Student Arithmetic Geometry Seminar

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Contents

1 January 19: Martin Olsson

Today, we will provide an introduction to what this seminar will be about.

1.1 Motivating Paper

We are motivated by Deligne–Mumford's paper "The irreducibility of moduli of curves of given genus." The main theorem is as follows.

> **Theorem 1** (Deligne–Mumford)**.** Fix a nonnegative integer $g \geq 2$ and a field k. Then the moduli space $\mathcal{M}_g$ of curves of genus g over k is irreducible.

There is quite a bit which goes into this proof.

- Curve theory.
- The formalism of moduli problems.

- The Semistable reduction theorem.
- Jacobian varieties.
- The notion of stable curves and the compactification $\overline{\mathcal{M}}_g$.
- Deformation theory.
- The theory of stacks and coarse moduli spaces.
- An analytic description to get from $\mathbb{C}$.

From these ingredients, there are other directions to go in.

- Finer geometry of $\mathcal{M}_g$.
- Moduli of curves with marked points $\mathcal{M}_{g,n}$.
- Higher-dimensional directions, such as abelian varieties.

Let's go ahead and provide a sketch.

Sketch of Theorem 1. The result is known over $\mathbb{C}$ via some analytic methods. We want to get to other fields. We begin with the following exercise.

Proposition 2. Fix a complete discrete valuation ring $(R, \mathfrak{m})$ with fraction field K and residue field $\kappa := R/\mathfrak{m}$. Fix a smooth proper morphism $f\colon X \to \operatorname{Spec} R$. If $X_{\overline{K}}$ is connected, then $X_{\overline{\kappa}}$ is also connected.

Sketch. One can assume that κ is algebraically closed by some argument. Then the Formal functions theorem tells us that

$$H^0(X, \mathcal{O}_X) = \varprojlim_n H^0(X_n, \mathcal{O}_{X_n}),$$

where $X_n := X \times_V \operatorname{Spec} V/\mathfrak{m}^n$. Now, suppose for the sake of contradiction that $X_\kappa = X_\kappa^1 \sqcup X_\kappa^2$. Then by taking nilpotent thickenings, we see that $X_n = X_n^1 \sqcup X_n^2$, so

$$\varprojlim_n H^0(X_n, \mathcal{O}_{X_n}) = \varprojlim_n H^0(X_n^1, \mathcal{O}_{X_n^1}) \times \varprojlim_n H^0(X_n^2, \mathcal{O}_{X_n^2}).$$

So we are receiving a product of rings $A_1 \times A_2$, which are flat over R, so by viewing the global sections back in $H^0(X, \mathcal{O}_X)$, which should be $\overline{K}$ upon algebraic closure, we will receive our contradiction. ■

Now, the (coarse) moduli space $\mathcal{M}_g$ fails to be either smooth or proper, but some theory of stacks allows us to reduce to this case. Namely, the point is to find some compactification $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$ where $\overline{\mathcal{M}}_g$ is a smooth proper $\mathbb{Z}$-stack with $\mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ dense in each fiber. As such, we can imagine using Proposition 2 to pull back the result from $\overline{\mathcal{M}}_g$ to $\mathcal{M}_g$. ■

1.2 Thinking about the Moduli Spaces

Let's describe what $\mathcal{M}_g$ is. By the functor of points description, we merely need to describe maps $S \to \mathcal{M}_g$, which we declare to be in natural bijection with genus-g curves $\pi\colon C \to S$; i.e., π is a proper, flat, and smooth morphism whose geometric fibers are genus-g curves.

Example 3. There is a family of curves over a field k given by the equations

$$y^2 = (x - a_1) \cdots (x - a_n),$$

where $a_1, \ldots, a_n$ are allowed to vary. Viewing the $a_\bullet$s as giving a point in affine space, we see that we are (approximately speaking) producing a rational map $\mathbb{A}^n \to \mathcal{M}_g$, where perhaps we need to check that we have a curve of the correct genus.

Now, using a functor of points description, smoothness of $\mathcal{M}_g$ over $\operatorname{Spec}\mathbb{Z}$ is requiring the following (in the sense of formal smoothness): for any surjection $A' \twoheadrightarrow A$ with kernel J such that $J^2 = 0$, any morphism $\operatorname{Spec} A \to \mathcal{M}_g$ induces a unique dashed arrow.

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[d] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

As such, we can unwind the functor of points description of $\mathcal{M}_g$ to prove something like smoothness, which is quite remarkable.

Analogously, we can describe $\overline{\mathcal{M}}_g$ via the functor of points: maps $S \to \overline{\mathcal{M}}_g$ are "stable curves" $\pi\colon C \to S$, which will be a proper flat morphism whose geometric fibers by $\overline{s} \to S$ satisfy the following.

- The geometric fibers are nodal curves, meaning that the completion at any closed point is $\kappa(\overline{s})\,[\![x]\!]$ or $\kappa(\overline{s})\,[\![x,y]\!]/(xy)$.
- Every rational component (namely, irreducible component whose normalization is $\mathbb{P}^1$) has three distinguished points. (These three points are desirable, for example, to ensure that its automorphism group is trivial.)

Now, we can also check being proper via the functor of points description, using the valuative criterion. Namely, for a complete discrete valuation ring R with fraction field K, we would like to know that any map $\operatorname{Spec} K \to \overline{\mathcal{M}}_g$ induces a unique dashed arrow.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \overline{\mathcal{M}}_g \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "!"] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

And again, we can directly turn this into a geometry problem of curves by tracking through the moduli interpretation, which is the Semistable reduction theorem.

2 January 26th: Martin Olsson

Today we discuss the setup for our moduli problems.

2.1 The Yoneda Lemma

Here is the statement.

Theorem 4 (Yoneda)**.** Fix a category $\mathcal{C}$, and let $\operatorname{PSh}(\mathcal{C})$ denote the category of presheaves on $\mathcal{C}$.

(a) For $X \in \mathcal{C}$, the functor $h_X\colon A \mapsto \operatorname{Mor}(A,X)$ is a presheaf $h_X\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

(b) There is a natural bijection
$$\operatorname{Mor}_{\operatorname{PSh}(\mathcal{C})}(h_X, F) \to FX.$$

(c) The construction $h_\bullet$ forms a fully faithful embedding $h_\bullet\colon \mathcal{C} \to \operatorname{PSh}(\mathcal{C})$.

Proof. We won't bother to show (a). For (b), the forward map sends $\eta\colon h_X \Rightarrow F$ to $\eta_X(\mathrm{id}_X)$, and one can check that this is a bijection. To show (c), we note that we need to show
$$\operatorname{Mor}_{\mathcal{C}}(X,Y) \simeq \operatorname{Mor}_{\operatorname{Pre}(\mathcal{C})}(h_X, h_Y),$$
but this simply follows by taking $F = h_Y$ in (b). ∎

Remark 5. Most of the time, we will take $\mathcal{C} = \mathrm{Sch}(S)$ for a fixed base scheme S.

Anyway, we can now make the following definition.

Definition 6 (representable)**.** Fix a category $\mathcal{C}$. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is *representable* if and only if $F \simeq h_X$ for some object $X \in \mathcal{C}$. In the sequel, we may want to fix the isomorphism $F \simeq h_X$, which can be specified by an element $\xi \in FX$.

Here are some examples.

Example 7. Take $\mathcal{C} := \mathrm{Sch}$, and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined by $FY := \Gamma(Y, \mathcal{O}_Y)^n$. We claim F is represented by $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. Indeed, we see

$$\mathrm{Mor}_{\mathrm{Sch}}(Y, \mathbb{A}^n_{\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x_1, \ldots, x_n], \mathcal{O}_Y(Y)) \simeq \mathcal{O}_Y(Y)^n,$$

as desired. To specify this isomorphism, Theorem 4 tells us that it is enough to track through the identity map in $\mathrm{Mor}(\mathbb{A}^n_{\mathbb{Z}}, \mathbb{A}^n_{\mathbb{Z}}) \simeq \Gamma(\mathbb{A}^n_{\mathbb{Z}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{Z}}})^n$, which we can track through is $(x_1, \ldots, x_n)$. Of course, we could choose other isomorphisms, such as determined by the element $(x_n, \ldots, x_1) \in \Gamma(\mathbb{A}^n_{\mathbb{Z}}, \mathcal{O}_{\mathbb{A}^n_{\mathbb{Z}}})^n$.

Example 8. Take $\mathcal{C} := \mathrm{Sch}$, and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined by

$$FY := \{(f_\bullet) \in \Gamma(Y, \mathcal{O}_Y)^n : (f_1, \ldots, f_n) = \Gamma(Y, \mathcal{O}_Y)\}.$$

Then F is represented by $\mathbb{A}^n_{\mathbb{Z}} \setminus \{0\}$.

Example 9. Take $\mathcal{C} := \mathrm{Sch}$, and consider the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ defined by setting FY to the collection of surjections $\mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{L}$ up to isomorphism, where $\mathcal{L}/Y$ is a line bundle. Then F is represented by $\mathbb{P}^n_{\mathbb{Z}}$. The representing element in $F(\mathbb{P}^n_{\mathbb{Z}})$ is given by the surjection $(x_0, \ldots, x_n)\colon \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$.

Remark 10. By identifying a line with the corresponding quotient space, we see that the above interpretation of $\mathbb{P}^n_k$ agrees with the interpretation as "lines in k^{n+1}."

Example 11. Fix a field k and homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree N, and consider $V(f) \subseteq \mathbb{P}^n_k$. A map $Y \to V(f)$ will certainly produce a map $Y \to \mathbb{P}^n_{\mathbb{Z}}$, so our answer should be a subfunctor of Example 9. But then we want to determine which surjections $(s_0, \ldots, s_n)\colon \mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{L}$ (which really consists of $n+1$ global sections of $\mathcal{L}$ globally generating), but then to live in $V(f)$, we request that $f(s_0, \ldots, s_n) = 0$.

Remark 12. Note $\mathbb{A}^n_{\mathbb{Z}} \subseteq \mathbb{P}^n_{\mathbb{Z}}$ (e.g. going to the standard affine open $x_0 \neq 0$). So we expect to have an inclusion $h_{\mathbb{A}^n_{\mathbb{Z}}} \subseteq h_{\mathbb{P}^n_{\mathbb{Z}}}$, and one can track through that it simply corresponds to the provided map $(s_0, \ldots, s_n)\colon \mathcal{O}_Y^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$ having s_0 be an isomorphism. In this case, after identifying $\mathcal{O}_Y$ with $\mathcal{L}$ via s_0, we see that the other sections are indeed providing an element of $\Gamma(Y, \mathcal{O}_Y)^n$.

Example 13 (Hilbert scheme)**.** Fix a flat separated X-scheme S. Then the Hilbert functor assigns a Y-scheme S to flat proper (locally of finite presentation) subschemes $Z \subseteq X \times_S Y$. It turns out that this functor is representable if X is quasi-projective, which is a result due to Grothendieck.

2.2 Functors Not Representable

Representable functors have the tendency to be sheaves. For the Zariski topology, here is the relevant definition.

Definition 14 (Zariski sheaf)**.** A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a *Zariski sheaf* if and only if we have the usual equalizer diagram

$$FY \to \prod_{\alpha\in\kappa} FY_\alpha \leftleftarrows \prod_{\alpha,\beta\in\kappa} F(Y_\alpha \cap Y_\beta),$$

where $\{Y_\alpha\}_{\alpha\in\kappa}$ is a Zariski open cover of Y.

Example 15. If $F \simeq h_X$ is representable, then F is a Zariski sheaf because morphisms glue.

Non-Example 16. Define the functor F as taking a scheme Y and taking the quotient of the set of $(n+1)$-tuples $(s_0, \ldots, s_n) \in \Gamma(Y, \mathcal{O}_Y)$ by multiplication by $\Gamma(Y, \mathcal{O}_Y^\times)$. This is not a sheaf, but if we do sheafification, we will recover Example 9. Thus, this functor is not representable.

As another remark, we have the following.

Lemma 17. If X is a scheme, and L/K is a field extension, then the map $X(\operatorname{Spec} K) \to X(\operatorname{Spec} L)$ is injective.

Proof. Indeed, a map $\operatorname{Spec} K \to X$ is simply a point $x \in X$ together with an inclusion $\kappa(x) \to K$, which by a similar description for $\operatorname{Spec} L \to X$ is uniquely determined by that map $\operatorname{Spec} L \to X$. ■

Example 18. Define the functor F by taking a scheme Y and returning elliptic curves over Y (up to isomorphism). But there are distinct elliptic curves over $\mathbb{Q}$ which become isomorphic over $\overline{\mathbb{Q}}$. For example, $y^2 = x^3 + D$ and $y^2 = x^3 - D$ is one such example, which become isomorphic over $\mathbb{Q}(i)$ where we can send $(x, y) \mapsto (-x, iy)$. Thus, F is not representable by Lemma 17.

Remark 19. One can argue similarly for curves of genus $g \geq 2$.

3 February 16th: Martin Olsson

I missed one week, and then one week was cancelled. I am a little too tired to take detailed notes.

Remark 20. Recall that the Yoneda lemma asserts that representing a functor $F\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set}$ amounts to specifying an object X in addition to an element $\xi \in FX$ so that the induced map $h_X \Rightarrow F$ (given by ξ) is representable.

Example 21. The Hilbert functor $\mathrm{Hilb}_{\mathbb{P}^n}$ is represented by some scheme H. In particular, we are provided an isomorphism $\sigma\colon h_H \Rightarrow \mathrm{Hilb}_{\mathbb{P}^n}$, so passing id_H through implies we are being given a flat proper subscheme $\mathcal{Z} \subseteq \mathbb{P}^n_H$ so that $\mathrm{Hilb}_H(S)$ is merely given by $h_H(S)$, which consists of pullbacks of $\mathcal{Z}$ along the morphisms $f\colon S \to H$. The point is that we can really see the element of $\mathrm{Hilb}_H(S)$ provided to us by some $f \in h_H(S)$ as $f^*\mathcal{Z}$.

Example 22. The functor $\mathcal{M}_{0,n}$ takes a scheme S and then asks for curves $f\colon C \to S$ (i.e., f is proper and flat where all fibers are isomorphic to $\mathbb{P}^1$), and we mark n points (in fact, S-points) somewhere on C. The issue here is that we are considering these up to isomorphism, but it turns out that an automorphism of C is basically determined by where it sends three points. When $S = \operatorname{Spec} k$, this is classical; in general, one needs to make some more global argument. So, for example, one finds that $\mathcal{M}_{0,3}$ is $\operatorname{Spec}\mathbb{Z}$ (where the universal curve is $\mathbb{P}^1_{\mathbb{Z}}$ with the marked points $\{0, 1, \infty\}$).

Example 23. Continuing from the above example, we see $\mathcal{M}_{0,4}$ is $\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$ basically by trying to mark a fourth point of $\mathbb{P}^1_{\mathbb{Z}}$ (outside $\{0, 1, \infty\}$). Here are universal curve is given by

$$\mathbb{P}^1 \times \left(\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}\right).$$

Then our first three marked points are $\{0\} \times \mathcal{M}_{0,4}$ and $\{1\} \times \mathcal{M}_{0,4}$ and $\{\infty\} \times \mathcal{M}_{0,4}$, and the last marked point is given by what we chose in $\mathcal{M}_{0,4}$ to begin with, so it is given by the diagonal embedding.

Remark 24. One takes the compactification $\overline{\mathcal{M}}_{0,n}$ of $\mathcal{M}_{0,n}$ by taking nodal curves (without automorphisms) instead of just curves. Using this one can realize $\overline{\mathcal{M}}_{0,4}$ by $\mathbb{P}^1$, and we can track through what the points $\{0, 1, \infty\}$ mean. For example, our universal curve is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at three points in $\{(0, 0), (1, 1), (\infty, \infty)\}$.

Remark 25. Continuing, it turns out that $\overline{\mathcal{M}}_{0,5}$ ought to be the universal curve over $\overline{\mathcal{M}}_{0,4}$. The point is that choosing 5 points amounts to choosing 4 points first (which is $\mathcal{M}_{0,4}$) and then the last point goes up a dimension. This continues inductively.

4 February 23rd: Charley Hutchison

Today we are talking about deformation of curves. Work over a field k. For our purposes, a deformation of a morphism $X_0 \to \operatorname{Spec} k$ is a pullback square of the form

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

where $X \to S$ should be flat. For our purposes, we will mostly work with "infinitesimal deformations" where S is an Artinian local ring such as $S = \operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$; we'll write $k[\varepsilon] := k[\varepsilon]/\left(\varepsilon^2\right)$.

Example 26. Let $A \twoheadrightarrow B$ be a closed embedding where $B = A/I$. Then getting a deformation by $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ is asking for an ideal $\widetilde{I} \subseteq A[\varepsilon]$ such that

$$\frac{A[\varepsilon]}{\widetilde{I}} \otimes_{k[\varepsilon]} k \cong A/I.$$

Namely, we have just turned our scheme-theoretic diagram into a ring-theoretic problem. For example, given an A-module map $\varphi\colon I \to B$, one can define

$$I_\varphi \coloneqq \{x + \varepsilon y : x \in I, y \in A, \varphi(x) \equiv y \pmod{I}\},$$

and one can see that $I_\varphi \subseteq A[\varepsilon]$ will do the trick. In fact, all possible deformations will take this form: one can take such an ideal $\widetilde{I}$ and then define a map $I \to B$ by lifting a given $x \in I$ to some $\widetilde{x} + \varepsilon\widetilde{y}$ and then outputting $\widetilde{y}$. (This is well-defined by flatness.)

The above example classifies our "affine" infinitesimal deformations of a closed embedding. Flatness is crucial to the discussion, though it is not totally obvious how. Namely, a Tor^1 computation tells us that a module M over $k[\varepsilon]$ succeeds in being flat if and only if

$$0 \to k\varepsilon \to k[\varepsilon] \to k \to 0$$

succeeds in staying exact after applying $- \otimes_{k[\varepsilon]} M$. The point is that we are able to build a large diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r] \arrow[d] & \widetilde{I} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & A[\varepsilon] \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A[\varepsilon]/\widetilde{I} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and note that the Nine lemma tells us that the top row is exact if and only if the bottom row is exact, and these conditions correspond to flatness of $\widetilde{I}$ and $A[\varepsilon]/\widetilde{I}$ respectively.

Example 27. Take $A \coloneqq k[x]$ and $I \coloneqq (f)$. Then we see that a deformation corresponds to an A-linear map $(f) \to k[x]/(f)$. For example, one can map $f \mapsto f'$.

Approximately speaking, one may restate this as follows: given closed subscheme $Z_0 \subseteq X_0$ (everything is affine here) defined by the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{X_0}$, deformations living inside X_0 are parameterized by

$$\operatorname{Hom}_{X_0}\left(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{X_0}/\mathcal{I}\right),$$

which is just global sections of the normal bundle $\mathcal{N}_{Z_0/X_0}$. This is fairly compelling geometrically: an infinitesimal deformation of Z_0 is a global vector field "normal" to Z_0.

Remark 28. Given a smooth separated scheme X, one can give X an affine cover and then run the above classification construction. Now, there is some gluing data corresponding to taking local deformations to global ones, and this data satisfies some cocycle condition. As such, it will turn out that our deformations correspond to classes in $H^1(X, \mathcal{T}_{X/k})$.

5 March 1st: Reed Jacobs

Today we're talking about Siegel modular varieties and the analytic theory of abelian varieties.

5.1 Siegel Modular Varieties

Fix a positive integer $g \geq 1$. Our "upper half-plane" is given by

$$\mathbb{H}^g := \left\{\tau \in \mathbb{C}^{g\times g} : \tau = \tau^\intercal, \operatorname{Im}\tau > 0\right\},$$

where $\operatorname{Im}\tau > 0$ means that the matrix $\operatorname{Im}\tau$ is positive-definite and in particular invertible. Notably, $\dim \mathbb{H}^g = \frac{1}{2}g(g+1)$ because we are looking at symmetric matrices.

To build our lattices, fix some positive integers $(e_1, \ldots, e_g)$ such that $e_i \mid e_{i+1}$ for each i, and we define

$$\Omega_\tau := \begin{bmatrix} \tau \\ E \end{bmatrix},$$

where $E := \operatorname{diag}(e_1, \ldots, e_g)$. Then our lattice Λ_τ is the $\mathbb{Z}$-span of Ω_τ.

Notably, $\mathbb{C}^g/\Lambda_\tau$ is a complex torus, and it has a Riemann form given by $H\colon (x,y) \mapsto x\operatorname{Im}(\tau)^{-1}\overline{y}^{-\intercal}$; thus, it is an abelian variety. What is important is that we are essentially getting all complex tori equipped with Riemann forms (i.e., abelian varieties). Approximately speaking, we will use the Riemann form to produce lots of holomorphic functions on our complex torus. To get us set up, we do note that any lattice $\Lambda \subseteq \mathbb{C}^g$ with Riemann form H can be given a basis such that

$$H_\mathbb{R} = \begin{bmatrix} & E \\ -E & \end{bmatrix}$$

where E is some diagonal matrix in the above form, and $\Lambda = E\mathbb{Z}^{2g}$. (This is some version of the Smith normal form.) Now, the point is that being a Riemann form enforces some algebraic conditions on our lattice. Another adjustment of bases actually puts us in the above form, completing our argument.

From here, the next natural question is when two Es as above may give the same abelian variety. Set $w := \left[\begin{smallmatrix} & E \\ -E & \end{smallmatrix}\right]$ as before, and one finds that $\operatorname{Sp}(w)$ acts on $\mathbb{H}^g$ by some analogue of fractional linear transformations. Explicitly,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\tau := (A\tau + BE)(C\tau + DE)^{-1}E.$$

Namely, one can check that $C\tau + DE$ is forced to be invertible. As such, one finds that the moduli space of g-dimensional principally polarized abelian varieties is given by $\mathcal{A}_g = \operatorname{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}^g$.

In analogy to classical modular forms, which can be viewed as invariant differentials on $\operatorname{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, one can define Siegel modular forms to be invariant differentials on $\operatorname{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}^g$. There is some problem in making this generalization because passing to an arithmetic subgroup because the notion of weight is a little complicated, and somehow we are really trying to look at a line bundle on a stack, which has its own sort of complications.

6 March 8th: Feiyang Lin

Today we are discussing curves of genus 6. As a warning, we are doing intersection theory with rational coefficients.

6.1 Tautological Chow Rings

Our goal is to show that the chow ring of $\mathcal{M}_6$ is "tautological."

Theorem 29. The Chow ring $A^*(\mathcal{M}_6)$ of $\mathcal{M}_6$ is tautological.

Let's review what being tautological means. Let $\mathcal{C}_g$ be the universal curve on $\mathcal{M}_g$ so that we have a projection $\pi\colon \mathcal{C}_g \to \mathcal{M}_g$. By gluing together the canonical bundle, we get a canonical bundle $\omega_\pi \in \operatorname{Pic} \mathcal{C}_g$, which produces a Chern class $\kappa \in c_1(\omega_\pi)$. This can then be pushed repeatedly to $\kappa_i := \pi_*(\kappa^{i+1})$. Alternatively, we can just take $\mathbb{E} := \pi_*\omega_\pi$ to be a vector bundle of rank g on $\mathcal{M}_g$, and its Chern class is similarly $\lambda_i := c_i(\mathbb{E})$ on $A^i(\mathcal{M}_g)$.

With all of this information, we let $R^*(\mathcal{M}_g)$ be the $\mathbb{Q}$-subalgebra of $A^*(\mathcal{M}_g)$ generated by the $\kappa_\bullet$ and $\lambda_\bullet$. Work of many people have shown that the following.

- High degree: $R^i(\mathcal{M}_g) = 0$ for $i \geq g-1$.
- "Top" degree: $R^{g-2}(\mathcal{M}_g)$ is generated by a single nonzero class $[\mathcal{H}_g]$.
- For $g \leq 5$, one actually has $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$ is isomorphic to $\mathbb{Q}[\kappa_1]/\left(\kappa_1^{g-1}\right)$.
- Even in degree 6, one finds that

$$R^*(\mathcal{M}_6) \cong \frac{\mathbb{Q}[\kappa_1,\kappa_2]}{(127\kappa_1^3 - 2304\kappa_1\kappa_2, 113\kappa_1^4 - 3686\kappa_2^2)}.$$

6.2 Building a Filtration

Our goal is to show that $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_6)$. The outline will be to build a filtration

$$Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_n = \mathcal{M}_6.$$

Then given surjections $R^*(\mathcal{M}_6) \twoheadrightarrow A^*(Z_{i+1}/Z_i)$ along with knowing that the complement of $Z_{i+1} \setminus Z_i$ lives in $R^*(\mathcal{M}_6)$, we will know that$A^*(\mathcal{M}_6)$ is contained in $R^*(\mathcal{M}_6)$, as needed. Indeed, this is by an inductive argument: one has a right-exact sequence

$$A^*(Z_i) \to A^*(Z_{i+1}) \to A^*(Z_i \setminus Z_{i+1}) \to 0,$$

and we know that $R^*(\mathcal{M}_6)$ surjects on to the left and right terms of the exact sequence by construction, so it surjects onto the middle term.

So it remains to construct the magical filtration. We will use the following result.

Theorem 30 (Vistoli). Let X be a smooth DM stack with action by $G \in \{\mathrm{GL}_n, \mathrm{SL}_n\}$. Then the quotient $q\colon X \to X/G$ gives rise to a surjection $A^*(X/G) \to A^*(X)$ (by pullback), and $A^*(X/G)$ is generated by any arbitrary set of lifts from $A^*(X)$.

We are not going to prove this. But here is our filtration.

Theorem 31 (Penev–Vakil). The stack $\mathcal{M}_6$ is a disjoint union of the following.

- The "Mukai–general" curves of genus 6.
- Bi-elliptic curves: double-covers of elliptic curves.
- Smooth planar quintic curves.
- Trigonal curves: triple-covers of $\mathbb{P}^1$.
- Hyperelliptic curves.

Wait, what is a Mukai–general curve? The point is to write down curves which are not among any of the other families. For example, not being trigonal nor hyperelliptic means that it has no g^1_3 or g^2_5; it turns out that this implies that we have a g^1_4. Brill–Noether theory now produces a line bundle $\mathcal{L}$ on $\mathcal{C}$ such that $h^0(\mathcal{L}) = 2$; letting $\mathcal{M} := K_C \otimes \mathcal{L}^\vee$ be the Serre dual, we see that $h^0(\mathcal{M}) = h^1(\mathcal{L}) = 3$.

Quickly, we claim that we can find a unique vector bundle $\mathcal{E}$ of rank 2 fitting in a nontrivial extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$$

such that $h^0(\mathcal{E}) = 5$ and is globally generated. The point is that one gets a map

$$H^0(\mathcal{M}^{\otimes 2}) = \operatorname{Ext}^1(\mathcal{M}, \mathcal{L}) \to H^0(\mathcal{M})^\vee \otimes H^1(\mathcal{M})$$

which is dual to the multiplication map $H^0(\mathcal{M}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{M}^{\otimes 2})$. Now, $\mathcal{M}$ is a g^2_6 and hence produces a map $C \to \mathbb{P}^2$, allowing us to move everything over to $\mathbb{P}^2$. Then one can go through the possibilities of what this map looks like. Note that $\mathcal{E}$ will end up being globally generated because both $\mathcal{L}$ and $\mathcal{M}$ are.

At this point in the talk, I lost the ability to take notes and had to pay attention.

6.3 March 15th: Ruoxi Li

I was not tall enough for this talk.

6.4 March 22nd: Vivasvat Vatatmaja

Today, we work over a base field k everywhere, which we will not comment again on. We begin with a definition.

Definition 32. Let G will be an algebraic group, and let X is a scheme (or possibly an algebraic space). Suppose X has a G-action given by $j: G \times X \to X \times X$ via $(g, x) \mapsto (gx, x)$.

- The action is *proper* if j is proper.
- The action is *free* if j is a closed embedding.
- Considering the map $j^{-1}\Delta_X \to \Delta_X$, we see that the action has trivial stabilizer if and only if this map has trivial fibers.

Here is our main theorem for today.

Theorem 33. The integral chow ring $A^*(\mathcal{M}_{1,1})$ of $\mathcal{M}_{1,1}$ is $\mathbb{Z}[t]/(12t)$.

In particular, we will need to discuss how to even make sense of the integral Chow ring. Here, $\mathcal{M}_{1,1}$ is the moduli space of curves of genus 1 with a marked point, and we will show that it can be realized as a quotient stack X/G.

So we want to understand the Chow ring of a quotient stack X/G.

Definition 34. Fix a group G acting on an algebraic space X and index i. Then choose a representation V of G such that G acts freely on an open subset $U \subsetneq V$ (and we take U to be the largest such open subset), where the codimension of $V \setminus U$ is $\dim X - i$. Then we define $X_G := (X \times U)/G$ and set $A^G_i(X) := A_{i+\dim V - \dim G}(X_G)$.

Remark 35. One can show that $A_i^G(X)$ does not depend on the choices. This is the "double fibration" argument. Indeed, choose another representation V' of G, and we let U' again be the locus where G acts freely. To compare the two, we note that G acts diagonally on $V \oplus V'$, and its locus W of free action contains $(U' \oplus V) \cup (V' \oplus U)$. Now, a codimension argument explains that

$$A_{i+\dim V+\dim V'-\dim G}((X \times W)/G) = A_{i+\dim V+\dim V'-\dim G}((X \times (U \oplus V'))/G).$$

Now, $(X \times (U \oplus V'))/G$ is a vector bundle on $(X \times U)/G$, so one finds that

$$A_{i+\dim V+\dim V'-\dim G}((X \times (U \oplus V'))/G) \cong A_{i+\dim V-\dim G}((X \times U)/G),$$

and a symmetric argument compares this to V'.

Remark 36. Naïvely, one might want to take G-invariant subvarieties to define our Chow ring. One does see that $[Y]_G \in A_\bullet^G(X)$ by an explicit construction, and in fact, one can show that any $\alpha \in A_i^G(X)$ can be realized in some representation via G-invariant subvarieties.

Remark 37. One can show that we have flat pullback and proper pushforward and so on.

Example 38. Let $G := \mathbb{G}_m$ act on a point $X := \{*\}$. Here, $\mathbb{G}_m$ acts on any d-dimensional vector space V, and it acts freely on the set of nonzero values $U \subseteq V$. So $U/\mathbb{G}_m$ is $\mathbb{P}^{d-1}$, so $A_i^G(X) = A_i\left(\mathbb{P}^{d-1}\right)$ for d large enough, for which classical computation provides $\mathbb{Z}t^i$, so $A^G(X) = \mathbb{Z}[t]$.

Example 39. Let $G := \mathrm{GL}_n$ act on a point $X := \{*\}$. Then we let V be the vector space of $(n \times p)$ matrices where p is very large, and we see that G acts by matrix multiplication on V, and it acts freely on the open subset U of matrices of maximal rank. Thus, U/G is $\mathrm{Gr}(n,p)$, and one finds that

$$A_i^G(X) = \mathbb{Z}[c_1, \ldots, c_n]$$

again by some classical computations.

We now take as our definition

$$\mathcal{A}_i^G(X/G) := A_{i-g}^G(X).$$

At this point, we return to $\mathcal{M}_{1,1}$. The point is to use Weierstrass equations. Consider $\mathbb{P}^9$ as the space of cubic forms in the variables (x,y,z), and we let X be the subspace of forms which are proportional to $y^2z - \left(x^3 + e_1x^2z + e_2xz^2 + e_3z^3\right)$ so that

$$G := \left\{ \begin{bmatrix} 1 & & B \\ & A & \\ & & A^{-2} \end{bmatrix} : A \neq 0 \right\}$$

acts on X via some change of coordinates representation hitting all Weierstrass equations. By restricting X further to the open subset U where $x^3 + e_1x^2 + e_2x + e_3$ has three distinct roots, we find that our curves are forced to be smooth. In particular, we have $\mathcal{M}_{1,1} \cong U/G$. We are now able to do a computation to prove our result.