

# Dimension Theory for the Impatient

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## Abstract

This document collects a variety of dimension-computing results from Eisenbud's *Commutative Algebra: with a View Toward Algebraic Geometry*. All references are to this book.

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# 1 Definitions

## 1.1 Kinds of Dimension

**Definition (Dimension).** The *Krull dimension* of a ring  $R$ , denoted  $\dim R$ , is the supremum of the length  $r$  of a chain of distinct primes

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

**Definition (Dimension, ideals).** Fix a ring  $R$  and an ideal  $I \subseteq R$ . Then we define the *dimension* of an ideal  $I$  to be  $\dim I := \dim R/I$ .

**Lemma.** Fix an ideal  $I$  of a ring  $R$ . Then  $\dim I$  is equal to the length of the longest chain of primes

$$I \subseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

in  $R$ .

**Definition (Codimension).** Fix  $I$  a proper ideal of a ring  $R$ .

- If  $I = \mathfrak{p}$  is a prime ideal of  $R$ , then we define the *codimension* as  $\operatorname{codim} \mathfrak{p} := \dim R_{\mathfrak{p}}$ .
- More generally, we define the *codimension* as

$$\operatorname{codim} I := \min_{\mathfrak{p} \supseteq I} \operatorname{codim} \mathfrak{p},$$

where the minimum is over all prime ideals  $\mathfrak{p}$  containing  $I$ .

**Lemma.** Fix a prime ideal  $\mathfrak{p}$  of a ring  $R$ . Then  $\operatorname{codim} \mathfrak{p}$  is equal to the length of the longest chain of primes

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{p},$$

where  $\mathfrak{p}$  is included in the chain; i.e.,  $\operatorname{codim} \mathfrak{p} = d$  here.

**Definition (Dimension, modules).** Given an  $R$ -module  $M$ , we define the *dimension* of  $M$  as  $\dim M := \dim R / \operatorname{Ann} M$ .

## 1.2 Kinds of Rings

**Definition (Regular).** Fix a local ring  $R$  of dimension  $d := \dim R$ . Further, let  $\mathfrak{m}$  be the maximal ideal of  $R$ . Then  $R$  is *regular* if and only if there exist elements  $\{f_1, \dots, f_d\} \subseteq R$  such that

$$\mathfrak{m} = (f_1, \dots, f_d).$$

**Remark (Corollary 10.14).** Regular local rings are integral domains.

**Definition (Discrete valuation ring).** A *discrete valuation ring* is an integral domain  $R$  equipped with a valuation  $\nu: K(R)^\times \rightarrow \mathbb{Z}$ .

**Proposition** (Proposition 11.1). Fix a Noetherian ring  $R$ . The following are equivalent.

- $R$  is a discrete valuation ring.
- $R$  is a field or regular local ring of dimension 1.

**Definition** (Dedekind). A *Dedekind domain* is a Noetherian normal domain of dimension 1.

## 2 Theorems

### 2.1 First Results

**Proposition.** Fix  $I$  an ideal of a ring  $R$ . Then

$$\dim I + \operatorname{codim} I \leq \dim R.$$

**Remark.** Equality for the above holds when  $R$  is an affine domain, by Corollary 13.4.

**Lemma.** Fix ideals  $I$  and  $J$  in a ring  $R$ . If  $I \subseteq J$ , then

$$\dim I \geq \dim J \quad \text{and} \quad \operatorname{codim} I \leq \operatorname{codim} J.$$

Similarly, if  $\mathfrak{p}$  and  $\mathfrak{q}$  are primes with  $\mathfrak{p} \subseteq \mathfrak{q}$ , then  $\operatorname{codim} \mathfrak{p} \leq \operatorname{codim} \mathfrak{q}$ , with equality if and only if  $\mathfrak{p} = \mathfrak{q}$ .

### 2.2 Localization, Completion, Polynomials

**Theorem.** Fix a ring  $R$ . Then

$$\dim R = \max_{\mathfrak{p} \in \operatorname{Spec} R} \dim R_{\mathfrak{p}}.$$

In other words, dimension is a local quantity.

**Corollary (Corollary 10.12).** Fix a local Noetherian ring  $R$  and a finitely generated module  $M$ . Then  $\dim R = \dim \hat{R}$ .

**Remark.** Corollary 10.12 is strictly weaker than Corollary 12.15.

**Corollary (Corollary 10.13).** Fix a Noetherian ring  $R$  with finite dimension. Then  $\dim R[x] = \dim R + 1$ .

### 2.3 Comparing Rings

**Proposition (Proposition 9.2).** Fix a ring homomorphism  $\varphi: R \rightarrow S$  which makes  $S$  into an integral  $R$ -algebra. Then, for any  $\mathfrak{p} \in \operatorname{Spec} R$  such that  $\ker \varphi \subseteq \mathfrak{p}$ , there exists  $\mathfrak{q} \in \operatorname{Spec} S$  such that

$$\mathfrak{p} = \varphi^{-1}(\mathfrak{q}).$$

In fact, for any ideal  $I \subseteq S$ , we have  $\dim S/I = \dim R/\varphi^{-1}(I)$ . In particular, if  $\varphi: R \rightarrow S$  is injective, then  $\dim R = \dim S$ .

**Lemma.** Fix a ring  $R$  and a multiplicatively closed subset  $U \subseteq R$ . Further, set  $S := R[U^{-1}]$  with the natural map  $\varphi: R \rightarrow S$ . Then, for any prime  $\mathfrak{p} \subseteq R[U^{-1}]$ , we have

$$\operatorname{codim} \varphi^{-1}(\mathfrak{p}) = \operatorname{codim} \mathfrak{p}.$$

**Theorem (Theorem 10.10).** Fix two local rings  $R$  and  $S$  with maximal ideals  $\mathfrak{m}$  and  $\mathfrak{n}$ , respectively. Given a map  $\varphi : R \rightarrow S$  of local rings so that  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ , we have

$$\dim S \leq \dim R + \dim S/\mathfrak{m}S$$

In fact, if  $S$  is a flat as an  $R$ -module, then we have equality.

## 2.4 Generating Elements

**Theorem (Theorem 10.2).** Fix a Noetherian ring  $R$ . Given an ideal  $(x_1, \dots, x_s) \in R$ , suppose  $\mathfrak{p}$  is a minimal prime over  $(x_1, \dots, x_s)$ . Then

$$\text{codim } \mathfrak{p} \leq s.$$

**Corollary (Corollary 10.5).** Fix a prime ideal  $\mathfrak{p}$  of a Noetherian ring  $R$  with codimension  $r$ . Then there are elements  $x_1, \dots, x_r$  such that  $\mathfrak{p}$  is minimal over  $(x_1, \dots, x_r)$ , and in fact  $\text{codim}(x_1, \dots, x_r) = r$ .

**Proposition (Proposition 10.8).** Fix a local ring  $R$  with maximal ideal  $\mathfrak{m}$ . Then  $\dim R$  is the minimal  $d \in \mathbb{N}$  such that there exist generators  $f_1, \dots, f_d$  so that

$$\mathfrak{m}^n \subseteq (f_1, \dots, f_d) \subseteq \mathfrak{m}$$

for some  $n$ .

**Corollary.** Let  $R$  be a Noetherian regular local ring with maximal ideal  $\mathfrak{m}$ . Then

$$\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim R.$$

## 2.5 Dimension for Modules

**Proposition (Proposition 10.8).** Fix a Noetherian local ring  $R$  with maximal ideal  $\mathfrak{m}$  and an  $R$ -module  $M$ . Then  $\dim M$  is equal to the minimal  $d$  such that there is some proper ideal  $(f_1, \dots, f_d) \subseteq R$  with finite colength on  $M$ .

**Corollary (Corollary 10.9).** Fix a Noetherian local ring  $R$  with maximal ideal  $\mathfrak{m}$  and an  $R$ -module  $M$ . Given  $x \in \mathfrak{m}$ , we have

$$\dim M/xM \geq \dim M - 1.$$

**Corollary (Corollary 12.5).** Fix a Noetherian local ring  $R$ . Given a finitely generated  $R$ -module  $M$  with parameter ideal  $\mathfrak{q}$ ,

$$\dim M = \dim \widehat{M}_{\mathfrak{q}} = \dim(\text{gr}_{\mathfrak{q}} M)_{\mathfrak{P}}.$$

Here,  $\mathfrak{P} \subseteq \text{gr}_{\mathfrak{q}} M$  is the irrelevant ideal.

## 2.6 The Hilbert Function

**Definition (Hilbert–Samuel function).** Fix a local Noetherian ring  $R$  with finitely generated  $R$ -module  $M$  and some prime of finite colength  $\mathfrak{q}$ . Then we define the *Hilbert–Samuel function* by

$$H_{\mathfrak{q},M}(n) := \ell(\mathfrak{q}^n M / \mathfrak{q}^{n+1}).$$

**Lemma.** Fix a local Noetherian ring  $R$  with maximal ideal  $\mathfrak{m}$ . Further suppose that there is a map  $k \hookrightarrow R$  such that the composite

$$k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$$

is an isomorphism. Then, for any finitely generated  $R$ -module  $M$  of finite length,

$$\ell_R(M) = \dim_k M.$$

**Theorem (Theorem 12.4).** Fix a local Noetherian ring  $R$  with unique maximal ideal  $\mathfrak{m}$ . Further, take a finitely generated module  $M$  and an ideal  $\mathfrak{q}$  of finite colength on  $M$ . Then

$$\dim M = 1 + \deg P_{\mathfrak{q},M}.$$

**Corollary (Corollary 13.7).** Fix a Noetherian graded ring  $R := R_0 \oplus R_1 \oplus \cdots$ . Then  $\dim R$  is the supremum of  $\dim R_{\mathfrak{p}}$  for all homogeneous prime ideals  $\mathfrak{p}$ .

Thus, if  $R_0$  is a field, then

$$\dim R = 1 + \deg P_R,$$

where  $P_R$  is the Hilbert polynomial for  $R$ .

## 2.7 Affine Domains

**Theorem (Theorem 13.3).** Fix an affine ring  $R$  of dimension  $d$ . Given a chain

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq R$$

with  $d_j := \dim I_j$  such that  $\{d_j\}_{j=0}^m$  is strictly decreasing and  $d_m > 0$ . Then there is a subring  $S \subseteq R$  such that

- (a)  $S \cong k[x_1, \dots, x_d]$ ,
- (b)  $R$  is finite over  $S$ , and
- (c) any ideal  $I_j$  has  $S \cap I_j = (x_{d_j+1}, \dots, x_d)$ .

**Theorem (Theorem A).** Fix an affine domain  $R$  over a field  $k$ . Then

$$\dim R = \text{transcendence degree}_k R.$$

**Corollary (Corollary 13.4).** Fix an affine domain  $R$ . Given an ideal  $I \subseteq R$ , we have

$$\dim I + \text{codim } I = \dim R.$$

**Corollary (Corollary 13.5).** Suppose that we have an inclusion  $R \subseteq T$  of affine domains over  $k$ . Then

$$\dim T = \dim R + \dim K(R) \otimes_R T.$$

**Corollary (Corollary 13.11).** Fix an affine domain  $R$ . If  $f \in R \setminus \{0\}$  is not a unit, then

$$\dim R/(f) = \dim R - 1.$$