Student Number Theory Seminar

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Today we are talking a bit about orbital integrals. As some motivation, orbital integrals are the object of interest on the geometric side of the Arthur–Selberg trace formula. It is frequently desirable to compute these orbital integrals in order to use the trace formula to extract spectral information.

1.1 Orbital Integrals

Today, F is a nonarchimedean local field. We let $\mathfrak{o}$ denote its ring of integers (with uniformizer π), and we let κ be its residue field (with cardinality q). We further fix a classical Lie group G over K, with Lie algebra $\mathfrak{g}$.

A regular semisimple element $\gamma \in \mathfrak{g}(\mathfrak{o})$ gives rise to an orbit $\mathcal{O}_\gamma$. We want to compute its volume, which is the stable orbital integral of γ. Over the past two decades, we have developed two methods for the computation: analytic and geometric, where the geometric method uses the theory of Bruhat–Tits buildings.

For some $\mathfrak{g}$, we consider the map $\varphi_n \colon \mathfrak{g}(\mathfrak{o}) \to \mathfrak{o}^n$ given by sending some m to the coefficients of its characteristic polynomial. Notably, $\mathcal{O}_\gamma$ is the fiber of χ_γ for some regular semisimple γ. Notably, φ_n ugprades to a morphism of schemes, so one can hope to compare what happens on the geometric and special fibers. For example, one has the following result.

Theorem 1 (Weil)**.** For a smooth scheme $\mathfrak{X}$ over $\mathfrak{o}$, one has

$$\int_{\mathfrak{X}(\mathfrak{o})} |\omega| = \frac{|\mathfrak{X}(\kappa)|}{q^n}.$$

However, one cannot hope for our map φ_n to be smooth over $\mathfrak{o}$. Thus, the strategy is to stratify our integral over $\varphi_n^{-1}(\chi_\gamma)(\mathfrak{o})$ and then smoothen over each stratum. The stratification depends on the theory of finitely generated $\mathfrak{o}$-modules, and the smoothening is by some π-adic congruence condition.

Let's begin by discussing $\mathfrak{gl}_n$. For simplicity, we will suppose that $\chi_\gamma(xx)$ is irreducible, and we let L be a free $\mathfrak{o}$-module of rank n. Now, we can see

$$\mathcal{O}_\gamma = \{m \in \operatorname{End}_{\mathfrak{o}}(L) : \chi_m(x) = \chi_\gamma(x)\}.$$

Letting d be the valuation of $\det \gamma$, we also define $M \subseteq L$ to be the image of some m. The point is that we can look at the quotient L/M and separate it according to the theory of finitely generated modules over $\mathfrak{o}$

(which is a principal ideal domain). Splitting up the integral accordingly permits an explicit computation of the orbital integrals.

2 September 11th: Yuchan Lee

Today we discuss the smoothening method in some more detail. For today, F will continue to be a nonarchimedean local field with characteristic 0 (or bigger than n, where n is fixed later). As usual, $\mathfrak{o} \subseteq F$ is the ring of integers, $\pi \in F$ is a uniformizer, $\kappa = \mathfrak{o}/(\pi)$ is the residue field, and we set $q := \#\kappa$.

Fix $\gamma \in \mathfrak{gl}_n(\mathfrak{o})$ which is regular and semisimple; equivalently, we want γ to have separable characteristic polynomial χ_γ. Let d be the order of $\det \gamma \in F$. Some kind of parabolic induction in our computation allows us to assume that χ_γ is irreducible. To review, we are interested in the conjugacy classes

$$\mathcal{O}_\gamma := \left\{ g^{-1}hg : g \in \mathrm{GL}_n(F) \right\}.$$

We want to compute $SO_\gamma := \mathrm{vol}(\mathcal{O}_\gamma)$.