616: Homotopy Theory

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Homotopy Theory

What we didn't do is make the construction at all usable in practice!
This time we will remedy this.

—Kiran S. Kedlaya, [Ked21]

1.1 January 22

In this course, we would like to do computations.

1.1.1 CW Complexes

We would like to compute the homology and cohomology of interesting spaces. CW complexes are a good mix between being simple enough to work with while being flexible enough to have many interesting examples.

Definition 1.1 (CW complex)**.** A *CW complex* is a topological space X equipped with an ascending chain of subspaces

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq$$

satisfying the following.

(i) $X = \bigcup_{k \geq -1} X_k$.

(ii) X_{k+1} is obtained inductively be adjoining $(k+1)$-disks $\{D_\alpha^{k+1}\}_\alpha$ to X_k as follows: some $x \in S_\alpha^k$ is identified with its image in X_k via some continuous map $f\colon \bigsqcup_\alpha S_\alpha^k \to X_k$.

(iii) The topology is on X is given by asserting $U \subseteq X$ is open if and only if $U \cap X_k$ is open for all k.

We call the interior $e_\alpha^n := (D_\alpha^n)^\circ$ an n-*cell*, and we call X_n the n-*skeleton*. If the total number of cells if finite, then X is *finite*; if merely each X_n is finite, then X is *finite type*.

Example 1.2. Compact manifolds have the structure of a CW-complex with only finitely many disks.

Remark 1.3. Note that we only ever identify points on the new disks for X_{k+1} along boundaries. In particular, we see that X can be seen as a union of the spaces e_α^n as n and α vary; thus, $e_\alpha^n \cap e_{\alpha'}^{n'} = \varnothing$ whenever $(n, \alpha) \neq (n', \alpha')$.

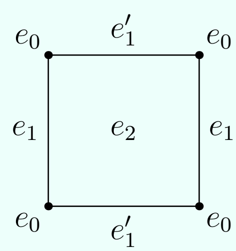
The point is that a CW complex allows us to work combinatorially with many topological spaces.

Example 1.4. We realize S^n as a CW complex as follows: simply attach D^n to a point $\{*\}$ by attaching the entire boundary to the point. Note that this is certainly continuous by definition of the quotient topology, and one can check that this is a bijection by doing casework at and away from $\{*\}$. Thus, we have a continuous bijection between compact Hausdorff spaces, so we have produced a homeomorphism.

Example 1.5. Here is another way to realize S^n. We do this inductively: note $n = 0$ has no content (simply take a point). Then supposing that we have already produced S^{n-1}, we simply attach a top and bottom hemisphere of n-cells to produce S^n.

Example 1.6. We can realize $\mathbb{RP}^n$ as a CW complex. Simply realize $\mathbb{RP}^n$ as S^n modulo the antipodal map, so we see that we can identify $\mathbb{RP}^n$ as $\mathbb{RP}^{n-1}$ union with a single n-cell (as in the previous example) but identified "twice-over" on the boundary.

Example 1.7. We realize $T^2 = S^1 \times S^1$ as a CW complex. Think of T^2 as a rectangle with the bottom/top and left/right edge pairs identified. Then we have a single point e_0 as our 0-cell, two 1-cells corresponding to the two edges (after identification), and then there is the 2-cell e^2 embedded. Here is the picture.



Example 1.8. We realize $\mathbb{CP}^n$ as a CW complex. The idea is that we can think about complex lines in $\mathbb{C}^{n+1}$ as intersecting

$$S^{2n+1} = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots + |z_n|^2 = 1 \right\},$$

and $p, q \in S^{n+1}$ both live on the same line if and only if $p/q \in S^1$. This explains that $\mathbb{CP}^{n+1}$ should be topologically a quotient S^{2n+1}/S^1.

We are now ready to give our CW structure, which we do inductively, beginning by noting that $\mathbb{CP}^0$ is a point. We would now like to attach D^{2n} to $\mathbb{CP}^{n-1}$ to produce $\mathbb{CP}^n$. To begin, note we can realize S^{2n} inside S^{2n+1} as above by adding the requirement $z_n \in \mathbb{R}$, and then we can realize D^{2n} inside this S^{2n} by further requiring $z_n \geq 0$. Then note that D^{2n} surjects onto S^{2n+1}/S^1 (by using the S^1-action), so we induce a homeomorphism of a quotient of D^{2n} onto S^{2n+1}/S^1. We take a moment to note that this quotient behaves as follows: distinct points $(z_0, \ldots, z_n)$ and $(w_0, \ldots, w_n)$ in D^{2n} are identified if and only if $z_n = w_n = 0$ (because we are only multiplying by scalars in S^1) and further $(z_0, \ldots, z_{n-1})$ and $(w_0, \ldots, w_{n-1})$ define the same line in $\mathbb{CP}^{n-1}$. In this way, we see that $\mathbb{CP}^{2n}$ is realized as $\mathbb{CP}^{2n-1}$ together with a D^{2n} attached as described above.

Bibliography

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Ked21] Kiran S. Kedlaya. *Notes on Class Field Theory*. 2021. URL: https://kskedlaya.org/papers/cft-ptx.pdf.

List of Definitions