Student Automorphic Forms Seminar

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Fall 2024

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1 September 4th: Nir Elber

Today we're talking about the local theory of Tate's thesis.

1.1 A Little Global Theory

In order to not lose perspective in the Fourier analysis that makes up the body of this talk, we discuss a little global theory. The goal of Tate's thesis is to derive analytic properties of L-functions such as the following.

Definition 1. We define the *Riemann ζ-function* as

$$\zeta(s) := \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Definition 2. Given a Dirichlet character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, we define the *Dirichlet L-function* by

$$L(s,\chi) := \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}.$$

Definition 3. For a number field K, we define the Dedekind ζ-function of a number field K

$$\zeta_K(s) := \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1-\mathrm{N}(\mathfrak{p})^{-s}}.$$

For now, "analytic properties" means deriving a meromorphic continuation, which in practice means deriving a functional equation.

Remark 4. There is a common generalization of the above two L-functions called a "Hecke L-function," but streamlined definitions would require a discussion of the adéles, which we are temporarily avoiding.

In fact, Hecke had proven functional equations before Tate, but Tate's arguments modernize better, which is why we will talk about them.

By way of example, we state the functional equation for the Riemann ζ-function $\zeta(s)$.

Theorem 5 (Riemann)**.** Define the completed Riemann ζ-function by

$$\Xi(s) := \pi^{-s/2}\Gamma(s)\zeta(s).$$

Then $\Xi(s)$ admits an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation $\Xi(s) = \Xi(1-s)$.

In some sense, the real goal of Tate's thesis is to explain the presence of the mysterious factor $\pi^{-s/2}\Gamma(s)$ which permits the functional equation $\Xi(s) = \Xi(1-s)$. To understand this, we write

$$\Xi(s) = \pi^{-s/2}\Gamma(s) \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

and the idea is that we should view this product over all places of $\mathbb{Q}$: the factor $\pi^{-s/2}\Gamma(s)$ belongs to the archimedean place of $\mathbb{Q}$!

Idea 6 (Tate)**.** Completed L-functions should be products over all places.

Very roughly, Idea 6 allows us to reduce global functional equations into products of local ones.

1.2 Local Z-integrals

We are interested in proving equations of the type $Z(s) \approx Z(1-s)$ for some suitable function Z. We will emply the following trick: we will show that both $Z(s)$ and $Z(1-s)$ live in the same one-dimensional space of functions and then study the scale factor between the two functions later.

To define our function Z, we take motivation from the definition of

$$\Gamma(s) = \int_{\mathbb{R}^+} e^{-t}t^s\,\frac{dt}{t}.$$

We know that this should correspond to the archimedean place of $\mathbb{Q}$, but we would like to extend this definition to the finite places. As such, we make the following observations.

- $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the archimedean place.

- The function $t \mapsto e^{-t}$ is an additive character $\mathbb{R} \to \mathbb{C}^\times$.
- The function $t \mapsto t^{-s}$ is a multiplicative character $\mathbb{R}^\times \to \mathbb{C}^\times$.
- The measure dt/t is a Haar measure of $\mathbb{R}^+$.

To begin our generalizations, we recall the definition of a local field, which will place $\mathbb{R}$ in the correct context.

Definition 7 (local field)**.** A *local field* is a locally compact nondiscrete topological field. It turns out that local fields of characteristic zero are exactly the finite extensions of $\mathbb{R}$ and $\mathbb{Q}_p$.

Next up, to place dt/t in the correct context, we should define a Haar measure.

Definition 8 (Haar measure)**.** Fix a locally compact topological group G. Then a *left-invariant Haar measure* $d\mu_\ell(g)$ is a Radon measure such that $\mu_\ell(gS) = \mu_\ell(S)$ for each $g \in G$ and measureable set S. In terms of intergrals, this is equivalent to

$$\int_G f(gh)\,dh = \int_G f(h)\,dh$$

for each $g \in G$ and integrable function f. It turns out that the Haar measure is unique up to scalar.

Remark 9. In general, a left-invariant Haar measure need not be right-invariant. However, this is frequently true: for example, if G is abelian or "reductive" (such as $G = \mathrm{GL}_n$), then left-invariant Haar measures are right-invariant.

Example 10. The Lebesgue measure dt is a Haar measure on $\mathbb{R}$. The measure $dt/|t|$ is a Haar measure on $\mathbb{R}^+$ and $\mathbb{R}^\times$.

Example 11. Fix a prime p. There is a unique Haar measure μ on $\mathbb{Q}_p$ such that $\mu(\mathbb{Z}_p) = 1$. For example, we find that $\mu(a + p\mathbb{Z}_p) = \frac{1}{p}$ for each $a \in \mathbb{Q}_p$.

Remark 12. Local fields turn out be normed, though this is not immediately obvious from the definition. Even though $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$ all have natural norms (extended from $\mathbb{Q}$), here is a hands-free way to obtain this norm from a local field K: choosing a Haar measure dt on K, we define the norm $|a|$ of some $a \in K$ as the scalar such that

$$d(at) = |a|\,dt.$$

It is not at all obvious that $|\cdot|$ is a norm (in particular, why does it satisfy the triangle inequality?), but it is true. As an example, $|\cdot|$ is the square of the Euclidean norm on $\mathbb{C}$. For $\mathbb{Q}_p$, we find $|p| = 1/p$.

Example 13. We are now able to say that $dt/|t|$ is a Haar measure of $K^\times$ for any local field K.

At this point, we may expect that our generalization of $\Gamma(s)$ to a generic local field K to be

$$Z(\psi,\omega) = \int_K \psi(t)\omega(t)\,\frac{dt}{|t|},$$

where $\psi\colon K \to \mathbb{C}^\times$ and $\omega\colon K^\times \to \mathbb{C}^\times$ are characters. However, this is a little too rigid for our purposes.

Notably, by taking linear combinations of additive characters, Fourier analysis explains that understanding Γ very well should permit understanding integrals of the form

$$\int_{\mathbb{R}+} f(t)t^s \, \frac{dt}{t},$$

where $f\colon \mathbb{R} \to \mathbb{C}$ is some sufficiently nice function. It will help to have the flexibility that this extra f permits.

Definition 14. Fix a local field K. For a nice enough function f and character $\omega\colon K^\times \to \mathbb{C}^\times$, we define the *local Z-integral*

$$Z(f,\omega) := \int_{K^\times} f(t)\omega(t) \, \frac{dt}{|t|}.$$

We will not dwell on this, but perhaps we should explain what is required by "nice enough" function f. The keyword is "Schwartz–Bruhat."

Definition 15 (Schwartz–Bruhat)**.** Fix a local field K.

- If $K \in \{\mathbb{R}, \mathbb{C}\}$, we say $f\colon K \to \mathbb{C}$ is *Schwartz–Bruhat* if and only if it is infinitely differentiable and for all of its derivatives to decay rapidly (namely, $f(t)p(t) \to 0$ as $|t| \to \infty$ for any polynomial p).
- For other K, we say $f\colon K \to \mathbb{C}$ is *Schwartz–Bruhat* if and only if it is locally constant and compactly supported.

We let $S(K)$ denote the vector space of Schwartz–Bruhat functions on K, and we let $S(K)'$ denote its dual (i.e., the vector space of distributions).

Example 16. The function $t \mapsto e^{-t^2}$ is a Schwartz–Bruhat function $\mathbb{C} \to \mathbb{C}$.

Example 17. Fix a prime p. Then the indicator function $1_{\mathbb{Z}_p}$ on $\mathbb{Q}_p$ is Schwartz–Bruhat.

Importantly, the definition of Schwartz–Bruhat will promise that

1.3 Fourier Analysis

We now take a moment to review where we are standing. We were hoping to prove a statement like $Z(s) \approx Z(1-s)$, where $Z(s)$ perhaps has some kind analytic properties. However, we currently have a function $Z(f,\omega)$, so it's not at all obvious how to replace f and ω with "dual" entries or how to make sense of analytic properties. In this subsection, we address both of these concerns; they are both related to character theory.

Going in order of difficulty, it is a little easier to explain how to add analytic structure to Z. After taking norms, we can find some $s \in \mathbb{C}$ such $|\omega| = |\cdot|^s$ so that $\eta := \omega \, |\cdot|^{-s}$ outputs to S^1, and

$$\omega = \eta \, |\cdot|^s \, .$$

Thus, we see that we can decompose characters $K^\times \to \mathbb{C}^\times$ into a unitary part η and an "unramified" part $|\cdot|^s$, and the unramified part now has a complex parameter $s \in \mathbb{C}$. Namely, $\omega \, |\cdot|^{-s}$ now outputs to S^1; i.e., this character is unitary. Fix a local field K.

Thus, we can view $Z(f,\eta)$ as having three parameters as $Z(f,\eta,s) := Z\left(f, \eta \, |\cdot|^s\right)$, where we now require that η is unitary. Because we already have some notion of smoothness in the parameter $s \in \mathbb{C}$, it remains to understand smoothness in the parameter of unitary character η.

Example 18. In the archimedean case, the parameter η is not so interesting.

- The characters $\mathbb{R}^\times \to S^1$ take the form $t \mapsto t^{-a} |t|^s$ where $a \in \{0,1\}$ and $s \in \mathbb{C}$.
- The characters $\mathbb{C}^\times \to S^1$ take the form $z \mapsto z^a \overline{z}^b |z|^s$ where $a, b \in \mathbb{Z}$ and $s \in \mathbb{C}$.

We now understand that our functional equation for Z should arise from $Z(f, \eta, s)$. We hope to take $s \mapsto 1 - s$, and it seems reasonable (by looking at functional equations for $L(s, \chi)$) to replace η with η^{-1}.

However, we still need to replace f with some dual function. In the archimedean case, we expect this to be the Fourier transform defined by

$$\widehat{f}(s) := \int_{\mathbb{R}} f(t) e^{2\pi i s t}\, dt.$$

Such a definition will more or less carry through for arbitrary local fields, but we will have to go through approximately the same dictionary that defined $Z(f, \omega)$ from $\Gamma(s)$. In particular, the functions $t \mapsto e^{2\pi i s t}$ list the characters of $\mathbb{R}$, so the above is the integration of our function f against the list of characters. To this end, we pick up the following theorem.

Theorem 19. Fix a local field K. Then there exists a nontrivial character $\psi \colon K \to S^1$. Any choice of ψ defines a bijection $K \to \widehat{K}$ by sending $a \in K$ to the character $\psi_a(t) := \psi(at)$.

Remark 20. For a general locally compact abelian group G, one can give $\widehat{G} := \operatorname{Hom}(G, \mathbb{C})$ a locally compact topology. With this topology in hand, one actually finds that the map $K \to \widehat{K}$ given by $a \mapsto \psi_a$ is an isomorphism of locally compact abelian groups. However, we do not currently have the need to work in this level of generality.

We are now ready to define our Fourier transform.

Definition 21. Fix a local field K, and choose a nontrivial character $\psi \colon K \to S^1$. For $f \in S(K)$, we define the *Fourier transform*

$$\mathcal{F}_\psi f(t) := \int_K f(t) \psi(st)\, dt.$$

Example 22. For $K = \mathbb{R}$, choose $\psi(t) := e^{2\pi i t}$. Then the Fourier transform (up to normalization) agrees with the usual one

$$\mathcal{F}_\psi f(t) := \int_{\mathbb{R}} f(t) e^{2\pi i s t}\, dt.$$

It turns out that one has the usual properties for the Fourier transform, such as $\mathcal{F}_\psi \mathcal{F}_\psi f(t) = f(-t)$.

We are now ready to state the local functional equation.

Theorem 23. Fix a local field K, and choose a nontrivial character $\psi \colon K \to S^1$. For every $f \in S(K)$, the function $Z(f, \omega, s)$ has a meromorphic continuation to $s \in \mathbb{C}$ (with well-understood poles) and satisfies a functional equation of the form

$$Z(f, \omega, s) = \gamma(\psi, \omega, s, dx) Z\left(\mathcal{F}_\psi f, \omega^{-1}, 1 - s\right),$$

where $\gamma(\psi, \omega, s, dx)$ is meromorphic in $s \in \mathbb{C}$ (with well-understood poles).

Tate's original proof of Theorem 23 was more or less by an explicit computation: by some argument interchanging integrals, one can relate $Z(f, \omega, s)$ with $Z(g, \omega, s)$ for separate $f, g \in S(K)$, which allows us to reduce the proof to a single f. Then one chooses some f such that $\mathcal{F}_\psi f = f$ (e.g., one chooses a Gaussian when $K = \mathbb{R}$) and does an explicit computation to check the result.

1.4 Multiplicity One

We are not going to prove Theorem 23 in detail, but we will gesture towards an argument which uses a bit more functional analysis and less explicit computation. The key point is that $Z(-,\omega)\colon S(K)\to\mathbb{C}$ is a distribution with the curious property that

$$\int_{K^\times} f(at)\omega(t)\,\frac{dt}{t} = \omega^{-1}(a)\int_{K^\times} f(t)\omega(t)\,\frac{dt}{|t|}.$$

Thus, we see that $Z(-,\omega)$ is an eigendistribution of sorts. To make this more explicit, we let $K^\times$ act on $S(K)$ by translation: for $a\in K$ and $f\in S(K)$, we define $(r(a)f)(t) := f(at)$. Then $K^\times$ acts on the distributions $S(K)'$ accordingly: for $a\in K$ and $\lambda\in S(K)'$ and $f\in S(K)$, we can define

$$\langle r'(a)\lambda, f\rangle := \left\langle \lambda, r(a)^{-1}f\right\rangle.$$

Tracking everything through, we see that $Z(-,\omega)\in S(K)'$ is an eigendistribution for the action of $K^\times$ on $S(K)'$ with eigenvalue given by the character ω.

Notation 24. Let $S(K)'(\omega)$ denote the space of $K^\times$-eigendistributions with eigenvalue ω.

Now, we see that $Z(-,\omega)\in S(K)'(\omega)$, and one can check that $Z\left(\mathcal{F}_\psi-,\omega^{-1}\right)$ is also in $S(K)'(\omega)$. Thus, a statement like Theorem 23 will follow from the following "representation-theoretic" multiplicity one result.

Theorem 25. Fix a local field K and character $\omega\colon K\to\mathbb{C}^\times$. Then

$$\dim S(K)'(\omega) = 1.$$

Let's explain the sort of inputs that go into proving Theorem 25, but we will not say more.

- One can verify that $Z(-,\omega)$ provides some vector in $S(K)'(\omega)$, so this space is at least nonempty.
- Let $C_c^\infty(K)$ denote the set of compactly supported Schwartz functions on K. Duality provides an exact sequence
$$0\to S(K)_0'\to S(K)'\to C_c^\infty(K)'\to 0,$$
where $S(K)_0'$ denotes the distributions supported at 0. Taking eigenspaces, we get an exact sequence
$$0\to S(K)_0'(\omega)\to S(K)'(\omega)\to C_c^\infty(K)'(\omega).$$
- One can show that $C_c^\infty(K)'(\omega)$ is one-dimensional with basis given by $f\mapsto \int_K f(t)\omega(t)\,dt/|t|$.
- Understanding $S(K)_0'$ comes down to some casework. For example, we look at the nonarchimedean case. If ω is trivial, we have the distribution $f\mapsto f(0)$; otherwise, $S(K)_0'(\omega)$ is zero.

2 September 11: Reed Jacobs

Today we are talking about the global theory of Tate's thesis. This will mostly be an excuse to discuss the adéles.

2.1 Adéles

For today, K is a global field; in particular, it is a number field (in characteristic 0) or a finite extension of $\mathbb{F}_p(t)$ (in postive characteristic $p>0$). Here is our main character for today.

Definition 26 (adéle ring)**.** Fix a global field K, and let V_K denote the set of places of K. Then the *adéle ring* $\mathbb{A}_K$ is the restricted direct product

$$\mathbb{A}_K := \prod_{v \in V_K} (K_v, \mathcal{O}_v).$$

Here, this notation means that $\mathbb{A}_K$ consists of infinite tuples $(a_v)_v \in \prod_{v \in V_K} K_v$ where $a_v \in \mathcal{O}_v$ for all but finitely many places $v \in V_K$.

Remark 27. Even though $\mathcal{O}_v$ may not have a definition for infinite v, we see that this does not matter because there are only finitely many infinite places anyway.

Remark 28. We see that $\mathbb{A}_K$ becomes a ring under the pointwise operations. In fact, it becomes a topological ring when given the subspace topology of the product topology on $\prod_v K_v$. In particular, upon taking the intersection, we see that our basic open sets look like

$$\prod_{v \in V_K} U_v,$$

where $U_v \subseteq K_v$ is open, but $U_v = \mathcal{O}_v$ for all but finitely many $v \in V_K$.

Here are some quick properties of the topology.

Proposition 29. Fix a global field K. Then $\mathbb{A}_K$ is a locally compact topological ring.

Proof. The fact that the ring operations are continuous can be checked pointwise on the level of the product $\prod_v K_v$. Perhaps we should also check that $\mathbb{A}_K$ is Hausdorff, for which we note that this follows from being a subspace of $\prod_v K_v$ again.

To be locally compact, we have to do a little more work. This follows more or less Fix some $x \in \mathbb{A}_K$, and we want some open neighborhood $U \subseteq \mathbb{A}_K$ of x with compact closure. By translation, we may assume that $x = 0$. Then the open subset

$$\prod_{\substack{v \in V_K \\ v \text{ infinite}}} B_v(0,1) \times \prod_{\substack{v \in V_K \\ v \text{ finite}}} \mathcal{O}_v,$$

where $B_v(0,1)$ is the open ball around $0 \in K_v$. This is open by construction, and its closure $\prod_{v \in V_K} \overline{B_v(0,1)}$ is a product of compact sets and hence compact. ■

Remark 30. Do note that $\prod_v K_v$ fails to be locally compact because the open subsets are too big! This is one reason why we want to work with the adéles instead: once has a chance of doing some reasonable topology on $\mathbb{A}_K$.

To misquote Tate, we remark that one can really only extract arithmetic information from $\mathbb{A}_K$ by putting K inside it. Here is this embedding.

Proposition 31. Fix a global field K. Then the embedding $i\colon K \hookrightarrow \mathbb{A}_K$ defined by

$$i\colon a \mapsto (a)_v$$

is discrete and cocompact.

Proof. We will not prove cocompactness; it is equivalent to the finiteness of the class group and Dirichlet's unit theorem. Alternatively, one can use Minkowski theory to show this directly.

Discreteness is a little easier to show. For $a \in K$, we must show that $i(a)$ has an open neighborhood disjoint from the rest of $i(K)$. By translating, we may take $a = 0$. Now, define the open neighborhood of 0 given by

$$U := \prod_{\substack{v \in V_K \\ v \text{ infinite}}} B_v(0,1) \times \prod_{\substack{v \in V_K \\ v \text{ finite}}} \mathcal{O}_v.$$

Now, for any $b \in K^\times$, the product formula asserts that $\prod_v |b|_v = 1$, but $\prod_v |b_v|_v < 1$ for any $(b_v)_v \in U$, so $U \cap i(K) = \{0\}$, as required. ■

The above cocompactness tells us that the quotient $\mathbb{A}_K/K$ will be interesting to us; similarly, the group $\mathbb{A}_K^\times/K^\times$ will be interesting to us.

2.2 Some Pontryagin Duals

We have some time, so let's expand the discussion of Remark 20.

Definition 32. Fix a locally compact abelian group G. Then we define the group $G := \operatorname{Hom}(G, S^1)$ to be the *Pontryagin dual*, which we turn into a topological group by giving it the compact-open topology. It turns out that $\widehat{G}$ is a locally compact abelian group.

Example 33. Theorem 19 explains that local fields K_v are self-dual.

Example 34. One can check that a character $\mathbb{Z} \to S^1$ has equivalent data to an element of S^1, which essentially proves $\widehat{\mathbb{Z}} \cong S^1$.

Example 35. On the other hand, a continuous homomorphism $S^1 \to S^1$ must be exponentiation by some integer, so $\widehat{S^1} \cong \mathbb{Z}$.

Remark 36. It is a general fact that the Pontryagin dual of $\widehat{G}$ is isomorphic to G. There is at least a map $G \to \operatorname{Hom}\left(\widehat{G}, S^1\right)$ given by $g \mapsto \operatorname{ev}_g$, where $\operatorname{ev}_g \colon \widehat{G} \to S^1$ is given by $\operatorname{ev}_g(\chi) := \chi(g)$.

We even have a nice duality, as in Theorem 19.

Theorem 37. Fix a global field K. Then the locally compact abelian group $\mathbb{A}_K$ is self-dual. More precisely, there exists a nontrivial character $\psi \colon \mathbb{A}_K \to S^1$. Then for any choice of ψ, then the map $\mathbb{A}_K \to \widehat{\mathbb{A}_K}$ given by

$$a \mapsto (\psi_a \colon t \mapsto \psi(at))$$

is an isomorphism of locally compact abelian groups.

Sketch. This follows roughly from Theorem 19, essentially by trying to take a product of the self-duality results for each local field in the restricted direct product. A detailed proof would require understanding the topology of the Pontryagin dual in more detail, so we will avoid it. ■

Here is an application.

Corollary 38. Fix a global field K. Then the Pontryagin dual of K is $\mathbb{A}_K/K$.

Proof. Fix a nontrivial character $\psi\colon \mathbb{A}_K/K \to S^1$ (note the quotient!), and we construct that the composite

$$K \hookrightarrow \mathbb{A}_K \overset{\psi}{\cong} \widehat{\mathbb{A}_K},$$

where the last map is given by Theorem 37. Now, this composite actually outputs to $\widehat{\mathbb{A}_K/K}$: any $a \in K$ produces a character ψ_a satisfying $\psi_a(t) = \psi(at) = 1$ for each $t \in K$, so ψ_a descends to a character of $\mathbb{A}_K/K$. We claim that the above map is an isomorphism, which we do by combining three observations.

- We can check that $\widehat{\mathbb{A}_K/K}$ is a vector space over K.
- Note $\mathbb{A}_K/K$ is compact, so $\widehat{\mathbb{A}_K/K}$ is discrete by some fact of Pontryagin duals.
- Note $\widehat{\mathbb{A}_K/K}/K \subseteq \widehat{\mathbb{A}_K}/K \cong \mathbb{A}_K/K$, and this last space is compcat.

The last two observations imply $\widehat{\mathbb{A}_K/K}$ is finite, but then the first observation requires it to be trivial. ■

2.3 A Poisson Summation Formula

The presence of a duality permits a well-behaved Fourier analysis. For our technicalities, we explain what our Schwartz functions are.

Definition 39 (Schwartz)**.** A *Schwartz–Bruhat* function f on the quotient $\mathbb{A}_K/K$ is a function of the form $\prod_\nu f_\nu$ such that each f_ν is Schwartz, and $f_\nu = 1_{\mathcal{O}_\nu}$ for all but finitely many ν. We also permit finite $\mathbb{C}$-linear combinations of these functions to be Scwhartz–Bruhat.

And here is our corresponding Fourier transform; it is our global analogue of Definition 21.

Definition 40. Fix a global field K, and choose a nontrivial character $\psi\colon \mathbb{A}_K/K \to S^1$. For any Schwartz–Bruhat function f, then we define the *Fourier transform* to be

$$\mathcal{F}_\psi f(y) := \int_{x\in\mathbb{A}_K} f(x)\psi(xy)\,dx.$$

Remark 41. For the discussion which follows shortly, it will be worth our time to write out what the Fourier transform is in general for a locally compact abelian group G. Given a nice enough function $f\colon G \to \mathbb{C}$ (such as Schwartz–Bruhat), the Fourier transform $\widehat{f}$ is a function on $\widehat{G}$ defined by

$$\widehat{f}(\chi) := \int_G f(g)\chi(g)\,dg.$$

For example, the identification of characters of $\mathbb{A}_K$ with $\mathbb{A}_K$ recovers the above definition. One finds that taking the Fourier transform twice almost recovers the original function.

Notably, the Fourier transform depends on a choice of Haar measure. These can be found by taking the product of the Haar measures on the local fields; by scaling, we can get the usual identity $\mathcal{F}_\psi\mathcal{F}_\psi f(x) = f(-x)$.

Riemann's original proof of Theorem 5 has the result more or less follow from some functional equation coming from a Poisson summation formula. As such, we desire a Poisson summation formula in our context as well. The point of a Poisson summation formula is to relate sums for a function f over the discrete cocompact subgroup $\mathbb{Z} \subseteq \mathbb{Q}$ with the same sums of the Fourier transform. Because $K \subseteq \mathbb{A}_K$ is discrete and cocompact, it is natural to have the following statement.

Theorem 42 (Poisson summation)**.** Fix a global field K, and choose a nontrivial character $\psi\colon \mathbb{A}_K/K \to S^1$. For any Schwartz–Bruhat function f, we have

$$\sum_{\gamma\in K} f(\gamma) = \sum_{\gamma\in K} \mathcal{F}_\psi f(\gamma).$$

Proof. Define $F\colon \mathbb{A}_K \to \mathbb{C}$ by

$$F(x) := \sum_{\gamma\in K} f(x+\gamma).$$

Note that F descends to a function on $\mathbb{A}_K/K$. Quickly, note that the convergence of these sums follows from our definition of Schwartz–Bruhat; the convergence is fast enough to ensure that we get a continuous function on $\mathbb{A}_K/K$. We will spend the rest of the proof ignoring convergence issues.

The point is to compute the Fourier transform of F and apply Fourier inversion. However, F has descended to a function on $\mathbb{A}_K/K$, so its Fourier transform should be a function on the Pontryagin dual of $\mathbb{A}_K/K$, which is K by Corollary 38. As such, we let $D \subseteq \mathbb{A}_K$ be a fundamental domain for $\mathbb{A}_K/K$, which we give a volume of 1 to fix our Haar measure, and we compute

$$\begin{aligned}
\mathcal{F}_\psi F(y) &= \int_{\mathbb{A}_K/K} F(x)\psi(xy)\,dx \\
&= \int_D F(x)\psi(xy)\,dx \\
&= \int_D \sum_{\gamma\in K} f(x+\gamma)\psi(xy)\,dx \\
&= \sum_{\gamma\in K} \int_D f(x+\gamma)\psi(xy)\,dx \\
&= \sum_{\gamma\in K} \int_D f(x)\psi((x-\gamma)y)\,dx \\
&\overset{*}{=} \sum_{\gamma\in K} \int_D f(x)\psi(xy)\,dx \\
&= \mathcal{F}_\psi f(y),
\end{aligned}$$

where $\overset{*}{=}$ holds because $\psi|_K = 1$. The general theory of Pontryagin duality allows us to apply a Fourier inversion formula for $\widehat{K} \cong \mathbb{A}_K/K$ to see $F(x)$ equals

$$\sum_{y\in K} \widehat{F}(y)\overline{\psi(xy)} = \sum_{y\in K} \mathcal{F}_\psi f(y)\overline{\psi(xy)}.$$

(Note that this is basically the Fourier transform of the Fourier transform, where we are plugging into the general definition of Remark 41.) Plugging in $x = 0$ completes the proof! ■

Remark 43 (Reed)**.** In the case where K is the function field of a curve C over a finite field $\mathbb{F}_q$, one is able to turn Theorem 42 into the Riemann–Roch theorem for C. We will not explain this in detail.

Remark 44. One can basically mimic the proof of Theorem 5 to prove a functional equation for "global Z-integrals" which look like

$$Z(f,\chi) := \int_{\mathbb{A}_K^\times} f(x)\chi(x)\,d^\times x,$$

where $d^\times x$ is some suitably defined Haar measure on $\mathbb{A}_K^\times$. The key analytic input into the proof of Theorem 5 is a Poisson summation formula used to produce the symmetry; analogously, the key input into the global functional equation is the "adélic Poisson summation formula" Theorem 42 used to produce the symmetry.

3 September 18th: Nir Elber

I filled in for Saud Molaib, who has a last-minute cancellation.

3.1 Algebraic Groups

3.2 Reductive Groups

3.3 Borel Subgroups

4 September 25th: Justin Wu

Today we are discussing how to put a topology on the adéles and related groups.

We begin by fixing some notation. Today, F will be a global field, and $V(F)$ is its set of places. Throughout, $S \subseteq V(K)$ will be a finite subset, usually including the finite set ∞ of infinite places. We define the adéle ring as the usual restricted direct product

$$\mathbb{A}_F = \prod_{v\in V(K)} (F_v, \mathcal{O}_v).$$

For our finite set S, we define

$$\begin{aligned}
\mathbb{A}_F^S &:= \prod_{v\notin S} (F_v, \mathcal{O}_v) \\
F_S &:= \mathbb{A}_{F,S} := \prod_{v\in S} F_v \\
\widehat{\mathcal{O}}_F^S &:= \prod_{v\notin S} \mathcal{O}_v.
\end{aligned}$$

For example, $\mathbb{A}_F^\varnothing = \mathbb{A}_F$, and $\widehat{\mathcal{O}}_F^\infty$ is compact.

Remark 45. There is not a unified consensus on notation. We are agreeing with Getz and Hahn.

4.1 Topology from Topological Rings

For an affine scheme X over F, we would like to give a topology on $X(\mathbb{A}_F)$. It is not totally obvious how to do this. The classical example is that $\mathbb{A}_F$ is a topological ring, and we may want to give a topology on its units $\mathbb{A}_F^\times$, but $\mathbb{A}_F^\times$ will fail to be a topological group if given the subspace topology from $\mathbb{A}_F$: the inversion map $\mathbb{A}_F^\times \to \mathbb{A}_F^\times$ won't be continuous!

The correct way to proceed is to use the closed embedding $j\colon \mathbb{G}_m \to \mathbb{A}^2$ given by $x \mapsto \left(x, x^{-1}\right)$, which is a closed embedding onto

$$\left\{(x,y) \in \mathbb{A}^2 : xy = 1\right\}.$$

Then we can give $\mathbb{A}_F^\times$ the subspace topology from $\mathbb{A}^2$. In particular, inversion when passed through this embedding j merely permutes the two coordinates, so it succeeds at being continuous, even for a general ring R! Here is the general result.

Theorem 46. Fix a Hausdorff topological ring R. There is a unique way to give a topology on the collection of sets $X(R)$ as X varies over the collection of finite type affine R-schemes satisfying the following.

(a) Functoriality: scheme maps $X \to Y$ produce continuous maps $X(R) \to Y(R)$.

(b) Fiber products: given maps $X \to Z$ and $Y \to Z$, the topology on $(X \times_Z Y)(R)$ is the topology induced by the fiber product $X(R) \times_{Z(R)} \times Y(R)$.

(c) Closed embeddings: given a closed embedding $X \to Y$ of schemes, one gets a closed embedding $X(R) \to Y(R)$ of topological spaces.

(d) Affine space: the topology on $\mathbb{A}^1(R)$ is the topology on R.

Sketch. We show uniqueness, but we won't show existence. For any affine R-scheme X of finite type, we know that there is some $r > 0$ and a closed embedding $X \hookrightarrow \mathbb{A}^r$. Then the topology no $\mathbb{A}^r(R)$ is fixed by the topology on $\mathbb{A}$ and taking fiber products, so the topology on $X(R) \subseteq \mathbb{A}^r(R)$ is fixed by having a closed embedding. ■

Example 47. We see that the group GL_n has a closed embedding into $M_n \times M_n$ via $g \mapsto (g, g^{-1})$. This allows us to give a topology on $\mathrm{GL}_n(R)$ for any topological ring R.

Example 48. For any affine group G of finite type, we can find a closed embedding $G \subseteq \mathrm{GL}_n$ to produce a topology on $G(R) \subseteq \mathrm{GL}_n(R)$. For the adéles, one can check that the topology is given by

$$G(\mathbb{A}_F) = \prod_{v \in V(K)} (G(F_v), G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)).$$

4.2 Hyperspecial Subgroups

We would like to understand the subgroups $G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$ of Example 48 in a way which is ambivalent to the embedding. This is a local object, so we begin by telling a local story.

Definition 49 (unramified)**.** Fix a reductive group G over a nonarchimedean local field F_v. We say that G is *unramified* if and only if G is quasi-split, and there is a finite unramified extension E_w/F_v such that G_{E_w} is split.

Theorem 50. Fix a reductive group G over a nonarchimedean local field F_v. Then G is unramified if and only if there is a smooth model $\mathcal{G}$ of G over $\mathcal{O}_v$ such that $\mathcal{G}_s$ is reductive.

Sketch. We sketch the backwards direction. If we have a model $\mathcal{G}$, then we basically pass to the special fiber to find our Borel subgroup and maximal torus; notably, the maximal torus may only split over a finite extension of the residue field, but then this will only require a finite unramified extension in order to split the torus back in G. ■

The remarkable part of conclusion of the above theorem is not exactly the mere existence of the model but instead having the reductive special fiber. For example, given any embedding $G \subseteq \mathrm{GL}_n$, one can look at the Zariski closure of the composite

$$G \to \mathrm{GL}_{n,F_v} \to \mathrm{GL}_{n,\mathcal{O}_v}$$

to produce a model of G over $\mathcal{O}_v$; in fact, there is a way to smooth out this model as well. However, there is no reason why the special fiber of this thing would be controlled.

This notion of models allows us to discuss subgroups like $G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$.

Definition 51 (hyperspecial)**.** Fix a reductive group G over a nonarchimedean local field F_v. Then a subgroup $H \subseteq G(F_v)$ is *hyperspecial* if and only if there is a smooth model $\mathcal{G}$ of G over $\mathcal{O}_v$ such that $\mathcal{G}$ has reductive special fiber, and $H = \mathcal{G}(\mathcal{O}_v)$.

In fact, we can even give an almost topological description.

Theorem 52. Fix a reductive group G over a nonarchimedean local field F_v.

(a) Every compact subgroup $K \subseteq G(F_v)$ is contained in some maximal compact subgroup.

(b) Every maximal compact subgroup $K \subseteq G(F_v)$ equals $\mathcal{G}(\mathcal{O}_v)$ for some smooth model $\mathcal{G}$ of G.

(c) All maximal compact subgroups are open.

We now transition back to our global theory. The miracle is that we achieve being hyperspecial almost everywhere.

Theorem 53. Fix a reductive group G over a global field F, and find a smooth model $\mathcal{G}$ of G over $\mathcal{O}_F^S$ for some finite subset $S \subseteq V(K)$. Then the subgroup $\mathcal{G}(\mathcal{O}_v)$ is hyperspecial for all but finitely many places $v \notin S$.

The main point is that $\mathcal{G}$ is reductive at the generic fiber, and it turns out that one can spread out the reductivity to a Zariski dense open subset of the special fibers.

Being hyperspecial almost everywhere allows us to recast our topology on $G(\mathbb{A}_F)$.

Proposition 54. Fix an affine scheme X over a global field F with a smooth model $\mathcal{X}$ over $\mathcal{O}_F^S$. Then there is a natural homeomorphism

$$X(\mathbb{A}_F) \to \prod_{v \in V(K)} (X(F_v), \mathcal{X}(\mathcal{O}_v)).$$

Corollary 55. Fix a reductive group G over a global field F. For any compact open subgroup $K \subseteq G(\mathbb{A}_F^S)$ for a finite subset $S \subseteq V(K)$, the projection of K onto $G(F_v)$ is hyperspecial for all but finitely many places $v \notin S$.

Proof. We understand what (compact) open subsets of the restricted direct product look like once given a model, so this follows from the above proposition. ■

4.3 The Adélic Quotient

For this subsection, take F to be a number field. For this seminar, we are essentially interested in the representation theory of $G(\mathbb{A}_F)$ for reductive groups G. To do number theory, we should really embed $G(F) \to G(\mathbb{A}_F)$ via $F \hookrightarrow \mathbb{A}_F$, so perhaps we are really interested in the quotient $G(F)\backslash G(\mathbb{A}_F)$.

Howver, $G(F)\backslash G(\mathbb{A}_F)$ is much too large. For example, it will almost always fail to be compact due to the presence of characteres.

Example 56. Take $G = \mathrm{GL}_n$. Then there is a surective composite

$$\mathrm{GL}_n(\mathbb{A}_F) \overset{\det}{\to} \mathrm{GL}_1(\mathbb{A}_F) \overset{|\cdot|}{\to} \mathbb{R}^+.$$

However, $\mathrm{GL}_n(F)$ goes to the identity!

Thus, to make $G(F)\backslash G(\mathbb{A}_F)$ smaller, we want to restrict $G(\mathbb{A}_F)$ down to

$$G(\mathbb{A}_F)^1 = \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi),$$

where $X^*(G)$ is the set of characters $G \to \mathbb{G}_m$. Note that $\mathbb{G}_m(F)$ is trivial under $|\cdot|$, so $G(F) \subseteq G(\mathbb{A}_F)^1$.

Another reason the quotient $G(F)\backslash G(\mathbb{A}_F)$ is too large due to the presence of the center. With this in mind, we define $A_G \subseteq G(F_\infty)$ as follows: consider a maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}} Z_G$; then we take the connected component of the identity of the $\mathbb{R}$-points of this thing.

Example 57. We can compute that the maximal $\mathbb{Q}$-split torus of $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ is $\mathbb{Q}^\times \subseteq F^\times$. Thus, $A_{\mathbb{G}_m} = \mathbb{R}^+$. More generally,

$$A_{\mathrm{GL}_n} = \{c1_n : c > 0\} \subseteq \mathrm{GL}_n(F_\infty).$$

We are now ready to define the adelic quotient.

Definition 58 (adelic quotient)**.** Fix a reductive group G over a global field F. Then the *adelic quotient* is the set

$$[G] \coloneqq A_G G(F)\backslash G(\mathbb{A}_F).$$

For a number field F, this is the same as $G(F)\backslash G(\mathbb{A}_F)^1$.

Remark 59 (Tamagawa numbers)**.** It turns out that $G(F)\backslash G(\mathbb{A}_F)^1$ may fail to be compact, but it will at least always have finite measure with respect to the Haar measure on $G(\mathbb{A}_F)^1$.