250B: Commutative Algebra

Or, Eisenbud With Details

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Contents

THEME 1: FUN WITH FILTRATIONS

1.1 February 24

So it's the day after death.

1.1.1 Midterm Review

Let's start talking about the second problem on the midterm. Note that $X^2 = 0$ for a matrix $X \in \mathbb{C}^{2\times 2}$ if and only if the characteristic polynomial is X^2 if and only if

$$\det X = \operatorname{tr} X = 0,$$

so the set of matrices with $X^2 = 0$ is generated by these two conditions. To see that these make a principal ideal, we check that

$$\frac{\mathbb{C}[a,b,c,d]}{(a+d, ad-bc)} \cong \frac{\mathbb{C}[a,b,c]}{(-d^2-bc)}$$

is an integral domain, which is not very hard.

1.1.2 Filtration of Rings

Today we are talking about the Artin–Rees lemma, which requires us talking about filtrations.

Definition 1.1 (Filtration, rings)**.** Fix R a ring. Then a *filtration* of R is a sequence of ideals

$$R = I_0 = I_1 \supseteq I_2 \supseteq \cdots$$

such that $I_i I_j \subseteq I_{i+j}$.

Example 1.2. Fix $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ a graded ring. Then we can set

$$I_p := \bigoplus_{i \geq p} R_i$$

so that

$$R = I_0 \supseteq I_1 \supseteq \cdots$$

is a filtration.

Definition 1.3 (I-adic filtration)**.** Fix R a ring and $I \subseteq R$ an ideal. Then

$$R \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \cdots$$

is a filtration. This is called the *I-adic filtration*.

Example 1.4. More concretely, consider $R = k[x_1, \ldots, x_n]$ graded by degree. Then we set I_m to be the union of $\{0\}$ the set of all polynomials with degree at least m. This manifests Example 1.2, but it is also the $(x_1, \ldots, x_n)$-adic filtration.

Generally speaking, if we have a filtration

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots,$$

we might be interested in the "bottom" of this filtration

$$\bigcap_{i=0}^{\infty} I_i.$$

Surely this is an ideal, but it might not be 0. Regardless, today we will be interested in the case where this is 0.

1.1.3 Associated Graded Rings

Filtrations give rise to the following definition.

Definition 1.5 (Associated graded ring)**.** Fix a filtration $\mathcal{J}$ notated

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots.$$

Then we define $R_i := I_i/I_{i+1}$ and define

$$\operatorname{gr}_{\mathcal{J}} R := \bigoplus_{k \geq 0} I_k/I_{k+1}$$

to be the *associated graded ring.* If $\mathcal{J}$ is the I-adic filtration, we denote this by $\operatorname{gr}_I(R)$. If the filtration is obvious, we will omit the subscript entirely.

A priori, the associated graded ring is only some very large module, but we can give it a ring structure as follows: if we have terms $[a] \in I_p/I_{p+1}$ and $[b] \in I_q/I_{q+1}$, then we can lift them to $a \in I_p$ and $b \in I_q$ so that $ab \in I_pI_q \subseteq I_{p+q}$, which is unique up to representative of I_{p+q+1}.

In particular, if $a \equiv a' \pmod{I_{p+1}}$ and $b \equiv b' \pmod{I_{q+1}}$, then

$$aa' \equiv bb' \pmod{I_{p+q+1}},$$

which is something we can check by hand by looking at $aa' + a(b' - b) + b'(a - a') - bb'$.

Example 1.6. We work in $R := k[[x]]$, which is local with maximal ideal $I := (x)$. Then $I^n = (x^n)$ gives rise an I-adic filtration. We can compute

$$I^n/I^{n+1} \cong \{ax^n : a \in k\} \cong kx^n$$

because we are taking (0 or) a very long polynomial with minimal degree x^n and then killing all higher degree terms. So our filtration reads as

$$\operatorname{gr}_I R = R/I \oplus I/I^2 \oplus I^2/I^{\oplus} \cdots = k \oplus kx \oplus kx^2 \oplus \cdots = k[x].$$

We can check that the multiplication rule actually matches.

Example 1.7. Fix $R = \mathbb{Z}$ and $I = (p)$ a prime ideal. Then $I^n/I^{n+1} = p^n\mathbb{Z}/p^{n+1}\mathbb{Z} = p^n(\mathbb{Z}/p\mathbb{Z})$, so we can represent anyone in $\operatorname{gr}_I R$ by

$$a_0 + a_1 p + a_2 p^2 + \cdots$$

where $a_0, a_1, \ldots \in \mathbb{Z}/p\mathbb{Z}$. Checking our ring structure, we can identify this with the "finite-length" p-adic integers $\mathbb{Z}_p$. What we get at the end is $\mathbb{Z}/p\mathbb{Z}[x]$, which requires some care with the ring structure: think of $a_1 p \cdot b_1 p$ as living in the $a_1 b_1 p^2$ coordinate, so this is essentially a polynomial ring.

We have the following warning.

Warning 1.8. There is no natural ring homomorphism $R \to \operatorname{gr}_I R$.

However, there is a natural map of sets. Explicitly, for our filtration

$$R \supseteq I \supseteq I^2 \supseteq \cdots,$$

we want to find an element of the associated graded ring. In analogy to picking up the "initial" nonzero homogeneous part of a polynomial, we pick up $f \in R$ and define

$$\operatorname{in} f = f + I^{n+1},$$

where n is the largest possible such that $f \in I^n$. Of course, there is something of a problem when f lives in all of I^n, in which case we set $\operatorname{in} f := 0$.

Let's think about how this plays with our ring structure. Taking $f, g \in R$, if R is a domain, then we get that $\operatorname{in}(fg) = \operatorname{in}(f)\operatorname{in}(g)$. However, if f and g are zero-divisors, then we might be in trouble when $fg = 0$.

And now for some examples.

Example 1.9. Fix $X \subseteq \mathbb{A}^n(k)$ a Zariski closed set with $X = Z(J)$ such that $J \subseteq k[x_1, \ldots, x_n] =: R$ is an ideal. Taking $p \in X$ to correspond to a maximal ideal $\mathfrak{m} \subseteq A(X)$, we claim that

$$\operatorname{gr}_I R$$

is the ring corresponding to the "tangent cone to p at X."

As an example, consider the curve $y^2 = x^2(x+1)$, which splits at 0. At a point which is not $(0,0)$, we will have a line and therefore will expect to get a polynomial ring.

However, let's focus on what happens at $(0,0)$. Analytically, we find that

$$\frac{y^2}{x^2} = x + 1.$$

Very close to $(0,0)$, we get that

$$\left(\frac{dy}{dx}\right)^2 = 1$$

so that the slope is ± 1.

Let's try to think more algebraically. We have the following lemma.

Lemma 1.10. Work in the context of the above example. Then

$$\operatorname{gr}_I(R/J) = (\operatorname{gr}_I R)/\operatorname{in} J.$$

Proof. This is on the homework. ■

The point of this lemma is that $\operatorname{gr}_I R$ we know to be a polynomial ring. With $I = (x, y)$ as in the example we are working out, we find that $\operatorname{in}(x)^2 = \operatorname{in}(y)^2$ because our ideal J is $y^2 - x^2(x+1)$. Namely, our associated ring looks like functions generated by the lines $\operatorname{in} x = \operatorname{in} y$ and $\operatorname{in} x = -\operatorname{in} y$, which is what we expected.

In contrast, the cusp $y^2 = x^3 - x$ will give a double point, generated only at $\operatorname{in}(x)^2$. Here, we will be generated by $(\operatorname{in} y)^2$, which is what our cusp looks like intuitively.

1.1.4 Filtration of Modules

Consider the following construction.

Definition 1.11 (Hilbert function, rings)**.** Fix R a local Noetherian ring where I is the maximal ideal. Then we define

$$\dim_{R/I}(\operatorname{gr}_I R)_n = \dim_{R/I}\left(I^n/I^{n+1}\right) = H_R(n).$$

Note that this definition is well-formed because R/I is a field.

We would like to generalize this to modules. We have the following series of definitions.

Definition 1.12 (Filtration, modules)**.** Given an R-module M, a *filtration* is a descending chain

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

This is an I*-filtration* if and only if $IM_n \subseteq M_{n+1}$.

There is no multiplicative condition on the filtration because M has no multiplication.

Definition 1.13 (Associated graded module)**.** Fix an R-module M, with a filtration $\mathcal{J}$, denoted by

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

Then we define

$$\operatorname{gr}_{\mathcal{J}} M := M/M_1 \oplus M_1/M_2 \oplus \cdots .$$

We remark that $\operatorname{gr}_{\mathcal{J}} M$ is a graded $\operatorname{gr}_I R$-module, which is not too hard to check by hand.

Definition 1.14 (Stable)**.** An I-filtration of an R-module M, denoted by

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

is I*-stable* if and only if $IM_j = M_{j+1}$ for sufficiently large j.

It's a math class, so let's try to prove something today.

Proposition 1.15. Fix $I \subseteq R$ an ideal. Further, take M to be a finitely generated R-module, $\mathcal{J}$ to be a stable I-filtrations by finitely generated R-modules. Then $\operatorname{gr}_{\mathcal{J}} M$ is a finitely generated $\operatorname{gr}_I R$-module.

Proof. We definition-chase. Let our filtration be

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

For sufficiently large n, we have that $I^k M_n = M_{n+k}$. Thus, it suffices to take generators for $M_0, M_1, \ldots, M_n$ to generate the entire associated graded module. ■

This lets us construct our Hilbert function for modules.

Definition 1.16 (Hilbert function, modules)**.** Fix R a local Noetherian ring where I is the maximal ideal with M a finitely generated R-module. Then we define

$$H_M(n) = \dim_{R/I}\left(I^n M/I^{n+1}M\right).$$

Note that this definition is well-formed because M is finitely generated.

1.1.5 The Artin–Rees Lemma

We are finally ready to provide our main result.

Theorem 1.17 (Artin–Rees lemma)**.** Fix R a Noetherian ring and $I \subseteq R$ an ideal with M a finitely generated R-module granted a stable I-filtration $\mathcal{J}$ denoted by

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

Then given a submodule $N \subseteq M$, the induced filtration by $N_k := M_k \cap N$ is also a stable I-filtration.

Proof. To prove this, we need to introduce the blow-up ring.

Definition 1.18 (Blow-up ring)**.** Fix R a ring and $I \subseteq R$ an ideal. Then we define the *blow-up ring* $\mathrm{B}_I R$ by

$$\mathrm{B}_I R := R \oplus I \oplus I^2 \oplus \cdots .$$

Concretely, think about $\mathrm{B}_I R$ as getting its ring structure from $k[t]$ by something like $k[It]$. This also gives us our grading. In particular, is that $\mathrm{B}_I R / I \, \mathrm{B}_I R \cong \mathrm{gr}_I R$ after tracking everything through.

Example 1.19. Fix $R := k[x, y]$ and consider $(0, 0) \in \mathbb{A}^2(k)$ with associated maximal ideal $I := (x, y) \subseteq R$. In this case, our blow-up ring looks like $k[x, y][tx, ty]$. To look at points, we need to look at the "graded" spectrum of $\mathrm{B}_I R$. Here are some ways to do this.

- Look at $Z \subseteq \mathbb{A}^2(k) \times \mathbb{P}^1(k)$ to be points (p, ℓ) such that $p \in \ell$. We can project $Z \twoheadrightarrow \mathbb{A}^2(k)$ in the natural way. As long as $p \neq 0$, there is exactly one pre-image. But if $p = 0$, then our pre-image contains all the lines in $\mathbb{P}^1(k)$! So we have created some "blowing up" at the origin.

- Alternatively, focus on $k[x, y][tx, ty]$. Set $u = tx$ and $v = ty$ so that we are essentially looking at the ring

$$\frac{k[x, y, u, w]}{(xw - yv)},$$

which correspond to the 2×2 singular matrices. Taking the quotient by the "line action" of matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}.$$

Most of the time, this quotient process will give us 0, but rarely we will have an entire line after doing the quotient.

We remark that there is also a notion of the blow-up module.

Definition 1.20 (Blow-up ring)**.** Fix R a ring and $I \subseteq R$ an ideal. Further, fix $\mathcal{J}$ an I-filtration. Then we define the *blow-up module* $\mathrm{B}_I M$ by

$$\mathrm{B}_I M := M_0 \oplus M_1 \oplus M_2 \oplus \cdots ,$$

which we can check to be a graded $\mathrm{B}_I R$-module.

In line with this, we have the following proposition.

Proposition 1.21. Fix R a Noetherian ring and $I \subseteq R$ an ideal with M a finitely generated R-module granted an I-filtration $\mathcal{J}$ denoted by

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

Then $\operatorname{B}_{\mathcal{J}} M$ is finitely generated as a $\operatorname{B}_I R$-module if and only if $\mathcal{J}$ is I-stable.

Proof. We omit this proof. It is largely definition-chasing. ■

We are now ready to attack the proof of the Artin–Rees lemma. Let $\mathcal{J}'$ be the induced filtration for N. From the definition, we see that $\operatorname{B}_{\mathcal{J}'} N \subseteq \operatorname{B}_{\mathcal{J}} M$ is a $\operatorname{B}_I R$-submodule. Now, $\operatorname{B}_{\mathcal{J}'} N$ is a submodule of the finitely generated module $\operatorname{B}_{\mathcal{J}} M$ under the Noetherian ring $\operatorname{B}_I R$, so we are done. ■

Here is a nice application.

Theorem 1.22 (Krull intersection)**.** Fix R a Noetherian ring with an ideal I and finitely generated module M. Then

$$N := \bigcap_{s \geq 0} I^s M$$

satisfies that there is some $x \in I$ such that $(1 - r)N = 0$.

Proof. By construction, we see that $IN = N$, which in particular holds because the standard I-filtration of M is stable. Then we showed as a lemma to Nakayama's lemma back in **??** that there is an element $r \in I$ with $(1 - r)N = 0$. ■

Corollary 1.23. Fix R a Noetherian ring with a proper ideal I. Further, if R is local or a domain, then

$$\bigcap_{s \geq 0} I^s = 0.$$

Proof. Set

$$J := \bigcap_{s \geq 0} I^s.$$

By the proof of the theorem, we get $IJ = J$, which finishes by Nakayama's lemma. When R is a domain, then the theorem gives us some $r \in I$ such that $(1 - r)J = 0$, but R being a domain will force $J = 0$ from this. ■

Remark 1.24. The condition that R is Noetherian is necessary.

We close with an exercise.

Exercise 1.25. Fix R a local Noetherian ring. If $\operatorname{gr}_I R$ is a domain, then R is a domain.

Proof. The main idea is that $\operatorname{in} f = 0$ implies $f = 0$, essentially by the corollary above. ■