times ” in the sense that two $\alpha, \alpha' \in ML^\times$ with $N_{ML/L}(\alpha) = N_{ML/L}(\alpha') = p$ have some $\chi \in ML^\times$ such that

$$S_\alpha = \chi \cdot S_{\alpha'} := \{\chi \cdot \eta_\sigma : \eta_\sigma \in S_{\alpha'}\}.$$

Proof. Suppose $\alpha, \alpha' \in ML^\times$ satisfy $N_{ML/L}(\alpha) = N_{ML/L}(\alpha') = p$. The key point is that

$$N_{ML/L}(\alpha/\alpha') = p/p = 1,$$

so [Lemma 12](#) promises us some $\gamma \in ML^\times$ such that $\alpha/\alpha' = \gamma/\sigma_K^f(\gamma)$. As such, we see that

$$\frac{\sigma(\alpha)}{\alpha} = \frac{\sigma(\alpha/\alpha')}{\alpha/\alpha'} \cdot \frac{\sigma(\alpha')}{\alpha'} = \frac{(\sigma\gamma/\gamma)}{\sigma_K^f(\sigma\gamma/\gamma)} \cdot \frac{\sigma(\alpha')}{\alpha'}.$$

As such, we set $\chi := (\sigma\gamma/\gamma)$.

To finish, we check that $S_\alpha \subseteq \chi \cdot S_{\alpha'}$, and the other inclusion is similar. Well, if $\eta_\sigma \in S_\alpha$, then

$$\frac{\chi\eta_\sigma}{\sigma_K^f(\chi\eta_\sigma)} = \frac{\chi}{\sigma_K^f(\chi)} \cdot \frac{\eta_\sigma}{\sigma_K^f(\eta_\sigma)} = \frac{(\sigma\gamma/\gamma)}{\sigma_K^f(\sigma\gamma/\gamma)} \cdot \frac{\sigma(\alpha')}{\alpha'} = \frac{\sigma(\alpha)}{\alpha},$$

so $\chi\eta_\sigma \in S_{\alpha'}$. This finishes. ■

We now return to describing triples.

Corollary 15. Fix everything as in the set-up, and fix $\alpha \in ML^\times$ such that $N_{ML/L}(\alpha) = p$. Then, for any triple $(\alpha'_1, \alpha'_2, \beta')$ corresponding to the fundamental class, there exist elements $\eta'_{c(\sigma_K)}, \eta'_{c(\sigma_x)} \in ML^\times$ with

$$\frac{\eta'_{c(\sigma_K)}}{\sigma_K^f(\eta'_{c(\sigma_K)})} = \frac{\sigma_K(\alpha)}{\alpha} \quad \text{and} \quad \frac{\eta'_{c(\sigma_x)}}{\sigma_K^f(\eta'_{c(\sigma_x)})} = \frac{\sigma_x(\alpha)}{\alpha}$$

such that

$$(\alpha'_1, \alpha'_2, \beta') = \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta'_{c(\sigma_K)}), \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{c(\sigma_x)}), \frac{\sigma_K(\eta'_{c(\sigma_x)})}{\eta'_{c(\sigma_x)}} \cdot \frac{\eta'_{c(\sigma_K)}}{\sigma_x(\eta'_{c(\sigma_K)})} \right).$$

In other words, all triples corresponding to the fundamental class come from the recipe described in [Corollary 9](#).

Proof. By [Corollary 9](#), we can certainly find some elements $\eta_{c(\sigma_K)}, \eta_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{\eta_{c(\sigma_K)}}{\sigma_K^f(\eta_{c(\sigma_K)})} = \frac{\sigma_K(\alpha)}{\alpha} \quad \text{and} \quad \frac{\eta_{c(\sigma_x)}}{\sigma_K^f(\eta_{c(\sigma_x)})} = \frac{\sigma_x(\alpha)}{\alpha},$$

for which

$$(\alpha_1, \alpha_2, \beta) := \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}), \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta_{c(\sigma_x)}), \frac{\sigma_K(\eta_{c(\sigma_x)})}{\eta_{c(\sigma_x)}} \cdot \frac{\eta_{c(\sigma_K)}}{\sigma_x(\eta_{c(\sigma_K)})} \right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\text{Gal}(L/K), L^\times)$. In particular, $(\alpha_1, \alpha_2, \beta)$ and $(\alpha'_1, \alpha'_2, \beta')$ both correspond to the same cohomology class and hence in the same equivalence class of triples, so we know that there exist $m_1, m_2 \in L^\times$ such that

$$\alpha'_1 = \alpha_1 \cdot N_{L/L_1}(m_1), \quad \alpha'_2 = \alpha_2 \cdot N_{L/L_2}(m_2), \quad \beta' = \beta \cdot \frac{\sigma_K(m_2)}{m_2} \cdot \frac{m_1}{\sigma_x(m_1)}.$$

As such, we set $\eta'_{c(\sigma_K)} := \eta_{c(\sigma_K)} \cdot m_1$ and $\eta'_{c(\sigma_x)} := \eta_{c(\sigma_x)} \cdot m_2$, and these can be checked to work. For example, $\eta'_{c(\sigma_K)}$ satisfies

$$\frac{\eta'_{c(\sigma_K)}}{\sigma_K^f(\eta'_{c(\sigma_K)})} = \frac{\sigma_K(\alpha)}{\alpha} \quad \text{and} \quad \frac{\eta'_{c(\sigma_x)}}{\sigma_K^f(\eta'_{c(\sigma_x)})} = \frac{\sigma_x(\alpha)}{\alpha}$$

by [Lemma 12](#). The rest of the checks are similar. ■

Corollary 16. Fix everything as in the set-up, and let $\pi_1 \in L_1^\times$ be a uniformizer. If the triple $(\alpha_1, \alpha_2, \beta)$ is a triple corresponding to the fundamental class, then

$$\alpha_1 \equiv \pi_1 \pmod{N_{L/L_1}(L^\times)}.$$

Proof by triples. Note that L/L_1 is an unramified extension, so all elements of absolute value 1 are norms, so there is in fact a class of elements containing all uniformizers in $L_1^\times / N_{L/L_1}(L^\times)$. Further, because α_1 is also only defined up to a multiple in $N_{L/L_1}(L^\times)$, to show that the classes in $L^\times / N_{L/L_1}(L^\times)$ coincide, it thus suffices to exhibit a single triple $(\alpha_1, \alpha_2, \beta)$ such that $\alpha_1 \in L_1^\times$ is a uniformizer.

This is a matter of force. To begin, we can use [Corollary 9](#) to find some α with $N_{ML/L}(\alpha) = p$ and $\eta_{c(\sigma_K)}, \eta_{c(\sigma_x)} \in ML^\times$ giving the triple $(\alpha_1, \alpha_2, \beta)$ as described. The idea is to force $\eta_{c(\sigma_K)}$ to have valuation zero.

Let v_{ML} be the fixed valuation of ML extending the standard valuation $v_{\mathbb{Q}_p}$ on \mathbb{Q}_p , and let v_L be its restriction to L . Because ML/L is an unramified, the image of v_{ML} and v_L in \mathbb{Q} is the same. In particular, we can find some $m_1 \in L_1^\times$ such that

$$v_{ML}(\eta_{c(\sigma_K)}) = v_L(m_1).$$

Thus, we replace $\eta_{c(\sigma_K)}$ with $\eta_{c(\sigma_K)}/m_1$, and we still satisfy the conditions of [Corollary 9](#) by [Lemma 12](#) while getting $v_{ML}(\eta_{c(\sigma_K)}) = 0$. Now, the corresponding α_1 looks like

$$\alpha_1 = \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{c(\sigma_K)}).$$

In particular, defining $v_{L_1} := v_L|_{L_1}$, it follows

$$v_{L_1}(\alpha_1) = v_{ML}(\alpha_1) = v_{ML}(\alpha),$$

However, $N_{ML/L}(\alpha) = p$ by construction, so we see that

$$[ML : L]v_{ML}(\alpha) = v_{ML}(p) = v_{\mathbb{Q}_p}(p) = 1.$$

Explicitly, we see that

$$[ML : L] = [\mathbb{Q}(\zeta_{N'}) : \mathbb{Q}(\zeta_m)] = \frac{[\mathbb{Q}(\zeta_{N'}) : \mathbb{Q}_p]}{[\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p]} = \frac{n}{f} = \varphi(p^\nu).$$

However, L_1/K has ramification degree $\varphi(p^\nu)$ (from the maximal totally ramified subextension $\mathbb{Q}_p(\zeta_{p^\nu})$), so its uniformizers are the elements of valuation $1/\varphi(p^\nu)$. Thus, we have computed that α_1 has the correct valuation and hence is a uniformizer. ■

Proof by the Artin map. We take a moment to say that there is an alternate derivation of [Corollary 16](#) using the Artin map: one can show that, if $u \in Z^2(L/K)$ is a representative of the fundamental class of an abelian extension L/K , then

$$\begin{aligned} \text{Gal}(L/K) &\rightarrow K^\times / N(L^\times) \\ \sigma &\mapsto \prod_{g \in \text{Gal}(L/K)} u(g, \sigma) \end{aligned}$$

is the inverse Artin map. In particular, from our explicit formula for α_1 , we see

$$\alpha_1 = \prod_{g \in \text{Gal}(L/L_1)} u(g, \bar{\sigma}_K) = \theta_{L/L_1}^{-1}(\bar{\sigma}_K).$$

However, $\bar{\sigma}_K$ is the Frobenius automorphism of L/L_1 because the extension L_1/K is totally ramified, implying that the residue field of L_1 is the same as $K = \mathbb{Q}_p$. Thus, $\theta_{L/L_1}^{-1}(\bar{\sigma}_K)$ is the class containing the uniformizers of L_1^\times . ■

We close with a sanity check.

Corollary 17. Fix everything as in the set-up, and let T_α denote the set of triples $(\alpha_1, \alpha_2, \beta)$ generated by some element $\alpha \in ML^\times$ with $N_{ML/L}(\alpha) = p$ via [Corollary 9](#). Then T_α is independent of α .

Proof. The main idea is to use (the proof of) [Corollary 14](#). Fix $\alpha, \alpha' \in ML^\times$ with $N_{ML/L}(\alpha) = N_{ML/L}(\alpha') = p$, and we need to show that $T_\alpha = T_{\alpha'}$. By symmetry, it will be enough for $T_\alpha \subseteq T_{\alpha'}$.

Following the proof of [Corollary 14](#), note that $N_{ML/L}(\alpha/\alpha') = 1$, so we are promised $\gamma \in ML^\times$ such that $\alpha/\alpha' = \gamma/\sigma_K^f(\gamma)$. Then we showed that any $\sigma \in \{\sigma_K, \sigma_x\}$ can set

$$\chi_\sigma := \frac{\sigma(\gamma)}{\gamma}$$

to give $S_{\alpha, \sigma}x = \cdot S_{\alpha', \sigma}$, where $S_{\alpha, \sigma}$ is the set of possible η_σ defined in [Corollary 14](#).

We now proceed directly with the proof. Suppose that we have some triple $(\alpha_1, \alpha_2, \beta) \in T_\alpha$, which we know that we can write down as

$$(\alpha_1, \alpha_2, \beta) = \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K}), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta_{\sigma_x}), \quad \frac{\sigma_K(\eta_{\sigma_x})}{\eta_{\sigma_x}} \cdot \frac{\eta_{\sigma_K}}{\sigma_x(\eta_{\sigma_K})} \right)$$

for some $\eta_{\sigma_K} \in S_{\alpha, \sigma_K}$ and $\eta_{\sigma_x} \in S_{\alpha, \sigma_x}$. We need to show that $(\alpha_1, \alpha_2, \beta) \in T_{\alpha'}$. Well, by [Corollary 14](#), we can set

$$I'_\sigma := \eta_\sigma / \chi_\sigma \in S_{\alpha', \sigma}$$

for $\sigma \in \{\sigma_K, \sigma_x\}$. We now compute

$$\begin{aligned} \alpha_1 &= \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma_K}) \\ &= \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i(\chi_\sigma) \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma'_K}) \\ &= \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i \left(\frac{\sigma_K \gamma}{\gamma} \right) \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma'_K}) \\ &= \alpha \cdot \frac{\sigma_K^f(\gamma)}{\gamma} \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma'_K}) \\ &= \alpha' \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta_{\sigma'_K}), \end{aligned}$$

where the last equality holds by definition of γ . Similarly, we see

$$\begin{aligned}
 \alpha_2 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta_{\sigma_x}) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\chi_{\sigma_x}) \cdot \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(\frac{\sigma_x(\gamma)}{\gamma}\right) \cdot \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}) \\
 &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}),
 \end{aligned}$$

where the product telescopes in the last equality because σ_x has order $\varphi(p^\nu)$. Lastly, we set

$$\begin{aligned}
 \beta &= \frac{\sigma_K(\eta_{\sigma_x})}{\eta_{\sigma_x}} \cdot \frac{\eta_{\sigma_K}}{\sigma_x(\eta_{\sigma_K})} \\
 &= \frac{\sigma_K(\chi_{\sigma_x})}{\chi_{\sigma_x}} \cdot \frac{\chi_{\sigma_K}}{\sigma_x(\chi_{\sigma_K})} \cdot \frac{\sigma_K(\eta'_{\sigma_x})}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x(\eta'_{\sigma_K})} \\
 &= \frac{\sigma_K \sigma_x \gamma / \sigma_K \gamma}{\sigma_x \gamma / \gamma} \cdot \frac{\sigma_K \gamma / \gamma}{\sigma_x \sigma_K \gamma / \sigma_x \gamma} \cdot \frac{\sigma_K(\eta'_{\sigma_x})}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x(\eta'_{\sigma_K})} \\
 &= \frac{\sigma_K(\eta'_{\sigma_x})}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x(\eta'_{\sigma_K})}.
 \end{aligned}$$

Thus,

$$(\alpha_1, \alpha_2, \beta) = \left(\alpha' \cdot \prod_{i=0}^{f-1} \sigma_K^i(\eta'_{\sigma_K}), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\eta'_{\sigma_x}), \quad \frac{\sigma_K(\eta'_{\sigma_x})}{\eta'_{\sigma_x}} \cdot \frac{\eta'_{\sigma_K}}{\sigma_x(\eta'_{\sigma_K})} \right) \in T_{\alpha'},$$

which finishes. ■