

# 18.757: Representation Theory of Lie Groups

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# CONTENTS

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*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

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# THEME 1

## THE ANALYTIC THEORY

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### 1.1 September 4

Welcome to the class.

#### 1.1.1 Administrative Notes

Here are some administrative notes.

- There will be problem sets every two weeks, due on Fridays. They are not expected to be too time-consuming.
- Technically, this course is a sequel to 18.745–18.755, but one can get away with a bit less. In particular, we will assume familiarity with some basic notions in Lie theory, things about simple complex Lie algebras (as related to compact Lie groups), the theory of roots and weights, and this theory of finite-dimensional representations. For example, things like the Peter–Weyl theorem may come up.
- We will largely follow [Etingof's lecture notes](#).

This course is about the representations of Lie groups, especially those which are not necessarily compact. For example, we may focus on real reductive Lie groups such as  $SL_n(\mathbb{R})$ , and there is a new feature here that we must care about infinite-dimensional representations.

One of our motivations comes from quantum physics, where one finds groups acting on infinite-dimensional Hilbert spaces. Another motivation is number-theoretic: one uses this theory to set up the archimedean theory of automorphic forms.

#### 1.1.2 Finite Groups

Let's recall some background. As one does, let's begin with the representation theory of finite groups. We split this into a few theorems.

**Theorem 1.1 (Maschke).** Let  $G$  be a finite group. Then every finite-dimensional complex representation is unitary (by averaging any given Hermitian form) and semisimple.

**Theorem 1.2 (Peter–Weyl).** Let  $G$  be a finite group. Then there is a decomposition

$$\mathbb{C}[G] = \bigoplus_{V \in \text{IrRep}(G)} \text{End}_{\mathbb{C}}(V)$$

of  $\mathbb{C}$ -algebras. It follows that the characters  $\{\text{tr}_V\}_{V \in \text{IrRep}(G)}$  form an orthonormal basis of the class functions  $G \rightarrow \mathbb{C}$ .

**Remark 1.3.** Let  $G$  be a finite group. For  $f \in \mathbb{C}[G]$ , there is a dimension formula

$$f(1) = \frac{1}{|G|} \sum_{V \in \text{IrRep}(G)} \dim V_i \cdot \text{tr}_{V_i} f.$$

Indeed, this follows from writing  $\langle \varphi, \psi \rangle = \sum_V \text{tr}_V(\varphi \psi')$  (where  $\psi' : g \mapsto \psi(g^{-1})$ ) and then noting that  $\langle \varphi, \psi' \rangle = \frac{1}{|G|}(\varphi * \psi')(1)$ .

### 1.1.3 Compact Groups

We now move up to the representation theory of compact connected Lie groups. Here is a generalization of Maschke's theorem.

**Theorem 1.4 (Weyl's unitarian trick).** Let  $G$  be a compact Lie group. Then the representations of  $G$  are semisimple, and the irreducible representations of  $G$  are finite-dimensional and unitary.

In order to compute the representations of  $G$ , one wants to pass to the Lie algebra  $\mathfrak{g} = \text{Lie } G$ . One needs to be slightly careful about this.

**Proposition 1.5.** Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . Consider the functor  $F : \text{Rep}(G) \rightarrow \text{Rep}(\mathfrak{g})$  sending a representation  $\rho$  to the differential representation  $d\rho_1$ .

- (a) If  $G$  is connected, then  $F$  is fully faithful.
- (b) If  $G$  is connected and simply connected, then  $F$  is also essentially surjective and hence an equivalence.

**Example 1.6.** Take  $G = \text{U}(1)$ . Because  $G$  is abelian, all irreducible representations are one-dimensional. These representations  $\text{U}(1) \rightarrow \mathbb{C}^\times$  are indexed by  $n \in \mathbb{Z}$ , given by  $z \mapsto z^n$ . Note that these are not in bijection with the representations of  $\text{Lie } G$  because  $G$  is not simply connected!

**Example 1.7.** Take  $G = \text{SU}(n)$  so that  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ . A weight is a character of the maximal torus  $T$ , for which we can take to be the subgroup of diagonal matrices. Explicitly,

$$T = \{\text{diag}(z_1, \dots, z_n) : z_1 \cdots z_n = 1\},$$

so the weight lattice is  $\mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$ , and a weight  $\lambda = (\lambda_1, \dots, \lambda_n)$  is dominant when the entries are increasing.

**Example 1.8.** For example, with  $\text{SU}(2)$ , we have an isomorphism  $\mathbb{Z}^2 / \mathbb{Z}(1, 1) \rightarrow \mathbb{Z}$  given by  $(\lambda_1, \lambda_2) \mapsto \lambda_2 - \lambda_1$ , and the dominant weights are the nonnegative integers. One finds that weight  $n$  corresponds to the  $n$ th symmetric power of the standard representation of  $\text{SU}(2)$ .

Here is a generalization of the Peter–Weyl theorem.

**Theorem 1.9 (Peter–Weyl).** Let  $G$  be a compact Lie group. The canonical map

$$\bigoplus_{\text{dominant } \lambda} V_{\lambda} \otimes V_{\lambda}^* \rightarrow C^{\infty}(G)$$

is an embedding with dense image; here  $V_{\lambda}$  refers to the irreducible representation corresponding to the dominant weight  $\lambda$ .

**Example 1.10 (Fourier analysis).** With  $G = \mathrm{U}(1)$ , then one can calculate that  $V_n \otimes V_n^* \rightarrow C^{\infty}(G)$  has image given by the  $n$ th power map  $\mathrm{U}(1) \rightarrow \mathrm{U}(1)$ . Thus, we are asserting that the collection of such polynomials are dense in the collection of all smooth functions  $\mathrm{U}(1) \rightarrow \mathbb{C}$ . If we identify  $\mathrm{U}(1)$  with  $\mathbb{R}/\mathbb{Z}$  via the exponential map, then this is asserting that the exponentials  $z \mapsto e^{2\pi i n z}$  have dense span in the collection of all smooth functions  $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$ .

One can characterize the image of the Peter–Weyl map.

**Definition 1.11 (finite).** A function  $f \in C^{\infty}(G)$  is  $G$ -finite if and only if the span of  $\{gf : g \in G\}$  is finite-dimensional. We let  $C_{\mathrm{fin}}(G)$  be the space of  $G$ -finite functions.

**Remark 1.12.** It turns out that the map

$$\bigoplus_{\text{dominant } \lambda} V_{\lambda} \otimes V_{\lambda}^* \rightarrow C^{\infty}(G)$$

has image given by the space of  $G$ -finite functions. (This is often proven as an input to the proof of the Peter–Weyl theorem; it is much easier to show!)

**Remark 1.13.** The vector space  $C_{\mathrm{fin}}(G)$  has two ring structures: there is pointwise multiplication (in  $\mathbb{C}$ ) and also convolution given by

$$(\varphi * \psi) : g \mapsto \int_G \varphi(x) \psi(x^{-1}g) \, dx,$$

where  $dx$  is a Haar measure for  $G$  normalized so that  $\int_G dx = 1$ . (Because  $G$  is compact,  $dx$  is bi-invariant.)

**Remark 1.14.** The convolution algebra is non-unital, so one sometimes upgrades to the algebra of distributions, where we have the unit  $\delta_1$ . Similar remarks hold for  $C^{\infty}(G)$  and even  $C(G)$ .

**Remark 1.15 (algebraic groups).** In fact,  $C_{\mathrm{fin}}(G)$  also has a comultiplication given by pulling back along the multiplication map  $m : G \times G \rightarrow G$ . Namely, the comultiplication is the composite

$$C_{\mathrm{fin}}(G) \xrightarrow{m^*} C_{\mathrm{fin}}(G \times G) = C_{\mathrm{fin}}(G) \otimes C_{\mathrm{fin}}(G).$$

Thus, we have a Hopf algebra, which allows us to associate a complex algebraic group  $G_{\mathrm{alg}}$  to  $G$ , and it turns out that unitary representations of  $G$  all arise from algebraic representations of  $G_{\mathrm{alg}}$ . Conversely, one can take a complex algebraic group  $G_{\mathrm{alg}}$  and then find a maximal compact subgroup  $G \subseteq G_{\mathrm{alg}}(\mathbb{C})$  which is unique up to conjugacy.

### 1.1.4 Unitary Representations

We will be interested in the unitary representations of Lie groups  $G$ , which we no longer assume to be compact.

**Remark 1.16.** If  $G$  is simple and not compact, then all unitary representations are infinite-dimensional. Proceeding by contraposition, suppose that  $G$  is simple and admits a finite-dimensional unitary representation  $\rho: G \rightarrow \mathrm{SU}(n)$ . Because  $G$  is simple, this is an embedding, and because  $\mathrm{SU}(n)$  is compact, we conclude that  $G$  must then also be compact.

Thus, we see that we will be interested in infinite-dimensional representations. Of course, one still must add in topologies everywhere, though this point is more technical now that our vector spaces are not finite-dimensional. For example, for unitary representations, we are looking for actions of  $G$  on Hilbert spaces, though we will find occasion to look at more general topological vector spaces.

Our main source of examples of representations arise from more general group actions.

**Example 1.17.** If  $G$  acts on a “geometric space”  $X$ , then we receive an induced action of  $G$  on classes of functions on  $X$ . For example, when  $X$  is a reasonably nice topological space, then we can think about  $G$  acting on  $L^2(X)$ ; when  $X$  is a manifold, we can think about  $G$  acting on  $C^\infty(X)$ .

**Example 1.18.** The action of  $G$  on  $G$  itself by left multiplication gives rise to some “regular” representations.

**Example 1.19.** Many matrix groups such as  $\mathrm{SL}(n, \mathbb{R})$  admit standard group action on a vector space. Note that this standard action may not be unitary!

Let’s begin building our language.

**Definition 1.20** (subrepresentation, irreducible). Fix a group  $G$  and topological vector space  $V$ . If  $\rho: G \rightarrow \mathrm{GL}(V)$  is a continuous representation, then a *subrepresentation* is a closed subspace  $W \subseteq V$  which is  $G$ -invariant. We say that  $\rho$  is *irreducible* if and only if there are no proper nontrivial subspaces.

Note the hypothesis that  $W \subseteq V$  is closed for our subrepresentations!

**Example 1.21.** The action of  $\mathrm{SL}_n(\mathbb{R})$  on  $\mathbb{R}^n$  has no nontrivial proper subrepresentations and hence is irreducible. However, this representation is not unitary.

**Remark 1.22.** The action of  $G$  on itself makes  $L^2(G)$  a representation of  $G$ . However, this representation frequently fails to be irreducible. For example,  $L^2(G)$  has many automorphisms, so it cannot be irreducible by a suitable version of Schur’s lemma. In some cases, we can see this more concretely: taking  $G = \mathbb{R}$ , then we know  $\mathbb{R}$  is isomorphic to its dual, so  $L^2(\mathbb{R}) \cong L^2(\mathbb{R}^{2V})$ , and this right-hand side has more obvious subrepresentations given by the subspace of functions which vanish on a given subset of positive measure.

## 1.2 September 9

Today, we will discuss some general nonsense of topological vector spaces.



### 1.2.1 Examples of Representations

Last time, we ended with the following example, which we recall here.

**Example 1.23 (Heisenberg group).** Fix a positive integer  $n \geq 1$ . Then we define the Heisenberg group  $H_n$  as the matrix group

$$H_n := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{bmatrix} : a \in \mathbb{R}^{1 \times n}, b \in \mathbb{R}^{n \times 1}, c \in \mathbb{R} \right\}.$$

It turns out that  $H_n$  admits a natural action on  $L^2(\mathbb{R}^n)$ , which is an irreducible representation. Quickly, the  $a$ -coordinate will act by translation on the  $\mathbb{R}^n$ , and the  $b$ -coordinate will act by a character  $b \mapsto e^{2\pi i \langle b, - \rangle}$ . One finds a similar action on  $C^\infty(\mathbb{R}^n)$  and  $\mathcal{S}(\mathbb{R}^n)$ .

Check it

**Remark 1.24.** There is a finite analogue, where all the  $\mathbb{R}$ s are replaced with a finite field  $\mathbb{F}_p$ , and the character  $b \mapsto e^{2\pi i \langle b, - \rangle}$  is replaced with  $b \mapsto e^{2\pi i \langle b, - \rangle/p}$ . Equivalently, we find  $H_n(\mathbb{F}_p) = \mathbb{F}_p \times V \times V^\vee$  (where  $V = \mathbb{F}_p^n$ ); then  $H_n(\mathbb{F}_p)$  admits a natural action on  $V$ , where  $V$  acts by translation, and  $\xi \in V^\vee$  acts by multiplying with the function  $\psi_\xi(x) := e^{2\pi i \langle x, \xi \rangle/p}$ . The central  $\mathbb{F}_p \subseteq H$  now acts by scalar multiplication as  $a \mapsto e^{2\pi i a/p}$ .

Let's check that this representation is irreducible. It is enough to check that  $\text{End}_{H_n}(\mathbb{C}[V])$  is  $\mathbb{C}$ . Well, commuting with the  $V$  leaves us with

$$\text{End}_V(\mathbb{C}[V]) = \text{End}_{\mathbb{C}[V]}(\mathbb{C}[V]) = \mathbb{C}[V].$$

Further, one sees that commuting with the scalar multiplication by elements in  $V^\vee$  restricts the possible endomorphisms all the way down to scalars.

Example 1.23 appears in the book, and the argument is not too different from the one given in the remark.

**Example 1.25.** The group  $\text{SL}_2(\mathbb{R})$  acts on  $\mathbb{R}^2$  and therefore has a unitary representation on  $L^2(\mathbb{R}^2)$ . This cannot possibly be irreducible because  $\text{SL}_2(\mathbb{R})$  commutes with the extra scalar action on  $\mathbb{R}^2$ , so  $L^2(\mathbb{R}^2)$  has too many endomorphisms. To make this representation smaller, we can choose  $s \in \mathbb{C}$ , which produces a character on  $\mathbb{R}^+$  by  $\chi_s: t \mapsto t^s$ ; then we can define  $L^2(\mathbb{R}^2, \chi_s)$  to be the functions which commute with this character (namely,  $f(t^{-1}x) = t^s f(x)$ ). (Geometrically, this is basically the sections of a line bundle on  $\mathbb{RP}^1$  given by the character.) We will soon see that almost all  $s$  produces an irreducible representation.

For example, for  $s = 0$ , then  $L^2(\mathbb{R}^2, \chi_0)$  consists of the functions on  $\mathbb{RP}^1$ . This is not irreducible because it has a subrepresentation given by the constant functions. But  $L^2(\mathbb{R}^2, \chi_0)/(\mathbb{C} \cdot 1)$  is still not irreducible: it turns out to be the sum of two irreducible representations  $L^+$  and  $L^-$ , where  $L^+$  is the closure of  $z\mathbb{C}[z]$ , and  $L^-$  is the closure of  $z^{-1}\mathbb{C}[z^{-1}]$ , where  $z$  is a standard coordinate on  $\mathbb{RP}^1$ . This can be related to Fourier series by embedding  $\mathbb{RP}^1$  into  $\mathbb{CP}^1$ , which is basically a circle. We will prove all these claims later.

**Remark 1.26.** Here is an amusing way to view  $L^2(\mathbb{R}^2, \chi_s)$ : this amounts to sections of a line bundle on  $\mathbb{RP}^1$ , and after removing  $\infty$ , we see that we are looking at functions on  $\mathbb{R}$ . One can check that these are the functions which transform by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} f(z) = f\left(\frac{az+b}{cz+d}\right) |cz+d|^s.$$

Fix a Lie group  $G$  acting on an orientable manifold  $X$ . If  $G$  preserves a volume form  $dx$  on  $X$ , then  $C_c^\infty(X)$  will have an invariant pairing

$$\langle \varphi, \psi \rangle := \int_X \varphi(x) \overline{\psi(x)} dx.$$

More generally, one can work with half-densities.

**Definition 1.27 (density).** An  $s$ -density on a smooth manifold  $X$  is a section of a line bundle whose sections on an affine patch are just functions but which has transformations between coordinate charts  $(x_n) \mapsto (x'_n)$  given by

$$f(x_1, \dots, x_n) \mapsto f(x'_1, \dots, x'_n) \left| \det \left( \frac{\partial x'_i}{\partial x_j} \right) \right|^s.$$

The point of working with half-densities is that we can define the standard inner product between them in the usual way.

**Example 1.28.** Half-densities of  $\mathrm{SL}_2(\mathbb{R})$  acting on  $\mathbb{RP}^1$  (in the obvious way) amounts to considering  $L^2(\mathbb{R}^2, \chi_{-1})$ .

## 1.2.2 Topological Vector Spaces

We spend a moment reviewing what we need about locally convex topological spaces; we refer to Appendix A for a more in-depth treatment.

**Convention 1.29.** All topological vector spaces are over  $\mathbb{C}$  and are Hausdorff.

**Definition 1.30 (locally convex).** A topological vector space  $V$  is *locally convex* if and only if  $0$  has an open neighborhood basis of convex sets.

**Remark 1.31.** Equivalently, by Corollary A.23 a topological vector space  $V$  if and only if its topology is generated by a collection of seminorms.

All representations in this class will actually be given by “Fréchet spaces.”

**Definition 1.32 (Fréchet).** A topological vector space  $V$  is *Fréchet* if and only if it is locally convex, has a countable basis of neighborhoods of  $0$ , and is sequentially complete.

**Remark 1.33.** By Proposition A.27, having a countable basis of neighborhoods of  $0$  is equivalent to being metrizable. (In fact, one can choose the metric to be translation-invariant.) Once  $V$  is metrizable, being sequentially complete is equivalent to being complete.

We will also frequently take our vector spaces  $V$  to be separable.

**Definition 1.34 (separable).** A topological space  $X$  is *separable* if and only if it admits a countable basis.

**Convention 1.35.** In this course, all Fréchet spaces are separable unless otherwise specified.

**Non-Example 1.36.** For  $p < 1$ , the space  $L^p([0, 1])$  fails to be locally convex. In fact, the only open nonempty convex subset is the whole space!

**Example 1.37.** For a topological space  $X$ , let  $C(X)$  be the space of continuous functions, where  $\{f_i\} \rightarrow f$  if and only if we have uniform convergence on compact sets. If  $X$  is (Hausdorff) compact, then  $C(X)$  is a Banach space (given by  $\|\cdot\|_\infty$ ). However, if  $X$  is merely a (Hausdorff, second countable) locally compact topological space, then  $C(X)$  is merely a Fréchet space: write  $X$  as a countable union  $\bigcup_i K_i$  of compact sets, and then we can use the seminorms  $\|\cdot\|_{K_i}$ . We refer to Example A.55 for details.

**Remark 1.38.** We may also write  $C_0(X)$  for  $C(X)$ , for no particular reason.

**Example 1.39.** If  $X$  is a manifold, then we can consider the topological space  $C^k(X)$ , where  $\{f_i\} \rightarrow f$  converges if and only if the first  $k$  derivatives of these functions converge uniformly on compact sets. Similarly, we see that  $C^k(X)$  is Banach when  $k$  is finite and  $X$  is compact; otherwise, it is merely Fréchet. (The seminorms are now given by  $\|\partial^\alpha(\cdot)\|_\infty$  as  $K$  varies over a covering collection of compact closed balls.) The argument is basically the same as the one in Example A.55, so we omit it. The same sort of argument shows that the space  $S(\mathbb{R}^n)$  of Schwartz functions is Fréchet.

**Non-Example 1.40.** If  $X$  fails to be compact, then the subspace  $C_c^\infty(X)$  of  $C^\infty(X)$  may fail to be a Fréchet space because it fails to be complete.

Sometimes, we will find ourselves in a circumstance where we can restrict to a nice class of spaces.

**Definition 1.41 (Banach).** A topological vector space  $V$  is a *Banach space* if and only if its topology is given by a norm, and it is complete with respect to that norm.

For example, Hilbert spaces are Banach spaces.

Here is one benefit of working with a Banach space.

**Lemma 1.42.** Fix a topological group  $G$  acting on a Banach space  $V$ . Then the action  $G \times V \rightarrow V$  is continuous if and only if the induced map  $\rho: G \rightarrow \text{Aut}(V)$  is continuous in the strong topology, in which  $\{E_i\} \rightarrow E$  if and only if  $\{E_i v\} \rightarrow E v$  converges for all  $v \in V$ .

*Proof.* The forward direction has little content: given a net  $\{g_i\} \rightarrow g$ , we know that  $\rho(g_i)v \rightarrow \rho(g)v$  for each  $v$  by continuity, so it follows that  $\rho(g_i) \rightarrow \rho(g)$ .

The reverse direction follows from the Uniform boundedness principle, which claims that  $|E_i v|$  being bounded for every  $v$  implies that  $\|E_i\|$  is bounded. Namely, to show that  $G \times V \rightarrow V$  is continuous, we choose a net  $\{(g_i, v_i)\} \rightarrow (g, v)$  in  $G \times V$ , and we would like to show that  $g_i v_i \rightarrow g v$ . By translation, we may assume that  $g = 1$ . We are given that  $\rho(g_i) \rightarrow 1$  in the strong topology, which implies that  $\rho(g_i)w \rightarrow w$  for all  $w$ , so we may also assume that  $v = 0$  by translation. Because now  $v_i \rightarrow 0$ , it is enough to show that  $\|\rho(g_i)\|$  is bounded, which is what follows from the Uniform boundedness principle. ■

**Remark 1.43.** A Banach space  $V$  can alternatively give  $\text{End } V$  the norm topology, but then the map  $G \rightarrow \text{Aut } V$  need not be continuous with the norm topology. For example, the action of  $\mathbb{R}$  on  $L^2(\mathbb{R})$  by translation  $T_a f(x) := f(x + a)$  fails to be continuous: as  $a \rightarrow 0$ , we see  $T_a \rightarrow \text{id}$ , but  $\|T_a - \text{id}\| = 1$  for all  $a \neq 0$ .

## 1.3 September 11

Today, we work with measures.

### 1.3.1 Partition of Unity

It will be helpful to have the following incarnation of partition of unity.

**Lemma 1.44** (Partition of unity). Let  $X$  be metric space. For any countable open cover  $\{U_i\}_{i \in \mathbb{N}}$  of pre-compact open subsets of  $X$ , there are continuous functions  $\{f_i\}_{i \in \mathbb{N}}$  from  $X$  to  $[0, 1]$  such that  $f_i|_{X \setminus U_i} = 0$  and

$$\sum_{i \in \mathbb{N}} f_i = 1.$$

*Proof.* Let's begin by proving a version of Urysohn's lemma: for any nonempty closed subset  $A \subseteq X$ , the function  $d_A: X \rightarrow \mathbb{R}$  given by

$$d_A(x) := \inf_{a \in A} d(x, a)$$

is well-defined and (Lipschitz) continuous because  $d_A(x) \leq d_A(y) + d(x, y)$  for any  $x, y \in X$ . Furthermore, we see that  $d_A(x) = 0$  if and only if  $x \in A$ : if  $d_A(x) = 0$ , then there is a sequence of points  $\{x_i\}$  in  $A$  for which  $d(x, x_i) \rightarrow 0$ , but this implies that  $x_i \rightarrow x$ , so  $x \in A$  because  $A$  is closed.

Now, for each  $i$ , we start with the function  $g_i := d_{A_i}/(1 + d_{A_i})$  where  $A_i := X \setminus U_i$ , so we see that  $\text{im } g_i \subseteq [0, 1]$ , and  $g_i(x) = 0$  if and only if  $x \notin U_i$ . We now define  $G(x) := \sum_i 2^{-i} g_i$  and set  $f_i := 2^{-i} g_i / G$ . The function  $G$  is continuous by the Weierstrass  $M$ -test, and it is always positive because the  $U_i$ s cover  $X$ ; thus, the functions  $f_i$  are well-defined, positive, and continuous. Lastly, we see that

$$\sum_{i \in \mathbb{N}} f_i = \frac{1}{G} \sum_{i \in \mathbb{N}} 2^{-i} g_i,$$

which is 1 by construction of  $G$ . ■

Because we will have occasion to retreat to topological spaces instead of manifolds, we pick up the following result.

**Lemma 1.45.** Let  $X$  be a locally compact second countable (Hausdorff) topological space  $X$ . Then  $X$  is metrizable.

*Proof.* By the Urysohn metrization theorem, it is enough to check that  $X$  is Hausdorff, second countable, and regular. However, regularity follows from being locally compact and Hausdorff. ■

### 1.3.2 Measures on a Space

Let's say a bit about measures.

**Definition 1.46** (measure). Fix a locally compact second countable topological space  $X$ . Then a *measure* is an element of the topological dual  $C(X)^*$ . In other words, it is a continuous linear functional on  $C(X)$ . Given  $\mu \in C(X)^*$ , we may write  $\int_X f d\mu$  for  $\mu(f)$ . We denote this space by  $\text{Meas}_c(X)$ , and give it the weak topology, where  $\{\mu_i\} \rightarrow \mu$  if and only if  $\{\mu_i(f)\} \rightarrow \mu(f)$  for all  $f \in C(X)$ .

**Remark 1.47.** In other words, the topology is given by the family of seminorms  $\mu \mapsto |\mu(f)|$ . Thus,  $C(X)^*$  is Hausdorff and locally convex by Corollary A.23.

**Example 1.48.** Suppose that  $X$  is an orientable manifold with volume form  $\omega$ . Then  $C_c(X)$  embeds into  $C(X)^*$  by sending  $g \in C_c(X)$  to the functional

$$g\omega: f \mapsto \int_X fg\omega.$$

This is linear by the linearity of the integral, so it remains to check that it is continuous; by Corollary A.42, it is enough to show that  $|g\omega|$  is bounded above by a continuous seminorm. Let  $K$  be the support of  $g$ , which is compact, and we claim that  $g\omega \leq (\|g\|_\infty \int_K \omega) \|\cdot\|_K$ , which will be enough. Well, for any  $f$ , we see  $g\omega(f)$  is bounded by

$$\left| \int_K fg\omega \right| \leq \|g\|_\infty \int_K \omega \cdot \|f\|_K.$$

**Example 1.49.** For any  $x \in X$ , there is a  $\delta$ -distribution  $\delta_x: C(X) \rightarrow \mathbb{C}$  given by  $\delta_x(f) := f(x)$ . To show that this measure is continuous, we use Corollary A.42, which tells us that it is enough to check that  $|\delta_x| \leq \|\cdot\|_K$  for some compact  $K \subseteq X$ . Well, taking  $K = \{x\}$ , we see  $|\delta_x(f)| = |f(x)| = \|f\|_K$  for any  $f$ .

**Example 1.50.** If  $f: X \rightarrow Y$  is a continuous map, then there is a linear map  $f^*: C(Y) \rightarrow C(X)$  given by  $f^* := (- \circ f)$ . (It sends continuous maps to continuous maps because  $f$  is continuous.) Continuing, we get a map  $f_*: C(X)^* \rightarrow C(Y)^*$  by sending  $\mu \in C(X)^*$  to  $(\mu \circ f^*) \in C(Y)^*$ , where  $f^*$  is continuous by Example A.57. We may write  $\mu \circ f^*$  as  $f_*\mu$ . Note  $f_*$  is continuous: if  $\{\mu_i\} \rightarrow \mu$  in  $C(X)^*$ , then any  $g \in C(Y)$  has  $\{\mu_i(g \circ f)\} \rightarrow \mu(g \circ f)$ , so  $\{f_*\mu_i(g)\} \rightarrow f_*\mu(g)$ .

**Remark 1.51.** If  $i$  is the inclusion  $K \subseteq X$  of a compact set, then  $i_*$  is just the restriction map  $C(K)^* \rightarrow C(X)^*$  given by  $i_*\mu(g) := \mu(g|_K)$ .

**Remark 1.52.** If  $X$  is compact, then  $C(X)^*$  has a natural (operator) norm given by

$$\|\mu\| := \sup_{\|f\|_\infty=1} |\mu(f)|.$$

However,  $C(X)^*$  is usually not equipped with the norm topology: for example, if  $\{x_i\} \rightarrow x$ , then  $\{\delta_{x_i}\} \rightarrow \delta_x$  in the weak topology because  $\{f(x_i)\} \rightarrow f(x)$  for any continuous  $f$ , but  $\|\delta_x - \delta_{x_i}\| = 2$  whenever  $x \neq x_i$ .

### 1.3.3 Supports

Here is our main definition.

**Definition 1.53 (support).** Fix a locally compact second countable topological space  $X$ . Then the *support*  $\text{supp } \mu$  of a functional  $\mu$  on  $C(X)$  is defined by  $x \notin \text{supp } \mu$  if and only if there is an open neighborhood  $U$  of  $x$  for which any  $f$  satisfying  $f|_{X \setminus U} = 0$  also has  $\mu(f) = 0$ .

**Lemma 1.54.** Fix a locally compact second countable topological space  $X$ . For any functional  $\mu \in C(X)^*$ , the subset  $\text{supp } \mu$  is closed. Furthermore, if  $\mu$  is continuous, then  $\text{supp } \mu$  is compact.

*Proof.* To see that  $\text{supp } \mu$  is closed, we show that its complement is open. Indeed, note that any  $x \notin \text{supp } \mu$  admits an open neighborhood  $U$  of  $x$  satisfying some condition in the definition; but then any  $y \in U$  still admits this open neighborhood, so  $U \cap \text{supp } \mu = \emptyset$ .

It remains to show that  $\text{supp } \mu$  is compact. We will show that  $\text{supp } \mu$  failing to be compact implies that  $\mu: C(X) \rightarrow \mathbb{F}$  fails to be continuous. For this, it is enough to check that there is a compact set  $K$  containing  $\text{supp } \mu$ . Well, because  $X$  is locally compact and second countable, we can find a countable collection of precompact open subsets covering  $X$ , so upon taking suitable unions and closures, we see that there is a sequence  $\{K_i\}_{i \in \mathbb{N}}$  of compact sets covering  $X$ . (This construction is also used in Example A.55.)

We now use the assumption that  $\text{supp } \mu$  fails to be compact, which means that it cannot be contained in any  $K_i$ . Then we are granted  $x_i \in \text{supp } \mu \setminus K_i$  for each  $i$ , which means that any open neighborhood  $U_i$  of  $x_i$  admits a function  $f_i$  such that  $f_i|_{X \setminus U_i} = 0$  and  $\mu(f_i) \neq 0$ ; we go ahead and shrink  $U_i$  to fully avoid  $K_i$ . Additionally, by adjusting  $f_i$  by a scalar, we may assume that  $\mu(f_i) = 1$ . Now, the point is to consider the sum

$$\sum_i f_i,$$

which converges uniformly on compact sets: as in Example A.55, it is enough to check uniform convergence on just the  $K_i$ s, and we see that this is a finite sum when restricted to any  $K_n$  because  $f_i$  is nonzero only inside  $U_i$ , so  $f_i|_{K_n} = 0$  for  $i > n$ .

On the other hand, we see that continuity of  $\mu$  would imply that

$$\mu\left(\sum_i f_i\right) = \sum_i \mu(f_i)$$

is a finite number, which is false because  $\mu(f_i) = 1$  for each  $i$ ! Thus,  $\mu$  fails to be continuous. ■

**Remark 1.55.** Lemma 1.54 explains why there is a  $c$  in our notation  $\text{Meas}_c(X)$ .

**Lemma 1.56.** Fix a locally compact second countable topological space  $X$ . For any  $\mu \in C(X)^*$  and  $f \in C(X)$  for which  $f|_{\text{supp } \mu} = 0$ , we have  $\mu(f) = 0$ .

*Proof.* For each  $x \notin \text{supp } \mu$ , we are granted an open neighborhood  $U_x$  of  $x$  such that any function  $g$  vanishes outside  $U$  has  $\mu(g) = 0$ . Because  $X$  is second countable, the open cover  $\{U_x\}_{x \notin \text{supp } \mu}$  of  $X \setminus \text{supp } \mu$  can be refined to a countable subcover. Let  $\{U_i\}_{i \in \mathbb{N}}$  be this countable subcover, and we let  $\{f_i\}_{i \in \mathbb{N}}$  be the partition of unity granted by Lemma 1.44 (which applies by Lemma 1.45). But now we see that

$$f = \sum_{i \in \mathbb{N}} f_i f.$$

Thus,  $\mu(f) = 0$ : it's enough to see that  $\mu(f_i f) = 0$  for all  $i \in \mathbb{N}$  because  $f_i f$  vanishes outside  $U$ . ■

As an application, we are able to push forward measures!

**Lemma 1.57.** Fix a locally compact second countable topological space  $X$ . Given  $\mu \in C(X)^*$  such that  $\text{supp } \mu$  is contained in a compact set  $K$ , there is a unique measure  $\nu \in C(K)^*$  such that

$$\nu(f) := \mu(\tilde{f}),$$

whenever  $\tilde{f} \in C(X)^*$  satisfies  $\tilde{f}|_K = f$ .

*Proof.* Let's begin with uniqueness. This will be accomplished as soon as we show that any function  $f \in C(K)$  is the restriction of some function  $\tilde{f} \in C(X)$ . This amounts to the Tietze extension theorem. To avoid issues with the normality hypothesis, we write  $X$  as an ascending union  $\bigcup_{i \in \mathbb{N}} K_i$  of compact sets with  $K_0 := K$ . Then compact Hausdorff spaces are normal, so  $f \in C(K)$  may inductively be extended to each  $K_n$

via the Tietze extension theorem, so  $f$  admits an extension  $\tilde{f}$  to  $C(X)$ . We remark that the Tietze extension theorem allows this extension to be done with  $\|\tilde{f}\|_\infty = \|f\|_\infty$ .

It remains to show existence. For this, we should show that the given formula for  $\nu$  is well-defined, linear, and continuous.

- Well-defined: we use Lemma 1.56. Indeed, if  $\tilde{f}_1$  and  $\tilde{f}_2$  in  $C(X)$  restrict to the same function  $f \in C(K)$ , then  $\mu(\tilde{f}_1 - \tilde{f}_2) = 0$  by Lemma 1.56.
- Linear: for any functions  $f$  and  $g$  in  $C(X)^*$ , we see that  $\nu(af|_K + bg|_K) = a\nu(f|_K) + b\nu(g|_K)$  because this check lifts to  $\mu$ .
- Continuous: because  $\mu$  is continuous, it is bounded by Lemma A.41, so there is an open neighborhood  $U$  of 0 for which  $\sup_{f \in U} |\mu(f)| < \infty$ . However, basic open neighborhoods of 0 look like finite intersections of open sets of the form

$$\{f \in C(X) : \|f|_E\|_\infty < \varepsilon\}$$

for some compact subset  $E \subseteq X$  and some  $\varepsilon > 0$ . By taking the union of the relevant  $E$ s (and the minimum of the  $\varepsilon$ s), we may assume that  $U$  actually has the above form. Additionally, because  $\mu$  only depends on  $f|_K$ , we may as well assume that  $E \subseteq K$ .

Now, the formula  $\|f|_E\|_\infty < \varepsilon$  also cuts out an open subset of  $C(K)$ , and we claim that

$$\sup_{f \in U} |\mu(f)| \stackrel{?}{=} \sup_{f \in U \cap C(K)} |\nu(f)|,$$

which shows that  $\nu$  is bounded and hence continuous by Lemma A.41. Well, any  $f \in U \cap C(K)$  can be extended to some  $\tilde{f} \in U$  by our Tietze extension theorem argument, so the result follows by construction of  $\nu$ . ■

**Notation 1.58.** Fix a locally compact second countable topological space  $X$ , and let  $i: K \rightarrow X$  be the inclusion of a compact subset. Given  $\mu \in C(X)^*$  such that  $\text{supp } \mu$  is contained in  $K$ , we let  $i^*\mu$  denote the measure constructed in Lemma 1.57. This is called the pullback measure.

**Remark 1.59.** We note that  $i^*$  is continuous. Well, if  $\mu_n \rightarrow \mu$ , we need to check that  $i^*\mu_n \rightarrow i^*\mu$ . Checking this on the weak topology is not hard: given any function  $f \in C(K)$ , we need to know that  $i^*\mu_n(f) \rightarrow i^*\mu(f)$ , which follows because  $\mu_n(\tilde{f}) \rightarrow \mu(\tilde{f})$  for any lift  $\tilde{f} \in C(X)$  of  $f$ .

**Lemma 1.60.** Fix a locally compact second countable topological space  $X$ , and let  $i: K \rightarrow X$  be the inclusion of a compact subset.

- If  $\mu \in C(X)^*$  has  $\text{supp } \mu \subseteq K$ , then  $i_*i^*\mu = \mu$ .
- If  $\mu \in C(K)^*$ , then  $\text{supp } i_*\mu \subseteq K$  and  $i^*i_*\mu = \mu$ .
- The map  $i_*: C(K)^* \rightarrow C(X)^*$  is a homeomorphism onto the space of measures  $\mu$  with  $\text{supp } \mu \subseteq K$ .

*Proof.* Note (c) follows from (a) and (b). To see (a), we note that any  $f \in C(X)$  has  $i_*i^*\mu(f) = i^*\mu(f|_K)$ , which is  $\mu(f)$  by definition of the pullback. To see (b), we start by assuming  $\text{supp } i_*\mu \subseteq K$ . Then we may take any  $f \in C(K)$ , lift it to  $\tilde{f} \in C(X)$ , and we compute that  $i^*i_*\mu(f) = i^*\mu(\tilde{f})$  is  $\mu(\tilde{f}|_K) = \mu(f)$ .

It remains to show that  $\text{supp } i_*\mu \subseteq K$ . Well, for any  $x \notin K$ , we need to show  $x \notin \text{supp } i_*\mu$ , for which we use the open neighborhood  $U := X \setminus K$ . Indeed, if  $f|_{X \setminus U} = 0$ , then  $f|_K = 0$ , so  $i_*\mu(f) = 0$  follows. ■

### 1.3.4 Sequential Completeness

We now turn towards showing that  $C(X)^*$  is sequentially complete.

**Lemma 1.61.** Fix a locally compact second countable topological space  $X$ . If  $\{\mu_i\}$  is a Cauchy sequence in  $C(X)^*$ , then there is a compact set  $K$  such that  $\text{supp } \mu_i \subseteq K$  for all  $i$ .

*Proof.* This argument is similar to Lemma 1.54, but the bookkeeping is more painful.

Suppose there is no such compact set  $K$ , and we will show that  $\{\mu_i\}$  has a subsequence which fails to be Cauchy. Before doing anything, note we are trying to see if  $\bigcup_{i \in \mathbb{N}} \text{supp } \mu_i$  is contained in a compact set. If we ever have  $\text{supp } \mu_i$  contained in  $\bigcup_{j < i} \text{supp } \mu_j$ , then we may as well remove  $\mu_i$  from the Cauchy sequence because it will not affect the final containment in a compact set.

Now, we will run an inductive construction as follows. Start with the compact set  $K_0 := \emptyset$  and  $\nu_0 := \mu_0$ . Now, for every  $n \geq 0$ , we take  $(K_n, \mu_n)$  and build a triple  $(f_n, K_{n+1}, \nu_{n+1})$  as follows.

1. Observe that there is a point  $x_n \in \text{supp } \nu_n \setminus K_n$ . Thus, we are granted a function  $f_n$  and a small open neighborhood  $U_n$  of  $x_n$  such that  $f_n$  vanishes outside  $U_n$  and  $\mu_n(f_n) \neq 0$ . By shrinking  $U_n$ , we may avoid  $K_n$  and assume that  $U_n$  is precompact. By rescaling, we may assume that  $\mu_n(f_1 + \cdots + f_n) = n$ .
2. Now, expand  $K_n$  (by  $\overline{U_n}$ ) to a compact set  $K_{n+1}$  containing  $U_n$  and  $\text{supp } \nu_n$ . Because  $\bigcup_{i \in \mathbb{N}} \text{supp } \mu_i$  is contained in no compact set, we may find  $\nu_{n+1}$  such that  $\text{supp } \nu_{n+1} \setminus K_{n+1}$ .

The above process continues indefinitely by the hypothesis on the  $\mu_i$ s.

The point is that the series  $\sum_n f_n$  converges uniformly on the compacts  $\{K_n\}$  because  $f_n|_{K_m} = 0$  for  $n > m$ . Letting this limiting function be  $f$ , we then see that  $\nu_n(f)$  only has the nonzero terms  $\nu_n(f_1 + \cdots + f_n) = n$ : the point is that  $f_m$  is only nonzero on  $U_m$ , so if  $m > n$ , then  $\text{supp } \nu_n$  is contained in  $K_{n+1}$ , so  $\nu_n(f_m) = 0$ . Thus, we see that

$$|\nu_n(f) - \nu_m(f)| = |n - m|$$

fails to go to 0 as  $n, m \rightarrow \infty$ , so the sequence  $\{\nu_n\}$  fails to be Cauchy. ■

**Remark 1.62.** This argument must do something a little tricky because the statement is false if  $\{\mu_i\}$  is replaced by a Cauchy net. Indeed, if that were true, then Remark 1.64 explains that  $C(X)^*$  would be complete, which is not true in general (see Example 1.65).

Here is an application of our work with supports.

**Lemma 1.63.** Fix a locally compact second countable topological space  $X$ . The space  $C(X)^*$  is sequentially complete.

*Proof.* We proceed in steps. The first half of the proof handles the case where  $X$  is compact by using the Uniform boundedness principle, and the second half of the proof reduces to this case by using Lemma 1.61.

1. Let's start by handling the case where  $X$  is compact. The general case will then follow by applying Lemma 1.61. Fix a Cauchy sequence  $\{\mu_i\}_{i \in \mathbb{N}}$  for which we want to find a limit. Well, for each  $f \in C(X)$ , the nature of the weak topology makes  $\{\mu_i(f)\}$  a Cauchy sequence in  $\mathbb{R}$ , so it admits a limiting value which we label

$$\mu(f) := \lim_{i \rightarrow \infty} \mu_i(f).$$

Thus, we have defined a function  $\mu: C(X) \rightarrow \mathbb{C}$ . Note linearity of the  $\mu_i$ s and taking the limit implies that  $\mu$  is linear. Additionally, having  $\mu_i(f) \rightarrow \mu(f)$  for all functions  $f$  implies that we will have  $\mu_i \rightarrow \mu$  as soon as we actually check that  $\mu \in C(X)^*$ .



2. When  $X$  is compact, it still remains to check that  $\mu$  as defined above is actually continuous. The point is that, because  $X$  is compact,  $C(X)$  is a Banach space with norm given by  $\|\cdot\|_\infty$  (see Example A.55). By Corollary A.42, it is enough to check that  $\|\mu\|$  is bounded, for which we will use the Uniform boundedness principle. As such, we will be able to use the Uniform boundedness principle: because each sequence  $\{\mu_i(f)\}$  is Cauchy for every  $f$ , we see that these sequences are always bounded, so the operator norms  $\|\mu_i\|$  must also be bounded. In particular, there is a large constant  $C$  for which  $|\mu_i(f)| \leq C \|f\|_\infty$  for all  $f$ , so we see that  $|\mu(f)| \leq C \|f\|_\infty$  for all  $f$ , so  $\|\mu\| \leq C$ .
3. We now turn to the general case. Once again, fix a Cauchy sequence  $\{\mu_i\}_{i \in \mathbb{N}}$  for which we want to find a limit. We use Lemma 1.61, which allows us to produce a compact subset  $K \subseteq X$  such that we may assume that  $\text{supp } \mu_i \subseteq K$  for each  $i$ . Let  $j: K \rightarrow X$  be the inclusion.

To reduce to the compact case, we note that Lemma 1.60 shows that the  $\mu_i$ s are in the image of the topological embedding  $j_*: C(K)^* \rightarrow C(X)^*$ , so it is enough to find a limit in  $C(K)^*$ . This is exactly the compact case! ■

**Remark 1.64.** The argument for  $X$  in the compact case applies to general Cauchy nets, so it follows that  $C(X)^*$  is complete. More generally, the argument shows that if  $\{\mu_i\}$  is a Cauchy net for which

$$\bigcup_i \text{supp } \mu_i$$

is contained in a compact set, then the limit exists.

It is worthwhile to have an example where  $C(X)^*$  is not complete.

**Example 1.65.** Give  $X = \mathbb{N}$  the discrete topology. Then we show that  $C(X)^*$  is not complete.

*Proof.* Here,  $C(X)$  is just functions  $\mathbb{N} \rightarrow \mathbb{C}$ , which are sequences in  $\mathbb{C}$ . We let  $\delta_n$  be the sequence  $\{\delta_n(i)\}_{i \in \mathbb{N}}$ .

Next, we calculate  $C(X)^*$ . For any  $\mu \in C(X)^*$ , we see that  $\mu(f)$  only depends on  $f|_{\text{supp } \mu}$  by Lemma 1.56. But  $\text{supp } \mu$  is some compact set in  $\mathbb{N}$ , so it is some finite set  $\{n_1, \dots, n_r\}$ . Thus, we see that  $\mu(f)$  is

$$\mu(f|_K) = \sum_{i=1}^r f(n_i) \mu(\delta_{n_i}).$$

Thus, we see that  $\mu$  only depends on the sequence  $\{\mu(\delta_n)\}_{n \in \mathbb{N}}$ , which has only finitely many nonzero terms. Conversely, any such sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  defines a measure by

$$\mu(f) := \sum_{n \in \mathbb{N}} f(n) \mu_n,$$

which is a finite sum because only finitely many of the  $\mu_n$ s are nonzero. It is not hard to see that these constructions are linear and inverse, so our measures are in bijection with finitely supported sequences in this manner. (For example, any measure  $\mu$  produces the sequence  $\{\mu(\delta_n)\}$ , which then produces the measure  $f \mapsto \sum_n f(n) \mu(\delta_n)$ , which is the original measure. The other inverse check is easier.)

To show that  $C(X)^*$  fails to be complete, we will show that  $C(X)^*$  is a dense subset of the dual space  $C(X)^\vee$  (where  $C(X)^\vee$  continues to have the weak topology). To explain why this completes this proof, note that  $C(X)^* \neq C(X)^\vee$  for dimension reasons, so  $C(X)^*$  cannot be a closed subset, so  $C(X)^*$  will have some Cauchy net not admitting a limit, as required.

It remains to show that  $C(X)^* \subseteq C(X)^\vee$  is dense, where  $C(X)^\vee$  has the weak topology. Well, basic open subsets in the weak topology look like  $\mu + U$  where  $\mu \in C(X)^\vee$  and  $U$  has the form

$$\bigcap_{i=1}^n \underbrace{\{\nu : |\nu(f_i)| < \varepsilon_i\}}_{U_i :=}$$

for some finitely many functions  $\{f_1, \dots, f_n\}$  and small real numbers  $\varepsilon_i > 0$ . We may as well assume that these functions are linearly independent, or else we can make the list of functions smaller. We would like to find some element of  $C(X)^*$  in  $\mu + U$ .

Because  $\{f_1, \dots, f_n\}$  are linearly independent, they span an  $n$ -dimensional subspace. We now claim that there is a subset  $K \subseteq \mathbb{N}$  of size  $n$  for which the sequences  $\{f_i(k)\}_{k \in K}$  continue to be linearly independent in  $\mathbb{C}^K$ . Well, the sequences  $\{f_i(n)\}_{n \in \mathbb{N}}$  form an  $n$ -dimensional subspace of  $\mathbb{C}^{\mathbb{N}}$ , so one can apply Gaussian elimination to this  $n \times |\mathbb{N}|$  matrix to put it into row-reduced Echelon form, from which the subset  $K$  can be read off as the pivots. (Row operations do not change the dimension of the span of any given collection of columns!)

Now, by rearranging, we may as well take  $K = \{1, 2, \dots, n\}$ . Then we can find constants  $\{c_1, \dots, c_n\}$  solving the system of  $n$  equations

$$\mu(f_i) = \sum_{j=1}^n c_j f_i(j)$$

because the matrix  $\{f_i(j)\}_{i,j}$  is invertible by the previous paragraph. These constants now define a measure  $\nu$  in  $C(X)^*$  satisfying  $\mu(f_i) = \nu(f_i)$  for each  $i$ , so  $\nu \in \mu + U$  follows. ■

**Remark 1.66.** What goes wrong in Example 1.65 is that  $C(\mathbb{N})^*$  cannot be first countable, which we can also check directly. Indeed, suppose that we have an open basis of neighborhoods  $\{U_i\}_{i \in \mathbb{N}}$  of 0. Now, each  $U_i$  is still absorbing by Example A.17, so we can find a real number  $\varepsilon_i > 0$  for which  $\varepsilon_i \delta_i \in U_i$  for each  $i$ . But now  $\varepsilon_i \delta_i \rightarrow 0$  in  $C(\mathbb{N})^*$  by definition of the weak topology while the function  $f: \mathbb{N} \rightarrow \mathbb{C}$  given by  $f(i) := \varepsilon_i^{-1}$  has  $(\varepsilon_i \delta_i)(f) = 1$  for all  $i$ . This is a contradiction!

Here is the theoretical culmination of our work with measures.

**Notation 1.67.** Fix a locally compact second countable topological space  $X$ . Then we set  $\text{Meas}_c^0(X) \subseteq \text{Meas}_c(X)$  to be the subspace spanned by  $\{\delta_x\}_{x \in X}$ .

**Proposition 1.68.** Fix a locally compact second countable topological space  $X$ . Then  $\text{Meas}_c^0(X)$  is sequentially dense in  $\text{Meas}_c(X)$ .

*Proof.* Quickly, we reduce to the case where  $X$  is compact. Fix some  $\mu \in \text{Meas}_c(X)$  that we want to approximate, and set  $K := \text{supp } \mu$ . Then Lemma 1.60 shows that  $\mu$  is in the image of the topological embedding  $j_*: \text{Meas}_c(K) \rightarrow \text{Meas}_c(X)$ , where  $j: K \rightarrow X$  is the embedding. Note further that  $j_*$  restricts to a linear map  $\text{Meas}_c^0(K) \rightarrow \text{Meas}_c^0(X)$  by Example 1.50: it's enough to check this on generators, for which we note  $j_* \delta_k = \delta_k$ . Thus, for any open neighborhood  $U \subseteq \text{Meas}_c(X)$  of  $\mu$ , it is enough to find  $\nu$  in the intersection  $U \cap \text{Meas}_c^0(K)$  (meaning  $j^*U$ ) and then apply  $j_*$  to complete the argument.

Thus, we may assume that  $X$  is compact. The idea is to basically approximate  $\mu$  using Riemann sums. Because we only have access to continuous functions, our Riemann sums will have to use a partition of unity. To explain the idea, we suppose that we are given a finite open cover  $\mathcal{U}$  of  $X$ . Then Lemma 1.44 (which applies by Lemma 1.45) grants us continuous nonnegative functions  $\{f_U\}_{U \in \mathcal{U}}$  such that  $\text{supp } f_U \subseteq U$  for each  $U \in \mathcal{U}$  and that  $\sum_{U \in \mathcal{U}} f_U = 1$ . We then define

$$\mu_{\mathcal{U}} := \sum_{U \in \mathcal{U}} \mu(f_U) \delta_{x_U} \in \text{Meas}_c^0(X),$$

where  $x_U \in U$  is some basepoint. (Technically,  $\mu_{\mathcal{U}}$  also depends on the functions  $f_{\bullet}$  and basepoints  $x_{\bullet}$ , but we will suppress this from the notation.) The hope is that if the open sets  $\mathcal{U}$  are small enough, then we can show that  $\mu$  and  $\mu_{\mathcal{U}}$  are close. To get some idea of what we need, we note that any  $f \in C(X)$  has  $\mu(f) - \mu_{\mathcal{U}}(f)$

equal to

$$\begin{aligned}\mu(f) - \mu_{\mathcal{U}}(f) &= \sum_{U \in \mathcal{U}} \mu(ff_U) - \sum_{U \in \mathcal{U}} \mu(f_U)f(x_U) \\ &= \mu\left(\sum_{U \in \mathcal{U}} ff_U - f(x_U)f_U\right),\end{aligned}$$

which one may hope to be small because the functions  $ff_U - f(x_U)f_U$  should be small and have small support. Indeed, we now note that  $\mu$  being continuous means that it is bounded, so because  $C(X)$  is Banach with the norm  $\|\cdot\|_{\infty}$  (recall  $X$  is compact!), we are able to write

$$\begin{aligned}|\mu(f) - \mu_{\mathcal{U}}(f)| &\leq \|\mu\| \sup_{x \in X} \sum_{U \in \mathcal{U}} |f(x) - f(x_U)| f_U(x) \\ &\leq \|\mu\| \sup_{\substack{U \in \mathcal{U} \\ x \in U}} |f(x) - f(x_U)|.\end{aligned}$$

To continue, we see that we are going to need to produce good, uniform approximations of  $f$  with our open covers.

We are now ready to define our open covers  $\mathcal{U}$ . Fix some  $\delta > 0$ . Because  $X$  is a compact metric space, we may take  $\mathcal{U}$  to be a collection of open balls of radius  $\delta$ . Let  $\mu_{\delta}$  be the resulting measure, and we claim that  $\{\mu_{\delta}\} \rightarrow \mu$  as  $\delta \rightarrow 0^+$ . This will complete the proof because, for example, we also see that  $\{\mu_{1/n}\} \rightarrow \mu$  as  $n \rightarrow \infty$ .

To check that  $\{\mu_{\delta}\} \rightarrow \mu$ , we fix a function  $f \in C(X)$ , and we must show that  $\{\mu_{\delta}(f)\} \rightarrow \mu(f)$ . Because  $X$  is compact(!), the function  $f$  is actually uniformly continuous, so for any  $\varepsilon > 0$ , we can choose any  $\delta > 0$  small enough so that  $d(x, y) < \delta$  implies that  $|f(x) - f(y)| < \varepsilon$ . Then we see that

$$\sup_{\substack{U \in \mathcal{U} \\ x \in U}} |f(x) - f(x_U)| < \varepsilon,$$

so  $|\mu(f) - \mu_{\mathcal{U}}(f)| < \varepsilon$ . The claim follows. ■

### 1.3.5 The Algebra of Measures

The point of Proposition 1.68 is that we can define maps on  $C(X)^*$  by just defining them on  $\delta_x$ s and extending by continuity. Here is an application of this.

**Lemma 1.69.** Fix locally compact second countable spaces  $X$  and  $Y$ . Then there is a unique bilinear map

$$\boxtimes: \text{Meas}_c(X) \times \text{Meas}_c(Y) \rightarrow \text{Meas}_c(X \times Y)$$

such that  $\delta_x \boxtimes \delta_y = \delta_{(x,y)}$  and which is sequentially continuous.

*Proof.* Observe that the given relation implies that  $\boxtimes$  is automatically uniquely defined on  $\text{Meas}_c^0(X) \times \text{Meas}_c^0(Y)$  by linearity, which is dense in  $\text{Meas}_c(X) \times \text{Meas}_c(Y)$  by Proposition 1.68, so there is certainly at most one map  $\boxtimes$  satisfying the given relation (by Proposition A.30). Thus, we see that the main problem is existence.

By Proposition A.30, it is enough to check that  $\boxtimes: \text{Meas}_c^0(X) \times \text{Meas}_c^0(Y) \rightarrow \text{Meas}_c(X \times Y)$  is sequentially continuous. Thus, fix Cauchy sequences  $\{\mu_i\} \rightarrow \mu$  and  $\{\nu_j\} \rightarrow \nu$ , and we want to check that  $\{\mu_i \boxtimes \nu_j\} \rightarrow \mu \boxtimes \nu$  as  $i, j \rightarrow \infty$ . Quickly, we remark that we can reduce to the case where  $X$  and  $Y$  are compact: let  $K \subseteq X$  be a compact subset containing the supports of  $\mu$  and the  $\mu_i$ s, and define  $L \subseteq Y$  similarly. (These exist by Lemma 1.61.) Then Lemma 1.60 allows us to restrict all measures in sight on  $X$  to  $K$  and restrict all measures in sight on  $Y$  to  $L$  (and all measures on  $X \times Y$  in sight to  $K \times L$ ), thereby reducing to the case that  $X$  and  $Y$  are compact.

The benefit of working with compact spaces is that we are able to use the Stone–Weierstrass theorem. Indeed, note that  $C(X) \otimes C(Y) \subseteq C(X \times Y)$  is a subalgebra separating points, so it is dense. Thus, though we want to check that  $\{\mu_i \boxtimes \nu_j(h)\} \rightarrow \mu \boxtimes \nu(h)$  for any  $h \in C(X \times Y)$ , we may find some  $h' \in C(X) \otimes C(Y)$  for which  $\|h - h'\|_\infty < \varepsilon$  and then observe that

$$|\mu \boxtimes \nu(h) - \mu_i \boxtimes \nu_j(h)| \leq |\mu \boxtimes \nu(h') - \mu_i \boxtimes \nu_j(h')| + (\|\mu\| \cdot \|\nu\| + \|\mu_i\| \cdot \|\nu_j\|) \|h - h'\|_\infty.$$

All the norms are uniformly bounded by the Uniform boundedness principle, so we see that it is now enough to check that  $\{\mu_i \boxtimes \nu_j(h')\} \rightarrow \mu \boxtimes \nu(h')$ .

By linearity, we may further assume that  $h' = fg$  for some  $f \in C(X)$  and  $g \in C(Y)$ . In this case, we see that  $\delta_x \boxtimes \delta_y(fg) = f(x)g(y)$  is a relation bilinear in both coordinates, so we see that  $\mu' \boxtimes \nu'(fg) = \mu'(f)\nu'(g)$  for any  $\mu' \in C(X)$  and  $\nu' \in C(Y)$ . Now,  $\{\mu_i(f)\} \rightarrow \mu(f)$  and  $\{\nu_j(g)\} \rightarrow \nu(g)$  are convergent sequences by the definition of the weak topology, so we conclude that  $\{\mu_i(f)\nu_j(g)\} \rightarrow \mu(f)\nu(g)$  and so  $\{\mu_i \boxtimes \nu_j(fg)\} \rightarrow \mu \boxtimes \nu(fg)$ . ■

**Remark 1.70.** It is worth remembering that, in the compact case, we calculated  $(\mu \boxtimes \nu)(f, g) = \mu(f)\nu(g)$ . Accordingly, this will be true whenever  $f$  and  $g$  are compactly supported.

**Remark 1.71.** One may hope to immediately reduce the claim to the compact case by using a result like  $\text{Meas}_c(X) = \text{colim}_{K \subseteq X} C(K)$ . While it is true that these things are isomorphic as vector spaces, it turns out that they are not homeomorphic because the correct topology to put on the colimit is weaker than the weak topology. Accordingly, if we used this weaker topology on  $\text{Meas}_c(X)$ , then all arguments go through.

**Notation 1.72.** Fix locally compact second countable spaces  $X$  and  $Y$ . Given  $\mu \in \text{Meas}_c(X)$  and  $\nu \in \text{Meas}_c(Y)$ , we define  $\mu \boxtimes \nu \in \text{Meas}_c(X \times Y)$  via Lemma 1.69.

Intuitively,

$$\int_{X \times Y} h d(\mu \boxtimes \nu) = \int_X \int_Y h d\nu d\mu.$$

Here is the sort of thing that we can show with this definition. The main idea of all these proofs is to find a way to reduce to  $\delta$ s.

**Lemma 1.73.** Fix locally compact second countable spaces  $X, Y$ , and  $Z$ . Up to identifying  $(X \times Y) \times Z$  and  $X \times (Y \times Z)$ , we have that  $\boxtimes$  is associative.

*Proof.* The first part of the argument of Lemma 1.69 shows that there is a unique tri-linear, sequentially continuous map

$$\boxtimes: \text{Meas}_c(X) \times \text{Meas}_c(Y) \times \text{Meas}_c(Z) \rightarrow \text{Meas}_c(X \times Y \times Z)$$

such that  $\delta_x \boxtimes \delta_y \boxtimes \delta_z = \delta_{(x,y,z)}$ . (Namely, linearity extends this map to the  $\text{Meas}_c^0$ s, which are then sequentially dense.) However, the two maps  $(\mu_X, \mu_Y, \mu_Z) \mapsto (\mu_X \boxtimes \mu_Y) \boxtimes \mu_Z$  and  $(\mu_X, \mu_Y, \mu_Z) \mapsto \mu_X \boxtimes (\mu_Y \boxtimes \mu_Z)$  both satisfy this relation. ■

**Notation 1.74.** Fix a locally compact second countable topological group  $G$ . Then we define the binary operator  $*$  on  $\text{Meas}_c(G)$  by

$$\mu * \nu := m_*(\mu \boxtimes \nu),$$

where  $m: G \times G \rightarrow G$  is the multiplication map.

**Remark 1.75.** Because  $m_*$  and  $\boxtimes$  are both sequentially continuous, we conclude that  $*$  is also sequentially continuous.

**Example 1.76.** For  $g, h \in G$ , we compute that  $\delta_g * \delta_h$  is  $m_*(\delta_g \boxtimes \delta_h) = \delta_{gh}$ .

**Example 1.77.** Suppose  $G$  is a Lie group. If  $\mu = \varphi dg$  and  $\nu = \psi dg$  are given by compactly supported functions, then we claim that  $\mu * \nu = (\varphi * \psi) dg$ .

*Proof.* To begin, we note from Remark 1.70 that

$$(\mu \boxtimes \nu)(f_1, f_2) = \mu(f_1)\nu(f_2) = \int_G f_1(g)\varphi(g) dg \int_G f_2(g)\psi(g) dg$$

for any pair of compactly supported functions  $f_1, f_2 \in C_c(G)$ . Because  $C_c(G) \otimes C_c(G)$  is dense in  $C(G \times G)$ , we conclude that

$$(\mu \boxtimes \nu)(f) = \int_{G \times G} f(g, h)\varphi(g)\psi(h) dg dh.$$

Taking the pushforward, we conclude

$$(\mu * \nu)(f) = \int_{G \times G} f(gh)\varphi(g)\psi(h) dg dh.$$

By sending  $g \mapsto gh^{-1}$  (via the invariance of the Haar measure), we see that

$$(\mu * \nu)(f) = \int_G f(g) \left( \int_G \varphi(gh^{-1}) \psi(h) dg \right) dg,$$

so the result follows. ■

**Example 1.78.** Suppose  $G$  is a Lie group. If  $\mu = \varphi dg$  for some compactly supported function  $\varphi$ , then  $\varphi * \nu =$ .

**Example 1.79.** For any  $\mu \in \text{Meas}_c(G)$  and  $f \in C(G)$ , we claim that  $\delta_g * \mu(f) = \mu(x \mapsto f(gx))$ . Intuitively, this corresponds to the formula

$$\int_{G \times G} f(xy) d\delta_g(x) d\mu(y) = \int_G f(gy) d\mu(y).$$

Anyway, the map sending  $f$  to  $g^*f: x \mapsto f(gx)$  is continuous by Example A.57, so we are checking for an equality of measures. Because  $\mu \mapsto \delta_g * \mu$  is sequentially continuous, as is  $\mu \mapsto g_*\mu$ , it is enough to check this on the dense subspace  $\text{Meas}_c^0(X)$ , so by linearity, it is enough to check this for  $\mu = \delta_h$  for  $h \in G$ , where this now has no content.

The point of defining  $\text{Meas}_c(G)$  is that it provides some continuous extension of the group algebra. For example, the group algebra has a natural action on any representation.

**Lemma 1.80.** Fix a Fréchet representation  $V$  of a locally countable second countable topological group  $G$ . Then there is a unique map

$$\cdot: \text{Meas}_c(G) \times V \rightarrow V$$

for which  $\delta_g \cdot v = g \cdot v$ .

*Proof.* Because  $\text{Meas}_c^0(G)$  is sequentially dense in  $\text{Meas}_c(G)$  by Proposition 1.68, so it is enough to show continuity by Proposition A.30.

Now, for two convergent nets  $\{\mu_i\} \rightarrow \mu$  and  $\{v_j\} \rightarrow v$ , we would like to show that  $\{\rho(\mu_i)v_j\} \rightarrow \rho(\mu)v$ . Thus, for any seminorm  $p$  on  $V$ , we must show that  $\{p(\rho(\mu_i)v_j - \rho(\mu)v)\} \rightarrow 0$ . However, we note that

$$p(\rho(\mu_i)v_j - \rho(\mu)v) \leq p(\rho(\mu_i - \mu)v) + p(\rho(\mu_i)(v_j - v)).$$

The action map  $\rho(\mu_i): V \rightarrow V$  is certainly continuous, so the right-hand term vanishes as  $j \rightarrow \infty$ .

It remains to deal with the left-hand term  $p(\rho(\mu_i - \mu)v)$ . By replacing  $\mu_i - \mu$  with  $\mu_i$ , we see that we may now assume that  $\mu = 0$ . Thus, given that  $\{\mu_i\} \rightarrow 0$ , we want to show that  $\{p(\rho(\mu_i)v)\} \rightarrow 0$ . We will do this by upper-bounding the map  $\mu \mapsto p(\rho(\mu)v)$  by a continuous function vanishing at  $\mu = 0$ , which will complete the proof. We do this in steps.

1. For any  $\mu \in \text{Meas}_c^0(X)$ , we note that we may expand  $\mu = \sum_{g \in G} c_g \delta_g$ , from which we define  $|\mu| := \sum_{g \in G} |c_g| \delta_g$ . One can check directly with the weak topology that  $|\cdot|$  is continuous.
2. The same expansion  $\mu = \sum_{g \in G} c_g \delta_g$  allows us to see

$$|p(\rho(\mu)v)| \leq \sum_{g \in G} |c_g| p(\rho(g)v).$$

With this in mind, we define  $f \in C(G)$  by  $f(g) := p(\rho(g)v)$ , and then  $|p(\rho(\mu)v)| \leq |\mu|(f)$ . However, the function  $\mu \mapsto |\mu|(f)$  is continuous by the construction of the weak topology, so we are done. ■

**Remark 1.81.** For any  $v \in V$ , we claim that  $(\mu * \nu) \cdot v = \mu \cdot (\nu \cdot v)$ . Indeed, this is true when  $(\mu, \nu)$  can be found in the sequentially dense subspace  $\text{Meas}_c^0(G) \times \text{Meas}_c^0(G)$ , so the result follows from the uniqueness part of Proposition A.30.

**Remark 1.82.** In this situation, we note that the action map  $\rho: \text{Meas}_c^0(G) \rightarrow \text{End}(V)$  is continuous, where the target has been given the strong topology: this amounts to checking that  $\{\mu_i\} \rightarrow \mu$  implies  $\{\rho(\mu_i)v\} \rightarrow \rho(\mu)v$  for all  $v$ . By continuity (via a suitable modification of Proposition A.30), we then have a continuous map

$$\text{Meas}_c(G) \rightarrow \text{End}(V).$$

(Technically, we should check that  $\text{End}(V)$  is complete when given the strong topology; let us content ourselves by allowing non-continuous endomorphisms in  $\text{End}(V)$  for this statement, and then this is not so hard. This is also implied by the Uniform boundedness principle if  $V$  is Banach.) Note that this improves Lemma 1.42!

### 1.3.6 Finite Vectors

In this subsection, we show that certain subspaces of controlled vectors are in fact dense. We will get some utility out of the following construction.

**Definition 1.83 (approximate identity).** Fix a locally compact second countable topological group  $G$  with Haar measure  $dg$ . Then an *approximate identity* is a sequence of functions  $\{\varphi_i\}$  in  $C_c(G)$  satisfying the following two properties.

- We have  $\int_G \varphi_i dg = 1$  for all  $i$ .
- For any open neighborhood  $U$  of 1, we have  $\text{supp } \varphi_i \subseteq U$  for all but finitely many  $i$ .

If  $G$  is a Lie group, we further require that the  $\varphi_i$  are smooth.

**Remark 1.84.** Let's explain why approximate identities exist. Let  $\{U_i\}_{i \in \mathbb{N}}$  be a neighborhood basis for  $1 \in G$ , which is allowed to be countable because  $G$  is first countable. By shrinking the  $U_i$ s, we may as well assume that it is a descending sequence of precompact open subsets. Then a suitable version of Urysohn's lemma (on the  $\overline{U_i}$ s) provides a nonnegative function  $\varphi_i$  in  $C(U_i) \subseteq C_c(G)$ , and scaling allows us to assume that  $\int_G \varphi_i dg = 1$ . (If  $G$  is a Lie group, then the  $U_n$ s can be precompact open balls, and we can define the  $\varphi_i$ s as smooth bump functions.) Now,  $\{\varphi_i\}$  is an approximate identity because the  $\{U_i\}$ s form a descending neighborhood basis of 1.

**Remark 1.85.** The key property of approximate identities is that  $\{\varphi_i dg\} \rightarrow \delta_1$  in the weak topology. Indeed, for  $f \in C(G)$ , we need to show that  $\int_G f \varphi_i dg \rightarrow f(1)$ ; by translation, we may assume that  $f(1) = 0$ . Well, by continuity of  $f$ , any  $\varepsilon > 0$  has an open neighborhood  $U$  of 1 such that  $|f(x)| < \varepsilon$  for all  $x \in U$ . Choosing  $U$  to be precompact, we see that any  $i$  large enough has  $\text{supp } \varphi_i \subseteq U$ , so we may write

$$\left| \int_G f \varphi_i dg \right| \leq \int_U \varepsilon \varphi_i dg = \varepsilon.$$

**Remark 1.86.** Here is a quick application: if  $G$  is compact and  $\varphi \in C(G)$ , then we claim that  $\varphi dg$  acts on a finite-dimensional representation  $V$  by the operator  $\int_G \varphi(g) \rho(g) dg$ . This map  $C(G) \rightarrow \text{End}(V)$  is certainly continuous (which can now be checked on matrix coefficients), and it converges to the correct values for  $\delta$ -distributions by checking on approximate identities (by uniform continuity), so we conclude by density via Proposition A.30.

Here is our first class of well-behaved vectors.

**Definition 1.87 (finite).** Let  $K$  be a compact second countable topological group. Fix a continuous representation  $V$  of  $K$ . Then a vector  $v \in V$  is  $K$ -finite if and only if the vector space

$$\text{span}\{gv : g \in K\}$$

is finite-dimensional. We let  $V_{\text{fin}}$  denote the space of  $K$ -finite vectors.

**Remark 1.88.** Note that  $V_{\text{fin}}$  is in fact a subspace of  $V$  invariant under  $K$ . For example, certainly  $0 \in V$ , and if  $v, w \in V_{\text{fin}}$ , then

$$\text{span}\{agv + bhw : a, b \in \mathbb{F}, g, h \in K\}$$

is finite-dimensional, so  $\text{span}\{v, w\} \subseteq V_{\text{fin}}$ . Lastly,  $v \in V_{\text{fin}}$  implies that  $gv \in V_{\text{fin}}$  because they have the same  $K$ -span.

**Remark 1.89.** We claim that  $v \in V$  is  $K$ -finite if and only if it is  $K^\circ$ -finite. Indeed,  $K$  has only finitely many connected components (because  $K$  is compact), so we can find a finite subset  $S \subseteq K$  representing  $K/K^\circ$ . (Note also that  $K^\circ$  is normal: for any  $g \in K$ , the conjugation automorphism is continuous while fixing the identity and therefore must send  $K^\circ$  to itself.) Now, if  $v$  is  $K$ -finite, then of course it is  $K^\circ$ -finite. On the other hand, if  $v$  is  $K^\circ$ -finite, then the normality of  $K^\circ \subseteq K$  implies that  $sv$  is still  $K^\circ$ -finite for all  $s \in S$ , so the space

$$\text{span}\{gv : g \in K\} = \text{span}\{gs v : g \in K^\circ, s \in S\}$$

continues to be finite-dimensional.

**Proposition 1.90.** Fix a compact second countable topological group  $K$ . For any Fréchet representation  $V$  of  $K$ , the subspace  $V_{\text{fin}}$  is sequentially dense in  $V$ .

*Proof.* By Remark 1.89, we may assume that  $K$  is connected. Now, the trick is to convolve with an approximate identity. Let  $\{\varphi_i\}$  be an approximate identity for  $K$ , which exists by Remark 1.84. For any  $v \in V$ , we note that  $\{(\varphi_i dg)\} \rightarrow \delta_1$  (by Remark 1.85) implies that  $\{(\varphi_i dg) \cdot v\} \rightarrow v$ , where the action map is the one given by Remark 1.82.

It now remains to approximate  $(\varphi_i dg) \cdot v$  by a  $K$ -finite vector. But this is easier because we know how to approximate functions. Indeed, by the Peter–Weyl theorem, we know that the  $K$ -finite vectors of  $L^2(K)$  are given by the dense subspace

$$\bigoplus_{\rho \in \text{IrRep}(K)} \rho \otimes \rho^*$$

of  $L^2(K)$  and are in particular given by continuous functions. Thus, we may find a  $K$ -finite  $\psi_i \in C(K)$  for which  $\|\varphi_i - \psi_i\|_2 < 1/i$  for each  $i$ . It follows that  $\varphi_i - \psi_i \rightarrow 0$  in  $L^2(K)$ , so we also have this limit in  $L^1(K)$  because  $K$  is compact (via Cauchy–Schwarz), so  $\varphi_i f - \psi_i f \rightarrow 0$  in  $L^1(K)$  for any  $f \in C(K)$ , so  $\varphi_i dg - \psi_i dg \rightarrow 0$  in  $\text{Meas}_c(K)$ , so  $\psi_i dg \rightarrow \delta_1$  in  $\text{Meas}_c(K)$ .

Thus, we once again have  $(\psi_i dg) \cdot v \rightarrow v$  as  $i \rightarrow \infty$ , and we can now show that  $(\psi_i dg) \cdot v$  is  $K$ -finite. Indeed, for any  $h \in G$ , we note that

$$h \cdot (\psi_i dg \cdot v) = (\delta_h * \psi_i dg) \cdot v$$

by the associativity of our action (which, as usual, can be checked on  $\delta$ -distributions). But now  $\delta_h * \psi_i dg = h_*(\psi_i dg)$  by Example 1.79, which we claim is  $h\psi_i dg$ . Indeed, for any  $f \in C(G)$ , we see that

$$h_*(\psi_i dg)(f) = \int_G f(hg)\psi_i(g) dx = \int_G f(g)\psi_i(h^{-1}g) dg.$$

Thus, the span of the vectors  $h \cdot (\psi_i dg \cdot v)$  equals the span of the vectors  $(h\psi_i dg) \cdot v$ , which is finite-dimensional because  $\psi_i$  is  $K$ -finite! ■

**Corollary 1.91.** Fix a compact second countable topological group  $K$ . Any irreducible representation of  $K$  is finite-dimensional.

*Proof.* Proposition 1.90 shows that there is a nonzero  $K$ -finite vector, which spans a finite-dimensional subrepresentation. But then this subrepresentation must be the whole space by irreducibility! ■

### 1.3.7 Smooth Vectors

For our next application, we pass to Lie groups.

**Definition 1.92 (smooth).** Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , and fix a Fréchet representation  $V$  of  $G$ . For each  $X \in \mathfrak{g}$  and  $v \in V$ , we define

$$Xv := \lim_{t \rightarrow 0} \frac{\exp(tX)v - v}{t}$$

provided that the limit exists. If  $Xv$  exists for all  $X \in \mathfrak{g}$ , then we say that  $v$  is  $C^1$ . Proceeding inductively, for each  $j \geq 1$ , we see that  $v$  is  $C^{j+1}$  if and only if  $v$  is  $C^j$  and  $Xv$  is  $C^j$  for all  $X \in \mathfrak{g}$ . Lastly, we say that  $v$  is *smooth* or  $C^\infty$  if and only if  $v$  is in  $C^j$  for all  $j \geq 1$ .

**Remark 1.93.** It turns out to be equivalent to require that the map  $G \rightarrow V$  defined by  $g \mapsto gv$  is smooth. Because we will avoid talking about differentiation in Fréchet spaces, this remark does not have mathematical meaning currently: what does it mean for the map  $G \rightarrow V$  to be smooth? We will instead work with the above more concrete definition.



**Remark 1.94.** Equivalently, the existence of  $Xv$  is asking for the function  $g \mapsto gv$  to have a derivative in the direction of  $X \in \mathfrak{g}$  at  $1 \in G$ . To avoid talking about derivatives in Fréchet spaces too much, we remark that this implies that the map  $g \mapsto \ell(gv)$  has a derivative in the direction of  $X \in \mathfrak{g}$  for any continuous linear functional  $\ell \in V^*$ , so  $v$  being smooth implies that  $g \mapsto \ell(gv)$  is smooth for each  $\ell \in V^*$ .

**Example 1.95.** Suppose that  $G$  is compact, and consider the Banach space representation  $V = C(G)$  where  $g$  acts by  $(gf)(x) := f(xg)$ . Then we see that  $Xf$  should be the uniform limit of the functions  $(\exp(tX)f - f)/t$  as  $t \rightarrow 0$ , which on points  $x \in G$  must be the directional derivative at  $x \in G$  in the direction  $X \in \mathfrak{g}$  (say, in smooth charts of  $G$  defined locally by  $\exp: \mathfrak{g} \rightarrow G$ ). Thus,  $f \in C(G)$  is  $C^1$  if and only if  $f$  is continuously differentiable. Iterating this argument, we find that  $f$  is  $C^j$  if and only if it has  $j$  derivatives with the  $j$ th derivative being continuous, so  $f$  is smooth if and only if  $f: G \rightarrow \mathbb{C}$  is smooth as a map of manifolds.

**Example 1.96.** Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , and consider the action of  $G$  on  $C^\infty(G)$  given by  $(gf)(x) := f(xg)$ . Then we see that we have  $X \in \mathfrak{g}$  acting by derivative in the  $X$  direction (according to the extended left-invariant vector field generated by  $X$ ), which is a continuous map once  $C^\infty(G)$  has been given the correct Fréchet topology. A direct calculation with the formula

$$Xf(x) = \left. \frac{d}{dt} f(x \exp(tx)) \right|_{t=0}$$

can verify that  $[X, Y]f = X(Yf) - Y(Xf)$  by a consideration of the Taylor expansion of  $f \circ \exp$ , so we see that the action of  $\mathfrak{g}$  gives a Lie algebra representation in this case.

The purpose of defining  $V_{\text{sm}}$  is that it (by definition) gives us an action by  $\mathfrak{g}$ . Here are the checks on this action.

**Lemma 1.97.** Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , and fix a Fréchet representation  $V$  of  $G$ . Then  $V_{\text{sm}} \subseteq V$  is a subspace, and the map  $\mathfrak{g} \rightarrow \text{End}(V_{\text{sm}})$  given by  $X: v \mapsto Xv$  is a Lie algebra homomorphism.

*Proof.* By definition, we see that  $X0 = 0$  and  $X(av + bw) = aXv + bXw$  for each  $X \in \mathfrak{g}$  and scalars  $a$  and  $b$  and  $v, w \in V$ . Thus, we see that  $0 \in V_{\text{sm}}$  and that  $V_{\text{sm}}$  is closed under linear combination (because the vectors in each  $C^j$  are by induction), so  $V_{\text{sm}}$  is a subspace. Additionally, this means that  $X$  acts linearly on  $V_{\text{sm}}$ .

It remains to show that the map  $\mathfrak{g} \rightarrow \text{End}(V_{\text{sm}})$  is a morphism of Lie algebras. For this, we will use a trick, attempting to pass to Example 1.96. The point is that we can separate out vectors in  $V$  by evaluating on continuous linear functionals. For any fixed  $\ell \in V^*$ , we claim that the map  $e_\ell: V_{\text{sm}} \rightarrow C^\infty(G)$  given by sending  $v$  to  $e_\ell(v): g \mapsto \ell(gv)$  preserves the  $\mathfrak{g}$ -action. (The output is in  $C^\infty(G)$  by Remark 1.94.) Well, for any  $X \in \mathfrak{g}$  and  $v \in V$ , we must check that  $Xe_\ell(v) = e_\ell(Xv)$ , for which we pick up some  $g \in G$  and evaluate

$$\begin{aligned} Xe_\ell(v)(g) &= \left. \frac{d}{dt} e_\ell(v)(g \exp(tX)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \ell(g \exp(tX)v) \right|_{t=0} \\ &\stackrel{*}{=} \ell \left( g \cdot \left. \frac{d}{dt} \exp(tX)v \right|_{t=0} \right) \\ &= \ell(g \cdot Xv) \\ &= e_\ell(Xv)(g), \end{aligned}$$

where the key equality  $\stackrel{*}{=}$  holds by continuity of  $\ell$  and  $g$ .

We now begin our checks.

- Scalar multiplication: note the map  $\mathfrak{g} \rightarrow \text{End}(V_{\text{sm}})$  preserves scalar multiplication (namely,  $X(av) = a(Xv)$ ) by rearranging the limit  $X(av)$  as

$$a \cdot \lim_{at \rightarrow 0} \frac{\exp(taX)v - v}{t} = a(Xv).$$

- Additive: for  $X, Y \in \mathfrak{g}$ , we need to show that  $(X + Y)v = Xv + Yv$ . By Corollary A.51, it is enough to show that  $\ell((X + Y)v) = \ell(Xv) + \ell(Yv)$  for every  $\ell \in V^*$ . However,  $\ell(Xv) = (Xe_\ell(v))(1)$  as discussed in the previous paragraph, so the result follows by seeing  $(X + Y)e_\ell = Xe_\ell + Ye_\ell$  by the additivity of taking derivatives.
- Bracket: for  $X, Y \in \mathfrak{g}$ , we need to show that  $[X, Y]v = XYv - YXv$ . Once again, by Corollary A.51, it is enough to show that  $\ell([X, Y]v) = \ell(XYv - YXv)$  for every  $\ell \in V^*$ . However,  $\ell(Xv) = (Xe_\ell(v))(1)$  as discussed in the previous paragraph, so the result follows by seeing  $[X, Y]e_\ell = XYe_\ell - YXe_\ell$  by definition of the Lie bracket on vector fields. ■

**Lemma 1.98.** Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , and fix a Fréchet representation  $V$  of  $G$ . If  $v \in V^1$  is  $C^1$ , then for any  $g \in G$  and  $X \in \mathfrak{g}$ , we have

$$g(Xv) = (\text{Ad}_g X)(gv),$$

where  $\text{Ad}_\bullet : G \rightarrow \text{GL}(\mathfrak{g})$  is the adjoint representation.

*Proof.* The main point is that applying  $g$  to a vector in  $V$  is a linear map, so it passes through derivatives. More precisely, we know  $g(Xv)$  equals

$$g \left( \left. \frac{d}{dt} \exp(tX)v \right|_{t=0} \right) = \left. \frac{d}{dt} (g \exp(tX)g^{-1})(gv) \right|_{t=0}.$$

Now,  $g \exp(tX)g^{-1} = \exp(t \text{Ad}_g X)$  by properties of the exponential (and the adjoint), so the result follows. ■

**Remark 1.99.** As an application, we note that  $v \in C^1$  implies that  $gv$  is in  $C^1$  because Lemma 1.98 tells us that we have all our derivatives. By an induction, we conclude that  $V_{\text{sm}} \subseteq V$  is  $G$ -invariant!

We now move towards a density result.

**Lemma 1.100.** Fix a Fréchet representation  $V$  of the Lie group  $G$ . For any  $\varphi \in C_c^\infty(G)$  and  $v \in V$ , the vector  $\varphi dg \cdot v$  is smooth if  $v$  is smooth. In fact, for every  $X \in \mathfrak{g}$ , we have

$$X(\varphi dg \cdot v) = (X\varphi dg) \cdot v.$$

*Proof.* By induction on  $v$  being  $C^j$  for various  $j$ , it is enough to just show the last claim for a given  $v \in V$  and  $\varphi \in C_c^\infty(G)$  and  $X \in \mathfrak{g}$ . To begin, we write

$$\begin{aligned} X(\varphi dg \cdot v) &= \left. \frac{d}{dt} \exp(tX)(\varphi dg \cdot v) \right|_{t=0} \\ &= \left. \frac{d}{dt} \delta_{\exp(tX)} \cdot (\varphi dg \cdot v) \right|_{t=0} \\ &= \left. \frac{d}{dt} (\delta_{\exp(tX)} * \varphi dg) \cdot v \right|_{t=0}, \end{aligned}$$

where we have used Remark 1.81 in the last step. Now, because the action of measures on  $V$  is continuous in  $V$  (by Lemma 1.80), we may move the  $v$  out to see that this equals

$$\left. \frac{d}{dt} (\delta_{\exp(tX)} * \varphi dg) \right|_{t=0} \cdot v.$$

It remains to show that  $\left. \frac{d}{dt} (\delta_{\exp(tX)} \cdot \varphi dg) \right|_{t=0} = X\varphi dg$ . It is enough to check this on any given function  $f \in C(G)$  because  $\text{Meas}_c(G)$  has the weak topology, so we see that

$$(\delta_{\exp(tX)} \cdot \varphi dg)(f) = \int_G f(\exp(tX)g) \varphi(g) dg$$

by Example 1.79, which is just

$$(\delta_{\exp(tX)} \cdot \varphi dg)(f) = \int_G f(g) \varphi(\exp(-tX)g) dg$$

because the Haar measure is left-invariant. We now see that taking the derivative with respect to  $t$  at  $t = 0$  will pass through the integral sign (for example, the Dominated convergence theorem applies because  $\varphi$  is compactly supported), revealing that

$$\left. \frac{d}{dt} (\delta_{\exp(tX)} \cdot \varphi dg)(f) \right|_{t=0} = \int_G f(g) X\varphi(g) dg,$$

which of course is  $(X\varphi dg)(f)$ ! ■

**Proposition 1.101.** Fix a Fréchet representation  $V$  of the Lie group  $G$ . Then  $V_{\text{sm}}$  is dense in  $V$ .

*Proof.* The argument is similar to Proposition 1.90. Fix a vector  $v \in V$  which we would like to approximate. By Remarks 1.84 and 1.85, we are granted smooth functions  $\{\varphi_i\}_{i \in \mathbb{N}}$  of compact support for which  $\{\varphi_i dg\} \rightarrow \delta_1$  in the weak topology. By the continuity of Remark 1.82, we see that  $\{\varphi_i dg \cdot v\} \rightarrow v$ . However, we are now done because the vectors  $\varphi_i dg \cdot v$  are smooth by Lemma 1.100! ■

Here is an application back to finite-dimensional representations!

**Corollary 1.102.** Fix a finite-dimensional continuous representation  $V$  of a Lie group  $G$ . For any  $v \in V$  and  $\ell \in V^*$ , the map  $G \rightarrow \mathbb{C}$  defined by  $g \mapsto \ell(gv)$  is smooth.

*Proof.* By definition of smooth vectors, it is enough to show that  $V_{\text{sm}} = V$ . But for finite-dimensional spaces, subspaces are automatically closed, so this follows from Proposition 1.101. ■

## 1.4 September 16

Today, we walk the roads laid by Harish-Chandra.

### 1.4.1 Admissible Representations

For today,  $G$  is a connected Lie group,  $K \subseteq G$  is a compact Lie subgroup, and  $\mathfrak{g}$  is the Lie algebra of  $G$ . We will soon take  $G$  to be reductive and  $K$  to be a maximal compact subgroup, but we will not have to do this for quite a while.

**Notation 1.103.** Fix a Fréchet representation  $V$  of a compact Lie subgroup  $K$ . Given an irreducible representation  $\rho$  of  $K$ , we may write  $V_\rho$  for the image of the canonical map  $\text{Hom}_K(\rho, V) \otimes \rho \rightarrow V$ .

**Remark 1.104.** The canonical map is given by evaluation: we send  $\varphi \otimes v \in V_\rho \otimes \rho$  to  $\varphi(v) \in V$ . Note that this map is  $K$ -invariant: for any  $g \in K$ , we have  $g(\varphi \otimes v) = \varphi(gv) = g\varphi(v)$ .

**Lemma 1.105.** Fix a Fréchet representation  $V$  of a compact Lie group  $K$ . Then the image of the canonical map

$$\bigoplus_{\rho \in \text{IrRep}(K)} V_\rho \rightarrow V$$

is exactly  $V_{\text{fin}}$ .

*Proof.* We have two inclusions to show. Quickly, to show that the image of the canonical map is contained in  $V_{\text{fin}}$ , we note that it is enough to show that  $\varphi(v) \in V_{\text{fin}}$  for any  $\varphi: \rho \rightarrow V$  and  $v \in V$ . Indeed, for any  $g \in K$ , we see that  $g\varphi(v) = \varphi(gv)$ , so

$$\text{span}\{g\varphi(v) : g \in K\} \subseteq \text{im } \varphi,$$

which is finite-dimensional because the domain of  $\varphi$  is finite-dimensional.

Thus, it remains to show that the canonical map surjects onto  $V_{\text{fin}}$ . Fix some  $v \in V$  and consider the subspace  $W$  spanned by the vectors  $gv$  as  $g$  varies over  $G$ . It is enough to show that  $W$  is in the image of our canonical map.

The main point is to reduce to the case that  $W$  is irreducible. We know that  $W$  is finite-dimensional, and because  $K$  is compact, we know that finite-dimensional representations of  $K$  are semisimple, so we may write  $W$  as a direct sum of irreducible representations. For example, we may write

$$W = \bigoplus_{i=1}^n W_i,$$

where the  $W_i$ s are irreducible representations of  $K$ , and so we see that it is enough to show that each  $W_i \subseteq W \subseteq V$  is in the image of the canonical map. Thus, we may assume that  $W$  is irreducible.

Now, choose  $\rho \in \text{IrRep}(K)$  so that  $W \cong \rho$ . Then we see that there is a map  $\varphi: \rho \rightarrow V$  with image exactly  $W$  (given by the isomorphism  $\rho \rightarrow W$ ), so  $W$  is given by the image of the canonical map on the vectors  $\{\varphi \otimes v : v \in \rho\}$ . ■

Now that we have basically realized the  $V_\rho$ s as summands of  $V$ , one may hope to find a projector back down to  $V$ .

**Lemma 1.106.** Fix a Fréchet representation  $V$  of a compact Lie group  $K$ . For an irreducible representation  $\rho$  of  $K$ , let  $\chi_\rho: K \rightarrow \mathbb{C}$  denote its character. Then the operator  $\bar{\chi}_\rho dg: V \rightarrow V$  fixes  $V_\rho \subseteq V$  and annihilates  $V_{\rho'} \subseteq V$  for any other irreducible representation  $\rho'$ .

*Proof.* The main point is to show that  $\bar{\chi}_\rho dg$  fixes  $V_\rho \subseteq V$ . Well, select some  $\varphi \in V_\rho$  and  $v \in \rho$ , and we want to show that  $\chi_\rho dg$  fixes  $\varphi(v)$ . Because  $\varphi$  is  $K$ -invariant, we see that  $\varphi$  commutes with the action of  $\text{Meas}_c^0(K)$  and therefore commutes with  $\text{Meas}_c(K)$  by sequential density. Thus,

$$\chi_\rho dg \cdot \varphi(v) = \varphi(\bar{\chi}_\rho dg \cdot v).$$

It remains to show that  $\bar{\chi}_\rho dg$  fixes  $v \in \rho$ . Well, by Remark 1.86, we know that  $\bar{\chi}_\rho dg$  acts by the operator  $\int_G \bar{\chi}_\rho(g) \rho(g) dg$ , which is the identity by Example B.42.

Thus, we have shown that  $\bar{\chi}_\rho dg$  fixes  $V_\rho$ . Then for any other distinct irreducible representation  $\rho'$  of  $K$ , we analogously know that  $\bar{\chi}_{\rho'} dg$  fixes  $V_{\rho'}$ , so

$$\bar{\chi}_\rho dg|_{V_{\rho'}} = (\bar{\chi}_\rho dg * \bar{\chi}_{\rho'} dg)|_{V_{\rho'}},$$

where we have used the associativity of Remark 1.81. However, a calculation with the definition of  $*$  (via Example 1.77) shows that

$$\bar{\chi}_\rho dg * \bar{\chi}_{\rho'} dg = \overline{\chi_\rho * \chi_{\rho'}} dg,$$

which vanishes when  $\rho \not\cong \rho'$  by Corollary B.41. ■

As an application, we refine Proposition 1.101.

**Proposition 1.107.** Suppose  $K$  is a compact Lie subgroup of a Lie group  $G$ . For any Fréchet representation  $V$  of  $G$ , we claim that  $V_\rho \cap V_{\text{sm}}$  is dense in  $V_\rho$  for each irreducible representation  $\rho$  of  $V$ .

*Proof.* Choose some  $v \in V_\rho$  to approximate by smooth vectors. Following Proposition 1.101, we choose an approximate identity  $\{\varphi_i\}_i$  so that  $\{\varphi_i dg\} \rightarrow \delta_1$ . Thus, because  $\bar{\chi}_\rho dg$  fixes  $v$  automatically, we see that

$$\lim_{i \rightarrow \infty} (\varphi_i dg * \bar{\chi}_\rho dg) \cdot v = v,$$

where we have silently used the associativity of Remark 1.81. Using the ambient compactness of everything (we may as well pass to the compact subset of  $G$  given by the product of the support of  $\varphi_i$  and  $K$ ), it now follows that

$$\varphi_i dg * \bar{\chi}_\rho dg = (\varphi_i * \bar{\chi}_\rho) dg$$

by Example 1.77. (Note  $\bar{\chi}_\rho$  is technically only in  $L^1(G)$ , but this convolution result still holds via a continuity argument.) Thus, Lemma 1.100 allows us to complete the proof as soon as we know that  $\varphi_i * \bar{\chi}_\rho$  is smooth, which follows because  $\varphi_i$  is smooth (and differentiation under the integral sign in local coordinates). ■

**Remark 1.108.** By summing along Lemma 1.105, we see that Proposition 1.107 implies that  $V_{\text{fin}} \cap V_{\text{sm}}$  is dense in  $V_{\text{fin}}$  and hence dense in  $V$  by Proposition 1.90.

Finite-dimensionality arguments are important above, so we introduce a suitable size constraint.

**Definition 1.109 (admissible).** Fix a Lie group  $G$  with compact Lie subgroup  $K$ . A Fréchet representation  $V$  is  $K$ -admissible if and only if

$$\dim V_\rho < \infty$$

for each irreducible representation  $\rho$  of  $K$ . We may just say that  $V$  is *admissible* if  $K$  is understood from context.

**Example 1.110.** Take  $G = \text{SL}(2, \mathbb{R})$  and  $K = \text{O}(2)$ . Let  $L_s$  be the  $s$ -densities of  $\text{SL}(2, \mathbb{R})$  acting on  $\mathbb{RP}^1$ . It turns out that the action of  $K$  on  $\mathbb{RP}^1$  is transitive, and one can find that  $L_s$  is admissible because  $\dim \text{Hom}_K(\rho, L_s) \leq 1$  for all irreducible representations  $\rho$  of  $K$ .

We will shortly see that the admissible representations are the ones that admit passage to algebra by using the  $K$ -finite vectors. In fact, admissibility is essentially the hypothesis that allows us to just use  $K$ -finite vectors without worrying too much about smoothness.

**Lemma 1.111.** Fix a Lie group  $G$  with compact Lie subgroup  $K$ , and fix a Fréchet representation  $V$  of  $G$ .

- (a) If  $V$  is admissible, then  $V_{\text{fin}} \subseteq V_{\text{sm}}$ .
- (b) The subspace  $V_{\text{fin}} \cap V_{\text{sm}}$  is invariant under the action of  $\mathfrak{g}$ .

*Proof.* We show (a) and (b) separately.

- (a) By Lemma 1.105, it is enough to check that  $V_\rho \subseteq V_{\text{sm}}$  for each irreducible representation  $\rho$  of  $V$ . Well, Proposition 1.107 assures us that  $V_\rho \cap V_{\text{sm}}$  is dense in  $V_\rho$ . But  $V$  is admissible, so equality follows because  $V_\rho$  is finite-dimensional!

- (b) Fix some  $v \in V_{\text{fin}} \cap V_{\text{sm}}$ , and we want to show that  $Xv \in V_{\text{fin}} \cap V_{\text{sm}}$  as well for any  $X \in \mathfrak{g}$ . Certainly  $Xv \in V_{\text{sm}}$  by definition of smoothness, so the main point is to show that  $Xv \in V_{\text{fin}}$ . Well, for any  $g \in K$ , Lemma 1.98 tells us that  $g(Xv) = (\text{Ad}_g X)(gv)$ , so it follows that

$$\text{span}\{g(Xv) : g \in K\} \subseteq \text{span}\{Y(gv) : Y \in \mathfrak{g} \text{ and } g \in K\}.$$

However, the latter space is finite-dimensional because it is the image of the finite-dimensional space  $\text{span}\{gv : g \in G\}$  under a finite-dimensional space  $\mathfrak{g}$  of operators. Thus,  $Xv \in V_{\text{fin}}$ . ■

**Remark 1.112.** One does not expect  $V_{\text{fin}}$  to admit an action by  $G$  in general, so it is perhaps remarkable that we managed to retain a  $\mathfrak{g}$ -action.

### 1.4.2 $(\mathfrak{g}, K)$ -Modules

Given a Lie group  $G$  with compact Lie subgroup  $K \subseteq G$ , we now see that any Fréchet representation  $V$  yields an interesting subspace  $V_{\text{fin}} \subseteq V$  with both a  $K$ -action and a  $\mathfrak{g}$ -action. We now codify this sort of information into an algebraic object.

**Definition 1.113 (Harish-Chandra pair).** A *Harish-Chandra pair* is a pair  $(\mathfrak{g}, K)$  of a Lie algebra  $\mathfrak{g}$  and a compact Lie group  $K$  equipped with a map  $\varphi: \text{Lie } K \rightarrow \mathfrak{g}$  and an action map  $\text{Ad}_\bullet: K \rightarrow \text{GL}(\mathfrak{g})$  such that

$$d \text{Ad}_e(X)(Y) = [X, Y]$$

for any  $X \in \text{Lie } K$  and  $Y \in \mathfrak{g}$ .

**Example 1.114.** If  $G$  is a Lie group with compact Lie subgroup  $K$ , then the embedding  $\text{Lie } K \subseteq \text{Lie } G$  makes  $(\text{Lie } G, K)$  into a Harish-Chandra pair. Indeed, we can define the action map  $\text{Ad}_\bullet: K \rightarrow \text{GL}(\text{Lie } G)$  as the restriction of the usual adjoint map  $\text{Ad}_\bullet: G \rightarrow \text{GL}(\text{Lie } G)$ .

**Definition 1.115  $(\mathfrak{g}, K)$ -module.** Fix a Lie algebra  $\mathfrak{g}$  and compact Lie group  $K$  for which  $(\mathfrak{g}, K)$  is a Harish-Chandra pair.

- (a) A  $(\mathfrak{g}, K)$ -module is a vector space  $M$  with actions by  $K$  and  $\mathfrak{g}$  such that  $M$  is a direct sum of finite-dimensional continuous representations of  $K$ , and the two induced actions of  $\text{Lie } K$  coincide: for any  $X \in \text{Lie } K$ , we have

$$\varphi(X)v = \left. \frac{d}{dt} \exp(tX)v \right|_{t=0}.$$

Furthermore, we require that  $g(Xv) = (\text{Ad}_g X)(gv)$  for all  $g \in K$  and  $X \in \mathfrak{g}$  and  $v \in M$ .

- (b) A  $(\mathfrak{g}, K)$ -module  $M$  is *admissible* if and only if  $\dim \text{Hom}_K(\rho, M)$  is finite for all irreducible representations  $\rho$  of  $K$ .
- (c) An admissible  $(\mathfrak{g}, K)$ -module  $M$  is *Harish-Chandra* if and only if it is finitely generated over  $U\mathfrak{g}$ .

**Example 1.116.** Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$  and compact Lie subgroup  $K$ . Then Example 1.114 tells us that the embedding  $\text{Lie } K \subseteq \mathfrak{g}$  makes  $(\mathfrak{g}, K)$  into a Harish-Chandra pair. Now, for any Fréchet representation  $V$  of  $G$ , we claim

$$V_{\text{fin}} \cap V_{\text{sm}}$$

is a  $(\mathfrak{g}, K)$ -module. Indeed, it has a  $\mathfrak{g}$ -action by Lemma 1.111 and a  $K$ -action by Remark 1.99. The identity  $Xv = \left. \frac{d}{dt} \exp(tX)v \right|_{t=0}$  for  $X \in \text{Lie } K$  follows by definition of the  $\mathfrak{g}$ -action, and  $g(Xv) = (\text{Ad}_g X)(gv)$  follows by Lemma 1.98.

**Example 1.117.** Continuing from the set-up of Example 1.116, we note that if  $V$  is admissible, then  $V_{\text{fin}} \cap V_{\text{sm}} = V_{\text{fin}}$  by Lemma 1.111. Furthermore,  $V_{\text{fin}}$  has  $\dim \text{Hom}_K(\rho, V_{\text{fin}}) < \infty$  for all irreducible representations  $\rho$  of  $K$  because  $V$  is admissible, so we conclude that  $V_{\text{fin}}$  is an admissible  $(\mathfrak{g}, K)$ -module.

**Remark 1.118.** One should really think of a  $(\mathfrak{g}, K)$ -module as something algebraic. Complex representations of  $\mathfrak{g}$  more or less reduce to representations of  $\mathfrak{g}_{\mathbb{C}}$ . Furthermore, finite-dimensional representations of  $K$  are basically the algebraic ones, which also pass to  $K_{\mathbb{C}}$ . Thus, in our definition of  $(\mathfrak{g}, K)$ -module, we may as well make everything complex and require the  $K$ -action on  $M$  to be algebraic.

As an aside, we note that the inclusion of the hypothesis  $g(Xv) = (\text{Ad}_g X)(gv)$  is just a byproduct of us not currently requiring  $K$  to be connected.

**Lemma 1.119.** Fix a Lie algebra  $\mathfrak{g}$  and compact Lie group  $K$  for which  $(\mathfrak{g}, K)$  is a Harish-Chandra pair. Suppose that we have a vector space  $M$  with actions by  $\mathfrak{g}$  and  $K$  such that  $M$  is a direct sum of finite-dimensional continuous representations of  $K$ , and

$$\varphi(X)v = \left. \frac{d}{dt} \exp(tX)v \right|_{t=0}$$

for any  $X \in \mathfrak{g}$  and  $v \in M$ . If  $K$  is connected, then we have

$$g(Xv) = (\text{Ad}_g X)gv$$

for any  $g \in K$  and  $X \in \mathfrak{g}$  and  $v \in M$ .

*Proof.* Let  $H \subseteq K$  be the subset of  $g \in K$  for which the identity holds for any  $X \in \mathfrak{g}$  and  $v \in M$ . We would like to show that  $H = K$ . Because  $K$  is connected, it is enough to show that  $H$  is closed under multiplication and contains an open neighborhood of the identity; we will run these checks separately.

1. We show that  $H$  is closed under multiplication. Well, if  $g, h \in H$ , then any  $X \in \mathfrak{g}$  and  $v \in M$  has

$$\begin{aligned} (gh)(Xv) &= g(h(Xv)) \\ &= g(\text{Ad}_h X)(hv) \\ &= (\text{Ad}_g \text{Ad}_h X)(ghv) \\ &= (\text{Ad}_{gh} X)(gh)v, \end{aligned}$$

so  $gh \in H$ .

2. We show that  $H$  contains an open neighborhood of the identity. Because  $\exp: \text{Lie } K \rightarrow K$  is a local diffeomorphism, it is enough to check that  $\exp(Y) \in H$  for any  $Y \in \text{Lie } K$ . For this, we fix some  $X \in \mathfrak{g}$  and  $v \in V$  (for now) and define the two paths  $\gamma_1, \gamma_2: \mathbb{R} \rightarrow V$  by

$$\begin{cases} \gamma_1(t) := \exp(tY)X \exp(-tY)v, \\ \gamma_2(t) := (\text{Ad}_{\exp(tY)} X)v. \end{cases}$$

We will complete the proof by showing that  $\gamma_1 = \gamma_2$ . For this, we may a few preparatory remarks. By replacing  $M$  with  $\text{span } K\mathfrak{g}Kv$ , we may assume that  $M$  is finite-dimensional for this check. Then every finite-dimensional representation of a compact Lie group is real analytic (indeed, this follows by tracking through the representation on the exponential map, which itself is real analytic), so  $\gamma_1$  and  $\gamma_2$  are real analytic, so we may compare them only using their Taylor expansions.

It remains to compute some derivatives.

- By passing through  $\varphi$ , we know that the operator  $\exp(tY)$  is the same as  $\exp(t\varphi(Y))$ , so

$$\gamma_1(t) = \exp(t\varphi(Y))X \exp(-t\varphi(Y))v,$$

which expands to the Taylor series

$$\gamma_1(t) = \sum_{i,j=0}^{\infty} \frac{(-1)^j}{i!j!} t^{i+j} \varphi(Y)^i X \varphi(Y)^j v.$$

- By construction of  $\text{Ad}_\bullet$ , we know that  $\gamma'_2(0) = [\varphi(Y), X]v$ . Then the fact that  $\text{Ad}_\bullet$  is a group homomorphism implies that  $\gamma'_2(t) = \text{Ad}_{\exp(tY)}[\varphi(Y), X]v$ , so an induction gives

$$\gamma_2^{(n)}(t) = [\cdots [\varphi(Y), [\varphi(Y), X]] \cdots]v,$$

where there are  $n$  opening braces  $[$  and  $n$  closing braces  $]$ .

A term-by-term comparison of the preceding Taylor series completes the proofs. Indeed, in the expansion of the  $2^n$  terms of  $[\cdots [\varphi(Y), [\varphi(Y), X]] \cdots]$ , we see that are basically choosing some number  $i$  times to use the  $\varphi(Y)X$  term or the  $-X\varphi(Y)$  term in the expansion of  $[\varphi(Y), X]$ , so we receive the term  $\varphi(Y)^i X \varphi(Y)^{n-i}$  an absolute total of  $\binom{n}{i}$  times; this count must then be corrected by the sign  $(-1)^{n-i}$ . ■

Here are the typical sources of  $(\mathfrak{g}, K)$ -modules.

**Example 1.120.** Fix a compact connected real Lie group  $K$ , and we set  $G := K_{\mathbb{C}}$ , which we think of as a real Lie group via restriction of scalars. Then we set  $\mathfrak{g}$  to be  $\text{Lie } G = \mathfrak{k} \otimes \mathbb{C}$ , where  $\mathfrak{k} := \text{Lie } K$ ; note  $\mathfrak{g}_{\mathbb{C}}$  is isomorphic to  $\mathfrak{g} \oplus \mathfrak{g}$ . Using the diagonal action  $\mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ , we receive a Harish-Chandra pair. In this way, we see that a Harish-Chandra module is the same as a  $(\mathfrak{g}, \mathfrak{g})$ -module which is admissible for the diagonal copy of  $\mathfrak{g}$ .

**Example 1.121.** Consider the  $\mathfrak{g}$ -bimodule  $U\mathfrak{g}$ , where the first copy of  $\mathfrak{g}$  acts by  $X: Y \mapsto XY$ , and the second copy of  $\mathfrak{g}$  acts by  $X: Y \mapsto -YX$ . This turns out to not be admissible.

### 1.4.3 Weakly Analytic Vectors

We are going to want access to some weaker smoothness conditions.

**Definition 1.122 (matrix coefficient).** Fix a Fréchet representation  $V$  of a Lie group  $G$ . For any  $v \in V$  and  $\ell \in V^*$ , we define the *matrix coefficient*  $G \rightarrow \mathbb{C}$  by

$$g \mapsto \ell(gv).$$

**Definition 1.123 (weakly analytic).** Fix a Fréchet representation  $V$  of a Lie group  $G$ . A vector  $v \in V$  is *weakly analytic* if and only if the matrix coefficient map  $g \mapsto \ell(gv)$  is real analytic for all  $\ell \in V^*$ .

Here is the main result we will need.

**Theorem 1.124 (Harish-Chandra analyticity).** Fix a semisimple Lie group  $G$  with compact subgroup  $K$ . For each admissible Fréchet representation  $V$  of a Lie group  $G$ , every  $v \in V_{\text{fin}}$  is weakly analytic.

To prove this, one uses elliptic regularity, so we spend a moment establishing some theory.



**Definition 1.125** (differential operator). Fix a smooth manifold  $X$  of dimension  $n$ . Then a *differential operator*  $d$  is a section of the vector bundle which is locally on a chart  $(x_i)$  is given by finite sums

$$\sum_{i: \{1, \dots, n\} \rightarrow \mathbb{N}} \varphi_i \partial x_1^{i_1} \cdots \partial x_n^{i_n}.$$

The *order* of the operator is the maximum of  $\sum_j i_j$  as  $i: \{1, \dots, n\} \rightarrow \mathbb{N}$  varies such that  $\varphi_i \neq 0$ .

**Remark 1.126.** The differential operators form a graded vector bundle. We define the principal symbol  $\sigma(d)$  to be the entry in the associated graded ring.

**Remark 1.127.** We find that  $\sigma(d)$  is a function on  $T^*X$ , and it is a polynomial of degree  $N$ .

**Definition 1.128** (elliptic). The operator  $d$  is *elliptic* if and only if  $\sigma(d)(x, \xi) \neq 0$  for all  $x$  and  $\xi$  nonzero.

## 1.5 September 18

Today, we talk about elliptic operators.

### 1.5.1 Applications of Weakly Analytic Vectors

Here is the local version of our statement.

**Theorem 1.129** (Elliptic regularity). Fix an open subset  $U \subseteq \mathbb{R}^n$  and a smooth function  $f$  on  $U$ . Suppose that  $d$  is an elliptic differential operator with real analytic coefficients and for which  $df = 0$ . Then  $f$  is real analytic.

**Example 1.130.** Take  $n = 1$  so that  $d$  looks like  $\sum_{i=0}^n f_i \partial_x^i$ . Being elliptic amounts to requiring that  $\varphi_n(x) \neq 0$  for all  $x$ , so after cancelling out, we are solving the differential equation

$$f^{(n)} + \varphi_{n-1} f^{(n-1)} + \cdots + \varphi_0 = 0,$$

and we are asking for  $f$  to be real analytic when the  $\varphi_\bullet$ s are. This follows from some theory of first-order differential equations.

**Example 1.131.** With  $U = \mathbb{R}^n$ , we see  $d$  looks like  $\sum_I f_I \partial_x^I$ . Suppose that all the  $f_I$ s are constant except for  $f_0$ , which we ask to be Schwartz. Then by shifting  $f$ , we are trying to solve  $df = f_0$ , which one can solve by taking a Fourier transform.

By a partition of unity argument, one can prove a global statement.

**Corollary 1.132.** Fix a real analytic manifold  $X$  and a smooth function  $f$  on  $X$ . Suppose that  $d$  is an elliptic differential operator with real analytic coefficients and for which  $df = 0$ . Then  $f$  is real analytic.

And here is our application.

*Proof of Theorem 1.124.* We proceed in steps.

1. We construct a differential operator  $C$  on  $G$ . Fix a left-invariant Riemannian metric on  $G$ ; this amounts to the choice of some  $B \in \text{Sym}^2(\mathfrak{g}^*)$ . By suitably averaging over  $K$ , we can assume that it is also right-invariant by  $K$ . Then  $B^{-1}$  is a  $K$ -invariant element of  $\text{Sym}^2(\mathfrak{g})$ , which then maps to some  $K$ -invariant quadratic element  $C$  of  $U\mathfrak{g}$ . Now,  $\mathfrak{g}$  acts by first-order differential operators on  $G$ , so  $U\mathfrak{g}$  acts by higher-order differential operators; for example,  $C$  is a differential operator.
2. We claim that  $C$  is elliptic. Upon identifying  $T^*G \cong G \times \mathfrak{g}^*$  via right translations, the principal symbol of  $C$  is exactly the quadratic element  $B^{-1}$  that we started with: we find  $\sigma(d_C)(g, \xi) = B^{-1}(\xi, \xi)$ , which of course is positive for every  $g$  and  $\xi$ .
3. We complete the proof. Fix  $v$  and  $\xi$ , and we note that  $C(f_{v,\xi}) = f_{C(v),\xi}$  by a direct calculation. Further, the action of  $C$  commutes with the  $K$ -action, so it preserves the finite-dimensional  $K$ -isotypic components. Accordingly, we fix some irreducible representation  $\rho$  of  $K$ , let  $P$  be the minimal polynomial of  $C$  acting on  $V_\rho$ , and we see that  $P(C)f_{v,\xi} = 0$ . But  $P(C)$  is still elliptic (its symbol is just the  $(\deg P)$ th power of the symbol of  $C$ ), so we are done by Corollary 1.132. ■

This has many nice applications.

**Corollary 1.133.** Fix a Lie group  $G$  with compact subgroup  $K \subseteq G$ , and let  $V$  be an admissible Fréchet representation.

- (a) The action of  $G$  on  $V$  is uniquely determined by its action of  $\mathfrak{g}$  on  $V_{\text{fin}}$ .
- (b) Closed subrepresentations of  $V$  are in bijection with  $(\mathfrak{g}, K)$ -submodules of  $V_{\text{fin}}$ .
- (c) The functor  $(-)_{\text{fin}}: \text{Rep}(G) \rightarrow \text{Mod}(\mathfrak{g}, K)$  is faithful.

*Proof.* We show these independently.

- (a) Because  $V_{\text{fin}} \subseteq V$  is dense, the action is determined by the action on the subspace  $V_{\text{fin}}$ . But the action of  $g$  on some  $v \in V_{\text{fin}}$  is then determined by the Taylor series of  $\ell(gv)$  as  $\ell$  varies over all  $\ell \in V^*$  (because  $g \mapsto \ell(gv)$  is real analytic!). These Taylor series coefficients can now be read off of the derivatives.
- (b) In one direction, we can take a closed subrepresentation  $W \subseteq V$  and produce the  $(\mathfrak{g}, K)$ -submodule  $W_{\text{fin}} \subseteq V_{\text{fin}}$ .

In the other direction, for any  $W \subseteq V_{\text{fin}}$  which is a  $(\mathfrak{g}, K)$ -module, we claim that the closure  $\overline{W} \subseteq V$  is a  $G$ -submodule. Well, for  $g \in G$  and  $w \in W$ , we would like to check that  $gw \in \overline{W}$ , for which it is enough to check that  $\ell(gw) = 0$  for every linear functional  $\ell$  vanishing on  $W$ . But the function  $g \mapsto \ell(gw)$  is real analytic, so we may determine it by determining its Taylor expansion at  $g = 1$ , which passes to the Lie algebra action on  $W_{\text{fin}}$  and can be seen to vanish.

Lastly, we need to check that these maps are mutually inverse, which we omit.

- (c) This follows from density of  $V_{\text{fin}}$  in  $V$ . ■

**Remark 1.134.** One cannot expect  $(-)_{\text{fin}}$  to be fully faithful because the same pair  $(\mathfrak{g}, K)$  can arise from different groups  $G$ .

**Remark 1.135.** The proofs of (a) and (b) do not use the  $K$ -action anywhere, so we can actually go down to just  $\mathfrak{g}$ -modules. This produces a stronger statement when  $K$  fails to be connected.

We will also be able to prove a variant of Schur's lemma.

**Lemma 1.136 (Dixmier).** Fix an associative  $\mathbb{C}$ -algebra  $A$ , and suppose that  $M$  is a simple  $A$ -module with at most countable dimension. Then  $\text{End}_A(M) = \mathbb{C}$ .

*Proof.* For brevity, set  $D := \text{End}_A(M)$ . Because  $M$  is simple, we know that  $D$  is a division algebra over  $\mathbb{C}$ : any nonzero element  $\varphi: M \rightarrow M$  must have trivial kernel and full image and therefore be invertible. Further, for any nonzero  $m \in M$ , we see that  $m$  generates  $M$ , so the map  $D \rightarrow M$  given by  $\varphi \mapsto \varphi(m)$  is injective, so  $\dim D \leq \dim M$ . It follows that  $D$  is a division algebra over  $\mathbb{C}$  with at most countable dimension over  $\mathbb{C}$ .

Now, suppose for the sake of contradiction that  $D \neq \mathbb{C}$ . Then because  $\mathbb{C}$  is algebraically closed,  $D$  contains an element  $t \in D$  which is transcendental over  $\mathbb{C}$ . However,  $\mathbb{C}(t)$  has uncountable dimension over  $\mathbb{C}$ ; for example, the elements  $\left\{ \frac{1}{t-a} \right\}_{a \in \mathbb{C}}$  are linearly independent. ■

**Corollary 1.137 (Schur's lemma).** Fix a Lie group  $G$  with compact subgroup  $K \subseteq G$ , and let  $V$  be an admissible Fréchet representation. If  $V$  is irreducible, then  $\text{End}_G(V) = \mathbb{C}$ .

*Proof.* By Corollary 1.133, we see that  $V_{\text{fin}}$  is an irreducible  $(\mathfrak{g}, K)$ -module, from which we see that it is enough to show that  $\text{End}_{\mathfrak{g}}(V_{\text{fin}}) = \mathbb{C}$ . The result now follows from Lemma 1.136 with the algebra  $U\mathfrak{g}$ . ■

**Remark 1.138.** If  $M$  is finite-dimensional, then there is another proof: any endomorphism  $\varphi: M \rightarrow M$  still has an eigenvalue  $\lambda$ , and the eigenspace  $M[\lambda]$  is a nonzero submodule of  $M$ , so  $M[\lambda] = M$ , so  $\varphi = \lambda \text{id}_M$ .

## 1.5.2 The Admissibility Theorem

Here is our next goal.

**Theorem 1.139 (Harish-Chandra).** Fix a real semisimple group  $G$  with maximal compact subgroup  $K$ . If  $V$  is an irreducible unitary representation, then

$$\dim \text{Hom}_K(\rho, V) \leq \dim \rho$$

for each irreducible representation  $\rho$  of  $K$ . In particular,  $V$  is admissible.

**Remark 1.140.** There are Banach space representations of  $\text{SL}(2, \mathbb{R})$  which are not admissible! This has to do with the negative solution to the so-called “Invariant subspace problem.”

This is the first time we actually want  $K$  to be a maximal compact subgroup, which is why we are asking for  $G$  to be semisimple.

Let's begin by recalling some definitions.

**Definition 1.141 (semisimple).** Fix a connected real Lie group  $G$ . Then  $G$  is *semisimple* if and only if  $G$  embeds into some  $\text{GL}(n, \mathbb{R})$  (i.e.,  $G$  is linear) and has semisimple Lie algebra.

**Definition 1.142 (split).** Fix a connected real Lie group  $G$ . Then  $G$  is *split* if and only if  $G$  contains a maximal torus  $T$  isomorphic to  $(\mathbb{R}^\times)^n$ .

**Example 1.143.** The group  $\mathrm{SL}(n, \mathbb{R})$  is split and semisimple with maximal torus given by the diagonal matrices.

**Example 1.144.** The group  $\mathrm{SO}(p, q)$  is semisimple, but it only succeeds at being split when  $p - q \in \{-1, 0, +1\}$ .

Real groups come with a Cartan involution.

**Definition 1.145 (Cartan involution).** Fix a semisimple real Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . Consider the Killing form  $(-, -)$  of a real semisimple Lie algebra  $\mathfrak{g}$ . A *Cartan involution* is an involution  $\theta$  of  $G$  for which the form  $C(-, -) := (-, \theta-)$  is positive-definite.

**Remark 1.146.** It turns out that such involutions exist and are unique up to conjugation.

**Remark 1.147.** If  $(\theta(x), \theta(y)) = (x, y)$ , then we find that  $C(x, y) = C(y, x)$ .

**Remark 1.148.** The subgroup  $K := G^\theta$  acts on  $\mathfrak{g}$  while preserving  $C$ , so it follows that  $K$  embeds into some  $\mathrm{SO}(n)$  and is therefore a compact subgroup. It turns out that  $K$  is a maximal compact subgroup.

**Example 1.149.** With  $G = \mathrm{SL}(n)$ , one finds that  $\theta(g) := g^{-\top}$  is a Cartan involution, so one finds that  $K = \mathrm{SO}(n)$ .

**Example 1.150.** If  $K$  is a compact group, then we can try out  $G = K_{\mathbb{C}}$ , and one can check that we get a Cartan involution  $\theta$  which acts by complex conjugation on  $G$  and therefore on  $\mathrm{Lie} G = \mathrm{Lie} K \otimes \mathbb{C}$  so that  $K = G^\theta$  again.

## 1.6 September 23

Today we continue discussing the admissibility theorem.

### 1.6.1 Real Structure Theory

Last class, we defined the notion of a real semisimple group, but it will be convenient to work in the larger class of real reductive groups. We will take the following convention.

**Definition 1.151 (reductive).** A *real reductive group* is one of the form  $G(\mathbb{R})$  where  $G$  is some connected algebraic group defined over  $\mathbb{R}$  for which  $G(\mathbb{C})$  is complex and reductive. Then a complex reductive group is a direct product of a semisimple group with a torus.

**Remark 1.152.** By descent, it is enough to only provide the data of the connected complex reductive algebraic group  $G_{\mathbb{C}}$  along with an involution  $\theta$  of  $G$ . The real reductive group is then attained as  $G_{\mathbb{C}}^\theta$ .

**Remark 1.153.** For a complex reductive group  $G$ , the commutator  $G'$  is semisimple, and the map

$$G' \times Z(G)^\circ \rightarrow G$$

is surjective with finite kernel.

**Example 1.154.** Technically, we required our real semisimple groups to be connected, which is not preserved under base change. This means that there are real semisimple groups which are not reductive! For example, consider  $\mathrm{PGL}(n, \mathbb{R})$  is not a real semisimple Lie group according to our definitions because it fails to be connected—we should technically work with  $\mathrm{PGL}(n, \mathbb{R})^+$  instead, but that is not an algebraic subgroup and therefore not reductive. Approximately speaking, this is a failure of the group  $\mathrm{PGL}$  to be simply connected.

**Remark 1.155.** In general, given a complex algebraic group  $G_{\mathbb{C}}$  with an involution  $\theta$ , the number of connected components of  $G_{\mathbb{C}}^\theta$  is finite and typically small. This can be studied with Galois cohomology: passing to the simply connected cover  $\tilde{G}$ , it inherits an extended involution (which we also call  $\theta$ ), and then the map  $\tilde{g} \mapsto \tilde{g}^{-1}\theta(\tilde{g})$  is a 2-cocycle in  $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z(\tilde{G}))$ .

**Example 1.156.** Here is a more serious problem: the group  $\mathrm{SL}(n, \mathbb{R})$  is not simply connected—it has fundamental group  $\mathbb{Z}/2\mathbb{Z}$ . Thus, it admits a covering by some  $G \rightarrow \mathrm{SL}(n, \mathbb{R})$ , but this group  $G$  fails to be algebraic and thus fails to be reductive. Indeed,  $\mathrm{Lie} G = \mathrm{Lie} \mathrm{SL}(n, \mathbb{R}) = \mathfrak{sl}(n, \mathbb{R})$ , so any finite-dimensional representation  $G \rightarrow \mathrm{GL}(N, \mathbb{R})$ , which induces  $\mathfrak{sl}(n, \mathbb{R}) \rightarrow \mathfrak{gl}(N, \mathbb{R})$ , but this now extends to  $\mathrm{SL}(n, \mathbb{C}) \rightarrow \mathrm{GL}(N, \mathbb{C})$ . Thus, our representation of  $G$  factors through  $\mathrm{SL}(n, \mathbb{R})$  and thus cannot be faithful.

We are going to assume the following bit of structure theory.

**Theorem 1.157 (Cartan).** Fix a real semisimple complex Lie group  $G$ . Then  $G$  admits a Cartan involution  $\theta: G \rightarrow G$ , and  $\theta$  is unique up to conjugation.

As discussed in Remark 1.148, the fixed points  $K := G^\theta$  form a compact subgroup of  $G$ ; it turns out that  $K^\theta$  is maximal compact.

**Remark 1.158.** Once we are given  $\theta$ , we may extend it to a complex involution  $\theta_{\mathbb{C}}$  of  $G_{\mathbb{C}}$ , and then we see  $K_{\mathbb{C}} = G_{\mathbb{C}}^{\theta_{\mathbb{C}}}$ . There is also a complex anti-linear involution  $\theta_0$  so that  $G_{\mathbb{C}}^{\theta_0} = G$ , which then provides a bijection between real forms and such anti-linear involutions.

## 1.6.2 Proof of the Admissibility Theorem

We now turn our attention towards the proof of Theorem 1.139. The first step is to establish a density result and a Schur's lemma.

**Lemma 1.159.** Fix an associative algebra  $A$ , and let  $V$  be a representation with  $\mathrm{End}_A(V) = \mathbb{C}$ . Then given any linearly independent vectors  $v_1, \dots, v_n$  and  $w_1, \dots, w_n \in V$ , there is an  $a \in A$  for which  $a(v_i) = w_i$  for each  $i$ .

*Proof.* Fix the vector  $v := (v_1, \dots, v_n)$  in  $V^{\oplus n}$ , and we need to show that  $Av = V^{\oplus n}$ . Well, writing  $V^{\oplus n} = \mathbb{C}^{\oplus n} \otimes V$ , it follows from  $\mathrm{End}_A(V) = \mathbb{C}$  that every submodule looks like  $U \otimes V$  where  $U \subseteq \mathbb{C}^n$  is a subspace: the tensor subcategory  $\langle V \rangle^{\otimes} \subseteq \mathrm{Mod}_{\mathbb{C}}(A)$  is seen to be equivalent to  $\mathrm{Vec}_{\mathbb{C}}$ . However,  $U \subsetneq \mathbb{C}^n$  allows us to find a nonzero vector  $u' \in U^\perp$ , which in turn will have  $u' \cdot v = 0$ , which contradicts the linear independence of  $\{v_1, \dots, v_n\}$ . ■

**Proposition 1.160 (Schur's lemma).** Fix an irreducible Hilbert space representation  $V$  of  $\mathbb{C}$  of a Lie group  $G$ . Then any bounded self adjoint operator is a scalar.

*Sketch.* The spectral theorem presents  $V$  as  $L^2(X, \mu)$  where  $T$  corresponds to multiplication by a measurable function  $\varphi$ .

Now, if  $\varphi$  is not a scalar, then  $X$  is not a point, so we can choose some  $f \in C_c^\infty(G)$  for which  $X_0 := (f \circ \varphi)^{-1}(\{0\})$  and  $X_1 := (f \circ \varphi)^{-1}(\{1\})$  both have positive measure, so  $X_0$  is a closed subspace invariant under  $G$ , which is our contradiction. ■

Here is our density result.

**Corollary 1.161 (Density).** Fix an irreducible Hilbert space representation  $V$  of  $\mathbb{C}$  of a Lie group  $G$ . Then for any linearly independent vectors  $v_1, \dots, v_n \in V$  and some  $w_1, \dots, w_n \in V$  and some error bound  $\varepsilon > 0$ , there is  $\varepsilon > 0$  and  $\mu \in \text{Meas}_c(G)$  such that

$$\|\varphi(v_i) - w_i\| < \varepsilon.$$

*Proof.* Once again, fix  $v := (v_1, \dots, v_n)$ , and we would like to check that the closure of  $\text{Meas}_c(G)$  acting on  $v$  achieves all  $V^{\oplus n}$ . Additionally, we still have that all closed subrepresentations of  $V^{\oplus n}$  take the form  $U \otimes V$  for some subset  $U$ . Indeed, if  $W \subseteq V$  is a closed subrepresentation, then there is a bounded self-adjoint projection operator  $\text{pr}: V \rightarrow W$ . But Schur's lemma shows that endomorphisms of  $V^{\oplus n}$  look like matrices, so the result follows after a little work. ■

Next time, we will show the following.

**Proposition 1.162.** Fix a finite-dimensional irreducible representation  $V$  of  $G$ . Then  $\dim \text{Hom}(\rho, V) \leq \dim \rho$  for irreducible representation  $\rho$  of  $K$ .

We now move towards the completion of the proof.

**Lemma 1.163.** For any  $\mu \in \text{Meas}_c(G)$ , there is a finite-dimensional irreducible representation of  $G$  for which the action of  $\mu$  on  $V$  is nonzero.

*Proof.* By smooth approximation to  $\delta_1$ , we may assume that  $\mu = \varphi dg$  for some  $\varphi \in C_c^\infty(G)$ . Indeed, if  $\mu$  acted by 0 on all finite-dimensional irreducible representations, then  $\mu * f dg$  would also act by 0 always.

Now, consider the ring  $\mathbb{C}[G_{\mathbb{C}}]$  of polynomial functions on  $G_{\mathbb{C}}$ , which because  $G$  is reductive, we see that  $\mathbb{C}[G_{\mathbb{C}}]$  must be a sum of finite-dimensional irreducible representations. Now, by Stone–Weierstrass, there is a polynomial  $P \in \mathbb{C}[G_{\mathbb{C}}]$  such that  $|P(g) - f(g)| < \varepsilon$  for every  $g \in \text{supp } f$ . We will thus take the representation generated by  $P$ .

It remains to check that the action of  $\mu$  on  $\overline{P}$  is nonzero, but this is given by

$$\int_G f(g) \overline{P}(g) dg,$$

which is approximately  $\int_G f(g) \overline{f}(g) dg$ , which is of course positive. ■

To conclude, we need the following identity, which we are not going to prove because it is just some pure algebra.

**Theorem 1.164 (Amitsur–Levitzki–Kolchin).** Choose some  $n \times n$  matrices  $A_1, \dots, A_{2n}$ . Then

$$\sum_{\sigma \in S_{2n}} (-1)^{|\sigma|} A_{\sigma(1)} \cdots A_{\sigma(2n)} = 0.$$

## 1.7 September 25

Today we finish our proof of the Admissibility theorem.

### 1.7.1 End of the Proof of Admissibility

Let's explain how we can finish the proof.

*Proof of Theorem 1.139.* Fix a unitary admissible representation  $V$  of  $G$ , and we would like to check that  $\dim \operatorname{Hom}(\rho, V) \leq \dim \rho$  for any irreducible representation  $\rho$  of  $K$ . Of course, this is equivalent to checking that  $\dim V_\rho \leq (\dim \rho)^2$ .

For this, we let  $e_\rho := \chi_\rho dg \in \operatorname{Meas}_c(K)$  be the projector onto the  $\rho$ -isotypic component, which we silently pushforward to  $\operatorname{Meas}_c(G)$ . Now, set

$$\operatorname{Meas}_c(G)^\rho := e_\rho * \operatorname{Meas}_c(G) * e_\rho$$

to be a subalgebra (without the same identity element). Now, for any nonzero  $\mu \in \operatorname{Meas}_c(G)^\rho$ , we see that  $\mu$  only has nonzero action on  $\rho$ -isotypic components, so Lemma 1.163 forces  $\mu$  to act in a nontrivial way on  $W_\rho$  for some irreducible representation  $W$ . The moral is that  $\operatorname{Meas}_c(G)^\rho$  will act faithfully on

$$\bigoplus_{W \in \operatorname{IrRep}(G)} W_\rho,$$

where  $W$  varies over finite-dimensional irreducible representations of  $G$ . Thus, Proposition 1.162 tells us that  $\dim W_\rho \leq (\dim \rho)^2$ . We now apply Theorem 1.164, which tells us that any collection of measures  $\mu_1, \dots, \mu_{2(\dim \rho)^2}$  in  $\operatorname{Meas}_c(G)^\rho$  has

$$\sum_{\sigma \in S_{2n}} (-1)^{|\sigma|} A_{\sigma(1)} \cdots A_{\sigma(2n)}$$

act by 0 on  $V$  and hence on  $V_\rho$ . (Here,  $n = (\dim \rho)^2$ .)

Now, suppose for the sake of contradiction that  $\dim \operatorname{Hom}_K(\rho, V) > (\dim \rho)$  so that we may find a set  $S \subseteq V_\rho$  of  $2(\dim \rho)^2 + 1$  linearly independent vectors, and then Corollary 1.161 allows us to find  $\mu_i$ s approximating matrices  $M_i$  of span  $S$  such that

$$\sum_{\sigma \in S_{2n}} (-1)^{|\sigma|} \mu_{\sigma(1)} \cdots \mu_{\sigma(2n)} \neq 0$$

because now the space is larger. (For example, one can take the  $M_i$ s to be distinct elementary matrices.) On the other hand, projecting down to  $\operatorname{Meas}_c(G)^\rho$  produces a contradiction of the action on  $V_\rho$ . ■

It remains to prove Proposition 1.162, which we will do later.

### 1.7.2 Infinitesimal Equivalence

It will be worthwhile to explain how much a  $(\mathfrak{g}, K)$ -module remembers a representation of  $G$ .

**Definition 1.165** (infinitesimally equivalent). Fix a real connected semisimple Lie group  $G$  with maximal compact subgroup  $K \subseteq G$ . Two admissible representations  $V$  and  $W$  of  $G$  are *infinitesimally equivalent* if and only if  $V_{\text{fin}}$  and  $W_{\text{fin}}$  are isomorphic as  $(\mathfrak{g}, K)$ -modules.

**Remark 1.166.** Certainly if  $V$  and  $W$  are isomorphic representations, then this descends to an isomorphism of the  $(\mathfrak{g}, K)$ -modules  $V_{\text{fin}}$  and  $W_{\text{fin}}$ .

**Remark 1.167.** If  $V$  is unitary and irreducible (and thus admissible by Theorem 1.139), we may grant  $V$  a  $G$ -invariant Hermitian form (which is unique up to scalar by Schur's lemma because a Hermitian form amounts to the data of an isomorphism  $\bar{V} \rightarrow V^*$ ). The same sort of argument shows that  $V_{\text{fin}}$  admits a unique Hermitian pairing by a Schur's lemma argument, but then  $V$  is the completion of  $V_{\text{fin}}$ , so we can recover  $V$  from  $V_{\text{fin}}$ . Thus, infinitesimal equivalence implies isomorphism in this case.

**Remark 1.168.** There are principal series representations for  $\text{SL}(2, \mathbb{R})$  which are infinitesimally equivalent but not isomorphic: one can use  $L^p(\mathbb{RP}^1)$  for  $p > 1$  or  $C^k(\mathbb{RP}^1)$  or  $C^\infty(\mathbb{RP}^1)$  and so on, which all have the same underlying  $(\mathfrak{g}, K)$ -module but complete it to a Fréchet space in different ways.

**Remark 1.169.** In general, it is a difficult problem to take a  $(\mathfrak{g}, K)$ -module  $M$  and ask if there actually is a representation  $V$  with  $M \cong V_{\text{fin}}$ . This is answered positively (with some growth conditions) by the Casselman–Wallach globalization theorem.

### 1.7.3 Proof of Algebraic Finiteness

We now turn to the proof of Proposition 1.162. To begin, we take complexifications: a finite-dimensional representation of  $G$  is algebraic, so we get an algebraic representation of  $G_{\mathbb{C}}$ . Our proof will require two big inputs. The first input is a structure result on our representations.

**Theorem 1.170 (Borel–Weil).** Fix a complex semisimple Lie algebra  $G$ . For each algebraic representation  $V$  of  $G$  of highest weight  $\lambda$ , then  $V_\lambda$  is the global sections of a line bundle  $\mathcal{L}_\lambda$  on  $G/B$ , where  $B$  is the Borel subgroup. Equivalently,  $V_\lambda$  is contained in  $\text{Ind}_B^G \chi$  for some character  $\chi$ .

*Proof.* Because  $B = T \rtimes U$ , it is enough to find ourselves in  $\text{Ind}_U^G \chi$  for some character  $\chi$  on  $T$ . Now, write

$$\mathbb{C}[G] = \bigoplus_{\lambda} V_{\lambda} \otimes V_{\lambda}^{\vee}$$

by a variant of the Peter–Weyl theorem. But  $V_{\lambda}^* = V_{\lambda^{\vee}}$ , and the  $U$ -invariants is 1-dimensional by the highest-weight theory, so we take  $\chi$  to be  $\chi_{\lambda}$ . ■

**Example 1.171.** With  $G = \text{SL}(2, \mathbb{C})$ , the Borel subgroup  $B$  is the upper-triangular matrices, so  $G/B$  is  $\mathbb{CP}^1$ . It turns out that the usual finite-dimensional representations  $\text{Sym}^n V_{\text{std}}$ .

**Remark 1.172.** In general  $G/B$  will be some kind of flag variety. Sometimes,  $\mathcal{L}_\lambda$  will come from a further quotient  $G/P$  where  $P \supseteq B$  is a parabolic subgroup, and it turns out that taking these sorts of pullbacks do not change the global sections.

Our second input explains that  $K$  is large, which we will prove next class.

**Proposition 1.173.** Fix a real semisimple Lie algebra  $G$  with maximal compact subgroup  $K$ . Then  $K_{\mathbb{C}}$  has an open dense orbit in  $G_{\mathbb{C}}/B_{\mathbb{C}}$ . In fact, there are only finitely many orbits.

**Example 1.174.** Consider  $G = \text{SL}(3, \mathbb{R})$  and  $K = \text{SO}(3, \mathbb{R})$ . Then  $G_{\mathbb{C}}/B_{\mathbb{C}}$  is the moduli space of maximal flags of  $\mathbb{C}^3$ , which amounts to the data of a point and a line in  $\mathbb{CP}^2$  (where the point is on the line). Letting  $q$  be our quadratic form, it turns out that our orbits are given by how the point and line intersect the conic; for example, the line could be tangent to the conic with the point elsewhere, the line could be tangent with the point on the conic, the line could intersect the conic, and so on.



*Proof of Proposition 1.162.* We have realized  $V$  as some  $V_\lambda$  living inside some  $\text{Ind}_{B_{\mathbb{C}}}^{G_{\mathbb{C}}} \chi$  for some character  $\chi$ . We now pull back along the open dense orbit

$$K_{\mathbb{C}} \rightarrow G_{\mathbb{C}}/B_{\mathbb{C}},$$

which is injective by the density. The point is that we have identifications

$$\text{Ind}_{B_{\mathbb{C}}}^{G_{\mathbb{C}}} \chi = \text{Ind}_{K_{\mathbb{C}} \cap B_{\mathbb{C}}}^{K_{\mathbb{C}}} \chi \subseteq \mathbb{C}[K_{\mathbb{C}}],$$

of  $K$ -representations, but we are now done because the Peter–Weyl theorem bounds the multiplicity of an irreducible representation of  $K$  in  $\mathbb{C}[K_{\mathbb{C}}]$ . ■

## THEME 2

# AROUND THE CENTER

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### 2.1 September 30

There is no class on Thursday.

#### 2.1.1 A Little More Structure Theory

It remains to prove Proposition 1.173.

*Proof of Proposition 1.173.* The orbits are locally closed by Chevalley's theorem, so it is enough to merely show there are finitely many orbits. We are going to deduce our result from the Bruhat decomposition, which asserts that  $B \backslash G/B$  is in bijection with the Weyl group  $W$ , which is  $N_G(T)/T$  for some maximal torus  $T \subseteq G$ .

Note  $\theta$  acts on  $G$  and hence on  $X := G/B$ . (For example, one can realize  $X$  as the collection of Borel subgroups of  $G$ .) Now, let  $Z \subseteq X^2$  be the graph of  $\theta$ , and note that  $Z$  is the collection of fixed points of the automorphism

$$\iota: (x, y) \mapsto (\theta(y), \theta(x))$$

of  $X^2$ .

Now, the Bruhat decomposition provides a decomposition of  $B \backslash X$  as the Weyl group, so we let  $X_w$  be the orbit corresponding to a given  $w \in W$ . We now claim that  $Z \cap X_w^2$  is a finite union of orbits of  $K$ ; by looping over all  $w$ , this shows that  $Z$  has finitely orbits of  $K$ , but  $Z \cong X$  as spaces with  $K$ -action, so the proposition follows. For this, it is enough to show that the induced map

$$\mathrm{Lie} K \rightarrow T_{(x,y)} Z \cap T_{(x,y)} X_w^2$$

for any given  $(x, y)$ : this will imply that all orbits are open, which means that there are finitely many, by quasicompactness. To show this claim, we note that  $T_x X = \mathfrak{g}/\mathfrak{b}_x$ , where  $\mathfrak{b}_x$  is the Lie algebra of the corresponding Borel algebra, so  $T_{(x,y)} X^2 = (\mathfrak{g}/\mathfrak{b}_x) \oplus (\mathfrak{g}/\mathfrak{b}_y)$ . Thus,  $T_{(x,y)} Z = (\mathfrak{g}/\mathfrak{b}_x \oplus \mathfrak{g}/\mathfrak{b}_y)^\iota$ , and  $T_{(x,y)} X_w^2$  is a  $G$ -orbit, so it is the (diagonal) image of  $\mathfrak{g}$ . We thus see that the intersection is

$$\{(X, X) \in (\mathfrak{g}/\mathfrak{b}_x \oplus \mathfrak{g}/\mathfrak{b}_y) : X \in \mathfrak{g}\}^\iota,$$

which means that  $X$  is fixed by  $\iota$ , which is equivalent to  $X \in \mathrm{Lie} K$ . ■

**Remark 2.1.** This argument does nothing to allow us to compute the number of orbits of  $K \backslash G/B$ . Indeed, the main claim achieves finiteness via compactness! However, one can calculate the number of orbits in terms of subsets of some quotient of the Weyl group.

### 2.1.2 Cartan Involutions

To close up our discussion of the admissibility theorem, we explain a little about Cartan involutions and the classification of real semisimple groups.

**Theorem 2.2.** Every semisimple real Lie group  $G$  has a Cartan involution  $\theta$  which is unique up to conjugacy. Furthermore, the pair  $(G, \theta)$  is in bijection with its complexification  $(G_{\mathbb{C}}, \theta_{\mathbb{C}})$ , which in turn is in bijection with complex involutions of  $G_{\mathbb{C}}$ .

**Theorem 2.3.** Let  $G_{\mathbb{C}}$  be a semisimple connected complex Lie group. Then real forms of  $G_{\mathbb{C}}$  are in bijection with complex involutions; the map sends a real form  $G$  of  $G_{\mathbb{C}}$  to the complexification of the Cartan involution.

This is a case of a Galois descent problem, in the special case of the extension  $\mathbb{R} \subseteq \mathbb{C}$ . Here is the way Galois descent is typically done: given a Galois extension  $F \subseteq E$  and an object  $Y_E$ , we would like to understand objects (e.g., schemes)  $X$  over  $F$  with  $X_E \cong Y_E$ . Supposing that there is one such object  $X$  over  $F$ , one typically finds that the other objects are classified by  $H^1(\text{Gal}(E/F), \text{Aut } X)$ .

**Example 2.4.** In our situation, we suppose that we are given a real Lie group  $G$ , and we would like to classify real Lie groups  $G'$  with  $G_{\mathbb{C}} \cong G'_{\mathbb{C}}$ . Well,  $G$  has a Cartan involution  $\theta$ , so a choice of isomorphism  $G_{\mathbb{C}} \cong G'_{\mathbb{C}}$  grants  $G'$  the involution  $\theta' := a \circ \theta$ , where  $a$  denotes the conjugation (which is an automorphism of  $G$ ). We thus see that  $(\theta')^2 = a a^{\theta}$ , so  $a^{\theta} = a^{-1}$  is the required 1-cocycle. Notably, the cocycles  $a_1$  and  $a_2$  produce the same real form if and only if they are the same up to some twisted conjugacy

$$a_2 = b a_1 b^{-\theta}.$$

**Example 2.5.** If  $G$  is a real semisimple Lie group, then we can try to classify conjugacy classes of maximal tori in  $G$ . Well, fix maximal torus  $T$  in  $G$ . Then any other torus  $T'$  is conjugate to  $T$  up in  $G_{\mathbb{C}}$ , so we can write  $T'_{\mathbb{C}} = g T_{\mathbb{C}} g^{-1}$  for some  $g \in G_{\mathbb{C}}$ . Now, we define  $x := g^{-1} g^{\theta}$  to normalize  $T$ ; we remark also that  $\theta(x) = x^{-1}$ . Changing  $g$  only changes  $x$  up to twisted conjugacy, sending  $x$  to (say)  $y^{-1} x y^{\theta}$ . It turns out that such  $x$  classify our maximal tori.

**Example 2.6.** We can run the same argument on conjugacy classes: given a single complex conjugacy class in  $G_{\mathbb{C}}$ , one can ask how the complex conjugacy class splits into real conjugacy classes. Well, given  $g_1, g_2 \in G$  equipped with  $x \in G_{\mathbb{C}}$  for which  $g_2 = x g_1 x^{-1}$ , we see that  $x^{-1} \theta(x)$  commutes with  $g_1$ , and we expect to be classified by such elements up to twisted conjugacy. Such a splitting is seen as early as the regular semisimple elements of  $\text{SL}(2, \mathbb{R})$ .

*Proof of Theorem 2.2.* We first show the existence of our Cartan involutions in the complex case. Let  $K$  be a maximal compact subgroup of  $G$  so that  $G = K_{\mathbb{C}}$ . Now,  $\mathbb{C}[G]$  is  $C(K)_{\text{fin}}$ , which is algebraically just

$$\bigoplus_{\lambda} V_{\lambda} \otimes V_{\lambda}^*.$$

Now, each  $V_{\lambda} \otimes V_{\lambda}^*$  has a  $\mathbb{C}$ -antilinear involution given by the conjugate transpose. Composing this with the swapping involution  $V_{\lambda} \otimes V_{\lambda}^* \cong V_{\lambda}^* \otimes V_{\lambda}$ , which yields a complex involution  $\theta$  for which

$$\theta f(g) = \overline{f(g)}$$

for all  $g \in K$ . We now receive an involution  $\theta$  of  $G$  for which  $G^{\theta} = K$ , which we claim is the required Cartan involution: writing  $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ , we see that the Killing form  $(-, -)_{\mathfrak{g}}$  when restricted to  $\mathfrak{k}$  is twice the Killing form  $(-, -)_{\mathfrak{k}}$  (because the Killing form is the trace of the composite of the adjoint maps) and when restricted to  $i\mathfrak{k}$  is negative twice the Killing form  $(-, -)_{\mathfrak{k}}$  (for the same reason). Positive-definiteness of  $(-, \theta -)$  follows.

It remains to construct Cartan involutions in the real case. Given a real Lie group  $G$ , we know that  $G_{\mathbb{C}}$  has a Cartan involution  $\theta$ , and  $G_{\mathbb{C}}$  also has a maximal compact subgroup  $K_{\mathbb{C}}$ . We would like to choose  $\theta_0$  so that  $K_c = G_{\mathbb{C}}^{\theta_0}$  and so that  $\theta_0$  and  $\theta$  commute. Then  $\sigma := \theta\theta_0$  is an involution and provides a Cartan involution on  $G$ .

To do so, we consider  $X := G_{\mathbb{C}}/K_{\mathbb{C}}$ . This quotient has an invariant metric (because we can take any metric on  $G_{\mathbb{C}}$  and then average it over  $K_{\mathbb{C}}$ ). We now claim that this metric has nonpositive sectional curvature.

**Example 2.7.** The quotient  $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$  is hyperbolic 3-space.

We will also be able to show that  $X$  is simply connected. We now need two geometric facts.

**Theorem 2.8 (Toponogov).** If  $X$  is a simply connected Riemannian manifold with constant nonpositive sectional curvature, then every two points are connected with a unique geodesic, and the sum of the angles of any triangle is less than  $180^\circ$ .

**Proposition 2.9.** Let  $X$  be a space where the sum of the angles of any triangle is less than  $180^\circ$ . Then any compact group acting on  $X$  has a fixed point.

The second fact follows by taking the image of an orbit and then taking the circumcenter to be the fixed point. View  $X$  as parameterizing maximal compact subgroups in  $G_{\mathbb{C}}$ . Then for each  $x \in X$ , we note that there is a geodesic segment connecting  $x$  to  $\theta(x)$ , we let  $y$  be the midpoint, and it follows that  $\theta(y) = y$  while  $\mathrm{Stab} y$  is conjugate to  $K_c$ . Then we can construct  $\theta_0$  using  $y$ , and this construction works. ■

**Remark 2.10.** Consider the Riemann curvature tensor for the  $G$ -invariant metric on  $M := G/K$ . Then there is  $R \in \Omega^2(\mathrm{End} T_M)$  given by  $Z \mapsto ([X, Y], Z)$  at the identity  $T_1 M = \mathfrak{g}/\mathfrak{k}$ . One can use the Cartan involution  $\theta$  on  $\mathfrak{g}$  to decompose  $\mathfrak{g}$  into  $\{\pm 1\}$  eigenspaces, which allows one to decompose  $R$ .

## 2.2 October 7

Today we continue with the analytic theory.

### 2.2.1 Another Proof of Real Cartan Involutions

Let's give an alternative proof of Theorem 2.2. We will once again assume that they exist in the complex case.

**Lemma 2.11.** Let  $G$  be a semisimple real Lie group with a Cartan involution  $\theta$ . For any other involution  $\sigma$ , there is a conjugate involution  $\sigma' = g\sigma g^{-1}$  commuting with  $\theta$ .

*Proof.* We do some clever linear algebra. As usual, let  $(-, -)$  be the Killing form, and let  $B_\theta$  be the bilinear form  $B_\theta(X, Y) := (X, \theta Y)$  to be negative-definite (by definition of the Cartan involution). Now, set  $s := \theta\sigma$ . We note the following properties of  $s$ .

- Note  $s$  is self-adjoint for  $B_\theta$  by a direct calculation:

$$B_\theta(sX, Y) = (\theta\sigma X, \theta Y) \stackrel{*}{=} (\sigma X, Y) \stackrel{*}{=} (X, \sigma Y) = B_\theta(X, sY).$$

Here, the key equalities  $\stackrel{*}{=}$  follow because  $\theta$  and  $\sigma$  are involutions preserving the group structure, so they must preserve the Killing form.

- Note  $\theta s \theta^{-1} = s^{-1} = \sigma s \sigma^{-1}$  by a direct calculation.

It follows that the action of  $s$  may be diagonalized with real eigenvalues, say as

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_{r_i},$$

where the eigenvalues of  $s$  are  $\{r_i\}_i$ . Notably,  $|s|^t$  (acting on  $\mathfrak{g}_{r_i}$  by  $|r_i|^t$ ) can be thought of as the product of various  $\exp(|t\delta|)$ s where  $\delta$  is the log of  $r_i$ , so the linear operator  $|s|^t$  even belongs to  $G$ . We thus achieve  $\theta |s|^t \theta^{-1} = |s|^{-t}$  and similar for  $\sigma$ .

Now, define  $\sigma'$  by  $s := |s| \sigma'$  where  $\sigma'$  acts by sign of  $r_i$  on the various  $\mathfrak{g}_{r_i}$ s. One can calculate that  $\sigma'$  commutes with both  $\theta$  and  $\sigma$ , so the involution

$$|s|^{1/2} \sigma |s|^{-1/2} = |s| \sigma = \sigma' s \sigma = \sigma' \theta \sigma \sigma = \sigma' \theta$$

is conjugate to  $\sigma$  and will commute with  $\theta$ . ■

*Proof of Theorem 2.2.* We already know that  $G_{\mathbb{C}}$  admits a Cartan involution  $\sigma$  with  $G = G_{\mathbb{C}}^{\sigma}$ . We also know that  $G_{\mathbb{C}}$  admits a compact form  $G_{\mathbb{C}}^{\theta_c} = K_{c_i}$ , where  $\theta_c$  is some other compact form. Lemma 2.11 allows us to assume that  $\sigma$  and  $\theta_c$  commute (by merely conjugating  $\sigma$ ), so  $\theta_c$  descends to an action on  $G$ . Now, the Killing form of  $G$  is half the killing form of  $G_{\mathbb{C}}$ , so  $\theta_c|_G$  is easily checked to be the desired Cartan involution.

Next, let's show the uniqueness: if we have two Cartan involutions  $\theta$  and  $\theta'$  on  $G$ , then by Lemma 2.11, we are allowed to assume that  $\theta\theta' = \theta'\theta$ . Thus, either  $\theta = \theta'$  or (by considering diagonalization) there is  $X \in \mathfrak{g}$  for which  $\theta(X) = -\theta'(X)$ . But then  $B_{\theta}(X, X) = -B_{\theta'}(X, X)$ , so both cannot be negative definite.

Lastly, we need to show the result about twists. We have already described a map sending  $G$  to its Cartan involution of  $G_{\mathbb{C}}$ , unique up to conjugation. The reverse map sends the Cartan involution  $\sigma$ , which we may as well assume to commute with  $\theta$ , to the real group  $G_{\mathbb{C}}^{\sigma\theta}$ . Perhaps we should check that this reverse map is well-defined. The main difficulty is that replacing  $\sigma$  with a conjugate commuting with  $\theta$  is not known to be unique because we had a choice of  $\theta$ . Well, given two Cartan involutions  $\theta$  and  $\theta'$  commuting with  $\sigma$ , up to conjugation above, we may assume that

$$\theta|_{G^{\sigma}} = \theta'|_{G^{\sigma}}.$$

We now claim that  $\theta$  is uniquely determined by the restriction to  $G^{\sigma}$ . Well, by taking products, we may assume that  $G_{\mathbb{C}}$  is either almost simple (namely,  $\mathfrak{g}$  is simple) or is the sum of two factors permuted by  $\sigma$ . In both cases, we note that  $\mathfrak{g}^{\sigma=-1}$  is a simple  $\mathfrak{k}_{\mathbb{C}}$ -module: in the second case, we find that  $\mathfrak{g}^{\sigma=-1}$  is simple and so isomorphic to  $\mathfrak{g}^{\sigma}$  as a  $\mathfrak{g}^{\sigma}$ -module, so the result follows; in the first case, any submodule  $V \subsetneq \mathfrak{g}^{\sigma=-1}$  makes  $V \oplus \mathfrak{g}^{\sigma}$  a Lie ideal of  $\mathfrak{g}$ , which is a contradiction. To complete the proof of the claim, we decompose

$$\mathfrak{g} = \mathfrak{g}^{\sigma} \oplus \mathfrak{g}^{\sigma=-1}.$$

Now,  $B_{\theta}|_{\mathfrak{g}^{\sigma=-1}}$  is a  $G^{\sigma}$ -invariant pairing (here we use the fact that they already agree on  $\mathfrak{g}^{\sigma}$ ), so it is unique up to scalar by Schur's lemma, and by definiteness, it is unique up to a positive scalar. It follows that

$$\theta|_{\mathfrak{g}^{\sigma=-1}} = r\theta'|_{\mathfrak{g}^{\sigma=-1}}$$

for some  $r > 0$ , but then  $r^2 = 1$  because we have an involution, so we are done. ■

**Remark 2.12.** These proofs can be upgraded to the reductive case.

## 2.2.2 More on Twisting

Here are some motivating examples.

**Example 2.13.** Consider  $G_{\mathbb{C}} = \mathrm{SL}(n)$ . Then we have involutions  $\sigma$  given by conjugating by

$$\underbrace{(1, \dots, 1)}_p, \underbrace{(-1, \dots, -1)}_q$$

where  $p + q = n$  (and  $q$  is even). One finds that  $G^{\sigma} = \mathrm{SL}(p) \times \mathrm{SL}(q)$  by a direct calculation, which we note is the complexification of  $\mathrm{SU}(p) \times \mathrm{SU}(q)$ , which is a maximal compact subgroup of the real form  $\mathrm{SU}(p, q)$ .

**Example 2.14.** For  $n \geq 3$ , there is an automorphism on  $G_{\mathbb{C}} = \mathrm{SL}(n)$  given by  $g \mapsto g^{-\top}$ . This produces the split forms.

**Remark 2.15.** In general, there is an exact sequence

$$1 \rightarrow G_{\mathrm{ad}} \rightarrow \mathrm{Aut} G \rightarrow \mathrm{Out} G \rightarrow 1.$$

Considerations around fixing a maximal torus shows that  $\mathrm{Out} G$  are given by automorphisms of the Dynkin diagram; for example, this group is only nontrivial in the simply laced case.

**Definition 2.16 (inner twist).** Two twists  $G$  and  $G'$  over  $\mathbb{R}$  are *inner twists* if and only if the corresponding involution has trivial image in  $\mathrm{Out} G$ .

**Definition 2.17 (quasisplit).** A form  $G$  is *quasisplit* if and only if the corresponding involution has trivial image in  $G_{\mathrm{ad}}$ .

**Remark 2.18.** It turns out that a form is quasisplit if and only if there is a Borel subgroup defined over  $\mathbb{R}$ .

**Example 2.19.** One can see that the split form is quasisplit.

**Example 2.20.** A complex group  $G$  viewed as a real group is quasisplit but not split. Indeed,  $G$  is the fixed points of  $G_{\mathbb{C}} = G \times G$  given by swapping the two factors, which is an outer automorphism.

**Remark 2.21.** Given two (regular semisimple) elements  $g, g' \in G$  conjugate in  $G_{\mathbb{C}}$  (i.e., they are stably conjugate), then we can consider the set

$$\{x \in G_{\mathbb{C}} : xgx^{-1}\},$$

which is a torsor for a maximal torus  $T := Z_{G_{\mathbb{C}}}(g)$ . Note that  $g$  and  $g'$  are actually conjugate if and only if this torsor admits a real point. It then turns out that the  $G_{\mathbb{C}}$ -conjugacy class of  $g$  in  $G$  becomes some subset of  $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); T)$ . The answer turns out to be the kernel of the map

$$H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); T) \rightarrow H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); G)$$

The left group can be seen to consist of the fixed points of the involution  $t \mapsto \bar{t}^{-1}$ , which amounts to the connected components of  $T_{\mathbb{C}}^{\sigma}$ , where  $\sigma$  is  $t \mapsto \theta_c(\bar{t}^{-1})$ . For example, if  $G$  is compact, then  $T$  is compact, and it turns out that this map is injective.

## 2.3 October 7

Today we discuss globalization.

### 2.3.1 Globalization

Next class we will prove the following.

**Theorem 2.22 (Harish-Chandra globalization).** Fix a real semisimple Lie group  $G$ , and set  $\mathfrak{g} := \text{Lie } G_{\mathbb{C}}$  and  $K$  to be a maximal compact subgroup as usual, and we assume that  $K$  is connected. For any irreducible unitary  $(\mathfrak{g}, K)$ -module  $M$ , then there is a unitary representation  $V$  of  $G$  for which  $M = V_{\text{fin}}$  and  $V = \overline{M}$ .

**Remark 2.23.** In particular, it follows that  $M$  is admissible by Theorem 1.139.

**Remark 2.24.** One can purely algebraically prove that any irreducible  $(\mathfrak{g}, K)$ -module is admissible. The key tool is the action of the Casimir elements (in  $Z\mathfrak{g} \subseteq U\mathfrak{g}$ ) on  $M$ . The point is that the Schur lemma forces every element in  $Z\mathfrak{g}$  to act by a scalar, which produces an “infinitesimal character”  $\chi: Z\mathfrak{g} \rightarrow \mathbb{C}$ . This will be useful once we understand the structure of  $Z\mathfrak{g}$ .

*Proof of Theorem 2.22.* Let  $(-, -)$  be the invariant unitary inner product on  $M$ . Viewing  $M$  as a representation of  $K$ , we may decompose

$$M = \bigoplus_{\nu} M_{\nu},$$

where  $\nu$  varies over the various dominant integral weights of  $K$  (and hence of  $K_{\mathbb{C}}^{\circ}$ ), and  $M_{\nu}$  is the corresponding isotypic component of the highest weight representation  $V_{\nu}$ . As such, we may write  $M_{\nu} = V_{\nu} \otimes W_{\nu}$ .

The main claim is that there is a quadratic form  $q$  on  $\mathfrak{h}^*$  such that

$$|Xv|^2 \stackrel{?}{\leq} |v|^2 \cdot q(\nu) |X|^2,$$

for any  $X \in \mathfrak{g}$  and  $v \in M$ ; here,  $|X|^2$  is computed with respect to the bilinear form  $-1 \cdot (-, \theta -)$  of a fixed Cartan involution  $\theta$ . Let's show this. Because  $(-, -)$  is invariant under  $\mathfrak{g}$ , we know  $(X'v', w') + (v', X'w') = 0$  for any  $v', w' \in M$  and  $X' \in \mathfrak{g}$ , so we see that  $|Xv|^2$  is

$$(Xv, Xv) = - (X^2v, v),$$

which we will do by fixing an orthonormal basis  $\{X_i\}_i$  of  $\mathfrak{g}$  and estimating  $\sum_i (X_i^2v, v)$  instead. Here, the orthonormal basis of  $\mathfrak{g}$  is taken with respect to the bilinear form  $-1 \cdot (-, \theta -)$ . In fact, decomposing  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\theta$  acts by  $+1$  on  $\mathfrak{k}$  and acts by  $-1$  on  $\mathfrak{p}$ , we see that

$$\sum_i X_i^2 = C_{\mathfrak{g}} - 2C_{\mathfrak{k}},$$

where  $C_{\mathfrak{g}}$  is the Casimir element of  $\mathfrak{g}$ , and  $C_{\mathfrak{k}}$  is the Casimir element of  $\mathfrak{k}$ . (Here,  $C_{\mathfrak{k}}$  means the Casimir element of the adjoint action of  $\mathfrak{k}$  on  $\mathfrak{g}$ . This equality follows because the Killing form on  $\mathfrak{k}$  is negative-definite by definition of the Cartan!) Now,  $C_{\mathfrak{g}}$  commutes with the  $\mathfrak{g}$ -action on  $M$ , so by Schur's lemma, we see that  $C_{\mathfrak{g}}$  must act by a scalar.

Let's discuss the Casimir action. Continuing, one can check that a Casimir element  $C_{\mathfrak{k}}$  acts on a highest weight representation  $V_{\nu}$  by the scalar  $(\nu + \rho)^2 - (\rho)^2$ , where  $(-, -)$  is some choice of invariant bilinear form. Indeed, write  $C_{\mathfrak{k}}$  as, up to some constant,

$$\sum_{\alpha} e_{\alpha} f_{\alpha} + \sum_{\alpha} f_{\alpha} e_{\alpha} + \sum_i h_i^2,$$

but this can be rearranged to  $2 \sum_{\alpha} f_{\alpha} e_{\alpha} + \sum_{\alpha} h_{\alpha}^2 + \sum_i h_i^2$ , and then it is a matter of calculating the action on a singular vector  $v$  of  $V_{\nu}$ , which is now easier because  $e_{\alpha} v = 0$ . Summing completes the calculation. The main claim now follows after an expansion of this action.

We now complete the argument. Set  $V$  to be the completion of  $M$  for the norm topology given by  $(-, -)$ . To upgrade the  $\mathfrak{g}$ -action to a  $G$ -action, it basically remains to check that

$$\exp(X)v = \sum_{n \geq 0} \frac{X^n v}{n!}$$

converges to  $V$  for any  $v \in V$ . By continuity, it is enough to check this for  $v \in M$ , and by decomposing into locally finite pieces, it is enough to check this for  $v \in M_{\nu}$ , which roughly speaking follows from the lemma as well as the fact that the  $X^n v$  is supported in weights of the form  $\nu + \alpha$  where  $\alpha$  is a sum of the weights of  $K$  acting on  $\mathfrak{g}$ . (Indeed, this latter statement follows by decomposing  $\mathfrak{g} \otimes M_{\nu}$  into weight spaces of  $K$ .) In particular, the inequality

$$|Xv| \leq |X| \cdot |v| \cdot \sqrt{q(\nu)},$$

so we find that

$$|X^n v| \leq |X|^n |v| \cdot c(c+R)(c+2R) \cdots (c+(n-1)R),$$

where  $c = \sqrt{q(\nu)}$  and  $R$  is the maximum among the  $\sqrt{q(\alpha)}$  where  $\alpha$  is one of weights appearing in the action of  $K$  on  $\mathfrak{g}$ . It follows that  $\exp(X)v$  converges for  $|X| < R^{-1}$ .

Once we know that  $\exp(X)v$  converges for small  $X$ , the Campbell–Hausdorff formula extends this to an action of the universal cover  $\tilde{G}$ . To see that our action actually factors through  $G$ , we let  $\tilde{K} \subseteq \tilde{G}$  be the preimage of  $K$ . So because the action of  $\tilde{K}$  factors through  $K$ , it follows that the action of  $\tilde{G}$  factors through  $G$  because  $\pi_1(G) = \pi_1(K^{\circ})$ .<sup>1</sup> (Indeed,  $\pi_1(G)$  is the kernel of the map  $\tilde{G} \rightarrow G$  and similarly for  $K$ .) ■

### 2.3.2 Verma Modules

We should say something about Verma modules.

**Definition 2.25.** Fix a complex semisimple Lie algebra  $\mathfrak{g}$ , and fix a Cartan subalgebra  $\mathfrak{h}$  and decomposition  $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ , and we set  $\mathfrak{b} := \mathfrak{n}_+ \oplus \mathfrak{h}$ . For a weight  $\lambda \in \mathfrak{h}^*$ , we define the *Verma module*

$$M_{\lambda} := \text{ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}.$$

**Remark 2.26.** Equivalently, one can think of  $M_{\lambda}$  as  $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_{\lambda}$ . As such,  $M_{\lambda}$  is generated by the singular vector  $v_{\lambda}$ , which lives in weight  $\lambda$ .

**Remark 2.27.** It follows from the PBW theorem that

$$U\mathfrak{n}_- \cong M_{\lambda},$$

where the isomorphism is given by  $X \mapsto Xv_{\lambda}$ .

**Remark 2.28.** It follows that  $\text{Hom}_{\mathfrak{g}}(M_{\lambda}, M)$  is given by the collection of singular vectors  $v \in M$  of weight  $\lambda$ . For example, if  $\alpha^{\vee}$  is a simple coroot with  $\langle \lambda^{\vee}, \alpha \rangle$  equal to a nonnegative integer  $m$ , then  $f_{\alpha}^{m+1} v_{\lambda}$  is a vector in  $M_{\lambda}$  of weight  $\lambda - (m+1)\alpha$ . Furthermore, one can check that  $f_{\alpha}^{m+1} v_{\lambda}$  is singular!

<sup>1</sup> In fact, one can show that  $G/K^{\circ}$  is contractible, which follows from the polar decomposition. Indeed, writing  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  as above, then one can show that  $\exp(\mathfrak{p})$  is a Euclidean space. Thus, one finds that  $G/K^{\circ}$  is some Euclidean space.



## 2.4 October 14

Today we discuss the Harish-Chandra isomorphism.

### 2.4.1 The Harish-Chandra Isomorphism

We now try to describe the center of  $U\mathfrak{g}$ .

**Notation 2.29.** Fix a Lie algebra  $\mathfrak{g}$ . Then we let  $Z\mathfrak{g}$  denote the center  $Z(U\mathfrak{g})$ .

Well, for  $z \in Z\mathfrak{g}$ , we see that  $z$  must act on  $M_\lambda$  by a scalar (namely, the scalar given by the image of  $v_\lambda$ ).

**Definition 2.30.** Fix a complex reductive Lie algebra  $\mathfrak{g}$ , and fix a Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . Then we define the *shifted Weyl action* of  $W$  on  $\mathfrak{h}^*$  by

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where  $\rho$  is half the sum of the positive roots.

**Theorem 2.31 (Harish-Chandra isomorphism).** Fix a complex reductive Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . For  $z \in Z\mathfrak{g}$ , let  $P_z(\lambda)$  be the scalar by which  $z$  acts on  $M_\lambda$ .

- (a) For  $z \in Z\mathfrak{g}$ , the function  $P_z(\lambda)$  is a polynomial in  $\lambda \in \mathfrak{h}^*$ , invariant under the shifted Weyl action.
- (b) The map  $z \mapsto P_z$  produces an isomorphism  $Z\mathfrak{g} \cong \mathbb{C}[\mathfrak{h}^*]^W$ .

**Remark 2.32.** Here is the source of the shifted Weyl action: the relevant irreducible modules arise as quotients of Verma modules, which means that they arise from inductions from the Borel subgroup of  $G$ . However, these inductions (which are sections of some line bundles) only make sense as  $\frac{1}{2}$ -densities, which requires the Weyl action to acquire a  $\frac{1}{2}$  shift. Slightly more explicitly, the story boils down to  $SL_2$ , where the Weyl action comes from a Fourier transform, and the shift is present for exactly this  $\frac{1}{2}$ -density reason.

**Remark 2.33.** Here,  $\mathbb{C}[\mathfrak{h}^*]$  means the polynomials on  $\mathfrak{h}^*$ , note polynomials with coefficients in  $\mathfrak{h}^*$ .

Here is a related tool.

**Theorem 2.34 (Chevalley restriction).** Fix a complex reductive Lie group  $G$  with Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . Then the restriction map  $\mathbb{C}[\mathfrak{g}] \rightarrow \mathbb{C}[\mathfrak{h}]$  restricts to an isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[\mathfrak{h}]^W.$$

**Example 2.35.** With  $\mathfrak{g} = \mathfrak{gl}(n)$ , we can show that  $\mathbb{C}[\mathfrak{g}]^G$  is generated by the polynomials  $X \mapsto \text{tr } \wedge^i X$  as  $i$  varies. Accordingly, these are precisely the symmetric polynomials when restricted to the diagonal matrices  $\mathfrak{h} \subseteq \mathfrak{g}$ .

*Proof of Theorem 2.34.* Let  $T \subseteq G$  be the Lie subgroup corresponding to  $\mathfrak{h} \subseteq \mathfrak{g}$ . Identifying  $W$  with  $N(T)/T$  in the usual way, we see that  $\mathbb{C}[\mathfrak{g}]^G$  of course maps to  $\mathbb{C}[\mathfrak{h}]^W$ . We now check injectivity and surjectivity separately.

- Injective: if  $P \in \mathbb{C}[\mathfrak{g}]^G$  vanishes on  $\mathfrak{h}$ , then  $P$  vanishes on every semisimple element in  $\mathfrak{g}$  because any semisimple element can be placed in a Cartan subalgebra and then conjugated to  $\mathfrak{h}$ . Thus,  $P = 0$  because semisimple elements are dense in  $\mathfrak{g}$ .
- Surjective: note  $\mathbb{C}[\mathfrak{h}]$  is spanned by the polynomials  $\lambda^n$  where  $\lambda$  is a dominant integral weight (indeed, we may even take  $\lambda$  to consist of fundamental weights, which provide a basis of  $\mathfrak{h}^*$ ), so  $\mathbb{C}[\mathfrak{h}]^W$  is spanned by the polynomials

$$Q_{\lambda,n} := \sum_{w \in W} (w\lambda)^n.$$

We now extend this polynomial to an element of  $\mathbb{C}[\mathfrak{g}]^G$ . Well, define  $P_{\lambda,n}$  by

$$P_{\lambda,n}(x) := \text{tr}(x^n; V_\lambda),$$

where  $V_\lambda$  is the finite-dimensional irreducible representation of highest weight  $\lambda$ . The representations  $V_\lambda$  are finite-dimensional, so  $P_{\lambda,n}$  is in fact a polynomial, and we can see that it is invariant under the  $G$ -action because we took a trace. When restricted to  $\mathfrak{h}$ , we see that  $P_{\lambda,n}$  is a sum of  $Q_{\lambda,n}$  plus some  $Q_{\mu,n}$ s for  $\mu < \lambda$  (which are the other weights of  $\lambda$ ). Thus, one can induct to show that  $Q_\lambda$  is in the image. ■

**Remark 2.36.** Here is one way to view the construction at the end of the argument: from Peter–Weyl, we expect  $\mathbb{C}[G]$  to decompose into sums of  $V_\lambda \otimes V_\lambda^*$ , but  $V_\lambda \otimes V_\lambda^*$  corresponds to the function given by the character.

**Corollary 2.37.** Fix a complex reductive Lie group  $G$  with Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . For any dominant integral weight  $\lambda$ , the group  $\text{Hom}_G(V_\lambda, \mathbb{C}[\mathfrak{g}])$  is finitely generated over  $\mathbb{C}[\mathfrak{g}]^G$ .

*Proof.* Note  $\text{Hom}_G(V_\lambda, \mathbb{C}[\mathfrak{g}]) = (V_\lambda^* \otimes \mathbb{C}[\mathfrak{g}])^G$  which then injects to  $V_\lambda^* \otimes \mathbb{C}[\mathfrak{h}]$  by the same injectivity argument given in Theorem 2.34. By some geometric invariant theory, we see that  $\mathbb{C}[\mathfrak{h}]$  is finitely generated over  $\mathbb{C}[\mathfrak{h}]^W$ .<sup>2</sup> The result now follows because  $V_\lambda^*$  is also finitely generated. ■

*Proof of Theorem 2.31.* For (a), fix some  $z \in Z\mathfrak{g}$ , and let's begin by showing that  $P_z(\lambda)$  is a polynomial in  $\lambda \in \mathfrak{h}^*$ . Well, recall that  $M_\lambda$  has the single generator  $v_\lambda$ , which is a singular vector of weight  $\lambda$ . Now, the PBW theorem tells us that elements of  $U\mathfrak{g}$  can be written uniquely as a polynomial in the basis  $\{f_\alpha\}_\alpha \sqcup \{h_i\}_i \sqcup \{e_\alpha\}_\alpha$ . In particular,  $z$  takes the form  $P$  for some polynomial  $P$  in the  $h_i$ s plus some monomials using at least one  $f_\alpha$  or  $e_\alpha$ . However,  $z$  must be in weight 0 because it is central: indeed, commuting with various  $h \in \mathfrak{h}$  is the same as saying that  $z$  has weight 0. Thus, all other monomials outside  $P(h)$  contain at least one of  $e_{\alpha_i}$ , which automatically kills  $v_\lambda$ . We conclude that

$$P_z(\lambda) = P(\lambda),$$

and now the right-hand side is definitionally a polynomial in  $\lambda$ .

We next check that  $P_z(\lambda)$  is invariant under the shifted Weyl action. On the homework, we showed that there is a canonical map  $M_{s_i \cdot \lambda} \rightarrow M_\lambda$  whenever  $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ . We conclude that  $P_z(\lambda) = P_z(s_i \cdot \lambda)$  in this case. Letting  $\lambda$  vary over all dominant integral weights (which are Zariski dense), the invariance follows.

It remains to check that we have actually provided an isomorphism  $Z\mathfrak{g} \rightarrow \mathbb{C}[\mathfrak{h}^*]^W$ . For this, we will use Theorem 2.34. By the PBW theorem, there is an isomorphism  $\mathbb{C}[\mathfrak{g}^*] \rightarrow U\mathfrak{g}$  of  $G$ -modules (where we are using the adjoint action). Thus, we get an isomorphism

$$\mathbb{C}[\mathfrak{g}^*]^G \rightarrow (U\mathfrak{g})^G = Z\mathfrak{g}$$

<sup>2</sup> More generally, if a finite group  $\Gamma$  acts on a complex vector space  $V$ , then any  $f \in \mathbb{C}[V]$  is the root of the polynomial  $\prod_{\gamma \in \Gamma} (t - \gamma(f)) = 0$ , so  $f$  is the root of a polynomial in  $\mathbb{C}[V]^\Gamma[t]$ .

of vector spaces. Applying Theorem 2.34 (and noting that there is an isomorphism  $\mathfrak{g} \cong \mathfrak{g}^*$  by using an invariant form), we receive an isomorphism  $\mathbb{C}[\mathfrak{g}^*]^G \rightarrow \mathbb{C}[\mathfrak{h}^*]^W$ . We now claim that the composite

$$\mathbb{C}[\mathfrak{h}^*]^W \leftarrow \mathbb{C}[\mathfrak{g}^*]^G \rightarrow (U\mathfrak{g})^G = Z\mathfrak{g} \rightarrow \mathbb{C}[\mathfrak{h}^*]^W$$

is the identity “up to lower-order terms.” Namely, we claim that a homogeneous polynomial  $P \in \mathbb{C}[\mathfrak{h}^*]^W$  of degree  $d$  goes to the same polynomial  $P$  only with lower-order terms, which then completes the argument because it forces the full composite to be an isomorphism. To prove the claim, we see that it is enough to check it on polynomials of  $\mathbb{C}[\mathfrak{g}^*]$  which have been averaged over  $S_n$ . However, this averaging process does not change the highest-order term, and the shifted action by  $W$  also does not change the highest-order term, so we are done! ■

## 2.5 October 16

Today we say more about the structure theory.

### 2.5.1 Admissibility of Irreducible $(\mathfrak{g}, K)$ -Modules

Here is our main result.

**Proposition 2.38.** Fix a semisimple complex connected Lie group  $G$ , and let  $\theta: G \rightarrow G$  be an involution. Set  $K := G^\theta$ , which we let have Lie algebra  $\mathfrak{k} = \mathfrak{g}^\theta$ , and then we decompose  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\mathfrak{p} := \mathfrak{k}^\perp$ .

- (a) For generic  $x \in \mathfrak{p}$ , we may arrange for  $x$  to be semisimple and for the dimension of  $\mathfrak{g}_x$  (which is the centralizer of  $x$ ) is minimal. Then  $\mathfrak{a} := \mathfrak{g}_x \cap \mathfrak{p}$  is abelian and semisimple.
- (b) For generic  $x \in \mathfrak{p}$ , the orbit  $K(a)$  contains a dense Zariski open subset in  $\mathfrak{p}$ .

**Example 2.39.** If  $G$  is a real group, then  $G_{\mathbb{C}} \cong G \times G$  has an automorphism  $\theta$  given by switching the two factors. Then  $K = G$ , so  $\mathfrak{k} = \mathfrak{g}$ , and  $\mathfrak{p} = \mathfrak{g}$  as well.

*Proof.* If  $\theta$  is the identity, then  $\mathfrak{p} = 0$ , and there is nothing to do. Thus, we may assume  $\mathfrak{p} \neq 0$ , and then we go ahead and choose  $x \in \mathfrak{p}$  generic. For example,  $x \neq 0$ , so there is a unique Jordan decomposition  $x = x_s + x_u$ , thereby implying  $x_s, x_u \in \mathfrak{p}$ . For example, if  $\mathfrak{p}$  has no semisimple elements, then we are required to have  $\mathfrak{p}$  consist solely of nilpotent elements, but this is false because the Killing form vanishes on nilpotent elements while the Killing form is non-degenerate on  $\mathfrak{p}$ .

We thus see that we can choose generic  $x \in \mathfrak{p}$  to be semisimple with  $\mathfrak{h} := \mathfrak{g}_x$  of minimal dimension. Certainly  $\mathfrak{h}$  is reductive, so we can write  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}_0$  where  $\mathfrak{h}'$  is semisimple and  $\mathfrak{h}_0$  is abelian. Decomposing  $\mathfrak{h}$  into  $\theta$ -eigenspaces, we claim that  $\theta$  is trivial on  $\mathfrak{h}'$ . This implies (a) because it shows that  $\mathfrak{g}_x \cap \mathfrak{p} = \mathfrak{h}_0$  is abelian.

It remains to show the claim. Otherwise, we can find nonzero semisimple  $y \in (\mathfrak{h}')^-$  (as before), meaning  $y \in (\mathfrak{h}' \cap \mathfrak{p})$ . But then  $x$  and  $y$  are commuting semisimple elements, so a generic (semisimple!) linear combination  $x - ty \in \mathfrak{p}$  has centralizer  $\mathfrak{g}_x \cap \mathfrak{g}_y$  (simply choose  $t$  so that the eigenvalues of  $x$  and  $ty$  do not overlap). But then  $\mathfrak{g}_{x+ty}$  is strictly smaller than  $\mathfrak{h}$ , contradicting the minimality.

This follows by a tangent space calculation. It is enough to show that the action map  $\rho: K \times \mathfrak{a} \rightarrow \mathfrak{p}$  is surjective on the differentials at some point (because this provides us with an open neighborhood). For this, we compute the differential at  $(1, x) \in K \times \mathfrak{a}$ . This is some map  $d\rho: \mathfrak{k} \oplus \mathfrak{a} \rightarrow \mathfrak{p}$ , and it sends  $(y, b)$  to  $[y, x] + b$  (because conjugation action goes to the Lie bracket). To show that this map surjects onto  $\mathfrak{p}$ , we note that the adjoint action of  $x$  on  $\mathfrak{g}$  has an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_x \oplus \text{ad}_x \mathfrak{g}$$

because  $\mathfrak{g}_x$  is the eigenspace of  $\text{ad}_x$  with eigenvalue 0. Intersecting such a decomposition with  $\mathfrak{p}$ , we see that  $\text{ad}_x$  swaps  $\mathfrak{k}$  and  $\mathfrak{p}$  (because  $x \in \mathfrak{p}$ ), so we get

$$\mathfrak{p} = \mathfrak{a} \oplus \text{ad}_x \mathfrak{k},$$

so the surjectivity follows. ■

**Remark 2.40.** Here, we can take “generic” to mean “Zariski open.” In fact, the argument shows that generic elements of  $\mathfrak{p}$  are semisimple.

**Corollary 2.41.** Fix a semisimple real connected group  $G$ .

- (a) For any irreducible representation  $V_\nu$  of  $K$ , the space  $\text{Hom}_K(V_\nu, \mathbb{C}[\mathfrak{p}])$  is finitely generated over  $\mathbb{C}[\mathfrak{g}]^G$ .
- (b) Any irreducible  $(\mathfrak{g}, K)$ -module is admissible.

*Proof.* Let’s start with (a). It is worthwhile to explain how  $\mathbb{C}[\mathfrak{g}]^G$  acts on  $\text{Hom}_K(V_\nu, \mathbb{C}[\mathfrak{p}])$ . Well, a function  $\mathbb{C}[\mathfrak{g}]^G$  can be restricted to  $\mathfrak{p}$ , and it remains  $K$ -invariant, so it may act by “scaling.”

Note that  $\text{Hom}_K(V_\nu, \mathbb{C}[\mathfrak{p}])$  can be identified with  $K$ -invariant polynomial maps  $\mathfrak{p} \rightarrow V_\nu^*$  by definition. However, by Proposition 2.38, the Zariski density of  $K(\mathfrak{a})$  implies that such a map is uniquely determined by its restriction to  $\mathfrak{a}$ . Thus, we have an embedding

$$\text{Hom}_K(V_\nu, \mathbb{C}[\mathfrak{p}]) \hookrightarrow \mathbb{C}[\mathfrak{a}] \otimes V_\nu^*.$$

Now, Theorem 2.34, we may as well choose a Cartan  $\mathfrak{h}$  containing  $\mathfrak{a}$ , and then  $\mathbb{C}[\mathfrak{g}]^G$  is identified with  $\mathbb{C}[\mathfrak{h}]^W$ . Because  $\mathfrak{a} \subseteq \mathfrak{h}$ , we know that  $\mathbb{C}[\mathfrak{a}]$  is finitely generated over  $\mathbb{C}[\mathfrak{h}]$  and so  $\mathbb{C}[\mathfrak{h}]^W$  as well, so the result follows.

We now turn to (b). Fix an irreducible  $(\mathfrak{g}, K)$ -module  $M$ . For a given weight  $\nu$  of  $K$ , we would like to show that the isotypic component  $M_\nu$  is finite-dimensional. Fix some  $v \in M_\nu$ , which we may as well assume is nonzero. Because  $M$  is irreducible, the induced map  $U\mathfrak{g} \rightarrow M$  given by  $X \mapsto Xv$  is surjective, but this map factor through  $U\mathfrak{g} \otimes_{U\mathfrak{k}} V_\nu$  because  $v \in M_\nu$ . Now, we have an isomorphism

$$\mathbb{C}[\mathfrak{p}^*] \otimes V_\nu \rightarrow U\mathfrak{g} \otimes_{U\mathfrak{k}} V_\nu$$

of  $K$ -modules by the PBW theorem (with a suitably ordered basis of  $\mathfrak{g}$ ). Additionally, Schur’s lemma tells us that  $Z\mathfrak{g}$  acts by scalars according to some character  $\chi$ , so we also have a surjection

$$\mathbb{C}_\chi \otimes_{Z\mathfrak{g}} (U\mathfrak{g} \otimes_{U\mathfrak{k}} V_\nu) \twoheadrightarrow M.$$

Now,  $(U\mathfrak{g} \otimes_{U\mathfrak{k}} V_\nu)$  has a filtration given by  $\mathbb{C}[\mathfrak{p}^*] \otimes V_\nu$ , and one can find that the action of  $Z\mathfrak{g}$  is the same one as the action of  $\mathbb{C}[\mathfrak{g}]^G$  on  $\mathbb{C}[\mathfrak{p}^*] \otimes V_\nu$  from (a). Thus, we have a surjection

$$\mathbb{C} \otimes_{\mathbb{C}[G]} (\mathbb{C}[\mathfrak{p}] \otimes V_\nu) \twoheadrightarrow \text{gr}(\mathbb{C}_\chi \otimes_{Z\mathfrak{g}} (U\mathfrak{g} \otimes_{U\mathfrak{k}} V_\nu)).$$

It remains to show that the left-hand object is admissible, which follows from (a): for any  $\mu$ , we see that

$$\text{Hom}_K(V_\mu, \mathbb{C} \otimes_{\mathbb{C}[G]} (\mathbb{C}[\mathfrak{p}] \otimes V_\nu)) = \text{Hom}_K(V_\mu \otimes V_\nu^*, \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{g}]^G} \mathbb{C}[\mathfrak{p}])$$

but this last space is finite-dimensional by (a). ■

**Remark 2.42.** Philosophically, the point is that  $K$ -equivariant  $\mathbb{C}[\mathfrak{p}^*]$ -modules can be used to study  $(\mathfrak{g}, K)$ -modules. Roughly speaking,  $(\mathfrak{g}, K)$ -modules can be “de-quantized” (i.e., take a suitable associated graded module) to  $K$ -equivariant coherent sheaves on  $\mathfrak{p}^*$ . Accordingly,  $(\mathfrak{g} \oplus \mathfrak{g}, G)$ -modules can be studied via  $G$ -equivariant

**Example 2.43.** A finitely generated  $(\mathfrak{g}, K)$ -module  $M$  has a support  $\text{supp } M \subseteq \mathfrak{p}^*$ . For an appropriately defined filtration of  $M$ , the module  $\text{gr } M$  is a finitely generated  $K$ -invariant module over  $\mathbb{C}[\mathfrak{p}^*]$ . It turns out that  $\text{supp } \text{gr } M$  is a closed subset of  $\mathfrak{p}^*$  not depending on the filtration, so it follows that the dimension of the support is independent of the filtration.

## 2.5.2 The Chevalley–Sheppard–Todd Theorem

Let's return to studying the center.

**Definition 2.44.** Fix a complex vector space  $V$ . Then a *complex reflection* is an automorphism  $g \in \text{GL}(V)$  of finite order such that  $V^{\langle g \rangle} \subseteq V$  has codimension 1.

**Theorem 2.45.** Fix a complex semisimple Lie algebra  $\mathfrak{g}$ , and let  $W$  act on  $\mathfrak{h}^*$  as usual.

- (a) Then  $\mathbb{C}[\mathfrak{h}]$  is free as a module over  $\mathbb{C}[\mathfrak{h}]^W$ .
- (b) Then  $\mathbb{C}[\mathfrak{h}]^W$  is a polynomial algebra.

**Remark 2.46** (Chevalley–Sheppard–Todd). In fact, one can show that a subgroup  $W \subseteq \text{GL}(V)$  is generated by complex reflections if and only if  $\mathbb{C}[V]^W$  is isomorphic to a polynomial algebra.

**Example 2.47.** Letting the Weyl group  $W$  act on  $\mathfrak{h}^*$ , Theorem 2.45 implies that  $\mathbb{C}[\mathfrak{h}^*]^W$  is a free commutative algebra.

**Example 2.48.** By Theorem 2.34, we see that  $\mathbb{C}[\mathfrak{g}]^G$  is also a free commutative algebra.

*Proof of Theorem 2.45.* The main content lies in (a).

1. We will use the Demazure operators. Given a complex reflection  $s_\alpha \in W$ , we see that  $\mathfrak{h}^{s_\alpha}$  is the kernel of  $\alpha$ . Thus, for a polynomial  $P \in \mathbb{C}[\mathfrak{h}]$ , we note that  $s(P) - P$  automatically vanishes on the kernel of the linear functional  $\alpha$ , so we may define

$$D_s P := \frac{sP - P}{\alpha},$$

which we know must continue to be a polynomial in  $\mathbb{C}[\mathfrak{h}]$ . The point is that  $P$  is invariant under  $W$  is equivalent to saying that  $sP = P$  for all  $s$ , which is equivalent to  $D_s P = 0$  for all  $s$ . As such,  $D_s(QP) = QD_s(P)$  as long as  $Q \in \mathbb{C}[\mathfrak{h}]^W$ .

2. Now, because  $W$  is a finite group,  $\mathbb{C}[\mathfrak{h}]$  is finitely generated over  $\mathbb{C}[\mathfrak{h}]^W$  (for example, this ring extension is integral because  $f \in \mathbb{C}[\mathfrak{h}]$  is a root of the monic polynomial  $\prod_{w \in W} (x - wf)$  in  $\mathbb{C}[\mathfrak{h}]^W$ ), so the quotient

$$A := \frac{\mathbb{C}[\mathfrak{h}]}{(\mathbb{C}[\mathfrak{h}]_+^W)}$$

is a finite-dimensional graded vector space; here, the  $_+$  means that we are considering polynomials with no constant term, and we are taking the quotient by the generated ideal. As such, we may choose homogeneous polynomials  $P_1, \dots, P_d$  whose images produce a basis in this quotient. A graded version of Nakayama's lemma then implies that they generate  $\mathbb{C}[\mathfrak{h}]$  over  $\mathbb{C}[\mathfrak{h}]^W$ .<sup>3</sup>

<sup>3</sup> More precisely, this follows by some induction on degrees: write a given polynomial in  $\mathbb{C}[\mathfrak{h}]$  as a linear combination in the  $P_i$ s with coefficients in  $\mathbb{C}[\mathfrak{h}]_+^W$  and  $\mathbb{C}[\mathfrak{h}]$ . Then we can inductively write the coefficients in  $\mathbb{C}[\mathfrak{h}]_+^W$  as linear combinations, and the process terminates by a degree argument.

3. It remains to check that there are no relations among these generators. As such, suppose that we have polynomials  $Q_1, \dots, Q_d \in \mathbb{C}[\mathfrak{h}]^W$  for which

$$\sum_i Q_i P_i = 0.$$

We may as well assume that this relation is minimal. The point is to use the Demazure operators to produce a smaller relation. In particular, for each  $i$ , we claim that we can find some  $D$  in the polynomial algebra of Demazure operators for which

$$D(P_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } \deg P_j \leq \deg P_i \text{ or } i \neq j. \end{cases}$$

Then we may apply the Demazure operators to our relation, choosing  $i$  to be the  $P_i$  with maximal degree and  $Q_i P_i$  nonzero; this then produces a smaller relation!

4. It remains to show the main claim of the previous paragraph, constructing the operator  $D$ . The key step is to show that a homogeneous polynomial  $P \in \mathbb{C}[\mathfrak{h}]$  admits some  $D = D_{s_1} \cdots D_{s_n}$  for which  $DP = 1$ . To see this, note that  $A^W$  only has the constants (because higher-degree  $W$ -invariant polynomials are killed in the quotient of  $A$ ). Thus, there is nothing to do when  $P$  is  $W$ -invariant. On the other hand, if  $\bar{P} \in A$  is not  $W$ -invariant, then there is a reflection  $s$  with  $s\bar{P} \neq \bar{P}$ , so  $D_s P \neq 0$ , so  $s\bar{P} - \bar{P} = \bar{\alpha} \cdot \bar{D}_s \bar{P}$ . We also see that  $\deg D_s P < \deg P$  for degree reasons, so we may induct down.

To complete the proof of the main claim, let  $\text{Dem}$  be the algebra of Demazure operators, and we note that the canonical pairing

$$\text{Dem}_d \times A_d \rightarrow \mathbb{C}$$

is perfect by the previous paragraph. Namely,  $A_d \hookrightarrow \text{Dem}_d^*$  by the previous paragraph, so by duality,  $\text{Dem}_d \rightarrow A_d^*$ , which provides the required operator  $D$  by unwinding this statement.

We now explain how (a) implies (b). This follows from the following criterion of Serre.

**Proposition 2.49 (Serre).** Fix  $A$  to be a  $\mathbb{N}_{\geq 0}$ -graded commutative algebra with  $A_0 = \mathbb{C}$ . Then  $A$  is free (as an algebra) if and only if there is a finite free resolution for  $\mathbb{C}$  as an  $A$ -module.

More generally, if  $\text{Spec } A$  is smooth if and only if it has finite homological dimension, meaning that each maximal ideal  $\mathfrak{m}$  grants  $A/\mathfrak{m}$  a finite projective resolution. To see why this criterion is helpful, note that we may simply choose the finite free resolution of  $\mathbb{C}$  for  $\mathbb{C}[\mathfrak{h}]$  and then simply restrict our ring to  $\mathbb{C}[\mathfrak{h}]^W$ , thereby producing a free resolution for  $\mathbb{C}$  over  $\mathbb{C}[\mathfrak{h}]^W$ . ■

**Remark 2.50.** Once we know that  $\mathbb{C}[\mathfrak{h}]^W$  is free, we automatically know that it is free of rank  $|W|$ . For example, one can show this by passing to the fraction fields and using Galois theory. Alternatively, the map  $\text{Spec } \mathbb{C}[\mathfrak{h}] \rightarrow \text{Spec } \mathbb{C}[\mathfrak{h}]^W$  is an integral map, so the Krull dimension must be preserved.

**Remark 2.51.** This ring  $A$  in the proof turns out to be the cohomology ring  $H^*(G/B)$ . One can calculate that  $\dim A_n$  is the number of Weyl elements of given length  $n$ . This is the sort of thing which can be checked by hand.

## 2.6 October 21

We began class by completing the proof of Theorem 2.45.

### 2.6.1 Serre's Criterion

Let's sketch the proof of Proposition 2.49. If  $A$  is a polynomial algebra  $\mathbb{C}[x_1, \dots, x_n]$ , then we can use the Koszul complex. More precisely, if  $A = \mathbb{C}[x]$ , then we have the resolution

$$0 \rightarrow \mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x] \rightarrow \mathbb{C} \rightarrow 0,$$

so if  $A = \mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[x_1] \otimes \dots \otimes \mathbb{C}[x_n]$ , then we can just take the tensor product of the free resolutions. In a more basis-free way, we can write the Koszul resolution as

$$0 \rightarrow \wedge^n V \otimes A \rightarrow \dots \rightarrow \wedge^2 V \otimes A \rightarrow V \otimes A \rightarrow A,$$

where  $V$  is some  $n$ -dimensional vector space, and  $A = k[V^*]$ .

Of course, we are more interested in the reverse direction. Quickly, if  $\dim A < \infty$  and  $A \neq \mathbb{C}$ , then there cannot be a finite free resolution of  $\mathbb{C}$  by considering Euler characteristics. Namely, for a graded vector space  $V$ , write  $P_V(t) := \sum_i t^i \dim V_i$ . Then any free resolution

$$0 \rightarrow V_n \otimes A \rightarrow \dots \rightarrow V_1 \otimes A \rightarrow V_0 \otimes A \rightarrow \mathbb{C} \rightarrow 0$$

must have

$$P_A(t) \left( \sum_{i=0}^n (-1)^i P_V(t) \right) = 1$$

by counting dimensions. We conclude that  $P_A(t) = 1$ , so  $A = \mathbb{C}$ .

Thus, we may suppose that  $A$  is infinite-dimensional. Then we select some  $a \in A$  which is not a zero divisor and which is not in the ideal  $I$  of positive-degree elements, and define  $B := A/(a)$ . The point is to show that  $k$  admits a finite free resolution over  $B$  implies that it admits a finite free resolution over  $A$ ; from here, an induction with some minimal generators shows that  $B$  being free implies that  $A$  is free. Indeed, take

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{C} \rightarrow 0$$

and apply  $-\otimes B$ . Of course, this doesn't have to be exact, but we can show that it has a direct summand which is a resolution of  $k$ ; the key input is that  $\text{Ext}^2(k, k) = 0$ .

### 2.6.2 Regular Elements

We recall the following definition.

**Definition 2.52 (regular).** An element  $x \in \mathfrak{g}$  is *regular* if and only if its centralizer  $\mathfrak{g}_x$  has

$$\dim \mathfrak{g}_x = \text{rank } r.$$

**Remark 2.53.** Suppose  $x$  is semisimple. Observe that  $\mathfrak{g}_x$  is the kernel of the map  $[X, -]$ , so by placing  $X$  in some Cartan subalgebra  $\mathfrak{h}$ , we find that  $\dim \mathfrak{g}_x \geq \dim \mathfrak{h}$ , so  $\dim \mathfrak{g}_x \geq \text{rank } \mathfrak{g}$ . Thus, we are saying that  $\dim \mathfrak{g}_x$  is as small as possible.

**Example 2.54.** In the case where  $\mathfrak{g} = \mathfrak{sl}(n)$ , one can put a given  $x \in \mathfrak{g}$  in Jordan normal form, and a calculation of the centralizer by hand shows that  $x$  is regular if and only if it has a single Jordan block for each eigenvalue, which is equivalent to the minimal polynomial being equal to the characteristic polynomial.

**Remark 2.55.** One can check that being regular is a Zariski open condition. For example, the only irregular element in  $\mathfrak{sl}(2)$  is zero.

**Proposition 2.56.** Fix a semisimple Lie algebra  $\mathfrak{g}$ . Let  $\mathfrak{h} \subseteq \mathfrak{g}$  be a Cartan subalgebra, with roots  $\Phi$  and simple roots  $\Delta$ . Then the element

$$e := \sum_{\alpha \in \Delta} e_{\alpha}$$

is regular.

*Proof.* Setting  $h := 2\rho^{\vee}$ , we find that  $\langle h, \alpha_i \rangle = 2$  for each simple root  $\alpha_i$ . As such,  $[h, e] = 2e$ . Now, recall that the elements  $h_i = [e_i, f_i]$  form basis of  $\mathfrak{h}$ , so we may find  $f$  of the form  $\sum_i f_i$  for which  $[e, f] = h$ . Now,  $[h, f_i] = -2f_i$  for each  $i$ , so  $[h, f] = -2f$ .

Thus,  $(e, f, h)$  is some  $\mathfrak{sl}(2)$ -triple, so we may calculate the centralizer of  $e$  using the representation theory of  $\mathfrak{sl}(2)$ . Note that only even-weight representations can appear because  $h$  acts on  $\mathfrak{g}_{\alpha}$  with nonzero even eigenvalues (namely, 2 for the simple roots) anyway. Then  $\mathfrak{g}_e$  is the number of summands in the decomposition into  $\mathfrak{sl}(2)$ -representations, which now equals the dimension of the weight-0 subspace. But the weight-0 subspace is simply the centralizer of  $h$ , which is exactly  $\mathfrak{h}$  because  $h$  acts on each  $\mathfrak{g}_{\alpha}$  by nonzero eigenvalues anyway. ■

**Remark 2.57.** It turns out that all regular nilpotent elements are conjugate to  $e$ .

## 2.7 October 23

We continue.

### 2.7.1 The Kostant Slice

Our story continues with the Kostant slice.

**Definition 2.58 (principal triple).** Fix a semisimple Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . Then we define the  $\mathfrak{sl}(2)$ -triple  $(e, f, h)$  so that  $e := \sum_i e_i$  and  $h := 2\rho^{\vee}$ . This is the *principal*  $\mathfrak{sl}(2)$  triple.

**Example 2.59.** If  $\mathfrak{g} = \mathfrak{sl}(n)$ , then  $e$  is the nilpotent matrix supported only just above the diagonal, where it has 1. In other words,  $e$  is a single Jordan block with eigenvalue 0. Note that the centralizer  $\mathfrak{g}_e$  consists of powers of  $e$ , which can be checked directly.

**Definition 2.60 (Kostant slice).** Fix a semisimple Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ , and let  $(e, f, h)$  be the principal  $\mathfrak{sl}(2)$ -triple. Then we define the *Kostant slice* to be  $S := f + \mathfrak{g}_e$ .

**Theorem 2.61.** Fix a semisimple Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ , and let  $(e, f, h)$  be the principal  $\mathfrak{sl}(2)$ -triple with Kostant slice  $S$ .

- (a) There is an isomorphism  $S \rightarrow \mathfrak{t}/W$  given by restriction, meaning that the map  $\mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[S]$  is an isomorphism.
- (b) There is an isomorphism  $S \rightarrow (f + \mathfrak{b})/U$  given by inclusion, and the action map  $(U \times S) \rightarrow (f + \mathfrak{b})$  induces an isomorphism, where  $\mathfrak{b}$  is the positive Borel subalgebra.

**Remark 2.62.** This is related to Whittaker models.



**Remark 2.63.** The Borel  $\mathfrak{b} \subseteq \mathfrak{g}$  is unique once we require that it contains  $e$ . (Such a Borel certainly exists by using the simple roots in  $e$ .) This  $\mathfrak{b}$  turns out to be the sum of eigenspaces where  $h$  acts with nonnegative eigenvalues; this subspace actually only depends on  $e$  because it is the subspace on which  $e^m$  acts by zero, where  $m$  is chosen to give us half the total dimension.

**Example 2.64.** If  $\mathfrak{g} = \mathfrak{sl}(n)$ , then  $S$  consists of  $f$  plus the upper-triangular matrices which are constant on the diagonals. One can use this description to prove (a) by hand.

Here is an application.

**Corollary 2.65.** Fix a semisimple Lie algebra  $\mathfrak{g}$ . Then  $\mathbb{C}[\mathfrak{g}]_+^G$  is a radical ideal whose zero set consists of the nilpotent elements.

*Proof.* Let's begin by explaining why  $\mathbb{C}[\mathfrak{g}]_+^G$  cuts out the nilpotents. In other words, we have to show that  $x \in \mathfrak{g}$  is nilpotent if and only if  $P(x) = 0$  for all  $P \in \mathbb{C}[\mathfrak{g}]_+^G$ .

1. Well, give  $x$  a Jordan decomposition  $x = x_s + x_n$ , so we may find a Cartan subalgebra  $\mathfrak{h}$  and Borel subalgebra  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$  for which  $x_s \in \mathfrak{h}$  and  $x_n \in \mathfrak{u}$ . (The fastest way to find  $\mathfrak{b}$  is to use Theorem 2.72, which we will prove later.) Then we claim that

$$P(x) \stackrel{?}{=} P(x_s).$$

Indeed, consider conjugation by  $\exp(t\rho^\vee)$ , which we can calculate sends the simple root  $e_\alpha$  to  $te_\alpha$ ; for a positive root  $\alpha$ , it sends  $e_\alpha$  to  $t^{\langle \rho^\vee, \alpha \rangle} e_\alpha$ . Writing  $x_n$  as a linear combination of the positive roots shows that conjugation by  $\exp(t\rho^\vee)$  fixes  $x_s$  (because conjugation within the torus does nothing) while adjusting  $x_n$  by some positive power of  $t$ . Sending  $t \rightarrow 0$  proves the claim.

2. We now see that we are done by the Chevalley restriction theorem:  $x$  is nilpotent if and only if  $x_s = 0$ , which is equivalent to having  $P(x_s) = 0$  for all  $P \in \mathbb{C}[\mathfrak{h}]_+^G$ ; in turn, it suffices to look at  $P \in \mathbb{C}[\mathfrak{h}]_+^W$  because the projection  $\mathfrak{h} \rightarrow \mathfrak{h}/W$  has well-behaved fibers. This completes the proof by the Chevalley restriction theorem because  $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[\mathfrak{g}]_+^G$ .

It remains to show that  $\mathbb{C}[\mathfrak{g}]_+^G$  is reduced.

1. Recall from Theorem 2.45 that this ideal admits homogeneous generators  $\{P_1, \dots, P_r\}$ , which are the polynomial generators for  $\mathbb{C}[\mathfrak{g}]^G$ . On the other hand,  $\dim \mathcal{N}$  is the dimension of  $\dim \mathfrak{g} - r$ : certainly  $\dim \mathcal{N} \geq \dim \mathfrak{g} - r$  because  $\mathcal{N}$  is cut out by  $r$  equations, but there is a surjective “moment map”  $T^*(G/B) \rightarrow \mathcal{N}$  (note  $T^*(G/B)$  consists of pairs  $(\mathfrak{b}, x)$  of some Borel  $\mathfrak{b} \subseteq \mathfrak{g}$  and  $x \in \text{rad } \mathfrak{b}$ , so the map sends  $(\mathfrak{b}, x) \mapsto x$ ). However,  $\dim T^*(G/B) = 2 \dim G/B$ , which is  $\dim \mathfrak{g} - r$ . Note that this also implies that  $\mathcal{N}$  is irreducible because it admits a surjection from the irreducible scheme  $T^*(G/B)$ .
2. It follows that the quotient  $\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]_+^G$  is Cohen–Macaulay: there is some general fact that if  $A$  is Cohen–Macaulay, then  $A/(a)$  is Cohen–Macaulay for any  $a \in A$  which is not a zero divisor. In more words, we can see that  $\mathcal{N}$  is a complete intersection by the considerations of the previous paragraph, which implies it is Cohen–Macaulay.

This then implies that every element has support of full dimension, where we are viewing elements as skyscraper modules. In particular, suppose there is nonzero nilpotent  $\varphi \in \mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]_+^G$ . Now, by Theorem 2.61, we see that the differentials  $dP_i|_f$  are linearly independent, so the zero set of  $P_i$  is smooth (and hence reduced) on a neighborhood of  $f$ , so  $\varphi$  must vanish in this open neighborhood; because  $\mathcal{N}$  is nilpotent, and the vanishing locus is closed, it follows that  $\varphi = 0$ . ■

**Remark 2.66.** It is conjectured that the scheme-theoretic pre-image of 0 under the map  $(x, y) \rightarrow [x, y]$  is reduced.

**Remark 2.67.** One does not expect  $\mathcal{N}$  to be smooth, but the open orbit consisting of the regular nilpotent elements is smooth because it is the open orbit of the action by a smooth algebraic group. (This is some general fact about group actions.)

## 2.7.2 Proof of the Kostant Slice

**Lemma 2.68.** Fix free homogeneous generators  $P_1, \dots, P_r$  of  $\mathbb{C}[\mathfrak{g}]^G$ , and consider the maps  $dP_i: \mathfrak{g} \rightarrow \mathfrak{g}$  given by the differential. Then  $dP_i|_x \in T_x \mathfrak{g}_x$  for all regular  $x$ , and these elements form a basis.

*Proof.* Here,  $dP_i$  is technically a map  $\mathfrak{g} \rightarrow \mathfrak{g}^*$ . The fact that  $dP_i(x) \in \mathfrak{g}_x$  follows because the polynomial  $P_i$  is  $G$ -invariant. For example,  $P_i(\text{Ad}_{\exp(y)} x) = P_i(x)$  for all  $y$ , so we conclude that  $dP_i$  vanishes on  $[\mathfrak{g}, x]$ . However, under the identification  $\mathfrak{g} = \mathfrak{g}^*$ , we see that  $[\mathfrak{g}, x]$  is the complement of  $\mathfrak{g}_x$ : an element  $z$  commutes with  $x$  if and only if  $([x, z], y) = 0$  for all  $y$ , which in turn is equivalent to  $(z, [x, y]) = 0$  for all  $y$ , which means that  $z$  is in the complement of  $[\mathfrak{g}, x]$ . (The implications also reverse.)

Consider the map  $\mathfrak{g} \rightarrow \mathfrak{h}/W$  induced by Chevalley restriction. For a fixed  $x$ , we may write  $\mathfrak{g} = V \oplus [\mathfrak{g}, x]$  for some complement  $V$  (say, under the Killing form); the previous paragraph reveals that  $V = \mathfrak{g}_x$ . Then we claim that the  $dP_i|_x$ s span if and only if the composite

$$V \xrightarrow{x} \mathfrak{g} \rightarrow \mathfrak{h}/W$$

is an isomorphism on tangent spaces, where the left map is given by  $v \mapsto (x + v)$ . Indeed, this is for size reasons:  $\dim V = \dim \mathfrak{g}_x = r$  because  $x$  is regular, and the tangent spaces of  $\mathfrak{h}/W$  are also  $r$ -dimensional for regular  $x$  (because  $\dim \mathfrak{h} = r$ ). By Chevalley restriction, the  $dP_i$ s span the target tangent space, so the above isomorphism will then guarantee that they span (equivalently, form a basis of) the source tangent space.

We are only going to show the spanning away from a codimension-2 subset of the regular locus  $\mathfrak{g}^{\text{reg}}$  in  $\mathfrak{g}$ . To see that this is enough, we consider the vector bundle  $\mathcal{V}$  over  $\mathfrak{g}^{\text{reg}}$  with fibers given by  $\mathfrak{g}_x$ .<sup>4</sup> Then the spanning of the sections  $dP_i$  amount to saying that the induced map  $\pi: \mathbb{C}^r \rightarrow \mathcal{V}$  is an isomorphism away from a codimension-2 subset of  $\mathfrak{g}^{\text{reg}}$ . However, this means that the polynomial  $\wedge^r \pi: \mathfrak{g}^{\text{reg}} \rightarrow \mathbb{C}$  is nonzero away from a codimension-2 subset, so it must be nonzero everywhere!

Thus, we are allowed to ignore a codimension-2 subset of regular elements in  $\mathfrak{g}$ . We now handle our  $x \in \mathfrak{g}$  in cases.

- We work with regular semisimple elements  $x$ . Here, we may choose  $\mathfrak{h}$  to contain  $x$ . Because  $x$  is regular, we see  $\mathfrak{h} = \mathfrak{g}_x$ . We thus would like to check that

$$\mathfrak{h} \xrightarrow{x} \mathfrak{g} \rightarrow \mathfrak{h}/W$$

is an isomorphism on tangent spaces. In other words, we would like to show that the projection  $\mathfrak{h} \rightarrow \mathfrak{h}/W$  is an isomorphism on tangent spaces at  $x$ .

Indeed, this quotient is étale at  $x$  by regularity, so this is not so bad. Alternatively, we can show the spanning directly. Note that  $W$  acts freely on the orbit of  $x$  by the regularity, so regularity provides us with an isomorphism  $T_x^*(\mathfrak{t}/W) \rightarrow T_x^*(\mathfrak{h})$ , which means that any  $\xi \in \mathfrak{h}^*$  can be lifted to an invariant polynomial in  $\mathbb{C}[\mathfrak{t}]^W$ . (Indeed, choose  $P_0$  with  $dP_0|_x = \xi$  and vanishing at the rest of the orbit of  $Wx$ , and then  $P = |W|^{-1} \sum_w w P_0$  will work.)

- We now work with generic elements  $x$  which are regular but not semisimple. The Jordan decomposition allows us to write  $x = s + n$  for semisimple  $s$  and nilpotent  $n$ ; we go ahead and extend  $s$  to a Cartan subalgebra  $\mathfrak{h}$ . Now, we cannot have that  $s$  is regular, so a generic such  $x$  will admit a unique root  $\alpha$  for which  $\langle s, \alpha \rangle = 0$ . Because  $s$  and  $n$  are forced to commute, we can see from the root decomposition of  $\mathfrak{g}$

<sup>4</sup> In short, this vector bundle arises by considering the kernel of the natural action of  $\mathfrak{g}$  on the trivial bundle  $\mathfrak{g}$  over  $\mathfrak{g}$  given by the adjoint. This is a vector bundle over the regular locus.

that  $n$  equals  $e_\alpha$  up to a scalar, so we may as well assume  $n = e_\alpha$ . To compute  $\mathfrak{g}_x$ , we note that  $\mathfrak{g}_x \subseteq \mathfrak{g}_s$  by the uniqueness of the Jordan decomposition, and

$$\mathfrak{g}_s = \alpha^\perp \oplus \mathfrak{sl}(2)_\alpha.$$

It follows that  $\mathfrak{g}_x = \alpha^\perp \oplus \mathbb{C}f_\alpha$  by further asking which elements of  $\mathfrak{g}_s$  (given above) commute with  $n = e_\alpha$ .

By a similar argument as in the previous point, we find that  $\mathfrak{h}/\langle s_\alpha \rangle \rightarrow \mathfrak{h}/W$  has non-degenerate differential at the image of  $x$  (namely, at  $s$ ). On the other hand, the map  $\mathfrak{g}_x \rightarrow \mathfrak{h}/\langle s_\alpha \rangle$  is the identity on  $\alpha^\perp$ , but we have an additional piece from  $f_\alpha$  which makes up for our last lost dimension after reducing the story to  $\mathfrak{sl}(2)_\alpha$ . ■

**Example 2.69.** With  $\mathfrak{g} = \mathfrak{sl}(n)$ , then  $P_i(X) = \text{tr } X^{i+1}$  (for  $i \in \{1, 2, \dots, n-1\}$ ). We claim that  $dP_i(X) = (i+1)X^i$ . Because diagonalizable matrices are Zariski open, we may restrict our attention to diagonal matrices, where

$$P_i(\text{diag}(a_1, \dots, a_n)) = \sum_j a_j^{i+1}.$$

Thus,  $dP_i = \sum_j (i+1)a_j^i da_j$ , which under the identification  $\mathfrak{g} \rightarrow \mathfrak{g}^*$  goes to the claimed matrix.

*Proof of Theorem 2.61(a).* We show the parts separately. For the injectivity, it is enough to check that  $G(S)$  contains a dense open subset in  $\mathfrak{g}$ , for which it is enough to check that  $T(G \times S) \rightarrow T\mathfrak{g} = \mathfrak{g}$  is surjective at any given point. For example, at  $(1, f) \mapsto f$ , this map is simply  $\mathfrak{g} \oplus \mathfrak{g}_e \rightarrow T_f \mathfrak{g} = \mathfrak{g}$  given by  $(x, s) \mapsto ([f, x] + s)$ . The surjectivity now follows from the representation theory of  $\mathfrak{sl}(2)$ : every vector in a representation is either in the image of  $\text{ad}_f$  or in the kernel of  $\text{ad}_e$  (which is  $\mathfrak{g}_e$ ).

We now turn to surjectivity, for which we use Lemma 2.68. We may as well consider the composite  $\mathbb{C}[\mathfrak{t}]^W \cong \mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[S]$ , and we will use graded Nakayama. Note that a grading is equivalent data to an action by  $\mathbb{G}_{m, \mathbb{C}}$ , meaning that the  $n$ th graded piece has an action by  $(-)^n$ .

As such, we would like to give  $\mathbb{C}[S]$  an action by  $\mathbb{G}_m$  as well. The obvious scaling won't work because it moves  $f$ . To fix this, lift  $\mathfrak{sl}(2) \rightarrow \mathfrak{g}$  to some representation  $\text{SL}(2) \rightarrow G$ , and we will have  $t$  act by the image of  $\text{Ad}(\text{diag}(\sqrt{t}, 1/\sqrt{t}))$ . Explicitly,

$$t \cdot s := \text{Ad}_{\text{diag}(\sqrt{t}, 1/\sqrt{t})}(ts).$$

Technically, this formula does not look algebraic because of the square roots, but it is: the principal representation  $\mathfrak{g}$  of  $\mathfrak{sl}(2)$  has only even weights, so the action of  $\text{SL}(2)$  will factor through  $\text{SL}(2)/\{\pm 1\} = \text{PGL}(2)$ . As such, the action on  $\mathbb{C}[S]$  is given by  $t$  acting by  $\text{Ad}(\text{diag}(t, 1))(ts)$ .

Now,  $\mathbb{C}[S]$  is a polynomial algebra with generators in degrees  $m_i + 1$ , where  $\{2m_1, \dots, 2m_r\}$  are the highest weights of  $\mathfrak{g}$  as the principal representation. Indeed, just  $m_i$  would come from acting on  $S$  by  $ts$ , but the extra adjoint shifts by 1.

We now are equipped with an inclusion  $\mathbb{C}[\mathfrak{t}]^W \hookrightarrow \mathbb{C}[S]$  of positively graded polynomial algebras, and Lemma 2.68 tells us that the differential map  $S \rightarrow \mathfrak{t}/W|_f$  is an isomorphism. Unwinding the definition of the tangent space, we have an isomorphism

$$\frac{\mathbb{C}[\mathfrak{t}]_+^W}{(\mathbb{C}[\mathfrak{t}]^W)_+^2} \xrightarrow{\sim} \frac{\mathbb{C}[S]_+}{(\mathbb{C}[S])_+^2},$$

which now upgrades to an isomorphism of the algebras by graded Nakayama (i.e., doing an induction on the degree). ■

**Remark 2.70.** The proof of (a) shows that  $\deg P_i = m_i + 1$ . Indeed, we know that the generators of each of the polynomial algebras  $\mathbb{C}[S]$  and  $\mathbb{C}[\mathfrak{t}]^W$  must have the same degree!

**Remark 2.71.** Let's contextualize part (b). Under the identification  $\mathfrak{g} \cong \mathfrak{g}^*$ , the space  $\mathfrak{b}$  goes to  $\mathfrak{u}^\perp$  because  $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ . As such,  $f + \mathfrak{b}$  is identified with

$$\{\xi : (\xi, x) = (f, x) \text{ for } x \in \mathfrak{u}\}.$$

Now, note that  $(f, x) = 0$  for  $x \in [\mathfrak{u}, \mathfrak{u}]$ : one can write  $x$  into the eigenspaces of roots  $e_\alpha$ , but this decomposition can feature no simple roots, so  $(f, x) = 0$  follows. Accordingly, the functional  $\psi := (f, -)$  is a Lie algebra homomorphism  $\mathfrak{u} \rightarrow \mathbb{C}$ , and the pre-image of  $\psi$  along the projection  $\mathfrak{g} \twoheadrightarrow \mathfrak{u}^*$  is  $f + \mathfrak{b}$ . As such,

$$\mathbb{C}[f + \mathfrak{b}] = \mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}[\mathfrak{u}^*]} \mathbb{C}_\psi.$$

One can think of this latter algebra as a degeneration of  $U\mathfrak{g} \otimes_{U\mathfrak{u}} \mathbb{C}_\psi$ , whose  $U$ -invariants are given by  $\text{End}_{\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{u}} \mathbb{C}_\psi)$ .

## 2.8 October 28

We continue.

### 2.8.1 The Jacobson–Morosov Theorem

For no reason in particular, we begin with a discussion of the Jacobson–Morosov theorem.

**Theorem 2.72 (Jacobson–Morosov).** Fix a complex semisimple Lie algebra  $\mathfrak{g}$ , and choose a nilpotent element  $e \in \mathfrak{g}$ . Then there is an  $\mathfrak{sl}(2)$ -triple  $(e, f, h)$  in  $\mathfrak{g}$ .

*Proof.* We follow [CG10, Section 3.7]. Throughout,  $(-, -)$  denotes the Killing form on  $\mathfrak{g}$ .

1. We show that there is a semisimple element  $h \in \mathfrak{g}$  for which  $[e, h] = 2e$ . The semisimplicity is not much of a constraint: if we can find any  $h \in \mathfrak{g}$  for which  $\text{ad}_h(e) = -2e$ , then we may consider the Jordan decomposition  $h = h_s + h_n$  of  $h$ , which is induced by the Jordan decomposition of  $\text{ad}_h \in \mathfrak{gl}(\mathfrak{g})$ . In particular, because  $e$  is an eigenvector of  $\text{ad}_h$  with eigenvalue  $-2$ , the same will be true of the semisimple element  $h_s$ .

Thus, it remains to explain why  $2e$  is in the image of  $\text{ad}_e : \mathfrak{g} \rightarrow \mathfrak{g}$ . Well,  $\text{ad}_e$  has kernel  $\mathfrak{g}_e$ , and we claim that its image is contained in  $\mathfrak{g}_e^\perp$ ; this implies that  $\text{ad}_e$  restricts to an injection  $\mathfrak{g}/\mathfrak{g}_e \rightarrow \mathfrak{g}_e^\perp$ , which is then an isomorphism for dimension reasons. To prove the claim, we choose some  $x \in \mathfrak{g}$  and  $y \in \mathfrak{g}_e$  and compute

$$(\text{ad}_e x, y) = -(x, [e, y]) = 0.$$

2. We explain how to construct  $f$  in the case that  $\mathfrak{g}_e$  is nilpotent. We have to show that there is some  $f$  for which  $[e, f] = h$ , and then we may adjust  $f$  to satisfy  $[h, f] = -2f$  formally: note that

$$\text{ad}_h \circ \text{ad}_e = \text{ad}_e \circ (\text{ad}_h + 2),$$

so  $\text{ad}_e$  sends eigenvectors of  $h$  with eigenvalue  $\lambda$  to eigenvectors of  $h$  with eigenvalue  $\lambda + 2$ , so one merely needs to replace the given  $f$  with  $\text{ad}_e f = h$  with the component of  $f$  for which the eigenvalue of  $\text{ad}_h$  acting on  $f$  is  $-2$ .

We thus want to show that  $h$  is in the image of  $\text{ad}_e : \mathfrak{g} \rightarrow \mathfrak{g}$ . As discussed in the preceding step, it is enough to show that  $h \in \mathfrak{g}_e^\perp$ . For this, consider the subspace  $\mathbb{C}h \oplus \mathfrak{g}_e$  of  $\mathfrak{g}$ . This is a Lie subalgebra because  $[h, \mathfrak{g}_e] \subseteq \mathfrak{g}_e$ : for any  $x \in \mathfrak{g}_e$ , we see that  $[e, [h, x]]$  equals

$$[[e, h], x] - [[e, x], h] = [2e, x] - [0, h],$$

which vanishes. In fact, because  $\mathfrak{g}_e$  is nilpotent, the fact that  $[h, \mathfrak{g}_e] \subseteq \mathfrak{g}_e$  shows that  $\mathbb{C}h \oplus \mathfrak{g}_e$  is solvable. As such, there is a basis of  $\mathfrak{g}$  so that the adjoint representation restricted to  $\mathbb{C}h \oplus \mathfrak{g}_e$  is upper-triangular. Because  $h$  is semisimple, it must go to a semisimple element, so it goes to a diagonal element under the new basis of  $\mathfrak{g}$ . On the other hand,  $\mathfrak{g}_e$  is nilpotent, so it must go to strictly upper triangular elements. It follows that  $\text{ad}_h \circ \text{ad}_x$  is still strictly upper triangular for any  $x \in \mathfrak{g}_e$ , so  $(h, x) = 0$  follows.

3. We explain how to construct  $f$  in the general case. We will do this by induction on  $\mathfrak{g}$ . If  $\mathfrak{g}_e$  is already nilpotent (for example, if  $\mathfrak{g} = 0$ ), then we are done by the previous step. Otherwise, the Jordan decomposition implies that  $\mathfrak{g}_e$  contains a nonzero semisimple element  $x$ . By standard weight theory, the centralizer  $\mathfrak{g}_x$  is reductive (indeed, the Killing form restricted to  $\mathfrak{g}_x$  can be shown to be non-degenerate).

We now claim that  $e$  is contained in the semisimple part of  $\mathfrak{g}_x$ , which will complete the proof by passing (inductively) to  $\mathfrak{g}_x$  because  $\mathfrak{g}_x$  is strictly smaller than  $\mathfrak{g}$ . (Indeed,  $\text{ad}_x$  is semisimple, so  $\mathfrak{g}_x = \mathfrak{g}$  implies that  $x = 0$ .) Well,  $\mathfrak{g}_x$  splits into its center (which is a torus) and its semisimple part, so because  $e$  is nilpotent, it cannot have any component from the torus (by the Jordan decomposition). ■

**Remark 2.73.** In the  $\mathfrak{sl}(2)$ -triple  $(e, f, h)$ , we see that  $f$  must be nilpotent, and  $h$  must be semisimple. Indeed,  $(e, f, h)$  arise from a morphism  $\varphi: \mathfrak{sl}(2) \rightarrow \mathfrak{g}$ . Thus,  $\text{ad} \circ \varphi$  is just some representation of  $\mathfrak{sl}(2)$ , and the representation theory of  $\mathfrak{sl}(2)$  shows that  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  must go to a nilpotent element, and  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  must go to a semisimple element.

**Remark 2.74.** Given  $h$ , we can see that  $f$  is unique: if  $f$  and  $f'$  both complete the  $\mathfrak{sl}(2)$ -triple, then we have two algebra morphisms  $\varphi, \varphi': \mathfrak{sl}(2) \rightarrow \mathfrak{g}$  with  $\varphi(e) = \varphi'(e)$  and  $\varphi(h) = \varphi'(h)$ . It follows that  $\ker \varphi \subseteq \mathfrak{sl}(2)$  is some Lie ideal containing both  $e$  and  $h$ , so it must totally vanish, thereby implying that  $f = \varphi(f) = \varphi'(f) = f'$ .

**Remark 2.75.** Let's explain how unique  $h$  is. Given two semisimple elements  $h$  and  $h'$  which extend to  $\mathfrak{sl}(2)$ -triples  $(e, f, h)$  and  $(e, f', h')$ , we see that  $h - h' \in \mathfrak{g}_e \cap \text{im ad}_e$ . Now, decompose  $\mathfrak{g}$  according to the eigenspaces  $\mathfrak{g}[i]$  of  $\text{ad}_h$ , which we see must have only integral eigenvalues  $i$ . It follows that  $h - h'$  is contained in

$$\mathfrak{u} := \bigoplus_{i>0} \mathfrak{g}_e[i] = \mathfrak{g}_e \cap \text{im ad}_e.$$

It turns out that the corresponding Lie subgroup  $U \subseteq G_{\text{ad}}$  has  $h + \mathfrak{u} = U \cdot h$  (roughly speaking,  $[\mathfrak{u}, h] = \mathfrak{u}$  implies that the orbit is open, and the orbit is closed because  $U$  is unipotent), so there is some  $u \in U$  for which  $u \cdot h = h'$ . Thus, we see that  $h$  is unique up to multiplication by an element of  $\exp(\mathfrak{g}_e)$ .

**Remark 2.76.** Here is an amusing application: for any representation  $V$  of  $\mathfrak{g}$ , we see that a nilpotent element  $e \in \mathfrak{g}$  must act by a nilpotent operator on  $V$ . Indeed, Theorem 2.72 allows us to pass the statement to  $\mathfrak{sl}(2)$ , where it follows from the representation theory of  $\mathfrak{sl}(2)$ .

We will use Theorem 2.72 to construct a canonical filtration on  $\mathfrak{g}$ .

**Definition 2.77.** Fix a representation  $V$  of a complex semisimple Lie algebra  $\mathfrak{g}$ , and choose a nilpotent element  $e \in \mathfrak{g}$ . Then extend  $e$  to an  $\mathfrak{sl}(2)$ -triple  $(e, f, h)$ . By the representation theory of  $\mathfrak{sl}(2)$ , the eigenvalues of  $h$  acting on  $V$  must all be integers, so we define  $V[i]$  to be the eigenspace with eigenvalue  $i \in \mathbb{Z}$ . We then define the filtration

$$V_{\geq i} := \bigoplus_{j \geq i} V[j].$$

**Remark 2.78.** The given filtration does not depend on the choice of  $h$ . (Of course, its definition shows that it does not depend on the choice of  $f$ .) Indeed, any other  $\mathfrak{sl}(2)$ -triple  $(e, f', h')$  admits some  $u \in G_{\text{ad}}$  as in Remark 2.75 for which  $h' = uhu^{-1}$ . However,  $\text{Ad}_u$  preserves the filtration on  $\mathfrak{g}$  because the Lie subgroup  $U$  of Remark 2.75 has its Lie algebra  $\mathfrak{u}$  contained in  $\mathfrak{g}_{\geq 1}$ , so for any  $x \in \mathfrak{u}$ , we see that  $\text{ad}_x$  and therefore  $\text{Ad}_{\exp(x)}$  preserves the filtration on  $\mathfrak{g}$ . The same claim then follows for the representation  $V$  by passing to  $\mathfrak{gl}(V)$ .

**Remark 2.79.** If  $V$  is finite-dimensional, then the filtration is exhaustive. Indeed,  $h$  has only finitely many eigenvalues.

## 2.8.2 Completing the Kostant Slice

We now prove Theorem 2.61(b).

*Proof of Theorem 2.61.* It suffices to construct an algebraic bijection  $U \times S \rightarrow f + \mathfrak{b}$ . There is certainly a natural map  $U \times S \rightarrow \mathfrak{g}$  given by  $(u, f + x) \mapsto u \cdot (f + x)$ . Observe that the action of  $h$  on  $\mathfrak{g}$  induces the filtration  $\{\mathfrak{g}_{\geq i}\}_{i \in \mathbb{Z}}$ . Here are our checks on this action map.

- Well-defined: we claim that the natural map outputs to  $f + \mathfrak{b}$ . To start, we note that  $\mathfrak{g}_e \subseteq \mathfrak{b}$ , and  $U$  preserves the positive filtration on  $\mathfrak{g}$ , so  $[\mathfrak{u}, \mathfrak{g}_e] \subseteq \mathfrak{b}$ , so  $U \cdot \mathfrak{g}_e \subseteq \mathfrak{b}$ . Thus, it remains to show that  $U \cdot f \subseteq f + \mathfrak{b}$ , for which it suffices to show that  $[\mathfrak{u}, f] \subseteq \mathfrak{b}$ . This is true for weight reasons:  $f \in \mathfrak{g}_{-2}$  while  $\mathfrak{u} \subseteq \mathfrak{g}_{\geq 2}$  because  $h$  acts with only even weights.
- Surjective: given  $x \in \mathfrak{b}$ , we need to find  $u \in U$  with  $u \cdot (f + x) \in f + \mathfrak{g}_e$ . (The surjectivity then follows by inverting  $u$ .) As above, we will use the natural filtration on  $\mathfrak{g}$ ; for example, we know that  $\mathfrak{b} = [f, \mathfrak{u}] + \mathfrak{g}_e$  because  $\mathfrak{b} = \mathfrak{g}_{\geq 0}$  and  $\mathfrak{u} = \mathfrak{g}_{\geq 2}$ . Additionally, we can see that this filtration has  $[f, \mathfrak{u}] \cap \mathfrak{g}_e = 0$  by comparing the filtration with the representation theory of  $\mathfrak{sl}(2)$ .

We now show by induction on  $d$  that there is some  $u \in U$  with  $u \cdot (f + x)$  in  $f + \mathfrak{g}_e + \mathfrak{g}_{\geq d}$ . There is nothing to do for  $d = 0$  because then  $\mathfrak{g}_{\geq 0} = \mathfrak{b}$ . Thus, if we suppose that we already have  $u \cdot (f + x) \in f + \mathfrak{g}_e + \mathfrak{g}_{\geq d}$ , we need to find a way to kill the piece  $x_d$  in degree  $d$ . As such, we write

$$x_d := x^\circ + [f, y]$$

where  $x^\circ \in \mathfrak{g}_e$  and  $y \in \mathfrak{g}_{d-2}$ . Then one can check that replacing  $u$  with  $u \exp(-y)$  will work.

- We omit the proof of injectivity. In short, if  $u \cdot (f + x) = (f + x')$ , then one can find  $y \in \mathfrak{u}$  with  $[f + x, y] = 0$ , which can be seen to produce a problem after going down to  $\mathfrak{sl}(2)$ . ■

## 2.8.3 Hamiltonian Reduction

One can view Theorem 2.61 as a variant of Hamiltonian reduction. To explain this, we must define a Poisson variety.

**Definition 2.80 (Poisson variety).** A *Poisson variety* is variety  $X$  for which the ring of global functions  $\mathcal{O}(X)$  is equipped with a Lie bracket  $\{-, -\}$  such that  $\{f, -\}$  is a derivation for all  $f$ . In other words,

$$\{f, gh\} = h\{f, g\} + g\{f, h\}$$

**Remark 2.81.** An alternating bracket  $\{-, -\}$  on  $\mathcal{O}(X)$  can be described by  $\pi \in \Gamma(X, \wedge^2 T^*X)$ , from which the bracket is defined by

$$\{f, g\} := \langle \pi, df \wedge dg \rangle.$$

This construction also makes it clear that an alternating bracket  $\{-, -\}$  gives rise to a unique  $\pi$ . Of course, there are some conditions (namely,  $[\pi, \pi] = 0$ ) in order to have a Poisson structure on  $X$ .

**Example 2.82.** If  $\pi \in \Gamma(X, \wedge^2 T^*X)$  is non-degenerate (i.e., if  $\dim X$  is some even integer  $2m$  for which  $\wedge^m \pi$  vanishes nowhere), then we may define  $\omega := \pi^{-1}$  to be a class in  $\wedge^2 T^*X = \Omega^2 X$ . In this case,  $[\pi, \pi] = 0$  is equivalent to  $d\omega = 0$ , meaning that a non-degenerate Poisson structure on  $X$  is equivalent to having a symplectic manifold  $(X, \omega)$ .

**Example 2.83.** As an instance of the above example, we see that the cotangent space  $T^*M$  of a smooth manifold  $M$  admits a symplectic structure  $\omega := d\eta$ , where

$$\eta := \sum_{ij} p_i dq_j,$$

where the  $p_i$ s and  $q_j$ s are suitably defined local coordinates. Thus,  $T^*M$  admits the structure of a Poisson variety.

**Example 2.84.** The affine variety  $X := \mathfrak{g}^*$  admits a Poisson structure by taking  $\{-, -\}$  to be the usual bracket on linear functionals. For example, any  $G$ -orbit in  $\mathfrak{g}^*$  gains a symplectic structure as in the previous example, meaning that all  $G$ -orbits are even-dimensional!

**Remark 2.85.** One can unwind the proof that  $G$ -orbits in  $\mathfrak{g}$  have even dimension as follows: the tangent space of an orbit at some  $x \in \mathfrak{g}$  is  $\mathfrak{g}/[\mathfrak{g}, x]$ , which admits a well-defined alternating non-degenerate pairing

$$\omega(y, z) := ([y, z], x),$$

where  $(-, -)$  is the Killing form on  $\mathfrak{g}$ . It then follows that the tangent space is even-dimensional, from which the result follows.

Group actions on Poisson varieties frequently give rise to moment maps.

**Definition 2.86 (moment map).** Fix a Lie group  $G$  acting on a Poisson variety  $X$ . Then a *moment map* is a map  $\mu: \mathfrak{g} \rightarrow \mathcal{O}(X)$  making the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mu} & \mathcal{O}(X) \\ & \searrow & \downarrow \\ & & \text{Vect}(X) \end{array}$$

commute; here, the map  $\mathfrak{g} \rightarrow \text{Vect}(X)$  is induced by the  $G$ -action, and the map  $\mathcal{O}(X) \rightarrow \text{Vect}(X)$  is given by  $f \mapsto df \perp \pi$ .

**Remark 2.87.** One can equivalently describe  $\mu$  as the data of a  $G$ -equivariant map  $X \rightarrow \mathfrak{g}^*$  satisfying some commutativity constraints.

**Example 2.88.** If  $G$  acts on a manifold  $M$ , then the action extends to  $X := T^*M$ , and there is a moment map  $\mu: X \rightarrow \mathfrak{g}^*$  given by

$$\mu(x, \xi): a \mapsto \langle a \cdot x, \xi \rangle,$$

where  $a \cdot x$  refers to the induced  $\mathfrak{g}$ -action.

**Theorem 2.89 (Hamiltonian reduction).** Let  $G$  act freely on a manifold  $M$ , and let  $N$  be the quotient. Then it turns out that

$$T^*N = \mu^{-1}(\{0\})/G,$$

which is referred to as Hamiltonian reduction.

**Remark 2.90.** The condition that  $G$  acts freely on  $M$  means that the induced projection  $M \rightarrow N$  is a principal  $G$ -bundle.

**Remark 2.91.** Let's relate Theorem 2.89 to Theorem 2.61. Indeed, Theorem 2.61(b) amounts to the statement that

$$\mathfrak{g}^*/(U, \psi_f) = S,$$

where  $\psi_f: x \mapsto [f, x]$ . This is analogous to a calculation of a tangent space appearing in Theorem 2.89.

### 2.8.4 Whittaker Things

We close our discussion today by defining some objects related to Whittaker modules.

**Definition 2.92 (Whittaker generator).** Fix a complex semisimple Lie algebra  $\mathfrak{g}$ , and let  $(e, f, h)$  be the principal  $\mathfrak{sl}(2)$ -triple. Then the *Whittaker generator* is  $W := U\mathfrak{g} \otimes_{U\mathfrak{u}} \mathbb{C}\psi$ , where  $\psi: \mathfrak{u} \rightarrow \mathbb{C}$  is defined by  $\psi(x) := [x, f]$ .

**Remark 2.93.** To check that  $\psi$  is a character, we note that  $\psi$  vanishes on  $[\mathfrak{u}, \mathfrak{u}]$  for weight reasons.

**Definition 2.94 (Whittaker module).** Fix a complex semisimple Lie algebra. Then a  $\mathfrak{g}$ -module  $M$  is *Whittaker* if and only if  $M \otimes_{U\mathfrak{u}} \mathbb{C}_{-\psi}$  has the action of  $\mathfrak{u}$  to be locally nilpotent. In other words, each vector in  $M \otimes_{U\mathfrak{u}} \mathbb{C}_{-\psi}$  is killed by some power of the  $\mathfrak{u}$ -action.

**Theorem 2.95 (Kostant).** Fix a complex semisimple Lie algebra  $\mathfrak{g}$ , and let  $(e, f, h)$  be the principal  $\mathfrak{sl}(2)$ -triple.

- (a) There is an isomorphism  $Z\mathfrak{g} \rightarrow \text{End}(W)$ , and  $\text{Ext}_{\mathfrak{g}}^i(W, W) = 0$  for all  $i > 0$ .
- (b) The category of Whittaker modules is equivalent to the category of  $Z\mathfrak{g}$ -modules. The functor takes a Whittaker module  $M$  to  $\text{Hom}(W, M)$ .



## THEME 3

# EXAMPLES

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### 3.1 October 30

I have regained access to my computer. Thank you, Hui!

#### 3.1.1 Extensions of the Kostant Slice

Let  $e$  be any nilpotent. By the Jacobson–Morosov theorem, we can fit  $e$  into an  $\mathfrak{sl}(2)$ -triple in  $\mathfrak{g}$ . Then

$$S_f := f + \mathfrak{g}_e$$

is another slice to the orbit of  $Gf$ , meaning that  $T_f \mathfrak{g} = T_f Gf \oplus T_f S_f$ . Of course, one can compute  $T_f Gf = [f, \mathfrak{g}]$  and  $T_f S_f = \mathfrak{g}_e$ .

It is interesting to consider the kernel  $S_f^\circ$  of the composite  $S_f \subseteq \mathfrak{g} \rightarrow \mathfrak{h}/W$ . The kernel of the map  $\mathfrak{g} \rightarrow \mathfrak{h}/W$  is just  $\mathcal{N}$ , so we see  $S_f^\circ = S_f \cap \mathcal{N}$ . If  $\mathfrak{g}$  is simple, then  $G$  acts on  $\mathcal{N}$  with finitely many orbits. For symplectic reasons, all orbits are even-dimensional. It turns out that there is a unique open orbit, consisting of the regular nilpotents; and then it turns out that there is a unique orbit of codimension 2, called the “subregular” nilpotents.

**Example 3.1.** Let  $J_m$  be the  $m \times m$  nilpotent Jordan block of size  $m$ . With  $\mathfrak{g} = \mathfrak{sl}(n)$ , then the regular orbit is the orbit of  $J_n$ , and the subregular orbit is the orbit of  $J_{n-1} \oplus J_1$ .

Now, in the subregular case, it turns out that  $\dim S_f^\circ = 2$ , and it can be seen to be a rational surface singularity of the type  $\mathbb{A}^2/\Gamma = \text{Spec } \mathbb{C}[x, y]^\Gamma$ , where  $\Gamma \subseteq \text{SL}(2)$  is a finite subgroup.

**Theorem 3.2.** There is a bijection between all the following items.

- (i) Finite subgroups of  $\text{SL}(2)$ .
- (ii) Rational surface singularities.
- (iii) Simply laced Dynkin graphs.
- (iv) Simply laced simple Lie algebras.

*Sketch.* We have already explained how to go from (iv) to (i); it turns out that adding in non-simply-laced simple Lie algebras does not do much.

- Starting with a finite subgroup  $\Gamma \subseteq \mathrm{SL}(2)$ , we build a graph whose vertices are the nontrivial irreducible representations of  $\Gamma$ , and we draw an edge between vertices  $\rho$  and  $\rho'$  if and only if  $\mathrm{Hom}(\rho \otimes \mathrm{std}, \rho') \neq 0$ . (Note that this condition is symmetric because  $\mathrm{std}$  is self-dual, and  $\dim \mathrm{Hom}(-, -)$  is symmetric.) It turns out that this is a simply laced Dynkin graph, and it provides a bijection between (i) and (iii).

For example, with the subgroup

$$\Gamma = \left\{ \begin{bmatrix} \zeta^a & \\ & \zeta^{-a} \end{bmatrix} : a \in \mathbb{Z} \right\},$$

where  $\zeta$  is some root of unity, we recover type  $A$  graphs. Dihedral groups recover type  $D$  graphs. It turns out that  $A_5 \subseteq \mathrm{SL}(2)$  recovers  $E_8$ ; other platonic solids yield  $E_6$  and  $E_7$ .

- A rational surface singularity  $S$  (with singularity at  $p$ ) admits a minimal resolution  $\pi: \tilde{S} \rightarrow S$ . It turns out that  $\pi^{-1}(\{p\})$  is some union of projective lines. One builds a graph by declaring the lines to be vertices, and two vertices are connected if and only if the lines intersect. ■

**Remark 3.3.** One can recover the resolution  $\tilde{S}$  from the  $S := S_f^\circ \cap \mathcal{N}$  arising from a simply laced Lie algebra  $\mathfrak{g}$  by resolving  $\tilde{N}$  as  $T^*(G/B)$ . (This is the “Springer resolution.”) Here,  $T_x(G/B)$  can be thought of as  $\mathfrak{g}/\mathfrak{b}_x$ , so  $T_x^*(G/B)$  can be identified with  $\mathfrak{b}_x^\perp$ , which is simply the unipotent radical  $\mathfrak{u}_x$ .

**Remark 3.4.** It turns out that the composite

$$T^*(G/B) \rightarrow \mathcal{N} \subseteq \mathfrak{g}^*$$

is the moment map. It is interesting to compute the pre-image of  $J_1 \oplus J_{n-1}$  in the case  $\mathfrak{sl}(n)$  where we expect some  $\mathbb{P}^1$ s intersected as described by the type- $A$  graph.

**Remark 3.5.** A non-simply laced diagram  $D$  can be “unfolded” (for example,  $G_2$  is simply  $D_4$  modulo an  $S_3$ -action), which allows us to extend some of the results to include non-simply-laced cases.

**Remark 3.6.** One can generalize the isomorphism  $S \rightarrow (f + \mathfrak{b})/U = \mathfrak{g}^*/(U, \psi)$ . In particular, if  $e$  is even (i.e.,  $h$  acts only by even eigenvalues in  $\mathfrak{g}$ ), then the linear functional  $\psi_f: x \mapsto (f, x)$  as before has weight 2 and thus vanishes on  $[\mathfrak{u}, \mathfrak{u}]$  for weight reasons. Thus, there continues to be an isomorphism  $S \rightarrow \mathfrak{g}^*/(U, \psi_f)$ . For example, for  $\mathfrak{sl}(n)$ , the condition on  $e$  is equivalent to all blocks having sizes of the same parity.

**Remark 3.7.** One can generalize the isomorphism  $S \rightarrow (f + \mathfrak{b})/U = \mathfrak{g}^*/(U, \psi)$  without hypotheses on  $e$ . The idea is that

$$\omega(x, y) := (f, [x, y])$$

turns out to be non-degenerate (and alternating) on  $\mathfrak{g}_1$ . Then we can choose a Lagrangian  $\mathcal{L} \subseteq \mathfrak{g}_1$  and define  $\mathfrak{u} := \mathcal{L} \oplus \mathfrak{g}_{\geq 2}$ . We still have  $\psi_f|_{[\mathfrak{u}, \mathfrak{u}]} = 0$ , and the story goes through. (As a sanity check, one can verify that  $\dim \mathfrak{g} - \dim S = 2 \dim \mathfrak{u}$ .)

**Remark 3.8 (Kostant–Rallis slice).** Suppose  $\mathfrak{g}$  admits an involution  $\theta$ , so we can decompose  $\mathfrak{g}$  into  $\mathfrak{k} \oplus \mathfrak{p}$  as the  $(+1)$ - and  $(-1)$ -eigenspaces for  $\theta$ . For a generic nilpotent  $e$  in  $\mathfrak{p}$ , one can extend it to an  $\mathfrak{sl}(2)$ -triple with  $f \in \mathfrak{p}$  and  $h \in \mathfrak{k}$ . It turns out that there is a bijection

$$f + (\mathfrak{g}_e \cap \mathfrak{p}) \rightarrow \mathfrak{p}/K.$$

The left-hand side  $S_f := f + (\mathfrak{g}_e \cap \mathfrak{p})$  is known as the Kostant–Rallis slice. It turns out that  $\mathfrak{p}/K$  is further isomorphic (as in Chevalley restriction) to  $\mathfrak{a}/W^\theta$ , where  $\mathfrak{a} \subseteq \mathfrak{p}$  is a maximal abelian subalgebra, and  $W^\theta$  is the subgroup of the Weyl group preserving  $\mathfrak{a}$ .

### 3.1.2 Representations of $\mathrm{SL}(2, \mathbb{R})$

Set  $G := \mathrm{SL}(2, \mathbb{R})$  and  $K := \mathrm{SO}(2, \mathbb{R})$ . Then we may let  $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{C})$  with  $K_{\mathbb{C}} = \mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^\times$ . We are interested in classifying irreducible  $(\mathfrak{g}, K)$ -modules.

Well, let  $V$  be some irreducible  $(\mathfrak{g}, K)$ -module. It is convenient to give  $\mathfrak{g}$  a basis with  $h = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ; this generates  $\mathfrak{k}_{\mathbb{C}}$ , and it is multiplied by  $i$  to make it conjugate to the usual value of  $h$ . For completeness, we note

$$e = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad \text{and} \quad f = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & -1 \end{bmatrix}$$

as well. Notably, complex conjugation acts by  $(e, f, h) \mapsto (-h, f, e)$ , as one would expect from the compact involution. Now, the  $\mathfrak{k}$ -action on  $V$  must integrate to  $K$ , which means that it admits a grading  $V = \bigoplus_{n \in \mathbb{Z}} V_n$  so that  $t \in K$  acts by  $t^n$ , which then means that  $h$  acts with eigenvalue  $n \in \mathbb{Z}$ .

Now, by standard  $\mathfrak{sl}(2)$ -theory, we see that  $e: V \rightarrow V$  acts by weight 2 and  $f: V \rightarrow V$  acts by weight  $-2$ , so if  $V$  is irreducible, then it must either have all even or all odd weights. To gain further information, we recall the Casimir element  $C := ef + fe + \frac{1}{2}h^2$  must act by a scalar  $c \in \mathbb{C}$  on  $V$ . Thus, by the PBW theorem, one can conclude that  $\dim V_n \leq 1$  for each  $n$ : any element of  $U\mathfrak{g}$  of weight 0 can be written as a polynomial in  $\mathbb{C}[h, C]$ , so

$$(U\mathfrak{g} \cdot x_n)_n = (U\mathfrak{g})_0 \cdot x_n = \mathbb{C}x_n$$

for any given  $x_n \in V_n$ , so  $V_n = \mathbb{C}x_n$  follows by irreducibility.

Additionally, we note that if  $V_n = 0$  for any  $n$ , we conclude that  $V = V_{<n} \oplus V_{>n}$  because both spaces are  $\mathfrak{sl}(2)$ -invariant, so we must have  $V = V_{<n}$  or  $V = V_{>n}$ . (In fact, the same claim holds whenever  $e$  or  $f$  have a kernel.) It follows that

$$\{n \in \mathbb{Z} : V_n \neq 0\}$$

is either a segment (i.e., evens or odds in  $[a, b]$ ) or a ray (i.e., even or odds in  $[a, \infty)$  or  $(-\infty, a]$ ) or everything (i.e., even or odds in  $\mathbb{Z}$ ). The case of a segment corresponds to the finite-dimensional representations of  $\mathfrak{sl}(2, \mathbb{R})$ , so we allow ourselves to ignore them. The remaining possibilities can all be realized. Let's start with the rays.

- The ray  $(-\infty, m]$  is realized by a Verma module of highest weight  $m$  with Borel given by  $\mathrm{span}\{h, e\}$ . To be irreducible, one needs  $m \leq 0$  because  $m \geq 0$  admit finite-dimensional quotients. Then one can check by hand that these are actually irreducible because any vector generates: any vector can be pushed along  $e$  until it is supported in  $V_m$ , and then it can be pushed back along  $f$  to generate the rest of the representation.
- The ray  $[m, \infty)$  is realized by a Verma module of highest weight  $m$  with Borel given by  $\mathrm{span}\{h, f\}$ . To be irreducible, one needs  $m \geq 0$  because  $m \leq 0$  admit finite-dimensional representations. Once again, one checks by hand that these representations are actually irreducible.

These are called the discrete series representations, and they are labeled by their highest weight as  $L_m$ .

Lastly, we consider the cases  $2\mathbb{Z}$  and  $1 + 2\mathbb{Z}$ . Here, we may choose  $v_n \in V_n$  for which  $fv_n = v_{n-2}$  for all  $n$ , from which we can use the Casimir element to compute

$$ev_n = ef v_{n+2} = \left( c - \frac{(n+1)^2}{4} \right) v_{n+2}.$$

Thus, one sees that this provides an irreducible module if and only if  $c \neq \frac{1}{4}(n+1)^2$  for all  $n$ . For convenience, we can write  $c = s^2/4$ , and we now have the formula

$$efv_{n+2} = \frac{(s-n-1)(s+n+1)}{4}v_n$$

where now we are required to have  $s$  not among the weights of  $V$ . One rescale the basis to  $w_n := c_n v_n$  so that the formulae look like

$$\begin{cases} ew_n = \frac{1}{2}(s-1-n)w_{n+2}, \\ fw_n = \frac{1}{2}(s+1+n)w_{n-2}. \end{cases}$$

We call this module  $P_s^+$  in the even-weight case and  $P_s^-$  in the odd-weight case. In total, we have collected the following result.

**Theorem 3.9.** The irreducible  $(\mathfrak{g}, K)$ -modules for  $(\mathfrak{g}, K) = (\mathfrak{sl}(2, \mathbb{C}), \mathrm{SO}(2, \mathbb{R}))$  are as follows.

- Finite-dimensional modules  $V_m$ .
- Discrete series: Verma modules  $L_m$  for  $m \neq 0$ .
- Principal series:  $P_s^+$  (for  $s \notin 1 + 2\mathbb{Z}$ ) and  $P_s^-$  (for  $s \notin 2\mathbb{Z}$ ).

There are isomorphisms  $P_s^\pm = P_{-s}^\pm$  and no other isomorphisms.

## 3.2 November 4

Today, we continue discussing the representation theory of  $\mathrm{SL}(2, \mathbb{R})$ .

### 3.2.1 Unitary Irreducible Representations

Last class, we classified the irreducible  $(\mathfrak{g}, K)$ -modules for  $G = \mathrm{SL}(2, \mathbb{R})$ , where  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and  $K = \mathrm{SO}(2, \mathbb{R})$ . Let's calculate which are unitary.

**Proposition 3.10.** The unitary irreducible  $(\mathfrak{g}, K)$ -modules for  $(\mathfrak{g}, K) = (\mathfrak{sl}(2, \mathbb{C}), \mathrm{SO}(2, \mathbb{R}))$  are as follows.

- The trivial representation.
- Discrete series.
- Unitary principal series:  $P_s^\pm$  for  $s \in i\mathbb{R}$ .
- Complementary principal series:  $P_s^+$  for  $|s| < 1$ .

*Proof.* Our explicit choice of basis on  $\mathfrak{sl}(2, \mathbb{C})$  satisfies that  $\bar{h} = -h$  and  $\bar{e} = f$  and  $\bar{f} = e$ . We now examine Theorem 3.9 to find which  $M$  admit an invariant Hermitian form  $(-, -)$  for which

$$(xv, w) + (v, \bar{x}w) = 0$$

for all  $v, w \in M$ . In other words, we need to satisfy the equations

$$\begin{cases} (hv, w) = (v, hw), \\ (ev, w) = -(v, fw), \\ (fv, w) = -(v, ew). \end{cases}$$

Give  $M$  a weight basis  $\{w_n\}_{n \in \mathbb{Z}}$ , where  $w_n \in M[n]$  is either nonzero or a basis vector. Then the first equation implies  $(w_n, w_m) = 0$  whenever  $n \neq m$ , so it remains to examine  $(w_n, w_n)$ . Well, observe

$$\begin{aligned} (ew_n, ew_n) &= -(few_n, w_n) \\ &= \left( \frac{(n+1)^2}{4} - c \right) (w_n, w_n) \end{aligned}$$

because  $-fe = \frac{(h+1)^2}{2} - c$ , where  $c$  is a Casimir eigenvalue. We are now ready to do our classification.

- If  $M$  is finite-dimensional, then we see that  $M$  is unitary if and only if  $M$  is trivial. From the formulae, we can see this because  $e$  is nilpotent, so it cannot have the above properties. From first principles, one can note that it is not possible to embed  $\mathrm{PSL}(2, \mathbb{R})$  into a compact group, so there cannot be a finite-dimensional unitary irreducible representation.
- If  $M$  has discrete series  $L_{\pm m}^{\pm}$ , then  $c = (m-1)^2/4$  by a direct calculation with the Verma module, so the quantity

$$\frac{(n+1)^2}{4} - c$$

is always positive (whenever  $w_n$  and  $ew_n$  are both positive), so our Hermitian invariant form can be chosen to be positive-definite. Namely, one defines  $(w_n, w_n)$  inductively.

- As in the previous point, we can see that  $P_s^{\pm}$  is unitary if and only if

$$\frac{(n+1)^2}{4} - \frac{s^2}{4}$$

is real and positive for all  $n$  for which  $w_n \neq 0$  and  $ew_n \neq 0$ . (In this case,  $w_n \neq 0$  implies  $ew_n \neq 0$ , so we may ignore this extra condition.) Notably, we have  $s \notin 11 + 2\mathbb{Z}$  when  $P_s^+$  admits even weights, and  $s \notin 2\mathbb{Z}$  when  $P_s^-$  admits odd weights.

The condition certainly holds when  $s \in i\mathbb{R}$ ; otherwise,  $s$  is real. With odd weights, we see that there are no real  $s$  because  $(-1+1)^2/4 = 0$ . With even weights, we see that we merely need to achieve  $s^2 < 1$ , which is equivalent to  $|s| < 1$ . ■

### 3.2.2 Realizations

We now give realizations for our irreducible  $(\mathfrak{g}, K)$ -modules. There is nothing to say for the finite-dimensional representations. For  $P_s^{\pm}$ , we found these earlier in the semester inside even or odd functions on  $\mathbb{R}^2 \setminus \{0\}$ , transforming in a specified way under dilations via

$$f(tx) = t^{s-1}f(x),$$

where  $t$  is a positive real number. We computed the  $K$ -finite vectors in a problem set, and it does indeed agree with the described principal series.

Let's discuss the unitary structure on these principal series.

- For  $s \in i\mathbb{R}$ , these representations should be unitary, and one can see it on these functions as the usual inner product on  $L^2(\mathbb{RP}^1)$  (which should technically be  $\frac{1-s}{2}$ -densities): we have

$$\langle f, g \rangle := \int_{\mathbb{RP}^1} f \bar{g},$$

where the point is that  $f\bar{g}(tx) = t^{-2}(x)$ , so  $f\bar{g}$  succeeds at being a volume form on  $\mathbb{RP}^2$ .

- For the complementary series  $P_s^+$  for  $0 < s < 1$ , we realize  $P_s^+$  in functions  $\mathbb{R}$  as  $f(x) dx^{(1-s)/2}$  (where the action is given by fractional linear transformations), where the inner product is

$$\langle f, g \rangle := \int_{\mathbb{R}^2} f(x) \bar{g}(y) \frac{dx dy}{|x - y|^{1+s}}.$$

One can formally check that this pairing is invariant: simply direct check dilations, translations, and  $x \mapsto 1/x$ . To explain where this formula comes from, it can be rewritten as  $\langle f * \bar{g}, |x|^{-1-s} \rangle$ . The Fourier transform of this is  $\langle \mathcal{F}f \cdot \overline{\mathcal{F}g}, \mathcal{F}|x|^{-1-s} \rangle$ . We can think of  $|x|^s$  as a tempered distribution satisfying the differential equation  $sxy = y'$ , and taking the Fourier transform of this differential equation reveals that the Fourier transform of  $|x|^{-1-s}$  is  $|x|^{-s}$  times a constant.

Lastly, we discuss the discrete series. Intuitively, we want our functions on the “circle”  $\mathbb{RP}^1$  to extend to holomorphic functions on the full disk. For example,  $L_m^+$  is realized as functions on the unit disk

$$D := \{z \in \mathbb{C} : |z| < 1\}.$$

(Formally, we should really work with the densities  $f(du)^{m/2}$ .) Note  $\mathrm{SL}(2, \mathbb{R})$  acts on  $D$  (one can transform  $D$  to the upper-half plane, and then the upper half-plane admits a natural action by fractional linear transformations), preserving the metric  $\frac{du d\bar{u}}{(1-|u|^2)^2}$ . The invariant pairing is thus given by

$$\left( f(du)^{m/2}, g(du)^{m/2} \right) := \int_D fg \mu.$$

The same story works for  $L_m^-$ . It turns out that taking  $K$ -finite vectors actually recovers the discrete series.

### 3.2.3 The Rest of this Course

Let's outline what will be done in the remainder of the course.

**Definition 3.11 (discrete series).** Fix a Lie group  $G$ . An irreducible representation  $V$  of  $G$  is *discrete series* if and only if the matrix coefficients

$$\ell_{v,w}(g) := (gv, w)$$

lie in  $L^2(G)$  for all  $v, w \in V$ .

**Remark 3.12.** When  $v$  and  $w$  are  $K$ -finite, we see that  $\ell_{v,w}$  is real analytic, so this is a “global” condition about the growth of its coordinates. Note that it is enough to check the condition for  $K$ -finite vectors because they form a dense subset.

**Remark 3.13.** Suppose  $V$  is discrete series. Fixing some nonzero vector  $w$ , we find that there is a nonzero map  $V \rightarrow L^2(G)$  given by  $v \mapsto \ell_{v,w}$ . Thus, because  $V$  is irreducible,  $V$  embeds into  $L^2(G)$ , so  $V$  is unitary.

For an admissible representation  $V$  of  $G$ , we have a character which is best understood as a distribution on  $G$ . We will show later that the restriction to regular semisimple elements is an analytic function, and the distribution is given locally by some  $L^1$ -function.

From here, we will turn our attention to two questions: classification and character formulae.

- For classification, there is the Langlands classification, which explains how to construct representations from the discrete series by induction. It then remains to classify the discrete series, for which there are two approaches: one can use  $D$ -modules and Beilinson–Bernstein localization, which classifies them from  $K_{\mathbb{C}}$ -orbits in  $G_{\mathbb{C}}/B_{\mathbb{C}}$ . Alternatively, one can view these as a special case of the local Langlands conjecture for a local field  $F$ , which provide a correspondence between irreducible representations of  $G(F)$  and homomorphisms from the Galois group of  $F$  to the Langlands dual group of  $G$  (and some extra data).
- For character formulae, we note that principal series are basically inductions of finite-dimensional representations, so the characters are given by the Frobenius formula. (One may call these “standard representations.”) One would then like to express irreducible characters as linear combinations of these standard ones. For complex groups, this is equivalent to the Kazhdan–Lusztig conjectures; in general, these amount to the Lusztig–Vogan generalizations.

Let's begin discussing characters.

**Remark 3.14.** It turns out that any irreducible  $(\mathfrak{g}, K)$ -module can be realized inside a Hilbert space, but then the  $G$ -action may not be unitary. For example, the principal series  $P_s^\pm$  was realized as densities  $f(x) dx^{(1-s)/2}$ , where  $dx$  can be chosen to be some  $K$ -invariant differential. Upon identifying the various  $s$  as providing the same vector space, we recover the given realizations.

**Definition 3.15 (character).** Fix an admissible representation  $V$  of a Lie group  $G$ . Then we define the distribution  $\chi_V$  on  $G$  by

$$\langle \chi_V, f dg \rangle = \text{tr}(f dg; V)$$

for each  $f \in C_c^\infty(G)$ .

**Remark 3.16.** For this to make sense, we need to show that the operator  $f dg$  on  $V$  is of trace class.

**Theorem 3.17.** Fix an admissible representation  $V$  of a Lie group  $G$ . Then  $\chi_V$  is an analytic function when restricted to the regular semisimple locus of  $G$ .

We will prove this next class. There are two ingredients. First,  $\chi_V$  is conjugation-invariant. Second, the map  $U\mathfrak{g} \rightarrow \text{Diff}(G)$  (by having  $G$  act on itself by left translations) allows  $Z\mathfrak{g}$  to act on smooth functions, so there is  $\lambda \in \mathfrak{h}^*/W$  for which the maximal ideal  $\mathfrak{m}_\lambda \subseteq Z\mathfrak{g}$  has

$$\langle f dg, zf \rangle = 0$$

for all  $z \in \mathfrak{m}_\lambda$ . The idea is that the map  $T \times G/T \rightarrow G$  given by  $(t, x) \mapsto (xtx^{-1})$  is a local isomorphism at a given  $g \in T$ . Then the second ingredient turns out to provide a holonomic system of differential equations on our distribution, from which being analytic follows. We will provide more details next class.

## 3.3 November 6

Today, we move on to character theory. We started class with some facts about the nilpotent cone, which I have moved to the appropriate section of the notes.

### 3.3.1 Constructing Characters

Our characters will be a certain kind of trace, so we need to explain why we can take some traces of infinite-dimensional representations.

**Definition 3.18 (Hilbert–Schmidt).** Fix a bounded operator  $A$  on a Hilbert space  $V$ , and choose a topological orthonormal basis  $\{v_i\}$  of  $V$ . Then we say that  $A$  is *Hilbert–Schmidt* if and only if

$$\sum_i |Av_i|^2 < \infty.$$

**Remark 3.19.** If  $A$  is defined on a dense subset of  $V$  but still satisfies the inequality, then it will extend to a Hilbert–Schmidt operator on  $V$ .

**Definition 3.20 (trace class).** Fix a bounded operator  $A$  on a Hilbert space  $V$ , and choose a topological orthonormal basis  $\{v_i\}$  of  $V$ . Then  $A$  is *trace class* if and only if

$$\sum_i (Ev_i, v_i) < \infty.$$

**Remark 3.21.** If  $A$  and  $B$  are both Hilbert–Schmidt, then  $AB$  is trace class.

**Lemma 3.22.** Fix  $M$  an irreducible  $(\mathfrak{g}, K)$ -module, and expand  $M = \bigoplus_\nu (W_\nu \otimes V_\nu)$  into irreducibles  $V_\nu$  of  $K$ . Then there is a constant  $C_M$  for which

$$\dim W_\nu \leq C_M \dim V_\nu.$$

*Proof.* Start with some  $V_\nu \subseteq V$ , so we receive a surjection  $U\mathfrak{g} \otimes_{U\mathfrak{k}} V_\nu \twoheadrightarrow V$  because  $V$  is irreducible. Writing  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , the PBW theorem grants that  $\text{Sym } \mathfrak{p} \otimes_{\mathbb{C}} V_\nu \twoheadrightarrow V$ , and taking the associated graded produces a finitely generated module  $\text{gr } M$  over  $S\mathfrak{g}$ , which we note has  $K$ -equivariant support, so it is contained in  $\mathfrak{k}^\perp \cap \mathcal{N}$ . (It is contained in  $\mathcal{N}$  by considerations with the infinitesimal character.)

We now note that  $G$  acts on  $\mathcal{N}$  with finitely many orbits; the same argument shows that  $K_{\mathbb{C}}$  acts on  $\mathcal{N} \cap \mathfrak{p}$  with finitely many orbits. Indeed, start with some faithful representation  $\mathfrak{g} \subseteq \mathfrak{sl}(n)$ , and we may write  $\mathfrak{sl}(n) = \mathfrak{g} \oplus \mathfrak{g}'$ , where  $\mathfrak{g}'$  is some  $\mathfrak{g}$ -invariant complement. Now,  $\mathcal{N} = \mathcal{N}_{\mathfrak{sl}(n)} \cap \mathfrak{g}$ , and we have the result for  $\mathfrak{sl}(n)$  by hand, so it is enough to show that any orbit  $\mathcal{O}$  for  $\text{SL}(n)$  has  $\mathcal{O} \cap \mathfrak{g}$  divide up into finitely many  $G$ -orbits (by the compactness of  $G$ ). For this, it is enough to show that each  $G$ -orbit in  $\mathcal{O} \cap \mathfrak{g}$  is open, which amounts to showing that

$$T_x \mathcal{O} \cap \mathfrak{g} \stackrel{?}{=} T_x(G(x)).$$

But  $T_x \mathcal{O} = [x, \mathfrak{sl}(n)]$ , which we may expand as  $[x, \mathfrak{g}] \oplus [x, \mathfrak{g}']$ , so both sides above are  $[x, \mathfrak{g}]$ . We similarly find that  $K_{\mathbb{C}}$  acts on  $\mathcal{N} \cap \mathfrak{p}$  by finitely many orbits by redoing the above argument: it is enough to check that a  $G$ -orbit  $\mathcal{O} \subseteq \mathfrak{g}$  has  $\mathcal{O} \cap \mathfrak{p}$  split into finitely many  $K$ -orbits, which follows from the same tangent space calculation because  $[x, \mathfrak{g}] = [x, \mathfrak{k}] \oplus [x, \mathfrak{p}]$ .

We now return to the proof. It is enough to check that any  $K$ -equivariant  $S\mathfrak{p}$ -module  $N$  has

$$[N : V_\nu] \leq C_N \dim V_\nu.$$

By dividing up the support of  $N$  by taking subquotients, we may assume that  $N$  is a torsion-free module supported on the closure of a  $K$ -orbit  $\mathcal{O} \subseteq \mathcal{N} \cap \mathfrak{p}$ . But then  $N$  can be embedded as sections of an equivariant bundle on the orbit, which embeds into  $\mathbb{C}[K]^r$ , and the result now follows from Peter–Weyl. ■

**Remark 3.23.** The  $G$ -orbits  $\mathcal{O}$  in  $\mathfrak{g}^*$  are symplectic, so they are in particular even-dimensional. It even turns out that  $\mathfrak{p} \cap \mathcal{O}$  is a Lagrangian subvariety.

**Proposition 3.24.** Fix an irreducible Hilbert space representation  $V$  of a Lie group  $G$ . Then any  $f \in C_c^\infty(G)$  acts by a trace class operator on  $V$ .

*Proof.* The idea is to use the Casimir operators  $C_k$  for  $K$ . In particular, we may write

$$V_{\text{fin}} = \bigoplus_\nu W_\nu \otimes V_\nu,$$

where  $\nu$  varies over irreducible representations  $V_\nu$  of  $K$ , and  $W_\nu$  is the multiplicity space.



By Lemma 3.22, we have a constant  $C$  for which

$$\dim W_\nu \leq C \dim V_\nu$$

in our situation. By the representation theory of Lie algebras, one can check that  $\dim V_\nu$  is  $O(|\nu|^d)$ , where  $d$  is the number of positive roots of  $\mathfrak{k}$ . Now, the Casimir element  $C_\mathfrak{k}$  acts by a scalar  $c_\nu$ , which is positive and of the order  $|\nu|^2$ .

We now claim that the operator  $(1 + C_\mathfrak{k})^{-N}$  is Hilbert–Schmidt for large  $N$ , where we are only defining this operator a priori on  $V_{\text{fin}}$ . Indeed, by combining the above bounds, it is enough to see that

$$\sum_{\nu \in \Lambda_\mathfrak{k}^+} \frac{(\dim V_\nu)^2}{|\nu|^{2N}} \leq C \sum_{\nu \in \Lambda_\mathfrak{k}^+} \frac{|\nu|^{2d}}{|\nu|^{2N}}$$

converges, which occurs as soon as (say)  $N > d + 1$ .

Now, write our  $f \in C_c^\infty(G)$  as

$$f = (1 + C_\mathfrak{k})^{-N} \cdot ((1 + C_\mathfrak{k})^{-N} \cdot (1 + C_\mathfrak{k})^{2N} f).$$

Notably,  $(1 + C_\mathfrak{k})^{2N} f$  is just some function in  $C_c^\infty(G)$ . This is then passed through a product of Hilbert–Schmidt operators, so it is trace class by Remark 3.21. ■

**Definition 3.25 (character).** Fix a Hilbert space representation  $V$  of  $G$ . Then we define the *character*  $\chi_V$  to be the distribution on  $G$  for which

$$\langle \chi_V, f dg \rangle := \text{tr}(f dg; V).$$

**Remark 3.26.** If  $V$  and  $V'$  are infinitesimally equivalent, then their characters agree. Indeed, everything in the definition can be passed down to  $V_{\text{fin}} \cong V'_{\text{fin}}$ , so the result follows from continuity.

## 3.4 November 13

Today, we continue discussing characters. We began class by proving an estimate from last class, which I have added there.

### 3.4.1 Regularity of Characters

We are interested in proving the following regularity result.

**Proposition 3.27.** Fix a Hilbert space representation  $V$  of  $G$ . Then  $\chi_V$  is analytic when restricted to the regular semisimple locus  $G^{\text{rs}}$  of  $G$ .

In fact, we will show the following stronger result.

**Proposition 3.28.** Fix a conjugation-invariant  $Z\mathfrak{g}$ -eigendistribution  $\delta$ . Then  $\delta|_{G^{\text{rs}}}$  is analytic.

Here, being an eigendistribution refers to the  $Z\mathfrak{g}$ -action given by extending the action  $\mathfrak{g} \cong \text{Vect}(G)^G$  to an action  $Z\mathfrak{g} \subseteq U\mathfrak{g} \rightarrow \text{Diff } G$ , and  $\text{Diff } G$  is able to act on  $\text{Dist } G \subseteq C_c^\infty(G)^*$ .

This proposition will in turn follow from the following analytic result.

**Proposition 3.29.** A distribution on a complex manifold  $X$  satisfying a holonomic system of differential equations is given by an analytic function.

*Proof.* Omitted. It is perhaps easier than our elliptic result from earlier in the course. Roughly speaking, one can do some favorable coordinate changes in order to make the holonomic system simply asserting that the distribution has certain derivatives vanishing, which amounts to asking for the function to be constant in those directions. ■

*Proof of Proposition 3.28.* We will check that  $\delta$  is cut out by a holonomic system of differential equations. Fix some regular semisimple  $g$ , and identify  $T_g^*G = \mathfrak{g}$  as usual, and we see that  $\mathbb{C}[T_gG] = \mathbb{C}[\mathfrak{g}]$ . Now, for each  $z \in Z\mathfrak{g}$ , we get some  $z \in \mathbb{C}[\mathfrak{g}]$ , and our distribution is cut out by  $z - \lambda$  for some  $\lambda$ , and we would like to show that these cut out a holonomic system when combined with the differential operators given by  $[x, \mathfrak{g}]$  (corresponding to being conjugation-invariant). Choosing  $x$  to be regular semisimple reveals that our roots must be orthogonal to  $\mathfrak{t}^\perp$  and therefore live in  $\mathfrak{t}$ .

Thus, the main part is to compute  $\bar{z}|_{\mathfrak{t}}$ . Note we may as well take  $\bar{z}$  to be given by one of the various  $G$ -invariant polynomials  $P$ . By the Harish-Chandra isomorphism, we may as well compare this to the differential of  $\mathfrak{t} \rightarrow \mathfrak{t}/W$ , so we may consider the action of  $Z$  on  $\mathbb{C}[G]^G$  spanned by the known characters  $\chi_{V_\lambda}$ . More precisely, we note that we have a composite

$$\mathbb{C}[G]^G \cong \mathbb{C}[T]^W \subseteq \mathbb{C}[T].$$

Note  $\mathbb{C}[G]^G$  has an action by  $Z\mathfrak{g}$ , so on  $T$  we have a differential-invariant differential operator  $d_p \in \mathbb{C}[\mathfrak{t}^*] \subseteq \text{Diff } T$ . On the other hand, we have a Weyl character formula

$$\chi_{V_\lambda} = \frac{1}{\delta} \sum_w t^{w(\lambda+\rho)},$$

and the operator  $d_p$  acts by  $\delta^{-1}d_p\delta$ . Looping over all  $\bar{z}_{p,x}|_{\mathfrak{t}}$  gives all possible  $\bar{d}_{p,x}$ , so we get all  $W$ -invariant polynomials, which have no common zero anyway. ■

## 3.5 November 18

We continue our discussion of characters.

### 3.5.1 Characters of the Principal Series

Let's work out some characters.

**Example 3.30.** We work out the characters  $\chi_{L_{\pm m}}(g)$ , where  $g$  is a compact toral elements of  $\text{SL}(2, \mathbb{R})$ .

*Proof.* We run through the classification of Theorem 3.9. Let's start with the case that  $g$  is compact, and we are working with some  $L_{\pm m}$ . Then we give each  $g \in K$  a diagonalization  $\text{diag}(\lambda, \bar{\lambda})$  where  $\lambda \in S^1$ . Then the character is given by

$$\chi(g) = \frac{\lambda^{\pm m}}{1 - \lambda^{-2}}.$$

Technically speaking, the series used for this calculation does not converge because  $|\lambda| = 1$ , but this expression does make sense when integrating against a function  $\varphi \in C_c^\infty(S^1)$  satisfying  $\varphi(\pm 1) = 0$ . ■

On the other hand, if we have some regular semisimple  $g$  which is not compact, then  $\chi_{L_m}(g) = \chi_{L_{-m}}(g)$ : note that  $\text{SL}(2, \mathbb{R})$  admits an outer automorphism given by  $g \mapsto g^{-\tau}$  (equivalently, this is conjugation by an element in  $\text{GL}(2, \mathbb{R})$  with determinant  $-1$ ), and this outer automorphism exhibits an isomorphism  $L_m \rightarrow L_{-m}$ . Furthermore, for  $g$  which is not compact, we see that  $g^{-\tau}$  is conjugate to  $g$ , which is not the case for

compact  $g$ . (At a high level, this is true because we can find a matrix in  $GL(2, \mathbb{R})$  with determinant  $-1$  which commutes with  $g$ .) Thus, we see that

$$\chi_{L_m} + \chi_{L_{-m}} + \chi_{V_{m-2}} = \chi_{P_{\pm}}(s)$$

for suitable principal series, so it remains to compute the character for the principal series.

Thus, we now turn to the principal series  $V_s^{\pm}$ . In fact, we will more generally. Let  $G$  be a split real reductive group with torus  $T$ , and let  $B \subseteq G$  be a Borel, and let  $U \subseteq B$  be the unipotent radical. Let  $\theta$  be some character on  $T$ , and we consider the induction  $V_{\theta}$  as functions on  $G/U$  satisfying the transition  $f(xt) = \theta(t)^{-1}f(x)$ . We will compute our character by chasing around the following diagram.

$$\begin{array}{ccc} & G \times G/B & \\ \swarrow & & \searrow \\ G & & \frac{G/U \times G/U}{T} \end{array}$$

Here, the left map is the projection, and the right map is given by the action map  $G \times G/U \rightarrow G/U \times G/U$ . One then pulls  $f$  back and pushes forward to produce an integral operator on the quotient  $(G/U)^2/T$ . To compute the trace of this integral operator, one merely compute the integral along the diagonal  $G/B \hookrightarrow (G/U)^2/T$ .

To do this, we now observe that the pre-image of the diagonal up in  $G \times G/B$  is given by

$$\tilde{G} = \{(g, B') : g \in G, g \in B'\},$$

where we view  $G/B$  as the space of Borel subgroups. Thus, we are now chasing around the following diagram.

$$\begin{array}{ccc} & \tilde{G} & \\ p \swarrow & & \searrow \pi \\ T & & G \end{array}$$

We are interested in integrating over the fibers of  $\pi$  to compute the integral over the diagonal up in  $\tilde{G}$ , so we need to say something about  $\pi$ . Over  $\mathbb{C}$ , we note that  $\pi_{\mathbb{C}}$  is generically  $|W|$ -to-1. Indeed, for  $g$  regular and semisimple, we are asking for Borel subgroups containing  $g$ , which more or less corresponds to choosing a flag from  $g$ , but if  $g$  is regular semisimple, then we are basically just reordering the eigenvalues of  $g$ .

We are now able to compute our character  $\chi$ . Let's explain this. Suppose that  $f$  is supported on the regular semisimple elements, and we want to compute  $\text{tr}(f dg; V_{\theta})$ , where  $V_{\theta}$  consists of functions on  $G/U$  for which  $f(xt) = \theta(t)f(x)$ . Then

$$\begin{aligned} \text{tr}(f dg; V_{\theta}) &= \int_{\tilde{G}} p^*(\theta) \pi^*(f) d\tilde{g} \\ &= \int_G \tilde{f} dg, \end{aligned}$$

where

$$\tilde{f}(g) = f(g) \sum_{\tilde{g}_w \in \pi^{-1}(g)} \theta(p(\tilde{g}_w)) \frac{d\tilde{g}_w}{dg}.$$

Thus,

$$\chi(g) = \sum_{\tilde{g}_w \in \pi^{-1}(g)} \theta(p(\tilde{g}_w)) \frac{d\tilde{g}_w}{dg}.$$

To compute  $d\tilde{g}_w/dg$ , we note that  $dg/d\tilde{g}_w$  is simply  $|\det d\pi_g|$ . To compute this determinant, write  $\tilde{g}_w = (g, B_w)$ , and we note that the bundle sequence  $B \rightarrow \tilde{G} \rightarrow G/B$  induces a morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{b}_w & \longrightarrow & T_{(g, B_w)} \tilde{G} & \longrightarrow & \mathfrak{g}/\mathfrak{b}_w \longrightarrow 0 \\ & & \parallel & & \downarrow d\pi & & \downarrow \\ 0 & \longrightarrow & \mathfrak{b}_w & \longrightarrow & T_g G & \longrightarrow & \mathfrak{g}/\mathfrak{b}_w \longrightarrow 0 \end{array}$$

of short exact sequences, and the right-hand map turns out to be given by  $\text{Ad}_g - 1$ . (Observe that this short exact sequence explains why  $d\tilde{g}$  is well-defined: one can trivialize  $\wedge^{\text{top}} T_{\tilde{g}} \tilde{G}$  because it is  $\wedge^{\text{top}}(\mathfrak{g}/\mathfrak{b}_w) \otimes \wedge^{\text{top}}(\mathfrak{b}_w) = \wedge^{\text{top}} \mathfrak{g}$ .) Notably,  $|d\pi|$  can be computed on the trivial line bundle  $\Omega_{\tilde{G}}^{\text{top}}$  as

$$\det(\text{Ad}_g - 1; \mathfrak{g}/\mathfrak{b}_w) = \prod_{\alpha \in \Phi_w^+} (1 - e^\alpha(g)).$$

This determinant may have a different answer for different  $w \in W$ , but this turns out to be okay because one can absorb the difference into  $\theta$ . Notably, applying a simple reflection  $s_\alpha$  replaces  $1 - e^\alpha$  with  $1 - e^{-\alpha}$ , which is  $-e^{-2\alpha}(1 - e^\alpha)$ . But  $e^{-2\alpha} = e^{s_\alpha(2\rho)}/e^{2\rho}$ , so we see that we can write

$$\chi = \left( \sum_{w \in W} e^{2(\nu)} \right) \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})^{-1},$$

where  $\nu$  is chosen so that  $\theta = \nu \cdot e^{-\rho}$ .

**Example 3.31.** For  $\text{SL}(2, \mathbb{R})$  and  $g = \text{diag}(a, a^{-1})$ , we get

$$\pm \frac{|a|^\nu + |a|^{-\nu}}{|a - a^{-1}|}.$$

**Example 3.32.** Suppose that  $V_\nu$  consists of global sections of some line bundle  $\mathcal{L}_\nu$  (considered as half-densities), where  $\mathcal{L}_\nu$  is the usual induction from  $G/U$ . If  $\nu$  is unitary, then one finds that  $|\nu(g)| = 1$  for all  $g \in T$ , so  $V_\nu$  is unitary, where the inner product is given by

$$\langle f_1, f_2 \rangle = \int_{G/B} f_1(x) \overline{f_2(x)} dx.$$

**Remark 3.33** (parabolic induction). Some of this story works more generally. Let  $P \subseteq G$  be a parabolic subgroup, and let  $P = LU$  be the Levi decomposition (where  $L$  is the Levi, and  $U$  is the unipotent radical). Given an admissible representation  $\rho$  of  $L$ , we may write  $i_P^G(\rho)$  to be the global sections of the corresponding equivariant sheaf  $\mathcal{F}_\rho$  (considered as half-densities), which concretely are the usual functions  $G/U \rightarrow \rho$  satisfying some  $L$ -invariant transition:

$$f(x\ell) = \rho^{-1}(\ell)f(x)e^{-\rho_P(\ell)}.$$

Now, if  $\rho$  is unitary, one finds  $i_P^G(\rho)$  is still unitary.

### 3.5.2 The Cartan Decomposition

As usual,  $G$  is a real reductive group.

**Proposition 3.34** (Cartan decomposition). Let  $G$  be a real group with maximal compact  $K$ , and let  $A$  be a maximal split torus. Then

$$G = KA^+K,$$

where  $A^+ = \{\exp(-x) : x \in \mathfrak{a}^+\}$ , where  $\mathfrak{a}^+$  is the positive part of  $\mathfrak{a}$  (for the Cartan involution).

**Remark 3.35.** Here, the  $^+$  in  $A^+$  is being used because  $\mathfrak{a}^+$  consists of those  $a \in \mathfrak{a}$  for which  $\langle \alpha, a \rangle > 0$  for each root  $\alpha$  of in  $\text{Lie } N$ .

*Sketch.* One starts with the polar decomposition  $G = PK$ , which arises because  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  by definition of  $\mathfrak{p}$ . To sketch this polar decomposition, one chooses an embedding  $G \hookrightarrow \mathrm{SL}(n, \mathbb{C})$  with compatible Cartan involution, approximately allowing us to pass to  $\mathrm{SL}(n, \mathbb{C})$ . Note then that  $K$  is  $\mathrm{SU}(n) \cap G$  and  $P$  is the intersection  $P_{\mathrm{SL}(n)} \cap G$ . Then given  $g \in G$ , one can let  $p$  be a square root of  $g\theta(g^{-1})$ , which exists as soon as one checks some positivity property of the eigenvalues. Then one checks that  $p \in P$  and  $k := p^{-1}g$  is in  $K$ .

We now pass to the Cartan decomposition. Well, as above, we may let  $A$  be a maximal split torus in  $G$ , and we may assume that  $\mathfrak{a} \subseteq \mathfrak{p}$ . In light of the polar decomposition, we basically have to show that

$$\mathfrak{a}^+ = \mathfrak{a}/W(\mathfrak{a}, K) \rightarrow \mathfrak{p}/K$$

is surjective; in fact, it is an isomorphism. This is something that one can check more or less directly. Once we have this claim, one writes any  $g \in G$  as  $xu$  where  $x \in P$  and  $u \in K$ , and then we know that  $x$  can be written as  $u'a(u')^{-1}$  for  $u' \in K$  and  $a \in A^+$ , so the Cartan decomposition follows. ■

**Example 3.36.** Take  $G = \mathrm{SL}(2, \mathbb{R})$  so that  $K = \mathrm{SO}(2, \mathbb{R})$ . Then  $G/K$  is just the upper-half plane  $\mathcal{H}$ , where the natural action by  $G$  preserves the negative curvature metric. Then we would like to show that  $KA^- \twoheadrightarrow G/K$ , which amounts to showing that we can get any point in  $\mathcal{H}$  by first scaling it appropriately by  $A^- = \{\mathrm{diag}(a, 1/a) : a > 0\}$  and then taking an orbit along  $K$ . Roughly speaking, the action by  $A$  translates the necessary distance, and then  $K$  rotates along points equidistant from  $i$ .

**Example 3.37.** Take  $G = \mathrm{SL}(n, \mathbb{R})$  so that  $K = \mathrm{SO}(n, \mathbb{R})$ . Then  $G/K$  parameterizes positive-definite quadratic forms: for every pair of quadratic forms, one can find a basis which is orthogonal for both. More explicitly, if we let  $Q_1$  be the standard quadratic form, then any other form  $Q_2$  can be diagonalized by an orthonormal basis for  $Q_1$ . (This identification is known as the principal axis reduction.) This manipulation has basically shown that  $KA^-$  surjects onto  $G/K$ .

**Example 3.38.** The polar decomposition  $G = PK$  of  $G = \mathrm{SL}(n, \mathbb{R})$  has  $K = \mathrm{SO}(n, \mathbb{R})$ . Then  $\mathfrak{p}$  consists of self-adjoint operators, and  $P$  consists of the positive self-adjoint operators.

**Remark 3.39.** Because  $K$  is compact, asymptotics of a function  $f$  on  $G$  is controlled by  $f|_{A^-}$ . For matrix coefficients, this reduces to an algebraic calculation of some eigenvalues of  $\mathfrak{a}$  on  $M/\mathfrak{u}^n M$ .

## 3.6 November 20

Today, we begin discussing the discrete series.

### 3.6.1 Exponents

We now aim to explain what is meant by “discrete series.” Fix a real group  $G$ , and we choose a maximal split torus  $A \subseteq G$ . It turns out that the Lie algebra  $\mathfrak{a}$  is maximal abelian in  $\mathfrak{k}^\perp$ , where  $K \subseteq G$  is a maximal compact subgroup. The centralizer  $\mathfrak{z}(\mathfrak{a})$  is some Levi, so we may choose a real parabolic  $P$  with Levi subgroup  $L$  with the correct Lie algebra. It turns out that the semisimple part  $M$  of  $L$  is compact: non-compact reductive groups have a nontrivial split torus, which would violate the maximality of  $A$ .

Now, the Levi decomposition allows us to write  $P = MAN$  where  $N$  is the unipotent radical of  $P$ . Namely, note  $L$  is reductive, so  $\mathfrak{l}$  is the sum of its semisimple part  $\mathfrak{m}$  and its center  $\mathfrak{a}$ , so  $L = MA$  follows.

**Remark 3.40.** Note  $G$  is split if and only if  $\dim \mathfrak{a} = \text{rank } \mathfrak{g}$ ; this then implies that  $\mathfrak{m} = 0$  and that  $P$  is a Borel subgroup. On the other hand,  $P$  being Borel is equivalent to  $\mathfrak{a}$  containing a regular element, which is equivalent to  $G$  being quasi-split. Recall that any group is an inner twist of a quasi-split group, but not every group can be an inner twist of a split group; for example,  $G = \text{SU}(n, n)$  has  $G_{\mathbb{C}} = \text{SL}(2n, \mathbb{C})$  is quasi-split but not split.

**Definition 3.41.** Let  $V$  be an admissible representation. Then an *exponent* of  $V$  is an eigenvalue of  $\mathfrak{a}$  acting on  $M/\mathfrak{n}^d M$  for some  $d > 0$ , where  $M = V_{\text{fin}}$ .

The following lemma makes sense of this definition in the irreducible case.

**Lemma 3.42.** Let  $M$  be an irreducible admissible  $(\mathfrak{g}, K)$ -module. Then  $M$  is finitely generated over  $\mathfrak{n}$ , so  $M/\mathfrak{n}^d M$  is finite-dimensional for all  $d > 0$ .

*Proof.* We may complexify everything and pass to  $\text{gr } M$ , induced by some filtration on  $M$  which we may assume to be finite-dimensional at all steps, compatible with the PBW filtration, and  $K$ -invariant. Then  $\text{gr } M$  is a coherent sheaf on  $\mathfrak{g}$  supported on  $\mathfrak{k}^{\perp} \cap \mathcal{N}$ . (Support being in  $\mathcal{N}$  follows because  $M$  is irreducible and therefore has an infinitesimal character.) Certainly we know that  $\text{gr } M$  is finitely generated over  $\text{gr } U\mathfrak{g} = S\mathfrak{g}$  by the irreducibility, so it is enough to check that the projection

$$\mathfrak{k}^{\perp} \cap \mathcal{N} \rightarrow \mathfrak{g}/\mathfrak{p}$$

is finite. Because this map is invariant under scaling, it is enough to merely check that the kernel is zero, so we need to compute the intersection  $\mathcal{N} \cap \mathfrak{k}^{\perp} \cap \mathfrak{p}$ .

To this end, choose a maximal abelian subspace  $\mathfrak{a} \subseteq \mathfrak{k}^{\perp}$ , and the Cartan involution  $\theta$  from  $\mathfrak{k}$  admits  $\theta|_{\mathfrak{a}} = -1$ . Now,  $\theta$  sends  $\mathfrak{p}$  to the opposite parabolic  $\mathfrak{p}^{-}$ , so  $\mathfrak{p} \cap \mathfrak{k}^{\perp}$  is contained in  $\mathfrak{p} \cap \mathfrak{p}^{-}$ , which is  $\mathfrak{a} \oplus \mathfrak{m}$ . But  $\theta$  acts on these two factors by  $(-1)$  and  $(+1)$ , respectively, so we conclude that  $\mathfrak{p} \cap \mathfrak{k}^{\perp} = \mathfrak{a}$ . However,  $\mathfrak{a}$  is toral, so  $\mathcal{N} \cap \mathfrak{a} = 0$ . ■

## 3.7 November 25

Today we continue discussing the discrete series.

### 3.7.1 The Discrete Series

Throughout,  $G$  is a real reductive group, and we choose letters  $P = MAN$  as usual.

**Theorem 3.43.** Let  $G$  be a real reductive group, and let  $V$  be an irreducible admissible representation of  $G$ . Then the following are equivalent.

- (i) All matrix coefficients of  $V$  are in  $L^2(G)$ .
- (ii) All exponents  $\lambda$  of  $V$  have  $\lambda < -\rho$ , where  $\rho$  is the half-sum of the roots of  $\mathfrak{a}$  in  $\mathfrak{n}$ . Here,  $\lambda < -\rho$  means that  $-\rho - \lambda$  is a nonzero sum of roots of  $\mathfrak{a}$  in  $\mathfrak{n}$  with nonnegative coefficients.

**Remark 3.44.** By definition of  $\rho$ , one has that  $e^{2\rho}(a) = \det(e^a; \mathfrak{n})$  for  $a \in \mathfrak{a}$ .

**Remark 3.45.** The notation  $\lambda < -\rho$  is justified as the usual partial order on weights. Indeed, there is some non-reduced root system given by the action of  $\mathfrak{a}$ .

**Example 3.46.** Take  $G = \mathrm{SL}(2, \mathbb{R})$ , and we will recover the discrete series. Here,  $K = \mathrm{SO}(2)$ . Let  $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$  be a Borel subalgebra of  $\mathfrak{g}$  in (distinct from the one giving rise to  $\mathfrak{k}$ ), and one finds that  $L_m^+$  is the induction  $U_{\mathfrak{g}} \otimes_{\mathfrak{b}} \mathbb{C}_m$ . It then turns out that  $L_m^+/\mathfrak{n}^d L_m^+$  has  $\mathfrak{a}$ -weights  $\{m, m+2, \dots, m(d-1)\}$ , so it satisfies the criteria of Theorem 3.43 if and only if  $m > 1$ .

**Example 3.47.** Consider the group  $\mathrm{SL}(2, \mathbb{R})$ . Then

*Proof of Theorem 3.43.* For simplicity, we will assume that the group is split, but it is not so important. The proof proceed in steps.

1. We set up some calculations. Note that  $\dim A = \mathrm{rank} G$  and  $\dim K = \#\Phi^+$ ; the second equality follows by hunting for  $\theta$ -invariant subspaces in the decomposition  $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ , where we know that  $\theta$  swaps  $\mathfrak{u}_-$  and  $\mathfrak{u}_+$  and acts by  $-1$  on  $\mathfrak{a}$ .

The moral is that the map  $c: K \times A \times K \rightarrow G$  should be generically a diffeomorphism. To see this, we note that the differential at some  $a \in A$  is the map  $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k} \rightarrow \mathfrak{g}$  defined by

$$dc_a: (x, y, z) \mapsto \mathrm{Ad}_a(x) + y + z.$$

Indeed, we can apply right translations to move the tangent space for  $z$  to the identity, everything commutes at the  $y$  coordinate, and the  $x$  coordinate needs to use left translation to move to the identity, but this adds an adjoint action.

Now, recall that  $\theta$  acts on  $\mathfrak{a}$  by  $-1$ , so  $\theta$  will send  $\mathfrak{u}^+ \oplus \mathfrak{a}$  to  $\mathfrak{u}^- \oplus \mathfrak{a}$ . It follows that

$$\mathfrak{k} = \{n + \theta(n) : n \in \mathfrak{u}\}$$

by hunting for  $\theta$ -invariants. As such, we note that  $\mathrm{Ad}_a(n) \rightarrow 0$  as  $a \rightarrow \infty$ , so we may ignore this term. It follows that

$$\det dc|_a \sim \det(\mathrm{Ad}_a; \mathfrak{u}^-) = e^{2\rho(a)}.$$

2. We bound the asymptotics of the matrix coefficients (on  $A^+$ ) in terms of the exponents. To start, choose smooth  $K$ -finite vectors  $v \in V$  and  $v^* \in V^\vee$ ; they vanish on all but finitely many  $K$ -isotypic components. Now, a given  $e^a \in A$  will act by a bounded operator, so

$$m_{v, v^*}(e^a) = \langle v^*, gv \rangle \lesssim_{v, v^*} \|e^a\|.$$

But now  $a \in \mathfrak{a}^+$  can be expanded in local coordinates as a sum of simple roots, so

$$m_{v, v^*}(e^a) \lesssim_{v, v^*} e^{R\langle a, \rho \rangle}$$

for some  $R > 0$ .

Now, for any  $n \in \mathfrak{n}$ , we may write  $v' = nv$ , and then there is a better bound. To see this, we may as well assume that  $n \in \mathfrak{g}_\alpha$  for some root  $\alpha$ , and then we write

$$\begin{aligned} m_{v', v^*}(e^a) &= \langle v^*, e^a nv \rangle \\ &= \langle v^*, \mathrm{Ad}_{e^a}(n)(av) \rangle \\ &= \langle v^*, e^{\alpha(a)} nav \rangle \\ &= e^{\alpha(a)} m_{(v^*)', v}(e^a), \end{aligned}$$

where  $(v^*)' = -n(v^*)$ . Because  $\alpha(a) < 0$ , we have a smaller bound: our right-hand side is bounded above by  $e^{\langle a, R\rho + \alpha \rangle}$  up to a constant.

Iterating the above process, we see that we can adjust our vectors so that the matrix coefficient is smaller than an arbitrary exponential.

3. For technical reasons, we complete the argument only in the case of  $G = \mathrm{SL}(2, \mathbb{R})$ . (For higher-rank situations, one has to handle interactions between roots, which is not difficult but is technical.) In particular, for any  $r$ , we can choose  $v$  and  $v^*$  so that  $m_{v,v^*}(e^a) \lesssim e^{r\langle a, \rho \rangle}$ .

Now, the first step tells us that  $V$  is in  $L^2$  if and only if some matrix coefficient satisfies the bound

$$m_{v,v^*}(e^a) \lesssim e^{ta}$$

for some  $t < -1$ . To see this, note that being in  $L^2$  is equivalent to having some nontrivial map to  $L^2(G)$ , which may as well be chosen as a matrix coefficient. Thus, we are left to checking the convergence of

$$\int_G |m_{v,v^*}(g)|^2 dg.$$

Using the Cartan decomposition and the first step, this becomes

$$\int_{K \times A^+ \times K} |m_{v^*,v}(k_1 a k_2)|^2 e^{2\rho(a)} dk_1 da dk_2.$$

The integrals over  $K$  do not change convergence because they are compact, so we are only interested in the convergence of the integral over  $A^+$ , whence the claim upon computing  $e^{2\rho}$ .

We now turn to understanding the exponents of  $V_{\mathrm{fin}}/\mathfrak{u}^n V_{\mathrm{fin}}$ . This space is finite-dimensional and admits a  $\mathfrak{a}$ -action, which we left to an  $A$ -action on  $V_{\mathrm{sm}}/\mathfrak{u}^n V_{\mathrm{sm}}$ . Now, the image of

$$V_{\mathrm{fin}}/\mathfrak{u}^n V_{\mathrm{fin}} \rightarrow V_{\mathrm{sm}}/\mathfrak{u}^n V_{\mathrm{sm}}$$

is forced to be  $A$ -invariant, and the  $A$ -action of the image must integrate the  $\mathfrak{a}$ -action. Now, by definition, these exponents  $\lambda$  are the eigenvalues of the  $\mathfrak{a}$ -action, so they become the eigenvalues of the  $A$ -action, so we are left to ask for the exponents to be bounded above by  $-1$  by the previous paragraph. This completes the proof. ■

**Definition 3.48 (discrete series).** Fix a real reductive group  $G$ . Then an irreducible admissible representation  $V$  of  $G$  is in the *discrete series* if and only if it satisfies one of the equivalent conditions of Theorem 3.43.

**Remark 3.49.** For Theorem 3.43, it is enough to merely check the exponents acting on  $V_{\mathrm{fin}}/\mathfrak{u} V_{\mathrm{fin}}$ . Indeed, the point is that the eigenvalues are decreasing as we pass to higher powers of  $\mathfrak{u}$ . To see this, we argue inductively: additional eigenvalues on  $V_{\mathrm{fin}}/\mathfrak{u}^{i+1} V_{\mathrm{fin}}$  will come from  $\mathfrak{u}^i V_{\mathrm{fin}}/\mathfrak{u}^{i+1} V_{\mathrm{fin}}$ . But the surjection

$$\mathfrak{u} \otimes \mathrm{gr}_i V_{\mathrm{fin}} \rightarrow \mathrm{gr}_{i+1} V_{\mathrm{fin}}$$

tells us that the eigenvalues  $\lambda$  for  $\mathfrak{u}^i V_{\mathrm{fin}}/\mathfrak{u}^{i+1} V_{\mathrm{fin}}$  can be written as  $\alpha + \lambda'$  where  $\alpha$  is a root of  $A$  acting on  $\mathfrak{u}$ , and  $\lambda'$  is an eigenvalue for  $\mathfrak{u}^{i-1} V_{\mathrm{fin}}/\mathfrak{u}^i V_{\mathrm{fin}}$ .

**Definition 3.50 (tempered).** Fix a real reductive group  $G$ . Then an irreducible admissible representation  $V$  of  $G$  is *tempered* if and only if all exponents  $\lambda$  of  $V$  have  $\lambda \leq -\rho$ , where notation is as in Theorem 3.43.

**Remark 3.51.** By the same argument as in Theorem 3.43, it turns out that being tempered is equivalent to  $V$  being found in  $L^{2+\varepsilon}(G)$  for all  $\varepsilon > 0$ .

**Example 3.52.** It turns out that unitary induction sends tempered representations to tempered representations. However, it does not send discrete series representations to discrete representations.

**Remark 3.53.** It turns out that a representation is tempered if and only if it is a summand of some unitary induction of a discrete series representation. Note that these inductions are unitary and hence semisimple.



### 3.7.2 The Langlands Classification

There is a classification of all admissible representations (up to infinitesimal equivalence) in terms of tempered representations. This is known as the Langlands classification. Let's say more about this.

**Theorem 3.54 (Langlands classification).** Fix a real reductive group  $G$ . For every admissible irreducible representation  $V$ , there is a unique (up to conjugacy) triple  $(P, \rho, \lambda)$ , where  $P \subseteq G$  is a parabolic,  $\rho$  is a tempered distribution of the Levi  $L \subseteq P$ , and a dominant real-valued character  $\lambda: L \rightarrow \mathbb{R}^+$ . Here, dominant means that  $\langle \lambda, \alpha^\vee \rangle > 0$  for each positive coroot  $\alpha^\vee$ .

The Langlands classification takes the triple  $(P, \rho, \lambda)$  to an irreducible quotient of the parabolic induction of  $\rho \otimes e^\lambda$ . We also remark that there is a left adjoint  $r_P^G$  to parabolic induction  $i_P^G$  which simply takes  $V_{\text{fin}}$  to  $V_{\text{fin}}/\mathfrak{u}V_{\text{fin}}$ .

**Example 3.55.** It turns out that  $V_{\text{fin}}/\mathfrak{u}V_{\text{fin}}$  is always nonzero when  $V_{\text{fin}}$  is. Indeed,  $V_{\text{fin}}$  being nonzero yields an embedding  $M \hookrightarrow i_P^G N$ . But the vanishing of  $V_{\text{fin}}/\mathfrak{u}V_{\text{fin}}$  will force  $N$  to be finite-dimensional, which causes problems by passing to finite-dimensional theory. This is related to the Casselman submodule theorem.

The proof of the Langlands classification rests on the following combinatorial lemma.

**Lemma 3.56 (Langlands).** Fix a real reductive group  $G$ , and let  $\Lambda$  be the weight lattice. Further, set

$$(\mathfrak{t}_{\mathbb{R}}^*)^+ := \{\lambda : (\lambda, \alpha) \geq 0 \text{ for all positive } \alpha\},$$

and

$$(\mathfrak{t}_{\mathbb{R}}^*)^- := \left\{ \sum_i a_i \alpha_i : a_i \leq 0 \text{ for all } i \right\}.$$

Then every  $\lambda \in \mathfrak{t}_{\mathbb{R}}^*$  can be written uniquely as a sum  $\lambda_1 + \lambda_2$  such that  $\lambda_1 \in (\mathfrak{t}_{\mathbb{R}}^*)^+$  and  $\lambda_2 \in (\mathfrak{t}_{\mathbb{R}}^*)^-$  and  $(\lambda_1, \lambda_2) = 0$ .

*Sketch.* Simply let  $\lambda_1$  be the closest point to  $\lambda$  in  $(\mathfrak{t}_{\mathbb{R}}^*)^+$ . ■

*Sketch of Theorem 3.54.* We argue for split groups, for simplicity. Choose a minimal exponent  $\lambda$  of  $V_{\text{fin}}$  in the standard partial order.

1. To determine the desired parabolic, we write  $\text{Re } \lambda = \lambda_1 - \lambda_2$  by Lemma 3.56, and we choose  $P$  as large as possible so that  $\lambda_1$  is a character of  $L$ , meaning that  $\text{Lie } L$  is generated by  $\mathfrak{a}$  and the roots  $\alpha$  for which  $(\alpha, \lambda_1) = 0$ .
2. Next, to recover the tempered representation of  $L$ , consider the submodule of  $r_P^G(V_{\text{fin}})$  where the center of  $L$  acts by  $\lambda_1$ , and this turns out to be tempered up to twisting by  $e^{\lambda_1}$ . ■

## 3.8 December 2

Today we begin talking about localization.

### 3.8.1 Introducing $\mathcal{D}$ -Modules

For motivation, recall that irreducible admissible  $(\mathfrak{g}, K)$ -modules admit an infinitesimal character  $\chi_\lambda$  (for some  $\lambda \in \mathfrak{t}^*/(W, \cdot)$ , indexed via the Harish-Chandra isomorphism), so we may in general attempt to understand modules over

$$U_\lambda \mathfrak{g} := U \mathfrak{g} \otimes_{Z \mathfrak{g}} \mathbb{C}_\lambda.$$

Our study of the nilpotent cone showed that  $\text{gr } U_\lambda \mathfrak{g} = \mathbb{C}[\mathcal{N}]$ , where  $\mathcal{N} \subseteq \mathfrak{g}^*$  is the nilpotent cone. Notably,  $\mathcal{N}$  is a Poisson variety, so we may view  $U_\lambda \mathfrak{g}$  as a quantization: the Poisson bracket  $\{\cdot, \cdot\}$  of  $\mathbb{C}[\mathcal{N}]$  turns out to be given by

$$\{\varphi, \psi\} = [\tilde{\varphi}, \tilde{\psi}] \pmod{U_\lambda^{<d+e-1}},$$

where  $\tilde{\varphi}$  and  $\tilde{\psi}$  are lifts to  $U_\lambda \mathfrak{g}$  of degrees  $d$  and  $e$ , respectively.

The structure of  $\mathcal{N}$  is fairly complicated, but it can be understood via geometry. In particular,  $\tilde{\mathcal{N}}$  admits a resolution

$$\pi: T^*(G/B) \rightarrow \mathcal{N}.$$

To define this map, recall  $T^*(G/B)$  can be identified with pairs  $(x, B)$ , where  $B \in G/B$  and  $x \in \text{rad Lie } B$ ; as such,  $\pi$  is given by  $(x, B) \mapsto x$ . This resolution  $\pi$  is indeed birational and proper, and  $\pi$  preserves the Poisson structures present. (Namely,  $T^*(G/B)$  is symplectic, which gives rise to a nice Poisson structure.)

We would now like to quantize the previous paragraph to hopefully understand  $U_\lambda \mathfrak{g}$ . In general, a cotangent bundle  $T^*X$  for a smooth algebraic variety  $X$  can be quantized by considering the (algebraic) differential operators on  $X$ .

**Definition 3.57 (differential operators).** Fix a smooth algebraic variety  $X$ . Then we define  $\mathcal{D}_M$  to be the ring sheaf of differential operators on  $X$ . Explicitly, on local coordinates  $(x_1, \dots, x_n)$ , the sections look like

$$\sum_I f_I d_{x_I}^I,$$

where  $I = (i_1, \dots, i_n)$  is some sequence and  $f_I \in \mathcal{O}$ .

**Remark 3.58.** One can alternatively construct  $\mathcal{D}_X$  as embedded in the sheaf  $\text{End } \mathcal{O}$  as admitting a filtration  $\bigcup_{i \geq 0} \mathcal{D}_X^{\leq i}$ , where  $\mathcal{D}_X^{\leq 0} = \mathcal{O}$  and  $d \in \mathcal{D}_X^{\leq i}$  if and only if  $[d, f] \in \mathcal{D}_X^{\leq i-1}$  for all  $f \in \mathcal{O}$ .

**Remark 3.59.** In characteristic 0, it turns out that  $\mathcal{D}_X$  is generated by  $\mathcal{O}$  and the sheaf of vector fields. As such, one can view  $\mathcal{D}_X$  as a sort of universal enveloping algebra from the differentials.

**Example 3.60.** One can check that

$$\mathcal{D}_{\mathbb{A}^n} = \frac{\mathbb{C}\langle x_1, \dots, x_n, d_1, \dots, d_n \rangle}{([d_i, x_j] = 1_{i=j}, [x_i, x_j] = 0, [d_i, d_j] = 0)_{ij}}.$$

Here, the numerator is the free non-commutative ring on the  $x_\bullet$ s and  $d_\bullet$ s.

It now turns out that  $\text{gr } \mathcal{D} = \mathcal{O}_{T^*X}$ , so we may indeed view  $\text{gr } \mathcal{D}$  as a quantization of  $T^*X$ .

We are thus motivated to study  $\mathcal{D}$ -modules.

**Definition 3.61 ( $\mathcal{D}$ -module).** Fix a smooth algebraic variety  $X$ . Then a (left)  $\mathcal{D}$ -module is a sheaf of (left) modules over  $\mathcal{D}_X$ . The category  $\text{Mod}(\mathcal{D}_X)$ .

**Example 3.62.** Fix an algebraic vector bundle  $(\mathcal{E}, \nabla)$  on  $X$  with a flat connection. Then  $\mathcal{E}$  upgrades into a  $\mathcal{D}$ -module by having a vector field  $z$  act by

$$z: \sigma \mapsto \nabla_z(\sigma).$$

**Example 3.63.** Given an open embedding  $j: U \hookrightarrow X$ , there is a functor  $j_*: \text{Mod}(\mathcal{D}_U) \rightarrow \text{Mod}(\mathcal{D}_X)$  defined as expected.

### 3.8.2 Holonomic $\mathcal{D}$ -Modules

As with Harish-Chandra bimodules, we have a notion of support.

**Definition 3.64** (singular support). Fix a smooth algebraic variety  $X$ , and let  $\mathcal{M}$  be a coherent  $\mathcal{D}$ -module, meaning that  $\mathcal{M}(U)$  is finitely generated over  $\mathcal{D}(U)$  on each affine  $U \subseteq X$ . A *good filtration* for  $\mathcal{M}$  is an ascending filtration  $\{\mathcal{M}_{\leq i}\}_{i \in \mathbb{Z}}$  satisfying the following.

- Exhaustive: the filtration  $\{\mathcal{M}_{\leq i}\}_{i \in \mathbb{Z}}$  is exhaustive and satisfies  $\bigcap_i \mathcal{M}_{\leq i} = 0$ .
- Compatible with  $\mathcal{D}$ :  $\mathcal{D}_{\leq i} \mathcal{M}_{\leq j} \subseteq \mathcal{M}_{\leq i+j}$  for all  $i$  and  $j$ .
- Finite type:  $\text{gr } \mathcal{M}$  is finitely generated over  $\text{gr } \mathcal{D}$ .

Then the *singular support* of  $\mathcal{M}$  is  $\text{supp } \text{gr } \mathcal{M}$ .

**Example 3.65.** Here is one way to construct good filtrations: start with a finitely generated  $\mathcal{O}$ -submodule  $V \subseteq \mathcal{M}$  of generators for  $\mathcal{M}$  over  $\mathcal{D}$ , and then simply define

$$\mathcal{M}_{\leq i} := \mathcal{D}_{\leq i} V.$$

**Remark 3.66.** Even though  $\text{gr } \mathcal{M}$  depends on the choice of (good) filtration, it turns out that the (set-theoretic) singular support does not. (Even the scheme-theoretic support may depend on the filtration!)

It turns out that the support is always pretty large.

**Proposition 3.67** (Bernstein's inequality). Fix a smooth algebraic variety  $X$ . For any  $\mathcal{D}$ -module  $\mathcal{M}$ , we have

$$\dim \text{supp } \mathcal{M} \geq \dim X.$$

**Example 3.68.** For example, on  $X = T^* \mathbb{A}^1$ , so it suffices to check that  $\dim \text{supp } \mathcal{M} > 0$ . However,  $\dim \text{supp } \mathcal{M} = 0$  means that  $\mathcal{M}$  is given by some finite-dimensional module over

$$\mathcal{D} = \frac{\mathbb{C}\langle x, \partial \rangle}{([\partial, x] = 1)}.$$

But then  $\text{tr id} = \dim \mathcal{M}$  must vanish!

The modules with small support turn out to be interesting.

**Definition 3.69** (holonomic). Fix a  $\mathcal{D}$ -module  $\mathcal{M}$  on a smooth algebraic variety  $M$ . Then  $\mathcal{M}$  is *holonomic* if and only if

$$\dim \text{supp } \mathcal{M} = \dim M.$$

**Example 3.70.** For  $(\mathcal{E}, \nabla)$  on some  $X$ , because  $\mathcal{E}$  is a vector bundle, one can choose the filtration so that  $\text{gr } \mathcal{E}$  is supported in degree 0. It follows that the singular support is just the zero section of  $X \rightarrow T^* X$ , so  $(\mathcal{E}, \nabla)$  is holonomic!

**Theorem 3.71.** Fix a smooth algebraic variety  $M$ . Then the irreducible holonomic  $\mathcal{D}$ -modules on  $X$  are in bijection with equivalence classes of pairs  $(Y, (\mathcal{E}, \nabla))$ , where  $Y \subseteq X$  is a smooth irreducible locally closed subvariety, and  $(\mathcal{E}, \nabla)$  is an irreducible vector bundle with flat connection on  $Y$ .

*Proof.* We will say nothing interesting about the proof, but we do say something about how this construction goes. In the case where  $Y \subseteq X$  is open, one can consider the  $\mathcal{D}$ -module  $j_*(\mathcal{E}, \nabla)$ , where  $j: Y \hookrightarrow X$  is the inclusion. One then shows that  $j_*(\mathcal{E}, \nabla)$  admits a unique irreducible subquotient.

In general, one has a chain  $Y \subseteq U \subseteq X$ , where  $i: Y \hookrightarrow U$  is closed, and  $j: U \hookrightarrow X$  is open. It turns out that  $(\mathcal{E}, \nabla)$  can be extended to a  $\mathcal{D}$ -module on  $U$  which is supported on  $Y$  (via some  $i_*$  construction). Then we apply  $j_*$  and find the unique irreducible submodule. ■

### 3.8.3

We can now return to our motivation.

**Theorem 3.72 (Localization).** Fix a complex reductive group  $G$  with Borel subgroup  $B$ . Then taking global sections produces an equivalence

$$\Gamma: \text{Mod}(\mathcal{D}_{G/B}) \rightarrow \text{Mod}(U_0\mathfrak{g}),$$

where the  $0$  denotes the infinitesimal character of the trivial representation. In other words,  $U_0\mathfrak{g} = U\mathfrak{g}/Z\mathfrak{g}$ .

**Remark 3.73.** There is also a generalization for  $U_\lambda\mathfrak{g}$ , but we need to modify our  $\mathfrak{g}$ -modules slightly.

The moral is that  $\mathcal{D}$ -modules do in fact provide the correct quantization of  $T^*(G/B)$ , in the same way that  $U_\lambda\mathfrak{g}$  is a quantization for  $\mathcal{N}$ . The above theorem follows from the following two theorems.

**Theorem 3.74.** Fix a complex reductive group  $G$  with parabolic subgroup  $P$ . Then taking global sections produces an equivalence

$$\Gamma: \text{Mod}(\mathcal{D}_{G/P}) \rightarrow \text{Mod}(\Gamma(\mathcal{D}_{G/P})).$$

**Theorem 3.75.** Fix a complex reductive group  $G$  with Borel subgroup  $B$ . Then there is a ring isomorphism

$$U_0\mathfrak{g} \cong \Gamma(\mathcal{D}_{G/B}).$$

**Remark 3.76.** Roughly speaking, the map  $U_0\mathfrak{g} \rightarrow \Gamma(\mathcal{D}_{G/B})$  is a quantization of the Springer resolution  $T^*(G/B) \rightarrow \mathcal{N}$ .

**Example 3.77.** Take  $G = \text{SL}(2)$  so that  $G/B = \mathbb{P}^1$ . Then there is a unique (up to isomorphism)  $\mathcal{D}$ -module  $\delta_0$  supported at  $0$ . In short,  $\delta_0$  has basis given by  $\{t^{-n}\}_{n>0}$ , where  $t\partial_t$  is an eigenvector with eigenvalue  $-1$ . From  $\delta_0$ , it turns out that one gets the irreducible Verma module  $M_{-2}$ ; note that  $M_{-2}$  has trivial central character, as explained by the Harish-Chandra isomorphism, so  $M_{-2}$  is in fact a  $U_0\mathfrak{g}$ -module.

To upgrade to  $(\mathfrak{g}, K)$ -modules, it will be helpful to have groups act on our  $\mathcal{D}$ -modules.

**Definition 3.78 (equivariant).** Fix an algebraic group  $H$  acting on a smooth variety  $X$ . Let  $a: H \times X \rightarrow X$  be the action map. A sheaf  $\mathcal{F}$  on  $X$  is  $H$ -equivariant if and only if there is an isomorphism  $a^*\mathcal{F} \rightarrow \text{pr}_2^*\mathcal{F}$  satisfying a cocycle condition on  $H \times H \times X$ . An equivariant sheaf of modules is also required to commute with the module structure.

**Remark 3.79.** Alternatively, one may state this as having  $H$  suitably acting on sections. Thus, we may say that an  $H$ -equivariant  $\mathcal{D}$ -module  $\mathcal{M}$  is one in which the  $H$ -action on the sections respects the  $\mathcal{D}$ -module structure, and we require that the two actions of  $\mathrm{Lie} H$  coincide. Here, one action of  $\mathrm{Lie} H$  is given by

$$\mathrm{Lie} H \rightarrow \mathrm{Vect} X \subseteq \mathcal{D}_X \rightarrow \mathrm{End} \mathcal{M},$$

and the other is given by differentiating the  $H$ -action on  $\mathcal{M}$ .

**Example 3.80.** If  $X$  is a point, then an  $H$ -equivariant  $\mathcal{D}$ -module is a representation of  $\pi_0 H$  because the action of  $\mathrm{Lie} H$  must be trivial.

**Corollary 3.81.** Fix a complex reductive group  $G$ . Then there is an equivalence

$$\Gamma: \mathrm{Mod}^K(\mathcal{D}_{G/B}) \rightarrow \mathrm{Mod}(\mathfrak{g}, K)_0.$$

Here, the left-hand side is the category of  $K$ -equivariant  $\mathcal{D}$ -modules on  $G/B$ . The right-hand side is the category of  $(\mathfrak{g}, K)$ -modules with trivial central character.

**Example 3.82.** Suppose  $H$  acts on  $X$  with finitely many orbits. (For example,  $K$  acts on  $G/B$  with finitely many orbits.) It turns out that irreducible  $H$ -equivariant  $\mathcal{D}$ -modules are in bijection with pairs  $(\mathcal{O}, \psi)$  where  $\mathcal{O}$  is an orbit  $Hx$ , and  $\psi$  is a representation of  $\pi_0(\mathrm{Stab}(x))$ . This allows us to classify irreducibles in  $\mathrm{Mod}(\mathfrak{g}, K)_0$ .

## APPENDIX A

# FUNCTIONAL ANALYSIS

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In this appendix, we introduce the small amount of functional analysis we will need in order to get going with infinite-dimensional vector spaces. In other words, we need to set up the theory of Fréchet spaces. Throughout this appendix,  $\mathbb{F}$  denotes one of the fields  $\mathbb{R}$  or  $\mathbb{C}$ . Our exposition is largely stolen from [Con90].

### A.1 Locally Convex Spaces

We begin with the following definition.

**Definition A.1 (topological vector space).** Fix a topological field  $k$ . Then a *topological vector space* is a vector space  $V$  over  $k$  equipped with a topology so that addition map  $+: V \times V \rightarrow V$  and scalar multiplication map  $\cdot: k \times V \rightarrow V$  are both continuous. In these notes, all topological vector spaces will be assumed to be Hausdorff.

A Fréchet space will be a complete topological vector space admitting two notable definitions: having its topology is defined by a countable family of seminorms, or being locally convex and metrizable. As such, let's quickly recall the definition of a seminorm.

**Definition A.2 (seminorm).** Fix a vector space  $V$  over  $\mathbb{F}$ . Then a *seminorm* is a function  $p: V \rightarrow \mathbb{R}$  satisfying the following.

- Subadditive: we have  $p(x + y) \leq p(x) + p(y)$  for any  $x, y \in V$ .
- Homogeneous: we have  $p(\lambda x) = |\lambda| p(x)$  for any  $x \in V$  and  $\lambda \in \mathbb{F}$ .

**Remark A.3.** The homogeneity implies that  $p(0) = 0$ , which is sometimes included in the definition. Similarly, the subadditivity now implies that  $2p(x) = p(x) + p(-x)$  is at least  $p(0) = 0$ , so  $p$  is automatically nonnegative; this is also sometimes included in the definition.

It is worthwhile to have a few ways to check continuity.

**Lemma A.4.** Fix a topological vector space  $V$  over  $\mathbb{F}$ , and let  $p: V \rightarrow \mathbb{R}$  be a seminorm. Then the following are equivalent.

- (i)  $p$  is continuous.
- (ii)  $\{v : p(v) < 1\}$  is open.
- (iii)  $p$  is continuous at 0.

*Proof.* This is [Con90, Proposition 1.3]. Of course (i) implies (ii). We show the remaining implications independently.

- We show (ii) implies (iii). For any net  $\{x_i\}$  converging to 0, we want to show that  $\{p(x_i)\} \rightarrow 0$ , which is true because any  $x_i$  in the open neighborhood  $\varepsilon U$  of 0 has  $p(x_i) < \varepsilon$ .
- We show (iii) implies (i). Note that any net  $\{x_i\}$  converging to some  $x$  has

$$|p(x_i) - p(x)| \leq p(x_i - x),$$

and  $p(x_i - x) \rightarrow 0$  because  $x_i - x \rightarrow 0$ . ■

We now start talking about convex sets, but we will relate our definitions back to seminorms.

**Definition A.5 (convex).** Fix a vector space  $V$  over  $\mathbb{F}$ . A subset  $A \subseteq V$  is *convex* if and only if any two  $a, b \in A$  has

$$ta + (1 - t)b \in A$$

for any  $t \in [0, 1]$ .

**Example A.6.** Let  $p: V \rightarrow \mathbb{R}$  be a seminorm. Then we claim that  $A := \{v \in V : p(v) < 1\}$  is convex. Indeed, for  $a, b \in A$  and  $t \in [0, 1]$ , we see that

$$p(ta + (1 - t)b) = tp(a) + (1 - t)p(b),$$

which is still less than 1, so  $ta + (1 - t)b \in A$ .

**Example A.7 (convex hull).** For any subset  $A \subseteq V$ , we may define the convex hull

$$\text{conv}(A) := \left\{ \sum_{i=1}^n t_i a_i : \{a_i\}_i \subseteq A, \{t_i\}_i \in [0, 1], t_1 + \cdots + t_n = 1 \right\}.$$

Note that  $\text{conv}(A)$  is convex: for two points  $\sum_i t_i a_i$  and  $\sum_j s_j b_j$  and  $t \in [0, 1]$ , the sum  $\sum_i tt_i a_i + \sum_j (1 - t)s_j b_j$  still has  $\sum_i tt_i + \sum_j (1 - t)s_j = t + (1 - t) = 1$ . In fact, if  $B$  is convex and contains  $A$ , then  $\text{conv}(A) \subseteq B$  because the sums  $\sum_i t_i a_i$  can be checked to be in  $B$  by induction.

Convex sets on their own turn out to not be good enough for our purposes, so we will need extra adjectives.

**Definition A.8 (balanced).** Fix a vector space  $V$  over  $\mathbb{F}$ . A subset  $A \subseteq V$  is *balanced* if and only if  $\lambda A \subseteq A$  for all  $\lambda \in \mathbb{F}$  such that  $|\lambda| \leq 1$ .

**Example A.9.** Let  $p: V \rightarrow \mathbb{R}$  be a seminorm. Then we claim that  $A := \{v \in V : p(v) < 1\}$  is balanced. Indeed, for  $a \in A$  and  $\lambda$  with  $|\lambda| \leq 1$ , we see that  $p(\lambda a) = |\lambda|p(a)$ , which is still less than 1, so  $\lambda a \in A$ .

**Example A.10.** For any subset  $A \subseteq V$ , the subset

$$\text{bal}(A) := \bigcup_{|\lambda| \leq 1} \lambda A$$

is balanced. Indeed, for any  $\mu$  with  $|\mu| \leq 1$ , we see that  $\mu \text{bal}(A) = \bigcup_{\lambda} \mu \lambda A$  is contained in  $\text{bal}(A)$  because  $|\lambda \mu| \leq 1$  whenever  $|\lambda| \leq 1$ . Of course, we always have  $A \subseteq \text{bal}(A)$ , and one can see that any balanced set containing  $A$  must contain each  $\lambda A$  and hence contain  $\text{bal}(A)$ .

It turns out that passing to balanced convex sets is not too big of a burden.

**Lemma A.11.** Fix a topological vector space  $V$  over  $\mathbb{F}$ . Any convex open neighborhood of 0 contains a balanced convex open neighborhood of 0.

*Proof.* Let  $U$  be a convex open neighborhood of 0. The point is to use the continuity of scalar multiplication: the continuity of

$$\cdot: \mathbb{F} \times V \rightarrow V$$

provides a basic open neighborhood  $B(0, \varepsilon) \times U'$  of  $(0, 0)$  of  $\mathbb{F} \times V$  such that  $B(0, \varepsilon)U' \subseteq U$ . We claim that  $\text{conv}(B(0, \varepsilon)U')$  is the desired open neighborhood. Here are our checks; set  $U'' := B(0, \varepsilon)U'$  for brevity.

- Because  $U$  is already convex, Example A.7 explains that  $U'' \subseteq U$  implies that  $\text{conv}(U'') \subseteq U$ .
- Convex: note  $\text{conv}(U'')$  is convex by Example A.7.
- Open: because scalar multiplication by a nonzero number is a homeomorphism, we see that  $U'' := B(0, \varepsilon)U'$ , which is

$$\bigcup_{0 < |\lambda| < \varepsilon} \lambda U',$$

is open. Then once  $U'$  is open, we see that  $\text{conv}(U')$  can be written as a union

$$\bigcup_{n \geq 1} \left( \bigcup_{t_1 + \dots + t_n = 1} t_1 U'' + \dots + t_n U'' \right),$$

which is open because the sum of two open subsets is open (indeed, the sum of open sets is a union of translates of just one of the open sets).

- Balanced: the previous step realized  $U''$  as a union  $\bigcup_{0 < |\lambda| < \varepsilon} \lambda U'$ , which can be shown to be balanced exactly as in Example A.10. ■

**Remark A.12.** Here is a sample application: suppose that 0 has a neighborhood basis of convex sets. Then Lemma A.11 implies that 0 also admits a neighborhood basis of balanced convex sets: for each  $U$  in the neighborhood basis, the lemma produces a smaller open subset which is still convex but now also balanced. Similarly, admitting a countable neighborhood basis of convex sets can be upgraded to admitting a countable neighborhood basis of balanced convex sets.

The previous remark allows us to make the following definition.

**Definition A.13 (locally convex).** A topological vector space  $V$  is *locally convex* if and only if 0 admits a neighborhood basis of convex sets.



**Remark A.14.** By translation, it is equivalent to require  $V$  to have a basis of convex sets. (Namely, if  $\mathcal{U}$  is the neighborhood basis of  $0$ , then  $\bigcup_{v \in V} v + \mathcal{U}$  is the basis of  $V$ .) By Remark A.12, we can also upgrade these notions to having bases of balanced convex sets.

It is notable that the previous two definitions avoid the mention of any topology. In order to continue not doing any topology, we pick up the following definition, which provides a linear algebraic stand-in for “contains an open neighborhood of the origin.”

**Definition A.15 (absorbing).** Fix a vector space  $V$  over  $\mathbb{F}$ . A subset  $A \subseteq V$  is *absorbing* if and only if any  $v \in V$  admits some  $\varepsilon > 0$  such that  $tv \in A$  for all  $t \in [0, \varepsilon)$ .

For example, we see that  $0$  is contained in any absorbing subset.

**Remark A.16.** Of course, if  $A$  is absorbing, and  $A \subseteq B$ , then  $B$  is absorbing: for each  $v$ , the  $\varepsilon$  which worked for  $A$  continues to work for  $B$ .

**Example A.17.** Let’s explain the remark given before the definition. If  $V$  is a topological vector space over  $\mathbb{F}$ , then we claim that any open neighborhood  $U$  of  $0$  is absorbing. This will follow by the continuity of scalar multiplication: for any  $v \in V$ , the map  $\mathbb{R} \rightarrow V$  given by  $t \mapsto tv$  is continuous. Thus, because  $0 \in U$ , there must be  $\varepsilon > 0$  such that  $(-\varepsilon, \varepsilon)v \subseteq U$ .

**Example A.18.** Let  $p: V \rightarrow \mathbb{R}$  be a seminorm, and set  $A := \{v \in V : p(v) < 1\}$ . Then we claim that  $(-a) + A$  is absorbing for all  $a \in A$ ; for example, setting  $a = 0$  will imply that  $A$  is absorbing. Now, for any  $v \in V$ , we need to show that  $a + tv \in A$  for small  $t$ . Well,  $p(a + tv) \leq p(a) + |t|p(v)$ , so taking any  $t$  with  $p(v)|t| < (1 - p(a))$  will do. (In particular, any  $t$  will work if  $p(v) = 0$ .)

We are now ready to construct some seminorms.

**Notation A.19.** Fix a vector space  $V$  over  $\mathbb{F}$ . For any absorbing subset  $A \subseteq V$ , we define  $\|\cdot\|_A : V \rightarrow \mathbb{R}_{\geq 0}$  by

$$\|v\|_A = \inf \{t \geq 0 : v \in tA\}.$$

Here are some basic facts about this construction.

**Remark A.20.** Because  $A$  is absorbing, we see that any  $v \in V$  does in fact have some  $t > 0$  for which  $(1/t)v \in A$  and hence  $v \in tA$ , so the infimum is a real number.

**Remark A.21.** Suppose further that  $A$  is convex. Then we claim that  $v \in tA$  whenever  $t > \|v\|_A$ ; note if  $v = 0$ , then  $\|0\|_A = 0$ , so there is nothing to do. The main point is to note that  $sv \in A$  implies that  $s'v \in A$  for any  $s' \in [0, s]$  by convexity. Thus, if  $t > \|v\|_A$ , then we know there is  $s < t$  such that  $v \in sA$ , so  $(1/s)v \in A$  while  $1/t < 1/s$ , so  $(1/t)v \in A$ , so  $v \in tA$ .

Let’s put all our adjectives together, finally explaining the relationship between seminorms and convex sets.

**Proposition A.22.** Fix a vector space  $V$  over  $\mathbb{F}$  and a subset  $A \subseteq V$ .

- (a) There is a seminorm  $p: V \rightarrow \mathbb{R}$  such that  $A = \{v \in V : p(v) < 1\}$  if and only if  $A$  is nonempty, convex, balanced, and  $(-a) + A$  is absorbing for all  $a \in A$ .
- (b) If  $A$  is merely convex, balanced, and absorbing, then there is a seminorm  $p: V \rightarrow \mathbb{R}$  such that  $\{v : p(v) < 1\} \subseteq A$ .

*Proof.* This is [Con90, Proposition 1.14]. The forward direction of (a) follows from combining Examples A.6, A.9 and A.18. It remains to show the reverse direction of (a) and (b). For both of these, we will take  $p := \|\cdot\|_A$ , which is a well-defined function by Remark A.20 (once we know that  $A$  in (a) is absorbing). We will run our checks in a few pieces.

- In (a), we show that  $A$  is absorbing. It is enough to check that  $0 \in A$ . Well, being nonempty, there is some  $a \in A$ . Because  $A$  is balanced, we see  $-a \in A$ , and because  $A$  is convex, it follows that  $0 \in A$ .
- If  $A$  is absorbing and balanced, we check that  $p(\lambda v) = |\lambda|p(v)$  for  $v \in V$  and  $\lambda \in \mathbb{F}$ . Well,  $\lambda v \in tA$  if and only if  $v \in \frac{t}{|\lambda|}A$ , which is equivalent to  $v \in \frac{t}{|\lambda|}A$  because  $A$  is balanced! It follows that

$$\{t \geq 0 : \lambda v \in tA\} = |\lambda| \left\{ \frac{t}{|\lambda|} : v \in \frac{t}{|\lambda|}A \right\},$$

so the check follows.

- If  $A$  is convex and absorbing, then we check that  $p(v+w) \leq p(v) + p(w)$  for  $v, w \in V$ . The geometric input is that  $tA + sA \subseteq (t+s)A$  for any  $t, s > 0$ ; this follows by convexity because

$$tA + sA = (t+s) \left( \frac{t}{t+s}A + \frac{s}{t+s}A \right)$$

is contained in  $(t+s)A$  by convexity. Now, for the check, we note that having  $p(v) < t$  and  $p(w) < s$  implies that  $v \in tA$  and  $w \in sA$  by Remark A.21, so  $v+w \in (t+s)A$ , so  $p(v+w) \leq t+s$ . Sending  $t \rightarrow p(v)$  and  $s \rightarrow p(w)$  completes the check.

- We complete the proof of (b). The above checks show that  $p$  is a seminorm, so it remains to check that  $\{v : p(v) < 1\} \subseteq A$ . This follows from  $A$  being balanced: if  $p(v) < 1$ , then there is  $t < 1$  such that  $v \in tA$ , and  $tA \subseteq A$  because  $A$  is balanced.
- We complete the proof of (a). The previous check shows that  $\{v : p(v) < 1\} \subseteq A$ , so it remains to check the other inclusion. Well, for any  $a \in A$ , we see that  $(-a) + A$  is absorbing, so  $a + ta \in A$  for small  $t > 0$ . It follows that  $a \in (1+t)^{-1}A$ , so  $p(a) < (1+t)^{-1} < 1$  follows. ■

**Corollary A.23.** Fix a topological vector space  $V$  over  $\mathbb{F}$ . The following are equivalent.

- (i)  $V$  is locally convex.
- (ii) The topology on  $V$  is induced by a family of seminorms.

*Proof.* We show the implications separately.

- Suppose that  $V$  is locally convex, so  $0$  admits a neighborhood basis  $\mathcal{U}$  of balanced convex sets by Remark A.14. By Example A.17, we see that each  $U \in \mathcal{U}$  has  $(-a) + U$  absorbing for all  $a \in U$ , so Proposition A.22 provides a seminorm  $p_U : V \rightarrow \mathbb{R}$  such that  $U = \{v : p_U(v) < 1\}$ . Note that  $p_U$  is continuous by Lemma A.4.

Lastly, we should check that the topology given by the seminorms  $\{p_U\}$  is the correct one. Well, this topology has basis given by finite intersections of sets of the form

$$\{v \in V : p_U(v) \in (a, b)\},$$

where  $(a, b) \subseteq \mathbb{R}$ . The continuity of the  $p_U$ s implies that any such subset is open in  $V$ . Conversely, any open neighborhood of  $0$  in  $V$  contains some  $U \in \mathcal{U}$  and therefore contains  $\{v \in V : p(v) \in (-1, 1)\}$ , so a comparison of the neighborhood bases (via translation) implies that the open neighborhood of  $0$  is still open.

- Suppose that  $V$  has its topology generated by a family of seminorms  $\{p_i\}$ . Well, because  $p_i(0) = 0$  for each  $i$ , an open neighborhood basis of  $0$  can be given by finite intersections of sets of the form  $p_i^{-1}((-\varepsilon, \varepsilon))$ . Of course, this is just

$$\varepsilon\{v \in V : p_i(v) < 1\},$$

which we note is convex by Example A.6. Thus,  $0$  admits a neighborhood basis of convex sets. ■

Now that we have an understanding of locally convex spaces, we may define Fréchet spaces.

**Definition A.24 (complete).** Fix a topological vector space  $V$ . Then a *Cauchy net* in  $V$  is a net  $\{v_i\}$  such that each neighborhood  $U$  of  $0$  has some  $N$  for which  $v_i - v_j \in U$  for all  $i, j > N$ . We say that  $V$  is *complete* if every Cauchy net converges.

**Remark A.25.** If  $V$  is metrizable, then we may as well work with Cauchy sequences and being sequentially complete.

**Definition A.26 (Fréchet).** A topological vector space  $X$  is *Fréchet* if and only if it is locally convex, metrizable, and complete.

The bizarre addition here is metrizable. This condition fits in with the other ones as follows.

**Proposition A.27.** Fix a locally convex topological vector space  $V$  over  $\mathbb{F}$ . Then the following are equivalent.

- (i)  $V$  has its topology induced by a translation-invariant metric.
- (ii)  $V$  is metrizable.
- (iii)  $V$  has a countable neighborhood basis of  $0$ .
- (iv) The topology on  $V$  is induced by a countable family of seminorms.

*Proof.* The implication (i) to (ii) has no content, and (ii) to (iii) follows by taking the neighborhood basis of open subsets given by  $\{v : d(v, 0) < 1/n\}$  for positive integers  $n$ . Next, (iii) implies (iv) by the proof of the forward direction of Corollary A.23, which built one seminorm for each balanced convex subset in the neighborhood basis of  $0$ .

Lastly, we have to show that (iv) implies (i). Well, given the countable family of seminorms  $\{p_i\}_{i \geq 1}$ , we define the function  $d: V \times V \rightarrow \mathbb{R}$  by

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x - y)}{1 + p_i(x - y)}.$$

Here are our checks on  $d$ .

- We check that  $d$  is a metric. The summation always converges because  $\frac{p_i(x-y)}{1+p_i(x-y)} \leq 1$  always. Continuing,  $d(x, x) = 0$  follows because  $p_i(0) = 0$  for all  $i$ , and the triangle inequality follows from the subadditivity of each of the  $p_i$ s.

It remains to check the positivity of  $d$ . Well, if  $x \neq y$ , then because  $V$  is Hausdorff, we see that  $p_i(x - y) > 0$  for some  $p_i$ . (Otherwise, the constant net  $\{x - y\}$  would converge to both  $0$  and  $x - y$ .) Thus,  $d(x, y) > 0$ .

- We check that  $d$  is translation-invariant. Well, for any  $a \in V$ , we see that  $d(x + a, y + a)$  is a function of  $(x + a) - (y + a) = (x - y)$  and will equal  $d(x, y)$ .

- Lastly, we check that  $d$  induces the topology on  $V$ . It is enough to check that these two topologies have the same convergent nets. Well, a net  $\{x_i\}$  converges to some  $x \in V$  if and only if  $p_\bullet(x_i - x) \rightarrow 0$  for all seminorms  $p_\bullet$ . This surely implies that  $d(x_i, x) \rightarrow 0$ , and conversely,  $d(x_i, x) \rightarrow 0$  will require that  $p_\bullet(x_i - x) \rightarrow 0$  for each  $p_\bullet$ . ■

## A.2 Cauchy Sequences

Because we will have occasion to work with more general topological spaces than metric spaces, we go ahead and pick up some facts about Cauchy sequences and completeness.

**Lemma A.28.** Fix a continuous linear map  $f: V \rightarrow V'$  of topological vector spaces. Then  $f$  sends Cauchy nets to Cauchy nets.

*Proof.* Suppose that  $\{a_i\}_{i \in I}$  is a Cauchy net of  $V$ , and we would like to show that  $\{f(a_i)\}_{i \in I}$  is a Cauchy net of  $V'$ . Well, for any open neighborhood  $U' \subseteq V'$  of 0, we are granted an open neighborhood  $U \subseteq V$  of 0 for which  $x \in U$  implies that  $f(x) \in U'$ . In particular, there is  $N \in I$  for which  $i, j > N$  implies that  $a_i - a_j \in U$  and thus  $f(a_i) - f(a_j) \in U'$ , as required. ■

**Definition A.29 (Cauchy sequence).** Fix a topological vector space  $V$ . Then a *Cauchy sequence* is a Cauchy net where the index set is a  $\mathbb{N}$ .

**Proposition A.30.** Fix a sequentially dense linear subspace  $W \subseteq V$ , and choose a sequentially continuous linear map  $f: W \rightarrow V'$  of topological vector spaces.

- There is at most one sequentially continuous linear map  $g: V \rightarrow V'$  such that  $g|_W = f$ .
- If  $V'$  is sequentially complete, then there is a sequentially continuous linear map  $g: V \rightarrow V'$  such that  $g|_W = f$ .

*Proof.* We start with (a). By linearity of restriction, it is enough to show that any continuous linear map  $g: V \rightarrow V'$  for which  $g|_W = 0$  has  $g = 0$ . Well, for any  $v \in V$ , the density of  $W \subseteq V$  grants a Cauchy sequence  $\{w_i\}$  in  $W$  for which  $\{w_i\} \rightarrow v$ . Thus, the continuity of  $g$  requires that  $\{g(w_i)\} \rightarrow g(v)$ , so  $g(v) = 0$ .

It remains to show (b), which is trickier. We begin by defining  $g$ , and then we will run many checks on it. Because  $W \subseteq V$  is sequentially complete, each  $v \in V$  admits a Cauchy sequence  $\{w_i\}$  of  $W$  such that  $\{w_i\} \rightarrow v$ . Then Lemma A.28 implies that  $\{f(w_i)\}$  is still a Cauchy sequence in  $V'$ , so we define

$$g(v) := \lim_{i \rightarrow \infty} f(w_i).$$

(The limit exists because  $V'$  is sequentially complete!) It remains to run many checks on  $g$ .

- Well-defined: if  $\{w_i\}$  and  $\{w'_i\}$  are two Cauchy sequences approaching a given  $v \in V$ , then we need to check that  $\lim f(w_i) = \lim f(w'_i)$ . Well, we see that  $\{w_i - w'_i\} \rightarrow 0$ , so  $\{f(w_i) - f(w'_i)\} \rightarrow f(0)$  by continuity of  $f$ , so the claim follows.
- Linear: given scalars  $a^1$  and  $a^2$  and  $v^1, v^2 \in V$ , we should check that  $g(a^1 v^1 + a^2 v^2) = a^1 g(v^1) + a^2 g(v^2)$ . Well, we simply pick up Cauchy sequences  $\{w_i^1\}$  approaching  $v^1$  and  $\{w_i^2\}$  approaching  $v^2$ . Then  $\{a^1 w_i^1 + a^2 w_i^2\}$  approaches  $a^1 v^1 + a^2 v^2$ , so we can pass everything through the definition of  $g$ .
- Continuous: given a convergent sequence  $\{v_i\} \rightarrow v$  in  $V$ , we would like to check that  $\{g(v_i)\} \rightarrow g(v)$ . By linearity of  $g$ , we may assume that  $v = 0$  (by subtracting out  $v$  everywhere). Now, for any open neighborhood  $U' \subseteq V'$  of 0, we would like to show that  $g(v_i) \in U'$  for large  $i$ . For technical reasons, we choose an open neighborhood  $U'_0 \subseteq U'$  of 0 such that  $U'_0 + U'_0 \subseteq U'$ , which is possible by the continuity

of addition. The point of doing this is that the closure of  $U'_0$  is contained in  $U'$ : for any Cauchy net  $\{u'_i\}$  in  $U'_0$  converging to some  $u'$ , we see that  $u' - u'_i \in U'_0$  for any  $i$  large enough, so  $u' \in U'_0 + U'_0$ .

Now, by continuity of  $f$ , we are granted an open neighborhood  $U \subseteq V$  of 0 such that  $x \in U \cap W$  implies  $f(x) \in U'_0$ . For any  $i$  large enough, we see that  $v_i \in U_0$ . Then for each  $i$ , we fix a Cauchy sequence  $\{w_{ij}\}$  of  $W$  approaching  $v_i$ ; for any  $j$  large enough, we see  $w_{ij} \in U_0$  as well. Thus,  $\{f(w_{ij})\} \rightarrow g(v_i)$  by definition of  $g$ , so  $g(v_i)$  is in the closure of  $U'_0$ , so  $g(v_i) \in U'$ , as required. ■

**Remark A.31.** By replacing all sequences with nets in the proof, we can upgrade the statement to show that continuous linear maps extend uniquely from dense linear subspaces when the target is complete.

## A.3 The Open Mapping Theorem

In this section, we review the proof of the Open mapping theorem in order to extend the usual proof (for Banach spaces) to the setting of Fréchet spaces.

As usual, our proof will have to rely on the Baire category theorem. Before introducing any strange terminology, let's start with a statement on just metric spaces.

**Lemma A.32.** Let  $X$  be a nonempty complete metric space. Then a countable intersection of dense open subsets is dense.

*Proof.* Let  $\{U_i\}_{i \in \mathbb{N}}$  be our collection of dense open subsets. We would like to show that their intersection  $\bigcap_{i \in \mathbb{N}} U_i$  intersects any open subset  $V$  of  $X$ . The idea is to recursively choose nearby elements in  $U_i \cap V$  for each  $i$ , and then use completeness of  $X$  to finish the proof. We proceed in steps.

1. We build a sequence of points  $\{x_n\}_{n \in \mathbb{N}}$  recursively, as follows. To start us off, we note  $V \cap U_0$  is nonempty and open (by density of  $U_0$ ), so we are granted a point  $x_0 \in U_0$  and  $\varepsilon_0$  such that  $B(x_0, \varepsilon_0) \subseteq V \cap U_0$ . For the recursion, we suppose that we are given such an open neighborhood  $B(x_n, \varepsilon_n)$ , and then because  $U_{n+1}$  is open and dense, we are provided a point  $x_{n+1}$  in the intersection and some  $\varepsilon_{n+1} < \varepsilon_n/3$  such that

$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}.$$

2. We claim that the sequence  $\{x_n\}_{n \in \mathbb{N}}$  is (rapidly) Cauchy. Indeed, note  $\varepsilon_{n+1} < \varepsilon_n/2$  for each  $n$ , so  $\varepsilon_n < 2^{-n}$  follows by an induction. Thus,  $d(x_n, x_{n+1}) < 2^{-n}$  for each  $n$ , so our sequence is rapidly Cauchy. To finish checking that it is Cauchy, we note that whenever  $i < j$ , we have

$$d(x_i, x_j) \leq \sum_{k=i}^{j-1} \underbrace{d(x_k, x_{k+1})}_{< 2^{-k}},$$

which is upper-bounded by  $2^{-i+1}$ .

3. Now, we let  $x$  be a limit point of  $\{x_n\}_{n \in \mathbb{N}}$ . (This is where we used completeness!) Because our sequence is eventually in  $B(x_n, \varepsilon_n)$  for any given  $n$ , we see that  $x \in B(x_n, \varepsilon_n)$  for each  $n$ . Thus,  $x \in V \cap U_0$  by the first step of the construction, and  $x \in U_n$  for each  $n \geq 1$  by the recursive step of the construction. ■

The previous lemma now upgrades to the Baire category theorem.

**Theorem A.33 (Baire category).** Let  $X$  be a nonempty complete metric space. Let  $\{U_i\}_{i \in \mathbb{N}}$  be a countable collection of dense open subsets. Then the intersection  $\bigcap_{i \in \mathbb{N}} U_i$  is not contained in a countable union of nowhere dense subsets.

*Proof.* Suppose for the sake of contradiction that we have

$$\bigcap_{i \in \mathbb{N}} U_i \subseteq \bigcup_{j \in \mathbb{N}} A_j,$$

where each  $A_j$  is nowhere dense. It thus follows that

$$\bigcap_{i \in \mathbb{N}} U_i \cap \bigcap_{j \in \mathbb{N}} X \setminus \overline{A_j}$$

is empty. We claim that  $X \setminus \overline{A_j}$  is open and dense, which yields the desired contradiction by Lemma A.32. Certainly  $X \setminus \overline{A_j}$  is open; for density, note that  $\overline{A_j}$  contains no open subset, which means that the complement intersects any open subset. ■

**Corollary A.34.** Let  $X$  be a nonempty complete metric space. Then  $X$  is not the countable union of nowhere dense subsets.

*Proof.* This follows from taking  $U_i = X$  for each  $i$  in Theorem A.33. ■

We now proceed with the Open mapping theorem. We will isolate the application of the Baire category theorem to the following lemma: Corollary A.34 shows that the hypothesis is satisfied.

**Lemma A.35.** Let  $f: X \rightarrow Y$  be a linear map of locally convex topological vector spaces. Suppose that  $\text{im } f$  is not the union of nowhere dense subsets. Then  $f(U)$  contains an open neighborhood of 0 for each open neighborhood  $U$  of 0.

*Proof.* This more or less follows from unwinding the hypothesis. Because  $V$  is locally convex, we may shrink  $U$  to make it convex and balanced (by Lemma A.11); we will only use this at the end of the proof. The hypothesis is applied as follows: because  $U$  is open, it is absorbing (by Example A.17), so  $V = \bigcup_{i \in \mathbb{N}} iU$ , so

$$\text{im } f = \bigcup_{i > 0} iU.$$

Thus, the hypothesis implies that one of the  $iU$  fails to be nowhere dense; because multiplication by  $i > 0$  is a homeomorphism, we see that  $U$  is also fails to be nowhere dense, so it contains an open subset  $V$ .

We now upgrade  $V$  into an open neighborhood of 0. Well, simply set  $V' := \frac{1}{2}(V - V)$ . Then  $f(V')$  is contained in  $\frac{1}{2}(U - U)$  by linearity, which is  $\frac{1}{2}(U + U)$  because  $U$  is balanced, which is contained in  $U$  because  $U$  is convex. ■

It remains to use the hypothesis that  $X$  is complete, which is done in the following lemma.

**Lemma A.36.** Let  $f: X \rightarrow Y$  be a continuous linear map of metrizable locally convex topological spaces. Suppose that  $X$  is complete and that  $\overline{f(U)}$  contains an open neighborhood of 0 for each open neighborhood  $U$  of 0. Then  $f$  is open.

*Proof.* The idea is to use the completeness of  $X$  to construct points of  $U$  which go to a required open neighborhood. We proceed in steps.

1. We are going to show that  $f(U)$  contains an open neighborhood of 0 for each open neighborhood  $U$  of 0, so let's spend a moment to explain why this is enough. For each open subset  $U' \subseteq X$  and  $x \in U'$ , we note that  $f(U' - x)$  contains an open neighborhood  $V_x$  of the origin. Thus,  $f(U')$  contains the open neighborhood  $f(x) + V_x$ , so  $f(U')$  equals

$$\bigcup_{x \in U'} f(x) + V_x,$$

which is open because it is a union of open subsets.

2. We unwind the hypothesis on  $f$ . By shrinking our open neighborhood  $U$  of 0, we may assume that  $U$  is convex and balanced (by Lemma A.11), so there is a seminorm  $p$  on  $X$  for which  $U$  is  $B(0, 1)$  for some translation-invariant metric  $d$  on  $X$ , chosen via Proposition A.27. Similarly, by hypothesis on  $f$ , we know that  $f(\overline{U})$  contains some open neighborhood  $V$  of 0, which we may again shrink until it is  $B(0, 2)$  for some translation-invariant metric  $d$  on  $Y$ . It will be worthwhile to remove the closure from this statement. Well, for any  $y \in Y$ , we see that  $y \in d(y, 0)V$  and so  $y \in \overline{q(y) \cdot f(U)}$ , so any  $\varepsilon > 0$  has some  $x \in X$  for which  $d(x, 0) < d(y, 0)$  and  $d(y, f(x)) < \varepsilon$ .
3. We will actually show that  $V \subseteq f(U)$ , so choose some  $y \in V$ . The completeness of  $X$  will be used via a limiting process to produce an element of  $U$  mapping to  $y$ . To start us off, fix some  $\varepsilon > 0$  (to be fixed at the end of the proof), and we take  $x_0 := 0$ . Now, if we are given  $x_0 + \cdots + x_n$ , we may select  $x_{n+1}$  so that  $d(x_{n+1}, 0) < d(y - f(x_0 + \cdots + x_n), 0)$  and

$$d(y - f(x_0 + \cdots + x_n), f(x_{n+1})) < \varepsilon/2^{n+1}$$

by the above paragraph.

4. We complete the proof. Now, by construction,  $d(x_n, 0) < \varepsilon/2^{n-1}$  for all  $n \geq 2$ , so the sequence of partial sums is rapidly Cauchy. As in the proof of Lemma A.32, it follows that these partial sums converge to some  $x \in X$ .

We claim that this  $x$  is the desired element. To start, we see that  $f(x_0 + \cdots + x_n) \rightarrow y$  as  $n \rightarrow \infty$  by construction, so  $f(x) = y$  by continuity of  $f$ !

It remains to check that  $x \in U$ . Well,  $d(x, 0)$  is bounded by

$$\sum_{n=0}^{\infty} d(x_n, 0) < d(x_0, 0) + d(x_1, 0) + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}},$$

and  $d(x_1, 0) < d(y, 0) < 2$  and  $\sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}} = \varepsilon$ , so  $x \in U$  for  $\varepsilon$  small enough. ■

**Theorem A.37 (Open mapping).** Let  $f: X \rightarrow Y$  be a continuous linear map of Fréchet spaces. If  $f$  is surjective, then  $f$  is open.

*Proof.* By Corollary A.34, we see that  $X$  is not a countable union of nowhere dense subsets. The result now follows from combining Lemmas A.35 and A.36. ■

**Corollary A.38.** Let  $f: X \rightarrow Y$  be a bijective continuous linear map of Fréchet spaces. Then  $f$  has continuous inverse.

*Proof.* Let  $g$  be the inverse map. Checking that  $g$  is continuous is equivalent to checking that  $f$  is open, which follows from Theorem A.37. ■

## A.4 The Hahn–Banach Theorems

In this section, we review the proof of the Hahn–Banach theorem. This section will be filled with plenty of nonsense. Ultimately, we are interested in extending continuous linear functionals on Fréchet spaces, but along the way, we will show that linear functionals separate convex sets.

For one of our applications, we will require a weaker notion than a seminorm.

**Definition A.39.** Fix a vector space  $V$  over  $\mathbb{F}$ . A function  $p: V \rightarrow \mathbb{R}_{\geq 0}$  is a half-seminorm if and only if it is sublinear and satisfies  $p(tx) = tp(x)$  for  $t > 0$  and  $x \in V$ .

The lack of input for  $p(-x)$  prevents  $p$  from being a seminorm.

As with Banach spaces, we check if a linear functional is continuous by checking if it is bounded, but the definition of bounded needs to be adjusted.

**Definition A.40 (bounded).** Fix a topological vector space  $V$  over  $\mathbb{F}$ . A linear functional  $\ell: V \rightarrow \mathbb{F}$  is *bounded* if and only if there is an open neighborhood  $U$  of 0 and a constant  $c > 0$  such that  $|\ell(x)| \leq c$  for all  $x \in U$ .

**Lemma A.41.** Fix a topological vector space  $V$  over  $\mathbb{F}$  and a linear functional  $\ell$  on  $V$ . Then the following are equivalent.

- (i)  $\ell$  is continuous.
- (ii)  $\ell$  is bounded.
- (iii)  $\ell$  is continuous at 0.

*Proof.* We use Lemma A.4. The implication from (i) to (ii) is direct; the proof that (iii) implies (i) is identical to the proof in Lemma A.4. To show (ii) implies (iii), we note  $|\ell|$  is a seminorm, and by considering nets, we see that it is enough to check that  $|\ell|$  is continuous at 0, which follows from Lemma A.4(ii) and the fact that  $\ell$  is bounded. ■

**Corollary A.42.** Fix a topological vector space  $V$  over  $\mathbb{F}$ , and let  $p: V \rightarrow \mathbb{R}_{\geq 0}$  be a continuous half-seminorm. Given a linear functional  $\ell: V \rightarrow \mathbb{F}$ , if  $-p(-v) \leq \ell(v) \leq p(v)$  for all  $v$ , then  $\ell$  is continuous.

*Proof.* Because  $-p(-v) \leq \ell(v) \leq p(v)$  for all  $v$ , we see that the open subset  $U := \{v \in V : p(v) < 1\}$  bounds  $\ell$  (with the constant 1), so we are done by Lemma A.41. ■

Thus, it will be important to be able to extend linear functionals along with an upper bound against a seminorm. By considerations with Zorn's lemma, we will find that the hard part is extending the linear functional one step, which is the content of the next lemma.

**Lemma A.43.** Fix a vector space  $V$  over  $\mathbb{R}$ , a half-seminorm  $p$  on  $V$ , and a linear functional  $\ell$  on a subspace  $W \subseteq V$  such that  $-p(-w) \leq \ell(w) \leq p(w)$  for all  $w \in W$ . Given any  $v' \in V$ , there is an extension  $\ell'$  to a linear functional on a subspace  $W'$  containing  $v'$  such that  $-p(-v) \leq \ell'(v) \leq p(v)$  for all  $v$ .

*Proof.* If  $v' \in W$  already, then there is nothing to do. Otherwise, for any real number  $c$ , we see that we may extend  $\ell$  to a linear functional  $\ell'$  on  $W' := W + \mathbb{R}v'$  by setting  $\ell'(v') := c$ . Namely, we have

$$\ell'(w + tv') = \ell(w) + tc$$

for any  $w \in W$  and  $t \in \mathbb{R}$ .

We would like to show that we can choose  $c$  so that  $-p(-v) \leq \ell'(v) \leq p(v)$  for all  $v$  of the form  $w + tv'$ . This requires a little trickery. By scaling, it is enough to only check with  $t \in \{\pm 1\}$  (because  $t = 0$  follows by hypothesis). Thus, we need both  $\ell(w) + c \leq p(w + v')$  and  $\ell(w) - c \leq p(w - v')$  for all  $w \in W$ . Now, such a  $c$  exists if and only if

$$\sup_{w \in W} (\ell(w) - p(w - v')) \stackrel{?}{\leq} \inf_{w \in W} (p(w + v') - \ell(w)).$$



For this, we should check that  $\ell(w) - p(w - v') \leq p(w' + v') - \ell(w')$  for any  $w, w' \in W$ , which is equivalent to  $\ell(w + w') \leq p(w - v') + p(w' + v')$ . This last inequality follows because  $\ell(w) \leq p(w)$  and the subadditivity of  $p$ . ■

**Proposition A.44.** Fix a vector space  $V$  over  $\mathbb{R}$ , a half-seminorm  $p$  on  $V$ , and a linear functional  $\ell$  on a subspace  $W \subseteq V$  such that  $-p(-v) \leq \ell(v) \leq p(v)$  for all  $v \in W$ . Then  $\ell$  extends to a linear functional  $\ell'$  on  $V$  such that  $-p(-v) \leq \ell'(v) \leq p(v)$  for all  $v \in V$ .

*Proof.* After Lemma A.43, the rest of this proof is largely formal nonsense. We use Zorn's lemma on the partially ordered set  $\mathcal{P}$  of pairs  $(V', \ell')$ , where  $V'$  is an intermediate subspace, and  $\ell'$  is a functional on  $V'$  bounded above by  $p$ ; the ordering is given by  $(V', \ell') \leq (V'', \ell'')$  if and only if  $V' \subseteq V''$  and  $\ell''|_{V'} = \ell'$ . Our application of Zorn's lemma is in two steps.

- We claim that  $\mathcal{P}$  has a maximal element, for which we use Zorn's lemma. First, note  $\mathcal{P}$  is nonempty because it has  $(W, \ell)$ . Secondly, any ascending chain  $\{(W_i, \ell_i)\}_i$  in  $\mathcal{P}$  has upper bound given by setting  $V' := \bigcup_i W_i$  and defining  $\ell'$  as the union of the  $\ell_i$ s. We can see that  $V'$  is still a vector space, and the nature of the partial ordering verifies that  $\ell'$  is a well-defined functional extending  $\ell$ . Thus, so  $(V', \ell')$  is indeed an upper bound for our chain.
- Let  $(V', \ell')$  be a maximal element of  $\mathcal{P}$ . We claim that  $V' = V$ , which will complete the proof. We already have  $V' \subseteq V$ , so it remains to show the other inclusion. Well, for any  $v \in V$ , we see that  $(V', \ell')$  can be extended up to  $V' + \mathbb{R}v$  by Lemma A.43, so the maximality of  $(V', \ell')$  requires  $V' + \mathbb{R}v = V'$ . Thus,  $v \in V'$ , so  $V \subseteq V'$  follows. ■

In order to work with the base field  $\mathbb{C}$ , we use a trick.

**Notation A.45.** Fix a vector space  $V$  over  $\mathbb{C}$ . Given a real linear functional  $\ell: V \rightarrow \mathbb{R}$ , we define  $C\ell: V \rightarrow \mathbb{C}$  by

$$C\ell(x) := \ell(x) - i\ell(ix).$$

On the other hand, for any complex linear functional  $\ell$  on  $V$ , we define its real part as  $\text{Re } \ell := \frac{1}{2}(\ell + \bar{\ell})$ .

Here are a few basic facts about this construction.

**Lemma A.46.** Fix a vector space  $V$  over  $\mathbb{C}$ .

- If  $\ell$  is a real linear functional on  $V$ , then  $C\ell: V \rightarrow \mathbb{C}$  is a complex linear functional. Furthermore,  $\text{Re } C\ell = \ell$ .
- If  $\ell$  is a complex linear functional on  $V$ , then  $C \text{Re } \ell = \ell$ .

*Proof.* We show these separately.

- First, we check that  $C\ell: V \rightarrow \mathbb{C}$  is a (complex) linear functional. Indeed,  $C\ell$  is already real linear (it is a sum of real linear functionals), so it is enough to check that  $C\ell(ix) = iC\ell(x)$  for any  $x \in V$ , which is true because quantities equal  $\ell(ix) + \ell(x)$ .

Secondly, for any  $x \in V$ , we see  $\text{Re } C\ell(x) = \text{Re}(C\ell(x)) = \ell(x)$ .

- This is a direct calculation. For any  $x \in V$ , we see  $C \text{Re } \ell(x) = \text{Re } \ell(x) - i \text{Re } \ell(ix)$ , which further expands to

$$\frac{\ell(x) + \overline{\ell(x)} - i\ell(ix) - i\overline{\ell(ix)}}{2},$$

which is  $\ell(x)$  after the dust settles. ■

**Theorem A.47 (Hahn–Banach existence).** Fix a vector space  $V$  over  $\mathbb{F}$ , a half-seminorm  $p$  on  $V$ , and a linear functional  $\ell$  on a subspace  $W \subseteq V$  such that  $\ell \leq p$ . Then  $\ell$  extends to a linear functional  $\ell'$  on  $V$  such that  $|\ell'| \leq p$ .

*Proof.* The case of  $\mathbb{F} = \mathbb{R}$  follows from Proposition A.44, so we may take  $\mathbb{F} = \mathbb{C}$ . For this, we freely use Lemma A.46. Even in this case, Proposition A.44 provides some real functional  $\ell'$  on  $V$  extending  $\operatorname{Re} \ell$  such that  $|\ell'| \leq p$ . Then  $C\ell'$  extends  $C\operatorname{Re} \ell = \ell$ , so it remains to check that  $|C\ell'| \leq p$ . Well, for any  $x \in V$ , there is a unit complex number  $\mu$  such that  $C\ell'(\mu x) = |C\ell'(x)|$ . But this means  $C\ell'(\mu x)$  equals  $\operatorname{Re} C\ell'(\mu x)$ , which is just  $\ell'(\mu x)$ , which we now know to be upper-bounded by  $p(\mu x)$  by construction. However,  $p(\mu x) = p(x)$  because  $p$  is a half-seminorm! ■

We are now ready to give some applications. Let's start with showing that linear functionals separate convex sets.

**Theorem A.48 (Hahn–Banach separation).** Fix a Fréchet space  $V$  over  $\mathbb{F}$ . Given disjoint convex subsets  $X$  and  $Y$  of  $V$  where  $X \subseteq V$  is open, there is a continuous linear functional  $\ell$  on  $V$  and a constant  $c$  so that

$$\operatorname{Re} \ell(x) < c \leq \operatorname{Re} \ell(y)$$

for all  $x \in X$  and  $y \in Y$ .

*Proof.* We proceed in steps.

1. We quickly reduce to the case of  $V = \mathbb{R}$ : given the statement over  $\mathbb{R}$ , whatever real functional  $\ell$  the statement would produce even with  $\mathbb{F} = \mathbb{C}$  can be upgraded to a complex linear functional by working with  $C\ell$ , which then still satisfies the required inequality because  $\operatorname{Re} C\ell = \ell$  by Lemma A.46.
2. Thus, we may assume that  $\mathbb{F} = \mathbb{R}$ . As another quick reduction, we remark that it is enough to find a continuous linear functional which merely achieves

$$\ell(x) \overset{?}{<} \ell(y)$$

for all  $x \in X$  and  $y \in Y$ . Then we note that  $\ell$  is surjective, so Theorem A.37 shows that  $\ell(X)$  is open! Thus,  $\ell(x)$  has a supremum  $c$  which it does not achieve, and this constant  $c$  suffices for the statement.

3. It remains to construct such a continuous linear functional  $\ell$ ; equivalently, we want to have  $\ell(x - y) < 0$  for all  $x \in X$  and  $y \in Y$ . We will construct  $\ell$  via Proposition A.44. By translation, we may assume that  $0 \in X$ , which means that we want  $Y$  to take on large values. To this end, we define our starting functional on  $\mathbb{R}y_0$  for some (nonzero!) basepoint  $y_0 \in Y$  to send  $y_0 \mapsto 1$ .
4. To apply Proposition A.44, we must still build a half-seminorm, which we do with Proposition A.22. Because we want to upper-bound  $\ell(X - Y)$ , it would make sense to  $X - Y$  as our convex set, but we need to have an open neighborhood of 0. Instead, we take

$$U := X - (-y_0 + Y).$$

This is a sum of convex sets and hence convex; similarly, this is a union of open sets and hence an open neighborhood of 0. Thus, Proposition A.22 provides us with a half-seminorm  $p$  on  $V$  such that  $p: V \rightarrow \mathbb{R}$  has  $U = \{v \in V : p(v) < 1\}$ , and we remark that  $p$  is continuous by Lemma A.4. (Technically, we do not know if  $U$  is balanced, but convexity of  $U$  along with  $0 \in U$  shows that  $tU \subseteq U$  for all  $t \in [0, 1)$ , which is good enough to show that  $p$  is a half-seminorm.)

5. We are now ready to apply Proposition A.44. Quickly, we check that  $-p(-y_0) \leq 1 \leq p(y_0)$ , which holds because  $y_0 \notin U$  (because this would imply  $X \cap Y \neq \emptyset$ ). Thus, proposition A.44 provides us with a linear functional  $\ell$  on  $V$  such that  $\ell(x_0 - y_0) = 1$  and  $-p(-v) \leq \ell(v) \leq p(v)$ . In particular,  $\ell$  is continuous by Lemma A.41.

6. We complete the proof. It remains to check that  $\ell(x) < \ell(y)$  for all  $x \in X$  and  $y \in Y$ . Well, for any  $x \in X$  and  $y \in Y$ , we see that  $x - (-y_0 + y) \in U$ , so  $p(y_0 + x - y) < 1$ , but  $\ell(y_0) = 1$ , so  $\ell(y_0 + x - y) < \ell(y_0)$ , so

$$\ell(x) < \ell(y),$$

as required. ■

Here are some applications. If one has a compact convex set, then one can improve the separation result.

**Corollary A.49.** Fix a Fréchet space  $V$  over  $\mathbb{F}$ . Given disjoint convex subsets  $K$  and  $A$  where  $K \subseteq V$  is compact and  $A \subseteq V$  is closed, there is a continuous linear functional  $\ell$  on  $V$  and constants  $c$  and  $d$  so that

$$\operatorname{Re} \ell(k) \leq c < d \leq \operatorname{Re} \ell(a)$$

for  $k \in K$  and  $a \in A$ .

*Proof.* We apply Theorem A.48. For this, we need to produce an open convex set. Well, because  $K$  is compact, the main claim that there is an open neighborhood  $U$  of  $0$  such that  $(K + U) \cap A = \emptyset$ . Let's explain how this completes the proof. To start, note that we may shrink  $U$  to be convex so that  $K + U$  is open and convex. Then Theorem A.48 provides us with a continuous linear functional  $\ell$  and constant  $d$  for which

$$\operatorname{Re} \ell(k + u) < d \leq \operatorname{Re} \ell(a)$$

for all  $k + u \in K + U$  and  $a \in A$ . However,  $K$  is compact, so the continuous function  $k \mapsto \operatorname{Re} \ell(k)$  must achieve its supremum, which we take to be  $c$ , and we are done after noting that we must have  $c < d$ .

It remains to show the main claim, which is purely topological in nature. Suppose that every open neighborhood  $U$  of  $0$  has  $(K + U) \cap A \neq \emptyset$ ; we will show that  $K \cap A \neq \emptyset$ . Consider the collection  $\{K \cap (A - U)\}$  of nonempty open subsets of  $K$ , which we note is closed under intersections, so we can extend it to a filter and then an ultrafilter  $\mathcal{F}$ . The compactness of  $K$  then shows that  $\mathcal{F} \rightarrow k$  for some  $k \in K$ ; by translation, we may assume  $k = 0$ . Unwinding, this means that  $\mathcal{F}$  contains every open neighborhood  $U$  of  $k$ , so  $K \cap (A - U)$  has nontrivial intersection with  $U$ , so  $A$  has nontrivial intersection with  $U + U$ , so we see  $0 \in \overline{A}$  (by shrinking  $U$ ). But because  $A$  is closed, we conclude  $0 \in A \cap K$ ! ■

In the special case where  $K$  is a point, we are able to say something about the existence of continuous functionals.

**Corollary A.50.** Fix a Fréchet space  $V$  over  $\mathbb{F}$ . Given a vector  $v \in V$  and subspace  $W \subseteq V$ , there is a continuous linear functional  $\ell$  on  $V$  with  $\ell|_W = 0$  and  $\ell(v) = 1$  if and only if  $v \notin \overline{W}$ .

*Proof.* Of course, if such an  $\ell$  exists, then  $\overline{W} \subseteq \ker \ell$ , so  $v \notin \overline{W}$ .

In the other direction, we use Corollary A.49 with  $K = \{v\}$  and  $A = \overline{W}$  to get a linear functional  $\ell$  such that  $\operatorname{Re} \ell(v) < \operatorname{Re} \ell(w)$  for all  $w \in \overline{W}$ . However,  $\operatorname{Re} \ell$  is a (real) linear functional, so it must vanish on  $W$  if it is not surjective, so  $\operatorname{Re} \ell|_W = 0$ , so  $\ell|_W = 0$  by Lemma A.46. Thus, we see  $\operatorname{Re} \ell(v) < 0$ , so we can complete the proof by dividing  $\ell$  by  $\ell(v)$ . ■

At long last, we are ready to extend some functionals.

**Corollary A.51.** Fix a Fréchet space  $V$  over  $\mathbb{F}$ . Any continuous linear functional  $\ell$  on a subspace  $W \subseteq V$  extends to a continuous linear functional on  $V$ .

*Proof.* If  $\ell = 0$ , then there is nothing to do. Otherwise, we use Corollary A.50. Choose a vector  $w_0 \in W$  such that  $\ell(w_0) = 1$ . Thus, we see that  $w_0 \notin \ker \ell$ , so Corollary A.50 gives us a continuous linear functional  $\ell'$  on the full space  $V$  for which  $\ell'|_{\ker \ell} = 0$  and  $\ell'(w_0) = \ell(w_0) = 1$ . Because  $W = \mathbb{R}w_0 + \ker \ell$ , we conclude that  $\ell'$  does in fact extend  $\ell$ , so we are done. ■

**Remark A.52.** It may appear that we can use Lemma A.41 to immediately extend continuity from  $W$  up to  $V$ . However, there is something tricky going on: a priori,  $\ell$  on  $W$  is only bounded by a continuous seminorm from  $W$ , and it is not obvious that such a seminorm should also extend to a suitable continuous seminorm on  $V$ !

## A.5 Examples of Fréchet Spaces

To gain some experience with our new definition, we provide some examples of Fréchet spaces.

**Definition A.53 (Banach).** A topological vector space  $V$  is *Banach* if and only if its topology is given by a norm, and  $V$  is complete.

**Example A.54.** Any Banach space is Fréchet. For example, one can take the countable family of seminorms to just be the single (actual) norm and conclude by Corollary A.23.

Of course, we are going to be interested in examples of Fréchet spaces which are not necessarily Banach spaces.

**Example A.55.** Fix a Hausdorff topological space  $X$ .

- (a) Then the space  $C(X)$  of continuous functions  $X \rightarrow \mathbb{C}$  has a topology where  $\{f_i\} \rightarrow f$  if and only if  $f_i \rightarrow f$  uniformly on compact sets. Furthermore,  $C(X)$  is locally convex.
- (b) If  $X$  is compact, then  $C(X)$  is a Banach space.
- (c) If  $X$  is locally compact and second countable, then  $C(X)$  is Fréchet.

*Proof.* Let's begin by explaining how to give  $C(X)$  a topology so that it is always locally convex. Consider the collection  $\mathcal{K}$  of compact subsets of  $X$ , which we give a partial ordering by inclusion. For each compact set  $K$ , we define a seminorm  $\nu_K: C(X) \rightarrow \mathbb{R}$  by

$$\nu_K(f) := \max_{x \in K} |f(x)|.$$

To see that  $\nu_K$  is a seminorm, we note that  $\nu_K$  is surely homogeneous, and  $\nu_K$  is subadditive because  $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ .

We may thus give  $C(X)$  the topology given by these seminorms. Quickly, note that  $C(X)$  is Hausdorff because  $f \neq g$  has some  $x \in X$  for which  $f(x) \neq g(x)$ , and then we have  $\nu_{\{x\}}(f - g) > 0$ . Thus, we see that this topology is locally convex for free! Lastly, observe that  $\{f_i\} \rightarrow f$  if and only if  $\{\nu_K(f_i - f)\} \rightarrow 0$  for all compact subsets  $K \subseteq X$ ; by translation, we may assume that  $f = 0$ . But of course, having  $\{\nu_K(f_i - f)\} \rightarrow 0$  is equivalent to having uniform convergence on compact sets.

Next up, let's suppose that  $X$  is compact, and we will show that  $C(X)$  is Banach. Well, we claim that the topology is actually induced by the single seminorm  $\nu_X$ , which is actually a norm. To see that the topology is induced by  $\nu_X$ , note that  $\nu_K \leq \nu_X$  for all  $K$ , so having  $\{\nu_K(f_i - f)\} \rightarrow 0$  for all  $K \subseteq X$  is equivalent to having  $\{\nu_X(f_i - f)\} \rightarrow 0$ . To see that  $\nu_X$  is a norm, it remains to check positivity, for which we note that any nonzero continuous function  $f$  is nonzero at some  $x \in X$ , so  $\nu_X(f) \geq |f(x)|$  is positive.

Lastly, we assume that  $X$  is locally compact and second countable. This means that  $X$  will admit a countable cover by pre-compact open subsets, so taking the closure shows that  $X$  receives a countable cover  $\{K_j\}_{j \in \mathbb{N}}$  by compact sets. By replacing each  $K_j$  with  $\bigcup_{j' \leq j} K_{j'}$ , we may assume that

$$K_0 \subseteq K_1 \subseteq \cdots.$$

Accordingly, we claim that the topology is generated by the seminorms  $\{\nu_{K_j}\}_{j \in \mathbb{N}}$ . Well, for any compact set  $K$ , we see  $\{\nu_K(f_i - f)\} \rightarrow 0$  is implied by having this convergence for any larger compact set, so it will be implied by  $\{\nu_{K_n}(f_i - f)\} \rightarrow 0$  for some  $n$  sufficiently large. ■

**Remark A.56.** Suppose  $X$  is compact. Then it turns out that  $C(X)$  is separable if and only if  $X$  is a metric space. Let's sketch the backwards direction: being metrizable implies that  $X$  is second countable, so there is a countable dense subset  $\{x_i\}_{i \in \mathbb{N}}$  of  $X$ . Consider the algebra  $\mathcal{A}$  over  $\mathbb{C}$  of functions generated by 1 and  $x \mapsto d(x_i, x)$ . Note  $x \mapsto d(x_i, x)$  is continuous, so  $\mathcal{A} \subseteq C(X)$ ; additionally,  $\mathcal{A}$  separates points by construction. It follows that  $\mathcal{A}$  is dense in  $C(X)$ ; considering corresponding algebra over  $\mathbb{Q}$  provides us with a countable dense subset of  $C(X)$ .

Here is an example of the sort of check we can do.

**Example A.57.** Let  $f: X \rightarrow Y$  be a continuous map. Then the map  $(- \circ f): C(Y) \rightarrow C(X)$  is well-defined and continuous. We may write this operation as  $f^*: C(Y) \rightarrow C(X)$ .

*Proof.* The map is well-defined because the composite of continuous maps is continuous. It remains to show continuity. Well, suppose that we have a converging net  $\{\ell_i\} \rightarrow \ell$  in  $C(Y)$ . We would like to show that  $\{\ell_i \circ f\} \rightarrow (\ell \circ f)$  in  $C(X)$ . Convergence is equivalent to converging uniformly on compacts, so we choose a compact set  $K \subseteq X$ , and we would like to check that

$$\|(\ell_i \circ f - \ell \circ f)|_K\|_\infty \xrightarrow{?} 0.$$

Well, this norm is bounded by  $\|(\ell_i - \ell)|_{f(K)}\|_\infty$ , which goes to 0 because  $f(K)$  is compact and  $\{\ell_i\} \rightarrow \ell$  uniformly on compact sets. ■

## A.6 Categorical Properties

For my own personal use, we review some categorical properties of certain full subcategories of the category of topological vector spaces, which is equipped with continuous linear maps for its morphisms. We recall that all topological vector spaces are assumed to be Hausdorff.

Because they are easier, let's begin with some limits.

**Definition A.58 (kernel).** Fix a continuous map  $f: V \rightarrow W$  of topological vector spaces. Then we define the *kernel*  $\ker f$  as the vector space kernel equipped with the subspace topology.

**Remark A.59.** Because  $\ker f$  is a subspace, we see that it continues to be a Hausdorff topological space. In fact, it is the categorical kernel: it is already the kernel in the category of vector space, so any map  $V' \rightarrow V$  vanishing under  $W$  factors via a linear map into  $\ker f$ . But  $\ker f$  is just a subspace, so the induced map  $W \rightarrow V'$  is continuous.

**Remark A.60.** Note that  $\ker f \subseteq V$  is closed because  $f$  is continuous, so if  $V$  is complete, we see that  $\ker f$  is also complete.

**Remark A.61.** If  $V$  is locally convex, then the subspace topology inherits the exact set of seminorms, so  $\ker f$  is also locally convex. Once we combine this argument with Remark A.60, we see that  $V$  being Fréchet implies that  $\ker f$  is Fréchet.

**Definition A.62 (product).** Fix a family  $\{V_i\}_{i \in I}$  of topological vector spaces. Then we define the *product*  $\prod_{i \in I} V_i$  as the vector space product equipped with the product topology.

**Remark A.63.** When checking that  $\prod_{i \in I} V_i$  is a categorical product, we note that the maps produced by being the vector space product and being the topological product are the same, so the result follows.

**Remark A.64.** If each  $V_i$  is complete, then we claim that the product  $\prod_{i \in I} V_i$  is still complete. Indeed, given any Cauchy net  $\{(v_{ij})_i\}_{j \in I}$ , we see that each projection  $\{v_{ij}\}_{j \in I}$  continues to be a Cauchy net and therefore has a limit  $\{v_{ij}\} \rightarrow v_j$ . But now having each projection converge means that  $\{(v_{ij})_i\}_{j \in I} \rightarrow (v_j)_j$ .

**Remark A.65.** If each  $V_i$  is locally convex with its topology determined by some family of seminorms  $\{p_{i\alpha}\}_\alpha$ , then the definition of the product means that the topology on the product are determined by the seminorms

$$\prod_{i \in I} V_i \xrightarrow{\text{pr}_j} V_j \xrightarrow{p_j^\alpha} \mathbb{R}.$$

This argument, when combined with Remark A.64, shows that a countable product of Fréchet spaces continues to be Fréchet.

**Proposition A.66.** Fix a diagram  $\{V_i\}_{i \in I}$  of topological vector spaces. Then the limit

$$\lim_{i \in I} V_i$$

exists in the category of topological vector spaces. If each  $V_i$  is locally convex, so is the limit. If  $I$  is countable and each  $V_i$  is Fréchet, then so is the limit.

*Proof.* Existence follows from Remarks A.59 and A.63. The remaining claims follow from Remarks A.61 and A.65. ■

**Example A.67.** We recover Example A.55. Fix a locally compact topological space  $X$ . Then we claim that

$$C(X) \stackrel{?}{=} \lim_{K \subseteq X} C(K),$$

where the limit is taken over all compact subsets  $K \subseteq X$ . Indeed, there is a canonical map  $f \mapsto (f|_K)_K$  from the left to the right; conversely, any tuple  $(f_K)_K$  on the right extends to a unique function  $f \in C(X)$  by construction of the limit. Lastly, we note that the map that we just described is actually a homeomorphism because a convergent net on the left  $\{f_i\} \rightarrow f$  converges if and only if  $\{f_i|_K\} \rightarrow f|_K$  uniformly for all compacts  $K \subseteq X$ , which is exactly the topology on the right-hand side.

We conclude that  $C(X)$  is locally convex; if  $X$  is second countable, then we can replace the limit over all compact subsets with a countable ascending chain, thereby showing that the limit is also Fréchet.

We now turn to some colimits, which are harder.

**Definition A.68 (quotient).** Fix a closed subspace  $W$  of a topological vector space  $V$ . Then we form the *quotient*  $V/W$  to be the vector space  $V/W$  to be equipped with the quotient topology induced by the projection  $V \twoheadrightarrow V/W$ .

**Remark A.69.** We claim that this quotient is the categorical quotient. Indeed, for any topological vector space  $V'$  and some map  $f: V \rightarrow V'$  for which  $W \subseteq \ker f$ , we get an induced map  $\bar{f}: V/W \rightarrow V'$ , which we need to be continuous. Well, a map  $\bar{f}: V/W \rightarrow V'$  is continuous if and only if the composite  $V \twoheadrightarrow V/W \rightarrow V'$  is continuous, which is true.

**Lemma A.70.** Fix a closed subspace  $W$  of a topological vector space  $V$ .

- (a) If  $V$  is locally convex, then  $V/W$  is locally convex.
- (b) If  $V$  is Fréchet, then  $V/W$  is Fréchet.

*Proof.* We show the parts separately.

- (a) For each seminorm  $p$  on  $V$ , we produce a continuous seminorm  $\bar{p}: V/W \rightarrow \mathbb{R}$  by

$$\bar{p}(v + W) := \inf_{w \in W} p(v + w).$$

Because  $p$  is a seminorm, we see that  $\bar{p}$  is also a seminorm. Additionally, we note  $\bar{p}$  is continuous: using Lemma A.4, it is enough to see that  $\{v + W : \bar{p}(v + W) < 1\}$  is open, which is true because this subset equals

$$\{v + W : \bar{p}(v + W) < 1\} \stackrel{?}{=} \{v : p(v) < 1\} + W.$$

Certainly any vector  $v$  for which  $p(v) < 1$  has  $\bar{p}(v) < 1$ . Conversely, if  $\bar{p}(v + W) < 1$ , then there is  $w \in W$  for which  $p(v + w) < 1$ .

Lastly, we have to check that these seminorms actually produce the correct topology on  $V/W$ . Well, the open sets of  $V/W$  are exactly projections of the open sets of  $V$ , so we are done because the previous paragraph explains that the open sets produced by our seminorms  $\bar{p}$  are exactly the projections of the open sets produced by the seminorms  $p$  of  $V$ .

- (b) The argument in (a) shows that  $V/W$  has its topology determined by countable many seminorms, so it remains to show that  $V/W$  is complete. Because  $V/W$  is metrizable (Proposition A.27), it is enough to check that Cauchy sequences in  $V/W$  converge. As such, choose a Cauchy sequence  $\{v_i + W\}_{i \in \mathbb{N}}$ , which means that  $\bar{p}(v_i - v_j + W) \rightarrow 0$  as  $i, j \rightarrow \infty$ . It is enough to show that any subsequence of this Cauchy sequence converges, so we may as well assume that  $\bar{p}(v_{i+1} - v_i + W) < 2^{-i}$  for each  $i$ . By definition of  $\bar{p}$ , we may thus, find vectors  $\{w_i\}$  for which

$$\bar{p}(v_{i+1} + w_{i+1} - (v_i + w_i)) < 2^{-i+1}.$$

Thus, the sequence  $\{v_i + w_i\}_{i \in \mathbb{N}}$  is rapidly Cauchy in  $V$ , so it has a limit to some vector  $v \in V$ . Projecting back down, we find that  $\{v_i + w_i\} \rightarrow v$  implies that  $\{v_i + W\} \rightarrow v + W$ , so we are done. ■

For our coproduct, we will retreat to locally convex vector spaces.

**Definition A.71 (coproduct).** Fix a family  $\{V_i\}_{i \in I}$  of locally convex vector spaces. Then we define the *coproduct* as given by the coproduct  $\bigoplus_{i \in I} V_i$  of vector spaces, and we equip it with a topology whereby an open neighborhood basis of the origin takes the form

$$\text{conv} \left( \sum_{i \in I} j_i(U_i) \right),$$

where  $j_i: V_i \rightarrow \bigoplus_{i \in I} V_i$  is the inclusion,  $U_i \subseteq V_i$  is an open neighborhood of 0 for each  $i$ . (This sum denotes the collection of any finite sum of vectors.)

**Remark A.72.** Let's verify that we have produced a categorical coproduct: given continuous inclusions  $V_i \rightarrow W$  for each  $i \in I$  into some locally convex vector space  $W$ , we need to check that the algebraically induced linear map  $\bigoplus_{i \in I} V_i \rightarrow W$  is continuous. Well, it's enough to check continuity on an open neighborhood basis of the identity, so we note that the pre-image of some convex open neighborhood  $U$  of  $W$  consists of the finitely supported sequences  $(v_i)_{i \in I}$  where  $v_i \in j_i^{-1}(U)$  for each  $i$ . The given open neighborhood basis does in fact cover such an open subset!

## APPENDIX B

# THE PETER–WEYL THEOREM

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In this appendix, we give a proof of the Peter–Weyl theorem, which is frequently used in the analytic part of the notes. This is taken verbatim from my notes for Math 261A.

## B.1 Introduction

### B.1.1 The Statement

In this appendix, we prove the Peter–Weyl theorem, following [Eti24, Sections 34–36]. The Peter–Weyl theorem is interested in the representation theory of compact real Lie groups. To explain the result, we temporarily pass to finite groups. With the correct perspective, the representation theory of finite groups  $G$  can be summarized as providing an isomorphism of  $(G \times G)$ -representations

$$\bigoplus_{V \in \text{IrrRep}(G)} V \otimes V^* \cong \mathbb{C}[G],$$

where  $\text{IrrRep}(G)$  denotes the set of isomorphism classes of finite-dimensional complex irreducible  $G$ -representations. The Peter–Weyl theorem is the analogous statement for compact real Lie groups.

To be precise, we must set up some notation. To begin, the morally correct formulation of  $\mathbb{C}[G]$  is  $\text{Ind}_1^G 1$ , which is functions on  $G$  with right translation as the group action. When moving to  $G$  compact, we need to add a finiteness condition to our functions on  $G$ .

**Definition B.1.** Fix a compact real Lie group  $G$ , and let  $dg$  be a Haar measure (see appendix B.2 for the definition). Then  $L^2(G)$  consists of (equivalence classes of) functions  $f: G \rightarrow \mathbb{C}$  such that

$$\int_G |f(g)|^2 dg < \infty.$$

Here, two functions  $f_1, f_2: G \rightarrow \mathbb{C}$  are equivalent if and only if  $f_1$  and  $f_2$  are the same  $dg$ -almost everywhere.

**Remark B.2.** The reason why  $L^2(G)$  is desirable is that it is a Hilbert space with positive-definite Hermitian inner product given by

$$\langle f_1, f_2 \rangle := \int_G f_1(g) \overline{f_2(g)} dg.$$



**Remark B.3.** Note that  $L^2(G)$  is a representation of  $G \times G$  by suitable translations: given  $f \in L^2(G)$  and  $(h_1, h_2) \in G \times G$ , we may define the action  $((h_1, h_2)f)(g) := f(h_2^{-1}gh_1)$ . Invariance of the Haar measure ensures  $(h_1, h_2)f \in L^2(G)$ .

Now that we have an analogue of  $\mathbb{C}[G]$ , we would like a candidate map  $V \otimes V^* \rightarrow L^2(G)$  for each  $V$ . Well, given a complex finite-dimensional representation  $V$ , one can check that  $\xi_V: V \otimes V^* \rightarrow L^2(G)$  given by

$$\xi_V(v \otimes \varphi) := (g \mapsto \varphi(gv))$$

is  $(G \times G)$ -invariant. We are now ready to state our theorem, whose proof is in Appendix B.3.

**Theorem B.4 (Peter–Weyl).** Fix a compact real Lie group  $G$ . Then the induced map

$$\xi_G: \bigoplus_{V \in \text{IrrRep}(G)} (V \otimes V^*) \rightarrow L^2(G)$$

has dense image.

**Remark B.5.** We cannot expect the map  $\xi_G$  to be an isomorphism of vector spaces. For example, the right-hand side is a Hilbert space, but the left-hand side has no obviously natural metric.

**Remark B.6.** Aside from the construction of the Haar measure in appendix B.2.2, our proof of Theorem B.4 will not require our groups to be anything more than compact Hausdorff topological groups.

### B.1.2 Layout and Notation

In appendix B.2, we discuss Haar measures on real Lie groups, culminating in a proof of complete reducibility of representations via Weyl’s unitarian trick. Then in appendix B.3, we prove the Peter–Weyl theorem. Lastly, in appendix B.4, we discuss two complements: unimodular groups and characters.

Throughout,  $G$  is a real Lie group, frequently but not always compact. Typically,  $V$  and  $W$  are finite-dimensional complex continuous representations of  $G$ ; for any representation  $V$ , we let  $\rho_V: G \rightarrow \text{GL}(V)$  be the structure morphism.

## B.2 The Haar Measure

In this section, we discuss the Haar measure for real Lie groups. Throughout, we will assume some basic integration theory on manifolds, such as covered in [Lee13, Chapter 16].

### B.2.1 Review of Integration

To review, given an open submanifold  $U \subseteq \mathbb{R}^m$  and a top differential form  $\omega \in \Gamma(U, \wedge^m T^*U)$ , one may write  $\omega = f(x_1, \dots, x_m) dx_1 \wedge \dots \wedge dx_m$  and then define the integral

$$\int_U \omega := \int_U f(x_1, \dots, x_m) dx_1 \wedge \dots \wedge dx_m.$$

For example, one can check that this definition behaves with respect to diffeomorphisms preserving orientation. As such, given an orientable manifold  $M$  of dimension  $m$ , one may define the integral with respect to a compactly supported top differential form  $\omega \in \Gamma(M, \wedge^m T^*M)$  as follows: choose a smooth atlas

$\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$  and a partition of unity  $\{\psi_\alpha\}_{\alpha \in \kappa}$  with respect to the open cover  $\{U_\alpha\}_{\alpha \in \kappa}$ ; then one may define the integral

$$\int_M \omega := \sum_{\alpha \in \kappa} \int_{U_\alpha} \psi_\alpha \omega,$$

where the integral on the right suitably takes place in Euclidean space after pulling back through  $\varphi_\alpha: U_\alpha \hookrightarrow \mathbb{R}^m$ . We will not check that this construction is well-defined because this is an exercise in differential topology; see [Lee13, Propositions 16.3–16.5]. We take a moment to remark that one may weaken the hypothesis that  $\omega$  is compactly supported provided that the above sum converges absolutely.

### B.2.2 The Definition

One can integrate on orientable manifolds over top differential forms, as described in [Lee13, Section 16.1]. Appendix B.2.1 explains to us how to integrate on manifolds. The Riesz representation theorem explains that integration is essentially the same as measuring by integrating against indicator functions: in particular, a top differential form  $\omega$  on  $M$  gives rise to the measure  $U \mapsto \int_M 1_U |\omega|$ , for suitably interpreted indicator  $1_U$  and  $|\omega|$ . Thus, we will be comfortable passing between top differential forms and the measures they produce with impunity.

**Definition B.7 (Haar measure).** Fix a real Lie group  $G$  of dimension  $m$ . Then a *left Haar measure* is the measure  $dg$  arising from a nonzero left-invariant top differential form  $\omega \in \Gamma(G, \wedge^m T^*G)$ . A *right Haar measure* is defined analogously. If further  $G$  is compact, then we say that  $\omega$  is *normalized* if and only if

$$\int_G dg = 1.$$

**Example B.8.** In the Lie group  $\mathbb{R}^m$ , the top differential form  $dx_1 \wedge \cdots \wedge dx_m$  is invariant under translation and thus produces both a left and a right Haar measure.

**Example B.9.** In the Lie group  $\mathrm{GL}_n(\mathbb{R})$ , the top differential form  $dX / \det X$  can be checked (via change-of-variables formulae of multivariable calculus) to produce both a left and a right Haar measure.

**Remark B.10.** Let's quickly explain why we are allowed to normalize  $\omega$  when  $G$  is compact. By scaling, it is enough to check that  $0 < \int_G dg < \infty$ . Well, by construction of the measure  $dg$  on  $G$ , one can see that any point is contained in a neighborhood of positive finite volume, and because  $G$  is compact, we see that  $G$  may be covered in finitely many such neighborhoods.

Let's check that such measures exist; they are also unique up to scalar.

**Proposition B.11.** Fix a real Lie group  $G$  of dimension  $m$ .

- (a) There exists a non-vanishing left-invariant top differential form  $\omega$ .
- (b) If  $\omega_1$  and  $\omega_2$  are nonzero left-invariant top differential forms, then there exists  $c \in \mathbb{R}^\times$  such that  $c\omega_1 = \omega_2$ .

*Proof.* We show the claims separately.

- (a) Fixing a basis  $\{X_1, \dots, X_m\}$  of  $T_1 G$ , one finds that we may extend this basis to a full global frame of  $TG$  by defining

$$X_i(g) := d\ell_g(X_i),$$

where  $\ell_g: G \rightarrow G$  denotes left multiplication by  $g$ . (We showed this in class; see [Lee13, Theorem 8.37] for more details.) In this way, one obtains a global frame  $\{X_1^*, \dots, X_m^*\}$  of  $T^*G$ , so we see that  $X_1^* \wedge \dots \wedge X_m^*$  is a non-vanishing left-invariant global section of  $\wedge^m T^*G$ .

- (b) It suffices to show that any left Haar measure  $\eta$  is a nonzero multiple of the left Haar measure constructed in (a). Well, for each  $g \in G$ , we see that  $\eta(g)$  is an element of the one-dimensional vector space  $\wedge^m T^*G|_g$ , so there exists a scalar  $c(g)$  such that  $\eta(g) = c(g)\omega(g)$ . It remains to check that the constructed function  $c: G \rightarrow \mathbb{R}$  is constant. Well, for any  $g \in G$ , we see that  $\eta(g) = d\ell_g(\eta(1))$  and  $\omega(g) = d\ell_g(\omega(1))$ , so it follows that  $c(g) = c(1)$ . ■

**Remark B.12.** In fact, Haar measures exist for arbitrary locally compact topological groups. Here, it does not make sense to talk about top differential forms, but one may still talk about left-invariant (Radon) measures. See (for example) [DE14, Chapter 1].

### B.2.3 Weyl's Unitarian Trick

As an application of the Haar measure, we show that representations of compact groups are completely reducible by imitating the usual proof for finite groups.

**Proposition B.13.** Fix a finite-dimensional complex representation  $V$  of a compact real Lie group  $G$ . Then there is a positive-definite  $G$ -invariant Hermitian inner product  $\langle \cdot, \cdot \rangle$  on  $V$ .

*Proof.* We begin by choosing any positive-definite Hermitian inner product  $\langle \cdot, \cdot \rangle_0$  on  $V$ . Then we fix a left Haar measure  $dg$  on  $G$ , and we define the function  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$  by

$$\langle v, w \rangle := \int_G \langle gv, gw \rangle_0 dg.$$

This integral is finite because  $G$  is compact and the function  $g \mapsto \langle gv, gw \rangle_0$  is continuous. The fact that  $\langle \cdot, \cdot \rangle$  is bilinear and Hermitian follows from the corresponding facts for  $\langle \cdot, \cdot \rangle_0$ . The fact that  $\langle \cdot, \cdot \rangle$  is  $G$ -invariant follows from the left-invariance of  $dg$ . Lastly, for positive-definiteness, we note that the function  $g \mapsto \langle gv, gv \rangle_0$  is continuous, nonnegative, and positive at  $g = 1$ ; thus, its integral must be positive. ■

**Corollary B.14.** Fix a compact real Lie group  $G$ . Then every finite-dimensional complex representation  $V$  is completely reducible.

*Proof.* We induct on  $\dim V$ , where the case of  $\dim V \leq 1$  has no content. Given some  $V$ , if  $V$  is already irreducible, then there is nothing to do; otherwise, suppose that we have a nonzero proper subrepresentation  $W \subseteq V$ . Then we use Proposition B.13 to choose a positive-definite  $G$ -invariant Hermitian inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , and we set

$$W^\perp := \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

Linear algebra implies  $W^\perp$  is a subspace such that  $W \oplus W^\perp \rightarrow V$ . Thus, we will be able to apply the inductive hypothesis to  $W$  and  $W^\perp$  to conclude the proof as soon as we know that  $W^\perp$  is a subrepresentation. Well, for any  $g \in G$  and  $v \in W^\perp$ , we note that  $\langle gv, w \rangle = \langle v, g^{-1}w \rangle = 0$  for all  $w \in W$ , so  $gv \in W^\perp$ . ■

## B.3 The Proof of the Peter–Weyl Theorem

Now that we have some measure theory on  $G$ , we are able to meaningfully discuss the Peter–Weyl theorem. Throughout,  $G$  is a compact real Lie group with normalized Haar measure  $dg$ . For example, Corollary B.14 implies that its finite-dimensional complex representations are completely reducible, and Example B.25 explains that left and right Haar measures align.

### B.3.1 A Little Spectral Theory

Roughly speaking, we will derive Theorem B.4 via an eigenspace decomposition on the Hilbert space  $L^2(G)$ . Thus, we need to make sense of such decompositions. For our purposes, it will be enough to work with self-adjoint compact operators. Let's begin by defining "compact."

**Definition B.15 (compact).** Fix a Hilbert space  $X$ . A bounded operator  $T: X \rightarrow X$  is *compact* if and only if  $T$  is a limit of operators  $\{T_n\}_{n \in \mathbb{N}}$  of finite rank in the norm topology.

Let's explain why this is called "compact."

**Lemma B.16.** Fix a compact operator  $T$  on a Hilbert space  $X$ . If the sequence  $\{v_m\}_{m \in \mathbb{N}}$  is bounded, then the sequence  $\{Tv_m\}_{m \in \mathbb{N}}$  has a convergent subsequence.

*Proof.* By compactness, we are promised a sequence  $\{T_n\}_{n \in \mathbb{N}}$  of finite-rank operators such that  $T_n \rightarrow T$ ; say  $\|T - T_n\| < \frac{1}{n}$  for each  $n$  for concreteness. Note that the conclusion is certainly true for each  $T_n$ .

Now, suppose that  $\{v_{0,m}\}_{m \in \mathbb{N}}$  is a bounded sequence in  $X$ ; by scaling, we may assume that  $\|v_{0,m}\| \leq 1$  for all  $m$ . We construct our subsequence inductively as follows: for each  $n \geq 1$ , let  $\{v_{n,m}\}_{m \in \mathbb{N}}$  be a subsequence of  $\{v_{n-1,m}\}_{m \in \mathbb{N}}$  such that the sequence  $\{T_n v_{n,m}\}_{m \in \mathbb{N}}$  converges.

We now define  $v_m := v_{m,m}$ , and we claim that  $\{Tv_m\}_{m \in \mathbb{N}}$  is Cauchy, which will complete the proof. This is a rather tedious computation. Because  $\{v_m\}_{m \in \mathbb{N}}$  is a subsequence of  $\{v_{n,m}\}_{m \in \mathbb{N}}$  for all  $n$ , it is enough to check that

$$\|Tv_{n,m} - Tv_{n,m'}\| \stackrel{?}{<} \frac{3}{n}$$

for all sufficiently large  $m$  and  $m'$ . Well, the triangle inequality yields

$$\|Tv_{n,m} - Tv_{n,m'}\| \leq \|T_n v_{n,m} - T_n v_{n,m'}\| + \|(T - T_n)(v_{n,m} - v_{n,m'})\|.$$

The left term can be bounded by  $1/n$  for large enough  $m$  and  $m'$  because the sequence  $\{T_n v_{n,m}\}_{m \in \mathbb{N}}$  is Cauchy. Then the right term can be bounded by  $\frac{2}{n}$  because  $\|T - T_n\| < \frac{1}{n}$  and  $\|v_{n,m} - v_{n,m'}\| \leq 2$ . Summing completes the argument. ■

We now provide some interesting examples of compact operators.

**Lemma B.17.** Fix a compact orientable smooth manifold  $M$ , and choose a non-vanishing top differential form  $\omega$  such that  $\int_M \omega = 1$ . For any continuous kernel  $k: M \times M \rightarrow \mathbb{C}$ , the operator  $T$  on  $L^2(M)$  defined by

$$Tf(y) := \int_M k(x, y) f(x) \omega(x)$$

is bounded and compact.

*Proof.* Quickly, note  $T$  is bounded because  $k$  is bounded as a continuous function on a compact space. It remains to check compactness. The main idea is to use Riemann sums to construct our finite-rank operators. We have two steps.

1. We construct the operators  $T_n$ . Choose a metric  $d$  on  $M$ , for example induced by a Riemannian metric. Fixing some  $n \geq 1$  for now, we may tile  $M$  by finitely many disjoint open subsets  $\{V_{n,i}\}_{i=1}^{N_n}$  whose closures cover  $M$ , each of diameter at most  $\frac{1}{n}$ ; we also go ahead and choose some  $p_{n,i} \in V_{n,i}$  for each  $i$ . Then we define the "Riemann sum" operator

$$T_n f(y) := \sum_{i=1}^{N_n} k(p_{n,i}, y) \int_{V_{n,i}} f(x) \omega(x),$$

where  $y \in \overline{V_{n,j}}$  (and if  $y$  lives in multiple closures, simply choose one  $j$ ). Note that  $T_n$  has finite rank because the output function is uniquely determined by the integrals  $\int_{V_{n,i}} f(x) \omega(x)$ .

2. We complete the proof by checking that  $T_n \rightarrow T$  as  $n \rightarrow \infty$ . Choose some  $f \in L^2(M)$  with  $\|f\|_2 = 1$ , and we must check that  $T_n f \rightarrow T f$  (uniformly in  $f$ ). Well, the triangle inequality yields

$$\|T_n f - f\|_2^2 \leq \sum_{i,j=1}^{N_n} \int_{V_{j,n}} \int_{V_{i,n}} |k(x,y) - k(p_{n,i}, p_{n,j})|^2 |f(x)|^2 \omega(x) \omega(y).$$

Because  $k$  is uniformly continuous (it is continuous on a compact metric space), we see that there exists a sequence  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  of real numbers converging to 0 such that  $d_{M \times M}((x,y), (x',y')) < \frac{1}{n}$  implies  $|k(x,y) - k(x',y')| < \varepsilon_n$ . Thus,  $|k(x,y) - k(p_{n,i}, p_{n,j})|^2 < \varepsilon_n^2$ , so the entire integral above is bounded by  $\varepsilon_n^2$  and hence converges to 0 uniformly in  $f$ . ■

We are now ready to prove our spectral theorem.

**Theorem B.18 (Hilbert–Schmidt).** Fix a compact self-adjoint operator  $T: X \rightarrow X$  on a Hilbert space  $X$ . Then the eigenvalues  $\sigma(T)$  of  $T$  form a countable set of  $\mathbb{R}$ . Furthermore, each nonzero eigenspace  $X[\lambda]$  is finite-dimensional, and the subspace

$$\bigoplus_{\lambda \in \sigma(T)} X[\lambda]$$

is dense in  $X$ .

*Proof.* The idea is to strip off eigenvectors from  $X$  one at a time.

We proceed in steps.

1. Quickly, observe that if we can achieve the conclusion for  $T^2$ , then we can achieve the conclusion for  $T$  as well:  $T$  will preserve the eigenspace decomposition achieved for  $T^2$ , so we find that it has the same kernel, and each nonzero eigenspace of  $T$  can be found inside exactly one nonzero eigenspace of  $T^2$ . The benefit of working with  $T^2$  is that all eigenvalues are now expected to be positive.
2. We describe how to strip off a single eigenvector. If  $\|T\| = 0$ , there is nothing to do, so we assume otherwise. By compactness, we are granted a sequence of finite-rank operators  $\{T_n\}_{n \in \mathbb{N}}$  such that  $T_n \rightarrow T$ ; by projecting onto the subspace of self-adjoint operators, we may assume that the  $T_n$ s are self-adjoint. Let's exhibit  $\|T\|^2$  as an eigenvalue: because  $T_n^2$  is finite-rank, the largest eigenvalue of  $T_n^2$  has magnitude  $\|T_n\|^2$ , but the spectral theorem requires it to be a positive real number; say  $v_n \in X$  is this eigenvector with  $\|v_n\| = 1$ . Then Lemma B.16 allows us to pass to a convergent subsequence so that  $T^2 v_n \rightarrow v$  for some  $v \in X$ ; thus,  $T_n^2 v_n \rightarrow v$  as well, so  $v_n \rightarrow \|T\|^{-2} v$ , so we see  $T^2 v = \|T\|^2 v$ .
3. We complete the proof. The process in the previous paragraph can be iterated by passing to the orthogonal complement of each discovered eigenvector. In this way, we get a sequence  $\{\lambda_i^2\}$  of decreasing positive real numbers which are eigenvalues of  $T^2$ . If the iterative process terminates, then there are only finitely many eigenvalues each with finite dimension, and we complete the argument. Otherwise, note that we must have  $\lambda_i \rightarrow 0$  by Lemma B.16 applied to the corresponding orthonormal eigenvectors  $\{w_i\}$ .

In this latter case, it remains to check that the sum of these eigenspaces (together with the kernel) is dense in  $X$ . For this, we show that any unit vector  $v$  orthogonal to all the  $w_i$ s is in the kernel of  $T^2$ . Well, using the described iterative process, we find that  $\|T^2 v\|$  is bounded above by  $\lambda_i^2$  for all  $i$ , so  $T^2 v = 0$  follows. ■

### B.3.2 The Proof

In this subsection, we will prove Theorem B.4. We begin by providing a more group-theoretic formulation of  $\text{im } \xi_G$ .

**Lemma B.19.** Fix a compact real Lie group  $G$ . For  $f \in L^2(G)$ , the following are equivalent.

- (i) We have  $f \in \text{im } \xi_G$ .
- (ii) The elements  $\{(h_1, 1)f : h_1 \in G\}$  span a finite-dimensional vector space over  $\mathbb{C}$ .
- (iii) The elements  $\{(1, h_2)f : h_2 \in G\}$  span a finite-dimensional vector space over  $\mathbb{C}$ .
- (iv) The elements  $\{(h_1, h_2)f : h_1, h_2 \in G\}$  span a finite-dimensional vector space over  $\mathbb{C}$ .

*Proof.* Of course (iv) implies (ii) and (iii). To see that (i) implies (iv), it is enough to note that any element in the domain is supported on only finitely many  $(V \otimes V^*)$ s by definition of the direct sum and thus belongs to a finite-dimensional subrepresentation of the domain.

It remains to show that (ii) or (iii) implies (i). We will show that (ii) implies (i); the other implication is symmetric. Let  $V$  be the irreducible representation spanned by  $v := f$ , where  $G$  acts by right translation, which we note is finite-dimensional by hypothesis. (We are implicitly using Corollary B.14 to show that  $V$  is irreducible.) Now, we use the inclusion  $\iota_\bullet : V \subseteq L^2(G)$  to define  $\varphi \in V^*$  by  $\varphi(v) := \iota_v(1)$ . Then

$$\xi_V(v \otimes \varphi)(g) = \varphi(gf) = f(g),$$

so  $f \in \text{im } \xi_V$ , as required. ■

Before beginning the proof of Theorem B.4, we need one more notion. We will prove by Theorem B.4 by directly approximating a given  $f \in L^2(G)$ . Thus, it will be helpful to have well-behaved operators which tell us how to approximate.

**Definition B.20 (approximate identity).** Fix a compact real Lie group  $G$  with normalized Haar measure  $dg$ . Then an *approximate identity* is a sequence of nonnegative functions  $\{e_n\}_{n \in \mathbb{N}}$  in  $L^2(G)$  satisfying the following: for any open neighborhood  $U$  of 1, all but finitely many of the  $e_n$ s are supported in  $U$ , but  $\int_G e_n(g) dg = 1$ .

**Remark B.21.** To tie this definition with the preceding paragraph, we claim that any continuous  $f \in L^2(G)$  has  $(e_n * f) \rightarrow f$  uniformly as  $n \rightarrow \infty$ . Indeed, because  $G$  is compact,  $f$  becomes uniformly continuous, so for  $\varepsilon > 0$ , we may choose an open neighborhood of  $U \subseteq G$  such that  $h_1 h_2^{-1} \in U$  implies  $|f(h_1) - f(h_2)| < \varepsilon$ . Then choose  $N$  such that  $n > N$  implies  $e_n$  is supported on  $U$ , so any  $h \in G$  has

$$|(e_n * f)(h) - f(h)| = \int_U |e_n(g)f(g^{-1}h) - e_n(g)f(h)| dg + \int_U |e_n(g)f(h) - f(h)| dg \leq \varepsilon + 0.$$

We are now ready for the proof.

*Proof of Theorem B.4.* In short, the idea is to take an approximate identity and then use the corresponding convolution operators to build a fine decomposition of  $L^2(G)$  via our spectral theorem. We proceed in steps.

1. We set up some notation. Choose an approximate identity  $\{e_n\}_{n \in \mathbb{N}}$ . By replacing each  $e_n$  with the function  $g \mapsto \frac{1}{2}(e_n(g) + e_n(g^{-1}))$ , we may assume that  $e_n(g) = e_n(g^{-1})$ . We now use  $e_n$  to define the convolution operator  $E_n : L^2(G) \rightarrow L^2(G)$  by  $E_n f := (e_n * f)$ .
2. We claim that the closure of  $\text{im } \xi_G$  contains the image of  $E_n$ . For this, we use Theorem B.18: note  $E_n$  is compact by Lemma B.17, and one can check that  $E_n$  is self-adjoint because  $e_n(g_n^{-1}) = e_n(g)$ . Thus, we are given a dense subspace of  $L^2(G)$  given by the eigenspace decomposition

$$\ker E_n \oplus \bigoplus_{\lambda \in \sigma(E_n) \setminus \{0\}} L^2(G)[\lambda],$$

where each nonzero eigenspace  $L^2(G)[\lambda]$  is finite-dimensional. It now suffices to show that  $L^2(G)[\lambda]$  is contained in the closure of  $\text{im } \xi_G$ , which Lemma B.19 explains is the same as showing that  $L^2(G)[\lambda]$  is stable under some translation action by  $G$ . This follows because  $E_n: L^2(G) \rightarrow L^2(G)$  is invariant under the right translation action by  $G$ , so  $G$  will preserve the eigenspaces!

3. We complete the proof. Fix some  $f \in L^2(G)$  which we would like to approximate by  $\text{im } \xi_G$ ; continuous functions are dense in  $L^2(G)$ , so we may assume that  $f$  is continuous. By the previous step, we know that each  $B_n f$  has some  $f_n \in \text{im } \xi_G$  such that  $\|E_n f - f_n\|_2 < \frac{1}{n}$ . We now check  $f_n \rightarrow f$ :

$$\lim_{n \rightarrow \infty} \|f - f_n\|_2 \leq \lim_{n \rightarrow \infty} \|f - E_n f\|_2 + \lim_{n \rightarrow \infty} \|E_n f - f_n\|_2.$$

The left limit vanishes by Remark B.21, and the right limit vanishes by construction of the  $f_n$ s. ■

## B.4 Complements

In this section, we write out a few complements to our above discussion which are not immediately relevant to our main theorem.

### B.4.1 Unimodular Groups

We will spend a quick subsection investigating the bizarre phenomenon seen in Examples B.8 and B.9 that the left and right Haar measures agree.

**Definition B.22 (unimodular).** Fix a real Lie group  $G$  of dimension  $m$ . Then the group  $G$  is *unimodular* if and only if every left Haar measure is also right-invariant.

**Remark B.23.** Proposition B.11 explains that left-invariant and right-invariant top differential forms both form one-dimensional vector spaces, so  $G$  is unimodular if and only if these are the same vector subspace of  $\Gamma(G, \wedge^m T^*G)$ .

The definition of being unimodular is purely “measure-theoretic,” so we seek out a more algebraic definition.

**Lemma B.24.** Fix a real Lie group  $G$  of dimension  $m$  with Lie algebra  $\mathfrak{g}$ . Then  $G$  is unimodular if and only if the representation  $\wedge^m \text{Ad}_\bullet: G \rightarrow |\wedge^m \mathfrak{g}^*|$  is trivial, where the absolute value signifies the image through  $|\cdot|: \mathbb{R}^\times \rightarrow \mathbb{R}^+$ .

*Proof.* The group  $G$  is unimodular if and only if the left Haar measure given by the left-invariant top differential form  $\omega = X_1 \wedge \cdots \wedge X_m$  constructed in Proposition B.11 is also right-invariant, possibly with a sign. Because  $\omega$  is already left-invariant, this is equivalent to  $\omega$  being invariant under the conjugation action of  $G$  (possibly up to a sign), which is equivalent to the adjoint action of  $G$  on  $\wedge^m \mathfrak{g}^*$  being trivial (again, possibly up to a sign). ■

**Example B.25.** If  $G$  is a compact real Lie group of dimension  $m$  with Lie algebra  $\mathfrak{g}$ , then we claim that  $G$  is unimodular. Indeed, note that the image of the character  $G \rightarrow \text{GL}(\wedge^m \mathfrak{g}^*)$  must be a compact subgroup of  $\mathbb{R}^\times$ . But the only compact subgroups of  $\mathbb{R}^\times$  are subgroups of  $\{\pm 1\}$ , so Lemma B.24 explains that these groups are unimodular.

With this in mind, we note that one can pass checking unimodularity to the Lie algebra.

**Definition B.26 (unimodular).** Fix a real Lie algebra  $\mathfrak{g}$  of dimension  $m$ . Then we say that  $\mathfrak{g}$  is *unimodular* if and only if the adjoint representation  $\wedge^m \text{ad}_\bullet: \mathfrak{g} \rightarrow \mathfrak{gl}(\wedge^m \mathfrak{g})$  is trivial.

**Proposition B.27.** Fix a connected real Lie group  $G$  of dimension  $m$  with Lie algebra  $\mathfrak{g}$ . Then  $G$  is unimodular if and only if  $\mathfrak{g}$  is unimodular.

*Proof.* If  $G$  is unimodular, then the adjoint representation  $G \rightarrow \text{GL}(\wedge^m \mathfrak{g}^*)$  is trivial by Lemma B.24 and noting that  $G$  is connected. As such, taking the differential representation  $\mathfrak{g} \rightarrow \mathfrak{gl}(\wedge^m \mathfrak{g}^*)$  will continue to be trivial, and it will continue to be trivial after taking a dual, so  $\mathfrak{g}$  becomes unimodular.

Conversely, suppose that  $\mathfrak{g}$  is unimodular so that the differential representation  $\mathfrak{g} \rightarrow \mathfrak{gl}(\wedge^m \mathfrak{g}^*)$  is trivial after taking a dual. We would like to show that  $G$  acts trivially on  $\wedge^m \mathfrak{g}^*$ , which will complete the proof by Lemma B.24. Well, let  $H \subseteq G$  be the closed Lie subgroup of  $G$  acting trivially on  $\wedge^m \mathfrak{g}^*$ , and a short computation reveals that the Lie algebra  $\mathfrak{h}$  of  $H$  is precisely

$$\mathfrak{h} = \ker(\mathfrak{g} \rightarrow \mathfrak{gl}(\wedge^m \mathfrak{g}^*)).$$

Thus, we see that  $\mathfrak{h} = \mathfrak{g}$  by hypothesis on  $\mathfrak{g}$ , so  $H = G$  because  $G$  is connected. ■

We are now ready for some examples.

**Example B.28.** If  $\mathfrak{g}$  is abelian, then  $\text{ad}_\bullet: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is already trivial, so  $\mathfrak{g}$  is unimodular. This explains Example B.8.

**Example B.29.** If  $\mathfrak{g}$  is semisimple of dimension  $m$ , then  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ : after breaking  $\mathfrak{g}$  into simple Lie algebras, this follows because a simple algebra can have no nonzero proper ideals. Now, we claim that  $\mathfrak{g}$  is unimodular: note that  $\wedge^m \mathfrak{g}$  is one-dimensional, so upon choosing a basis vector, the representation  $\mathfrak{g} \rightarrow \mathfrak{gl}(\wedge^m \mathfrak{g})$  becomes some representation  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathbb{R})$ . But now  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$  means that any element of  $\mathfrak{g}$  can be written as  $[X, Y]$ , whereupon we note

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X) = 0$$

because  $\rho(X)$  and  $\rho(Y)$  are just some real numbers.

**Example B.30.** Suppose that both  $\mathfrak{g}$  and  $\mathfrak{h}$  are unimodular Lie algebras of dimensions  $m$  and  $n$ , respectively. Then we claim that  $\mathfrak{g} \oplus \mathfrak{h}$  is unimodular. Indeed, by expanding out bases of our exterior powers, we see that  $\wedge^{m+n}(\mathfrak{g} \oplus \mathfrak{h}) = \wedge^m \mathfrak{g} \oplus \wedge^n \mathfrak{h}$ , and the adjoint action onto each of those factors is already known to be trivial.

**Example B.31.** Any reductive Lie algebra is the sum of an abelian and a semisimple Lie algebra, so any reductive Lie algebra is unimodular by combining the previous three examples. This explains Example B.9.

## B.4.2 Characters

In this subsection, we use the Peter–Weyl theorem to understand the characters of a compact group. For brevity, if  $V$  is a representation, we will write  $\rho_V$  for its structure map.

**Definition B.32 (character).** Fix a real Lie group  $G$  and a representation  $\rho: G \rightarrow \text{GL}(V)$ . Then we define the character  $\chi_V: G \rightarrow \mathbb{C}$  by  $\chi_V(g) := \text{tr } \rho(g)$ .



**Remark B.33.** As usual, for representations  $V$  and  $W$  one can check that  $\chi_{V \oplus W} = \chi_V + \chi_W$  and  $\chi_{V \otimes W} = \chi_V \cdot \chi_W$ .

**Remark B.34.** If  $G$  is compact, we claim that  $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ . Use Proposition B.13 to give  $V$  a  $G$ -invariant positive-definite Hermitian inner product  $\langle \cdot, \cdot \rangle$ . Then  $\varphi \in V^*$  has  $(g\varphi)(v) = \varphi(g^{-1}v)$ , so the first equality follows upon expanding a basis. The second equality will follow upon showing that  $g$  has eigenvalues which have absolute value 1, which follows because the image of  $G \rightarrow \mathrm{GL}(V)$  must be compact.

The following example explains why the Peter–Weyl theorem may be helpful to us.

**Example B.35.** Note that  $V \otimes V^* \cong \mathrm{End}(V)$ , so one may ask where  $\xi_V$  sends  $\mathrm{id}_V$ . Well, give  $V$  a basis  $\{v_1, \dots, v_n\}$  so that  $V^*$  has the dual basis  $\{v_1^*, \dots, v_n^*\}$ , and then we see  $\xi_V$  sends  $\mathrm{id}_V \in \mathrm{End}(V)$  corresponds to  $\sum_i (v_i \otimes v_i^*) \in V \otimes V^*$ , which goes to  $g \mapsto \sum_i v_i^*(gv_i)$ , which is the trace!

Here is our main result.

**Theorem B.36.** Fix a compact real Lie group  $G$ . Let  $L^2(G)^G$  denote the set of functions  $f \in L^2(G)$  which are invariant under the action by  $\{(g, g) : g \in G\}$ . Then the collection of characters  $\chi_V$  form an orthonormal basis of  $L^2(G)$  as  $V$  varies over finite-dimensional complex irreducible representations.

*Proof.* We begin by remarking that  $\chi_V \in L^2(G)^G$  by properties of the trace. We have two checks.

- Orthonormal: for representations  $V$  and  $W$ , we use Remarks B.33 and B.34 to compute

$$\langle \chi_V, \chi_W \rangle = \int_G \chi_V(g) \overline{\chi_W(g)} dg = \int_G \chi_{V \otimes W^*}(g) dg.$$

If  $V \cong W$ , then we see that the middle term is 1 (by Remark B.34). Otherwise, if  $V \not\cong W$ , we upgrade it to the endomorphism

$$R := \int_G \rho_{V \otimes W^*}(g) dg,$$

which we note lives in  $(V \otimes W^*)^G$  by construction so that  $\langle \chi_V, \chi_W \rangle$  is the trace of  $R$ . Now, if  $V \not\cong W$  then  $(V \otimes W^*)^G$  vanishes by Schur's lemma, so  $R = 0$ .

- Spanning: we use Theorem B.4. On one hand, we note that  $(\mathrm{im} \xi_G)^G$  is precisely the span of the characters because  $(V \otimes V^*)^G$  is spanned by the identity, which goes to  $\chi_V \in L^2(G)$  via Example B.35. Thus, we are left with showing that  $(\mathrm{im} \xi_G)^G$  is dense in  $L^2(G)$ . The idea is to consider the projection operator  $P: L^2(G) \rightarrow L^2(G)^G$  given by

$$Pf(h) := \int_G f(g^{-1}hg) dg.$$

Notably,  $P$  is bounded: for any  $f \in L^2(G)$ , the invariance of the Haar measure shows  $\|Pf\|_2 \leq \|f\|_2$ . Additionally,  $P$  fixes  $L^2(G)^G$ . Thus, for any  $f \in L^2(G)^G$ , we may find a sequence  $\{f_n\}_{n \in \mathbb{N}} \in \mathrm{im} \xi_G$  approaching  $f$ , and then we see  $f_n \rightarrow f$  implies  $Pf_n \rightarrow f$ , so  $f$  is in the closure of  $(\mathrm{im} \xi_G)^G$ . ■

**Example B.37 (Fourier series).** Consider  $G = S^1$ , which is abelian so that  $L^2(G)^G = L^2(G)$ . The representations of  $S^1$  are one-dimensional, where the map  $S^1 \rightarrow \mathbb{C}^\times$  are given by  $z \mapsto z^n$  for some fixed integer  $n \in \mathbb{Z}$ . Thus, we are asserting that polynomials form an orthonormal basis in the collection of functions on  $S^1$ . Passing to the cover  $\mathbb{R} \rightarrow S^1$ , this shows that the functions  $t \mapsto e^{nit}$  form an orthonormal basis of the  $2\pi$ -periodic functions in  $L^2(\mathbb{R})$ . This recovers Fourier series on  $\mathbb{R}/2\pi\mathbb{Z}$ .

### B.4.3 Matrix Coefficients

In this subsection, we explain how the orthogonality of characters can be upgraded to produce orthogonality of matrix coefficients. As an application, we compute the convolution of characters.

**Definition B.38 (matrix coefficient).** Fix a finite-dimensional continuous representation  $V$  of a compact group  $G$ . Further, fix an invariant unitary inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , and let  $\{v_1, \dots, v_n\}$  be an ordered orthonormal basis. Then we define the *matrix coefficients* to be the map  $G \rightarrow \mathbb{C}$  defined by  $\rho(g)_{ij} := \langle v_i, \rho(g)v_j \rangle$ .

It turns out that these matrix coefficient maps are smooth, which is shown in Proposition 1.101.

**Example B.39.** By definition, we see that

$$\chi_\rho(g) = \sum_{i=1}^n \rho(g)_{ii}.$$

Here is our main result.

**Proposition B.40.** Fix a compact Lie group  $G$ . Further, choose finite-dimensional irreducible representations  $V$  and  $W$  of  $G$ , and equip them with invariant unitary inner products and ordered bases  $\{v_1, \dots, v_m\}$  and  $\{w_1, \dots, w_n\}$ , respectively.

(a) If  $V \not\cong W$ , then we always have

$$\int_G \rho_V(g)_{ij} \overline{\rho_W(g)_{k\ell}} dg = 0.$$

(b) We have

$$\int_G \rho_V(g)_{ij} \overline{\rho_V(g)_{k\ell}} dg = \begin{cases} 1/\dim V & \text{if } (i, j) = (k, \ell), \\ 0 & \text{else.} \end{cases}$$

*Proof.* We argue as in Theorem B.36. To begin, we see

$$\int_G \rho_V(g)_{ij} \overline{\rho_W(g)_{k\ell}} dg = \int_G \langle v_i \otimes w_k, (\rho_V(g) \otimes \rho_{\overline{W}}(g))(v_j \otimes w_\ell) \rangle dg$$

by considering the coefficients of the tensor product operator. However,  $\rho_V \otimes \rho_{\overline{W}} = \rho_{V \otimes W^*}$  by elaborating on Remarks B.33 and B.34, so we see that we are trying to compute a matrix coefficient of the operator

$$R := \int_G \rho_{V \otimes W^*}(g) dg.$$

By construction, the image of  $R$  lives in  $(V \otimes W^*)^G$  (it is an average!), so this automatically vanishes when  $V \not\cong W$ , which gives (a).

To prove (b), we now assume that  $V = W$ . Observe that  $Rg = gR = R$ , so  $R$  is a  $G$ -invariant map and will therefore preserve the isotypic components of  $(V \otimes V^*)$ . On the other hand, the image must be  $G$ -invariant, and  $\text{Hom}_G(V, V)$  consists only of the scalars, which  $R$  certainly preserves, so we see that  $R$  projects onto the scalars. For example, by expanding out our bases, we see that the scalars are all scalar multiples of the identity map  $v := \sum_i v_i \otimes v_i^*$ , and because our inner product is invariant, we see that  $R$  must be the orthogonal projection onto  $v$ . Thus,

$$Rv' = \frac{\langle v', v \rangle}{\langle v, v \rangle} v.$$

For example, when  $(i, j) \neq (k, \ell)$ , the fact that we are a projection automatically makes the matrix coefficient vanish. And when  $(i, j) = (k, \ell)$ , the above evaluates to  $1/\langle v, v \rangle = 1/\dim V$ . ■

**Corollary B.41.** Fix a compact Lie group  $G$ . Further, choose finite-dimensional irreducible representations  $V$  and  $W$  of  $G$ . Then

$$\chi_V * \chi_W = \langle \chi_V, \chi_W \rangle \chi_V.$$

*Proof.* Fix some input  $g$ , and choose unitary invariant inner products and orthonormal bases for  $V$  and  $W$  as usual. Then we see that  $(\chi_V * \chi_W)(g)$  is

$$\int_G \chi_V(gh^{-1}) \chi_W(h) dh = \sum_{i,k=1}^n \int_G \rho_V(gh^{-1})_{ii} \rho_W(h)_{kk} dh.$$

To continue, we should expand out the matrix coefficient for  $\rho_V(gh^{-1})$  to give

$$\sum_{i,j,k=1}^n \rho_V(g)_{ij} \int_G \rho_V(h^{-1})_{ji} \rho_W(h)_{kk} dh.$$

As usual, we note that  $\rho_V(h^{-1})_{ji} = \overline{\rho_V(h)_{ij}}$ , which is seen by expanding a basis. Then the integral vanishes by Proposition B.40 unless  $V \cong W$ , which explains the coefficient  $\langle \chi_V, \chi_W \rangle$  in the right-hand side of the statement of the corollary.

Otherwise, in the event that  $V \cong W$ , we may as well identify them and use the same orthonormal bases. Then Proposition B.40 tells us that the integral still vanishes unless  $(j, i) = (k, k)$ , in which case the value is  $1/\dim V$ . Thus, we are left with the sum

$$\sum_{i=1}^n \rho_V(g)_{ii} \cdot \frac{1}{\dim V} = \chi_V(g),$$

as required. ■

**Example B.42.** Here is a similar calculation. If  $V$  is an irreducible representation of the compact Lie group  $G$ , then we claim that the operator

$$\int_G \chi_V(g) \rho(g^{-1}) dg$$

is  $\text{id}_V$ . Indeed, a direct calculation with the invariance of the Haar measure shows that this operator is  $G$ -invariant, so it must be a scalar. To compute that this scalar is the identity, we can check that the trace of this operator equals  $\dim V$ . Indeed, Corollary B.41 explains the trace is equal to  $(\chi_V * \chi_V)(1) = \dim V$ .

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