# 215A: Algebraic Topology

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# Contents

*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

Theme 1

# Introduction

## 1.1 August 24

It begins.

### 1.1.1 Logistics

Here are the logistical notes.

- The professor is Ian Agol, whose office is Evans 921. Office hours are Tuesdays after class, Monday at 3PM, Wednesday at 9AM, or by appointment.
- There is a `bCourses`.
- Homework will be weekly, and it will make up the entire grade.
- The prerequisites are Math 113 and 202A or equivalent. From point-set topology in particular we will want notions of compactness, connectedness, metric spaces, and a few topologies like the identification topology with respect to a continuous map.

### 1.1.2 Overview

We will cover chapters 0–3 of [Hat01].

- Chapter 0 consists of "geometric notions." Particularly important are the notion of homotopy and CW complexes.
- Chapter 1 is on fundamental groups.
- Chapter 2 is on homology. This is an abelian extension of fundamental groups.
- Chapter 3 is on cohomology. Poincaré duality relates cohomology with homology.

Chapter 4 is typically covered in Math 215B, on homotopy theory.

Let's talk a bit about the interests of the course. Topology as a whole is interested in "spaces up to deformation." In this class, deformation will mean homotopy mostly, but there are finer notions of interest like homeomorphism. As for the spaces, we will focus on spaces which are locally homogeneous in some sense, like manifolds (which are locally homeomorphic to $\mathbb{R}^n$). These notions come up naturally throughout mathematics; for example, integrals of holomorphic functions are roughly independent of path chosen. Poincaré himself was interested in differential equations, whose configuration spaces could be manifolds.

In this class, we will attach invariants to our topological spaces to be able to understand how to differentiate between our spaces (up to deformation). We focus on the following invariants.

- Fundamental groups and covering spaces. This has a close tie to Galois theory, an analogy made process by the étale fundamental group in algebraic geometry.
- Cohomology. The origins are from complex analysis and Stokes's theorem, but cohomology itself has vast generalizations and manifestations throughout mathematics, leading to the field of homological algebra. However, there are applications to algebraic geometry, number theory, and so on. The most notable application here is the proof of the Weil conjectures.
- Higher homotopy groups. Our approach will not begin with this viewpoint, but it is possible.

### 1.1.3 Homotopy and Homotopy Type

Let's jump in chapter 0.

**Notation 1.1.** We set $I := [0,1]$ for convenience.

**Definition 1.2** (deformation retract). Fix a subspace $A$ of a topological space $X$. Then a *deformation retract* is a family of functions $f_\bullet \colon X \times I \to X$ where $f_0 = \operatorname{id}_X$ and $\operatorname{im} f_1 = A$ and $f_t|_A = \operatorname{id}_A$ for all $t \in I$.

**Example 1.3** (mapping cylinder). Fix a continuous function $f \colon X \to Y$. Then the *mapping cylinder* $M_f$ is the space $(X \times I) \sqcup Y$ quotiented by $(x,1) \sim f(x)$. Then $M_f$ has a deformation retraction to $Y$ by $f_t(x) := (x,t)$. Visually, we have attached $Y$ to a thickening of $X$.

**Example 1.4.** Define $f \colon S^1 \to S^1$ by $f(z) := z^2$. Then $M_f$ has $S^1$ on one domain side and $S^1$ covered twice on the target side. With a little deformation, this is a Möbius strip. Approximately speaking, one should cut the cylinder in half and then rearrange. One can see that the Möbius strip deformation retracts to $S^1$ by squishing the width of the cylinder to the central line.

A deformation retract is a special case of a homotopy. Here is the definition of a homotopy.

**Definition 1.5** (homotopy). Two continuous maps $f_0, f_1 \colon X \to Y$ are *homotopic* if and only if there is a continuous function $F_\bullet \colon X \times I \to Y$ such that $F_0 = f_0$ and $F_1 = f_1$. Here, $F$ is called a *homotopy*, and we write $f_0 \sim f_1$.

**Example 1.6.** A subspace $A \subseteq X$ has a deformation retract if and only if $\operatorname{id}_X$ is homotopic to some $r \colon X \to X$ with $\operatorname{im} r = A$ and $r|_A = \operatorname{id}_A$. Indeed, the deformation retract is exactly the needed homotopy.

Homotopy allows us to define homotopy equivalence.

**Definition 1.7** (homotopy equivalence). A continuous map $f \colon X \to Y$ is a *homotopy equivalence* if and only if there is a continuous map $g \colon Y \to X$ such that $(g \circ f) \sim \operatorname{id}_X$ and $(f \circ g) \sim \operatorname{id}_Y$. We then say that $X$ and $Y$ have the same *homotopy type* and write $X \simeq Y$.

**Remark 1.8.** It is not enough to merely require $(g \circ f) \sim \operatorname{id}_X$. For example, let $X := \{x\}$ be a point.

**Remark 1.9.** One can check directly that $\sim$ is an equivalence relation on spaces. The main check here is that one can compose homotopies.

We will often find that our algebraic invariants are only able to detect homotopy equivalence, which is why homotopy equivalence will be so important to us.

**Example 1.10.** Example 1.4 shows that the Möbius strip is homotopic to $S^1$.

**Example 1.11.** More generally, a deformation retract is a homotopy equivalence. The inverse homotopy equivalence is the inclusion.

**Example 1.12** (dunce cap)**.** Take the disc $D^2$ and glue the edges together as follows: mark three points $A$, $B$, and $C$, and glue $AB$ to $AC$ to $CB$ (in those orientations). Then the resulting space is homotopic to a point.

We have a special name for being homotopic to a point.

**Definition 1.13** (contractible)**.** A topological space $X$ is *contractible* if and only if it is homotopic to a point.

These notions allow us to define a homotopy category, whose objects are homotopy classes of topological spaces and morphisms are continuous maps. In some sense, our algebraic invariants are trying to distinguish between objects in this category. It turns out that this category is not concrete, meaning that there is no way to realize its objects as sets reasonably. Approximately speaking, this means that there can be no canonical representing topological space for each homotopy class, but topologists try anyway.

**Remark 1.14.** There are a number of results called "topological rigidity" theorems which give homeomorphism $X \cong Y$ given merely $X \simeq Y$ and some extra hypotheses. For example, this holds for closed surfaces by a classification result.

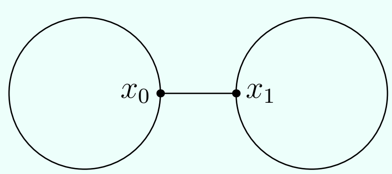
**Example 1.15.** Attach two $S^1$s by a line to make a space $X$, and attach them along an edge to make a space $Y$. These spaces are homotopic, but they are not homeomorphic (removing a point from $X$ may disconnect it, but this is not the case for $Y$).

### 1.1.4 CW Complexes

Here is our definition.

**Definition 1.16** (CW complex)**.** Let $X^0$ be a discrete set of points, and define $X^n$ inductively by $X^{n+1} := X^n \cup \{e_\alpha^{n+1}\}$, where $\varphi_\alpha : \partial e_\alpha^{n+1} \to X^n$ is a homeomorphism telling us how to union. Here, $e_\alpha^{n+1}$ is a copy of the $n$-ball $B^n$, so the $\varphi_\alpha$ are explaining how to identify the edges.

**Example 1.17.** Here is a CW complex.




Namely, $X^0 = \{x_0, x_1\}$, and $X^1$ is the edges.

**Example 1.18.** Take a point $\{*\}$ for $X^0$, and define $\varphi_n$ to be some loop based on $\{*\}$. Then the resulting space is some infinite union of circles intersecting at $\{*\}$. Notably, this space is not compact and in fact should not even be embedded into the plane or $\mathbb{R}^3$ because such an embedding is unlikely to be a homeomorphism.

**Example 1.19.** The sphere $S^n := D^n/\partial D^n$ is a CW structure with only two cells: it is $e^0 \cup e^n$. Notably, the CW structure here has $X^0 = X^1 = \cdots = X^{n-1}$.

**Example 1.20.** Alternatively, one can define $S^n$ inductively as follows: take $S^0$ to be two points, and define $S^{n+1}$ to be $S^n$ as an equator unioned with two $(n+1)$-cells making hemispheres attached to the equator. One can then define $S^\infty$ to be the union of all the $S^\bullet$ where we identify $S^n \hookrightarrow S^{n+1}$ via the equator. This is a CW complex of infinite dimension. It turns out that $S^\infty$ is contractible, though $S^n$ is not for any finite $n$.

**Example 1.21.** Define real projective space $\mathbb{RP}^n$ as the set of vectors $x \in \mathbb{R}^{n+1} \setminus \{0\}$ where we identify $x$ with $\lambda x$ for any $\lambda \in \mathbb{R}^\times$. Notably, by setting the last coordinate equal to $0$, we expect to get $\mathbb{RP}^{n-1}$. But if the last coordinate is equal to zero, we can scale it uniquely to $1$, and then the remaining coordinates may vary arbitrarily. In total, we find

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup \mathbb{R}^{n-1}.$$

Thus, we get the cell structure $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$.

**Remark 1.22.** The CW structure is not unique. For example, one can separate out edges by putting a point in the middle of them.

One can show that the CW complex is compact if and only if it has finitely many cells.

## 1.2 August 29

Last time we discussed homotopies, homotopy equivalence, and CW complexes. To review, the goal of algebraic topology is to define (algebraic) invariants of topological spaces and then perhaps figure out when two spaces are equivalent (for suitable definition of equivalent). In theory, our invariants would be able to entirely classify some subset of spaces we are looking at, but it is rather rare. To execute this plan, we need a source of spaces (mostly CW complexes and ways to combine them) and then methods to tell if spaces are equivalent.

### 1.2.1 Operations on Spaces

Let's discuss how to make new spaces from old ones. Thankfully, our operations will send CW complexes to CW complexes, though there is something to check.

**Definition 1.23** (product)**.** Fix CW complexes $X$ and $Y$. Then we form the *product* $X \times Y$ (at the level of CW complexes) using as $(n+m)$-cells $e_\alpha^m \times f_\beta^n$ where $e_\alpha^m$ is an $m$-cell of $X$ and $f_\beta^m$ is an $n$-cell of $Y$. Notably, the $n$-skeleton is

$$(X \times Y)^n = \bigcup_{k+\ell=n} X^k \times Y^\ell,$$

and one can attach in the obvious way. This produces a CW structure.

**Remark 1.24.** It is possible that $X \times Y$ with its CW structure need not be the same as the product topology. There is an example in the appendix of [Hat01], but we won't care so much for this course.

**Definition 1.25** (subcomplex)**.** Fix a CW complex $X$. Then a *subcomplex* is a closed subspace $A \subseteq X$ which is a union of cells of $X$ and also a CW complex.

**Definition 1.26** (quotient)**.** Fix a subcomplex $A$ of a CW complex $X$. Then $X/A$ is also a CW complex. Here, the definition of $X/A$ is somewhat technical: its cells are the cells of $X \setminus A$ and then a $1$-cell from $A$, and one attaches in the obvious way (inductively) via the quotient map $X^{n-1} \to X^{n-1}/A^{n-1}$.

**Definition 1.27** (suspension)**.** Fix a CW complex $X$. Then the *suspension* is the quotient

$$SX := \frac{X \times I}{\{(0,x) \sim (0,x') \text{ and } (1,x) \sim (1,x')\}}.$$

**Example 1.28.** Take $X = S^0$, which is two points. Then $X \times I$ is two lines, and we then identify the endpoints of the two lines accordingly to produce a circle $S^1$. More generally, $SS^n = S^{n+1}$ essentially by just gluing two $S^n$s onto the equator of $S^{n+1}$.

**Definition 1.29** (join)**.** Fix CW complexes $X$ and $Y$. Then the *join* $X * Y$ is the product $X \times Y \times I$ (as CW complexes) modded out by the equivalence relation identifying $(x,y,0) \sim (x,y',0)$ and $(x,y,1) \sim (x',y,1)$.

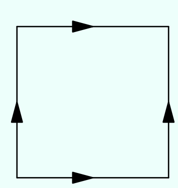
**Example 1.30** (simplex)**.** Consider $X = Y = I = \Delta^1$. Then $X * Y$ is the cube modded out by crushing $Y$ on one end and crushing $X$ on the other end, forming a tetrahedron, which is $\Delta^3$. More generally, $\Delta^n * \Delta^m = \Delta^{n+m+1}$.

**Example 1.31.** One has $S^0 * S^0 = S^1$, and more generally $S^n * X = SX$. Essentially, we are gluing two copies of $X$ onto an equator, which is the suspension.

**Definition 1.32** (wedge product)**.** Fix CW complexes $X$ and $Y$ and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then we form the *wedge product* $X \vee Y$ as $X \sqcup Y$ identifying $x_0 \sim y_0$.

**Definition 1.33** (smash product)**.** Fix CW complexes $X$ and $Y$ and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then the *smash product* is $(X \times Y)/(X \vee Y)$, where $X \vee Y$ is embedded into $X \times Y$ as $x \mapsto (x, y_0)$ and $y \mapsto (y, x_0)$.

**Example 1.34.** One can check that $S^1 \times S^1$ is a torus. To form the smash product, we are crushing the boundary of the square as follows.



More generally, $S^m \wedge S^n = S^{m+n}$.

**Definition 1.35** (attach)**.** Fix a subcomplex $A$ of a CW complex $X_1$ and a map $f\colon A \to X_0$ to another CW complex $X_0$. Then $X_0 \sqcup_f X_1$ is the space $X_0 \sqcup X_1$ modded out by the equivalence relation $a \sim f(a)$ for all $a \in A$.

**Example 1.36.** An attaching map $\varphi_\alpha\colon \partial D^n \to X^{n-1}$ of a CW complex are attachments $X^{n-1} \sqcup_{\varphi_\alpha} D^n$ in the above sense.

### 1.2.2 Homotopy Extension

We are going to, over time, prove the following results. To begin, quotients preserve homotopy type.

**Proposition 1.37.** Fix a subcomplex $A$ of a CW complex $X$. If $A$ is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

**Example 1.38.** Fix a connected graph $X$, which is a one-dimensional CW complex. Fix a spanning tree $T \subseteq X$, which is contractible (any tree can be contracted one edge at a time), so $X \to X/T$ is a homotopy equivalence. Then $X/T$ becomes a wedge of loops corresponding (roughly) to the number of "independent" cycles. Notably, this collapsing is far from canonical, essentially unique up to choosing the spanning tree and then an order of edges. In some sense, because the homotopy group of a wedge of loops is a free group, we are able to study automorphisms of the free group in this way.

**Proposition 1.39.** Fix a subcomplex $A$ of a CW complex $X_1$. Given homotopic maps $f, g\colon A \to X_0$, then $X_0 \sqcup_f X_1 = X_0 \sqcup_g X_1$.

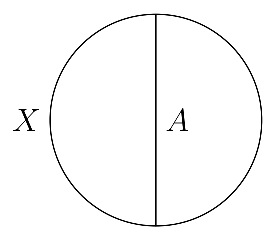
The idea of the above result is that if we can move the attaching maps $f$ and $g$ around, we should not really be adjusting the homotopy type.

To prove these results, we want access to the homotopy extension property.

**Definition 1.40** (homotopy extension property)**.** Fix a subspace $A$ of a topological space $X$. Then the pair $(X, A)$ has the *homotopy extension property* if and only if all $F_0\colon X \to Y$ and small homotopy $f_\bullet\colon A \times I \to Y$ with $F_0|_A = f_0$, then there is an extended homotopy $F_\bullet\colon X \times I \to Y$ where $F_t|_A = f_t$ for all $t \in I$.

It will turn out that a subcomplex $A$ of a CW complex $A$ makes $(X, A)$ have the homotopy extension property, but this will take some work to prove.

By way of example, make $Y$ the following "theta graph," and the left edge is $X$, and $A$ is the middle interval.




Here, $A \subseteq X$ is going to have the homotopy extension property. For example, one can contract $A$ to a point and imagine dragging neighborhoods of $A \cap X$ in $X$ (and in fact all of $Y$) along for the ride.

One way to think about the homotopy extension property is that we have a map $X \cup (A \times I) \to Y$ (by taking the union $F_0$ and $f_\bullet$), and we and to extend it to a full map $X \times I \to Y$. With this in mind, we would thus like to have to retract $r\colon (X \times I) \to (X \cup (A \times I))$ and then composing. By taking $Y = X \times I$, one sees that having such a retraction $r$ is in fact equivalent to the homotopy extension property.

So we want to find the retraction $r\colon (X \times I) \to (X \cup (A \times I))$.

**Lemma 1.41.** Fix a subspace $A$ of a topological space $X$. Then $(X, A)$ has the homotopy extension property if and only if $A$ has a "mapping cylinder neighborhood." In other words, there is a space $B$ and map $f\colon B \to A$ such that $M_f$ is homeomorphic to a neighborhood of $A$.

Approximately speaking, what's going on here is that the mapping cylinder allows us some squishing region through which to extend homotopies. Then the above criteria can be checked for CW pairs $(X, A)$ by tracking through attachments. Namely, a reparameterization of the attaching map has mapping cylinder which has the property needed above. Rigorously, one inducts on the $n$-skeleton of a CW complex $X$, using the homotopy extension property for cells of $X$ not in $A$ (and not caring about cells already in $A$).

Read Hatcher

## 1.3 August 31

We now shift gears and talk about our first algebraic invariant: the fundamental group.

### 1.3.1 The Fundamental Group

Let's start with an example.

**Example 1.42.** Fix a loop $\gamma\colon S^1 \to (\mathbb{C} \setminus \{0\})$ which is continuously differentiable. Then complex analysis tells us that

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz$$

counts the number of times that $\gamma$ "winds" around the integer. We might call this the "linking number" of $\gamma$. Notably, one can check that continuously varying $\gamma$ does not adjust the linking number, so this linking number is homotopy invariant.

The fundamental group is a generalization of this notion.

**Definition 1.43** (fundamental group)**.** Let $X$ be a topological space, and fix a basepoint $x_0 \in X$. Then the *fundamental group* $\pi_1(X, x_0)$ is the set of homotopy equivalence classes

$$\pi_1(X, x_0) := \{[f] \text{ such that } f\colon I \to X \text{ has } f(0) = f(1) = x_0\}.$$

We will give $\pi_1(X, x_0)$ a group structure below.

**Remark 1.44.** There is also a $\pi_0(X)$, which consists of homotopy classes of points $[x]$ for $x \in X$, where $[x]$ denotes the path-connected component of $X$. If we let $\Omega(X, x_0)$ denote the topological space of loops $f\colon I \to X$ such that $f(0) = f(1) = x_0$, then we find $\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$.

**Remark 1.45.** If we don't want to care about basepoints, one can look at $C\left(S^1, X\right)$, which is the set of maps $S^1 \to X$. This can be given a topology via the compact-open topology. Approximately speaking, these will correspond to conjugacy classes in $\pi_1(X, x_0)$ provided that $X$ is path-connected. For example, the homotopy class of a constant loop $S^1 \to X$ consists of the contractible loops in $X$; note there is something to check here in that one wants to know that a contractible loop (not relative to the basepoints) is in fact contractible relative to the basepoint.

**Example 1.46.** Let $X = \{x_0\}$ be a point. Then $\pi_1(X, x_0) = 1$ because there is only path $I \to X$.

**Example 1.47.** Let $X$ be a convex subset of $\mathbb{R}^n$ for some $n > 0$. Then for any $x_0 \in X$ has $\pi_1(X, x_0) = 1$. Indeed, use the convex hypothesis to shrink any path down to the constant path.

We can give $\pi_1(X, x_0)$ a product via composition.

**Definition 1.48** (composition)**.** Let $X$ be a topological space, and fix a basepoint $x_0 \in X$. Given paths $f, g\colon I \to X$ such that $f(1) = g(0)$, we define the path $(f \cdot g)\colon I \to X$ via

$$(f \cdot g)(t) := \begin{cases} f(2t) & \text{if } 0 \le t \le 1/2, \\ g(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Note that $f \cdot g$ is well-defined at $t = 1/2$ because $f(1) = g(0)$.

The point of the above definition is to "squish" a path to do both $f$ and $g$ in the interval $I$, but at twice the speed. One has the following checks.

- The class $[f \cdot g]$ does not depend on the choice of representatives $f$ and $g$. Essentially, if $f_1 \sim f_2$ and $g_1 \sim g_2$, then one can use these two homotopies to glue together to make a new homotopy $(f_1 \cdot g_1) \sim (f_2 \sim g_2)$.
- We have $[(f \cdot g) \cdot h] = [f \cdot (g \cdot h)]$, so composition associates. The point is that these are basically reparameterizations of each other.
- There is an identity path given by $e_{x_0}(t) := x_0$. The identity check is done again by some idea of reparameterization.
- For a given path $f\colon I \to X$, we can define $\overline{f}\colon I \to X$ by $\overline{f}(t) := f(1-t)$ and then check that

  $$f \cdot \overline{f} \sim e_{f(0)},$$

  so $\left[\overline{f}\right]$ provides the inverse path for $[f]$ in $\pi_1(X, x_0)$. The point is that $f \cdot \overline{f}$ is

  $$\left(f \cdot \overline{f}\right)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le 1/2, \\ f(2-2t) & \text{if } 0 \le t \le 1/2. \end{cases}$$

  One can then provide a homotopy by

  $$h_s(t) := \begin{cases} f(2t) & \text{if } 0 \le t \le s/2, \\ f(s) & \text{if } s/2 \le t \le 1 - s/2, \\ f(2-2t) & \text{if } 1 - s/2 \le t \le 1, \end{cases}$$

  so $h_0 = e_{f(0)}$ and $h_1 = f \cdot \overline{f}$.

For these checks, it is helpful to have lemmas establishing continuity of piecewise functions and establishing that reparameterization does not affect homotopy class.

**Remark 1.49.** Staring hard at our definition of composition, one sees that our reparameterization business is really just choosing various piecewise affine maps $I \to I$ with slopes in $2^{\mathbb{Z}}$ and breaks at the dyadic rationals $2^{\mathbb{Z}}\mathbb{Z} \subseteq \mathbb{Q}$. These maps form a group called the Thompson group.

**Remark 1.50** (fundamental groupoid)**.** Fix a topological space $X$, and define a category where the objects are points $x \in X$, and the morphisms $x \to y$ are paths (up to homotopy fixing endpoints). The above checks now show that this is in fact a category, where each morphism has an inverse. This category is called the *fundamental groupoid*. Modding out by isomorphism, our objects are now path components in $X$, and choosing a particular component produces the fundamental group in its endomorphisms.

**Remark 1.51.** Verifying that $\pi_1(X, x_0)$ only required reparameterization. So as in Remark 1.50, we could also look at the category where paths are only considered up to reparameterization, and the above checks still go through. This is related to the notion of "thin homotopy."

**Lemma 1.52.** Fix a topological space $X$. Further, fix a path $p\colon I \to X$. Then $f \mapsto (\overline{p} \cdot f \cdot p)$ provides an isomorphism $\pi_1(X, p(1)) \to \pi_1(X, p(0))$.

*Proof.* This is well-defined because $f_1 \sim f_2$ implies $\overline{p} \cdot f_1 \sim \overline{p} \cdot f_2$ implies $\overline{p} \cdot f_1 \cdot p \sim \overline{p} \cdot f_2 \cdot p$. This is a group homomorphism because

$$\overline{p} \cdot f \cdot g \cdot p \sim \overline{p} \cdot f \cdot p \cdot \overline{p} \cdot g \cdot p.$$

Lastly, this is an isomorphism because $\overline{p}$ provides the inverse map. ■

**Remark 1.53.** The above result roughly says that we can indeed look at the fundamental groupoid only in terms of the path-connected components.

Thus, we see that $\pi_1(X, x_0)$ is well-defined up to base-point provided that $X$ is path-connected. However, the isomorphism between base-points is only defined up to path between those basepoints! Roughly speaking, the problem is that elements of $\pi_1(X, x_0)$ should really only be thought of up to inner automorphism because we can pre- and post-compose by some loop at $x_0$.

**Lemma 1.54.** If $X$ is homeomorphic to $Y$ by $\varphi\colon X \to Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$ for any $x_0 \in X$.

*Proof.* Use $\varphi$. ■

### 1.3.2 The Fundamental Group of $S^1$

Here is our result.

**Theorem 1.55.** Fix any $x \in S^1$. Then $\pi_1\left(S^1, x\right) \cong \mathbb{Z}$. In fact, there is an isomorphism $\Phi\colon \mathbb{Z} \to \pi_1\left(S^1, x\right)$ given by

$$n \mapsto \left[t \mapsto (\cos 2\pi nt, \sin 2\pi nt)\right].$$

*Sketch without covering spaces.* We show injectivity and surjectivity independently.

- Think of $S^1$ as embedded in $\mathbb{C}$ as $\{z : |z| = 1\}$ and take a smooth path $f\colon I \to S^1$, lift it to a map $\widetilde{f}\colon I \to \mathbb{R}$ via

$$\widetilde{f}(t) := \int_0^t d\theta,$$

  where $d\theta$ is some differential form $S^1$ (say, $x\,dy - y\,dx$). Then $\widetilde{f}(1)$ is intuitively contained in $2\pi\mathbb{Z}$ and is homotopy invariant. Now, $f$ is not smooth, then we can use some small homotopy to make $f$ smooth and then use the above argument. This provides an inverse map to $\Phi$ and thus shows that $\Phi$ is injective.

- For surjectivity, one can use uniform continuity of any path $f\colon I \to S^1$ and the compactness of $S^1$ in order to divide up $I$ into intervals on which $f$ can be written as a composition of well-behaved paths, which eventually allows us to force $f$ to make piecewise linear. Once $f$ is piecewise linear, we go interval-by-interval and fix $f$ to be constant speed. Eventually $f$ becomes one of the $\Phi(n)$ for some $n$. ■

For the covering space approach, the point is that we understand the fundamental group of $\mathbb{R}$ well, and we have a fairly well-behaved "covering map" $p\colon \mathbb{R} \to S^1$ given by $p(\theta) := (\cos 2\pi\theta, \sin 2\pi\theta)$. The main claim, then, is that any path $\omega\colon I \to S^1$ has a unique lift $\widetilde{\omega}\colon I \to \mathbb{R}$ such that $\widetilde{\omega}(0) = \omega(0)$ and $p \circ \widetilde{\omega} = \omega$. The point is that once we lift, we can use a homotopy up in $\mathbb{R}$ (fixing the endpoints of $\widetilde{\omega}$), which will then go back down to a homotopy on $S^1$ if we are careful. Anyway, this lifting process can essentially be done as described in the surjectivity check above.

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# List of Definitions