Sato–Tate Groups of Generic Superelliptic Curves

Nir Elber

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman [Shu16]

Chapter 0
Introduction

0.1 Notation

Elements.

- V and W are vector spaces, frequently $\mathbb{Q}$-Hodge structures.
- $\mathbb{Q}(n)$ is the Tate twist.
- $\mathrm{H}^{\bullet}_{\mathrm{B}}$ is Betti cohomology, $\mathrm{H}^{\bullet}_{\mathrm{dR}}$ is de Rham cohomology, and $\mathrm{H}^{\bullet}_{\text{ét}}$ is étale cohomology.
- $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras.

Groups.

- If V is a $\mathbb{Q}$-Hodge structure, then $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$ are the Mumford–Tate and Hodge groups, respectively.
- $\mathbb{S}$ is the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{R}}$.
- Given a number field F, we define the torus $\mathrm{T}_F := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{m,\mathbb{Q}}$.
- Given a CM or totally real number field F, we define the subtorus $\mathrm{U}_F \subseteq \mathrm{T}_F$ by
  $$\mathrm{U}_F := \{x \in \mathrm{T}_F : x\overline{x} = 1\},$$
  where $\overline{x}$ is complex conjugation when F is CM and the identity when F is totally real.
- Given an algebraic group G, G° denotes the connected component, $Z(G)$ denotes its center, and G^{der} denotes the derived subgroup.

Categories.

- For a field F, Vec_F is the category of vector spaces over F.
- $\mathrm{HS}_{\mathbb{Q}}$ is the category of $\mathbb{Q}$-Hodge structures.

Organization is thematic. As such, dependencies are not always strictly linear, though we do our best to not require any content from a later chapter; at times, it is motivational to mention some content from a later chapter, but this is kept to a minimum. Additionally, some omitted proofs may require content from later chapters even if not mentioned.

Chapter 1
A Little Hodge Theory

Once we explicitely know a Mumford-Tate group, we can let it work for us.

—Moonen, [Moo, (5.5)]

In this chapter, we define the notion of a Hodge structure as well as some related groups (the Mumford–Tate group and the Hodge group). Our exposition follows Moonen's unpublished notes [Moo; Moo99] and Lombardo's master's thesis [Lom13, Chapter 3]. Throughout, we find motivation from geometry (and in particular the cohomology of complex varieties), but we will review cohomology only later.

1.1 Hodge Structures

Cohomology of a variety frequently comes with some extra structure. On the étale site, we will later get significant utility of the fact that étale cohomology is a Galois representaion. On the analytic site, the corresponding structure is called a "Hodge structure."

1.1.1 Definition and Basic Properties

Here is our defintion.

Definition 1.1 (Hodge structure)**.** A $\mathbb{Q}$-*Hodge structure* is a finite-dimensional vector space $V \in \mathrm{Vec}_{\mathbb{Q}}$ such that $V_{\mathbb{C}}$ admits a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V_{\mathbb{C}}^{p,q}$$

where $V_{\mathbb{C}}^{p,q} = \overline{V_{\mathbb{C}}^{q,p}}$. For fixed $m \in \mathbb{Z}$, if $V_{\mathbb{C}}^{p,q} \neq 0$ unless $p+q=m$, we say that V is *pure of weight* m. We let $\mathrm{HS}_{\mathbb{Q}}$ denote the category of $\mathbb{Q}$-Hodge structures, where a morphism of Hodge structures is a linear map preserving the decomposition over $\mathbb{C}$. In the sequel, it may be helpful to note that one can bring this definition down to $\mathbb{Z}$ as well.

Example 1.2. We give the "Tate twist" $\mathbb{Q}(1) := 2\pi i\mathbb{Q}$ a Hodge structure of weight -2 where the only nonzero entry is $\mathbb{Q}(1)^{-1,-1} = \mathbb{Q}(1)$.

Example 1.3. Given a complex projective smooth variety X, the Betti cohomology $\mathrm{H}^n_{\mathrm{B}}(X,\mathbb{Q})$ admits a Hodge structure via the comparison isomorphisms: we find that

$$\mathrm{H}^n_{\mathrm{B}}(X,\mathbb{C}) \simeq \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X),$$

where $\mathrm{H}^{p,q}(X) := \mathrm{H}^q(X,\Omega^p_{X/\mathbb{C}})$. This construction is even functorial: a morphism of complex projective smooth varieties $\varphi\colon X \to Y$ induces a morphism of Hodge structures $\varphi^*\colon \mathrm{H}^n_{\mathrm{B}}(Y,\mathbb{Q}) \to \mathrm{H}^n_{\mathrm{B}}(X,\mathbb{Q})$.

Perhaps one would like to check that the category $\mathrm{HS}_{\mathbb{Q}}$ is abelian. The quickest way to do this is to realize $\mathrm{HS}_{\mathbb{Q}}$ as a category of representations of some group. The relevant group is the Deligne torus.

Notation 1.4 (Deligne torus)**.** Let $\mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ denote the Deligne torus. We also let $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ denote the *weight cocharacter* given by $w(r) := r \in \mathbb{C}$ on $\mathbb{R}$-points.

Remark 1.5. One can realize $\mathbb{S}$ more concretely as

$$\mathbb{S}(R) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathrm{GL}_2(R) : a^2+b^2 \in R^\times \right\},$$

where R is an $\mathbb{R}$-algebra. Indeed, there is a ring isomorphism from $R \otimes_{\mathbb{R}} \mathbb{C}$ to $\left\{ \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] : a,b \in R \right\}$ by sending $1 \otimes 1 \mapsto \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right]$ and $1 \otimes i \mapsto \left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$. For example, one can define two characters $z,\overline{z}\colon \mathbb{S}_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ given by $z\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a+bi$ and $\overline{z}\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a-bi$ so that $(z,\overline{z})$ is an isomorphism $\mathbb{S}_{\mathbb{C}} \to \mathbb{G}^2_{m,\mathbb{C}}$. Thus, the character group $X^*(\mathbb{S})$ is a free $\mathbb{Z}$-module of rank 2 with basis $\{z,\overline{z}\}$, and the action of complex conjugation $\iota \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ simply swaps z and $\overline{z}$.

Example 1.6. The following cocharacters of $\mathbb{S}$ will be helpful.

- We define the *weight cocharacter* $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ given by $w(r) := r \in \mathbb{C}$ on $\mathbb{R}$-points.
- We define $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ given by $\mu(z) := (z,1)$ on $\mathbb{C}$-points.

Here is the relevance of $\mathbb{S}$ to Hodge structures.

Lemma 1.7. Fix some $V \in \mathrm{Vec}_{\mathbb{Q}}$. Then a Hodge structure on V has equivalent data to a representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$.

Proof. Remark 1.5 informs us that the character group $X^*(\mathbb{S})$ of group homomorphisms $\mathbb{S} \to \mathbb{G}_m$ is a rank-2 free $\mathbb{Z}$-module generated by $z\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a+bi$ and $\overline{z}\colon \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \mapsto a-bi$ on $\mathbb{C}$-points.[1] Without too many details, upon passing to the Hopf algebra, one is essentially looking for units in $\mathbb{R}\left[a,b,\left(a^2+b^2\right)^{-1}\right]$, of which there are not many. Note that there is a Galois action by $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on these two characters $\{z,\overline{z}\}$, given by swapping them. Let $\iota \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ denote complex conjugation, for brevity.

Now, a representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ must have $V_{\mathbb{C}}$ decompose into eigenspaces according to the characters $X^*(\mathbb{S})$, so one admits a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\chi \in X^*(\mathbb{S})} V_{\mathbb{C}}^{\chi}.$$

However, one also needs $V_{\mathbb{C}}^{\iota\chi} = \overline{V_{\mathbb{C}}^{\chi}}$ because ι swaps $\{\chi,\iota\chi\}$. By Galois descent, this is enough data to (conversely) define a representation $h\colon \mathbb{S} \to \operatorname{Gal}(V)_{\mathbb{R}}$.

[1] Alternatively, note one has an isomorphism $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \cong \mathbb{C}^\times \times \mathbb{C}^\times$ by sending $(z,w) \mapsto z \otimes w$. Then these two characters are $(z,w) \mapsto z$ and $(z,w) \mapsto w$.

To relate the previous paragraph to Hodge structures, we recall that $X^*(\mathbb{S})$ is a rank-2 free $\mathbb{Z}$-module, so write $\chi_{p,q} := z^{-p}\overline{z}^{-q}$ so that $\iota\chi_{p,q} = \chi_{q,p}$. Setting $V_{\mathbb{C}}^{p,q} := V_{\mathbb{C}}^{\chi_{p,q}}$ now explains how to relate the previous paragraph to a Hodge structure, as desired. ■

Remark 1.8. The weight of a Hodge structure on some $V \in \mathrm{HS}_{\mathbb{Q}}$ can be read off of h as follows: note the weight cocharacter $h \circ w$ equals the $(-m)$th power map if and only if the weight is m.

Thus, we see immediately the category $\mathrm{HS}_{\mathbb{Q}}$ is abelian. Additionally, representation theory explains how to take tensor products and duals.

Example 1.9. We see that $V \in \mathrm{HS}_{\mathbb{Q}}$ has $V^\vee$ inherit a Hodge structure by setting $(V^\vee)^{p,q} := (V^{-p,-q})^\vee$.

Example 1.10. We are now able to define the Tate twists $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$, where negative powers indicates taking a dual. In particular, one can check that $\mathbb{Q}(n) \otimes \mathbb{Q}(m) = \mathbb{Q}(n+m)$ for any $n, m \in \mathbb{Z}$.

Notation 1.11. For any Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ and integer $m \in \mathbb{Z}$, we may write

$$V(m) := V \otimes \mathbb{Q}(m).$$

We conclude this section by explaining one important application of Hodge structures.

Definition 1.12 (Hodge class)**.** Fix a $\mathbb{Q}$-Hodge structure V. A *Hodge class* of V is an element of $V \cap V^{0,0}$.

Remark 1.13. Looking at the construction in the proof of Lemma 1.7, we see that $v \in V$ is a Hodge class if and only if it is fixed by the corresponding representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$.

Example 1.14. Fix a complex projective smooth variety X of dimension n and some even nonnegative integer $2p \geq 0$. Then one has Hodge classes given by elements of

$$\mathrm{H}^{2p}_{\mathrm{B}}(X, \mathbb{Q}) \cap \mathrm{H}^{p,p}(X)(p).$$

Now, any algebraic subvariety $Z \subseteq X$ of codimension k defines a linear functional on $\mathrm{H}^{2n-2k}_{\mathrm{dR}}(X, \mathbb{C})$ defined by

$$\omega \mapsto \int_Z \omega,$$

which one can check is supported on $\mathrm{H}^{k,k}$. Thus, by Poincaré duality, one finds that Z produces a Hodge cycle in $\mathrm{H}^{2k}_{\mathrm{B}}(X, \mathbb{Q})$.

In light of the above example, one has the following conjecture.

Conjecture 1.15 (Hodge)**.** Fix a complex projective smooth variety X. Then any Hodge class can be written as a linear combination of classes arising from algebraic subvarieties.

Remark 1.16. Here are some remarks on what is known about the Hodge conjecture, though it is admittedly little in this level of generality.

- The Hodge classes in $\mathrm{H}^2_{\mathrm{B}}(X)(1)$ come from algebraic subvarieties.
- The cup product of any two classes arising from algebraic subvarieties continues to be Hodge and arises from algebraic subvarieties.

For example, if one can show that all Hodge classes are cup products of Hodge classes of codimension 1 on a variety X, then one knows the Hodge conjecture for X.

We are not interested in proving (cases of) the Hodge conjecture in this thesis, so we will not say much more.

1.1.2 Polarizations

Here is an important example of a morphism of Hodge structures.

Definition 1.17 (polarization)**.** Fix a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m given by the representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. A *polarization* on V is a morphism $\varphi\colon V \otimes V \to \mathbb{Q}(-m)$ of Hodge structures such that the induced bilinear form on $V_{\mathbb{R}}$ given by

$$\langle v, w\rangle := (2\pi i)^m \varphi(h(i)v \otimes w)$$

is symmetric and positive-definite. If V admits a polarization, we may say that V is *polarizable*, and we let $\mathrm{HS}^{\mathrm{pol}}_{\mathbb{Q}} \subseteq \mathrm{HS}_{\mathbb{Q}}$ be the full subcategory of polarizable $\mathbb{Q}$-Hodge structures.

Remark 1.18. The positive-definiteness condition on $\langle \cdot, \cdot \rangle$ implies that φ is non-degenerate. Indeed, one may check non-degeneracy upon base-changing to $\mathbb{R}$ (because this is equivalent to inducing an isomorphism of vector spaces $V \to V^\vee$, which can be checked by fixing some $\mathbb{Q}$-bases and computing a determinant). Then we see that $\langle \cdot, \cdot \rangle$ being non-degenerate implies that

$$\varphi(v \otimes w) = (2\pi i)^{-m} \langle h(-i)v, w\rangle$$

is non-degenerate because $h(-i)\colon V \to V$ is an isomorphism of vector spaces (because $h(-i)^4 = \mathrm{id}_V$).

Remark 1.19. The symmetry condition on $\langle \cdot, \cdot \rangle$ implies a symmetry or alternating condition on φ. Indeed, we compute

$$\begin{aligned}
\varphi(v \otimes w) &= (2\pi i)^{-m} \langle h(-i)v, w\rangle \\
&= (2\pi i)^{-m} \langle w, h(-i)v\rangle \\
&= \varphi(h(i)w \otimes h(-i)v) \\
&= h_{\mathbb{Q}(-m)}(i)\varphi(w \otimes h(-1)v) \\
&= 1\varphi\left(w \otimes (-1)^m w\right) \\
&= (-1)^m \varphi(w \otimes v).
\end{aligned}$$

Thus, φ is symmetric when m is even, and φ is alternating when m is odd.

Let's give some constructions of polarizable Hodge structures.

Example 1.20. It will turn out that $\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ of any abelian variety A (over $\mathbb{C}$) is polarizable, explaining the importance of this notion for our application. Because we are reviewing abelian varieties in chapter 3, we will not say more here.

Example 1.21. If V is polarizable and pure of weight m, then any Hodge substructure $W \subseteq V$ is still polarizable (and pure of weight m). Indeed, one can simply restrict the polarization to W, and all the checks go through. For example, positive-definiteness of $\langle\cdot,\cdot\rangle$ means $\langle v, v\rangle > 0$ for all nonzero $v \in V$, so the same will be true upon restricting to W.

Example 1.22. If V and W are polarizable and pure of weight m, then $V \oplus W$ is also polarizable. Indeed, letting φ and ψ be polarizations on V and W respectively, we see that $(\varphi \oplus \psi)$ defined by

$$(\varphi \oplus \psi)((v, w), (v', w')) \coloneqq \varphi(v, v') + \psi(w, w')$$

succeeds at being a polarization: certainly it is a morphism of Hodge structures to $\mathbb{Q}(-m-n)$, and one can check that the corresponding bilinear form on $V \oplus W$ simply splits into a sum of the forms on V and W and is therefore symmetric and positive-definite.

Example 1.23. If V and W are polarizable and pure of weights m and n respectively, then $V \otimes W$ is also polarizable. Indeed, as in Example 1.22, let φ and ψ be polarizationson V and W respectively, and then we find that $(\varphi \otimes \psi)$ can be defined on pure tensors by

$$(\varphi \otimes \psi)(v \otimes w, v' \otimes w') \coloneqq \varphi(v, v')\psi(w, w').$$

One checks as before that this gives a polarization on $V \otimes W$: we certainly have a morphism of Hodge structures, and the corresponding bilinear form is the product of the bilinear forms on V and W and is therefore symmetric and positive-definite.

Example 1.24. If V is polarizable and pure of weight m with polarization φ, and $W \subseteq V$ is a Hodge substructure (which is polarizable by Example 1.21), then we claim $W^\perp$ (taken with respect to $\langle\cdot,\cdot\rangle$) is also a Hodge substructure and hence polarizable by Example 1.21. Well, for any $w' \in W_{\mathbb{R}}^\perp$ and $z \in \mathbb{S}(\mathbb{R})$, we must check that $h(z)w' \in W_{\mathbb{R}}^\perp$. For this, we note that any $w \in W$ has

$$\begin{aligned}
\langle w, h(z)w'\rangle &= (2\pi i)^{-m}\varphi(h(i)w \otimes h(z)w') \\
&= h_{\mathbb{Q}(-m)}(1/z)(2\pi i)^{-m}\varphi(h(i/z)w \otimes w') \\
&= h_{\mathbb{Q}(-m)}(1/z)\langle h(i/z)w, w'\rangle \\
&= 0,
\end{aligned}$$

where the last equality holds because $W \subseteq V$ is a Hodge substructure.

Note that one does not expect any Hodge substructure to have a complement, so Example 1.24 is a very important property of polarizations.

1.1.3 The Albert Classification

The presence of a polarization places strong restrictions on the endomorphisms of a Hodge structure. To explain how this works, we begin by reducing to the irreducible case: given a polarizable Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$, we begin by noting that V can be decomposed into irreducible Hodge substructures

$$V = \bigoplus_{i=1}^{N} V_i^{\oplus m_i},$$

where V_i is an irreducible Hodge structure (i.e., an irreducible representation of $\mathbb{S}$) and $m_i \geq 0$ is some nonnegative integer. Then standard results on endomorphisms of representations tell us that

$$\operatorname{End}_{\mathrm{HS}}(V) = \bigoplus_{i=1}^{N} M_{m_i}(\operatorname{End}_{\mathrm{HS}}(V_i)),$$

and Schur's lemma implies that $\operatorname{End}_{\mathrm{HS}}(V_i)$ is a division algebra. The point of the above discussion is that we may reduce our discussion of endomorphisms to irreducible Hodge structures. We remark that polarizability of V implies that irreducible Hodge substructures continue to be polarizable by Example 1.21.

We are thus interested in classifying what algebras may appear as $\operatorname{End}_{\mathrm{HS}}(V)$ for irreducible Hodge structures $V \in \mathrm{HS}_{\mathbb{Q}}$. To this end, we note that $\operatorname{End}_{\mathrm{HS}}(V)$ comes with some extra structure.

Definition 1.25 (Rosati involution)**.** Let φ be a polarization on a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$. The *Rosati involution* is the function $(\cdot)^\dagger\colon \operatorname{End}_{\mathbb{Q}}(V) \to \operatorname{End}_{\mathbb{Q}}(V)$ defined by

$$\varphi(dv \otimes w) = \varphi(v \otimes d^\dagger w)$$

for all $d \in \operatorname{End}_{\mathrm{HS}}(V)$ and $v, w \in V$.

Remark 1.26. In light of Remark 1.18, we see that $d^\dagger$ is simply the adjoint of $d\colon V \to V$ associated to φ viewed as a non-degenerate bilinear pairing. For example, we immediately see that $(\cdot)^\dagger$ induces a well-defined linear operator $\operatorname{End}_{\mathbb{Q}}(V) \to \operatorname{End}_{\mathbb{Q}}(V)$.

Here are the important properties of the Rosati involution.

Lemma 1.27. Fix a Hodge structure $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with polarization φ and associated Rosati involution $(\cdot)^\dagger$.

(a) If $d \in \operatorname{End}_{\mathrm{HS}}(V)$, then $d^\dagger \in \operatorname{End}_{\mathrm{HS}}(V)$.

(b) Anti-involution: for any $d, e \in \operatorname{End}_{\mathbb{Q}}(V)$, we have $d^{\dagger\dagger} = d$ and $(de)^\dagger = e^\dagger d^\dagger$.

(c) Positive: for any nonzero $d \in \operatorname{End}_{\mathbb{Q}}(V)$, we have $\operatorname{tr} dd^\dagger > 0$.

Proof. We show the claims in sequence.

(a) This follows because φ is a morphism of Hodge structures. Formally, we would like to check that $d^\dagger$ commutes with the action of $\mathbb{S}$. Let $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be the representation corresponding to the Hodge structure. Well, for any $g \in \mathbb{S}(\mathbb{C})$ and $v, w \in V$, we compute

$$\begin{aligned}
\varphi(v \otimes d^\dagger h(g) w) &= \varphi(dv \otimes h(g) w) \\
&= h_{\mathbb{Q}(-m)}(g)\varphi\left(h(g^{-1}) dv \otimes w\right) \\
&\overset{*}{=} h_{\mathbb{Q}(-m)}(g)\varphi\left(d h(g^{-1}) v \otimes w\right) \\
&= h_{\mathbb{Q}(-m)}(g)\varphi\left(h(g^{-1}) v \otimes d^\dagger w\right) \\
&= \varphi(v \otimes h(g) d^\dagger w)
\end{aligned}$$

where $\overset{*}{=}$ holds because d is a morphism of Hodge structures. The non-degeneracy of φ given in Remark 1.18 now implies that $d^\dagger h(g) = h(g) d^\dagger$, so we are done.

(b) This is a purely formal property of adjoints.

(c) The point is to reduce this to the case where V is a matrix algebra over $\mathbb{R}$ and $(\cdot)^\dagger$ is the transpose. Indeed, this positivity can be checked after a base-change to $\mathbb{R}$. As such, we let $\langle\cdot,\cdot\rangle$ be the symmetric positive-definite bilinear form assocated to φ defined by

$$\langle v, w\rangle := (2\pi i)^{-m}\varphi(h(i)v \otimes w)$$

for any $v, w \in V_\mathbb{R}$. We thus see that $(\cdot)^\dagger$ is also the adjoint operator with respect to $\langle\cdot,\cdot\rangle$: we know

$$(2\pi i)^{-m}\langle h(i)dv, w\rangle = (2\pi i)^{-m}\langle h(i)v, d^\dagger w\rangle$$

for any $v, w \in V_\mathbb{R}$, which is equivalent to always having $\langle dv, w\rangle = \langle v, d^\dagger w\rangle$. Now, we may fix an orthornomal basis of $V_\mathbb{R}$ with respect to $\langle\cdot,\cdot\rangle$ so that $\operatorname{End}_\mathbb{R}(V_\mathbb{R})$ is identified with $M_n(\mathbb{R}^{\dim V})$ and $(\cdot)^\dagger$ is identified with the transpose. Then $\operatorname{tr} dd^\intercal$ is the sum of the squares of the matrix entries of d and is therefore positive when d is nonzero. ■

We are now ready to state the Albert classification, which classifies division algebras over $\mathbb{Q}$ equipped with a positive anti-involution.

Theorem 1.28 (Albert classification)**.** Let D be a division algebra over $\mathbb{Q}$ equipped with a Rosati involution $(\cdot)^\dagger\colon D \to D$. Further, let F be the center of D, and let $F^\dagger$ be the subfield fixed by $(\cdot)^\dagger$. Then D admits exactly one of the following types.

- Type I: D is a totally real number field so that $D = F = F^\dagger$, and $(\cdot)^\dagger$ is the identity.
- Type II: D is a totally indefinite quaternion division algebra over F where $F = F^\dagger$, and $(\cdot)^\dagger$ corresponds to the transpose on $D \otimes_\mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R})$.
- Type III: D is a totally definite quaternion division algebra over F where $F = F^\dagger$, and $(\cdot)^\dagger$ corresponds to the canonical involution on $D \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{H}$ (where $\mathbb{H}$ is the quaternions).
- Type IV: D is a division algebra over F, where F is a totally imaginary quadratic extension of $F^\dagger$, and $(\cdot)^\dagger$ is the complex conjugation automorphism of F. In other words, F is a CM field, and $F^\dagger$ is the maximal totally real subfield.

Proof. This is a rather lengthy computaion. We refer to [Mum74, Section 21, Application I]. ■

1.2 Monodromy Groups

In this section, we define the Mumford–Tate group and the Hodge group.

1.2.1 The Mumford–Tate Group

We are now ready to define the Mumford–Tate group. Intuitively, it is the monodromy group of the associated representation of a Hodge structure.

Definition 1.29 (Mumford–Tate group)**.** For some $V \in \mathrm{HS}_\mathbb{Q}$, the *Mumford–Tate group* $\mathrm{MT}(V)$ is the smallest algebraic $\mathbb{Q}$-group containing the image of the corresponding representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$.

Remark 1.30. Because $\mathbb{S}$ is connected, we see that h is also connected. Namely, $\mathrm{MT}(V)^\circ \subseteq \mathrm{MT}(V)$ will be an algebraic $\mathbb{Q}$-group containing the image of h if $\mathrm{MT}(V)$ does too, so equality is forced.

Example 1.31. Suppose that $V \in \mathrm{HS}_{\mathbb{Q}}$ is pure of weight m.

- If $m = 0$, then we claim that $\mathrm{MT}(V) \subseteq \mathrm{SL}(V)$. It is enough to check that h outputs into $\mathrm{SL}(V)$.
- If $m \neq 0$, then we claim that $\mathrm{MT}(V)$ contains $\mathbb{G}_{m,\mathbb{Q}}$. It is enough to check that $\mathrm{MT}(V)_{\mathbb{C}}$ contains $\mathbb{G}_{m,\mathbb{C}}$. Well, for any $z \in \mathbb{C}$ $h(z, \overline{z})$ acts on the component $V^{p,q} \subseteq V_{\mathbb{C}}$ by $z^{-p}z^{-q} = z^{-m}$, so $\mathrm{MT}(V)_{\mathbb{C}}$ must contain the scalar z^{-m} for all $z \in \mathbb{C}$. The conclusion follows.

Because Hodge structures are defined after passing to $\mathbb{C}$, it will be helpful to have a definition of $\mathrm{MT}(V)$ as a monodromy group corresponding to a morphism over $\mathbb{C}$.

Lemma 1.32. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$, and let $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ be the corresponding representation. Then $\mathrm{MT}(V)$ is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ such that $\mathrm{MT}(V)_{\mathbb{C}}$ contains the image of $h_{\mathbb{C}} \circ \mu$.

Proof. Let M' be the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ containing $h_{\mathbb{C}} \circ \mu$. We want to show that $M' = M$.

- To show $M' \subseteq \mathrm{MT}(V)$, we must show that $\mathrm{MT}(V)_{\mathbb{C}}$ contains the image of $h_{\mathbb{C}} \circ \mu$. Well, $\mathrm{MT}(V)_{\mathbb{R}}$ contains the image of h, so $\mathrm{MT}(V)_{\mathbb{C}}$ contains the image of $h_{\mathbb{C}}$, which contains the image of $h_{\mathbb{C}} \circ \mu$.
- Showing $\mathrm{MT}(V) \subseteq M'$ is a little harder. We must show that M' contains the image of $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. It is enough to check that M' contains the image of $h_{\mathbb{C}}$ because then we can descend everything to $\mathbb{R}$, and because $\mathbb{C}$ is algebraically closed, we see that $\mathbb{C}$-points are certainly dense enough so that it is enough to chek that $M'(\mathbb{C})$ contains the image $h(\mathbb{S}(\mathbb{C}))$.

  The point is that M' is defined over $\mathbb{Q}$, so $M'_{\mathbb{C}}$ is stable under the action of complex conjugation, which we denote by ι. Similarly, h being defined over $\mathbb{R}$ guarantees that it commutes with complex conjugation. In particular, we already know that M' contains the points of the form $h(z, 1)$ for $(z, 1) \in \mathbb{S}(\mathbb{C})$. Thus, we see that M' also contains the points

  $$\iota(h(z, 1)) = h(\iota(z, 1)) = h(1, z)$$

  because everything is defined over $\mathbb{R}$. (This last equality follows by tracking through the action of ι on $\mathbb{S}(\mathbb{C})$.) We conclude that M' contains $h(z, w)$ for any $(z, w) \in \mathbb{S}(\mathbb{C})$, so we are done. ■

Roughly speaking, the point of the group $\mathrm{MT}(V)$ is that $\mathrm{MT}(V)$ is an algebraic $\mathbb{Q}$-group remembering everything one wants to know about the Hodge structure. One way to rigorize this is as follows.

Proposition 1.33. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$. Suppose $T \in \mathrm{HS}_{\mathbb{Q}}$ can be written as

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^{\vee})^{\otimes n_i} \right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $W \subseteq T$ is a Hodge substructure if and only if the action of $\mathrm{MT}(V)$ on T stabilizes W.

Proof. For each $W \in \mathrm{HS}_{\mathbb{Q}}$, we let h_W denote the corresponding representation. In the backwards direction, we note that $\mathrm{MT}(V)$ stabilizing W implies that $h(s)$ stabilizes $W_{\mathbb{R}}$ for any s. We can thus view $W_{\mathbb{R}} \subseteq T_{\mathbb{R}}$ as a subrepresentation of $\mathbb{S}$, so taking eigenspaces reveals that W can be given the structure of a Hodge substructure of T.

The converse will have to use the construction of T. Indeed, suppose that $W \subseteq T$ is a Hodge substructure, and let $M \subseteq \mathrm{GL}(V)$ be the smallest algebraic $\mathbb{Q}$-group stabilizing $W \subseteq T$. We would like to show that $\mathrm{MT}(V) \subseteq M$. By definition of $\mathrm{MT}(V)$, it is enough to show that h factors through $M_{\mathbb{R}}$, meaning we must show that $h(s)$ stabilizes W for each $s \in \mathbb{S}$. Well, $h(s)$ will act by characters on the eigenspaces $W_{\mathbb{C}}^{p,q} \subseteq W_{\mathbb{C}}$, so $h(s)$ does indeed stabilize W. ■

Corollary 1.34. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$. Suppose $T \in \mathrm{HS}_{\mathbb{Q}}$ can be written as

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^\vee)^{\otimes n_i}\right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $t \in T$ is a Hodge class if and only if it is fixed by $\mathrm{MT}(V)$.

Proof. We apply Proposition 1.33 to $\mathbb{Q}(0) \oplus T$. Then we note that $\operatorname{span}_{\mathbb{Q}}\{(1,t)\} \subseteq \mathbb{Q}(0) \oplus T$ is a Hodge substructure if and only if it is preserved by $\mathrm{MT}(V)$. We now tie each of these to the statement.

- On one hand, we see that being a one-dimensional Hodge substructure implies that $(1,t)$ must have bidegree (p,p) for some $p \in \mathbb{Z}$, but we have to live in $(0,0)$ because our 1 lives in $\mathbb{Q}(0)$. Thus, this is equivalent to being a Hodge class.

- On the other hand, being preserved by $\mathrm{MT}(V)$ implies that $\mathrm{MT}(V)$ acts by scalars on $(1,t)$, but $\mathrm{MT}(V)$ acts trivially on $\mathbb{Q}(0)$, so all the relevant scalars must be 1. Thus, this is equivalent to being fixed by $\mathrm{MT}(V)$. ■

We thus see that understanding the Mumford–Tate group is important from the perspective of the Hodge conjecture (Conjecture 1.15). It will be helpful to note that this characterizes $\mathrm{MT}(V)$ in some cases.

Proposition 1.35. Fix a field k of characteristic 0. Let $H \subseteq \mathrm{GL}_{n,k}$ be a reductive subgroup. Suppose H' is the algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}_{n,k}$ defined by fixing all H-invariants occuring in any tensor representation

$$T = \bigoplus_{i=1}^{N} \left(V^{\otimes m_i} \otimes (V^\vee)^{\otimes n_i}\right),$$

where $m_i, n_i \geq 0$ are nonnegative integers. Then $H = H'$.

Proof. Note $H \subseteq H'$ is automatic, so the main content comes from proving the other inclusion. Proving this would step into the (rather deep) theory of algebraic groups, which we will avoid. Instead, we will mention that the key input is Chevalley's theorem, which asserts that any subgroup H of G is the stabilizer of some line in some representation of G. We refer to [Del18, Proposition 3.1]; see also [Mil17, Theorem 4.27]. ■

Corollary 1.36. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ such that $\mathrm{MT}(V)$ is reductive. Then $\mathrm{MT}(V)$ is exactly the algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ fixing all Hodge classes.

Proof. Corollary 1.34 explains that the Hodge classes are exactly the vectors fixed by $\mathrm{MT}(V)$, so this follows from Proposition 1.35. ■

Remark 1.37. Corollary 1.36 is true without a reductivity assumption (see [Del18, Proposition 3.4]), but we will not need this in our applications. (On the other hand, one does not expect Proposition 1.35 to be true without any assumptions on H.) Namely, we will be interested in abelian varieties, whose Hodge structures are polarizable by Example 1.20, and we will shortly see that this implies that $\mathrm{MT}(V)$ is reductive in Lemma 1.44.

1.2.2 The Hodge Group

In computational applications, it will be frequently be easier to compute a smaller monodromy group related to $\operatorname{MT}(V)$.

Definition 1.38 (Hodge group)**.** Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight. Then the *Hodge group* $\operatorname{Hg}(V)$ is the smallest algebraic $\mathbb{Q}$-subgroup $\operatorname{GL}(V)$ containing the image of $h|_{\mathbb{U}}$, where $\mathbb{U} \subseteq \mathbb{S}$ is defined as the kernel of the norm character $z\overline{z} \colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$.

Remark 1.39. Even though z and $\overline{z}$ are only defined as characters $\mathbb{S}_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$, the norm character $z\overline{z}$ is defined as a character $\mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ because it is fixed by complex conjugation. For example, we see that

$$\mathbb{U}(\mathbb{R}) = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus, we see that $\mathbb{U}$ stands for "unit circle." While we're here, we remark that $\mathbb{U}(\mathbb{C}) \subseteq \mathbb{S}(\mathbb{C})$ is identified with the subset $\{(z, 1/z) : z \in \mathbb{C}^\times\}$.

Remark 1.40. The same argument as in Remark 1.30 shows that the connectivity of $\mathbb{U}$ implies the connectivity of $\operatorname{Hg}(V)$.

Intuitively, $\operatorname{Hg}(V)$ removes the scalars that might live in $\operatorname{MT}(V)$ by Example 1.31. These scalars are an obstruction to $\operatorname{MT}(V)$ being a semisimple group, and we will see in Proposition 1.77 that $\operatorname{Hg}(V)$ will thus frequently succeed at being semisimple. Let's rigorize this discusison.

Lemma 1.41. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m, and let $h \colon \mathbb{S} \to \operatorname{GL}(V)_{\mathbb{R}}$ be the corresponding representation.

(a) We have $\operatorname{Hg}(V) \subseteq \operatorname{SL}(V)$.

(b) Thus,

$$\operatorname{MT}(V) = \begin{cases} \operatorname{Hg}(V) & \text{if } m = 0, \\ \mathbb{G}_{m,\mathbb{Q}} \operatorname{Hg}(V) & \text{if } m \neq 0, \end{cases}$$

where the almost direct product in the second case is given by embedding $\mathbb{G}_{m,\mathbb{Q}} \to \operatorname{GL}(V)$ via scalars.

Proof. We show the claims in sequence.

(a) It is enough to check that $\operatorname{SL}(V)$ contains the image of $h|_{\mathbb{U}}$. In other words, we want to check that $\det h(z) = 1$ for all $z \in \mathbb{U}(\mathbb{R})$. By extending scalars, it is enough to compute the determinant as an operator on $V_{\mathbb{C}}$. For this, we note that $h(z)$ acts on the component $V^{p,q} \subseteq V_{\mathbb{C}}$ by the scalar $z^{-p}\overline{z}^{-q}$, so the determinant of $h(z)$ acting on $V^{p,q} \oplus V^{q,p}$ is

$$\left(z^{-p}\overline{z}^{-q}\right)^{\dim V^{p,q}} \cdot \left(z^{-q}\overline{z}^{-p}\right)^{\dim V^{q,p}} = (z\overline{z})^{-(p+q)\dim V^{p,q}}$$

because $\dim V^{p,q} = \dim V^{q,p}$. This simplifies to $(z\overline{z})^{-\frac{1}{2}m\dim(V^{p,q}\oplus V^{q,p})}$ because V is pure of weight m, so the result follows by summing over all pairs (p,q).[2]

(b) Before doing anything serious, we remark that $\mathbb{G}_{m,\mathbb{Q}} \operatorname{Hg}(V)$ is in fact an almost direct product. Namely, we should check that the intersection $\mathbb{G}_{m,\mathbb{Q}} \cap \operatorname{Hg}(V)$ is finite (even over $\mathbb{C}$). Well, by (a), $\operatorname{Hg}(V) \subseteq \operatorname{SL}(V)$. Thus, it is enough to notice that $\mathbb{G}_{m,\mathbb{Q}} \cap \operatorname{SL}(V)$ is finite because V is finite-dimensional over $\mathbb{C}$: over $\mathbb{C}$,

[2] If m is even, this argument does not work verbatim for the component $(m/2, m/2)$. Instead, one can compute the determinant of $h(z)$ acting on $V^{m/2,m/2}$ directly as $(z\overline{z})^{-\frac{1}{2}m\dim V^{m/2,m/2}}$.

the intersection consisits of scalar matrices $\lambda\,\mathrm{id}_V$ such that $\lambda^{\dim V}=1$, so the intersefction is the finite algebraic group $\mu_{\dim V}$.

We now proceed with the argument. Because $\mathbb{U}\subseteq\mathbb{S}$, we of course have $\mathrm{Hg}(V)\subseteq\mathrm{MT}(V)$, and if $m\neq 0$, then Example 1.31 implies that $\mathbb{G}_{m,\mathbb{Q}}\subseteq\mathrm{MT}(V)$ so that $\mathbb{G}_{m,\mathbb{Q}}\,\mathrm{Hg}(V)\subseteq\mathrm{MT}(V)$. It is therefore enough to check the given equalities after base-changing to $\mathbb{R}$. Namely, using Lemma 1.32, we should check that $\mathrm{Hg}(V)(\mathbb{C})$ contains the image of $h_{\mathbb{C}}\circ\mu$ when $m=0$, and $\mathbb{C}^\times\,\mathrm{Hg}(V)(\mathbb{C})$ contains the image of $h_{\mathbb{C}}\circ\mu$ when $m\neq 0$. Well, for any $z\in\mathbb{C}^\times$, we may write $z=re^{i\theta}$ where $r\in\mathbb{R}^+$ and $\theta\in\mathbb{R}$. Then we compute

$$\begin{aligned} h(\mu(z)) &= h(z,1) \\ &= h\left(re^{i\theta},1\right) \\ &= h\left(\sqrt{r}e^{i\theta/2},\sqrt{r}e^{-i\theta/2}\right)h\left(\sqrt{r}e^{i\theta/2},\frac{1}{\sqrt{r}e^{i\theta/2}}\right). \end{aligned}$$

Now, $h\left(\sqrt{r}e^{i\theta/2},\sqrt{r}e^{-i\theta/2}\right)$ is a scalar as computed in Example 1.31, and $\left(\sqrt{r}e^{i\theta/2},\frac{1}{\sqrt{r}e^{i\theta/2}}\right)$ lives in $\mathbb{U}(\mathbb{C})=\{(z,w):zw=1\}$. Thus, we see that $h(\mu(z))$ is certainly contained in $\mathbb{C}^\times\,\mathrm{Hg}(V)(\mathbb{C})$, completing the proof in the case $m\neq 0$. In the case where $m=0$, the scalar $h\left(\sqrt{r}e^{i\theta/2},\sqrt{r}e^{-i\theta/2}\right)$ is actually the identity, so we see that $h(\mu(z))\in\mathrm{Hg}(V)(\mathbb{C})$. ∎

It is worthwhile to note that there is also a tensor characterization of $\mathrm{Hg}(V)$.

Proposition 1.42. Fix $V\in\mathrm{HS}_{\mathbb{Q}}$ of pure weight. Suppose $T\in\mathrm{HS}_{\mathbb{Q}}$ is of pure weight n and can be written as

$$T=\bigoplus_{i=1}^{N}\left(V^{\otimes m_i}\otimes(V^\vee)^{\otimes n_i}\right),$$

where $m_i,n_i\geq 0$ are nonnegative integers. Then $W\subseteq T$ is a Hodge substructure if and only if the action of $\mathrm{Hg}(V)$ on T stabilizes W.

Proof. Of course, if $W\subseteq T$ is a Hodge substructure, then W is preserved by the action of $\mathrm{MT}(V)$, so W will be preserved by the action of $\mathrm{Hg}(V)\subseteq\mathrm{MT}(V)$.

Conversely, if $\mathrm{Hg}(V)$ stabilizes W, then we would like to show that $W\subseteq T$ is a Hodge substructure, which by Proposition 1.33 is the same as showing that $\mathrm{MT}(V)$ stabilizes W. For this, we use Lemma 1.41, which tells us that $\mathrm{MT}(V)\subseteq\mathbb{G}_{m,\mathbb{Q}}\,\mathrm{Hg}(V)$. Namely, because $\mathrm{Hg}(V)$ already stabilizes W, it is enough to note that of course the scalars $\mathbb{G}_{m,\mathbb{Q}}$ stabilize the subspace $W\subseteq T$. ∎

Corollary 1.43. Fix an irreducible Hodge structure $V\in\mathrm{HS}_{\mathbb{Q}}$ of pure weight. Observe that the inclusion $\mathrm{Hg}(V)\subseteq\mathrm{GL}(V)$ makes V into a representation of $\mathrm{Hg}(V)$. Then V is irreducible as a representation of $\mathrm{Hg}(V)$.

Proof. By Proposition 1.42, a $\mathrm{Hg}(V)$-submodule is a Hodge substructure, but there are no nonzero proper Hodge substructures because V is an irreducible Hodge structure. ∎

1.3 Computational Tools

In this section, we provide some discussion which will help the computations used later in this thesis.

1.3.1 Bounding with Known Classes

Here, we use endomorphisms and the polarization to bound the size of $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$.

Lemma 1.44. Fix a polarizable Hodge structure $V \in \mathrm{HS}_\mathbb{Q}$ of pure weight. Then $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$ are reductive.

Proof. By [Mil17, Corollary 19.18], it is enough to find faithful semisimple representations of $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$. We claim that the inclusions $\mathrm{MT}(V) \subseteq \mathrm{GL}(V)$ and $\mathrm{Hg}(V) \subseteq \mathrm{GL}(V)$ provide this representation: certainly this representation is faithful, and it is faithful because any subrepresentation is a Hodge substructure by Propositions 1.33 and 1.42. ■

Lemma 1.45. Fix $V \in \mathrm{HS}_\mathbb{Q}$. Let $D := \mathrm{End}_{\mathrm{HS}}(V)$ be the endomorphism algebra of V. Then $\mathrm{MT}(V)$ is an algebraic $\mathbb{Q}$-subgroup of

$$\mathrm{GL}_D(V) := \{g \in \mathrm{GL}(V) : g \circ d = d \circ g \text{ for all } d \in D\}.$$

Proof 1. Noting that $\mathrm{GL}_D(V)$ is an algebraic $\mathbb{Q}$-group (it is a subgroup of $\mathrm{GL}(V)$ cut out by the equations given by commuting with a basis of D), it is enough to show that $\mathrm{GL}_D(V)$ contains the image of the representation $h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$. Well, by definition D consists of morphisms commuting with the action of $\mathbb{S}$, so the image of h must commute with D. ■

Proof 2. Motivated by Corollary 1.36, one expects to find Hodge classes corresponding to the condition of commuting with D. Well, there is a canonical isomorphism $V \otimes V^\vee \to \mathrm{End}_\mathbb{Q}(V)$ of $\mathbb{S}$-representations, so by tracking through how representations of $\mathbb{S}$ correspond to Hodge structures, we see that $f\colon V \to V$ preserves the Hodge structure if and only if it is fixed by $\mathbb{S}$, which is equivalent to the corresponding element $f \in V \otimes V^\vee$ being fixed by $\mathbb{S}$, which is equivalent to f being a Hodge class by Remark 1.13. This completes the proof of the lemma upon comparing with Corollary 1.34. ■

Remark 1.46. Of course, we also have $\mathrm{Hg}(V) \subseteq \mathrm{GL}_D(V)$ because $\mathrm{Hg}(V) \subseteq \mathrm{MT}(V)$.

Lemma 1.47. Fix $V \in \mathrm{HS}_\mathbb{Q}$ pure of weight m with polarization φ. Then $\mathrm{MT}(V)$ is an algebraic $\mathbb{Q}$-subgroup of

$$\mathrm{GSp}(\varphi) := \{g \in \mathrm{GL}(V) : \varphi(gv \otimes gw) = \lambda(g)\varphi(v \otimes w) \text{ for fixed } \lambda(g) \in \mathbb{Q}\}.$$

Proof 1. Once again, we note that $\mathrm{GSp}(\varphi)$ is an algebraic $\mathbb{Q}$-group cut out by equations of the form

$$\varphi(gv \otimes gw)\varphi(v' \otimes w') = \varphi(v \otimes w)\varphi(gv' \otimes gw')$$

as $v, w, v', w' \in V$ varies over a basis. Thus, it is enough to check that $\mathrm{GSp}(\varphi)$ contains the image of $h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$. Well, for any $z \in \mathbb{S}(\mathbb{R})$, we note that

$$\varphi(h(z) \otimes h(z)w) = h_{\mathbb{Q}(-m)}(z)\varphi(v \otimes w)$$

for any $v, w \in V_\mathbb{R}$ because φ is a morphism of Hodge structures. ■

Proof 2. Once again, Corollary 1.36 tells us to expect the polarization to produce a Hodge class corresponding to the above equations cutting out $\mathrm{MT}(V)$.

This construction is slightly more involved. We begin by constructing two Hodge classes.

- Note $\varphi\colon V \otimes V \to \mathbb{Q}(-m)$ is a morphism of Hodge structrures, so it is an $\mathbb{S}$-invariant map and hence given by an $\mathbb{S}$-invariant element of $V^\vee \otimes V^\vee(-m)$. Thus, $\varphi \in V^\vee \otimes V^\vee(-m)$ is a Hodge class by Remark 1.13.

- Because φ is non-degenerate, it induces an isomorphism $V(m) \to V^\vee$. Now, $\operatorname{End}_{\mathbb{Q}}(V)$ is canonically isomorphic to $V \otimes V^\vee$, which we now see is isomorphic (via φ) to $V \otimes V(m)$. We let $C \in V \otimes V(m)$ be the image of $\mathrm{id}_V \in \operatorname{End}_{\mathbb{Q}}(V)^{\mathbb{S}}$ in $V \otimes V(m)$, which we note is a Hodge class again by Remark 1.13. (Here, C stands for "Casimir.")

In total, we see that we have produced a Hodge class $C \otimes \varphi$. It remains to show that $g \in \mathrm{GL}(V)$ fixing $C \otimes \varphi$ implies that $g \in \mathrm{GSp}(\varphi)$, which will complete the proof by Corollary 1.34.

Well, suppose $g(C \otimes \varphi) = C \otimes \varphi$. Note $g(C \otimes \varphi) = gC \otimes g\varphi$, which can only equal $C \otimes \varphi \in (V \otimes V) \otimes_{\mathbb{Q}} (V^\vee \otimes V^\vee)$ if there is a scalar $\lambda \in \mathbb{Q}^\times$ such that $gC = \lambda C$ and $g\varphi = \lambda^{-1}\varphi$. This second condition amounts to requiring

$$\varphi\left(g^{-1}v \otimes g^{-1}w\right) = \lambda^{-1}\varphi(v \otimes w)$$

for any $v, w \in V$, which rearranges into $g \in \mathrm{GSp}(\varphi)$. ■

Remark 1.48. The construction given in the above proof is described in [GGL24, Remark 8.3.4]. They also show the converse claim that any $g \in \mathrm{GSp}(\varphi)$ fixes $C \otimes \varphi$.

To see this, one has to do an explicit computation with C. For this, let $\{v_1, \ldots, v_n\}$ be a basis of V, and $\{v_1^*, \ldots, v_n^*\}$ be the dual basis of $V(m)$ taken with respect to φ. Then $C = \sum_{i=1}^n v_i \otimes v_i^*$. Similarly, we see that $\{gv_1, \ldots, gv_n\}$ is a basis of V with a dual basis $\{(gv_1)^*, \ldots, (gv_n)^*\}$ so that $C = \sum_{i=1}^n (gv_i) \otimes (gv_i)^*$. Now, on one hand, if g has multiplier λ, then $g\varphi = \lambda^{-1}\varphi$. On the other hand, $\varphi(gv_i, gv_j^*) = \lambda 1_{i=j}$, so $(gv_i)^* = \lambda^{-1} gv_i^*$, which allows us to compute $gC = \lambda C$. In total, $g(C \otimes \varphi) = C \otimes \varphi$.

Remark 1.49. One can check that $\mathrm{GSp}(\varphi)$ does not depend on the choice of polarization. Roughly speaking, the point is that the choice of a different polarization amounts to some choice of an element in $D^\times$ which we can track through.

In light of the above two lemmas, we pick up the following notation.

Notation 1.50. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with $D := \operatorname{End}_{\mathrm{HS}}(V)$ and polarization φ. Then we define

$$\mathrm{GSp}_D(\varphi) := \mathrm{GL}_D(V) \cap \mathrm{GSp}(\varphi).$$

By Lemmas 1.45 and 1.47, we see that $\mathrm{MT}(V) \subseteq \mathrm{GSp}_D(\varphi)$.

Remark 1.51. In "most cases," we expect that generic Hodge structures V should have the equality $\mathrm{MT}(V) = \mathrm{GL}_D(V)$, and if V admits a polarization φ, then we expect the equality $\mathrm{MT}(V) = \mathrm{GSp}_D(\varphi)$. To rigorize this intuition, one must discuss Shimura varieties, which we will avoid doing for now.

We can also apply Lemmas 1.45 and 1.47 to bound $\mathrm{Hg}(V)$.

Notation 1.52. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ pure of weight m with $D := \operatorname{End}_{\mathrm{HS}}(V)$ and polarization φ. Then we define

$$\mathrm{Sp}(\varphi) := \{g \in \mathrm{GL}(V) : \varphi(gv \otimes gw) = \varphi(v \otimes w)\},$$

and

$$\mathrm{Sp}_D(\varphi) := \mathrm{GL}_D(V) \cap \mathrm{Sp}(\varphi).$$

Remark 1.53. Let's explain why $\operatorname{Hg}(V) \subseteq \operatorname{Sp}_D(\varphi)$. By Lemma 1.45, we see that $\operatorname{Hg}(V) \subseteq \operatorname{MT}(V) \subseteq \operatorname{GL}_D(V)$, so it remains to check that $\operatorname{Hg}(V) \subseteq \operatorname{Sp}(\varphi)$. Proceeding as in Lemma 1.47, it is enough to check that the image of $h|_{\mathbb{U}}$ lives in $\operatorname{Sp}(\varphi)_{\mathbb{R}}$, for which we note that any $z \in \mathbb{U}(\mathbb{R})$ has

$$\varphi(h(z)v \otimes h(z)w) = h_{\mathbb{Q}(-m)}(z)\varphi(v \otimes w),$$

but $h_{\mathbb{Q}(-m)}(z) = |z|^{-2m} \operatorname{id}_{\mathbb{Q}(-m)}$ is the identity because $z \in \mathbb{U}(\mathbb{R})$.

Thus far, our tools have been upper-bounding $\operatorname{MT}(V)$ and $\operatorname{Hg}(V)$. Here is a tool which sometimes provides a lower bound.

Lemma 1.54. Fix $V \in \operatorname{HS}_{\mathbb{Q}}$ of pure weight, and let $D := \operatorname{End}_{\operatorname{HS}}(V)$ be the endomorphism algebra of V. Then

$$D = \operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{MT}(V)} = \operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{Hg}(V)}.$$

Proof. As discussed in the second proof of Lemma 1.45, the Hodge calsses of $\operatorname{End}_{\mathbb{Q}}(V) \cong V \otimes V^\vee$ are exactly the endomorphisms of the Hodge structure, so the first equality follows from Corollary 1.34.

The second equality is purely formal: note that the scalar subgroup $\mathbb{G}_{m,\mathbb{Q}} \subseteq \operatorname{GL}(V)$ acts trivially on $V \otimes V^\vee \cong \operatorname{End}_{\mathbb{Q}}(V)$. Thus, we use Lemma 1.41 to compute

$$\begin{aligned}
\operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{Hg}(V)} &= \operatorname{End}_{\mathbb{Q}}(V)^{\mathbb{G}_{m,\mathbb{Q}} \operatorname{Hg}(V)} \\
&= \operatorname{End}_{\mathbb{Q}}(V)^{\mathbb{G}_{m,\mathbb{Q}} \operatorname{MT}(V)} \\
&= \operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{MT}(V)},
\end{aligned}$$

as required. ■

Remark 1.55. To understand Lemma 1.54 as providing a lower bound, note that if $\operatorname{MT}(V)$ is "too small," then there will be many invariant elements in $\operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{MT}(V)}$, perhaps exceeding D. On the other hand, the upper bound $\operatorname{MT}(V) \subseteq \operatorname{GL}_D(V)$ corresponds to the inequality $D \subseteq \operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{MT}(V)}$.

1.3.2 Sums

For later use in computations, it will be helpful to have a few remarks on computing the Mumford–Tate and Hodge groups of a sum. Here the Hodge group really shines: given two Hodge structures $V_1, V_2 \in \operatorname{MT}(V)$ pure of nonzero weight, Lemma 1.41 tells us that $\operatorname{MT}(V_1)$ and $\operatorname{MT}(V_2)$ and $\operatorname{MT}(V_1 \oplus V_2)$ are all equal to some smaller group times scalars. It will turn out to be reasonable to hope that

$$\operatorname{Hg}(V_1 \oplus V_2) \stackrel{?}{=} \operatorname{Hg}(V_1) \times \operatorname{Hg}(V_2),$$

but then the introduction of scalars makes the hope $\operatorname{MT}(V_1 \oplus V_2) \stackrel{?}{=} \operatorname{MT}(V_1) \times \operatorname{MT}(V_2)$ unreasonable!

With this in mind, let's begin to study Hodge groups of sums of Hodge structures.

Lemma 1.56. Fix Hodge structures $V_1, \ldots, V_k \in \operatorname{Hg}_{\mathbb{Q}}$ pure of the same weight.

(a) The subgroup $\operatorname{Hg}(V_1 \oplus \cdots \oplus V_k) \subseteq \operatorname{GL}(V_1 \oplus \cdots \oplus V_k)$ is contained in $\operatorname{Hg}(V_1) \times \cdots \times \operatorname{Hg}(V_k) \subseteq \operatorname{GL}(V_1 \oplus \cdots \oplus V_k)$.

(b) For each i, the projection map $\operatorname{pr}_i \colon \operatorname{Hg}(V_1 \oplus \cdots \oplus V_k) \to \operatorname{Hg}(V_i)$ is surjective.

Proof. For each i, let h_i denote the representations of $\mathbb{S}$ corresponding to the Hodge structures V_i, and let $h := (h_1, \ldots, h_k)$ be the representation $\mathbb{S} \to \operatorname{GL}(V)$ where $V := V_1 \oplus \cdots \oplus V_k$. We show the claims in sequence.

(a) We must show that $\operatorname{Hg}(V_1) \times \cdots \times \operatorname{Hg}(V_k)$ contains the image of $h|_{\mathbb{U}}$. Well, for any $z \in \mathbb{U}(\mathbb{R})$ and index i, we see that $h_i(z) \in \operatorname{Hg}(V_i)$, so
$$h(z) = \operatorname{diag}(h_1(z), \ldots, h_k(z))$$
lives in $\operatorname{Hg}(V_1) \times \cdots \times \operatorname{Hg}(V_k)$, as required.

(b) Fix an index i. It is enough to show that smallest algebraic $\mathbb{Q}$-group containing the image of pr_i also contains the image of $h_i|_{\mathbb{U}}$. Well, by definition of h, we see that h_i is equal to the composite
$$\mathbb{S} \stackrel{h}{\to} \operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_k) \stackrel{\operatorname{pr}_i}{\to} \operatorname{GL}(V_i),$$
from which the claim follows. ■

Remark 1.57. All the claims in Lemma 1.56 are true if Hg is replaced by MT everywhere. One simply has to replace $\mathbb{U}$ with $\mathbb{S}$ in the proof.

Lemma 1.56 makes $\operatorname{Hg}(V_1 \oplus V_2) \stackrel{?}{=} \operatorname{Hg}(V_1) \times \operatorname{Hg}(V_2)$ appear to be a reasonable expectation. However, we note that we cannot in general expect this to be true: roughly speaking, there may be Hodge cycles on $V_1 \oplus V_2$ which are not seen on just V_1 or V_2. Here is a degenerate example.

Example 1.58. Fix a Hodge structure $V \in \operatorname{HS}_{\mathbb{Q}}$ of pure weight, and let $n \geq 1$ be a positive integer. Letting $h\colon \mathbb{S} \to \operatorname{GL}(V)_{\mathbb{R}}$ be the corresponding representation, we get another Hodge structure $h^n\colon \mathbb{S} \to \operatorname{GL}\left(V^{\oplus n}\right)$. We claim that the diagonal embedding of $\operatorname{Hg}(V)$ into $\operatorname{GL}(V)^n \subseteq \operatorname{GL}\left(V^{\oplus n}\right)$ induces an isomorphism
$$\operatorname{Hg}(V) \to \operatorname{Hg}\left(V^{\oplus n}\right).$$
On one hand, we note that $\operatorname{Hg}\left(V^{\oplus n}\right)$ lives inside the diagonal embedding of $\operatorname{Hg}(V)$: note $\operatorname{Hg}\left(V^{\oplus n}\right) \subseteq \operatorname{Hg}(V)^n$ by Lemma 1.56, and $\operatorname{Hg}\left(V^{\oplus n}\right)$ must live inside the diagonal embedding of $\operatorname{GL}(V) \subseteq \operatorname{GL}\left(V^{\oplus n}\right)$ becuase all components of $h^n\colon \mathbb{S} \to \operatorname{GL}\left(V^{\oplus n}\right)_{\mathbb{R}}$ are equal. On the other hand, the surjectivity of the projections $\operatorname{Hg}\left(V^{\oplus n}\right) \to \operatorname{Hg}(V)$ from Lemma 1.56 implies that $\operatorname{Hg}\left(V^{\oplus n}\right)$ must equal the diagonal embedding of $\operatorname{Hg}(V)$ (instead of merely being contained in it).

One can upgrade this example as follows.

Lemma 1.59. Fix Hodge structures $V_1, \ldots, V_k \in \operatorname{Hg}_{\mathbb{Q}}$ pure of the same weight, and let $m_1, \ldots, m_k \geq 1$ be positive integers. Then the diagonal embeddings $\Delta_i\colon \operatorname{GL}(V_i) \to \operatorname{GL}\left(V_i^{\oplus m_i}\right)$ induce an isomorphism
$$\operatorname{Hg}(V_1 \oplus \cdots \oplus V_k) \to \operatorname{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right).$$

Proof. We proceed in steps. The proof is a direct generalization of the one given in Example 1.58. For each i, let $h_i\colon \mathbb{S} \to \operatorname{GL}(V_i)_{\mathbb{R}}$ be the representation corresponding to the Hodge structure, and set $h := (h_1^{m_1}, \ldots, h_k^{m_k})$.

1. We claim that $\operatorname{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right)$ lives in the image of $(\Delta_1, \ldots, \Delta_k)$. Indeed, the image is some algebraic $\mathbb{Q}$-subgroup of $\operatorname{GL}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right)$, so we would like to check that this algebraic $\mathbb{Q}$-subgroup contains the image of $h|_{\mathbb{U}}$. Well, for any $z \in \mathbb{U}(\mathbb{R})$, we see that
$$h(z) = (\Delta_1(h_1(z)), \ldots, \Delta_k(h_k(z)))$$
lives in the image of $(\Delta_1, \ldots, \Delta_k)$.

2. For each i, let H_i be the projection of $\operatorname{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right)$ onto one of the V_i components as in Lemma 1.56; the choice of V_i component does not matter by the previous step. By Lemma 1.56, we see that $H_i = \operatorname{Hg}(V_i)$. However, the previous step now requires
$$\operatorname{Hg}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right) = \Delta_1(H_1) \times \cdots \times \Delta_k(H_k),$$
so we are done. ■

Remark 1.60. As usual, this statement continues to be true for MT replacing Hg. One can either see this by applying Lemma 1.41 or by redoing the proof with $\mathbb{S}$ replacing $\mathbb{U}$.

The point of the lemma is that we can reduce our computation of Hodge groups to Hodge structures which are the sum of pairwise non-isomoprhic irreducible Hodge strucutures. Let's make a few remarks about this situation for completeness. Let $V_1, \ldots, V_k$ be pairwise non-isomorphic irreducuble Hodge structures which are pure of the same weight, and set $V := V_1 \oplus \cdots \oplus V_k$. Here are some remarks on $\mathrm{Hg}(V_1 \times \cdots \times V_k)$, summarizing everything we have done so far.

- We know that $\mathrm{Hg}(V) \subseteq \mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k)$.
- We know that the projections of $\mathrm{Hg}(V)$ onto each factor $\mathrm{Hg}(V_i)$ are surjective.
- For each i, we may view V_i as a representation of $\mathrm{Hg}(V_i)$ via the inclusion $\mathrm{Hg}(V_i) \subseteq \mathrm{GL}(V_i)$. Then Corollary 1.43 tells us that V_i is an irreducible representation of $\mathrm{Hg}(V_i)$.
- One can also apply Lemma 1.54 to the full space V to see that

$$\begin{aligned}\mathrm{End}_{\mathrm{Hg}(V)}(V) &= \mathrm{End}_{\mathrm{HS}}(V) \\ &= \prod_{i=1}^{k} \mathrm{End}_{\mathrm{HS}}(V_i) \\ &= \prod_{i=1}^{k} \mathrm{End}_{\mathrm{Hg}(V_i)}(V_i).\end{aligned}$$

The following results take the above situation and provides some criteria to have

$$\mathrm{Hg}(V) \stackrel{?}{=} \mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_k).$$

Before stating the lemma, we remark that all groups in sight are connected by Remark 1.40, and we already have one inclusion by Lemma 1.56, so it suffices to pass to an algebraic closure and work with Lie algebras instead of the Lie groups. The following lemma is essentially due to Ribet [Rib76, pp. 790–791].

Lemma 1.61 (Ribet)**.** Work over an algebraically closed field of characteristic 0. Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)$. For each index i, let $\mathrm{pr}_i \colon (\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)) \to \mathfrak{gl}(V_i)$ be the ith projection, and set $\mathfrak{g}_i := \mathrm{pr}_i(\mathfrak{g})$. Suppose the following.

(i) Each $\mathfrak{g}_i$ is nonzero and simple.

(ii) For each pair (i, j) of distinct indices, the projection map $(\mathrm{pr}_i, \mathrm{pr}_j) \colon \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is surjective.

Then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$.

Proof. We proceed by induction on k. If $k \in \{0, 1\}$, then there is nothing to say. For the induction, we now assume that $k \geq 2$ and proceed in steps.

1. For our set-up, we let J be the kernel of $\mathrm{pr}_k \colon \mathfrak{g} \to \mathfrak{g}_n$. By definition, $J \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ takes the form $I \oplus 0$ for some subspace $I \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$. Formally, one may let I be the set of vectors v such that $(v, 0) \in J$ and argue for the equality $J = I \oplus 0$ because all vectors in J take the form $(v, 0)$.

   The main content of the proof goes into showing that I is actually an ideal. To set ourselves up to prove this claim, let $\mathfrak{n} \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ denote its normalizer. We would like to show that $\mathfrak{n} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$, for which we use the inductive hypothesis.

2. For each pair of distinct indices $i, j < k$, we claim that the projection $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{n} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is surjective. Well, choose $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$, and we need to find an element in $\mathfrak{n}$ with X_i and X_j at the correct coordinates.

   To begin, we note that (ii) yields some $(X_1, \ldots, X_k) \in \mathfrak{g}$ such that with the correct $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$ coordinates. We would like to show that $X := (X_1, \ldots, X_{k-1})$ lives in $\mathfrak{n}$, which will complete this step. Well, select any $Y := (Y_1, \ldots, Y_{k-1})$ in I, and we see $(Y, 0) \in J$, so
$$[(X, X_k), (Y, 0)] = ([X, Y], 0)$$
   lives in J too (recall J is an ideal), so we conclude $[X, Y] \in I$. We conclude that X normalizes I, so $X \in \mathfrak{n}$.

3. We take a moment to complete the proof that $I \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ is an ideal. It is enough to check that the normalizer $\mathfrak{n}$ of I in $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ equals all of $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$. For this, we use the inductive hypothesis. The previous step shows that $\mathfrak{g}_i = \mathrm{pr}_i(\mathfrak{n})$ for each i, and we know by (i) that each $\mathfrak{g}_i$ is already nonzero and simple. Lastly, the previous step actually checks condition (ii) for the inductive hypothesis, completing the proof that $\mathfrak{n} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$.

4. We claim $I = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$. Because $I \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$ is an ideal of a sum of simple algebras, we know that
$$I = \bigoplus_{i \in S} \mathfrak{g}_i$$
   for some subset $S \subseteq \{1, \ldots, k-1\}$ of indices. Thus, to achieve the equality $I \stackrel{?}{=} \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_{k-1}$, it is enough to check that each projection $\mathrm{pr}_i\colon I \to \mathfrak{g}_{k-1}$ is surjective. Unravelling the definition of I, it is enough to check that each $X_i \in \mathfrak{g}_i$ has some $(X_1, \ldots, X_k) \in \mathfrak{g}$ with the correct X_i coordinate and $X_k = 0$. This last claim follows from hypothesis (ii) of $\mathfrak{g}$!

5. We now finish the proof of the lemma. Certainly $\mathfrak{g} \subseteq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$, so it is enough to compute dimensions to prove the equality. By the short exact sequence
$$0 \to J \to \mathfrak{g} \to \mathfrak{g}_n \to 0,$$
   it is enough to show that $\dim J = \dim \mathfrak{g}_1 + \cdots + \dim \mathfrak{g}_{k-1}$. However, this follows from the previous step because $\dim J = \dim I$. ■

In practice, it is somewhat difficult to check (ii) of Lemma 1.61. Here is an automation.

Lemma 1.62 (Moonen–Zarhin)**.** Work over an algebraically closed field of characteristic 0. Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)$. For each index i, let $\mathrm{pr}_i\colon (\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k)) \to \mathfrak{gl}(V_i)$ be the ith projection, and set $\mathfrak{g}_i := \mathrm{pr}_i(\mathfrak{g})$. Suppose the following.

(i) Each $\mathfrak{g}_i$ is nonzero and simple.

(ii) Fix a simple Lie algebra $\mathfrak{l}$, and define $I(\mathfrak{l}) := \{i : \mathfrak{g}_i \cong \mathfrak{l}\}$. If $\#I(\mathfrak{l}) > 1$, we require the following to hold.

- All automorphisms of $\mathfrak{l}$ are inner.
- One can choose isomorphisms $\mathfrak{l} \to \mathfrak{g}_i$ for each $i \in I(\mathfrak{l})$ such that the representations $\mathfrak{l} \to \mathfrak{g}_i \to \mathfrak{gl}(V_i)$ are all isomorphic.
- The diagonal inclusion
$$\prod_{i \in I(\mathfrak{l})} \mathrm{End}_{\mathfrak{g}_i}(V_i) \to \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{i \in I(\mathfrak{l})} V_i\right)$$
  is surjective.

Then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$.

Proof. We will show that (ii) in the above lemma implies (ii) of Lemma 1.61, which will complete the proof. We will proceed by contraposition in the following way. Fix a pair (i,j) of distinct indices, and we are interested in the map $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$. Supposing that $(\mathrm{pr}_i, \mathrm{pr}_j)$ fails to be surjective (which is a violation of (ii) of Lemma 1.61), we will show that (ii) cannot be true. In particular, we will assume the first two points of (ii) and show then that the third point of (ii) is false.

Roughly speaking, we are going to use the first two points of (ii) to find an $\mathfrak{h}$ and then produce an endomorphism of $\bigoplus_{i\in I(\mathfrak{h})} V_i$ which does not come from gluing together endomorphisms of the V_is. Having stated the outline, we proceed with the proof in steps.

1. We claim that the image $\mathfrak{h}$ of the map $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is the graph of an isomorphism $\mathfrak{g}_i \to \mathfrak{g}_j$. For this, we use the hypothesis that $(\mathrm{pr}_i, \mathrm{pr}_j)$ fails to be surjective. Well, we claim that the projections $\mathfrak{h} \to \mathfrak{g}_i$ and $\mathfrak{h} \to \mathfrak{g}_j$ are isomorphisms, which implies that $\mathfrak{h}$ is the graph of the composite isomorphism
$$\mathfrak{g}_i \leftarrow \mathfrak{h} \to \mathfrak{g}_j.$$
By symmetry, it is enough to merely check that $\mathfrak{h} \to \mathfrak{g}_i$ is an isomorphism. On one hand, $\mathfrak{h} \to \mathfrak{g}_i$ is surjective because $\mathrm{pr}_i\colon \mathfrak{g} \to \mathfrak{g}_i$ is surjective by construction of $\mathfrak{g}_i$. On the other hand, the kernel of the projection $\mathfrak{h} \to \mathfrak{g}_i$ will be an ideal of $\mathfrak{h}$ of the form $0 \oplus I$ where $I \subseteq \mathfrak{g}_j$ is some subspace. In fact, becasue the projection $\mathfrak{h} \to \mathfrak{g}_j$ is also surjective, we see that $I \subseteq \mathfrak{g}_j$ must be an ideal, so the simplicity of $\mathfrak{g}_j$ grants two cases.
   - If $I = 0$, then $\mathrm{pr}_i\colon \mathfrak{h} \to \mathfrak{g}_i$ becomes injective and is thus an isomorphism, completing this step.
   - If $I = \mathfrak{g}_j$, then $\mathfrak{h}$ fits into a short exact sequence
   $$0 \to (0 \oplus \mathfrak{g}_j) \to \mathfrak{h} \to \mathfrak{g}_i \to 0,$$
   so $\dim \mathfrak{h} = \dim(\mathfrak{g}_i \oplus \mathfrak{g}_j)$, implying the inclusion $\mathfrak{h} \subseteq \mathfrak{g}_i \oplus \mathfrak{g}_j$ is an equality. However, this cannot be the case because we assumed that $(\mathrm{pr}_i, \mathrm{pr}_j)\colon \mathfrak{g} \to \mathfrak{g}_i \to \mathfrak{g}_j$ fails to be surjective!
2. We construct an isomorphism of $\mathfrak{g}$-representations $V_i \to V_j$. For this, we use the first two points of (ii). Let's begin by collecting some data.
   - The previous step informs us that $\mathfrak{g}_i \cong \mathfrak{g}_j$. In fact, because this isomorphism is witnessed by the projections $\mathrm{pr}_i\colon \mathfrak{g} \to \mathfrak{g}_i$ and $\mathrm{pr}_j\colon \mathfrak{g} \to \mathfrak{g}_j$, we see that we are granted an isomorphism $f\colon \mathfrak{g}_i \to \mathfrak{g}_j$ such that $\mathrm{pr}_j = f \circ \mathrm{pr}_i$.
   - We now let $\mathfrak{l}$ be a simple Lie algebra isomorphic to both(!) $\mathfrak{g}_i$ and $\mathfrak{g}_j$. The second point of (ii) grants isomorphisms $f_i\colon \mathfrak{l} \to \mathfrak{g}_i$ and $f_j\colon \mathfrak{l} \to \mathfrak{g}_j$ of Lie algebras and an isomorphism $d\colon V_i \to V_j$ of $\mathfrak{l}$-representations.

   We now construct our isomorphism from d. Because d is only an isomorphism of $\mathfrak{l}$-representations, we are only granted that $(X_1, \ldots, X_k) \in \mathfrak{g}$ satisfies $f(X_i) = X_j$ and hence
   $$\begin{aligned} d\left((f_i f_j^{-1} f)(X_i) v_i\right) &= d\left(f_i\left(f_j^{-1} f(X_i)\right) v_i\right) \\ &= f_j\left(f_j^{-1} f(X_i)\right) d(v_i) \\ &= X_j d(v_i) \end{aligned}$$
   for all $v_i \in V_i$. We would be done if we could remove the pesky automorphism $f_i f_j^{-1} f\colon \mathfrak{g}_i \to \mathfrak{g}_i$. This is possible because all automorphisms of $\mathfrak{g}_i \cong \mathfrak{l}$ are inner (!), so one may simply "change bases" to remove the inner automorphism. Explicitly, find $a \in \mathrm{GL}(V_i)$ such that $f_i f_j^{-1} f(X) = aXa^{-1}$ for all $X \in \mathfrak{g}_i$, and then we define $e := d \circ a$. Then we find that any $v_i \in V_i$ has
   $$\begin{aligned} e(X_i v_i) &= d\left(aX_i a^{-1} \cdot av\right) \\ &= d\left((f_i f_j^{-1} f)(X_i) \cdot av\right) \\ &= X_j d(av) \\ &= X_j e(v). \end{aligned}$$

3. We complete the proof. The previous step provides a morphism $e\colon V_i \to V_j$ of $\mathfrak{g}$-representations. We thus note that the composite
$$\bigoplus_{i'\in I(\mathfrak{l})} V_{i'} \twoheadrightarrow V_i \xrightarrow{e} V_j \hookrightarrow \bigoplus_{i'\in I(\mathfrak{l})} V_{i'}$$
is an endomorphism which does not come from the diagonal inclusion of $\prod_{i\in I(\mathfrak{l})} \operatorname{End}_{\mathfrak{g}_i}(V_i)$. This completes the proof by showing that the third point of (ii) fails to hold. ■

Remark 1.63. We should remark on some history. Lemma 1.61 is due to Ribet [Rib76, pp. 790–791], but the given formulation is due to Moonen and Zarhin [MZ95, Lemma 2.14]. In the same lemma, Moonen and Zarhin prove Lemma 1.62, and they seem to be the first to recognize the utility of this lemma for computing Hodge groups. For example, Lombardo includes this result in his master's thesis [Lom13, Lemma 3.3.1] and includes a generalized version in another paper as [Lom16, Lemma 3.7], where it is used to compute Hodge groups of certain products of abelian varieties.

Remark 1.64. Let's explain how Lemma 1.62 is typically applied, which is admittedly somewhat different from the application used in this thesis. In the generic case, one expects (i), for example if $\operatorname{Hg}(V) = \operatorname{Sp}_D(\varphi)^\circ$ for D of Types I–III as in Remark 1.51. In this case, one can also check the first condition of (ii) by a direct computation, the second condition of (ii) has no content, and the third condition of (ii) comes from Lemma 1.54. For more details, we refer to (for example) the applications given in [Lom13; Lom16].

1.3.3 The Lefschetz Group

For motivational reasons, we mention the Lefschetz group $\operatorname{L}(V)$, which contains $\operatorname{Hg}(V)$ but is more controlled. Here is our definition.

Definition 1.65 (Lefschetz group)**.** Fix a polarizable Hodge structure $V \in \operatorname{HS}_{\mathbb{Q}}$ of pure weight. Then we define
$$\operatorname{L}(V) := \operatorname{Sp}_D(\varphi),$$
where $D := \operatorname{End}_{\operatorname{HS}}(V)$, and φ is a polarization.

Thus, Remark 1.53 that $\operatorname{Hg}(V) \subseteq \operatorname{L}(V)$.

Remark 1.66. Let's interpret $\operatorname{L}(V)$ geometrically. Roughly speaking, $\operatorname{L}(V)$ is a form of $\operatorname{Hg}(V)$ which only keeps track of endomorphisms and the polarization instead of keeping track of all Hodge classes. As such, we generically expect $\operatorname{Hg}(V) = \operatorname{L}(V)$ to hold, but we do not expect it to hold always. (Technically, there are generic cases when we do not expect this equality; for example, if V is irreducible of Type III in ths sense of the Albert classificaiotn Theorem 1.28, then $\operatorname{L}(V)$ is not connected, so we cannot have equality.) Furthermore, when $\operatorname{Hg}(V) = \operatorname{L}(V)$, we expect to have strong control on the Hodge classes of V; for example, the Hodge conjecture is known in many such cases [Mur84, Theorem 3.1].

Computationally, one reason why $\operatorname{L}(V)$ is more controlled is that it is much easier to compute. For example, L behaves well in sums.

Lemma 1.67. Fix pairwise non-isomorphic irreducible polarizable Hodge structures $V_1, \ldots, V_k$ of the same pure weight, and let $m_1, \ldots, m_k \geq 1$ be integers. Then the diagonal embeddings $\Delta_i\colon \operatorname{GL}(V_i) \to \operatorname{GL}\left(V_i^{\oplus m_i}\right)$ induce an isomorphism
$$\operatorname{L}(V_1) \times \cdots \times \operatorname{L}(V_k) \to \operatorname{L}\left(V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}\right).$$

Proof. The main idea is to compute some endomorphism algebras and polarizations. We proceed in steps. Set $V := V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$ for brevity.

1. We work with endomorphisms. We may view Hodge structures as $\mathbb{S}$-representations, whereupon we find that
$$\operatorname{End}_{\mathrm{HS}}(V) = \operatorname{End}_{\mathrm{HS}}(V_1)^{m_1 \times m_1} \times \cdots \times \operatorname{End}_{\mathrm{HS}}(V_k)^{m_k \times m_k}.$$
In particular, we see that any f commuting with $\operatorname{End}_{\mathrm{HS}}(V)$ implies that f must preserve each $V_i^{\oplus m_i}$ (because there is a separate algebra $\operatorname{End}_{\mathrm{HS}}\left(V_i^{\oplus m_i}\right)$ for each i). Further, $f|_{V_i^{\oplus m_i}}$ must come from the diagonal embedding $\operatorname{End}_{\mathrm{HS}}(V_i) \to \operatorname{End}_{\mathrm{HS}}\left(V_i^{\oplus m_i}\right)$ because $\operatorname{End}_{\mathrm{HS}}(V_i)^{m_i \times m_i}$ may swap any of the m_i copies of V_i.
We conclude that f commutes with endomorphisms implies that
$$f = (\Delta_1 f_1, \ldots, \Delta_k f_k),$$
where $\Delta_i \colon \operatorname{End}(V_i) \to \operatorname{End}\left(V_i^{\oplus m_i}\right)$ is the diagonal embedding, and each f_i commutes with $\operatorname{End}_{\mathrm{HS}}(V_i)$. Conversely, the computation of $\operatorname{End}_{\mathrm{HS}}(V)$ above allows us to conclude that any f in the above form commutes with $\operatorname{End}_{\mathrm{HS}}(V)$.

2. We work with the polarization. Choose polarizations $\varphi_1, \ldots, \varphi_k$ on $V_1, \ldots, V_k$ (respectively), and we note that these polarizations glue into a polarization φ on V. With this choice of polarization, we see that $f = (\Delta_1 f_1, \ldots, \Delta_k f_k)$ as in the previous step preserves φ if and only if each factor $\Delta_i f_i$ preserves the polarization $\varphi|_{V_i^{\oplus m_i}}$, which is equivalent to f_i preserving the polarization φ_i. In total, we thus see that $f \in \mathrm{L}(V)$ if and only $f_i \in \mathrm{L}(V_i)$ for each i, so we are done. ■

Lemma 1.67 tells us that we can always reduce the computation of the Lefschetz group to irreducible components. In this way, it now suffices to compute $\mathrm{L}(V)$ by working with V according to the Albert classification (Theorem 1.28). All these computations are recorded in [Mil99, Section 2]. Because we will only be interested in Type IV in the sequel, we will only record the part of this computation we need for completeness.

Lemma 1.68. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight with $D := \operatorname{End}_{\mathrm{HS}}(V)$ and polarization φ. Suppose $D = F$ is a CM field. Then
$$\mathrm{L}(V)_{\mathbb{C}} \cong \mathrm{GL}_{[V:F]}(\mathbb{C})^{\frac{1}{2}[F:\mathbb{Q}]}.$$

Proof. We proceed in steps. Let $F^\dagger \subseteq F$ be the maximal totally real subfield, and choose embeddings $\rho_1, \ldots, \rho_{e_0} \colon F^\dagger \hookrightarrow \mathbb{R}$, where $e_0 := \frac{1}{2}[F : \mathbb{Q}]$. For each i, we will let σ_i and τ_i be complex conjugate embeddings $F \hookrightarrow \mathbb{C}$ restricting to ρ_i.

1. We begin by explaining the exponent $e_0 = \frac{1}{2}[F : \mathbb{Q}]$. Note V is a free $F^\dagger$-module of rank $[V : F]$, so $V_{\mathbb{R}}$ is a free module over
$$F^\dagger \otimes \mathbb{R} = \prod_{i=1}^{e_0} F^\dagger_{\rho_i},$$
where $F^\dagger_\rho = \mathbb{R}$ refers to the $F^\dagger \otimes \mathbb{R}$ module where F acts by ρ. The above decomposition of $F \otimes \mathbb{R}$ implies a decomposition
$$V_{\mathbb{R}} = V_1 \oplus \cdots \oplus V_{e_0},$$
where each V_i of a vector space over $F^\dagger_{\rho_i}$, all the same dimension.
We now understand the effect of endomorphisms and the polariaztion on our decomposition. Thus, we see that $f \colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ commutes with $F^\dagger \otimes \mathbb{R}$ if and only if f preserves each factor V_i (due to the decomposition of $F^\dagger \otimes \mathbb{R}$) and commute with the action of $F^\dagger_{\rho_i}$ on each V_i. Similarly, we see that the

polarization φ makes the V_is orthogonal: for each $d \in F^\dagger$, we see that any $v_i \in V_i$ and $v_j \in V_j$ has

$$\begin{aligned}\rho_i(d)\varphi(v_i, v_j) &= \varphi(dv_i, v_j)\\ &= \varphi(v_i, \overline{d}v_j)\\ &= \varphi(v_i, dv_j)\\ &= \rho_j(d)\varphi(v_i, v_j),\end{aligned}$$

so $i \neq j$ implies that $\varphi(v_i, v_j) = 0$. Thus, we see that φ must restrict to non-degenerate skew-symmetric bilinear forms on each V_i individually. In total, $f \colon V_\mathbb{R} \to V_\mathbb{R}$ preserves φ if and only if $f|_{V_i}$ preserves $\varphi|_{V_i}$ for each i. In total, we see that

$$\mathrm{L}(V)_\mathbb{R} = \mathrm{Sp}_{F\otimes_{\rho_1}\mathbb{R}}(\varphi|_{V_1}) \times \cdots \times \mathrm{Sp}_{F\otimes_{\rho_k}\mathbb{R}}(\varphi|_{V_{e_0}}).$$

2. It remains to show that $\mathrm{Sp}_{F\otimes_{\rho_i}\mathbb{R}}(\varphi|_{V_i})_\mathbb{C}$ is isomorphic to $\mathrm{GL}_{[V:F]}(\mathbb{C})$; here, note $[V : F] = [V_i : F^\dagger_{\rho_i}]$. For this, we abstract the situation somewhat: suppose that a vector space V over $\mathbb{R}$ has been equipped with an action by $\mathbb{C} \subseteq \mathrm{End}_\mathbb{R}(V)$, and furthermore, φ is a skew-Hermitian form on V. Then we want to show $\mathrm{Sp}_\mathbb{C}(\varphi)_\mathbb{C} \cong \mathrm{GL}_{[V:\mathbb{R}]}(\mathbb{C})$.

   The trick is that we can keep track of commuting with the action of $\mathbb{C}$ on V by merely commuting with the action of $i \in \mathbb{C}$. Thus, let $J \colon V \to V$ be this map, which satisfies $J^2 = -1$. Now, the action of $J_\mathbb{C}$ on $V_\mathbb{C}$ must diagonalize into eigenspaces $V_i \oplus V_{-i}$ with eigenvalues i and $-i$ respectively; note that we must have $\dim V_i = \dim V_{-i}$ in order for the characateristic polynomial of J to have real coefficients. The point is that $f \in \mathrm{End}(V_\mathbb{C})$ commutes with the action of $\mathbb{C}$ if and only if it commutes with the action of J, which we can see is equivalent to f preserving the decomposition $V_i \oplus V_{-i}$.

   We now study the polarization φ. Note that φ vanishes on $V_{\pm i} \oplus V_{\pm i}$: for any $v, v' \in V_{\pm i}$, we see that

$$\begin{aligned}\pm i\varphi(v, v') &= \varphi(Jv, v')\\ &= \varphi(v, -Jv')\\ &= \mp i\varphi(v, v'),\end{aligned}$$

   from which $\varphi(v, v') = 0$ follows. For example, this implies that any $f \in \mathrm{End}(V_\mathbb{C})$ commuting with the J-action will automatically preserve φ on $V_{\pm i} \times V_{\pm i}$. Additionally, we see that φ must restrict to a non-degenerate bilinear form on $V_i \times V_{-i}$.

   We are now ready to claim that restriction defines an isomorphism $\mathrm{Sp}_\mathbb{C}(\varphi)_\mathbb{C} \to \mathrm{GL}_\mathbb{C}(V_i)$. This restriction does actually output to $\mathrm{GL}_\mathbb{C}(V_i)$ because $g \in \mathrm{Sp}_\mathbb{C}(\varphi)_\mathbb{C}$ must preserve the decomposition $V_i \oplus V_{-i}$. To see the injectivity, we note that preserving φ requires

$$\varphi(v, gw) = \varphi\left(g^{-1}v, w\right)$$

   for all $v \in V_i$ and $w \in V_{-i}$; thus, the non-degeneracy of φ implies that $g \in \mathrm{Sp}_\mathbb{C}(\varphi)_\mathbb{C}$ is uniquely determined by its action on V_i. Conversely, for the surjectivity, we see that we can take any element in $\mathrm{GL}(V_i)$ and use the previous sentence to extend it uniquely to an element of $\mathrm{Sp}_\mathbb{C}(\varphi)_\mathbb{C}$. ■

1.4 The Center

Our last computational tool concerns the center of $\mathrm{Hg}(V)$. This discussion is somewhat more involved, so we will spend a full section here.

Let's begin with some motivation. Fix a Hodge struture $V \in \mathrm{HS}_\mathbb{Q}$. In the application of this thesis, we will use Lemma 1.62 to compute $\mathrm{Hg}(V)^{\mathrm{der}}$: note $\mathrm{Hg}(V)^{\mathrm{der}}$ is semisimple and hence its Lie algebra can be written as the sum of simple Lie algebras which may be amenable to the lemma. Because $\mathrm{Hg}(V)$ is reductive by Lemma 1.44, it remains to compute the center $Z(\mathrm{Hg}(V))$; recall $\mathrm{Hg}(V)$ is connected by Remark 1.40, so we may as well compute the connected component $Z(\mathrm{Hg}(V))^\circ$. As usual, the same discussion holds for $\mathrm{MT}(V)$, but we note that $Z(\mathrm{MT}(V))^\circ$ tends to be nontrivial because usually $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{MT}(V)$ by Example 1.31.

In Proposition 1.77, we find that $Z(\operatorname{Hg}(V))^\circ$ is trivial unless V has irreducible factors of Type IV in the sense of the Albert classification (Theorem 1.28). As such, we spend the rest of the section focusing on computations in Type IV. Computations are well-understood when V comes from an abelian variety with complex multiplication, so the main contribution here is that these arguments generalize with only slight modifications.

1.4.1 General Comments

The following lemma begins our discussion.

Lemma 1.69. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight, and set $D := \operatorname{End}_{\mathrm{HS}}(V)$ with $F := Z(D)$. Viewing D as a $\mathbb{Q}$-group, we have

$$Z(\operatorname{Hg}(V)) \subseteq \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F},$$

where $\operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ embeds into $\operatorname{GL}(V)$ via the D-action on V.

Proof. Here, F is a product of number fields because it is a commutative semisimple $\mathbb{Q}$-algebra. Recall from Lemma 1.54 that

$$D = \operatorname{End}_{\mathbb{Q}}(V)^{\operatorname{Hg}(V)},$$

which upgrades to an equality of algebraic subgroups of $\operatorname{End}_{\mathbb{Q}}(V)$ because $\mathbb{Q}$-points are dense in these algebraic groups by combining [Mil17, Corollary 17.92] and [Mil17, Definition 12.59]. In particular, we see that $\operatorname{Hg}(V)$ commutes with $D^\times$, so the double centralizer theorem enforces $Z(\operatorname{Hg}(V)) \subseteq D^\times$ even as algebraic groups. However, $Z(\operatorname{Hg}(V))$ now commutes fully with $D^\times$, so in fact $Z(\operatorname{Hg}(V)) \subseteq Z(D)^\times$, which is what we wanted. ■

Remark 1.70. One also has $Z(\operatorname{MT}(V)) \subseteq \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ because $\operatorname{MT}(V) \subseteq \mathbb{G}_{m,\mathbb{Q}} \operatorname{Hg}(V)$ by Lemma 1.41, and the scalars $\mathbb{G}_{m,\mathbb{Q}}$ already live in $\operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$.

Lemma 1.69 is that it places the center $Z(\operatorname{Hg}(V))$ in an explicit torus $\operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$. Subgroups of tori are well-understood by (co)character groups, so this puts us in good shape. This torus will be important enough to have its own notation.

Notation 1.71. Fix a commutative semisimple $\mathbb{Q}$-algebra F (i.e., a product of number fields). Then we define the torus

$$\mathrm{T}_F := \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}.$$

Remark 1.72. Writing F as a product of number fields $F_1 \times \cdots \times F_k$, we find

$$\mathrm{T}_F = \mathrm{T}_{F_1} \times \cdots \times \mathrm{T}_{F_k}$$

because $F = F_1 \times \cdots \times F_k$ is an equality of $\mathbb{Q}$-algebras.

Remark 1.73. Let's compute the character group $\mathrm{X}^*(\mathrm{T}_F)$. By Remark 1.72, it's enough to do this computation when F is a field. The choice of a primitive element $\alpha \in F$ with minimal monic polynomial $f(x)$ yields an isomorphism $F \cong \mathbb{Q}[x]/(f(x))$. Upon base-changing to $\overline{\mathbb{Q}}$, we get a ring isomorphism

$$F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \prod_{i=1}^{n} \frac{\overline{\mathbb{Q}}[x]}{(x-\alpha_i)},$$

where $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are the roots of $f(x)$ in $\overline{\mathbb{Q}}$. Each root α_i provides a unique embedding $F \hookrightarrow \overline{\mathbb{Q}}$, so we see that $(\mathrm{T}_F)_{\overline{\mathbb{Q}}} \cong \mathbb{G}^n_{m,\overline{\mathbb{Q}}}$, where the n maps $(\mathrm{T}_F)_{\overline{\mathbb{Q}}} \to \mathbb{G}_{m,\overline{\mathbb{Q}}}$ are given by the embedding $\sigma_i \colon F \hookrightarrow \overline{\mathbb{Q}}$ defined by $\sigma_i(\alpha) := \alpha_i$. In total, we find that $\mathrm{X}^*(\mathrm{T}_F)$ is a free $\mathbb{Z}$-module spanned by the embeddings $\Sigma_F := \{\sigma_1, \ldots, \sigma_n\}$, and it has the natural Galois action. Dually, $\mathrm{X}_*(\mathrm{T}_F)$ has the dual basis $\Sigma_F^\vee = \{\sigma_1^\vee, \ldots, \sigma_n^\vee\}$.

In the light of the above remark, we will want the following notation.

Notation 1.74. Given a number field F, we let Σ_F denote the collection of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$. Given a product of number fields $F := F_1 \times \cdots \times F_k$, we define $\Sigma_F := \Sigma_{F_1} \sqcup \cdots \sqcup \Sigma_{F_k}$.

The point of the above notation is that $\mathrm{X}^*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F]$ as Galois modules.

It is possible to upgrade Lemma 1.69 in the presence of a polarization.

Lemma 1.75. Fix a polarizable $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight, and set $D := \mathrm{End}_{\mathrm{HS}}(V)$ with $F := Z(D)$. Then

$$Z(\mathrm{Hg}(V)) \subseteq \left\{ g \in \mathrm{T}_F : gg^\dagger = 1 \right\},$$

where $(\cdot)^\dagger$ is the Rosati involution.

Proof. As usual, everything in sight upgrades to algebraic groups. Let φ be a polarization. Fix some $g \in \mathrm{Hg}(V)$; note that Lemma 1.69 implies $g \in \mathrm{T}_F$, so it makes sense to write down $g^\dagger$.

Now, by the non-degeneracy of φ, it is enough to show that

$$\varphi\left(gg^\dagger v \otimes w\right) \stackrel{?}{=} \varphi(v \otimes w)$$

for any $v, w \in V$. Well, the definition of $(\cdot)^\dagger$ tells us that the left-hand side equals $\varphi\left(g^\dagger v \otimes g^\dagger w\right)$, which equals $\varphi(v \otimes w)$ because $\mathrm{Hg}(V) \subseteq \mathrm{Sp}(\varphi)$ by Remark 1.53. ■

Once again, this torus is important enough to earn its own notation.

Notation 1.76. Fix a commutative semisimple $\mathbb{Q}$-algebra F with involution $(\cdot)^\dagger$. Then we define the torus

$$\mathrm{U}_F := \left\{ g \in \mathrm{T}_F : xx^\dagger = 1 \right\}.$$

Here is an application of Lemma 1.75.

Proposition 1.77. Fix polarizable $V \in \mathrm{HS}_{\mathbb{Q}}$ of pure weight. Suppose that V has no irreducible Hodge substructures with endomorphism algebra of Type IV in the sense of the Albert classification (Theorem 1.28). Then $Z(\mathrm{Hg}(V))$ is finite, and $\mathrm{Hg}(V)$ is semisimple.

Proof. Quickly, recall from Lemma 1.44 that $\mathrm{Hg}(V)$ is reductive, so the finitness of $Z(\mathrm{Hg}(V))$ implies that $Z(\mathrm{Hg}(V))^\circ = 1$ and thus $\mathrm{Hg}(V) = \mathrm{Hg}(V)^{\mathrm{der}}$, making $\mathrm{Hg}(V)$ is semisimple. (See also [Mil17, Proposition 19.10].) As such, we will focus on the first claim.

Set $D := \operatorname{End}_{\mathrm{HS}}(V)$ with $F := Z(D)$ so that $\operatorname{Hg}(V) \subseteq \mathrm{U}_F$ by Lemma 1.75. It is therefore enough to check that U_F is finite. Well, F is a product of number fields, and upon comparing with Theorem 1.28, we see that avoiding Type IV implies that F is a product of totally real fields. Totally real fields have only two units, so finiteness of U_F follows. ■

Thus, we see that we have pretty good control outside of Type IV factors, so we will spend the rest of this section on Type IV. For some applications outside Type IV, see (for example) [Lom16].

1.4.2 Type IV: The Signature

The arguments in the next two subsections are motivated by the computation of [Yu15, Lemma 4.2] and [Yan94, Proposition 1.1]. For this subsection, $V \in \mathrm{HS}_{\mathbb{Q}}$ is a Hodge structure whose irreducible factors are of Type IV in the sense of the Albert classification (Theorem 1.28). In our application to abelian varieties, it will also be enough to assume that the Hodge structure of V is concentrated in $V^{0,1}$ and $V^{1,0}$, so we do so.

By assumption, we know that $D := \operatorname{End}_{\mathrm{HS}}(V)$ is a division algebra over its center $F := Z(D)$, where F is a CM algebra (i.e., a product of CM fields), and the Rosati involution $(\cdot)^\dagger$ restricts to complex conjugation on F. In particular, $F^\dagger$ is the product of the maximal totally real subfields of F.

The basic approach of this subsection is that Lemma 1.69 requires $Z(\operatorname{Hg}(V))^\circ \subseteq \mathrm{T}_F$, and one can compute subtori using the machinery of (co)character groups. In particular, we recall that $\mathrm{X}^*(\Sigma_F) = \mathbb{Z}[\Sigma_F]$ and $\mathrm{X}_*(\Sigma_F) = \mathbb{Z}[\Sigma_F^\vee]$ as Galois modules. We will need a way to work directly with the Hodge structure on V. It will be described by the following piece of combinatorial data. Recall that a CM algebra is a product of CM fields.

Definition 1.78 (signature)**.** Fix a CM algebra F, and recall that Σ_F is the set of homomorphisms $F \hookrightarrow \overline{\mathbb{Q}}$. Then a *signature* is a function $\Phi\colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ such that the sum

$$\Phi(\sigma) + \Phi(\overline{\sigma})$$

is constant with respect to $\sigma \in \Sigma_F$; here, $\overline{\sigma}$ denotes the complex conjugate embedding to σ. We may call the pair (F, Φ) a *CM signature*.

Remark 1.79. One can also view Φ as an element of $\mathbb{Z}[\Sigma_F]$ as

$$\Phi := \sum_{\sigma \in \Phi} \Phi(\sigma)\sigma.$$

Remark 1.80. The case that $\Phi(\sigma) + \Phi(\overline{\sigma})$ always equals 1 corresponds to Φ being a CM type.

Remark 1.81. If we expand F as a product of CM fields $F = F_1 \times \cdots \times F_k$, then $\Sigma_F = \Sigma_{F_1} \sqcup \cdots \sqcup \Sigma_{F_k}$. Thus, we see that a signature of F has only a little more data than a signature on each of the $\Sigma_{F_\bullet}$s individually; in particular, one should make sure that $\Phi(\sigma) + \Phi(\overline{\sigma})$ remains equal across the different fields.

The idea is that we can keep track of a signature with a Hodge structure.

Lemma 1.82. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ with $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$ such that $\operatorname{End}_{\mathrm{HS}}(V)$ contains a CM algebra F. Then the function $\Phi\colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ defined by

$$V^{1,0} \cong \bigoplus_{\sigma \in \Sigma_F} \mathbb{C}_\sigma^{\Phi(\sigma)}$$

is a signature, which we will call the induced signature. This is an isomorphism of F-representations, where $\mathbb{C}_\sigma$ is a complex F-representation via the embedding σ.

Proof. In short, the condition that $\Phi(\sigma)+\Phi(\overline{\sigma})$ is constant comes from the condition $V^{0,1}=\overline{V^{1,0}}$. To see this, note that V is a free module over F, so $V_\mathbb{C}$ is a free module over $F\otimes\mathbb{C}$ of finite rank. As such, we may set $d:=[V:F]$ so that $V\cong F^d$ as F-representations, and then we find

$$V_\mathbb{C}\cong\bigoplus_{\sigma\in\Sigma_F}\mathbb{C}_\sigma^d.$$

Now, $V_\mathbb{C}=V^{1,0}\oplus V^{0,1}$, and because F acts by endomorphisms of Hodge structures, we get a well-defined action of F on $V^{1,0}$ and $V^{0,1}$ individually. In particular, the definition of Φ also grants

$$V^{0,1}\cong\bigoplus_{\sigma\in\Sigma_F}\mathbb{C}_\sigma^{d-\Phi(\sigma)}$$

as F-representations, so

$$\overline{V^{0,1}}\cong\bigoplus_{\sigma\in\Sigma_F}\mathbb{C}_{\overline{\sigma}}^{d-\Phi(\sigma)}$$

To complete the proof, we note that $V^{0,1}=\overline{V^{1,0}}$ continues to be true as F-representations, so we must have $\Phi(\sigma)=d-\Phi(\overline{\sigma})$ for all σ. The result follows. ■

Of course, we cannot expect this signature Φ to remember everything about the Hodge structure. For example, if $\operatorname{End}_{\mathrm{HS}}(V)$ contains a larger CM algebra F' than F, then the signature induced by F' knows more about the Hodge structure than the one induced by F. However, in "generic cases," this signature is expected to suffice. For our purposes, we will take generic to mean that there are no more endomorphisms than the ones promised by F; i.e., this explains why we will assume $Z(\operatorname{End}_{\mathrm{HS}}(V))=F$ in the sequel.

We now relate our signature to cocharacters of $Z(\operatorname{Hg}(V))^\circ$. For this, it will be helpful to realize $Z(\operatorname{Hg}(V))$ as some kind of monodromy group. The trick is to consider the determinant.

Lemma 1.83. Fix $V\in\mathrm{HS}_\mathbb{Q}$ of pure weight such that $Z(\operatorname{End}_{\mathrm{HS}}(V))$ equals an algebra F. Then $Z(\operatorname{Hg}(V))^\circ$ equals the largest algebraic $\mathbb{Q}$-subgroup of T_F containing the image of $(\det_F\circ h)\colon\mathbb{U}\to(\mathrm{T}_F)_\mathbb{R}$.

Proof. The point is that taking the determinant will kill $\operatorname{Hg}(V)^{\mathrm{der}}$ because $\operatorname{Hg}(V)\subseteq\mathrm{GL}_F(V)$. There are two inclusions we must show.

- We show that $Z(\operatorname{Hg}(V))^\circ$ contains the image of $(\det F\circ h|_\mathbb{U})$. Well, $\operatorname{Hg}(V)$ contains the image of $h|_\mathbb{U}$, so it is enough to show that $Z(\operatorname{Hg}(V))^\circ$ contains the image of $\det_F\colon\operatorname{Hg}(V)\to\mathrm{T}_F$. For this, we recall that $\operatorname{Hg}(V)$ is connected (by Remark 1.40), so

$$\operatorname{Hg}(V)=Z(\operatorname{Hg}(V))^\circ\operatorname{Hg}(V)^{\mathrm{der}}.$$

  Note that $\det_F$ is simply $(\cdot)^{\dim_F V}$ on the torus $Z(\operatorname{Hg}(V))^\circ$, so that piece surjects onto $Z(\operatorname{Hg}(V))^\circ$! Thus, it is enough to check that $\det_F\colon\operatorname{Hg}(V)^{\mathrm{der}}\to\mathrm{T}_F$ is trivial, which is true by the definition of the derived subgroup upon noting that $\det_F$ is a homomorphism with abelian target.

- Suppose that $T\subseteq\mathrm{T}_F$ contains the image of $(\det_F\circ h|_\mathbb{U})$. Then we claim that T contains $Z(\operatorname{Hg}(V))^\circ$. Let $H\subseteq\mathrm{GL}_F(V)$ be the pre-image of T under $\det_F\colon\mathrm{GL}_F(V)\to\mathrm{T}_F$. Then H must contain the image of $h|_\mathbb{U}$, so it contains $\operatorname{Hg}(V)$ by defintion. In particular, H contains $Z(\operatorname{Hg}(V))^\circ$! Now, T contains $\det_F(H)$, so T contains $\det_F(Z(\operatorname{Hg}(V))^\circ)$, but the previous point check remarked that this simply equals $Z(\operatorname{Hg}(V))^\circ$, so we are done. ■

Proposition 1.84. Fix $V\in\mathrm{HS}_\mathbb{Q}$ with $V_\mathbb{C}=V^{0,1}\oplus V^{1,0}$ such that $Z(\operatorname{End}_{\mathrm{HS}}(V))$ equals a CM algebra F. Let $\Phi\colon\Sigma_F\to\mathbb{Z}_{\geq0}$ be the induced signature. Then the induced representation $(\det_F\circ h)\colon\mathbb{U}\to(\mathrm{T}_F)_\mathbb{R}$ sends the generator of $\mathrm{X}_*(\mathbb{U})$ to

$$-\sum_{\sigma\in\Sigma_F}(\Phi(\sigma)-\Phi(\overline{\sigma}))\sigma^\vee.$$

Proof. This boils down to computing the map $\det_F \circ h|_{\mathbb{U}}$. We proceed in steps.

1. To set ourselves up, recall that
$$\mathbb{U}_{\mathbb{C}} = \{(z, 1/z) : z \in \mathbb{G}_{m,\mathbb{C}}\},$$
so one has an isomorphism cocharacter $z^\vee : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{U}_{\mathbb{C}}$ given by $z^\vee \mapsto z \mapsto (z, 1/z)$. Thus, we have left to show that
$$\det_F \circ h_{\mathbb{C}} \circ z^\vee \stackrel{?}{=} -\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee.$$
We may check this equality on geometric points.

2. We describe the map $h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V)_{\mathbb{C}}$. By definition, $h(z, w) \in \mathrm{GL}(V)$ acts by z^{-1} on $V^{1,0}$ and by w^{-1} on $V^{0,1}$. Thus, the definition of Φ grants that $h(z, w)$ diagonalizes. To be more explicit, for each $\sigma \in \Sigma_F$, we define $V_\sigma^{p,q}$ to be the σ-eigenspace for the F-action on $V^{p,q} \subseteq V_{\mathbb{C}}$. Then we see that $h(z, w)$ acts on $V_\sigma^{1,0}$ by the scalar z^{-1} and on $V^{0,1}$ by the scalar w^{-1}.

3. We describe the map $(\det_F \circ h_{\mathbb{C}}) : \mathbb{S}_{\mathbb{C}} \to (\mathrm{T}_F)_{\mathbb{C}}$. Realizing geometric points in $(\mathrm{T}_F)_{\mathbb{C}}$ as tuples in $\mathbb{C}^{\Sigma_F}$, we see that $\det_F$ simply takes the determinant of the matrix $h_{\mathbb{C}}(z, w)|_{V_\sigma}$ to the σ-component in $(\mathrm{T}_F)_{\mathbb{C}}$. (One finds this by tracking through the definition of $\det_F$ as a morphism of algebraic groups.) As such, we see that
$$\det h_{\mathbb{C}}(z, w)|_{V_\sigma} = z^{-\Phi(\sigma)} w^{-\Phi(\overline{\sigma})}$$
because Φ is a signature.

4. We complete the proof. The previous step shows that $(\det_F \circ h_{\mathbb{C}} \circ z^\vee)(z)$ goes to the element
$$\left(z^{-\Phi(\sigma)+\Phi(\overline{\sigma})}\right)_{\sigma \in \Sigma(F)} \in \mathbb{C}^{\Sigma_F}.$$
This completes the proof upon noting that the cocharacter $\sigma^\vee : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{T}_F$ simply maps into the σ-component of $\mathbb{C}^{\Sigma_F}$ on geometric points. ■

Remark 1.85. Notably, the given element sums to 0, which corresponds to the fact that $\mathrm{Hg}(V) \subseteq \mathrm{SL}(V)$ as seen in Lemma 1.41. Indeed, by diagonalizing the F-action on V, we see that $(\mathrm{T}_F \cap \mathrm{SL}(V))^\circ$ consists of the $g \in \mathrm{T}_F$ such that the product of the elements in g equals 1.

Proposition 1.84 easily translates into a computation of the cocharacter group $\mathrm{X}_*(\mathrm{Hg}(V))^\circ$. In the next few results, saturated simply means that the induced quotient is torsion-free.

Corollary 1.86. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ with $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$ such that $Z(\mathrm{End}_{\mathrm{HS}}(V))$ equals a CM algebra F. Let $\Phi : \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the induced signature. Then $Z(\mathrm{Hg}(V))^\circ \subseteq \mathrm{T}_F$ has cocharacter group equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$ containing
$$\sum_{\sigma \in \Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee.$$

Proof. This is immediate from combining Lemma 1.83 and Proposition 1.84 with the equivalence of categories X_* between algebraic tori and Galois modules. See [Mil17, Theorem 12.23] for the proof that X^* is an equivalence, which is similar. ■

Corollary 1.87. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ with $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$ such that $Z(\mathrm{End}_{\mathrm{HS}}(V))$ equals a CM algebra F. Let $\Phi : \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the signature defined in Lemma 1.82. Then $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$ has cocharacter group equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F) = \mathbb{Z}[\Sigma_F^\vee]$ containing
$$\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee.$$

Proof. This follows from Corollary 1.86. By Lemma 1.41, it is enough to add in the cocharacter given by the scalars $\mathbb{G}_{m,\mathbb{Q}} \to \mathrm{T}_F$, which is $\sum_{\sigma\in\Sigma_F} \sigma^\vee$. Thus, the fact that Φ is a signature implies that

$$\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee$$

certainly lives in $\mathrm{X}_*(\mathrm{MT}(V)) \subseteq \mathrm{X}_*(\mathrm{T}_F)$.

Conversely, if X is some saturated Galois submodule containing $\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee$, then we would like to show that $\mathrm{X}_*(\mathrm{MT}(V)) \subseteq X$. Well, X is a Galois submodule, so it must contain the complex conjugate element $\sum_{\sigma\in\Sigma_F} \Phi(\overline{\sigma})\sigma^\vee$. On one hand, this then sums with the given element to produce

$$\sum_{\sigma\in\Sigma_F} \sigma^\vee \in X$$

because X is saturated. On the other hand, we can take a difference to see that

$$\sum_{\sigma\in\Sigma_F} (\Phi(\sigma) - \Phi(\overline{\sigma}))\sigma^\vee \in X.$$

We conclude that X contains the cocharacter of the scalars $\mathbb{G}_{m,\mathbb{Q}} \subseteq \mathrm{T}_F$ and the cocharacter lattice of $Z(\mathrm{Hg}(V))^\circ \subseteq \mathrm{T}_F$, so we conclude that X must also contain the cocharacter lattice of $Z(\mathrm{MT}(V))^\circ$. ■

Remark 1.88. One can also prove the above corollary by following the proof of Corollary 1.86. For example, this approach provides a monodromy interpretation of $Z(\mathrm{MT}(V))^\circ$ analogous to Lemma 1.83. Here, one replaces the generator of $\mathrm{X}_*(\mathbb{U})$ with the cocharacter $\mu \in \mathrm{X}_*(\mathbb{S})$, and one finds that $\det_F \circ h_\mathbb{C}$ sends μ to $\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee$. One is then able to prove statements analogous to Proposition 1.84 and Corollary 1.86.

Let's pause for a moment with an explanation of how one can use Corollary 1.87 to compute $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$. The approach for $Z(\mathrm{Hg}(V))^\circ$ is similar but only a little more complicated.

We will only compute over a Galois extension $L/\mathbb{Q}$ containing all factors of F. In this case, the F-action on V_L diagonalizes, so one can identify $(\mathrm{T}_F)_L \subseteq \mathrm{GL}(V)_L$ as the diagonal torus for some basis of V_L. In particular, for each $\sigma \in \Sigma_F$, the cocharacter $\sigma^\vee$ corresponds to one of the standard cocharacters for the diagonal torus of $\mathrm{GL}(V)_L$. Now, Corollary 1.87 tells us that $\mathrm{X}_*(Z(\mathrm{MT}(V))^\circ) \subseteq \mathrm{X}_*(\mathrm{T}_F)$ equals the saturation of the sublattice spanned by the vectors

$$g\left(\sum_{\sigma\in\Sigma_F} \Phi(\sigma)\sigma^\vee\right) = \sum_{\sigma\in\Sigma_F} \Phi(\sigma)(g\sigma)^\vee,$$

where g varies over $\mathrm{Gal}(L/F)$. By computing a basis of the saturation of this sublattice, we get a family of 1-parameter subgroups of the diagonal torus of $\mathrm{GL}(V)_L$ which together generate $Z(\mathrm{MT}(V))^\circ$. This more or less computes $Z(\mathrm{MT}(V))^\circ$.

1.4.3 Type IV: The Reflex

In the sequel, we will be most interested in equations cutting out $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$. One could imagine proceeding as above to compute $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$ via 1-parameter subgroups and then afterwards finding the desired equations. This is somewhat computationally intensive, so instead we will turn our attention to computing character groups. As in [Yu15, Lemma 4.2], this will require a discussion of the reflex.

Definition 1.89 (reflex signature)**.** Fix CM fields F and F^* and CM signatures (F,Φ) and (F^*,Φ^*). We say that these CM signatures are *reflex* if and only if there is a Galois extension $L/\mathbb{Q}$ containing F and F^* such that each $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ has

$$\Phi(\sigma|_F) = \Phi^*\left(\sigma^{-1}|_{F^*}\right).$$

In this situation, we may call (F^*,Φ^*) a *reflex signature* for (F,Φ).

Remark 1.90. We check that (F, Φ) and (F^*, Φ^*) does not depend on the choice of Galois extension L. Indeed, suppose that we have another Galois extension $L'/\mathbb{Q}$ containing F and F^*; let L'' be a Galois extension containing both L and L'. By symmetry, it is enough to check that (F, Φ) are reflex with respect to L if and only if they are reflex with respect to L''. Well, for any $\sigma \in \operatorname{Gal}(L''/\mathbb{Q})$, we see that $\Phi(\sigma|_F) = \Phi^*\left(\sigma^{-1}|_{F^*}\right)$ is equivalent to $\sigma|_L \in \operatorname{Gal}(L/\mathbb{Q})$ satisfying $\Phi(\sigma|_L|_F) = \Phi^*\left(\sigma|_L^{-1}|_{F^*}\right)$, so we are done after remarking that restriction $\operatorname{Gal}(L''/\mathbb{Q}) \to \operatorname{Gal}(L/\mathbb{Q})$ is surjective.

Remark 1.91. We check that reflex signatures certainly exist: one can choose any Galois closure L of F and then define $\Phi^* \colon \operatorname{Gal}(L/\mathbb{Q}) \to \mathbb{Z}_{\geq 0}$ by $\Phi^*(\sigma) := \Phi\left(\sigma^{-1}|_L\right)$.

Remark 1.92. In the theory of abelian varieties with complex multiplication, it is customary to make F^* as small as possible, which makes it unique. This is useful for moduli problems. However, this is not our current interest, and we are not requiring that the reflex signature be unique because it will be convenient later to take large extensions.

The point of introducing the reflex is that it provides another monodromy interpretation of $Z(\mathrm{MT}(V))^\circ$. To achieve this, we need the reflex norm.

Definition 1.93 (reflex norm)**.** Fix CM fields F and F^* and reflex CM signatures (F, Φ) and (F^*, Φ^*). Then we define the *reflex norm* as the map $\mathrm{N}_{\Phi^*} \colon F^* \to \overline{\mathbb{Q}}$ by

$$\mathrm{N}_{\Phi^*}(x) := \prod_{\sigma \in \Sigma_{F^*}} \sigma(x)^{\Phi^*(\sigma)}.$$

Note that this is a character in $\mathrm{X}^*(\mathrm{T}_{F^*})$.

Technically, this definition does not require us to remember that (F^*, Φ^*) is reflex to (F, Φ), but we will want to know this in the following checks.

Lemma 1.94. Fix CM fields F and F^* and reflex CM signatures (F, Φ) and (F^*, Φ^*).

(a) If (F_1^*, Φ_1^*) is a CM signature restricting to (F^*, Φ^*), then (F, Φ) and (F_1^*, Φ_1^*) are still reflex, and

$$\mathrm{N}_{\Phi_1^*} = \mathrm{N}_{\Phi^*} \circ \mathrm{N}_{F_1^*/F^*}.$$

(b) The image of N_{Φ^*} lands in F.

Proof. Here, "restricting" simply means that F_1^* contains F^* and $\Phi_1^*(\sigma) = \Phi^*(\sigma|_{F^*})$ for all $\sigma \in \Sigma_{F_1^*}$.

(a) That (F, Φ) and (F_1^*, Φ_1^*) are still reflex follows from the definition: choose a Galois extension L containing F and F_1^*, and then each $\sigma \in \operatorname{Gal}(L/\mathbb{Q})$ has

$$\begin{aligned}\Phi(\sigma|_F) &= \Phi^*\left(\sigma^{-1}|_{F^*}\right) \\ &= \Phi_1^*\left(\sigma^{-1}|_{F_1^*}\right).\end{aligned}$$

To check the equality of reflex norms, we extend each $\sigma \in \Sigma_{F^*}$ to some $\widetilde{\sigma} \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and then we directly compute

$$\begin{aligned}
\mathrm{N}_{\Phi^*}\left(\mathrm{N}_{F_1^*/F^*}(x)\right) &= \prod_{\sigma\in\Sigma_{F^*}} \sigma\left(\mathrm{N}_{F_1^*/F^*}(x)\right)^{\Phi^*(\sigma)} \\
&= \prod_{\substack{\sigma\in\Sigma_F^* \\ \tau\in\mathrm{Hom}_{F^*}(F_1^*,\overline{\mathbb{Q}})}} \widetilde{\sigma}\tau(x)^{\Phi^*(\sigma)} \\
&= \prod_{\substack{\sigma\in\Sigma_F^* \\ \tau\in\mathrm{Hom}_{F^*}(F_1^*,\overline{\mathbb{Q}})}} \widetilde{\sigma}\tau(x)^{\Phi_1^*(\widetilde{\sigma}\tau)} \\
&= \mathrm{N}_{\Phi_1^*}(x),
\end{aligned}$$

where the last step holds by noting that $\widetilde{\sigma}\circ\tau$ parameterizes Σ_{F^*}.

(b) We begin by reducing to the case where $F^*/\mathbb{Q}$ is Galois. Indeed, the previous step tells us that extending F^* merely passes to a norm subgroup of F^*, but norm subgroups are Zariski dense in T_{F^*}, so it suffices to check the result on such norm subgroups. Thus, we may assume that $F^*/\mathbb{Q}$ is Galois, contains F, and thus $\Phi^*(\sigma)=\Phi\left(\sigma^{-1}|_F\right)$. Now, for any $g\in\mathrm{Gal}(F^*/F)$, we see $\Phi^*(\sigma)=\Phi^*\left(g^{-1}\sigma\right)$, so

$$\begin{aligned}
g\left(\mathrm{N}_{\Phi^*}(x)\right) &= \prod_{\sigma\in\mathrm{Gal}(F^*/\mathbb{Q})} g\sigma(x)^{\Phi^*(\sigma)} \\
&= \prod_{\sigma\in\mathrm{Gal}(F^*/\mathbb{Q})} \sigma(x)^{\Phi^*\left(g^{-1}\sigma\right)} \\
&= \mathrm{N}_{\Phi^*}(x),
\end{aligned}$$

as required. ■

At long last, we move towards our monodromy interepretation using the reflex. The following argument generalizes [Yu15, Lemma 4.2].

Lemma 1.95. Fix reflex CM signatures (F,Φ) and (F^*,Φ^*). Suppose that F^* contains F and is Galois over $\mathbb{Q}$. For each $g\in\mathrm{Gal}(F^*/\mathbb{Q})$, the reflex norm $\mathrm{N}_{\Phi^*}\colon \mathrm{T}_{F^*}\to\mathrm{T}_F$ sends the cocharacter $g^\vee\in\mathrm{X}_*(\mathrm{T}_{F^*})$ to

$$\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)(g^\vee)=\sum_{\sigma\in\Sigma_F}\Phi(\sigma)(g\sigma)^\vee.$$

Proof. Notably, N_{Φ^*} outputs to T_F by Lemma 1.94. To begin, we expand

$$\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)(g^\vee)=\sum_{\sigma\in\Sigma_{F^*}}\Phi^*(\sigma)\mathrm{X}_*(\sigma)(g^\vee).$$

We now check $\mathrm{X}_*(\sigma)(g^\vee)=\left(g\sigma^{-1}\right)^\vee$: for any $\tau\in\mathrm{X}^*(\mathrm{T}_{F^*})$, we compute the perfect pairing

$$\langle\tau,\mathrm{X}_*(\sigma)(g^\vee)\rangle=\langle\tau\sigma,g^\vee\rangle,$$

which is the indicator function for $\tau\sigma=g$ and hence equals $\left\langle\cdot,(g\sigma^{-1})^\vee\right\rangle$. We are now able to write

$$\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)(g^\vee)=\sum_{\sigma\in\Sigma_{F^*}}\Phi^*(\sigma)\left(g\sigma^{-1}\right)^\vee.$$

Replacing σ with σ^{-1}, we are done upon recalling $\Phi^*\left(\sigma^{-1}\right)=\Phi(\sigma|_F)$ and collecting terms which together restrict to the same embedding of F. ■

Proposition 1.96. Fix $V \in \mathrm{HS}_{\mathbb{Q}}$ with $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$ such that $Z(\mathrm{End}_{\mathrm{HS}}(V))$ equals a CM algebra $F = F_1 \times \cdots \times F_k$. Let $\Phi \colon \Sigma_F \to \mathbb{Z}_{\geq 0}$ be the induced signature, which we decompose as $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_k$ where $(F_\bullet, \Phi_\bullet)$ is a CM signature for all $F_\bullet$. Suppose F^* is a CM field equipped with CM signatures $\Phi_1^*, \ldots, \Phi_k^*$ such that (F_i, Φ_i) and (F^*, Φ_i^*) are reflex for all i. Then $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$ is the image of

$$(\mathrm{N}_{\Phi_1^*}, \ldots, \mathrm{N}_{\Phi_k^*}) \colon \mathrm{T}_{F^*} \to \mathrm{T}_F.$$

Proof. Note that norms are surjective on these algebraic tori, so Lemma 1.94 tells us that the image of N_{Φ^*} will not change if we pass to an extension of F^*. As such, we will go ahead and assume that F^* contains F and is Galois over $\mathbb{Q}$.

In light of Corollary 1.87, it is enough to show that the image of $\mathrm{X}_*(\mathrm{N}_{\Phi^*})$ (which we note is already a Galois submodule) has saturation equal to the smallest saturated Galois submodule of $\mathrm{X}_*(\mathrm{T}_F)$ containing $\sum_{\sigma \in \Sigma_F} \Phi(\sigma)\sigma^\vee$. This follows from the computation of Lemma 1.95 upon letting g vary over $\mathrm{Gal}(F^*/\mathbb{Q})$. ■

Let's explain how Proposition 1.96 is applied to compute equations cutting out $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$, where $F = F_1 \times \cdots \times F_k$ is a CM algebra. As before, we will only compute over an extension $L = F^*$ of F which is Galois over $\mathbb{Q}$; let $\Phi_1^*, \ldots, \Phi_k^*$ be the signatures on L making (L, Φ_i^*) and (F_i, Φ_i) reflex for each i. Note, we know that $(\mathrm{T}_F)_L \subseteq \mathrm{GL}(V)_L$ may embed as a diagonal torus.

An equation cutting out $Z(\mathrm{MT}(V))_L^\circ$ in the (subtorus of the) diagonal torus $(\mathrm{T}_F)_L \subseteq \mathrm{GL}(V)_L$ then becomes a character of $(\mathrm{T}_F)_L$ which is trivial on $Z(\mathrm{MT}(V))^\circ$. In other words, these equations are given by the kernel of

$$\mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(Z(\mathrm{MT}(V))^\circ).$$

We now use Proposition 1.96. We know that $Z(\mathrm{MT}(V))^\circ \subseteq \mathrm{T}_F$ is the image of $(\mathrm{N}_{\Phi_1^*}, \ldots, \mathrm{N}_{\Phi_k^*}) \colon \mathrm{T}_L \to \mathrm{T}_F$, so the kernel of the above map is the same as the kernel of

$$\mathrm{X}^*\left((\mathrm{N}_{\Phi_1^*}, \ldots, \mathrm{N}_{\Phi_k^*})\right) : \mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(\mathrm{T}_L).$$

To compute this kernel cleanly, note Lemma 1.95 computes $\mathrm{X}_*\left(\mathrm{N}_{\Phi_i^*}\right)$ for each i, so we see $\mathrm{X}^*\left(\mathrm{N}_{\Phi_i^*}\right)$ can be computed as the transpose of the matrix of $\mathrm{X}_*\left(\mathrm{N}_{\Phi^*}\right)$. Attaching these matrices together gives a matrix representation for the above map, and we get our equations by computing the kernel of this matrix.

Remark 1.97. In practice, one can expand $V = V_1 \oplus \cdots \oplus V_k$ into irreducible Hodge substructures and then work with $E := F_1 \times \cdots \times F_k$ where $F_i := Z(\mathrm{End}_{\mathrm{HS}}(V_i))$ for each i. Technically speaking, F may only embed into E "diagonally" because some $V_\bullet$s may be isomorphic to each other. However, this does not really affect anything we do because we may as well work with the image of $Z(\mathrm{MT}(V))^\circ$ under the inclusion $\mathrm{T}_F \subseteq \mathrm{T}_E$. Working with T_E is more convenient because it can actually be identified with the diagonal torus of $\mathrm{GL}(V)_E$ instead of merely a diagonally embedded subtorus.

Chapter 2

Absolute Hodge Classes

2.1 Review of Cohomology

2.2 The Definition

Theorem 2.1 (Principle B)**.**

2.3 A Category of Motives

Chapter 3
Abelian Varieties

In this chapter, we gather together all the results about abelian varieties we need. Many of the results in the earlier sections discussed here can be found in any reasonable text on abelian varieties such as [Mum74; Mil08; EGM]. Results in the later sections are more specialized, and we will provide references when appropriate. Ultimately, our goal is to define ℓ-adic monodromy groups, explain why one might care about them, and indicate how one might compute them.

3.1 Definitions and Constructions

In this section, we set up the theory of abelian varieties rather quickly. We will usually only indicate proofs that work in the complex analytic situation because the general theory usually requires intricate algebraic geometry.

3.1.1 Starting Notions

Let's begin with a definition.

Definition 3.1 (abelian variety)**.** Fix a ground scheme S. An *abelian scheme* A over S is a smooth projective geometrically integral group scheme over S. An *abelian variety* A is an abelian scheme over a field.

Remark 3.2. Throughout, we will work with abelian varieties instead of abelian schemes as much as possible. However, one should be aware that many of the results generalize.

Here, a group variety refers to a group object in the category of varieties over K.

Remark 3.3. With quite a bit of work, one can weaken the hypotheses of being an abelian variety quite significantly. For example, arguments involving group varieties are able to show that being connected and geometrically reduced implies geometrically integral, and it is a theorem that one can replace projectivity with properness. See [SP, Remark 0H2U] for details.

Here are the starting examples.

Example 3.4 (elliptic curves)**.** Any (smooth) cubic equation cuts out a genus-1 curve in $\mathbb{P}^2$. If the curve has points defined over K, this defines an elliptic curve, which can be shown to be an abelian variety. The interesting part comes from defining the group structure. One way to do this is to show that the map $E \to \operatorname{Pic}^0_{E/K}$ given by $x \mapsto [x] - [\infty]$ is an isomorphism of schemes and then give E the group structure induced by $\operatorname{Pic}^0_{E/K}$. (Here, $\operatorname{Pic}^0_{E/K}$ is the moduli space of line bundles over E of degree 0. Smoothness of the curve makes this in bijection with divisors of degree 0.)

Example 3.5. Fix a positive integer $g \geq 0$. If $\Lambda \subseteq \mathbb{C}^g$ is a polarizable sublattice, then $\mathbb{C}^g/\Lambda$ defines an abelian variety over $\mathbb{C}$. Here, polarizable means that there is an alternating map $\varphi\colon \Lambda \times \Lambda \to \mathbb{Z}$ such that the pairing

$$\langle x, y\rangle \coloneqq \psi_{\mathbb{R}}(x, iy)$$

on $\Lambda_{\mathbb{R}}$ is symmetric and positive-definite. (As worked out in [Mil20b, Section I.2], this is equivalent data to a polarization on the Hodge structure $\Lambda = \mathrm{H}_1^{\mathrm{B}}(A, \mathbb{Z})$.) The requirement of polarizability is used to show that the quotient $\mathbb{C}^g/\Lambda$ is actually projective; see [Mum74, Section 3, Theorem].

It is notable that we have not required our abelian varieties A to actually be abelian even though (notably) both examples above are abelian. Indeed, abelian varieties are always abelian groups, which follows from an argument using the Rigidity theorem. We will not give this argument in full because we will not use it, but we state a useful corollary.

Proposition 3.6. Let $\varphi\colon A \to B$ be a smooth map of abelian varieties over a field K. Then φ is the composition of a homomorphism and a translation.

Proof. By composing with a translation, we may assume that $\varphi(0) = 0$. Then one applies the Rigidity theorem to the map $\widetilde{\varphi}\colon A \times A \to B$ defined by

$$\widetilde{\varphi}(a, a') \coloneqq \varphi(a + a') - \varphi(a) - \varphi(a')$$

to find that $\widetilde{\varphi}$ is constantly 0, completing the proof. See [Mil08, Corollary I.1.2] for details. ∎

Corollary 3.7. The group law on an abelian variety A is commutative.

Proof. The inversion map $i\colon A \to A$ on an abelian variety sends the identity to itself, so Proposition 3.6 tells us that i must be a homomorphism. It follows that the group law is commutative. ∎

In particular, we find that morphisms between abelian varieties are rather strutured: we are allowed to basically only ever consider homomorphisms!

It will turn out that considering abelian varieties up to isomorphism is too strong for most purposes, so we introduce the following definition.

Definition 3.8 (isogeny)**.** A morphism $\varphi\colon A \to B$ of abelian varieties over a field K is an *isogeny* if and only if it is a homomorphism satisfying any one of the following equivalent conditions.

(a) φ is surjective with finite kernel.

(b) $\dim A = \dim B$, and φ is surjective.

(c) $\dim A = \dim B$, and φ has finite kernel.

(d) φ is finite, flat, and surjective.

The *degree* of the isogeny is $\#\ker\varphi$ (thought of as a group scheme).

Remark 3.9. Let's briefly indicate why (a)–(d) above are equivalent; see [Mil08, Proposition 7.1] for details. A spreading out argument combined with the homogeneity of abelian varieties implies that

$$\dim B = \dim A + \dim \varphi^{-1}(\{b\})$$

for any b in the image of φ; this gives the equivalence of (a)–(c). Of course (d) implies (a) (one only needs the finiteness and surjectivity); to show (a) implies (d), we note flatness follows by "miracle flatness" because all fibers have equal dimension, and finiteness follows because finite kernel upgrades to quasi-finiteness.

Intuitively, an isogeny is a "squishy isomorphism."

Example 3.10. Any dominant morphism of elliptic curves sending the identity to the identity is an isogeny.

Example 3.11. In the complex analytic setting, an isogeny of two abelian varieties $A = \mathbb{C}^g/\Lambda$ and $B = \mathbb{C}^g/\Lambda'$ amounts (up to change of basis) an inclusion of lattices $\Lambda' \subseteq \Lambda$.

Example 3.12. Fix any abelian variety A. For any nonzero integer n, the multiplication-by-n endomorphism $[n]_A \colon A \to A$ is an isogeny. To see this, note that it is enough to check that $A[n] := \ker[n]_A$ is finite. In the complex analytic situation where $A = \mathbb{C}^g/\Lambda$, this follows because $\frac{1}{n}\Lambda/\Lambda$ is finite; in general, one must show that $A[n] := \ker[n]_A$ is zero-dimensional, which is somewhat tricky. See [SP, Lemma 0BFG] for details. We remark that one can compute $\deg[n]_A = d^{2\dim A}$, which is again not so hard to see in the complex analytic situation.

Motivated by the complex analytic setting (and the "squishy isomorphism" intuition), one might hope that one can recover weak-ish inverses for isogenies. This turns into an important property of abelian varieties.

Lemma 3.13. Fix an isogeny $\varphi \colon A \to B$ of abelian varieties of degree d. Then there exists an "inverse isogeny" $\beta \colon B \to A$ such that

$$\begin{cases} \alpha \circ \beta = [d]_B, \\ \beta \circ \alpha = [d]_A. \end{cases}$$

Proof. By some theory regrading group scheme quotients, it is enough to check that φ factors through $[d]_A$, which holds because $\ker \varphi$ has order d as a group scheme and thus vanishes under $[d]_A$. ■

Remark 3.14. As usual, we remark that the above lemma is easier to see in the complex analytic situation, but the key point of trying to factor through $[d]_A$ remains the same.

Lemma 3.13 motivates the following definition, and it codifies our intuition viewing isogenies as squishy isomorphisms.

Definition 3.15 (isogeny category)**.** Fix a field K. We define the *isogeny category* of abelian varieties over K as having objects which are abelian varieties over K, and a morphism $A \to B$ in the isogeny category is an element of $\operatorname{Hom}_K(A, B)_{\mathbb{Q}}$.

We close our discussion of isogenies with one last remark on the size of kernels.

Remark 3.16. If $\varphi\colon X \to Y$ is a finite separable morphism of varieties, then a spreading out argument shows that the number of geometric points in a general fiber of φ equals the degree of φ. Applied to isogenies, the homogeneity of abelian varieties is able to show that the number of geometric points in the fiber of any separable isogeny equals the degree.

Example 3.17. Here is an application of Remark 3.16: if $\operatorname{char} K \nmid n$, then one can show that $A[n]$ has $n^{2\dim A}$ geometric points. Again, this is not so hard to see in the complex analytic setting. The hypothesis $\operatorname{char} K \nmid n$ is needed to show that $[n]_A$ is separable; in general, the argument is trickier and can (for example) use some intersection theory [Mil08, Theorem I.7.2].

Now that we have a reasonable category, one can ask for decompositions. Here is the relevant result and definition.

Theorem 3.18 (Poincaré reducibility)**.** Fix an abelian subvariety B of an abelian variety A defined over a field K. Then there is another abelian subvariety $B' \subseteq A$ such that the multiplication map induces an isogeny $B \times B' \to A$.

Proof. As usual, we argue only in the complex analytic case. Here write $A = V/\Lambda$ for complex affine space V, and we find that $B = W/(\Lambda \cap W)$ for some subspace $W \subseteq V$. Now, the polarization induces a Hermitian form on V, so we can define $W' := W^\perp$ so that $B' := W'/(\Lambda \cap W')$ will do the trick. For more details, see [Mil20b, Theorem 2.12] for more details. ■

Definition 3.19 (simple)**.** Fix a field K. An abelian variety A over K is *simple* if and only if it is irreducible in the isogeny category.

Remark 3.20. Theorem 3.18 implies that any abelian variety can be decomposed uniquely into a product of simple abelian varieties, of course up to isogeny and permutation of factors.

3.1.2 The Jacobian

In this thesis, the abelian varieties of interest to us will be Jacobians. There are a few approaches to their definition, which we will not show are equivalent, but we refer to [Mil08, Chapter III] for details. The most direct definition is as a moduli space.

Definition 3.21 (Jacobian)**.** Fix a smooth proper curve C over a field K such that $C(K)$ is nonempty. Then the *Jacobian* $\operatorname{Jac} C$ is the group variety $\operatorname{Pic}^0_{C/K}$, where $\operatorname{Pic}^0_{C/K}$ is the moduli space of line bundles on C with degree 0.

Remark 3.22. We will not check that we have defined an abelian variety, nor that we have even defined a scheme. There are interesting questions regarding the representability of moduli spaces, which we are omitting a discussion of. Milne provides a reasonably direct construction in [Mil08, Section III.1], but we should remark that one expects representability to be true in a broader context. In particular, there are formal ways to check (say) properness of $\operatorname{Pic}^0_{C/K}$, from which it does follow that we have defined an abelian variety.

Remark 3.23. One can actually weaken the smoothness assumption on C to merely being "compact type." This is occasionally helpful when dealing with moduli spaces because it allows us to work a little within the boundary of the moduli space of curves.

Remark 3.24. Notably, Example 3.4 tells us that the Jacobian of a curve is E itself.

Note that the assumption $C(K) \neq \varnothing$ allows us to choose some point $\infty \in C(K)$ and then define a map $C(K) \to \operatorname{Jac} C$ by $p \mapsto [p] - [\infty]$. This map turns out to be a regular closed embedding [Mil08, Proposition 2.3]. It is psychologically grounding to see that this map is universal in some sense.

Proposition 3.25. Fix a smooth proper curve C over a field K such that $C(K) \neq \varnothing$. Choose $\infty \in C(K)$, and consider the map $\iota\colon C \to \operatorname{Jac} C$ given by $\iota(p) := [p] - [\infty]$. For any abelian variety A over K and smooth map $\varphi\colon C \to A$ such that $\varphi(\infty) = 0$, there exists a unique map $\widetilde{\varphi}\colon \operatorname{Jac} C \to A$ making the following diagram commute.

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \operatorname{Jac} C \arrow[d, dashed, "\widetilde{\varphi}"] \\ & A \end{tikzcd}$$

Proof. We will not need this, so we won't even point in a direction of a proof. We refer to [Mil08, Proposition III.6.1]. ∎

It is worthwhile to provide a complex analytic construction of the Jacobian. Given a curve C, line bundles are in bijection with divisor classes, and divisor classes of degree 0 can all be written in the form $\sum_{i=1}^k([P_i]-[Q_i])$ for some points $P_1, Q_1, \ldots, P_k, Q_k \in C(\mathbb{C})$. One can take such a divisor and define a linear functional on $\mathrm{H}^1(C, \Omega_C^1)$ by

$$\omega \mapsto \sum_{i=1}^k \int_{Q_i}^{P_i} \omega.$$

The construction of this linear functional is not technically well-defined up to divisor class; instead, one can check that changing the divisor class adjusts the linear functional exactly by the choice of a cycle in $\mathrm{H}_1^{\mathrm{B}}(C, \mathbb{Z})$ embedded into $\mathrm{H}^1(C, \Omega_C^1)^\vee$ via the integration pairing. In this one way, one finds that

$$\operatorname{Jac} C(\mathbb{C}) = \frac{\mathrm{H}^1(C, \Omega_C^1)^\vee}{\mathrm{H}_1^{\mathrm{B}}(C, \mathbb{Z})}.$$

In particular, we have realized $\operatorname{Jac} C$ explicitly as a complex affine space modulo some lattice, exactly as in Example 3.5. (One sees that $\operatorname{rank}_{\mathbb{Z}} \mathrm{H}_1^{\mathrm{B}}(C, \mathbb{Z}) = \dim_{\mathbb{R}} \mathrm{H}^1(C, \Omega_C^1)^\vee$ by the Betti-to-de Rham comparison isomorphism.) This construction makes it apparent that

$$\mathrm{H}_1^{\mathrm{B}}(\operatorname{Jac} C(\mathbb{C}), \mathbb{Z}) \cong \mathrm{H}_1^{\mathrm{B}}(C, \mathbb{Z}).$$

This is in fact a general property.

Proposition 3.26. Fix a smooth proper curve C over a field K such that $C(K) \neq \varnothing$. Choose $\infty \in C(K)$, and consider the map $\iota\colon C \to \operatorname{Jac} C$ given by $\iota(p) := [p] - [\infty]$. Then the induced map

$$\iota^*\colon \mathrm{H}^1(\operatorname{Jac} C) \to \mathrm{H}^1(C)$$

is an isomorphism, where H is any of the Weil cohomology theories of section 2.1.

Proof. The proof requires analyzing each cohomology theory individually. Above we outlined the proof when H is Betti cohomology, and we note that the result follows for de Rham cohomology by the comparison isomorphism. ∎

Corollary 3.27. Fix a smooth proper curve C over a field K such that $C(K) \neq \varnothing$. Then $\dim \operatorname{Jac} C$ equals the genus of the curve C.

Proof. Again, this is easy to see in the complex analytic case from the explicit construction. In general, one can read off the dimension of an abelian variety A from $\dim \mathrm{H}^1(A)$ and then apply Proposition 3.26. ■

3.1.3 The Dual

Even though we will technically not need it, we take a moment to discuss duality and polarizations of abelian varieties; we do want to understand these notions so that we can make sense of the Weil pairing. Motivated by the utility of the Picard group in defining the Jacobian, we make the following definition.

Definition 3.28 (dual abelian variety)**.** Fix an abelian variety A over a field K. Then we define the *dual abelian variety* $A^\vee$ as the group scheme $\operatorname{Pic}^\circ_{A/K}$ over K.

Remark 3.29. As usual, we will not check that $A^\vee$ is an abelian variety or even a scheme, but it is. (The ingredients that go into these arguments will not be relevant for us.) We refer to [EGM, Chapter 6] for these arguments, in addition to the useful fact that $\dim A = \dim A^\vee$.

Remark 3.30. It is worthwhile to note that, in the complex analytic situation, there already is a notion of a dual abelian variety. If $A = V/\Lambda$ is an abelian variety, then $A^\vee = V^*/\Lambda^*$, where V^* is the vector space of conjugation-semilinear functionals $V^* \to \mathbb{C}$, and Λ^* consists of the functionals which are integral on Λ. It is rather tricky to explain how this definition relates to the one above, so we will not do so and instead refer to [Ros86, Section 4].

It is worth our time to explain why this is called duality. To begin, there is a duality for morphisms.

Lemma 3.31. Fix a homomorphism $f\colon A \to B$ of abelian varieties over a field K. Then there is a dual homomorphism $f^\vee\colon B^\vee \to A^\vee$.

Proof. We define the homomorphism on geometric points. Then a point of $B^\vee(\overline{K})$ is a line bundle $\mathcal{L}$ on $B_{\overline{K}}$, which we can pull back to a line bundle $f^*\mathcal{L}$ on $A_{\overline{K}}$, which is a point of $A^\vee(\overline{K})$. ■

Lemma 3.32. Fix an abelian variety A over a field K. Then there is a canonical isomorphism $A \to A^{\vee\vee}$.

Proof. We sketch the construction of the map and refer to [EGM, Theorem 7.9] for details. Because $A^\vee$ is a moduli space of line bundles, there is a universal Poincaré line bundle $\mathcal{P}_A$ on $A \times A^\vee$. Unravelling the definition of $A^\vee$, we see that morphisms $S \to A^\vee$ correspond to line bundles on $A \times S$. Turning this around, we thus see that we can view $\mathcal{P}_A$ as a family of line bundles on $A^\vee$ parameterized by A and thus providing a map $A \to A^{\vee\vee}$. This map is the required isomorphism. ■

Most of the utility one achieves from the dual is that it allows us to the complex-analytic notion of a polarization into algebraic geometry. As in Remark 3.30, we view $A = V/\Lambda$ as a complex torus, and the dual abelian variety $A^\vee$ can be realized concretely as some V^*/Λ^*. Now, a polarization of A refers to a polarization of $\Lambda = \mathrm{H}_1^{\mathrm{B}}(A, \mathbb{Z})$, which as mentioned in Example 3.5 has equivalent data to an alternating form $\psi\colon \Lambda \otimes \Lambda \to \mathbb{Z}$ such that the bilinear form

$$\langle x, y\rangle := \psi_{\mathbb{R}}(x, iy)$$

on $\Lambda_{\mathbb{R}}$ is symmetric and positive-definite. But now we see that this choice of ψ determines a map $A \to A^\vee$ given by $v \mapsto \psi(v, \cdot)$.

Thus, we would like our polarizations some kind of map $A \to A^\vee$. However, we need to keep track of all the adjectives that ψ had in order to make this an honest definition. For example, perhaps we want to keep track of the constraint that ψ is alternating. To do so, we use cohomology. We will shortly explain in Proposition 3.59 that the cup product provides an isomorphism $\wedge^2\mathrm{H}^1(A,\mathbb{Z}) \to \mathrm{H}^2(A,\mathbb{Z})$, which induces an isomorphism

$$\operatorname{Hom}_{\mathbb{Z}}\left(\wedge^2\Lambda,\mathbb{Z}\right) \cong \mathrm{H}^2(A,\mathbb{Z})$$

upon taking duals. Thus, ψ being an alternating form can be traced backed to it coming from a class in $\mathrm{H}^2(A,\mathbb{Z})$.

Continuing, perhaps we want to keep track of the constaint that $\langle\cdot,\cdot\rangle$ is symmetric. This is equivalent to having $\psi_{\mathbb{R}}(ix,iy) = \psi(x,y)$, which turns out to be equivalent to $\psi_{\mathbb{C}} \in \mathrm{H}^2(A,\mathbb{C})$ living in the $(1,1)$ component. Well, it turns out that the exponential short exact sequence

$$0 \to \mathbb{Z} \overset{2\pi i}{\to} \mathcal{O}_A \overset{\exp}{\to} \mathcal{O}_A^\times \to 0$$

induces a (first Chern class) map $c_1 \colon \mathrm{H}^1(A,\mathcal{O}_A^\times) \to \mathrm{H}^2(A,\mathbb{Z})$, which is an isomorphism onto the $(1,1)$ component. Thus, the condition that $\langle\cdot,\cdot\rangle$ is symmetric can be traced back to $\psi_{\mathbb{C}}$ coming from a class in $\mathrm{H}^1(A,\mathcal{O}_A^\times)$, which has equivalent data to a line bundle $\mathcal{L}$.

Lastly, it turns out that positive-definiteness of $\langle\cdot,\cdot\rangle$ corresponds to the $\mathcal{L}$ being ample. On the other hand, given a line bundle $\mathcal{L}$ on A, we remark that there already is a natural way to construct a map $A \to A^\vee$ from a line bundle. This gives our definition.

Definition 3.33 (polariaztion)**.** Fix an abelian variety A over a field K. A *polarization* is a morphism $\varphi\colon A \to A^\vee$ such that there is an ample line bundle $\mathcal{L}$ on $A_{\overline{K}}$ giving the equality

$$\varphi(x) = t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

for any $x \in A_{\overline{K}}$. We say that φ is *principal* if and only if it is an isomorphism, and we say that A is a *pricipally polarized*.

Remark 3.34. It turns out that the construction of the above map does correspond to the map $A \to A^\vee$ defined complex-analytically.

Remark 3.35. It turns out that polarizations are isogenies.

Remark 3.36. Here is the sort of thing that one can do with this definition. One may also want to define a Rosati involution on $\operatorname{End}(A)_{\mathbb{Q}}$, analogous to the Rosati involution on polarized Hodge structures. Well, given a (principal) polarization $\varphi\colon A \to A^\vee$, we can define a Rosati involution $(\cdot)^\dagger$ on $\operatorname{End}(A)_{\mathbb{Q}}$ by sending any $f \in \operatorname{End}(A)_{\mathbb{Q}}$ to

$$f^\dagger := \varphi^{-1} \circ f^\vee \circ \varphi.$$

If λ is a principal polarization, then this Rosati involution descends to $\operatorname{End}(A)$. One can check that $(\cdot)^\dagger$ continues to be a positive anti-involution, but it is not easy; see for example [EGM, Theorem 12.26]. This allows us to apply the Albert classification Theorem 1.28 to our situation.

Example 3.37. For any smooth proper curve C such that $C(K) \neq \varnothing$, it turns out that the Jacobian $\operatorname{Jac} C$ is principally polarized. It is not too hard to describe the line bundle which gives the polarization: let $\iota\colon C \to \operatorname{Jac}(C)$ be an embedding given be one of the points in $C(K)$, and then the line bundle is given by the divisor

$$\underbrace{f(C) + \cdots + f(C)}_{g-1},$$

where g is the genus of C. See [EGM, Theorem 14.23] or [Mil08, Theorem 6.6] for more details.

Analogous to the complex-analytic setting $A = V/\Lambda$, we may still want to be able to define an alternating form on $\Lambda = \mathrm{H}_1^{\mathrm{B}}(A, \mathbb{Z})$. We will achieve a satisfying version of this in Lemma 3.64, but for now, let us point that this is not immediately obvious how to do this because we currently have no analogue for Λ in the general setting. However, we note that the alternating form Λ is able to induce an alternating form on V, and we can access a dense subset of V by taking torsion. Thus, for now, we will aim to provide a pairing

$$A[n](K^{\mathrm{sep}}) \times A[n](K^{\mathrm{sep}}) \to \mathbb{Z}/n\mathbb{Z}$$

for each integer n such that $\operatorname{char} K \nmid n$. Unwinding how we took a polarization to a map $A \to A^\vee$, we note that we may as well define the above map using a polarization $\varphi\colon A \to A^\vee$ by instead defining a pairing

$$A[n](K^{\mathrm{sep}}) \times A^\vee[n](K^{\mathrm{sep}}) \to \mathbb{Z}/n\mathbb{Z}$$

and then pre-composing with $A \to A^\vee$. More generally, given an isogeny $f\colon A \to B$, we will be able to show that there is a perfect pairing

$$(\ker f) \times (\ker f^\vee) \to \mathbb{G}_m,$$

upon which we find the desired pairing by taking $f = [n]_A$ and taking K^{sep}-points.

Proposition 3.38 (Weil pairing)**.** Fix an isogeny $f\colon A \to B$ of abelian varieties over K. Then there is a perfect pairing

$$(\ker f) \times (\ker f^\vee) \to \mathbb{G}_m.$$

Proof. We provide an explicit construction of the pairing on K^{sep}-points, but we will not check that it is perfect, for which we refer to [Ton15, Theorem 8.1.3]. Select $x \in (\ker f)(K^{\mathrm{sep}})$ and $y^\vee \in (\ker f^\vee)(K^{\mathrm{sep}})$. The point $y^\vee$ corresponds to a line bundle $\mathcal{L}$ on $B^\vee_{K^{\mathrm{sep}}}$. Being in the kernel of f grants a trivialization $\sigma\colon f^*\mathcal{L} \to \mathcal{O}_{A_{K^{\mathrm{sep}}}}$, which is unique up to a scalar. Now, note that $t_a^* f^*\mathcal{L} = f^* t_{f(a)}^*\mathcal{L} = f^*\mathcal{L}$ because $a \in \ker f$, so there is another trivialization of $f^*\mathcal{L}$ given by $t_a^*\beta\colon \mathcal{L} \to \mathcal{O}_{A_{K^{\mathrm{sep}}}}$. We now define our Weil pairing as $t_a^*\beta \circ \beta^{-1}$, which we realize as an element of $\mathbb{G}_m(K^{\mathrm{sep}})$ by noting that $t_a^*\beta\circ\beta^{-1}$ is an automorphism of $\mathcal{O}_{A_{K^{\mathrm{sep}}}}$ and is therefore a scalar. ∎

Corollary 3.39. Fix an abelian variety A over a field K, and let $\varphi\colon A \to A^\vee$. For each positive integer n, there is a Galois-invariant perfect symplectic pairing

$$e_\varphi\colon A[n](K^{\mathrm{sep}}) \times A[n](K^{\mathrm{sep}}) \to \mu_n.$$

Furthermore, for any positive integer m, the following diagram commutes.

$$\begin{tikzcd} A[nm](K^{\mathrm{sep}}) \arrow[d, "m"'] & \times & A[nm](K^{\mathrm{sep}}) \arrow[r, "e_\varphi"] \arrow[d, "m"'] & \mu_{mn} \arrow[d, "m"] \\ A[n](K^{\mathrm{sep}}) & \times & A[n](K^{\mathrm{sep}}) \arrow[r, "e_\varphi"] & \mu_n \end{tikzcd}$$

Proof. We described above how to construct the pairing from the one given in Proposition 3.38 by setting $f = [n]_A$ and then using the polarization φ. The remaining properties of e_φ (such as Galois-invariance) can be checked using the explicit construction given in Proposition 3.38. ∎

3.1.4 Applying Hodge Theory

We now explain the utility of chapter 1 to our application. Here is the main result.

Theorem 3.40 (Riemann)**.** The functor $A\mapsto \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$ provides an equivalence of categories between the isogeny category of abelian varieties defined over $\mathbb{C}$ and the category of polarizable $\mathbb{Q}$-Hodge structures V such that $V_{\mathbb{C}}=V^{0,1}\oplus V^{1,0}$.

Proof. Writing $A=\mathbb{C}^g/\Lambda$ for a polarizable lattice Λ, we see that the given functor takes A to $\Lambda\otimes_{\mathbb{Z}}\mathbb{Q}$. It is thus not hard to see that this functor is fully faithful. To see that it is essentially surjective, we begin with any polarizable $\mathbb{Q}$-Hodge structure V and find a polarizable sublattice Λ in order to produce the desired abelian variety A/Λ. Admittedly, most of the work for this theorem was already done in Example 1.20 when we showed that the previous sentence actually gives an abelian variety! ■

The moral of the story is that we can keep track of abelian varieties A over $\mathbb{C}$ by only keeping track of their Hodge structures $\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$. With this in mind, we allow ourselves the following notation.

Notation 3.41. Fix an abelian variety A over $\mathbb{C}$. Then we define the *Mumford–Tate group* of A to be

$$\mathrm{MT}(A)\coloneqq\mathrm{MT}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right).$$

We define $\mathrm{Hg}(A)$ and $\mathrm{L}(A)$ similarly.

Here is the main corollary of Theorem 3.40 that we will want.

Corollary 3.42. Fix an abelian variety A over $\mathbb{C}$. Then the natural map

$$\mathrm{End}_{\mathbb{C}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathrm{End}_{\mathbb{Q}}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)^{\mathrm{MT}(A)}$$

is an isomorphism.

Proof. By Lemma 1.54, we see that the right-hand side is simply $\mathrm{End}_{\mathrm{HS}}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)$. The result now follows from Theorem 3.40. ■

As another aside, we go ahead and restate the Albert classification (Theorem 1.28) for our abelian varieties.

Proposition 3.43. Fix a simple abelian variety A of dimension g, defined over a field K of characteristic 0, and set $D\coloneqq\mathrm{End}_K(A)_{\mathbb{Q}}$ and $F\coloneqq Z(D)$. Letting $(\cdot)^\dagger$ be the Rosati involution on D, we also let $F^\dagger$ be the $(\cdot)^\dagger$-invariants of F. Further, set $d\coloneqq\sqrt{[D:F]}$ and $e\coloneqq[F:\mathbb{Q}]$ and $e_0\coloneqq[F^\dagger:\mathbb{Q}]$. Then we have the following table of restrictions on (g,d,e,e_0).

Type	e	d	Restriction
I	e_0	1	$e\mid g$
II	e_0	2	$2e\mid g$
III	e_0	2	$2e\mid g$
IV	$2e_0$	d	$e_0d^2\mid g$

Proof. Recall that D is amenable to the Albert classification as discussed in Remark 3.36. The middle two columns follow from the discussion of the various types; for example, in Type I, we see $d=1$ because $D=F$, and $e=e_0$ because F is totally real. To receive the dimension restrictions, we note that some descent argument allows us to reduce to the case where $K=\mathbb{C}$, where we receive an inclusion $D\subseteq\mathrm{End}(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q}))$ by Theorem 3.40.[1] This is an inclusion of division $\mathbb{Q}$-algebras, so we see that $\dim_{\mathbb{Q}}D\mid 2g$; this implies

$$d^2e\mid 2g,$$

[1] It is still possible to get an inclusion like this in general. It requires a discussion of the ℓ-adic representations, which we engage in later.

which rearranges into the required restrictions. ∎

Remark 3.44. The requirement that $\operatorname{char} F = 0$ is necessary in the table; the restrictions are somewhat different (and weaker!) in positive characteristic.

While we're here, we state the main theorem of [Del18] on absolutely Hodge cycles.

Theorem 3.45 (Deligne)**.** Fix an abelian vareity A defined over a number field K. Then all Hodge classes on A are absolutely Hodge.

We will not attempt a proof of this result, but we will remark that Theorem 2.1 allows us to reduce this result to the case of abelian varieties with many endomorphisms, which is more amenable. There is still much work to be done!

3.1.5 Complex Multiplication

Even though it is not strictly necessary for our exposition, we take a moment to discuss some theory surrounding complex multiplication. We refer to [Mil20b] throughout for more details. The relevance of this discussion to us mostly arises because we have defined analogous notions in sections 1.4.2 and 1.4.3.

Intuitively, complex multiplication simply means that an abelian variety has many endomorphisms. To explain this properly, we note that the endomorphism algebra of a simple abelian variety A is a division $\mathbb{Q}$-algebra described in Proposition 3.43; if we drop the assumption that A is simple, then it could be a product of matrix algebras of such division $\mathbb{Q}$-algebras. This motivates the following definition to properly account for such matrix algebras.

Definition 3.46 (reduced degree)**.** Write a semisimple algebra D over a field K as a product $D_1 \times \cdots \times D_k$ of simple algebras. Then we define the *reduced degree* as

$$[D : K]_{\text{red}} := \sum_{i=1}^{k} \sqrt{[D_i : F_i]} \cdot [D_i : K],$$

where $F_i := Z(B_i)$ for each i

Remark 3.47. It is not technically obvious that $[D_i : F_i]$ is a square, but this follows from the theory of central simple algebras. Roughly speaking, one can show that $D_i \otimes \overline{D_i} \cong M_n(\overline{D_i})$ for some $n \geq 0$, from which the result follows; see [Mil20a, Corollary IV.2.16].

Remark 3.48. Given an inclusion $B \subseteq \operatorname{End}_K(V)$, one receives a bound

$$[B : K]_{\text{red}} \leq [V : K].$$

Roughly speaking, this follows by breaking up B into simple pieces (which are matrix algebras of division algebras) and then looking for these pieces in $\operatorname{End}_K(V)$. See [Mil20b, Proposition I.1.2]

In light of the previous remark, we are now able to make a definition.

Definition 3.49 (complex multiplication)**.** Fix an abelian variety A over a field K. Then A has *complex multiplication over K* if and only if

$$[\operatorname{End}_K(A)_{\mathbb{Q}} : \mathbb{Q}]_{\text{red}} = 2 \dim A.$$

Namely, we see that $2\dim A$ is as large as possible by Remark 3.48, by taking V to be H^1 for some Weil cohomology H.[2]

Remark 3.50. The key benefit of the reduced degree is that it is additive: given abelian varieties A and A', we claim

$$[\operatorname{End}(A\oplus A')_{\mathbb{Q}}:\mathbb{Q}]_{\mathrm{red}}\stackrel{?}{=}[\operatorname{End}(A)_{\mathbb{Q}}:\mathbb{Q}]_{\mathrm{red}}+[\operatorname{End}(A')_{\mathbb{Q}}:\mathbb{Q}]_{\mathrm{red}}.$$

Indeed, by breaking everything into simple pieces, we may assume that A and A' are both powers of a simple abelian variety. If they are powers of different simple abelian varieties, then this is a direct computation. Otherwise, they are powers of the same simple abelian variety, in which case all central simple algebras in sight are matrix algebras over the same division algebra, and the result follows by another computation.

Remark 3.51. A computation with Proposition 3.43 shows that a simple abelian variety A has complex multiplication only in Type IV when $d=1$; i.e., we require $\operatorname{End}_K(A)$ to be a CM field. Combining this with Remark 3.50, we find that an abelian variety A has complex multiplication if and only if each of its factors does.

Remark 3.52. If an abelian variety A with complex multiplication is a sum of non-isomorphic simple abelian varieties, then its endomorphism algebra is simply a product of CM fields. In general, one can show that it is still the case that any abelian variety A with complex multiplication has a CM algebra of degree $2\dim A$ contained in its endomorphism algebra. However, this requires a little structure theory of semisimple algebras; see [Mil20b, Proposition 3.6].

Complex multiplication places strong constraints on the Mumford–Tate group.

Proposition 3.53. Fix an abelian variety A over $\mathbb{C}$. Then A has complex multiplication if and only if $\operatorname{MT}(A)$ is a torus.

Proof. We show the two implications separately.

- In one direction, if A has complex multiplication, then Remark 3.52 grants a CM algebra $E\subseteq\operatorname{End}_{\mathbb{C}}(A)_{\mathbb{Q}}$ with $[E:\mathbb{Q}]=2\dim A$. Then $\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$ is a free module over E of rank 1, so we see that $\mathrm{GL}_F\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)$ is isomorphic to T_F. We conclude by Lemma 1.45.
- In the other direction, suppose $\operatorname{MT}(A)$ is a torus. Find a maximal torus T containing $\operatorname{MT}(A)$. Then Corollary 3.42 tells us that
$$\operatorname{End}_{\mathbb{C}}(A)_{\mathbb{Q}}=\operatorname{End}_{\mathbb{Q}}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)^{\operatorname{MT}(A)},$$
which then contains $\operatorname{End}_{\mathbb{Q}}\left(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\right)^T$. However, the latter is a commutative semisimple $\mathbb{Q}$-algebra of dimension $2g$: it suffices to check this after base-changing to $\mathbb{C}$, whereupon we may identify T with the diagonal torus, from which the claim follows. This completes the proof. ■

One benefit of complex multiplication is that it lets move difficult geometric questions into combinatorial ones. To see this, we need to define the following combinatorial gadget.

Definition 3.54. Fix an abelian variety A with complex multiplication defined over $\mathbb{C}$. Choose a CM algebra $E\subseteq\operatorname{End}_{\mathbb{C}}(A)_{\mathbb{Q}}$ with $\dim E=2\dim A$. Then we define the *CM type* of A to be the CM signature (E,Φ) given by Lemma 1.82. Note that $\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$ is then a one-dimensional E-vector space, so $\operatorname{im}\Phi\subseteq\{0,1\}$, so we can realize Φ as a subset of $\operatorname{Hom}(E,\mathbb{C})$.

[2] Outside the complex-analytic case, it may look like one wants to use the ℓ-adic result Theorem 3.75 over a general field. However, it turns out to be enough to merely achieve the injectivity of the map Theorem 3.75, which is easier.

Remark 3.55. Note that we are not requiring $E = Z(\operatorname{End}_{\mathbb{C}}(A)_{\mathbb{Q}})$, though this is automatically the case when the simple components of A all have multiplicity 1. Of course, there still is a CM signature coming from the case $E = Z(\operatorname{End}_{\mathbb{C}}(A)_{\mathbb{Q}})$.

Remark 3.56. There is a still a way to recover the CM type even when A is not defined over $\mathbb{C}$. For example, one can note that H^{10} is supposed to be the Lie algebra $\operatorname{Lie} A$, so one can instead recover Φ from the E-action on $\operatorname{Lie} A$.

Remark 3.57. One can read the simplicity of A off of the CM type (E, Φ). To begin, one needs E to be a field for A to be simple. Now that E is a field, we know that $A \sim B^r$ where B is an abelian variety with complx multiplication; say that it has CM type (E', Φ'). Then the Hodge structure on A is determined by the Hodge structure on B. Tracking this through as in [Lan11, Theorem 3.6] shows that A is simple if and only if any Galois extension $L/\mathbb{Q}$ of E has that

$$\{\sigma \in \operatorname{Gal}(L/\mathbb{Q}) : \Phi\sigma = \Phi\} = \operatorname{Gal}(L/E),$$

where Φ is being suitably thought of as an element of $\mathbb{Z}[\operatorname{Hom}(E, L)]$.

Remark 3.58. It turns out that there is (essentially) exactly one abelian variety with CM type (E, Φ), up to isogeny over the algebraic closure. See [Mil20b, Proposition 3.12].

Remark 3.58 tells us that we are basically allowed to only pay attention to the CM type in the theory of complex multiplication.

3.2 The ℓ-Adic Representation

In this subsection, we now define the ℓ-adic representation and give some of its basic properties.

3.2.1 The Construction

A priori, an abelian variety A gives rise to many ℓ-adic Galois representations via each of its cohomology groups $\mathrm{H}^{\bullet}_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)$. However, it turns out that we are allowed to only care about one of them.

Proposition 3.59. Fix an abelian variety A over a field K, and let H be a Weil cohomology theory. Then the cup product defines an isomorphism between the exterior algebra $\wedge \mathrm{H}^1(A)$ and the cohomology ring $\mathrm{H}^*(A)$.

Proof. In the complex analytic case, we proceed as in [Mil20b, Proposition 2.6]. Write $A = \mathbb{C}^g/\Lambda$ for a lattice Λ. Fixing some index p, we will show that the cup product defines an isomorphism

$$\wedge^p \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Z}) \to \mathrm{H}^p_{\mathrm{B}}(A, \mathbb{Z}).$$

Well, we note that A is homeomorphic to $\left(S^1\right)^{2g}$, so the Künneth formula allows us to reduce the question to S^1, where the result is true by a direct computation. In the general case, one notes that the group structure on A induces a Hopf bialgebra structure on both $\wedge \mathrm{H}^1(A)$ and $\mathrm{H}^*(A)$; then one can appeal to some structure theory to deduce the equality. See [EGM, Corollary 6.13] or more precisely [EGM, Corollary 13.32]. ■

Thus, in ℓ-adic cohomology (where $\operatorname{char} K \nmid \ell$ in K), we see that one can understand all cohomology groups of A by merely understanding $\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Z}_\ell)$. Analogous to the complex analytic case, we will be able to work with the dual "homology group" more concretely.

Let's spend some time giving a more elementary description of $\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Z}_\ell)^\vee$. We refer to [EGM, Corollary 10.38] and the surrounding discussion for more details. We will do this by passing to the fundamental group. In particular, note that there is a Galois-invariant isomorphism

$$\mathrm{H}^1(A_{K^{\text{sep}}}, \mathbb{Z}_\ell) \cong \operatorname{Hom}\left(\pi_1(A_{K^{\text{sep}}}, a), \mathbb{Z}_\ell\right),$$

where $a \in A(K^{\text{sep}})$ is some basepoint. We will go ahead and choose $a = 0$.

Remark 3.60. Let's take a moment to explain this isomorphism. By taking limits, it is enough to show this isomorphism with $\mathbb{Z}_\ell$ replaced by μ_n where $\operatorname{char} K \nmid n$. Then one knows that $\mathrm{H}^1(A_{K^{\text{sep}}}, \mu_n)$ is in bijection with Galois coverings with Galois group μ_n by using the short exact sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1.$$

This completes the proof upon unravelling the definition of π_1 on the right-hand side.

We now use the fact that A is an abelian variety to compute $\pi_1(A_{K^{\text{sep}}}, 0)$: one can show that any étale covering of A is still an abelian variety and hence is an isogeny onto A (for suitable choice of group law). Thus, Lemma 3.13 promises that the multiplication-by-n maps $[n]_A \colon A \to A$ provide a cofinal sequence of Galois étale coverings of A (at least when $\operatorname{char} K \nmid n$), allowing us to compute that the ℓ-part of $\pi_1(A_{K^{\text{sep}}}, 0)$ equals

$$\varprojlim A\left[\ell^\bullet\right](K^{\text{sep}}).$$

In conclusion, we see that $\mathrm{H}^1(A_{K^{\text{sep}}}, \mathbb{Z}_\ell)$ is naturally isomorphic to

$$\left(\varprojlim A\left[\ell^\bullet\right](K^{\text{sep}})\right)^\vee$$

as Galois representations. We are now allowed to define the Tate module.

Definition 3.61 (Tate module)**.** Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Then we define the ℓ-adic Tate module as

$$T_\ell A := \varprojlim A\left[\ell^\bullet\right](K^{\text{sep}}),$$

and we define the rational ℓ-adic Tate module as $V_\ell A := T_\ell A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 3.62. Intuitively, $T_\ell A$ should be thought of as an ℓ-adic stand-in for $\mathrm{H}_1(A)$.

The discussion above suggesets that $T_\ell A$ should be a free $\mathbb{Z}_\ell$-module of rank 2. Let's check this directly. By taking limits, it is enough to show the following.

Lemma 3.63. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. For each $\nu \geq 0$, there is a group isomorphism

$$A\left[\ell^\nu\right](K^{\text{sep}}) \cong \mathbb{Z}/\ell^{2\nu \dim A}\mathbb{Z}.$$

Proof. The two groups have the same size by Example 3.17, so the result follows for $\nu \in \{0, 1\}$ automatically. For $\nu \geq 2$, we induct using the short exact sequence

$$0 \to A[\ell](K^{\text{sep}}) \to A\left[\ell^{\nu+1}\right](K^{\text{sep}}) \xrightarrow{\ell} A\left[\ell^\nu\right](K^{\text{sep}}) \to 0$$

and some cardinality arguments. For example, one can finish by applying the classification of finite abelian groups. ■

One benefit of a more concrete object is that it is easier to work with directly. For example, we can now find a perfect pairing on $\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Z}_\ell)$.

Lemma 3.64. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Choose a polarization $\varphi\colon A \to A^\vee$. Then the Weil pairing induces a Galois-invariant perfect symplectic pairing

$$e_\varphi\colon \mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell(-1).$$

Proof. By taking duals, it is enough to induce a Galois-invariant perfect symplectic pairing

$$e_\varphi\colon T_\ell A \otimes_{\mathbb{Q}_\ell} T_\ell A \to \mathbb{Z}_\ell(1).$$

This follows by taking a limit of the Weil pairing given in Corollary 3.39. Recall that $\mathbb{Z}_\ell(1)$ is the Galois representation $\varprojlim \mu_{\ell^\bullet}$. ■

One can also see the Galois action more explicitly: being careful about the Galois action on cohomology and the Tate module, we see that the induced Galois representation

$$\rho_\ell\colon \ \mathrm{Gal}(K^{\text{sep}}/K) \to \mathrm{GL}(T_\ell A)$$

is simply given by the Galois action on the points in the limit $A\left[\ell^\bullet\right](K^{\text{sep}})$.

3.2.2 The ℓ-Adic Monodromy Group

Now that we have a representation, we may as well define a monodromy group.

Definition 3.65 (ℓ-adic monodromy group)**.** Fix an abelian A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Then the *$\ell$-adic monodromy group* $G_\ell(A)$ is the smallest algebraic $\mathbb{Q}_\ell$-group containing the image of the Galois representation

$$\mathrm{Gal}(K^{\text{sep}}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)\right).$$

Remark 3.66. By taking duals, we see that one produces an isomorphic Galois representation by working with $T_\ell A$ instead. Note that this dual is not very expensive: by using the Weil pairing of Lemma 3.64, we can remove the dual in exchange for a twist, writing

$$\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Z}_\ell) \cong T_\ell A(-1).$$

Remark 3.67. Unlike $\mathrm{MT}(V)$ and $\mathrm{Hg}(V)$, we do not expect $G_\ell(A)$ to be connected in general. However, being an algebraic $\mathbb{Q}_\ell$-group, it will only have finitely many connected components. Thus, we see that the pre-image of $G_\ell(A)^\circ$ in $\mathrm{Gal}(K^{\text{sep}}/K)$ is an open subgroup of finite index, so there is a unique minimal field extension K_A^{conn}/K such that $G_\ell(A_{K_A^{\text{conn}}}) = G_\ell(A)^\circ$. Thus, our group becomes connected, only at the cost of a field extension.

The interesting geometric objects arising from Hodge theory were the Hodge classes, which Remark 1.13 explains were exactly the vectors fixed by the group action. Analagously, we pick up the following definition.

Definition 3.68 (Tate class)**.** Fix an abelian A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Then a *Tate class* is a vector of some tensor construction

$$\bigoplus_{i=1}^k \mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)^{\otimes n_i} \otimes \mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)^{\vee \otimes m_i}(p_i),$$

where the $n_\bullet$s, $m_\bullet$s, and $p_\bullet$s are some nonnegative integers, fixed by the action of $\mathrm{Gal}(K^{\text{sep}}/K)$

Remark 3.69. We remark as in Corollary 1.34 that a vector v as above is a Tate class if and only if it is fixed by the indcued action by $G_\ell(A)$. Indeed, the subset of $\mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)\right)$ fixing v is some algebraic $\mathbb{Q}_\ell$-subgroup, so if it contains the image of $\mathrm{Gal}\left(K^{\text{sep}}/K\right)$, then it contains $G_\ell(A)$. We also take a moment to note that Proposition 1.35 explains that one can now cut out $G_\ell(A)$ by requiring it to hold all the Tate classes invariant, as discussed in Corollary 1.36.

Analogous to Conjecture 1.15, one has a Tate class, which we will only state for abelian varieties.

Conjecture 3.70 (Tate). Fix an abelian variety A over a number field K, and fix a prime number ℓ. Then any Tate class can be written as a $\mathbb{Q}_\ell$-linear combination of classes arising from algebraic subvarieties of powers of A.

Remark 3.71. Of course, there are Tate classes and there is a Tate conjecture for more general varieties.

We conclude this section with a few bounds on the ℓ-adic monodromy group, analogous to the discussion for Mumford–Tate groups in section 1.3.1. Let's begin with endomorphisms.

Lemma 3.72. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Set $D := \operatorname{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$G_\ell(A) \subseteq \left\{g \in \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)\right) : g \circ d = d \circ g \text{ for all } d \in D\right\}.$$

Proof. We proceed as in Lemma 1.45. The right-hand group is an algebraic $\mathbb{Q}_\ell$-group, so it suffices to check that it contains the image of $\mathrm{Gal}(K^{\text{sep}}/K)$. Well, for any $g \in \mathrm{Gal}(K^{\text{sep}}/K)$, we see that

$$g \circ d = d \circ g$$

is an equality which holds on the level of endomorphisms of A because d is defined over K (which g fixes). ■

Lemma 3.73. Fix an abelian variety A over a field K, and suppose ℓ is a prime such that $\operatorname{char} K \nmid \ell$. Choose a polarization $\varphi\colon A \to A^\vee$. Then there is a perfect symplectic pairing e_φ such that

$$G_\ell(A) \subseteq \left\{g \in \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{K^{\text{sep}}}, \mathbb{Q}_\ell)\right) : e_\varphi(gv \otimes gw) = \lambda(g) e_\varphi(v \otimes w) \text{ for fixed } \lambda(g) \in \mathbb{Q}_\ell\right\}.$$

Proof. We proceed as in Lemma 1.47. The right-hand group is an algebraic $\mathbb{Q}_\ell$-group, so it suffices to check that it contains the image of $\mathrm{Gal}(K^{\text{sep}}/K)$. Well, for any $g \in \mathrm{Gal}(K^{\text{sep}}/K)$, we see that

$$e_\varphi(gv \otimes gw) = g e_\varphi(v \otimes w)$$

by the Galois-invariance of Lemma 3.64. Now, we note that $\mathrm{Gal}(K^{\text{sep}}/K)$ acts on $\mathbb{Q}_\ell(-1)$ through the cyclotomic character, so the right-hand side equals a scalar $\lambda(g)$ times $e_\varphi(v \otimes w)$, so we are done. ■

Remark 3.74. There are of course alternate proofs of Lemmas 3.72 and 3.73 by finding Tate classes and then appealing to Remark 3.69. One uses the same classes constructed in the alternate proofs of Lemmas 1.45 and 1.47.

Lastly, we would like to recover the bound of Corollary 3.42 on endomorphisms, sharpening Lemma 3.72. However, the proof is not so easy: the proof of Corollary 3.42 had to translate endomorphisms of the Hodge structure back to endomorphisms of the abelian variety via Theorem 3.40. Recovering the equivalence of Theorem 3.40 is rather difficult: this result is due to Faltings [Fal86, Theorem 3], in his proof of Mordell's conjecture.

Theorem 3.75 (Faltings)**.** Fix an abelian variety A over a number field K, and suppose ℓ is a prime. Then the induced map

$$\operatorname{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \operatorname{End}_{\operatorname{Gal}(\overline{K}/K)}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$$

is an isomorphism.

We will definitely not attempt to summarize a proof here, but we will remark that it is not even totally obvious that this map is injective! Speaking from experience, this makes for a reasonable topic for a final term paper in a first course in algebraic geometry.

Remark 3.76. Via the isomorphism

$$\operatorname{End}_{\mathbb{Q}_\ell}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right) \cong \mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell) \otimes \mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)^\vee,$$

we see that Theorem 3.75 can be viewed as asserting that all the Tate classes in the above space arise from endomorphisms of A. This verifies Conjecture 3.70.

Remark 3.77. We have snuck in the hypothesis that K is a number field into the statement of Theorem 3.75. It is also true for finite fields, where it is due to Tate [Tat66]. However, it is not expected to be true in general!

We are now able to provide a satisfying analogue to Lemma 1.54.

Corollary 3.78. Fix an abelian variety A over a number field K, and suppose ℓ is a prime. Then the natural map

$$\operatorname{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \operatorname{End}_{G_\ell(A)}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$$

is an isomorphism.

Proof. Remark 3.76 explains that the endomorphisms of A are exactly the Tate classes, so the result follows from the discussion in Remark 3.69. ■

Remark 3.79. The above corollary allows us to prove the following analogue of Proposition 3.53 (by the same proof!): A has CM defined over a number field K if and only if $G_\ell(A)$ is a torus.

While we're here, we remark enough property of $G_\ell(A)$ due to Faltings.

Theorem 3.80 (Faltings)**.** Fix an abelian variety A over a number field K, and suppose ℓ is a prime. Then $G_\ell(A)$ is reductive.

Proof. By [Mil17, Corollary 19.18], it is enough to find a faithful semisimple representation of $G_\ell(A)$. As in Lemma 1.44, we see that the inclusion

$$G_\ell(A) \subseteq \operatorname{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$$

is semisimple by [Fal86, Theorem 3], so we are done. ■

Remark 3.81. Over finite fields, Tate [Tat66] has proven that the Galois representation $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ is semisimple. Because the Galois group is (topologically) generated by the Frobenius, this amounts to checking that the endomorphism Frob_q has semisimple action.

To finish up our discussion of computational tools for $G_\ell(A)$, we repeat the results Lemmas 1.56 and 1.59 for our new context. Their proofs are exactly the same, replacing $\mathbb{U}$ (or $\mathbb{S}$) with $\operatorname{Gal}(\overline{F}/F)$ and then making the same minimality arguments for our monodromy groups.

Lemma 3.82. Fix abelian varieties $A_1, \ldots, A_k$ over a field F.

(a) The subgroup
$$G_\ell(A_1 \times \cdots \times A_k) \subseteq \mathrm{GL}(\mathrm{H}^1_{\text{ét}}((A_1 \times \cdots \times A_k)_{F^{\text{sep}}}, \mathbb{Q}_\ell))$$
is contained in $G_\ell(A_1) \times \cdots \times G_\ell(A_k)$.

(b) For each i, the projection map $\mathrm{pr}_i \colon G_\ell(A_1 \times \cdots \times A_k) \to G_\ell(A_i)$ is surjective.

Lemma 3.83. Fix abelian varieties $A_1, \ldots, A_k$ over a field F, and let $m_1, \ldots, m_k \geq 1$ be positive integers. Then the diagonal embeddings $\Delta_i \colon \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{i,F^{\text{sep}}}, \mathbb{Q}_\ell)\right) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A^{m_i}_{i,F^{\text{sep}}}, \mathbb{Q}_\ell)\right)$ induce an isomorphism
$$G_\ell(A_1 \times \cdots \times A_k) \to G_\ell\left(A_1^{m_1} \times \cdots \times A_k^{m_k}\right).$$

3.2.3 The Mumford–Tate Conjecture

Over the next two subsections, we will explain some tools used to compute $G_\ell(A)$. In this subsection, we will discuss $G_\ell(A)^\circ$. Suppose that A is defined a number field K.

A motivic perspective would have us hope that all the monodromy groups attached to A are essentially the same. However, as explained in Remark 3.67, we only expect $G_\ell(A)$ to be connected after an extension K. Thus, for example, one can only hope that $\mathrm{MT}(A)$ knows about $G_\ell(A)^\circ$. We may now state the following conjecture.

Conjecture 3.84 (Mumford–Tate). Fix an abelian variety A over a number field K. For all primes ℓ, we have
$$\mathrm{MT}(A)_{\mathbb{Q}_\ell} = G_\ell(A)^\circ$$
as subgroups of $\mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A, \mathbb{Q}_\ell)\right)$. Here, $\mathrm{MT}(A)$ is embedded into this group by the Betti-to-étale comparison isomorphism.

Our work in chapter 1 provides many tools for computing $\mathrm{MT}(A)$, so Conjecture 3.84 would allow us to translate this knowledge into a computation of $G_\ell(A)^\circ$.

Even though Conjecture 3.84 is not fully proven, there is a lot known. Let's review a small amount. For example, both groups are reductive by Lemma 1.44 and Theorem 3.80. Additionally, Theorem 3.75 provides a suitable analogue of Theorem 3.40, telling us that both groups $\mathrm{MT}(A)$ and $G_\ell(A)$ cut out endomorphisms in $\mathrm{End}(A)$.

Continuing, one inclusion of Conjecture 3.84 is known, due to Deligne [Del18, Corollary 6.2].

Theorem 3.85 (Deligne). Fix an abelian variety A over a number field K. For all primes ℓ, we have
$$G_\ell(A)^\circ \subseteq \mathrm{MT}(A)_{\mathbb{Q}_\ell}.$$

In particular, it becomes enough to compare numerical invariants of the two groups (such as rank) to argue for an equality. For example, the following independence result is due to Larsen and Pink [LP95, Theorem 4.3].

Theorem 3.86 (Larsen–Pink). Fix an abelian variety A over a number field K. If $\mathrm{MT}(A)_{\mathbb{Q}_\ell} = G_\ell(A)^\circ$ holds for any prime ℓ, then it holds for all primes ℓ.

One even knows that the centers of the groups coincide, due to Vaisu [Vas07, Theorem 1.3.1].

Theorem 3.87 (Vaisu)**.** Fix an abelian variety A over a number field K. For each prime ℓ, we have

$$Z(\mathrm{MT}(A))^\circ_{\mathbb{Q}_\ell} = Z(G_\ell(A))^\circ.$$

Vaisu [Vas07] has in fact shown quite a bit about the Mumford–Tate conjecture; see in particular [Vas07, Theorem 1.3.4].

Much is known about products, especially products with restricted endomorphism types. For example, Theorem 3.87 immediately implies the Mumford–Tate conjecture for abelian varieties with complex multiplication by combiing with Proposition 3.53 and Remark 3.79.

Remark 3.88. In fact, the Mumford–Tate conjecture for abelian varieties with complex multiplication is much older: it is originally due to Pohlmann [Poh68, Theorem 4], but Ribet in [Rib04] has pointed out that the result is a corollary of results due to Shimura and Tanimaya [ST61], and [Yu15] has recently explicated this argument.

Include proof if include CM theorem

Continuing, by combining [Ich91; Lom16], one is able to compute both $\mathrm{MT}(A)$ and $G_\ell(A)^\circ$ for many abelian varieties of Types I–III and control contributions coming from Type IV; this permits a proof of the Mumford–Tate conjecture for products of abelian varieties of dimension at most 3. More generally, the following result is due to Commelin [Com18, Theorem 1.2].

Theorem 3.89 (Commelin)**.** Fix abelian varieties A and B over a number field K. If the Mumford–Tate conjecture holds for both A and B, then it holds for $A \times B$.

To give a taste for how some of these results are proven, we show the following, which follows from [Vas07, Theorem 1.3.4].

Proposition 3.90. Fix an abelian variety A over a number field K. Suppose that $A = A_1 \times \cdots \times A_k$, where the A_is are simple abelian varieties such that $\mathrm{End}_{\overline{K}}(A_i)$ equals a CM field F_i such that $\dim A_i = \dim F_i$ for each i. Set $F := \mathrm{End}_{\overline{K}}(A)$, and then we claim the Mumford–Tate conjecture holds for A, and

$$\mathrm{Hg}(A)^{\mathrm{der}} = \mathrm{L}(A)^{\mathrm{der}}.$$

Proof. For special ℓ, we will actually compute $\mathrm{MT}(A)^{\mathrm{der}}$ and $G_\ell(A)^{\circ,\mathrm{der}}$ "simultaneously" to show that they are equal to the suitable version of $\mathrm{GSp}_F(\varphi)^{\mathrm{der}}$ or $\mathrm{GSp}_F(e_\varphi)^{\mathrm{der}}$. By adding in what we know about the centers from Theorem 3.87 (and the independence of ℓ given in Theorem 3.86), the Mumford–Tate conjecture follows for A. The outline is to base-change to $\mathbb{C}$, where the Lie algebra of $\mathrm{L}(A)^{\mathrm{der}}$ becomes a product of $\mathfrak{sl}_2(\mathbb{C})$s, from which we can appeal to Lemma 1.62. We proceed in steps

1. In practice, it will be convenient to only write down the computation for $\mathrm{MT}(A)^{\mathrm{der}}$, but we will indicate along the way the changes that need to be made for $G_\ell(A)^{\circ,\mathrm{der}}$. Quickly, we note that we may assume that the $A_\bullet$s are pairwise non-isomorphic because all groups involved are immune to adding a factor which already appears (see Lemma 1.59 for MT, Lemma 1.67 for L, and Lemma 3.83 for G_ℓ).

   Now, for brevity, set $V := \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ so that $\mathrm{Hg}(A) = \mathrm{Hg}(V)$ and $\mathrm{L}(A) = \mathrm{L}(V)$; we remark that V is a free module over $F := F_1 \times \cdots \times F_k$ of rank 2. Similarly, we write $V_i\colon \mathrm{H}^1_{\mathrm{B}}(A_i, \mathbb{Q})$ for each i. Recall that $\mathrm{Hg}(V) \subseteq \mathrm{L}(V)$ by Lemma 1.47 (the ℓ-adic case uses Lemma 3.73). Thus, to achieve the equality of the derived subgroups, it is enough to achieve the equality after base-changing to an algebraic closure.

2. We recall some part of the computation from Lemma 1.68. Fix a polarization φ on V. For each i, let $\rho_{i1}, \ldots, \rho_{i,e_{0i}}$ be the embeddings of $F_i^\dagger$ into $\mathbb{R}$. Then each i finds a decomposition of V_i into

$$V_i = V_{i1} \oplus \cdots \oplus V_{ie_{0i}}$$

   so that

$$\mathrm{L}(V_i)_{\mathbb{R}} = \mathrm{Sp}_{F_i \otimes_{\rho_{i1}} \mathbb{R}}(\varphi|_{V_{i1}}) \times \cdots \times \mathrm{Sp}_{F_i \otimes_{\rho_{ie_{0i}}} \mathbb{R}}(\varphi|_{V_{ie_{0i}}}).$$

Thus, Lemma 1.67 allows us to expand

$$\mathrm{L}(V)_{\mathbb{R}} = \prod_{i=1}^{k} \mathrm{Sp}_{F_i\otimes_{\rho_{i1}}\mathbb{R}}(\varphi|_{V_{i1}}) \times \cdots \times \mathrm{Sp}_{F_i\otimes_{\rho_{ie_{0i}}}\mathbb{R}}(\varphi|_{V_{ie_{0i}}}).$$

We now also recall from Lemma 1.68 that each $\mathrm{Sp}_{F_i\otimes_{\rho_{ij}}\mathbb{R}}(\varphi|_{V_{ij}})_{\mathbb{C}}$ is isomorphic to $\mathrm{GL}_2(\mathbb{C})$; in particular, this group is connected.

3. We would like to show that the inclusion

$$\mathrm{Hg}(V)^{\mathrm{der}}_{\mathbb{C}} \subseteq \prod_{i=1}^{k} \mathrm{Sp}_{F_i\otimes_{\rho_{i1}}\mathbb{R}}(\varphi|_{V_{i1}})^{\mathrm{der}}_{\mathbb{C}} \times \cdots \times \mathrm{Sp}_{F_i\otimes_{\rho_{ie_{0i}}}\mathbb{R}}(\varphi|_{V_{ie_{0i}}})^{\mathrm{der}}_{\mathbb{C}}$$

is an isomorphism. All groups involved are connected, so we may check this inclusion on the level of the Lie algebra, so we would like for the inclusion

$$\mathrm{Lie}\,\mathrm{Hg}(V)^{\mathrm{der}}_{\mathbb{C}} \subseteq \prod_{i=1}^{k} \mathrm{Lie}\,\mathrm{Sp}_{F_i\otimes_{\rho_{i1}}\mathbb{R}}(\varphi|_{V_{i1}})^{\mathrm{der}}_{\mathbb{C}} \times \cdots \times \mathrm{Lie}\,\mathrm{Sp}_{F_i\otimes_{\rho_{ie_{0i}}}\mathbb{R}}(\varphi|_{V_{ie_{0i}}})^{\mathrm{der}}_{\mathbb{C}}$$

is surjective. For this, we use Lemma 1.62. Here are our checks; for brevity, set $\mathfrak{hg}(V) := \mathrm{Lie}\,\mathrm{Hg}(V)^{\mathrm{der}}_{\mathbb{C}}$, and let $\mathfrak{sl}_2(\mathbb{C})_{ij}$ be the factor $\mathrm{Lie}\,\mathrm{Sp}_{F_i\otimes_{\rho_{ij}}\mathbb{R}}(\varphi|_{V_{ij}})^{\mathrm{der}}_{\mathbb{C}}$, which we note is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

(i) We claim that $\mathfrak{hg}(V)$ surjects onto $\mathfrak{sl}_2(\mathbb{C})_{ij}$, which we note is nonzero and simple. Because the $\mathfrak{hg}(V)$ is semisimple, its image in $\mathfrak{sl}_2(\mathbb{C})_{ij}$ continues to be reductive. Now, reductive subgroups of $\mathfrak{sl}_2(\mathbb{C})$ are either tori of all of $\mathfrak{sl}_2(\mathbb{C})$, so we merely need to check that the image cannot be a torus. If the image in some $\mathfrak{sl}_2(\mathbb{C})_{ij}$ is a torus, then because the Galois action $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ permutes the set $\{V_{ij}\}_j$ (but will fix $\mathrm{Hg}(V)$), we see that the image in $\mathfrak{sl}_2(\mathbb{C})_{ij}$ will continue to be a torus for all j. Explicitly, we note that the image of $\mathrm{Hg}(V)$ in $\prod_j \mathrm{Sp}_{F_i\otimes_{\rho_{ij}}\mathbb{R}}(\varphi|_{V_{ij}})_{\mathbb{C}}$ needs to be preserved under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, so if the projection is commutative in one factor, then it is commutative in all factors because the $\mathbb{Q}$-points are dense.

There is no Galois action. This doesn't make sense.

But then we note that $\mathrm{Hg}(V)$ projects onto $\mathrm{Hg}(V_i)$ by Lemma 1.56, which then has an inclusion into $\mathrm{L}(V_i)_{\mathbb{C}} = \prod_j \mathrm{Sp}_{F_i\otimes_{\rho_{ij}}\mathbb{R}}(\varphi|_{V_{ij}})_{\mathbb{C}}$, where we see it must have commutative image by the preceding paragraph. In particular, $\mathrm{Hg}(V_i)$ must be a torus, so A_i has complex multiplication by Proposition 3.53, which is a contradiction to its definition.

(ii) The first point of (ii) is automatic from the construction. The second point follows because all the $\mathfrak{sl}_2(\mathbb{C})_{ij}$s include as the standard representation into $\mathfrak{gl}(V_{ij})$. The last point amounts to checking that

In practice, it will be convenient to only write down the computation for $\mathrm{MT}(A)^{\mathrm{der}}$, but we will indicate along the way the changes that need to be made for $G_\ell(A)^{\circ,\mathrm{der}}$. Quickly, we note that we may assume that the $A_\bullet$s are pairwise non-isomorphic because all groups involved are immune to adding a factor which already appears (see Lemma 1.59 for MT, Lemma 1.67 for L, and Lemma 3.83 for G_ℓ).

Now, for brevity, set $V := \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})$ so that $\mathrm{MT}(A) = \mathrm{MT}(V)$ and $\mathrm{L}(A) = \mathrm{L}(V)$; we remark that V is a free module over $F := F_1 \times \cdots \times F_k$ of rank 2. Similarly, we set $V_i := \mathrm{H}^1_{\mathrm{B}}(A_i,\mathbb{Q})$. Recall that $\mathrm{MT}(V) \subseteq \mathrm{L}(V)$ by Lemma 1.47 (the ℓ-adic case uses Lemma 3.73). Thus, to achieve the equality of the derived subgroups, it is enough to achieve the equality after base-changing to an algebraic closure. ■

3.2.4 Computing ℓ-Adic Monodromy

The previous subsection explains that one expects to be able to compute $G_\ell(A)^\circ = \mathrm{MT}(A)$. We now explain how to use a computation of $G_\ell(A)^\circ$ to compute $G_\ell(A)$ in full. The idea is to use the Galois action on Tate classes. Our exposition follows [GGL24, Sections 8.1–8.2]. We begin with some notation.

Notation 3.91. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\operatorname{char} K \nmid \ell$. We will write $V := \mathrm{H}^1_{ét}(A_{\overline{K}}, \mathbb{Q}_\ell)$. For each $n \geq 0$, we define W_n to be the spce of Tate classes in the nth tensor power, writing

$$W_n := \left(V^{\otimes n} \otimes V^{\vee \otimes n}\right)^{G_\ell(A)^\circ}.$$

We also write $W := \bigoplus_{n \geq 0} W_n$ for brevity.

Roughly speaking, the point is that the spaces $W_\bullet$ of Tate classes are able to keep track of $G_\ell(A)^\circ$.

Lemma 3.92. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\operatorname{char} K \nmid \ell$, and define V and $W_\bullet$ as in Notation 3.91.

(a) If $G \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ fixes W, then $G \subseteq G_\ell(A)^\circ$.

(b) There is a finite-dimensional subspace $W' \subseteq W$ such that $G \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ fixes W' if and only if $G \subseteq G_\ell(A)^\circ$.

Proof. This essentially follows from Proposition 1.35.

(a) Recall $G_\ell(A)^\circ$ is reductive by Theorem 3.80. Thus, by Proposition 1.35, we know that if $G \subseteq \mathrm{GL}(V)$ fixes every $G_\ell(A)^\circ$-invariant in any

$$\bigoplus_{i=1}^k \left(V^{\otimes m_i} \otimes V^{\vee \otimes n_i}\right),$$

then $G \subseteq G_\ell(A)^\circ$. However, we claim that all $G_\ell(A)^\circ$-invariants in the above space can be found in W, which will complete the proof. Indeed, by Theorem 3.87, we see that the scalars $\mathbb{G}_{m,\mathbb{Q}_\ell}$ can be found in $G_\ell(A)^\circ$; however, these scalars act by the character $z \mapsto z^{m_i - n_i}$ on $V^{\otimes m_i} \otimes V^{\vee \otimes n_i}$, so any $G_\ell(A)^\circ$-invariant subspace must then have $m_i = n_i$.

(b) The above argument provides countably many equations (in the form of invariant tensors) which cut out $G_\ell(A)^\circ$. However, any algebraic subgroup of $\mathrm{GL}(V)$ will be cut out by finitely many equations, so we can choose W' to be the span of any such subset of finitely many defining equations. ■

Remark 3.93. The proof of (b) in fact gives an effective way to compute the subspace W': simply write down enough tensor elements to cut out $G_\ell(A)^\circ \subseteq \mathrm{GL}(V)$.

We would now like to upgrade from $G_\ell(A)^\circ$ to $G_\ell(A)$.

Lemma 3.94. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\operatorname{char} K \nmid \ell$, and define V and $W_\bullet$ as in Notation 3.91. For each $n \geq 0$, the subspace W_n is stabilized by $G_\ell(A)$.

Proof. We already know that $G_\ell(A)^\circ$ acts trivially on W_n, so this will follow purely formally from the fact that $G_\ell(A)^\circ$ is a normal subgroup of $G_\ell(A)$.

We would like to show that each $g \in G_\ell(A)$ stabilizes W_n. Well, W_n exactly consists of the $G_\ell(A)^\circ$-invariants inside $V^{\otimes n} \otimes V^{\vee \otimes n}$, so it suffices to show that gW_n is stabilized by $G_\ell(A)^\circ$. Well, for any $g_0 \in G_\ell(A)^\circ$, we see that

$$g_0 g W_n = g \cdot g^{-1} g_0 g W_n,$$

so we conclude by noting that $g^{-1} g_0 g \in G_\ell(A)^\circ$ because $G_\ell(A)^\circ \subseteq G_\ell(A)$ is a normal subgroup. ■

Combining the above two lemmas, we see that we get a faithful representation

$$G_\ell(A)/G_\ell(A)^\circ \to \mathrm{GL}(W).$$

This faithful representation allows us to compute $G_\ell(A)$: we are looking for elements of $\mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Q}_\ell)\right)$ which produce the automorphisms of W seen in the image of the above faithful representation. Tracking through this sort of reasoning produces our main result.

Proposition 3.95. Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\operatorname{char} K \nmid \ell$, and define V and $W_\bullet$ as in Notation 3.91. Then $G_\ell(A)$ equals the group

$$\bigcup_{\sigma\in\mathrm{Gal}(\overline{K}/K)} \{g \in \mathrm{GL}\,(V) : g|_W = \sigma|_W\}\,.$$

In fact, each set in the union is a connected component of $G_\ell(A)$.

Proof. We begin by noting that $\mathrm{Gal}(\overline{K}/K)$ does in fact preserve W: indeed, one has a composite

$$\mathrm{Gal}(\overline{K}/K) \to G_\ell(A) \to \mathrm{GL}(W),$$

where the first map is well-defined by the definition of $G_\ell(A)$, and the second map is well-defined by summing Lemma 3.94.

Now, we have two inclusions to show.

- Suppose $g \in G_\ell(A)$. Then we must find $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $g|_W = \sigma|_W$. Well, $G_\ell(A)$ is by definition the Zariski closure of the image of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{GL}(V)$, so the open subset $gG_\ell(A)^\circ$ of $G_\ell(A)$ must contain $\sigma|_V$ for some $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Now, $G_\ell(A)^\circ$ acts trivially on W, so we see that $g|_W = \sigma|_W$.
- Suppose $g \in \mathrm{GL}(V)$ satisfies $g|_W = \sigma|_W$. Then we would like to show that $g \in G_\ell(A)$. The argument in the previous point grants $g_0 \in G_\ell(A)$ such that $g_0|_V = \sigma|_V$, so in particular, $g|_W = g_0|_W$. Thus, gg_0^{-1} acts trivially on W, so $gg_0^{-1} \in G_\ell(A)^\circ$, so it follows that $g \in G_\ell(A)$.

Lastly, it remains to discuss connected components. Well, note that $g, g' \in G_\ell(A)$ live in the same connected component if and only if $g'g^{-1} \in G_\ell(A)$, which is equivalent to $g'g^{-1}$ acting trivially on W, which is equivalent to $gG_\ell(A)^\circ = g'G_\ell(A)^\circ$. ■

Remark 3.96. A careful reading of the above proof shows that we only needed the following facts about W: it is stable under $G_\ell(A)$, and $g \in \mathrm{GL}\,(V)$ lives in $G_\ell(A)^\circ$ if and only if it fixes W. Thus, we see that we can replace W with any $G_\ell(A)$-subrepresentation $W' \subseteq W$ which cuts out $G_\ell(A)^\circ$ in the sense of Lemma 3.94. This allows us to make W' quite small (e.g., finite-dimensional).

Remark 3.97. It is worth comparing Proposition 3.95 with the definition of the twisted Lefschetz group, defined in [BK15, Definition 5.2]. Roughly speaking, the twisted Lefschetz group is simply the construction of Proposition 3.95 with W replaced by the subspace of W generated by endomorphisms and the polarization; see [GGL24, Remark 8.3.5] for precise discussion of the relation. In this way, one expects the twisted Lefschetz group to equal $G_\ell(A)$ in generic cases, but Remark 3.96 explains that one may need to remember more Hodge classes in exceptional cases.

Proposition 3.95 suggests that one can find representatives of each connected component in $G_\ell(A)$ by looping over all $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and finding some $g \in \mathrm{GL}(V)$ such that $g|_W = \sigma|_W$. This is currently not so computable because $\mathrm{Gal}(\overline{K}/K)$ is an infinite group, and W is an infinite-dimensional vector space. Remark 3.96 explains how to replace W with a finite-dimensional subrepresentation, so it remains to explain how to reduce $\mathrm{Gal}(\overline{K}/K)$ to a finite quotient.

Definition 3.98 (connected monodromy field)**.** Fix an abelian variety A defined over a field K, and let ℓ be a prime such that $\operatorname{char} K \nmid \ell$. Then we define the *connected monodromy field* K_A^{conn} so that the open subgroup $\mathrm{Gal}(\overline{K}/K_A^{\mathrm{conn}})$ is the pre-image of the connected component $G_\ell(A)^\circ$ in the Galois representation

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Q}_\ell)\right).$$

Remark 3.99. Note that such a field K_A^{conn} exists and is finite over K by Galois theory: note $G_\ell(A)^\circ \subseteq G_\ell(A)$ is a finite-index subgroup (because the quotient is a discrete algebraic group), so the pre-image $U \subseteq \mathrm{Gal}(\overline{K}/K)$ of $G_\ell(A)^\circ$ similarly must be open and finite index and hence closed and finite index.

Thus, we see that the Galois reprentation to $\mathrm{GL}(W)$ factors through the finite group $\mathrm{Gal}(K_A^{\mathrm{conn}}/K)$. In this way, we are able to reduce the computation suggested by Proposition 3.95 from the infinite group $\mathrm{Gal}(\overline{K}/K)$ to the finite quotient $\mathrm{Gal}(K_A^{\mathrm{conn}}/K)$.

Remark 3.100. Let's describe how one might compute K_A^{conn} in practice. BY combining the definition of K_A^{conn} with Lemma 3.92, we see that $\mathrm{Gal}(\overline{K}/K_A^{\mathrm{conn}})$ is the kernel of the representation

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(W),$$

so one could imagine computing the open subgroup $\mathrm{Gal}(\overline{K}/K_A^{\mathrm{conn}})$ by computing the above representation. As usual, we remark that Lemma 3.92 allows us to replace W with a finite-dimensional subrepresentation W' "cutting out" $G_\ell(A)^\circ$.

3.3 The Sato–Tate Conjecture

Now that we have a good handle on monodromy groups, we describe one of their applications: the Sato–Tate conjecture. These notions are not central for the results we want to prove, so we will be somewhat sketchy throughout.

3.3.1 The Weil Conjectures

Roughly speaking, the Sato–Tate conjecture is about counting points on an abelian variety A over finite fields $\mathbb{F}_q$ as q varies. In this subsection, we will briefly describe the Weil conjectures because they explain why these point-counts ought to be related to cohomology; these conjectures are now theorems due to Deligne [Del74; Del80].

Theorem 3.101 (Weil conjectures)**.** Fix a smooth projective variety X over a finite field $\mathbb{F}_q$ of dimension n. Consider the formal power series.

$$\zeta_X(T) := \exp\left(\sum_{r=1}^{\infty} \#X\left(\mathbb{F}_{q^r}\right)\frac{T^r}{r}\right)$$

(a) Rationality: one can write

$$\zeta_X(T) = \frac{P_1(T)\cdots P_{2n-1}(T)}{P_2(T)\cdots P_{2n}(T)}$$

for polynomials $P_\bullet(T) \in 1 + T\mathbb{Z}[T]$.

(b) Riemann hypothesis: the roots of the polynomial $P_\bullet(T)$ are complex numbers with roots of magnitude $q^{-\bullet/2}$.

It is worth explaining a bit of the proof of these conjectures for abelian varieties. Our exposition is an abbreviated form of the exposition in (say) [Mil08, Chapter II].

Fix an abelian variety A of dimension g over a finite field $\mathbb{F}_q$. The main point is to find a way to compute $\#A(\mathbb{F}_q)$, and then Theorem 3.101 will follow. Viewing $A(\mathbb{F}_q)$ is the set of fixed (geometric) points of the Frobenius endomorphism $\mathrm{Frob}_q \colon A \to A$, one would like to use the Lefschetz fixed point formula to conclude. In particular, we should be able to read off the value of $\#A(\mathbb{F}_q)$ from a suitably defined characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$.

To be explicit, one finds that the characteristic polynomial $P(T)$ of $\operatorname{Frob}_\mathfrak{p}$ acting on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ satisfies

$$P(1) = \#A(\mathbb{F}_q).$$

Thus, by factoring $P(T) = \prod_{i=1}^{2g}(T - \alpha_i)$, one finds that

$$\#A(\mathbb{F}_{q^r}) = \prod_{i=1}^{2g}(1 - \alpha_i^r),$$

which proves the rationality conjecture of Theorem 3.101 after some manipulation. In brief, one finds that $P(T) = P_1(T)$, and in general, the polynomial $P_i(T)$ has roots given by multiplying i of the roots in the set $\{\alpha_1, \ldots, \alpha_{2g}\}$ together.

Remark 3.102. Comparing the previous paragaph with the proof of the rationality conjecture from the Lefschetz trace formula

$$\#A(\mathbb{F}_q) = \sum_{i=0}^{2g}(-1)^i \operatorname{tr}\left(\operatorname{Frob}_q \mid \mathrm{H}^i_{\text{ét}}(A_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)\right),$$

one sees that what allows us to recover all the polynomials $P_\bullet(T)$ from merely $P_1(T)$ is that the higher cohomology of A is generated by the cohomology in degree 1 by Proposition 3.59.

It remains to prove the Riemann hypothesis conjecture of Theorem 3.101. This amounts to checking that the roots of $P(T)$ have magnitude $1/\sqrt{p}$, which eventually corresponds to the following fact.

Proposition 3.103. Fix an abelian variety A over a finite field $\mathbb{F}_q$, and consider the induced Frobenius endomorphism Frob_q. Then

$$\operatorname{Frob}_q \circ \operatorname{Frob}_q^\dagger = [q]_A.$$

Proof. Proving this requires more tools than we would like to introduce at this time, so we refer to [Mil08, Lemma III.1.2]. ■

3.3.2 The Sato–Tate Group

In this section, we will define the Sato–Tate group and state the Sato–Tate conjecture. Our exposition loosely follows [Sut19]. Fix an abelian variety A defined over a number field K, and choose a prime ℓ. We also let $\rho_\ell\colon \operatorname{Gal}(\overline{K}/K) \to \operatorname{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)\right)$ denote the associated Galois representation.

Intuitively, the Sato–Tate conjecture asserts that the Frobenius elements $\rho_\ell(\operatorname{Frob}_\mathfrak{p})$ equidistribute in $G_\ell(A)$ as $\mathfrak{p}$ varies over the maximal ideals of $\mathcal{O}_K$. This conjecture does not make sense verbatim, so we will have to work a bit to write down something formal. Consider the following points.

- To begin, we note that $\operatorname{Frob}_\mathfrak{p}$ only makes sense as a conjugacy class, and it only makes sense as a conjugacy class when ρ_ℓ vanishes on the relevant inertia subgroup of $\operatorname{Gal}(\overline{K}/K)$.

  Two remarks are thus in order. First, to vanish on the inertia subgroup, we must exclude a finite set of primes $\mathfrak{p}$ where A has bad reduction. (We are using the Néron–Ogg–Shafarevich criterion [BLR90, Theorem 5].) Second, we will simply regard $\rho_\ell(\operatorname{Frob}_\mathfrak{p})$ as a conjugacy class as well. Thus, we really want to say that conjugacy classes equidistribute in a suitable space of conjugacy classes.

- It turns out that $\rho_\ell(\operatorname{Frob}_\mathfrak{p})$ is not a totally random element of $G_\ell(A)$. Indeed, by Proposition 3.103, we see that the multiplier of $\operatorname{Frob}_\mathfrak{p}$ acting on $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ equals $\mathrm{N}(\mathfrak{p})$. Thus, we would like to rescale $\operatorname{Frob}_\mathfrak{p}$ back down by $1/\sqrt{\mathrm{N}(\mathfrak{p})}$.

  Once again, this requires two remarks. First, after rescaling, we will be working in the smaller subgroup

  $$G_\ell^1(A) := G_\ell(A) \cap \operatorname{Sp}(e_\varphi),$$

where φ is a choice of polarization on A. Second, the rescaling cannot happen in $\mathbb{Q}_\ell$ because $\mathbb{Q}_\ell$ does not have enough square roots. As such, we must choose an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, allowing us to consider the elements $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ in the complex Lie group $G_\ell^1(A)_\iota(\mathbb{C})$.[3]

- Another piece of structure to keep track of is that $\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ is semisimple, as discussed in Remark 3.81. This means that the subgroup topological generated by $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ (which we now see has all eigenvalues equal to 1 after the normalization in the previous step) will be compact! A standard result in the structure theory of complex Lie groups is that they have maximal compact subgroups unique up to conjugacy, so one can find an element in our conjugacy class $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ in any given maximal compact subgroup of $G_\ell^1(A)_\iota(\mathbb{C})$.

With the above preparations, we are now ready to state the Sato–Tate conjecture.

Definition 3.104 (Sato–Tate group)**.** Fix an abelian variety A defined over a number field K, and choose a prime ℓ and an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Then we define the *Sato–Tate group* $\mathrm{ST}(A)$ to be a maximal compact subgroup of the complex Lie group $G_\ell^1(A)_\iota$, where $G_\ell^1(A)$ is the subset of $G_\ell(A)$ with multiplier equal to 1.

Conjecture 3.105 (Sato–Tate)**.** Fix an abelian variety A defined over a number field K, and choose a prime ℓ and an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. For each nonzero prime ideal $\mathfrak{p}$ of K such that A has good reduction at $\mathfrak{p}$, choose the conjugacy class $x_\mathfrak{p} \in \mathrm{Conj}(\mathrm{ST}(A))$ containing the conjugacy class $\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$. Then the conjugacy classes $\{x_\mathfrak{p}\}$ equidistribute with respect to the pushforward of the Haar measure along $\mathrm{ST}(A) \to \mathrm{Conj}(\mathrm{ST}(A))$.

The relevance of the Sato–Tate conjecture for us is that it will let us numerically check that we have the correct ℓ-adic monodromy group; precisely how this is done will be explained in the subsequent subsections.

We will spend the rest of the present subsection making some remarks about Conjecture 3.105.

Remark 3.106. Not much is known about Conjecture 3.105. Roughly speaking, all known proofs prove something akin to modularity for not just the Galois representation attached to A but also its symmetric powers (and maybe more!).

- If A has complex multiplication, then this essentially follows from the Fundamental theorem of complex multiplication.
- For elliptic curves, the state of the art is [Bar+14; Bar+11], where the Sato–Tate conjecture is proven for elliptic curves over totally real and CM fields.
- These potential automorphy techniques were extended to some classes of abelian varieties by Johansson in [Joh17, Theorem 1].

One obnoxious defect of Conjecture 3.105 is that we must make choices regarding ℓ and ι. The choice ι is not so egregious because everything ought to descend to something algebraic, but it is quite unclear that $\mathrm{ST}(A)$ and even $G_\ell^1(A)$ does not depend crucially on ℓ. One expects $G_\ell(A)^\circ$ to not depend on ℓ by the Mumford–Tate conjecture (Conjecture 3.84). The relevant conjecture for the full group $G_\ell(A)$ is the Algebraic Sato–Tate conjecture [BK15, Conjecture 2.1].

[3] Another reason for passing to $\mathbb{C}$ is that groups in $\mathbb{C}$ have access to a good measure theory.

Conjecture 3.107 (Algebraic Sato–Tate)**.** Fix an abelian variety A defined over a number field K. Then there exists an algerbaic subgroup $\operatorname{AST}(A) \subseteq \operatorname{GL}_{2g}(\mathbb{Q})$ such that

$$\operatorname{AST}(A)_{\mathbb{Q}_\ell} = G_\ell^1(A)$$

for all primes ℓ.

This conjecture, being similar in spirit to the Mumford–Tate conjecture, has quite a bit known. For example, Banaszak and Kedlaya have shown this conjecture for products of abelian varieties of dimensions at most 3 [BK15, Theorem 6.11]. Roughly speaking, their proof boils down to the fact that one has $\operatorname{Hg}(A) = \operatorname{L}(A)^\circ$ in these small dimensions, which permits a direct computation of $\operatorname{AST}(A)$ along the lines of Proposition 3.95 (see Remark 3.96).

Remarkably, Farfán and Commelin have shown that the Algebraic Sato–Tate conjecture implies the Mumford–Tate conjecture in [CC22].

Theorem 3.108 (Farfán–Commelin)**.** Fix an abelian variety A defined over a number field K. If A satisfies the Mumford–Tate conjecture (Conjecture 3.84) that $G_\ell(A)^\circ = \operatorname{MT}(A)$ for all primes ℓ, then A satisfies the Algebraic Sato–Tate conjecture (Conjecture 3.107) that there exists an algebraic group $\operatorname{AST}(A) \subseteq \operatorname{GL}_{2g}(\mathbb{Q})$ such that $\operatorname{AST}(A)_{\mathbb{Q}_\ell} = G_\ell^1(A)$ for all primes ℓ.

Proof. The proof requires a discussion of Tannakian formalism, so we will not include it. We remark that they actually prove that the Mumford–Tate conjecture is equivalent to a more refined version of the Algebraic Sato–Tate conjecture with $\operatorname{AST}(A)$ equal to the "motivic Galois group" of A. ■

Include proof if include abelian motives

3.3.3 Some Examples

In this subsection, we compute some basic Sato–Tate groups. The general outline is to compute the Hodge or Mumford–Tate groups first, check the Mumford–Tate conjecture to get G_ℓ°, and then compute some Galois action to get G_ℓ. We begin with some elliptic curves.

Example 3.109 (no complex multiplication)**.** Consider the elliptic curve $E\colon y^2 = x^3 + x + 1$ over $\mathbb{Q}$. One can compute that $\operatorname{End}_{\mathbb{C}}(E) = \mathbb{Z}$, so E does not have complex multiplication. Thus, $\operatorname{Hg}(E) \subseteq \operatorname{SL}_{2,\mathbb{Q}}$ needs to be a connected reductive subgroup which is not a torus (see Proposition 3.53); however, the only Lie subalgebras of $\mathfrak{sl}_2(\mathbb{C})$ are either commutative or all of $\mathfrak{sl}_2(\mathbb{C})$, so we conclude that $\operatorname{Hg}(E) = \operatorname{SL}_{2,\mathbb{Q}}$. Thus, $\operatorname{MT}(E) = \operatorname{GL}_{2,\mathbb{Q}}$.

The same computation (with Remark 3.79) allows us to conclude that $G_\ell(E) = \operatorname{GL}_{2,\mathbb{Q}_\ell}$ for all primes ℓ, thus proving the Mumford–Tate conjecture (Conjecture 3.84) in this case. We thus find $G_\ell^1(E) = \operatorname{SL}_{2,\mathbb{Q}_\ell}$, so upon choosing $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we see that $G_\ell^1(E)_\iota = \operatorname{SL}_{2,\mathbb{C}}$, so choosing a maximal compact subgroup finds $\operatorname{ST}(E) = \operatorname{SU}_2$.

Example 3.110 (complex multiplication)**.** Consider the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$. Then we see that $\operatorname{End}_{\mathbb{C}}(E) = \mathbb{Z}[\zeta_3]$, where ζ_3 acts by $(x, y) \mapsto (\zeta_3 x, y)$, so E has complex multiplication. Thus, $\operatorname{Hg}(E) \subseteq \operatorname{SL}_{2,\mathbb{Q}(\zeta_3)}$ is a torus (by Proposition 3.53), but it cannot be trivial (by Corollary 3.42), so we conclude that $\operatorname{Hg}(E)$ is the diagonal torus of $\operatorname{SL}_{2,\mathbb{Q}(\zeta_3)}$.

For primes ℓ which split completely in $\mathbb{Q}(\zeta_3)$, the same computation (with Remark 3.79 and Corollary 3.78) where ℓ splits completely in $\mathbb{Q}_\ell$ reveals $G_\ell(E) = \mathbb{G}_{m,\mathbb{Q}_\ell}^2$ equals the diagonal torus in $\operatorname{GL}_{2,\mathbb{Q}(\zeta_3)}$, proving the Mumford–Tate conjecture (Conjecture 3.84) in this case. We thus find $G_\ell^1(E) \cong \mathbb{G}_{m,\mathbb{Q}_\ell}$, so upon choosing $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we see that $G_\ell^1(E) \cong \mathbb{G}_{m,\mathbb{Q}_\ell}$, so choosing a maximal compact subgroup finds $\operatorname{ST}(E) \cong \operatorname{U}_1$.

Example 3.111 (potential complex multiplication)**.** Consider the elliptic curve $E\colon y^2 = x^3 + 1$ but now over $\mathbb{Q}$. Example 3.110 computed that $\operatorname{MT}(E) \cong \mathbb{G}_{m,\mathbb{Q}}$ and $G_\ell(E)^\circ = \mathbb{G}_{m,\mathbb{Q}_\ell}$ (for primes $\ell \equiv 1 \pmod 3$). In this case, we see that there are endomorphisms not defined over $\mathbb{Q}$ and hence not fixed by $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so $K_E^{\text{conn}} \neq \mathbb{Q}$; instead, these endomorphisms are defined over $K_E^{\text{conn}} = \mathbb{Q}(\zeta_3)$. We thus see that $G_\ell(E) \subseteq \operatorname{GL}_{2,\mathbb{Q}_\ell}$ normalizes its index-2 subgroup $G_\ell(E)^\circ$ (which is the diagonal torus), so $G_\ell(E)$ must be the diagonal torus together with the nontrivial Weyl element in $\operatorname{GL}_{2,\mathbb{Q}_\ell}$, which we write as $\mathbb{G}_{m,\mathbb{Q}_\ell}^2 \rtimes S_2$. We thus find $G_\ell^1(E) \cong \mathbb{G}_{m,\mathbb{Q}_\ell} \rtimes S_2$, so $\operatorname{ST}(E) \cong \mathrm{U}_1 \rtimes S_2$.

Remark 3.112. In the above example, we appealed to the fact that the only elements normalizing the diagonal torus are the Weyl elements, which is a bit ad-hoc and will not work in higher dimensions. Roughly speaking, Proposition 3.95 provides the machine which works in higher dimensions, where we know that the Galois representation now factors through $\operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$, and we are allowed to replace W with merely $W_1 \oplus W_2$, which can be computed to be generated by the endomorphisms and polarization.

We take a moment to remark that the above examples generalize to work with all elliptic curves, doing casework on having no complex multiplication, complex multiplication, and potential complex multiplication.

We now introduce the main example of the present thesis.

Proposition 3.113. Fix $\lambda \in \mathbb{C} \setminus \{0,1\}$, and define A to be the Jacobian of the normalization of the proper curve C with affine chart $y^9 = x(x-1)(x-\lambda)$. If A does not have complex multiplication, then

$$\begin{cases} \operatorname{MT}(A)_{\mathbb{C}}^{\text{der}} \cong \operatorname{SL}_2(\mathbb{C})^3 \\ Z(\operatorname{MT}(A))_{\mathbb{C}}^\circ \cong \mathbb{G}_m^4. \end{cases}$$

We use this to compute $\operatorname{ST}(A_K)$ if $\lambda \in K$ and K contains K_A^{conn}.

Proof. We proceed in steps.

1. To begin, we do some prelimarinary algebraic geometry, along the lines of [Moo10, Section 1]. The curve C comes equipped with a natural map $x\colon C \to \mathbb{P}^1$, with Galois with cyclic Galois group μ_9, where μ_9 acts on C by multiplication of the y-coordinate. As such, a computation with the Riemann–Hurwitz formula reveals that the genus is $g = 7$, so $\dim A = 7$. From here, we can find the differentials

$$\left\{ \frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8} \right\}$$

   are all holomorphic on C, and they are linearly independent, so we see that this is a basis of the space of differentials in $\mathrm{H}^0(C, \Omega^1_{C/\mathbb{C}}) = \mathrm{H}^0(A, \Omega^1_{A/\mathbb{C}})$. We remark that the above is also an eigenbasis for the induced μ_9-action on $\mathrm{H}^0(A, \Omega^1_{A/\mathbb{C}})$.

2. We decompose A into pieces. Note that C projects onto the elliptic curve $C_0\colon y^3 = x(x-1)(x-\lambda)$ via the map $(x,y) \mapsto \left(x, y^3\right)$, so C_0 is a factor of A. One can see that the basis of differentials of C_0 is given by dx/y^2, which pulls back to the differential dx/y^6 on A. In this way, we see that the quotient $A_1 \coloneqq A/C_0$ will have $\mathrm{H}^0(A_1, \Omega^1_{A_1/\mathbb{C}})$ have a basis given by

$$\left\{ \frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8} \right\}.$$

   Note that we do not yet know if A_1 is simple!

3. We compute some endomorphism algebras. Note C_0 has $\mu_3 \subseteq \operatorname{Aut}(C_0)$ where ζ_3 acts by multiplication on the y-coordinate, so C_0 has complex multiplication by $F_0 \coloneqq \mathbb{Q}(\zeta_3)$.

We conclude this step by showing that A_1 is simple. This will follow from the fact that A does not have complex multiplication. Note the μ_9-action on A fixes C_0 (we can be seen on the level of the Hodge structure), so it must also fix A_1, so we see $\mathbb{Q}(\zeta_9) \subseteq \operatorname{End}_{\mathbb{C}}(A_1)_{\mathbb{Q}}$. Thus, A_1 contains an isotypic component B^r (where B is simple) such that

$$\mathbb{Q}(\zeta_9) \subseteq \operatorname{End}_{\mathbb{C}}(B^r) = M_r\left(\operatorname{End}_{\mathbb{C}}(\mathrm{H}^1_{\mathrm{B}}(B,\mathbb{C}))\right).$$

As such, we set $D := \operatorname{End}_{\mathbb{C}}(B)$ and $F := Z(D)$ so that $d := \sqrt{[D:F]}$ and $e := [F:\mathbb{Q}]$ satisfy $6 \mid rde$ (because $\mathbb{Q}(\zeta_9)$ is contained in a maximal subfield of $M_r(D)$) and $r^2d^2e \leq 2\dim A_1 = 12$. If we had $r^2d^2e = 12$, then A_1 would have complex multiplication, which contradicts the fact that A does not have complex multiplication. Thus, we must instead have $rde = r^2d^2e = 6$, which implies that $r = d = 1$ and so $A_1 = B$ with $\operatorname{End}_{\mathbb{C}}(A_1)$ given exactly by $F_1 := \mathbb{Q}(\zeta_9)$.

4. We compute some signatures. We begin with C_0. Letting $\tau_i \in \operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$ be given by $\tau_i(\zeta_3) := \zeta_3^i$ for $i \in \{1,2\}$, we see that the signature $\Phi_0\colon \operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) \to \mathbb{Z}_{\geq 0}$ of E_0 is thus given by $\Phi_0(\tau_1) = 1$ and $\Phi_0(\tau_2) = 0$ because the second step provided an (eigen)basis of $\mathrm{H}^{10}(C_0) = \mathrm{H}^0(C_0, \Omega^1_{C_0/\mathbb{C}})$.

   We next consider A_1. The second step provided a basis of $\mathrm{H}^{10}(A_1) = \mathrm{H}^0(A_1, \Omega^1_{A_1/\mathbb{C}})$. As such, we define $\sigma_i \in \operatorname{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ to be the automorphism given by $\sigma_i(\zeta_9) := \zeta_9^i$ for $i \in \{1,2,4,5,7,8\}$, and we are able to compute that our signature $\Phi_1\colon \operatorname{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q}) \to \mathbb{Z}_{\geq 0}$ is given by

$$\Phi(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{7,8\}, \\ 1 & \text{if } i \in \{4,5\}, \\ 2 & \text{if } i \in \{1,2\}. \end{cases}$$

5. We compute $\mathrm{MT}(A)^{\mathrm{der}}$; note that this equals $\mathrm{Hg}(A)^{\mathrm{der}}$ by Lemma 1.41. By Lemma 1.56, we have an inclusion

$$\mathrm{Hg}(A) \to \mathrm{Hg}(C_0) \oplus \mathrm{Hg}(A_1)$$

   which surjects onto each factor. Now, C_0 has complex multiplication, so $\mathrm{Hg}(C_0)$ is a torus by Proposition 3.53, so $\mathrm{Hg}(A)^{\mathrm{der}}$ has trivial projection onto $\mathrm{Hg}(C_0)$. We conclude that the above inclusion upgrades into an isomorphism $\mathrm{Hg}(A)^{\mathrm{der}} \to \mathrm{Hg}(A_1)^{\mathrm{der}}$.

   To compute $\mathrm{Hg}(A_1)^{\mathrm{der}}$, we use Proposition 3.90 to see that this equals $\mathrm{L}(A_1)^{\mathrm{der}}$, so we complete this step by noting that $\mathrm{L}(A_1)^{\mathrm{der}}_{\mathbb{C}} \cong \mathrm{SL}_2(\mathbb{C})^3$ by the computation in Lemma 1.68.

6. We compute $Z(\mathrm{MT}(A))^\circ_{\mathbb{C}}$. We use Proposition 1.96 and in particular the discussion following the proof. Indeed, set $L := \mathbb{Q}(\zeta_9)$, which we note is a Galois extension of $\mathbb{Q}$ containing F_0F_1. Then we note that $Z(\mathrm{MT}(A))^\circ \subseteq \mathrm{T}_F$, where $F := F_0 \times F_1$ has $(\mathrm{T}_F)_L$ embedded into $\mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A,L)\right)$ as a subtorus of the diagonal torus. Explicitly, we can choose an F-eigenbasis of $\mathrm{H}^1_{\mathrm{B}}(A,L) = \mathrm{H}^1_{\mathrm{B}}(C_0,L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_1,L)$ as

$$\{u_1, u_2, v_1, v_1', v_2, v_2', v_4, v_4', v_5, v_5', v_7, v_7', v_8, v_8'\},$$

   where the subscript partially indicates the F-eigenvalue. Then we see that $(\mathrm{T}_F)_L \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A,L)\right)$ embeds as

$$\{\operatorname{diag}(\mu_1,\mu_2,\lambda_1,\lambda_1,\lambda_2,\lambda_2,\lambda_4,\lambda_4,\lambda_5,\lambda_5,\lambda_7,\lambda_7,\lambda_8,\lambda_8) : \mu_\bullet, \lambda_\bullet \in \mathbb{G}_{m,L}\}.$$

   The discussion following Proposition 1.96 explains that equations cutting out $Z(\mathrm{MT}(A))^\circ_L \subseteq (\mathrm{T}_F)_L$ can be viewed as elements of the kernel of the map

$$\mathrm{X}^*\left((\mathrm{N}_{\Phi_0^*}, \mathrm{N}_{\Phi_1^*})\right) : \mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(\mathrm{T}_L).$$

Using the established bases for these lattices, we see that our map can be written as the matrix

$$\begin{array}{c} \\ \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_7 \\ \sigma_8 \end{array} \begin{array}{c} \begin{array}{cc|cccccc} \mu_1 & \mu_2 & \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 & \lambda_7 & \lambda_8 \end{array} \\ \left[\begin{array}{cc|cccccc} 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \end{array}\right] \end{array}.$$

Then one can compute a basis of the kernel of the matrix, which tells us that $Z(\mathrm{MT}(A))_L^\circ \subseteq (\mathrm{T}_F)_L$ is cut out by the equations

$$\begin{aligned} \lambda_1\lambda_8 &= \lambda_2\lambda_7, \\ \lambda_1\lambda_8 &= \lambda_4\lambda_5, \\ \mu_1\mu_2\lambda_7 &= \lambda_5\lambda_8, \\ \lambda_1\lambda_4\lambda_7 &= \lambda_2\lambda_5\lambda_8. \end{aligned}$$

Thus, we see that $Z(\mathrm{MT}(A))_{\mathbb{C}}^\circ \cong \mathbb{G}_{m,\mathbb{C}}^4$ with isomorphism given by the cocharacters $(\mu_1, \lambda_1, \lambda_4, \lambda_8)$.

7. We use the previous steps to compute $G_\ell^1(A)$ when ℓ splits completely in K_A^{conn}. Recall we notably know the Mumford–Tate conjecture that $G_\ell(A)^\circ = \mathrm{MT}(A)_{\mathbb{Q}_\ell}$ by Proposition 3.90. Thus, we choose ℓ to split completely in K_A^{conn} so that $\mathbb{Q}(\zeta_9) \subseteq \mathbb{Q}_\ell$, allowing us to engage in the diagonalization of the previous step. For example, the computation in Lemma 1.68 reveals that the isomorphism between $\mathrm{L}(A)^{\mathrm{der}}$ and SL_2^3 is defined over L (indeed, one merely needs to be able to take L-eigenspaces), so we find that
$$G_\ell(A)^{\mathrm{der}} = \left\{\operatorname{diag}\left(1_2, g_1, g_2, g_4, g_4^{-\intercal}, g_2^{-\intercal}, g_1^{-\intercal}\right) : g_1, g_2, g_3 \in \mathrm{SL}_{2,\mathbb{Q}_\ell}\right\}.$$
Continuing, we add in the equation $\det g = 1$ to the equations cutting out $Z(G_\ell(A_L))^\circ \subseteq (\mathrm{T}_F)_{\mathbb{Q}_\ell}$ given in the previous step. This reveals that $Z\left(G_\ell^1(A_L)\right)^\circ \subseteq (\mathrm{T}_F)_{\mathbb{Q}_\ell}$ is cut out by the equations
$$\begin{aligned} \mu_1\mu_2 &= 1, \\ \lambda_1\lambda_8 &= 1, \\ \lambda_2\lambda_7 &= 1, \\ \lambda_4\lambda_5 &= 1, \\ \lambda_2 &= \lambda_1\lambda_4. \end{aligned}$$
In particular, we see that $Z\left(G_\ell^1(A)\right)^\circ \cong \mathbb{G}_{m,\mathbb{Q}_\ell}^3$ given by the cocharacters $(\mu_1, \lambda_1, \lambda_4)$. In total, we find $G_\ell^1(A) \subseteq \mathrm{GL}_{14,\mathbb{Q}_\ell}$ equals
$$\left\{\operatorname{diag}\left(\mu_1, \mu_1^{-1}, \lambda_1 g_1, \lambda_1\lambda_4 g_2, \lambda_4 g_4, \lambda_4^{-1} g_4^{-\intercal}, \lambda_1^{-1}\lambda_4^{-1} g_2^{-\intercal}, \lambda_1^{-1} g_1^{-\intercal}\right) : \mu_\bullet, \lambda_\bullet \in \mathbb{G}_{m,\mathbb{Q}_\ell}, g_\bullet \in \mathrm{SL}_{2,\mathbb{Q}_\ell}\right\}.$$

8. At last, we compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}. By Theorem 3.108, we see that $\mathrm{ST}(A)$ does not depend on the choice ℓ, so we may as well choose ℓ to split completely in K_A^{conn}. Then we simply base-change the result of the previous step to $\mathbb{C}$, and then we may take maximal compact subgroups to see ST is
$$\left\{\operatorname{diag}\left(\mu_1, \mu_1^{-1}, \lambda_1 g_1, \lambda_1\lambda_4 g_2, \lambda_4 g_4, \lambda_4^{-1} g_4^{-\intercal}, \lambda_1^{-1}\lambda_4^{-1} g_2^{-\intercal}, \lambda_1^{-1} g_1^{-\intercal}\right) : \mu_\bullet, \lambda_\bullet \in \mathrm{U}_1, g_\bullet \in \mathrm{SU}_2\right\}.$$
(It is not too hard to see that the product of maximal compact subgroups continues to be a maximal compact subgroup.) This completes the computation. ■

Remark 3.114. Note that $\mathrm{M}T(A) \neq \mathrm{L}(A)$ because the centers are different! This continues to be visible in the Sato–Tate group: the first four equations $\mu_1\mu_2 = \lambda_1\lambda_8 = \lambda_2\lambda_7 = \lambda_4\lambda_5 = 1$ can be explained by the polarization (see Lemma 1.75), but the last equation $\lambda_2 = \lambda_1\lambda_4$ corresponds to an exceptional Hodge class not generated by endomorphisms or the polarization.

The hypothesis that A fails to have CM is necessary, as we will see in the following two examples.

Proposition 3.115. Define A to be the Jacobian of the proper curve C with affine chart $y^9 = x^3 - 1$. Then $\mathrm{MT}(A)_\mathbb{C}$ is a torus isomorphic to $\mathbb{G}_{m,\mathbb{C}}^4$. We use this to compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}.

Proof. We proceed in steps, following Proposition 3.113.

1. To begin, we once again note that C has genus 7, so A has dimension 7, and we have a basis of holomorphic differentials given by

$$\left\{\frac{dx}{y^4}, \frac{dx}{y^5}, \frac{dx}{y^6}, \frac{dx}{y^7}, \frac{dx}{y^8}, \frac{x\,dx}{y^7}, \frac{x\,dx}{y^8}\right\}.$$

   This time around, we see that $\mu_3 \times \mu_9$ acts on C by coordinate-wise multiplication on $(x, y) \in C$.

2. We decompose A into pieces.

   - Note C projects onto $C_0 \colon y^3 = x^3 - 1$ by $(x, y) \mapsto (x, y^3)$. (This is the quotient of C by $\mu_3 \times 1$.) We see that C_0 is an elliptic curve, and it has complex multiplicatino by μ_3; for example, μ_3 can act by multiplication on y. One can compute that C_0 has a basis of holomorphic differentials given by dx/y^2, which pulls back to the differential dx/y^6 on C.
   - Note C projects onto the proper curve C_1 with affine chart $y^9 = x^3(x-1)$ by $(x, y) \mapsto (x^3, xy)$, so A has $A_1 := \operatorname{Jac} C_1$ as a factor.[4] (This is the quotient of C by $\mu_3 \subseteq \mu_3 \times \mu_9$ embedded by $\zeta \mapsto (\zeta, \overline{\zeta})$.) One can compute that C_1 is genus 3 using the Riemann–Hurwitz formula, and then we can compute that it has a basis of holomorphic differentials given by $\{x^2\,dx/y^8, x^2\,dx/y^7, x\,dx/y^5\}$, which pull back to the differentials $\{dx/y^8, x\,dx/y^7, dx/y^5\}$ on C (up to a scalar).
     Note that C_1 has an action by μ_9 by multiplying on the y-coordinate, so $\mathbb{Q}(\zeta_9) \subseteq \operatorname{End}_\mathbb{C}(A_1)_\mathbb{Q}$. However, $\dim A_1 = 3$, so we see that A_1 has complex multiplication. We will check that A_1 is simple shortly.
   - Note C projects onto the proper curve C_2 with affine chart $y^9 = x^6(x-1)$ by $(x, y) \mapsto (x^3, x^2y)$, so A has $A_2 := \operatorname{Jac} C_2$ as a factor. (This is the quotient of C by $\mu_3 \subseteq \mu_3 \times \mu_9$ embedded by $\zeta \mapsto (\zeta, \zeta)$.) One can compute that C_2 has genus 3 using the Riemann–Hurwitz formula, and then we can compute that it has a basis of holomorphic differentials given by $\{x^5\,dx/y^8, x^4\,dx/y^7, x^2\,dx/y^4\}$, which pull back to the differentials $\{x\,dx/y^8, dx/y^7, dx/y^4\}$ on C (up to a scalar).
     Note that C_2 has an action by μ_9 by multiplying on the y-coordinate, so $\mathbb{Q}(\zeta_9) \subseteq \operatorname{End}_\mathbb{C}(A_2)_\mathbb{Q}$. However, $\dim A_2 = 3$, so we see that A_2 has complex multiplication. We will check that A_2 is simple shortly.

   We spend a moment checking that A is isogenous to $C_0 \times A_1 \times A_2$. The above computations have provided a map $C_0 \times A_1 \times A_2 \to A$, so it is enough to check that this is an isomorphism after base-changing to $\mathbb{C}$. The computations above have shown that this map provides an isomorphism

$$\mathrm{H}^0\left(A, \Omega^1_{A/\mathbb{C}}\right) \to \mathrm{H}^0\left(C_0, \Omega^1_{C_0/\mathbb{C}}\right) \oplus \mathrm{H}^0\left(A_1, \Omega^1_{A_1/\mathbb{C}}\right) \oplus \mathrm{H}^0\left(A_2, \Omega^1_{A_2/\mathbb{C}}\right).$$

   (We take a moment to remark that the right-hand side is even a decomposition of $\mathrm{H}^0\left(A, \Omega^1_{A/\mathbb{C}}\right)$ into μ_3-eigenspaces!) This corresponds to an isomorphism on one piece of the Hodge structure, which

[4] Technically, we should take normalizations everywhere. We will omit these normalizations.

we note upgrades to an isomorphism of Hodge structures because the relevant Hodge structures are concentrated in $(0,1)$ and $(1,0)$, which are complex conjugates. We conclude that A is isogenous to $C_0 \times A_1 \times A_2$ by Theorem 3.40.

3. We compute some signatures. For our notation, we let $F_0 := \mathbb{Q}(\zeta_3)$ have the embeddings $\{\tau_1, \tau_2\}$, where $\tau_\bullet \in \operatorname{Gal}(F_0/\mathbb{Q})$ sends $\zeta_3 \mapsto \zeta_3^\bullet$; similarly, we let $F_1 = F_2 := \mathbb{Q}(\zeta_9)$ have the embeddings $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_7, \sigma_8\}$ where $\sigma_\bullet \in \operatorname{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ sends $\zeta_9 \mapsto \zeta_9^\bullet$. Here are our signatures.

   - On C_0, we see that H^{10} is spanned by dx/y^2, so with μ_3 acting on y, we get the signature $\Phi_0(\tau_1) = 1$ and $\Phi_0(\tau_2) = 0$.
   - On C_1, we see that H^{10} has basis given by $\left\{x^2\,dx/y^8, x^2\,dx/y^7, x\,dx/y^5\right\}$. Thus, with μ_9 acting on y, we get the signature
   $$\Phi_1(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{5,7,8\}, \\ 1 & \text{if } i \in \{1,2,4\}. \end{cases}$$
   One can check that Φ_1 satisfies the check of Remark 3.57, proving that A_1 is simple.
   - On C_1, we see that H^{10} has basis given by $\left\{x\,dx/y^8, dx/y^7, dx/y^4\right\}$. Thus, with μ_9 acting on y, we get the signature
   $$\Phi_2(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{4,7,8\}, \\ 1 & \text{if } i \in \{1,2,5\}. \end{cases}$$
   One can check that Φ_2 satisfies the check of Remark 3.57, proving that A_1 is simple.

   The above computation allows us to conclude that we have decomposed A into simple abelian varieties with complex multiplication.

4. We compute $\mathrm{MT}(A)_\mathbb{C}$. Because A has complex multiplication, we see that $\mathrm{MT}(A)$ is a torus by Proposition 3.53 embedded in T_F, where $F := F_0 \times F_1 \times F_2$. As such, we may use Proposition 1.96 and the surrounding discussion following the proof to compute equations cutting out $\mathrm{MT}(A) \subseteq \mathrm{T}_F$. In particular, set $L := \mathbb{Q}(\zeta_9)$, which we note is a Galois extension of $\mathbb{Q}$ containing $F_0F_1F_2$. Then we note that $\mathrm{H}^1_\mathrm{B}(A, L) = \mathrm{H}^1_\mathrm{B}(C_0, L) \oplus \mathrm{H}^1_\mathrm{B}(A_1, L) \oplus \mathrm{H}^1_\mathrm{B}(A_2, L)$ can be given a basis
   $$\{u_1, u_2, v_1, v_2, v_4, v_5, v_7, v_8, w_1, w_2, w_4, w_5, w_7, w_8\},$$
   where the subscript partially indicates the F-eigenvalue. Then we see that $(\mathrm{T}_F)_L \subseteq \mathrm{GL}\left(\mathrm{H}^1_\mathrm{B}(A, L)\right)$ embeds as
   $$\left\{\operatorname{diag}(\mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_7, \lambda_8, \kappa_1, \kappa_2, \kappa_4, \kappa_5, \kappa_7, \kappa_8) : \mu_\bullet, \lambda_\bullet, \kappa_\bullet \in \mathbb{G}_{m,L}\right\}.$$
   The discussion following Proposition 1.96 explains that equations cutting out $Z(\mathrm{MT}(A))^\circ_L \subseteq (\mathrm{T}_F)_L$ can be viewed as elements of the kernel of the map
   $$\mathrm{X}^*\left((\mathrm{N}_{\Phi_0^*}, \mathrm{N}_{\Phi_1^*}, \mathrm{N}_{\Phi_2^*})\right) : \mathrm{X}^*(\mathrm{T}_F) \to \mathrm{X}^*(T_L).$$
   Using the established bases for these lattices, we see that our map can be written as the matrix
   $$\begin{array}{cc} & \begin{array}{cc|cccccc|cccccc} \mu_1 & \mu_2 & \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 & \lambda_7 & \lambda_8 & \kappa_1 & \kappa_2 & \kappa_4 & \kappa_5 & \kappa_7 & \kappa_8 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_7 \\ \sigma_8 \end{array} & \left[\begin{array}{cc|cccccc|cccccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

Then one can compute a basis of the kernel of the matrix, which tells us that $\mathrm{MT}(A)_L \subseteq (\mathrm{T}_F)_L$ is cut out by the following equations. To begin, it turns out that $(A_1)_L$ and $(A_2)_L$ are isogenous, which we can see from the six equations

$$\begin{aligned}\lambda_1 &= \kappa_5,\\ \lambda_2 &= \kappa_1,\\ \lambda_4 &= \kappa_2,\\ \lambda_5 &= \kappa_7,\\ \lambda_7 &= \kappa_8,\\ \lambda_8 &= \kappa_4.\end{aligned}$$

(Namely, these equations imply an isomorphism of $\mathrm{MT}(A)$-representations $\mathrm{H}^1_{\mathrm{B}}(A_1, L) \cong \mathrm{H}^1_{\mathrm{B}}(A_2, L)$ and hence an isomorphism of Hodge structures, which gives the isogeny by Theorem 3.40.) Then there are the equations given by the polarization (via Lemma 1.75)

$$\begin{aligned}\mu_1\mu_2 &= \kappa_1\kappa_8,\\ \kappa_1\kappa_8 &= \kappa_2\kappa_7,\\ \kappa_1\kappa_8 &= \kappa_4\kappa_5.\end{aligned}$$

Lastly, there is the exceptional equation

$$\mu_1\kappa_7 = \kappa_5\kappa_8.$$

In total, we find that $\mathrm{MT}(A)_L$ is a torus isomorphic to $\mathbb{G}^4_{m,L}$ via the cocharacters $(\kappa_1, \kappa_2, \kappa_4, \kappa_8)$.

5. We use the previous step to compute $G^1_\ell(A_K)$ when ℓ splits completely in $K := K^{\mathrm{conn}}_A$. Recall that we know the Mumford–Tate conjecture that $G_\ell(A)^\circ = \mathrm{MT}(A)_{G_\ell}$ by Remark 3.88. Thus, we choose ℓ to split completely in K^{conn}_A so that $L \subseteq \mathbb{Q}_\ell$, allowing us to engage in the diagonalization of the previous step. Now, to compute $G^1_\ell(A_K)$ from $G_\ell(A_K)$, we simply need to add in the equation that the multipler is 1. This reveals that $G^1_\ell(A_{K^{\mathrm{conn}}_A}) \subseteq (\mathrm{T}_F)_{\mathbb{Q}_\ell}$ is cut out by the following equations. As before, we have the six equations

$$\begin{aligned}\lambda_1 &= \kappa_5,\\ \lambda_2 &= \kappa_1,\\ \lambda_4 &= \kappa_2,\\ \lambda_5 &= \kappa_7,\\ \lambda_7 &= \kappa_8,\\ \lambda_8 &= \kappa_4\end{aligned}$$

given by the isogeny $(A_1)_L \sim (A_2)_L$, and we have the equations given by the polarization

$$\begin{aligned}\mu_1\mu_2 &= 1,\\ \kappa_1\kappa_8 &= 1,\\ \kappa_2\kappa_7 &= 1,\\ \kappa_4\kappa_5 &= 1.\end{aligned}$$

Lastly, there is still the exceptional equation

$$\mu_1\kappa_7 = \kappa_5\kappa_8.$$

In total, we find that $G^1_\ell(A)$ is a torus isomorphic to $\mathbb{G}^3_{m,L}$ via the cocharacters $(\kappa_1, \kappa_2, \kappa_4)$. In total, we see $G^1_\ell(A_K)^\circ \subseteq \mathrm{GL}_{14}$ is

$$\left\{\operatorname{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4}, \frac{\kappa_1\kappa_4}{\kappa_2}, \kappa_4^{-1}, \kappa_1, \kappa_2, \kappa_2^{-1}, \kappa_1^{-1}, \kappa_4, \kappa_1, \kappa_2, \kappa_4, \kappa_4^{-1}, \kappa_2^{-1}, \kappa_1^{-1}\right) : \kappa_\bullet \in \mathbb{G}_{m,\mathbb{Q}_\ell}\right\}.$$

6. At last, we compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}. By Theorem 3.108, we see that ST does not depend on the choice of ℓ, so we may as well choose ℓ to split completely in K_A^{conn}. Then we may simply base-change the result of the previous step to $\mathbb{C}$, and then we may take maximal compact subgroups to see ST is
$$\left\{\operatorname{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4},\frac{\kappa_1\kappa_4}{\kappa_2},\kappa_4^{-1},\kappa_1,\kappa_2,\kappa_2^{-1},\kappa_1^{-1},\kappa_4,\kappa_1,\kappa_2,\kappa_4,\kappa_4^{-1},\kappa_2^{-1},\kappa_1^{-1}\right):\kappa_\bullet\in \mathrm{U}_1\right\}.$$
Once again, we remark that the product of maximal compact subgroups continues to be maximal compact. ■

Proposition 3.116. Define A to be the Jacobian of the proper curve C with affine chart $y^9 = x\left(x^2+1\right)$. Then $\mathrm{MT}(A)_{\mathbb{C}}$ is a torus isomorphic to $\mathbb{G}_{m,\mathbb{C}}^4$. We use this to compute $\mathrm{ST}(A_K)$ where K contains K_A^{conn}.

Proof. This argument is essentially the same as Proposition 3.115, so we will be a bit briefer.

1. Once again, we see that C has genus 7, so A has dimension 7, and we have a basis of holomorphic differentials given by
$$\left\{\frac{dx}{y^4},\frac{dx}{y^5},\frac{dx}{y^6},\frac{dx}{y^7},\frac{dx}{y^8},\frac{x\,dx}{y^7},\frac{x\,dx}{y^8}\right\}.$$
This time around, we see that μ_{18} acts on C by $\zeta_{18}\cdot(x,y)=(-x,-\zeta_9 y)$.

2. We decompose A into pieces.
   - As usual, C_0 projects onto $y^3 = x\left(x^2+1\right)$ by $(x,y)\mapsto\left(x,y^3\right)$. (This is the quotient of C by μ_3.) The Riemann–Hurwitz formula yields that C_0 is an elliptic curve with complex multiplication by μ_3 acting on the y-coordinate. We see that C_0 has a basis of holomorphic differentials given by dx/y^2, which pulls back to dx/y^6 ln C.
   - Now, C projects onto the proper curve C_1 with affine chart $y^9 = x^5(x+1)$ by $(x,y)\mapsto\left(x^2,xy\right)$, so A has $A_1 := \operatorname{Jac} C_1$ as a factor. (This is the quotient of C by μ_2.) The Riemann–Hurwitz formula implies that C_1 has genus 3, and then we can compute that it has a basis of holomorphic differentials given by $\left\{x^4\,dx/y^8,x^3\,dx/y^7,x^2\,dx/y^5\right\}$, which pulls back to $\left\{x\,dx/y^8,dx/y^7,dx/y^5\right\}$ on C (up to scalar).
     Note that C_1 has an action by μ_9 acting on the y-coordinate, so $\mathbb{Q}(\zeta_9)\subseteq\operatorname{End}_{\mathbb{C}}(A_1)_{\mathbb{Q}}$. We will check in the next step that A_1 is simple by computing its signature and applying Remark 3.57.

   We can see on the level of differentials that the induced map $C_0\times A_1\to A$ is injective, so we let A_2 be the cokernel. In terms of Hodge structures, we can see from the computation that
$$\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})=\mathrm{H}^1_{\mathrm{B}}(C_0,\mathbb{Q})\oplus\mathrm{H}^1_{\mathrm{B}}(A_1,\mathbb{Q})\oplus\mathrm{H}^1_{\mathrm{B}}(A_2,\mathbb{Q})$$
   is a decomposition of μ_{18}-representations because the left two spaces on the right-hand side are stable under the μ_{18}-action. We conclude that $\mathbb{Q}(\zeta_9)\subseteq\operatorname{End}_{\mathbb{C}}(A_2)_{\mathbb{Q}}$ as well.

3. We compute some signatures. As before, we let $F_0 := \mathbb{Q}(\zeta_3)$ have $\{\tau_1,\tau_2\}=\operatorname{Gal}(\mathbb{Q}(F_0)/\mathbb{Q})$ where $\tau_\bullet\colon\zeta_3\mapsto\zeta_3^\bullet$, and we let $F_1=F_2:=\mathbb{Q}(\zeta_9)$ have $\{\sigma_1,\ldots,\sigma_8\}=\operatorname{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ has $\sigma_\bullet\colon\zeta_9\mapsto\zeta_9^\bullet$.
   - On C_0, we look at the μ_9-eigenbasis of H^{10} to conclude that our signature has $\Phi_0(\tau_1)=1$ and $\Phi_0(\tau_2)$.
   - On C_1, we look at the μ_9-eigenbasis of H^{10} to conclude that our signature is
$$\Phi_1(\sigma_i)=\begin{cases}0 & \text{if } i\in\{4,7,8\},\\ 1 & \text{if } i\in\{1,2,5\}.\end{cases}$$
   One can check that Φ_1 satisfies the check of Remark 3.57, proving that A_1 is simple.

- On A_2, we take the remaining differentials from A to find that our signature is

$$\Phi_2(\sigma_i) = \begin{cases} 0 & \text{if } i \in \{5,7,8\}, \\ 1 & \text{if } i \in \{1,2,4\}. \end{cases}$$

  Again, one checks that Φ_2 satisfies the check of Remark 3.57

4. At this point, we recognize that our signatures are the same as in Proposition 3.115 up to swapping Φ_1 and Φ_2. Thus, up to some reordering of letters, the exact same computation goes through. Let's provide the result.

   To be explicit, we give $\mathrm{H}^1_{\mathrm{B}}(A,L) = \mathrm{H}^1_{\mathrm{B}}(C_0,L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_1,L) \oplus \mathrm{H}^1_{\mathrm{B}}(A_2,L)$ a basis

$$\{u_1,u_2,v_1,v_2,v_4,v_5,v_7,v_8,w_1,w_2,w_4,w_5,w_7,w_8\},$$

   where the subscript partially indicates the F-eigenvalue, where $F := F_0 \times F_1 \times F_2$. Then we set $L := \mathbb{Q}(\zeta_9)$, and we see $(\mathrm{T}_F)_L \subseteq \mathrm{GL}\left(\mathrm{H}^1_{\mathrm{B}}(A,L)\right)$ embeds as

$$\{\operatorname{diag}(\mu_1,\mu_2,\kappa_1,\kappa_2,\kappa_4,\kappa_5,\kappa_7,\kappa_8,\lambda_1,\lambda_2,\lambda_4,\lambda_5,\lambda_7,\lambda_8) : \mu_\bullet,\kappa_\bullet,\lambda_\bullet \in \mathbb{G}_{m,L}\}.$$

   With this choice of lettering, the equations that end up cutting out $\mathrm{MT}(A)_L \subseteq (\mathrm{T}_F)_L$ are exactly the same, so $\mathrm{MT}(A)_L \cong \mathbb{G}^4_{m,L}$ via the cocharacters $(\kappa_1,\kappa_2,\kappa_4,\kappa_8)$.

   One is now able to compute $G^1_\ell(A)$ in the case where ℓ splits completely in $K := K_A^{\mathrm{conn}}$. One finds the exact same equations via the same computation, so we find $G^1_\ell(A_K) \subseteq \mathrm{GL}_{14}$ is given by

$$\left\{\operatorname{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4},\frac{\kappa_1\kappa_4}{\kappa_2},\kappa_1,\kappa_2,\kappa_4,\kappa_4^{-1},\kappa_2^{-1},\kappa_1^{-1},\kappa_4^{-1},\kappa_1,\kappa_2,\kappa_2^{-1},\kappa_1^{-1},\kappa_4\right) : \kappa_\bullet \in \mathbb{G}_{m,\mathbb{Q}_\ell}\right\}.$$

   Base-changing to $\mathbb{C}$ and taking a maximal compact subgroup, we find $\mathrm{ST}(A_K)$ is

$$\left\{\operatorname{diag}\left(\frac{\kappa_2}{\kappa_1\kappa_4},\frac{\kappa_1\kappa_4}{\kappa_2},\kappa_1,\kappa_2,\kappa_4,\kappa_4^{-1},\kappa_2^{-1},\kappa_1^{-1},\kappa_4^{-1},\kappa_1,\kappa_2,\kappa_2^{-1},\kappa_1^{-1},\kappa_4\right) : \kappa_\bullet \in \mathrm{U}_1\right\},$$

   as required. ■

3.3.4 Moment Statistics

In this subsection, we explain how to numerically verify the Sato–Tate conjecture (Conjecture 3.107). Fix an abelian variety A of dimension g defined over a number field K, and choose a prime ℓ and embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$; for example, this allows us to define the usual ℓ-adic representation $\rho_\ell\colon \operatorname{Gal}(\overline{K}/K) \to \mathrm{GL}\left(\mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Q}_\ell)\right)$.

The main idea is that the map sending $g \in \mathrm{ST}(A)$ to the characteristic polynomial of $g \in \mathrm{GL}_{2g}(\mathbb{C})$ is well-defined up to conjugacy classes, so it defines a (continuous) map $\operatorname{Conj}(\mathrm{ST}(A)) \to \mathbb{C}^{2g+1}$, where $\mathbb{C}^{2g+1}$ simply lists out the coefficients of the characteristic polynomial. In this way, we can push the Haar measure on $\mathrm{ST}(A)$ all the way to $\mathbb{C}^{2g+1}$ to compute what the distribution of the characteristic polynomial will be.

Of course, in practice, it may be difficult to compute the characteristic polynomial of

$$\left[\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right] \in \operatorname{Conj}(\mathrm{ST}(A))$$

for some prime $\mathfrak{p}$ of K such that A has good reduction at $\mathfrak{p}$. For our application, we will only be interested in superelliptic curves, for which this can be computed in SageMath [Aru+19]. To help out the computation a bit more, we make two quick remarks.

Remark 3.117. Let $P(T)$ be the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ acting on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}_\mathfrak{p}}},\mathbb{Q}_\ell)$. Then we remark that $P(1)$ has a geometric interpretation as $\#A(\mathbb{F}_\mathfrak{p})$.

Remark 3.118. It suffices to only consider primes $\mathfrak{p}$ which are totally split in K because such primes have density 1. This is helpful because primes that split $\mathfrak{p}$ completely have residue fields isomorphic to $\mathbb{F}_p$ where $p \in \mathbb{Z}$ is the prime sitting below $\mathfrak{p}$, so we are frequently able to reduce the computation to something only involving integral coefficients.

As before, let's begin with some elliptic curve examples. Here, we note that the characteristic polynomial of $\frac{1}{\mathrm{N}(\mathfrak{p})}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$ will have degree 2, with leading coefficient 1, and the condition on the multiplier (from Proposition 3.103) implies that the constant coefficient is 1. Thus, we see that the only interesting coefficient of the characteristic polynomial is given by the trace.

Lemma 3.119. The map $\operatorname{tr}\colon \operatorname{Conj}(\mathrm{SU}_2) \to [-2,2]$ is a homeomorphism, and the pushforward of the normalized Haar measure of SU_2 onto $\operatorname{Conj}(\mathrm{SU}_2) = [-2,2]$ is given by the semicircle measure $\frac{1}{2\pi}\sqrt{4-t^2}\,dt$.

Proof. We show the claims separately.

1. We show that $\operatorname{tr}\colon \operatorname{Conj}(\mathrm{SU}_2) \to [-2,2]$ is a well-defined homeomorphism. Note that $\operatorname{tr}\colon \operatorname{Conj}(\mathrm{SU}_2) \to \mathbb{C}$ is continuous, and all spaces in sight are compact and Hausdorff, so it is enough to check that tr is a bijection.

   A priori, tr is only defined as a map $\operatorname{tr}\colon \operatorname{Conj}(\mathrm{SU}_2) \to \mathbb{C}$. To begin, we note that any element of SU_2 is diagonalizable by a unitary matrix, and the corresponding diagonal matrix must then look like $\operatorname{diag}(\lambda, \overline{\lambda})$ where $|\lambda|^2 = 1$. By writing $\lambda = e^{i\theta}$, we see that the trace of this element is $2\cos\theta$, so we see that $\operatorname{tr}\colon \operatorname{Conj}(\mathrm{SU}_2) \to [-2,2]$ is a well-defined surjection.

   It remains to check that tr is injective. Because each conjugacy class is represented by a diagonal matrix, it is enough to check that $g_1 := \operatorname{diag}(\lambda_1, \overline{\lambda}_1)$ and $g_2 := \operatorname{diag}(\lambda_2, \overline{\lambda}_2)$ have $\operatorname{tr} g_1 = \operatorname{tr} g_2$ only if g_1 and g_2 are conjugate. Well, write $\lambda_\bullet = e^{i\theta_\bullet}$, and then we see that
   $$2\cos\theta_1 = 2\cos\theta_2,$$
   which implies that $\{\pm\theta_1\} = \{\pm\theta_2\}$, so $\{\lambda_1, \overline{\lambda}_1\} = \{\lambda_2, \overline{\lambda}_2\}$. We now do casework: if $\lambda_1 = \lambda_2$, then we see that $g_1 = g_2$ on the nose; otherwise, $\lambda_1 = \overline{\lambda}_2$, and we see that
   $$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \overline{\lambda}_1 \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \begin{bmatrix} \lambda_2 & \\ & \overline{\lambda}_2 \end{bmatrix},$$
   so g_1 is conjugate to g_2.

2. We now compute the required measures. A linear algebra argument with the condition $gg^\dagger = 1_2$ shows that any element of SU_2 can be written uniquely in the form
   $$\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}$$
   where $|\alpha|^2 + |\beta|^2 = 1$. In this way, we see that SU_2 is isomorphic to the unit group of the quaternions $\mathbb{H}$, so SU_2 is diffeomorphic to S^3 and inherits a Haar measure by pullback. Explicitly, one finds that SU_2 inherits an action on S^3 by rotations, so the Lebesgue measure on S^3 is invariant under the group. Note that we have yet to normalize the Haar mesure on SU_2.

   We would now like to compute the volume of SU_2 with given trace t. Writing $\alpha = a + bi$ and $\beta = c + di$, we see that we are forcing $a = \frac{1}{2}t$, which then requires the remaining coordinates to live in a sphere of radius $\sqrt{1 - \frac{1}{4}t^2}$. Thus, we see that our normalized Haar measure is
   $$\frac{\sqrt{1-\frac{1}{4}t^2}\,dt}{\int_{-2}^{2}\sqrt{1-\frac{1}{4}t^2}\,dt}.$$
   A quick substitution with $t = 2\cos\theta$ in the bottom integral reveals that it equals π, whereupon we find that the desired measure is $\frac{1}{2\pi}\sqrt{4-t^2}\,dt$ after some rearranging. ■

Remark 3.120. In the sequel, it is occasionally more convenient to identify $\operatorname{Con}(\mathrm{SU}_2)$ with the collection of diagonal matrices $\operatorname{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ where $\theta \in [0,\pi)$. Then we see that the trace is $2\cos\theta$, so we produce a measure of $\frac{2}{\pi}\sin^2\theta\, d\theta$ on $[0,\pi)$.

Example 3.121 (no complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + x + 1$ over $\mathbb{Q}$ studied in Example 3.109. Then we recall that $\mathrm{ST}(E) = \mathrm{SU}_2$, so we may use the computation of Lemma 3.119 to see that the Sato–Tate conjecture (Conjecture 3.105) implies that the values

$$\left\{\operatorname{tr}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right\}_{\mathfrak{p}\text{ prime}}$$

equidistribute according to the semicircle measure $\frac{1}{2\pi}\sqrt{4-t^2}\,dt$ on $[-2,2]$.

Example 3.122 (complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$ studied in Example 3.110. Then we recall that $\mathrm{ST}(E) \cong \mathrm{U}_1$ embedded as $z \mapsto \operatorname{diag}(z,\overline{z})$. We may write U_1 as $\mathrm{U}_1 = \left\{e^{i\theta} : \theta \in [0,2\pi)\right\}$, so we can equip this group with the normalized Haar measure $\frac{1}{2\pi}\,d\theta$. (The map $e^{i\theta} \mapsto \theta$ is a homeomorphism away from a set of measure 0.) Noting the trace of $\operatorname{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ is $2\cos\theta$, we see the Sato–Tate conjecture (Conjecture 3.105) implies that the values

$$\left\{\operatorname{tr}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right\}_{\mathfrak{p}\text{ prime}}$$

equidistribute according to the measure $\frac{1}{\pi}\cdot\frac{1}{\sqrt{4-t^2}}\,dt$ on $[-2,2]$.

Example 3.123 (potential complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$ studied in Example 3.111. Then we recall that $\mathrm{ST}(E) \cong \mathrm{U}_1 \rtimes S_2$, where $\mathrm{U}_1 \subseteq \mathrm{GL}_{2,\mathbb{C}}$ is embedded as $z \mapsto \operatorname{diag}(z,\overline{z})$, and $S_2 = \{1, w\}$ acts by switching the coordinates. Again, we give $\mathrm{U}_1 = \left\{e^{i\theta} : \theta \in [0,2\pi)\right\}$ the normalized Haar measure $\frac{1}{2\pi}\,d\theta$, so $\mathrm{U}_1 \rtimes S_2$ gets the normalized Haar measure $\frac{1}{4\pi}\,d\theta$. For $u = \operatorname{diag}\left(e^{i\theta}, e^{-i\theta}\right) \in U$, we note that the trace of $(u,1) \in \mathrm{U}_1 \rtimes S_2$ is simply $2\cos\theta$ while the trace of $(u,w) \in \mathrm{U}_1 \rtimes S_2$ vanishes. Thus, we see the Sato–Tate conjecture (Conjecture 3.105) implies that the values

$$\left\{\operatorname{tr}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right\}_{\mathfrak{p}\text{ prime}}$$

equidistribute according to the measure $\frac{1}{2\pi}\cdot\frac{1}{\sqrt{4-t^2}}\,dt + \frac{1}{2}\delta_0\,dt$ on $[-2,2]$. Here, δ_0 refers to the δ-distribution concentrated at 0.

We now return to the Jacobian of (the normalization of the proper curve with affine chart) $y^9 = x(x-1)(x-\lambda)$. It will be helpful to take products of Haar measures in the sequel. The following result is an easier form of [DE14, Proposition 1.5.6].

Lemma 3.124. Fix a locally compact topological group G. Suppose that there are closed subgroups $H, K \subseteq G$ such that $G = HK$ and $K \subseteq C_G(H)$. Letting dh and dg be left Haar measures on H and K, respectively, we find that $dk\,dh$ is a left Haar measure on G.

Proof. We are tasked with showing that the integral

$$\int_H \int_K f(hk)\,dk\,dh$$

is left-invariant for G. It is left-invariant for H with no content, so it suffices to show the same for K. This follows after some manipulation because K commutes with H. ■

Remark 3.125. In fact, [DE14, Proposition 1.5.6] shows something much stronger: one can replace the strong group-theoretic condition that $K \subseteq C_G(H)$ with merely that K is compact. In fact, a careful reading of the proof there reveals that we may even replace the condition that K is compact with merely having $H \cap K$ compact and $\Delta_G|_K = 1$, where Δ_G is the modular function on G.

Here is our application.

Proposition 3.126. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x(x-1)(x-\lambda)$, where λ lives in a number field. Suppose that A does not have complex multiplication. We compute a Haar measure on $\operatorname{ST}(A_K)$ whenever K contains K_A^{conn}.

Proof. The Sato–Tate computation of Proposition 3.113 (combined with the conjugacy class computation of Lemma 3.119) reveals that an element of $\operatorname{Conj}(\operatorname{ST}(A))$ can be written as

$$\operatorname{diag}\left(\begin{bmatrix} e^{i\alpha_0} & \\ & e^{-i\alpha_0}\end{bmatrix}, \begin{bmatrix} e^{i\alpha_1+i\theta_1} & \\ & e^{i\alpha_1-i\theta_1}\end{bmatrix}, \begin{bmatrix} e^{i\alpha_1+i\alpha_4+i\theta_2} & \\ & e^{i\alpha_1+i\alpha_4-i\theta_2}\end{bmatrix}, \begin{bmatrix} e^{i\alpha_4+i\theta_4} & \\ & e^{i\alpha_4-i\theta_4}\end{bmatrix},\right.$$
$$\left.\begin{bmatrix} e^{-i\alpha_4+i\theta_4} & \\ & e^{-i\alpha_4-i\theta_4}\end{bmatrix}, \begin{bmatrix} e^{-i\alpha_1-i\alpha_4+i\theta_2} & \\ & e^{-i\alpha_1-i\alpha_4-i\theta_2}\end{bmatrix}, \begin{bmatrix} e^{-i\alpha_1+i\theta_1} & \\ & e^{-i\alpha_1-i\theta_1}\end{bmatrix}\right)$$

where $\alpha_\bullet \in [0, 2\pi)$ and $\theta_\bullet \in [0,\pi)$. Technically, the map $(\alpha_\bullet, \theta_\bullet)\colon [0,2\pi)^4 \times [0,\pi)^3 \to \operatorname{Conj}(\operatorname{ST}(A))$ is the finite-to-one because $Z(\operatorname{ST}(A))^\circ \cap \operatorname{ST}(A)^{\text{der}}$ is finite, but this will make no effect on our computations as long as we normalize to have total volume 1 and only integrate against genuine functions on $\operatorname{Conj}(\operatorname{ST}(A))$.

Anyway, we see that the trace is given by

$$\begin{aligned} &2\cos\alpha_0 + 2\cos(\alpha_1+\theta_1) + 2\cos(\alpha_1-\theta_1) + 2\cos(\alpha_1+\alpha_4+\theta_2) + 2\cos(\alpha_1+\alpha_4-\theta_2) \\ &\quad + 2\cos(\alpha_4+\theta_4) + 2\cos(\alpha_4-\theta_4). \end{aligned}$$

We finish by remarking that Lemma 3.124 gives our Haar measure as

$$\frac{1}{(2\pi)^3}\,d\alpha_0\,d\alpha_1\,d\alpha_4 \cdot \frac{1}{\pi^2}\left(2\sin^2\theta_1 \cdot 2\sin^2\theta_2 \cdot 2\sin^2\theta_4\right) d\theta_1\,d\theta_2\,d\theta_4,$$

which is what we wanted. (Note we used Remark 3.120 for the Haar measure on SU_2.) ■

Proposition 3.127. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x^3 - 1$. Suppose that A does not have complex multiplication. We compute a Haar measure on $\operatorname{ST}(A_K)$ whenever K contains K_A^{conn}.

Proof. The Sato–Tate computation of Proposition 3.115 reveals that an element of $\operatorname{Conj}(\operatorname{ST}(A))$ can be written as

$$\operatorname{diag}\left(e^{i\alpha_2-i\alpha_1-\alpha_4}, e^{i\alpha_1+i\alpha_4-i\alpha_2}, e^{-i\alpha_4}, e^{i\alpha_1}, e^{i\alpha_2}, e^{-\alpha_2}, e^{-\alpha_1}, e^{i\alpha_4}, e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_4}, e^{-i\alpha_4}, e^{-i\alpha_2}, e^{-i\alpha_1}\right)$$

where $\alpha_\bullet \in [0, 2\pi)$. For example, we see that the trace is given by

$$2\cos\cos(\alpha_1-\alpha_2+\alpha_4) + 4\cos\alpha_1 + 4\cos\alpha_2 + 4\cos\alpha_4$$

We finish by remarking that Lemma 3.124 gives our Haar measure as

$$\frac{1}{(2\pi)^3}\,d\alpha_1\,d\alpha_2\,d\alpha_4,$$

which is what we wanted. ■

Remark 3.128. As remarked at the end of the proof of Proposition 3.116, we can run the exact same computation with working the curve given by $y^9 = x\left(x^2+1\right)$ because the resulting Sato–Tate group is the same up to reordering the basis.

Remark 3.129. For the previous examples, there are more interesting coefficients in the characteristic polynomial than merely the trace. Hoever, they are rather lengthy to write down, so we have chosen not to.

It still remains to explain how we numerically verify the Sato–Tate conjecture. The idea is that we can try to compute

$$\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})$$

for various primes $\mathfrak{p}$ and then compare it with what is expected from

$$\int_{\mathrm{Conj}(\mathrm{ST}(A))} \operatorname{tr} g\, dg,$$

where dg refers to the pushforward of the Haar measure from $\mathrm{Conj}(\mathrm{ST}(A))$. One usually expects the above integral to vanish, so one can either look at other coefficients of the characteristic polynomial or at powers of $\operatorname{tr} g$. In the sequel, we will compute with only powers of $\operatorname{tr} g$ for simplicity, but we do remark that one can typically recover the other coefficients via a combination of Vieta's formulae and Newton's sums.

As usual, let's begin with elliptic curves. Here, explicit formulae are possible.

Example 3.130 (no complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + x + 1$ over $\mathbb{Q}$ studied in Examples 3.109 and 3.121. Fix some integer $m \geq 0$. Using the given Haar measure (from Remark 3.120), we find that one expects the average of $\left\{\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m\right\}_{\mathfrak{p} \text{ prime}}$ to be

$$\int_0^\pi (2\cos\theta)^m \frac{2}{\pi} \sin^2\theta\, d\theta = \begin{cases} \frac{1}{m/2+1}\binom{m}{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where the last equality is verified by expanding $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and $4\sin^2\theta = 2 - e^{2i\theta} - e^{-2i\theta}$.

Example 3.131 (complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}(\zeta_3)$ studied in Examples 3.110 and 3.122. Fix some integer $m \geq 0$. Using the given Haar measure, we find that one expects the average of $\left\{\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}} \iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m\right\}_{\mathfrak{p} \text{ prime}}$ to be

$$\int_0^{2\pi} (2\cos\theta)^m \frac{1}{2\pi}\, d\theta = \begin{cases} \binom{m}{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where the last equality is verified by expanding $2\cos\theta = e^{i\theta} + e^{-i\theta}$.

Example 3.132 (complex multiplication)**.** We continue with the elliptic curve $E\colon y^2 = x^3 + 1$ over $\mathbb{Q}$ studied in Examples 3.111 and 3.123. Fix some integer $m \geq 0$. Using the given Haar measure, we find that one expects the average of $\left\{\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m\right\}_{\mathfrak{p}\text{ prime}}$ to be

$$\int_0^{2\pi} (2\cos\theta)^m \frac{1}{4\pi}\,d\theta = \begin{cases} \frac{1}{2}\binom{m}{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

where the last equality is verified by expanding $2\cos\theta = e^{i\theta} + e^{-i\theta}$.

We now return to $y^9 = x(x-1)(x-\lambda)$. Here, we do not attempt to give explicit formulae, but we list the first few expected values, which were computed using SageMath.

Example 3.133. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x(x-1)(x-10)$. SageMath can verify that A does not have complex multiplication. For $m \in \{0, 1, \ldots, 6\}$, we use Proposition 3.126 to find that we expect the aveage of $\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m$ as $\mathfrak{p}$ varies over primes K (for K containing K_A^{conn}) to be as follows.

m	0	1	2	3	4	5	6
expected	1	0	8	0	186	0	7160
actual	1.0	0.0	7.8	0.2	180	16	6400

Here, the "actual" amounts have been rounded to two significant digits, and they were computed by averaging over primes $p < 216289$ which were $1 \pmod 9$; the condition $p \equiv 1 \pmod 9$ corresponds to splitting completely in $\mathbb{Q}(\zeta_9)$ (see Remark 3.118). These "actual" amounts suggest that $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, a fact which we will verify in the next chapter.

Example 3.134. Let A be the Jacobian of the normalization of the proper curve with affine chart $y^9 = x^3 - 1$, where λ lives in a number field. For $m \in \{0, 1, \ldots, 6\}$, we use Proposition 3.127 to find that we expect the aveage of $\left(\operatorname{tr} \frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\iota\rho_\ell(\mathrm{Frob}_\mathfrak{p})\right)^m$ as $\mathfrak{p}$ varies over primes K (for K containing K_A^{conn}) to be as follows.

m	0	1	2	3	4	5	6
expected	1	0	26	0	2118	0	239300
actual	1.0	0.0	25	6.0	2000	890	220000

Here, the "actual" amounts have been rounded to two significant digits, and they were computed by averaging over primes $p < 100000$ which were $1 \pmod 9$; the condition $p \equiv 1 \pmod 9$ corresponds to splitting completely in $\mathbb{Q}(\zeta_9)$ (see Remark 3.118). These "actual" amounts suggest that $K_A^{\mathrm{conn}} = \mathbb{Q}(\zeta_9)$, a fact which we will verify in the next chapter.

Remark 3.135. If one runs the same computation as in the previous example with $y^9 = x\left(x^2+1\right)$, one should further restrict primes past $p \equiv 1 \pmod 9$ in order to see the correct moment statistics. This is because now $K_A^{\mathrm{conn}} \neq \mathbb{Q}(\zeta_9)$.

Chapter 4

The Fermat Curve

In ths chapter, we will study the Galois representation attached to the projective $\mathbb{Q}$-curve $X_N^1 \subseteq \mathbb{P}^1_{\mathbb{Q}}$ cut out by the equation

$$X_N \colon X^N + Y^N + Z^N = 0,$$

where $N \geq 3$ is some nonnegative integer. For the rest of this chapter, we will fix N and thus denote this curve by $X \subseteq \mathbb{P}^1_{\mathbb{Q}}$. It is worthwhile to summarize the basic steps of the computation.

4.1 Homology and Cohomology

The exposition of this section follows [Ots16, Sections 2 and 3]. We will spend this section setting up some notation and proving basic facts about how these objects relate to each other.

4.1.1 The Group Action

Throughout, it will be helpful to note that the finite alegbraic $\mathbb{Q}$-group

$$G_N \coloneqq \frac{\mu_N \times \mu_N \times \mu_N}{\Delta \mu_N}$$

acts on X_N; here, $\Delta\mu_N \subseteq \mu_N \times \mu_N \times \mu_N$ refers to the diagonally embedded copy of μ_N. As with X_N, we will denote this group by G for the rest of the chapter, and we will let $\zeta \coloneqq \zeta_N$ be a primitive Nth root of unity.

Notably, the action map $G \times X \to X$ is defined over $\mathbb{Q}$ even though $G(\mathbb{Q})$ is trivial. For brevity, we will denote elements of G by $g_{[r:s:t]} \coloneqq [\zeta^r : \zeta^s : \zeta^t]$. We will also have occasion to study the character group $\widehat{G} \coloneqq \widehat{G}_N$, which we identify with

$$\widehat{G}_N = \left\{(a,b,c) \in (\mathbb{Z}/N\mathbb{Z})^3 : a+b+c = 0\right\}.$$

Explicitly, given a triple (a,b,c), we let $\alpha_{(a,b,c)}$ denote the corresponding character, which sends $g_{[r:s:t]} \mapsto \zeta^{ra+bs+tc}$.

In the sequel, we will have many vector spaces induced by X via (co)homology, which therefore have a G-action by functoriality. With this in mind, we make the following definition.

Definition 4.1. Given a $\mathbb{Q}(\zeta)$-vector space H with a G-action, we define

$$\mathrm{H}_\alpha \coloneqq \{v \in \mathrm{H} : g \cdot v = \alpha(g)v\}$$

to be the α-eigenspace for each $\alpha \in \widehat{G}$.

One inconvenience of this definition is that the vector spaces H of interest are frequently defined over $\mathbb{Q}$, but H_α is not.

Thus, we note that some $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\widehat{G}$ as follows: say $\tau(\zeta) = \zeta^u$ for some $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, and then

$$(\tau\alpha)([\zeta^r : \zeta^s : \zeta^t]) = \alpha\left([\zeta^{u^{-1}r}, \zeta^{u^{-1}s} : \zeta^{u^{-1}t}]\right),$$

so we see that $\tau\alpha = u^{-1}\alpha$, where the multiplication $u^{-1}\alpha$ is understood to happen where α is a triple in $(\mathbb{Z}/N\mathbb{Z})^3$. With this in mind, given $\alpha \in \widehat{G}$, we let $[\alpha] \subseteq \widehat{G}$ be the collection of characters of the form $u\alpha$ as $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ varies; for example, $-\alpha \in [\alpha]$. The point of this discussion is that we are able to build a decomposition

$$\mathbb{Q}[G] \cong \prod_{[\alpha] \in G/(\mathbb{Z}/N\mathbb{Z})^\times} \mathbb{Q}([\alpha]),$$

where $\mathbb{Q}([\alpha])$ is the image of the map $\mathbb{Q}[G] \to \mathbb{C}$ given by the characters in $[\alpha]$. We are now ready to make the following definition.

Definition 4.2. Given some $\mathbb{Q}$-vector space H with a G-action, we are now ready to define

$$\mathrm{H}_{[\alpha]} := \left\{ v \in \mathrm{H} : v \otimes 1 \in \bigoplus_{\beta \in [\alpha]} (\mathrm{H} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_\alpha \right\}.$$

The discussion of the Galois action of the previous paragraph implies that $\mathrm{H}_{[\alpha]}$ is a generalized eigenspace of the G-action on H. In particular, we find that $\mathrm{H}_{[\alpha]} \otimes \overline{\mathbb{Q}} = \bigoplus_{\beta \in [\alpha]} \mathrm{H}_\beta$, so $\mathrm{H} = \bigoplus_{[\alpha]} \mathrm{H}_{[\alpha]}$.

4.1.2 Differential Forms

In this subsection, we will define a few differential forms. A reasonable reference for this subsection is [Lan11, Section 1.7]. A computation with the Riemann–Hurwitz formula shows that the genus of X is $\frac{(N-1)(N-2)}{2}$, so we know that there are many holomorphic differential forms. On the other hand, we know that the space of differential forms lives in $\mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C})$, which is equipped with a G-action. Anyway, we are now ready to define our differential form.

Definition 4.3. Fix notation as above. For $a \in \mathbb{Z}/N\mathbb{Z}$, let $[a]$ be a representative in $\{0, 1, \ldots, N-1\}$. For any $\alpha_{(a,b,c)} \in \widehat{G}$, we define the differential form

$$\omega_{\alpha_{(a,b,c)}} := x^{[a]} y^{[b]-N} \frac{dx}{x}$$

in the affine patch $x^N + y^N + 1 = 0$ of X. In the sequel, we may also denote this differential form by $\omega_{(a,b,c)}$.

Remark 4.4. Because $x^N + y^N + 1 = 0$ implies $x^{N-1}\,dx = -y^{N-1}\,dy$, we also see that

$$\omega_{(a,b,c)} = -x^{[a]-N} y^{[b]} \frac{dy}{y}.$$

Further, we can pass to the affine patch $1 + v^N + u^N = 0$ of X by substituting $(x, y) = (1/u, v/u)$, for which we note $d(1/u)/(1/u) = -du/u$ so that

$$\omega_{(a,b,c)} = -u^{N-[a]-[b]} v^{[b]-N} \frac{du}{u}.$$

From Remark 4.4, we see that $\omega_{(a,b,c)}$ always succeeds at being meromorphic with poles only at points of the form $[X : Y : 0]$, and it is closed (i.e., has vanishing residues) if and only if $0 \notin \{a, b, c\}$. Further, we see that $\omega_{(a,b,c)}$ succeeds at being holomorphic provided that we also have $[a] + [b] < N$, which we note is equivalent to $[a] + [b] + [c] = N$.

We have now provided $\frac{(N-1)(N-2)}{2}$ holomorphic differentials of X, so we would like to check that we have actually found a basis of $\mathrm{H}^0(X(\mathbb{C}), \Omega^1_{X/\mathbb{C}})$. Well, these differential forms are nonzero by construction,[1] and they are linearly independent because they are all eigenvectors for the G-action.

Lemma 4.5. Fix notation as above. For each $\alpha \in \widehat{G}$, the differential form ω_α is an eigenvector for the G-action with eigenvalue α.

Proof. Say $\alpha = \alpha_{(a,b,c)}$ for some $a, b, c \in \mathbb{Z}/N\mathbb{Z}$. Then for any $g_{[r:s:0]} \in G$, we note

$$\begin{aligned} (g_{[r:s:0]})^* \omega_{(a,b,c)} &= (\zeta^r x)^{[a]} (\zeta^s y)^{[b]-N} \frac{d(\zeta^r x)}{(\zeta^r x)} \\ &= \zeta^{ar+bs} \cdot x^{[a]} y^{[b]-N} \frac{dx}{x} \\ &= \alpha_{(a,b,c)}(g_{[r:s:0]}) \omega_{(a,b,c)}. \end{aligned}$$

The reason to $g_{[r:s:0]}$ in the above computation is that we need the G-action to stay in the affine patch of points of the form $[X : Y : 1]$. ■

Remark 4.6. Thus, we see that our differential forms must be linearly independent because they are eigenvectors with different eigenvalues. As such, we have constructed eigenbases of $\mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C})$ and $\mathrm{H}^0(X(\mathbb{C}), \Omega^1_{X/\mathbb{C}})$.

While we're here, we compute the Poincaré pairing of our basis of differential forms.

Lemma 4.7. Fix notation as above. Choose $\alpha, \alpha' \in \widehat{G}$ such that $\alpha = (a, b, c)$ and $\alpha' = (a', b', c')$ have nonzero entries. Then the Poincaré pairing

$$P \colon \mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C}) \times \mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}), \mathbb{C})) \to \mathbb{C}$$

given by $(\omega, \eta) \mapsto \frac{1}{2\pi i} \int_X (\omega \wedge \eta)$ sends $(\omega_\alpha, \omega_{\alpha'})$ to

$$P(\omega_\alpha, \omega_{\alpha'}) = \begin{cases} 0 & \text{if } \alpha \neq -\alpha', \\ (-1)^N \frac{N}{N-[a]-[b]} & \text{if } \alpha = -\alpha'. \end{cases}$$

Proof. We use the Poincaré residue, which implies that

$$P(\omega, \eta) = \sum_{x \in X(\mathbb{C})} \operatorname{Res}_x \left(\eta \int \omega \right),$$

where the sum is over poles, and $\int \omega$ refers to any choice of local primitive for ω in the neighborhood of x. To use this, we note that the computation of Remark 4.4 implies that ω_α and $\omega_{\alpha'}$ can only have poles at the points $[1 : -\zeta^s : 0]$ for some $s \in \mathbb{Z}/N\mathbb{Z}$, and in this neighborhood, we may write

$$\omega_\alpha = -u^{N-[a]-[b]} v^{[b]-N} \frac{du}{u}$$

[1] Later, Remark 4.11 will give another way to prove this via periods.

and similarly for $\omega_{\alpha'}$. In particular, we see that

$$-\frac{1}{N-[a]-[b]}u^{N-[a]-[b]}v^{[b]-N}$$

makes a reasonable primitive for ω_α, so the Poincaré residue yields

$$P(\omega_\alpha,\omega_{\alpha'})=\sum_{s\in\mathbb{Z}/N\mathbb{Z}}\operatorname{Res}_{(-\zeta^s,0)}\left(-\frac{1}{N-[a]-[b]}u^{N-[a]-[b]}v^{[b]-N}\cdot -u^{N-[a']-[b']}v^{[b']-N}\,\frac{du}{u}\right).$$

Now, if $\alpha\neq\alpha'$, then we see that we are computing the residues of some monomial times du/u, but the power of u in the monomial is nonzero, so the residues all vanish. Lastly, we need to discuss what happens with $\alpha=-\alpha'$, where we see

$$\begin{aligned}P(\omega_\alpha,\omega_{-\alpha})&=\sum_{s\in\mathbb{Z}/N\mathbb{Z}}\operatorname{Res}_{(-\zeta^s,0)}\left(-\frac{1}{N-[a]-[b]}u^{N-[a]-[b]}v^{[b]-N}\cdot -u^{N-[-a]-[-b]}v^{[-b]-N}\,\frac{du}{u}\right)\\&=\sum_{s\in\mathbb{Z}/N\mathbb{Z}}\operatorname{Res}_{(-\zeta^s,0)}\left(-\frac{1}{N-[a]-[b]}u^{N-[a]-[b]}v^{[b]-N}\cdot u^{[a]+[b]-N}v^{-[b]}\,\frac{du}{u}\right)\\&=\frac{1}{N-[a]-[b]}\sum_{s\in\mathbb{Z}/N\mathbb{Z}}\operatorname{Res}_{(-\zeta^s,0)}\left(v^{-N}\,\frac{du}{u}\right)\\&=\frac{1}{N-[a]-[b]}\sum_{s\in\mathbb{Z}/N\mathbb{Z}}(-\zeta^s)^{-N}\\&=(-1)^N\frac{N}{N-[a]-[b]},\end{aligned}$$

as desired. ■

4.1.3 Some Group Elements

In this subsection, we define a few elements of $\mathbb{Q}[G]$ which we will then use in the next subsection. We begin with the three elements

$$t\coloneqq\sum_{g\in G}g,\qquad v\coloneqq\sum_{s\in\mathbb{Z}/N\mathbb{Z}}g_{[0:s:0]},\qquad\text{and}\qquad h\coloneqq\sum_{r\in\mathbb{Z}/N\mathbb{Z}}g_{[r:0:0]}.$$

We take a moment to note that these elements satisfy the relations $tg=gt=t$ for any $g\in G$, and $t=hv=vh$, and $v^2=Nv$ and $h^2=Nh$. In the sequel, we will get a lot of mileage out of the idempotent

$$p\coloneqq\frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}}(1-g_{[r:0:0]})(1-g_{[0:s:0]}).$$

Let's check that p is idempotent.

Lemma 4.8. Fix notation as above.

(a) Then p is idempotent.

(b) For any $r,s\in\mathbb{Z}/N\mathbb{Z}$, we have $(1-g_{[r:0:0]})(1-g_{[0:s:0]})p=(1-g_{[r:0:0]})(1-g_{[0:s:0]})$.

Proof. Both claims hinge upon the fact that a direct expansion of $(1-g_{[r:0:0]})(1-g_{[0:s:0]})$ implies

$$p=\frac{1}{N^2}\left(N^2-Nh-Nv+t\right).$$

We now show the claims separately.

(a) This is a direct computation: write

$$\begin{aligned}
p^2 &= \frac{1}{N^4}\left(N^2 - Nh - Nv + t\right)\left(N^2 - Nh - Nv + t\right) \\
&= \frac{1}{N^4}\left(N^4 + N^2h^2 + N^2v^2 + t^2 - 2N^3h - 2N^3v + 2N^2t + N^2hv - 2Nht - 2Nvt\right) \\
&= \frac{1}{N^4}\left(N^4 + N^3h + N^3v + N^2t - 2N^3h - 2N^3v + 2N^2t + N^2t - 2N^2t - 2N^2t\right) \\
&= \frac{1}{N^4}\left(N^4 - N^3h - N^3v + N^2t\right) \\
&= p.
\end{aligned}$$

(b) We will compute as in (a): note $h(1-g_{[r:0:0]}) = 0$ and $v(1-g_{[0:s:0]}) = 0$, so

$$\begin{aligned}
(1-g_{[r:0:0]})(1-g_{[0:s:0]})p &= (1-g_{[r:0:0]})(1-g_{[0:s:0]}) \cdot \frac{1}{N^2}\left(N^2 - Nh - Nv + hv\right) \\
&= (1-g_{[r:0:0]})(1-g_{[0:s:0]}) \cdot \frac{N^2}{N^2} + 0 + 0 + 0 \\
&= (1-g_{[r:0:0]})(1-g_{[0:s:0]}),
\end{aligned}$$

as required. ■

4.1.4 Homology

In this subsection, we will study $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q})$. By the end, we will define a 1-cycle $\gamma \coloneqq \gamma_N$ such that $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}[G]\cdot[\gamma]$. Morally, this means that we can understand our homology by focusing on this one cycle.

To begin, we need some path in $X(\mathbb{C})$, so we define $\delta\colon [0,1] \to X(\mathbb{C})$ by

$$\delta(t) \coloneqq \left[t^{1/N} : (1-t)^{1/N} : \zeta_{2N}^{-1}\right].$$

Notably, $\delta(0) = [0:1:\zeta_{2N}^{-1}]$ and $\delta(1) = [1:0:\zeta_{2N}^{-1}]$, so $g = [\zeta^r : \zeta^s : 1]$ has $g_*\delta(0) = [0:\zeta^s:\zeta_{2N}^{-1}]$ and $g_*\delta(1) = [\zeta^r : 0 : \zeta_{2N}^{-1}]$. The point of this computation is that we see

$$(1 - g_{[r:0:0]} - g_{[0:s:0]} + g_{[r:s:0]})_*\delta \in \mathrm{Z}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}).$$

We are now ready to define γ.

Definition 4.9. Fix notation (and in particular δ) as above. Then we define

$$\gamma \coloneqq \frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} (1-g_{[r:0:0]})(1-g_{[0:s:0]})_*\delta.$$

Note $\gamma = p_*\delta$.

The above computation shows that $\gamma \in \mathrm{Z}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q})$. We will want to know to its periods later. Note that the following result is essentially a special case of [Del18, Lemma 7.12].

Lemma 4.10. Fix notation as above. Suppose $(a,b,c) \in (\mathbb{Z}/N\mathbb{Z})^3$ has no nonzero entries. Then

$$\int_\gamma \omega_{(a,b,c)} = \zeta_{2N}^{[a]/N+[b]/N-1}\frac{\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[b]}{N}\right)}{\Gamma\left(\frac{[a]}{N}+\frac{[b]}{N}\right)}.$$

Proof. This is a direct computation. Denote the integral by $P(\gamma,\omega_{(a,b,c)})$. By adjunction, $\int_{p_*\delta}\omega_{(a,b,c)} = \int_\delta p^*\omega_{(a,b,c)}$. This allows us to compute

$$\begin{aligned} P(\gamma,\omega_{(a,b,c)}) &= \frac{1}{N^2}\int_\delta \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} (1-g_{[r:0:0]})(1-g_{[0:s:0]})^*\omega_{(a,b,c)} \\ &= \frac{1}{N^2}\int_\delta \sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right)\omega_{(a,b,c)} \\ &= \left(\frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right)\right)\int_\delta \omega_{(a,b,c)} \\ &= \left(\frac{1}{N^2}\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right)\right)\zeta_{2N}^{[a]/N+[b]/N-1}\int_0^1 t^{[a]/N}(1-t)^{[b]/N-1}\,\frac{dt}{t}. \end{aligned}$$

The last integral (famously) equals the Beta function, and it evaluates to $\Gamma\left(\frac{[a]}{N}\right)\Gamma\left(\frac{[b]}{N}\right)\Gamma\left(\frac{[a]+[b]}{N}\right)^{-1}$. We take a moment to check that

$$\sum_{r,s\in\mathbb{Z}/N\mathbb{Z}} \left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right) \stackrel{?}{=} N^2.$$

Well, $\left(1-\zeta^{ar}\right)\left(1-\zeta^{bs}\right) = 1-\zeta^{ar}-\zeta^{bs}+\zeta^{ar+bs}$, and because $a,b\neq 0$, we see that summing over r and s causes the terms not equal to 0 to vanish. Thus, we are left with N^2. ■

Remark 4.11. Because the right-hand side is nonzero, Lemma 4.10 implies that the differential forms $\omega_{(a,b,c)}$ are nonzero.

We are now ready to show that $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}[G]\cdot[\gamma]$.

Lemma 4.12. Fix notation as above. Then $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{Q}) = \mathbb{Q}[G]\cdot[\gamma]$.

Proof. It is enough to show that $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C}) = \mathbb{C}[G]\cdot[\gamma]$. Note that there is a canonical pairing

$$\begin{array}{ccc} \mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})\times \mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}),\mathbb{C}) & \to & \mathbb{C} \\ (c,\omega) & \mapsto & \int_c\omega \end{array}$$

which is perfect by the de Rham theorem. We already have a basis $\{\omega_{(a,b,c)}\}_{a,b,c\neq 0}$ of $\mathrm{H}^1_{\mathrm{dR}}(X(\mathbb{C}),\mathbb{C})$, so we will find a dual basis for $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})$ inside $\mathbb{C}[G]\cdot[\gamma]$. Well, for $g\in G$ and $\alpha\in\widehat{G}$, we see

$$\int_{g^*\gamma}\omega_\alpha = \int_\gamma g^*\omega_\alpha$$

equals $\alpha(g)P(\gamma,\omega_\alpha)$, where $P(\gamma,\omega_\alpha) \coloneqq \int_\gamma\omega_\alpha$ is the (nonzero!) period computed in Lemma 4.10. With this in mind, we define

$$c_\alpha \coloneqq \frac{1}{N^2P(\gamma,\omega_\alpha)}\sum_{g\in G}\alpha(g)^{-1}g^*[\gamma]$$

for each $\alpha=\alpha_{(a,b,c)}$ with $a,b,c\neq 0$. Then we see that $\int_{c_\alpha}\omega_\beta = 1_{\alpha=\beta}$ by the orthogonality relations, so $\{c_\alpha\}$ is a dual basis of $\mathrm{H}_1^{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})$, and it lives in $\mathbb{C}[G]\cdot[\gamma]$ by its construction. ■

4.2 Galois Action

We now use the notation set up in the previous section to write out the Galois action on the space of some absolute Hodge cycles attached to X. Throughout this section, p is a nonnegative index. We take a moment to note that the action of G on X upgrades into an action of G^{2p} on X^{2p}. Our exposition closely follows [GGL24, Subsection 8.5]. As in section 4.1.1, we will identify $\widehat{G}^{2p}$ with some subset of tuples in $(\mathbb{Z}/N\mathbb{Z})^{6p}$. And for a vector space H defined over $\mathbb{Q}(\zeta)$ (respectively, $\mathbb{Q}$) and character $\alpha \in \widehat{G}^{2p}$, we define H_α (respectively, $\mathrm{H}_{[\alpha]}$) as the corresponding α-eigenspace (respectively, $[\alpha]$-generalized eigenspace).

4.2.1 Hodge Cycles on X^{2p}

To understand the geometry of X, we will only be interested in tensor powers of $\mathrm{H}^1(X)$ (for a choice of cohomology theory H), which by the Künneth formula embed as

$$\mathrm{H}^1(X)^{\otimes 2p} \subseteq \mathrm{H}^{2p}\left(X^{2p}\right).$$

When H is de Rham cohomology H_{dR}, we thus see we are interested in when the image of an element in $\mathrm{H}^1_{\mathrm{dR}}(X)^{\otimes p}$ succeeds at being a Hodge cycle. Well, note that the action of G on $\mathrm{H}^1_{\mathrm{dR}}(X,\mathbb{C})$ extends to an action of G^{2p} on $\mathrm{H}^1_{\mathrm{dR}}(X,\mathbb{C})^{\otimes 2p}$. This action diagonalizes with one-dimensional eigenspaces by extending Remark 4.6. We will use properties of the diagonalization to read off when we have an element of bidegree (p,p) in $\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p},\mathbb{C}\right)$.

Following [Del18, Proposition 7.6], it will be useful to have the following definition.

Definition 4.13 (weight)**.** Given a function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$, we define its *weight map* as the function $\langle f\rangle\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Q}$ defined by

$$\langle f\rangle \coloneqq \frac{1}{N}\sum_{a\in\mathbb{Z}/N\mathbb{Z}} f(ua)[a]$$

For $p \geq 0$, we note that we may identify $\widehat{G}^{2p}$ with a tuple in $(\mathbb{Z}/m\mathbb{Z})^{2p}$, and then we define the *weight* $\langle\alpha\rangle$ of a character $\alpha \in \widehat{G}^{2p}$ as $\langle 1_\alpha\rangle(1)$, where $1_\alpha\colon \mathbb{Z}/N\mathbb{Z}\to\mathbb{Z}$ is the multiplicity of an element in $\mathbb{Z}/N\mathbb{Z}$ in the tuple α.

Remark 4.14. The point of this definition is as follows: given $\alpha \in \widehat{G}$ with $\alpha = (a,b,c)$ having nonzero entries, we note that ω_α has two possible cases.

- If $[a]+[b]+[c] = N$ so that $\langle\alpha\rangle = 1$, then $\omega_{(a,b,c)}$ is holomorphic so that $\omega_\alpha \in \mathrm{H}^{10}(X)$.
- If $[a]+[b]+[c] = 2N$ so that $\langle\alpha\rangle = 2$, then ω_α is not holomorphic so that $\omega_\alpha \in \mathrm{H}^{01}(X)$.

In all cases, we find $\omega_\alpha \in \mathrm{H}^{2-\langle\alpha\rangle,\langle\alpha\rangle-1}(X)$.

We now upgrade Remark 4.14 to $\mathrm{H}^1_{\mathrm{dR}}(X,\mathbb{C})^{\otimes 2p}$.

Lemma 4.15. Choose $\alpha \in \widehat{G}^{2p}$ as $\alpha = (\alpha_1,\ldots,\alpha_{2p})$ having nonzero entries. Then

$$\omega_\alpha \coloneqq \omega_{\alpha_1}\otimes\cdots\otimes\omega_{\alpha_{2p}}$$

embedded in $\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p},\mathbb{C}\right)$ is of bidegree $(4p-\langle\alpha\rangle,\langle\alpha\rangle-2p)$.

Proof. Because the Künneth isomorphism upgrades to an isomorphism of Hodge structures, it is enough to note that $\omega_{\alpha_i} \in H^{\langle\alpha\bullet}$ (see Remark 4.14) implies ω_α has bidegree

$$\left(4p-\sum_{i=1}^{2p}\langle\alpha_i\rangle, \sum_{i=1}^{2p}\langle\alpha_i\rangle - 2p\right).$$

The proposition follows because weight is additive. ■

Proposition 4.16. Choose $\alpha \in \widehat{G}^{2p}$ as $\alpha = (\alpha_1, \ldots, \alpha_{2p})$ having nonzero entries. Then $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}\right)_{[\alpha]}$ is one-dimensional over $\mathbb{Q}([\alpha])$, and the following are equivalent.

(a) $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}\right)_{[\alpha]}(p)$ consists entirely of Hodge classes.

(b) We have $\langle u\alpha \rangle = 3p$ for all $u \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Proof. We begin by embedding

$$\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{C}\right)_{u\alpha}$$

into

$$\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right) = \bigoplus_{\substack{q_1, \ldots, q_{2p} \\ q_1 + \cdots + q_{2p} = 2p}} \mathrm{H}^{q_1}_{\mathrm{dR}}(X, \mathbb{C}) \otimes \cdots \otimes \mathrm{H}^{q_{2p}}_{\mathrm{dR}}(X, \mathbb{C}),$$

where this last equality holds by the Künneth isomorphism. Quickly, we reduce to the case where $q_1 = \cdots = q_{2p} = 1$: for each $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, we note that $u\alpha$ has nonzero entries. On the other hand, the G-action on $\mathrm{H}^0(X) = \mathbb{C}$ is always trivial, so we note that if any of the $q_\bullet$s are not equal to 1, then one of them must equal 0, meaning that

$$\left(\mathrm{H}^{q_1}_{\mathrm{dR}}(X, \mathbb{C}) \otimes \cdots \otimes \mathrm{H}^{q_{2p}}_{\mathrm{dR}}(X, \mathbb{C})\right)_{u\alpha} = \mathrm{H}^{q_1}_{\mathrm{dR}}(X, \mathbb{C})_{u\alpha_1} \otimes \cdots \otimes \mathrm{H}^{q_{2p}}_{\mathrm{dR}}(X, \mathbb{C})_{u\alpha_{2p}}$$

is the zero vector space. Thus, we see that

$$\mathrm{H}^{2p}_{\mathrm{dR}}\left(X^{2p}, \mathbb{C}\right)_{[\alpha]} = \bigoplus_{u \in (\mathbb{Z}/N\mathbb{Z})^\times} \left(\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{C})^{\otimes 2p}\right)_{u\alpha}.$$

The comparison isomorphism now implies that $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]}$ has dimension $[\mathbb{Q}([\alpha]) : \mathbb{Q}]$ over $\mathbb{Q}$ and thus one dimension over $\mathbb{Q}([\alpha])$.

It remains to show that (a) and (b) are equivalent. Well, the $\mathbb{Q}$-vector space $\mathrm{H}^{2p}_{\mathrm{B}}\left(X^{2p}, \mathbb{Q}\right)_{[\alpha]}(p)$ will consist of Hodge classes if and only if $\left(\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{C})^{\otimes 2p}\right)_{u\alpha}$ is of bidegree (p, p), which is equivalent to $\langle u\alpha \rangle = 3p$ by Lemma 4.15. ■

4.2.2 An Absolute Hodge Cycle

4.2.3 Computations on de Rham Component

4.2.4 End of the Computation

4.3 Fermat Hypersurfaces

We would be remiss without mentioning something about Fermat hypersurfaces. Thus, we will state (but not prove) a few facts about what is known for Fermat hypersurfaces. There is much known here, but the proofs tend to be somewhat harder than what one does with the Fermat curves, which is why we have avoided the theory.

Chapter 5

Families of Curves

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List of Definitions