Review for Midterm 1

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Abstract

This document condenses the major definitions and results from class and a couple extra things covered in the exercises.

Contents

1 Definitions

1.1 Basic Notions

Definition 1 (language)**.** A *language* $\mathcal{L}$ consists of the sets $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{C}$ of symbols. Here, $\mathcal{F}$ are functions, $\mathcal{R}$ are relations, and $\mathcal{C}$ are constants. Notably, there is an arity function $n\colon (\mathcal{F} \cup \mathcal{R}) \to \mathbb{N}$.

Definition 2 (structure)**.** Fix a language $\mathcal{L}$. Then an $\mathcal{L}$*-structure* $\mathcal{M}$ consists of the following data.

- Domain: a nonempty set M.
- Functions: for each $f \in \mathcal{F}$, there is a function $f^{\mathcal{M}}\colon M^{n(f)} \to M$.
- Relations: for each $R \in \mathcal{R}$, there is a relation $R^{\mathcal{M}} \subseteq M^{n(r)}$.
- Constants: for each $c \in \mathcal{C}$, there is a constant $c^{\mathcal{M}} \in M$.

The various $(-)^{\mathcal{M}}$ data are called *interpretations*.

Definition 3 (homomorphism, embedding, isomorphism)**.** Fix a language $\mathcal{L}$. Then an $\mathcal{L}$*-homomorphism* $\eta\colon \mathcal{M} \to \mathcal{N}$ of $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ is a one-to-one map $\eta\colon M \to N$ preserving the interpretations, as follows.

- Functions: for each $f \in \mathcal{F}$, we have $\eta \circ f^{\mathcal{M}} = f^{\mathcal{N}} \circ \eta^{n(f)}$.
- Relations: for each $R \in \mathcal{R}$, if $\overline{m} \in R^{\mathcal{M}}$, then $\eta^{n(R)}(m) \in R^{\mathcal{N}}$.
- Constants: for each $c \in \mathcal{C}$, we have $\eta\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$.

If $\eta\colon M \to N$ is one-to-one and the relations condition is an equivalence, then η is an $\mathcal{L}$*-embedding*. If $\eta\colon M \to N$ is the identity $M \subseteq N$, then we say that $\mathcal{M}$ is an $\mathcal{L}$-substructure. In addition, if η is onto, then η is an $\mathcal{L}$*-isomorphism*.

Definition 4 (term)**.** Let $\mathcal{L}$ be a language. The set of $\mathcal{L}$*-terms* is the smallest set $\mathcal{T}$ satisfying the following.

- Constants: for each $c \in \mathcal{C}$, we have $c \in \mathcal{T}$.
- Variables: $x_i \in \mathcal{T}$ for each $i \in \mathbb{N}$. Notably, we have only countably many variables.
- Functions: if $t_1, \ldots, t_n \in \mathcal{T}$ where $n = n(f)$ for some $f \in \mathcal{F}$, then $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Given an $\mathcal{L}$-structure $\mathcal{M}$ and term $t \in \mathcal{T}$ with variables $x_1, \ldots, x_n$ and elements $a_1, \ldots, a_n \in M$, we define $t^{\mathcal{M}}(\overline{a})$ in the obvious way.

Definition 5 (formula)**.** The set of $\mathcal{L}$*-formulae* is the smallest set satisfying the following.

- Any atomic $\mathcal{L}$-formula φ is an $\mathcal{L}$-formula.
- For any $\mathcal{L}$-formulae φ and ψ, then $\neg\varphi$ and $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\mathcal{L}$-formulae.
- For any variable v_i for $i \in \mathbb{N}$, then $\exists v_i \varphi$ is an $\mathcal{L}$-formula.

Definition 6 (sentence)**.** Fix a language $\mathcal{L}$. An $\mathcal{L}$-formula with no free variables is a *sentence*.

Definition 7 (truth)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Further, fix an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and a tuple $\overline{a} \in M^n$. Then we define *truth* as $\mathcal{M} \vDash \varphi(\overline{a})$ to mean that φ is true upon plugging in $\overline{a}$, where our definition is inductive on atomic formulae as follows.

- $\mathcal{M} \vDash (t_1 = t_2)(\overline{a})$ if and only if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.
- $\mathcal{M} \vDash R(t_1, \ldots, t_n)$ if and only if $\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_2^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$.

We define truth inductively on formulae now as follows.

- $\mathcal{M} \vDash (\varphi \wedge \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ and $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash (\varphi \vee \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ or $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash \neg\varphi(\overline{a})$ if and only if we do not have $\mathcal{M} \vDash \varphi(\overline{a})$.
- $\mathcal{M} \vDash \exists v \varphi(\overline{a}, v)$ if and only if there exists $b \in M$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$.

In this case, we say that $\mathcal{M}$ *satisfies, models, etc.* $\varphi(\overline{a})$ and so on.

Definition 8 (definable)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $B \subseteq M$. Then a subset $X \subseteq M^n$ is B-*definable* if and only if there is a formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_k)$ and tuple $\overline{b} \in B^k$ such that $\overline{a} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{a}, \overline{b})$. The tuple $\overline{b}$ might be called the *parameters*. We may abbreviate M-definable to simply *definable*.

Definition 9 (algebraic closure, definable closure)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $A \subseteq M$.

- The *definable closure* $\operatorname{dcl}(A)$ of A is the set of all $b \in M$ such that there is a formula $\varphi(\overline{x}, y)$ and $\overline{a} \in A$ such that
$$\{b' \in M : \mathcal{M} \vDash \varphi(\overline{a}, b')\}$$
is the set $\{b\}$.
- The *algebraic closure* $\operatorname{dcl}(A)$ of A is the set of all $b \in M$ such that there is a formula $\varphi(\overline{x}, y)$ and $\overline{a} \in A$ such that
$$\{b' \in M : \mathcal{M} \vDash \varphi(\overline{a}, b')\}$$
is a finite set containing $\{b\}$.

1.2 Theories

Definition 10 (theory)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Then the *theory* $\operatorname{Th}_{\mathcal{L}}(\mathcal{M})$ of $\mathcal{M}$ is the set of all sentences φ such that $\mathcal{M} \vDash \varphi$.

Definition 11 (elementary equivalence)**.** Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. Then we say that $\mathcal{M}$ and $\mathcal{N}$, written $\mathcal{M} \equiv \mathcal{N}$, are *elementarily equivalent* if and only if $\operatorname{Th}(\mathcal{M}) = \operatorname{Th}(\mathcal{N})$.

Definition 12 (elementary substructure)**.** Fix a language $\mathcal{L}$ and two structures $\mathcal{M}$ and $\mathcal{N}$. Then we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, written $\mathcal{M} \leq \mathcal{N}$ if and only if $\mathcal{M}$ is a substructure of $\mathcal{N}$ and $\mathcal{M}_M \equiv \mathcal{N}_M$.

Definition 13 (theory)**.** Fix a language $\mathcal{L}$. Then an $\mathcal{L}$-*theory* T is a set of $\mathcal{L}$-sentences. For an $\mathcal{L}$-structure $\mathcal{M}$, we say that $\mathcal{M}$ *models* T, written $\mathcal{M} \vDash T$, if and only if $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathcal{M}$. We let $\operatorname{Mod}(T)$ denote the class of all models $\mathcal{M}$ of T, and we call it an *elementary class*.

Definition 14 (logically implies)**.** Fix a language $\mathcal{L}$ and theory T. Then we say that T *logically implies* a sentence φ, written $T \vDash \varphi$, if and only if any $\mathcal{L}$-structure $\mathcal{M}$ modelling T has $\mathcal{M} \vDash \varphi$.

Definition 15 (diagram)**.** Fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$. The *diagram* $\operatorname{Diag}(\mathcal{M})$ is the set φ of atomic $\mathcal{L}_M$-sentences (in the expanded language $\mathcal{L}_M$) or negations of atomic sentences such that $\mathcal{M} \vDash \varphi$. The *elementary diagram* $\operatorname{elDiag} \mathcal{M}$ is the theory $\operatorname{Th}_{\mathcal{L}_M}(\mathcal{M}_M)$.

1.3 Adjectives for Theories

Definition 16 (satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *satisfiable* if and only if it has a model $\mathcal{M}$.

Definition 17 (finitely satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *finitely satisfiable* if and only if any finite subset of T is satisfiable.

Definition 18 (witness)**.** Fix a theory T of a language $\mathcal{L}$. Then T has *witnesses* (or *Henkin constants*) if and only if each formula $\varphi(x)$ in one free variable x has a constant symbol c such that $\exists x \varphi(x) \to \varphi(c)$ lives in T.

Definition 19 (Skolem functions)**.** An $\mathcal{L}$-theory T has *built-in Skolem functions* if and only if any $\mathcal{L}$-formula $\varphi(\overline{x}, y)$ has a function symbol f_φ such that

$$\forall \overline{x}((\exists y\, \varphi(\overline{x}, y)) \to \varphi(\overline{x}, f_\varphi(\overline{x}))).$$

The theory T has *definable Skolem functions* if and only if any $\mathcal{L}$-formula $\varphi(\overline{x}, y)$ has a function f with definable graph satisfying the above property.

Definition 20 (κ-categorical)**.** A theory T of a language $\mathcal{L}$ is κ-*categorical* if and only if T has exactly one isomorphism class of models of cardinality κ.

Definition 21 (complete)**.** An $\mathcal{L}$-theory T is *complete* if and only if $T \vDash \varphi$ or $T \vDash \neg\varphi$ for any $\mathcal{L}$-sentence φ.

Definition 22 (model-complete)**.** A theory T is *model-complete* if and only if any chain of models $\mathcal{M} \subseteq \mathcal{N}$ of models of T is in fact an elementary embedding.

Definition 23 (strongly minimal)**.** A theory T is *strongly minimal* if and only if any definable subset of any model of T is either finite or cofinite.

Definition 24 (o-minimal)**.** A theory T of ordered sets is o-*minimal* if and only if T, restricted to the language $\{<\}$, is DLO, and all definable subsets of any model of T is a finite union of points and intervals.

1.4 Ultraproducts

Definition 25 (filter)**.** Fix a set I. Then a *filter* $\mathcal{F}$ on I is a subset of $\mathcal{P}(I)$ satisfying the following.

(a) $I \in \mathcal{F}$.

(b) Finite intersection: for $X, Y \in \mathcal{F}$, we have $X \cap Y \in \mathcal{F}$.

(c) Containment: if $X \in \mathcal{F}$ and $Y \subseteq I$ contains X, then $Y \in \mathcal{F}$ also.

Definition 26 (ultrafilter)**.** Fix a set I. Then an *ultrafilter* $\mathcal{F}$ on I is a nontrivial filter on I such that each subset $X \subseteq I$ has one of $X \in \mathcal{F}$ or $I \setminus X \in \mathcal{F}$. Equivalently, $\mathcal{U}$ is maximal among the partially ordered set of nontrivial filters on I, ordered by inclusion.

Remark 27. For any nontrivial filter $\mathcal{F}$ on a set I, there exists an ultrafilter $\mathcal{U}$ containing $\mathcal{F}$.

Definition 28 (ultraproduct)**.** Fix a language $\mathcal{L}$ and some $\mathcal{L}$-structures $\{\mathcal{M}_\alpha\}_{\alpha \in I}$. The *ultraproduct* is the $\mathcal{L}$-structure defined as follows.

- The universe M is $\prod_{\alpha \in I} M_\alpha$ modded out by the equivalence relation $\sim$ given by $(a_\alpha) \sim (b_\alpha)$ if and only if
$$\{\alpha \in I : a_\alpha = b_\alpha\} \in \mathcal{U}.$$
- Functions are interpreted component-wise.
- For an n-ary relation R, $R^{\mathcal{M}}((a_{1\alpha}), \dots, (a_{n\alpha}))$ if and only if the set of α such that $R^{M_\alpha}(a_{1\alpha}, \dots, a_{n\alpha})$ is in $\mathcal{U}$.

1.5 Ehrenfeucht–Fraïssé games

Definition 29 (unnested)**.** An atomic $\mathcal{L}$-formula φ is *unnested* if and only if it takes one of the following forms.

- Equalities: $t_i = t_j$ or $x_i = c$ where the $t_\bullet$ are variables or constants.
- Relations: $R(t_1, \dots, t_n)$ where the $t_\bullet$ are variables or constants.
- Functions: $f(t_1, \dots, t_n) = t_{n+1}$ where the $t_\bullet$ are variables or constants.

Definition 30. Fix a language $\mathcal{L}$ with two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, and we fix a natural number n. The *Ehrenfeucht–Fraïssé game* $EF_n(\mathcal{A}, \mathcal{B})$ of length n is played as follows.

- Player I picks $\mathcal{A}$ or $\mathcal{B}$ and chooses some $a_1 \in A$ or $b_1 \in B$. Then Player II chooses an element $b_1 \in B$ or $a_1 \in A$ from the opposite universe to the one Player I chose.
- Then the above move is repeated until we have two n-tuples $(a_1, \dots, a_n)$ or $(b_1, \dots, b_n)$.
- Player II wins if, for any unnested atomic formula $\psi(x_1, \dots, x_n)$, we have $\mathcal{A} \vDash \psi(\overline{a})$ is equivalent to $\mathcal{B} \vDash \psi(\overline{b})$. Otherwise, Player I wins.

1.6 Cell Decomposition

Definition 31 (cell)**.** Fix a model $\mathcal{R}$ of an o-minimal theory T. Then a *cell* is defined as follows.

- A 0-cell is a point.
- A 1-cell in $\mathcal{R}$ is a set of the form (a, b) where $-\infty \le a < b \le \infty$.
- From n, an $(n+1)$-cell in $\mathcal{R}^{n+1}$ is a set of one of the following forms.
  - We can have
    $$\{(x_1, \ldots, x_n, y) : (x_1, \ldots, x_n) \in X \text{ and } y = f(x_1, \ldots, x_n)\}$$
    where $X \subseteq \mathcal{R}^n$ is an n-cell and $f \colon X \to \mathcal{R}$ is continuous and definable.
  - We can have $(-\infty, f)_X$ or $(f, g)_X$ or $(g, \infty)_X$ where
    $$(f, g)_X \coloneqq \{(x_1, \ldots, x_n, y) : f(\overline{x}) < y < f(\overline{y})\}$$
    where X is an n-cell and $f, g \colon X \to \mathcal{R}$ is continuous and definable with $f(\overline{x}) < g(\overline{x})$ always (where $(-\infty, f)_X$ and $(g, \infty)_X$ are defined analogously).
  - Lastly, we can have all of $\mathcal{R}^n$.

2 Examples

Example 32. Any finite structure can be aximoatized by a single $\mathcal{L}$-formula. The point is to write down explicitly what all the interpretations are.

Example 33. Let T be any theory in any language (such that $<$ is definable) with $\mathbb{N} \models T$. Then $\mathbb{N}$ has arbitrarily large elements, so compactness produces a model of T which is elementarily superstructure to $\mathbb{N}$ but with an element larger than any element of $\mathbb{N}$.

Example 34. The class of torsion groups is not elementarily definable in the language $\mathcal{L} = \{e, *\}$ of groups. The idea is that torsion groups can have elements of arbitrarily large order, so any theory T containing every torsion group as a model will also have as a model of

Example 35. The theory DLO of dense linear orders is complete, $\aleph_0$-categorical, not $\aleph_1$-categorical, and eliminates quantifiers. This theory is o-minimal.

Example 36. The theory DAG of divisible abelian groups eliminates quantifiers, is not $\aleph_0$-categorical, but it is κ-categorical for any $\kappa \geq \aleph_1$. This theory is strongly minimal.

Example 37. The theory ACF is not complete (though ACF_p is), κ-categorical for any infinite κ, and eliminates quantifiers.

Example 38. The theory of discrete linear orders without endpoints is complete (by the Ehrenfeucht–Fraïssé game) but not κ-categorical for any infinite κ.

Example 39. The theory Tor_2 of 2-torsion groups is κ-categorical for any infinite κ (because it has finite models) but not complete. In contrast, the theory of infinite 2-torsion groups is complete.

Example 40. Let $\mathcal{U}$ be a non-principal ultrafilter on the set $\mathcal{P}$ of primes. Then we have a field isomorphism

$$\mathbb{C} \cong \prod_{\mathcal{U}} \overline{\mathbb{F}_p}.$$

Example 41. The theory RCF of real closed fields eliminates quantifiers and is thus o-minimal.

Example 42. The theory ODAG of ordered divisible abelian groups eliminates quantifiers and is thus o-minimal.

Example 43. The theory of sets with infinitely many equivalence classes of size 2 and 3 (and all classes have this size) does not eliminate quantifiers, but it does eliminate quantifiers after adding predicates corresponding to the size of the equivalence class. This theory is $\aleph_0$-categorical but not $\aleph_1$-categorical.

Example 44. The theory of sets with infinitely many equivalence classes all of infinite size is $\aleph_0$-categorical, but it is not κ-categorical for any $\kappa \geq \aleph_1$. This theory eliminates quantifiers.

3 Theorems

We begin by listing some quick implications and coherence checks between our definitins.

- Finitely satisfiable implies satisfiable by compactness.
- If T is finitely axiomatizable, then there is a finite subset of T axiomatizing T.
- A theory T is $\forall$-axiomatizable if and only if it goes down substructures.
- A theory T is $\forall\exists$-axiomatizable if and only if any chain of models $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots$ has its union a model of T.
- If a theory T is κ-categorical for infininite κ and has no finite models, then T is complete.
- If T eliminates quantifiers, and there is a common substructure to any model of T, then T is complete.
- If T eliminates quantifiers, then T is model-complete.
- If T is model-complete (e.g., T eliminates quantifiers), then T is $\forall\exists$-axiomatizable.
- If T eliminates quantifiers, and $\mathcal{L}$ has no relation symbols, then T is strongly minimal.

3.1 Building Models

Theorem 45 (compactness)**.** Fix a language $\mathcal{L}$ and theory T. If T is finitely satisfiable, then T is satisfiable. Furthermore, T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}| + \aleph_0$.

Theorem 46 (Łoś)**.** Fix a language $\mathcal{L}$ and $\mathcal{L}$-structures $\{\mathcal{M}_\alpha\}_{\alpha\in I}$, and expand $\mathcal{L}$ to the language $\mathcal{L}' := \mathcal{L}_{\prod_{\alpha\in I} M_\alpha}$. Now, let $\mathcal{U}$ be an ultrafilter on I so that $\mathcal{M} := \prod_{\mathcal{U}} M_\alpha$ is an $\mathcal{L}'$-structure. Then for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ has $\mathcal{M} \vDash \varphi\left(a_1^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}}\right)$ if and only if

$$\{\alpha \in I : \mathcal{M}_\alpha \vDash \varphi(a_1, \ldots, a_n)\} \in \mathcal{U}.$$

Lemma 47 (Tarski–Vaught test)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and a subset $A \subseteq M$. The following are equivalent.

- There is an elementary substructure $\mathcal{A} \leq \mathcal{M}$ with universe A.
- Any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y)$ and n-tuple $\overline{a} \in A^n$ has $\mathcal{M} \vDash \exists y\, \varphi(\overline{a}, y)$ if and only if there is some $b \in A$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$.

Theorem 48 (Löwenheim–Skolem)**.** Fix a language $\mathcal{L}$ and infinite structure $\mathcal{M}$.

- Downward: For all subsets $A \subseteq M$, there exists an elementary substructure $\mathcal{N} \leq \mathcal{M}$ containing A with $|N| = |A| + |\mathcal{L}| + \aleph_0$.
- Upward: For any cardinal $\kappa \geq |M| + |\mathcal{L}|$, there exists an $\mathcal{L}$-structure $\mathcal{N}$ with cardinality κ and $\mathcal{M} \leq \mathcal{N}$.

3.2 Analyzing Structure

Proposition 49. Fix an $\mathcal{L}$-theory T which is κ-categorical for cardinality κ. If T has only infinite models, then T is complete.

Proposition 50. Fix a finite language $\mathcal{L}$. For any structures $\mathcal{A}$ and $\mathcal{B}$, Player II has a winning strategy in the $EF_n(\mathcal{A}, \mathcal{B})$ game for all $n > 0$ if and only if $\mathcal{A} \vDash \psi$ is equivalent to $\mathcal{B} \vDash \psi$ for all sentences ψ.

Theorem 51 (cell decomposition)**.** Fix a model $\mathcal{R}$ of an o-minimal theory T.

(a) Given a finite collection $X_1, \ldots, X_m \subseteq \mathcal{R}^n$ of definable subsets, then there is a cell decomposition $\mathcal{C}$ of $\mathcal{R}^n$ such that each $X_\bullet$ is a union of some of these cells.

(b) Any definable function $f : \mathcal{R}^n \to \mathcal{R}$ is piecewise continuous. In other words, there is a cell decomposition $\mathcal{C}$ of $\mathcal{R}^n$ such that f is continuous upon restriction to each cell.

Can I use this?

Theorem 52. Fix an $\mathcal{L}$-theory T and an $\mathcal{L}$-formula $\varphi(\overline{x})$. The following are equivalent.

- There is a quantifier-free formula $\psi(\overline{x}, y)$ such that $T \vDash \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(y))$. (This y is only needed when $\mathcal{L}$ has no constant symbols.)
- If $\mathcal{M}$ and $\mathcal{N}$ are models of T with a common substructure $\mathcal{A}$ of T, then for any $\overline{a} \in \mathcal{A}$, we have $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi(\overline{a})$.

Corollary 53. Let T be an $\mathcal{L}$-theory. Suppose that, for any quantifier-free formula $\varphi(\overline{x}, y)$, if $\mathcal{M}$ and $\mathcal{N}$ are models of T with a common substructure $\mathcal{A}$ of T, then for any $\overline{a} \in \mathcal{A}$, we have $\mathcal{M} \vDash \exists y\, \varphi(\overline{a}, b)$ if and only if $\mathcal{N} \vDash \exists y\, \varphi(\overline{a}, y)$. Then T eliminates quantifiers.