12A: Introduction to Logic

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Theme 1: Introduction

1.1 January 19

Let's go ahead and get going. Today we are hyping the course.

1.1.1 Symbolic Logic

In this course, we are more interestsed in symbolic logic. More broadly, we are interested in what kind of reasoning is "logical," and we will do this by abstracting what a good argument is.

In symbolic logic, we have lots of symbols for our reasoning words. Here is a table of such symbols.

word	symbol
not	$\neg$
and	$\wedge$
or	$\vee$
if, then	$\to$
for all	$\forall$
there exists	$\exists$

In this course we will be able to give a rigorous definition for what a valid formula is in the language of these symbols. The truth value of a statement will have no dispute.

Example 1.1. The formula

$$(\forall x \mathsf{Red}(x) \vee \exists y \mathsf{Square}(y)) \to \exists z(\mathsf{Red}(z) \wedge \mathsf{Square}(z))$$

asserts that "if everything is red and something is square, then there is something which is both red and square." This is a good, true assertion: that something which is square must also be red, finishing.

1.1.2 Advertisements

This sort of reasoning has applications in lots of fields.

- Logic is a main branch of philosophy. For example, we will study the syllogistic reasoning of Aristotle.[1]
- Logic is at the base of mathematics, and careful logical reasoning informs foundational mathematics (e.g., Gödel's incompleteness theorems or the independence of the Continuum hypothesis). For example, we will have to understand (basic) mathematical proofs in this course. We will talk about foundational mathematics a bit at the end of the course.
- Logic and its methods (e.g., λ-calculus) impacts how one does computer programming. For a concrete example, logic is used in SQL to give statements for database queries. As another example, formal hardware and software verification comes down to very careful logical analysis.
- One approach to artificial intelligence is by trying to create a machine which spits out true facts from old ones, for which the formal language of first-order logic is quite important.

[1] Here is an example of a syllogism: Suppose that all men are mortal and that Socrates is a man. Then it follows Socrates is mortal.

- The kind of epistemic logic of trying to reason about what people know and do not know is important in game theory and hence has applications to economics. This can quickly get complicated: for example, we might want to keep track of the fact that (e.g., in poker) Player 1 knows that Player 2 knows that Player 3 has an ace card, for this fact might affect Player 2's behavior.
- Linguistics is interested in what sentences mean, for which one had to keep track of formal semantics to determine truth values.
- In cognitive science, how hard it is to understand/learn something turns out to be directly proportional to the length of the shortest logically equivalent propositional formula. In other words, longer formulae are harder to get in one's head.

1.1.3 Logistics

Let's talk about logistics.

- There is a class Piazza, which hopefully will get some use.
- The course outline in the syllabus is more of a guess than a promise; we may get ahead or behind it, but the syllabus will be updated frequently to match.
- There is a textbook. It is more like a math textbook: one is expected to read things multiple times instead of in an English class where one tries to skim as much as possible. While the material is dense, the course has been designed to try to make the course accessible to everyone.
- All reading (including the textbook) will be freely available online.
- It is better to skim the reading before lecture to not completely be lost during lecture. It is also recommended to do another pass on the reading after lecture.
- There are weekly problem sets, released on Mondays (startig next Monday) and due on Sunday midnights. They will be graded via GradeScope, and there are regrade requests (as usual).
- In theory, the problem sets will not depend on a great deal on the lecture Friday before the deadline.
- The class will be curved upwards depending on its difficulty, at the very end.
- Please come to office hours instead of struggling needlessly on one's own.

Next class we will talk about propositional logic.

1.2 January 21

Today we are talking about propositional logic.

1.2.1 Propostions and Connectives

Roughly, propositional logic is about reasoning with propositions with propositional connectives.

Remark 1.2. Propositional logic might also be called sentential logic or boolean logic.

Here are propositions.

Proposition

Definition 1.3 (Proposition)**.** In this class, a *proposition* will simply be any declarative sentence.

Example 1.4. The sentence "Paris the capital of France" is a proposition.

Non-Example 1.5. The question “Is Paris the capital of France?” is not a proposition.

Remark 1.6. Logic can handle questions, but we will not discuss it. It is called inquisitive logic.

Here are propositional connectives.

Connective

Definition 1.7 (Connective)**.** A *propositional connective* is some word or phrase that we can use to build new propositions from old ones.

There are lots of propositional connectives. Have some examples.

Example 1.8. Suppose p and q are propositions. For example, p can be “the theif entered through the back door,” and q can be “the thief left through the side door.” Then we have the following.

- “p or q” is a proposition.
- “p and q” is a proposition.
- “If p then q” is a proposition.
- “Not p” is a proposition. (Grammatically, “it is not the case that p.”)

These are read by replacing p and q with the propositions they represent.

Example 1.9. Propositional connectives do not have to be absolute. For example, “it is more likely that p than it is that q” is a proposition.

1.2.2 Reasoning

So thus far we have propositions and their connectives. How might we reason with them? Here’s an example.

Example 1.10. If we happen to know that “p or q,” and we know that “it is not the case that p,” then q must be true. This is essentially process of elimination; e.g., we might imagine running this argument with p that “the infection is viral” and q that “the infection is bacterial.”

The reasoning in the above argument feels quite true, even without making p or q concrete propositions. The reasoning makes us feel good.

Warning 1.11. Suppose we have the following premises.

- p or q.
- p.

It does not follow that q is false. In short, p or q permits both p and q to be true.

Here is another example of reasoning.

Example 1.12. We have the following premises.

- If p, then q.
- Not q.

Then it follows that not p. Concretely, we can set p to be "the reactor is cooling down" and q to be "the blue light is on." Then the fact that the blue light is not on would imply that the reactor is not cooling down.

We do have to be careful with conditionals, however.

Warning 1.13. The premise "if p then q" does not imply that "if q then p."

And here is some reasoning with less concrete connectives.

Example 1.14. Suppose we are given that p is more likely than q. Then it follows that p is more likely than q and r, for any other event r. In essence, trying to make more events happen is harder. For concreteness, think through this argument with the following concrete premises:

- Set p to be "the US will sign the treaty."
- Set q to be "Russia will sign the treaty."
- Set r to be "China will sign the treaty."

Having more people sign the treaty is harder.

Reasoning can be hard, and sometimes our intuition might be wrong. Here is some bad reasoning.

Non-Example 1.15. Suppose we have the following premises.

- If p, then q.
- Not p.

Then it does not follow that q. Concretely, set p to be "the patient is taking her medicine" and q to be "the patient will get better." Then the fact that patient is not taking her medicine does not imply that the patient will not get better: perhaps the patient will get better for some other reason.

The above reasoning is bad because we went from true premises to false conclusions. This is what we try to avoid.

Here is more bad reasoning.

Non-Example 1.16. Suppose we have the following premises.

- It is more likely that p than q.
- It is more likely that p than r.

Then it does not follow that p is more likely than q or r. For example, take p to be the event that a dice roll is odd, q to be the event that a dice roll is $\{1,2\}$, and r to be the vent that a dice roll is $\{3,4\}$. The probability that q or r exceeds the probability that p in this case.

As an aside, our bad reasoning might still give good premises at the end. The reason that we like good reasoning better is that every single time we will get good premises from good ones; with bad reasoning, we run the risk of getting bad results at the end.

Example 1.17. The argument that the proposition "grass is green" directly implies "the sky is blue" is not valid reasoning because these two propositions have effectively nothing to do with each other. Nevertheless, "the sky is blue" is a true conclusion.

1.2.3 Truth-Functional Connectives

Earlier we gave examples of lots of different propositional connectives. It turns out that we only care about very few of these: the truth-functional propositional connectives.

We need to have a reasonable notion of truth. We adopt the following conventions.

Convention 1.18. In this course, we take the following.

- All propositions are either true or false.
- No proposition are both true and false.

These probably seem obvious, but we need to be careful.

Example 1.19. The following propositions are bad in that they have unclear truth value.

- The proposition "ice cream is delicious" is a proposition of taste (this depends on the person), so we will ignore it.
- "Bob is bald" is a bit vague because "bald" is not well-defined, so we will ignore it.
- "This proposition is false" is self-referential and more or less breaks down truth (if the proposition is true, then the proposition declares its own falsehood), so we will ignore it. Similar is "this proposition is true." (This is known as the liar paradox.)

One way to escape these problems is to simply declare that they are not propositions. We choose to ignore the altogether.

Nevertheless, we go forwards with out notion of truth value.

Truth value

Definition 1.20 (Truth value)**.** The *truth value* of a proposition is "true" if the proposition is true and "false" if it is false.

Very quickly, we note that the connectives we've talked about come in two classes.

Unary, binary

Definition 1.21 (Unary, binary)**.** A propositional connective is *unary* (respectively, *binary*) if and only if it acts on one (repsectively, two) proposition.

Example 1.22. The connective "not" is a unary connective. The connective "We know that" is a unary connective.

Definition 1.23. More generally, the *arity* of a connective is the number of propositions the connective acts on.

Example 1.24. The arity of "not" is 1.

Natural language tends to focus on unary and binary connectives.

We are now ready to define what a truth-functional connective is. Here is the definition for unary connectives.

Truth-functional

Definition 1.25 (Truth-functional)**.** A unary connective $\#$ is *truth-functional* means that $\#p$ has truth value which is a function of (i.e., is completely determined by) the truth value of p.

Example 1.26. The connective "not" is a truth-functional connective: the truth value of p tells us what the truth value of "not p" is.

Non-Example 1.27. The connective "the police know that" is not a truth-functional connective: a statement being true or false does not immediately tell us whether or not the police know it. Hopefully if p is false, then the police do not know p; but if p is true, perhaps the police simply do not it yet. The point is that the truth value of "the police know that p" is simply not a function of p.

Here is truth-functionality for binary connectives.

Truth-functional

Definition 1.28 (Truth-functional)**.** A binart connective $\#$ is *truth-functional* means that $p\#q$ has truth value which is a function of (i.e., is completely determined by) the truth values of p and q.

Example 1.29. The connective taking the propositions p, q to "p and q" is truth-functional.

1.3 January 24