Special Values

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1 Special Values of Dirichlet L-Functions

This talk was given by Rui. Roughly speaking, the style of these sorts of special values results is that someone observes some equalities, then one works out examples, we make a general conjecture, and eventually it is proven.

1.1 Some Examples

Let's begin by discussing the simplest L-function: the Riemann ζ-function.

Definition 1. The *Riemann ζ-function* ζ is defined by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$.

Example 2. Here is what is known about some small special values. Euler showed that $\zeta(s) = \frac{\pi^2}{6}$, and Apéry showed that $\zeta(3)$ is irrational.

Remark 3. In general, there is a conjecture that the values $\{\pi, \zeta(3), \zeta(5), \ldots\}$ forms an algebrically independent set. Roughly speaking, this is expected by the Grothendieck period conjecture.

Today, we will be happy working in only slightly larger generality, with Dirichlet L-functions.

Definition 4 (Dirichlet character)**.** Fix a positive integer N. Then a *Dirichlet character* $\pmod N$ is a character $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. The Dirichlet character η is *primitive* if and only if it does not factor through $(\mathbb{Z}/D\mathbb{Z})^\times$ for any divisor $D \mid N$. Further, we say that η is even (respectively, odd) if and only if $\eta(-1) = 1$ (respectively, $\eta(-1) = -1$).

Definition 5 (Dirichlet L-function)**.** Given a Dirichlet character $\eta \pmod N$, we define the *Dirichlet L-function* $L(\eta, s)$ by

$$L(\eta, s) \coloneqq \sum_{n=1}^{\infty} \frac{\eta(n)}{n^s},$$

where implicitly $\eta(n) = 0$ whenver $\gcd(n, N) > 1$.

Example 6. Let $\eta\colon (\mathbb{Z}/2\mathbb{Z})^\times \to \mathbb{C}^\times$ be the nontrivial character. Then $L(\eta, 1) = \frac{\pi}{4}$.

Example 7. Let $\eta\colon (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times$ be the character defined by $\eta(3) = \eta(5) = -1$. This can be proven via a trick. Consider the power series

$$f(x) \coloneqq x - \frac{1}{3}x - \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots,$$

from which one finds $f'(x) = 1 - x^2 - x^4 + x^6 + \cdots = \frac{1-x^2-x^4+x^6}{1-x^8}$. Then one can integrate $f'(x)$ to get

$$f(x) = \frac{\sqrt{2}}{4} \log \left| \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{x} + 1} \right|.$$

It follows that $L(\eta, 1) = \frac{\sqrt{2}}{2} \log\left(\sqrt{2} + 1\right)$.

The previous example is an example of the class number formula, and right now it looks like a miracle. To give a taste for what is remarkable here, we note that $1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]^\times$ is a fundamental unit. As such, we expect some interesting arithmetic to be going on.

Here is a more general result.

Theorem 8. Suppose $\eta \pmod N$ is a primitive nontrivial Dirichlet character.

(a) If η is even, then for any positive integer m, we have

$$L(\eta, 2m) \equiv \pi^{2m} \pmod{\overline{\mathbb{Q}}^\times}.$$

(b) If η is odd, then for any positive integer m, we have

$$L(\eta, 2m-1) \equiv \pi^{2m-1} \pmod{\overline{\mathbb{Q}}^\times}.$$

The above is an instance of Deligne's conjecture.

For another general result, we note that an even primitive quadratic character $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ has kernel which is an index-2 subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times \cong \operatorname{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, so it corresponds to a quadratic extension F of $\mathbb{Q}$. In fact, the fact that η is even tells us that complex conjugation fixes F, so F is totally real. It then turns out that

$$L(\eta, 1) \equiv \sqrt{\operatorname{disc} \mathcal{O}_F} \cdot \log |u_F| \pmod{\mathbb{Q}^\times},$$

where u_F is a fundamental unit of $\mathcal{O}_F$. This also comes from the class number formula, and it is an instance of Beilinson's conjecture.

1.2 Funtional Equations

As usual, to write down a suitable functional equation for our L-functions, we must add some archimedean factors.

Definition 9 (completed Dirichlet L-function)**.** Let $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Then we define $d \in \{0,1\}$ by $\eta(-1) = (-1)^d$ and then

$$L_\infty(\eta, s) := \pi^{-\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right).$$

Then the *completed Dirichlet L-function* is $\Lambda(\eta, s) := L_\infty(\eta, s)L(\eta, s)$.

Remark 10. Recall that $\Gamma(s)$ is defined by

$$\Gamma(s) := \int_0^\infty e^{-t}t^s\,\frac{dt}{t}$$

for s such that $\operatorname{Re} s > 0$. One also knows that $\Gamma(s)$ admits a meromorphic continuation with understood poles, and it has a functional equation $\Gamma(s+1) = s\Gamma(s)$.

And here is our functional equation.

Theorem 11. Let $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Then $L(\eta, s)$ admits a meromorphic continuation (with poles only at $s \in \{0,1\}$ only when η is trivial) to all $\mathbb{C}$ and satisfies a functional equation

$$\Lambda(\eta, s) = \varepsilon(\eta, s)\Lambda\left(\eta^{-1}, 1-s\right),$$

where $\varepsilon(\eta, s)$ is some appropriately normalized Gauss sum.

We will not prove this today (it is mildly technical). Instead, we will use it to show a partial version of Theorem 8. With that said, we will need to do something in the direction of a meromorphic continuation because we will try to understand negative integer values of $L(\eta, s)$.

By expanding out the series, we see that

$$\Gamma(s)L(\eta, s) = \int_0^\infty \sum_{\substack{n\geq 1\\ \gcd(n,N)=1}} \eta(n)e^{-nt}t^s\,\frac{dt}{t} = \int_0^\infty \frac{1}{1-e^{-Nt}}\sum_{n=0}^{N_1}\eta(n)e^{-nt}\,\frac{dt}{t}.$$

One now plugs into the general machine that produces analytic continuation and functional equations.

Lemma 12. Choose a smooth Schwarz function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then

$$L(f, s) := \frac{1}{\Gamma(s)}\int_0^\infty f(t)t^s\,\frac{dt}{t}$$

has an analytic continuation to all $\mathbb{C}$ and satisfies $L(f, -n) = (-1)^n f^{(n)}(0)$ for all $n \geq 0$.

Proof. To control singularities, we let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth bump function satisfying $\varphi|_{[0,1]} = 1$ and $\varphi_{[2,\infty)} = 0$. Thus, if we expand $f = f_1 + f_2$ with $f_1 = \varphi f$ and $f_2 = (1-\varphi)f$, we see that

$$\int_0^\infty f_2(t)t^s\,\frac{dt}{t} = \int_1^\infty f_2(t)t^s\,\frac{dt}{t},$$

and the rapid decay of f grants this term an analytic continuation to all $\mathbb{C}$, and it even satisfies

$$L(f_2,-n)=\left(\frac{1}{\Gamma(-s)}\int_1^\infty f_2(t)t^s\,\frac{dt}{t}\right)\Bigg|_{s=-n}=0.$$

Thus, we are allowed to ignore f_2 piece. For the f_1 part, we inductively integrate by parts. For example, our first integration by parts produces

$$L(f,s)=\underbrace{\frac{1}{\Gamma(s)}f_1(t)\frac{t^s}{s}\Bigg|_0^\infty}_{0}-\frac{1}{s\Gamma(s)}\int_0^\infty f(t)t\cdot t^s\,\frac{dt}{t}=-L(f_1',s+1).$$

Thus, we have moved out s to $s+1$, and we can iteratively produce the needed continuation from the argument above. The result on the special value follows from a computation. ■

One can now use the lemma to see that

$$L(\eta,-n)\in\mathbb{Q}(\eta).$$

Then one can use the functional equation Theorem 11 to prove Theorem 8 after tracking everything through. I apologize, but I chose not to write down the details.

2 The Kummer Congruence and p-Adic Analysis on $\mathbb{Z}_p$

This talk was given by Mitch. We would like to motivate p-adic L-functions and prove the Kummer congruences, which are used in their construction.

2.1 The Kummer Congruence

Last time, we had an equality of the form

$$\int_{\mathbb{R}^+}\underbrace{\frac{1}{1-e^{-Nt}}\sum_{n=1}^{N-1}\eta(n)e^{-nt}}_{f_\eta(t):=}\cdot t^s\,\frac{dt}{t}=\Gamma(s)L(\eta,s),$$

where $\eta \pmod N$ is some primitive Dirichlet character. The moral of the story is that we see that we are taking a Mellin transform of some function $f_\eta(t)$, so it may be interesting to study these functions on their own terms.

For example, if $\eta=1$ is the trivial character, then one finds that

$$tf_1(t)=\frac{t}{1-e^{-t}}=\sum_{m=0}^\infty B_m\frac{t^m}{m!},$$

where $\{B_m\}_{m\geq 0}$ are the Bernoulli numbers. (Indeed, this is a definition of the Bernoulli numbers.) More generally, one can expand

$$tf_\eta(t)=\sum_{m=0}^\infty B_{\eta,m}\frac{t^m}{m!}$$

to define "twisted" Bernoulli numbers.

For our special values result, we found an identity

$$L(f,-n)=(-1)^nf^{(n)}(0),$$

where $\Gamma(s)L(f,s)$ refers to the Mellin transform, which eventually implies a special values result

$$L(\eta,-n)=-\frac{(-1)^{n+1}B_{\eta,n+1}}{n+1}$$

after some rearrangement. Parity arguments actually allow us to more or less ignore the sign $(-1)^{n+1}$. Namely, when n is even, then $L(\eta,-n)=0$ for even n (unless η is trivial); and when n is odd, then $L(\eta,-n)=0$ for $n\geq 1$ odd.

We are now ready to state our Kummer congruences.

Theorem 13 (Kummer congruence)**.** Fix a nontrivial primitive Dirichlet character $\eta \pmod N$. Fix a prime p coprime to N. Choose nonnegative integers n_1, n_2, and k such that $n_1,n_2\geq k$ and $n_1\equiv n_2 \pmod{(p-1)p^{k-1}}$. Then

$$-\frac{B_{\eta,n_1+1}}{n_1+1}\equiv -\frac{B_{\eta,n_2+1}}{n_2+1} \pmod{p^k}.$$

If η is trivial, then we also need to require $p\nmid (n_1-1)(n_2-1)$.

The moral of the story is that the special values of $L(\eta,s)$ (at integers) admit some kind of continuity in $\mathbb{Z}_p$. This will motivate us to define a p-adic L-function which interpolates these values. This interpolation will turn out to be a profittable thing to do, essentially due to Euler systems.

Remark 14. Here is a historical remark. For reasons related to Fermat's last theorem, Kummer was interested in the notion of a "regular prime." Namely, an odd prime p is found to be regular if and only if $p\nmid \#\operatorname{Cl}(\mathbb{Q}(\zeta_p))$, which turns out to be equivalent to the prime p not dividing any of the numerators of $B_2,B_4,\ldots,B_{p-3}$.

2.2 Using the p-Adic L-Function

Let's begin to describe what a p-adic L-function should be. Fix a prime p and some (space of) characters $\eta^{(p)}\colon (\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$ where $\eta^{(p)} \pmod N$ is a Dirichlet character with $p\nmid N$. Additionally, we fix some $\eta_p\colon(\mathbb{Z}/p^r\mathbb{Z})^\times\to\mathbb{C}^\times$, and we would like to "interpolate" the values

$$L\left(\eta^{(p)}\eta_p,-n\right)$$

as $n\geq 0$ varies. More precisely, we will find that L should be thought of as a measure where η_p is an input.

For our construction, we choose some $f_{\eta_p,n}\colon\mathbb{Z}_p^\times\to\overline{\mathbb{Q}}_p^\times$ given by $f_{\eta_p,n}(a)\coloneqq\eta_p(a)a^n$. Then we will be able to appropriately interpolate with this function.

Remark 15. Note that $f_{\eta_p,n}$ can be thought of as a Galois representation of $\operatorname{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$.

Now, we note

$$L^{\{p\}}\left(\eta^{(p)}\eta_p,s\right)\coloneqq\prod_{\substack{q\text{ prime}\\ \gcd(q,Np)=1}}\frac{1}{1-\eta^{(p)}(q)q^{-s}}=\begin{cases}L\left(\eta^{(p)}\eta_p,s\right) & \text{if }\eta_p\neq 1,\\ L\left(\eta^{(p)}\eta_p,s\right)\left(1-\eta(p)p^{-s}\right) & \text{if }\eta_p=1.\end{cases}$$

Morally, the $L^{\{p\}}$ product simply removes any problems at p, which are relevant while we are working p-adically. The interpolation now appeals to the following result.

Theorem 16. Fix a primitive Dirichlet character $\eta^{(p)} \pmod N$ with $p\nmid N$. Then there is a p-adic measure $d\mu_{\eta^{(p)}}$ such that for any Dirichlet character $\eta_p \pmod{p^k}$ admits

$$\int_{\mathbb{Z}_p^\times}\eta_p(x)x^n\,d\mu_{\eta^{(p)}}(x)=L^{\{p\}}(\eta^{(p)}\eta_p,-n).$$

Remark 17. It is worth comparing this statement to Tate's thesis, where we represent some (completed) L-function of a Hecke character χ as the Mellin transform against a character. The bizarre measure $\mu_{\eta^{(p)}}$ can be seen as incorporating the bizarre prime-to-p parts of the character χ.

We have not bothered to define p-adic integration, but let's explain why this implies Theorem 13 first.

Proof that Theorem 16 implies Theorem 13. This proof is rather formal. Write η as $\eta^{(p)}\eta_p$, where $\eta^{(p)}$ as conductor prime to p, and η_p has conductor which is a power of p. Now, for n large (say, $n \geq k$), we see that

$$L^{\{p\}}\left(\eta^{(p)}\eta_p, -n\right) = \left(1 - \eta(p)p^{-n}\right) L\left(\eta^{(p)}\eta_p, -n\right) \equiv L\left(\eta^{(p)}\eta_p, -n\right) \pmod{p^k}$$

if η_p is trivial, and the statement is still true when η_p is nontrivial. Thus, after plugging in our special values result as $-\frac{B_{\eta,n+1}}{n+1} = L(\eta, -n)$, and in light of Theorem 16, we would like to show

$$\int_{\mathbb{Z}_p^\times} \eta_p(x)x^{n_1}\, d\mu_{\eta^{(p)}}(x) \stackrel{?}{\equiv} \int_{\mathbb{Z}_p^\times} \eta_p(x)x^{n_2}\, d\mu_{\eta^{(p)}}(x) \pmod{p^k}$$

whenever $n_1 \equiv n_2 \pmod{(p-1)p^{k-1}}$. This last equivalence holds on the level of the integrands because we are looking $\pmod{p^k}$. ■

2.3 Integration

Let's say something about how $\mu_{\eta^{(p)}}$ functions.

Remark 18. Do note that we are not looking for the usual Haar measure: small cosets receive size $1/p^\bullet$, which is large p-adically. Additionally, this will have basically no hope of incorporating the prime-to-p information discussed in Remark 17.

So let's rebuild some functional analysis so that we can value our measures in $\mathbb{Q}_p$.

Definition 19 (Banach space)**.** Fix a complete valued p-adic field K. A *Banach space* over K is a complete normed vector space B over K whose norm $\|\cdot\|$ satisfies the triangle inequaltiy

$$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|.$$

Example 20. Fix a compact topological space X. Then the space $C^0(X, \mathbb{Q}_p)$ of continuous functions $X \to \mathbb{Q}_p$ is a Banach space over $\mathbb{Q}_p$. The norm is given by $\|\cdot\|_\infty$.

Definition 21 (orthonormal basis)**.** Fix a Banach space B overa complete valued field p-adic K. Then an *orthonormal basis* is a set $\{e_i\} \subseteq B$ such that $\|e_i\| = 1$ for all i, and any vector v admits a unique expansion

$$v = \sum_i x_i e_i,$$

which converges in the sense $x_i \to 0$ where $\|v\| = \max_i |x_i|$.

Remark 22. We are not requiring that $\{e_i\}$ be countable. The condition that $x_i \to 0$ also includes a hypothesis that only finitely many of the $x_\bullet$s are nonzero.

Our key example will be $C^0(\mathbb{Z}_p, \mathbb{Q}_p)$. Here is a nice basis of this space.

Example 23. For nonnegative integers $n \geq 0$, define the function $\binom{x}{n} : \mathbb{Z}_p \to \mathbb{Q}_p$ as the polynomial

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-(n-1))}{n!}.$$

This is continuous because polynomials are continuous. We also remark that $\left\|\binom{x}{n}\right\|_\infty = 1$, which can be seen by checking on the dense subset $\mathbb{Z} \subseteq \mathbb{Z}_p$.

Proposition 24. The functions $\left\{\binom{x}{n}\right\}_{n \geq 0}$ form an orthonormal basis of $C^0(\mathbb{Z}_p, \mathbb{Q}_p)$.

Proof. We won't bother to write out the proof explicitly, but let's explain why we might expect this. Suppose we have an expansion

$$f(x) = \sum_n a_n(f)\binom{x}{n}$$

already. Then one has a system of equations involving the values of f on integers to solve for the $a_\bullet(f)$s. ■