256A: Algebraic Geometry

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Fall 2022

Contents

THEME 1
INTRODUCTION

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 24

A feeling of impending doom overtakes your soul.

1.1.1 Administrative Notes

Here are housekeeping notes.

- Here are some housekeeping notes. There is a syllabus on `bCourses`.
- We hope to cover Chapter II of [Har77], mostly, supplemented with examples from curves.
- There are lots of books.
  - We use [Har77] because it is short.
  - There is also [Vak17], which has more words.
  - The book [Liu06] has notes on curves.
  - There are more books in the syllabus. Professor Tang has some opinions on these.
- Some proofs will be skipped in lecture. Not all of these will appear on homework.
- Some examples will say lots of words, some of which we won't have good definitions for until later. Do not be afraid of words.

Here are assignment notes.

- Homework is 70% of the class.
- Homework is due on noon on Fridays. There will be 6–8 problems, which means it is pretty heavy. The lowest homework score will be dropped.

- Office hours exist. Professor Tang also answers emails.
- The term paper covers the last 30% of the grade. They are intended to be extra but interesting topics we don't cover in this class.

1.1.2 Motivation

We're going to talk about schemes. Why should we care about schemes? The point is that schemes are "correct."

Example 1.1. In algebraic topology, there is a cup product map in homology, which is intended to algebraically measure intersections. However, intersections are hard to quantify when we aren't dealing with, say manifolds.

Here is an example of algebraic geometry helping us with this rigorization.

Theorem 1.2 (Bézout)**.** Let C_1 and C_2 be curves in $\mathbb{P}^2(k)$, for some algebraically closed k, where C_1 and C_2 are defined by homogeneous polynomials f_1 and f_2. Then the "intersection number" between the curves C_1 and C_2 is $(\deg f_1)(\deg f_2)$.

This is a nice result, for example because it automatically accounts for multiplicities, which would be difficult to deal with directly using (say) geometric techniques. Schemes will help us with this.

Example 1.3. Moduli spaces are intended to be geometric objects which represent a family of geometric objects of interest. For example, we might be interested in the moduli space of some class of elliptic curves.

It turns out that the correct way to define these objects is using schemes as a functor; we will see this in this class.

Remark 1.4. One might be interested in when a functor is a scheme. We will not cover this question in this class in full, but it is an interesting question, and we will talk about this in special cases.

1.1.3 Elliptic Curves

For the last piece of motivation, let's talk about elliptic curves, over a field k.

Definition 1.5 (Elliptic curve)**.** An *elliptic curve* over k is a smooth projective curve of genus 1, with a marked k-rational point.

Remember that we said that we not to be afraid of words. However, we should have some notion of what these words mean: being a curve means that we are one-dimensional, being smooth is intuitive, and having genus 1 roughly means that base-changing to a complex manifold has one hole. Lastly, the k-rational point requires defining a scheme as a functor.

Here's another (more concrete) definition of an elliptic curve.

Definition 1.6 (Elliptic curve)**.** An *elliptic curve* over k is an affine variety in $\mathbb{A}^2(k)$ cut out be a polynomial of the form

$$y^2 + a_1xy + a_3y^2 = x^3 + a_2x^2 + a_4x + a_6$$

with nonzero discriminant plus a point $\mathcal{O}$ at infinity.

Remark 1.7. Why are these equivalent? Well, the Riemann–Roch theorem approximately lets us take a smooth projective curve of genus 1 and then write it as an equation; the marked point goes to $\mathcal{O}$. In the reverse direction, one merely needs to embed our affine curve into projective space and verify its smoothness and genus.

Instead of working with affine varieties, we can also give a concrete description of an elliptic curve using projective varieties.

Definition 1.8 (Elliptic curve)**.** An *elliptic curve* over k is a projective variety in $\mathbb{P}^2(k)$ cut out be a polynomial of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with nonzero discriminant.

We get the equivalence of the previous two definitions via the embedding $\mathbb{A}_2(k) \hookrightarrow \mathbb{P}^2(k)$ by $(x, y) \mapsto [x : y : 1]$; the point at infinity $\mathcal{O}$ is $[0 : 1 : 0]$.

1.1.4 Crackpot Varieties

In order to motivate schemes, we should probably mention varieties, so we will spend some time in class discussing affine and projective varieties. For convenience, we work over an algebraically closed field k.

Definition 1.9 (Affine space)**.** Given a field k, we define *affine* n*-space over* k, denoted $\mathbb{A}^n(k)$. An *affine variety* is a subset $Y \subseteq \mathbb{A}^n(k)$ of the form

$$Y = V(S) := \{p \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } f \in S\},$$

where $S \subseteq k[x_1, \ldots, x_n]$.

Remark 1.10. The set $S \subseteq k[x_1, \ldots, x_n]$ in the above definition need not be finite or countable. In certain cases, we can enforce this condition; for example, if $n = 1$, then $k[x]$ is a principal ideal domain, so we may force $\#S = 1$.

Note that we have defined vanishing sets $V(S)$ from subsets $S \subseteq k[x_1, \ldots, x_n]$. We can also go from vanishing sets to subsets.

Definition 1.11. Fix a field k and subset $Y \subseteq \mathbb{A}^n(k)$. Then we define the ideal

$$I(Y) := \{f \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } p \in Y\}.$$

Remark 1.12. One should check that this is an ideal, but we won't bother.

So we've defined some geometry. But we're in an algebraic geometry class; where's the algebra?

Theorem 1.13 (Hilbert's Nullstellensatz)**.** Fix an algebraically closed field k and ideal $J \subseteq k[x_1, \ldots, x_n]$. Then

$$I(V(J)) = \operatorname{rad} I,$$

where $\operatorname{rad} I$ is the radical of I.

Remark 1.14. The Nullstellensatz has no particularly easy proof.

The point of this result is that it ends up giving us a contravariant equivalence of posets of radical ideals and affine varieties.

Why do we care? In some sense, we prefer to work with ideals because it "remembers" more information than merely the points on the variety. To see this, note that elements $f \in k[x_1, \ldots, x_n]$ we are viewing as giving functions on $\mathbb{A}^n(k)$. However, when we work on a variety $Y \subseteq \mathbb{A}^n(k)$, then sometimes two functions will end up being identical on Y. So the correct ring of functions on Y is

$$k[x_1, \ldots, x_n]/I(Y),$$

so indeed keeping track of the (algebraic) ideal $V(Y)$ gets us some extra (geometric) information.

We will use this discussion as a jumping-off point to discuss affine schemes and then schemes. Affine schemes will have the following data.

- A commutative ring A, which we should think of as the ring of functions on a variety.
- A topological space $\operatorname{Spec} A$, which has more information than merely points on the variety.
- A structure sheaf of functions on $\operatorname{Spec} A$.

Remark 1.15. Our topological space $\operatorname{Spec} A$ will contain more points than just the points on the variety. In some sense, these extra points make the topology more apparent.

Remark 1.16. Going forward, one might hope to remove requirements that the field k is algebraically closed (e.g., to work with a general ring) or talk about ideals which are not radical. This is the point of scheme theory.

1.2 August 26

Let's finish up talking varieties, and then we'll move on to affine schemes.

1.2.1 Projective Varieties

We're going to briefly talk about projective varieties. Let's start with projective space.

Definition 1.17 (Projective space)**.** Given a field k, we define *projective n-space over k*, denoted $\mathbb{P}^n(k)$ as

$$\frac{k^{n+1} \setminus \{(0, \ldots, 0\}}{\sim},$$

where $\sim$ assigns two points being equivalent if and only if they span the same 1-dimensional subspace of k^{n+1}. We will denote the equivalence class of a point $(a_0, \ldots, a_n)$ by $[a_0 : \ldots : a_n]$.

To work with varieties, we don't quite cut out by general polynomials but rather by homogeneous polynomials.

Definition 1.18 (Projective variety)**.** Given a field k and a set of some homogeneous polynomials $T \subseteq k[x_1, \ldots, x_n]$, we define the *projective variety* cut out by T as

$$V(T) := \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in T\}.$$

Example 1.19. The elliptic curve corresponding to the affine algebraic variety in $\mathbb{A}^2(k)$ cut out by $y^2 - x^3 - 1$ becomes the projective variety in $\mathbb{P}^2(k)$ cut out by

$$Y^2Z - X^3 - Z^3 = 0.$$

Remark 1.20. One can give projective varieties some Zariski topology as well, which we will define later in the class.

What to remember about projective varieties is that we can cover $\mathbb{P}^2(k)$ (say) by affine spaces as

$$\begin{aligned}\mathbb{P}^2(k) &= \{[X:Y:Z] : X,Y,Z \in k \text{ not all } 0\} \\ &= \{[X:Y:Z] : X,Y,Z \in k \text{ and } X \neq 0\} \\ &\quad \cup \{[X:Y:Z] : X,Y,Z \in k \text{ and } Y \neq 0\} \\ &= \{[1:y:z] : y,z \in k\} \\ &\quad \cup \{[x:1:z] : x,z \in k\} \\ &\simeq \mathbb{A}^2(k) \cup \mathbb{A}^2(k).\end{aligned}$$

The point is that we can decompose $\mathbb{P}^2(k)$ into an affine cover.

Example 1.21. Continuing from Example 1.19, we can decompose $Z\left(Y^2Z - X^3 - Z^3\right)$ into having an affine open cover by

$$\underbrace{\left\{(x,y) : y^2 - x^3 - 1 = 0\right\}}_{z\neq 0} \cup \underbrace{\left\{(x,z) : z - x^3 - z^3 = 0\right\}}_{y\neq 0} \cup \underbrace{\left\{(y,z) : y^2z - 1 - z^3 = 0\right\}}_{x\neq 0}.$$

Notably, we get almost everything from just one of the affine chunks, and we get the point at infinity by taking one of the other chunks.

Remark 1.22. It is a general fact that we only need two affine chunks to cover our projective curve.

1.2.2 The Spectrum

The definition of a(n affine) scheme requires a topological space and its ring of functions. We will postpone talking about the ring of functions until we discuss sheaves, so for now we will focus on the space.

Definition 1.23 (Spectrum)**.** Given a ring A, we define the *spectrum*

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}.$$

Example 1.24. Fix a field k. Then $\operatorname{Spec} k = \{(0)\}$. Namely, non-isomorphic rings can have homeomorphic spectra.

Exercise 1.25. Fix a field k. We show that

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(\pi) : \pi \text{ is monic, irred.}, \deg \pi > 0\}.$$

Proof. To begin, note that (0) is prime, and (π) is prime for irreducible non-constant polynomials π because irreducible elements are prime in principal ideal domains. Additionally, we note that all the given primes are distinct: of course (0) is distinct from any prime of the form (π), but further, given monic non-constant irreducible polynomials α and β, having

$$(\alpha) = (\beta)$$

forces $\alpha = c\beta$ for some $c \in k[x]^\times$. But $k[x]^\times = k^\times$, so $c \in k^\times$, so $c = 1$ is forced by comparing the leading coefficients of α and β.

It remains to show that all prime ideals $\mathfrak{p} \subseteq k[x]$ take the desired form. Well, $k[x]$ is a principal ideal domain, so we may write $\mathfrak{p} = (\pi)$ for some $\pi \in k[x]$. If $\pi = 0$, then we are done. Otherwise, $\deg \pi \geq 0$, but $\deg \pi > 0$ because $\deg \pi = 0$ implies $\pi \in k[x]^\times$. By adjusting by a unit, we may also assume that π is monic. And lastly, note that (π) is prime means that π is prime, so π is irreducible. ■

Example 1.26. If k is an algebraically closed field, then the only nonconstant irreducible polynomials are linear (because all nonconstant polynomials have a root and hence a linear factor), and of course any linear polynomial is irreducible. Thus,

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(x - \alpha) : \alpha \in k\}.$$

Set $\mathfrak{m}_\alpha := (x - \alpha)$ so that $\alpha \mapsto \mathfrak{m}_\alpha$ provides a natural map from $\mathbb{A}^1_k$ to $\operatorname{Spec} k[x]$. In this way we can think of $\operatorname{Spec} k[x]$ as $\mathbb{A}^1_k$ with an extra point (0).

Remark 1.27. Continuing from Example 1.26, observe that we can also recover function evaluation at a point $\alpha \in \mathbb{A}^1_k$: given $f \in k[x]$, the value of $f(\alpha)$ is the image of f under the canonical map

$$k[x] \twoheadrightarrow \frac{k[x]}{\mathfrak{m}_\alpha} \cong k,$$

where the last map is the forced $x \mapsto \alpha$. Observe running this construction at the point $(0) \in \operatorname{Spec} k[x]$ makes the "evaluation" map just the identity.

Example 1.28. Similar to $k[x]$, we can classify $\operatorname{Spec} \mathbb{Z}$: all ideals are principal, so our primes look like (p) where $p = 0$ or is a rational prime. Namely, essentially the same proof gives

$$\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) : p \text{ prime}, p > 0\}.$$

The condition $p > 0$ is to ensure that all the points on the right-hand side are distinct; certainly we can write all nonzero primes $(p) \subseteq \mathbb{Z}$ for some nonzero (p), and we can adjust p by a unit to ensure $p > 0$. Conversely, $(p) = (q)$ with $p, q > 0$ forces $p \mid q$ and $q \mid p$ and so $p = q$.

We might hope to have a way to view $\operatorname{Spec} k[x]$ as points even when k is not algebraically closed.

Example 1.29. Set $k = \mathbb{Q}$. There is a map sending a nonconstant monic irreducible polynomial $\pi \in \mathbb{Q}[x]$ to its roots in $\overline{\mathbb{Q}}$, and note that this map is injective because one can recover a polynomial from its roots. Further, all the roots of π are Galois conjugate because π is irreducible, and a Galois orbit S_α of a root α corresponds to the polynomial

$$\pi(x) = \prod_{\beta \in S_\alpha} (x - \beta),$$

where $\pi(x) \in \mathbb{Q}[x]$ because its coefficients are preserved the Galois action. Thus, there is a bijection between the nonconstant monic irreducible polynomials $\pi \in \mathbb{Q}[x]$ and Galois orbits of elements in $\overline{\mathbb{Q}}$.

So far, all of our examples have been "dimension 0" (namely, a field k) or "dimension 1" (namely, $\mathbb{Z}$ and $k[x]$). Here is an example in dimension 2.

Exercise 1.30. Let k be algebraically closed. Any $\mathfrak{p} \in \operatorname{Spec} k[x, y]$ is one of the following types of prime.

- Dimension 2: $\mathfrak{p} = (0)$.
- Dimension 1: $\mathfrak{p} = (f(x, y))$ where f is nonconstant and irreducible.
- Dimension 0: $\mathfrak{p} = (x - \alpha, y - \beta)$, where $\alpha, \beta \in k$.

Proof. We follow [Vak17, Exercise 3.2.E]. If $\mathfrak{p} = (0)$, then we are done. If $\mathfrak{p}$ is principal, then we can write $\mathfrak{p} = (f)$ where $f \in k[x,y]$ is a prime element and hence irreducible. Observe that if f is irreducible, then f is also a prime element because $k[x,y]$ is a unique factorization domain.

Lastly, we suppose that $\mathfrak{p}$ is not principal. We start by finding $f, g \in \mathfrak{p}$ with no nonconstant common factors. Because $\mathfrak{p} \neq 0$, we can find $f_0 \in \mathfrak{p} \setminus \{0\}$, and assume that (f_0) is maximal with respect to this (namely, $f_0 \notin (f_0')$ for any $f_0' \in \mathfrak{p}$). Because $\mathfrak{p}$ is not principal, we can find $g_0 \in \mathfrak{p} \setminus (f_0)$. Now, we can use unique prime factorization of f_0 and g_0 to find some $d \in k[x,y]$ such that

$$f_0 = fd \qquad \text{and} \qquad g_0 = gd$$

where f and g share no common factors. (Namely, $\nu_\pi(d) = \min\{\nu_\pi(f_0), \nu_\pi(g_0)\}$ for all irreducible factors $\pi \in k[x,y]$.) Note $d \notin \mathfrak{p}$ by the maximality of f_0, so $f, g \in \mathfrak{p}$ is forced.

Continuing, embedding f and g into $k(x)[y]$ and using the Euclidean algorithm there, we can write

$$af + bg = 1$$

where $a, b \in k(x)[y]$, because f and g have no common factors in $k(x)[y]$. (Any common factor would lift to a common factor in $k[x,y]$.[1]) Clearing denominators, we see that we can find $h(x) \in k[x] \cap \mathfrak{p}$, but by factoring $h(x)$ using the fact that k is algebraically closed, we see that we can actually enforce $(x - \alpha) \in \mathfrak{p}$ for some $\alpha \in k$.

By symmetry, we can force $(y - \beta) \in \mathfrak{p}$ for some $\beta \in \mathfrak{p}$ as well, so $(x - \alpha, y - \beta) \subseteq \mathfrak{p}$. However, we see that $(x - \alpha, y - \beta)$ is maximal because of the isomorphism

$$\frac{k[x,y]}{(x - \alpha, y - \beta)} \to k$$

by $x \mapsto \alpha$ and $y \mapsto \beta$. Thus, $\mathfrak{p} = (x - \alpha, y - \beta)$ follows. ■

Remark 1.31. The intuition behind Exercise 1.30 is that the prime ideal $(x - \alpha, y - \beta)$ "cuts out" the zero-dimensional point $(\alpha, \beta) \in \mathbb{A}^2_k$. Then the prime ideal (f) cuts out some one-dimensional curve in $\mathbb{A}^2_k$, and the prime ideal (0) cuts out the entire two-dimensional plane. We have not defined dimension rigorously, but hopefully the idea is clear.

Remark 1.32. It is remarkable that the number of equations we need to cut out a variety of dimension d is $2 - d$. This is not always true.

The point is that we seem to have recovered $\mathbb{A}^1_k$ by looking at $\operatorname{Spec} k[x]$ and $\mathbb{A}^2_k$ by looking at $\operatorname{Spec} k[x,y]$, so we can generalize this to arbitrary rings cleanly, realizing some part of Remark 1.16.

Definition 1.33 (Affine space)**.** Given a ring R, we define *affine n-space over R* as

$$\mathbb{A}^n_R \coloneqq \operatorname{Spec} R[x_1, \ldots, x_n].$$

So far all the rings we've looked at so far have been integral domains, but it's worth pointing out that working with general rings allows more interesting information.

Example 1.34. We classify $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$. Notably, all prime ideals here must correspond to prime ideals of $k[\varepsilon]$ containing $\left(\varepsilon^2\right)$ and hence contain $\operatorname{rad}\left(\varepsilon^2\right) = (\varepsilon)$, which allows only the prime (ε). (We will make this correspondence precise later.) So $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ has a single point.

[1] If $d(x,y)/e(x)$ divides both f and g in $k(x)[y]$, where d and e share no common factors, then $d \mid fe, ge$ in $k[x,y]$. Unique prime factorization now forces $d \mid f, g$ in $k[x,y]$.

Remark 1.35. In some sense, $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ will be able to let us talk about differential information algebraically: ε is some very small nonzero element such that $\varepsilon^2 = 0$. So we can study a "function" $f \in k[x]$ locally at a point p by studying $f(p+\varepsilon)$. Rigorously, $f(x) = \sum_{i=0}^d a_i x^i$ has

$$f(x+\varepsilon) = \sum_{i=0}^d a_i (x+\varepsilon)^i = \sum_{i=0}^d a_i x^i + \sum_{i=1}^d i a_i x^{i-1}\varepsilon = f(x) + f'(x)\varepsilon.$$

One can recover more differential information by looking at $k[\varepsilon]/\left(\varepsilon^n\right)$ for larger n.

1.2.3 The Zariski Topology

Thus far we've defined our space. Here's our topology.

Definition 1.36 (Zariski topology)**.** Fix a ring A. Then, for $S \subseteq A$, we define the *vanishing set*

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\}$$

Then the *Zariski topology* on $\operatorname{Spec} A$ is the topology whose closed sets are the $V(S)$.

Intuitively, we are declaring A as the (continuous) functions on $\operatorname{Spec} A$, and the evaluation of the function $f \in A$ at the point $\mathfrak{p} \in \operatorname{Spec} A$ is $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27). Then the vanishing sets of a continuous function must be closed, and without easy access to any other functions on $\operatorname{Spec} A$, we will simply declare that these are all of our closed sets.

In the affine case, we can be a little more rigorous.

Example 1.37. Set $A := k[x_1, \ldots, x_n]$, where k is algebraically closed. Then, given $f \in k[x_1, \ldots, x_n]$, we want to be convinced that $V(\{f\})$ matches up with the affine k-points $(a_1, \ldots, a_n)$ which vanish on f. Well, $(a_1, \ldots, a_n)$ corresponds to the prime ideal $(x_1 - a_1, \ldots, x_n - a_n) \in \operatorname{Spec} A$, and

$$\{f\} \subseteq (x_1 - a_1, \ldots, x_n - a_n)$$

is equivalent to f vanishing in the evaluation map

$$k[x_1, \ldots, x_n] \twoheadrightarrow \frac{k[x_1, \ldots, x_n]}{(x_1 - a_1, \ldots, x_n - a_n)} \to k,$$

which is equivalent to $f(a_1, \ldots, a_n) = 0$. So indeed, f vanishes on $(a_1, \ldots, a_n)$ if and only if the corresponding maximal ideal is in $V(\{f\})$.

With intuition out of the way, we should probably check that the sets $V(S)$ make a legitimate topology. To begin, here are some basic properties.

Lemma 1.38. Fix a ring A.

(a) If subsets $S, T \subseteq A$ have $S \subseteq T$, then $V(T) \subseteq V(S)$.

(b) A subset $S \subseteq A$ has $V(S) = V((S))$.

(c) An ideal $\mathfrak{a} \subseteq A$ has $V(\mathfrak{a}) = V(\operatorname{rad} I)$.

Proof. We go in sequence.

(a) Note $\mathfrak{p} \in V(T)$ implies that $T \subseteq \mathfrak{p}$, which implies $S \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(S)$.

(b) Surely $S \subseteq (S)$, so $V((S)) \subseteq V(S)$. Conversely, if $\mathfrak{p} \in V(S)$, then $S \subseteq \mathfrak{p}$, but then the generated ideal (S) must also be contained in $\mathfrak{p}$, so $\mathfrak{p} \in V((S))$.

(c) Surely $\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a}$, so $V(\operatorname{rad} \mathfrak{a}) \subseteq V(I)$. Conversely, if $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{p} \subseteq \mathfrak{a}$, so

$$\mathfrak{p} \subseteq \bigcap_{\mathfrak{q} \supseteq \mathfrak{a}} \mathfrak{q} = \operatorname{rad} \mathfrak{a},$$

so $\mathfrak{p} \in V(\operatorname{rad} \mathfrak{a})$. ■

Remark 1.39. In light of (b) and (c) of Lemma 1.38, we can actually write all closed subsets of $\operatorname{Spec} A$ as $V(\mathfrak{a})$ for a radical ideal $\mathfrak{a}$. We will use this fact freely.

And here are our checks.

Lemma 1.40. Fix a ring A.

(a) $V(A) = \varnothing$ and $V((0)) = \operatorname{Spec} A$.

(b) Given ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$, then $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(c) Given a collection of ideals $\mathcal{I} \subseteq \mathcal{P}(A)$, we have

$$\bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a}) = V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right).$$

Proof. We go in sequence.

(a) All primes are proper, so no prime $\mathfrak{p}$ has $A \subseteq \mathfrak{p}$, so $V(A) = \varnothing$. Also, 0 is an element of all ideals, so all $\mathfrak{p} \in \operatorname{Spec} A$ have $(0) \subseteq \mathfrak{p}$, so $V((0)) = \operatorname{Spec} A$.

(b) Note $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$, so $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ follows. Conversely, take $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, and suppose $\mathfrak{p} \notin V(\mathfrak{a})$ so that we need $\mathfrak{p} \in V(\mathfrak{b})$. Well, $\mathfrak{p} \notin V(\mathfrak{a})$ implies $\mathfrak{a} \not\subseteq \mathfrak{p}$, so we can find $a \in \mathfrak{a} \setminus \mathfrak{p}$. Now, for any $b \in \mathfrak{b}$, we see

$$ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p},$$

so $a \notin \mathfrak{p}$ forces $b \in \mathfrak{p}$. Thus, $\mathfrak{b} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{b})$.

(c) Certainly any $\mathfrak{b} \in \mathcal{I}$ has $\mathfrak{b} \subseteq \sum_{\mathfrak{a} \in \mathcal{I}} I$, so $V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right) \subseteq \bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a})$ follows.

Conversely, suppose $\mathfrak{p} \in \bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a})$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{a} \in \mathcal{I}$, so $\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a} \subseteq \mathfrak{p}$ follows. Thus, $\mathfrak{p} \in V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right)$. ■

Remark 1.41. For ideals $I, J \subseteq A$, note that $IJ \subseteq I \cap J$. Additionally, $I \cap J \subseteq \operatorname{rad}(IJ)$: if $f \in I \cap J$, then $f^2 \in (I \cap J)^2 \subseteq IJ$. It follows from Lemma 1.38 that

$$V(IJ) \supseteq V(I \cap J) \supseteq V(\operatorname{rad}(IJ)) = V(IJ),$$

so $V(I) \cup V(J) = V(IJ) = V(I \cap J)$. So V does respect some poset structure.

It follows that the collection of vanishing sets is closed under finite union and arbitrary intersection, so they do indeed specify the closed sets of a topology.

1.2.4 Easy Nullstellensatz

While we're here, let's also generalize Definition 1.11 in the paradigm that $\operatorname{Spec} A$ is the analogue for affine space.

Definition 1.42. Fix a ring A. Then, given a subset $Y \subseteq \operatorname{Spec} A$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Remark 1.43. To see that this is the correct definition, note we want $f \in I(Y)$ if and only if f vanishes at all points $\mathfrak{p} \in Y$. We said earlier that the value of f at $\mathfrak{p}$ should be $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27), so f vanishes at $\mathfrak{p}$ if and only if $f \in \mathfrak{p}$. So we want

$$I(Y) = \{f \in A : f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in Y\} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

As before, we'll write in a few basic properties of I.

Lemma 1.44. Fix a ring A, and fix subsets $X, Y \subseteq \operatorname{Spec} A$.

(a) If $X \subseteq Y$, then $I(Y) \subseteq I(X)$.

(b) The ideal $I(X)$ is radical.

Proof. We go in sequence.

(a) Note

$$I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = I(X).$$

(b) Suppose that $f^n \in I(X)$ for some positive integer n, and we need to show $f \in I(X)$. Then $f^n \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, so $f \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, so $f \in I(X)$. ■

And here is our nice version of Theorem 1.13.

Proposition 1.45. Fix a ring A.

(a) Given an ideal $\mathfrak{a} \subseteq A$, we have $I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$.

(b) Given a subset $X \subseteq \operatorname{Spec} A$, we have $V(I(X)) = \overline{X}$.

(c) The functions V and I provide an inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. We go in sequence.

(a) Observe

$$I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} = \operatorname{rad} \mathfrak{a}.$$

(b) Using Lemma 1.40, we find

$$\overline{X} = \bigcap_{V(\mathfrak{a}) \supseteq X} V(\mathfrak{a}) = V\left(\sum_{V(\mathfrak{a}) \supseteq X} \mathfrak{a}\right).$$

Now, $X \subseteq V(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in X$, which is equivalent to $\mathfrak{a} \subseteq I(X)$. Thus,

$$\overline{X} = V\left(\sum_{\mathfrak{a} \subseteq I(X)} \mathfrak{a}\right) = V(I(X)).$$

(c) Note that V sends (radical) ideals to closed subsets of $\operatorname{Spec} A$ by the definition of the Zariski topology. Also, I sends (closed) subsets of $\operatorname{Spec} A$ to radical ideals by Lemma 1.44. Additionally, for a closed subset $X \subseteq \operatorname{Spec} A$, we have

$$V(I(X)) = \overline{X} = X,$$

and for a radical ideal $\mathfrak{a}$, we have

$$I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a} = \mathfrak{a},$$

so I and V are in fact mutually inverse. ■

Remark 1.46. Given $X \subseteq \operatorname{Spec} A$, we claim $I(X) = I(\overline{X})$. Well, these are both radical ideals, so it suffices by Proposition 1.45 (c) to show $V(I(X)) = V(I(\overline{X}))$, which is clear because these are both $\overline{X}$.

Remark 1.47. Intuitively, what makes proving Proposition 1.45 so much easier than Theorem 1.13 is that we've added extra points to our space in order to track varieties better.

1.2.5 Some Continuous Maps

As a general rule, we will make continuous maps between our spectra by using ring homomorphisms. Here is the statement.

Lemma 1.48. Given a ring homomorphism $\varphi\colon A \to B$, the pre-image function $\varphi^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$ induces a continuous function $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$.

Proof. We begin by showing $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is well-defined: given a prime $\mathfrak{q} \subseteq \operatorname{Spec} B$, we claim that $\varphi^{-1}\mathfrak{q}$ is a prime in $\operatorname{Spec} A$. Well, if $ab \in \varphi^{-1}\mathfrak{q}$, then $\varphi(a)\varphi(b) \in \mathfrak{q}$, so $\varphi(a) \in \mathfrak{q}$ or $\varphi(b) \in \mathfrak{q}$. So indeed, $\varphi^{-1}\mathfrak{q}$ is prime.

We now show that $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous. It suffices to show that the pre-image of a closed set $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ under φ^{-1} is a closed set. For concreteness, we will make $\operatorname{Spec}\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ our pre-image map so that we want to show $(\operatorname{Spec}\varphi)^{-1}(V(\mathfrak{a}))$ is closed. Well,

$$\begin{aligned}(\operatorname{Spec}\varphi)^{-1}(V(\mathfrak{a})) &= \{\mathfrak{q} \in \operatorname{Spec} B : (\operatorname{Spec}\varphi)(\mathfrak{q}) \in V(\mathfrak{a})\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{a} \subseteq (\operatorname{Spec}\varphi)(\mathfrak{q})\} \\ &= \left\{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}\right\}.\end{aligned}$$

Now, if $\mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}$, then any $a \in \mathfrak{a}$ has $\varphi(a) \in \mathfrak{q}$, so $\varphi(\mathfrak{a}) \subseteq \mathfrak{q}$. Conversely, if $\varphi(\mathfrak{a}) \subseteq \mathfrak{q}$, then any $a \in \mathfrak{a}$ has $\varphi(a) \in \mathfrak{q}$ and hence $a \in \varphi^{-1}\mathfrak{q}$, so $\mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}$ follows. In total, we see

$$(\operatorname{Spec}\varphi)^{-1}(V(\mathfrak{a})) = \{\mathfrak{q} \in \operatorname{Spec} B : \varphi(\mathfrak{a}) \subseteq \mathfrak{q}\} = V(\varphi(\mathfrak{a})),$$

which is closed. ■

In fact, we have defined a (contravariant) functor.

Proposition 1.49. The mapping Spec sending rings A to topological spaces $\operatorname{Spec} A$ and ring homomorphisms $\varphi\colon A \to B$ to continuous maps $\operatorname{Spec}\varphi = \varphi^{-1}$ assembles into a functor $\operatorname{Spec}\colon \operatorname{Ring}^{\mathrm{op}} \to \operatorname{Top}$.

Proof. Thus far our data is sending objects to objects and morphisms to (flipped) morphisms, so we just need to run the functoriality checks.

- Identity: note that $\operatorname{Spec}\operatorname{id}_A$ sends a prime $\mathfrak{p} \in \operatorname{Spec} A$ to

$$(\operatorname{Spec}\operatorname{id}_A)(\mathfrak{p}) = \operatorname{id}_A^{-1}(\mathfrak{p}) = \{a \in A : \operatorname{id}_A a \in \mathfrak{p}\} = \mathfrak{p},$$

  so indeed, $\operatorname{Spec}\operatorname{id}_A = \operatorname{id}_{\operatorname{Spec} A}$.

- Functoriality: given morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, as well as a prime $\mathfrak{r} \in \operatorname{Spec} C$, we compute

$$\begin{aligned}(\operatorname{Spec}(\psi \circ \varphi))(\mathfrak{r}) &= (\psi \circ \varphi)^{-1}(\mathfrak{r}) \\ &= \{a \in A : \psi(\varphi(a)) \in \mathfrak{r}\} \\ &= \{a \in A : \varphi(a) \in (\operatorname{Spec}\psi)(\mathfrak{r})\} \\ &= \{a \in A : a \in (\operatorname{Spec}\varphi)((\operatorname{Spec}\psi)(\mathfrak{r}))\} \\ &= (\operatorname{Spec}\varphi \circ \operatorname{Spec}\psi)(\mathfrak{r}).\end{aligned}$$

  So indeed, $\operatorname{Spec}(\psi \circ \varphi) = \operatorname{Spec}\varphi \circ \operatorname{Spec}\psi$. ■

Here is a quick example.

Definition 1.50 (k-points)**.** Given a ring A and field k, a *k-point* of $\operatorname{Spec} A$ is a ring homomorphism $\iota\colon A \to k$.

Remark 1.51. To see that Definition 1.50 does indeed cut out a single point, note $\iota\colon A \to k$ induces $\operatorname{Spec}\iota\colon \operatorname{Spec} k \to \operatorname{Spec} A$ and therefore picks out a single point of $\operatorname{Spec} A$ because $\operatorname{Spec} k = \{(0)\}$.

Remark 1.52. To see that Definition 1.50 is reasonable, let $A = k[x_1, \ldots, x_n]$ so that $\operatorname{Spec} A = \mathbb{A}_k^n$. Then a map $\iota\colon A \to k$ is determined by $a_i := \iota(x_i)$, so we expect this ι to correspond to the point $(a_1, \ldots, a_n)$. Indeed, Remark 1.51 says we should compute

$$(\operatorname{Spec}\iota)((0)) = \iota^{-1}((0)) = \ker\iota = (x_1 - a_1, \ldots, x_n - a_n),$$

which does indeed correspond to the point $(a_1, \ldots, a_n)$.

Here is a more elaborate example: closed subsets can be realized as spectra themselves!

Exercise 1.53. Fix a ring A and ideal $\mathfrak{a} \subseteq A$. Letting $\pi\colon A \twoheadrightarrow A/\mathfrak{a}$ be the natural projection, we have that

$$\operatorname{Spec}\pi\colon \operatorname{Spec} A/\mathfrak{a} \to V(\mathfrak{a})$$

is a homeomorphism.

Proof. To be more explicit, we claim that the maps

$$\begin{aligned}\operatorname{Spec} A/\mathfrak{a} &\cong V(\mathfrak{a}) \\ \mathfrak{q} &\mapsto \pi^{-1}\mathfrak{q} \\ \pi(\mathfrak{p}) &\leftarrow\!\shortmid \mathfrak{p}\end{aligned}$$

are continuous inverses. Here are our well-definedness and continuity checks.

- That $\mathfrak{q} \mapsto \pi^{-1}\mathfrak{q}$ is continuous follows from Lemma 1.48. Note $\pi^{-1}\mathfrak{q}$ contains $\mathfrak{a}$ because any $a \in \mathfrak{q}$ has $\pi(a) = [0]_\mathfrak{a} \in \mathfrak{q}$.
- For any $\mathfrak{p}$ containing $\mathfrak{a}$, we need to show that $\pi(\mathfrak{p})$ is prime. Of course, if $\mathfrak{p}$ is proper, then $\pi(\mathfrak{p})$ is proper as well. For the primality check, note $[a]_\mathfrak{a} \cdot [b]_\mathfrak{a} \in \pi(\mathfrak{p})$ implies $ab \in \mathfrak{p} + \mathfrak{a} = \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, so $[a]_\mathfrak{a} \in \mathfrak{p}$ or $[b]_\mathfrak{a} \in \mathfrak{p}$.
- To show that $\mathfrak{p} \mapsto \pi\mathfrak{p}$ is continuous, note that a closed set $V(\overline{S}) \subseteq \operatorname{Spec} A/\mathfrak{a}$ has pre-image
$$\pi^{-1}(V(\overline{S})) = \{\mathfrak{p} : \pi\mathfrak{p} \supseteq \overline{S}\}.$$
Now, set $S = \pi^{-1}(\overline{S})$. Now $\pi\mathfrak{p} \supseteq \overline{S}$ if and only if each $a \in S$ has $\pi(a) \in \pi\mathfrak{p}$, which is equivalent to $a \in \mathfrak{a} + \mathfrak{p} = \mathfrak{p}$. Thus,
$$\pi^{-1}(V(\overline{S})) = V(S),$$
which is closed.

Here are our inverse checks.

- Given $\mathfrak{p} \in V(\mathfrak{a})$, note
$$\pi^{-1}(\pi\mathfrak{p}) = \{a \in A : \pi(a) \in \pi\mathfrak{p}\} = \{a \in A : a \in \mathfrak{a} + \mathfrak{p}\} = \mathfrak{a} + \mathfrak{p} = \mathfrak{p}.$$
- Given $\mathfrak{q} \in \operatorname{Spec} A/\mathfrak{a}$, note
$$\pi\left(\pi^{-1}\mathfrak{q}\right) = \pi\left(\{a \in A : \pi(a) \in \mathfrak{q}\}\right).$$
Because $\pi\colon A \twoheadrightarrow A/\mathfrak{a}$ is surjective, the output here is just $\mathfrak{q}$. ■

A similar story exists for open sets, but we must be more careful. Here are our open sets.

Definition 1.54 (Distinguished open sets)**.** Given a ring A and element $f \in A$, we define the *distinguished open set*
$$D(f) := (\operatorname{Spec} A) \setminus V(\{f\}) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}.$$

Intuitively, these are the points on which f does not vanish.

Remark 1.55. In fact, the distinguished open sets form a base: any open set takes the form $(\operatorname{Spec} A) \setminus V(S)$ for some $S \subseteq A$, so we write
$$(\operatorname{Spec} A) \setminus V(S) = \{\mathfrak{p} : S \not\subseteq \mathfrak{p}\} = \bigcup_{f \in S} \{\mathfrak{p} : f \notin \mathfrak{p}\} = \bigcup_{f \in S} D(f).$$

Here is our statement.

Exercise 1.56. Fix a ring A and element $f \in A$. Letting $\iota\colon A \to A_f$ be the localization map,
$$\operatorname{Spec} \iota\colon \operatorname{Spec} A_f \to D(f)$$
is a homeomorphism.

Proof. The arguments here are analogous to Exercise 1.53. To be explicit, we will say that our maps are
$$\begin{aligned} \operatorname{Spec} A_f &\cong D(f) \\ \mathfrak{q} &\mapsto \iota^{-1}\mathfrak{q} \\ \mathfrak{p}A_f &\leftarrow\!\shortmid \mathfrak{p} \end{aligned}$$
and reassure the reader that the checks are fairly routine. For example, the inverse checks are done in [Eis95, Proposition 2.2]. ■

Remark 1.57. Not every open set is a distinguished open set. For example, taking k algebraically closed,

$$\mathbb{A}_k^2 \setminus \{(0,0)\} \subseteq \mathbb{A}_k^2$$

is an open set not in the form $D(f)$; equivalently, we need to show $V(\{f\}) \neq \{(x,y)\}$ for any $f \in k[x,y]$. Intuitively, this is impossible because a curve cuts out a one-dimensional variety of $\mathbb{A}_k^2$, not a zero-dimensional point.

Rigorously, we are requiring $f \in k[x,y]$ to have $f \in \mathfrak{p}$ if and only if $\mathfrak{p} = (x,y)$. However, f is certainly nonzero and nonconstant, so f has an irreducible factor π, which means that $f \in (\pi)$, where (π) is prime because $k[x,y]$ is a unique factorization domain.

1.3 August 29

Today we talk about the structure sheaf. To review, so far we have defined the spectrum $\operatorname{Spec} A$ of a ring A and given it a topology. The goal for today is to define its structure sheaf. Here is a motivating example.

Example 1.58. Set $A := \mathbb{C}[x_1, \ldots, x_n]$ so that $\operatorname{Spec} A = \mathbb{A}_k^n$. Recall that $\{D(f)\}_{f \in A}$ is a base for the Zariski topology, and we would like the functions on this ring to be A_f, the rational polynomials which allow some f in the denominator. In other words, these are rational functions on $\mathbb{C}^n$ whose poles are allowed on $V(\{f\})$ only.

1.3.1 Sheaves

Sheaves are largely a topological object, so we will forget that we are interested in the Zariski topology for now. Throughout, X will be a topological space.

Notation 1.59. Given a topological space X, we let $\operatorname{Op} X$ denote the poset (category) of its open sets.

Namely, the objects of $\operatorname{Ob} X$ are open sets, and

$$\operatorname{Mor}(V,U) = \begin{cases} \{*\} & V \subseteq U, \\ \varnothing & \text{else.} \end{cases}$$

Here is our definition.

Definition 1.60 (Presheaf)**.** A *presheaf* $\mathcal{F}$ on a topological space X valued in a category $\mathcal{C}$ is a contravariant functor $\mathcal{F} \colon (\operatorname{Ob} X)^{\mathrm{op}} \to \mathcal{C}$. More concretely, $\mathcal{F}$ has the following data.

- Given an open set $U \subseteq X$, we have $\mathcal{F}(U) \in \mathcal{C}$.
- Given open sets $V \subseteq U \subseteq X$, we have a restriction map $\operatorname{res}_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$.

This data satisfies the following coherence conditions.

- Identity: given an open set $U \subseteq X$, $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(U)}$.
- Functoriality: given open sets $W \subseteq V \subseteq U$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\operatorname{res}_{U,V}"] \arrow[dr, "\operatorname{res}_{U,W}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,W}"] \\ & \mathcal{F}(W) \end{tikzcd}$$

Notation 1.61. We might call an element $f \in \mathcal{F}(U)$ a *section over* U.

Notation 1.62. Given $f \in \mathcal{F}(U)$, we might write $f|_V := \operatorname{res}_{U,V} f$.

Remark 1.63. In principle, one can have any target category $\mathcal{C}$ for our presheaf. However, we will only work Set, Ab, Ring, Mod_R in this class. In particular, we will readily assume that $\mathcal{C}$ is a concrete category.

Now that we've defined an algebraic object, we should discuss its morphisms.

Definition 1.64 (Presheaf morphism)**.** Fix a topological space X. A *presheaf morphism* between $\mathcal{F}$ and $\mathcal{G}$ is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$. In other words, for each open set $U \subseteq X$, we have a morphism $\eta_U\colon \mathcal{F}(U) \to \mathcal{F}(V)$; these morphisms make the following diagram commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"'] & \mathcal{G}(V) \end{tikzcd}$$

We've talked about presheaves a lot; where are sheaves?

Definition 1.65 (Sheaf)**.** Fix a topological space X. A presheaf $\mathcal{F}\colon (\operatorname{Ob} X)^{\mathrm{op}} \to \mathcal{C}$ is a *sheaf* if and only if it satisfies the following for any open set $U \subseteq X$ with an open cover $\mathcal{U}$.

- Identity: if $f_1, f_2 \in \mathcal{F}(U)$ have $f_1|_V = f_2|_V$ for all $V \in \mathcal{U}$, then $f_1 = f_2$.
- Gluability: if we have $f_V \in \mathcal{F}(V)$ for all $V \in \mathcal{U}$ such that
$$f_{V_1}|_{V_1 \cap V_2} = f_{V_2}|_{V_1 \cap V_2}$$
for all $V_1, V_2 \in \mathcal{U}$, then there is $f \in \mathcal{F}(U)$ such that $f|_V = f_V$ for all $V \in \mathcal{U}$.

Sheaves of functions will be our key example here. Intuitively, any type of function which can be determined "locally" will form a sheaf; for example, here are continuous functions.

Exercise 1.66. Fix topological spaces X and Y. For each $U \subseteq X$, let $\mathcal{F}(U)$ denote the set of continuous functions $f\colon U \to Y$, and equip these sets with the natural restriction maps. Then $\mathcal{F}$ is a sheaf.

Proof. To begin, here are the functoriality checks.

- Identity: for any $f \in \mathcal{F}(U)$, we have $f|_U = f$.
- Functoriality: if $W \subseteq V \subseteq U$, any $f \in \mathcal{F}(U)$ will have $(f|_V|_W)(w) = f(w) = (f|_W)(w)$ for any $w \in W$, so $f|_V|_W = f|_W$ follows.

Here are sheaf checks. Fix an open cover $\mathcal{U}$ of an open set $U \subseteq X$.

- Identity: suppose $f_1, f_2 \in \mathcal{U}$ have $f_1|_V = f_2|_V$ for all $V \in \mathcal{U}$. Now, for all $x \in U$, we see $x \in U_x$ for some $U_x \in \mathcal{U}$, so
$$f_1(x) = (f_1|_{U_x})(x) = (f_2|_{U_x})(x) = f_2(x),$$
so $f_1 = f_2$ follows.

- Gluability: suppose we have $f_V \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that $f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$ for each $V_1, V_2 \in \mathcal{U}$. Now, for each $x \in \mathcal{U}$, find $U_x \in \mathcal{U}$ with $x \in U_x$ and set

$$f(x) := f_{U_x}(x).$$

Note this is well-defined: if $x \in U_x$ and $x \in U_{x'}$, then $f_{U_x}(x) = f_{U_x}|_{U_x\cap U_x'}(x) = f_{U_x'}|_{U_x\cap U_x'}(x) = f_{U_x'}(x)$. Additionally, we see that, for each $V \in \mathcal{U}$ and $x \in V$, we have

$$f|_V(x) = f(x) = f_V(x)$$

by construction, so we are done.

Lastly, we need to check that f is continuous. Well, for any open set $V_0 \subseteq Y$, we can compute

$$f^{-1}(V_0) = \{x \in U : f(x) \in V_0\} = \bigcup_{V\in\mathcal{U}} \{x \in V : f(x) \in V_0\} = \bigcup_{V\in\mathcal{U}} \{x \in V : f_V(x) \in V_0\} = \bigcup_{V\in\mathcal{U}} f_V^{-1}(V_0),$$

which is open as the arbitrary union of open sets because $f_V : V \to Y$ is a continuous function. ■

In contrast, sheaves have trouble keeping track of "global" information.

Example 1.67. For each $U \subseteq \mathbb{R}$, let $\mathcal{F}(\mathbb{R})$ denote the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, and equip these sets with the natural restriction maps. Then $\mathcal{F}$ is not a sheaf: for each open set $(n-1, n+1)$ for $n \in \mathbb{N}$, the function $f_{(n-1,n+1)} := \mathrm{id}_{(n-1,n=1)}$ is bounded and continuous, but the glued function $f = \mathrm{id}_{\mathbb{R}}$ is not bounded on all of $\mathbb{R}$. (We glued using Exercise 1.66, which does force the definition of f.)

Remark 1.68. For most of our examples, the identity axiom is easily satisfied: intuitively, the identity axiom says that two sections are equal if and only if they agree locally. However, gluability is usually the tricky one: it requires us to build a function from local behavior.

Remark 1.69. Note that the section $f \in \mathcal{F}(U)$ promised by the gluability axiom is unique by the identity axiom.

Ok, so we've defined the sheaf as an algebraic object, so here are its morphisms.

Definition 1.70 (Sheaf morphism)**.** A *sheaf morphism* is a morphism of the (underlying) presheaves.

As an aside, we note that we can succinctly write the sheaf conditions in an exact sequence.

Lemma 1.71. Fix a topological space X and presheaf $\mathcal{F} : (\operatorname{Ob} X)^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{C}$ is an abelian category or Grp. Then $\mathcal{F}$ is a sheaf if and only if the sequence

$$\begin{array}{ccccc} 0 \to \mathcal{F}(U) & \to & \prod_{V\in\mathcal{U}} \mathcal{F}(V) & \to & \prod_{V_1,V_2\in\mathcal{U}} \mathcal{F}(V_1\cap V_2) \\ f & \mapsto & (f|_V)_{V\in\mathcal{U}} & & \\ & & (f_V)_{V\in\mathcal{U}} & \mapsto & (f_{V_1}|_{V_1\cap V_2} - f_{V_2}|_{V_1\cap V_2})_{V_1,V_2} \end{array} \tag{1.1}$$

is exact.

Proof. In one direction, suppose that $\mathcal{F}$ is a sheaf, and we will show that (1.1) is exact for any open cover $\mathcal{U}$ of an open set U.

- Exact at $\mathcal{F}(U)$: suppose $f_1, f_2 \in \mathcal{F}(U)$ have the same image in $\prod_{V\in\mathcal{U}} \mathcal{F}(V)$. This means that

$$f_1|_V = f_2|_V$$

for all $V \in \mathcal{U}$, so the identity axiom tells us that $f_1 = f_2$.

- Exact at $\prod_{V\in\mathcal{U}}\mathcal{F}(V)$: of course any $f\in\mathcal{F}(U)$ goes to $(f|_V)_{V\in\mathcal{U}}$, which goes to

$$f|_{V_1}|_{V_1\cap V_2}-f|_{V_2}|_{V_1\cap V_2}=f|_{V_1\cap V_2}-f|_{V_1\cap V_2}=0\in\prod_{V_1,V_2\in\mathcal{U}}\mathcal{F}(V_1\cap V_2)$$

  and therefore lives in the kernel. Conversely, suppose $(f_V)_{V\in\mathcal{U}}$ vanishes in $\prod_{V_1,V_2}\mathcal{F}(V_1\cap V_2)$. Rearranging, this means that

$$f_{V_1}|_{V_1\cap V_2}=f_{V_2}|_{V_1\cap V_2},$$

  so the gluability axiom tells us that we can find $f\in\mathcal{F}(U)$ such that $f|_V=f_V$. This finishes.

Conversely, suppose that $\mathcal{F}$ makes (1.1) always exact, and we will show that $\mathcal{F}$ is a sheaf. Fix an open cover $\mathcal{U}$ of an open set U.

- Identity: suppose that $f_1,f_2\in\mathcal{F}(U)$ have $f_1|_V=f_2|_V$ for any $V\in\mathcal{U}$. This means that f_1 and f_2 have the same image in $\prod_{V\in\mathcal{U}}\mathcal{F}(V)$, so the exactness of (1.1) at $\mathcal{F}(U)$ enforces $f_1=f_2$.
- Gluability: suppose that we have $f_V\in\mathcal{F}(V)$ for each $V\in\mathcal{U}$ in such a way that $f_{V_1}|_{V_1\cap V_2}=f_{V_2}|_{V_1\cap V_2}$ for all $V_1,V_2\in\mathcal{U}$. Then the image of $(f_V)_{V\in\mathcal{U}}$ in $\prod_{V_1,V_2\in\mathcal{U}}\mathcal{F}(V_1\cap V_2)$ is

$$(f_{V_1}|_{V_1\cap V_2}-f_{V_2}|_{V_1\cap V_2})_{V_1,V_2}=(0)_{V_1,V_2},$$

  so exactness of (1.1) forces there to be $f\in\mathcal{F}(U)$ such that $f|_V=f_V$ for each $V\in\mathcal{U}$. This finishes. ■

Remark 1.72. One might want to continue this left-exact sequence. To see this, we will have to talk about cohomology, which is a task for later in life.

1.3.2 Sheaf on a Base

In light of our sheaf language, we are trying to define a "structure" sheaf $\mathcal{O}_{\operatorname{Spec}A}$ on $\operatorname{Spec}A$, and we wanted to have

$$\mathcal{O}_{\operatorname{Spec}A}(D(f))=A_f.$$

We aren't going to be able to specify a presheaf with this data, but we can specify a sheaf. In some sense, the presheaf is unable to build up locally in the way that a sheaf can, so having the data on a base like $\{D(f)\}_{f\in A}$ need not be sufficient to define the full presheaf.

But as alluded to, we can do this for sheaves. We begin by defining a sheaf on a base.

Definition 1.73 (Sheaf on a base)**.** Fix a topological space X and a base $\mathcal{B}$ for its topology. Then a *sheaf on a base valued in* $\mathcal{C}$ is a contravariant functor $F\colon\mathcal{B}^{\mathrm{op}}\to\mathcal{C}$ satisfying the following identity and gluability axioms: for any $B\in\mathcal{B}$ with a basic cover $\{B_i\}_{i\in I}$, we have the following.

- Identity: if we have $f_1,f_2\in F(B)$ such that $f_1|_{B_i}=f_2|_{B_i}$ for all B_i, then $f_1=f_2$.
- Gluability: if we have $f_i\in F(B_i)$ for each i such that $f_i|_B=f_j|_B$ for each $B\subseteq B_i\cap B_j$, then there is $f\in F(B)$ such that $f|_{B_i}=f_i$ for each i.

Example 1.74. Given a topological space X and a base $\mathcal{B}$, any sheaf $\mathcal{F}\colon(\operatorname{Op}X)^{\mathrm{op}}\to\mathcal{C}$ "restricts" to a sheaf on a base $\mathcal{F}_{\mathcal{B}}$ by setting $\mathcal{F}_{\mathcal{B}}(B):=\mathcal{F}(B)$ for all $B\in\mathcal{B}$ and reusing the same restriction maps. The identity and gluability axioms follow from their (stronger) sheaf counterparts; checking this amounts writing down the axioms.

Morphisms are constructed in the obvious way.

Definition 1.75 (Sheaf on a base morphisms)**.** Fix a topological space X and a base $\mathcal{B}$ for its topology. Then a *morphism* between two sheaves F and G on the base $\mathcal{B}$ is a natural transformation of the (underlying) contravariant functors.

Example 1.76. Given a topological space X and a base $\mathcal{B}$, any sheaf morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ restricts in the obvious way to a morphism $\eta_{\mathcal{B}}\colon \mathcal{F}_{\mathcal{B}} \to \mathcal{G}_{\mathcal{B}}$ (namely, $(\eta_{\mathcal{B}})_B = \eta_B$) on the corresponding sheaves on a base. Checking this amounts to saying out loud that the diagram on the left commutes for any $B' \subseteq B$ because it is the same as the diagram on the right.

$$\begin{tikzcd}
\mathcal{F}_{\mathcal{B}}(B) \arrow[r, "(\eta_{\mathcal{B}})_B"] \arrow[d, "\operatorname{res}_{B,B'}"'] & \mathcal{G}_{\mathcal{B}}(B) \arrow[d, "\operatorname{res}_{B,B'}"] \\
\mathcal{F}_{\mathcal{B}}(B') \arrow[r, "(\eta_{\mathcal{B}})_{B'}"'] & \mathcal{G}_{\mathcal{B}}(B')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(B) \arrow[r, "\eta_B"] \arrow[d, "\operatorname{res}_{B,B'}"'] & \mathcal{G}(B) \arrow[d, "\operatorname{res}_{B,B'}"] \\
\mathcal{F}(B') \arrow[r, "\eta_{B'}"'] & \mathcal{G}(B')
\end{tikzcd}$$

We are interested in showing that we can build a sheaf from a sheaf on a base uniquely, but it will turn out to be fruitful to spend a moment to discuss how this behaves on morphisms first for the uniqueness part of this statement.

Lemma 1.77. Fix a topological space X with a base $\mathcal{B}$ for its topology. Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X with values in $\mathcal{C}$ and a morphism of the (underlying) sheaves on a base $\eta_{\mathcal{B}}\colon \mathcal{F}_{\mathcal{B}} \to \mathcal{G}_{\mathcal{B}}$, there is a unique sheaf morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ such that $(\eta_{\mathcal{B}})_B = \eta_B$ for each $B \in \mathcal{B}$.

Proof. We show uniqueness before existence.

- Uniqueness: fix any open $U \subseteq X$, and we will try to solve for $\eta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$. Well, fix a basic open cover $\mathcal{U}$ of U; then, for any $B \in \mathcal{U}$, we need the following diagram to commute.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,B}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,B}"] \\
\mathcal{F}(B) \arrow[r, "\eta_B=(\eta_{\mathcal{B}})_B"'] & \mathcal{G}(B)
\end{tikzcd}$$

  In particular, for any $f \in \mathcal{F}(U)$, we need $\eta_U(f)|_B = (\eta_{\mathcal{B}})_B(f|_B)$. Thus, $\eta_U(f)|_B$ is fully specified by the data provided by $\eta_{\mathcal{B}}$, so the identity axiom for $\mathcal{G}$ forces $\eta_U(f)$ to be unique.

- Existence: to begin, fix any open $U \subseteq X$, and we will define $\eta_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$. As alluded to above, we let $\mathcal{U}$ be the set of basis elements which are contained in U so that $\mathcal{U}$ is a (large) basic cover of U.

  Then, picking up $f \in \mathcal{F}(U)$, we will try to use the gluability axiom by setting $g_B := (\eta_{\mathcal{B}})_B(f|_B)$ for each $B \in \mathcal{U}$. In particular, for any $B, B' \in \mathcal{U}$, any basic $B_0 \subseteq B \cap B'$ has

$$(g_B|_{B\cap B'})|_{B_0} = g_B|_{B_0} = (\eta_{\mathcal{B}})_B(f|_B)|_{B_0} = (\eta_{\mathcal{B}})_{B_0}(f|_B|_{B_0}) = \eta_{B_0}(f|_{B_0}) = g_{B_0},$$

  which is also $(g_{B'}|_{B\cap B'})_{B_0}$ by symmetry, so the identity axiom applied to $B \cap B'$ implies $g_B|_{B\cap B'} = g_{B'}|_{B\cap B'}$. Thus, the gluability axiom applied to U gives us a unique $g \in \mathcal{G}(U)$ such that

$$g|_B = (\eta_{\mathcal{B}})_B(f|_B)$$

  for each basic set $B \subseteq U$. We define $\eta_U(f) := g$.

  It remains to show that η does in fact assemble into a sheaf morphism. Fix open sets $V \subseteq U$, and we need the following diagram to commute.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\
\mathcal{F}(V) \arrow[r, "\eta_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

Well, pick up any $f \in \mathcal{F}(U)$. Then, for any basic $B \subseteq V \subseteq U$, we see that

$$\eta_U(f)|_B = (\eta_\mathcal{B})_B(f|_B) = (\eta_\mathcal{B})_B(f|_V|_B),$$

so the uniqueness of $\eta_V(f|_V)$ forces $\eta_V(f|_V) = \eta_U(f)|_U$. This finishes. ■

1.3.3 Extending a Sheaf on a Base

We dedicate this subsection to the following result, describing how to extend a sheaf on a base to a full sheaf.

Proposition 1.78. Fix a topological space X with a base $\mathcal{B}$ for its topology. Given a sheaf on a base $F\colon \mathcal{B}^{\text{op}} \to \mathcal{C}$, there is a sheaf $\mathcal{F}$ and isomorphism (of sheaves on a base) $\iota\colon F \to \mathcal{F}_\mathcal{B}$ satisfying the following universal property: any sheaf $\mathcal{G}$ with a morphism (of sheaves on a base) $\varphi\colon F \to \mathcal{G}_\mathcal{B}$ has a unique sheaf morphism $\psi\colon \mathcal{F} \to \mathcal{G}$ making the following diagram commute.

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{F}_\mathcal{B} \arrow[d, "\psi_\mathcal{B}"] \\ & \mathcal{G}_\mathcal{B} \end{tikzcd} \tag{1.2}$$

Proof. We begin by providing a construction of $\mathcal{F}$. For each open set $U \subseteq X$, define

$$\mathcal{F}(U) \coloneqq \varprojlim_{B \subseteq U} F(B) = \left\{ (f_B)_{B\subseteq U} \in \prod_{B \subseteq U} F(B) : f_B|_{B'} = f_{B'} \text{ for each } B' \subseteq B \subseteq U \right\}.$$

(Namely, we are implicitly assuming that our target category has limits.) Observe that, when $V \subseteq U$, the natural surjection

$$\prod_{B\subseteq U} \mathcal{F}(B) \to \prod_{B \subseteq V} \mathcal{F}(B)$$

induces a map $\mathcal{F}(U) \to \mathcal{F}(V)$. Indeed, an element $(f_B)_{B\subseteq U} \in \mathcal{F}(U)$ gets sent to $(f_B)_{B\subseteq V}$, and it is still the case that $B' \subseteq B \subseteq V$ implies $f_B|_{B'} = f_{B'}$ because actually $B' \subseteq B \subseteq U$. Thus, $(f_B)_{B\subseteq V} \in \mathcal{F}(V)$, so we have a well-defined map

$$\begin{aligned} \operatorname{res}_{U,V}\colon \quad \mathcal{F}(U) &\to \mathcal{F}(V) \\ (f_B)_{B\subseteq U} &\mapsto (f_B)_{B\subseteq V} \end{aligned}$$

which will serve as our restrictions. We start by checking that these data assemble into a presheaf.

- When $U = V$, we are sending $(f_B)_{B\subseteq U} \in \mathcal{F}(U)$ to itself, so $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(I)}$.
- Given $W \subseteq V \subseteq U$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\operatorname{res}_{U,V}"] \arrow[dr, "\operatorname{res}_{U,W}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,W}"] \\ & \mathcal{F}(W) \end{tikzcd} \qquad \begin{tikzcd} (f_B)_{B\subseteq U} \arrow[r, mapsto] \arrow[dr, mapsto] & (f_B)_{B\subseteq V} \arrow[d, mapsto] \\ & (f_B)_{B\subseteq W} \end{tikzcd}$$

  commutes, which is our functoriality check.

We now show that these data make a sheaf. Fix an open set $U \subseteq X$ with an open cover $\mathcal{U}$. To help our constructions, given any open subset $V \subseteq X$, let $\mathcal{B}_V$ denote the collection of basis elements B contained in V; notably $\mathcal{B}_V$ is a basic cover for V. Then, for any open $U' \subseteq U$, we let

$$\mathcal{S}_{U'} \coloneqq \bigcup_{V \subseteq U} \mathcal{S}_{U'\cap V}.$$

Notably, $\mathcal{S}_{U'}$ is a basic cover for U' such that any $B \in \mathcal{S}_{U'}$ is contained in some element of $\mathcal{U}$.

- Identity: suppose that $(f_B)_{B\subseteq U}, (g_B)_{B\subseteq U} \in \mathcal{F}(U)$ restrict to the same element on any $V \in \mathcal{U}$. Now, fix any $B_0 \subseteq U$, and we will show $f_{B_0} = g_{B_0}$.

  Now consider $\mathcal{S}_{B_0}$: for each $B' \in \mathcal{S}$, we can find $V \in \mathcal{U}$ so that $B' \subseteq V$, for which we know
  $$(f_B)_{B\subseteq V} = (g_B)_{B\subseteq V}.$$
  In particular $f_{B_0}|_{B'} = f_{B'} = g_{B'} = g_{B_0}|_{B'}$ for any $B \in \mathcal{S}$, so the identity axiom for the sheaf on a base F forces $f_{B_0} = g_{B_0}$.

- Gluability: suppose we are given some $(f_{V,B})_{B\subseteq V} \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that
  $$(f_{V,B})_{B\subseteq V\cap V'} = (f_{V,B})_{B\subseteq V}|_{V\cap V'} = (f_{V',B})_{B\subseteq V}|_{V\cap V'} = (f_{V',B})_{B\subseteq V\cap V'}$$
  for any $V, V' \in \mathcal{U}$. In other words, for any basic $B \subseteq V \cap V'$, we have $f_{V,B} = f_{V',B}$.

  Now, for any basic $B_0 \subseteq U$, we will solve for f_{B_0}. Using $\mathcal{S}_{B_0}$, note that any $B \in \mathcal{S}_{B_0}$ has some $V_B \in \mathcal{U}$ such that $B \subseteq V_B$, so we will use $f_{V_B,B}$ at this point. Note that if $B \subseteq V_B'$ as well, then $f_{V_B,B} = f_{V_B',B}$, so our $f_{V_B,B}$ is independent of V_B. Continuing, if we have $B \subseteq B_1 \cap B_2$, then
  $$f_{V_{B_1},B_1}|_B = f_{V_{B_1},B} = f_{V_{B_2},B} = f_{V_{B_2},B_2}|_B,$$
  so gluability applied to our sheaf F on a base promises us a unique f_{B_0} such that $f_{B_0}|_B = f_{V_B,B}$ for any $B \in \mathcal{S}_{B_0}$.

  We now need to show that the $(f_B)_{B\subseteq U}$ assemble into an element of $\mathcal{F}(U)$. Namely, if we have $B_0' \subseteq B_0$, we need to show that $f_{B_0}|_{B_0'} = f_{B_0'}$. Well, for any $B \in \mathcal{S}_{B_0'}$, we compute
  $$f_{B_0}|_{B_0'}|_B = f_{B_0}|_B = f_{V_B,B} = f_{B_0}|_B,$$
  so the uniqueness of f_{B_0} gives the equality.

For our next step, we define $\iota_{B_0} \colon F(B) \to \mathcal{F}_{\mathcal{B}}(B_0)$ by
$$\iota_{B_0}(f) \coloneqq (f|_B)_{B\subseteq B_0}.$$
Here are the checks on ι.

- Well-defined: note $\iota_{B_0}(f)$ is an element of $\mathcal{F}_{\mathcal{B}}(B_0)$ because $B' \subseteq B \subseteq B_0$ will have $f|_B|_{B'} = f|_{B'}$.

- Natural: if $B \subseteq B'$, then note that the diagrams
  $$\begin{tikzcd} F(B_0) \arrow[r, "\iota_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{F}_{\mathcal{B}}(B_0) \arrow[d, "\mathrm{res}_{B,B'}"] \\ F(B_0') \arrow[r, "\iota_{B'}"'] & \mathcal{F}_{\mathcal{B}}(B_0') \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & (f|_B)_{B\subseteq B_0} \arrow[d, mapsto] \\ f|_{B_0'} \arrow[r, mapsto] & (f|_B)_{B\subseteq B_0'} \end{tikzcd}$$
  commute, finishing.

- Injective: suppose that $f, g \in F(B_0)$ have the same image in $\mathcal{F}_{\mathcal{B}}(B_0)$. This means that $(f|_B)_{B\subseteq B_0} = (g|_B)_{B\subseteq B_0}$, so $f = f|_{B_0} = g|_{B_0} = g$, so we are done.

- Surjective: fix some $(f_B)_{B\subseteq B_0} \in \mathcal{F}_{\mathcal{B}}(B_0)$. Notably, for any basic $B_1, B_2 \subseteq B_0$ with some basic $B \subseteq B_1 \cap B_2$, we have
  $$f_{B_1}|_B = f_B = f_{B_2}|_B,$$
  so gluability applied to F promises $f \in F(B_0)$ such that $f|_B = f_B$ for all basic $B \subseteq B_0$. So $\iota_{B_0}(f) = (f_B)_{B\subseteq B_0}$.

We now begin showing that $\mathcal{F}$ satisfies the universal property. Fix some sheaf $\mathcal{G}$ on X with a morphism $\varphi\colon F \to \mathcal{G}_\mathcal{B}$.

In light of Lemma 1.77, it suffices to show the existence and uniqueness of a morphism $\psi_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$ on the base $\mathcal{B}$ making (1.2) commute. Namely, the existence of $\psi_\mathcal{B}$ promises a full sheaf morphism $\psi\colon \mathcal{F} \to \mathcal{G}$ extending via Lemma 1.77; for uniqueness, two possible $\psi, \psi'\colon \mathcal{F} \to \mathcal{G}$ with $\psi_\mathcal{B}$ and $\psi'_\mathcal{B}$ both commuting will enforce $\psi_\mathcal{B} = \psi'_\mathcal{B}$ and then $\psi = \psi'$ by the uniqueness of Lemma 1.77.

Continuing with the proof, we note that the fact that ι is an isomorphism means that the commutativity of (1.2) is equivalent to the diagram

$$\begin{tikzcd} F & \mathcal{F}_\mathcal{B} \arrow[l, "\iota^{-1}"'] \arrow[d, "\psi_\mathcal{B}"] \\ & \mathcal{G}_\mathcal{B} \arrow[from=1-1, "\varphi"'] \end{tikzcd}$$

commuting. However, the commutativity of this diagram is equivalent to setting $\psi_\mathcal{B} := \varphi \circ \iota^{-1}$. Thus, uniqueness of $\psi_\mathcal{B}$ is immediate, and existence of $\psi_\mathcal{B}$ amounts to noting the composition of natural transformations remains a natural transformation. ■

Remark 1.79. The universal property implies that the pair $(\mathcal{F}, \iota)$ is unique up to unique isomorphism, for a suitable notion of unique isomorphism. Namely, the usual abstract nonsense arguments with universal properties is able to show that if we have another sheaf $\mathcal{F}'$ with isomorphism $\iota'\colon F \to \mathcal{F}'_\mathcal{B}$ satisfying the universal property, then $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic. (This isomorphism $\eta\colon \mathcal{F} \cong \mathcal{F}'$ is unique if we ask for the corresponding diagram

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \mathcal{F}_\mathcal{B} \arrow[d, "\eta_\mathcal{B}"] \\ & \mathcal{F}'_\mathcal{B} \end{tikzcd}$$

to commute.)

Remark 1.80. One can also define $\mathcal{F}(U)$ as compatible systems of stalks, but we have not defined stalks yet.

1.3.4 Affine Schemes (Finally!)

We are now ready to define the structure sheaf. We use Proposition 1.78. Note that $D(f) \subseteq D(g)$ implies $V(g) \subseteq V(f)$, so $I(V(g)) \supseteq I(V(f))$, so $f \in \operatorname{rad} g$, so $f^n = ag$ for some positive integer n. This implies that g is a unit in A_f, so there is a natural restriction map

$$\operatorname{res}_{D(g),D(f)}\colon A_g \to A_f.$$

That we have a presheaf on a base is then routine to check.

It remains to check our coherence properties. Fix an open cover

$$D(f) = \bigcup_{i \in \mathcal{I}} D(f_i),$$

which implies using the previous argument that $f_j^{n_j} = c_j f$ for some positive integer n_j and element $c_j \in A$, and in fact f^n for some large n is in the ideal generated by the f_j; one can check that this is equivalent. Now here are our checks.

- Identity: we may assume that $\mathcal{I}$ is finite because the equivalent condition at the end allows us to remove all but finitely many of the f_i.

Now, for identity, it suffices to pick up $s \in D(f)$ with $s|_{D(f_i)} = 0$ (in A_{f_i}) for all f_i, and we want to show $s = 0$. Namely, we know $f_i^{n_i} s = 0$ (in A_f) for some n_i everywhere. However, as stated above, we may write

$$f^n = \sum_{i \in \mathcal{I}} a_i f_i,$$

and then taking this equation to a very large power and multiplying through by s forces $f^n s = 0$.

- Gluability: we will check this next time.

The above data then assembles into our structure sheaf.

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We finish defining the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ today.

Remark 1.81. Let X be a topological space. One can write the conditions for F being a sheaf on a base $\mathcal{B}$ as the exactness of

$$0 \to F(B_0) \to \prod_{B \in \mathcal{U}} F(B) \to \prod_{B, B' \in \mathcal{U}} F(B \cap B'),$$

where $\mathcal{U}$ is a basic cover for B_0. Notably, the map into $F(B \cap B')$ is well-defined by using the gluability axiom.

Remark 1.82. One complaint about sheaves on a base is that we have to choose a base. To be more canonical, we will discuss stalks today, which treats all points the same.

1.4.1 Finishing the Structure Sheaf

Recall that we defined

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$$

with some restriction maps. We need to show that our data will assemble into a sheaf on a base, and from last class, we still need to check the gluability axiom for our structure sheaf. As usual, pick a basic cover

$$D(f) = \bigcup_{i \in \mathcal{I}} D(f_i).$$

We also give sections $s_i \in A_{f_i}$ which restrict properly.

For psychological reasons, we may assume $f = 1$ by replacing our ring A with A_f and the open set $D(f)$ with $\operatorname{Spec} A_f$. Notably, $D(f_i) = (A_f)_{f_i}$, so no information is really changing.

Also, we will freely take $\mathcal{I}$ to be finite. To begin, there is a finite subset $\mathcal{I}' \subseteq \mathcal{I}$ such that we still have a cover

$$\operatorname{Spec} A = \bigcup_{i \in \mathcal{I}'} D(f_i),$$

which essentially holds by some compactness result. More formally, $\{D(f_i)\}_{i \in \mathcal{I}}$ covering $\operatorname{Spec} A$ is equivalent to $(f_i)_{i \in \mathcal{I}} = A$, from which we can extract our finite subcover. Thus, we should be able to build up our section $s \in A$ from the finite subcover $\mathcal{I}'$, and then for any $i \in \mathcal{I} \setminus \mathcal{I}'$, we can build s' from the finite subcover from $\mathcal{I} \cup \{i\}$, and we get $s = s'$ from the identity axiom, from which

$$s|_{D(f_i)} = s'|_{D(f_i)} = s_i,$$

finishing.

Now, with $\mathcal{I}$ finite, we may find some very large n so that we can write

$$s_i = a_i/f_i^n \in A_{f_i}$$

for some very large n. The coherence among our sections is requiring that $s_i|_{D(f_i)\cap D(f_j)} = s_j|_{D(f_i)\cap D(f_j)}$, which means

$$(f_if_j)^m \left(f_j^n a_i - f_i^n a_j\right) = 0$$

for all i and j, for some very large and uniform m. Then setting N to be large enough, we can take very large powers everywhere so that we can write

$$s_i = a_i'/f_i^N$$

for a perhaps different a_i', where now we have $f_j^N a_i - f_i^N a_j = 0$ for all i and j.

Now, note that $D(f_i) = D\left(f_i^N\right)$, so we still have

$$\operatorname{Spec} A = \bigcup_{i\in\mathcal{I}} D\left(f_i^N\right),$$

so we can write

$$1 = \sum_{i\in\mathcal{I}} b_i f_i^N.$$

As an aside, this is, roughly speaking, a partition of unity to give functions on each of our affine open sets $D(f_i)$. As such, we set

$$a := \sum_{i\in\mathcal{I}} b_i a_i'$$

so that, for any $j \in \mathcal{J}$,

$$f_j^N a = \sum_{i\in\mathcal{I}} b_i f_j^N a_i' = \sum_{i\in\mathcal{I}} b_i f_i^N a_j' = a_j',$$

so $a = a_j'/f_j^N = s_j$ in A_{f_j}, which is what we wanted.

Having finished the last of our checks, we give our definition.

Definition 1.83 (Affine scheme)**.** Fix a ring A. An *affine scheme* is the topological space $\operatorname{Spec} A$ (given the Zariski topology) together with the sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ such that

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$$

for each $f \in A$.

1.4.2 Stalks

To define a morphism of schemes, we will want to discuss stalks.

Remark 1.84. We might expect a morphism of affine schemes to be merely a continuous map together with a natural transformation of the structure sheaves, but this will not be enough data. Namely, we want all of our morphisms of affine schemes to be induced by ring homomorphisms, which will require adding a little data to what our morphisms do.

The extra data in those morphisms will come from stalks.

Definition 1.85 (Stalk)**.** Fix a presheaf $\mathcal{F}$ on a topological space X. For a point $p \in X$, we define the *stalk of $\mathcal{F}$ at $p \in X$* to be the direct limit

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U).$$

Concretely, elements of $\mathcal{F}_p$ are ordered pairs (U, s) where $s \in \mathcal{F}(U)$ with $p \in U$, modded out by an equivalence relation $\sim$. Namely, $(U, s) \sim (U', s')$ if and only if there is $W \subseteq U \cap U'$ such that $s|_W = s'|_W$.

Definition 1.86 (Germ)**.** Fix a presheaf $\mathcal{F}$ on a topological space X. For a point $p \in X$ and section $f \in \mathcal{F}(U)$ with $p \in U$, the *germ of f at p* is the element

$$[(U, f)] \in \mathcal{F}_p.$$

Notation 1.87. I will write the germ of $f \in \mathcal{F}(U)$ at $p \in U$ as $f|_p$. This notation is not standard.

Here are some examples.

Lemma 1.88. Fix a ring A. Then $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} = A_\mathfrak{p}$.

Proof. Check this directly using the concrete description of our germs. In particular, denominators are allowed to be anything in $A \setminus \mathfrak{p}$, which is precisely what gives $A_\mathfrak{p}$. ■

Notably, $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ is always a local ring. This will be important.

Example 1.89. Let X be the topological space $\mathbb{C}$ (or any Riemann surface), and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions $X \to \mathbb{C}$. Then, for any $p \in X$, we have

$$\mathcal{O}_{X,p} = \left\{ \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ with positive radius of convergence} \right\}$$

essentially by complex analysis. Namely, we are using the fact that holomorphic functions are analytic.

Here is why we care about stalks.

Idea 1.90. Stalks remember everything about a sheaf.

Here is the rigorization of Idea 1.90.

Proposition 1.91. Fix presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X with a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$.

(a) For any $p \in X$, there is a natural map $\varphi_p \colon \mathcal{F}_p \to \mathcal{G}_p$.

(b) Suppose $\mathcal{F}$ is a sheaf. Then there is a natural embedding

$$\mathcal{F}(U) \hookrightarrow \prod_{p \in U} \mathcal{F}_p$$

taking $f \in \mathcal{F}(U)$ to its germs $(f|_p)_{p \in U}$. An element $(f_p)_{p \in U}$ is in the image if and only if, for each $p \in U$, there is an open set U_p containing p such that w can find $\widetilde{f_p} \in \mathcal{F}(U_p)$ such that all $q \in U_p$ have $f_q = (\widetilde{f_p})|_q$. Intuitively, we are saying that all stalks in a small neighborhood come from a single section.

(c) Suppose $\mathcal{G}$ is a sheaf. Given morphisms $\varphi_1, \varphi_2 \colon \mathcal{F} \to \mathcal{G}$ such that $(\varphi_1)_p = (\varphi_2)_p$ for all $p \in X$, we have $\varphi_1 = \varphi_2$.

Proof. We go in sequence.

(a) This follows from category theory. Alternatively, we can write this explicitly as being induced by

$$\varphi_p \colon [(U, s)] \mapsto [U, \varphi_U(s)].$$

We omit check that this is well-defined.

(b) The map is an embedding by the identity axiom. The classification of the image comes from the gluability axiom, roughly speaking.

(c) Fix an open set $U \subseteq X$ so that we need $(\varphi_1)_U = (\varphi_2)_U$. Now, the point is that any $\varphi\colon \mathcal{F} \to \mathcal{G}$ will make the following diagram commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\varphi"'] & \prod_{p\in U} \mathcal{F}_p \arrow[d, "\prod \varphi_p"] \\ \mathcal{G}(U) \arrow[r, hook] & \prod_{p\in U} \mathcal{G}_p \end{tikzcd}$$

Namely, the value of $\varphi_U(f)$ is uniquely determined by the top row of the diagram for all $f \in \mathcal{F}(U)$. ■

Remark 1.92. The sheaf conditions on (b) and (c) are unnecessary.

1.4.3 Morphisms Between Sheaves

We are going to want to do category theory on sheaves, so let's begin. Throughout our target category for our sheaves will be abelian (and concrete).

Remark 1.93. We will be able to show that the category of sheaves on an abelian category is an abelian category. However, we will not do this in detail because we are not sadistic.

Definition 1.94 (Presheaf kernel)**.** Given a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X, we define the *presheaf kernel* as

$$(\ker\varphi)(U) := \ker(\varphi_U)$$

for each $U \subseteq X$, where restriction maps are induced by $\mathcal{F}$. Then $\ker\varphi$ is our *presheaf kernel.*

Lemma 1.95. Given a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X, the presheaf kernel is a sheaf.

Proof. Omitted. ■

Having a kernel gives us a definition.

Definition 1.96 (Injective)**.** A morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if and only if the kernel presheaf $\ker\varphi$ is identically zero. Equivalently, we are asking for φ_U to be injective everywhere.

In our stalk philosophy, we might hope we can detect injectivity at stalks. Indeed, we can.

Proposition 1.97. Fix a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$. Given that $\mathcal{F}$ is a sheaf, we have that φ is

Proof. Use the embedding

$$\mathcal{F}(U) \hookrightarrow \prod_{p\in U} \mathcal{F}_p$$

for each $U \subseteq X$. ■

There is also the following result.

Proposition 1.98. Fix a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$. Then φ is an isomorphism if and only if φ_U is an isomorphism for all $U \subseteq X$ if and only if φ_p is an isomorphism for all $p \in X$.

Sketch. The hard part is showing that if we have isomorphisms at each stalk, then we have an isomorphism of full sheaves. We already know about injectivity, so let's focus on surjectivity. Well, for any $g \in \mathcal{G}(U)$, we get a system of compatible germs $(g|_p)_{p\in U}$, and because φ_p is an isomorphism, we may set

$$f_p \coloneqq \varphi_p^{-1}(f|_p).$$

Then we claim that f_p is a set of compatible germs, which gives rise to a section $f \in \mathcal{F}(U)$. Indeed, for each $p \in U$, we can find $U_p \subseteq U$ small enough so that $f_p \in \mathcal{F}(U_p)$ and $\varphi_{U_p}(f_p) = g|_{U_p}$. Then we simply have to use gluability directly to take these $f_p \in \mathcal{F}(U_p)$ to give our section; for this, we need to check

$$f_p|_{U_p\cap U_q} = f_q|_{U_p\cap U_q}.$$

However, $\varphi_{U_p\cap U_q}$ is injective, so it suffices to throw this through φ and check equality there. However, both of these equal

$$g|_{U_p\cap U_q}$$

when passed through φ, so we are safe. Note we used the injectivity of φ at the end here! ■

Remark 1.99. We are avoiding surjectivity for the moment because it is a little trickier. In particular, a morphism φ will be able to be surjective without being each φ_U being surjective. However, surjectivity will still be equivalent to surjectivity on the stalks.

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List of Definitions