Isometries Are Linear

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The goal of this note is to prove the following theorem.

Theorem 1. Fix a real inner product space V. If $T\colon V \to V$ is a function which fixes the origin and preserves distances, then T is a linear transformation preserving the inner product (i.e., orthogonal), and all (complex) eigenvalues have magnitude 1. If V is finite-dimensional, then T is diagonalizable (over $\mathbb{C}$) with respect to some orthonormal basis.

Proof. We will proceed in steps. Over time, the proof will become gradually more algebraic. The main point is to show that T preserves inner products as early as possible.

0. We take a moment to actually write down the hypotheses on T. Fixing the origin means that $T(0) = 0$, and preserving distances means that the distance between two vectors v and w is the same as the distance between Tv and Tw. In other words, we require
$$\|Tv - Tw\| = \|v - w\|.$$

1. We claim that T preserves inner products. The main point is that
$$\|v + w\|^2 = \langle v + w, v + w\rangle = \|v\|^2 + \|2\|^2 + 2\langle v, w\rangle$$
implies that
$$\langle v, w\rangle = \frac{\|v + w\|^2 - \|v\|^2 - \|w\|^2}{2},$$
so we may recover the inner product from norms alone. However, $v + w$ is written with a $+$, and we only have access to differences a priori. But this is okay: write
$$\begin{aligned}\langle Tv, Tw\rangle &= -\langle Tv, -Tw\rangle \\ &= -\frac{\|Tv - Tw\| - \|Tv\|^2 - \|-Tw\|^2}{2} \\ &= -\frac{\|Tv - Tw\| - \|Tv - T(0)\|^2 - \|T(0) - Tw\|^2}{2} \\ &= -\frac{\|v - w\| - \|v\|^2 - \|-w\|^2}{2} \\ &= -\langle v, -w\rangle \\ &= \langle v, w\rangle.\end{aligned}$$

2. We claim that T is linear. Namely, given $v, w \in V$ and scalars $a, b \in \mathbb{R}$, we would like to show that $T(av + bw) - aTv - bTw = 0$. The idea is to instead compute the norm of this and break things down

into the inner product, where we know that things are linear. Explicitly,

$$\begin{aligned}\|T(av+bw)-aTv-bTw\|^2 &= \langle T(av+bw), T(av+bw)\rangle + a^2\langle Tv, Tv\rangle + b^2\langle Tw, Tw\rangle \\ &\qquad - 2a\langle T(av+bw), Tv\rangle - 2b\langle T(av+bw), Tw\rangle + 2an\langle Tv, Tw\rangle \\ &= \langle av+bw), av+bw)\rangle + a^2\langle v, v\rangle + b^2\langle w, w\rangle \\ &\qquad - 2a\langle av+bw, v\rangle - 2b\langle av+bw, w\rangle + 2an\langle v, w\rangle \\ &= \|(av+bw)-av-bw\|^2 \\ &= 0.\end{aligned}$$

3. We show that all eigenvalues have magnitude 1. Well, if v is a nonzero eigenvector in the complexification $V_{\mathbb{C}}$ of V with eigenvalue $\lambda \in \mathbb{C}$, then we see that $Tv = \lambda v$ implies that

$$\|v\| = \|Tv\| = \|\lambda v\| = |\lambda| \cdot \|v\|,$$

so $|\lambda| = 1$.

4. If $V_{\mathbb{C}}$ is finite-dimensional, we show that T is diagonalizable. We do this by induction on $\dim V_{\mathbb{C}}$. If $\dim V_{\mathbb{C}} = 0$, there is nothing to show. Otherwise, take $\dim V_{\mathbb{C}} > 0$. Because $T\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ is an operator on a complex vector space, it will have an eigenvector v. But then the space $\{v\}^\perp$ is preserved by T because T preserves inner products, so we may restrict T to $T\colon \{v\}^\perp \to \{v\}^\perp$. Because $\{v\}^\perp$ is of smaller dimension, we may diagonalize the restriction of T to $\{v\}^\perp$, and so the decomposition

$$V_{\mathbb{C}} \cong \{v\} \oplus \{v\}^\perp$$

permits a diagonalization to $V_{\mathbb{C}}$. ■