Smooth Manifolds for the Impatient

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Spring 2024

Abstract

This document collects a variety of definitions and results from the starting theory of smooth manifolds.

Contents

1 Definitions

1.1 Point-Set Topology

Definition 1 (Hausdorff)**.** A topological space X is *Hausdorff* if and only if any two distinct points $p, q \in X$ have disjoint open neighborhoods $U, V \subseteq M$; i.e., $p \in U$ and $q \in V$ but $U \cap V = \varnothing$.

Definition 2 (second-countable)**.** A topological space X is *second-countable* if and only if the topology on X has a countable base.

Definition 3 (locally Euclidean)**.** Fix a nonnegative integer n. A topological space X is *locally Euclidean of dimension* n if and only if each $p \in X$ has some open neighborhood $U \subseteq X$ and open subset $\widehat{U} \subseteq \mathbb{R}^n$ such that there is a homeomorphism $\varphi\colon U \to \widehat{U}$. The piar (U, φ) is called a *coordinate chart*.

Remark 4. It is a result of cohomology that if one has homeomorphic nonempty open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, then $m = n$.

Remark 5. In the above definition, one may assume that $\widehat{U}$ is an open ball, essentially by replacing $\widehat{U}$ with an open ball containing $\varphi(p)$ and replacing U with the preimage of this open ball. One can even assume that $\widehat{U}$ is all of $\mathbb{R}^n$ because $\mathbb{R}^n$ is homeomorphic to the open ball.

Remark 6. The above remark allows us to give any topological n-manifold a countable base of precompact open subsets.

Definition 7 (topological manifold)**.** A topological space X is a *topological* n*-manifold* if and only if it is Hausdorff, second-countable, and locally Euclidean of dimension n. If one weakens open subsets of $\mathbb{R}^n$ in locally Euclidean to open subsets of $\mathbb{H}^n$, then X is a *topological* n*-manifold with boundary*.

Definition 8 (connected)**.** A topological space X is *connected* if and only if the only subsets of X which are both open and closed are $\varnothing$ and X.

Definition 9 (path-connected)**.** A topological space X is *path-connected* if and only if any two points $p, q \in X$ have some path $\gamma\colon I \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Definition 10 (locally path-connected)**.** A topological space X is *locally path-connected* if and only if X has a base of path-connected open subsets.

Remark 11. Topological manifolds M are locally path-connected because $\mathbb{R}^n$ is path-connected. Thus, M is connected if and only if path-connected.

Remark 12. Having a countable base of precompact coordinate balls implies by an inductive argument that $\pi_1(M)$ is actually countable.

Definition 13 (locally compact)**.** A Hausdorff topological space X is *locally compact* if and only if each $x \in X$ has some open neighborhood U contained in a compact subset K.

Remark 14. Any topological manifold is locally compact because a basis of smooth charts can be refined into one where each basic open subset is precompact.

Definition 15 (paracompact)**.** A topological space X is *paracompact* if and only if any open cover $\mathcal{U}$ of M admits an open, locally finite refinement. Here, *locally finite* means that any $p \in X$ has some open neighborhood V such that $\#\{U \in \mathcal{U} : U \cap V \neq \varnothing\} < \infty$.

1.2 Smooth Structures

Definition 16 (diffeomorphism)**.** A *diffeomorphism* $F\colon U \to V$ between two open subsets of Euclidean space is a bijective smooth map with smooth inverse.

Definition 17 (smooth structure)**.** An *atlas* $\mathcal{A}$ on a topological n-manifold M is a collection of charts which cover M. The atlas $\mathcal{A}$ is *smooth* if and only if any two charts $(U, \varphi), (V, \psi) \in \mathcal{A}$ makes the transition map

$$\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}$$

into a diffeomorphism. A *smooth structure* is a maximal smooth atlas.

Remark 18. Any smooth atlas $\mathcal{A}$ on M is contained in a unique smooth structure $\overline{\mathcal{A}}$. Explicitly, one can construct $\overline{\mathcal{A}}$ as the collection of smooth charts which make the relevant transition maps smooth.

Definition 19 (smooth manifold)**.** A *smooth n-manifold* is a pair $(M, \mathcal{A})$ of a topological n-manifold M and smooth structure $\mathcal{A}$ on M. If M is merely a topological n-manifold with boundary, then, then M is a *smooth n-manifold with boundary*.

Definition 20 (interior, boundary)**.** Fix a smooth n-manifold M with boundary. A point $p \in M$ is an *interior point* if and only if any smooth chart (U, φ) on M makes $\varphi(p)$ in the interior of $\mathbb{H}^n$. A point $p \in M$ is a *boundary point* if and only if any smooth chart (U, φ) on M makes $\varphi(p)$ in the boundary of $\mathbb{H}^n$.

Remark 21. Any point in M is either an interior point or boundary point.

1.3 Smooth Maps

Definition 22 (smooth)**.** A map $F\colon M \to N$ is *smooth* if and only if any $p \in M$ has smooth charts (U, φ) on M and (V, ψ) on N such that $p \in U$ and $F(U) \subseteq V$ and the composite $\psi \circ F \circ \varphi^{-1}$ is smooth.

Remark 23. By compatbility of smooth charts, if F is smooth, then actually any smooth charts (U, φ) on M and (V, ψ) with $F(U) \subseteq V$ will make $\psi \circ F \circ \varphi^{-1}$ smooth.

Definition 24 (diffeomorphism)**.** A bijective smooth map F is a *diffeomorphism* if and only if F^{-1} is smooth.

Definition 25 (tangent vector)**.** Fix a smooth n-manifold M. Then a *tangent vector* or *derivation* at some $p \in M$ is an $\mathbb{R}$-linear map $v\colon C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$v(fg) = f(p)v(g) + g(p)v(f).$$

We let T_pM denote the vector space of derivations at p.

Definition 26 (differential)**.** Fix a smooth map $F\colon M \to N$ of smooth manifolds. Then the *differential of F at $p \in M$* is the linear map $dF_p\colon T_pM \to T_{F(p)}N$ defined by

$$dF_p(v)(f) := v(f \circ F).$$

Definition 27 (tangent bundle)**.** Fix a smooth n-manifold M. Then the *tangent bundle* TM is the smooth n-manifold

$$TM := \bigsqcup_{p \in M} T_pM$$

equipped with the smooth charts given by $TU \cong T\widehat{U} \cong \widehat{U} \times \mathbb{R}^n$ for any smooth chart (U, φ) where $\widehat{U} := \operatorname{im} \varphi$.

Definition 28 (constant rank)**.** Fix a smooth map $F\colon M \to N$. If $\operatorname{rank} dF_p$ is constant for all $p \in M$, then F has *constant rank* r. For example, if $\operatorname{rank} dF_p$ is always surjective, then F is a *submersion*; if $\operatorname{rank} dF_p$ is always injective, then F is an *immersion*.

Remark 29. These are local properties: if $\operatorname{rank} dF_p$ is injective (resp., surjective, invertible), then there is an open neighborhood $U \subseteq M$ of p such that $F|_U$ is an immersion (resp., submersion, diffeomorphism).

Definition 30 (embedding)**.** A smooth map $F\colon M \to N$ of smooth manifolds is a *smooth embedding* if and only if F is an immersion and a topological embedding.

Remark 31. Injective smooth immersions $F\colon M \to N$ become embeddings as soon as one can show that they are topological embeddings. For example, it is enough for F to be open, to be closed, to be proper (which implies closed), for M to be compact (which implies proper), or for $\partial M = 0$ and $\dim M = \dim N$.

Definition 32 (smooth covering)**.** A smooth map $\pi\colon E \to M$ of smooth manifolds is a *smooth covering map* if and only if each $p \in M$ has an *evenly covered open* neighborhood $U \subseteq M$ such that the restriction $\pi\colon \pi^{-1}(U) \to U$ maps its connected components diffeomorphically to U.

Remark 33. Suppose M is a connected smooth n-manifold. Then topological covering maps $\pi\colon E \to M$ gives E a unique smooth n-manifold structure making π a smooth covering map.

Remark 34. A local diffeomorphism π upgrades to a smooth covering map if it is proper. Proper is equivalent to finite fibers for smooth covering maps, so this condition is not necessary.

2 Coherences

The following result is used to build the Grassmannian and the tangent bundle.

Proposition 35. Fix a set M and a collection of subsets $\{U_\alpha\}_{\alpha\in\kappa}$ of subsets together with maps $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$ satisfying the following.

(i) Charts: φ_α is a bijection to an open subset of $\mathbb{R}^n$.

(ii) Smooth charts: the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open, and the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth.

(iii) Second-countable: $\{U_\alpha\}_{\alpha\in\kappa}$ has a countable subcover.

(iv) Hausdorff: for any distinct $p, q \in M$, either $p, q \in U_\alpha$ for some α, or $p \in U_\alpha$ and $q \in U_\beta$ for disjoint U_α and U_β.

Then M has a unique smooth n-manifold structure with smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$.

The following result explains how to use curves for differentials.

Proposition 36. Fix a smooth n-manifold without boundary. For any $p \in M$ and $v \in T_pM$, there is a smooth injective map $\gamma\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and $d\gamma_0\left(\frac{d}{dt}|_p\right) = v$.

The following result explains how to think about smooth submersions.

Proposition 37. Fix a smooth map $\pi\colon M \to N$ of smooth manifolds. Then π is a smooth submersion if and only if any point of M is in the image of a local setion $\sigma\colon N \to M$ of π.

3 Examples

3.1 Smooth Manifolds

Example 38. Note $\mathbb{R}^n$ is a smooth manifold: it is certainly a topological manifold, and it has smooth atlas given by $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^N})$.

Example 39. Note $S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is a smooth n-manifold. Indeed, it is the level set of the smooth function $|\cdot|^2 : \mathbb{R}^{n+1} \to \mathbb{R}$ at the regular value 1. For our smooth atlas, define the projection map $\pi_i^\pm\colon U_i^\pm \to B(0,1)$ where

$$U_i^\pm := \{x \in S^n : \pm x_i > 0\}.$$

Then one can check that π_i is a diffeomorphism, providing our smooth atlas.

Alternatively, one can define the stereographic projection $\sigma\colon (S^n \setminus \{(0,\ldots,0,1)\}) \to \mathbb{R}^n$ by

$$\sigma(x_1,\ldots,x_{n+1}) := \frac{(x_1,\ldots,x_n)}{1-x_{n+1}} \qquad \text{and} \qquad \sigma^{-1}(u_1,\ldots,u_n) := \frac{\left(2u_1,\ldots,2u_n,|u|^2-1\right)}{|u|^2+1},$$

which are both smooth because they are smooth as functions between Euclidean spaces.

Example 40. Given a real vector space $\mathbb{R}^{n+1}$, one can define the smooth n-manifold $\mathbb{RP}^n$ as equivalence classes of lines. The standard smooth atlas is given by the projection $\pi_i \colon U_i \to \mathbb{R}^n$ where we set $U_i := \{x \in \mathbb{RP}^n : x_i \neq 0\}$; the inverse of the projection is given by

$$(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n].$$

One can define $\mathbb{P}(V)$ for a general real or complex vector space V in an analogous way.

Example 41. Given smooth manifolds $M_1, \ldots, M_k$ without boundary, one can define the product $M_1 \times \cdots \times M_k$. Smooth charts are given by taking products of smooth charts on the individual $M_\bullet$s.

Example 42. Given a smooth n-manifold M, any open subset $U \subseteq M$ is also a smooth n-manifold. The smooth structure on U is given by restricting any smooth chart on M to U.

3.2 Smooth Maps

Example 43. Constant maps are smooth.

Example 44. The identity map is smooth. More generally, if $U \subseteq M$ is an open subset, then the inclusion $i \colon U \to M$ is a smooth embedding.

Example 45. A smooth chart (U, φ) on a smooth n-manifold M induces a diffeomorphism $\varphi \colon U \to \widehat{U}$ where $\widehat{U} := \operatorname{im} \varphi \subseteq \mathbb{R}^n$.

Example 46. For any $p \in \mathbb{R}^n$, the vector space $T_p\mathbb{R}^n$ is n-dimensional spanned by the derivations $\frac{\partial}{\partial x_i}\big|_p$.

Example 47. Fix a smooth n-manifold M. For any smooth chart (U, φ) on M, set $\widetilde{U} := \operatorname{im} \varphi$, and one finds that any $p \in U$ makes

$$T_pM \cong T_pU \cong T_{\varphi(p)}\widehat{U} \cong T_{\varphi(p)}\mathbb{R}^n.$$

As such, T_pM is n-dimensional spanned by the derivations

$$\left.\frac{\partial}{\partial x_i}\right|_p := (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}\right).$$

Example 48. Fix a smooth map $F\colon M \to N$ of smooth n-manifolds. Given $p \in M$, fix smooth charts (U, φ) on M and (V, ψ) such that $p \in U$ and $F(p) \in V$. (By restricting, we may assue that $F(U) \subseteq V$.) Let $\widetilde{F} := \psi \circ F \circ \varphi^{-1}$ and $\widehat{p} := \varphi(p)$ be the coordinate representations. Then

$$dF_p \left(\left. \frac{\partial}{\partial x_i} \right|_p \right) = d\left(F \circ \varphi^{-1}\right)_{\widehat{p}} \left. \frac{\partial}{\partial x_i} \right|_{\widehat{p}} = d\left(\psi^{-1} \circ \widehat{F}\right)_{\widehat{p}} \left. \frac{\partial}{\partial x_i} \right|_{\widehat{p}} = \left(d\psi_{\widehat{F}(\widehat{p})}\right)^{-1} d\widehat{F}_{\widehat{p}} \left(\left. \frac{\partial}{\partial x_i} \right|_{\widehat{p}} \right).$$

By the multivariate chain rule (and an explicit computation on $f \in C^\infty(N)$), this is

$$\left(d\psi_{\widehat{F}(\widehat{p})}\right)^{-1} \left(\sum_{j=1}^{\dim N} \frac{\partial \widehat{F}_j}{\partial x_i}(\widehat{p}) \left. \frac{\partial}{\partial y_j} \right|_{\widehat{F}(\widehat{p})} \right) = \sum_{j=1}^{\dim N} \frac{\partial \widehat{F}_j}{\partial x_i}(\widehat{p}) \left. \frac{\partial}{\partial y_j} \right|_{\widehat{F}(\widehat{p})}.$$

So dF_p is given by the Jacobian matrix.

Example 49. Taking $F = \mathrm{id}_M$ in the above example, we see

$$d\left(\psi \circ \varphi^{-1}\right)_{\varphi(p)} \left(\left. \frac{\partial}{\partial x_i} \right|_{\varphi(p)} \right) = \sum_{j=1}^{\dim M} \frac{\partial y_j}{\partial x_i}(\varphi(p)) \left. \frac{\partial}{\partial y_j} \right|_{\psi(p)}.$$

Rearranging, we see

$$\left. \frac{\partial}{\partial x_i} \right|_p = d\left(\psi^{-1}\right)_{\psi(p)} \circ d\left(\psi \circ \varphi^{-1}\right)_{\varphi(p)} \left(\left. \frac{\partial}{\partial x_i} \right|_{\varphi(p)} \right) = \sum_{j=1}^{\dim M} \frac{\partial y_j}{\partial x_i}(\varphi(p)) \left. \frac{\partial}{\partial y_j} \right|_{\psi(p)}.$$

Example 50. The projection map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth surjective submersion. Surjectivity has little content, and smoothness follows by checking on charts. Being a submersion is also checked on charts: for $p \in \mathbb{R}^{n+1} \setminus \{0\}$ given by $p := (z_0, \ldots, z_n)$ such that $z_i \neq 0$, it is enough to check that the composite

$$T_p\mathbb{R}^{n+1} \cong T_p\left(\mathbb{R}^{n+1} \setminus \{0\}\right) \xrightarrow{\pi} T_{\pi(p)}\mathbb{RP}^n \xrightarrow{\varphi_i} T_{\varphi_i(\pi(p))}U_i$$

given by $(x_0, \ldots, x_n) \mapsto (x_0/x_i, \ldots, \widehat{1}, \ldots, x_n/x_i)$ is a smooth submersion at p. But we see that $\frac{\partial}{\partial x_j}$ goes to $\frac{1}{z_i}\frac{\partial}{\partial x_j}$ for $j \neq i$, which is enough.

Example 51. The projection map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ is a smooth surjective submersion. Surjectivity has little content, and smoothness follows because we have local sections: any $p \in \mathbb{R}^{n+1} \setminus \{0\}$ is in the local section $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ given by $x \mapsto |p|\, x$. Namely, this section is smooth because it is smooth as a map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

Example 52. The projection map $\pi\colon S^n \to \mathbb{RP}^n$ is a smooth covering map: the projection $\mathbb{R}^{n+1} \to S^n$ is smooth, so it is enough to check that $\mathbb{R}^{n+1} \to \mathbb{RP}^n$ is smooth, which we know. Also, $\mathbb{R}^{n+1} \to \mathbb{RP}^n$ is a submersion, so it is surjective on differentials, so the induced map $S^n \to \mathbb{RP}^n$ must also be surjective on differentials. Lastly, S^n is compact, and this map is surjective, so we see that we have a proper local diffeomorphism, which is a smooth covering map.

4 Theorems

Theorem 53 (Partition of unity)**.** Fix a smooth n-manifold, possibly with boundary. Given any open cover $\mathcal{U}$, there is a partition of unity subordinate to $\mathcal{U}$. In other words, there are smooth functions $\psi_U : M \to \mathbb{R}$ satisfying the following.

(a) $\operatorname{im} \psi_U \subseteq [0, 1]$.

(b) $\operatorname{supp} \psi_U \subseteq U$.

(c) The collection $\{\operatorname{supp} \psi_U\}_{U \in \mathcal{U}}$ is locally finite.

(d) $\sum_{U \in \mathcal{U}} \psi_U = 1$.

Corollary 54 (Extension lemma)**.** Fix a smooth n-manifold, possibly with boundary. For any closed subset $A \subseteq M$ and smooth function $f \colon A \to \mathbb{R}^k$ (i.e., f has a smooth extension in a neighborhood of each point) and open neighborhood U of A, there is a smooth function $\widetilde{f} \colon M \to \mathbb{R}^k$ extending f such that $\operatorname{supp} \widetilde{f} \subseteq U$.

Theorem 55 (Rank)**.** Fix a smooth m-manifold M and n-manifold N, and fix a smooth map $F \colon M \to N$ of constant rank r. For any $p \in M$, there is a chart (U, φ) on M and a chart (V, ψ) on N such that $p \in U$ and $F(U) \subseteq V$ and

$$\left(\psi \circ F \circ \varphi^{-1}\right)(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

Remark 56. Fix a smooth n-manifold. For any closed subset $A \subseteq M$, there is a smooth nonnegative function $f \colon M \to \mathbb{R}$ such that $f^{-1}(\{0\}) = K$.

Theorem 57. Fix a smooth surjective submersion $\pi \colon M \to N$ of smooth manifolds. Suppose P is a smooth manifold.

(a) Check smoothness: a map $\overline{F} \colon N \to P$ is smooth if and only if $(\overline{F} \circ \pi) \colon M \to P$ is smooth.

(b) Produce smootheness: given a smooth map $F \colon M \to P$ constant on the fibers of π, there is a unique smooth map $\overline{F} \colon N \to P$ such that $F = \overline{F} \circ \pi$.