Student Arithmetic Geometry Seminar

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1 August 30: Martin Olsson

Today is an organizational meeting. There is no paper list yet (but soon), but almost all dates for talks have been taken anyway. The papers that Professor Olsson has in mind are along the lines of "how to do birational geometry with stacks."

1.1 Geometric Invariant Theory

There is a classical book by Mumford and Fogarty on geometric invariant theory. More recently there is some hope to do this theory over a more general base by Seshadri and some theory of "adequate" moduli spaces by Alper.

For today, we will over a Noetherian ring R, and let G be a smooth, affine, connected group scheme with reductive geometric fibers, which we may just call a reductive group scheme over R.

Example 1. The group $G = \mathrm{GL}_{n,R}$ and the other classical groups are an example, but $\mathbb{G}_{a,R}$ is not.

Example 2. We won't define reductive, but here is one way to access the notion: examples of reductive group schemes are the linearly reductive group schemes whose category of representations is semisimple, and these are all the examples in characteristic 0 (but not in characteristic p).

In Alper's story, a linearly reductive group scheme corresponds to a "good" moduli space, but a reductive group scheme corresponds to an "adequate" moduli space. (We have not said what "corresponds" means.)

For our affine story of geometric invariant theory, one has an affine R-scheme $X = \operatorname{Spec} A$ equipped with a G-action, which amounts to a morphism $G \times X \to X$ with some special properties. Everything in sight is affine, so we can also think about this as a morphism $A \to A \otimes \mathcal{O}_G$ with some special properties. In general, there is a quotient map $X \to [X/G]$, where $[X/G]$ is some stack, and then $[X/G]$ maps onto $Y := \operatorname{Spec} A^G$, and $\operatorname{Spec} A^G$ is perhaps the "affine quotient." Here is the visual.

$$X \to [X/G] \xrightarrow{\pi} Y.$$

Geometric invariant theory now roughly divides into two steps.

1. Find a substack $\mathcal{X}^s$ of $[X/G]$ with finite diagonal. (This roughly corresponds to finding the points with closed orbits by the G-action.)

2. Find the course moduli space of $\mathcal{X}^s$.

Let's see an example.

Example 3. We work over a field k; fix some integers $a_0, \ldots, a_n \in \mathbb{Z}$. Now, we let $\mathbb{G}_m$ act on $\mathbb{A}_k^{n+1} = \operatorname{Spec} k[x_0, \ldots, x_n]$ by

$$u * x_i := u^{a_i} x_i.$$

Note that having $a_i > 0$ for all i implies that $A^G = k$, so A^G is quite small!

We are able to execute the first step above, which we do in steps.

1. Given a geometric point $\overline{x}$ of $[X/G]$, let $G_{\overline{x}}$ denote the stabilizer (which is some subgroup scheme).

2. Then we let $\mathcal{U} \subseteq [X/G]$ be the maximal open subscheme containing the $\overline{x}$ for which $G_{\overline{x}}$ is finite.

3. It turns out that $\pi(\mathcal{X} \setminus \mathcal{U}) \subseteq Y$ is closed; we let this subset be Z.

4. It now turns out that $\mathcal{X}^s := \pi^{-1}(Y \setminus Z)$ will do the trick.

Let's work through an example. Continue with a field k, and now take two integers $a, b \in \mathbb{Z}$, and we are able to let $\mathbb{G}_m$ act on $A := \operatorname{Spec} k[x, y]$ by $u * x := u^a x$ and $u * y := u^b y$. On rings, this map is as follows.

$$\begin{array}{ccccc} k[x,y] & \otimes & k[u,1/u] & \leftarrow & k[x,y] \\ x & \otimes & u^a & \leftarrow\!\shortmid & x \\ y & \otimes & u^b & \leftarrow\!\shortmid & y \end{array}$$

Let's do some cases.

(a) Suppose $a = 0$ and $b \neq 0$; the case $a \neq 0$ and $b = 0$ is symmetric. Now, it turns out we can only check monomials, and we find that $A^{\mathbb{G}_m} = k[x]$. But there is some extra stacky information because having y nonzero makes our point have stabilizer μ_b (which is the bth roots of unity); when $y = 0$, our stabilizer is actually a full $\mathbb{G}_m$! Thus, we can compute that $\mathcal{U}$ is $[\operatorname{Spec} k[x, y, 1/y]/\mathbb{G}_m]$, so we will find that $\mathcal{X}^s$ is empty!

(b) Suppose $a > 0$ and $b < 0$, and set $g := \gcd(a, b)$. On monomials, we find

$$u * x^\alpha y^\beta = u^{a\alpha + b\beta} x^\alpha y^\beta,$$

so one can calculate that $A^{\mathbb{G}_m} = \operatorname{Spec} k[w]$, where $w := x^{b/g} y^{a/g}$.

Now, if $\alpha = 0$ or $\beta = 0$, then our stabilizer is small (something like μ_a or μ_b again), and our orbit fails to be closed. But if both are nonzero, then we are cutting out some subvariety which looks like

$$x^{b/g} y^{a/g} = \alpha\beta,$$

which is closed with stabilizer small. As a result, our $\mathcal{U}$ is everything minus the origin, so $Z = 0$, so we find that $\mathcal{X}^s$ consists of the points not on the axes modulo $\mathbb{G}_m$.

Remark 4. We close by making a quick remark on how to add a character to this story. With our group G, we may equip a character $\chi \colon G \to \mathbb{G}_m$, and then we can try to understand

$$A_\chi := \{g \in A : g * f = \chi(g) f\}.$$

2 September 6th: Martin Olsson

Today we are giving an introduction of algebraic stacks "for the working mathematician."

2.1 Algebraic Stacks

We work over a base scheme S. Here is an okay definition.

Definition 5 (algebraic stack)**.** An *algebraic stack* is a functor $\mathfrak{X} \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Groupoids}$ satisfying the following.

(a) Descent: $\mathfrak{X}$ is a sheaf for the étale topology.

(b) The diagonal of $\mathfrak{X}$ is representable by a scheme.

(c) $\mathfrak{X}$ admits a smooth cover by a scheme.

Remark 6. Here, $\mathrm{Groupoids}$ is a category of categories (where all morphisms are isomorphisms). One must be a rather careful to explain what a functor valued in groupoids actually is.

Remark 7. The "algebraic" part of "algebraic stack" arises from (b) and (c).

Intuitively, a stack should be thought of as a scheme with some "stacky points" that have some larger automorphism group; for example, quotient stacks can be thought of in this way. (This is somewhat similar to orbifolds in differential topology.) Our goal is to turn the above definition into this intuition.

One way to produce stacks is by group actions.

Definition 8 (principal homogeneous space)**.** Fix an affine group scheme G over S. Then a *principal homogeneous space* under G is a flat surjective scheme $P \to S$ with G-action such that the induced map $G \times_S P \to P \times_S P$ given by $(g, x) \mapsto (gx, x)$ is an isomorphism.

Example 9. There is an equivalence of groupoid categories between invertible sheaves over S and principal homogeneous spaces under $\mathbb{G}_m$. In one direction, we take the line bundle $\mathcal{L}$ to the scheme representing the functor $\mathrm{Isom}(\mathcal{L}, \mathcal{O}_S)$, where the $\mathbb{G}_m$-action arises from its action on $\mathcal{O}_S$. One can check that $\mathrm{Isom}(\mathcal{L}, \mathcal{O}_S)$ is isomorphic to

$$\mathrm{Spec}_S\left(\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}\right),$$

where the $\mathbb{G}_m$-action on $\mathcal{L}^{\otimes n}$ is given by the nth power of $\mathbb{G}_m$ on $\mathcal{L}$.

Example 10. Given a G-action on a scheme U, let's try to make sense of the algebraic stack $\mathfrak{X} := [U/G]$. Well, given a test scheme $T \to S$, we produce the groupoid of diagrams

$$\begin{array}{ccc} P & \xrightarrow{\rho} & U \\ {\scriptstyle G}\downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $P \to T$ is a principal homogeneous space under G. We will write down our objects as pairs (P, ρ). Intuitively, one can think of the object (P, ρ) as being related to its image in U, which is approximately a G-orbit in U.

Let's explain this last example a little more.

(a) Descent follows by some kind of faithfully flat descent for affine schemes.

(b) Approximately speaking, (b) is asking for the functor $\operatorname{Isom}_{\mathfrak{X}}((P,\rho),(P',\rho'))\colon \mathrm{Sch}_T^{\mathrm{op}} \to \mathrm{Set}$ to be representable. Here, our isomorphisms need to be isomorphisms of the principal homogeneous spaces (namely, commuting with the G-action) also commuting with the ρs.

(c) Quickly, note that there is a "tautological" principal homogeneous space $G \times U \to U$, which we will label (P_0,ρ_0). If G is smooth, then (c) is the statement that the map

$$\operatorname{Isom}_{U\times_S T}((P_0,\rho_0),(P,\rho)) \to T$$

is smooth and surjective, which is not obvious but true.

Here is are some more examples.

Example 11. We attempt to take a quotient stack $\mathfrak{X} := [U/\mathbb{G}_m]$. Then our objects in some groupoid $\mathfrak{X}(T)$ are principal homogeneous spaces over $\mathbb{G}_m$, which we now understand to be line bundles. Thus, for example, taking two line bundles $\mathcal{L}$ and $\mathcal{L}'$, (b) is asking for

$$\operatorname{Isom}_T(\mathcal{L},\mathcal{L}') \cong \operatorname{Isom}_T(\mathcal{L}'\otimes\mathcal{L}^\vee,\mathcal{O}_T)$$

to be representable. One can check this sometimes, I suppose.

Example 12. Take $G=\mu_p$ over a base field k of positive characteristic $p>0$. We try to consider $B\mu_p := [(\operatorname{Spec} k)/\mu_p]$. Notably, μ_p is flat but not smooth, so checking (c) may be trickier. The main point is to use the Kummer exact sequence

$$1 \to \mu_p \to \mathbb{G}_m \stackrel{(\cdot)^p}{\to} \mathbb{G}_m \to 1.$$

As such, principal homogeneous spaces over μ_p turn out to be pairs $(\mathcal{L},\lambda)$, where $\mathcal{L}$ is an invertible sheaf, and $\lambda\colon \mathcal{L}^{\otimes p} \to \mathcal{O}$ is an isomorphism. One can check that this identification shows $B\mu_p$ is the same as $[\mathbb{G}_m/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{G}_m$ by $u * v := u^p v$. The point is that we are now taking a quotient by a smooth scheme, so we have checked (c)! In general, one can always do this kind of trick when our group G is flat.

Example 13. One can enforce points with automorphisms by basically making our line bundles in our groupoids. For example, let's say we want to put a μ_2 stabilizer on $0 \in \mathbb{P}^1$ and put a μ_3 stabilizer on $\infty \in \mathbb{P}^1$. Then our functor $\mathfrak{X}$ should assign test schemes T to a map $T \to \mathbb{P}^1$ so that the produced line bundle $\mathcal{L}$ has assigned isomorphisms $\mathcal{L}_0^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^1,0}$ and $\mathcal{L}_\infty^{\otimes 3} \to \mathcal{O}_{\mathbb{P}^1,\infty}$. It is not obvious if we can realize $\mathfrak{X}$ as a quotient, though it turns out that we can.

3 September 13: Martin Olsson

A paper list has been released. I'm too tired to take notes today.