The Local Fundamental Class

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Abstract

We compute the local fundamental class of the extension $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ when p is an odd prime. This requires making a number of standard group cohomology constructions fully explicit in the process.

Contents

1 Set-Up

We will work over $\mathbb{Q}_p$ as our base field, where p is an odd prime. Set $N := p^\nu m$ where k and m integers with $p \nmid m$. This gives us the following tower of fields.

$$\begin{tikzcd}
 & \mathbb{Q}_p(\zeta_N) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}_p(\zeta_{p^\nu}) \arrow[dr, dash] & & \mathbb{Q}_p(\zeta_m) \arrow[dl, dash] \\
 & \mathbb{Q}_p &
\end{tikzcd}$$

To help us a little later, we will assume that the extension $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ is not totally ramified nor as unramified, for in this case we can understand the extension by viewing it as a cyclic extension. We provide some quick commentary on these extensions.

- The extension $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$ is unramified of degree $f := \operatorname{ord}_p(m)$; note we are assuming $1 < f < n$. Its Galois group is thus generated by the Frobenius element defined by $\overline{\sigma}_K : \zeta_m \mapsto \zeta_m^p$.
- The extension $\mathbb{Q}_p\left(\zeta_{p^\nu}\right)/\mathbb{Q}_p$ is totally ramified of degree $\varphi\left(p^\nu\right)$. Its Galois group is thus isomorphic to $\left(\mathbb{Z}/p^\nu\mathbb{Z}\right)^\times$, where the isomorphism takes $x \in \left(\mathbb{Z}/p^\nu\mathbb{Z}\right)^\times$ to

$$\sigma_x : \zeta_{p^\nu} \mapsto \zeta_{p^\nu}^{x^{-1}}.$$

The group $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$ is cyclic, so we will fix a generator x, which gives us a distinguished generator $\sigma_x \in \operatorname{Gal}(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p)$.

- Because $\mathbb{Q}_p(\zeta_{p^\nu})$ is totally ramified and $\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p$ is unramified, we have that the fields $\mathbb{Q}_p(\zeta_{p^\nu})$ and $\mathbb{Q}_p(\zeta_m)$ are linearly disjoint over $\mathbb{Q}_p$. As such, $\mathbb{Q}_p(\zeta_N) = \mathbb{Q}_p(\zeta_{p^\nu})\mathbb{Q}_p(\zeta_m)$ has

$$\begin{aligned}
\operatorname{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p(\zeta_{p^\nu})) &\simeq \operatorname{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) = \langle \overline{\sigma}_K \rangle \\
\operatorname{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p(\zeta_m)) &\simeq \operatorname{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \sigma_x \rangle \\
\operatorname{Gal}(\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p) &\simeq \operatorname{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p) \times \operatorname{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle \overline{\sigma}_K \rangle \times \langle \sigma_x \rangle.
\end{aligned}$$

  In light of these isomorphisms, we will upgrade $\overline{\sigma}_K$ to the automorphism of $\mathbb{Q}_p(\zeta_N)/\mathbb{Q}_p$ sending $\zeta_m \mapsto \zeta_m^p$ and fixing $\mathbb{Q}_p(\zeta_{p^\nu})$; we do analogously for σ_x. We also acknowledge that our degree is

$$n := [\mathbb{Q}_p(\zeta_N) : \mathbb{Q}_p] = [\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p] \cdot [\mathbb{Q}_p(\zeta_{p^\nu}) : \mathbb{Q}_p] = f\varphi(p^\nu).$$

For brevity, we will also set $L_1 := \mathbb{Q}_p(\zeta_{p^\nu})$ and $L_2 := \mathbb{Q}_p(\zeta_m)$, which makes the fields under L look like the following.

$$\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[dr, dash] & \\
L_1 \arrow[dr, dash] & & L_2 \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

Now, the main idea in the computation is to use an unramified extension of the same degree as $\mathbb{Q}_p(\zeta_N)$. As such, we set $N' := p^n - 1$ so that $[\mathbb{Q}_p(\zeta_{N'}) : \mathbb{Q}_p]$ is $\operatorname{ord}_{N'}(p) = n$. This modifies our diagram of fields as follows.

$$\begin{tikzcd}
 & \mathbb{Q}_p(\zeta_N, \zeta_{N'}) \arrow[dl, dash, "\mathrm{unr}"'] \arrow[dr, dash, "\mathrm{ram}"] & \\
\mathbb{Q}_p(\zeta_N) \arrow[d, dash, "\mathrm{unr}"'] \arrow[dr, dash, "\mathrm{ram}"] & & \mathbb{Q}_p(\zeta_{N'}) \arrow[dl, dash, "\mathrm{unr}"] \\
\mathbb{Q}(\zeta_{p^\nu}) \arrow[dr, dash, "\mathrm{ram}"'] & \mathbb{Q}_p(\zeta_m) \arrow[d, dash, "\mathrm{unr}"] & \\
 & \mathbb{Q}_p &
\end{tikzcd}$$

We have labeled the unramified extensions by "unr" and the totally ramified extensions by "ram."

For brevity, we set $K := \mathbb{Q}_p$ and $L := \mathbb{Q}_p(\zeta_N)$ and $M := \mathbb{Q}_p(\zeta_{N'})$ so that $ML = \mathbb{Q}_p(\zeta_N, \zeta_{N'})$. This abbreviates our diagram into the following.

$$\begin{tikzcd}
 & ML \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & M \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

As before, we provide some comments on the field extensions.

- The extension $\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p$ is unramified of degree n. As before, its Galois group is cyclic, generated by $\sigma_K \colon \zeta_{N'} \mapsto \zeta_{N'}^p$. Observe that σ_K restricted to $\mathbb{Q}_p(\zeta_m)$ is $\overline{\sigma}_K$, explaining our notation. In particular, σ_K has order n, but $\overline{\sigma}_K$ has order $f < n$.

- As before, note that $\mathbb{Q}_p(\zeta_{p^\nu})$ and $\mathbb{Q}(\zeta_{N'})$ are linearly disjoint because $\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p$ is totally ramified while $\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p$ is unramified. As such, we may say that

$$\begin{aligned}\operatorname{Gal}(ML/M) &\simeq \operatorname{Gal}(\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle\sigma_x\rangle \\ \operatorname{Gal}(ML/\mathbb{Q}_p(\zeta_{p^\nu})) &\simeq \operatorname{Gal}(M/K) = \langle\sigma_K\rangle \\ \operatorname{Gal}(ML/K) &\simeq \operatorname{Gal}(\mathbb{Q}_p(\zeta_{N'})/\mathbb{Q}_p) \times \operatorname{Gal}(\mathbb{Q}_p(\zeta_{p^\nu})/\mathbb{Q}_p) = \langle\sigma_K\rangle \times \langle\sigma_x\rangle.\end{aligned}$$

  Again, we will upgrade σ_K and σ_x to their corresponding automorphisms on any subfield of ML.

- We take a moment to compute

$$\operatorname{Gal}(ML/L) \simeq \left\{\sigma_K^{a_1}\sigma_x^{a_2} \in \operatorname{Gal}(ML/K) : \sigma_K^{a_1}\sigma_x^{a_2}|_L = \mathrm{id}_L\right\}.$$

  Because L is $\mathbb{Q}_p(\zeta_{p^\nu})\mathbb{Q}_p(\zeta_m)$, it suffices to fix each of these fields individually. Well, to fix $\mathbb{Q}_p(\zeta_{p^\nu})$, we need $\sigma_x^{a_2}$ to vanish, so we might as well force $a_2 = 0$. But to fix $\mathbb{Q}_p(\zeta_m)$, we need $\sigma_K^{a_1}|_{\mathbb{Q}(\zeta_m)} = \overline{\sigma}_k^{a_1}$ to be the identity, so we are actually requiring that $f \mid a_1$ here. As such,

$$\operatorname{Gal}(ML/L) = \langle\sigma_K^f\rangle.$$

These comments complete the Galois-theoretic portion of the analysis.

2 Idea

We will begin by briefly describe the outline for the computation. For a finite extension of finite fields L/K, let $u_{L/K} \in H^2(L/K)$ denote the fundamental class.

Now, take variables as in our set-up in section 1. The main idea is to translate what we know about the unramified extension M/K over to the general extension L/K. In particular, we are able to compute the fundamental class $u_{M/K} \in H^2(M/K)$, so we observe that

$$\operatorname{Inf}_{M/K}^{ML/K} u_{M/K} = [ML:M]u_{M/K} = n \cdot u_{ML/K} = [ML:L]u_{ML/L} = \operatorname{Inf}_{L/K}^{ML/K} u_{L/K}.$$

As such, we will be able to compute $u_{L/K}$ as long as we are able to invert the inflation map $\operatorname{Inf}\colon H^2(L/K) \to H^2(ML/K)$. This is not actually very easy to do in general,[1] but we are in luck because this inflation map here comes from the Inflation–Restriction exact sequence

$$0 \to H^2(L/K) \stackrel{\operatorname{Inf}}{\to} H^2(ML/K) \stackrel{\operatorname{Res}}{\to} H^2(ML/L).$$

The argument for the Inflation–Restriction exact sequence is an explicit computation on cocycles (involving some dimension shifting), but it can be tracked backwards to give the desired cocycle.

3 Computation

In this section we record the details of the computation.

3.1 Group Cohomology

Throughout this section, G will be a group (usually finite) and $H \subseteq G$ will be a subgroup (usually normal). We denote $\mathbb{Z}[G]$ by the group ring and $I_G \subseteq \mathbb{Z}[G]$ by the augmentation ideal, defined as the kernel of the map $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ which sends $g \mapsto 1$ for all $g \in G$.

We begin by recalling the statement of the Inflation–Restriction exact sequence.

[1] The difficulty comes from the fact that a generic cocycle might be off from an inflated cocycle by some truly hideous coboundary.

Theorem 1 (Inflation–Restriction)**.** Let G be a finite group with normal subgroup $H \subseteq G$. Given a G-module A, suppose that the $H^i(H, A) = 0$ for $1 \le i < q$ for some index $q \ge 1$. Then the sequence

$$0 \to H^q\left(G/H, A^H\right) \overset{\text{Inf}}{\to} H^q(G, A) \overset{\text{Res}}{\to} H^q(H, A)$$

is exact.

Sketch. The proof is by induction on q, via dimension shifting. For $q = 1$, we can just directly check this on 1-cocycles. The main point is the exactness at $H^q(G, A)$: if $c \in Z^1(G, A)$ has $\operatorname{Res}(c) \in B^1(H, A)$, then find $a \in A$ with

$$\operatorname{Res}(c)(a) \coloneqq h \cdot a - a.$$

As such, we define $f_a \in B^1(G, A)$ by $f_a(g) \coloneqq g \cdot a - a$, which implies that $c - f_a$ vanishes on H. It is then possible to stare at the 1-cocycle condition

$$(c - f_a)(gg') = (c - f_a)(g) + g \cdot (c - f_a)(g')$$

to check that $c - f_a$ only depends on the cosets of H (e.g., by taking $g' \in H$) and that $\operatorname{im}(c - f_a) \subseteq A^H$ (e.g., by taking $g \in H$).

For $q > 1$, we use dimension shifting via the following lemma.

Lemma 2 (Dimension shifting)**.** Let G be a group with subgroup $H \subseteq G$. Given a G-module A, all indices $q \ge 1$ have

$$\delta\colon H^q(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^{q+1}(H, A).$$

Sketch. Recall that we have the short exact sequence of $\mathbb{Z}[H]$-modules

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

In fact, this short exact sequence splits over $\mathbb{Z}$, so it will still be short exact after applying $\operatorname{Hom}_{\mathbb{Z}}(-, A)$, which gives the short exact sequence

$$0 \to A \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to \operatorname{Hom}_{\mathbb{Z}}(I_G, A) \to 0$$

of $\mathbb{Z}[H]$-modules. The result now follows from the long exact sequence of cohomology upon noting that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ is coinduced and hence acyclic for cohomology. ■

Using the above lemma, we have the following the commutative diagram with vertical arrows which are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & H^q\left(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H\right) \arrow[r] \arrow[d, "\delta"] & H^q(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[r] \arrow[d, "\delta"] & H^q(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \arrow[d, "\delta"] \\
0 \arrow[r] & H^{q+1}\left(G/H, A^H\right) \arrow[r] & H^{q+1}(G, A) \arrow[r] & H^{q+1}(H, A)
\end{tikzcd}$$

The top row is exact by the inductive hypothesis, so the bottom row is therefore also exact. ■

Our goal is to make the above proof explicit in the case of $q = 2$, which is the only reason we sketched the above proofs at all. We begin by making the dimension shifting explicit.

Lemma 3. Let G be a group with subgroup $H \subseteq G$, and let $\{g_\alpha\}_{\alpha\in\lambda}$ be coset representatives for $H\backslash G$. Now, given a G-module A, the maps

$$\begin{aligned} \delta_H \colon Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) &\to Z^2(H, A) \\ c &\mapsto \left[(h, h') \mapsto h \cdot c(h')(h^{-1} - 1)\right] \\ \left[h \mapsto \left((h'g_\bullet - 1) \mapsto h' \cdot u((h')^{-1}, h)\right)\right] &\leftarrow\!\shortmid u \end{aligned}$$

are group homomorphisms which descend to the isomorphism $\overline{\delta} \colon H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \simeq H^2(H, A)$ of Lemma 2. The map δ above is surjective, and the reverse map is a section; when $H = G$, these are isomorphisms.

Proof. We begin by noting that our short exact sequence can be written more explicitly as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(I_G, A) & \longrightarrow & 0 \\ & & a & \longmapsto & (z \mapsto \varepsilon(z)a) & & & & \\ & & & & f & \longmapsto & f|_{I_G} & & \end{array}$$

We now track through the induced boundary morphism $\delta \colon H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \to H^2(H, Q)$.

- We begin with $c \in Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$, which means that we have $c(h) \colon I_G \to A$ for each $h, h' \in H$, and we satisfy
$$c(hh') = c(h) + h \cdot c(h').$$
Tracking through the action of H on $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)$, this means that
$$c(hh')(g - 1) = c(h)(g - 1) + h \cdot c(h')(h^{-1}g - h^{-1})$$
for any $g \in G$.

- To pull c back to $C^1(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$, we need to lift $c(h) \colon I_G \to A$ to a $\widetilde{c}(h) \colon \mathbb{Z}[G] \to A$. Recalling that we only need to preserve group structure, we simply precompose $c(h)$ with the map $\mathbb{Z}[G] \to I_G$ given by $z \mapsto z - \varepsilon(z)$. That is, we define
$$\widetilde{c}(h)(z) := c(h)(z - \varepsilon(z)).$$

- We now push $\widetilde{c}$ through $d \colon C^1(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)) \to Z^2(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$. This gives
$$(d\widetilde{c})(h, h') = g\widetilde{c}(h') - \widetilde{c}(hh') + \widetilde{c}(h)$$
for any $h, h' \in H$. Concretely, plugging in some $z \in \mathbb{Z}[G]$ makes this look like
$$\begin{aligned} (d\widetilde{c})(h, h')(z) &= (h\widetilde{c}(h'))(z) - \widetilde{c}(hh')(z) + \widetilde{c}(h)(z) \\ &= h \cdot c(h')\left(h^{-1}z - \varepsilon(h^{-1}z)\right) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) \\ &= h \cdot c(h')\left(h^{-1}z - \varepsilon(z)\right) - c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)). \end{aligned}$$
Now, from the 1-cocycle condition on c, we recall
$$-c(hh')(z - \varepsilon(z)) + c(h)(z - \varepsilon(z)) = -h \cdot (c(h')(h^{-1}z - \varepsilon(z)h^{-1})),$$
so
$$\begin{aligned} (d\widetilde{c})(h, h')(z) &= h \cdot c(h')\left(\varepsilon(z)h^{-1} - \varepsilon(z)\right) \\ &= \varepsilon(z) \cdot \left(h \cdot c(h')\left(h^{-1} - 1\right)\right). \end{aligned}$$
In particular, we see that $d\widetilde{c} \in Z^2(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ pulls back to $(h, h') \mapsto h \cdot c(h')\left(h^{-1} - 1\right)$ in $Z^2(H, A)$. It is not too difficult to check that we have in fact defined a 2-cocycle, but we will not do so because it is not necessary for the proof.

Now, we do know that δ_H is a homomorphism abstractly on elements of our cohomology classes by the Snake lemma, but it is also not too hard to see that

$$\delta_H\colon Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) \to Z^2(H, A)$$

is in fact a homomorphism of groups directly from the construction. In short,

$$\delta_H(c+c')(h,h') = h' \cdot c(h)\left(h^{-1}-1\right) + h' \cdot c'(h)\left(h^{-1}-1\right) = (\delta_H(c)+\delta_H(c'))(h,h')$$

for any $h, h' \in H$.

It remains to prove the last sentence. We run the following checks; given $u \in Z^2(H, A)$, define $c_u \in C^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$ by

$$c_u(h)(h'g_\bullet - 1) = h' \cdot u\left((h')^{-1}, h\right).$$

Note that this is enough data to define $c_u(h)\colon I_G \to A$ because I_G is a free $\mathbb{Z}$-module generated by $\{g-1 : g \in G\}$.

- We verify that c_u is a 1-cocycle. This is a matter of force. Pick up $h, h' \in H$ and $g_\bullet h'' \in G$ and write

$$\begin{aligned}
&(hc_u(h'))(h''g_\bullet - 1) + c_u(hh')(h''g_\bullet - 1) + c_u(h)(h''g_\bullet - 1) \\
&= h \cdot c_u(h')\left(h^{-1}h''g_\bullet - h^{-1}\right) + c_u(hh')(h''g_\bullet - 1) + c_u(h)(h''g_\bullet - 1) \\
&= h \cdot \left(h^{-1}h''u\left((h'')^{-1}h, h'\right) - h^{-1}u(h,h')\right) + h''u\left((h'')^{-1}, hh'\right) + h''u\left((h'')^{-1}, h\right) \\
&= h''u\left((h'')^{-1}h, h'\right) - u(h,h') + h''u\left((h'')^{-1}, hh'\right) + h''u\left((h'')^{-1}, h\right).
\end{aligned}$$

  This is just the 2-cocycle condition for u upon dividing out by h'', so we are done.

- For $u \in Z^2(H, A)$, we verify that $\delta_H(c_u) = u$. Indeed, given $h, h' \in H$, we check

$$\begin{aligned}
\delta_H(c_u)(h,h') &= h \cdot c_u(h')\left(h^{-1}-1\right) \\
&= h \cdot h^{-1} \cdot u(h,h') \\
&= u(h,h').
\end{aligned}$$

So far we have verified that δ has section $u \mapsto c_u$ and hence must be surjective. Lastly, we take $H = G$ and show that $c_{\delta c} = c$ to finish. Indeed, for $g, g' \in G = H$, we write

$$\begin{aligned}
c_{\delta_H c}(g)(g'-1) &= g' \cdot (\delta_H c)\left((g')^{-1}, g\right) \\
&= g'(g')^{-1} \cdot c(g)(g'-1) \\
&= c(g)(g'-1),
\end{aligned}$$

which is what we wanted. ■

We also have used dimension shifting to show that $H^1\left(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H\right) \to H^2\left(G/H, A^H\right)$ is an isomorphism, but this requires a little more trickery. To begin, we discuss how to lift from $\operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ to $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$.

Lemma 4. Let G be a group with subgroup $H \subseteq G$. Fix a G-module A with $H^1(H, A) = 0$. Then, for any $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$, the function $h \mapsto h\psi\left(h^{-1}-1\right)$ is a cocycle in $Z^1(H, A) = B^1(H, A)$, so we can define a function $I_\bullet\colon \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H \to A$ such that

$$\psi(h-1) = h \cdot I_\varphi - I_\varphi$$

for all $h \in H$. In fact, given $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$, we can construct $\widetilde{\varphi} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H$ by

$$\widetilde{\varphi}(z) := \varphi(z - \varepsilon(z)) + \varepsilon(z) I_\varphi$$

so that $\widetilde{\varphi}|_{I_G} = \varphi$.

Proof. We will just run the checks directly.

- We start by checking $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ give 1-cocycles $c(h) := \varphi(h-1)$ in $Z^1(A, H)$. To begin, we note that $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ simply means that any $z - \varepsilon(z) \in I_G$ has
$$\psi(z-\varepsilon(z)) = (h\psi)(z-\varepsilon(z)) = h\psi\left(h^{-1}z - h^{-1}\varepsilon(z)\right)$$
for all $h \in H$. In particular, replacing h with h^{-1} tells us that
$$h\psi(z-\varepsilon(z)) = \psi(hz - h\varepsilon(z)).$$
Now, we can just compute
$$\begin{aligned}(dc)(h,h') &= hc(h') - c(hh') + c(h) \\ &= hc\left(h'-1\right) - c\left(hh'-1\right) + c\left(h-1\right) \\ &= c\left(hh'-h\right) - c\left(hh'-1\right) + c\left(h-1\right),\end{aligned}$$
where in the last equality we used the fact that $\psi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$. Now, $(dc)(h,h')$ manifestly vanishes, so we are done.
- Note that $\widetilde{\varphi} \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ because it is a linear combination of (compositions of) homomorphisms.
- Note that any $z \in I_G$ has $\varepsilon(z) = 0$, so
$$\widetilde{\varphi}(z) = \varphi(z-0) + 0 \cdot I_\varphi = \varphi(z),$$
so $\widetilde{\varphi}|_{I_G} = \varphi$.
- It remains to check that $\widetilde{\varphi}$ is fixed by H. This requires a little more effort. Recall that $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$ means that any $z - \varepsilon(z) \in I_G$ has
$$h\varphi(z-\varepsilon(z)) = \varphi\left(hz - h\varepsilon(z)\right)$$
for any $h \in H$. Now, we just compute
$$\begin{aligned}(h\widetilde{\varphi})(z) &= h\widetilde{\varphi}\left(h^{-1}z\right) \\ &= h\left(\varphi\left(h^{-1}z - \varepsilon(h^{-1}z)\right) + \varepsilon(h^{-1}z)I_\varphi\right) \\ &= \varphi\left(z - h\varepsilon(z)\right) + \varepsilon(z)\cdot hI_\varphi \\ &= \varphi\left(z - h\varepsilon(z)\right) + \varepsilon(z)\varphi(h-1) + \varepsilon(z)I_\varphi \\ &= \varphi(z-\varepsilon(z)) + \varepsilon(z)I_\varphi \\ &= \widetilde{\varphi}(z).\end{aligned}$$

The above checks complete the proof. ■

Remark 5. For motivation, the $\widetilde{\varphi}$ was constructed by tracking through the following diagram.

$$\begin{tikzcd} & \dfrac{C^0(H,A)}{B^0(H,A)} \arrow[r] \arrow[d] & \dfrac{C^0(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A))}{B^0(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A))} \arrow[r] \arrow[d] & \dfrac{C^0(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A))}{B^0(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A))} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z^1(H,A) = B^1(H,A) \arrow[r] & Z^1(H,\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A)) \arrow[r] & Z^1(H,\operatorname{Hom}_{\mathbb{Z}}(I_G,A)) & \end{tikzcd}$$

In short, take $\varphi \in Z^0(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)) = \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$, pull it back to $z \mapsto \varphi(z - \varepsilon(z))$. Pushing this down to $Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A))$ and pulling back to $Z^1(H, A)$ takes us to the 1-cocycle $h \mapsto h\varphi\left(h^{-1}-1\right)$. Here we use the $H^1(H, A) = 0$ condition above and adjust our lift $z \mapsto \varphi(z - \varepsilon(z))$ accordingly.

And now we can now make our dimension shifting explicit.

Lemma 6. Work in the context of Lemma 4 and assume that $H \subseteq G$ is normal. We track through the isomorphism

$$\delta\colon H^1\left(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H\right) \simeq H^2\left(G/H, A^H\right)$$

given by the exact sequence

$$0 \to A^H \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)^H \to \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H \to 0.$$

Proof. We begin with some $c \in H^1\left(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H\right)$. To track through the δ, we define

$$\widetilde{c}(gH) \coloneqq c(gH)(z - \varepsilon(z)) + I_{c(gH)}\varepsilon(z)$$

to be the lift given in Lemma 4. Now, we are given that $dc = 0$, which here means that any $z \in \mathbb{Z}[G]$ and $gH, g'H \in G/H$ will have

$$\begin{aligned}
0 &= (dc)(gH, g'H)(z - \varepsilon(z)) \\
0 &= (gH \cdot c(g'H) - c(gg'H) + c(gH))(z - \varepsilon(z)) \\
0 &= g \cdot c(g'H)\left(g^{-1}z - g^{-1}\varepsilon(z)\right) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)) \\
g \cdot c(g'H)\left(g^{-1} - 1\right)\varepsilon(z) &= g \cdot c(g'H)\left(g^{-1}z - \varepsilon(z)\right) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)) \\
g \cdot c(g'H)\left(g^{-1} - 1\right)\varepsilon(z) &= g \cdot c(g'H)\left(g^{-1}z - \varepsilon(g^{-1}z)\right) - c(gg'H)(z - \varepsilon(z)) + c(gH)(z - \varepsilon(z)).
\end{aligned}$$

We now directly compute that

$$\begin{aligned}
(d\widetilde{c})(gH, g'H)(z) &= (gH \cdot c(g'H) - c(gg'H) + c(gH))(z) \\
&= g \cdot c(g'H)\left(g^{-1}z - \varepsilon(g^{-1}z)\right) + gI_{c(g'H)}\varepsilon(z) \\
&\quad - c(gg'H)(z - \varepsilon(z)) - I_{c(gg'H)}\varepsilon(z) \\
&\quad + c(gH)(z - \varepsilon(z)) + I_{c(gH)}\varepsilon(z) \\
&= \left(g \cdot c(g'H)\left(g^{-1} - 1\right) + g \cdot I_{c(g'H)} - I_{c(gg'H)} + I_{c(gH)}\right)\varepsilon(z)
\end{aligned}$$

As such, we have pulled ourselves back to the 2-cocycle given by

$$\boxed{u(gH, g'H) \coloneqq g \cdot c(g'H)\left(g^{-1} - 1\right) + g \cdot I_{c(g'H)} - I_{c(gg'H)} + I_{c(gH)}}.$$

We quickly note that this is in fact independent of our choice of representative $g \in gH$: changing representative of g to gh for $h \in H$ will only affect the terms

$$h \cdot c(g'H)\left(h^{-1}g^{-1} - 1\right) + hI_{c(g'H)} = c(g'H)\left(g^{-1} - h\right) + c(g'H)(h - 1) + I_{c(g'H)} = c(g'H)\left(g^{-1} - 1\right) + I_{c(g'H)},$$

so we are indeed safe. This completes the proof. ■

We now make Theorem 1 explicit in the case of $q = 2$.

Lemma 7. Let G be a group with normal subgroup $H \subseteq G$. Fix a G-module A with $H^1(H, A) = 0$, and define the function $I_\bullet\colon \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H \to A$ of Lemma 4. Given $c \in Z^2(G, A)$ such that $\operatorname{Res}^G_H c \in B^2(H, A)$; in particular, suppose we have $b \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)$ such that all $h \in H$ have

$$\operatorname{Res}^G_H(\delta^{-1}c)(h) = (db)(h) = h \cdot b - h,$$

where δ^{-1} is the inverse isomorphism of Lemma 3. Then we find $u \in Z^2\left(G/H, A^H\right)$ such that

$$[\operatorname{Inf} u] = [c]$$

in $H^2(G, A)$.

Proof. The main point is that boundary morphisms δ commute with Res and Inf. By construction, we have that $\left(\operatorname{Res}_H^G \delta^{-1}c\right) - db = 0$ in $Z^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$. Pulling back to $Z^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$, we note that

$$c' := \left(\delta^{-1}c - db\right) \in Z^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, A))$$

vanishes on H by hypothesis. Because $\delta^{-1}c - db$ is a 1-cocycle, we are able to write

$$c'(gg') = c'(g) + gc'(g').$$

Letting g' vary over H, we see that $\delta^{-1}c - db$ is well-defined on G/H. On the other hand, for any $h \in H$ and $g \in G$, we note that $g^{-1}hg \in H$, so

$$c'(g) = c'\left(g \cdot g^{-1}hg\right) = c'\left(hg\right) = c'\left(h\right) + hc(g),$$

implying that $c'(g) \in \operatorname{Hom}_{\mathbb{Z}}(I_G, A)^H$.

We are now ready to apply Lemma 6, which we use on c', thus defining $u := \delta(c')$. Explicitly, we have

$$\boxed{u(gH, g'H) = g \cdot c'(g'H)\left(g^{-1} - 1\right) + g \cdot I_{c'(g'H)} - I_{c'(gg'H)} + I_{c'(gH)}}.$$

This is explicit enough for our purposes. Observe that $[\operatorname{Inf} u] = [c]$ because $[\operatorname{Inf} c'] = [\delta^{-1}c]$, and δ commutes with Inf. ■

3.2 Number Theory

Throughout, we will let $u_{L/K}$ denote a representative of the fundamental class in $H^2(L/K)$ rather than the actual cohomology class, mostly out of laziness.

We now return to the set-up in section 1 and track through Lemma 7 in our case. For reference, the following is the diagram that we will be chasing around; here $G := \operatorname{Gal}(ML/K)$ and $H := \operatorname{Gal}(ML/L)$.

$$\begin{tikzcd}
 & & H^2(\operatorname{Gal}(M/K), M^\times) \arrow[d, "\operatorname{Inf}"] & \\
0 \arrow[r] & H^2(\operatorname{Gal}(L/K), L^\times) \arrow[r, "\operatorname{Inf}"] & H^2(G, ML^\times) \arrow[r, "\operatorname{Res}"] & H^2(\operatorname{Gal}(ML/L), ML^\times) \\
0 \arrow[r] & H^1(G/H, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)^H) \arrow[r, "\operatorname{Inf}"] \arrow[u, "\delta"] & H^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)) \arrow[r, "\operatorname{Res}"] \arrow[u, "\delta"] & H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)) \arrow[u, "\delta"]
\end{tikzcd}$$

To begin, we know that we can write

$$u_{M/K}\left(\sigma_K^i, \sigma_K^j\right) = p^{\left\lfloor \frac{i+j}{n} \right\rfloor} = \begin{cases} 1 & i+j < n, \\ p & i+j \geq n. \end{cases}$$

Inflating this down to $H^2(G, ML^\times)$ gives

$$(\operatorname{Inf} u_{M/K})\left(\sigma_K^{a_1}\sigma_x^{a_2}, \sigma_K^{b_1}\sigma_x^{b_2}\right) = p^{\left\lfloor \frac{a_1+b_1}{n} \right\rfloor}.$$

Now, we use Lemma 2 to move down to $H^1(G, \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times))$ as

$$\delta^{-1}(\operatorname{Inf} u_{M/K})\left(\sigma_K^{a_1}\sigma_x^{a_1}\right)\left(\sigma_K^{b_1}\sigma_x^{b_2} - 1\right) = \sigma_K^{b_1}\sigma_x^{b_2} \cdot (\operatorname{Inf} u_{M/K})\left(\sigma_K^{[-b_1]}\sigma_x^{[-b_2]}, \sigma_K^{a_1}\sigma_x^{a_2}\right) = p^{\left\lfloor \frac{a_1+[-b_1]}{n} \right\rfloor},$$

where $[k]$ denote the integer $0 \leq [k] < n$ such that $k \equiv [k] \pmod{n}$.

Now, we need to show that the restriction to $H = \langle \sigma_k^f \rangle$ is a coboundary. That is, we need to find $b \in \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)$ such that

$$\delta^{-1}(\operatorname{Inf} u_{M/K})\left(\sigma_K^{fa_1}\right) = \frac{\sigma_K^{fa_1} \cdot b}{b}.$$

Because I_G is freely generated by elements of the form $g-1$ for $g \in G$, it suffices to plug in some arbitrary $\sigma_K^{b_1}\sigma_x^{b_2}-1$, which we see requires

$$\begin{aligned} p^{\left\lfloor \frac{fa_1+[-b_1]}{n} \right\rfloor} &= \frac{\left(\sigma_K^{fa_1}\cdot b\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)}{b\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)} \\ &= \frac{\sigma_K^{fa_1} b\left(\sigma_K^{b_1-fa_1}\sigma_x^{b_2}-1\right)}{\sigma_K^{fa_1} b\left(\sigma_K^{-fa_1}-1\right) b\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)}. \end{aligned}$$

We can see that b should not depend on b_2, so we define $\hat{b}\left(\sigma_K^a\right) = b\left(\sigma_K^a\sigma_x^\bullet - 1\right)$; the above is then equivalent to

$$\begin{aligned} p^{\left\lfloor \frac{fa_1+[-b_1]}{n} \right\rfloor} &= \frac{\sigma_K^{fa_1}\hat{b}\left(\sigma_K^{b_1-fa_1}\right)}{\sigma_K^{fa_1}\hat{b}\left(\sigma_K^{-fa_1}\right)\hat{b}\left(\sigma_K^{b_1}\right)} \\ p^{\left\lfloor \frac{fa_1+b_1}{n} \right\rfloor} &= \frac{\hat{b}\left(\sigma_K^{-b_1-fa_1}\right)}{\hat{b}\left(\sigma_K^{-fa_1}\right)\sigma_K^{-fa_1}\hat{b}\left(\sigma_K^{-b_1}\right)}, \end{aligned}$$

where we have negated b_1 in the last step. At this point, the right-hand side will look a lot more natural if we set $\tau := \sigma_K^{-1}$, which turns this into

$$\frac{\hat{b}\left(\tau^{fa_1}\right)\tau^{fa_1}\hat{b}\left(\tau^{b_1}\right)}{\hat{b}\left(\tau^{b_1 fa_1}\right)} = (1/p)^{\left\lfloor \frac{fa_1+b_1}{n} \right\rfloor}$$

after taking reciprocals. Thus, we see that $\hat{b}$ should be counting carries of τs. With this in mind, we note that $1-\zeta_{p^\nu} \in L$ is a uniformizer because $L/\mathbb{Q}_p\left(\zeta_{p^\nu}\right)$ is an unramified extension. It follows that

$$\left(1-\zeta_{p^\nu}\right)^{\varphi(p^\nu)} \in \mathrm{N}_{ML/L}\left(ML^\times\right).$$

Further, $\left(1-\zeta_{p^\nu}\right)^{\varphi(p^\nu)}$ is only a unit (in $\mathcal{O}_L^\times$) multiplied p, so in fact p is a norm from $ML^\times$ because ML/L is unramified and so all units in $\mathcal{O}_L^\times$ are norms from $ML^\times$. Thus, we find $\alpha \in ML^\times$ such that

$$\mathrm{N}_{ML/L}(\alpha) = p.$$

The point of doing all of this is so that we can codify our carrying by writing

$$\hat{b}\left(\tau^a\right) := \prod_{i=0}^{\lfloor a/f \rfloor - 1} \tau^{if}(\alpha)^{-1}.$$

Tracking out $\hat{b}$ backwards to b, our desired $b \in \operatorname{Hom}_{\mathbb{Z}}(I_G, ML^\times)$ is given by

$$\boxed{b\left(\sigma_K^{a_1}\sigma_x^{a_2}-1\right) = \prod_{i=0}^{\lfloor [-a_1]/f \rfloor - 1} \sigma_K^{-if}(\alpha)^{-1}}.$$

We take a moment to write out $c := \delta^{-1}(\operatorname{Inf} u_{M/K})/db$, which looks like

$$\begin{aligned} c\left(\sigma_K^{a_1}\sigma_x^{a_2}\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right) &= \frac{\delta^{-1}(\operatorname{Inf} u_{M/K})}{db}\left(\sigma_K^{a_1}\sigma_x^{a_2}\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right) \\ &= \frac{\delta^{-1}(\operatorname{Inf} u_{M/K})\left(\sigma_K^{a_1}\sigma_x^{a_2}\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)}{\left(\sigma_K^{a_1}\sigma_x^{a_2}b\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)/b\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)} \\ &= \frac{p^{\lfloor (a_1+[-b_1])/n\rfloor}}{\sigma_K^{a_1}\sigma_x^{a_2}b\left(\sigma_K^{b_1-a_1}\sigma_x^{b_2-a_2}-\sigma_K^{-a_1}\sigma_x^{-a_2}\right)/b\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)} \\ &= p^{\lfloor (a_1+[-b_1])/n\rfloor}\cdot \hat{b}\left(\sigma_K^{b_1}\right)\cdot\sigma_K^{a_1}\sigma_x^{a_2}\left(\frac{\hat{b}\left(\sigma_K^{-a_1}\right)}{\hat{b}\left(\sigma_K^{b_1-a_1}\right)}\right). \end{aligned}$$

Before proceeding, we discuss a few special cases.

- Taking $\sigma_K^{a_1}\sigma_x^{a_2}=\sigma_x$, we get

$$\begin{aligned} c\left(\sigma_x\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right) &= p^{\lfloor (0+[-b_1])/n\rfloor}\cdot \hat{b}\left(\sigma_K^{b_1}\right)\cdot\sigma_x\left(\frac{1}{\hat{b}\left(\sigma_K^{b_1}\right)}\right) \\ &= \hat{b}\left(\sigma_K^{b_1}\right)/\sigma_x\hat{b}\left(\sigma_K^{b_1}\right). \end{aligned}$$

  In particular, $c\left(\sigma_x\right)\left(\sigma_K^{-1}-1\right)=1$, provided that $f>1$. Additionally, $c(\sigma_x)\left(\sigma_x^{b_2}-1\right)=1$.
  Our general theory says that $h\mapsto c(\sigma_x)(h-1)$ is a 1-cocycle in $Z^1(H,ML^\times)$ (though we could also check this directly), so Hilbert's Theorem 90 promises us a magical element $I_{c(\sigma_x)}\in ML^\times$ such that

$$\frac{\sigma_K^{fb_1}I_{c(\sigma_x)}}{I_{c(\sigma_x)}}=\frac{\hat{b}\left(\sigma_K^{fb_1}\right)}{\sigma_x\hat{b}\left(\sigma_K^{fb_1}\right)}$$

  for all $\sigma_K^{fb_1}\in H$. This condition will be a little clearer if we write everything in terms of $\tau:=\sigma_K^{-1}$, which transforms this into

$$\frac{\tau^{fb_1}I_{c(\sigma_x)}}{I_{c(\sigma_x)}}=\frac{\hat{b}\left(\tau^{-fb_1}\right)}{\sigma_x\hat{b}\left(\tau^{-fb_1}\right)}=\prod_{i=0}^{b_1-1}\frac{\tau^{if}(\alpha^{-1})}{\sigma_x\tau^{if}(\alpha^{-1})}=\prod_{i=0}^{b_1-1}\frac{\sigma_x\tau^{if}(\alpha)}{\tau^{if}(\alpha)}.$$

  Because we are dealing with a cyclic group H, it is not too hard to see that it suffices merely for $b_1=1$ to hold, so our magical element $I_{c(\sigma_x)}$ merely requires

$$\boxed{\frac{\sigma_K^{-f}\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}}=\frac{\sigma_x(\alpha)}{\alpha}}$$

  after inverting τ back to σ_K.

- Taking $\sigma_K^{a_1}\sigma_x^{a_2}=\sigma_K$, we get

$$c\left(\sigma_K\right)\left(\sigma_K^{b_1}\sigma_x^{b_2}-1\right)=p^{\lfloor (1+[-b_1])/n\rfloor}\cdot\hat{b}\left(\sigma_K^{b_1}\right)\cdot\sigma_K\left(\frac{\hat{b}\left(\sigma_K^{-1}\right)}{\hat{b}\left(\sigma_K^{b_1-1}\right)}\right).$$

  In particular, $\sigma_K^{b_1}\sigma_x^{b_2}=\sigma_x^{-1}$ will give $c(\sigma_K)\left(\sigma_x^{-1}-1\right)=1$. We will also want $c(\sigma_K)\left(\sigma_K^{-b_1}-1\right)$ for $0\le b_1<f$. Using the fact that $f<n$ and $f>1$, it is not too hard to see that everything will cancel down to 1 except in the case where $b_1 = f-1$, where we get

$$c(\sigma_K)\left(\sigma_K^{-(f-1)}-1\right) = \sigma_K\left(\frac{1}{\hat{b}\left(\sigma_K^{-f}\right)}\right) = \sigma_K(\alpha).$$

Continuing as before, our general theory says that $h \mapsto c(\sigma_x)(h-1)$ is a 1-cocycle in $Z^1(H, ML^\times)$, though again we could just check this directly. It follows that Hilbert's Theorem 90 promises us a magical element $I_{c(\sigma_K)} \in ML^\times$ such that

$$\frac{\sigma_K^{fb_1} I_{c(\sigma_K)}}{I_{c(\sigma_K)}} = p^{\lfloor (1+[-fb_1])/n \rfloor} \cdot \hat{b}\left(\sigma_K^{fb_1}\right) \cdot \sigma_K\left(\frac{\hat{b}\left(\sigma_K^{-1}\right)}{\hat{b}\left(\sigma_K^{fb_1-1}\right)}\right)$$

for all $\sigma_K^{fb_1} \in H$. Using $f > 1$, this collapses down to

$$\frac{\sigma_K^{fb_1} I_{c(\sigma_K)}}{I_{c(\sigma_K)}} = \frac{\hat{b}\left(\sigma_K^{fb_1}\right)}{\sigma_K \hat{b}\left(\sigma_K^{fb_1-1}\right)}.$$

As before, this condition will be a little clearer if we set $\tau := \sigma_K^{-1}$, which turns the condition into

$$\frac{\tau^{fb_1} I_{c(\sigma_K)}}{I_{c(\sigma_K)}} = \frac{\hat{b}\left(\tau^{fb_1}\right)}{\sigma_K \hat{b}\left(\tau^{fb_1+1}\right)} = \prod_{i=0}^{b_1-1} \frac{\tau^{if}\left(\alpha^{-1}\right)}{\sigma_K \tau^{if}\left(\alpha^{-1}\right)} = \prod_{i=0}^{b_1-1} \frac{\sigma_K \tau^{if}(\alpha)}{\tau^{if}(\alpha)}.$$

(Notably, $\hat{b}\left(\tau^{fb_1}\right) = \hat{b}\left(\tau^{fb_1+1}\right)$ because $f > 1$.) Again, because H is cyclic generated by τ^f, an induction shows that it suffices to check this condition for $b_1 = 1$, which means that our magical element $I_{c(\sigma_K)} \in ML^\times$ is constructed so that

$$\boxed{\frac{\sigma_K^{-f}\left(I_{c(\sigma_K)}\right)}{I_{c(\sigma_K)}} = \frac{\sigma_K(\alpha)}{\alpha}}$$

where we have again inverted back from τ to σ_K.

- We will not actually need a more concrete description of this, but we remark that we can run the same story for any $g \in G$ through to get an element $I_{c(g)} \in ML^\times$ such that

$$\frac{\sigma_K^{fb_1} I_{c(g)}}{I_{c(g)}} = \frac{1}{c(g)(\sigma_K^{fb_1} - 1)}$$

for any $\sigma_K^{fb_1} \in H$. As usual, this follows from our general theory.

We are now ready to describe the local fundamental class. Piecing what we have so far, we know from Lemma 7 that we can write

$$u_{L/K}(g, g') := gc(g')\left(g^{-1}-1\right) \cdot \frac{gI_{c(g')} \cdot I_{c(g)}}{I_{c(gg')}}.$$

Here are the values that we care about for our specific computation.

- We write

$$\begin{aligned} u_{L/K}(\sigma_K, \sigma_x) &= \sigma_K c(\sigma_x)\left(\sigma_K^{-1}-1\right) \cdot \frac{\sigma_K I_{c(\sigma_x)} \cdot I_{c(\sigma_K)}}{I_{c(\sigma_K\sigma_x)}} \\ &= \frac{\sigma_K I_{c(\sigma_x)} \cdot I_{c(\sigma_K)}}{I_{c(\sigma_K\sigma_x)}}. \end{aligned}$$

- We write

$$\begin{aligned} u_{L/K}(\sigma_x, \sigma_K) &= \sigma_x c(\sigma_K)\left(\sigma_x^{-1} - 1\right) \cdot \frac{\sigma_x I_{c(\sigma_K)} \cdot I_{c(\sigma_x)}}{I_{c(\sigma_x \sigma_K)}} \\ &= \frac{\sigma_x I_{c(\sigma_K)} \cdot I_{c(\sigma_x)}}{I_{c(\sigma_x \sigma_K)}}. \end{aligned}$$

- In particular, we know that we can set β in a triple equal to

$$\begin{aligned} \beta &:= \frac{u_{L/K}(\sigma_K, \sigma_x)}{u_{L/K}(\sigma_x, \sigma_K)} \\ &= \frac{\sigma_K I_{c(\sigma_x)} \cdot I_{c(\sigma_K)} / I_{c(\sigma_K \sigma_x)}}{\sigma_x I_{c(\sigma_K)} \cdot I_{c(\sigma_x)} / I_{c(\sigma_x \sigma_K)}} \\ &\boxed{\beta = \frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)}}. \end{aligned}$$

  As a sanity check, we can hit this β with σ_K^{-f} to show that $\beta \in (ML)^H = L$; namely, $\sigma_K^{-f} I_{c(\sigma_K)} = \frac{\sigma_K \alpha}{\alpha} \cdot I_{c(\sigma_K)}$ and $\sigma_K^{-f} I_{c\sigma(x)} = \frac{\sigma_x \alpha}{\alpha} \cdot I_{c(\sigma_x)}$ by construction, so we can see that everything will appropriately cancel out.

- We will go ahead and compute α_1 and α_2, for completeness. For α_1, our element is given by

$$\begin{aligned} \alpha_1 &:= \prod_{i=0}^{f-1} u_{L/K}\left(\sigma_K^i, \sigma_K\right) \\ &= \prod_{i=0}^{f-1} \left(\sigma_K^i c\left(\sigma_K, \sigma_K^{-i} - 1\right) \cdot \frac{\sigma_K^i I_{c(\sigma_K)} \cdot I_{c\left(\sigma_K^i\right)}}{I_{c\left(\sigma_K^{i+1}\right)}}\right). \end{aligned}$$

  Recall from our general theory that $I_{c(g)}$ only depends on the coset of g in G/H, so we see that the product of the quotients $I_{c\left(\sigma_K^i\right)}/I_{c\left(\sigma_K^{i+1}\right)}$ will cancel out. As for the c term, we know from our computation that this is 1 until $i = f - 1$, which gives $\sigma_K(\alpha)$. As such, we collapse down to

$$\boxed{\alpha_1 = \sigma_K^f(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right)}.$$

- For α_2, our element is given by

$$\begin{aligned} \alpha_2 &:= \prod_{i=0}^{\varphi(p^\nu)-1} u_{L/K}\left(\sigma_x^i, \sigma_x\right) \\ &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i c(\sigma_x)\left(\sigma_x^{-i} - 1\right) \cdot \frac{\sigma_x^i I_{c(\sigma_x)} \cdot I_{c(\sigma_x^i)}}{I_{c(\sigma_x^{i+1})}}. \end{aligned}$$

  Recalling that σ_x has order $\varphi\left(p^\nu\right)$, our quotient term $I_{c(\sigma_x^i)}/I_{c(\sigma_x^{i+1})}$ will again cancel out. Additionally, the cocycle c always spits out 1 on these inputs, so we are left with

$$\boxed{\alpha_2 = \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right)}.$$

We summarize the results above in the following theorem.

Theorem 8. Fix everything as in the set-up. Then there exists some $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$ and elements in $I_{c(\sigma_K)}, I_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{\sigma_K^{-f}\left(I_{c(\sigma_K)}\right)}{I_{c(\sigma_K)}} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{\sigma_K^{-f}\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} = \frac{\sigma_x(\alpha)}{\alpha}.$$

Then the triple

$$(\alpha_1, \alpha_2, \beta) := \left(\sigma_K^f(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right), \quad \frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)}\right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\mathrm{Gal}(L/K), L^\times)$.

We remark that we can replace α with $\sigma_K^f(\alpha)$ (which still has norm p) while keeping all other variables the same; this gives us the following slightly prettier presentation. Note that we have multiplied the equations for $I_\bullet$ by σ_K^f on both sides.

Corollary 9. Fix everything as in the set-up. Then there exists some $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$ and elements in $I_{c(\sigma_K)}, I_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{I_{c(\sigma_K)}}{\sigma_K^f\left(I_{c(\sigma_K)}\right)} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{I_{c(\sigma_x)}}{\sigma_K^f\left(I_{c(\sigma_x)}\right)} = \frac{\sigma_x(\alpha)}{\alpha}.$$

Then the triple

$$(\alpha_1, \alpha_2, \beta) := \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right), \quad \frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)}\right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\mathrm{Gal}(L/K), L^\times)$.

3.3 Checks

In this section we run some checks and discuss some consequences of Theorem 8, in the form of Corollary 9. For these results, we recall that we set $L := \mathbb{Q}_p(\zeta_N)$ and $L_1 := \mathbb{Q}_p(\zeta_{p^\nu})$ and $L_2 := \mathbb{Q}_p(\zeta_m)$ so that $\overline{\sigma}_K = \sigma_K|_{L_1}$ generates $\mathrm{Gal}(L/L_1)$ and σ_x generates $\mathrm{Gal}(L/L_2)$.

In the discussion which follows, we will make repeated use of the fact that (using notation of Corollary 9)

$$\sigma_K^f\left(I_{c(\sigma_K)}\right) = \frac{\alpha}{\sigma_K(\alpha)} \cdot I_{c(\sigma_K)} \qquad \text{and} \qquad \sigma_K^f\left(I_{c(\sigma_x)}\right) = \frac{\alpha}{\sigma_x(\alpha)} \cdot I_{c(\sigma_x)}.$$

And here are our checks; we start by showing that our elements are in the right field.

Lemma 10. Fix a triple $(\alpha_1, \alpha_2, \beta)$ as in Corollary 9. Then the following are true.

(a) $\alpha_1 \in L_1^\times$.

(b) $\alpha_2 \in L_2^\times$.

(c) $\beta \in L^\times$.

Proof. We run the checks one at a time.

(a) It suffices to show that α_1 is fixed by $\mathrm{Gal}(M/L_1) = \langle \sigma_K \rangle$. As such, we simply compute

$$\begin{aligned}
\sigma_K(\alpha_1) &= \sigma_K\left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right)\right) \\
&= \sigma_K(\alpha) \cdot \prod_{i=0}^{f-1} \sigma_K^{i+1}\left(I_{c(\sigma_K)}\right) \\
&= \sigma_K(\alpha) \cdot \sigma_K^f\left(I_{c(\sigma_K)}\right) \prod_{i=1}^{f-1} \sigma_K^{i+1}\left(I_{c(\sigma_K)}\right) \\
&= \alpha \cdot I_{c(\sigma_K)} \prod_{i=1}^{f-1} \sigma_K^{i+1}\left(I_{c(\sigma_K)}\right) \\
&= \prod_{i=0}^{f-1} \sigma_K^{i+1}\left(I_{c(\sigma_K)}\right) \\
&= \alpha_1.
\end{aligned}$$

(b) It suffices to show that α_2 is fixed by $\mathrm{Gal}(M/L_2) = \langle \sigma_K^f, \sigma_x \rangle$. On one hand,

$$\begin{aligned}
\sigma_K^f(\alpha_2) &= \sigma_K^f\left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right)\right) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(\sigma_K^f I_{c(\sigma_x)}\right) \\
&= \left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(\frac{\alpha}{\sigma_x(\alpha)}\right)\right) \cdot \left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right)\right) \\
&= \left(\prod_{i=0}^{\varphi(p^\nu)-1} \frac{\sigma_x^i(\alpha)}{\sigma_x^{i+1}(\alpha)}\right) \cdot \alpha_2 \\
&= \alpha_2,
\end{aligned}$$

where the product telescopes because σ_x has order $\varphi\left(p^\nu\right)$.
On the other hand,

$$\begin{aligned}
\sigma_x(\alpha_2) &= \sigma_x\left(\prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right)\right) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^{i+1}\left(I_{c(\sigma_x)}\right) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right),
\end{aligned}$$

where we have again used the fact that σ_x has order $\varphi\left(p^\nu\right)$. This last product is α_2, so we are done.

(c) It suffices to show that β is fixed by $\mathrm{Gal}(M/L) = \langle \sigma_K^f \rangle$. Applying force, we see

$$\begin{aligned}
\sigma_K^f(\beta) &= \sigma_K^f\left(\frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)} \right) \\
&= \frac{\sigma_K\left(\sigma_K^f I_{c(\sigma_x)}\right)}{\sigma_K^f I_{c(\sigma_x)}} \cdot \frac{\sigma_K^f I_{c(\sigma_K)}}{\sigma_x\left(\sigma_K^f I_{c(\sigma_K)}\right)} \\
&= \frac{\sigma_K\left(\alpha/\sigma_x\alpha\right) \cdot \sigma_K\left(I_{c(\sigma_x)}\right)}{(\alpha/\sigma_x\alpha) \cdot I_{c(\sigma_x)}} \cdot \frac{(\alpha/\sigma_K\alpha) \cdot I_{c(\sigma_K)}}{\sigma_x(\alpha/\sigma_K\alpha) \cdot \sigma_x\left(I_{c(\sigma_K)}\right)} \\
&= \frac{\sigma_K\alpha}{\sigma_K\sigma_x\alpha} \cdot \frac{\sigma_x\alpha}{\alpha} \cdot \frac{\alpha}{\sigma_K\alpha} \cdot \frac{\sigma_x\sigma_K\alpha}{\sigma_x\alpha} \cdot \frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)} \\
&= \beta.
\end{aligned}$$

The above checks complete the proof. ■

Next we show the relations.

Lemma 11. Fix a triple $(\alpha_1, \alpha_2, \beta)$ as in Corollary 9. Then the following are true.

(a) $\mathrm{N}_{L/L_1}(\beta) = \alpha_1/\sigma_x\alpha_1$.

(b) $\mathrm{N}_{L/L_2}(\beta^{-1}) = \alpha_2/\overline{\sigma}_K\alpha_2$.

Proof. We go one at a time.

(a) Note $\mathrm{Gal}(L/L_1) = \langle \overline{\sigma}_K \rangle$. In particular, $\overline{\sigma}_K$ has order f, so we can just compute out

$$\begin{aligned}
\mathrm{N}_{L/L_1}(\beta) &= \prod_{i=0}^{f-1} \sigma_K^i(\beta) \\
&= \prod_{i=0}^{f-1} \sigma_K^i \left(\frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)} \right) \\
&= \prod_{i=0}^{f-1} \frac{\sigma_K^{i+1}\left(I_{c(\sigma_x)}\right)}{\sigma_K^i\left(I_{c(\sigma_x)}\right)} \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right) \Big/ \sigma_x\left(\prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right) \right) \\
&= \frac{\sigma_K^f\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right) \Big/ \sigma_x\left(\prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right) \right) \\
&= \frac{\alpha}{\sigma_x\alpha} \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right) \Big/ \sigma_x\left(\prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right) \right) \\
&= \alpha_1/\sigma_x\alpha.
\end{aligned}$$

(b) Note $\operatorname{Gal}(L/L_2) = \langle \sigma_x \rangle$, so we compute

$$\begin{aligned}
\mathrm{N}_{L/L_2}(\beta) &= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i(\beta) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i \left(\frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)} \right) \\
&= \prod_{i=0}^{\varphi(p^\nu)-1} \frac{\sigma_x^i\left(I_{c(\sigma_K)}\right)}{\sigma_x^{i+1}\left(I_{c(\sigma_K)}\right)} \cdot \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_K\left(I_{c(\sigma_x)}\right) \Bigg/ \prod_{i=0}^{\varphi(p^\nu)-1} I_{c(\sigma_x)} \\
&= \sigma_K\left(\prod_{i=0}^{\varphi(p^\nu)-1} I_{c(\sigma_x)} \right) \Bigg/ \prod_{i=0}^{\varphi(p^\nu)-1} I_{c(\sigma_x)} \\
&= \sigma_K\alpha_2/\alpha_2.
\end{aligned}$$

Taking the reciprocal finishes; in particular, $\overline{\sigma}_K\alpha_2 = \sigma_K\alpha_2$ is a legal expression because $\alpha_2 \in L^\times$.

The above checks complete the proof. ■

3.4 Consequences

With some checks out of the way, here are some actual consequences. To begin, we state Hilbert's Theorem 90.

Lemma 12. Suppose that L/K is a (finite) cyclic extension of fields such that $\Gamma := \operatorname{Gal}(L/K)$ is generated by $\sigma \in \Gamma$. Given some $\alpha \in L^\times$ such that $\mathrm{N}(\alpha) = 1$, there exists $\beta_0 \in L^\times$ such that $\alpha = \beta_0/\sigma\beta_0$. In fact, this β_0 is unique "up to a multiple in $K^\times$" in the sense that

$$\left\{\beta \in L^\times : \alpha = \beta/\sigma\beta\right\} = \left\{x\beta_0 : x \in K^\times\right\}.$$

Proof. That such a β_0 exists follows directly from Hilbert's Theorem 90. For the last sentence, of course any $\beta := x\beta_0 \in L^\times$ with $x \in K^\times$ will have

$$\frac{\beta}{\sigma\beta} = \frac{\beta_0}{\sigma\beta_0} = \alpha.$$

In the other direction, if $\beta \in L^\times$ has $\beta/\sigma\beta = \alpha$, then

$$\sigma(\beta/\beta_0) = (\sigma\beta)/(\sigma\beta_0) = \beta/\beta_0,$$

so $\beta/\beta_0 \in K^\times$ and $\beta = (\beta/\beta_0) \cdot \beta_0$. ■

And here are some quick consequences of this.

Corollary 13. Fix everything as in the set-up, and fix $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$. Choosing some $\sigma \in \{\sigma_K, \sigma_x\}$, the elements I_σ satisfying

$$\frac{I_\sigma}{\sigma_K^f\left(I_\sigma\right)} = \frac{\sigma(\alpha)}{\alpha}$$

are unique up to a multiple in $L^\times$, in the sense of Lemma 12.

Proof. Note that $\operatorname{Gal}(ML/L) = \langle \sigma_K^f \rangle$ is cyclic generated by σ_K^f and $\mathrm{N}_{ML/L}(\sigma\alpha/\alpha) = p/p = 1$, so we may simply apply Lemma 12 directly to get the result. ■

We might be worried that our choice α is affecting the set of $I_{c(\sigma_K)}$ or $I_{c(\sigma_x)}$, but in fact they are not, more or less.

Corollary 14. Fix everything as in the set-up, and choose $\sigma \in \{\sigma_K, \sigma_x\}$. Given $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$, define

$$S_\alpha := \left\{ I_\sigma \in ML^\times : \frac{I_\sigma}{\sigma_K^f(I_\sigma)} = \frac{\sigma(\alpha)}{\alpha} \right\}.$$

Then the set S_α is "unique up to a multiple in $ML^\times$" in the sense that two $\alpha, \alpha' \in ML^\times$ with $\mathrm{N}_{ML/L}(\alpha) = \mathrm{N}_{ML/L}(\alpha') = p$ have some $x \in ML^\times$ such that

$$S_\alpha = x \cdot S_{\alpha'} := \{x \cdot I_\sigma : I_\sigma \in S_{\alpha'}\}.$$

Proof. Suppose $\alpha, \alpha' \in ML^\times$ satisfy $\mathrm{N}_{ML/L}(\alpha) = \mathrm{N}_{ML/L}(\alpha') = p$. The key point is that

$$\mathrm{N}_{ML/L}(\alpha/\alpha') = p/p = 1,$$

so Lemma 12 promises us some $\gamma \in ML^\times$ such that $\alpha/\alpha' = \gamma/\sigma_K^f(\gamma)$. As such, we see that

$$\frac{\sigma(\alpha)}{\alpha} = \frac{\sigma(\alpha/\alpha')}{\alpha/\alpha'} \cdot \frac{\sigma(\alpha')}{\alpha'} = \frac{(\sigma\gamma/\gamma)}{\sigma_K^f(\sigma\gamma/\gamma)} \cdot \frac{\sigma(\alpha')}{\alpha'}.$$

As such, we set $x := (\sigma\gamma/\gamma)$.

To finish, we check that $S_\alpha \subseteq x \cdot S_{\alpha'}$, and the other inclusion is similar. Well, if $I_\sigma \in S_{\alpha'}$, then

$$\frac{xI_\sigma}{\sigma_K^f(xI_\sigma)} = \frac{x}{\sigma_K^f(x)} \cdot \frac{I_\sigma}{\sigma_K^f(I_\sigma)} = \frac{(\sigma\gamma/\gamma)}{\sigma_K^f(\sigma\gamma/\gamma)} \cdot \frac{\sigma(\alpha')}{\alpha'} = \frac{\sigma(\alpha)}{\alpha},$$

so $xI_\sigma \in S_\alpha$. This finishes. ■

We now return to describing triples.

Corollary 15. Fix everything as in the set-up, and fix $\alpha \in ML^\times$ such that $\mathrm{N}_{ML/L}(\alpha) = p$. Then, for any triple $(\alpha_1', \alpha_2', \beta')$ corresponding to the fundamental class, there exist elements $I'_{c(\sigma_K)}, I'_{c(\sigma_x)} \in ML^\times$ with

$$\frac{I'_{c(\sigma_K)}}{\sigma_K^f\left(I'_{c(\sigma_K)}\right)} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{I'_{c(\sigma_x)}}{\sigma_K^f\left(I'_{c(\sigma_x)}\right)} = \frac{\sigma_x(\alpha)}{\alpha}$$

such that

$$(\alpha_1', \alpha_2', \beta') = \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I'_{c(\sigma_K)}\right), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I'_{c(\sigma_x)}\right), \quad \frac{\sigma_K\left(I'_{c(\sigma_x)}\right)}{I'_{c(\sigma_x)}} \cdot \frac{I'_{c(\sigma_K)}}{\sigma_x\left(I'_{c(\sigma_K)}\right)} \right).$$

In other words, all triples corresponding to the fundamental class come from the recipe described in Corollary 9.

Proof. By Corollary 9, we can certainly find some elements $I_{c(\sigma_K)}, I_{c(\sigma_x)} \in ML^\times$ such that

$$\frac{I_{c(\sigma_K)}}{\sigma_K^f\left(I_{c(\sigma_K)}\right)} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{I_{c(\sigma_x)}}{\sigma_K^f\left(I_{c(\sigma_x)}\right)} = \frac{\sigma_x(\alpha)}{\alpha},$$

for which

$$(\alpha_1, \alpha_2, \beta) := \left(\alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right), \quad \prod_{i=0}^{\varphi(p^\nu)-1} \sigma_x^i\left(I_{c(\sigma_x)}\right), \quad \frac{\sigma_K\left(I_{c(\sigma_x)}\right)}{I_{c(\sigma_x)}} \cdot \frac{I_{c(\sigma_K)}}{\sigma_x\left(I_{c(\sigma_K)}\right)} \right)$$

corresponds to the fundamental class $u_{L/K} \in H^2(\mathrm{Gal}(L/K), L^\times)$. In particular, $(\alpha_1, \alpha_2, \beta)$ and $(\alpha_1', \alpha_2', \beta')$ both correspond to the same cohomology class and hence in the same equivalence class of triples, so we know that there exist $m_1, m_2 \in L^\times$ such that

$$\alpha_1' = \alpha_1 \cdot \mathrm{N}_{L/L_1}(m_1), \quad \alpha_2' = \alpha_2 \cdot \mathrm{N}_{L/L_2}(m_2), \quad \beta' = \beta \cdot \frac{\sigma_K(m_2)}{m_2} \cdot \frac{m_1}{\sigma_x(m_1)}.$$

As such, we set $I'_{c(\sigma_K)} := I_{c(\sigma_K)} \cdot m_1$ and $I'_{c(\sigma_x)} := I_{c(\sigma_x)} \cdot m_2$, and these can be checked to work. For example, $I'_{c(\sigma_K)}$ satisfies

$$\frac{I'_{c(\sigma_K)}}{\sigma_K^f\left(I'_{c(\sigma_K)}\right)} = \frac{\sigma_K(\alpha)}{\alpha} \qquad \text{and} \qquad \frac{I'_{c(\sigma_x)}}{\sigma_K^f\left(I'_{c(\sigma_x)}\right)} = \frac{\sigma_x(\alpha)}{\alpha}$$

by Lemma 12. The rest of the checks are similar. ■

Corollary 16. Fix everything as in the set-up, and let $\pi_1 \in L_1^\times$ be a uniformizer. If the triple $(\alpha_1, \alpha_2, \beta)$ is a triple corresponding to the fundamental class, then

$$\alpha_1 \equiv \pi_1 \pmod{\mathrm{N}_{L/L_1}(L^\times)}.$$

Proof by triples. Note that L/L_1 is an unramified extension, so all elements of absolute value 1 are norms, so there is in fact a class of elements containing all uniformizers in $L_1^\times / \mathrm{N}_{L/L_1}(L^\times)$. Further, because α_1 is also only defined up to an element $\mathrm{N}_{L/L_1}(L^\times)$, to show that the classes in $L^\times / \mathrm{N}_{L/L_1}(L^\times)$ coincide, it thus suffices to exhibit a single triple $(\alpha_1, \alpha_2, \beta)$ such that $\alpha_1 \in L_1^\times$ is a uniformizer.

This is a matter of force. To begin, we can use Corollary 9 to find some α with $\mathrm{N}_{ML/L}(\alpha) = p$ and $I_{c(\sigma_K)}, I_{c(\sigma_x)} \in ML^\times$ giving the triple $(\alpha_1, \alpha_2, \beta)$ as described. The idea is to force $I_{c(\sigma_K)}$ to have valuation zero.

Let v_{ML} be the fixed valuation of ML extending the standard valuation $v_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, and let v_L be its restriction to L. Because ML/L is an unramified, the image of v_{ML} and v_L in $\mathbb{Q}$ is the same. In particular, we can find some $m_1 \in L_1^\times$ such that

$$v_{ML}\left(I_{c(\sigma_K)}\right) = v_L(m_1).$$

Thus, we replace $I_{c(\sigma_K)}$ with $I_{c(\sigma_K)}/m_1$, and we still satisfy the conditions of Corollary 9 by Lemma 12 while getting $v_{ML}\left(I_{c(\sigma_K)}\right) = 0$. Now, the corresponding α_1 looks like

$$\alpha_1 = \alpha \cdot \prod_{i=0}^{f-1} \sigma_K^i\left(I_{c(\sigma_K)}\right).$$

In particular, defining $v_{L_1} := v_L|_{L_1}$, it follows

$$v_{L_1}(\alpha_1) = v_{ML}(\alpha_1) = v_{ML}(\alpha),$$

However, $\mathrm{N}_{ML/L}(\alpha) = p$ by construction, so we see that

$$[ML : L] v_{ML}(\alpha) = v_{ML}(p) = v_{\mathbb{Q}_p}(p) = 1.$$

Explicitly, we see that

$$[ML : L] = [\mathbb{Q}(\zeta_{N'}) : \mathbb{Q}(\zeta_m)] = \frac{[\mathbb{Q}(\zeta_{N'}) : \mathbb{Q}_p]}{[\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p]} = \frac{n}{f} = \varphi\left(p^\nu\right).$$

However, L_1/K has ramification degree $\varphi(p^\nu)$ (from the maximal totally ramified subextension $\mathbb{Q}_p(\zeta_{p^\nu})$), so its uniformizers are the elements of valuation $1/\varphi(p^\nu)$. Thus, we have computed that α_1 has the correct valuation and hence is a uniformizer. ■

Proof by the Artin map. We take a moment to say that there is an alternate derivation of Corollary 16 using the Artin map: one can show that, if $u \in Z^2(L/K)$ is a representative of the fundamental class of an abelian extension L/K, then

$$\begin{aligned} \operatorname{Gal}(L/K) &\to K^\times / \operatorname{N}(L^\times) \\ \sigma &\mapsto \prod_{g \in \operatorname{Gal}(L/K)} u(g, \sigma) \end{aligned}$$

is the inverse Artin map. In particular, from our explicit formula for α_1, we see

$$\alpha_1 = \prod_{g \in \operatorname{Gal}(L/L_1)} u(g, \overline{\sigma}_K) = \theta_{L/L_1}^{-1}(\overline{\sigma}_K).$$

However, $\overline{\sigma}_K$ is the Frobenius automorphism of L/L_1 because the extension L_1/K is totally ramified, implying that the residue field of L_1 is the same as $K = \mathbb{Q}_p$. Thus, $\theta_{L/L_1}^{-1}(\overline{\sigma}_K)$ is the class containing the uniformizers of $L_1^\times$. ■