Student Arithmetic Geometry Seminar

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Contents

1 August 30: Martin Olsson

Today is an organizational meeting. There is no paper list yet (but soon), but almost all dates for talks have been taken anyway. The papers that Professor Olsson has in mind are along the lines of "how to do birational geometry with stacks."

1.1 Geometric Invariant Theory

There is a classical book by Mumford and Fogarty on geometric invariant theory. More recently there is some hope to do this theory over a more general base by Seshadri and some theory of "adequate" moduli spaces by Alper.

For today, we will over a Noetherian ring R, and let G be a smooth, affine, connected group scheme with reductive geometric fibers, which we may just call a reductive group scheme over R.

Example 1. The group $G = \mathrm{GL}_{n,R}$ and the other classical groups are an example, but $\mathbb{G}_{a,R}$ is not.

Example 2. We won't define reductive, but here is one way to access the notion: examples of reductive group schemes are the linearly reductive group schemes whose category of representations is semisimple, and these are all the examples in characteristic 0 (but not in characteristic p).

In Alper's story, a linearly reductive group scheme corresponds to a "good" moduli space, but a reductive group scheme corresponds to an "adequate" moduli space. (We have not said what "corresponds" means.)

For our affine story of geometric invariant theory, one has an affine R-scheme $X = \operatorname{Spec} A$ equipped with a G-action, which amounts to a morphism $G \times X \to X$ with some special properties. Everything in sight is affine, so we can also think about this as a morphism $A \to A \otimes \mathcal{O}_G$ with some special properties. In general, there is a quotient map $X \to [X/G]$, where $[X/G]$ is some stack, and then $[X/G]$ maps onto $Y := \operatorname{Spec} A^G$, and $\operatorname{Spec} A^G$ is perhaps the "affine quotient." Here is the visual.

$$X \to [X/G] \xrightarrow{\pi} Y.$$

Geometric invariant theory now roughly divides into two steps.

1. Find a substack $\mathcal{X}^s$ of $[X/G]$ with finite diagonal. (This roughly corresponds to finding the points with closed orbits by the G-action.)
2. Find the course moduli space of $\mathcal{X}^s$.

Let's see an example.

Example 3. We work over a field k; fix some integers $a_0, \ldots, a_n \in \mathbb{Z}$. Now, we let $\mathbb{G}_m$ act on $\mathbb{A}_k^{n+1} = \operatorname{Spec} k[x_0, \ldots, x_n]$ by

$$u * x_i := u^{a_i} x_i.$$

Note that having $a_i > 0$ for all i implies that $A^G = k$, so A^G is quite small!

We are able to execute the first step above, which we do in steps.

1. Given a geometric point $\overline{x}$ of $[X/G]$, let $G_{\overline{x}}$ denote the stabilizer (which is some subgroup scheme).
2. Then we let $\mathcal{U} \subseteq [X/G]$ be the maximal open subscheme containing the $\overline{x}$ for which $G_{\overline{x}}$ is finite.
3. It turns out that $\pi(\mathcal{X} \setminus \mathcal{U}) \subseteq Y$ is closed; we let this subset be Z.
4. It now turns out that $\mathcal{X}^s := \pi^{-1}(Y \setminus Z)$ will do the trick.

Let's work through an example. Continue with a field k, and now take two integers $a, b \in \mathbb{Z}$, and we are able to let $\mathbb{G}_m$ act on $A := \operatorname{Spec} k[x, y]$ by $u * x := u^a x$ and $u * y := u^b y$. On rings, this map is as follows.

$$\begin{array}{ccccc} k[x,y] & \otimes & k[u,1/u] & \leftarrow & k[x,y] \\ x & \otimes & u^a & \leftarrow\!\shortmid & x \\ y & \otimes & u^b & \leftarrow\!\shortmid & y \end{array}$$

Let's do some cases.

(a) Suppose $a = 0$ and $b \neq 0$; the case $a \neq 0$ and $b = 0$ is symmetric. Now, it turns out we can only check monomials, and we find that $A^{\mathbb{G}_m} = k[x]$. But there is some extra stacky information because having y nonzero makes our point have stabilizer μ_b (which is the bth roots of unity); when $y = 0$, our stabilizer is actually a full $\mathbb{G}_m$! Thus, we can compute that $\mathcal{U}$ is $[\operatorname{Spec} k[x, y, 1/y]/\mathbb{G}_m]$, so we will find that $\mathcal{X}^s$ is empty!

(b) Suppose $a > 0$ and $b < 0$, and set $g := \gcd(a, b)$. On monomials, we find

$$u * x^\alpha y^\beta = u^{a\alpha + b\beta} x^\alpha y^\beta,$$

so one can calculate that $A^{\mathbb{G}_m} = \operatorname{Spec} k[w]$, where $w := x^{b/g} y^{a/g}$.

Now, if $\alpha = 0$ or $\beta = 0$, then our stabilizer is small (something like μ_a or μ_b again), and our orbit fails to be closed. But if both are nonzero, then we are cutting out some subvariety which looks like

$$x^{b/g} y^{a/g} = \alpha\beta,$$

which is closed with stabilizer small. As a result, our $\mathcal{U}$ is everything minus the origin, so $Z = 0$, so we find that $\mathcal{X}^s$ consists of the points not on the axes modulo $\mathbb{G}_m$.

Remark 4. We close by making a quick remark on how to add a character to this story. With our group G, we may equip a character $\chi \colon G \to \mathbb{G}_m$, and then we can try to understand

$$A_\chi := \{g \in A : g * f = \chi(g) f\}.$$

2 September 6th: Martin Olsson

Today we are giving an introduction of algebraic stacks "for the working mathematician."

2.1 Algebraic Stacks

We work over a base scheme S. Here is an okay definition.

Definition 5 (algebraic stack)**.** An *algebraic stack* is a functor $\mathfrak{X} \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Groupoids}$ satisfying the following.

(a) Descent: $\mathfrak{X}$ is a sheaf for the étale topology.

(b) The diagonal of $\mathfrak{X}$ is representable by a scheme.

(c) $\mathfrak{X}$ admits a smooth cover by a scheme.

Remark 6. Here, $\mathrm{Groupoids}$ is a category of categories (where all morphisms are isomorphisms). One must be a rather careful to explain what a functor valued in groupoids actually is.

Remark 7. The "algebraic" part of "algebraic stack" arises from (b) and (c).

Intuitively, a stack should be thought of as a scheme with some "stacky points" that have some larger automorphism group; for example, quotient stacks can be thought of in this way. (This is somewhat similar to orbifolds in differential topology.) Our goal is to turn the above definition into this intuition.

One way to produce stacks is by group actions.

Definition 8 (principal homogeneous space)**.** Fix an affine group scheme G over S. Then a *principal homogeneous space* under G is a flat surjective scheme $P \to S$ with G-action such that the induced map $G \times_S P \to P \times_S P$ given by $(g, x) \mapsto (gx, x)$ is an isomorphism.

Example 9. There is an equivalence of groupoid categories between invertible sheaves over S and principal homogeneous spaces under $\mathbb{G}_m$. In one direction, we take the line bundle $\mathcal{L}$ to the scheme representing the functor $\operatorname{Isom}(\mathcal{L}, \mathcal{O}_S)$, where the $\mathbb{G}_m$-action arises from its action on $\mathcal{O}_S$. One can check that $\operatorname{Isom}(\mathcal{L}, \mathcal{O}_S)$ is isomorphic to

$$\operatorname{Spec}_S\left(\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}\right),$$

where the $\mathbb{G}_m$-action on $\mathcal{L}^{\otimes n}$ is given by the nth power of $\mathbb{G}_m$ on $\mathcal{L}$.

Example 10. Given a G-action on a scheme U, let's try to make sense of the algebraic stack $\mathfrak{X} := [U/G]$. Well, given a test scheme $T \to S$, we produce the groupoid of diagrams

$$\begin{tikzcd} P \arrow[r, "\rho"] \arrow[d, "G"'] & U \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

where $P \to T$ is a principal homogeneous space under G. We will write down our objects as pairs (P, ρ). Intuitively, one can think of the object (P, ρ) as being related to its image in U, which is approximately a G-orbit in U.

Let's explain this last example a little more.

(a) Descent follows by some kind of faithfully flat descent for affine schemes.

(b) Approximately speaking, (b) is asking for the functor $\operatorname{Isom}_{\mathfrak{X}}((P,\rho),(P',\rho')) \colon \operatorname{Sch}_T^{\mathrm{op}} \to \operatorname{Set}$ to be representable. Here, our isomorphisms need to be isomorphisms of the principal homogeneous spaces (namely, commuting with the G-action) also commuting with the ρs.

(c) Quickly, note that there is a "tautological" principal homogeneous space $G \times U \to U$, which we will label (P_0, ρ_0). If G is smooth, then (c) is the statement that the map

$$\operatorname{Isom}_{U\times_S T}((P_0,\rho_0),(P,\rho)) \to T$$

is smooth and surjective, which is not obvious but true.

Here is are some more examples.

Example 11. We attempt to take a quotient stack $\mathfrak{X} := [U/\mathbb{G}_m]$. Then our objects in some groupoid $\mathfrak{X}(T)$ are principal homogeneous spaces over $\mathbb{G}_m$, which we now understand to be line bundles. Thus, for example, taking two line bundles $\mathcal{L}$ and $\mathcal{L}'$, (b) is asking for

$$\operatorname{Isom}_T(\mathcal{L}, \mathcal{L}') \cong \operatorname{Isom}_T(\mathcal{L}' \otimes \mathcal{L}^\vee, \mathcal{O}_T)$$

to be representable. One can check this sometimes, I suppose.

Example 12. Take $G = \mu_p$ over a base field k of positive characteristic $p > 0$. We try to consider $B\mu_p := [(\operatorname{Spec} k)/\mu_p]$. Notably, μ_p is flat but not smooth, so checking (c) may be trickier. The main point is to use the Kummer exact sequence

$$1 \to \mu_p \to \mathbb{G}_m \overset{(\cdot)^p}{\to} \mathbb{G}_m \to 1.$$

As such, principal homogeneous spaces over μ_p turn out to be pairs $(\mathcal{L}, \lambda)$, where $\mathcal{L}$ is an invertible sheaf, and $\lambda \colon \mathcal{L}^{\otimes p} \to \mathcal{O}$ is an isomorphism. One can check that this identification shows $B\mu_p$ is the same as $[\mathbb{G}_m/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{G}_m$ by $u * v := u^p v$. The point is that we are now taking a quotient by a smooth scheme, so we have checked (c)! In general, one can always do this kind of trick when our group G is flat.

Example 13. One can enforce points with automorphisms by basically making our line bundles in our groupoids. For example, let's say we want to put a μ_2 stabilizer on $0 \in \mathbb{P}^1$ and put a μ_3 stabilizer on $\infty \in \mathbb{P}^1$. Then our functor $\mathfrak{X}$ should assign test schemes T to a map $T \to \mathbb{P}^1$ so that the produced line bundle $\mathcal{L}$ has assigned isomorphisms $\mathcal{L}_0^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^1,0}$ and $\mathcal{L}_\infty^{\otimes 3} \to \mathcal{O}_{\mathbb{P}^1,\infty}$. It is not obvious if we can realize $\mathfrak{X}$ as a quotient, though it turns out that we can.

3 September 13: Martin Olsson

A paper list has been released. I'm too tired to take notes today.

4 September 20: Rose Lopez

Today we're talking about birational geometry of stacks.

4.1 Classical Birational Geometry

Here is the central definition.

Definition 14 (birational)**.** A *rational map* $f\colon X \to Y$ is a morphism on dense open subsets of X and Y. An isomorphism in this category (of varieties equipped with rational maps as morphisms) is a *birational map*.

One can always factor birational maps.

Theorem 15 (weak factorization)**.** Fix a birational map $f\colon X \to Y$ of smooth proper varieties over an algebraically closed field k of characteristic 0. Suppose that f is an isomorphism on $U \subseteq X$. Then f factors as

$$X_0 \to X_1 \to \cdots \to X_n = Y,$$

where each map $X_i \to X_{i+1}$ is either a blow up or the reverse of a blow up $X_{i+1} \to X_i$, where the blow-ups are all disjoint from U.

To do our geometry, it is helpful to keep track of some algebra.

Definition 16 (Burnside ring)**.** Fix a field k and nonnegative integer n. Then the *Burnside ring* $\operatorname{Burn}_{k,n}$ is the free abelian group generated by the isomorphism classes of finitely generated field extensions of k of transcendence degree n.

Notably, given a smooth projective irreducible k-variety X, we can produce a class $[X]$ in the Burnside ring as $[k(X)]$. By summing over all transcendence degrees, we get a graded Burnside ring Burn. Given $U \subseteq X \setminus D$ for a divisor $D = D_1 \cup \cdots \cup D_\ell$, one can define $[U]$ basically by subtracting out the classes of the divisors $D_i \times \mathbb{P}^1$ and adding back in $(D_i \cap D_j) \times \mathbb{P}^2$.

With our larger ring, we can find relations add $[X]+[Y] = [X \cup Y]$ and $[X]\cdot[Y] = [X \times Y]$ and $[U] = [U']$ if and only if there is an isomorphism between them coming from a birational projective morphism. The moral of the story is that Burn_n turns out to basically be generated by these U modulo certain relations given by cutting out other dense open subsets.

4.2 Stacky Birational Geometry

Given stacks $\mathcal{X}$ and $\mathcal{Y}$, a rational map $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism defined on a dense open subscheme of $\mathcal{X}$. Then an isomorphism on the dense opens is *birational*; however, our definition of birationally equivalent requires that the birational map is proper (defined by valuative criterion) and representable (by schemes). One can tell a lot of the same story; for example, there are blow-ups. Now that we are looking at schemes, we can also take root stacks in a way that Professor Olsson described last week. The moral of the story is that one can prove an analogue of Theorem 15, where we need to allow blow-ups and these root stacks.

5 September 27: Xiangru Zeng

Today we are discussing valuative criteria for the existence of moduli stacks. We will work over an algebraically closed field k of characteristic 0.

5.1 Valuative Criteria

Roughly speaking, one can frequently show that a moduli space is at least an algebraic stack. However, we will often want to upgrade this stack to a more geometric object, such as an algebraic space or scheme. Here is an example of one such result.

Theorem 17 (Keel–Mori)**.** Fix a Deligne–Mumford stack $\mathfrak{X}$ of finite type and separated over k. Then there is a coarse moduli space X equipped with a projection $\pi\colon \mathfrak{X} \to X$; furthermore, one can guarantee that $\pi_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$ and that the functor $\pi_*\colon \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(X)$ is exact.

Remark 18. The conclusion the above theorem defines what is called a "tame coarse moduli space." There is also a relative notion over a scheme S. The above theorem more or less says all our finite type separated moduli spaces are tame in characteristic 0.

To improve the above result, we would like to remove the requirement that the stack is separated, which more or less is a requirement of having finite automorphism groups.

For example, if a linearly reductive group G acts on some open quasiprojective variety $U \subseteq \mathbb{P}^n$ semistably, then geometric invariant theory still permits us to produce a quotient

$$U/G = \operatorname{Proj} \bigoplus_{n \geq 0} \Gamma(U, \mathcal{O}_U(n))^G.$$

Combining this example with the previous theorem, we produce the following definition.

Definition 19. A map $\pi\colon \mathfrak{X} \to X$ from an algebraic stack to an algebraic space is a *good moduli space* if and only if the following hold.

- π is quasiseparated and quasicompact.
- $\pi_*\mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X$.
- $\pi_*\colon \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(X)$ is exact.

It is not clear that this is a good definition, but here are some properties.

Theorem 20. Fix a good moduli space $\pi\colon \mathfrak{X} \to X$.

(a) π is surjective and universally closed.

(b) For two points $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}(k)$, we have $\pi(\mathfrak{x}_1) = \pi(\mathfrak{x}_2)$ if and only if $\overline{\{\mathfrak{x}_1\}} \cap \overline{\{\mathfrak{x}_2\}} \neq \varnothing$.

(c) If $\mathfrak{X}$ is Noetherian, then π is universal (with respect to being a good moduli space).

6 November 8: Drew Keisling

Today we are talking about the moduli space of vector bundles on a curve. As such, today we will work over a base field k which is algebraically closed and of characteristic 0, and X is a smooth projective curve over k with genus $g \geq 2$.

6.1 Moduli of Semistable Vector Bundles

We begin by defining our stack.

Definition 21. Fix a nonnegative integer r and integer d. Then define $\mathcal{M}_X(r,d)\colon \mathrm{Sch}_k \to \mathrm{Grpd}$ by sending a test k-scheme T to the collection of vector bundles $\mathcal{E}$ on $T \times X$ which are flat over T and have rank r and degree d. Recall that the degree is the degree of the line bundle $\det \mathcal{E} := \wedge^r \mathcal{E}$.

Remark 22. We want our vector bundles $\mathcal{E}$ to be flat over T so that we have a reasonably continuous family of vector bundles.

We now pick up some definitions.

Definition 23. Fix a vector bundle $\mathcal{E}$ on X.

- The *slope* of $\mathcal{E}$ equals
$$\mu(E) := \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}.$$
- The vector bundle $\mathcal{E}$ is *stable* if and only if there is a subbundle $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mu(\mathcal{E}') < \mu(E)$.
- The vector bundle $\mathcal{E}$ is *semistable* if and only if there is a subbundle $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mu(\mathcal{E}') \leq \mu(E)$.
- The vector bundle $\mathcal{E}$ is *polystable* if and only if $\mathcal{E}$ is a direct sum $\bigoplus_i \mathcal{E}_i$ such that each $\mathcal{E}_\bullet$ is stable and satisfies $\mu(\mathcal{E}_\bullet) = \mu(\mathcal{E})$.

It is not hard to see that stable implies polystable implies semistable.

Using slopes as above allows one to make inductive arguments, as follows.

Proposition 24. Fix a semistable vector bundle $\mathcal{E}$ on X. Then there is a filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_{n-1} \subseteq \mathcal{E}_n = \mathcal{E}$$

of semistable vector bundles such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable with slope $\mu(\mathcal{E})$. In fact, the summands $\mathcal{E}_i/\mathcal{E}_{i-1}$ are unique up to permutation (and isomorphism).

We are thus allowed to make the following definition.

Definition 25 (associated graded bundle)**.** Fix a semistable vector bundle $\mathcal{E}$ on X. Then the *associated graded polystable bundle* is

$$\operatorname{gr} \mathcal{E} := \bigoplus_i \frac{\mathcal{E}_i}{\mathcal{E}_{i-1}},$$

where $\{\mathcal{E}_\bullet\}$ is the ascending filtration of Proposition 24.

We are now ready to state a theorem on the moduli space of semistable vector bundles.

Theorem 26. Consider the functor $\mathcal{M}_X^{\mathrm{ss}}(r,d)$ of semistable vector bundles on X.

(a) It is an open subfunctor of $\mathcal{M}_X(r,d)$.

(b) The functor $\mathcal{M}_X^{\mathrm{ss}}(r,d)$ is an algebraic stack of finite type over k. It has affine diagonal, and it is smooth and irreducible.

(c) There exists a good moduli space $\pi\colon \mathcal{M}_X^{\mathrm{ss}}(r,d) \to M_X^{\mathrm{ss}}(r,d)$ such that the target is an irreducible, proper algebraic space over k of dimension $r^2(g-1)+1$. In fact, $M_X^{\mathrm{ss}}(r,d)$ is the collection of equivalence classes of vector bundles on X, where the equivalence relation identifies semistable vector bundles with the same associated graded bundle.

Proof. For (a), roughly speaking, one can cut out the complement by proper Quot schemes. For (c), one uses valuative criteria. ■

6.2 Projectivity of the Moduli Space

Our main theorem is as follows.

Theorem 27. The proper k-scheeme $M_X^{\mathrm{ss}}(r,d)$ is a projective variety.

Proof. Note that we already have properness, so this is really a matter of finding an ample line bundle.

As such, we are on the hunt for line bundles. For this, we describe the construction of determinantal line bundles. Suppose we have a stack $\mathcal{S}$ over k, and let $\mathcal{E}$ be a vector bundle on $X \times \mathcal{S}$ flat over $\mathcal{S}$. We are going to play with the two projections for our construction. To start, note that one can find a resolution

$$0 \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{E} \to 0$$

such that $K^0 := R^1 \operatorname{pr}_{\mathcal{S}*} \mathcal{E}^{-1}$ and $K^1 := R^1 \operatorname{pr}_{\mathcal{S}*} \mathcal{E}^0$ are vector bundles on $\mathcal{S}$ making $R \operatorname{pr}_{\mathcal{S}*} \mathcal{E}$ the complex of maps from K^0 to K^1. We are now ready to make a definition.

Definition 28 (determinental line bundle)**.** Fix notation as above with a stack $\mathcal{S}$ and vector bundle $\mathcal{E}$ on $X \times \mathcal{S}$. We now define

$$\det R \operatorname{pr}_{\mathcal{S}*} \mathcal{E} := \det K^0 \otimes \left(\det K^1\right)^\vee.$$

Remark 29. For example, if $\operatorname{rank} K^0 = \operatorname{rank} K^1$ (for example, this is implied by $\operatorname{rank} R \operatorname{pr}_{\mathcal{S}*} \mathcal{E} = 0$), then one can show that the induced map $K^0 \to K^1$ from $\mathcal{E}^{-1} \to \mathcal{E}^0$ induces a map $\det K^0 \to \det K^1$, giving a global section of $\det(R \operatorname{pr}_{\mathcal{S}*} \mathcal{E})^\vee$.

We now specialize the discussion of the previous pargraph to $\mathcal{S} = \mathcal{M}_X^{\mathrm{ss}}(r,d)$, and $\mathcal{E}_{\mathrm{univ}}$ is the universal vector bundle on $X \times \mathcal{M}_X^{\mathrm{ss}}(r,d)$ such that the fiber over some $\mathcal{E} \in \mathcal{M}_X^{\mathrm{ss}}(r,d)$ is simple $\mathcal{E}$. Using the above construction, we note that any vector bundle $\mathcal{V}$ on X produces a determinental line bundle

$$\mathcal{L}_{\mathcal{V}} := \left(\det R \operatorname{pr}_{\mathcal{S}*}(\operatorname{pr}_X^* \mathcal{V} \otimes \mathcal{E}_{\mathrm{univ}})\right)^\vee,$$

which can somehow be thought of as (the determinant of) a Fourier transform. One can check that $\mathcal{V} \mapsto \mathcal{L}_{\mathcal{V}}$ produces a group homomorphism $K_0(X) \to \operatorname{Pic} \mathcal{M}_X^{\mathrm{ss}}(r,d)$, so we are producing lots of line bundles in a nice functorial way, though the output only depends on the rank and degree.

Now that we have a line bundle $\mathcal{L}_{\mathcal{V}}$, we are on the hunt for global sections. Notably, the construction given above was able to produce one global section on $\mathcal{L}_{\mathcal{V}}$, but the choice of global section depends on the choice of $\mathcal{V}$ (even though the isomorphism class of $\mathcal{L}_{\mathcal{V}}$ only depends on the rank and degree of $\mathcal{V}$!). Thus, we can imagine that we are able to produce lots of global sections on the single line bundle $\mathcal{L}_{\mathcal{V}}$, so it has good chances of being ample.

Drew then recorded the meat of the argument, which I did not follow in detail. ■

7 November 22: Joseph Stahl

Today we are resolving singularities on stacks. Throughout, we work over a field k of characteristic 0.

7.1 The Main Theorem

Here is our main theorem. Intuitively, it is a stacky form of resolution of singularities.

Theorem 30. Let $\mathcal{Y}$ be a smooth toroidal Artin stack over a field k, and let $\mathcal{X} \subseteq \mathcal{Y}$ be a reduced, generically toroidal, closed substack. Then there exists a canoncail sequence of multi-weighted blow-ups

$$\mathcal{Y}_N \stackrel{\pi_N}{\to} \mathcal{Y}_{n-1} \to \cdots \to \mathcal{Y}_1 \stackrel{\pi_1}{\to} \mathcal{Y}_0 = \mathcal{Y}$$

with proper transforms $\mathcal{X}_i \subseteq \mathcal{Y}_i$ of $\mathcal{X}$ satisfying the following conditions.

(a) $\mathcal{X}_N$ is a smooth toroidal Artin stack over k.

(b) The composite Π is an isomorphism over the log smooth subset of $\mathcal{X}$.

(c) The inverse image of the complement of the log smooth subset of $\mathcal{X}$ is a simple normal crossing divisor of $\mathcal{X}_N$.

(d) Each π_i is birational, surjeective, and universally closed. These π_i also factor through some Y_i, where $\mathcal{Y}_i \to Y_i$ is a good moduli space.

Moreover, the construction sending the pair $\mathcal{X} \subseteq \mathcal{Y}$ to the sequence of blow-ups is functorial.

Remark 31. One can recover a scheme-theoretic resolution of singularties by de-stackifying the construction at the end.

This statement requires some explanation, such as "toroidal" and "multi-weighted blow-ups." For example, being toroidal adds the extra structure of a simple normal crossing divisor on $\mathcal{Y}$ (such as the empty divisor). Being generically toroidal requires something like being dense open with respect to this divisor.

Roughly speaking, the idea is to ireatively blow up the $\mathcal{Y}_\bullet$s along the worst singular locus of $\mathcal{X}_i$. Here is an example of this process.

Example 32. Let $X \subseteq \mathbb{A}^3$ be the Whitney umbrella cut out by $x^2 - y^2 z$. It is singular along the x-axis, but the origin is the wost singularity. Thus, we begin by blowing up at the origin via $\pi \colon \operatorname{Bl}_0 \mathbb{A}^3 \to \mathbb{A}^3$. The one finds that $\pi^{-1}(X \setminus \{0\})$ is $x^2 - y^2 z = z^2 \left(t^2 - u^2 z\right)$ (where $[t : u : v]$ are the coordinates of the $\mathbb{P}^2$ at the origin), so we still have the same kind of singularity!

To fix this, we need to not just do regular blow-ups but multi-weighted blow-ups.

A key ingredient in the proof is a certain way to measure how bad a singularity is. This is a local invariant, so the ambient smoothness allows us to merely define this in the case that $\mathcal{Y} = Y$ and $\mathcal{X} = X$ are schemes. Now, given a point $p \in Y$, we will define $\operatorname{inv}_p(X \subseteq Y)$ as a finite sequence of nondecreasing rational numbers (possibly empty or including ∞) that satisfies the following.

(a) The values of inv are well-ordered (lexicographically).

(b) If $X = Y$, then $\operatorname{inv}_p(X \subseteq Y)$ is empty.

(c) Having $\operatorname{inv}_p(X \subseteq Y) = \{0\}$ if and only if $p \notin X$.

(d) For $p \in X$, one has $\operatorname{inv}_p(X \subseteq Y) \geq (1, \ldots, 1)$, where the number of 1s is the codimension of X contained in Y; equality holds if and only if X is smooth and toroidal at p.

(e) The first entry of $\mathrm{inv}_p(X \subseteq Y)$ is the "log" order of the ideal I defining X in Y at p. This "log" order is somehow analogous to a valuation.

(f) inv is upper semi-continuous on Y. In other words, this function is allowed to jump up on closed subsets.

(g) There is some functoriality.

Now, the main theorem follows from iteratively applying the following claim.

Theorem 33. Fix $\mathcal{X} \subseteq Y$ as before, and suppose that $\mathcal{X}$ is singular. Then there is a multi-weighted blow-up $\pi\colon \mathcal{Y}' \to \mathcal{Y}$ with proper transform $\mathcal{X}' \subseteq \mathcal{Y}'$ of $\mathcal{X} \subseteq \mathcal{Y}$ satisfying the following.

(a) $\mathcal{Y}'$ is a smooth toroidal Artin stack over k.

(b) The maximum of $\mathrm{inv}_p(\mathcal{X}' \subseteq \mathcal{Y}')$ is strictly smaller than the maximum of $\mathrm{inv}_p(\mathcal{X} \subseteq \mathcal{Y})$.

(c) π is an isomorphism away from the closed subset of $p \in Y$ such that $\mathrm{inv}_p(\mathcal{X} \subseteq \mathcal{Y})$ equals the maximum.

Remark 34. We are requiring that $\mathcal{X}$ be singular so that it is possible for the invariant to get strictly smaller.

7.2 Multi-Weighted Blow-Ups

For the last quarter of the talk, the speaker gave an example where $X \subseteq \mathbb{A}^3$ is cut out by $x^2 + y^2z + z^3 = 0$; the toroidal structure is given by the divisor cut out by $yz = 0$. It turns out that the worst singularity is at (x^2, y^2z, z^3); roughly speaking, the point is to take the monomials which appear in our equation. Then there is some recipe using the theory of toric varieties to produce a multi-weighted blow-up.