Representation Theory of Finite Groups

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Abstract

We review the representation theory of finite groups. Emphasis is placed on providing a reference, so proofs are chosen to be short and memorable. There are no examples. All groups G, H, K, and so on are finite. All vector spaces are finite-dimensional and over $\mathbb{C}$.

Contents

1 Down to Irreducibles

In this section, we study some questions which directly reduce to irreducible representations: decompositions and computing homomorphisms.

1.1 Basic Constructions

Let's start at the beginning.

Definition 1 (representation)**.** Fix a group G. Then a (finite-dimensional) G-*representation* (over $\mathbb{C}$) is a finite-dimensional $\mathbb{C}$-vector space V equipped with a homomorphism $\rho\colon G \to \operatorname{Aut}(V)$. We occasionally call ρ itself the representation and write V_ρ to denote the "underlying" vector space.

Remark 2. Fix a group G. Then a G-representation (over $\mathbb{C}$) has equivalent data to a $\mathbb{C}[G]$-module. On one hand, a $\mathbb{C}[G]$-module V is a $\mathbb{C}$-module (i.e., a $\mathbb{C}$-vector space), and it comes with a G-action from the module structure. On the other hand, a G-representation $\rho\colon G \to \operatorname{Aut}(V)$ extends the $\mathbb{C}$-action $\mathbb{C} \to \operatorname{End}(V)$ to a ring morphism $\mathbb{C}[G] \to \operatorname{End}(V)$.

Example 3. The above remark has also told that any group G has the "regular representation" given by $\mathbb{C}[G]$.

This module-theoretic perspective tells us that we should define morphisms of representations (called "G-invariant") to be morphisms of $\mathbb{C}[G]$-modules so that the category of G-representations (over $\mathbb{C}$) is simply $\mathrm{Mod}_{\mathbb{C}[G]}$. This tells us that our category is abelian, so we may define subobjects (called "suprepresentations" or "invariant subspaces"), quotients (called "quotient representations"), direct sums, and tensor products in $\mathrm{Mod}_{\mathbb{C}[G]}$.

Example 4. For any G-representation ρ, the G-invariants

$$V_\rho^G := \{v \in V_\rho : \rho(g)v = v \text{ for all } g \in G\}$$

is a G-invariant subspace. Indeed, we can see directly that it is G-invariant, and it is the intersection of the kernels $\ker(\mathrm{id}_V - \rho(g))$ over all $g \in G$, so it is a subspace.

Definition 5 (regular representation)**.** Fix a group G. Because $\mathbb{C}[G]$ is itself a $\mathbb{C}[G]$-module, we see that $\mathbb{C}[G]$ is a G-representation. It is called the *regular representation*.

Another perspective is that representation theory is linear algebra with some extra bells and whistles, so we attach many definitions from linear algebra to our representations. For example, the dimension of a G-representation ρ is

$$\dim \rho := \dim V_\rho.$$

Remark 6. A quick benefit of a linear algebra perspective is that, for any G-representation ρ, the operator $\rho(g)$ is diagonalizable for any $g \in G$. Indeed, G is finite, so g and hence $\rho(g)$ has finite order. It thus follows that $\rho(g)$ is diagonalizable. To see this, it is enough to show that any vector in $V := V_\rho$ is a sum of eigenvectors of the operator $\varphi := \rho(g)$. Let n be the operator of φ. Then the minimal polynomial of φ is $x^n - 1$, which has no repeated roots when factored over $\mathbb{C}$, so φ is diagonalizable.

However, in contrast to both linear algebra and modules, we use the special structure to give Hom and $\otimes$ a special structure.

Definition 7. Fix G-representations ρ and ρ'. Then $\mathrm{Hom}_{\mathbb{C}}(V_\rho, V_{\rho'})$ has the structure of G-representation by defining

$$g\varphi := \rho'(g) \circ \varphi \circ \rho(g)^{-1}.$$

One can check directly that this provides $\mathbb{C}[G]$-module structure. As a special case, we define the dual as $\rho^\vee := \mathrm{Hom}_{\mathbb{C}}(V_\rho, \mathbb{C})$.

Remark 8. Let's explain the above definition. In the context of the previous definition, we claim that $\mathrm{Hom}_{\mathbb{C}}(V_\rho, V_{\rho'})^G = \mathrm{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'})$. Indeed, $\varphi\colon V_\rho \to V_{\rho'}$ is fixed by $g \in G$ if and only if

$$\rho(g) \circ \varphi \circ \rho'(g)^{-1} = \varphi$$

for all $g \in G$, which rearranges to $\rho(g) \circ \varphi = \varphi \circ \rho'(g)$.

Definition 9. Fix G-representations ρ and ρ'. Then $V_\rho \otimes_\mathbb{C} V_{\rho'}$ has the structure of G-representation by defining
$$g(v \otimes v') := gv \otimes gv'.$$
One can check directly that this provides $\mathbb{C}[G]$-module structure.

Here is a quick sanity check that our definitions have been set up correctly.

Lemma 10. Fix G-representations ρ and ρ'. Then $\operatorname{Hom}_\mathbb{C}(V_\rho, V_{\rho'}) \cong V_\rho^\vee \otimes_\mathbb{C} V_{\rho'}$.

Proof. There is a natural map
$$\eta\colon V_\rho^\vee \otimes_\mathbb{C} V_{\rho'} \to \operatorname{Hom}_\mathbb{C}(V_\rho, V_{\rho'})$$
by extending $\eta(\varphi \otimes v')\colon v \mapsto \varphi(v)v'$; further, η is G-linear because
$$\begin{aligned}(g\eta(\varphi\otimes v'))(v) &= \rho'(g)\circ\eta(\varphi\otimes v')\left(\rho(g)^{-1}v\right)\\ &= \rho'(g)\varphi\left(\rho(g)^{-1}v\right)v'\\ &= (g\varphi)(v)(\rho'(g)v')\\ &= \eta(g\varphi\otimes gv')(v).\end{aligned}$$

It remains to show η is bijective. Well, the domain and codomain of η both have dimension $(\dim\rho)(\dim\rho')$, so it suffices to show η is surjective. As such, fix bases $\{v_1,\ldots,v_n\}$ and $\{v_1',\ldots,v_{n'}'\}$ of V and V', respectively. For any linear map $\psi\colon V_\rho \to V_{\rho'}$, we let $\{a_{ii'}\}_{i,i'}$ be the associated matrix. Then we define $\varphi_i\colon V_\rho \to \mathbb{C}$ by extending $v_j \mapsto 1_{i=j}$ linearly, and we see
$$\psi(v_i) = \sum_{j'=1}^{n'} a_{ij'}v_{j'} = \sum_{j'=1}^{n} a_{ij'}\varphi_i(v_i)v_{j'} = \sum_{j=1}^{n}\sum_{j'=1}^{n} a_{jj'}\varphi_j(v_i)v_{j'}$$
for each v_i. Thus, we see
$$\psi = \eta\left(\sum_{j=1}^{n}\sum_{j'=1}^{n}\varphi_j\otimes a_{jj'}v_{j'}\right),$$
finishing. ■

1.2 Decomposing Representations

We are going to decompose representations into irreducible ones.

Definition 11 (irreducible)**.** A G-representation ρ is *irreducible* if and only if it is nonzero and has no nonzero proper subrepresentations.

We are going to want to decompose general representations into irreducible ones. It will be productive to discuss inner products.

Definition 12 (unitary)**.** A G-representation ρ is *unitary* for a Hermitian inner product $\langle\cdot,\cdot\rangle$ on V_ρ if and only if
$$\langle gv, gw\rangle = \langle v, w\rangle$$
for any $v, w \in V_\rho$.

It is a remarkable fact that we can think about any given representation as being unitary.

Proposition 13 (Weyl)**.** Let G be a finite group. For any representation ρ, there exists a Hermitian inner product $\langle\cdot,\cdot\rangle$ on V_ρ for which ρ is unitary.

Proof. Because V_ρ is a finite-dimensional $\mathbb{C}$-vector space, we can choose a basis of V to yield an isomorphism $V_\rho \cong \mathbb{C}^n$ where $n = \dim\rho$. Then we can certainly give V_ρ some inner product $\langle\cdot,\cdot\rangle_0$ in the form of the usual one on $\mathbb{C}^n$. To fix the G-invariance of this inner product, we define

$$\langle v, w\rangle := \frac{1}{\#G}\sum_{g\in G}\langle \rho(g)v, \rho(g)w\rangle_0.$$

A linear combination of Hermitian inner products remains conjugate-symmetric, bilinear, and positive, so $\langle\cdot,\cdot\rangle$ is conjugate-symmetric, bilinear, and positive. In fact, we can also see that $\langle\cdot,\cdot\rangle$ is non-degenerate: if $\langle v, w\rangle = 0$, then we must have $\langle \rho(g)v, \rho(g)w\rangle_0 = 0$ for each $g \in G$, so $v = w$ follows by setting g to be the identity. Lastly, we see that $\langle\cdot,\cdot\rangle$ makes ρ unitary because

$$\langle \rho(g')v, \rho(g')w\rangle = \frac{1}{\#G}\sum_{g\in G}\langle \rho(gg')v, \rho(gg')w\rangle_0 = \frac{1}{\#G}\sum_{g\in G}\langle \rho(g)v, \rho(g)w\rangle_0 = \langle v, w\rangle$$

for any $v, w \in V_\rho$ and $g' \in G$. ■

The following result explains why we care about being unitary.

Lemma 14. Fix a G-representation ρ unitary for $\langle\cdot,\cdot\rangle$. If $W \subseteq V_\rho$ is a G-invariant subspace, then the orthogonal complement

$$W^\perp := \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}$$

is also a G-invariant subspace, and $V_\rho \cong W \oplus W^\perp$ as G-representations.

Proof. To see that $W^\perp$ is G-invariant, we note that $v \in W^\perp$ implies that

$$\langle gv, w\rangle = \left\langle v, g^{-1}w\right\rangle = 0$$

for any $g \in G$ and $w \in W$; notably, we are using the fact that $g^{-1}w \in W$ as well. To see that $V_\rho \cong W \oplus W^\perp$, we define the map $\varphi\colon W \oplus W^\perp \to V_\rho$ by $\varphi\colon (w, w') \mapsto w + w'$. This map is G-linear, and it describes the usual orthogonal decomposition of a vector space (recall V_ρ is finite-dimensional), so it is an isomorphism of representations. ■

Theorem 15 (Maschke)**.** Any G-representation ρ is a direct sum of finitely many irreducible representations.

Proof. We induct on $\dim\rho$. If $\dim\rho = 0$, then ρ is the zero representation, which is the direct sum of no irreducible representations. Otherwise, given ρ with $\dim\rho > 0$, we have two cases.

- If ρ is irreducible, then we are done.
- If ρ is not irreducible, then ρ has a nonzero proper G-invariant subspace $W \subseteq V_\rho$. Then Proposition 13 combined with Lemma 14 allows us to decompose ρ as a direct sum of two proper subrepresentations arising from $W, W^\perp \subseteq V_\rho$. Thus, $\dim W, \dim W^\perp < \dim\rho$, so we may induct to finish. ■

Theorem 15 lets us define the "isotypical decomposition."

Definition 16 (isotypical decomposition)**.** Fix a G-representation ρ. Let $\rho_1, \ldots, \rho_k$ denote distinct irreducible representations of G. Then the *isotypical decomposition* of ρ consists of the nonnegative integers $n_1, \ldots, n_k$ such that

$$\rho \cong \bigoplus_{i=1}^{k} \rho_i^{n_i}.$$

Note that we have not yet shown that the isotypical decomposition is unique, only that it exists. This requires a bit more machinery; we will wait until Corollary 33 to provide a proof.

1.3 Morphisms Between Representations

An advantage to working with "simple" objects is that their morphisms are relatively controlled.

Theorem 17 (Schur's lemma)**.** Fix an irreducible G-representation ρ. Any G-invariant map $\varphi\colon V_\rho \to V_\rho$ is multiplication by a scalar.

Proof. Note φ is a linear operator on a $\mathbb{C}$-vector space, so it has an eigenvalue λ. Thus, $\ker(\varphi - \lambda \mathrm{id}_V)$ contains a nonzero vector, so it has a nonzero subrepresentation of V_ρ. Because V_ρ is irreducible, it follows that

$$\ker(\varphi - \lambda \mathrm{id}_V) = V_\rho,$$

so $\varphi(v) = \lambda v$ for all $v \in V_\rho$. ■

Theorem 17 has a number of important corollaries.

Example 18. Let G be a finite abelian group. We claim that all irreducible representations are one-dimensional. Indeed, for any G-representation ρ, we note that $\rho(g)\colon V_\rho \to V_\rho$ is a G-invariant map because G is abelian: we compute

$$\rho(g)(\rho(g')v) = \rho(gg')v = \rho(g'g)v = \rho(g')(\rho(g)v).$$

Thus, Theorem 17 implies that $\rho(g)$ must equal a scalar λ_g. In particular, any one-dimensional subspace of V_ρ is a nonzero G-invariant subspace of V_ρ, so if ρ is irreducible, then $\dim V_\rho = 1$ is forced.

Corollary 19. Fix irreducible G-representations ρ and ρ'. Then

$$\dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) = \begin{cases} 1 & \text{if } V_\rho \cong V_{\rho'}, \\ 0 & \text{else.} \end{cases}$$

Proof. We deal with the two cases separately.

- If $V_\rho \cong V_{\rho'}$, then after fixing such an isomorphism, we are computing
$$\dim \operatorname{End}_{\mathbb{C}[G]}(V_\rho).$$
Of course, scalars in $\mathbb{C}$ are morphisms, and these are distinct morphisms because ρ is irreducible and hence nonzero. However, Theorem 17 tells us these are the only morphisms, so $\dim \operatorname{End}_{\mathbb{C}[G]}(V_\rho) = \dim \mathbb{C} = 1$.
- If $V_\rho \not\cong V_{\rho'}$, we show $\operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) = 0$. Well, any morphism $\varphi\colon V_\rho \to V_{\rho'}$ is either not injective or not surjective. If φ is not injective, then $\ker \varphi \subseteq V_\rho$ is a nontrivial subrepresentation, so the irreducibility enforces $\ker \varphi = V_\rho$, so $\varphi = 0$.
  On the other hand, if φ is not surjective, then $\operatorname{im} \varphi \subsetneq V_{\rho'}$ is a proper subrepresentation, so irreducibility enforces $\operatorname{im} \varphi = 0$, so $\varphi = 0$. ■

For the next corollaries, we want the following lemma. Roughly speaking, the symmetry of the statement in Corollary 19 in ρ and ρ' can be extended to arbitrary representations, which we will use to great profit.

Lemma 20. Fix a group G. Let $\rho_1, \ldots, \rho_k$ be irreducible representations, and fix nonnegative integers $n_1, \ldots, n_k$ and $n_1, \ldots, n_k'$. Then any morphism

$$\varphi\colon \bigoplus_{i=1}^k \rho_i^{\oplus n_i} \to \bigoplus_{i=1}^k \rho_i^{\oplus n_i'}$$

is the sum of the induced maps $\rho_i^{\oplus n_i} \to \rho_i^{\oplus n_i'}$. Thus,

$$\operatorname{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{i=1}^k V_{\rho_i}^{\oplus n_i}, \bigoplus_{i=1}^k V_{\rho_i}^{\oplus n_i'}\right) \cong \bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{C}[G]}\left(V_{\rho_i}^{\oplus n_i}, V_{\rho_i}^{\oplus n_i'}\right).$$

Proof. Composing φ with inclusion and projection, for any indices a and b, we have induced maps

$$V_a^{\oplus n_a} \to \bigoplus_{i=1}^k \rho_i^{\oplus n_i} \xrightarrow{\varphi} \bigoplus_{i=1}^k \rho_i^{\oplus n_i'} \to V_b^{\oplus n_b'}.$$

Call this composite $\varphi_{b,a}$. It follows that we may write

$$\varphi(v_1, \ldots, v_k) = \left(\sum_{i=1}^k \varphi_{1,i}(v_i), \ldots, \sum_{i=1}^k \varphi_{k,i}(v_i)\right)$$

If $a \neq b$, then any G-invariant map $V_a \to V_b$ must vanish by Corollary 19, so the above sum actually collapses into

$$\varphi(v_1, \ldots, v_k) = (\varphi_{1,1}v_1, \ldots, \varphi_{k,k}v_k).$$

To show the last sentence, we note that there is a natural map η from the right to left by sending a k-tuple of maps $(\varphi_1, \ldots, \varphi_k)$ to the map

$$\eta(\varphi_1, \ldots, \varphi_k)\colon (v_1, \ldots, v_k) \mapsto (\varphi_1 v_1, \ldots, \varphi_k v_k).$$

A direct computation shows that η is G-linear. Now, η is injective because if $\eta(\varphi_1, \ldots, \varphi_k)$ vanishes, then it must vanish in each coordinate, forcing $(\varphi_1, \ldots, \varphi_k) = (0, \ldots, 0)$. Further, the above proof establishes that η is surjective, so η is an isomorphism. ■

Corollary 21. Fix a G-representations ρ and ρ' with isotypical decompositions $\rho \cong \bigoplus_{i=1}^k \rho_i^{\oplus n_i}$ and $\rho' \cong \bigoplus_{i=1}^k \rho_i^{\oplus n_i'}$. Then

$$\dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) = \sum_{i=1}^k n_i n_i'.$$

Proof. By Lemma 20, we see

$$\operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) \cong \operatorname{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{i=1}^k V_{\rho_i}^{\oplus n_i}, \bigoplus_{i=1}^k V_{\rho_i}^{\oplus n_i'}\right) \cong \bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{C}[G]}\left(V_{\rho_i}^{\oplus n_i}, V_{\rho_i}^{\oplus n_i'}\right).$$

Now, for each i, a morphism $V_{\rho_i}^{\oplus n_i} \to V_{\rho_i}^{\oplus n_i'}$ is an $n_i' \times n_i$ matrix of morphisms $V_{\rho_i} \to V_{\rho_i'}$ by tracking what happens to each coordinate, so we actually have

$$\operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) \cong \bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{C}[G]}(V_{\rho_i}, V_{\rho_i})^{\oplus n_i n_i'}.$$

Taking dimensions and applying Corollary 19 finishes. ■

Remark 22. Note the bilinear form $(\cdot,\cdot)$ defined on finite-dimensional G-representations by $(\rho,\rho') := \dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'})$ is automatically bilinear with respect to direct sums. (This is because Hom commutes with direct sums and products.) Here are some other properties.

- Corollary 21 tells us that this form is symmetric.
- If ρ and ρ' are irreducible, then by Corollary 19, we see (ρ,ρ') is 1 if $\rho \cong \rho'$ and 0 otherwise.
- If ρ has $(\rho,\rho') = 0$ for all ρ', then we claim $\rho = 0$. Indeed, give ρ an isotypical decomposition $\bigoplus_{i=1}^k \rho_i^{\oplus n_i}$. But then computing $(\rho,\rho_i) = n_i$ by Corollary 21 for each i enforces $n_i = 0$ always, so $\rho = 0$.

Corollary 23. Fix a G-representation ρ with isotypical decomposition $\rho \cong \bigoplus_{i=1}^k \rho_i^{\oplus n_i}$. Then

$$\operatorname{End}_{\mathbb{C}[G]}(V_\rho) \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C}),$$

where $M_{n_i}(\mathbb{C})$ is the matrix algebra.

Proof. By Lemma 20, we note that any G-invariant map $\varphi\colon V_\rho \to V_\rho$ is the sum of maps $\varphi_i\colon V_{\rho_i}^{\oplus n_i} \to V_{\rho_i}^{\oplus n_i}$, so we have an isomorphism

$$\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{C}[G]}\left(V_{\rho_i}^{\oplus n_i}, V_{\rho_i}^{\oplus n_i}\right) \to \operatorname{End}_{\mathbb{C}[G]}(V_\rho)$$

of $\mathbb{C}[G]$-modules. Because this isomorphism merely sends $(\varphi_1,\ldots,\varphi_k)$ to the summed morphisms, we see that it is also compatible with the ring structures on both sides, so this is an isomorphism of $\mathbb{C}[G]$-algebras.

It remains to show $\operatorname{End}_{\mathbb{C}[G]}\left(V_{\rho_i}^{\oplus n_i}, V_{\rho_i}^{\oplus n_i}\right)$ is isomorphic to $M_{n_i}(\mathbb{C})$. Well, we see that any morphism $\varphi\colon V_{\rho_i}^{\oplus n_i} \to V_{\rho_i}^{\oplus n_i}$ can be written as

$$\varphi(v_1,\ldots,v_{n_i}) = \left(\sum_{j=1}^n \varphi_{1j}(v_j), \ldots, \sum_{j=1}^n \varphi_{n_ij}(v_j)\right)$$

where the maps $\varphi_{ab}\colon V_{\rho_i} \to V_{\rho_i}$ are defined by the inclusion to $V_{\rho_i}^{\oplus n_i}$ followed by φ followed by projection. However, Theorem 17 tells us that each φ_{ab} is a scalar $\lambda_{ab} \in \mathbb{C}$, so the data of the above morphism φ is simply given by the matrix $(\lambda_{ab})_{a,b=1}^{n_i}$. ■

2 Character Theory

In this section, we introdue characters and use them to great profit.

2.1 Characters

One difficulty in understanding representations is that they are inherently multidimensional objects. To fix this, we introduce characters.

Definition 24 (character)**.** Fix a G-representation ρ. Then the *character* $\chi_\rho\colon G \to \mathbb{C}$ of ρ is defined as $\chi_\rho(g) := \operatorname{tr}\rho(g)$.

For example, one can compute the trace by providing V_ρ with any basis and then summing along the diagonal entries of the matrix associated to $\rho(g)$. This construction does not depend on the basis because the trace of a matrix does not change when the basis changes.

Example 25. Let $\rho\colon G \to \mathbb{C}[G]$ be the regular representation. Then we claim $\chi_\rho(g) = |G|1_{g=e}$. Indeed, note $\mathbb{C}[G]$ has the standard basis $\{h\}_{h\in G}$, and $\rho(g)$ acts by permuting them by left multiplication. Then, for any $g \in G$, the diagonal entry given by $h \in G$ is 1 if $gh = h$ (which is equivalent to $g = e$) and 0 otherwise. So $\chi_\rho(g) = \operatorname{tr}\rho(g) = |G|1_{g=e}$ follows.

Here are some basic properties.

Lemma 26. Fix a G-representations ρ.

(a) If $\dim\rho = 1$, then $\rho = \chi_\rho$ after identifying V_ρ with $\mathbb{C}$.

(b) χ_ρ is defined up to conjugacy class.

(c) $\chi_\rho(1) = \dim\rho$.

(d) We have
$$\dim V^G = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g).$$

Proof. Here we go.

(a) For any $g \in G$, we note $\rho(g)\colon \mathbb{C} \to \mathbb{C}$ is a morphism of vector spaces, so it is equal to its trace.

(b) For any $g, h \in G$, we compute
$$\chi_\rho\left(ghg^{-1}\right) = \operatorname{tr}\left(\rho(h)\circ\rho(g)\circ\rho(h)^{-1}\right) = \operatorname{tr}\left(\rho(g)\circ\rho(h)^{-1}\circ\rho(g)\right) = \operatorname{tr}\rho(g) = \chi_\rho(g),$$
so $\chi(g)$ is defined up to conjugacy class of g.

(c) Note $\chi_\rho(1) = \operatorname{tr}\rho(1) = \operatorname{tr}\operatorname{id}_{V_\rho}$. This is $\dim V_\rho$ by summing along the diagonal of the identity matrix.

(d) Define the linear map $\pi\colon V \to V$ by
$$\pi \coloneqq \frac{1}{|G|}\sum_{g\in G}\rho(g).$$
Notably, $\operatorname{tr}\pi = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)$ by the linearity of tr. We claim that π is a projection onto V^G. We have two checks.

- Note $\pi(v) \in V^G$ for any $v \in V$: indeed, we compute
$$g'\pi(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g'g)v = \frac{1}{|G|}\sum_{g\in G}\rho(g)v = \pi(v)$$
for any $g' \in G$.
- Note $\pi(v) = v$ for any $v \in V^G$: indeed, we compute
$$\pi(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g)v = \frac{1}{|G|}\sum_{g\in G}v = v.$$

It now follows that $\operatorname{tr}\pi = \dim V^G$. To see this concretely, we set $d \coloneqq \dim V^G$ and $n \coloneqq \dim\rho$, and we give V a basis by extending a basis $\{v_1, \ldots, v_d\}$ of V^G to a basis $\{v_1, \ldots, v_n\}$ of V. Letting $\{\pi_{ij}\}_{i,j=1}^n$ be the associated matrix, we note that $\pi(v_i) = v_i$ for each $1 \le i \le d$ implies that $\pi_{ii} = 1$ if $1 \le i \le d$; otherwise, for each $i > d$, we see $\pi_{ii} = 0$ because $\pi(v_i) \in V^G$ is a linear combination of the v_j with $1 \le j \le d$, which has no v_i component. Thus, summing along the diagonal confirms $\operatorname{tr}\pi = \dim V^G$. ■

We can also describe how characters behave with our other constructions.

Lemma 27. Fix G-representations ρ and ρ'.

(a) $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$.

(b) $\chi_{\rho\otimes\rho'} = \chi_\rho \cdot \chi_{\rho'}$.

(c) $\chi_{\rho^\vee}(g) = \chi_\rho\left(g^{-1}\right)$ for any g.

Proof. Here we go.

(a) For any $g \in G$, we compute

$$\chi_{\rho\oplus\rho'}(g) = \operatorname{tr}(\rho(g)\oplus\rho'(g)) \overset{*}{=} \operatorname{tr}\rho(g) + \operatorname{tr}\rho'(g) = \chi_\rho(g) + \chi_{\rho'}(g).$$

To see $\overset{*}{=}$ concretely, we note that we can give the underlying vector space $V_\rho \oplus V_{\rho'}$ a basis by concatenating the bases of V_ρ and $V_{\rho'}$, upon which the matrix associated to $\rho(g)\oplus\rho'(g)$ looks like

$$\begin{bmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{bmatrix},$$

whose trace is the sum of the traces of $\rho(g)$ and $\rho'(g)$.

(b) For any $g \in G$, we compute

$$\chi_{\rho\otimes\rho'}(g) = \operatorname{tr}(\rho(g)\otimes\rho'(g)) \overset{*}{=} \operatorname{tr}\rho(g) \cdot \operatorname{tr}\rho'(g) = \chi_\rho(g)\cdot\chi_{\rho'}(g).$$

To see $\overset{*}{=}$ concretely needs some work. Give V_ρ and $V_{\rho'}$ bases $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_{n'}\}$, respectively, and let the matrices associated to $\rho(g)$ and $\rho'(g)$ be $\{a_{ij}\}_{i,j=1}^n$ and $\{a'_{i'j'}\}_{i',j'=1}^n$, respectively. Now, $V_\rho \otimes V_{\rho'}$ has basis given by $v_i \otimes v'_{i'}$ where the i and i' vary, so we compute

$$(\rho(g)\otimes\rho'(g))(v_i\otimes v'_{i'}) = \rho(g)v_i \otimes \rho'(g)v'_{i'} = \left(\sum_{j=1}^{n} a_{ij}v_j\right)\otimes\left(\sum_{j'=1}^{n'} a'_{i'j'}v'_{j'}\right) = \sum_{j=1}^{n}\sum_{j'=1}^{n'} a_{ij}a'_{i'j'}(v_j\otimes v'_{j'}).$$

Thus, the diagonal entry (at (i,i')) here is $a_{ii}a_{i'i'}$. Summing over all diagonal entries, we conclude

$$\operatorname{tr}(\rho(g)\otimes\rho'(g)) = \sum_{i=1}^{n}\sum_{i'=1}^{n'} a_{ii}a_{i'i'} = \operatorname{tr}\rho(g)\cdot\operatorname{tr}\rho'(g).$$

(c) By Proposition 13, we may give V_ρ an inner product $\langle\cdot,\cdot\rangle$ making ρ a unitary representation. Then

$$\chi_{\rho^\vee}(g) = \operatorname{tr}\left(\varphi\mapsto\varphi\circ\rho(g)^{-1}\right) \overset{*}{=} \operatorname{tr}\left(\rho(g)^{-\intercal}\right) = \operatorname{tr}\rho(g)^{-1} = \chi_\rho\left(g^{-1}\right),$$

where $\overset{*}{=}$ amounts to giving $V_\rho^\vee$ a dual basis. ■

2.2 Orthogonality Relations

Characters get most of their structure from having an inner product.

Notation 28. For any functions $\varphi, \psi \colon G \to \mathbb{C}$, we define

$$\langle\varphi,\psi\rangle := \frac{1}{|G|}\sum_{g\in G}\varphi(g)\psi\left(g^{-1}\right).$$

One can directly check that $\langle\cdot,\cdot\rangle$ is an inner product on the $\mathbb{C}$-vector space $\operatorname{Mor}(G,\mathbb{C})$ (though not Hermitian!).

Remark 29. Fix a G-representation ρ. By Remark 6, we see that $\rho(g)$ is diagonalizable, and we know that its eigenvalues are roots of unity (of order dividing $|G|$) and in particular have magnitude 1. Thus, $\rho\left(g^{-1}\right)$ has eigenvalues conjugate to the eigenvalues of $\rho(g)$, with the correct multiplicities, so

$$\chi_\rho(g) = \operatorname{tr}\rho(g) = \overline{\operatorname{tr}\rho\left(g^{-1}\right)} = \overline{\chi_\rho\left(g^{-1}\right)}.$$

Thus, our inner product does look Hermitian when we work with characters of representations.

The following result explains how we will use this inner product to talk about representations.

Theorem 30. Fix G-representations ρ and ρ'. Then $\langle\chi_\rho, \chi_{\rho'}\rangle = \dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'})$.

Proof. We apply force. By Remark 8 and Lemma 10, we see

$$\dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) = \dim \operatorname{Hom}_{\mathbb{C}}(V_\rho, V_{\rho'})^G = \dim\left(V_\rho^\vee \otimes_{\mathbb{C}} V_{\rho'}\right)^G.$$

To relate to characters, we use Lemma 26 and then use Lemma 27 to compute

$$\dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho^\vee\otimes\rho'}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho\left(g^{-1}\right)\chi_{\rho'}(g).$$

Exchanging the roles of g and g^{-1} finishes the proof. ■

Corollary 31. Fix a G-representations ρ and ρ' with isotypical decompositions $\rho \cong \bigoplus_{i=1}^k \rho_i^{\oplus n_i}$ and $\rho' \cong \bigoplus_{i=1}^k \rho_i^{\oplus n_i'}$. Then

$$\langle\chi_\rho, \chi_{\rho'}\rangle = \sum_{i=1}^k n_i n_i'.$$

In particular, $\langle\chi_\rho, \chi_{\rho_i}\rangle = n_i$.

Proof. By Lemma 26, we see

$$\chi_\rho = \sum_{i=1}^k n_i\chi_{\rho_i} \qquad \text{and} \qquad \chi_{\rho'} = \sum_{i=1}^k n_i'\chi_{\rho_i},$$

so the bilinearity of our inner product yields

$$\langle\chi_\rho, \chi_{\rho'}\rangle = \sum_{i=1}^k\sum_{j=1}^k n_i n_j\langle\chi_{\rho_i}, \chi_{\rho_j}\rangle.$$

Now, Theorem 30 tells us that $\langle\chi_{\rho_i}, \chi_{\rho_j}\rangle = \dim \operatorname{Hom}_{\mathbb{C}[G]}(V_{\rho_i}, V_{\rho_j})$, which is $1_{i=j}$ by Corollary 19. So this collapses to

$$\langle\chi_\rho, \chi_{\rho'}\rangle = \sum_{i=1}^k n_i n_i',$$

which is what we wanted. The last sentence now follows by giving ρ_i an isotypical decomposition "ρ_i." ■

Remark 32. One can show Corollary 31 in the form of

$$\dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'}) = \sum_{i=1}^{k} n_i n_i'$$

directly by taking dimensions in Lemma 20. In particular, one can prove suitable versions of Corollaries 33 to 37 without needing to talk about characters at all!

Corollary 33. Fix a G-representation ρ. Then the isotypical decomposition of ρ is unique.

Proof. Suppose we have two isotypical decompositions of ρ. In other words, we may fix irreducible representations $\rho_1, \ldots, \rho_k$ and nonnegative integers $n_1, \ldots, n_k$ and $n_1', \ldots, n_k'$ such that

$$\bigoplus_{i=1}^{k} \rho_i^{\oplus n_i} \cong \rho \cong \bigoplus_{i=1}^{k} \rho_i^{\oplus n_i'}.$$

Then applying Corollary 31 to each of our isotypical decompositions yields

$$n_i = \langle \chi_\rho, \chi_{\rho_i} \rangle = n_i'$$

for each i, finishing. ■

Corollary 34. Fix a group G. Then a G-representation ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.

Proof. Let $\rho \cong \bigoplus_{i=1}^{k} \rho_i^{\oplus n_i}$ be an isotypical decomposition of ρ. Then Corollary 31 tells us

$$\langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^{k} n_i^2.$$

If ρ is irreducible, then there is only one nonzero term in the above sum, and it is equal to $1^2 = 1$, so $\langle \chi_\rho, \chi_\rho \rangle = 1$. Conversely, if the above sum is 1, then we have $n_i^2 \le 1$ for each i, and we have equality achieved exactly once, so $\rho \cong \rho_i$ for some irreducible representation ρ_i, which is what we wanted. ■

Corollary 35. Fix a group G. Then a G-representation ρ is irreducible if and only if $\rho^\vee$ is irreducible.

Proof. We compute

$$\langle \chi_{\rho^\vee}, \chi_{\rho^\vee} \rangle = \sum_{g \in G} \chi_{\rho^\vee}(g) \chi_{\rho^\vee}\left(g^{-1}\right) \stackrel{*}{=} \sum_{g \in G} \chi_\rho\left(g^{-1}\right) \chi_\rho(g) = \langle \chi_\rho, \chi_\rho \rangle,$$

where we used Lemma 27 in $\stackrel{*}{=}$. So the left-hand side equals 1 if and only if the right-hand side equals 1, from which Corollary 34 finishes. ■

Corollary 36 (First orthogonality relation)**.** Fix a group G. Given irreducible representations ρ and ρ', we have

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \begin{cases} 1 & \text{if } \rho \cong \rho', \\ 0 & \text{else.} \end{cases}$$

Proof. One can see this by comparing isotypical decompositions of ρ and ρ' and applying Corollary 31. Alternatively, one may use Theorem 30 and then Corollary 19. ■

Corollary 37. Fix a group G. There are only finitely many irreducible representations of G, and if they are $\rho_1, \ldots, \rho_k$, then

$$\sum_{i=1}^{k} (\dim \rho_i)^2 = |G|.$$

Proof. The point here is to compute the isotypical decomposition of the representation $\rho\colon G \to \mathbb{C}[G]$. Indeed, for any G-representation ρ', we see that

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \chi_{\rho'}\left(g^{-1}\right) = \frac{1}{|G|} \cdot |G| \chi_{\rho'}(1),$$

where we have used the computation in Example 25. To finish, Lemma 26 tells us $\chi_{\rho'}(1) = \dim \rho'$, so $\langle \chi_\rho, \chi_{\rho'} \rangle = \dim \rho'$.

Thus, if we let $\rho \cong \bigoplus_{i=1}^{k} \rho_i^{\oplus n_i}$ be the isotypical decomposition of $\mathbb{C}[G]$, Corollary 31 tells us that $n_i = \langle \chi_\rho, \chi_{\rho_i} \rangle = \dim \rho_i$. Taking dimensions, we see

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{k} n_i \dim \rho_i = \sum_{i=1}^{k} (\dim \rho_i)^2.$$

We now show that $\rho_i, \ldots, \rho_k$ are all the irreducible representations. For any irreducible G-representation ρ, if $\rho \not\cong \rho_i$ for each i, then Corollary 31 implies that $\dim \operatorname{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], \rho) = 0$, which is false as shown above. ■

We are now ready to explain why we care so much about characters.

Corollary 38. Fix G-representations ρ and ρ'. Then $\rho \cong \rho'$ if and only if $\chi_\rho = \chi_{\rho'}$.

Proof. There is nothing to say for the forward direction. In the reverse direction, let $\rho_1, \ldots, \rho_k$ denote the irreducible G-representations. Then we see

$$\langle \chi_\rho, \chi_{\rho_i} \rangle = \langle \chi_{\rho'}, \chi_{\rho_i} \rangle$$

for any i, so Corollary 31 lets us give ρ and ρ' the same isotypical decomposition

$$\bigoplus_{i=1}^{k} \rho_i^{\oplus \langle \chi_\rho, \chi_{\rho_o} \rangle},$$

so $\rho \cong \rho'$ follows. ■

Corollary 36 is the "first" orthogonality relation. We will prove the second one later.

2.3 Class Functions

Lemma 26 motivates the following definition.

Definition 39 (class function)**.** Fix a group G. Then a function $\varphi\colon G \to \mathbb{C}$ is a *class function* if and only if $\varphi\left(hgh^{-1}\right) = \varphi(g)$ for any $g, h \in G$. Note that the set of all class functions forms a $\mathbb{C}$-vector space.

It will turn out that characters of irreducible representations form an orthonormal basis of the vector space of all class functions. This is difficult to show directly, approximately speaking because it is not easy to show that there are "enough" representations. Instead, we will upgrade this "numerical" result into an isomorphism of algebras.

Theorem 40. Fix a group G. Then the action map

$$\mathbb{C}[G] \to \prod_{\rho \text{ irreducible}} \operatorname{End}_{\mathbb{C}}(V_\rho)$$

defines an isomorphism of algebras.

Proof. The action map does define a morphism of algebras (by definition of a representation), and Corollary 37 explains that the domain and codomain have the same dimension. Thus, to conclude, it is enough to show that the action map has trivial kernel.

Well, suppose that $x \in \mathbb{C}[G]$ acts by 0 on all irreducible representations ρ; we would like to show $x = 0$. To begin, note that x acting by 0 on all irreducibles implies that x acts by 0 on any representation by Theorem 15. For example, x must act by 0 on the regular representation $\mathbb{C}[G]$. In particular, $x \cdot 1 = 0$ in $\mathbb{C}[G]$, so we are done. ■

We now need to take this isomorphism of algebras down to a numerical result. This will be done by considering the center.

Lemma 41. Fix a group G, and let $\varphi\colon G \to \mathbb{C}$ be a function. Then the following are equivalent.

(a) φ is a class function.

(b) The element $e_\varphi := \sum_{g\in G} \varphi(g)g \in \mathbb{C}[G]$ is in the center of $\mathbb{C}[G]$.

Proof. We have two implications to show. For any $h \in G$, we compute

$$he_\varphi h^{-1} = \sum_{g\in G} \varphi(g)hgh^{-1} = \sum_{g\in G} \varphi\left(hgh^{-1}\right)g.$$

Now, e_φ is in the center if and only if $he_\varphi h^{-1} = e_\varphi$ for all $h \in G$. (The forward implication is by definition; the reverse implication is because any element of $\mathbb{C}$ commutes with e_φ already.) But comparing the g-coordinate of $he_\varphi h^{-1}$ above and e_φ reveals that this is equivalent to $\varphi\left(hgh^{-1}\right) = \varphi(g)$ for any $g, h \in G$, which is equivalent to φ being a class function. ■

Lemma 42. For any finite-dimensinal vector space V over $\mathbb{C}$, we have $Z(\operatorname{End}_{\mathbb{C}}(V)) = \{\lambda \mathrm{id}_V : \lambda \in \mathbb{C}\}$.

Proof. Certainnly any scalar operator $\lambda\,\mathrm{id}_V$ lives in $Z(\operatorname{End}_{\mathbb{C}}(V))$. Conversely, we note that $\operatorname{End}_{\mathbb{C}}(V)$ acts standardly on V, and in fact V is an irreducible module for this algebra. Thus, the same argument as in Theorem 17 implies that any $\operatorname{End}_{\mathbb{C}}(V)$-invariant endomorphism of V is a scalar. ■

Proposition 43. Fix a group G. Then the number of irreducible representations of G equals the number of conjugacy classes of G.

Proof. We compute the dimension of the center of both sides of Theorem 40. On the left, we see that Lemma 41 explains that $Z(\mathbb{C}[G])$ is isomorphic (as a $\mathbb{C}$-vector space) to the space of class functions, so $\dim Z(\mathbb{C}[G])$ is the number of conjugacy classes. On the right, we see that Lemma 42 explains that each factor in the produce has one-dimensional center, so $\dim Z\left(\prod_\rho \operatorname{End}_{\mathbb{C}}(V_\rho)\right)$ equals the number of irreducible representations of G. ■

Corollary 44. Fix a group G. Then the characters of irreducible representations form an orthonormal basis of the vector space of all class functions.

Proof. That these characters are orthonormal follows from Corollary 36, so it remains to show that these span the vector space of class functions. Well, Proposition 43 tells us that the number of irreducible representations equals the dimension of the space of class functions (that is, the number of conjufacy classes). ■

While we're here, we also prove the second orthogonality relation.

Corollary 45 (Second orthogonality relation)**.** Fix a group G. Let $\rho_1, \ldots, \rho_r$ be the irreducible representations of G. For any $g \in G$, we let $[g]$ denote the conjugacy class of G. Then each $g, h \in G$ has

$$\sum_{i=1}^{r} \chi_{\rho_i}(g)\chi_{\rho_i}\left(h^{-1}\right) = \begin{cases} |G|/|[g]| & \text{if } [g] = [h], \\ 0 & \text{else.} \end{cases}$$

Proof. Let the conjugacy classes of G be represented as $[g_1], \ldots, [g_r]$; note that this is equal to the number of irreducible representations by Proposition 43. The point here is to do linear algebra to achieve the result from Corollary 36. Indeed, define the $r \times r$ matrix

$$M := \begin{bmatrix} \sqrt{\frac{|[g_1]|}{|G|}}\chi_1(g_1) & \cdots & \sqrt{\frac{|[g_r]|}{|G|}}\chi_1(g_r) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{|[g_1]|}{|G|}}\chi_r(g_1) & \cdots & \sqrt{\frac{|[g_r]|}{|G|}}\chi_r(g_r) \end{bmatrix}.$$

The main claim is that M is a unitary matrix. Notably, Remark 29 tells us that

$$M^\dagger = \begin{bmatrix} \sqrt{\frac{|[g_1]|}{|G|}}\overline{\chi_1(g_1)} & \cdots & \sqrt{\frac{|[g_1]|}{|G|}}\overline{\chi_r(g_1)} \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{|[g_r]|}{|G|}}\overline{\chi_1(g_r)} & \cdots & \sqrt{\frac{|[g_r]|}{|G|}}\overline{\chi_r(g_r)} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{|[g_1]|}{|G|}}\chi_1\left(g_1^{-1}\right) & \cdots & \sqrt{\frac{|[g_1]|}{|G|}}\chi_r\left(g_1^{-1}\right) \\ \vdots & \ddots & \vdots \\ \sqrt{\frac{|[g_r]|}{|G|}}\chi_1\left(g_r^{-1}\right) & \cdots & \sqrt{\frac{|[g_r]|}{|G|}}\chi_r\left(g_r^{-1}\right) \end{bmatrix}.$$

Thus, Corollary 36 tells us that

$$(MM^\dagger)_{ik} = \sum_{j=1}^{r} M_{ij}M^\dagger_{jk} = \sum_{j=1}^{r} \frac{|[g_j]|}{|G|}\chi_i(g_j)\chi_k\left(g_j^{-1}\right) = \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\chi_k(g^{-1}) = 1_{i=k},$$

so $MM^\dagger$ is the identity matrix, as needed. In particular, $M^\dagger = M^{-1}$, so we also see that $M^\dagger M$ is the identity matrix, so

$$1_{i=k} = (M^\dagger M)_{ik} = \sum_{j=1}^{r} M^\dagger_{ij}M_{jk} = \frac{\sqrt{|[g_i]| \cdot |[g_k]|}}{|G|}\sum_{j=1}^{r}\chi_j(g_i^{-1})\chi_j(g_k).$$

Thus, if $i = k$, then we see the leftmost summation evaluates to $|G|/|[g_i]|$; otherwise, the leftmost summation vanishes. The summation can replace g_i and g_k with any representative of their respective conjugacy classes by Lemma 26, so we complete the proof. ■

Remark 46. The moral of the above proof is that the character table (which is the matrix $\{\chi_i([g_j])\}$) is "almost" unitary. Indeed, it becomes unitary after appropriately scaling the columns.