256B: Algebraic Geometry

Nir Elber

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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Theme 1
Building Cohomology

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

1.1 January 17

Let's just get started.

1.1.1 Course Notes

Here are some notes about the course.

- The professor is Paul Vojta, whose email is vojta@math.berkeley.edu.
- The course webpage is https://math.berkeley.edu/~vojta/256b.html.
- The textbook is [Har77].
- We will assume algebraic geometry on the level of Math 256A, which is a prerequisite for this course.
- This course focuses on (Zariski) cohomology of schemes, so we will spend most of our time going through [Har77, Chapter III]. We will also discuss smoothness, which lives in [Har77, Chapter III] as well. Along our way, we will want to discuss some topics in [Har77, Chapter II] in more detail, such as on divisors.
- Grading will be based on homework. Homework will be weekly or biweekly, due on Wednesdays (in general).

1.1.2 Abelian Categories

We'll assume some basic category theory (monomorphisms, epimorphisms, equalizers, coequalizers, etc.). Abelian categories are somewhat complex, so we provide their definition. Roughly speaking, our end goal is to do cohomology, which arises from homological algebra, and homological algebra lives in abelian categories.

Definition 1.1 (preadditive)**.** A *preadditive category* is a category $\mathcal{C}$ where the morphism set $\operatorname{Hom}_{\mathcal{C}}(A,B)$ forms an abelian group for any $A,B\in\mathcal{C}$, and composition distributes over addition. Explicitly, the composition map
$$\circ\colon\ \operatorname{Hom}_{\mathcal{C}}(B,C)\times\operatorname{Hom}_{\mathcal{C}}(A,B)\to\operatorname{Hom}_{\mathcal{C}}(A,C)$$
is bilinear.

It follows directly from having the preadditive structure that finite products and finite coproducts are canonically isomorphic. However, these (bi)products need not exist.

Definition 1.2 (additive)**.** An *additive category* is a preadditive category admitting all finite products/coproducts.

Definition 1.3 (abelian)**.** An *abelian category* is an additive category $\mathcal{C}$ in which the following hold.

- Every morphism admits a kernel and a cokernel; here, a (co)kernel is a (co)equalizer with the zero map.
- Every monomorphism is the kernel of some morphism.
- Every epimorphism is the cokernel of some morphism.

Let's give some examples.

Example 1.4. The following are abelian categories; we omit the checks.

- The category Ab of abelian groups is abelian.
- For a ring A, the category $\mathrm{Mod}(A)$ of A-modules is abelian. In particular, for a field k, the category $\mathrm{Vec}(k)$ of k-vector spaces is abelian.

Example 1.5. Here are more abelian categories, related to sheaves. All of their "abelian" hypotheses are done by passing to stalks or a similar local argument.

- For a topological space X, the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on X is abelian.
- Similarly, for a ringed space $(X,\mathcal{O}_X)$, the category $\mathrm{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules is abelian.
- For a scheme X, the category $\mathrm{QCoh}(X)$ of quasicoherent sheaves on X is abelian.
- Similarly, for a scheme X, the category $\mathrm{Coh}(X)$ of coherent sheaves on X is also abelian. Notably, we do not have infinite products here, but that's okay.

Example 1.6. For any abelian category $\mathcal{A}$, its opposite category $\mathcal{A}^{\mathrm{op}}$ is also abelian. One can see this by going through the conditions, all of which dualize.

1.1.3 Exact Functors

We will want to discuss exact functors in order to homological algebra in our abelian categories. Let's have at it.

Definition 1.7 (additive)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if and only if the map

$$F\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{D}}(FA,FB)$$

(of F acting on morphisms $A \to B$) is a group homomorphism, for any $A, B \in \mathcal{C}$. Flipping arrows and using Example 1.6 produces the same definition for contravariant functors.

Example 1.8. Fix a topological space X. Then the functor $\Gamma(X,-)\colon \operatorname{Ab}(X) \to \operatorname{Ab}$ of global sections $\mathcal{F} \mapsto \Gamma(X,\mathcal{F})$ is additive.

Remark 1.9. Being additive implies that the functor preserves biproducts. Roughly speaking, this holds because being a biproduct can be written as a set of equations for the object (and its inclusion/projection morphisms) to satisfy.

To define (left) exact for a functor, we need to define what it means to be exact.

Definition 1.10 (exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. Then a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* B if and only if $\ker g = \ker(\operatorname{coker} f)$ (up to some identification). Here, $\ker(\operatorname{coker} f)$ is intended to basically be the image.

Definition 1.11 (left exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) additive functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left-exact* if and only if a left exact sequence

$$0 \to A' \to A \to A''$$

produces a left exact sequence

$$0 \to FA' \to FA \to FA''.$$

Reversing the arrows produces the dual notion of right exactness.

Remark 1.12. Being left exact equivalently means that F preserves kernels, so by Remark 1.9 and a little category theory, F actually preserves all finite limits.

Example 1.13. The functor of global sections from Example 1.8 is left exact by [Har77, Exercise II.1.8].

To get us set up, let's approximately describe what we are trying to do. Basically, fix an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of abelian groups on a topological space X. Then there is a sequence of "cohomology" functors $\{H^i(X,-)\}_{i\in\mathbb{N}}$ with $H^0(X,-) = \Gamma(X,-)$ and a "long" exact sequence as follows

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{F}') \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{F}'') \arrow[dll] & \\ & H^1(X,\mathcal{F}') \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}'') \arrow[r] & \cdots \end{tikzcd}$$

where the maps $H^i(X,\mathcal{F}'') \to H^{i+1}(X,\mathcal{F}')$ take some work to describe.

Remark 1.14. These functors will have a number of magical properties, which will amount to the main theorems of this course. Let's give an example. Fix a projective scheme X over a field k, where $i\colon X \to \mathbb{P}^n_k$ is the promised closed embedding; let $\mathcal{I}$ be the corresponding ideal sheaf of this closed embedding. Then we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n_k} \to i_*\mathcal{O}_X \to 0,$$

which one can do cohomology to. In fact, one can take the tensor product of this exact sequence with the twisting sheaves $\mathcal{O}_{\mathbb{P}^n_k}(m)$; for example, we will prove that $H^1(\mathbb{P}^n_k, \mathcal{I}(m)) = 0$ for sufficiently large m, which eventually implies that the map

$$\Gamma\left(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)\right) \to \Gamma(X, \mathcal{O}_X(m))$$

is surjective for sufficiently large m. In other words, global sections of $\mathcal{O}_X(m)$ are all restrictions of global sections of $\mathcal{O}_{\mathbb{P}^n_k}(m)$!

1.2 January 19

We'll do some homological algebra today.

1.2.1 Homological Algebra on Complexes

Homological algebra is something that comes out of understanding complexes, which we will now define.

Definition 1.15 (complex)**.** Fix an abelian category $\mathcal{A}$. A *complex* $(A^\bullet, d^\bullet)$ is a collection $\{A^i\}_{i\in\mathbb{Z}}$ together with some morphisms $d^i\colon A^i \to A^{i+1}$ such that $d^{i+1}\circ d^i = 0$. We may abbreviate the differential $d^\bullet$ from the notation.

Remark 1.16. The above definition is usually a "cocomplex." We will have no need for the dual notion of a complex in this course.

Remark 1.17. By convention, if we state that we have a complex but only define A^i for a subset of $\mathbb{Z}$, then the full bona fide complex simply sets the undefined terms to zero.

Now that we have a complex, we should define a morphism.

Definition 1.18 (complex morphism)**.** Fix an abelian category $\mathcal{A}$. Given complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$, a morphism of complexes $\varphi^\bullet\colon A^\bullet \to B^\bullet$ is a collection of morphisms $\varphi^i\colon A^i \to B^i$ making the following diagram commute for each i.

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "\varphi^i"'] & A^{i+1} \arrow[d, "\varphi^{i+1}"] \\ B^i \arrow[r, "d^{i+1}"] & B^{i+1} \end{tikzcd}$$

Unsurprisingly, our definition of morphism provides us with a category of complexes, and in fact the category of complexes is an abelian category, where the point is that biproducts, kernels, and cokernels can all be computed pointwise at each term of the complex.

We are now ready to define cohomology.

Definition 1.19 (cohomology)**.** Fix a complex $(A^\bullet, d^\bullet)$ valued in an abelian category $\mathcal{A}$. Then we define the ith *cohomology* as

$$h^i(A^\bullet) := \frac{\ker d^i}{\operatorname{im} d^{i-1}}.$$

Here, $\operatorname{im} d^{i-1}$ has an induced map to $\ker d^i$ because $d^i \circ d^{i-1} = 0$.

Remark 1.20. Quickly, recall that the image $\operatorname{im} d^{i-1}$ is in fact $\ker(\operatorname{coker} d^{i-1})$.

Remark 1.21. In fact, cohomology is functorial: a morphism $f^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of complexes induces a morphism $h^i(f^\bullet) \colon h^i(A^\bullet) \to h^i(B^\bullet)$ on the ith cohomology, and one can check that this makes h^i into a functor. To be explicit, this morphism is induced by the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} d_A^{i-1} \arrow[r] \arrow[d, "f^i"] & \ker d_A^i \arrow[r] \arrow[d, "f^i"] & h^i(A^\bullet) \arrow[r] \arrow[d, dashed, "f^i"] & 0 \\
0 \arrow[r] & \operatorname{im} d_B^{i-1} \arrow[r] & \ker d_B^i \arrow[r] & h^i(B^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Namely, the morphisms on the left are well-defined because $f^\bullet$ is in fact a morphism.

The main result on these cohomology groups is the following.

Proposition 1.22. Fix an abelian category $\mathcal{A}$. Given a short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of complexes in $\mathcal{A}$, there are natural maps $\delta^i \colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ producing a long exact sequence as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & h^i(A^\bullet) \arrow[r] & h^i(B^\bullet) \arrow[r] & h^i(C^\bullet) \arrow[dll, "\delta^i"] \\
& h^{i+1}(A^\bullet) \arrow[r] & h^{i+1}(B^\bullet) \arrow[r] & h^{i+1}(C^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. To produce the long exact sequence, use the Snake lemma. The proof is somewhat technical, so I will refer directly to [Elb22, Theorem 4.82], though the proof there is for the dual notion of homology instead of cohomology. (Note that we can replace $\mathcal{A}$ with $\mathcal{A}^{\mathrm{op}}$ to recover the result.) The naturality of the $\delta^\bullet$ can be checked directly from its construction. ■

We would like to measure a morphism of complexes based on what it does to cohomology: namely, two morphisms of complexes may induce the same map on cohomology despite being technically distinct. One way this might happen is by being "chain" homotopic.

Definition 1.23 (chain homotopy)**.** Fix morphisms $f^\bullet, g^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of the chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. A *chain homotopy* is a sequence of maps $k^i \colon A^i \to B^{i-1}$ such that

$$f^i - g^i = k^{i+1} \circ d_A^i + d_B^{i-1} \circ k^i.$$

In this case, we say that $f^\bullet$ and $g^\bullet$ are chain homotopic.

Remark 1.24. One can check directly that being chain homotopic is an equivalence relation on chain morphisms.

And here is our result.

Proposition 1.25. Fix morphisms $f^\bullet, g^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. If $f^\bullet \sim g^\bullet$, then $h^i(f^\bullet) = h^i(g^\bullet)$ for all i.

Proof. By some embedding theorem, we may as well work in $\operatorname{Mod}(R)$ for some ring R. Now, fix some $\alpha \in \ker d_A^i$, and we want to show that

$$[f^i(\alpha) - g^i(\alpha)] = 0$$

in $h^i(B^\bullet)$. But now let $k^j \colon A^j \to B^{j-1}$ for $j \in \mathbb{Z}$ provide our chain homotopy, so we see

$$f^i(\alpha) - g^i(\alpha) = k^{i+1}\big(\underbrace{d_A^i(\alpha)}_{0}\big) + d_B^{i-1}\left(k^i(\alpha)\right)$$

vanishes in $h^i(B^\bullet)$, as desired. ■

1.2.2 Injective Resolutions

We would now like to use our homological algebra to say something concrete about functors, which requires building injective resolutions. Injective resolutions are built out of injectives, so here is that definition.

Definition 1.26 (injective)**.** Fix an object I in an abelian category $\mathcal{A}$. Then I is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is right exact.

Remark 1.27. The functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is already left-exact (and contravariant), so it is equivalent to ask for this functor to be fully exact. Unwinding the definition, we may equivalently ask for short exact sequences

$$0 \to A' \to A \to A'' \to 0$$

to produce short exact sequences

$$0 \to \operatorname{Hom}_{\mathcal{A}}(A'', I) \to \operatorname{Hom}_{\mathcal{A}}(A, I) \to \operatorname{Hom}_{\mathcal{A}}(A', I) \to 0,$$

but this is already left-exact, so we are really only concerned about surjectivity on the right. So we may equivalently ask for injections $A' \hookrightarrow A$ to produce surjections $\operatorname{Hom}_{\mathcal{A}}(A', I) \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(A, I)$; i.e., any map $A' \to I$ can be extended to a full map $A \to I$.

We also have the following dual notion.

Definition 1.28 (projective)**.** Fix an object P in an abelian category $\mathcal{A}$. Then P is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is right exact.

Remark 1.29. Exactly the dual arguments to Remark 1.27 say that being projective is equivalent to $\operatorname{Hom}_{\mathcal{A}}(P, -)$ being fully exact, or equivalently that any map $P \to A''$ can be pulled back to a map $P \to A$ whenever we have a surjection $A \twoheadrightarrow A''$.

And we now define our resolutions.

Definition 1.30 (resolution)**.** Fix an object A in an abelian category $\mathcal{A}$. A *coresolution* is an exact sequence

$$0 \to A \xrightarrow{\varepsilon} E^0 \to E^1 \to \cdots$$

in $\mathcal{A}$; we may write this as $0 \to A \to E^\bullet$. A *resolution* is an exact sequence

$$\cdots \to E_1 \to E_0 \xrightarrow{\varepsilon} A \to 0$$

in $\mathcal{A}$; again, we may write this as $E^\bullet \to A \to 0$. For any property $\mathcal{P}$ of objects in $\mathcal{A}$, we say that the resolution is $\mathcal{P}$ if and only if the Es are all $\mathcal{P}$.

Of interest to us right now are injective and projective resolutions, but we will find use for other kinds of resolutions.

We want to be able to build injective resolutions. The following provides the required adjective.

Definition 1.31 (enough injectives)**.** An abelian category $\mathcal{A}$ has *enough injective* if and only if any object $A \in \mathcal{A}$ has a monomorphism to an injective object.

And here is the relevant result.

Proposition 1.32. Fix an abelian category $\mathcal{A}$ with enough injectives. Then every object $A \in \mathcal{A}$ has an injective resolution.

Proof. By induction, it is enough to show that, for any map $f\colon A \to E$, there exists a map $g\colon E \to I$ where I is injective and the sequence $A \to E \to I$ is exact. Indeed, this will be enough because we can start with the sequence $0 \to A$, then extend to $0 \to A \to E^0$, then extend to $0 \to A \to E^0 \to E^1$, and so on.

Now, to show the claim of the previous paragraph, we note that we may find an injective object I and a monomorphism $\overline{g}\colon \operatorname{coker} f \to I$ because $\mathcal{A}$ has enough injectives. Then we note that the composite

$$A \to E \to \operatorname{coker} f \hookrightarrow I$$

produces the exact sequence $A \to E \to I$, as desired. ■

1.3 January 22

Today we will derive functors.

1.3.1 More on Injective Resolutions

A nice property of injective resolutions is that they are, in some sense, functorial in their object.

Proposition 1.33. Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$. Given injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$, one can find maps $g^i\colon E^i \to F^i$ for each i inducing a chain morphism of the injective resolutions.

Proof. This is an exercise is induction and using the injective. ■

In fact, this morphism is unique.

Proposition 1.34. Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$, and fix injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$. Then any two morphisms $f^\bullet$ and $g^\bullet$ of the injective resolutions, which agree on $A \to B$, are chain homotopic.

Proof. Set $h^\bullet := f^\bullet - g^\bullet$. Upon subtracting out g suitably, we see that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\delta"] \arrow[d, "0"'] & I^0 \arrow[r, "d_A^0"] \arrow[d, "h^0"'] & I^1 \arrow[r, "d_A^1"] \arrow[d, "h^1"'] & I^2 \arrow[r, "d_A^2"] \arrow[d, "h^2"'] & \cdots \\
0 \arrow[r] & B \arrow[r, "\varepsilon"] & J^0 \arrow[r, "d_B^0"] & J^1 \arrow[r, "d_B^1"] & J^2 \arrow[r, "d_B^2"] & \cdots
\end{tikzcd}$$

commutes, and we want to show that the morphism $h^\bullet$ of the injective resolutions is chain homotopic to the zero map.

Now, we see $h^0 \circ \delta = 0$, so we may as well factor h^0 through $\operatorname{coker} \delta \subseteq I^1$. But J^0 is an injective object, so the map $\overline{h}^0 \colon \operatorname{coker} \delta \to J^i$ extends to a map $k^1 \colon I^1 \to J^0$. For completeness, we also define $k^0 \colon I^0 \to J^{-1}$ be the zero map. Anyway, we now compute

$$d_B^{-1} \circ k^0 + k^1 \circ d_A^0 = h^0$$

by construction.

Further, we see

$$\left(h^1 - d_B^0 \circ k^1\right) \circ d_A^0 = h^1 \circ d_A^0 - d_B^0 \circ h^0 = 0$$

by the commutativity of our diagram. As such, we have a map $\left(h^1 - d_B^0 \circ k^1\right) \colon \operatorname{coker} d_A^0 \to J^1$ which can be extended to a map $k^2 \circ I^2 \to J^1$ by the injectivity of J^1. In particular, we see that $h^1 - d_B^0 \circ k^1 = k^2 \circ d_A^1$ by construction. Explicitly, let $\pi^1 \colon I^1 \to \operatorname{coker} d^0$ and $i^1 \colon \operatorname{coker} d^0 \to I^2$ be the obvious maps, and we compute

$$d_B^0 \circ k^1 + k^2 \circ d_A^1 = h^1 - \overline{h}^1 \circ \pi^1 + k^2 \circ d^1 = h^1 - k^2 \circ i^2 \circ pi^1 + k^2 \circ d_A^1 = h^1.$$

We now iterate the construction of k^{i+1} from k^i provided in this paragraph inductively to complete the proof. ■

Remark 1.35. The proofs of the previous two proposition nowhere require that the resolutions on A be injective. We will have no need to work in this generality though.

1.3.2 Right-Derived Functors

At long last, we can derive functors.

Definition 1.36 (right-derived functor)**.** Fix a left-exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories. For each $i \in \mathbb{N}$, we define the *right derived functors*

$$R^i F(A, I^\bullet) := h^i(FI^\bullet),$$

where $0 \to A \to I^\bullet$ is an injective resolution of the object A. This construction is functorial: given a morphism $\varphi \colon A \to B$ in $\mathcal{A}$ equipped with injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, we define the morphism

$$R^i F(\varphi, f^\bullet) \colon h^i(FI^\bullet) \to h^i(FJ^\bullet)$$

as $h^i(F(f^\bullet))$ for any extension $f^\bullet \colon I^\bullet \to J^\bullet$ of φ.

We would like to remove the dependencies on the injective resolutions. This requires a couple checks. To begin, we get rid of the dependency of $R^i F(\varphi)$ on $f^\bullet$.

Lemma 1.37. Fix objects A and B in an abelian category $\mathcal{A}$, and equip them with injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$. For any two morphisms $f^\bullet, g^\bullet \colon I^\bullet \to J^\bullet$ extending a given morphism $\varphi \colon A \to B$, we have

$$R^i F(\varphi, f^\bullet) = R^i F(\varphi, g^\bullet).$$

Proof. We know that $f^\bullet$ and $g^\bullet$ are chain homotopic by Proposition 1.34. This chain homotopy is preserved by an additive functor, so $Ff^\bullet$ and $Fg^\bullet$ are still chain homotopic, so Proposition 1.25 implies the conclusion upon taking cohomology. ■

Notation 1.38. Fix everything as in Definition 1.36. We will write $R^iF(\varphi)$ for $R^iF(\varphi, f^\bullet)$ because it is independent of the choice of $f^\bullet$ by Lemma 1.37 (and an $f^\bullet$ always exists by Proposition 1.33). For now, $R^iF(\varphi)$ still should depend on the choice of injective resolutions, but we will suppress it from the notation anyway.

Remark 1.39. Perhaps we should check functoriality of our construction.

- For an object A equipped with an injective resolution $0 \to A \to I^\bullet$, we can extend $\mathrm{id}_A \colon A \to A$ by $\mathrm{id}_{I^\bullet} \colon I^\bullet \to I^\bullet$. Passing through F and taking cohomology reveals $R^iF(\mathrm{id}_A) = \mathrm{id}_{R^iF(A,I^\bullet)}$.
- Fix morphisms $\varphi \colon A \to B$ and $\psi \colon B \to C$ extending to maps of injective resolutions $f^\bullet \colon I^\bullet \to J^\bullet$ and $g^\bullet \colon J^\bullet \to K^\bullet$, respectively. Then one want to extend $(\psi \circ \varphi) \colon A \to C$ to a morphism $I^\bullet \to K^\bullet$ is via $g^\bullet \circ f^\bullet$, and doing so establishes that

$$\begin{tikzcd} R^iF(A,I^\bullet) \arrow[r, "R^iF(\varphi)"] \arrow[dr, "R^iF(\psi\circ\varphi)"'] & R^iF(B,J^\bullet) \arrow[d, "R^iF(\psi)"] \\ & R^i(C,K^\bullet) \end{tikzcd}$$

  commutes, from which we can read off functoriality.

Remark 1.40. We can purchase that R^iF does not depend on the choice of injective resolution from Remark 1.39: running the functoriality check on $0 \to A \to I^\bullet$ mapping to $0 \to A \to J^\bullet$ and then back to $0 \to A \to I^\bullet$ reveals that the maps $R^iF(A, I^\bullet) \to R^iF(A, J^\bullet$ and $R^iF(A, J^\bullet) \to R^iF(A, I^\bullet)$ are mutually inverse, so we get the needed isomorphism.

Remark 1.41. Note R^iF is additive because all steps in the construction (passing through F and then taking cohomology) are additive.

We can even compute our 0th right-derived functor without tears.

Example 1.42. Fix an abelian category $\mathcal{A}$ with enough injectives. Then $F \simeq R^0F$. Indeed, on objects, fix an injective resolution $0 \to A \to I^\bullet$ for a given object $A \in \mathcal{A}$, and we see that

$$R^0F(A) = h^0(F(I^\bullet)) = \ker(FI^0 \to FI^1) = FA,$$

where the last equality follows from left-exactness of F. On morphisms $\varphi \colon A \to B$, we fix injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, and then we produce a morphism of left exact sequences as follows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[d, "f^1"] \\ 0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \end{tikzcd}$$

Passing through F retains left exactness (and commutativity), allowing us to conclude $R^0F(\varphi) = F\varphi$.

1.4 January 24

Today we continue deriving functors.

1.4.1 The Long Exact Sequence

Here is the main result on cohomology.

Theorem 1.43. Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, there are natural morphisms $\delta^i\colon R^iF(A'') \to R^{i+1}F(A')$ for $i \geq 0$ (i.e., the δ^i are natural in the short exact sequence) such that there is a long exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & R^0F(A') \arrow[r] & R^0F(A) \arrow[r] & R^0F(A'') \arrow[dll, "\delta^0"'] \\ & R^1F(A') \arrow[r] & R^1F(A) \arrow[r] & R^1F(A'') \arrow[r] & \cdots \end{tikzcd}$$

Proof. We use Proposition 1.22. The main obstacle is that we need to produce a short exact sequence of injective resolutions for A', A, and A''. We begin by fixing injective resolutions $0 \to A' \to I^\bullet$ and $0 \to A'' \to J^\bullet$, which we would like to glue together into an injective resolution for A as well. In particular, we would like a sequence of morphisms to go into the middle of the following diagram.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & A \arrow[r, dashed] \arrow[d] & I^0 \oplus J^0 \arrow[r, dashed] \arrow[d] & I^1 \oplus J^1 \arrow[r, dashed] \arrow[d] & \cdots \\ 0 \arrow[r] & A'' \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & 0 \end{tikzcd}$$

Here, the downward morphisms, except the ones on the far left, are all given by having a split short exact sequence. (Note $I^i \oplus J^i$ is injective for each i because the sum of injective objects must be injective; this can be seen directly from the definition of injective objects.)

Working inductively, the main point is as follows: suppose we have a diagram as follows, where we would like to induce the vertical morphism f making the diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "f'"'] & K \arrow[r] \arrow[d, dashed, "f"] & K'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I \arrow[r] & I \oplus J \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

Here, I and J are injective, and f' and f'' is injective; the Snake lemma will imply that f is injective too. Well, by summing, all one needs is maps $g'\colon K \to I$ and $g''\colon K \to J$ making the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "f'"'] & K \arrow[r] \arrow[dl, dashed, "g'"'] \arrow[dr, dashed, "g''"] & K'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I \arrow[r] & I \oplus J \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

For this, we see that g'' is given by composition, and g' is given because $K' \subseteq K$ and I is injective object.

We now explain how the previous step proves the result. We immediately produce the needed map $A \to I^0 \oplus J^0$. Now to go from having the map $I^i \oplus J^i \to I^{i+1} \oplus J^{i+1}$ to having the map $I^{i+1} \oplus J^{i+1}$, we use the above paragraph on the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \frac{I^{i+1}}{I^i} \arrow[r] \arrow[d, hook] & \frac{I^{i+1}}{I^i} \oplus \frac{J^{i+1}}{J^i} \arrow[r] \arrow[d, dashed] & \frac{J^{i+1}}{J^i} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & I^{i+2} \arrow[r] & I^{i+2} \oplus J^{i+2} \arrow[r] & J^{i+2} \arrow[r] & 0 \end{tikzcd}$$

This completes the construction of the needed short exact sequence of injective resolutions, from which the result follows upon using Proposition 1.22 on the short exact sequence of complexes

$$0 \to FI^\bullet \to FI^\bullet \oplus FJ^\bullet \to FJ^\bullet \to 0.$$

(This is still short exact because additive functors preserve split short exact sequences.) Note that we have not checked that the $\delta^\bullet$s are natural in the short exact sequence; this follows from the naturality of Proposition 1.22. ■

1.4.2 Acyclic Objects

We note the following computation.

Proposition 1.44. Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. If $I \in \mathcal{A}$ is injective, then $R^iF(I) = 0$ for all $i \geq 1$.

Proof. There is an injective resolution

$$0 \to I \to I \to 0 \to 0 \to \cdots$$

of I. Upon taking F, we see that $R^0F(I) = I/0$ and $R^1F(I) = 0/I$ and $R^iF(I) = 0/0$ for $i \geq 2$. This proves the result. ■

We now get the following definition.

Definition 1.45 (acyclic)**.** Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. We say an object $A \in \mathcal{A}$ is *acyclic for F* if and only if $R^iF(A) = 0$ for all $i \geq 1$.

Example 1.46. If $A \in \mathcal{A}$ is injective, then Proposition 1.44 implies that A is acyclic for any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$.

Here is the point of defining acyclic objects.

Proposition 1.47. Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. For any acyclic resolution $0 \to A \to I^\bullet$, there are canonical isomorphisms

$$R^iF(A) \cong h^i(FJ^\bullet).$$

Proof. Induct on i using the long exact sequences. For example, there is nothing to say for $i = 0$. To get up to $i = 1$, use the exact sequence

$$0 \to A \xrightarrow{\varepsilon} J^0 \to \operatorname{coker} \varepsilon \to 0$$

to produce the needed long exact sequence

$$0 \to FA \to FJ^0 \to F \operatorname{coker} \varepsilon \to R^1F(A) \to 0,$$

and $h^1(FJ^\bullet)$ becomes the needed quotient. This process continues upwards. ■

1.4.3 A Little δ-Functors

Here is our definition.

Definition 1.48 (δ-functor)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A *δ-functor* consists of the data of some additive functors $T^i\colon \mathcal{A} \to \mathcal{B}$ for each $i \in \mathbb{N}$ and some morphisms $\delta^i\colon T^iA'' \to T^{i+1}A$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ such that there is a long exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & T^0A' \arrow[r] & T^0A \arrow[r] & T^0A'' \arrow[dll, "\delta^0"'] & \\ & T^1A' \arrow[r] & T^1A \arrow[r] & T^1A'' \arrow[r] & \cdots \end{tikzcd}$$

Example 1.49. If $\mathcal{A}$ has enough injective, the derived functors provide examples of δ-functors by Theorem 1.43.

The following definition will be very helpful.

Definition 1.50 (initial)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A δ-functor $(T^\bullet, \delta_T^\bullet)$ is *initial* if and only if any other δ-functor $(U^\bullet, \delta_U^\bullet)$ together with a map $\varphi\colon T^0 \Rightarrow U^0$ has a unique sequence of natural transformations $\eta^\bullet\colon T^\bullet \Rightarrow U^\bullet$ extending φ and commute with the formation of the long exact sequences. Explicitly, a short exact sequence $0 \to A' \to A \to A'' \to 0$ induces the following morphism of long exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & T^0A' \arrow[r] \arrow[d, "f^0"'] & T^0A \arrow[r] \arrow[d, "f^0"'] & T^0A'' \arrow[r, "\delta_T^0"] \arrow[d, "f^0"'] & T^1A' \arrow[r] \arrow[d, "f^1"'] & T^1A \arrow[r] \arrow[d, "f^1"'] & \cdots \\ 0 \arrow[r] & U^0A' \arrow[r] & U^0A \arrow[r] & U^0A'' \arrow[r, "\delta_U^0"] & U^1A' \arrow[r] & U^1A \arrow[r] & \cdots \end{tikzcd}$$

Note that initial δ-functors are unique up to unique isomorphism when they exist.

1.5 January 26

Today we will finish our discussion of right-derived functors.

1.5.1 Initial δ-Functors

We will want to make some use of our discussion of δ-functors.

Definition 1.51 (effaceable)**.** Fix an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories. Then F is *effaceable* if and only if each $A \in \mathcal{A}$ has a monomorphism $u\colon A \to M$ such that $Fu = 0$.

We have the following result which will help us check that right-derived functors are initial.

Theorem 1.52. Fix δ-functor $(T^\bullet, \delta^\bullet)\colon \mathcal{A} \to \mathcal{B}$. If $T^\bullet$ is *effaceable* for all $i > 0$, then $(T^\bullet, \delta^\bullet)$ is initial.

Proof. Omitted. The proof is somewhat long and technical. We refer to [Wei94, Theorem 2.4.7] for most of the needed details. ∎

Corollary 1.53. Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ has enough injectives. If $F\colon \mathcal{A} \to \mathcal{B}$ is left exact, the right-derived functors $(R^\bullet F, \delta^\bullet)$ is effaceable and thus initial.

Proof. By Theorem 1.52, it remains to show being effaceable. Well, for any object $A \in \mathcal{A}$, we can find a map $u\colon A \to I$ where I is injective, so the map $R^iu\colon R^iU \to R^iI$ is the zero map for $i > 1$ because $R^iI = 0$ by Proposition 1.44. ■

Corollary 1.54. Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ has enough injectives. If $(T^\bullet, \delta^\bullet)$ is an initial δ-functor, then T^0 is left exact, and $T^\bullet \simeq R^iT^0$ for all $i \geq 0$.

Proof. For any short exact sequence
$$0 \to A' \to A \to A'' \to 0,$$
being a δ-fucntor implies the left exact sequence
$$0 \to T^0A' \to T^0A \to T^0A''.$$
Thus, T^0 is left exact. Now, the usual category theory arguments show that initial δ-functors (when they exist) are unique up to unique isomorphism, so Corollary 1.53 completes the proof. ■

1.5.2 Having Enough Injectives

Let's show that some abelian categories have enough injectives. We begin with Ab.

Definition 1.55 (divisible)**.** An abelian group A is *divisible* if and only if the multiplication-by-n map $n\colon A \to A$ is surjective for all nonzero integers n.

Example 1.56. The groups $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{R}$, and 0 are divisible.

Here is the point of this definition.

Proposition 1.57. An abelian group A is injective in Ab if and only if A is divisible.

Proof. We show our implications separately.

- Suppose A is injective, and fix some $a \in A$ and nonzero integer $n \in \mathbb{Z}$ so that we want to find $a' \in A$ with $a = na'$. Well, we have the morphism $n\mathbb{Z} \to A$ given by $n \mapsto a$, but $n\mathbb{Z} \subseteq \mathbb{Z}$ means that the injectivity of A forces $n\mathbb{Z} \to A$ to extend to $\mathbb{Z} \to A$, as follows.

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[dr, "n\mapsto a"'] & \mathbb{Z} \arrow[d, dashed] \\ & & A \end{tikzcd}$$

  Now, the image of 1 along $\mathbb{Z} \to A$ can be called a' and has $na' = a$ by construction.

- Suppose A is divisible. We will use Zorn's lemma. Well, for our setup, suppose that we have an inclusion $M' \subseteq M$ and a map $\varphi\colon M' \to A$ which we would like to extend up to M.

  Let Φ be the collection of extensions of $\varphi\colon M' \to A$ to some subgroup $N \subseteq M$ containing M', and order Φ by extension: we have $(N_1, \varphi_1) \preceq (N_2, \varphi_2)$ if and only if $N_1 \subseteq N_2$ and $\varphi_2|_{N_1} = \varphi_1$. Now, Φ is nonempty (it has (M', φ)), and its ascending chains are upper-bounded (the union of an extension of group homomorphisms will continue to be a group homomorphism), so Zorn's lemma provides Φ with a maximal element (M'', φ'').

We claim that $M'' = M$, which will complete the proof. Well, we will show a contrapositive: suppose $(N, \psi) \in \Phi$ has $N \neq M$; then we claim that (N, ψ) is not maximal. Well, given any $x \in M \setminus N$, we will extend ψ to $N + \mathbb{Z}x$. Set $H := \{n \in \mathbb{Z} : nx \in N\}$. We have two cases.

- Suppose $H = 0$. Then $N + \mathbb{Z}x = N \oplus \mathbb{Z}x$, so we can extend ψ by just setting $\psi(x) := 0$.
- Suppose $H = n\mathbb{Z}$ for some positive integer $n > 0$. Divisibility promises us some $a \in A$ such that $\psi(nx) = na$, so we would like to extend ψ by $\psi(x) = a$. Namely, we would like to define $\widetilde{\psi} \colon (N + \mathbb{Z}x) \to A$ by
$$\widetilde{\psi}(m + kx) := \psi(m) + ka.$$
Of course, this will be a group homomorphism extending $\widetilde{\psi}$ provided that it is well-defined. Well, suppose $m + kx = m' + k'x$, and we want to show that $\psi(m) + ka = \psi(m') + k'a$, or equivalently, $\psi(m - m') = (k' - k)a$. We now note that $(k' - k)x = m - m' \in N$, so $k' - k = n\ell$ for some integer ℓ by construction of n, so we computed
$$(k' - k)a = n\ell a = \psi(n\ell x) = \psi((k' - k)x) = \psi(m - m'),$$
as needed. ■

Theorem 1.58. Fix a ring R. The category $\mathrm{Mod}(R)$ has functorial injectives.

Proof. We proceed in steps.

1. As an intermediate step, set $J := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. Then we note that
$$\mathrm{Hom}_R(-, J) \simeq \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$$
by the hom–tensor adjunction. Additionally, if $M \neq 0$, we see that $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is nonzero by being an injective object, so the left-hand side is also nonzero. Lastly, the right-hand functor is exact, so the left-hand functor is exact, so we see that J is injective.
We now set $A^\vee := \mathrm{Hom}_R(A, J)$.
2. So we will want to show that the map
$$\mathrm{ev}_\bullet \colon A \to A^{\vee\vee}$$
given by $\mathrm{ev}_a \colon \varphi \mapsto \varphi(a)$ is injective. Well, let $K := \ker \mathrm{ev}_\bullet$, and we draw the following commutative diagram.
$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "\mathrm{ev}_\bullet"'] & M \arrow[d, "\mathrm{ev}_\bullet"] \\ K^{\vee\vee} \arrow[r] & M^{\vee\vee} \end{tikzcd}$$
Because $(-)^\vee$ is an exact functor, we see that the bottom row must be injective. But the diagonal composite is zero, so actually $\mathrm{ev}_\bullet \colon K \to K^{\vee\vee}$ must be fully the zero map. Thus, $K = 0$ by the check in the previous step.
3. We actually construct the needed injection. Note we have a surjection
$$\bigoplus_{x \in A^\vee} R \twoheadrightarrow A^\vee,$$
so we have an injection
$$A \hookrightarrow A^{\vee\vee} \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{x \in A^\vee} R, J\right) = J^{A^\vee}.$$

The right-hand side can be seen to be injective, so we are essentially done; notably, our construction is functorial in A. Explicitly, given a map $A \to B$, we induce a map $B^\vee \to A^\vee$, and taking fibers of this map induces a map $J^{A^\vee} \to J^{B^\vee}$. (Any coordinate in $A^\vee$ not in the image of $B^\vee$ can just get sent to 0.) ■

1.6 January 29

Today we continue to show that categories have enough injectives.

1.6.1 Exactness in Abelian Categories

Let's say a few more things about abelian categories.

Example 1.59. Fix an abelian category $\mathcal{A}$. Then $\mathcal{A}$ has an empty biproduct 0, which is both initial and final by its definition. We will not bother to write out the identification of biproducts in additive categories.

Remark 1.60. Fix an abelian category $\mathcal{A}$. Any morphism $\varphi\colon A \to B$ can be factored as $\nu \circ \eta$ where $\eta\colon A \to X$ is epic and $\nu\colon X \to B$ is monic. To see that this factorization exists, we can set $\eta = \operatorname{coker}(\ker \varphi)$ and $\nu = \ker(\operatorname{coker} \varphi)$. Additionally, the factorization $\nu \circ \eta$ is unique in the following sense: if $\eta'\colon A \to X'$ and $\nu'\colon X' \to B$ is another such factorization, there is a unique isomorphism $\psi\colon X \to X'$ making the following diagram commute.

$$\begin{tikzcd} & X \arrow[dd, "\psi"] \arrow[dr, "\nu"] & \\ A \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & & B \\ & X' \arrow[ur, "\nu'"'] & \end{tikzcd}$$

Remark 1.61. The previous remark implies that being an isomorphism is equivalent to being both monic and epic. Namely, one just factors the given morphism $\varphi\colon A \to B$ in the two ways $\mathrm{id}_B \circ \varphi = \varphi \circ \mathrm{id}_A$ to conclude that φ has an inverse.

The prior two remarks allow us to make sense of exactness in a meaningful way.

Definition 1.62 (exact). Fix morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, and factor these as $\varphi = \nu \circ \eta$ and $\psi = \mu \circ \varepsilon$ where ν and μ are epic and η and ε are monic. Here is the diagram.

$$\begin{tikzcd} & X \arrow[dr, "\nu"] & & Y \arrow[dr, "\varepsilon"] & \\ A \arrow[ur, "\eta"] \arrow[rr, "\varphi"'] & & B \arrow[ur, "\mu"] \arrow[rr, "\psi"'] & & C \end{tikzcd}$$

Then the sequence

$$A \to B \to C$$

is *exact* if and only if $\nu = \ker \varepsilon$; this is equivalent to asking for $\varepsilon = \operatorname{coker} \nu$.

The equivalence of these two notions follows by the uniqueness of the factorization. Note that this is approximately the correct notion because we really want to say that φ surjects onto the kernel of ψ. But then we note ν basically acts as the image of φ, and ε basically acts as the kernel of ψ.

1.6.2 Sheaves Have Enough Injectives

We now move up to sheaves.

Theorem 1.63. Fix a ringed space $(X, \mathcal{O}_X)$. Then the category $\operatorname{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules has functorial injectives.

Proof. Fix a sheaf $\mathcal{F} \in \operatorname{Mod}(\mathcal{O}_X)$. For each $x \in X$, recall that $\operatorname{Mod}(\mathcal{O}_{X,x})$ has functorial injectives by Theorem 1.58, so we let $I_x(\mathcal{F}_x)$ be an injective module into which $\mathcal{F}_x$ injects. Letting $j_x \colon \{x\} \to X$ denote the inclusion map, we then define

$$\mathcal{I} \coloneqq \prod_{x \in X} (j_x)_* I_x(\mathcal{F}_x).$$

Note that this is an $\mathcal{O}_X$-module because it is the product of $\mathcal{O}_X$-modules. Note that there is a naturally defined map $i \colon \mathcal{F} \to (j_x)_* I_x(\mathcal{F}_x)$ defined by the composite

$$\mathcal{F}(U) \to \mathcal{F}_x \to I_x(\mathcal{F}_x)$$

for each $x \in U$ (and we get the zero map for $x \notin U$). This map i is injective on stalks: we can see that $\mathcal{F}_x$ will embed into the coordinate $(j_x)_* I_x(\mathcal{F}_x)$. Additionally, this construction of i is functorial.

As such, it just remains to show that $\mathcal{I}$ is injective. Suppose that $\mathcal{G} \in \operatorname{Mod}(\mathcal{O}_X)$, and we compute

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \simeq \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* I_x(\mathcal{F}_x)) \simeq \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x(\mathcal{F}_x)).$$

Now, each $I_x(\mathcal{F}_x)$ is an injective object, so the functors $\operatorname{Hom}_{\mathcal{O}_{X,x}}(-, I_x(\mathcal{F}_x))$ is an exact functor for each $x \in X$, so the total functor above is exact, as needed. ■

Corollary 1.64. Fix a topological space X. Then the category $\operatorname{Ab}(X)$ of category of sheaves of abelian groups on X has functorial injectives.

Proof. Set $\mathcal{O}_X$ to be the constant sheaf $\underline{\mathbb{Z}}$ on X. Then $\mathcal{O}_X$ is a sheaf of rings, and $\mathcal{O}_X$-modules are exactly sheaves of abelian groups, so the result follows from Theorem 1.63. ■

1.6.3 Sheaf Cohomology

We can finally define sheaf cohomology.

Definition 1.65 (sheaf cohomology)**.** Fix a topological space X. Because $\operatorname{Ab}(X)$ has enough injectives (by Corollary 1.64) and $\Gamma(X, -) \colon \operatorname{Ab}(X) \to \operatorname{Ab}$ is left exact, we define the *sheaf cohomology functors* as

$$H^\bullet(X, -) \coloneqq R^\bullet \Gamma(X, -).$$

Remark 1.66. It is rather hard to compute $H^\bullet(X, -)$ directly from the definition. For example, it will be helpful to build a large class of acyclic objects and then use Proposition 1.47.

To realize the above remark, we have the following definition.

Definition 1.67 (flasque)**.** Fix a sheaf $\mathcal{F}$ on a topological space X. Then $\mathcal{F}$ is *flasque* if and only if its restriction maps are surjective.

1.7 January 31

Here we go.

1.7.1 Flasque Resolutions

We have "already" seen many examples of flasque sheaves, as explained in the following lemma.

Lemma 1.68. Fix a ringed space $(X, \mathcal{O}_X)$. Then any injective $\mathcal{O}_X$-module is flasque.

Proof. Fix an open subset $U \subseteq X$, and let $j: U \hookrightarrow X$ be the inclusion so that we may set $\mathcal{O}_U := j_!(\mathcal{O}_X|_U)$. Notably, we are realizing $\mathcal{O}_X$ as an $\mathcal{O}_U$-module.

Let's quickly review $j_!$. Explicitly, for a sheaf $\mathcal{F}$ on U, we defined $j_!\mathcal{F}$ as "extension by zero": it is the sheafification of the presheaf

$$\mathcal{O}_U^{\text{pre}}(W) :=\mapsto \begin{cases} \mathcal{F}(W) & \text{if } W \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

Notably, the stalks of $j_!\mathcal{F}$ are $\mathcal{F}_x$ if $x \in U$ and 0 otherwise, which we can see by working with the above sheaf. Notably, there is a canonical map $\mathcal{F} \to (j_!\mathcal{F})|_U$, which we can see is an isomorphism by checking on stalks.

We now proceed with the proof. For each open $V \subseteq U$, there is an injection $\mathcal{O}_V \hookrightarrow \mathcal{O}_U$ (we can see that this is an injection by checking on stalks). As such, for our injective sheaf $\mathcal{I}$, we get the following commutative diagram.

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I}) \arrow[d] \\ \mathcal{I}(U) \arrow[r] & \mathcal{I}(V) \end{tikzcd}$$

We claim that we can place vertical isomorphisms, which will complete the proof because the top row is surjective because $\mathcal{I}$ is an injective $\mathcal{O}_X$-module.

Well, for the vertical morphisms, we write

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) &= \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U^{\text{pre}}, \mathcal{I}) \\ &\overset{*}{=} \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_U^{\text{pre}}|_U, \mathcal{I}|_U) \\ &= \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{I}|_U) \\ &= \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U), \mathcal{I}(U)) \\ &= \mathcal{I}(U). \end{aligned}$$

Here, $\overset{*}{=}$ is just given by restricting an $\mathcal{O}_X$-morphism; it is injective because the map $\mathcal{O}_U \to \mathcal{I}$ can be determined by how it behaves on stalks, which are only seen on U, and it is surjective because one can extend a map $\mathcal{O}_U^{\text{pre}}|_U \to \mathcal{I}|_U$ to a full map $\mathcal{O}_U^{\text{pre}} \to \mathcal{I}$ by having the map $\mathcal{O}_U^{\text{pre}}(W) \to \mathcal{I}(W)$ just be zero (which of course is the only option!). ■

Anyway, let's put our flasque sheaves to good use.

Lemma 1.69. Fix a topological space X. Any flasque sheaf $\mathcal{F} \in \operatorname{Ab}(X)$ is acyclic for $H^\bullet(X, -)$.

Proof. This is a matter of dimension-shifting. We claim that $H^i(X, \mathcal{F}) = 0$ for all $i \geq 1$ and flasque sheaves $\mathcal{F}$. We proceed by induction on i, so we may assume the result for indices less than i. Now, find an injective sheaf with an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$. Letting $\mathcal{G}$ be the quotient, we produce the exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

The two middle terms are flasque (see Lemma 1.68), so the right term is flasque. Now, [Har77, Exercise 1.16] tells us that $\mathcal{G}$ is flasque, and the sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0$$

is exact. Now, the long exact sequence produces the exact sequence

$$H^{i-1}(X, \mathcal{I}) \to H^{i-1}(X, \mathcal{G}) \to H^i(X, \mathcal{F}) \to \underbrace{H^i(X, \mathcal{I})}_{0}.$$

The previous exact sequence shows that the map $H^{i-1}(X, \mathcal{I}) \to H^{i-1}(X, \mathcal{G})$ is surjective map for $i = 1$, and it continues to be surjective for other i by the induction (namely, both terms will be zero). Thus, the map $H^{i-1}(X, \mathcal{G}) \to H^i(X, \mathcal{F})$ is the zero map, so we conclude that $H^i(X, \mathcal{F}) = 0$. ■

And here is the promised sanity check.

Proposition 1.70. Fix a ringed space $(X, \mathcal{O}_X)$. Then $H^\bullet(X, -) = R^\bullet\Gamma(X, -)$, where now $\Gamma(X, -)$ is a functor $\operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Ab}$.

Proof. An injective resolution in $\operatorname{Mod}(\mathcal{O}_X)$ is a flasque resolution by Lemma 1.68 and hence an acyclic resolution in $\operatorname{Ab}(X)$ by Lemma 1.69. So Proposition 1.47 completes the proof. ■

Remark 1.71. A priori, the objects $H^\bullet(X, -)$ were just abelian groups, but Proposition 1.70 assures us that we can usually give this more structure. In particular, if X is an A-scheme for a ring A, then actually $\Gamma(X, -)$ is a functor $\operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(A)$, and the right-derived functors for this $\Gamma(X, -)$ agree with $H^\bullet(X, -)$ upon passing through the forgetful functor because the forgetful functor is exact (namely sending cohomology to cohomology).

1.7.2 Directed Colimits

We would like for our cohomology to vanish at high dimensions when X is a finite-dimensional scheme. The following lemma will be useful.

Lemma 1.72. Fix a Noetherian topological space X and a directed system $\{\mathcal{F}_\alpha\}_{\alpha \in \Lambda}$ of flasque sheaves. Then the directed limit $\varinjlim \mathcal{F}_\alpha$ is flasque.

Proof. Quickly, because colimits commute with colimits, we see that

$$\left(\varinjlim \mathcal{F}_\alpha\right)(U) = \varinjlim \mathcal{F}_\alpha(U).$$

(In particular, this is a sheaf, and then it satisfies the needed universal property by construction; the above equality requires X to be Noetherian.) Now, fix open subsets $V \subseteq U$. Then $\varinjlim \mathcal{F}_\alpha(U) \to \varinjlim \mathcal{F}_\alpha(V)$ is surjective because it is surjective on the components (and colimits commute with colimits), so the above description of our sections completes the proof. ■

1.8 February 2

Let's just get this over with.

1.8.1 More on Directed Colimits

We continue our discussion towards Grothendieck vanishing. We can now see that directed colimits commutes with cohomology.

Proposition 1.73. Fix a Noetherian topological space X. Given a directed system $\{\mathcal{F}_\alpha\}_{\alpha\in\Lambda}$ of sheaves in $\operatorname{Ab}(X)$, there is a natural isomorphism

$$\varinjlim H^\bullet(X,\mathcal{F}_\alpha) = H^\bullet\left(X, \varinjlim \mathcal{F}_\alpha\right)$$

compatible in the long exact sequence.

Proof. For convenience, let $\mathcal{C}$ be the category of directed systems in $\operatorname{Ab}(X)$ indexed by Λ. We would like to exhibit an isomorphism

$$\varinjlim H^\bullet(X,-) \simeq H^\bullet\left(X, \varinjlim -\right)$$

of δ-functors $\mathcal{C}\to\operatorname{Ab}$.

Quickly, we note that we can take a sheaf $\mathcal{F}$ and map it to its "sheaf of discontinuous sections" given by

$$U \mapsto \prod_{x\in U}\mathcal{F}_x.$$

This construction is functorial and can be repeated, so we get functorial flasque resolutions in $\mathcal{F}$.

In particular, let $\mathcal{G}_\alpha^\bullet$ be the produced flasque resolution of $\mathcal{F}_\alpha$. Thus, using Lemma 1.69 with Proposition 1.47 to compute our cohomology, we see

$$\varinjlim H^i(X,\mathcal{F}_\alpha) \simeq \varinjlim h^i(\Gamma(X,\mathcal{G}_\alpha^\bullet)).$$

Now, taking directed colimits is exact, so this is

$$h^i\left(\varinjlim \Gamma(X,\mathcal{G}_\alpha^\bullet)\right).$$

Taking global sections commutes with directed colimits (here we use that X is Noetherian with [Har77, Exercise 1.11]), so this is

$$h^i\left(\Gamma\left(X,\varinjlim \mathcal{G}_\alpha^\bullet\right)\right).$$

Now, taking these directed colimits commutes with taking stalks, so it will be exact on sheaves, so we have the resolution

$$0\to\varinjlim\mathcal{F}_\alpha\to\varinjlim\mathcal{G}_\alpha^\bullet,$$

so our last cohomology is the desired $H^i\left(X,\varinjlim\mathcal{F}_\alpha\right)$. Everything has been done on the level of resolutions, so we have produced a bona fide isomorphism of δ-functors. ■

Example 1.74. Cohomology also commutes with infinite direct sums because these are directed colimits (of the finite sums!).

1.8.2 Cohomology on Closed Subsets

Next up, one reduction we will want to make is to go down closed subschemes, so we have the following.

Lemma 1.75. Fix a closed subset $j\colon Y \to X$ of a topological space. Given a sheaf $\mathcal{F} \in \mathrm{Ab}(Y)$, there is a natural isomorphism

$$H^\bullet(Y, \mathcal{F}) = H^\bullet(X, j_*\mathcal{F})$$

compatible in the long exact sequence.

Proof. We are asking for an isomorphism of δ-functors $\mathrm{Ab}(Y) \to \mathrm{Ab}$. The point is that, by computing stalks, j_* is an exact functor, and by computing sections, j_* sends flasque sheaves to flasque sheaves. So we use the usual combination of Lemma 1.69 with Proposition 1.47 so that a flasque resolution $\mathcal{G}^\bullet$ of $\mathcal{F}$ produces the sequence of natural isomorphisms

$$H^i(Y, \mathcal{F}) \simeq h^i(\Gamma(Y, \mathcal{G}^\bullet)) = h^i(\Gamma(X, j_*\mathcal{G}^\bullet)) \simeq H^i(X, j_*\mathcal{F}).$$

Everything was done on the level of resolutions, so this is an isomorphism of δ-functors. ■

Remark 1.76. If $Y \subseteq X$ is not closed, j_* need not be exact.

With our closed subsets, we will want the notion of restricting sheaves.

Definition 1.77. Fix a topological space X and closed subset $i\colon Z \to X$ and open subset $j\colon U \to X$ where $U = X \setminus Z$. For a sheaf $\mathcal{F}$ on X, we set $\mathcal{F}_Z := i_*(\mathcal{F}|_Z)$ and $\mathcal{F}_U := j_!(\mathcal{F}|_U)$.

Remark 1.78. Fix everything as above. Computing stalks, we see that there is an exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0$$

of sheaves on X, provided $\mathcal{F} \in \mathrm{Ab}(X)$. We will not bother to give the construction of the maps; they can be given on the level of presheaves, where the left map is essentially an inclusion, and the right map is essentially a restriction.

1.9 February 5

Here we go.

1.9.1 Grothendieck Vanishing

Last class we began the proof of the following result, which I have moved to today because it will be our focus for today.

Theorem 1.79 (Grothendieck vanishing). Fix a Noetherian topological space X of dimension n. Given $\mathcal{F} \in \mathrm{Ab}(X)$, we have $H^i(X, \mathcal{F}) = 0$ for $i > n$.

Proof. We proceed by induction on the collection of pairs $(n, m) \in \{(-1, 0)\} \cup \mathbb{N} \times \mathbb{Z}^+$, where $n = \dim X$ and m is the number of irreducible components of X. For our induction, we order $\{(-1, 0)\} \cup \mathbb{N} \times \mathbb{Z}^+$ lexicographically; here $\dim \varnothing = -1$. In other words, we will induct on the dimension, and within that induction, we will induct on the number of irreducible components.

Anyway, we proceed in steps.

1. We begin by reducing to X being irreducible; fix X of dimension n, and assume all lower results (lower dimension, fewer irreducible components if of dimension n). We may assume that X is nonempty, so choose an irreducible component $Z \subseteq X$, and set $U \coloneqq X \setminus Z$. Notably, for this paragraph (making the reduction), we are assuming the statement for Z, so any sheaf $\mathcal{F} \in \mathrm{Ab}(X)$ has the exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0$$

by Remark 1.78. By the long exact sequence, it will be enough to show that $H^i(X, \mathcal{F}_Z) = H^i(X, \mathcal{F}_U) = 0$ for all $i > n$. For one, note that $H^i(X, \mathcal{F}_Z) = 0$ for all $i > n$ because $H^i(X, \mathcal{F}_Z) = H^i(Z, \mathcal{F}|_Z)$ by Lemma 1.75, and we have assumed the conclusion for Z.

So it remains to discuss $H^\bullet(X, \mathcal{F}_U)$. We begin by claiming that there is a sheaf $\mathcal{G}$ on $\overline{U}$ such that $\mathcal{F}_U = j_*\mathcal{G}$ where $j\colon \overline{U} \hookrightarrow X$ is the inclusion. Indeed, Remark 1.78 provides an exact sequence

$$0 \to (\mathcal{F}_U)_{X \setminus \overline{U}} \to \mathcal{F}_U \to (\mathcal{F}_U)_{\overline{U}} \to 0,$$

but $(\mathcal{F}_U)_{X \setminus \overline{U}} = 0$ by computing stalks: any nonzero stalk must have $p \in X \setminus \overline{U}$ and $(\mathcal{F}|_U)_p \neq 0$ and hence $p \in U$ also, but no such p suffices. Thus, we see

$$\mathcal{F}_U \cong (\mathcal{F}_U)_{\overline{U}} \cong j_*(\mathcal{F}_U|_{\overline{U}}),$$

so $\mathcal{G} \coloneqq \mathcal{F}_U|_{\overline{U}}$ will do the trick.

We are now ready to show that $H^i(X, \mathcal{F}_U) = 0$ for $i > n$. Well, by the previous paragraph, we see

$$H^i(X, \mathcal{F}_U) = H^i(X, j_*(\mathcal{F}_U|_{\overline{U}})) = H^i(\overline{U}, \mathcal{F}_U|_{\overline{U}})$$

by Lemma 1.75. But now, by inductive hypothesis, this vanishes for $i < n$ because $\overline{U}$ has one fewer irreducible component than X and no higher dimension.

2. We handle some base cases. When $\dim X = -1$, we have $X = \varnothing$, where there is nothing to do. We will also handle $\dim X = 0$ while we're here. The previous step allows us to assume that X is irreducible.

Quickly, we remark that the only closed subsets of X are $\{\varnothing, X\}$. Indeed, of course these sets are closed. Conversely, if $Z \subseteq X$ is a minimally closed subset, then minimality forces Z to be irreducible, but then $\varnothing \subseteq Z \subseteq X$ requires $Z \in \{\varnothing, X\}$ because $\dim X = 0$.

Thus, the previous paragraph implies that X has the indiscrete topology. In particular, all sheaves are flasque because evaluating a sheaf on $\varnothing$ makes a single point, so $H^i(X, -)$ vanishes for $i > 0$, as needed.

3. Fix a sheaf $\mathcal{F}$ of abelian groups on X; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case where $\mathcal{F}$ is finitely generated (as a sheaf—namely, there are finitely many sections such that the restrictions of those sections generate $\mathcal{F}(U)$ for all open $U \subseteq X$). Well, define the set

$$B \coloneqq \bigcup_{\text{open } U \subseteq X} \mathcal{F}(U),$$

and let A denote the collection of finite subsets. Notably, by restriction, A becomes a set which is directed by the collection of open sets on X. Anyway, for $\alpha \in A$, let $\mathcal{F}_\alpha$ denote the sheaf generated by the sections in α, and we conclude by noting that $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$, so

$$H^i(X, \mathcal{F}) = \varinjlim H^i(X, \mathcal{F}_\alpha)$$

by Proposition 1.73, which vanishes for $i > \dim X$ by assumption of this step.

4. Fix a finitely generated sheaf $\mathcal{F}$ of abelian groups on X; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case where $\mathcal{F}$ is generated by a single section (and its restrictions). Indeed, assuming we have the case of generated by a single section, we may proceed by induction: if $\mathcal{F}$ is generated

by n sections α (so that $\mathcal{F} = \mathcal{F}_\alpha$), let $\alpha' \subsetneq \alpha$ be a proper subset of sections, and let $\mathcal{F}_{\alpha'}$ be the sheaf generated by α'. Then we have the exact sequence

$$0 \to \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G} \to 0,$$

and we note that $\mathcal{G}$ is generated by fewer than n elements; explicitly, $\mathcal{G}$ is generated by the images of the sections in $\alpha \setminus \alpha'$. To be explicit, one can see that $\mathcal{F}_{\alpha\setminus\alpha'} \twoheadrightarrow \mathcal{G}$ by checking on stalks. Thus, $H^i(X, \mathcal{F}_{\alpha'}) = H^i(X, \mathcal{G}) = 0$ for $i > \dim X$ by assumption, so the long exact sequence enforces $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.

5. Fix a sheaf $\mathcal{F}$ of abelian groups on X generated by a single section $s \in \mathcal{F}(U)$ where $U \subseteq X$ is open; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case of subsheaves of $\underline{\mathbb{Z}}_U$.

   Well, we may assume that U is nonempty (or else $\mathcal{F} = 0$, and there is nothing to be done). Now, there is a map $\underline{\mathbb{Z}}_U \to \mathcal{F}$ given by sending $1 \mapsto s$ on U (working on the presheaf) and then appropriately restricting elsewhere. This map is surjective by hypothesis on $\mathcal{F}$ (indeed, it is surjective on the level of the presheaves), so we let $\mathcal{K}$ denote the kernel, providing the short exact sequence

$$0 \to \mathcal{K} \to \underline{\mathbb{Z}}_U \to \mathcal{F} \to 0.$$

   Now, $H^i(X, \mathcal{K}) = H^i(X, \underline{\mathbb{Z}}_U) = 0$ for $i > \dim X$ by assumption of this section, so $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$ by the long exact sequence.

6. In this reduction step, we will use that X is irreducible. Fix a subsheaf $\mathcal{F}$ of $\underline{\mathbb{Z}}_U$ for open $U \subseteq X$; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case $\mathcal{F} = \underline{\mathbb{Z}}_U$. If $\mathcal{F} = 0$, there is nothing to do.

   Otherwise, we may let d be the smallest positive integer such that $d \in \mathcal{F}_x$ as $x \in U$ varies (notably, some $\mathcal{F}_x$ is nonzero, so a d exists). Now,

$$V := \{x \in U : d \in \mathcal{F}_x\}$$

   is nonempty and open ($d \in \mathcal{F}_x$ means that $d \in \mathcal{F}(U')$ for some open $U' \subseteq U$), and $\mathcal{F}_x = d\mathbb{Z}$ for each $x \in V$, so $\mathcal{F}|_V = d\underline{\mathbb{Z}}$, so we have an equality $\mathcal{F}_V = d\underline{\mathbb{Z}}_V$. So we have an exact sequence

$$0 \to d\underline{\mathbb{Z}}_V \to \mathcal{F} \to \mathcal{F}/d\underline{\mathbb{Z}}_V \to 0$$

   of sheaves on X. By assumption, we have the result for $d\underline{\mathbb{Z}}_V$. Now, $\mathcal{F}/d\underline{\mathbb{Z}}_V$ is supported on $\overline{U \setminus V}$ by construction of V, and $\overline{U \setminus V}$ will have smaller dimension than X because X is irreducible, so we get the result for $\mathcal{F}/d\underline{\mathbb{Z}}_V$ by the inductive hypothesis on X. So the long exact sequence purchases the result for $\mathcal{F}$.

7. We complete the induction. We may assume that X is irreducible of dimension n, and we may assume that $\mathcal{F} = \underline{\mathbb{Z}}_U$. Well, we have an exact sequence

$$0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{X\setminus U}$$

   by Remark 1.78. Because X is irreducible, $X \setminus U$ has smaller dimension, $H^i(X, \underline{\mathbb{Z}}_{X\setminus U}) = 0$ for $i > \dim X - 1$ (also using Lemma 1.75). Additionally, $\underline{\mathbb{Z}}$ is flasque because X is irreducible—all open subsets of X are connected, so $\underline{\mathbb{Z}}(U) = \mathbb{Z}$ always—and hence acyclic by Lemma 1.69, so we conclude $H^i(X, \underline{\mathbb{Z}}_U) = 0$ for $i > \dim X$ by the long exact sequence. ■

While we're here, let's do an example computation to show Theorem 1.79 is sharp.

Exercise 1.80. Fix a field k, and set $X := \mathbb{A}^1_k$. Given distinct points $P, Q \in X$, set $U := X \setminus \{P, Q\}$, and we see

$$H^1(X, \underline{\mathbb{Z}}_U) \neq 0.$$

Proof. Let $j\colon U \hookrightarrow X$ denote the inclusion so that $\underline{\mathbb{Z}}_U = j_!(\underline{\mathbb{Z}})$. Note that U is irreducible, so any open subsets are connected, so we may as well have

$$\underline{\mathbb{Z}}_U(V) = \begin{cases} \mathbb{Z} & \text{if } V \subseteq U \text{ and } V \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Anyway, note that we have the exact sequence

$$0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{\{P,Q\}} \to 0$$

by Remark 1.78, so we get a long exact sequence

$$H^0(X, \underline{\mathbb{Z}}_U) \to H^0(X, \underline{\mathbb{Z}}) \to H^0(X, \underline{\mathbb{Z}}_{\{P,Q\}}) \to H^1(X, \underline{\mathbb{Z}}_U) \to H^1(X, \underline{\mathbb{Z}}).$$

Because X is irreducible, we see that $\underline{\mathbb{Z}}$ is flasque (all open subsets are connected), so the rightmost term vanishes by Lemma 1.69. Also, above we noted that $H^0(X, \underline{\mathbb{Z}}_U) = 0$, and computing global sections on the other sheaves implies that we have

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^1(X, \underline{\mathbb{Z}}_U) \to 0,$$

so the rightmost map cannot be surjective. ■

THEME 2

COHOMOLOGY ON SCHEMES

2.1 February 7

Today we will compute cohomology on affine schemes.

2.1.1 Cohomology on Affine Schemes

To build our cohomology on $\operatorname{Spec} A$, we pick up the following checks.

Proposition 2.1. Fix an injective A-module I, where A is Noetherian. Then $\widetilde{I}$ is a flasque sheaf on $\operatorname{Spec} A$.

Proof. This proof is somewhat annoying, so we omit and refer to [Har77, Proposition III.3.4]. The main idea is to do Noetherian induction on $\operatorname{Supp} \widetilde{I}$. ■

Proposition 2.2. Fix a Noetherian ring A. Then quasicoherent sheaves on $X := \operatorname{Spec} A$ are acyclic.

Proof. Fix an A-module M, and we want to show that $\widetilde{M}$ is acyclic. Well, fix an injective resolution $0 \to M \to I^\bullet$, which by Proposition 2.1 produces an acyclic resolution

$$0 \to \widetilde{M} \to \widetilde{I}^\bullet.$$

Lemma 1.69 followed by Proposition 1.47 allows us to compute cohomology using this resolution, but this is just

$$0 \to M \to I^\bullet$$

upon taking global sections, which is exact, so the cohomology of $\widetilde{M}$ must vanish. ■

Thus, we see quasicoherent sheaves on affine schemes are well-behaved. This turns out to characterize affine schemes.

Theorem 2.3 (Serre)**.** Fix a Noetherian scheme X. Then the following are equivalent.

(i) X is affine.

(ii) $H^i(X, \mathcal{F}) = 0$ for all quasicoherent sheaves $\mathcal{F}$ on X and indices $i > 0$.

(iii) $H^1(X, \mathcal{I}) = 0$ for all quasicoherent sheaves $\mathcal{I}$ of ideals on X.

Proof. Note (i) implies (ii) is Proposition 2.2, and (ii) implies (iii) with no content. So the main content of the argument is (iii) implies (i). We proceed in steps.

1. To set ourselves up, we recall [Har77, Exercise II.2.17], which is on the homework, which asserts that X is affine if and only if there is a finite set $\{f_1, \ldots, f_r\}$ of global sections generating $A := \Gamma(X, \mathcal{O}_X)$ such that the open subschemes
$$X_{f_i} := \{x \in X : (f_i)_x \notin \mathfrak{m}_x\}$$
are affine.

2. We claim that all closed points $p \in X$ have some $f \in A$ such that X_f is affine and $p \in X_f$. Well, let $U \subseteq X$ be some affine open neighborhood of $p \in X$, and let $Y := X \setminus U$. Note that we have a short exact sequence
$$0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(p) \to 0$$
of sheaves on X, where $\mathcal{I}_\bullet$ is the ideal sheaf of a closed subscheme, and $k(p)$ refers to the skyscraper sheaf at p. Notably exactness of this sequence can be checked on stalks; more explicitly, the left map is just an isomorphism on $X \setminus \{p\}$, and on $\{p\}$ this is $0 \to \mathfrak{m}_p \to \mathcal{O}_{X,p} \to k(p) \to 0$, which is exact because p is closed (!). Now, by (iii), we have an exact sequence
$$H^0(X, \mathcal{I}_Y) \to H^0(X, k(p)) \to \underbrace{H^1(X, \mathcal{I}_{Y \cup \{p\}})}_{0},$$
so we can find some $f \in \Gamma(X, \mathcal{I}_Y)$ such that $f \notin \mathfrak{m}_p$ by this surjectivity, so $p \notin X_f$ by construction. But now $f \in \mathcal{I}_Y$ means that $Y \cap X_f \neq \varnothing$, so $X_f \subseteq U$, and in fact, we see $U_f \subseteq X_f \subseteq U_f$, so X_f is affine (because U is affine means U_f is affine), so we have completed the proof of the claim.

3. We now need to produce lots of closed points on X. For this, we claim that any quasicompact scheme X has a closed point. Well, fix a finite affine open cover $\{U_i\}_{i=1}^n$ of X, doable because X is quasicompact, and we may suppose that no affine open subset is covered by the union of the other ones (for then we could remove this open subset from the finite collection). Then we see that
$$U_1 \setminus (U_2 \cup U_3 \cup \cdots \cup U_n)$$
is a closed nonempty subset of U_1, so it has a closed point $p \in U_1$ corresponding to a maximal ideal of $\Gamma(U_1, \mathcal{O}_{U_1})$ which contains the ideal cut out by the complement of $(U_2 \cup \cdots \cup U_n)$. We claim that p is still closed in X.

   Well,
$$X \setminus \{p\} = \bigcup_{i=1}^n (U_i \setminus \{p\}) = (U_1 \setminus \{p\}) \cup \bigcup_{i=2}^n U_i$$
is open, so we are okay.

4. We exhibit a finite set $\{f_1, \ldots, f_r\} \subseteq A$ of global sections such that $X_{f_\bullet}$ are affine and cover X. Well, let X_{cl} denote the set of closed points of X, and the second step shows that each $p \in X_{\text{cl}}$ has some $f_p \in A$ such that $p \in X_{f_p}$ and X_{f_p} is affine.

   We want $\{X_{f_p}\}_{p \in X_{\text{cl}}}$ to cover X, so we consider
$$Z := X \Bigg\backslash \bigcup_{p \in X_{\text{cl}}} X_{f_p},$$
which is a closed subset of X, so we give Z the reduced subscheme structure. Well, suppose for the sake of contradiction that Z is nonempty. Because X is quasicompact, so is the closed subset Z, so Z has a closed point $p \in Z$. But then p is still closed in X: we know $Z \setminus \{p\}$ is open in X, so there is an open $U \subseteq X$ such that $Z \setminus \{p\} = Z \cap U$, so $X \setminus \{p\} = (X \setminus Z) \cup U$. This is a contradiction because $p \in X_{f_p}$ and so cannot be in Z.

   We are now done: X is quasicompact, so we can reduce $\{X_{f_p}\}_{p \in X_{\text{cl}}}$ to a finite subcover, which completes this step and hence the proof.

5. We complete the proof. In particular, to plug into the first step, fix $\{f_1, \ldots, f_r\}$ as in the previous step, and we must show that $(f_1, \ldots, f_r) = A$.

   We want to show that the map $\alpha\colon \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X$ given by $(a_1, \ldots, a_r) \mapsto (a_1 f_1 + \cdots + a_r f_r)$ is surjective on global sections. Certainly α is surjective on stalks: any $x \in X$ can be placed in some X_{f_r}, and then $f_r\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is already an isomorphism. But now $\mathcal{K} := \ker \alpha$ produces the exact sequence

$$\Gamma\left(X, \mathcal{O}_X^{\oplus r}\right) \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X) \to H^1(X, \mathcal{K}).$$

   To complete this step, we would like to know that $H^1(X, \mathcal{K}) = 0$.

   Namely, we will claim that $H^1\left(X, \mathcal{K} \cap \mathcal{O}_X^{\oplus i}\right) = 0$ for each $i \in \{0, \ldots, r\}$ by induction. There is nothing to say for $i = 0$. Then if $H^1(X, \mathcal{F} \cap \mathcal{O}_X^{\oplus(i-1)}) = 0$, we have an exact sequence

$$0 \to \mathcal{K} \cap \mathcal{O}_X^{\oplus(i-1)} \to \mathcal{K} \cap \mathcal{O}_X^{\oplus i} \to \mathcal{Q}_i \to 0,$$

   where $\mathcal{Q}_i$ is the needed sheaf. One can see that $\mathcal{Q}_i$ is a subsheaf of $\mathcal{O}_X$ (namely, it is an ideal sheaf) by applying the Snake lemma to the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \cap \mathcal{O}_X^{\oplus(i-1)} \arrow[r] \arrow[d, hook] & \mathcal{K} \cap \mathcal{O}_X^{\oplus i} \arrow[r] \arrow[d, hook] & \mathcal{Q}_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus(i-1)} \arrow[r] & \mathcal{O}_X^{\oplus i} \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

   Thus, $H^1(X, \mathcal{Q}_i) = 0$ by our hypothesis (iii), so the long exact sequence implies that $H^1\left(X, \mathcal{K} \cap \mathcal{O}_X^{\oplus i}\right) = 0$, where we are now using the inductive hypothesis. ■

2.2 February 9

We spent most of the class completing the proof of Theorem 2.3. I have edited into those notes for continuity reasons.

Remark 2.4. Note the identity map $A \to \Gamma(X, \mathcal{O}_X)$ induces a map $\varphi\colon X \to \operatorname{Spec} A$ via the adjunction. So after we produce the sections $(f_1, \ldots, f_r)$ with X_{f_i} affine, it might appear that we are done because we might be able to glue the $X_{f_i} \cong \operatorname{Spec} A_{f_i}$ into making φ an isomorphism. But this does not work: indeed, φ need not even be surjective!

2.2.1 More Cohomology on Affine Schemes

Let's see an application of some of the work we've done.

Corollary 2.5. Fix a Noetherian scheme X. Any quasicoherent sheaf $\mathcal{F}$ on X can be embedded into a flasque quasicoherent sheaf.

Proof. The point is to reduce to the affine case and then use Proposition 2.1. Let $\{U_i\}_{i=1}^n$ be a finite affine open cover of X, where $U_i = \operatorname{Spec} A_i$. Because $\mathcal{F}$ is quasicoherent, we can find an A_i-module M_i such that $\mathcal{F}|_{U_i} \cong M_i$, and then we may embed M_i into an injective A_i-module I_i. So we have injections $\mathcal{F}|_{U_i} \to \widetilde{I}_i$ on U_i, so we can glue these into a map

$$\mathcal{F} \to \bigoplus_{i=1}^n (j_i)_* \widetilde{I}_i,$$

where $j_i\colon U_i \to X$ is the inclusion. Notably, $(j_i)_*$ sends quasicoherent sheaves to quasicoherent sheaves (note X is Noetherian), so $\bigoplus_{i=1}^n (j_i)_* \widetilde{I}_i$ is still quasicoherent, and the above map is injective because we can check injectivity on stalks and so on the affine open cover $\{U_i\}_{i=1}^n$. Lastly, $(j_i)_*$ sends flasque sheaves to flasque sheaves, so our sum is still flasque. ■

2.3 February 12

Today we will discuss Čech cohomology.

2.3.1 Čech Cohomology to Groups

For today, X will be a topological space, $\mathcal{F}$ will be a sheaf of abelian groups on X, and $\mathcal{U} := \{U_i\}_{i\in I}$ is an open cover of X, and we will fix a well-ordering on I. For indices $i_0, \ldots, i_p \in I$, we define the notation

$$U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

We can now define our complex.

Definition 2.6 (Čech complex)**.** Fix notation as above. For each $p \geq 0$, define

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

and the map $d\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}},$$

where the hat means an omission of the index.

Remark 2.7. One can check directly that $d^{p+1} \circ d^p = 0$, which we will not write out. Thus, $(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}), d^\bullet)$ is in fact a complex of abelian groups.

We now define a convention our indices. Given a class $\alpha \in C^p(\mathcal{U}, \mathcal{F})$, for arbitrary indices $i_0, \ldots, i_p \in I$ (perhaps not in order), we define

$$\alpha_{i_0,\ldots,i_p} := \begin{cases} 0 & \text{if there is a repeated index,} \\ (-1)^{\operatorname{sgn}\sigma} \alpha_{\sigma(i_0),\ldots,\sigma(i_p)} & \text{where } \sigma(i_0) < \cdots < \sigma(i_p). \end{cases}$$

Note that even if $\sigma(i_0) < \cdots < \sigma(i_p)$ fails to hold, multiplicativity of the sign means that we still have the equation

$$\alpha_{i_0,\ldots,i_p} = (-1)^{\operatorname{sgn}\sigma} \alpha_{\sigma(i_0),\ldots,\sigma(i_p)},$$

so our notation makes sense.

Remark 2.8. Our differential also still makes sense with these indices. By multiplicativity of the sign, it will suffice to prove the result by induction on the length of σ. Namely, supposing that

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}},$$

we will show that

$$(d^p\alpha)_{\sigma(i_0),\ldots,\sigma(i_{p+1})} := \sum_{j=0}^{p+1} (-1)^j \alpha_{\sigma(i_0),\ldots,\widehat{\sigma(i_j)},\ldots,\sigma(i_{p+1})},$$

where σ is a transposition $(\ell, \ell+1)$. Namely, the left-hand side is multiplied by -1, so we need the right-hand side to also be multiplied by -1. For the terms $j \notin \{\ell, \ell+1\}$, then we get our sign on each term. Lastly, $j \in \{\ell, \ell+1\}$ swap in the summation, so their signs also suitably swap with each other.

Remark 2.9. Even if there are repeated indices, we still achieve

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1}(-1)^j\alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}.$$

The left-hand side is zero, and the right-hand side is zero at almost every term, except perhaps when j is an index repeated exactly twice, but in this case, the two times j is an index repeated exactly twice will have cancelling signs, so the entire thing still vanishes.

Anyway, we can now define our cohomology.

Definition 2.10 (Čech cohomology)**.** Fix notation as above. We define $\check{H}^p(\mathcal{U},\mathcal{F}) := h^p(C^\bullet(\mathcal{U},\mathcal{F}))$.

Let's do so some sample computations.

Example 2.11. If $\mathcal{U} = \{X\}$, then we see that

$$C^p(\mathcal{U},\mathcal{F}) = \begin{cases} \Gamma(X,\mathcal{F}) & \text{if } p = 0, \\ 0 & \text{else,} \end{cases}$$

so $\check{H}^p(\mathcal{U},\mathcal{F})$ is the same.

Remark 2.12. Čech cohomology frequently does not actually produce a long exact sequence, so perhaps it is not technically a cohomology theory. Indeed, using Example 2.11, it is not the case that a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of sheaves of abelian groups on X will produce a short exact sequence

$$0 \to \Gamma(X,\mathcal{F}') \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'') \to 0.$$

Example 2.13. We always have $\check{H}^0(\mathcal{U},\mathcal{F}) = \Gamma(X,\mathcal{F})$. Indeed, $\Gamma(X,\mathcal{F})$ is by definition the kernel of the map

$$\prod_i \mathcal{F}(U_i) \stackrel{d^1}{\to} \prod_{i<j} \mathcal{F}(U_i \cap U_j),$$

where $(d^1\alpha)_{ij} = \alpha_i - \alpha_j$. But this is exactly $\Gamma(X,\mathcal{F})$ by the sheaf conditions: we can uniquely glue sections on the $U_\bullet$ which agree on the intersections.

Exercise 2.14. Fix a field k and $X := \mathbb{P}^1_k = \operatorname{Proj} k[x,y]$, and let $U := D_+(x)$ and $V := D_+(y)$ make up the standard affine open cover $\mathcal{U} := \{U,V\}$ of X. We compute the Čech cohomology of the sheaf $\mathcal{F} = \mathcal{O}_X(1)$.

Proof. We begin by computing our complex.

- We compute $C^0(\mathcal{U},\mathcal{O}_X(1)) = \Gamma(U,\mathcal{O}_X(1)) \times \Gamma(V,\mathcal{O}_X(1)) = xk[y/x] \oplus yk[x/y]$.
- We compute $C^1(\mathcal{U},\mathcal{O}_X(1)) = xk[y/x,x/y]$.
- For $p \geq 2$, we have $C^2(\mathcal{U},\mathcal{O}_X(1)) = 0$ because our cover has only two elements anyway.

The only nontrivial differential is the map $C^0(\mathcal{U},\mathcal{O}_X(1)) \to C^1(\mathcal{U},\mathcal{O}_X(1))$, which we see "restricts" x and y to their images in $xk[y/x,x/y] = \Gamma(D_+(xy),\mathcal{O}_X(1))$.

In total, we may compute

$$\check{H}^0(\mathcal{U}, \mathcal{O}_X(1)) = xk[y/x] \cap yk[x/y] = kx \oplus ky,$$

which is correctly the global sections. Continuing,

$$\check{H}^1(\mathcal{U}, \mathcal{O}_X(1)) = \frac{\ker d^1}{\operatorname{im} d^0} = \operatorname{coker} d^0 = 0$$

because d^0 is surjective: any element of $xk[y/x, x/y]$ can be separated into polynomials in x and polynomials in y, so it can be realized from $C^0(\mathcal{U}, \mathcal{O}_X(1))$. Lastly, we note $\check{H}^p(\mathcal{U}, \mathcal{O}_X(1)) = 0$ for $p \geq 2$ because $C^p(\mathcal{U}, \mathcal{O}_X(1)) = 0$ there. ■

2.4 February 14

Today we compare Čech and derived cohomology.

2.4.1 Čech Cohomology to Sheaves

For today, X will be a topological space, $\mathcal{F}$ will be a sheaf of abelian groups on X, and $\mathcal{U} := \{U_i\}_{i \in I}$ is an open cover of X, and we will fix a well-ordering on I. We fix notation as previous.

So we get a complex on sheaves as follows, upgrading our previous complex.

Definition 2.15 (Čech complex)**.** Fix notation as above. For each $p \geq 0$, define

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} j_* \mathcal{F}|_{U_{i_0, \ldots, i_p}},$$

where j is the needed inclusion, and the map $d\colon \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p \alpha)_{i_0, \ldots, i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}},$$

where the hat means an omission of the index.

One checks as usual that we do indeed have a complex, which again we will not write out.

Example 2.16. Fix everything as above. Then we can compute

$$\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F}).$$

We now begin doing our comparison.

Lemma 2.17. Fix notation as above. Then there is a natural transformation $\varepsilon\colon \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ given by $\varepsilon_V(s) := (s|_{U_i \cap V})_{U_i \in \mathcal{U}}$. In fact,

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$$

is an exact sequence of sheaves.

Proof. We won't bother to check that ε is in fact a morphism of sheaves. For the exactness, note the sequence

$$0 \to \Gamma(V, \mathcal{F}) \to \Gamma\left(V, \mathcal{C}^0(\mathcal{U}, \mathcal{F})\right) \to \Gamma\left(V, \mathcal{C}^1(\mathcal{U}, \mathcal{F})\right)$$

is exact for all V by the sheaf condition on $\mathcal{F}$. For exactness elsewhere, we need exactness of

$$\mathcal{C}^{p-1}(\mathcal{U},\mathcal{F}) \to \mathcal{C}^p(\mathcal{U},\mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F}).$$

This can be checked on stalks, which is done by hand. We won't write out the details. ■

Proposition 2.18. Fix everything as above. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for $p > 0$.

Proof. Fix some $p \geq 0$ for the time being. For $\mathcal{F}$ is flasque, the restrictions $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ will also be flasque, so the pushforward $j_*(\mathcal{F}|_{U_{i_0,\ldots,i_p}})$ will also continue to be flasque, so the product $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ will be flasque. Thus, Lemma 1.69 and Proposition 1.47 allow us to compute $H^p(X,\mathcal{F})$ via this resolution. To complete the proof, we note

$$H^p(X,\mathcal{F}) = h^p(\Gamma(X,\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}))) = h^\bullet(C^\bullet(\mathcal{U},\mathcal{F})) = \check{H}^p(\mathcal{U},\mathcal{F}).$$

The left-hand side vanishes by Lemma 1.69, so the right-hand side also vanishes. ■

2.4.2 The Čech Comparison Theorem

So we have some acyclic objects agreeing. We are now ready to construct the needed natural map.

Lemma 2.19. Fix everything as above. Then there is natural map $\check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$.

Proof. Fix an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Because the $\mathcal{I}^\bullet$ are injective, an inductive argument produces a morphism of the complexes $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \to \mathcal{I}^\bullet$. Then this morphism of complexes induces a morphism on cohomology, as desired. Choosing the injectives functorially in $\mathcal{F}$ promises that this map is natural.

Alternatively, one can check that this map does not depend on the choice of $\mathcal{I}$ by doing some homotopy computation using the argument of Proposition 1.34; naturality follows by choosing the injective resolutions to have maps between them a priori. Being explicit, we can produce a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{C}(\mathcal{U},\mathcal{F})^\bullet \arrow[r] \arrow[d] & \mathcal{I}^\bullet \arrow[d] \\ 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{C}(\mathcal{U},\mathcal{G})^\bullet \arrow[r] & \mathcal{J}^\bullet \end{tikzcd}$$

where the rightmost square commutes up to some homotopy. Taking cohomology produces the needed commuting square for our map to be natural. ■

We now check when this map is an isomorphism.

Theorem 2.20. Fix a Noetherian separated scheme X, and let $\mathcal{U}$ be an affine open cover of X, and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then for all $p \geq 0$, the natural map $\check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$ of Lemma 2.19 are isomorphisms.

Proof. We induct on p. For $p = 0$, one uses Example 2.13; we won't bother to check that the map is the natural one. Additionally, we remark that if $\mathcal{F}$ is flasque, we get the result for $p > 0$ by Proposition 2.18.

Now, fix some quasicoherent sheaf $\mathcal{F}$ for which we want to show that $\check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$ is an isomorphism. Then Corollary 2.5 allows us to embed $\mathcal{F}$ into a flasque quasicoherent sheaf $\mathcal{F}$; letting $\mathcal{Q}$ be the quotient, we get the exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0.$$

Note that X being separated implies that $U_{i_0,\ldots,i_p}$ is affine, so Proposition 2.18 implies $H^1(U_{i_0,\ldots,i_p},\mathcal{F}) = 0$, so

$$0 \to \mathcal{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{R}(U_{i_0,\ldots,i_p}) \to 0$$

is exact. Thus,

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

is an exact sequence of the Čech complexes, so we get a long exact sequence of Čech cohomology. Notably, we even have a morphism of the exact sequences of the above sequence with injective resolutions of $\mathcal{F}$ and $\mathcal{G}$ and $\mathcal{Q}$. Being explicit, there is going to be a morphism of short exact sequences as follows.

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U}, \mathcal{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{I}^\bullet) \arrow[r] & \Gamma(X, \mathcal{J}^\bullet) \arrow[r] & \Gamma(X, \mathcal{K}^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Now, if $p \geq 1$, then $\check{H}^p(\mathcal{U}, \mathcal{G}) = H^p(\mathcal{U}, \mathcal{G}) = 0$ by being flasque (see also Proposition 2.18) so we get the commutative diagram as follows.

$$\begin{tikzcd}
\check{H}^{p-1}(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d, "\varepsilon"'] & \check{H}^{p-1}(\mathcal{U}, \mathcal{Q}) \arrow[r] \arrow[d, "\varepsilon"'] & \check{H}^p(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\varepsilon"'] & 0 \\
H^{p-1}(X, \mathcal{G}) \arrow[r] & H^{p-1}(X, \mathcal{Q}) \arrow[r] & H^p(X, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

The two maps on the left are isomorphisms by the inductive hypothesis, so the map ε (on cokernels!) must be an isomorphism as well by a Five lemma. ■

2.5 February 16

Here we go.

2.5.1 Upgrading Čech Comparison

Here is a quick remark.

Remark 2.21. Fix a scheme X over $\operatorname{Spec} A$. Then the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of A-modules, so the cohomology $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$ are A-modules as well. Analogously, $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of quasicoherent A-modules, so the induced map Lemma 2.19 can be checked to be a map of A-modules by making everything into a morphism of A-modules. So Theorem 2.20 explains that $H^\bullet(X, \mathcal{F})$ is an A-module when X is a Noetherian separated A-scheme.

We will want to upgrade Theorem 2.20 somewhat; notably, Theorem 2.20 has some strong hypotheses on X and $\mathcal{F}$, which we will work to remove. We will succeed in removing them for H^1.

Our method will be based on allowing the open cover $\mathcal{U}$ to get finer. So we should define what is meant by a refinement.

Definition 2.22 (refinement)**.** A *refinement* of an open cover $\mathcal{U}$ on X is an open cover $\mathcal{V}$ such that there is a mp $\lambda\colon \mathcal{V} \to \mathcal{U}$ such that $V \subseteq \lambda(V)$ for each $V \in \mathcal{V}$. In practice, we may index $\mathcal{U}$ and $\mathcal{V}$ and view λ as a function on indices.

Refinements allow us to improve Čech cohomology. To make this precise, we need to get morphisms on Čech cohomology.

Lemma 2.23. Fix a sheaf $\mathcal{F}$ of abelian groups on X. Given a refinement $\lambda\colon \mathcal{V} \to \mathcal{U}$ where $\mathcal{U} := \{U_i\}_{i \in I}$ and $\mathcal{V} := \{V_j\}_{j \in J}$, we get a natural map of complexes $\lambda^\bullet\colon C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{V}, \mathcal{F})$ and hence a (very) natural map $\lambda^p\colon \check{H}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{H}^\bullet(\mathcal{V}, \mathcal{F})$.

Proof. Define $\lambda^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ by

$$(\lambda^p \alpha)_{j_0,\ldots,j_p} := \alpha_{\lambda(j_0),\ldots,\alpha(j_p)}.$$

One can check that λ^p upgrades to a morphism of complexes by checking that

$$\begin{tikzcd} C^p(\mathcal{U},\mathcal{F}) \arrow[r, "d^p_{\mathcal{U}}"] \arrow[d, "\lambda^p"'] & C^{p+1}(\mathcal{U},\mathcal{F}) \arrow[d, "\lambda^{p+1}"] \\ C^p(\mathcal{V},\mathcal{F}) \arrow[r, "d^p_{\mathcal{V}}"] & C^{p+1}(\mathcal{V},\mathcal{F}) \end{tikzcd}$$

commutes, so we upgrade to a morphism on cohomology. While we're here, we do a flurry of naturality checks.

- Note that λ^p is also natural in $\mathcal{F}$ because the diagram

  $$\begin{tikzcd} C^p(\mathcal{U},\mathcal{F}) \arrow[r, "C^p\varphi"] \arrow[d, "\lambda^p"'] & C^p(\mathcal{U},\mathcal{G}) \arrow[d, "\lambda^p"] \\ C^p(\mathcal{V},\mathcal{F}) \arrow[r, "C^p\varphi"] & C^p(\mathcal{U},\mathcal{G}) \end{tikzcd}$$

  commutes for any sheaf morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$.
- Note that λ is functorial in the refinement. Indeed, given a refinement $\mu \colon \mathcal{W} \to \mathcal{V}$ where $\mathcal{W} := \{W_k\}_{k \in K}$, then we see that

  $$\begin{tikzcd} C^\bullet(\mathcal{U},\mathcal{F}) \arrow[rr, "(\mu\circ\lambda)^\bullet"] \arrow[dr, "\lambda^\bullet"'] & & C^\bullet(\mathcal{W},\mathcal{F}) \\ & C^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur, "\mu^\bullet"'] & \end{tikzcd}$$

  commutes by an explicit computation.
- Note that the morphism $\lambda^\bullet \colon \check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{V},\mathcal{F})$ is independent of the choice of refinement λ, which can be seen by taking a common refinement of two choices $\lambda, \lambda' \colon \mathcal{U} \to \mathcal{V}$; we won't write out the relevant diagram for this check. ∎

Remark 2.24. Any two refinements of $\mathcal{U}$ have a common refinement by taking intersections, so we have a directed system, so we can construct a directed colimit

$$\varinjlim \check{H}^p(\mathcal{U},\mathcal{F})$$

for each p.

The following naturality check for Lemma 2.23 will be especially important.

Lemma 2.25. Fix a sheaf $\mathcal{F}$ of abelian groups on X. Given a refinement $\lambda \colon \mathcal{V} \to \mathcal{U}$ where $\mathcal{U} := \{U_i\}_{i \in I}$ and $\mathcal{V} := \{V_j\}_{j \in J}$, the following diagram commutes, where the unlabeled arrows are from Lemma 2.19.

$$\begin{tikzcd} \check{H}^\bullet(\mathcal{U},\mathcal{F}) \arrow[rr] \arrow[dr, "\lambda^\bullet"'] & & H^p(X,\mathcal{F}) \\ & \check{H}^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur] & \end{tikzcd}$$

Proof. Fix an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Then one can build a commutative diagram of resolutions

$$\begin{tikzcd} \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \arrow[rr] \arrow[dr] & & \mathcal{I}^\bullet \\ & \mathcal{C}^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur] & \end{tikzcd}$$

where the top arrow is induced by a choice of arrow in the bottom right. (Notably, $\lambda^\bullet$ is induced in basically the same way as Lemma 2.23.) So taking global sections and then cohomology produces the needed commutative diagram. ■

The point of Lemma 2.25 is that we can combine it with Remark 2.24 to produce a map

$$\varinjlim \check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F}).$$

And here is our result.

Proposition 2.26. Fix a sheaf $\mathcal{F}$ of abelian groups on X. Then the natural map

$$\varinjlim \check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$$

described above is an isomorphism for $p \in \{0,1\}$.

Proof. For $p=0$, then Example 2.13 tells us that everything involved is $\Gamma(X,\mathcal{F})$. So the main content will be with $p=1$.

So we take $p=1$. We would like to dimension-shift, but we will run into complications. Embed $\mathcal{F}$ into a flasque sheaf $\mathcal{I}$ and let $\mathcal{Q}$ be the quotient so that

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$$

is an exact sequence. Now, to access Čech cohomology, note any open cover $\mathcal{U}$ has an injection $C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{I})$, so we let $D^\bullet(\mathcal{U})$ be the quotient complex so that we have an exact sequence

$$0 \to C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{I}) \to D^\bullet(\mathcal{U}) \to 0$$

which we have checked in Lemma 2.23 is natural in the refinement $\mathcal{U}$. (Naturality in $D^\bullet$ is induced.) Thus, for a refinement $\lambda\colon \mathcal{V} \to \mathcal{U}$, we get a commutative diagram as follows with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{U},\mathcal{I}) \arrow[r] \arrow[d] & h^0(D(\mathcal{U})) \arrow[r] \arrow[d] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^1(\mathcal{U},\mathcal{I}) \arrow[d] \\ 0 \arrow[r] & \check{H}^0(\mathcal{V},\mathcal{F}) \arrow[r] & \check{H}^0(\mathcal{V},\mathcal{I}) \arrow[r] & h^0(D(\mathcal{V})) \arrow[r] & \check{H}^1(\mathcal{V},\mathcal{F}) \arrow[r] & \check{H}^1(\mathcal{V},\mathcal{I}) \end{tikzcd} \tag{2.1}$$

Note $\check{H}^1(\mathcal{U},\mathcal{I}) = \check{H}^1(\mathcal{V},\mathcal{I}) = 0$ by Proposition 2.18, so the ends are zero. Also, the left two maps are isomorphisms by the $p=0$ case (they are both $\Gamma(X,\mathcal{F})$). Now, the universal property of the cokernel produces the morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & h^0(D^\bullet(\mathcal{U})) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{U},\mathcal{Q}) \arrow[r, equal] \arrow[d, equal] & \Gamma(X,\mathcal{Q}) \arrow[d, equal] \\ 0 \arrow[r] & h^0(D^\bullet(\mathcal{V})) \arrow[r] & \check{H}^0(\mathcal{V},\mathcal{Q}) \arrow[r, equal] & \Gamma(X,\mathcal{Q}) \end{tikzcd}$$

basically because $\Gamma(X,-)$ is already known to be left exact. Thus, we see that the left vertical map above is injective, so the Five lemma in (2.1) tells us that $\check{H}^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(\mathcal{V},\mathcal{F})$ is injective.

We now take colimits over everything (which is an exact operation because the colimits are filtered) to draw the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(X,\mathcal{I}) \arrow[r] \arrow[d] & \varinjlim h^0(D^\bullet(\mathcal{U})) \arrow[r] \arrow[d, dashed] & \varinjlim \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \varinjlim \check{H}^1(\mathcal{U},\mathcal{I}) \arrow[d] \\ 0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] & \Gamma(X,\mathcal{I}) \arrow[r] & \Gamma(X,\mathcal{R}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{I}) \end{tikzcd}$$

As before, the ends vanish, and the middle arrow is induced. More precisely, the Horseshoe lemma produces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{C}^\bullet(\mathcal{U},\mathcal{I}) \arrow[r] \arrow[d] & \mathcal{C}^\bullet(\mathcal{U},\mathcal{Q}) \arrow[d] & \\ 0 \arrow[r] & \mathcal{I}^\bullet \arrow[r] & \mathcal{I}^\bullet \oplus \mathcal{J}^\bullet \arrow[r] & \mathcal{J}^\bullet \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

where $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and $0 \to \mathcal{I} \to \mathcal{J}^\bullet$ are both injective resolutions, which then produces the morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U},\mathcal{I}) \arrow[r] \arrow[d] & D^\bullet(\mathcal{U}) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \Gamma(X,\mathcal{I}^\bullet) \arrow[r] & \Gamma(X,\mathcal{I}^\bullet \oplus J^\bullet) \arrow[r] & \Gamma(X,J^\bullet) \arrow[r] & 0 \end{tikzcd}$$

by taking global sections. Taking cohomology (and then taking colimits) is what produces the needed vertical map making everything commute.

We now continue staring at (2.2). Now, we know that the first and second vertical arrows are isomorphisms, so the Five lemma dictates that it is enough to show that the third (middle) vertical arrow is an isomorphism. It is already injective, so we just need surjectivity. Morally, this is because sections of the quotient sheaf will (on an open cover) come from sections of the quotient presheaf, and $h^0(D^\bullet(\mathcal{U}))$ is exactly these sections on the open cover $\mathcal{U}$.

To be explicit, we work on stalks. Let $q\colon \mathcal{I} \to \mathcal{Q}$ denote the quotient map and some $s \in \Gamma(X,\mathcal{Q})$ which we would like to hit. Well, looking on stalks, each $x \in X$ has some open neighborhood U_x with a section $t_x \in \Gamma(U_x,\mathcal{I})$ such that $q_x((t_x)_x) = s|_{U_x}$. Setting $\mathcal{U} := \{U_x\}_{x\in X}$, we have that $((t_x)_x) \in C^0(\mathcal{U},\mathcal{I})$, so we can take this along q to get $(s|_{U_x})_{x\in X} \in C^0(\mathcal{U},\mathcal{Q})$, but in fact this lives in $D^\bullet(\mathcal{U})$ because it came from $C^\bullet(\mathcal{U},\mathcal{I})$, so we see that s was in fact in the image. ■

2.6 February 21

We spent most class completing a proof from the previous class, and I have edited directly into the notes of the previous class for continuity.

2.6.1 Cohomology on Projective Space

We are now moving towards the proof of Serre duality, for which we will want to have computed the cohomology of some line bundles on projective space. Throughout, we will take A to be a Noetherian ring (for example, a field), set $S := A[x_0,\ldots,x_r]$ to be the graded ring, and we set $X := \mathbb{P}^r_A = \operatorname{Proj} S$. Our goal is to compute the cohomology of the sheaves $\mathcal{O}_X(n) := \widetilde{S(n)}$. We also recall the following construction: for an $\mathcal{O}_X$-module $\mathcal{F}$, we can define

$$\Gamma_\bullet(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(X,\mathcal{F}(n)),$$

which is a $\mathbb{Z}$-graded S-module. Let's go ahead and state our desired theorem.

Theorem 2.27. Fix a Noetherian ring A, and set $S := A[x_0, \ldots, x_r]$ and $X := \mathbb{P}^r_A$.

(a) The natural map

$$S \to \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$$

is an isomorphism of $\mathbb{Z}$-graded S-modules.

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for all $0 < i < r$ and $n \in \mathbb{Z}$.

(c) $H^r(X, \mathcal{O}_X(-r-1)) = A$.

(d) For each $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-r-1-n)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of free A-modules of finite rank.

Remark 2.28. The standard affine open covering on $\mathbb{P}^n_A$ tells us that $H^i(X, \mathcal{F}) = 0$ for any $i > r$ and any quasicoherent sheaf $\mathcal{F}$. Notably, we are using Čech cohomology via the comparison theorem Theorem 2.20, which applies because X is in fact Noetherian and separated.

2.7 February 23

Here we go.

2.7.1 More on Cohomology on Projective Space

Today we prove Theorem 2.27.

Proof of Theorem 2.27. As suggested by the remark, our proof of Theorem 2.27 will use Čech cohomology. It will be helpful to glue everything together into

$$\mathcal{F} := \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n),$$

which is a $\mathbb{Z}$-graded quasicoherent sheaf of S-modules. Taking cohomology, which commutes with infinite sums because taking infinite sums is exact, we see that

$$H^i(X, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{O}_X(n))$$

for each i. Notably, one has an A-module structure everywhere by Remark 2.21.

Now, for our open cover $\mathcal{U}$ (for Čech cohomology), we take $U_j := D_+(x_j)$ for $0 \le j \le r$; note then that $U_{i_0, \ldots, i_p} := U_{i_0} \cap \cdots \cap U_{i_p} = D_+(x_{i_0} \cdots x_{i_p})$. In particular, $\mathbb{P}^r_A$ is Noetherian and separated (note A is Noetherian), so

$$\check{H}^\bullet(\mathcal{U}, \mathcal{O}_X(n)) \cong H^\bullet(X, \mathcal{O}_X(n))$$

by Theorem 2.20. Notably, we find that

$$\mathcal{F}(U_{i_0 \cdots i_p}) = S_{x_{i_0} \cdots x_{i_p}}$$

by tracking through the localizations on $\mathcal{O}_X(n)$, meaning that our Čech complex looks like

$$0 \to \prod_{i_0} S_{x_{i_0}} \stackrel{d^0}{\to} \prod_{i_0, i_1} S_{x_{i_0} x_{i_1}} \stackrel{d^1}{\to} \cdots \stackrel{d^{r-1}}{\to} S_{x_0 \cdots x_r} \to 0$$

of graded S-modules.

We now proceed with our arguments.

(a) We compute

$$\check{H}^0(\mathcal{U},\mathcal{F}) = \ker d^0 = \bigcap_i S_{x_i}.$$

Here, the intersections are being taken in $S_{x_0\cdots x_r}$, legal because the relevant localization maps are injective. Now, this last S-module is just S because (for example) $S = S_{x_0} \cap S_{x_1}$ by a direct computation.

(c) We begin by claiming that $\check{H}^r(\mathcal{U},\mathcal{F})$ is the free graded A-module with basis given by elements of the form $x_0^{\ell_0}\cdots x_r^{\ell_r}$ where ℓ_i are negative integers. Notably, in degree $-r-1$, we are searching for solutions to $\ell_0+\cdots+\ell_r = -r-1$, which is only $\ell_0=\cdots=\ell_r=-1$, so we will have rank 1, which is (c).

So it remains to show the claim. Well, looking at our Čech complex,

$$\check{H}^r(\mathcal{U},\mathcal{F}) = \operatorname{coker}\left(\prod_j S_{x_0\cdots\widehat{x}_j\cdots x_r} \stackrel{d^{r-1}}{\to} S_{x_0\cdots x_r}\right).$$

Now, $S_{x_0\cdots x_r}$ is a free A-module with basis given by terms of the form $x_0^{\ell_0}\cdots x_r^{\ell_r}$ where $\ell_0,\ldots,\ell_r\in\mathbb{Z}$, so we want the cokernel to kill all undesired terms. Well, tracking through d^{r-1}, we find that it is just inclusion (up to sign), so the image is the free A-module with basis given by terms of the form $x_0^{\ell_0}\cdots x_r^{\ell_r}$ with at least one nonnegative exponent, so the claim follows.

(d) Let's begin by describing the pairing. Given some $s\in H^0(X,\mathcal{O}_X(n))$ (which is just a global section), we produce a map $s\colon \mathcal{O}_X(-r-1-n)\to\mathcal{O}_X(-r-1)$. Moving up to cohomology, we get an A-linear map $H^r(X,\mathcal{O}_X(-r-1-n))\to H^r(X,\mathcal{O}_X(-r-1))$, which upon letting s vary produces the required bilinear pairing

$$H^0(X,\mathcal{O}_X(n)) \to \operatorname{Hom}_A(H^r(X,\mathcal{O}_X(-r-1-n)), H^r(X,\mathcal{O}_X(-r-1))).$$

It remains to check that this pairing is perfect. If $n<0$, then both terms in our pairing will vanish, so there will be nothing left to check. For example, $H^0(X,\mathcal{O}_X(n))=0$ by (a), and $H^r(X,\mathcal{O}_X(-r-1-n))=0$ by the claim in (c): we know $H^r(X,\mathcal{O}_X(-r-1-n))=0$ is a free A-module with basis given by $x_0^{\ell_0}\cdots x_r^{\ell_r}$ where the $\ell_\bullet$s are negative integers summing to $-r-1-n>-r-1$, but there is no such basis element.

So we may take $n\geq 0$. Tracking through the description of our pairing on Čech cohomology, we see that the basis element $x_0^{m_0}\cdots x_r^{m_r}\in H^0(X,\mathcal{O}_X(n))$ (of total degree n) will send the basis element $x_0^{\ell_0}\cdots x_r^{\ell_r}\in H^r(X,\mathcal{O}_X(-r-1-n))$ to the element $x_0^{m_0+\ell_0}\cdots x_r^{m_r+\ell_r}\in H^r(X,\mathcal{O}_X(-r-1))$ (which means 0 if any exponent is nonnegative). Let's explain this. To begin, we need to show that we actually have a well-defined map. On Čech cohomology, we are attempting to describe a map

$$\left(\ker d^0\right)_n \otimes_A \frac{C^r(\mathcal{U},\mathcal{F})_{-r-1-n}}{\left(\operatorname{im} d^{r-1}\right)_{-r-1-n}} \to \frac{C^r(\mathcal{U},\mathcal{F})_{-r-1}}{\left(\operatorname{im} d^{r-1}\right)_{-r-1}}.$$

We can now see that $\ker d^0=S$, so it does have basis elements in the form $x_0^{m_0}\cdots x_r^{m_r}$ of total degree n, and tensoring by this element will indeed send basis elements $x_0^{\ell_0}\cdots x_r^{\ell_r}\in H^r(X,\mathcal{O}_X(-r-1-n))$ as described. (Notably, we do go to 0 if any exponent is nonnegative because this is the image of d^{r-1}. Perhaps one might also want to note that if we input some element $x_0^{\ell_0}\cdots x_r^{\ell_r}$ with a nonnegative exponent, then the corresponding product will have a nonnegative exponent in the same spot.) Formally, perhaps one should go through the following commutative diagram, as follows.

$$\begin{tikzcd}
C^\bullet(\mathcal{U},\mathcal{O}_X(-r-1-n)) \arrow[r, "(-\otimes s)"] \arrow[d, equal] & C^\bullet(\mathcal{U},\mathcal{O}_X(-r-1)) \arrow[d, equal] \\
C^\bullet(\mathcal{U},\mathcal{F})_{-r-1-n} \arrow[r, dashed] & C^\bullet(\mathcal{U},\mathcal{F})_{-r-1}
\end{tikzcd}$$

We will continue the proof next class. ■

Remark 2.29. The choice of isomorphism in (c) is notably not canonical. In particular, it depends on our choice of basis element for the cokernel.

BIBLIOGRAPHY

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: 10.1017/CBO9781139644136. URL: https://doi-org.libproxy.berkeley.edu/10.1017/CBO9781139644136.

[Kle16] Felix Klein. *Elementary Mathematics from a Higher Standpoint*. Trans. by Gert Schubring. Vol. II. Springer Berlin, Heidelberg, 2016.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Elb22] Nir Elber. *Commutative Algebra*. 2022. URL: https://dfoiler.github.io/notes/250B/notes.pdf.

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