

Geometric Representation Theory Seminar

Nir Elber

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1 September 5th: David Nadler

The theme for this semester is “categorical representation theory,” which is now contained in “geometric representation theory,” which is the title of our seminar. Gurbir Dhillon [Dhi22] has provided an informal introduction.

1.1 Basic Definitions

We are interested in representations of groups. Here are some examples of groups of interest.

- Finite groups, such as the finite groups of Lie type.
- Groups over local fields, such as real Lie groups and p -adic groups.
- Algebraic groups, such as the classical groups.

Importantly, these are not merely sets with multiplication: they come from algebraic groups.

There are two motivations for representation theory.

- Groups are hard, so representations provide concrete objects to understand them.
- Representations are a natural setting for objects with symmetries.

With this in mind, a representation should be thought of as a map from our group G to some automorphism group $\text{Aut}_{\mathcal{C}}(c)$ of an object in a category. What kind of representation one has depends on \mathcal{C} and c .

Example 1. One frequently works with \mathcal{C} as the category of finite-dimensional complex vector spaces. Thus, a representation is a homomorphism $G \rightarrow \mathrm{GL}_n(\mathbb{C})$. Things can become more complicated working with other fields than \mathbb{C} , such as finite fields.

In order to use extra (say, analytic or algebraic) properties of our set-up, we will want to study functions on G . So we let $\mathrm{Fun}(G)$ denote the functions on G and $\mathrm{Dist}(G)$ denote the distributions on G . The group law on G then gives a co-algebra and algebra structure to $\mathrm{Fun}(G)$ and $\mathrm{Dist}(G)$, respectively. The point is that representations of G become comodules over $\mathrm{Fun}(G)$ or modules over $\mathrm{Dist}(G)$.

Remark 2. Thus far, we have made the following choices: the category \mathcal{C} and which functions/distributions we think about. In practice, we typically fix a field k for our coefficients (though \mathcal{C} may not be Vec_k), and we want to fix a type of functions.

Example 3. Consider $G = S^1$.

- Viewing G as a compact (real) Lie group, we could think about smooth representations, functions, and distributions, then it turns out that our representations decompose into one-dimensional irreducible representations.
- From the perspective of homotopy theory, we view G as a topological group, and we want to work with locally constant functions. Then our distributions of S^1 (valued in S^1) consists of the differential graded algebra of chain complexes on S^1 ; this is quasi-isomorphic to $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ where we require $\deg \varepsilon = -1$. So we want to understand $\mathbb{C}[\varepsilon]$ -modules, but putting ε in degree -1 means that it must act by 0. One can upgrade this equivalence to complexes.

1.2 Character Sheaves

As motivation for the subject, we explain how categorical representation theory contextualizes Lusztig's character sheaves. We are interested in studying representations of finite groups of Lie type, such as the classical groups of Lie type like $\mathrm{GL}_n(\mathbb{F}_q)$, so fix our ambient algebraic group G . There are basically two ways to construct representations.

- Induction from an abelian subgroup, or more generally a parabolic subgroup.
- Parity, due to Langlands; this is the theory of cuspidal representations.

In order to avoid having to pray too much, recall that we can understand representations of G by their characters, which make up a basis of the class functions. Thus, we are interested in constructing cuspidal characters, and then we will get everyone else by induction.

Lusztig does not try to construct characters directly on $G(\mathbb{F}_q)$ but instead construct character sheaves on the algebraic group G . Passing to the algebraic group allows us to use algebraic geometry and maybe even complex Lie theory and so on to produce sheaves, and then we will do some “decategorification” of the character sheaves to get our characters back (explicitly, one does the usual dimensions of Frobenius actions for étale cohomology). The punchline is that character sheaves can be produced by induction from tori, so we get the needed characters!

Remark 4. Philosophically, from the point of view of the seminar, what's going on is that geometric representation theory is controlled by a choice of topological field theory. At a core level, we made the problem “core” enough to access this by induction, namely by forgetting about \mathbb{F}_q .

2 September 12: Raymond

Categorical representation theory begins with an algebraic group G , and then we choose a group algebra for our functions or distributions. For example, one can work with the category of D -modules on G , which are in some sense categorical distributions. Today we are talking about D -modules. As a last remark, one note that one finds representations by groups acting on a variety X (such as a flag variety), and D -modules (as with cohomology) provides a way to linearize this action.

2.1 The Sheaf D_X

One may be interested in solving homogeneous linear differential equations in one time variable like

$$\left(\sum_{i=0}^N p_i(t) D^i \right) f = 0,$$

where D is the differential operator. (Here, t is a time variable.) Thus, our setting is going to look like polynomials in t and D which do not commute but instead have a product rule

$$D(tf) = f + tDf,$$

meaning $(Dt - tD) = 1$, or $[D, t] = 1$. So we get some non-commutative ring

$$D_{\mathbb{A}^1} = \frac{\mathbb{C}\langle t, d \rangle}{([d, t] = 1)}.$$

This is a Weyl algebra in one variable, but one can add more variables: perhaps one has n time variables t_1, \dots, t_n and n corresponding differentials $\partial_1, \dots, \partial_n$, and then we find that the t_i s should commute with each other, the ∂_i s should commute with each other, t_i and ∂_j should commute when $i \neq j$, and we should have a product rule for t_i and ∂_i . This builds a Weyl algebra.

Note that $D_{\mathbb{A}^1}$ has some modules on which acts.

Example 5. Note $D_{\mathbb{A}^1}$ has a left action on the ideal generated by ∂ . One can compute this $\mathbb{C}[\partial]$, which we can compute has $\partial \cdot \partial^n = \partial^{n+1}$ and $t \cdot \partial^n = -n\partial^{n-1}$ (by some product rule computation).

Let's try to think about More generally, we can pick up a smooth affine variety X (such as a flag variety) over a field k . Here are two interesting subsets.

- One can embed $\mathcal{O}_X(X) \subseteq \text{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$. This is supposed to correspond to our t s.
- Vector fields Vect_X : one can look at the subset of $\text{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$ of derivations F . Explicitly, we are asking for this to be k -linear and satisfy the Leibniz rule

$$F(fg) = fF(g) + F(f)g.$$

This is supposed to correspond to our ∂ s.

Then we define D_X to be the subalgebra of $\text{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$ generated by the above two subalgebras; one can check that $D_{\mathbb{A}^1}$ is as described above.

Remark 6. By considering affine open subschemes of X , we can upgrade D_X into a presheaf on the base, and in fact, one can see that this will sheafify appropriately. Thus, even if X is not affine, we have still produced a sheaf D_X of (non-commutative) k -algebras.

2.2 D -modules

This permits the following definition.

Definition 7 (D -module). Fix a smooth variety X over a field k . Then a D_X -module is a sheaf of D_X -modules which is also quasicoherent over $\mathcal{O}_X \subseteq D_X$. The category of D_X -modules is $\text{Mod}(D_X)$.

Here are some examples.

Example 8. Note \mathcal{O}_X is a (left) D_X -module because $D_X \subseteq \text{End}_k(\mathcal{O}_X)$ already. Locally on affines, one finds that \mathcal{O}_X is D_X / Vect_X . As such, a map $\mathcal{O}_X \rightarrow M$ of D_X -modules amounts to choosing where $1 \in \mathcal{O}_X$ goes, so we are picking out an element $f \in M$ which is killed by Vect_X (namely, killed by our differential operators).



Warning 9. Because D_X is non-commutative, there is a difference between left and right D_X -modules. Most of our modules are left.

Example 10. One can differentiate top differential forms in ω_X on the right by the Lie derivative. (Approximately speaking, this is dual to the action on \mathcal{O}_X via the integration pairing.)

Example 11. Fix a closed point $x \in X$, and then we consider the corresponding maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_X$. Then we have a D_X -module of “ δ -functions” at x given by $\delta_x := D_X / (D_X \mathfrak{m}_x)$. Here is some motivation for calling this δ -functions.

- Support: δ_x is supported set-theoretically just x ; scheme-theoretically, it is supported on the infinitesimal neighborhood of x . One sees this by taking stalks appropriately everywhere.
- Universal property: a map $\delta_x \rightarrow M$ is determined by where it sends $1 \in D_X$, and we need to make sure that the image $f \in M$ of 1 has $\mathfrak{m}_x f = 0$. Thus, f is really only allowed to have freedom at x .

Example 12. One can also think about δ -functions on closed smooth subschemes. Roughly speaking, one can break down the closed smooth subscheme (locally) into a product of things we should have as Vect_X s and things we should have as \mathfrak{m}_x s.

Given a smooth morphism $f: X \rightarrow Y$ of smooth schemes, one wants some functoriality between $\text{Mod}(D_X)$ and $\text{Mod}(D_Y)$.

- Pullback: one has a functor $f^*: \text{Mod}(D_Y) \rightarrow \text{Mod}(D_X)$ by tensoring over D_Y with the bimodule $D_{X \rightarrow Y} := \mathcal{O}_X \otimes_{\mathcal{O}_Y} D_Y$. Do note that the tensor products over D_Y is misleading because we are thinking about our $M \in \text{Mod}(D_Y)$ as $\mathcal{O}_X \otimes_{\mathcal{O}_Y} M$.

Approximately speaking, we are pulling back by thinking of M as functions, and we can pull back functions.

- Pushforward: one has a functor $f_*: \text{Mod}(D_X) \rightarrow \text{Mod}(D_Y)$.

3 September 19: Raymond

Here we go.

3.1 Kähler Differentials

We would like to go between left and right D_X -modules, for which we want to be tensoring by some bi-module. For a k -algebra A , there is a universal differential $d: A \rightarrow \Omega_{A/k}$ so that

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_A(\Omega_{A/k}, M)$$

for any A -module M ; here $\mathrm{Der}_k(A, M)$ are the k -linear derivations $A \rightarrow M$. In particular, taking $M = A$ above, we see that $\Omega_{A/k}$ is dual (over A) to Vect_A . This continues to be true after gluing, so Ω_X is dual to Vect_X for a scheme X .

Now, let Ω_X^{top} denote the top differential $\wedge^{\dim X} \Omega_X$. We have a D_X -module structure given by the Lie derivative: for $\xi \in \mathrm{Vect}_X$ and $\omega \in \Omega_X^{\mathrm{top}}$, we define $(\mathrm{Lie} \xi)(\omega) \in \Omega_X^{\mathrm{top}}$ given by

$$(\mathrm{Lie} \xi)(\omega)(\xi_1, \dots, \xi_n) = \xi(\omega(\xi_1, \dots, \xi_n)) - \sum_{i=1}^n \omega(\xi_1, \dots, [\xi, \xi_i], \dots, \xi_n),$$

which one can essentially believe as some kind of product rule. The following can be checked by some computations.

- $(\mathrm{Lie} f\xi)(\omega) = \mathrm{Lie} \xi(f\omega)$ for any function f .
- $\mathrm{Lie} \xi(f\omega) = \xi(f)\omega + f(\mathrm{Lie} \xi)(\omega)$.
- $\mathrm{Lie}[\xi_1, \xi_2](\omega) = [\mathrm{Lie} \xi_1, \mathrm{Lie} \xi_2](\omega)$, where the bracket means taking the commutator.

In particular, the first formula tells us that this is a right D_X -action: f and ξ have switched places, so it is perhaps more natural to write this as $\omega \cdot f\xi = \omega \cdot f \cdot \xi$. Even the second formula

Remark 13 (Nadler). There is a perfect pairing between functions in \mathcal{O}_X and top forms in Ω_X^{top} by integration. Thus, we can turn left modules into right modules by this sort of duality. More formally, we note that differentials act on $C_c^\infty(M)$ in the natural way on the left, so distributions act on $C_c^\infty(M)^*$ on the right. Given $\omega \in \Omega_X^{\mathrm{top}}$, we can produce a distribution by integration: $f \mapsto \int_M f \omega$, and then we can unwind the action of dual action of differentials to produce the Lie derivative: for vector field X , we have

$$\int_M df(X) \wedge \omega = \int_M f(-\mathcal{L}_X \omega).$$

Now, given a left D_X -module, we can produce a right D_X -module by $\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} -$; conversely, given a right D_X -module, we can produce a left D_X -module by $- \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{top}}$.

Example 14. A left D_X -module M is essentially a family of functions, so given a section m , we see that elements of $M \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{top}}$ look like $m\omega$ where ω is a top form. This now has our right action by something like

$$\xi \cdot (m\omega) = -\xi(m)\omega + m\xi\omega.$$

Namely, there are some complications in our actions.

Remark 15 (Eugene, Nadler). Let's explain this a bit more conceptually. Note that $\mathrm{Diff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}})$ has two separate right actions: Ω_X^{top} has a natural right action, and \mathcal{O}_X has a natural left action giving the total another right action. Thus, we see that $\mathrm{Diff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}}) \otimes_{D_X} -$ is turning left modules into right modules: choose one of the right actions to form the tensor product and use the other right action for our right module structure.

3.2 Pullback and Pushforward

Let's now return to our discussion of pullback and pushforward. Fix some $f: X \rightarrow Y$.

- Pullback: we define $D_{X \rightarrow Y} := \mathcal{O}_X \otimes_{f^* \mathcal{O}_Y} D_Y$ and then are able to define $f^* M := D_{X \rightarrow Y} \otimes_{D_X} M$ for D_Y -module M . Note $D_{X \rightarrow Y}$ can be thought of as $\text{Diff}(f^* \mathcal{O}_Y, \mathcal{O}_X)$.
- Pushforward: we will still use $D_{X \rightarrow Y}$, but we now must keep twisting around. Given our left D_X -module M , we make it into a right module by considering $\Omega_X^{\text{top}} \otimes_{\mathcal{O}_X} M$, which we are now able to tensor with the bimodule $D_{X \rightarrow Y}$ to make a right D_Y -module $(\Omega_X^{\text{top}} \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X \rightarrow Y}$. But we want a left D_Y -module, so we define

$$f_* M := ((\Omega_X^{\text{top}} \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X \rightarrow Y}) \otimes \Omega_X^{\text{top}, -1}.$$

Alternatively, one can think about this as tensoring by $\text{Diff}(f^* \Omega_Y^{\text{top}}, \Omega_X^{\text{top}})$.

3.3 The Convolution

Fix a group scheme G . We would like for $\text{Mod}(D_G)$ to be a monoidal category, which will turn into our group algebra. For $a \in G$, recall we define δ_a as $D_G / D_G \mathfrak{m}_a$. One can compute that

$$m_* (\pi_1^* \delta_a \otimes_{\mathcal{O}_{G \times G}} \pi_2^* \delta_b) = \delta_{m(a,b)},$$

so there is some suspicion of this module structure.