174: Category Theory

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Spring 2022

Contents

THEME 1

BASIC DEFINITIONS

Category theory is much easier once you realize that it is designed to formalize and abstract things you already know.

—Ravi Vakil, [Vak17]

1.1 January 19

Reportedly there is a lot of material that Bryce would like to cover today.

1.1.1 Our Definition

We're doing category theory, so let's define what a category is.

Definition 1.1 (Category)**.** A category $\mathcal{C}$ is a pair of objects and morphisms $(\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C})$ satisfying the following.

- $\operatorname{Ob}\mathcal{C}$ is a collection of *objects*. By abuse of notation, when we write $c \in \mathcal{C}$
- $\operatorname{Mor}\mathcal{C}$ is a collection of *morphisms*. Morphisms might also be called arrows or maps or functions or continuous functions or similar.

  A morphism is written $f : x \to y$ where $x, y \in \operatorname{Ob}\mathcal{C}$. Here, x is the *domain*, and y is the *codomain*.

In the above definition, we have some coherence conditions:

- For each $x \in \mathcal{C}$, there is a morphism $\mathrm{id}_x : x \to x$.
- Given any pair of morphisms $f : x \to y$ and $g : y \to z$, there exists a *composition* $gf : x \to z$. Importantly, the codomain of f is the domain of g.

Additionally, morphisms satisfy the following coherence conditions.

- Associativity: for any morphisms $f : a \to b$ and $g : b \to c$ and $h : c \to d$, we have that $h(gf) = (hg)f$.
- Identity: given any morphism $f : a \to b$, we have $\mathrm{id}_b f = f$ and $f \,\mathrm{id}_a = f$.

Yes, this is a long definition. For reference, it is on page 3 of Riehl.

The intuition to have here is that we have objects to be thought of as points a bunch of morphisms which are to be thought of arrows between them. Here is an example of some morphisms in a category.

$$\begin{tikzcd}
\bullet \arrow[loop left] \arrow[r] \arrow[d] & \bullet \arrow[loop above] \arrow[dl] \arrow[dr] & \\
\bullet \arrow[loop below] \arrow[rr] & & \bullet \arrow[loop below]
\end{tikzcd}$$

The loops are identity morphisms. As an aside, it is reasonable to think that definition of a category is overly abstract. Most of the time we will be thinking about some concrete category.

Before continuing, we bring in the following definition.

> **Definition 1.2** (Hom-sets)**.** Fix a category $\mathcal{C}$. Then, given objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ or $\mathrm{Hom}_{\mathcal{C}}(x, y)$ or $\mathrm{Hom}(x, y)$ or $\mathrm{Mor}(x, y)$ for the set of morphisms $f : x \to y$. I personally prefer $\mathrm{Mor}(x, y)$.

Note that two objects need not have a morphism between them. For example, the following is a category even though the two objects have a morphism between them.

$$\begin{tikzcd}
\bullet \arrow[loop left] & \bullet \arrow[loop right]
\end{tikzcd}$$

As a less contrived example, there is no morphism between $\mathbb{F}_2$ and $\mathbb{F}_3$ in the category of fields.

1.1.2 Examples

Let's talk about examples.

> **Example 1.3.** The category Set has objects which are all sets and its morphisms are the functions between sets.

> **Example 1.4.** The category Grp has objects which are all groups and its morphisms are group homomorphisms. Similarly, Ab has abelian groups.

> **Example 1.5.** The category Ring has objects which are all rings (with identity) and its morphisms are group homomorphisms.

> **Example 1.6.** The category Field has objects which are all fields and its morphisms are field/ring homomorphisms.

> **Example 1.7.** The category Vec_k has objects which are all k-vector spaces and its morphisms are k-linear transformations.

Those are the good examples. We like them because they are with familiar objects.

Here are some weirder examples.

> **Example 1.8** (Walking arrow)**.** The diagram
>
> $$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$
>
> induces a category with a single non-identity morphism.

Note that we will stop writing down all the identity morphisms and all induced morphisms because they're annoying to write out.

Example 1.9 (Walking isomorphism)**.** The diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

induces a category with two non-identity morphisms. We declare that any composition of the two non-identity morphisms is the identity.

There are also such things as a poset category, but for this we should define a poset first.

Definition 1.10 (Poset)**.** A *poset* $(\mathcal{P}, \leq)$ is a set $\mathcal{P}$ and a relation $\leq$ on $\mathcal{P}$ which satisfies the following; let $a, b, c \in \mathcal{P}$.

- Reflexive: $a \leq a$.
- Antisymmetric: $a \leq b$ and $b \leq a$ implies $a = b$.
- Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$.

Now, it turns out that all posets induce a category.

Example 1.11 (Poset category)**.** Given any poset $(\mathcal{P}, \leq)$, we can define the poset category as follows.

- The objects are elements of $\mathcal{P}$.
- For $x, y \in \mathcal{P}$, there is a morphism $x \to y$ if and only if $x \leq y$, and there is only one morphism.

Checking that the poset category is in fact a category is not very interesting. The identity law comes from reflexivity, where id_a witnesses $a \leq a$.

Additionally, transitivity defines our composition: if $a \leq b$ and $b \leq c$, then $a \leq c$, and the morphism representing $a \leq c$ is unambiguous because there is at most one morphism $a \to c$. This uniqueness is in fact crucial for our composition: if $f : a \to b$ and $g : b \to c$ and $h : c \to d$ are morphisms, then $h(gf) = (hg)f$ because they are both morphisms $a \to d$, of which there is at most one.

We continue with our examples. We will not check that these are actually categories formally; perhaps the reader can do the checks on their own time.

Example 1.12 (Groups)**.** Given a group G, we can define the category $\mathrm{B}G$ to have one object $*$ and morphisms $g : * \to *$ given by group elements $g \in G$. Composition in the category is group multiplication; the identity morphism id_* needed is the identity element of G; and the associativity check comes from associativity in G.

Example 1.13 (Pointer sets)**.** We define the category of pointed sets Set_* to consist of objects which are ordered pairs (X, x) where X is a set and $x \in X$ is an element. Then morphism are "based maps" $f : (X, x) \to (Y, y)$ to consist of the data of a function $f : X \to Y$ such that $f(x) = y$.

Example 1.14. Given any set S, we can define a category consisting of objects which are elements of S and morphisms which are only the required identity morphisms.

This last example generalizes.

Definition 1.15 (Discrete, indiscrete)**.** Fix a category $\mathcal{C}$. Then $\mathcal{C}$ is *discrete* if and only if the only morphisms are identity morphisms. Additionally, $\mathcal{C}$ is *indiscrete* if and only if $\mathrm{Mor}(x, y)$ has exactly one element for each pair of objects (x, y).

Warning 1.16. A total order with more than one element is not a category. Namely, if we have distinct objects x and y, then we cannot have both $x \le y$ and $y \le x$, so not both $\operatorname{Mor}(x,y)$ and $\operatorname{Mor}(y,x)$ inhabited.

1.1.3 Size Issues

Let's briefly talk about why we are calling $\operatorname{Ob}\mathcal{C}$ and $\operatorname{Mor}\mathcal{C}$ "collections." In short, we cannot have a set that contains all sets, but we would still like a category which contains all categories. There are a few ways around this; here are two.

- Grothendieck inaccessible categories: we essentially upper-bound the size of our sets and then let Set contain all of our sets.
- Proper classes: we add in things called "classes" to foundational mathematics we are allowed to be bigger than sets.

We will avoid doing anything like this in this course, so here is a definition making our avoidance concrete.

Definition 1.17 (Small, locally small)**.** Fix $\mathcal{C}$ a category. Then $\mathcal{C}$ is *small* if and only if $\operatorname{Mor}\mathcal{C}$ is a set. Alternatively, $\mathcal{C}$ is *locally small* if and only if $\operatorname{Mpr}(x,y)$ is a set.

Example 1.18. The category Set is locally small, but it is not small. To see that it is not small, note that $S \mapsto \operatorname{Mor}(\{*\}, S)$ is an injective map, so $\operatorname{Mor}\mathrm{Set}$ must be at least as big as Set.

It turns out that most of our categories will be locally small. It is a very nice property to have.

1.1.4 Isomorphism

In algebra (e.g., group theory), we are interested in when two objects are the same. In category theory, we focus on the morphisms between objects, so we need to be careful how we define this. Here is our definition.

Definition 1.19 (Isomorphism)**.** Fix a category $\mathcal{C}$. Then a morphism $f : x \to y$ is an *isomorphism* if and only if there is a morphism $g : y \to x$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. We call g the *inverse* of f and often notate it f^{-1}.

This is fairly intuitive: isomorphisms are those morphisms with a way to reverse them.

Observe that we called g "the" inverse of f, and we may do so because inverses are unique.

Proposition 1.20. Fix a category $\mathcal{C}$. Inverses of morphisms, if they exist, are unique.

Proof. Fix $f : x \to y$ some isomorphism, and suppose that we have found two inverse morphisms $g, h : y \to x$. Then

$$g = g\,\mathrm{id}_y = g(fh) = (gf)h = \mathrm{id}_x\,h = h,$$

so indeed the inverse morphisms that we found are the same. ■

Anyways, here are some examples.

Example 1.21. In Set, the isomorphisms are the bijective maps. For this we would have to show that bijective maps have inverse maps, which is not too hard to show.

Example 1.22. In Grp, the isomorphisms are group isomorphisms. Similarly, isomorphisms in Ring are ring isomorphisms.

As a warning, we will say now that lots of categories do not have a good categorical notion of injectivity or surjectivity, so we will not be able to say that isomorphisms are merely "bijective" morphisms.

1.2 January 21

By the way, this course is being run by Bryce (interested in category theory, homological algebra, and algebraic topology) and Chris (interested in representation theory and category theory).

1.2.1 Small Correction

Last class we discussed trying to a total order $(\mathcal{P}, \leq)$ into an indiscrete category. One way to do this is to say to give a morphism between two objects $a, b \in \mathcal{P}$ if and only if one of $a < b$ or $b < a$ or $a = a$ is true. Observe that the order does not actually matter here because any two objects have exactly one morphism anyways.

1.2.2 Groupoids

Reportedly, there will usually not be a lecture to begin out our discussion sections, but here is a lecture to begin out our first discussion section.

Last time we left off talking about indiscrete categories. Here is a nice fact.

Proposition 1.23. Fix $\mathcal{C}$ an indiscrete category. Then all maps are isomorphisms.

Proof. Fix any morphism $f : x \to y$. There is also a morphism $g : y \to x$, and we see that $gf \in \mathrm{Mor}(x, x)$. But $\mathrm{id}_x \in \mathrm{Mor}(x, x)$ as well, so we are forced to have $gf = \mathrm{id}_x$ by uniqueness of morphisms. Similar shows that $fg = \mathrm{id}_y$, finishing the proof. ■

Remark 1.24. This statement is also true for discrete categories but only because all identity morphisms are isomorphisms immediately.

The property of the proposition is nice enough to deserve a definition.

Definition 1.25 (Groupoid)**.** A category in which all morphisms are isomorphisms is called a *groupoid*.

Example 1.26. Viewing groups as one-element categories, we see that groups are groupoids because all elements (i.e., morphisms of the one-object set) have inverses and hence are isomorphisms.

Intuitively, a groupoid is a group but more "spread out."

1.2.3 Arrow Words

We close out with some miscellaneous definitions for our morphisms.

Definition 1.27 (Endo-, automorphism)**.** Fix a category $\mathcal{C}$. A morphism $f : x \to y$ is an *endomorphism* if and only if $x = y$. A morphism $f : x \to y$ is an *automorphism* if and only if it is an isomorphism and an endomorphism.

Example 1.28. In the category of abelian groups, the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 is an endomorphism but not an automorphism.

Definition 1.29 (Monic, epic)**.** Fix a category $\mathcal{C}$ and a morphism $f : x \to y$.

- We say f is a *monomorphism* (or is *monic*) if and only if $fg = fh$ implies $g = h$ for any morphisms $g, h : c \to x$. In other words, the map

$$\operatorname{Mor}(c, x) \overset{f\circ -}{\to} \operatorname{Mor}(c, y)$$

  is injective. (This map is called "post-composition.") We might write $f : x \hookrightarrow y$ for emphasis.

- We say f is an *epimorphism* (or is *epic*) if and only if $gf = hf$ implies $g = h$ for any morphisms $g, h : y \to c$. In other words, the map

$$\operatorname{Mor}(y, c) \overset{-\circ f}{\to} \operatorname{Mor}(x, c)$$

  is injective. (This map is called "pre-composition.") We might write $f : x \twoheadrightarrow y$ for emphasis.

Intuitively, the monomorphism condition looks like the injectivity condition (namely, $f(x) = f(y)$ implies $x = y$), so monic is supposed to be a generalization for injective.

Example 1.30. In the category of sets, monic is equivalent to injective, and epic is equivalent to surjective. Then it happens that being monic and epic implies being isomorphic. We will not fill in the details here.

Warning 1.31. It is not always true that being monic and epic implies being isomorphic. It is true in $\mathrm{Set}, \mathrm{Ab}, \mathrm{Grp}$ but not in, say, Ring as the below example shows.

Example 1.32. The inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring is both epic and monic but not an isomorphism. We run some checks.

- We show monic. Suppose $g, h : R \to \mathbb{Z}$ are morphisms with $fg = fh$. We claim $g = h$. Well, for any $r \in R$, we see $g(r) = f(g(r))$ and $h(r) = f(h(r))$ because f is merely an inclusion, so $g(r) = h(r)$ follows.

- We show epic. Suppose $g, h : \mathbb{Q} \to R$ are morphisms with $gf = hf$. We claim $g = h$. We start by noting any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}$ will have

$$g\left(n/m\right) \cdot g(m) = g(n)$$

  and similar for h. However, $g(m) = g(f(m)) = h(f(m)) = h(m)$ and $g(n) = h(n)$ for the same reason, so $g\left(\frac{n}{m}\right) = g(n)/g(m) = h(n)/h(m) = h\left(\frac{n}{m}\right)$, and we are done because any rational can be expressed as some $\frac{n}{m}$.

- Lastly, f is not an isomorphism because $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. For example, $2x - 1$ has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

And now discussion begins.

1.3 January 24

Chris is giving the lecture today. Reportedly, it might be rough around the edges, but I have full faith in its coherence.

1.3.1 Review

Let's quickly talk about two fun types of categories.

Definition 1.33 (Slice categories)**.** Fix a category $\mathcal{C}$ and an object $c \in \mathcal{C}$.

- We define the *slice category* $\rfloor/\mathcal{C}$ to have objects which are morphisms $f : c \to x$ for objects $x \in \mathcal{C}$. The morphisms from $f : c \to x$ to $g : c \to y$ is a morphism $h : x \to y$ such that $f = gh$. Namely, we require the following triangle to commute.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

- Dual to this is the *slice category* $\mathcal{C}/c$ where we reverse all the arrows. For example, our objects are morphisms $f : x \to c$, and morphisms from $f : x \to c$ to $g : y \to c$ are morphisms $f : x \to y$ such that $g = hf$.

There are also groupoids, which we have defined previously.

1.3.2 Subcategories

We have the following definition.

Definition 1.34 (Subcategory)**.** A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ whose objects and morphisms come from $\mathcal{C}$ and that the composition law is inherited. Explicitly, we require $\mathcal{D}$ to have the identity morphisms and be closed under composition of $\mathcal{C}$ (i.e., if $f : x \to y$ and $g : y \to z$ are morphisms in $\mathcal{D}$, then gf is also a morphism in $\mathcal{D}$.)

We are going to want ways to generate subcategories. Here is one way.

Definition 1.35 (Full subcategory)**.** Fix a category $\mathcal{C}$. Then we define the *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ to be defined by choosing some objects $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{C}$ and then choosing morphisms by taking all of them. Explicitly, for $x, y \in \operatorname{Ob}\mathcal{D}$, we have

$$\operatorname{Mor}_{\mathcal{D}}(x, y) = \operatorname{Mor}_{\mathcal{C}}(x, y).$$

Example 1.36. The category of abelian groups is a full subcategory of the category of groups. Namely, the category of abelian groups is made of the objects which are abelian groups and all arrows are simply all group homomorphisms, so no morphisms have been lost in this restriction.

Example 1.37. The category of finite sets is a full subcategory in the category of sets.

Example 1.38. Given a category $\mathcal{C}$, one can take the *maximal groupoid* of $\mathcal{C}$ to be the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the isomorphisms of $\mathcal{C}$. So as long as $\mathcal{C}$ has morphisms which are not isomorphisms, then the maximal groupoid will not be full.

Example 1.39. The category Rng is a subcategory of Ring, but it is not full. For example, in Ring, the map $\mathbb{Z} \overset{\times 2}{\mathbb{Z}}$ is not a morphism even though it is a morphism in Rng.

One has to be a bit careful with this, however.

Non-Example 1.40. The category Grp is not a subcategory of Set because one can endow the same set with different group structures.

1.3.3 Duality

Here is our main character.

Definition 1.41 (Opposite category)**.** Given a category $\mathcal{C}$, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to have objects which are objects of $\mathcal{C}$ and morphisms $f^{\mathrm{op}} : y \to x$ of $\mathcal{C}^{\mathrm{op}}$ are in one-to-one correspondence with morphisms $f : x \to y$ of $\mathcal{C}$. Lastly, composition is defined by, for $f^{\mathrm{op}} : y \to x$ and $g^{\mathrm{op}} : z \to y$, we have

$$f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}.$$

In pictures, the composition law reversed the diagram $x \xrightarrow{f} y \xrightarrow{g} z$ to

$$x \xleftarrow{f^{\mathrm{op}}} y \xleftarrow{g^{\mathrm{op}}} z.$$

Let's see some examples.

Example 1.42. Given a partial order $(\mathcal{P}, \leq)$, the opposite category is by (partial) ordering $\mathcal{P}$ simply by flipping the partial order: $b \leq_{\mathrm{op}} a$ if and only if $a \leq b$. Namely, the opposite category of a partial order remains a partial order.

Example 1.43. Fix a group G and form its category $\mathrm{B}G$. Now, when we reverse the arrows $(\mathrm{B}G)^{\mathrm{op}}$, we get a category corresponding to the group law G^{op} with group law defined by

$$h^{\mathrm{op}} g^{\mathrm{op}} = gh.$$

Namely, the opposite category of a group is still a group.

In fact, we have that $\mathrm{B}G \cong (\mathrm{B}G)^{\mathrm{op}}$ (for whatever $\cong$ means) by taking making our morphisms perform inversion by $\varphi : g \mapsto (g^{\mathrm{op}})^{-1}$. This map is bijective, and we can check the composition by writing

$$\varphi(gh) = ((gh)^{\mathrm{op}})^{-1} = (h^{\mathrm{op}} g^{\mathrm{op}})^{-1} = (g^{\mathrm{op}})^{-1} (h^{\mathrm{op}})^{-1} = \varphi(g)\varphi(h),$$

so everything works.

Example 1.44. Algebraic geometry says that $\mathrm{CRing}^{\mathrm{op}}$ is equivalent to the category of affine schemes AffSch. The point here is that the opposite category is potentially very different from the original category. (Mnemonically, the opposite of algebra is geometry.)

Now, here is the idea of duality.

Idea 1.45. Theorem statements that hold for categories will need to be true for their opposite category as well.

As an example, let's work with monomorphisms and epimorphisms. For example, $f : y \to z$ is monic if and only if the commutativity of the diagram

$$\begin{tikzcd} x \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & y \arrow[r, "f"'] & z \end{tikzcd}$$

forces $g = h$. Similarly, $f : x \to y$ is epic if and only if the commutativity of the diagram

$$\begin{tikzcd} x \arrow[r, "f"'] & y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & z \end{tikzcd}$$

forces $g = h$. But notice that flipping the epic diagram notes that epic condition is equivalent to the commutativity of the diagram

$$\begin{tikzcd} x \arrow[r, shift left, "g^{\mathrm{op}}"] \arrow[r, shift right, "h^{\mathrm{op}}"'] & y \arrow[r, "f^{\mathrm{op}}"'] & x \end{tikzcd}$$

forces $g = h$, which is the same thing as $g^{\mathrm{op}} = h^{\mathrm{op}}$. Thus, we have the following lemma.

Lemma 1.46. Fix a category $\mathcal{C}$. Then a morphism f is monic if and only if f^{op} is epic in $\mathcal{C}$.

Proof. This comes from the discussion above. ■

The point is that we can prove theorems about monic and epic maps simultaneously by working with (say) monomorphisms general categories and then dualizing to get the statement about epimorphisms.

Let's see this strategy in action. We have the following definition.

Definition 1.47 (Section, retraction)**.** Suppose that $s : x \to y$ and $r : y \to x$ are morphisms such that $rs = \mathrm{id}_x$; i.e., the composition

$$x \overset{s}{\to} y \overset{r}{\to} x$$

is id_x. Then we say that s is a *section* of r, and r is a *retraction* of s.

Think about these as having a one-sided inverse. We have the following lemma.

Lemma 1.48. A morphism s in $\mathcal{C}$ is a section of some morphism if and only if s^{op} is a retraction in $\mathcal{C}$.

Proof. Fix $s : x \to y$. The condition that there exists r so that $rs = \mathrm{id}_x$ is equivalent to there exists r^{op} such that $s^{\mathrm{op}} r^{\mathrm{op}} = \mathrm{id}_x^{\mathrm{op}}$, which translates into the lemma. ■

And now let's actually see a proof.

Proposition 1.49. A morphism s in $\mathcal{C}$ is a section of some morphism implies that s is a monomorphism.

Proof. Suppose that $s : x \to y$ is a section for the morphism $r : y \to x$ so that $rs = \mathrm{id}_x$. Now, suppose that $sg = sh$ so that we want to show $g = h$. But we see that

$$g = \mathrm{id}_x g = (rs)g = r(sg) = r(sh) = (rs)h = \mathrm{id}_x h = h,$$

so we are done. ■

So here is our dual statement, which we get for free.

Proposition 1.50. A morphism r in $\mathcal{C}$ is a retraction of some morphism implies that r is an epimorphism.

Proof. We note that r is a retraction in $\mathcal{C}$ implies that r^{op} is a section in $\mathcal{C}^{\mathrm{op}}$, so by the above, r^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$. Thus, it follows that r is an epimorphism in $\mathcal{C}$. ■

We've been saying "section of" and "retraction of" a lot, so we optimize out these words in the following definition.

Definition 1.51 (Split monorphism, split epimorphism)**.** We say that a morphism f is a *split monomorphism* if and only if it is a section of some morphism. Similarly, we say that f is a *split epimorphism* if and only if it is the retraction of some morphism.

So the above statements show that split monomorphisms are in fact monomorphisms, and split epimorphisms are in fact epimorphisms.

1.3.4 Yoneda Lite

So far we have said that monic is similar to injective and epic is similar to surjective. We would like to make these sorts of correspondences a little more concrete, so we add more abstraction.

Definition 1.52 (Post- and pre-composition)**.** Fix a morphism $f : x \to y$ of $\mathcal{C}$. Then, given an object $c \in \mathcal{C}$, we define the maps $f_* : \operatorname{Mor}(c, x) \to \operatorname{Mor}(c, y)$ and $f^*(y, c) \to \operatorname{Mor}(x, c)$ by

$$f_*(g) := fg \qquad \text{and} f^*(g) := gf.$$

The map f_* is called *post-composition* because we apply f after; the map f^* is called *pre-composition* because we apply it after.

Note that f_* and f^* are nice because they are all real functions of sets (for locally small categories) with which we can use to understand f. Here are some equivalent conditions.

Proposition 1.53. Fix f a morphism of the category $\mathcal{C}$. Then the following are true.

(a) f is an isomorphism if and only if f_* is bijective if and only if f^* is bijective.

(b) f is monic if and only if f_* is injective.

(c) f is epic if and only if f^* is injective (!).

(d) f is split monic if and only if f^* is surjective.

(e) f is split epic if and only if f_* is surjective.

Proof. We omit most of these; let's show (b). We have two directions. Suppose that f is monic. Then fix an object c, and we show that the map

$$f_* : \operatorname{Mor}(c, x) \to \operatorname{Mor}(c, y)$$

by $f_*(g) := fg$ is injective. But indeed, $f_*(g) = f_*(h)$ implies $fg = fh$ implies $g = h$ by monic, so injectivity follows.

Conversely, suppose f_* is monic. Then suppose that $fg = fh$ for some morphisms $g, h : c \to x$, and we show that $g = h$. But f_* is injective! So

$$f_*(g) = fg = fh = f_*(h)$$

forces $g = h$, and we are done. ■

THEME 2

FUNCTORS AND NATURAL TRANSFORMATIONS

Mathematics is the art of giving the same names to different things

—Henri Poincaré

2.1 January 26

We will start on new things.

2.1.1 Functors

In this class, we will repeatedly talk about the following idea.

Idea 2.1. Everything is a special case of everything else.

In other words, we will want to abstract old ideas from new ones, and this will happen a lot.
The first time we are going to see this is by trying to consider categories of

Remark 2.2. Yes, Russel's paradox prevents a category of all categories. Nevertheless, we will try. One way to get around this is to do size declarations: for example, we can consider the category of all small categories, as we are about to do.

Anyways, we would like to give some categorical structure to (say, small) categories. Well, what will be our morphisms between categories? They will be "functors."
Before defining functors, we should describe what a functor $F : \mathcal{C} \to \mathcal{D}$ should do.

- Viewing $\mathcal{C}$ as consisting of the data of objects and morphisms, an initial requirement might be that F takes objects to objects and morphisms to morphisms.
- We would also like F to preserve the "structure" of our categories, which essentially means we want to preserve composition in our categories. So we will require a "functoriality" condition to preserve this structure.

Let's try to get an intuitive feeling for how functoriality should behave.

Example 2.3. Fix an abelian group A. Then there is a map $\operatorname{Hom}(A,-)$ sending abelian groups Ab to sets Set. In fact, we get a map of morphisms as well, for a morphism $f : X \to Y$ provides a post-composition mapping

$$f_* : \operatorname{Hom}(A,X) \to \operatorname{Hom}(A,Y)$$

by $\varphi \mapsto f\varphi$. This association has some nice properties. For example, we have the following.

- We see $(\mathrm{id}_X)_* : \operatorname{Hom}(A,X) \to \operatorname{Hom}(A,X)$ sends $\varphi \mapsto \varphi$, so $(\mathrm{id}_X)_* = \mathrm{id}_{\operatorname{Hom}(A,X)}$.
- Given $f : X \to Y$ and $g : Y \to Z$, we have $gf : X \to Z$, and we can see that

$$(gf)_*(\varphi) = gf\varphi = g_*(f_*(\varphi)) = (g_*f_*)(\varphi),$$

  so we are "preserving composition" in some sense because we composed before and after.

Example 2.4. Given a topological space X, we can create the fundamental group $\pi_1(X)$. This mapping is nice because a continuous map $f : X \to Y$ will induce a map $\pi(f) : \pi_1(X) \to \pi_1(Y)$, and in fact we can check that $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X)}$ as well as preserving composition ($f : X \to Y$ and $g : Y \to Z$ gives $\pi_1(gf) = \pi_1(g)\pi_1(f)$).

With the above motivation, we are now ready to give the definition of a functor.

Definition 2.5 (Functor)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of "assignments" $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ and $\operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{D}$ satisfying the following coherence laws.

- Morphisms make sense: if $f : x \to y$ a morphism in $\mathcal{C}$, then Ff is a morphism with domain Fx and codomain Fy.
- Identity: given an object $c \in \mathcal{C}$, we require $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Composition: given morphisms $f : x \to y$ and $g : y \to z$ in $\mathcal{C}$, we require that $F(gf) = F(g)F(f)$.

2.1.2 More Examples

Let's do more examples.

Example 2.6 (Forgetful)**.** There is a functor $U : \mathrm{Grp} \to \mathrm{Set}$ which sends a group G to its underlying set G and a group homomorphism to the underlying function. In other words, we are simply forgetting the algebraic structure of the group. Because the composition law in groups is composition of functions, and identities in Grp do nothing like in Set.

Example 2.7 (Forgetful)**.** Here are more forgetful functors.

- $\mathrm{Ring} \to \mathrm{Grp}$ (by $R \mapsto R^\times$)
- $\mathrm{Field} \to \mathrm{Ring}$
- $\mathrm{Ring} \to \mathrm{Ab}$
- $\mathrm{Grp} \to \mathrm{Set}_*$ by sending $G \mapsto (G, e_G)$; namely, we point the set of G by its identity, which must be fixed by group homomorphisms anyways.

With all of our forgetful functors lying around, we have the following definition.

Definition 2.8 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if it has a forgetful functor to Set.

This is not terribly formal because we haven't defined what a forgetful functor means, but hopefully this is sufficiently intuitive: $\mathcal{C}$ should be sets with some extra structure.

Before our next example, we pick up the following example.

Definition 2.9 (Endofunctor)**.** A functor F is an *endofunctor* of its "domain" and "codomain" categories are the same category.

Example 2.10. There is an endofunctor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ sending a set X to its power set $\mathcal{P}(X)$. We send morphisms $f : X \to Y$ to $\mathcal{P}(f)$ by sending subsets $S_X \subseteq X$ in $\mathcal{P}(X)$ to the image $f(S_X) \in \mathcal{P}(Y)$. We will not check the functoriality conditions, but it can be done without too much effort.

And now for more examples.

Example 2.11. There is a functor $\mathrm{Top} \to \mathrm{Htpy}$ by sending a topological space X to the same space up to homotopy. Then we send continuous maps to continuous maps, up to homotopy.

Example 2.12. There is a "free" functor $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$ sending a set S to the abelian group

$$\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z}s.$$

Essentially, this is the free $\mathbb{Z}$-module generated by S; formally, $\mathbb{Z}[S]$ is made of finite $\mathbb{Z}$-linear combinations of elements of S.

Then we can take a function $f : S \to T$ to a group homomorphism $\mathbb{Z}[S] \to \mathbb{Z}[T]$ because we have described where to send the "basis elements" of S, and hence this f will uniquely determine the full map.

Example 2.13. Fix $\mathcal{C}$ a locally small category, and fix some $x \in \mathcal{C}$. Then there is a functor $\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ by sending

$$y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y) \qquad \text{and} \qquad (f : y \to z) \mapsto f_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z),$$

where $f_* : \varphi \mapsto f\varphi$ is again post-composition.

Example 2.14. There is an endofunctor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ by sending objects and morphisms to themselves.

2.1.3 Categories of Categories

While we're here, we note that we can create new functors from old ones by "composition."

Proposition 2.15. Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ functors. Then the naturally defined map $GF : \mathcal{C} \to \mathcal{E}$ is also a functor.

Proof. We do indeed send objects to objects, and a morphism $f : x \to y$ in $\mathcal{C}$ will be sent to $F(f) : Fx \to Fy$ and then

$$GF(f) : GFx \to GFy.$$

Further, we can check that $GF(\mathrm{id}_x) = G(\mathrm{id}_{Fx}) = \mathrm{id}_{GFx}$, so GF preserves identities. And then, given $f : x \to y$ and $g : y \to z$, we see that

$$GF(gf) = G(F(g)F(f)) = GF(g)GF(f),$$

which finishes the composition check. ■

The point of the above composition law, is that it lets us form a "category."

Definition 2.16. We define Cat to be the category of small categories where morphisms are functors. We define CAT to be the category of locally small categories where morphisms again are functors.

Remark 2.17. Fixing two small categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ can be identified with a function on merely the morphism sets $\operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{D}$, which is itself a set. Thus, Cat is a locally small category: $\mathrm{Cat} \in \mathrm{CAT}$.

2.1.4 Subcategories.

To finish out class, we have the following warning.

Warning 2.18. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We check that the naturally defined "image" $F(\mathcal{C})$ need not be a subcategory of $\mathcal{D}$.

Here is an example. Let $\mathcal{C}$ be the following category.

$$a \xrightarrow{f} b$$

$$a' \xrightarrow{f'} b'$$

Then let $\mathcal{D}$ be the following category.

$$0 \xrightarrow{x} 1 \xrightarrow{y} 2$$

Now we define $F : \mathcal{C} \to \mathcal{D}$ by $Ff = x$ and $Ff' = y$, which will make a perfectly fine functor. However, the composition $yx : 0 \to 2$ in $\mathcal{D}$ does not live in the image of F, so this image is not a subcategory.

To fix this problem, one often says something like "given a functor $F : \mathcal{C} \to \mathcal{D}$, consider the full subcategory of $F(\mathcal{C})$" to mean closing up $F(\mathcal{C})$'s potentially unclosed composition.

2.2 January 31

So class is in-person today.

2.2.1 Small Remark

A question was asked in the Discord server about dualizing. In theory, dualizing theorems should be very easy: simply state the theorem in the opposite category, provided we have shown the necessary machinery to make the theorem dualize as necessary.

2.2.2 Contravariance

Today we are talking about contravariance. A functor $F : \mathcal{C} \to \mathcal{D}$ is defined so far as what are called "covariant" functors. We would like to define contravariant functors. There are lots of equivalent ways to do this.

Definition 2.19 (Contravariance, I)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a mapping of objects and morphisms with the following coherence laws.

- If $f : a \to b$ in $\mathcal{C}$, then $Ff : Fb \to Fa$. (Note the reversal of direction!)
- Identity: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for each $c \in \mathcal{C}$.
- Contravariant (!) composition: if $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, then $F(gf) = F(f)F(g)$.

This in fact comes from dualizing.

Definition 2.20 (Contravariance, II)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

To be explicit, if we are given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, then a morphism $f : a \to b$ in $\mathcal{C}$ is first taken to a morphism $f^{\mathrm{op}} : b^{\mathrm{op}} \to a^{\mathrm{op}}$. And if we have another morphism $g : b \to c$ in $\mathcal{C}$, then we see the diagram

$$a \xrightarrow{f} b \xrightarrow{g} c$$

becomes

$$a^{\mathrm{op}} \xleftarrow{f^{\mathrm{op}}} b^{\mathrm{op}} \xleftarrow{g^{\mathrm{op}}} c^{\mathrm{op}}$$

becomes

$$Fa^{\mathrm{op}} \xleftarrow{Ff^{\mathrm{op}}} Fb^{\mathrm{op}} \xleftarrow{Fg^{\mathrm{op}}} Fc^{\mathrm{op}},$$

which gives our composition law.

We can also dualize in the opposite direction.

Definition 2.21 (Contravariance, III)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Warning 2.22. We will use Definition 2.20 as our definition of contravariance.

Example 2.23. We work with Vec_k the category whose objects are k-vector spaces and morphisms which are linear maps. Then we have a functor

$$-^* : \mathrm{Vec}_k^{\mathrm{op}} \to \mathrm{Vec}_k$$

by taking $V \mapsto V^*$. (Here, $V^* := \mathrm{Hom}_k(V, k)$.) As for morphisms, we need to take $f : V \to W$ to some map $f^* : W^* \to V^*$, which is

$$f^* : \varphi \mapsto \varphi f.$$

Example 2.24. We work with Poset the category whose objects are posets and morphisms which are order-preserving maps. I.e., a map $f : P \to Q$ is order-preserving if and only if $a \leq b$ in P implies $f(a) \leq f(b)$ in Q. Now we define the contravariant functor $\mathcal{O} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Poset}$ by taking

$$X \mapsto \{U : \text{open } U \subseteq X\},$$

where the order on the right is by inclusion. Then a continuous map $f : X \to Y$ becomes the order-preserving (!) map $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ by

$$\mathcal{O}(f)(U_Y) \coloneqq f^{-1}(U_Y).$$

Explicitly, open subsets $U_1 \subseteq U_2$ of Y have $f^{-1}(U_1) \subseteq f^{-1}(U_2)$ back in X.

Remark 2.25. We can use the above example to define a presheaf. "Presheaf" can have lots of meanings.

- A "presheaf" can be any contravariant functor.
- A "presheaf" can be any contravariant functor with codomain Set.
- A "presheaf" can be any contravariant functor from $\mathcal{O}(X)^{\mathrm{op}}$. It is Set-valued (respectively, $\mathcal{C}$-valued) if its codomain is Set (respectively, $\mathcal{C}$).

2.2.3 A Lemma

It's a math class, so we should probably prove something today.

Theorem 2.26. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ preserves isomorphisms.

Remark 2.27. By convention, all functors will be covariant, and if we want a contravariant functor, we will write $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. In other words, I will now stop writing "(covariant)."

Proof. Let $f : a \to b$ be an isomorphism in $\mathcal{C}$ with inverse g. We want to show that $F(f)$ is an isomorphism; we claim that $F(g)$ is its inverse. Indeed,

$$F(f)F(g) = F(fg) = F(\mathrm{id}_b) = \mathrm{id}_{F(b)} \qquad \text{and} \qquad F(g)F(f) = F(gf) = F(\mathrm{id}_a) = \mathrm{id}_{F(a)},$$

so indeed, $F(g)$ is an inverse of $F(f)$. So $F(f)$ is an isomorphism, and we are done. ■

This example can do things.

Example 2.28. Fix groups G, H and their one-object categories $\mathrm{B}G, \mathrm{B}H$. We claim that functors $F : \mathrm{B}G \to \mathrm{B}G$ contain exactly the data of a group homomorphism $G \to H$. To see that F induces a group homomorphism, suppose $\sigma, \tau \in G$, we have by funtoriality

$$F(\sigma\tau) = F(\sigma)F(\tau),$$

which is exactly what we need to be a group homomorphism. Conversely, if $f : G \to H$ is a group homomorphism, then f induces a functor: $f(\sigma\tau) = f(\sigma)f(\tau)$ by definition, and $f(\mathrm{id}_G) = \mathrm{id}_H$ is a result of group theory.

Example 2.29. A functor $F : \mathrm{B}G \to \mathcal{C}$ is precisely the data of a G-action of an object $c \in \mathcal{C}$. We send the one object $* \in \mathrm{B}G$ somewhere, say to an object $c \in \mathcal{C}$. Then each $\sigma \in G$ goes to some morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ (which is in fact an isomorphism because σ is an isomorphism $\mathrm{B}G$). So in total we get a map

$$G \to \mathrm{Aut}\, c,$$

which is exactly the data of a group action. This unifies group actions on all sorts of structures.

The above definition is special enough to have a name.

Definition 2.30 (Functorial group action)**.** A *functorial group action* of G on a category $\mathcal{C}$ is a functor $\mathrm{B}G \to \mathcal{C}$.

Remark 2.31. Technically we will be viewing these functors as providing left actions. To get a right action, we want a functor $(\mathrm{B}G)^{\mathrm{op}} \to \mathcal{C}$.

Note, as in the example, the functor contains the same data as a group homomorphism $G \to \mathrm{Aut}\, c$ for some $c \in \mathcal{C}$.

Remark 2.32. Bryce would like to make us aware that writing down $G \to \mathrm{Aut}\, c$ as a group homomorphism is only legal when $\mathcal{C}$ is locally small.

Example 2.33. Given a group G, a G-representation V of G is a functor $\mathrm{B}G \to \mathrm{Vec}_k$ where $* \in \mathrm{B}G$ goes to $V \in \mathrm{Vec}_k$.

2.2.4 The Hom Bifunctor

We have a little time left, so let's do something fun. Given a (locally small) $\mathcal{C}$ and an object $x \in \mathcal{C}$, we get two functors

$$\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set} \qquad \text{and} \qquad \mathrm{Mor}_{\mathcal{C}}(-, x) : \mathcal{C} \to \mathrm{Set}.$$

The former functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $\varphi : y \to z$ to $\varphi_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$ to $\varphi_* : f \mapsto \varphi f$. We can check this functor is covariant because

$$\varphi_* \psi_*(f) = \varphi\psi f = (\varphi\psi)_*(f).$$

Now, the latter functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(y, x)$ and $\varphi : y \to z$ to $\varphi^* : \mathrm{Mor}_{\mathcal{C}}(z, x) \to \mathrm{Mor}_{\mathcal{C}}(y, z)$ by $\varphi^* : f \mapsto \varphi f$. We can check this functor is contravariant because

$$\psi^* \varphi^* f = f \varphi \psi = (\varphi\psi) * f.$$

2.3 February 2

Today we are talking about product categories and the Hom bifunctor.

2.3.1 Hom Bifunctor

Here is our definition.

Definition 2.34 (Product category)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then we define the *product category* $\mathcal{C} \times \mathcal{D}$ as follows.

- We define $\operatorname{Ob}\mathcal{C} \times \mathcal{D}$ to be the collection of ordered pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
- We define $\operatorname{Mor}((c, d), (c', d'))$ to be the collection of ordered pairs (f, g) with $f : c \to c'$ a morphism in $\mathcal{C}$ and $g : d \to d'$ a morphism in $\mathcal{D}$.

Lastly, we define identity to be the identity on each object and composition by composition component-wise.

From yesterday, we have the following functors.

Definition 2.35 (Functors represented by objects)**.** Fix $\mathcal{C}$ a locally small category and $x \in \mathcal{C}$ an object. Then we have the functors

$$\operatorname{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \operatorname{Set} \qquad \text{and} \qquad \operatorname{Mor}_{\mathcal{C}}(-, x) : \mathcal{C}^{\text{op}} \to \operatorname{Set}.$$

The former functor is the *covariant functor represented by* x, and the latter is the *contravariant functor represented by* x.

We would like to codify the structure that having two functors gives us, so we have the following definition.

Definition 2.36 (Bifunctor)**.** A *bifunctor* is a functor whose domain is a product of categories.

In particular, here is our standard example.

Definition 2.37 (Hom bifunntor)**.** Fix $\mathcal{C}$ a locally small category. Then Hom *bifunctor* is the functor given by the functors representing a particular object $x \in \mathcal{C}$. Namely, we have

$$\operatorname{Mor}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \operatorname{Set}$$

by taking $(x, y) \mapsto \operatorname{Mor}_{\mathcal{C}}(x, y)$.

We will not check that this is actually a functor, but it is.

2.3.2 Category Isomorphism

We would like a notion of two categories being the same, but this is somewhat subtle. Here is a first approximation.

Definition 2.38 (Isomorphism)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if and only if there is an inverse functor $G : \mathcal{D} \to \mathcal{C}$ so that $GF = \operatorname{id}_{\mathcal{C}}$ and $FG = \operatorname{id}_{\mathcal{D}}$. In this case we say that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*.

Remark 2.39. As usual, isomorphisms are unique and whatnot.

Let's make this definition a little more concrete.

Proposition 2.40. An isomorphism $F : \mathcal{C} \to \mathcal{D}$ descends to a bijective (i.e., injective and surjective) map $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$.

Remark 2.41. We are attempting to care about set-theoretic issues in our phrasing because Bryce cares about set-theoretic issues.

Proof of Proposition 2.40. Let G be the inverse morphism for F. Then we claim that the induced map $G : \operatorname{Ob}\mathcal{D} \to \operatorname{Ob}\mathcal{C}$ will be the inverse for the induced map for F. This is clear because $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. ■

It turns out that isomorphisms are a little too strong: there are categories we want to be the same but are not actually isomorphic.

Example 2.42. The category

$$\begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd}$$

is not isomorphic to

$$\bullet$$

because there are a different number of objects, so there is no bijection.

2.3.3 Natural Transformation

To salvage our notion of categorical isomorphism, we need a notion of naturality. Naturality is more of something that we can feel as mathematicians rather than something we like to formalize.

Example 2.43. Any two trivial groups have a canonical isomorphism between them. In fact, there is only one homomorphism at all.

Non-Example 2.44. There is no "natural" or "canonical" isomorphism $\mathbb{Z}/3\mathbb{Z} \to A_3$, though the groups are isomorphic.

Non-Example 2.45. Given a two-dimensional $\mathbb{R}$-vector space named V, there is no canonical isomorphism $\mathbb{R}^2 \to V$.

We would like maps to preserve all the structure we could want. So here is our notion of naturality for functors.

Definition 2.46 (Natural transformation). Fix functors $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\eta : F \Rightarrow G$ consists of the data of a morphism $\eta_X : Fc \to Gc$ for each $c \in \mathcal{C}$ such that the following diagram always commutes for any morphism $f : c \to c'$ in $\mathcal{C}$.

$$\begin{tikzcd} Fc \arrow[r, "\eta_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\eta_{c'}"'] & Gc' \end{tikzcd}$$

The maps φ_c are called the *components* of φ.

Quote 2.47. Burn this square into your minds. It is the most important square in this class.

As usual, we start with examples.

Exercise 2.48. We work in Vec_k. Then we consider the functor $-^{**} : \mathrm{Vec}_k \to \mathrm{Vec}_k$ by $V \mapsto V^{**}$. Then we claim that there is a natural transformation from $-^{**}$ to id, using the natural transformation

$$\mathrm{ev}_V : V \to V^{**}$$

by $\mathrm{ev}_V(x) := (\lambda \in V^* \mapsto \lambda x)$.

Proof. We need to check that the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\mathrm{ev}_W"'] & W^{**} \end{tikzcd}$$

Very quickly, we recall that $f^{**} : V^{**} \to W^{**}$ is by

$$f(\varphi) = (\lambda \in W^* \mapsto \varphi(\lambda f)).$$

Namely, $\lambda : W \to k$, so $\lambda f : V \to k$ lives in V^*, so $\varphi(\lambda f) \in k$.

Now we check the commutativity of the square. Fix some $x \in V$ and a linear functional $\lambda : W \to k$. Then we can carefully compute, after many tears and groans, that

$$f^{**}(\mathrm{ev}_V(x))(\lambda) = \mathrm{ev}_V(x)(\lambda f) = \lambda f(x) = \mathrm{ev}_W(f(x))(\lambda).$$

Because λ was arbitrary, we see that $f^{**}\,\mathrm{ev}_V(\lambda) = \mathrm{ev}_W\, f(x)$, which then gives us $f^{**}\,\mathrm{ev}_V = \mathrm{ev}_W\, f$. ■

We have the following definition.

Definition 2.49 (Natural isomorphism)**.** A natural transformation $\eta : F \to C$ is a *natural isomorphism* if and only if its component morphisms are isomorphisms.

Example 2.50. In finVec_k, the above ev is a natural isomorphism because $\mathrm{ev}_V : V \Rightarrow V^{**}$ is an isomorphism when V is finite-dimensional.

Here is a quick proposition.

Proposition 2.51. Let $\varphi : F \Rightarrow G$ be a natural isomorphism. Then the inverse morphisms $\psi_c := \varphi_c^{-1}$ assemble to make a natural transformation $\psi : G \Rightarrow F$.

Proof. We will be brief. Given a morphism $f : x \to y$, we need to check that the following diagram commutes.

$$\begin{tikzcd} Gx \arrow[r, "\psi_x"] \arrow[d, "Gf"'] & Fx \arrow[d, "Ff"] \\ Gy \arrow[r, "\psi_y"'] & Fy \end{tikzcd}$$

In other words, we need to know that $\psi_y Ff = Gf\psi_x$. Well, we already know that

$$\varphi_y Ff = Gf\varphi_x$$

by naturality, so

$$Ff\psi_x = \psi_y\varphi_y Ff\psi_x = \psi_y Gf\varphi_x\psi_x = \psi_y Gf$$

after checking through. ■

2.4 February 7

2.4.1 Examples of Natural Transformations

We're talking about more natural transformations today. For our first example, consider the covariant power set functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ by $S \mapsto \mathcal{P}(S)$ and $f : S \to T$ to $\mathcal{P}(f)(U) := f(U)$ for $U \subseteq S$.

Exercise 2.52. We define a natural transformation $\eta_\bullet : \mathrm{id}_{\mathrm{Set}} \Rightarrow \mathcal{P}$ a function $\eta_S : S \to \mathcal{P}(S)$ by

$$\eta_S(x) := \{x\}$$

Proof. Fix $f : S \to T$ a morphism in Set. After plugging everything in, we need the following diagram to commute.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\eta_S"'] & T \arrow[d, "\eta_T"] \\ \mathcal{P}(S) \arrow[r, "\mathcal{P}(f)"'] & \mathcal{P}(T) \end{tikzcd}$$

To see this commutes, fix some $x \in S$, and we run it through the diagram as follows.

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\eta_S"'] & f(x) \arrow[d, mapsto, "\eta_T"] \\ \{x\} \arrow[r, mapsto, "\mathcal{P}(f)"'] & \{f(x)\} \end{tikzcd}$$

So indeed, the diagram does commute. ■

Remark 2.53. We may call the second diagram an "internal" diagram because it is looking internally at our objects.

For our next example, recall we defined a functorial G-action on some object $c \in \mathcal{C}$ by a functor $F : \mathrm{B}G \to \mathcal{C}$. Our goal is to define a G-equivariant map between objects.

Exercise 2.54. We track the data between two G-representations $F, G : \mathrm{B}G \to \mathrm{Vec}_k$ by a natural transformation $\eta_\bullet : \mathrm{Vec}_k \Rightarrow \mathrm{Vec}_k$.

Proof. Because $\mathrm{B}G$ has only one object $*$, we set $V := F(*)$ and $W := G(*)$ and need to check the commutativity of the following diagram, for some $g : * \to *$ in G.

$$\begin{tikzcd} V \arrow[r, "Fg"] \arrow[d, "\eta_*"'] & V \arrow[d, "\eta_*"] \\ W \arrow[r, "Fw"'] & W \end{tikzcd}$$

Note that the natural transformation $\eta_\bullet$ really only consists of the map η_*, which is a linear map $V \to W$ which respects the group action: $\eta_*(gv) = g\eta_*(v)$. ■

These G-equivariant maps can be turned into a category.

Definition 2.55 (G-representations)**.** We define the category of G-*representations* to be the category consisting of objects which are functors $F : \mathrm{B}G \to \mathrm{Vec}_k$ and morphisms which are natural transformations between the functors.

Exercise 2.56. We check that there is a category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose

Proof. To define our morphisms, suppose $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\eta_\bullet : F \Rightarrow G$ and $\nu_\bullet : G \Rightarrow H$. Lastly, we define our composition by

$$(\nu\eta)_X := \eta_X \nu_X.$$

To check that $(\eta\nu)_\bullet : F \Rightarrow H$ is in fact a natural transformation, we have the following ladder.

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"'] \arrow[dd, bend right, "(\nu\eta)_x"'] & Fy \arrow[d, "\eta_y"] \arrow[dd, bend left, "(\nu\eta)_y"] \\ Gx \arrow[r, "Gf"] \arrow[d, "\nu_x"'] & Gy \arrow[d, "\nu_y"] \\ Hx \arrow[r, "Hf"'] & Hy \end{tikzcd}$$

Each square commutes, so the 2×1 rectangle will also commute. We check associativity by drawing a 3×1 rectangle and seeing that it commutes.

To define our identity maps for our category, we take $(\mathrm{id}_F)_X := \mathrm{id}_{F(x)} : Fx \to Fx$. We can check that this works with our composition without too many tears. ■

Definition 2.57 (Functor category)**.** The category of the above exercise is the *functor category*, notated $\mathcal{D}^{\mathcal{C}}$.

Example 2.58. We have that $\mathrm{Rep}_G = \mathrm{Vec}_k^{BG}$.

2.4.2 Yoneda, Contravariant It Is

For the discussion that follows, we fix $\mathcal{C}$ locally small and $f : w \to x$ and $h : y \to z$ some morphisms in $\mathcal{C}$. From this we get the following square.

$$\begin{tikzcd} \mathrm{Mor}(x,y) \arrow[r, "h-"] \arrow[d, "-f"'] & \mathrm{Mor}(x,z) \arrow[d, "-f"] \\ \mathrm{Mor}(w,y) \arrow[r, "h-"'] & \mathrm{Mor}(w,z) \end{tikzcd}$$

We can check that this square commutes. Here is the internal square.

$$\begin{tikzcd} g \arrow[r, mapsto, "h-"] \arrow[d, mapsto, "-f"'] & hg \arrow[d, mapsto, "-f"] \\ gf \arrow[r, mapsto, "h-"'] & hgf \end{tikzcd}$$

Hooray, it commutes. The point is that $h-$ and $-f$ are going to induce natural transformations of our Mor functors.

- The functors $\mathrm{Mor}(x, -), \mathrm{Mor}(w, -) : \mathcal{C} \to \mathrm{Set}$. Then any morphism $f : w \to x$ induces a natural transformation $-f : \mathrm{Mor}(x, -) \Rightarrow \mathrm{Mor}(w, -)$. The naturality check is the commutativity of the above square.

- Similarly, the functors $\mathrm{Mor}(-,y), \mathrm{Mor}(-,z) : \mathcal{C} \to \mathrm{Set}$. Then any morphism $h : x \to y$ induces a natural transformation $h- : \mathrm{Mor}(-,y) \Rightarrow \mathrm{Mor}(-,w)$. The naturality check is again the commutativity of the above square.

We won't be more explicit about our squares because my head hurts.

Remark 2.59. Later in life we will talk about the Yoneda embedding, which is essentially about the embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}^{\mathcal{C}}$, which takes $x \mapsto \mathrm{Mor}(x,-)$ and $f : x \to y$ to the natural transformation $-f : \mathrm{Mor}(x,-) \Rightarrow \mathrm{Mor}(y,-)$. This will turn out to be a functor and very good. We will not say more for now.

2.4.3 Categorification

The category Set has some nice operations: we can talk about products $A \times B$, disjoint unions $A \sqcup B$, and functions $A^C = \{f : C \to A\}$. Note that these notations are suggestive of multiplication, addition (depending on whom you talk to), and exponentiation. For example,

$$\#(A \times B) = \#A \times \#B, \quad \#(A \sqcup B) = \#A + \#B, \quad \#\left(A^C\right) = \#A^{\#C}.$$

This gives us some notion of a "cardinality functor" $\# : \mathrm{FinSet} \to \mathbb{N}$, which we can check does some things.

This lets us define "categorification." We will not give a formal definition of this, but here are some instructive examples.

Example 2.60. The functor $\# : \mathrm{FinSet} \to \mathbb{N}$ is a decategorification functor. For example, we can categorify $a \times (b+c) = a \times b + a \times c$ in $\mathbb{N}$ to some natural isomorphism

$$A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C).$$

Example 2.61. There is a decategorification functor $\dim : \mathrm{fdRep}_G \to \mathbb{N}$.

2.4.4 Equivalence: Advertisement

Let's close class by defining an equivalence of categories. Recall that we called a functor $F : \mathcal{C} \to \mathcal{D}$ an isomorphism if and only if it has an inverse functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$.

This is a bad notion of saying two categories are the same.

Example 2.62. The categories of k-matrices and k-vector spaces are not isomorphic (they don't have the same), even though we often think about vector spaces as merely being some dimensional space.

Here is the fix

Definition 2.63 (Equivalence)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if and only if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$.

2.5 February 9

2.5.1 Equivalence

We can define a category Mat_k to have objects which are the natural numbers and morphisms which are $\mathrm{Mat}_k(n,m)$ equal to the $m \times n$ matrices with coefficients in k. In linear algebra, we want to think about each natural number n as a k-vector space of dimension n, and we want to think about each matrix $n \to m$ as a linear map. In other words, Mat_k should be "the same" as fdVec_k.

However, fdVec_k and Mat_k do not even have the same number of objects, so they cannot be isomorphic. We still want them to be the same, so we weaken our notion of isomorphism.

Definition 2.64 (Equivalence)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ if and only if $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$. If an equivalence between $\mathcal{C}$ and $\mathcal{D}$ exists, then $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, denoted $\mathcal{C} \simeq \mathcal{D}$.

We should probably start by showing that our notion of equivalence forms what we think of as an equivalence relation.

Remark 2.65 (Bryce)**.** Equivalence does not form an equivalence relation for size reasons.

Lemma 2.66. Fix categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. Then the following hold.

- Reflexive: $\mathcal{C} \simeq \mathcal{C}$.
- Symmetric: $\mathcal{C} \simeq \mathcal{D}$ implies $\mathcal{D} \simeq \mathcal{C}$.
- Transitive: $\mathcal{C} \simeq \mathcal{D}$ and $\mathcal{D} \simeq \mathcal{E}$ implies $\mathcal{C} \simeq \mathcal{E}$.

Proof. We will be brief.

- We have that $\mathrm{id}_{\mathcal{C}}$ provides the needed equivalence.
- If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence with $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$, then G witnesses $\mathcal{D} \simeq \mathcal{C}$.
- Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ witness $C \simeq D$, and fix $F' : \mathcal{D} \to \mathcal{E}$ and $G' : \mathcal{E} \to \mathcal{D}$ witness $D \simeq E$.

  In particular, we are promised natural isomorphisms $\varphi : G \simeq \mathrm{id}_{\mathcal{C}}$ and $\psi : FG \simeq \mathrm{id}_{\mathcal{D}}$ and $\varphi' : G'F' \simeq \mathrm{id}_{\mathcal{D}}$ and $\psi' : F'G' \simeq \mathrm{id}_{\mathcal{E}}$. We would like $GG'F'F \simeq \mathrm{id}_{\mathcal{C}}$, and then $F'FGG' \simeq \mathrm{id}_{\mathcal{E}}$ will follow in a very similar way.

  Well, for an object $c \in \mathcal{C}$, we define our natural transformation $\eta_\bullet$ as having component

  $$\eta_c \coloneqq \varphi_c \circ G\varphi'_{Fc},$$

  which takes $GG'F'Fc$ to GFc to c. We show naturality directly. Fix some morphism $f : x \to y$ in $\mathcal{C}$. We need the following diagram to commute.

  $$\begin{tikzcd} GG'F'Fx \arrow[r, "\eta_x"] \arrow[d, "GG'F'Ff"'] & x \arrow[d, "f"] \\ GG'F'Fy \arrow[r, "\eta_y"'] & y \end{tikzcd}$$

  To see that this commutes, here is an expanded diagram.

  $$\begin{tikzcd} GG'F'Fx \arrow[r, "G\varphi'_{Fx}"] \arrow[d, "GG'F'Ff"'] \arrow[rr, bend left, "\eta_x"] & GFx \arrow[r, "\varphi_x"] \arrow[d, "GFf"] & x \arrow[d, "f"] \\ GG'F'Fx \arrow[r, "G\varphi'_{Fy}"'] \arrow[rr, bend right, "\eta_y"'] & GFy \arrow[r, "\varphi_y"'] & y \end{tikzcd}$$

By definition of $\eta_\bullet$, it now suffices to show that the left and right squares commute. The right square commutes by naturality of φ_x. To see that the left square commutes, we note that it is what we get after applying G to the naturality square for φ' on the morphism $GFf : GFx \to GFy$.

Lastly, to see that η is a natural isomorphism, we note that each component $\eta_c = \varphi_c \circ G\varphi'_{Fc}$ is the composite of isomorphisms, where we are using that φ and φ' are natural isomorphisms and that functors preserve isomorphisms. ■

This is nice because oftentimes showing that two categories are equivalent is easier by showing a chain of equivalences instead of doing it directly. For example, in our proof that $\mathrm{Mat}_k \simeq \mathrm{fdVec}_k$, we will instead show that both of these categories are equivalent to $\mathrm{fdVec}_k^{\mathrm{basis}}$ of vector spaces with given basis.

2.5.2 Lazy Equivalence

We want to provide a tool for constructing equivalences without having to actually write down a natural transformation. By way of analogy, when showing an "isomorphism of sets" we often show that a given map is both injective and surjective. We will do something similar.

Definition 2.67 (Adjectives for functors). Fix categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $F : \mathcal{C} \to \mathcal{D}$. We consider the map $F^\circ : F : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(Fx, Fy)$. Then

- F is *full* if and only if F° is surjective.
- F is *faithful* if and only if F° is injective.
- F is *fully faithful* if and only if F is full and faithful.
- F is *essentially surjective* on objects if and only if each $d \in \mathcal{D}$ has some $c \in \mathcal{C}$ such that $Fc \cong d$ in $\mathcal{D}$.
- F is an *embedding* if and only if F is faithful and injective on objects.
- F is a *full embedding* if and only if F is an embedding and full.

Remark 2.68. Technically we might want to require that $\mathcal{C}$ and $\mathcal{D}$ be locally small, but there are ways of stating "surjective" and "injective" to note require the underlying domain and codomain to be sets.

Remark 2.69. Being "essentially surjective" will give problems with the axiom of choice later in life because we are not requiring any notion of uniqueness.

We note that a functor being "full" or "faithful" are both local conditions on particular sets of morphisms. For example, if a functor doesn't even hit an object which is outside the image of F, then we can't touch those morphism sets.

Example 2.70. Full and faithful does not imply injective on objects. For example, consider the natural functor F from the left category to the right category, which causes full-on collisions but not locally on the morphism sets.

$$\begin{tikzcd} a_1 \arrow[d] & a_2 \arrow[d] \arrow[r, phantom, "\overset{F}{\Longrightarrow}"] & a \arrow[d] \\ b_1 & b_2 & b \end{tikzcd}$$

Namely, the maps $\mathrm{Mor}_{\mathcal{C}}(a_\bullet, b_\bullet) \to \mathrm{Mor}_{\mathcal{C}}(a, b)$.

Let's finish class by proving something.

Proposition 2.71. The following are closed under composition.

- Full functors.
- Faithful functors.
- Essentially surjective functors.

Proof. We will be very brief.

- Read the proof of the below and replace all instances of the word "surjective" with "injective."
- Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are faithful functors. Then fix $x, y \in \mathcal{C}$, and we know that the induced maps
$$F^\circ : \operatorname{Mor}_{\mathcal{C}}(x, y) \to \operatorname{Mor}_{\mathcal{D}}(Fx, Fy) \qquad \text{and} \qquad G^\circ : \operatorname{Mor}_{\mathcal{D}}(Fx, Fy) \to \operatorname{Mor}_{\mathcal{D}}(GFx, GFy)$$
are both injective, so their composite is injective. To be explicit, if f and g have $(GF)f = (GF)g$, then $G(Ff) = G(Fg)$, so $Ff = Fg$ by injectivity of G°, so $f = g$ by
- Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are essentially surjective functors. Well, fix any $e \in \mathcal{E}$, and we are promised an object $d \in \mathcal{D}$ such that $Gd \cong e$. But now we are promised an object $c \in \mathcal{C}$ such that $Fc \cong d$, so $GFc \cong Fd \cong e$, which shows that GF is essentially surjective. ■

2.6 February 11

2.6.1 A Better Equivalence

Today we will be talking about the following theorem for our discussion.

Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are true.

(a) If F is an equivalence, then F is fully faithful and essentially surjective.

(b) Assuming a strong form of the axiom of choice, the converse holds.

Remark 2.73. The strong form of the Axiom of choice is for, not sets, but classes/categories depending on how we choose to construct our categories.

Proof of (a) in Theorem 2.72. We will want some lemmas.

Lemma 2.74. Fix a category $\mathcal{C}$. Further, fix a morphism $f : c \to d$ and isomorphisms $\varphi : c \cong c'$ and $\psi : d \cong d'$. Then there is a unique morphism $f' : c' \to d'$ such that one (or equivalently, all) of the following four squares commute.

$$\begin{tikzcd}
c' \arrow[r, shift left] \arrow[d, dashed, "f'"'] & c \arrow[l, shift left, "\varphi"] \arrow[d, "f"] \\
d' \arrow[r, shift left] & d \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

Here, the four squares are achieved by changing the direction of φ and ψ.

Proof. This is on the homework. ■

We now return to the proof of the theorem. In the easier direction, suppose that F is an equivalence with its inverse equivalence $G : \mathcal{D} \to \mathcal{C}$, witnessed by natural isomorphisms $\eta_\bullet : \mathrm{id}_\mathcal{C} \Rightarrow GF$ and $\varepsilon : GF \Rightarrow \mathrm{id}_\mathcal{D}$. We have the following checks.

- We show that F is essentially surjective. Indeed, for any object $d \in \mathcal{D}$, we set $c := Gd$. Then we see $FGd \cong d$ is witnessed by the component isomorphism ε_d.

- We show that F is faithful, for which we have to use Lemma 2.74. Indeed, suppose that we have morphisms $f, g : c \to d$ such that $Ff = Fg$. Then in fact $GFf = Gfg$, so the following diagrams will commute.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "GFf=GFg"'] & GFd \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "g"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "GFf=GFg"'] & GFd \end{tikzcd}$$

  It follows from Lemma 2.74 that there f and g are uniquely determined, so $f = g$.

We quickly remark that, by symmetry, G is also faithful.

- We show that F is full, which will use the lemma as well as the fact that G is faithful (!). Well, suppose that we have some morphism $g : Fc \to Fd$. Passing through to G, we get a morphism $Gg : GFg \to GFg$, so by Lemma 2.74, there is a unique morphism $f : c \to d$ so that the following diagram commutes.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, dashed, "f"'] & GFc \arrow[d, "Gg"] \\ d \arrow[r, "\eta_d"'] & GFd \end{tikzcd}$$

  Now, both GFf and Gg make the following diagram commute.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, dashed, "{Gg,GFf}"] \\ d \arrow[r, "\eta_d"'] & GFd \end{tikzcd}$$

  Thus, by Lemma 2.74, we see $GFf = Gg$, so $Ff = g$ by the faithfulness of G. This finishes. ■

Proof of (b) in Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a fully faithful and essentially surjective functor. We need to construct a $G : \mathcal{D} \to \mathcal{C}$ with some natural isomorphisms. We do this by hand.

- For each $d \in \mathcal{D}$, we callously choose[1] Gd to be any $c \in \mathcal{C}$ together with an isomorphism $\varepsilon_d : GFd \to d$. Indeed, such a d with isomorphism ε_d exists because F is essentially surjective.

- For each $f : d \to d'$ in $\mathcal{D}$, we use Lemma 2.74 to choose h to be the unique morphism making the following diagram commute.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[d, "f"'] & FGd \arrow[d, dashed, "h"] \\ d' \arrow[r, "\varepsilon_{d'}"'] & FGd' \end{tikzcd}$$

  But because F is fully faithful, there will be a unique morphism which we call Gf such that $F(Gf) = h$.

We would like to check that G is in fact our inverse equivalence. However, we don't even know if G is a functor yet.

[1] Note we are using some fuzzy form of the axiom of choice here. We will not say more about this.

- Fix $d \in \mathcal{D}$ and we compute $G(\mathrm{id}_d)$. We run through the definition. Well, we note that id_{FGd} makes the following diagram commute, so it will be the morphism generated by Lemma 2.74.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[d, "\mathrm{id}_d"'] & FGd \arrow[d, dashed, "\mathrm{id}_{FGd}"] \\ d \arrow[r, "\varepsilon_{d'}"'] & FGd' \end{tikzcd}$$

  But now we see that $F(\mathrm{id}_{Gd}) = \mathrm{id}_{FGd}$, so id_{Gd} must be the corresponding morphism promised by the fullness and faithfulness of F. In particular, by definition, $G(\mathrm{id}_d) = \mathrm{id}_{Gd}$.

- Suppose we have $f : d \to d'$ and $g : d' \to d''$. We want to show that $G(gf) = Gg \circ Gf$. For this, we have the following very big diagram.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"'] \arrow[d, "FGf"] \arrow[dd, bend right, "F(Gf\circ Gf)"'] & d \arrow[d, "g"] \arrow[dd, bend left, "gf"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] \arrow[d, "FGg"] & d' \arrow[d, "f"] \\ FGd'' \arrow[r, "\varepsilon_{d''}"'] & d'' \end{tikzcd}$$

  This diagram does commute, from which we see that the left arrow can be either $F(Gg \circ Gf)$ (by funtoriality of F) or $F(G(gf))$. So by Lemma 2.74, we have $F(Gg \circ Gf) = F(G(gf))$, so faithfulness of F implies $Gg \circ Gf = G(gf)$.

Now we construct our natural isomorphisms.

- By construction of the εs, the following diagram commutes.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FGf"'] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$

- For the other direction, we note that if $Fx \cong Fy$ in $\mathcal{D}$, then $x \cong y$, which we will prove on the homework.[2] In particular, to create an isomorphism $\eta_c : c \to GFc$, it suffices to create an isomorphism $Fc \to FGFc$, for which we use $F\eta_c := \varepsilon_{Fc}^{-1}$. For naturality, we suppose we have a morphism $f : c \to c'$, and we note that the following diagram commutes.

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[d, "Ff"'] & FGFc \arrow[r, "\varepsilon_{Fc}"] \arrow[d, "FGFf"] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "F\eta_{c'}"'] & FGFc' \arrow[r, "\varepsilon_{Fc'}"'] & Fc' \end{tikzcd}$$

  Indeed, the outer rectangle commutes by definition of the $\eta_\bullet$s, and the right square commutes by naturality of the $\varepsilon_\bullet$s. Then this forces the left square to commute by an argument by noting

$$\varepsilon_{Fc'} \circ F\eta_{c'} \circ Ff = \varepsilon_{Fc'} \circ FGFf \circ F\eta_c$$

  by the commutativity of the outer diagram, so we get the commutativity by inverting along $\varepsilon_{Fc'}$. ■

2.7 February 14

Here we go.

[2] Yes, I know.

2.7.1 Using Our Equivalence

Last time we proved the following theorem.

Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are true.

(a) If F is an equivalence, then F is fully faithful and essentially surjective.

(b) Assuming a strong form of the axiom of choice, the converse holds.

Let's use this for fun and profit.

Corollary 2.75 (Math 110). The categories Mat_k and fdVec_k are equivalent.

Proof. Fix $\mathcal{C} := \mathrm{fdVec}_k^{\mathrm{basis}}$ to be the category consisting of objects which are ordered pairs $(V, \mathcal{B})$ of vector space equipped with a given ordered basis and morphisms which are linear transformations. I will call these based vector spaces because I can.

Observe that we have a functor $\mathcal{C} \to \mathrm{Mat}_k$ by sending the based vector space $(V, \mathcal{B})$ to $\dim V$ and the linear transformation $T : (V, \mathcal{B}) \to (V', \mathcal{B}')$ to the corresponding matrix representation. We run the following checks.

- The functor F is fully faithful because (based) linear transformations $(V, \mathcal{B}) \to (V', \mathcal{B}')$ are in bijective correspondence with matrices in $k^{\dim V' \times \dim V}$, which is exactly $\mathrm{Mor}_{\mathrm{Mat}_k}$
- This is essentially surjective because it is surjective: the vector space k^n goes to $n \in \mathrm{Mat}_k$.

Thus, F is an equivalence.

To continue, we use the forgetful functor $U : \mathcal{C} \to \mathrm{fdVec}_k$ by simply forgetting the basis. This is fully faithful because look at it, and it is essentially surjective because it is actually surjective. Thus, U witnesses $\mathcal{C} \simeq \mathrm{fdVec}_k$. Applying transitivity, we see

$$\mathrm{Mat}_k \simeq \mathcal{C} \simeq \mathrm{fdVec}_k,$$

which finishes. ■

We have the following definition.

Definition 2.76 (Essential image). The *essential image* of a functor $F : \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ consisting of objects $d \in \mathcal{D}$ such that $d \cong Fc$ for some $c \in \mathcal{C}$.

We are saying "full subcategory" to just throw in all the morphisms, so we don't have to worry about potential composition problems in $\mathcal{D}$.

Corollary 2.77. A fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of $\mathcal{C}$ onto the essential image of F.

Proof. Apply Theorem 2.72, where being essentially surjective follows from the definition of the essential image. ■

2.7.2 Motivating Diagram Chasing

We're going to be talking about diagram-chasing for a little while. This is the technique by which we extract large amounts of information from a commutative diagram. Namely, we will get to formally define what a commutative diagram is and so on. For this, we will want to do a little graph theory.

Definition 2.78 (Path). Fix a category $\mathcal{C}$. Then a *path* in $\mathcal{C}$ is finite sequence of the form

$$(A_1, f_1, A_2, f_2, \dots, A_n, f_n, A_{n+1}),$$

where $A_1, \dots, A_{n+1} \in \operatorname{Ob}\mathcal{C}$ and $f_k \in \operatorname{Mor}(A_k, A_{k+1})$ for each k.

Remark 2.79. Equivalently, we could encode this path by the sequence of morphism $f_1, \dots, f_n$ such that $\operatorname{cod} f_k = \operatorname{dom} f_{k+1}$.

Let's see an example of the power of abstracting diagrams.

Definition 2.80 (Monoid). A *monoid in the category* Set is a set M with morphisms $\mu : M \times M \to M$ and $\eta : \{*\} \to M$ such that the following diagrams commute.

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mu \times \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times \mu"'] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mathrm{id}_M \times \eta"] \arrow[dr, "\mathrm{id}_M"'] & M \times M \arrow[d, dashed, "\mu"] & M \arrow[l, "\eta \times \mathrm{id}_M"'] \arrow[dl, "\mathrm{id}_M"] \\ & M & \end{tikzcd}$$

Remark 2.81. Our monoid is made by the binary operation $\cdot_\mu : (a, b) \mapsto \mu(a, b)$ and an identity element $e := \eta(*)$. The left-hand diagram gives associativity in our "monoid" where μ is our binary operation: if $a, b, c \in M$, then we have

$$(a \cdot_\mu b) \cdot_\mu c = a \cdot_\mu (b \cdot_\mu c).$$

The right-hand diagram promises us an identity element $e := \eta(*)$: if $m \in M$, then

$$m \cdot_\mu e = m = e \cdot_\mu m.$$

Remark 2.82. It is not technically necessary for us to use sets M, but if we don't, then we need a good notion of product and one-element set. For example, Top can work instead of Set if we want to keep track of topologies.

Example 2.83. A unital ring R is a monoid in the category of Ab (where our products are tensor products and one-element set is $\mathbb{Z}$). Namely, we have morphisms $\mu : R \otimes R \to R$ and $\eta : \mathbb{Z} \to R$ with the following commutative diagrams.

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu \times \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \times \mu"'] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\mathrm{id}_R \times \eta"] \arrow[dr, "\mathrm{id}_R"'] & R \otimes R \arrow[d, dashed, "\mu"] & R \arrow[l, "\eta \times \mathrm{id}_R"'] \arrow[dl, "\mathrm{id}_R"] \\ & R & \end{tikzcd}$$

The left-hand diagram shows that multiplication is an associative bilinear map, and the right-hand diagram promises an identity. We will not be more explicit.

2.7.3 Commutative Diagrams

We should probably define a diagram now.

Definition 2.84 (Diagram). Fix $\mathcal{J}$ and $\mathcal{C}$ categories. A *diagram* in $\mathcal{C}$ indexed by $\mathcal{J}$ is a functor $F : \mathcal{J} \to \mathcal{C}$.

Notably, we are not requiring this functor to be an embedding.

Example 2.85. A diagram of the shape $(0 \to 1)^2$ is a commutative square. To be explicit, our index category is as follows.

$$\begin{tikzcd} (0,0) \arrow[r, "\mathrm{id}\times f"] \arrow[d, "f\times\mathrm{id}"'] \arrow[dr, "f\times f"] & (0,1) \arrow[d, "f\times\mathrm{id}"] \\ (1,0) \arrow[r, "\mathrm{id}\times f"'] & (1,1) \end{tikzcd}$$

Namely, if we send this to $\mathcal{C}$, we some diagram as follows.

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c' \arrow[d] \\ d \arrow[r] & d' \end{tikzcd}$$

Because we embedded by a functor, we know that $c \to c' \to d'$ is the same as $c \to d \to d'$.

Example 2.86. We can think about triangles as images of squares which collapse a bit, as follows.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, equal] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Alternatively, we could just set the index category to be $\bullet \to \bullet \to \bullet$.

Definition 2.87 (Commutes)**.** A diagram $F : \mathcal{J} \to \mathcal{C}$ *commutes* if and only if, given $k, k' : i \to j$ in $\mathcal{J}$ has $Fk = Fk'$.

The point of this definition is that we don't want composition to matter too much in our index category. For example, if we have morphisms $0 \to 1$ and $1 \to 2$ in $\mathcal{J}$ which go to $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, we want to be sure we have $0 \to 2$ goes to fg without having to look too hard at $\mathcal{J}$.

Example 2.88. Any diagram over a preorder will commute for free because any two i, j has at most one element in $\operatorname{Mor}(i, j)$.

It's a math class, so we should probably prove something today.

Proposition 2.89. Functors preserve commutative diagrams.

Proof. Fix $\mathcal{J}, \mathcal{C}, \mathcal{D}$ all diagrams with a commutative diagram $K : \mathcal{J} \to \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$. Indeed, if $k, k' : i \to j$ in $\mathcal{J}$, then $Kk = Kk'$, so $JKk = JKk'$, so $JK : \mathcal{J} \to \mathcal{D}$ is indeed a commutative diagram. ■

And here is a nice result on commutative diagrams.

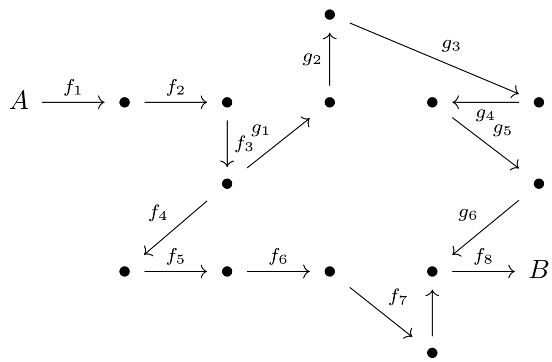
Lemma 2.90. Fix $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are paths in $\mathcal{C}$. Then if we have an equality of composites

$$f_k f_{k-1} \cdots f_{i+1} f_i = g_n g_{n-1} \cdots g_2 g_1,$$

then

$$f_m \cdots f_1 = f_m \cdots f_k g_n \cdots g_1 f_{i-1} \cdots f_1.$$

Here is the image for the above lemma: we are allowed to take either path from A to B, given that the f-parts and g-parts are commuting.



Proof. Look at it. Namely, we have composition is well-defined, so take the given equality and add the required compositions on either end. ■

2.8 February 16

Here we go.

2.8.1 House-Keeping

Let's start with the attendance question from last class because it was a little tricky.

Exercise 2.91. All nonempty indiscrete categories are equivalent.

Proof. The first part of this problem is remembering that indiscrete categories are ones that have all morphism sets are singletons. The second part of the problem is recognizing the following lemma.

Lemma 2.92. Fix $\mathcal{C}$ be a nonempty indiscrete category. Then $\mathcal{C}$ is equivalent to $\mathrm{B}e$, where e is the single-element group.

Proof. We use the functor $F : \mathcal{C} \to \mathrm{B}e$ sending all objects to $*$ and all morphisms to id_*. It is surjective on objects because there is only one object to hit, and $\mathcal{C}$ is nonempty. Further, F is fully faithful because, for any $c, c' \in \mathcal{C}$, the induced map

$$F : \mathrm{Mor}(c, c') \to \mathrm{Mor}(*, *)$$

is a bijection because both of these are singletons. It follows from Theorem 2.72 that F is an equivalence. ■

So transitivity promises that all indiscrete categories are equivalent, finishing the proof. ■

Remark 2.93. In fact, one can use essentially the same proof to show that any functor between indiscrete categories is an equivalence. In particular, the (weak) inverse to the equivalence generated by Lemma 2.92 is not canonical.

2.8.2 Diagram-Chasing Philosophy

We recall that we proved Lemma 2.90 last time, which philosophically means that we should not try to show equalities of morphisms where there is some overlap between the morphisms. For example, to compare all paths in the rectangle

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

above, we merely have to check the commutativity of the squares.

We would like to have some tools to prove that diagrams commute.

Remark 2.94. We remark that the following force commutative diagrams immediately.

- Any diagram indexed by a preorder commutes.
- Any diagram in a preorder commutes because any two morphisms between objects must be equal, so we get the commuting in the image of the index category.

2.8.3 Initial and Final Objects

Let's keep building up our theory.

Definition 2.95 (Initial, final)**.** Fix a category $\mathcal{C}$.

- An object $i \in \mathcal{C}$ is *initial* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(i, c)$.
- The dual notion is that an object $t \in \mathcal{C}$ is *final* or *terminal* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(c, t)$.

Remark 2.96. It is true that initial and final objects are unique up to unique isomorphism. We will not show this here because it might appear on the homework.

And here are many, many examples.

Example 2.97. We work in Set.

- We have $\varnothing$ is initial. Namely, there is only one function $\varnothing \to S$ for any set S by taking all elements of $\varnothing$ to whatever one's heart desires in S, and there is only one way to do this because any two such functions always have the same outputs.
- The singleton set $\{*\}$ is final. Indeed, any set S has a unique function $S \to \{*\}$ by sending all elements of S to $*$.

Example 2.98. In Top, the initial object is $\varnothing$ and the final object is $\{*\}$.

Example 2.99. We work in Set_*, which are ordered pairs (S, s) where $s \in S$. Morphisms $(S, s) \to (T, t)$ are functions $f : S \to T$ such that $f(s) = t$. Singleton sets $\{*\}$ is both initial and final. It's final for the same reason as in Set, and it is initial because any pointed set (S, s) has the unique morphism $* \mapsto s$.

Example 2.100. We work in Ab or Grp. Then the trivial group 0 is the initial and final object by sending identities to identities.

Non-Example 2.101. The object $\mathbb{Z}/2\mathbb{Z}$ is not initial in Ring: there is no morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. Funnily enough, there is at most one morphism from $\mathbb{Z}/2\mathbb{Z}$ to anywhere.

Non-Example 2.102. We work in Ring.

- The object $\mathbb{Z}$ is initial by sending $1 \mapsto 1_R$ (which is forced) for any ring R, and this uniquely determines the rest of the morphism.
- The zero ring 0 is final in Ring because there is only one function $R \to 0$ for any ring R, and it is in fact a ring homomorphism.

Example 2.103. The category Field has no initial or final object. There is no final object because all morphisms are injections, and we cannot embed all fields into one large field. There is no initial object because there are no morphisms between fields of different characteristic. (One can fix this problem by considering the fields of characteristic p, where $\mathbb{F}_p$ is the initial object.)

Quote 2.104. I hate this category, and you should too.

Example 2.105. Let $\mathcal{P}$ be a preorder category.

- We claim that global minimums are equivalent to initial objects. To be explicit, there is surely at most one morphism between any two elements, so the object $m \in \mathcal{P}$ is an initial object if and only if there is a morphism $m \to x$ for each $x \in \mathcal{P}$ if and only if $m \leq x$ for each x if and only if m is a global minimum.
- Dually, global maximums are equivalent to final objects.

These new definitions give us a quick criterion for diagram-chasing.

Lemma 2.106. Fix $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ be "parallel" paths in $\mathcal{C}$; i.e., $s := \operatorname{dom} f_1 = \dim g_1$ and $t := \operatorname{cod} f_n = \operatorname{cod} g_m$. If s is initial or t is final, then

$$f_n \cdots f_1 = g_m \cdots g_1.$$

Proof. We have two cases.

- Take s initial. Then $f_n \cdots f_1$ and $g_m \cdots g_1$ are both maps $s \to t$, of which there is a unique map by s being initial, so these are equal.
- Take t final. Then repeat the above sentence using the fact t is final instead of s being initial. ■

2.8.4 Concrete Categories

We have the following definition.

Definition 2.107 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if there is a fully faithful functor $U : \mathcal{C} \to \mathrm{Set}$. We call U the *forgetful functor.*

For example, this asserts that two morphisms $f, g : x \to y$ in $\mathcal{C}$ are equal if and only if their "restrictions" down in Set are equal, for which we can do an element-wise check on elements of sets.

Lemma 2.108. Fix $U : \mathcal{C} \to \mathcal{D}$ be faithful functors. A diagram in $\mathcal{C}$ commutes if and only if its image through U commutes.

Proof. Fix J our index category with the diagram $K : J \to \mathcal{C}$. We already know that K commuting implies that UK commutes by Proposition 2.89.

In the other direction, suppose that UK commutes. Then pick up $k, k' : i \to j$ in J so that $UKk = UKk'$, but then U being faithful forces

$$Kk = Kk',$$

which is exactly what we need to commute. ■

And here is why we care

Corollary 2.109. Commutativity of a diagram in a concrete category can be checked on "elements."

Proof. Essentially we use the forgetful functor in Lemma 2.108. To be explicit, checking on "elements" is doing the diagram-chase in Set, which we can then pull back to the original concrete category through the forgetful functor via Lemma 2.108. ■

In other words, we can diagram-chase by working everything in set.

2.8.5 Commutative Rectangles

We have the following warning.

Warning 2.110. Consider the following rectangle.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

We know that the squares commuting implies that the rectangle commutes. The converse is not true.

Example 2.111. We work in Ab. The outer rectangle of the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

will commute, but the inner squares do not. (The zero map is not the identity map.)

We can salvage Warning 2.110 as follows.

Lemma 2.112. Fix a rectangle as follows.

$$\begin{tikzcd}
a \arrow[r, "e"] \arrow[d, "f"'] & b \arrow[r, "k"] \arrow[d, "g"] & c \arrow[d, "h"] \\
a' \arrow[r, "j"'] & b' \arrow[r, "m"'] & c'
\end{tikzcd}$$

Suppose the outer rectangle commutes. Then the diagram commutes if

- the right square commutes and m is monic, or
- the left square commutes and e is epic.

Proof. We have separate cases.

- Suppose the right square commutes and m is monic. The right square commutes, so $hk = mg$. Similarly, the outer rectangle commutes, so $hke = mjf$. But then
$$mge = hke = mjf,$$
so $ge = jf$ because m is monic. This shows the left square commutes, so we are done.
- This holds by running the proof of the above in the opposite category, where the main point is that the left and right squares flip, and m being monic turns into e being epic. ■

2.9 February 18

Apparently I have to take notes today.

2.9.1 Motivating Horizontal Composition

A while ago we discussed vertical composition of natural transformations: if $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, then we can define a natural transformation $(\beta\alpha) : F \Rightarrow H$ by $(\beta\alpha)_c := \beta_c\alpha_c$. To quickly review, the naturality condition can be checked by drawing the following commutative diagram.

$$\begin{tikzcd}
Fc \arrow[r, "Ff"] \arrow[d, "\alpha_c"] \arrow[dd, bend right=60, "\beta_c\alpha_c"'] & Fd \arrow[d, "\alpha_d"'] \arrow[dd, bend left=60, "\beta_d\alpha_d"] \\
Gc \arrow[r, "Gf"] \arrow[d, "\beta_c"] & Gd \arrow[d, "\beta_d"'] \\
Hc \arrow[r, "Hf"'] & Hd
\end{tikzcd}$$

We are going to discuss horizontal composition because Eckmann–Hamilton would like to know your location. The set-up is as follows: suppose that we have functors $F, G : \mathcal{C} \to \mathcal{D}$ with $\alpha : F \Rightarrow G$ and $F', G' : \mathcal{D} \to \mathcal{E}$ with $\beta : F' \Rightarrow G'$. Here is the diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

Our goal is to define $(\beta * \alpha) : F'F \Rightarrow G'G$.

2.9.2 Whiskering

To define this horizontal composition, we define "whiskering." There are two kinds of whiskering.

- Here is the diagram for left whiskering.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathcal{E} \end{tikzcd}$$

  We would like to define $\alpha H : FH \Rightarrow GH$. Well, we simply define $(\alpha H)_c := \alpha_{Hc}$, which defines a natural transformation by noting the following diagram commutes for a morphism $f : c \to d$ in $\mathcal{C}$ by the naturality of α on $Hf : Hc \to Hd$. This gives the following commutative naturality square.

$$\begin{tikzcd} FHc \arrow[r, "FHf"] \arrow[d, "\alpha_{Hc}"'] & FHd \arrow[d, "\alpha_{Hd}"] \\ GHc \arrow[r, "GHf"'] & GHd \end{tikzcd}$$

- There is also a notion of right whiskering. Here is the diagram.

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{E} \arrow[r, "H'"] & \mathcal{X} \end{tikzcd}$$

  We define $H'\alpha : H'F \Rightarrow H'G$ by $(H'\alpha)_d := H'\alpha_d$. This is a natural transformation because we can pick up some morphism $f : c \to d$ in $\mathcal{D}$ and apply H' to the naturality diagram for α, giving the following commutative naturality square.

$$\begin{tikzcd} H'Fc \arrow[r, "H'Ff"] \arrow[d, "H'\alpha_c"'] & H'Fd \arrow[d, "H'\alpha_d"] \\ H'Gc \arrow[r, "H'Gf"'] & H'Gd \end{tikzcd}$$

2.9.3 Horizontal Composition

From whiskering, there are two ways to define horizontal composition. To review, here is our diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

- We start by whiskering on the left and then whisker on the right. So we start by noting we have $\beta F : F'F \Rightarrow G'F$ induced by whiskering the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

  Then we have $G'\alpha : G'F \Rightarrow G'G$ by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend right, "G'"'] & \mathcal{E} \end{tikzcd}$$

  In total, we see that $(G'\alpha)(\beta F) : F'F \Rightarrow G'G$. Note this is a natural transformation by vertical composition!

- We start by whiskering on the right and then whisker on the left. So we start by noting we have $F'\alpha : F'F \Rightarrow F'G$ by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] & \mathcal{E} \end{tikzcd}$$

  Then we have $\beta G : F'G \Rightarrow G'G$ induced by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "G"'] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

  In total, we see that $(\beta G)(F'\alpha) : F'F \Rightarrow G'G$, which is a natural transformation by vertical composition.

We now claim that the two horizontal compositions that we just defined are the same. We could just track an element through, or we could simply note that this is the naturality of β applied to the morphism $\alpha_c : Fc \to Gc$. Indeed, we are showing that the following diagram commutes.

$$\begin{tikzcd} F'F \arrow[r, Rightarrow, "F'\alpha"] \arrow[d, Rightarrow, "\beta F"'] & F'G \arrow[d, Rightarrow, "\beta G"] \\ G'F \arrow[r, Rightarrow, "G'\alpha"'] & G'G \end{tikzcd}$$

Now, applying naturality of β to $\alpha_c : Fc \to Gc$, we see that the following diagram commutes.

$$\begin{tikzcd} F'Fc \arrow[r, "F'\alpha_c"] \arrow[d, "\beta_{Fc}"'] & F'Gc \arrow[d, "\beta_{Gc}"] \\ G'Fc \arrow[r, "G'\alpha_c"'] & G'Gc \end{tikzcd}$$

But this diagram is exactly what we wanted, so we are done.

2.9.4 Horizontal and Vertical Composition

For our last note, we show that horizontal composition of vertical compositions is the same as vertical composition of horizontal compositions. Here is our diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \arrow[r, bend left=60, "F'"] \arrow[r, "G'" description] \arrow[r, bend right=60, "H'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{E} \end{tikzcd}$$

We claim that

$$(\beta'\alpha') * (\beta\alpha) \stackrel{?}{=} (\beta' * \beta)(\alpha'\alpha).$$

The point is to draw the following giant commuting square. The "morphisms" are induced by various kinds of whiskering in the diagram, and they all commute by uniqueness of horizontal composition.

$$\begin{tikzcd} F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\ G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\ H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H \end{tikzcd}$$

We now follow two paths. Consider the red path below.

$$\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\
G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\
H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H
\end{tikzcd}$$

By definition of horizontal composition, this is $(\beta' * \beta)(\alpha' * \alpha)$. Now consider the different red path below.

$$\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\
G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\
H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H
\end{tikzcd}$$

The top leg is $\beta\alpha$, and the right leg is $\beta'\alpha'$, so this total red path comes out to $(\beta'\alpha')(\beta\alpha)$.

So comparing our two red paths, we see that

$$(\beta'\alpha') * (\beta\alpha) = (\beta' * \beta)(\alpha' * \alpha),$$

which is what we wanted.

THEME 3

UNIVERSAL PROPERTIES

The Yoneda embedding, contravariant it is.

—Mike Stay, [Vak17]

3.1 February 23

Today we begin talking about universal properties and associated fun.

Convention 3.1. For today, all of our categories will be locally small. We will not care more about size issues.

3.1.1 A Functorial Initial and Final

We recall the following definition.

Definition 2.95 (Initial, final)**.** Fix a category $\mathcal{C}$.

- An object $i \in \mathcal{C}$ is *initial* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(i, c)$.
- The dual notion is that an object $t \in \mathcal{C}$ is *final* or *terminal* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(c, t)$.

The moral of our story is that being initial and terminal will encode our universal properties.

Here is a nice starting proposition and corollary.

Proposition 3.2. An object $c \in \mathcal{C}$ is initial if and only if $\# \operatorname{Mor}(c, x) = 1$ for each $x \in \mathcal{C}$. Similarly, c is terminal if and only if $\# \operatorname{Mor}(x, c) = 1$ for each $x \in \mathcal{C}$.

Proof. This is a restatement of the definition. For example, $\# \operatorname{Mor}(c, x) = 1$ is asserting there is a unique morphism from c to x for any object x. ■

Corollary 3.3. An object $c \in \mathcal{C}$ is initial if and only if the functor $\mathrm{Mor}(c, -) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ "represented" by c is naturally isomorphic to the (contravariant!) constant functor $\{*\} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ sending everyone in $\mathcal{C}$ to $\{*\}$.

To be explicit, the functor $\{*\} : \mathcal{C} \to \mathrm{Set}$ sends objects $c \in \mathcal{C}$ to $c \mapsto \{*\}$ and sends morphisms $f : c \to d$ to $f \mapsto \mathrm{id}_{\{*\}}$.

Proof. As before, we see that c is initial if and only if $\# \mathrm{Mor}(c, x) = 1$ for each x if and only if

$$\mathrm{Mor}(c, x) \cong \{*\}$$

because all singletons form an isomorphism class in Set. We label φ_x to be $\mathrm{Mor}(c, x) \cong \{*\} \cong \{*\}(x)$ to be the unique such isomorphism.

If φ_x assemble to a natural isomorphism, then we get the reverse direction. For the forwards direction, we have to check that the following diagram commutes for naturality: suppose $f : x \to y$ is a morphism in $\mathcal{C}$, and we want

$$\begin{tikzcd}
\mathrm{Mor}(c, x) \arrow[r, "\varphi_x"] \arrow[d, "f\circ-"'] & \{*\}(x) \arrow[d, "\mathrm{id}_{\{*\}}"] \\
\mathrm{Mor}(c, y) \arrow[r, "\varphi_y"] & \{*\}(y)
\end{tikzcd}$$

to commute. But this commutes for free because $\{*\}(y) = \{*\}$ is a terminal object, so all morphisms to it are the same. ∎

Corollary 3.4. An object $c \in \mathcal{C}$ is terminal if and only if the functor $\mathrm{Mor}(-, c) : \mathcal{C} \to \mathrm{Set}$ "represented" by c is naturally isomorphic to the constant functor $\{*\} : \mathcal{C} \to \mathrm{Set}$ sending everyone in $\mathcal{C}$ to $\{*\}$.

Proof. This is dual to the previous corollary. ∎

3.1.2 Representability

Here is our central definition.

Definition 3.5 (Representable)**.** Fix a category $\mathcal{C}$.

- A covariant functor $F : \mathcal{C} \to \mathrm{Set}$ is *representable* if and only if there exists some $c \in \mathcal{C}$ such that $F \simeq \mathrm{Mor}(c, -)$.
- A contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is *representable* if and only if there exists some $c \in \mathcal{C}$ such that $F \simeq \mathrm{Mor}(-, c)$.

In either case, we call c together with the promised natural isomorphism the *representation* of F.

Example 3.6. Corollary 3.3 says that c is initial if and only if $\{*\} : \mathcal{C} \to \mathrm{Set}$ is represented by c. Similar holds for the terminal case.

Here is our mantra.

Idea 3.7. A representable functor encodes a universal property of an object.

Remark 3.8. Bryce would like you to repeat Idea 3.7 every day before you go to sleep. He will know if you haven't.

Less formally, Idea 3.7 is saying that a universal property for an object c is a description of $\operatorname{Mor}(c, -)$ or $\operatorname{Mor}(-, c)$.

Let's see some examples.

Exercise 3.9. The identity functor $\mathrm{id}_{\mathrm{Set}} : \mathrm{Set} \to \mathrm{Set}$, is represented by singleton set $\{*\}$.

Proof. To be explicit we would like to show that

$$\operatorname{Mor}(\{*\}, X) \cong X$$

naturally by taking $f \mapsto f(*)$. So we define $\eta_X : \operatorname{Mor}(\{*\}, X) \to X$ by $f \mapsto f(*)$. This is an isomorphism because we have the inverse morphism $\eta_X^{-1} : x \mapsto (* \mapsto x)$. This is natural because, with a morphism $h : X \to Y$, we draw the following diagram.

$$\begin{tikzcd}
\operatorname{Mor}(\{*\}, X) \arrow[r, "\eta_X"] \arrow[d, "h\circ -"'] & X \arrow[d, "h"] \\
\operatorname{Mor}(\{*\}, Y) \arrow[r, "\eta_Y"'] & Y
\end{tikzcd}$$

This is natural by tracking some $f : \{*\} \to X$ through: along the top, it goes to $h(f(*))$, and along the bottom it goes to $(hf)(*) = h(f(*))$. ■

Exercise 3.10. The forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ is represented by $\mathbb{Z}$.

Proof. The content is to construct an isomorphism

$$\operatorname{Mor}(\mathbb{Z}, G) \cong G$$

for any group G. Well, to see this, we send $f \mapsto f(1)$ and more or less wave our hands to say that a group homomorphism $\mathbb{Z} \to G$ is uniquely determined by where it sends 1 because $f(n) = n \cdot f(1)$, and any such $f(1)$ is legal because we can set $f(n) = n \cdot f(1)$.

So let $\eta_G : \operatorname{Mor}(\mathbb{Z}, G) \to G$ be this isomorphism. For naturality, we need to show that the following diagram commutes for a given group homomorphism $\psi : G \to H$.

$$\begin{tikzcd}
\operatorname{Mor}(\mathbb{Z}, G) \arrow[r, "\eta_X"] \arrow[d, "\psi\circ -"'] & G \arrow[d, "U\psi"] \\
\operatorname{Mor}(\mathbb{Z}, H) \arrow[r, "\eta_Y"'] & H
\end{tikzcd}$$

Well, along the top, we send f to $f(1)$ to $\psi(f(1))$. Along the bottom, we send f to ψf to $(\psi f)(1)$ to $\psi(f(1))$. ■

Exercise 3.11. The forgetful functor $U : \mathrm{Ring} \to \mathrm{Set}$ is represented by $\mathbb{Z}[x]$.

Proof. The point is that we have an isomorphism

$$\operatorname{Mor}(\mathbb{Z}[x], R) \cong R$$

because the image of $\mathbb{Z}$ is fixed for any morphism $\varphi : \mathbb{Z}[x] \to R$, and where we send x is uniquely determined by a chosen element $r \in R$. ■

Remark 3.12. In some sense, $\mathbb{Z}$ and $\mathbb{Z}[x]$ are the "free" object in their respective categories.

And now for contravariant representable functors.

Exercise 3.13. The functor $\mathcal{P} : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ by sending $X \mapsto \mathcal{P}(X)$ and $f : X \to Y$ to $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is represented by $\Omega = \{0, 1\}$.

Proof. As usual, the content of the proof is our isomorphism

$$\mathrm{Mor}(X, \Omega) \cong \mathcal{P}(X).$$

Namely, we send $f : X \to \Omega$ to $f^{-1}(1)$. Conversely, given a subset $U \subseteq X$, we can track it by the morphism $1_{x\in U} : X \to \Omega$. ■

Exercise 3.14. Fix sets A and B. Consider the functor $\mathrm{Mor}(- \times A, B) : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$. We claim that this is represented by $\mathrm{Mor}(A, B)$.

Proof. Our isomorphism

$$\mathrm{Mor}(\mathrm{Mor}(A, B), C) \cong \mathrm{Mor}(C \times A, B)$$

is given by currying $f \mapsto ((a, b) \mapsto f(a)(b))$. The inverse mapping is $f \mapsto (a \mapsto b \mapsto f(a, b))$. ■

Remark 3.15. Many of the above representatives are "nice" in that it seems like they are unique in some sense. This will tie into universal properties.

3.2 February 25

We talk about the Yoneda lemma today.

3.2.1 The Yoneda Lemma

Today we discuss the following question.

Question 3.16. What information "goes into" a natural transformation of a representable functor?

Today we will prove the following theorem.

Theorem 3.17 (Yoneda lemma). Fix $\mathcal{C}$ a locally small category and $F : \mathcal{C} \to \mathrm{Set}$ a functor. Further, fix $c \in \mathcal{C}$. Then there is a "natural" bijection (natural in both c and F)

$$\varphi : \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \cong Fc.$$

Here the outer Mor is in the 2-category, talking about natural transformations $\mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$.

Natural here is in both c and F: if we fix one of them, then the isomorphism is functorial in the other.

Proof. We take this in parts.

- We construct φ. Suppose that $\alpha : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ is a natural transformation. Then we can produce an element of Fc by noting we have a map $\alpha_c : \mathrm{Mor}_{\mathcal{C}}(c, c) \to Fc$, so we can set

$$\boxed{\varphi(\alpha) := \alpha_c(\mathrm{id}_c)} \in Fc.$$

- We construct an inverse $\psi : Fc \to \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c,-),F)$. Well, picking up some $x \in Fc$, then we want a natural transformation $\psi(x) : \operatorname{Mor}_{\mathcal{C}}(c,-) \Rightarrow F$. So given another $d \in \mathcal{C}$, we want a morphism

$$\psi(x)_d : \operatorname{Mor}_{\mathcal{C}}(c,d) \to Fd.$$

  To do this, we pick up a morphism $f : c \to d$, and we want an element of Fd. Without any better ideas, we note we have a morphism $Ff : Fc \to Fd$, so we define

$$\boxed{\psi(x)_d(f) := (Ff)(x)}.$$

  We now check that $\psi(x)$ is in fact a natural transformation. Well, suppose that we have a map $g : d \to e$, we need the following square to commute.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}}(c,d) \arrow[r, "\psi(x)_d"] \arrow[d, "g\circ-"'] & Fd \arrow[d, "Fg"] \\ \operatorname{Mor}_{\mathcal{C}}(c,e) \arrow[r, "\psi(x)_e"'] & Fe \end{tikzcd}$$

  For this, we pick up some morphism $f : c \to d$.

  – Along the top, we go to $(Ff)(x)$ and then $(Fg \circ Ff)(x)$.
  – Along the bottom, we go to gf and then $(F(gf))(x) = (Fg \circ Ff)(x)$.

- We show that $\varphi \circ \psi$ is the identity. Well, pick up some $x \in Fc$. Then

$$\varphi(\psi(x)) = \psi(x)_c(\mathrm{id}_c) = (F\,\mathrm{id}_c)(x) = \mathrm{id}_{Fc}(x) = x.$$

- We show that $\psi \circ \varphi$ is the identity. Well, pick up some natural transformation $\alpha : \operatorname{Mor}_{\mathcal{C}}(c,-) \Rightarrow F$ and some object $d \in \mathcal{C}$ and some morphism $f : c \to d$, and we compute

$$\psi(\varphi(\alpha))_d(f) = \psi(\alpha_c(\mathrm{id}_c))_d(f) = (Ff)(\alpha_c(\mathrm{id}_c)) = (Ff \circ \alpha_c)(\mathrm{id}_c).$$

  At this point we look stuck, but naturality of $\alpha : \operatorname{Mor}_{\mathcal{C}}(c,-) \Rightarrow F$ saves us! We draw the following diagram.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}}(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f\circ-"'] & Fc \arrow[d, "Ff"] \\ \operatorname{Mor}_{\mathcal{C}}(c,d) \arrow[r, "\alpha_d"'] & Fd \end{tikzcd}$$

  Thus, we know $Ff \circ \alpha_c = \alpha_d \circ (- \circ f)$, so the above is

$$\psi(\varphi(\alpha))_d(f) = (\alpha_d \circ (f \circ -))(\mathrm{id}_c) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f).$$

  So $\psi(\varphi(\alpha))_d$ and α_d match as functions on $\operatorname{Mor}_{\mathcal{C}}(c,d)$, so they are equal. Thus, $\psi(\varphi(\alpha)) = \alpha$ as natural transformations. So we are done.

The above points establish the needed bijection.

It remains to check functoriality.

- We show that φ is functorial in c. We write φ^c for φ given by $c \in \mathcal{C}$. Suppose that we have a morphism $f : c \to c'$, and we want to show that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c,-),F) \arrow[r, "\varphi^c"] \arrow[d] & Fc \arrow[d] \\ \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c',-),F) \arrow[r, "\varphi^{c'}"'] & Fc' \end{tikzcd} \tag{$*$}$$

The right arrow is Ff. The left arrow requires some thinking: we pick up some natural transformation $\alpha : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ and want to produce a natural transformation $\mathrm{Mor}_{\mathcal{C}}(c', -) \Rightarrow F$. Visually, the map we want is moving

$$\big((\mathcal{C} \to \mathrm{Set}) \to \mathcal{D}\big) \to \big((\mathcal{C} \to \mathrm{Set}) \to \mathcal{D}\big).$$

Well, given an object $d \in \mathcal{C}$ and morphism $p : c \to d$, we can send $p' : c' \to d$ to

$$\beta_d(p') := \alpha_d(p'f),$$

which we can type-check actually lives in Fd.

We take a moment to verify that β is a natural transformation. For this, we need to check the naturality of the following square, for a morphism $g : d \to e$, that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(c', d) \arrow[r, "\beta_d"] \arrow[d, "g\circ-"'] & Fd \arrow[d, "Fg"] \\ \mathrm{Mor}_{\mathcal{C}}(c', e) \arrow[r, "\beta_e"'] & Fe \end{tikzcd}$$

Well, we pick up a morphism $p' : c' \to d$.

- Along the top, we go to $(Fg)(\beta_d(p')) = (Fg)(\alpha_d(p'f)) = (Fg \circ \alpha_d)(p'f)$. By naturality of α, we see $Fg \circ \alpha_d = \alpha_e \circ (g \circ -)$, so we have $\alpha_e(gp'f)$.
- Along the bottom, we go to $\beta_e(gp') = \alpha_e(gp'f)$.

So indeed, β is a natural transformation.

Finally, we check the naturality of $(*)$.

- Along the top, we go to $\varphi^c(\alpha) = \alpha_c(\mathrm{id}_c)$ and then to $(Ff)(\alpha_c(\mathrm{id}_c)) = (Ff \circ \alpha_c)(\mathrm{id}_c)$. By naturality of α, we see $Ff \circ \alpha_c = \alpha_{c'}(f \circ -)$, so we have $(Ff \circ \alpha_c)(\mathrm{id}_c) = \alpha_{c'}(f\,\mathrm{id}_c) = \alpha_{c'}(f)$.
- Along the bottom, we go to

$$\varphi^{c'}(\beta) = \beta_{c'}(\mathrm{id}_{c'}) = \alpha_{c'}(\mathrm{id}_{c'}\, f) = \alpha_{c'}(f).$$

These match, so the diagram commutes.

- We show φ is functorial in F. We write φ^F for φ given by $F : \mathcal{C} \to \mathrm{Set}$. Now, suppose that we have some natural transformation $\eta : F \Rightarrow G$, and we want to show that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \arrow[r, "\varphi^F"] \arrow[d] & Fc \arrow[d] \\ \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c, -), G) \arrow[r, "\varphi^G"'] & Gc \end{tikzcd}$$

The right arrow is η_c. The left arrow requires some thinking, as before. Fix some natural transformation $\alpha : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$, and we produce a natural transformation $\beta : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow G$. Well, given an object d and morphism $p : c \to d$, we are given an element $\alpha_d(f) \in Fd$, and we want an element in Gd. So we define

$$\beta_d(p) := \eta_d(\alpha_d(p)).$$

We quickly check that this $\beta : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow G$ actually assembles into a natural transformation. Given $f : d \to e$, we need to check the commutativity of the following diagram.

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(c, d) \arrow[r, "\beta_d"] \arrow[d, "f\circ-"'] & Gd \arrow[d, "Gf"] \\ \mathrm{Mor}_{\mathcal{C}}(c, e) \arrow[r, "\beta_e"'] & Ge \end{tikzcd} \qquad (**)$$

Well, pick up a morphism $p : c \to d$.

- Along the top, we go to $(Gf)(\beta_d(p)) = (Gf)(\eta_d(\alpha_d(p)))$. By naturality of η, this is $(\eta_e \circ Fe \circ \alpha_d)(p)$. By naturality of α, this is $(\eta_e \circ \alpha_e \circ (f \circ -))(p) = \eta_e(\alpha_e(fp))$.
- Along the bottom, we go to $\beta_e(fp) = \eta_e(\alpha_e(fp))$.

So indeed, β is a natural transformation.

Finally, we check the naturality of $(**)$.

- Along the top, we go to $\varphi^F(\alpha) = \alpha_c(\mathrm{id}_c)$ and then to $\eta_c(\alpha_c(\mathrm{id}_c))$.
- Along the bottom, we go to $\varphi^G(\beta) = \beta_c(\mathrm{id}_c) = \eta_c(\alpha_c(\mathrm{id}_c))$.

These match, so the diagram commute.

Thus, we have checked that φ is functorial in both c and F. I have a headache, so we will call it quits there. ■

3.3 February 28

Chris is back!

3.3.1 Yoneda Lemma Review

Today is more Yoneda lemma. We recall the statement.

Theorem 3.17 (Yoneda lemma)**.** Fix $\mathcal{C}$ a locally small category and $F : \mathcal{C} \to \mathrm{Set}$ a functor. Further, fix $c \in \mathcal{C}$. Then there is a "natural" bijection (natural in both c and F)

$$\varphi : \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \cong Fc.$$

Here the outer Mor is in the 2-category, talking about natural transformations $\mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$.

As seen in the proof, the bijection is by

$$\varphi(\eta) := \eta_c(\mathrm{id}_c) \qquad \text{and} \qquad \varphi^{-1}(x)_d := \big((f \in \mathrm{Mor}(c, d)) \mapsto (Ff)(x)\big).$$

We quickly remark that we can motivate the definition φ^{-1} by drawing the following naturality square with given internal diagram; here $f : c \to d$ is some morphism.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(c,c) \arrow[r, "f \circ -"] \arrow[d, "\varphi^{-1}(x)_c"'] & \mathrm{Mor}_{\mathcal{C}}(c,d) \arrow[d, "\varphi^{-1}(x)_d"] \\
Fc \arrow[r, "Ff"'] & Fd
\end{tikzcd}$$

Because we want $\varphi^{-1}(x)_c(\mathrm{id}_c) = x$, our definition of $\varphi^{-1}(x)_d(f)$ is forced.

As for naturality, we note that we can view Fc as the image of the functor $F : \mathcal{C} \to \mathrm{Set}$ on applying $c \in \mathcal{C}$, or alternatively we could view Fc as the image of the functor $\mathrm{ev}_c : \mathrm{Set}^{\mathcal{C}} \to \mathrm{Set}$ on applying F.

Remark 3.18 (Contravariant Yoneda)**.** There is also a contravariant version of the Yoneda lemma as well, which provides takes a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a functor and some object $c \in \mathcal{C}$. Then there is a "natural" bijection (natural in both c and F)

$$\varphi : \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(-, c), F) \cong Fc.$$

Again, this bijection is natural in F and c.

3.3.2 Yoneda Embeddings

We are going to describe an embedding

$$\text{よ} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

This is defined in essentially exactly the way that we want. We define $\text{よ}(c) := \mathrm{Mor}(-, c)$ and send morphisms $f : c \to d$ to the natural transformation $\text{よ}(f) : \mathrm{Mor}(-, c) \to \mathrm{Mor}(-, d)$ by $\text{よ}(f) : g \mapsto fg$. It is not too hard to see that $\text{よ}(f)$ is in fact a natural transformation and composes properly, so we will omit those checks now.

Anyways, here is our theorem.

Theorem 3.19 (Yoneda embedding)**.** Fix $\mathcal{C}$ a category. The Yoneda embedding $\text{よ} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a fully faithful embedding.

Proof. We have the following checks.

- We show that よ is faithful, so suppose we have objects $c, d \in \mathcal{C}$, and we want to show that the map

$$\text{よ} : \mathrm{Mor}_{\mathcal{C}}(c, d) \to \mathrm{Mor}(\text{よ}(c), \text{よ}(d))$$

  is injective. Namely, if $f, g : c \to d$, then $\text{よ}(f) : x \mapsto fx$ and $\text{よ}(g) : x \mapsto gx$, so $\text{よ}(f) = \text{よ}(g)$ forces

$$f = f\,\mathrm{id}_c = \text{よ}(f)(\mathrm{id}_c) = \text{よ}(g)(\mathrm{id}_c) = g\,\mathrm{id}_c = g.$$

- We show that よ is full. We will actually use Theorem 3.17. Well, suppose that we have a morphism

$$\eta : \mathrm{Mor}_{\mathcal{C}}(-, c) \Rightarrow \mathrm{Mor}_{\mathcal{C}}(-, d),$$

  and we want a morphism $f : c \to d$ such that $\text{よ}(f) = \eta$. Well, viewing $\mathrm{Mor}(-, d)$ as just some functor $F : \mathcal{C} \to \mathrm{Set}$, we are promised a bijection (by the contravariant version of Theorem 3.17)

$$\mathrm{Mor}\left(\mathrm{Mor}_{\mathcal{C}}(-, c), \mathrm{Mor}_{\mathcal{C}}(-, d)\right) \to \mathrm{Mor}(c, d).$$

  In particular, η under this bijection goes to some map $f = \eta_c(\mathrm{id}_c) \in \mathrm{Mor}(c, d)$. But in fact $\text{よ}(f) = (f \circ -)$ also has $(f \circ -)(\mathrm{id}_c) = f$, so because the above is a bijection, we have $\eta = \text{よ}(f)$.

- We show that よ is an embedding. For this, we suppose $c \neq d$ are distinct objects, and we need to show that $\mathrm{Mor}_{\mathcal{C}}(-, c) \neq \mathrm{Mor}_{\mathcal{C}}(-, d)$ as natural transformations. However, morphisms should "remember" their codomain in the data of a morphism, so $\mathrm{Mor}_{\mathcal{C}}(x, c)$ and $\mathrm{Mor}_{\mathcal{C}}(x, d)$ will be distinct automatically. ■

It might feel like cheating that we are forcing our morphisms to remember their codomain, but it is somewhat necessary: in the indiscrete category, we might accidentally try to force our morphisms to be witnessed by the same object, but then the above is not actually an embedding because all morphism sets would be equal.

Just for fun, here is an application.

Corollary 3.20 (Cayley's theorem)**.** Any group G is isomorphic to a subgroup of $\mathrm{Sym}(G) = \mathrm{Aut}_{\mathrm{Set}}(G)$.

Proof. To convert this to category theory, we use the category $\mathrm{B}G$. Well, Theorem 3.19 provides an embedding

$$\text{よ} : \mathrm{B}G \to \mathrm{Set}^{\mathrm{B}G^{\mathrm{op}}}.$$

Here, we can think of $\mathrm{Set}^{\mathrm{B}G^{\mathrm{op}}}$ as sets equipped with a right G-action: any such functor $\mathrm{B}G^{\mathrm{op}} \to \mathrm{Set}$ sends the object $* \in \mathrm{B}G$ to a set $S \in Set$ as well as morphisms/elements $g \in G$ to functions $g : S \to S$ satisfying

$$s \cdot (gh) = (s \cdot g) \cdot h$$

for $s \in S$ and $g, h \in G$.

Now, to use よ, we see that $* \in \mathrm{B}G$ goes to the functor よ$(*) = \mathrm{Mor}(-, *) : \mathrm{B}G^{\mathrm{op}} \to \mathrm{Set}$. Fixing any set $\bullet \in \mathrm{B}G$, we see that we really have the data of a morphism $\mathrm{B}G^{\mathrm{op}} \to \mathrm{Mor}(\bullet, *)$. In other words, we are giving $\mathrm{Mor}(\bullet, *)$ a right G-action: each $g \in \mathrm{B}G$ and $x \in \mathrm{Mor}(\bullet, *)$ will have a multiplication given by

$$x \cdot g \coloneqq g^{\mathrm{op}} x,$$

where we have composition by

$$(x \cdot g) \cdot h = h^{\mathrm{op}} g^{\mathrm{op}} x = (gh)^{\mathrm{op}} x = x \cdot (gh).$$

So indeed, よ$(*)$ is precisely the data of this right G-action on $\mathrm{Mor}(\bullet, *)$. In other words, よ$(*)$ is providing the data of the object $\mathrm{Mor}(\bullet, *)$ in the category of G^{op}-sets.

On the other hand, each morphism $g : * \to *$ of $\mathrm{B}G$ will go to よ$(g) : \mathrm{Mor}(\bullet, *) \Rightarrow \mathrm{Mor}(\bullet, *)$ by よ$(g) : x \mapsto gx$. To be explicit, our multiplication is by

$$よ(g)(x) \coloneqq gx.$$

In fact, each element よ(g) is a G^{op}-equivariant map on $\mathrm{Mor}(\bullet, *)$, where this object is thought of as a G^{op}-set. Indeed,

$$よ(g)(x \cdot h) = g(xh) = (gx)h = (よ(g)x) \cdot h,$$

which is what we wanted. In particular, よ injects $\mathrm{Mor}_{\mathrm{B}G}(*, *)$ to "G-equivariant" natural transformations $\mathrm{Mor}_{\mathrm{B}G}(-, *) \Rightarrow \mathrm{Mor}_{\mathrm{B}G}(-, *)$.

So to finish up, because よ is a fully faithful functor, we see that we are injective on morphism sets, so we can say

$$G = \mathrm{Mor}_{\mathrm{B}G}(*, *) \overset{よ}{\hookrightarrow} \mathrm{Mor}_{G^{\mathrm{op}}\text{-set}}\left(\mathrm{Mor}_{\mathrm{B}G}(-, *), \mathrm{Mor}_{\mathrm{B}G}(-, *)\right) \overset{*}{=} \mathrm{Aut}_{G^{\mathrm{op}}\text{-set}}\left(\mathrm{Mor}_{\mathrm{B}G}(-, *), \mathrm{Mor}_{\mathrm{B}G}(-, *)\right),$$

where $\overset{*}{=}$ holds because all elements of G are invertible and hence all the morphisms we are looking at are invertible. Now, we remark that the data of $\mathrm{Mor}(-, *)$ is really only the data of $\mathrm{Mor}(*, *)$ because $\mathrm{B}G$ has only one object, so we actually get to embed

$$G \hookrightarrow \mathrm{Aut}_{G^{\mathrm{op}}\text{-set}}\left(\mathrm{Mor}_{\mathrm{B}G}(*, *), \mathrm{Mor}_{\mathrm{B}G}(*, *)\right) = \mathrm{Aut}_{G^{\mathrm{op}}\text{-set}}(G, G).$$

Lastly, applying the forgetful functor from G-sets to just sets, we have an embedding $G \hookrightarrow \mathrm{Aut}_{\mathrm{Set}}(G)$, so we are done. ■

3.4 March 2

We continue. Chris did some review that the Yoneda embedding よ is full, which can be found in our proof of Theorem 3.19.

3.4.1 Unique Representation

For the attendance question, we have the following.

Proposition 3.21. Fix a category $\mathcal{C}$. Then $x \cong y$ implies that $\mathrm{Mor}(x, -) \simeq \mathrm{Mor}(y, 0)$.

Proof. The point is that the Yoneda embedding よ must induce an isomorphism よ(f) : よ$(x) \cong$ よ(y), which is what we wanted. ■

In fact, the converse is also true. We have the following definition.

Definition 3.22 (Creates, reflects isomorphisms)**.** A functor F *reflects isomorphisms* if and only if Ff being an isomorphism implies that f is an isomorphism. Similarly, a functor F *creates isomorphisms* if and only if $Fx \cong Fy$ forces $x \cong y$.

Example 3.23. The functor from $F : \mathrm{Be} + \mathrm{Be} \to \mathrm{Be}$ sending both points to $*$ will certainly reflect isomorphisms because we can only pull back. This F however does not reflect isomorphisms because $Fe_1 \cong Fe_2$ but e_1 is not isomorphic to e_2; there are no morphisms between them all.

We have the following result.

Proposition 3.24. If a functor F is fully faithful, then F creates and reflects isomorphisms.

Proof. This was on the homework. ■

Remark 3.25. Thus, noting that よ is fully faithful, we see that $\mathrm{Mor}(x, -) \simeq \mathrm{Mor}(y, 0)$ forces $x \cong y$.

The point of all of our discussion is as follows.

Proposition 3.26. Suppose two objects x and y represent a functor $F : \mathcal{C} \to \mathrm{Set}$ by natural isomorphisms $\eta : \mathrm{Mor}(x, -) \Rightarrow F$ and $\mu : \mathrm{Mor}(y, -) \Rightarrow F$. Then there is a canonical isomorphism $x \cong y$ by $\eta^{-1}\mu$.

Proof. Intuitively, $f : x \cong y$ induces $- \circ f : \mathrm{Mor}(y, -) \to \mathrm{Mor}(x, -)$, from we would like the following diagram to commute.

$$\begin{tikzcd}
\mathrm{Mor}(x,-) \arrow[dr, Rightarrow, "\eta"] & \\
& F \\
\mathrm{Mor}(y,-) \arrow[uu, Rightarrow, "-\circ f"] \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

As such, we see that we want f to induce a natural isomorphism $(- \circ f) = \eta^{-1}\mu : \mathrm{Mor}(y, -) \Rightarrow \mathrm{Mor}(x, -)$, which we can in fact pull back to f because of Theorem 3.17. ■

Idea 3.27. Thus, we can say that an object represents a functor is unique up to unique (commuting) isomorphism.

3.4.2 Universal Properties

With our uniqueness in hand, we are ready to talk about universal properties.

Definition 3.28 (Universal property I, element)**.** A *universal property* for an object $c \in \mathcal{C}$ is a representable functor $F : \mathcal{C} \Rightarrow \mathrm{Set}$ along with an element $x \in Fc$ such that x induces (by the Yoneda lemma) a natural isomorphism $\mathrm{Mor}(c, -) \Rightarrow F$. In such a triplet (c, F, x), we call x the *universal element*.

To be explicit, $x \in Fc$ is inducing a natural transformation $\mathrm{Mor}(c, -) \Rightarrow F$ by Theorem 3.17, so the condition we are requiring is that we have a natural isomorphism.

Exercise 3.29. We discuss $\mathbb{Z}[x]$ as the free ring on X.

Proof. We will represent the forgetful functor $U : \text{Ring} \to \text{Set}$. To start, we need to show that $\mathbb{Z}[x]$ does actually represent U, for which we need a natural isomorphism

$$\eta : \text{Mor}_{\text{Ring}}(\mathbb{Z}[x], -) \Rightarrow U.$$

In particular, fixing a ring R, we need an isomorphism $\text{Mor}_{\text{Ring}}(\mathbb{Z}[x], R) \Rightarrow UR$, which we do by sending a morphism $f : \mathbb{Z}[x] \to R$ to $f(x) \in UR$. This is indeed a bijection because we can uniquely determine a morphism $\mathbb{Z}[x] \to R$ by where we send x.

Lastly, our universal element is x. To see this, we track through Theorem 3.17 to compute

$$\eta_{\mathbb{Z}[x]}(\text{id}_{\mathbb{Z}[x]}) = \text{id}_{\mathbb{Z}[x]}(x) = x,$$

which is what we wanted. ■

Remark 3.30. In words, the above proof says that $\mathbb{Z}[x]$ is the universal ring that has a distinguished element x. Being the "universal ring" is usually called being the "free ring."

For our next example, we have the following definition.

Definition 3.31 (Tensor products, I)**.** Fix two k-vector spaces V and W. Then $V \otimes W$ is made of formal sums

$$\sum_{i=1}^{n} v_i \otimes w_i$$

where $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$. Further, $(v, w) \mapsto v \otimes w$ is k-bilinear.

Exercise 3.32. We discuss $V \otimes W$ by universal property.

Proof. The point is to consider the functor

$$\text{Bilin}(V, W, -) : \text{Vec}_k \to \text{Set}$$

taking $U \mapsto \text{Bilin}(V, W, U)$, where $\text{Bilin}(V, W, U)$ consists of the k-bilinear maps $V \times W \to U$. Tensor products are actually intended to represent this functor. So here is a better definition for tensor products.

Definition 3.33 (Tensor products, II)**.** Given vector spaces V and W, we define $V \otimes W$ as the object that represents $\text{Bilin}(V, W, -)$.

We do not actually know if $V \otimes W$ really exists, but we will do so shortly. Our universal element x is intended to live in $\text{Bilin}(V, W, V \otimes W)$, so we are characterizing $V \otimes W$ by the bilinear map $V \times W \to V \otimes W$. In particular, we can now more or less say that $V \otimes W$ is the "universal" vector space with respect to a bilinear map $V \times W \to V \otimes W$.

Unwinding a bit, we will name the map $V \times W \to V \otimes W$ as $\otimes$. In particular, we are hoping that this element induces a natural isomorphism

$$\text{Mor}(V \otimes W, -) \Rightarrow \text{Bilin}(V, W, -).$$

In particular, by the Yoneda lemma, we are hoping that bilinear maps $V \times W \to U$ are in natural bijection with linear maps $V \otimes W \to U$. ■

3.5 March 4

Today we're having both Bryce and Chris lecture today (not in that order). We're in luck.

3.5.1 More on Universal Properties

We recall our definition.

Definition 3.28 (Universal property I, element)**.** A *universal property* for an object $c \in \mathcal{C}$ is a representable functor $F : \mathcal{C} \Rightarrow \mathrm{Set}$ along with an element $x \in Fc$ such that x induces (by the Yoneda lemma) a natural isomorphism $\mathrm{Mor}(c, -) \Rightarrow F$. In such a triplet (c, F, x), we call x the *universal element*.

We continue our discussion of Exercise 3.32. Typically, one would think about the universal property as follows.

Definition 3.34 (Universal property, II)**.** An object $c \in \mathcal{C}$ *satisfies the universal property* given by a functor $F : \mathcal{C} \to \mathrm{Set}$ if we have a universal element x such that (c, F, x) is a universal property.

In particular, last time we talked about having the functor

$$\mathrm{Bilin}(V, W, -) : \mathrm{Vec}_k \to \mathrm{Set}$$

which we claim was represented by some object $V \otimes W$ and our universal element $\otimes \in \mathrm{Bilin}(V, W, -) \to V \otimes W$. By Theorem 3.17, we can unwind this to a natural transformation

$$\eta : \mathrm{Hom}(V \otimes W, -) \Rightarrow \mathrm{Bilin}(V, W, -),$$

which we are claiming is a natural isomorphism to be our universal property. Well, we have our object $\otimes$, so we now track everything through. Here is our diagram to unwind the isomorphism above for a morphism $\overline{f} : V \otimes W \to U$ corresponding to the bilinear map $f : V \times W \to U$.

$$\begin{tikzcd}
\mathrm{Hom}(V \otimes W, V \otimes W) \arrow[r, "\overline{f}\circ -"] \arrow[d, "\eta_{V\otimes W}"'] & \mathrm{Hom}(V \otimes W, U) \arrow[d, "\eta_U"] \\
\mathrm{Bilin}(V, W, V \otimes W) \arrow[r, "\overline{f}\circ -"'] & \mathrm{Bilin}(V, W, U)
\end{tikzcd}$$

So to unwind what η_U means, we plug into $\mathrm{id}_{V\otimes W}$. This makes the following diagram.

$$\begin{tikzcd}
\mathrm{id}_{V\otimes W} \arrow[r, mapsto, "\overline{f}\circ -"] \arrow[d, mapsto, "\eta_{V\otimes W}"'] & \overline{f} \arrow[d, mapsto, "\eta_U"] \\
\otimes \arrow[r, mapsto, "\overline{f}\circ -"'] & f
\end{tikzcd}$$

Namely, we are told that bilinear maps $f : V \times W \to U$ correspond uniquely to a morphism $\overline{f} : V \otimes W \to U$ (by Theorem 2.72) in such a way that the following diagram commutes.

$$\begin{tikzcd}
V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed, "\overline{f}"] \\
& U
\end{tikzcd}$$

So this is our usual universal property for the tensor product.

Remark 3.35. We will actually need to construct $V \otimes W$, which we did not do, in order to show that there exists a way to represent the functor

$$\mathrm{Bilin}(V, W, -) : \mathrm{Vec}_k \to \mathrm{Set}.$$

However, in practice, we only ever want to pay attention to the above universal property.

3.5.2 Category of Elements

Today's discussion will not be discussion. Bryce is, reportedly, sorry. For brevity, we will take the following convention.

Definition 3.36 (Universal)**.** We say that an object $c \in \mathcal{C}$ is *universal* if and only if it is either initial or final.

Of course, $V \otimes W$ will not turn out to be universal in Vec_k, but if we change our category, then it will be, which is nice.

With that said, here is our main character for today.

Definition 3.37 (Category of elements)**.** Fix $F : \mathcal{C} \to \mathrm{Set}$ a functor. Then the *category of elements* of F, denoted $\int F$ is made of the following data.

- Objects are pairs (c, x) where $c \in \mathcal{C}$ and $x \in Fc$. In practice, we should think about the object $x \in Fc$ on its own, but we will have to remember which c it comes from.
- Morphisms $(c, x) \to (d, y)$ made of morphisms $f : c \to d$ which preserve our "base points" as $(Ff)(x) = y$. Importantly, we are keeping track of the arrows in $\mathcal{C}$, not in Set; e.g., F might not be injective on arrows, so we will keep track of these definitions.
- Identities are identities lifted from $\mathcal{C}$.
- Composition is composition in $\mathcal{C}$.

Remark 3.38. There is a natural forgetful functor $\Pi : \int F \to \mathcal{C}$ by

$$\Pi(c, x) := c \qquad \text{and} \qquad \Pi(f) := f.$$

We bring this up because this is roughly why we are keeping track of the morphisms in $\mathcal{C}$ instead of Set.

There is also a contravariant version.

Definition 3.39 (Category of elements, contravariant)**.** Fix $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a functor. Then the *category of elements* of F, denoted $\int F$ is made of the following data.

- Objects are pairs (c, x) where $c \in \mathcal{C}$ and $x \in Fc$.
- Morphisms $(c, x) \to (d, y)$ made of morphisms $f : c \to d$ which preserve our "base points" as $(Ff)(y) = x$. This flips because F is contravariant.
- Identities are identities lifted from $\mathcal{C}$.
- Composition is composition in $\mathcal{C}$.

Let's see some examples.

Example 3.40. Let $\mathcal{C}$ be a concrete category with faithful (forgetful) functor $U : \mathcal{C} \to \mathrm{Set}$. We work through $\int U$.

- Objects are pairs (c, x) where $x \in Uc$.
- Morphisms are morphisms $f : (c, x) \to (d, y)$ such that $(Uf)(x) = y$.

In other words, $\int U$ is roughly the objects $c \in \mathcal{C}$ with an identified base point. Specifically, $\int(U : \mathrm{Top} \to \mathrm{Set}) = \mathrm{Top}_*$.

Example 3.41. Fix $\mathcal{C}$ a locally small category, which is how you know Bryce is lecturing, which permits a functor $\mathrm{Mor}(c, -) : \mathcal{C} \Rightarrow \mathrm{Set}$. We discuss $\int \mathrm{Mor}(c, -)$.

- Objects are pairs (d, f) where $f \in \mathrm{Hom}(c, d)$. So our objects are morphisms.
- A morphism $\varphi : (d, f) \to (e, g)$ is a morphism $\varphi : d \to e$ in $\mathcal{C}$ such that $\varphi f = (\varphi \circ -)(f) = g$.

In other words, this gives the category under $\mathcal{C}$, denoted $c/\mathcal{C}$. The contravariant version gives $\mathcal{C}/c$.

Exercise 3.42. Fix $F : C^{\mathrm{op}} \to \mathrm{Set}$ a contravariant functor. We recover $\int F$ as a comma category.

Proof. To set up our discussion, we recall that Theorem 3.17 provides us with a sufficiently natural bijection

$$\psi : Fc \cong \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(-, c), F).$$

Now, objects in $\int F$ will naturally be objects $x \in Fc$. We would to track morphisms $f : (c, x) \to (d, y)$ through here as well, which means that we are going to need a morphism $\psi(x) \to \psi(y)$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Roughly speaking, we are going to want the following diagram to commute.

$$\begin{tikzcd} \mathrm{Mor}(-, c) \arrow[r, Rightarrow, "\psi(x)"] \arrow[d, Rightarrow, "(f\circ -)"'] & F \\ \mathrm{Mor}(-, d) \arrow[ur, Rightarrow, "\psi(y)"'] & \end{tikzcd} \tag{$*$}$$

In particular, Theorem 3.19 tells us that all such morphisms between natural transformations take the form $(f \circ -)$ for some morphism f, from which we can track our base point.

The point of all this is that we are going to have a nice correspondence between $\int F$ and the comma category

$$\int F \cong よ \downarrow \widetilde{F},$$

where $\widetilde{F} : \{*\} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the constant functor taking $* \mapsto F$. Indeed, to quickly unwind our definition of the comma category, it is made of triplets $(c \in \mathcal{C}, * \in \{*\}, f : よ(c) \to F(*))$, where morphisms $h : (c, *, f) \to (c', *, f')$ require the following diagram to commute.

$$\begin{tikzcd} よ(c) \arrow[r, "f"] \arrow[d, "Fh"'] & F \arrow[d, equal] \\ よ(c') \arrow[r, "f'"'] & F \end{tikzcd}$$

Notably, we only have to check the $\mathrm{id}_F : F \to F$ morphism because this is the only morphism carried from $\widetilde{F} : \{*\} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. But this diagram above is exactly the one we asked for in $(*)$, so we are done. ■

Next time we will discuss the following result.

Proposition 3.43. Fix $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then $\int F$ has an initial object (c, x) if and only if F is representable by c with universal element x.

Proof of the forwards direction. In one direction, take $(c, x) \in \int F$ initial. We would like a natural isomorphism $\eta : \mathrm{Mor}(c, -) \Rightarrow F$. Well, by Theorem 3.17, we get some natural transformation η corresponding to x, where

$$\eta_d(f) := (Ff)(x)$$

by pushing through our definition in Theorem 3.17. For this to be a natural isomorphism, we need the components $\eta_d : \mathrm{Mor}(c, d) \to Fd$ to be isomorphisms. In other words, for each $d \in \mathcal{C}$ and $y \in Fd$, we need some $f : c \to d$ such that

$$(Ff)(x) = \eta_d(f) = y.$$

Equivalently, there is a unique morphism $f : (c, x) \to (d, y)$ in $\int F$, which is what we wanted. ■

Remark 3.44. In the dual case, F will be contravariant, and our initial object becomes final.

3.6 March 7

We continue.

3.6.1 Housekeeping

We begin by discussing a homework problem. Here is a definition.

Definition 3.45 (Divisible)**.** An abelian group A is *divisible* if and only if, for each $a \in A$ and $n \in \mathbb{Z} \setminus \{0\}$.

It happens that the category of divisible abelian groups has non-injective monomorphisms. For example, we have the following.

Exercise 3.46. The map $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Proof. Suppose that we have maps $f, g : A \to \mathbb{Q}$ such that $\pi f = \pi g$. We claim that $f = g$. Indeed, for any $a \in A \setminus \{0\}$, we need to show that $f(a) = g(a)$, for which so far we know that $\pi(f(a)) = \pi(f(g))$, so there is an integer n such that

$$f(a) = g(a) + n.$$

Suppose for the sake of contradiction that $n \neq 0$. Then, because A is divisible, there exists an element $b \in A$ such that $a = 2nb$, so we get to write

$$2nf(b) = f(2nb) = f(a) = g(a) + n = 2ng(b) + n,$$

so $f(b) = g(b) + \frac{1}{2}$. Pushing this though π, we get

$$b = b + \frac{1}{2},$$

so $\frac{1}{2} \in \mathbb{Z}$, which is our contradiction. ■

And here is the attendance question.

Exercise 3.47. We describe $\int F$ where $F : \{*\} \to \mathrm{Set}$ is some functor.

Proof. Set $X := F(*)$. The objects in $\int F$ are pairs $(*, c)$ where $c \in X$, and the morphisms are morphisms $f : * \to *$ such that $Ff(c) = Ff(d)$, but only $f = \mathrm{id}_*$ is permitted. So we have objects which are elements of X and only identities, so this is the discrete category on X. ■

3.6.2 A Representability Test

Last time we were showing the following result.

Proposition 3.43. Fix $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then $\int F$ has an initial object (c, x) if and only if F is representable by c with universal element x.

Last time we showed the forwards direction.

Proof of the backwards direction. Suppose that we have a natural isomorphism $\alpha : \mathrm{Mor}(c, -) \Rightarrow F$, and we need an object to be initial in $\int F$. Without much to do, we set

$$x := \alpha_c(\mathrm{id}_c) \in Fc,$$

and we claim that (c, x) is our desired initial element in $\int F$.

Well, pick up some object (d, y), and we want to show that there is a unique morphism $(c, x) \to (d, y)$. To be explicit, our data consist of $d \in \mathcal{C}$ and $y \in Fd$. The main claim is that, for any morphism $f : c \to d$, we have

$$\alpha_d(f) = (Ff)(f),$$

as we showed in the Yoneda lemma. Here is the relevant naturality diagram.

$$\begin{tikzcd}
\mathrm{Mor}(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f\circ-"'] & Fc \arrow[d, "Ff"] \\
\mathrm{Mor}(c,d) \arrow[r, "\alpha_d"] & Fd
\end{tikzcd}$$

Tracking through id_c in the diagram gives the result because $\alpha_c(\mathrm{id}_c)$ was defined to be x. It follows that we have a morphism $f : (c, x) \to (d, y)$ if and only if $(Ff)(x) = y$ if and only if $\alpha_d(f) = y$, which we know to be unique because α_d is an isomorphism. ■

From the way we have proven things, we actually have the following result.

Corollary 3.48. In fact, F is represented by c with universal element x if and only if $(c, x) \in \int F$ is initial.

Proof. If $(c, x) \in \int F$ is initial, then we showed last time that c represents our functor, and x is actually our universal property (by staring at our proof). Conversely, if F is represented by c, we conjured our universal element $x := \alpha_c(\mathrm{id}_c)$ to create our initial element (c, x). ■

3.6.3 Unique Representation

Because the Yoneda embedding (Theorem 3.19) creates isomorphisms, if $\mathrm{Mor}(c, -) \simeq \mathrm{Mor}(c', -)$, then $c \cong c'$, so our representing objects are isomorphic. We might hope for something more.

Remark 3.49. There is a technical notion of "evil" that basically says that sometimes in category theory our notion of equality is too strong. For example, isomorphism of categories is too strong, so we had equivalence of categories to fix this.

Example 3.50. "Cardinality" of a category is not preserved by equivalence, so it is evil. For example, any two indiscrete categories are equivalent, but they have different numbers of elements.

Anyways, we have the following result.

Proposition 3.51. For a functor $F : \mathcal{C} \to \text{Set}$, the full subcategory spanned by its representations in $\mathcal{C}$ is either empty or a contractible groupoid.

Wait, contractible groupoid?

Definition 3.52 (Contractible groupoid)**.** A *contractible groupoid* is a category where all morphism sets $\text{Mor}(c, d)$ has exactly one element.

Remark 3.53. The idea is that we can "collapse" our category inwards along unique isomorphisms.

We showed back in Exercise 2.91 that all contractible groupoids are equivalent to Be; here is the idea behind why we are bringing this up.

Idea 3.54. Unique isomorphisms tend to have contractible groupoids in the background.

So the idea behind introducing Proposition 3.51 is that there will be a unique morphism $f : c \to d$ that will also send the corresponding universal elements correctly in that $f : (c, x) \to (d, y)$. It is a good isomorphism.

Before continuing, here is a lemma.

Lemma 3.55. The full subcategory of $\mathcal{C}$ spanned by its final objects is either empty or a contractible groupoid.

Proof. We will be brief. If it is empty, we are done. Otherwise, for any two final objects t_1, t_2, there is exactly one morphism $t_1 \to t_2$ because t_2 is final. So we are done. ■

Remark 3.56. We can dualize the above lemma (by working in $\mathcal{C}^{\text{op}}$) to replace the word "final" with "initial" everywhere.

And now we prove Proposition 3.51.

Proof of Proposition 3.51. If F is not representable, then $\int F$ has no initial objects because initial objects induce representations. Otherwise, $\int F$ will have initial objects, but they form a contractible groupoid by Remark 3.56. ■

3.6.4 Typical Universal Properties

Because we are feeling benevolent today, here are some examples.

Exercise 3.57. Consider the contravariant functor $\mathcal{P} : \text{Set}^{\text{op}} \to \text{Set}$, which sends maps objects by $\mathcal{P} : S \mapsto \mathcal{P}(S)$ and morphisms by taking $f : S \to T$ to $f^{-1} : \mathcal{P}(T) \to \mathcal{P}(S)$. We discuss Proposition 3.43 with this functor.

Proof. Our objects are pairs (X, A) where X is a set and $A \subseteq X$ is a subset. Our morphisms $(X, A) \to (Y, B)$ are maps $f : X \to Y$ such that $f^{-1}(B) = A$.

Now, back in Exercise 3.13, we showed that $\Omega = \{0, 1\}$ represents $\mathcal{P}$ with universal element 1. Accordingly, we claim that $(\Omega, \{1\})$ is final (note $\mathcal{P}$ is contravariant) in $\int F$. Indeed, for any pair (X, A), there is a unique map $f : X \to \Omega$ such that $f^{-1}(\{1\}) = A$ which describes itself. ■

Exercise 3.58. Consider the functor $\operatorname{Bilin}(V, W, -) : \mathrm{Vec}_k$. We discuss Proposition 3.43.

Proof. To start, we note that our objects of $\int \operatorname{Bilin}(V, W, -)$ consists of a vector space U with a bilinear map $f : V \times W \to U$. A morphism $(U, f) \to (U', f')$ is a linear map $g : U \to U'$ such that $gf = f'$; i.e., the following diagram should commute.

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[dr, "f'"'] & U \arrow[d, "g"] \\ & U' \end{tikzcd}$$

Explicitly, we want

$$\operatorname{Bilin}(V, W, -)(g)(f) = f',$$

but $\operatorname{Bilin}(V, W, -)(g) = (g \circ -)$ by definition.

On the other hand, we know that $V \otimes W$ represents $\operatorname{Bilin}(V, W, -)$ with universal element $\otimes : V \times W \to V \otimes W$ by $(v, w) \mapsto v \otimes w$. Noting that this means $(V \otimes W, \otimes)$ ought to be initial, we are told that whenever we have a bilinear map $V \otimes W \to U$, there is a unique map $V \otimes W \to U$ such that the following diagram commutes.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed] \\ & U' \end{tikzcd}$$

This is the typical universal property. ■

Exercise 3.59. Consider the forgetful functor $U : \mathrm{Ring} \to \mathrm{Set}$. We discuss Proposition 3.43.

Proof. Our objects in $\int R$ consists of pairs (R, r) such that $r \in R$. Our morphisms $f : (R, r) \to (S, s)$ is a morphism $f : R \to S$ such that $f(r) = s$.

Now, back in Exercise 3.11, we showed that $\mathbb{Z}[x]$ should represent this functor with universal element x, so we want $(\mathbb{Z}[x], x)$ to be initial in $\int F$. In other words, for any pair (R, r), there is a unique morphism $\mathbb{Z}[x] \to R$ such that $x \mapsto r$. Indeed, this morphism must take $1 \mapsto 1$, so we are sending

$$\sum_{k=0}^{N} a_k x^k \longmapsto \sum_{k=0}^{N} a_k r^k,$$

which finishes. ■

Theme 4
Limits and Colimits

It's true that many pieces of categorical terminology do come from analysis, but maybe all that says is that analysis is an old and venerable subject.

—Tom Leinster, [Lei09]

4.1 March 9

The fun continues but now in a different form.

4.1.1 Products

Let's do some examples to start because we are feeling kind today.

Exercise 4.1. We consider products $A \times B$ in Set, defined as

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$$

in a categorical sense.

Remark 4.2. It turns out that products are limits.

Proof. We would like to make this definition more fit for category theory, for which we note that we have projection maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ by $\pi_A : (a, b) \mapsto a$ and $\pi_B : (a, b) \mapsto b$ respectively. In fact, we are universal in the following sense: for any object A and maps $\varphi_A : C \to A$ and $\varphi_B : C \to B$, there is a unique (!) map $\varphi : C \to A \times B$ making the following diagram commute.

$$\begin{tikzcd} & C \arrow[d, dashed, "\varphi"] \arrow[ddl, bend right, "\varphi_A"'] \arrow[ddr, bend left, "\varphi_B"] & \\ & A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A & & B \end{tikzcd}$$

To see uniqueness, we note that we must have

$$\pi_A(\varphi(c)) \coloneqq \varphi_A(c) \qquad \text{and} \qquad \pi_B(\varphi(c)) \coloneqq \varphi_B(c),$$

so we must define our map φ as

$$\varphi(c) \coloneqq (\varphi_A c, \varphi_B c).$$

In fact, we can see that this defined map does have $\pi_A \circ \varphi = \varphi_A$ and $\pi_B \circ \varphi = \varphi_B$, so the diagram does indeed commute. ■

This example should feel similar to universal properties: whenever something occurs in our diagram, we have some unique induced map. To see this more formally, we have the following auxiliary exercise.

Exercise 4.3. We exhibit the universal property for $A \times B$ in the category $\mathcal{C} \coloneqq \mathrm{Set}$.

Proof. We note that we are being granted a bijection between pairs of maps (φ_A, φ_B) and our maps φ. In other words, there is a bijection

$$\mathrm{Mor}(C, A) \times \mathrm{Mor}(C, B) \to \mathrm{Mor}(C, A \times B).$$

To turn this into a universal property, we consider the functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by

$$F : \mathrm{Mor}(-, A) \times \mathrm{Mor}(-, B).$$

In particular, we send morphisms $f : S \to T$ to $F(f) : \mathrm{Mor}(T, A) \times \mathrm{Mor}(T, B) \to \mathrm{Mor}(S, A) \to \mathrm{MOr}(S, B)$ by

$$F(f) \coloneqq (- \circ f) \times (- \circ f).$$

We won't check this is a functor, but you can if you like doing that kind of thing.

Now, we to get our universal property, we need to exhibit our natural isomorphism

$$\eta : F \Rightarrow \mathrm{Mor}(-, A \times B).$$

We already gave a bijection of sets in the previous exercise, so we just need to show that it is natural. Well, pick up some morphism $f : T \to S$, and we have the following diagram to check.

$$\begin{tikzcd} \mathrm{Mor}(S, A) \times \mathrm{Mor}(S, B) \arrow[r, "(-\circ f)\times(-\circ f)"] \arrow[d, "\eta_S"'] & \mathrm{Mor}(T, A) \times \mathrm{Mor}(T, B) \arrow[d, "\eta_T"] \\ \mathrm{Mor}(S, A \times B) \arrow[r, "-\circ f"'] & \mathrm{Mor}(T, A \times B) \end{tikzcd}$$

So now, pick up some pair $(h, g) \in \mathrm{Mor}(S, A) \times \mathrm{Mor}(S, B)$ and track through. Along the bottom, we go to $h \times g$ which then goes to $hf \times gf$. Along the top, we start with (h, g) then go to (hf, gf) which then goes to $hf \times gf$.

Now let's compute our universal element. For this, we need to find what we are getting out of Theorem 3.17, which is

$$\eta^{-1}_{A\times B}(\mathrm{id}_{A\times B}) = (\pi_A, \pi_B),$$

which is fairly intuitive. In particular, we can track $\eta_{A\times B}((\pi_A, \pi_B))$ through and get $\mathrm{id}_{A\times B}$, which will give what we want. ■

Here are some generalizing remarks.

Remark 4.4. The universal property of products in Exercise 4.3 did not depend on Set, but the construction did. Products are more of an ambient concept that might or might not happen in some given category.

Remark 4.5. We could just have easily defined arbitrary products

$$\prod_{\alpha\in\lambda} S_\alpha$$

for some sets $\{S_\alpha\}_{\alpha\in\lambda}$ by just increasing the number of terms. For example, our functor we want to represent is now

$$F : \prod_{\alpha\in\lambda} \mathrm{Mor}(-, S_\alpha).$$

We can also write out the analogous universal property in terms of Exercise 4.1.

Example 4.6. The product of the one term A equipped with the projection map $\mathrm{id}_A : A \to A$. Indeed, for any map $C \to A$, there is a unique map $C \to A$ making the following diagram commute.

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[dr] & A \arrow[d, equal] \\ & A \end{tikzcd}$$

Example 4.7. The product of no terms at all is the final object X. Indeed, whenever we have no morphisms going anywhere, there is a unique map to X making whatever diagram you want commute.

4.1.2 Coproducts

Next let's discuss coproducts. Let's just give the universal property.

Definition 4.8 (Coproduct)**.** Given two objects $A, B \in \mathcal{C}$, we define the *coproduct* object $A \coprod B$ to be equipped with maps $\iota_A : A \to A \coprod B$ and $\iota_B : B \to A \coprod B$ such that, whenever we have an object Z with maps $\varphi_A : A \to Z$ and $\varphi_B : B \to Z$, there is a unique map $A \coprod B \to Z$ making the following diagram commute.

$$\begin{tikzcd} A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "\varphi_A"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "\varphi_B"] \\ & A \coprod B \arrow[d, dashed] & \\ & Z & \end{tikzcd}$$

Example 4.9. The disjoint union $A \sqcup B$ is the coproduct in Set. Indeed, our maps are $\iota_A : a \mapsto (a, 0)$ and $\iota_B : b \mapsto (b, 1)$. To see the universal property, suppose that we have an object C with maps $\varphi_A : A \to C$ and $\varphi_B : B \to C$. To see the uniqueness of $\varphi : A \sqcup B \to C$, we see that we must have

$$\varphi(\iota_A a) = \varphi_A(a) \qquad \text{and} \qquad \varphi(\iota_B b) = \varphi_B(b)$$

which exhausts all possible cases for elements of $A \sqcup B$. It is then not too hard to check that this does satisfy $\varphi \circ \iota_A = \varphi_A$ and $\varphi \circ \iota_B = \varphi_B$ by construction.

Example 4.10. We can generalize to products with multiple terms. If we have one object, the coproduct of A is just A. Similarly, if we have no objects, then the coproduct will be an initial object.

4.1.3 More on Products

Now let's generalize our examples. We begin by making the product even more categorical. At a high level, we might have lots of objects $\{A_\alpha\}_{\alpha\in\lambda}$, we are given maps $\pi_\alpha : \prod A \to A_\alpha$ in some universal way.

$$
\begin{tikzcd}
 & \prod A \arrow[dl, "\pi_\alpha"'] \arrow[dr, "\pi_\beta"] & \\
A_\alpha & \cdots & A_\beta
\end{tikzcd}
$$

To make this more in terms of category theory, we note that we can formalize the bottom part of the diagram as the image of some functor

$$F : \mathcal{J} \to \mathcal{C}$$

for some discrete category $\mathcal{J}$. Namely, our objects A_α look like $F(\alpha)$ for various $\alpha \in \mathcal{J}$.

To put the product $\prod A$ on the same footing, we will similarly define the constant functor

$$C_x : \mathcal{J} \to \mathcal{C}$$

which sends all objects of $\mathcal{J}$ to x and all morphisms to id_x.

We would like to create arrows between our diagrams, we are asking for an arrow between our functors, so we are more or less asking for a natural transformation $\eta : C_x \Rightarrow F$. Namely, the component morphisms take some $\alpha \in \mathcal{J}$, we are being promised a morphism $\eta_\alpha : x \to A_\alpha$. If we wanted to check that η is a natural transformation, we would pick up a morphism $\mathrm{id}_\alpha : \alpha \to \alpha$ in $\mathcal{J}$, which gives rise to the following diagram.

$$
\begin{tikzcd}
x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\eta_\alpha"] & x \arrow[d, "\eta_\alpha"] \\
F(\alpha) \arrow[r, "\mathrm{id}_{F(\alpha)}"'] & F(\alpha)
\end{tikzcd}
$$

Notably, this commutes for free. If we wanted to add more structure to our products, we might want to change $\mathcal{J}$ to be not discrete and have F be a more general diagram. This gives rise to limits.

> **Definition 4.11** (Cone)**.** Fix an index category $\mathcal{J}$ and a category $\mathcal{C}$ with an object $c \in \mathcal{C}$. Then a *cone* is a natural transformation from the constant functor $C_c \Rightarrow F$, where $F : \mathcal{J} \to \mathcal{C}$ is some diagram.

The limit will be the object $\lim F \in \mathcal{C}$ which is a "universal" cone, in the same way that the product was universal with respect to a "discrete cone." We will not discuss this more formally today, but we will discuss it more next lecture.

4.2 March 11

We do more limits today.

4.2.1 Cones and Cocones

We recall the following definition. For today, we fix $\mathcal{J}$ as our index category.

> **Definition 4.11** (Cone)**.** Fix an index category $\mathcal{J}$ and a category $\mathcal{C}$ with an object $c \in \mathcal{C}$. Then a *cone* is a natural transformation from the constant functor $C_c \Rightarrow F$, where $F : \mathcal{J} \to \mathcal{C}$ is some diagram.

> **Definition 4.12** (Apex)**.** A cone over F with *summit* or *apex* c is a natural transformation $\lambda : c \Rightarrow F$ with components $\lambda_j : c \to Fj$. These components are called *legs*.

Informally, λ consists of morphisms $\lambda_j : c \to Fj$ which commute with the morphisms promised by $\mathcal{J}$. Namely, for any morphism $f : i \to j$, we have the following naturality square diagram;

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "\lambda_i"'] & c \arrow[d, "\lambda_j"] \\ Fi \arrow[r, "Ff"'] & Fj \end{tikzcd}$$

Collapsing the top makes this look more like a triangle.

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F_i \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

Of course, we also have a dual notion. We have the following definition.

Definition 4.13 (Nadir)**.** A cone under F (or "cocone") with *nadir* c is a natural transformation $\lambda : F \Rightarrow c$ with components $\lambda_j : Fj \to c$. These components are (still) called *legs*.

Remark 4.14. We use the word nadir because someone wanted to.

This time our picture looks like the following.

$$\begin{tikzcd} Fi \arrow[rr, "Ff"] \arrow[dr, "\lambda_i"'] & & Fj \arrow[dl, "\lambda_j"] \\ & c & \end{tikzcd}$$

Notably, the nadir c is under F this time.

Example 4.15. Fix our index category $\mathcal{J} = \mathbb{Z}$. For a cone $F : \mathcal{J} \to \mathcal{C}$ over c, our diagram looks like the following.

$$\begin{tikzcd} & & & c \arrow[dl] \arrow[d] \arrow[dr] & & & \\ \cdots \arrow[r] & F(-1) \arrow[rr] & & F(0) \arrow[r] & F(1) \arrow[r] & \cdots \end{tikzcd}$$

4.2.2 Limits and Colimits

Intuitively, our limits will be the universal apex for a cone. It is the best cone; in some sense, it is the smallest or "closest" apex to the diagram. The diagram looks like the following.

$$\begin{tikzcd} & c' \arrow[ddl, bend right, "\lambda'_i"'] \arrow[ddr, bend left, "\lambda'_j"] & \\ & c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

We have the following definition to induce our desired behavior.

Definition 4.16 (Cone functor)**.** Fix a diagram $F : \mathcal{J} \to \mathcal{C}$.

- We define the functor $\operatorname{Cone}(-, F) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by

$$c \mapsto \operatorname{Cone}(c, F) := \operatorname{Hom}_{\mathrm{Cat}}(c, F)$$

which are the natural transformations $\lambda : c \Rightarrow F$. Then a morphism $f : c \to d$ goes to the morphism $F(f) : \operatorname{Hom}_{\mathrm{Cat}}(d, F) \to \operatorname{Hom}_{\mathrm{Cat}}(c, F)$ so that a cone $\lambda : d \Rightarrow F$ gives rise to a cone

$$F(f)(\lambda) = \lambda_\bullet \circ f \in \operatorname{Hom}_{\mathrm{Cat}}(c, F).$$

- We define the functor $\operatorname{Cone}(-, F) : \mathcal{C} \to \mathrm{Set}$ by

$$c \mapsto \operatorname{Cone}(F, c) := \operatorname{Hom}_{\mathrm{Cat}}(F, c)$$

which are the natural transformations $\lambda : c \Rightarrow F$. Then a morphism $f : c \to d$ goes to the morphism $F(f) : \operatorname{Hom}_{\mathrm{Cat}}(F, c) \to \operatorname{Hom}_{\mathrm{Cat}}(F, d)$ so that a cone $\lambda : F \Rightarrow c$ gives rise to a cone

$$F(f)(\lambda) = f \circ \lambda_\bullet \in \operatorname{Hom}_{\mathrm{Cat}}(F, d).$$

Here is the image of $\operatorname{Cone}(f)$ creating a cone with apex c from a cone to apex d.

$$\begin{tikzcd}
& d \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\lambda_i \circ f"'] \arrow[ddr, bend left, "\lambda_i \circ f"] & \\
& c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

We will not check the functoriality of this functor, but surely it works: just look at it.

Our functors give the following definitions.

Definition 4.17 (Limit, colimit)**.** A *limit* of a diagram $F : \mathcal{J} \to \mathcal{C}$ is a representation of $\operatorname{Cone}(-, F)$; in other words, it is a natural isomorphism $\mathcal{C}(-, c) \simeq \operatorname{Cone}(-, F)$. (Note $\operatorname{Cone}(-, F)$ is the contravariant.) Dually, a *colimit* is a representation of $\operatorname{Cone}(F, -)$.

We will mostly be talking about limits and leave the discussion of colimits to the curious.

Note that, by Theorem 3.17, we see that a natural transformation

$$\alpha \in \operatorname{Hom}(\mathcal{C}(-, c), \operatorname{Cone}(-, F))$$

corresponds to some literal cone $\operatorname{Cone}(c, F)$. From our discussion of the category of elements, we note that we can also think of a limit in the following way.

Definition 4.18 (Limit, colimit)**.** A *limit* of a diagram $F : \mathcal{J} \to \mathcal{C}$ is a terminal object in $\int \operatorname{Cone}(-, F)$.

To review, our objects of $\int \operatorname{Cone}(-, F)$ look like pairs $(c, \lambda) \in \int \operatorname{Cone}(-, F)$ where $\lambda : c \Rightarrow F$. Then our morphisms $(c, \lambda) \to (c, \mu)$ have the data $f : c \to d$ such that

$$\operatorname{Cone}(f, F)(\mu) = \lambda.$$

In other words, we require $\mu_\bullet \circ f = \lambda_\bullet$, which is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd}
& d \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
& c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

Thus, (c, λ) being terminal in $\int \mathrm{Cone}(-, F)$ means that any pair of objects (d, μ) will have the morphism f be unique.

At this point, we can see that our limits are indeed unique up to unique isomorphism because our terminal objects are unique up to unique isomorphism. Alternatively, we can give the following argument.

Proposition 4.19. The limit of a diagram $F : \mathcal{J} \to \mathcal{C}$ is unique up to unique isomorphism.

Proof. The point is to stack our limits on top of each other. So that (c, λ) and (c', λ') are both limits of F. Then we place them in the following diagram and note that we have unique maps f and g induced by the diagram above.

$$\begin{tikzcd}
& c \arrow[d, dashed, "g"] \arrow[dddl, bend right, "\lambda_i"'] \arrow[dddr, bend left, "\lambda_j"] & \\
& c' \arrow[d, dashed, "f"] \arrow[ddl, "\lambda'_i"'] \arrow[ddr, "\lambda'_j"] & \\
& c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

By uniqueness, we see that $f \circ g$ must be the identity (there is only one such morphism from $c \to c$ making the diagram commute, and id_c works), so f and g are must be inverses by redoing the stacking with f to show $g \circ f = \mathrm{id}_d$. ■

Notation 4.20. From now on, we will write $\lim F$ for the limit of F and $\mathrm{colim}\, F$ for F.

Remark 4.21. Not all categories have all their limits and colimits. For example, Field does not have an initial object (we cannot inject into both $\mathbb{F}_2$ and $\mathbb{F}_3$), so Field is missing the limit of the diagram from the empty category.

We close with an example.

Exercise 4.22. We show that product are limits from the discrete category.

Proof. Fix our functor $F : \mathcal{J} \to \mathcal{C}$ with apex (P, π), which means that we have morphisms $\pi_j : P \to Fj$. Note that we have no commutativity among the $j \in \mathcal{J}$ because $\mathcal{J}$ has no non-identity morphisms.

To translate the universal property, we see that whenever we have another apex (c, λ), there is a unique morphism $f : c \to P$ making the following diagram commute.

$$\begin{tikzcd}
 & c \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\lambda_i"'] \arrow[ddr, bend left, "\lambda_j"] & \\
 & P \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

This is what we wrote down in Exercise 4.3. ■

4.3 March 14

The fun, as they say, never stops.

4.3.1 More Examples

Chris is back, so today is just examples.

Exercise 4.23. We discuss the limit of the diagram

$$A \xrightarrow{f} B.$$

Proof. The limit will be an object X with a map $\iota : X \to A$ such that, for any object Y, there is a unique map $Y \to X$ making the following diagram commute.

$$\begin{tikzcd}
Y \arrow[r, dashed] \arrow[d, "\varphi"'] & X \arrow[dl] \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Well, we simply set $X := A$ with $X \to A$ simply as the identity map $\mathrm{id}_A : X \to A$. Then we are forced to have $Y \to X$ be φ by the diagram commuting, which finishes. ■

Exercise 4.24. We exhibit a product where the projection maps are not epimorphisms.

Proof. This is somewhat hard because faithful functors preserve epimorphisms, so concrete categories won't work here. So we consider the following category.

$$\begin{tikzcd}
 & B \arrow[dl] \arrow[dr] \arrow[drr, dashed, bend left] & & \\
A & & C \arrow[r, shift left] \arrow[r, shift right] & D
\end{tikzcd}$$

It is not too hard to see that B is the product of A and C (the only object with map to both A and C is B itself, so it is our only object to check), but the map $B \to C$ is not an epimorphism because of the problems with $C \to D$. ■

4.3.2 Equalizers

Definition 4.25 (Equalizer)**.** An *equalizer* is a limit of the following diagram.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

We denote this by $\operatorname{eq}(f,g)$.

More concretely, we set up our diagram as follows.

$$\begin{tikzcd} E \arrow[d, "e"'] \arrow[dr, "e'"] & \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

By commutativity of the diagram, we want $fe = ge = e'$, so we will ignore the morphism e' entirely: it's induced by the rest of the diagram.

Now, for E to be universal, we are saying that, for any morphism $h : X \to A$, there is a unique morphism $X \to E$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[d, dashed, "h"'] \arrow[dr, "\varphi"] & & \\ E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

This is not an obvious limit; here is an example.

Exercise 4.26. We compute equalizers in Set.

Proof. As a starting example, we note that we do have a "trivial" cone with $X = \varnothing$. This does not use the other information of our limit, so we simply define

$$E \coloneqq \{a \in A : f(a) = g(a)\}$$

with inclusion morphism $\iota : E \subseteq A$. Certainly $f\iota = g\iota$ by construction.

Now, to show the universal property, any other object X with a morphism $h : X \to A$ such that $fh = gh$, we see that $h(x) \in E$ for each $x \in X$. Thus, h does map into E, so we have our induced map

$$\widetilde{h} : X \to E$$

by simply restricting the codomain. This morphism is unique because any such morphism $\widetilde{h}$ must have $\iota\widetilde{h} = h$, so $\widetilde{h}(x) = h(x)$ for each $x \in X$. ■

Remark 4.27. I think the same construction will work for equalizers in any concrete category.

Exercise 4.28. Working in Ab, we consider the equalizer of the following diagram, where $f : A \to B$ is some morphism.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

In particular, we claim that the equalizer is the kernel.

Proof. By essentially doing the same proof as in Set, the equalizer will be the set

$$E := \{a \in A : f(a) = 0(a) = 0\},$$

which is $\ker f$. ∎

Remark 4.29. It follows that $\mathrm{eq}(f, g) = \mathrm{eq}(f - g, 0) = \ker(f - g)$ by tracking through what we need for our diagrams to commute.

Here is a nice result on equalizers.

Proposition 4.30. Given two morphisms $f, g : A \to B$ and an equalizer $e : E \to A$, the map e is always monic

Proof. Fix two maps $h, k : X \to E$ such that $eh = ek$. This has the following diagram.

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Then we see that eh and ek both have $f(eh) = f(ek) = g(eh) = g(ek)$, so there is a unique map $x : X \to E$ such that $ex = eh = ek$. But then we see that h and k both work, so $h = k$ is forced. ∎

Remark 4.31. This is notably different from projections failing to be epic because we are really only told that $p_A f = p_B f$ or $p_B f = p_B g$ when looking at just one projection. However, we need both of these for $f = g$.

4.3.3 Coequalizers

Of course, there is also a dual notion of an equalizer.

Definition 4.32 (Coequalizer)**.** A *coequalizer* is a colimit of the following diagram.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

We denote this by $\mathrm{coeq}(f, g)$.

From essentially the same discussion as before, the only data we need for a cocone of the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is an object Q with a morphism $q : B \to Q$. The universal property is saying that any object X with a morphism $\varphi : B \to X$ has a unique induced morphism as follows.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "\varphi"'] & Q \arrow[d, dashed] \\ & & X \end{tikzcd}$$

And now for examples.

Exercise 4.33. We compute coequalizers in Set.

Proof. The "dual" to a subset is a quotient, so we have reason to believe that the coequalizer should be a quotient. Thus, we define the equivalence relation $\sim$ in B generated by $f(a) \sim g(a)$.[1] It will happen that the canonical projection map $B \twoheadrightarrow B/\sim$ is our coequalizer. ■

4.4 March 16

These notes were transcribed from Rhea's notes. Thank you, Rhea!

4.4.1 Limit Review

Let's review the kinds of limits we can do.

- The limit of an empty set is the final object.
- The limit of a discrete category is a product.
- The limit of the arrow
$$A \longrightarrow B$$
is A.
- The limit of the diagram
$$\bullet \rightrightarrows \bullet$$
is the equalizer.

We continue our discussion with diagrams of three points.

Exercise 4.34. We show that limit of the triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

is A.

Proof. Any apex L for the diagram will consist of maps $\iota_A : L \to A$ and $\iota_B : L \to B$ and $\iota : L \to C$ so that the following diagram commutes.

$$\begin{tikzcd} & L \arrow[dl, "\iota_A"'] \arrow[d, dashed, "\iota_B"] \arrow[dd, bend left, "\iota_C"] \\ A \arrow[r, dashed, "f"] \arrow[dr, "h"'] & B \arrow[d, dashed, "g"] \\ & C \end{tikzcd} \tag{$*$}$$

However, we note that $\iota_B = f\iota_A$ and $\iota_C = h\iota_A$ by the commutativity of the diagram, so in fact, we can make the cone by only specifying ι_A.

And in fact, for any choice $\iota_A : L \to A$, we can induce the above diagram to commute by forcing $\iota_B := f\iota_A$ and $\iota_C : h\iota_A$, which will cause $(*)$ to commute because all the internal triangles commute.

[1] For example, we can take the intersection of all equivalence relations $B \times B$ which contain the requirements $f(a) \sim g(a)$ for each $a \in A$.

We thus claim that A equipped with $\mathrm{id}_A : A \to A$ is our limit. This means that we want a unique induced arrow $\varphi : L \to A$ making the following diagram commute.

$$\begin{tikzcd} L \arrow[r, dashed, "\varphi"] \arrow[d, "\iota_A"'] & A \arrow[dl, "\mathrm{id}_A"] \\ A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Well, any such arrow $\varphi : L \to A$ must satisfy $\varphi = \mathrm{id}_A \varphi = \iota_A$, so φ is forced. And indeed, $\varphi = \iota_A$ causes the necessary triangle to commute, we are done. ■

Remark 4.35. At a high level, what is causing this diagram to commute is that we are reducing this limit to a limit on a one-object category, which we know how to do.

4.4.2 Pullbacks

For our next limit, we have the following definition.

Definition 4.36 (Cospan)**.** A *cospan* is a diagram of the following form.

$$A \longrightarrow B \longleftarrow C$$

Equivalently, a cospan is a diagram indexed by the following category.

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

As with equalizers, we can decrease the number of arrows we have to keep track of in a cone over a cospan. Indeed, an apex L over a cospan is equipped with maps $\varphi_A : L \to A$ and $\varphi_B : L \to B$ and $\varphi_C : L \to C$ such that the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "\varphi_C"] \arrow[d, "\varphi_A"'] \arrow[dr, "\varphi_B"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Now, the commutativity diagram now forces $\varphi_B = f\varphi_A = g\varphi_C$, so we can simply induce φ_B from the rest of the diagram. As such, we decrease the data of a cone over a cospan as merely consisting of the maps $\varphi_A : L \to A$ and $\varphi_C : L \to C$ forcing $f\varphi_A = g\varphi_C$; i.e., we require the following diagram to commute.

$$\begin{tikzcd} L \arrow[r, "\varphi_C"] \arrow[d, "\varphi_A"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Of course, a cone will induce a diagram of the above form by forgetting the morphism φ_B. Conversely, a diagram of the above form makes a cone by setting $\varphi_B := f\varphi_A = g\varphi_C$, which will satisfy the needed commutativity to be a cone by construction.

Anyways, here is our limit.

Definition 4.37 (Pullback)**.** A *pullback* $A \times_B C$ is the limit of a cospan, labeled as follows.

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_C"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The right angle next to $A \times_B C$ is how we diagrammatically notate pullbacks.

Warning 4.38. The pullback $A \times_B C$ also depends on the chosen maps $f : A \to B$ and $g : C \to B$, even though these maps are not included in the notation.

It turns out that pullbacks are actually nontrivial limits, so we will need to fix our category to compute them. Here's an example.

Exercise 4.39. We compute pullbacks in Set.

Proof. Fix our diagram as follows.

$$\begin{tikzcd} L \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

As a first attempt, we might try $L = X \times Y$ with π_X and π_Y being the usual projection. But this does not work because the diagram might not commute: there is no reason to have

$$f(x) = f(\pi_X(x, y)) = (f\pi_X)(x, y) = (g\pi_Y)(x, y) = g(\pi_Y(x, y)) = g(y)$$

for each $x \in X$ and $y \in Y$. However, without much better to do, we force this condition in the rudest way possible: we simply restrict our product to be

$$X \times_Z Y := \{(x, y) \in X \times Y : f(x) = g(y)\},$$

where $\pi_X : (x, y) \mapsto x$ and $\pi_Y : (x, y) \mapsto y$ are the usual projections. This does indeed make a valid cone because any $(x, y) \in X \times Y$ will have

$$(f\pi_X)(x, y) = f(\pi_X(x, y)) = f(x) = g(y) = g(\pi_Y(x, y)) = (g\pi_Y)(x, y),$$

so $f\pi_X = g\pi_Y$.

It remains to show that this $X \times_Z Y$ creates the universal cone. Well, fix a set W with morphisms $\varphi_X : W \to X$ and $\varphi_Y : W \to Y$ so that the following diagram commutes.

$$\begin{tikzcd} W \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\varphi_X"] \arrow[ddr, bend right, "\varphi_Y"'] & & \\ & X \times_Z Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

We need to show that there is a unique arrow φ. To show that it is unique, note that we need

$$\pi_X(\varphi(w)) = (\pi_X\varphi)(w) = \varphi_X(w) \qquad \text{and} \qquad \pi_Y(\varphi(w)) = (\pi_Y\varphi)(w) = \varphi_Y(w)$$

by the commutativity of the diagram. It follows that we are forced to have

$$\varphi(w) := (\varphi_X w, \varphi_Y w).$$

We now show that this works. Note that this φ is well-defined because each $w \in W$ has

$$f(\varphi_X w) = (f\varphi_X)(w) = (g\varphi_Y)(w) = g(\varphi_Y w),$$

so $(\varphi_X w, \varphi_Y w) \in X \times_Z Y$. Then we have $\pi_X \varphi = \varphi_X$ and $\pi_Y \varphi = \varphi_Y$ by construction, forcing the diagram to commute for free. ■

4.4.3 Pullbacks as Equalizers

It is perhaps not too surprising that we ended up with something that looks like a product, with some equalizing condition. In fact, we can realize pullbacks as an equalized product.

Proposition 4.40. Work in some category $\mathcal{C}$. Fix morphisms $f : X \to Z$ and $g : Y \to Z$ in some category. Further, assume that $X \times Y$ exists with the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. If $\operatorname{eq}(f\pi_X, g\pi_Y)$ exists, then it is equal to $X \times_Z Y$.

Proof. Set $E := \operatorname{eq}(f\pi_X, g\pi_Y)$ with equalizing map $e : E \to X \times Y$. Our required map $E \to X$ will be $\pi_X e$; similarly, the required map $E \to Y$ will be $\pi_Y e$. Now, we see that E with $\pi_X e : E \to X$ and $\pi_Y e : E \to Y$ makes the following diagram commute.

$$\begin{tikzcd}
E \arrow[r, "\pi_X e"] \arrow[d, "\pi_Y e"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Indeed, we have

$$f(\pi_X e) = (f\pi_X)e \stackrel{*}{=} (g\pi_Y)e = g(\pi_Y e),$$

where $\stackrel{*}{=}$ is by construction of the equalizer.

It remains to show that E is universal. Well, pick up some object W with maps $\varphi_X : W \to X$ and $\varphi_Y : W \to Y$ such that $f\varphi_X = g\varphi_Y$. We then claim that there is a unique morphism φ causing the following diagram to commute.

$$\begin{tikzcd}
W \arrow[drr, bend left, "\varphi_X"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, bend right, "\varphi_Y"'] & & \\
& E \arrow[r, "\pi_X e"'] \arrow[d, "\pi_Y e"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

We start with the existence of the map φ. For this, we expand the diagram as follows.

$$\begin{tikzcd}
W \arrow[rrr, bend left, "\varphi_X"] \arrow[r, dashed, "\varphi"] \arrow[drr, "\varphi_Y"'] & E \arrow[r, "e"] & X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\
& & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Note that the square does not commute anymore. We have two steps.

- We use the universal property of $X \times Y$. The maps φ_X and φ_Y induce a unique map $\psi : W \to X \times Y$ such that $\pi_X \psi = \varphi_X$ and $\pi_Y \psi = \varphi_Y$.

- We use the universal property of E. By construction,

$$(f\pi_X)\psi = f(\pi_X\psi) = f\varphi_X = g\varphi_Y = g(\pi_Y\psi) = (g\pi_Y)\psi,$$

so ψ equalizes $f\pi_X$ and $g\pi_Y$. As such, there is a unique map $\varphi : W \to E$ such that $e\varphi = \psi$. In particular, we see that

$$(\pi_X e)\varphi = \pi_X(e\varphi) = \pi_X\psi = \varphi_X \qquad \text{and} \qquad (\pi_Y e)\varphi = \pi_Y(e\varphi) = \pi_Y\psi = \varphi_Y,$$

so the required diagram commutes.

It remains to show that the map $\varphi : E \to X$ is unique. Suppose that we have two such maps φ_1 and φ_2. We again proceed in two steps.

- We use the universal property of $X \times Y$. Note that

$$\varphi_X = (\pi_X e)\varphi_\bullet = \pi_X(e\varphi_\bullet) \qquad \text{and}\varphi_Y = (\pi_Y e)\varphi_\bullet = \pi_Y(e\varphi_\bullet),$$

so both morphisms $e\varphi_\bullet$ are the needed unique morphism $W \to X \times Y$. So we see $e\varphi_1 = e\varphi_2$.

- We use the universal property of E. Note that

$$(f\pi_X)(e\varphi_\bullet) = f(\pi_X e\varphi_\bullet) = f\varphi_X = g\varphi_Y = g(\pi_Y e\varphi_\bullet) = (g\pi_Y)(e\varphi_\bullet),$$

so the universal property of E forces there to be a unique map φ such $e\varphi = e\varphi_1 = e\varphi_2$. But of course, φ_1 and φ_2 are such maps φ, so $\varphi_1 = \varphi_2$ follows.

This finishes checking that E is universal. ■

Remark 4.41 (Bryce)**.** As Bryce would like to point out, the existence proof might look like it shows that φ is unique immediately—we did use two uniqueness results, after all—but some care is required. Namely, we only know that the morphism φ is the unique morphism commuting with ψ and then happens to make the diagram commute, so φ might not be unique making the diagram commute.

Remark 4.42 (Bryce)**.** It will turn out that all limits can be realized as equalizers of products.

4.4.4 Direct and Inverse Limits

We close lecture with two definitions.

Definition 4.43 (Direct limit)**.** A *direct limit* is a colimit of the poset category $\mathbb{N}$. In other words, a direct limit is a colimit of a diagram of the following form.

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

Intuitively, we can think of direct limits as ascending unions.

Definition 4.44 (Inverse limit)**.** An *inverse limit* is a limit of the poset category $\mathbb{N}^{\mathrm{op}}$. In other words, an inverse limit is a limit of a diagram of the following form.

$$A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow \cdots$$

Dually, we can intuitively think of inverse limits as a descending intersection.

4.5 March 18

Once again, these notes were transcribed from Rhea's notes. Thank you, Rhea!

4.5.1 Direct and Inverse Limits

We continue where we left off with direct and inverse limits. We recall our definitions.

Definition 4.43 (Direct limit)**.** A *direct limit* is a colimit of the poset category $\mathbb{N}$. In other words, a direct limit is a colimit of a diagram of the following form.

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

Definition 4.44 (Inverse limit)**.** An *inverse limit* is a limit of the poset category $\mathbb{N}^{\text{op}}$. In other words, an inverse limit is a limit of a diagram of the following form.

$$A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow \cdots$$

It is reasonable to ask why, say, we take the colimit over $\mathbb{N}$ instead of the limit. Here is why.

Exercise 4.45. Fix a functor $F : \mathbb{N} \to \mathcal{C}$. Then the limit of the diagram F is $F0$, where the maps $F0 \to Fn$ are the induced ones.

Proof. For concreteness, we will let the morphism $i \to j$ in $\mathbb{N}$ be denoted $(i \to j)$. As usual, we begin by restating what it means for a cone to have apex c over our diagram F. In particular, the cone has data consisting of the object L and morphisms $\varphi_n : c \to F(n)$ such that the following diagram commutes.

$$\begin{tikzcd} c \arrow[d, "\varphi_0"'] \arrow[dr, "\varphi_1"] \arrow[drr, "\varphi_2"] \\ F0 \arrow[r] & F1 \arrow[r] & F2 \arrow[r] & \cdots \end{tikzcd} \tag{$*$}$$

In other words, we require that

$$\varphi_n = F(0 \to n)\varphi_0$$

for each $n \in \mathbb{N}$. Thus, we can retrieve the data of φ_n from merely knowing φ_0, and these data are unique determined. And conversely, from merely knowing φ_0, we can set $\varphi_n := F(0 \to n)\varphi_0$ to get over F with apex c because $(*)$ commutes for free.

As such, we claim that the limit of this diagram is $F0$, with $\varphi_0 := \mathrm{id}_{F0}$. Indeed, suppose that we have some cone over F with apex c, and we need to induce a unique morphism $\varphi : c \to F0$ such that the following diagram commutes.

$$\begin{tikzcd} c \arrow[d, dashed, "\varphi"] \arrow[dd, bend right, "\varphi_0"'] \\ F0 \arrow[d, equal] \\ F0 \arrow[r] & F1 \arrow[r] & F2 \arrow[r] & \cdots \end{tikzcd}$$

Well, for the diagram to commute, we need $\mathrm{id}_{F0}\,\varphi = \varphi_0$, so $\varphi = \varphi_0$ is forced. And certainly $\varphi = \varphi_0$ will work because it has

$$\varphi_n = F(0 \to n)\varphi_0 = F(0 \to n)\varphi$$

for any $n \in \mathbb{N}$, which is what we wanted. This finishes. ■

So limits over $\mathbb{N}$ are not very interesting, but colimits under $\mathbb{N}$ (i.e., direct limits) are, which are why they get a fancy name.

4.5.2 Pushouts

We are obligated to spend a few words saying that there is a dual to Definition 4.37.

Definition 4.46 (Span)**.** A *span* is a diagram of the following form.

$$A \longrightarrow B \longleftarrow C$$

Equivalently, a span is the image of the following index category.

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

Definition 4.47 (Pushout, fibered coproduct)**.** A *pushout* is a colimit of a span, labeled as follows.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\iota_C"] \\ B \arrow[r, "\iota_B"'] & B \oplus_A C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The right angle next to $B \oplus_A C$ signifies that this is a pushout.

As before, we have a warning that $B \oplus_A C$ depends on the morphisms f and g, even though these are not communicated in the definition.

The story for pushouts is exactly dual to the story for pullbacks, simply by placing everything in the opposite category. For example, as suggested, a cone under a span with nadir X merely needs the data of $\varphi_C : C \to X$ and $\varphi_B : B \to C$ with the coherence condition

$$\varphi_B f = \varphi_C g$$

because this morphism should be equal to φ_A and will cause the needed diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "\varphi_A"] & C \arrow[d, "\varphi_C"] \\ B \arrow[r, "\varphi_B"'] & X \end{tikzcd}$$

to commute. This explains why we only drew two arrows in Definition 4.47.

As such, we can provide a wordier version of the universal property for pushouts via the universal property for colimits: fix any object X with morphisms $\varphi_B : B \to X$ and $\varphi_C : C \to X$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\varphi_C"] \\ B \arrow[r, "\varphi_B"'] & X \end{tikzcd}$$

Then there is a unique morphism $\varphi : B \oplus_A C \to X$ which makes the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\iota_C"'] \arrow[ddr, bend left, "\varphi_C"] & \\ B \arrow[r, "\iota_B"] \arrow[drr, bend right, "\varphi_B"'] & B \oplus_A C \arrow[dr, dashed, "\varphi"] & \\ & & X \end{tikzcd}$$

Remark 4.48. It is possible to have both a pullback and a pushout in the same square. For example, consider the following diagram in Set.

$$\begin{tikzcd} A\cap B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A\cup B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Now, we see that $A \cap B$ is the pullback in this diagram via our computation in Exercise 4.39. We will not show that $A \cup B$ is in fact the pushout of this diagram, but it is true; roughly speaking, this is the coproduct $A \sqcup B$ modded out by identifying $\iota_A\pi_A = \iota_B\pi_B$.

Example 4.49. We work in Ab. Then, working in the standard pushout diagram, we set

$$B \oplus_A C \cong \frac{B \oplus C}{\langle (f(a), 0) - (0, g(a)) : a \in A\rangle}.$$

Then this is our pushout, as in the following diagram.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\iota_C"] \\ B \arrow[r, "\iota_B"] & B\oplus_A C \end{tikzcd}$$

This diagram commutes essentially by construction, and it is universal basically because we have modded out by the smallest amount possible in order to make this diagram commute.

4.5.3 Hom Sets of (Co)products

Fix $\mathcal{I}$ some discrete category and $A : \mathcal{I} \to \mathcal{C}$ some functor. As we discussed when talking about coproducts, a cone under A with nadir X will contain the data of morphisms

$$f_i : A_i \to X$$

for each $i \in \mathcal{I}$, and there are no commutativity conditions here because $\mathcal{I}$ has no non-identity morphisms.

Now, suppose that we have a coproduct object $\coprod_{i\in\mathcal{I}} A_i$ equipped with $\iota_j : A_j \to \coprod_{i\in\mathcal{I}} A_i$. Then, given an object X with a morphism $f : \coprod_{i\in\mathcal{I}} A_i$, we can generate a tuple of morphisms

$$f \in \mathrm{Mor}\left(\coprod_{i\in\mathcal{I}} A_i, X\right) \mapsto \{f\iota_i : A_i \to X\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathrm{Mor}(A_i, X).$$

However, because of the data of the morphisms $f_i : A_i \to X$ makes a cone under A, it seems like we can reverse the map.

Proposition 4.50. Fix a discrete category $\mathcal{I}$ and a diagram $A : \mathcal{I} \to \mathcal{C}$. Further, let $\coprod_{i\in\mathcal{I}} A_i$ be the coproduct of A equipped with inclusions $\iota_j : A_j \to \coprod_{i\in\mathcal{I}} A_i$.

Then there is a (canonical) isomorphism

$$\mathrm{Mor}\left(\coprod_{i\in\mathcal{I}} A_i, X\right) \cong \prod_{i\in\mathcal{I}} \mathrm{Mor}(A_i, X).$$

Proof. The forwards map is $f \mapsto \{f\iota_i\}_{i\in\mathcal{I}}$, as discussed preceding the statement. We call this map φ.

To show that φ is an isomorphism, we give it an explicit inverse. Well, given some tuple $\{f_i\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \operatorname{Mor}(A_i, X)$ provides the data of a cone under A with nadir X: indeed, we only need the morphisms $f_i : A_i \to X$ to have a cone. Then the universal property of the coproduct (!) promises a (unique) morphism $f : \coprod_{i\in\mathcal{I}} \to X$ making the following diagram commute.

$$\begin{tikzcd} A_j \arrow[d, "\iota_j"'] \arrow[dr, "f_j"] & \\ \coprod_{i\in\mathcal{I}} A_i \arrow[r, "f"'] & X \end{tikzcd} \tag{$*$}$$

So we have generated a morphism $f : \coprod_{i\in\mathcal{I}} A_i \to X$. We call this map $\psi\left(\{f_i\}_{i\in\mathcal{I}}\right)$.

It remains to show that φ and ψ are mutually inverse. We run the checks independently.

- We show $\varphi\psi$ is the identity. Well, we start with $\{f_i\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \operatorname{Mor}(A_i, X)$. Then $f := \psi\left(\{f_i\}_{i\in\mathcal{I}}\right)$ is chosen to make $(*)$ commute. In particular, we are told that

$$\varphi(f) = \{f\iota_i\}_{i\in\mathcal{I}} = \{f_i\}_{i\in\mathcal{I}}$$

  by construction of f. This finishes.

- We show $\psi\varphi$ is the identity. This time we start with $f : \coprod_{i\in\mathcal{I}} A_i \to X$. Then $\psi(\varphi(f))$ is the unique morphism g such that

$$g\iota_i = \varphi(f)_i = f\iota_i$$

  for each $i \in \mathcal{I}$. But we see that f will work here, so $g = f$ follows, finishing.

The above checks do witness φ and ψ to be isomorphisms. ■

Remark 4.51. In fact, the isomorphism in Proposition 4.50 is natural in X as well as A, for suitably defined notions of natural.

Of course, there is also an analogous story for products, by reversing all of our arrows.

Proposition 4.52. Fix a discrete category $\mathcal{I}$ and a diagram $A : \mathcal{I} \to \mathcal{C}$. Further, let $\prod_{i\in\mathcal{I}} A_i$ be the coproduct of A equipped with projections $\pi_j : \prod_{i\in\mathcal{I}} A_i \to A_j$.

Then there is a (canonical) isomorphism

$$\operatorname{Mor}\left(X, \prod_{i\in\mathcal{I}} A_i\right) \cong \prod_{i\in\mathcal{I}} \operatorname{Mor}(X, A_i).$$

Proof. We argue by duality. Moving all objects into $\mathcal{C}^{\text{op}}$ turns the product into a coproduct, so we are looking for an isomorphism

$$\operatorname{Mor}\left(\coprod_{i\in\mathcal{I}} A_i^{\text{op}}, X^{\text{op}}\right) \cong \prod_{i\in\mathcal{I}} \operatorname{Mor}(A_i^{\text{op}}, X^{\text{op}}),$$

which is exactly Proposition 4.50. Then, from this isomorphism, we merely have to push back from $\mathcal{C}^{\text{op}}$ to $\mathcal{C}$ to get the result.

We spend a moment to unravel this isomorphism. Note that

$$\left(\prod_{i\in\mathcal{I}} A_i\right)^{\text{op}} \cong \coprod_{i\in\mathcal{I}} A_i^{\text{op}},$$

where our new inclusion morphisms are $\pi_j^{\text{op}} : A_j^{\text{op}} \to \coprod_{i\in\mathcal{I}} A_i^{\text{op}}$. Now, given $f : X \to \prod_{i\in\mathcal{I}} A_i$, the isomorphism in Proposition 4.50 yields

$$\{f^{\text{op}}\pi_i^{\text{op}}\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \operatorname{Mor}(X^{\text{op}}, A_i^{\text{op}}),$$

which then comes back to

$$\{\pi_i f\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \operatorname{Mor}(A_i, X).$$

This finishes. We quickly remark that we could also just use the above mapping to argue more directly by the universal property of products, merely imitating the proof of Proposition 4.50. ■

4.5.4 Surjective Projection Maps

We close lecture with an application of the ideas in the previous subsection.

Theorem 4.53. Fix $\mathcal{C}$ be a category such that $\operatorname{Mor}(A, B) \neq \varnothing$ for each $A, B \in \mathcal{C}$. Further, fix a discrete category $\mathcal{I}$ and a diagram $A : \mathcal{I} \to \mathcal{C}$, and suppose that we have a product $\prod_{i\in\mathcal{I}} A_i$ with projection maps

$$\pi_j : \prod_{i\in\mathcal{I}} A_i \to A_j.$$

Then these morphisms π_j are split epimorphisms.

Proof. Fix some $j \in \mathcal{J}$, and we show that π_j is a split epimorphism. The point is to manually create a lifting morphism from A_j to $\prod_{i\in\mathcal{I}} A_i$.

Well, for each $k \in \mathcal{J}$, we find some $\eta_k \in \operatorname{Mor}(A_j, A_k)$ (which exists by hypothesis on $\mathcal{C}$), and we will further require that $\eta_j := \operatorname{id}_{A_j}$. The point is that we are promised that the following diagram commutes, for any $k \in \mathcal{I}$.

$$\begin{tikzcd} A_j \arrow[r, dashed, "\eta"] \arrow[dr, "\eta_k"'] & \prod_{i\in\mathcal{I}} A_i \arrow[d, "\pi_k"] \\ & A_k \end{tikzcd}$$

Thus, as shown in the above diagram, we are promised some morphism $\eta : \prod_{i\in\mathcal{I}} A_i \to A_j$ such that $\eta\pi_k = \eta_k$ for each $k \in \mathcal{J}$. In particular, we see that

$$\eta\pi_j = \eta_j = \operatorname{id}_{A_j},$$

so we have manually split π_j. ■

Remark 4.54 (Nir)**.** I am under the impression that we needed some strong form of choice to choose all the morphisms η_j.

Remark 4.55. Arguing by duality, Theorem 4.53 tells us that the inclusion morphisms of a coproduct are split monomorphisms.

The condition that all morphism sets are nonempty is not actually very strong.

Example 4.56. The category $\text{Set}_{\neq\varnothing}$ of nonempty sets has all nonempty morphism sets. Thus, the projection maps from a product of nonempty sets will all be split epimorphisms.

Example 4.57. Similarly, the category Grp has all morphism sets nonempty because any two groups G and H have the trivial morphism $\varphi : G \to H$ by $\varphi : g \mapsto e_H$.

Corollary 4.58. Fix $\mathcal{C}$ a concrete category with faithful functor $U : \mathcal{C} \to \mathrm{Set}$. Further, fix a diagram $A : \mathcal{I} \to \mathcal{C}$ over a discrete category $\mathcal{I}$ with a product $\prod_{i\in\mathcal{I}} A_i$ with projection maps

$$\pi_j : \prod_{i\in\mathcal{I}} A_i \to A_j.$$

If $UA_i \neq \varnothing$ for each i, then the maps π_j are epimorphisms.

Proof. Fix some index $j \in \mathcal{I}$. Because all the A_i have $UA_i \neq \varnothing$, we see that the induced map

$$U\pi_j : \prod_{i\in\mathcal{I}} UA_i \to UA_j$$

is a split epimorphism (and in particular, an epimorphism) by Example 4.56.

So to finish, it suffices to show that faithful functor will pull back epimorphisms to epimorphisms, which we siphon off to the following lemma.

Lemma 4.59. Fix $U : \mathcal{C} \to \mathcal{D}$ a faithful functor. If $\pi : A \to B$ is a morphism in $\mathcal{C}$ such that $U\pi$ is an epimorphism (or monomorphism), then π is also an epimorphism (or monomorphism).

Proof. We show that π is an epimorphism; the other case follows similarly or by arguing by duality. Suppose that we have an object X with morphisms $f, g : B \to X$. We want to show that $f\pi = g\pi$ implies $f = g$. But now, we see that

$$U(f)U(\pi) = U(f\pi) = U(g\pi) = U(g)U(\pi),$$

so because $U\pi$ is an epimorphism, $Uf = Ug$ follows. However, U is faithful, so we get $f = g$, finishing. ■

Remark 4.60 (Nir)**.** I do not think that we can strengthen Lemma 4.59 to making π a split epimorphism if $U\pi$ is an epimorphism. For example, the subcategory $\mathcal{C}$

$$\{1,2\} \xrightarrow{\pi} \{1\}$$

of Set will embed via a faithful functor $U : \mathcal{C} \to \mathrm{Set}$, upon which $U\pi$ will be a split epimorphism. However, π itself is not a split epimorphism because this category has no morphism $\{1\} \to \{1,2\}$ at all!

The above lemma finishes the proof. ■

In other words, we see that, in concrete categories, products of nonempty objects will have surjective projection maps. Arguing by duality, we also see that coproducts of nonempty objects will have injective inclusion maps.

4.6 March 28

Welcome back from spring break. We are still doing limits.

4.6.1 Complete Categories

To talk about limits more concretely, we will want to make our categories nicer.

Definition 4.61 (Complete)**.** A category $\mathcal{C}$ is *complete* (*cocomplete*) if and only if $\mathcal{C}$ has all small limits. In other words, each diagram $F : \mathcal{J} \to \mathcal{C}$ for $\mathcal{J}$ small has a (co)limit.

Remark 4.62. One can more generally talk about diagrams with index category that is not small.

The reason we are asking for only small (co)limits is for moral size reasons.

Theorem 4.63. A category $\mathcal{C}$ with products indexed by $\operatorname{Mor} \mathcal{C}$ is a preorder.

Proof. Fix any two morphisms $f, g : a \to b$ in $\mathcal{C}$. We would like to show that $f = g$, so suppose for the sake of contradiction that $f \neq g$. Well, we are granted a product

$$p \coloneqq \prod_{h \in \operatorname{Mor} \mathcal{C}} b$$

in $\mathcal{C}$, with projection maps $\pi_h : p \to b$.

But now consider morphisms from a to p. The issue here is that

$$\operatorname{Mor}\left(a, \prod_{h \in \operatorname{Mor} \mathcal{C}} b\right) \cong \prod_{h \in \operatorname{Mor} \mathcal{C}} \operatorname{Mor}(a, b).$$

Comparing sizes, there are at most $\operatorname{Mor} \mathcal{C}$ morphisms on the left and at least $2^{\operatorname{Mor} \mathcal{C}}$ morphisms on the right. So the right is strictly larger than the left, finishing. ■

Remark 4.64. It is not obvious that $\operatorname{Mor} \mathcal{C}$ is strictly smaller than $2^{\operatorname{Mor} \mathcal{C}}$ because $\operatorname{Mor} \mathcal{C}$ might not be a set, but such is life. Bryce muttered something about inaccessible cardinals, but I am not a set theorist and therefore did not record it.

Corollary 4.65. All complete small categories are preorders.

Proof. This follows from the preceding theorem and the definition of complete. ■

Let's try to give a more nontrivial example.

Proposition 4.66. The category Set is complete.

Proof. Fix a diagram $F : \mathcal{J} \to \mathrm{Set}$. For our limit, we will simply define

$$L \coloneqq \operatorname{Cone}(*, F)$$

to be the set of cones over F with apex $*$; note that there are no size issues because this is a subset of product of all $\operatorname{Mor}(*, Fi)$ for all $i \in \mathcal{J}$, which is okay because $\mathcal{J}$ is small.

For our projection maps, we define $\lambda_j : L \to F_j$ by

$$\lambda_j(\mu) \coloneqq \mu_j(*),$$

where this makes sense because $\mu : * \Rightarrow F$ has $\mu_j : * \to Fj$, so we can extract out our element by $\mu_j(*)$.

Quickly, we verify that we have defined a cone $\lambda : L \Rightarrow F$. Namely, for any morphism $f : i \to j$, we need to show that the following commutes.

$$\begin{tikzcd} & L \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

For this, we diagram-chase on our elements. Namely, we compute

$$(Ff)\lambda_i(\mu) = (Ff)\mu_i(*) = \mu_j(*)$$

because the μ we picked up is a cone already. But now this right-hand side is $(Ff)\lambda_j(\mu)$ by hypothesis.

We now show that $\lambda : L \Rightarrow F$ is our limit cone. Well, pick up any cone $\eta : X \Rightarrow F$, and we need to induce a unique morphism $\varphi : X \to L$ making the following diagram commute.

$$\begin{tikzcd} & X \arrow[ddl, "\eta_i"'] \arrow[ddr, "\eta_j"] & \\ & L \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd} \tag{1}$$

Well, given $x \in X$, we need to define $\varphi(x)$ to be a cone, which means that given $j \in \mathcal{J}$, we need a morphism $\varphi(x)_j : * \to Fj$. Looking around, the only morphism of this form that we have is

$$\varphi(x)_j(*) := \eta_j(x).$$

To check that φ is a cone, we need to run the check, for any $f : i \to j$ in $\mathcal{J}$, we have

$$(Ff)\varphi(x)_i(*) = (Ff)\eta_i(x) = \eta_j(x) = \varphi(x)_j(*)$$

by using the fact that η is already cone.

We now show that the diagram (1) commutes. Well, we have, for any $x \in X$,

$$\lambda_j\varphi(x) = \varphi(x)_j(*) = \eta_j(x)$$

by definition of λ_j and $\varphi(x)$.

It remains to show that φ is unique. Well, suppose that we have a morphism $\psi : X \to L$ making (1) commutes so that $\lambda_j\psi = \eta_j$ everywhere. Well, for any $x \in X$, we need to verify that $\psi(x) = \varphi(x)$, which means that for any $j \in \mathcal{J}$, we need to verify that

$$\psi(x)_j(*) = \varphi(x)_j$$

because $\psi(x)_j, \varphi(x)_j : * \to Fj$. However, by hypothesis, we have

$$\psi(x)_j(*) = \lambda_j\psi(x) = \eta_j(x)$$

by hypothesis on our commuting, so we see that ψ is in fact forced. ■

4.6.2 Limits through Functors

It's definition time!

Definition 4.67 (Preserves, reflects limits)**.** Fix a diagram $\mathcal{K} : \mathcal{J} \to \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

- The functor F *preserves limits* if and only if a limit cone $\lambda : L \Rightarrow \mathcal{K}$ in $\mathcal{C}$ gives another limit cone $F\lambda : FL \to F\mathcal{K}$ in $\mathcal{D}$.
- The functor F *reflects limits* if and only if a limit cone $F\lambda : FL \to F\mathcal{K}$ in $\mathcal{D}$ promises another limit cone $\lambda : L \Rightarrow \mathcal{K}$ in $\mathcal{C}$.

Here is the image for these two properties.

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[rrr, "F"] & & & \mathcal{D} & \\
 & L \arrow[rrr, Rightarrow, shift left, "\text{preserves}"] \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & & & FL \arrow[lll, Rightarrow, shift left, "\text{reflects}"] \arrow[dl, "F\lambda_i"'] \arrow[dr, "F\lambda_j"] & \\
\mathcal{K}i \arrow[rr, "\mathcal{K}f"'] & & \mathcal{K}j & F\mathcal{K}i \arrow[rr, "F\mathcal{K}f"'] & & F\mathcal{K}j
\end{tikzcd}
$$

And here is one more definition.

Definition 4.68 (Create limits)**.** Fix a diagram $\mathcal{K} : \mathcal{J} \to \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$. Then F *creates limits* if and only if F already reflects limits and any a limit $\eta : d \Rightarrow F\mathcal{K}$ in $\mathcal{D}$ induces a limit $\lambda : L \Rightarrow \mathcal{K}$ in $\mathcal{C}$ such that $F\lambda = \eta$.

Essentially, having a limit in $\mathcal{D}$ allows us to bring the limit upwards to $\mathcal{C}$. This is different from reflecting limits because reflecting limits already assumed that we had the objects present in $\mathcal{C}$ already while creating limits conjures the objects for us.

Remark 4.69. There are also dual notions for all the above definitions by adding the prefix co- everywhere.

Here is a quick result to get some practice with these words.

Proposition 4.70. Fix a functor $F : \mathcal{C} \to \mathcal{D}$ which creates limits for some class of diagrams in $\mathcal{C}$. Further, suppose that $\mathcal{D}$ has limits for these diagrams. Then $\mathcal{C}$ has limits for these diagrams, and F will preserve these limits.

Proof. Fix a diagram $\mathcal{K} : \mathcal{J} \to \mathcal{C}$. Then $F\mathcal{K}$ has a limit $\eta : d \Rightarrow F\mathcal{K}$, which because F creates limits will go up to a limit $\lambda : c \Rightarrow \mathcal{K}$ in $\mathcal{C}$.

It remains to show that F preserves the limit of $\mathcal{K}$. We already know that F will preserve the limit λ because we lifted by hand (and so λ will go down to η), so suppose that we have some perhaps distinct limit $\mu : c' \Rightarrow \mathcal{K}$ in $\mathcal{C}$. Then the uniqueness of limits promises us a unique isomorphism $c \cong c'$ which commutes with the various legs in λ and μ, so it follows

$$F\mu \simeq F\lambda = \eta,$$

so F still preserved our limit μ going to η. ■

We close by stating a few results.

Proposition 4.71. Fully faithful functors reflect limits and colimits.

Proof. This is supposedly on the homework. ■

Remark 4.72. The functor need not create limits or colimits. Intuitively, this is because a fully faithful functor is merely an embedding, so the codomain category might have lots of space elsewhere for limits that our embedding does not hit.

Proposition 4.73. Equivalences preserve, reflect, and create all limits and colimits.

Proof. We leave this as an exercise. ■

Remark 4.74. Philosophically, this is good because we expect equivalent categories to be "the same" and so they should have the same limits and colimits, for a good notion of same.

4.7 March 30

We continue rolling.

4.7.1 More on Functors through Limits

We add a new definition.

Definition 4.75 (Strictly creates limits)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ *strictly creates limits* for some class $\mathcal{K}$ of diagrams if and only if a diagram $K : \mathcal{J} \to \mathcal{C}$ in $\mathcal{K}$ with limit cone $\mu : d \Rightarrow FK$ in $\mathcal{D}$ has the following.

- There is a unique lift of μ to $\mathcal{C}$ to a cone over $\mathcal{K}$ in $\mathcal{C}$.
- The lift of μ is a limit cone.

This is different from merely creating limits because of the uniqueness.

And now a result.

Quote 4.76 (Bryce)**.** Any time I use the word "small," it's kinda sloppy.

Proposition 4.77. Fix $\mathcal{A}$ a small category and some category $\mathcal{C}$. The forgetful functor

$$U : \operatorname{Fun}(\mathcal{A}, \mathcal{C}) \to \operatorname{Fun}(\operatorname{Ob}\mathcal{A}, \mathcal{C})$$

strictly creates all limits and colimits that $\mathcal{C}$ admits. In fact, the lifts are computed pointwise: given $a \in \mathcal{A}$, then $\operatorname{ev}_a : \operatorname{Fun}(\mathcal{A}, \mathcal{C}) \to \mathcal{C}$ sending $F \mapsto Fa$ preserves limits and colimits.

To elaborate, $\operatorname{Ob} A \subseteq \mathcal{A}$ is the subcategory of $\mathcal{A}$ where we have forgotten about all the non-identity morphisms.

Proof. We kinda just do it. We note that

$$\operatorname{Fun}(\operatorname{Ob}\mathcal{A}, \mathcal{C}) \cong \prod_{\operatorname{Ob}\mathcal{A}} \mathcal{C}$$

in the nicest possible way because we are approximately thinking about a functor from a discrete set of $\mathcal{C}$ as just a $\operatorname{Ob}\mathcal{A}$-indexed tuple of things in $\mathcal{C}$. These are pretty much literally equal. Here are our projection maps π_a are pretty much evaluating the functor ev_a along $\operatorname{Ob} A$.

Now, fix a diagram $K : \mathcal{J} \to \mathrm{Fun}(\mathcal{A}, \mathcal{C})$ such that $\mathrm{ev}_a\, \mathcal{K} : \mathcal{J} \to \mathcal{C}$ has a limit for each $a \in \mathcal{A}$. In other words, we are working in the context of the limits that $\mathcal{C}$ admits. As such, we get lots of limit cones. Type-checking everything through, we have a diagram that looks like the following.

$$\begin{tikzcd} & \lim \mathrm{ev}_a K \arrow[dl, "\lambda_i^a"'] \arrow[dr, "\lambda_j^a"] & \\ \mathrm{ev}_a K_i \arrow[rr] & & \mathrm{ev}_a K_j \end{tikzcd}$$

Namely, this is a limit taking place in $\mathcal{C}$ that we asserted $\mathcal{C}$ should admit. We want to lift this upwards to get a limit for K. Namely, we want a functor

$$\lim K : \mathcal{A} \to \mathcal{C}.$$

Well, we simply define our behavior on objects by hand as

$$(\lim K)(a) \coloneqq \lim \mathrm{ev}_a K.$$

This verifies that we are lifting pointwise because $(\lim K)(a) = \mathrm{ev}_a(\lim K)$, so we are really just concerned with showing the uniqueness of our lift.

To continue assembling our data, for $f : a \to b$ in $\mathcal{A}$, we need

$$(\lim K)(f) : (\lim K)(a) \to (\lim K)(b).$$

In particular, we are exhibiting a morphism as follows.

$$\begin{tikzcd} \lim \mathrm{ev}_a K \arrow[r, dashed] \arrow[d, "\lambda_j^a"'] & \lim \mathrm{ev}_b K \arrow[d, "\lambda_j^b"] \\ \mathrm{ev}_a K_j \arrow[r, dashed, "K_j(f)"'] & \mathrm{ev}_b K_j \end{tikzcd}$$

We can exhibit a map along the bottom objects because $\mathrm{ev}_a K_j = K_j(a)$ and $\mathrm{ev}_b K_j = K_j(b)$, so we have some morphism of $K_j(f)$ of the bottom objects. Then this will induce a unique map upstairs

$$\lim \mathrm{ev}_a K \to \lim \mathrm{ev}_b K$$

by the universal property of our limits. We call this map $(\lim K)(f)$, and it will turn out to be functorial. To be more explicit, the composite maps

$$\lim \mathrm{ev}_a K \xrightarrow{\lambda_j^a} \lim \mathrm{ev}_a K_j \; K_j \; (f) \to \lim \mathrm{ev}_b K_j$$

make a cone, which will induce the desired morphism.

For a taste, let's show that $(\lim K)(\mathrm{id}_a)$ is still the identity. The main point is that the following diagram commutes.

$$\begin{tikzcd} \lim \mathrm{ev}_a K \arrow[r, dashed, "\mathrm{id}_{\lim \mathrm{ev}_a K}"] \arrow[d, "\lambda_j^a"'] & \lim \mathrm{ev}_a K \arrow[d, "\lambda_j^b"] \\ \mathrm{ev}_a K_j \arrow[r, dashed, "K_j(\mathrm{id}_a) = \mathrm{id}_{K_j(a)}"'] & \mathrm{ev}_a K_j \end{tikzcd}$$

In particular, the bottom objects are pretty much equal because their morphisms are the identities. Thus, the map above $\lim \mathrm{ev}_a K \to \lim \mathrm{ev}_a K$ is unique, and $\mathrm{id}_{\lim \mathrm{ev}_a K}$ works, so we are done.

Lastly, we talk about uniqueness. Essentially, by the uniqueness of the rest of our universal properties, we can back-solve for what our functor $\lim K$ should have been the entire time. ■

4.7.2 Limits in Set

We continue.

Theorem 4.78. Fix any diagram $F : \mathcal{J} \to \mathcal{C}$. Then we claim there is a natural isomorphism

$$\mathrm{Mor}(x, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \mathrm{Mor}(x, F)$$

in Set.

Roughly speaking, this is a superpowered version of Proposition 4.52.

Proof. Back in Proposition 4.66, we showed that we can construct a limit of a diagram $K : \mathcal{I} \to \mathrm{Set}$ by

$$\lim_{\mathcal{I}} K = \mathrm{Cone}(*, K).$$

In particular, we have a natural isomorphism

$$\lim_{\mathcal{J}} \mathrm{Mor}(x, F) \cong \mathrm{Cone}(*, \mathrm{Mor}(x, F)).$$

As such, we claim that

$$\mathrm{Cone}(*, \mathrm{Mor}(x, F)) \cong \mathrm{Cone}(x, F).$$

For this, we take a cone $\lambda : * \Rightarrow \mathrm{Cone}(*, \mathrm{Mor}(x, F))$ and recover a morphism $\lambda_j(*) : x \to Fj$, which we claim will assemble to a cone in $\mathrm{Cone}(x, F)$. To check the coherence, we pick up a morphism $f : i \to j$ and check the naturality square as

$$(Ff)\lambda_i(*) \stackrel{?}{=} \lambda_j(*),$$

which is true because λ was already a cone.

In the other direction, we take a cone $\mu : x \Rightarrow F$. We can then define the cone

$$\mu' : * \Rightarrow \mathrm{Mor}(x, F)$$

by sending $\mu'_j(*) := \mu_j$. These maps will be inverses, and we will get naturality by elbow grease.

Synthesizing, we now have an isomorphism

$$\lim_{\mathcal{J}} \mathrm{Mor}(x, F) \cong \mathrm{Cone}(x, F).$$

But now this is isomorphic to

$$\mathrm{Mor}(x, \lim_{\mathcal{J}} F)$$

because morphisms from x to the diagram to F are the same as morphisms from x to $\lim_{\mathcal{J}} F$. ■

Remark 4.79. We cannot talk very well about

$$\mathrm{Mor}(\lim_{\mathcal{J}} F, x)$$

because it is hard to map out of limits. On the other hand, it is true that

$$\mathrm{Mor}(\mathrm{colim}_{\mathcal{J}} F, y) \cong \lim_{\mathcal{J}} \mathrm{Mor}(F, y)$$

by believing very hard.

And now let's see an example because Bryce is feeling benevolent.

Example 4.80. Iterated products of an element $a \in \mathcal{C}$ are called powers. The above fact gives

$$\operatorname{Mor}\left(X, \prod_{\mathcal{I}} A\right) \cong \prod_{\mathcal{I}} \operatorname{Mor}(X, A).$$

We close with a last result.

Corollary 4.81. The covariant representable functors $\operatorname{Mor}(X, -) : \mathcal{C} \to \operatorname{Set}$ in $\mathcal{C}$.

Proof. This follows directly from the previous result, where preservation of limits requires some notion of naturality given in the previous proposition. ∎

4.8 April 1

Unsurprisingly, we are not having discussion today.

Exercise 4.82. In Set, we provide a nice isomorphism $\operatorname{Mor}(A, B \cap C)$ and $\operatorname{Mor}(A, B) \cap \operatorname{Mor}(A, C)$.

Proof. Note that we have a pull-back square as follows.

$$\begin{tikzcd} B \cap C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\ B \arrow[r] & B \cup C \end{tikzcd}$$

Passing through $\operatorname{Mor}(A, -)$, we have the following diagram, which is still a pull-back diagram by Theorem 4.78.

$$\begin{tikzcd} \operatorname{Mor}(A, B \cap C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Mor}(A, C) \arrow[d] \\ \operatorname{Mor}(A, B) \arrow[r] & \operatorname{Mor}(A, B \cup C) \end{tikzcd}$$

As such, we get a more or less canonical isomorphism

$$\operatorname{Mor}(A, B \cap C) \cong \operatorname{Mor}(A, B) \cap \operatorname{Mor}(A, C).$$

∎

4.8.1 Limits through Representable Functors

Last time we were talking about the following result, which we restate and add to.

Theorem 4.83. Fix a locally small category $\mathcal{C}$.

(a) Covariant representable functors preserve limits.

(b) The (covariant) Yoneda embedding よ $: \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \operatorname{Set})$ preserves and reflects limits.

Proof. Note that (a) is essentially Theorem 4.78. It remains to show (b). For one side, note that よ reflects limits because it is an embedding.

It remains to preserve limits. Suppose that we have a diagram $F : \mathcal{J} \to \mathcal{C}$ with a limit in $\mathcal{C}$. We need to show that

$$\text{よ}(\lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \text{よ}(F).$$

We provide an isomorphism on the corresponding diagrams. So we pick up n object $X \in \mathcal{C}$, and we see that

$$\text{よ}(\lim_{\mathcal{J}} F)(X) = \operatorname{Mor}(X, \lim_{\mathcal{J}} F)$$

by tracking through Theorem 3.19 through. Now, by part (a), we see

$$\operatorname{Mor}(X, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \operatorname{Mor}(X, F) = \lim_{\mathcal{J}}((\text{よ}(F))(X)).$$

However, we showed that limits are computed pointwise back in Proposition 4.77, so

$$\lim_{\mathcal{J}}((\text{よ}(F))(X)) \cong (\lim_{\mathcal{J}} \text{よ} F)(X).$$

This completes checking our isomorphisms of objects. Tracking through the naturality everywhere, we get to say the same thing for morphisms, finishing. ■

Quote 4.84 (Bryce)**.** Can I say the whole thing is trivial? No, part of it is trivial.

As usual, there is a dual story. Namely, we have

$$\operatorname{Mor}(\operatorname{colim}_{\mathcal{J}} F, X) \cong \operatorname{Cone}(F, X)$$

by the universal property of colimits. However, working in Set, we can take opposites to get

$$\lim_{\mathcal{J}^{\mathrm{op}}} \operatorname{Mpr}(F, X) = \operatorname{Cone}(*, \operatorname{Mor}(F, X)) \cong \operatorname{Cone}(F, C).$$

This is not quite the colimit we want because the colimit wants to work with cocones, not with cones, so flipping $\mathcal{J}$ is not doing the right flip.

Anyway, working with the dual gives us the following statement, which is completely dual.

Theorem 4.85. Fix a locally small category $\mathcal{C}$.

(a) Contravariant representable functors preserve colimits.

(b) The (contravariant) Yoneda embedding $\text{よ} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}, \operatorname{Set})$ preserves and reflects limits in $\mathcal{C}^{\mathrm{op}}$.

Note that we are talking about limits in $\mathcal{C}^{\mathrm{op}}$ because of the flipping of the diagram we just described.

4.8.2 Computing Limits

We would like to compute limits. So let's compute limits.

Theorem 4.86. Fix a diagram $F : \mathcal{J} \to \mathcal{C}$. Then $\lim_{\mathcal{J}} F$ exists if and only if the equalizer of c and d in the diagram

$$\begin{tikzcd}
 & & F(\operatorname{cod} h) \\
E \arrow[r, dashed] & \prod_{j \in \mathcal{J}} Fj \arrow[ur, "\pi_{\operatorname{cod} h}"] \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] \arrow[d, "\pi_{\operatorname{dom} h}"'] & \prod_{f \in \operatorname{Mor} \mathcal{J}} F(\operatorname{cod} f) \arrow[u, "\pi_h"'] \arrow[d, "\pi_f"] \\
 & F(\operatorname{dom} h) \arrow[r, "Fh"'] & F(\operatorname{cod} h)
\end{tikzcd}$$

exists, in which case the limit is this equalizer. Here, c is induced as a map from the top product diagram, and d is induced as a map from the bottom product diagram.

Remark 4.87. Thus, checking if a category is complete reduces to checking that we have (small) products and equalizers.

Proof. We first show this in Set. By tracking the diagram through, we have

$$c(x_j)_{j\in\mathcal{J}} = (x_{\mathrm{cod} f})_{f\in\mathrm{Mor}\,\mathcal{J}},$$

by looking at what we should do with π_f and tracking the commutativity of the diagram. Similarly,

$$d(x_j)_{j\in\mathcal{J}} = (Ff(x_{\mathrm{dom}\,f}))_{f\in\mathrm{Mor}\,\mathcal{J}},$$

which we can again track through some particular π_f projection map. Now, we can compute

$$\lim_{\mathcal{J}} F = \mathrm{Cone}(*, F) = \{\lambda : * \Rightarrow F\}.$$

In particular, we are dealing with some subset of $\prod_{j\in\mathcal{J}} Fj$, which we want to track through to be an equalizer. So in one direction, let $\chi : * \Rightarrow F$ be a cone; then $f \in \mathrm{Mor}\,\mathcal{J}$ will need to have

$$Ff(x_{\mathrm{dom} f}) = x_{\mathrm{cod} f},$$

which is exactly the equalizing condition we need because of our construction of c and d. We can track this backwards to describe a cone from this equalizing condition, so we are done with this case in Set.

We now show the general case. In one direction, suppose that an equalizer E of our diagram exists. Applying the covariant Yoneda embedding $よ : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. In particular, pushing through $\mathrm{Mor}(-, E)$, the fact that we preserve limits gives us the following diagram.

$$\mathrm{Mor}(-,E) \dashrightarrow \mathrm{Mor}\left(-, \prod_{j\in\mathcal{J}} Fj\right) \underset{d}{\overset{c}{\rightrightarrows}} \mathrm{Mor}\left(-, \prod_{f\in\mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f)\right)$$

Pulling out the products, we get a diagram that looks like the following.

$$\mathrm{Mor}(-,E) \dashrightarrow \prod_{j\in\mathcal{J}} \mathrm{Mor}\left(-, Fj\right) \underset{d}{\overset{c}{\rightrightarrows}} \prod_{f\in\mathrm{Mor}\,\mathcal{J}} \mathrm{Mor}\left(-, F(\mathrm{cod}\, f)\right)$$

Drawing in the big diagram, our c and d morphisms which we induced are precisely the c and d we need from the case of Set. In particular, because $よ$ reflects limits, this last limit is

$$\lim_{\mathcal{J}} よF,$$

which is what we wanted after pulling the $よ$ outside and using the fact that we have an embedding.

For the other direction, we just run the entire argument in reverse, starting with the limit, pushing it through $よ$, viewing that as the equalizer in Set, using the fact that $よ$ reflects limits to pull back the equalizer to the original category, finishing. I will not write this out because I only understood it for the five seconds after Bryce explained it. ∎

There is also a dual story, as follows.

Theorem 4.88. Fix a diagram $F : \mathcal{J} \to \mathcal{C}$. Then $\mathrm{colim}_{\mathcal{J}} F$ exists if and only if the coequalizer of c and d in the diagram

$$\coprod_{f\in\mathrm{Mor}\,\mathcal{J}} F(\mathrm{cod}\, f) \underset{d}{\overset{c}{\rightrightarrows}} \coprod_{j\in\mathcal{J}} Fj \dashrightarrow E$$

exists, in which case the limit is this equalizer. Here, c and d are induced analogously as previously.

Proof. Duality. ∎

Corollary 4.89. The categories Set and Grp and Ab and Top and Vec_k are all complete and cocomplete.

Proof. We run down.

- Set is complete because it has products (products) and equalizers (equalizers).
- Set is cocomplete because it has coproducts (disjoint union) and equalizers (take a quotient).
- Grp is complete because it has products (products) and equalizers (weird kernel things).
- Grp is cocomplete because it has coproducts (free products) and coequalizers (weird quotient things).
- Ab is complete because it has products (products) and coequalizers (quotient things).
- Ab is cocomplete because it has coproducts (direct sums) and coequalizers (kernel things).
- The story for Vec is the same as Ab.
- Top is complete because it is pretty much the same as Set.

This finishes. ■

4.9 April 4

Chris is back!

4.9.1 Elements in Categories

Fix a concrete category $\mathcal{C}$ with its forgetful functor $U : \mathcal{C} \to \mathrm{Set}$.

Example 4.90. In Set, we can view elements $x \in X$ for a set X as morphisms from $\{*\}$ to X.

Example 4.91. In Grp, we can view elements $g \in G$ for a group G as morphisms from $\mathbb{Z}$ to G, by essentially tracking where 1 goes.

Can we work more generally?

Non-Example 4.92. In FinGrp, there is no way for this to occur. Roughly speaking, we are asking for $\mathrm{Mor}(T, -) \simeq U$, but this functor U need not be representable.

It turns out that there is no way to make this precise for general categories. As such, we have the following definition.

Definition 4.93 (Generalized element)**.** Fix a terminal object T in a category $\mathcal{C}$. Then we call a *generalized element* of an object X to be a morphism $T \to X$. More generally, a *generalized element of shape* $Y \in \mathcal{C}$ is a morphism $Y \to X$.

It turns out that arguments about elements can often be (philosophically) transformed into arguments about generalized elements.

We are feeling benevolent, so here are some examples.

Example 4.94. Fix morphisms $f, g : A \to B$. We can translate the statement

$$f = g \iff f(x) = g(x) \text{ for all } a \in A$$

into asserting that

$$f = g \iff fx = gx \text{ for all } x : X \to A.$$

To see this that this is true, we note the forward direction is by substitution, and the reverse direction is by setting $X = A$ and $x = \mathrm{id}_A$ so that we are told $f = fx = gx = g$.

Example 4.95. A morphism $f : A \to B$ is a monomorphism if and only if, for each object X, we have $fg = fh$ for $g, h : X \to A$ implies $g = h$. This is merely saying that $fg = fh$ for "generalized elements," which correctly generalizes our notion of injectivity.

Non-Example 4.96. A morphism $f : A \to B$ is an epimorphism if and only if, for each object X, we have $gf = hf$ for $g, h : B \to X$ implies $g = h$. This does not look like surjectivity for sets: we would be asking that any morphism $g : X \to B$ has a morphism $h : X \to A$ making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ & X \arrow[u, "g"'] \arrow[ul, dashed, "h"] \end{tikzcd}$$

This is a fairly strong condition.

4.9.2 Sheaves

For our next example, we define a sheaf.

Definition 4.97 (Presheaf)**.** Fix a category $\mathcal{C}$. A $\mathcal{D}$-valued *presheaf* on $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. If $\mathcal{D}$ is omitted, we will assume that $\mathcal{D} = \mathrm{Set}$.

To define a sheaf, we need to talk about topological spaces, which is made of a set X and a topology T. Then we have a category $\mathcal{O}(X)$, which is a preorder by the containment of open sets; i.e., we have exactly one morphism $U \to V$ for open sets $U, V \subseteq X$ if and only if $U \subseteq V$.

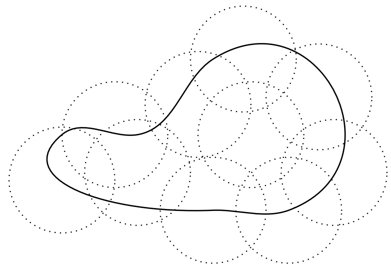
In particular, given a presheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Set}$, we see that $U \subseteq V$ induces a morphism $\mathcal{F}(V) \to \mathcal{F}(U)$, which we call restriction because we will think about this as restriction of functions. We will denote $f \in \mathcal{F}(V)$ as going to $f|_U \in \mathcal{F}(U)$; there is no ambiguity to the morphism $\mathcal{F}(V) \to \mathcal{F}(U)$ because the morphism $U \to V$ is unique.

Now, here is our definition.

Definition 4.98 (Sheaf)**.** Fix a topological space (X, T) to pick up a presheaf $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Set}$. Then F is a $(\mathrm{Set}$-valued) *sheaf* if and only if it satisfies the following extra conditions.

- Locality: fix an open set U and an open cover $\{U_\alpha\}_{\alpha \in \lambda}$. If $f|_{U_\alpha} = g|_{U_\alpha}$ for all α, we have $f = g$.
- Gluing: fix an open set U and an open cover $\{U_\alpha\}_{\alpha \in \lambda}$. If we have local elements $f_\alpha \in \mathcal{O}(U_\alpha)$ such that $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in \lambda$, then there is a global element $f \in \mathcal{O}(U)$ such that $f|_{U_\alpha} = f_\alpha$ for each α.

Visually, locality is saying that an open cover of an open set U can completely determine an element of V. Here is the image.



Now, if we have two elements f and g we are equal on the entire open set, then locality requires that $f = g$. Gluing is roughly saying that if we have a family $\{f_\alpha\}_{\alpha\in\lambda}$ on our open cover in such a way that we restrict properly to intersections, then we can glue our elements together.

Example 4.99. Given topological spaces X and Y, we can define the sheaf of continuous functions $X \to Y$ by taking an open set $U \subseteq X$ to the set of continuous functions $\mathcal{O}(U)$ from U to Y. This satisfies locality and gluing somewhat intuitively.

- Locality: if two continuous functions are equal when restricted to an open cover, then they are equal as functions: simply track where all the elements go.
- Gluing: continuous functions can be built up from local functions if they agree on intersection. This is called the pasting lemma.

Here is an alternate definition of sheaves.

Definition 4.100 (Sheaf)**.** Fix a topological space (X, T) to pick up a $\mathcal{C}$-valued presheaf $F : \mathcal{O}(X)^{\text{op}} \to \text{Set}$. For any open set $U \subseteq X$ with open cover $\{U_\alpha\}_{\alpha\in\lambda}$, the morphism

$$\mathcal{F}(U) \to \prod_{\alpha\in\lambda} \mathcal{F}(U_\alpha)$$

can be induced (component-wise) by the restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(U_\alpha)$. By restricting further, we can build two maps

$$\pi_{\text{res}\,\alpha}, \pi_{\text{res}\,\beta} \prod_{\alpha\in\lambda} \mathcal{F}(U_\alpha) \to \prod_{\alpha\in\lambda}\prod_{\beta\in\lambda} \mathcal{F}(U_\alpha \cap U_\beta)$$

by taking $\{f_\alpha\}_{\alpha\in\lambda}$ by either going to $\pi_{\text{res}\,\alpha} : \{f_\alpha\}_{\alpha\in\lambda} \mapsto f_\alpha|_{U_\alpha\cap U_\beta}$ or $\pi_{\text{res}\,\alpha} : \{f_\alpha\}_{\alpha\in\lambda} \mapsto f_\beta|_{U_\alpha\cap U_\beta}$. Now, $\mathcal{F}$ is a *sheaf* if and only if

$$\mathcal{F}(U) \longrightarrow \prod_{\alpha\in\lambda} \mathcal{F}(U_\alpha) \underset{\text{res}\,\beta}{\overset{\text{res}\,\alpha}{\rightrightarrows}} \prod_{\alpha\in\lambda}\prod_{\beta\in\lambda} \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer diagram.

We would like to show that our second definition correctly generalizes the first definition in the case where $\mathcal{C} = \text{Set}$.

Indeed, suppose that $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \text{Set}$ is a sheaf in the first notion. Then, to show that we have an equalizer diagram, we first show that the following diagram commutes.

$$\mathcal{F}(U) \longrightarrow \prod_{\alpha\in\lambda} \mathcal{F}(U_\alpha) \underset{\text{res}\,\beta}{\overset{\text{res}\,\alpha}{\rightrightarrows}} \prod_{\alpha\in\lambda}\prod_{\beta\in\lambda} \mathcal{F}(U_\alpha \cap U_\beta)$$

In the case of functions, pick up some $f \in \mathcal{F}(U)$, and which goes to

$$\{f|_{U_\alpha}\}_{\alpha\in\lambda}$$

in the middle. Then tracking through $\operatorname{res}\alpha$ and $\operatorname{res}\beta$, we are asking for

$$f|_{U_\alpha}|_{U_\alpha\cap U_\beta} = f|_{U_\beta}|_{U_\alpha\cap U_\beta}.$$

However, this is true by functoriality. This kind of argument holds more generally, even if we are not talking about a specific function $f \in \mathcal{F}(U)$.

We now show that we have an equalizer diagram.

$$\begin{tikzcd}
X \arrow[d, dashed] \arrow[dr] & & \\
\mathcal{F}(U) \arrow[r, "\iota"] & \prod_{\alpha\in\lambda}\mathcal{F}(U_\alpha) \arrow[r, shift left, "\operatorname{res}\alpha"] \arrow[r, shift right, "\operatorname{res}\beta"'] & \prod_{\alpha\in\lambda}\prod_{\beta\in\lambda}\mathcal{F}(U_\alpha\cap U_\beta)
\end{tikzcd}$$

In particular, we are more or less granted an infinite tuple of morphisms $\{f_\alpha\}_{\alpha\in\lambda}$ from $f_\alpha : X \to \mathcal{F}(U_\alpha)$. The gluing axiom, now, looks like the existence of the induced map above. In particular, requiring that $f_\alpha|_{U_\alpha\cap U_\beta} = f_\beta|_{U_\alpha\cap U_\beta}$, which is the hypothesis for gluing, and then we get a full $f : X \to \mathcal{F}(U)$ such that $\iota(f) = \{f|_{U_\alpha}\}_{\alpha\in\lambda}$ restricts as $f_\alpha = f|_{U_\alpha}$.

The locality axiom is the uniqueness in the equalizer diagram. Namely, if we have two such morphisms $f, g : X \to \mathcal{F}(U)$ such that $\iota f = \iota g$, then the uniqueness of the diagram forces $f = g$. This translates precisely into saying that $f|_{U_\alpha} = g|_{U_\alpha}$ for any α and β forces $f = g$, which is what locality wants.

4.10 April 6

Today is almost the last day of discussing limits.

4.10.1 Limit Functoriality

We begin by talking about functoriality of limits.

Proposition 4.101. Fix a category $\mathcal{C}$ with all limits of shape $\mathcal{J}$. Upon choosing a limit and associated limit cone for each diagram $\mathcal{J} \to \mathcal{C}$ defines the action of objects of a functor

$$\lim_{\mathcal{J}} : \operatorname{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}.$$

Proof. As promised, for each $F : \mathcal{J} \to \mathcal{C}$, we choose the limit cone $\lambda : \lim_{\mathcal{J}} F \Rightarrow F$. For the resulting argument, we need three diagrams F, G, H with cones λ, μ, η over F, G, H respectively.

We know $\lim_{\mathcal{J}}$ is supposed to do on objects, so we pick up a morphism $\alpha : F \Rightarrow G$ and take it to a morphism of limit objects. Well, note that we can consider the composite of natural transformations

$$\lim_{\mathcal{J}} F \overset{\lambda}{\Rightarrow} F \overset{\alpha}{\Rightarrow} G.$$

In particular, this cone over G induces a unique arrow

$$\lim_{\mathcal{J}} F \to \lim_{\mathcal{J}},$$

which we suggestively call $\lim_{\mathcal{J}} \alpha$. It remains to run our functoriality checks.

- Identity: we check $\lim_{\mathcal{J}} \operatorname{id}_F = \operatorname{id}_{\lim_{\mathcal{J}} F}$. We have the following diagram.

$$\begin{tikzcd}
\lim_{\mathcal{J}} F \arrow[r, dashed] \arrow[d, Rightarrow, "\lambda"'] & \lim_{\mathcal{J}} F \arrow[d, Rightarrow, "\lambda"] \\
F \arrow[r, Rightarrow, "\operatorname{id}_F"'] & F
\end{tikzcd}$$

If we place $\mathrm{id}_{\lim_{\mathcal{J}} F}$ on the top arrow, then the diagram commutes, so we conclude that

$$\lim_{\mathcal{J}} \mathrm{id}_F = \mathrm{id}_{\lim_{\mathcal{J}} F}$$

by uniqueness.

- Associativity: given morphisms $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, which looks like the following diagram.

$$\begin{tikzcd} \lim_{\mathcal{J}} F \arrow[r, dashed] \arrow[d, Rightarrow, "\lambda"'] & \lim_{\mathcal{J}} G \arrow[r, dashed] \arrow[d, Rightarrow, "\mu"] & \lim_{\mathcal{J}} H \arrow[d, Rightarrow, "\eta"] \\ F \arrow[r, Rightarrow, "\alpha"'] & G \arrow[r, Rightarrow, "\beta"'] & H \end{tikzcd}$$

This diagram commutes, so the arrow $\lim_{\mathcal{J}} F \to \lim_{\mathcal{J}} H$ can be made $(\lim_{\mathcal{J}} \beta)(\lim_{\mathcal{J}} \alpha)$ or $(\lim_{\mathcal{J}} \beta\alpha)$, so they are the same by uniqueness.

This finishes our functoriality check. ■

Remark 4.102. We might be able to have $\lim_{\mathcal{J}}$ output into cones instead of $\mathcal{C}$, but we won't bother.

Remark 4.103. We have to choose all the limit cones in advance, which is somewhat annoying.

4.10.2 Limits of Limits

We also talk a little about limits of limits. Note that a functor $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ more or less induces two different functors

$$F : \mathcal{I} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$$

by taking $i \in \mathcal{I}$ to the functor $j \mapsto F(i, j)$. If we wanted to be careful, we would note that we should send morphisms $f : j \to j'$ to $F(\mathrm{id}_i, f)$.

Remark 4.104. Formally speaking, we are kind of saying that

$$\mathrm{Fun}^2(\mathcal{I} \times \mathcal{J}) \simeq \mathrm{Fun}^2(\mathcal{I}, \mathrm{Fun}(\mathcal{J}, \mathcal{C})).$$

So the category of categories is Cartesian-closed. I only wrote down that sentence for its meme value.

Anyway, here is our statement.

Theorem 4.105. Fix a locally small category $\mathcal{C}$. If the limits

$$\lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} F(i, j) \qquad \text{and} \qquad \lim_{j \in \mathcal{J}} \lim_{i \in \mathcal{I}} F(i, j)$$

exist, then they isomorphic and isomorphic to $\lim_{\mathcal{I} \times \mathcal{J}} F$.

Formally, we are viewing the limit

$$\lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} F(i, j)$$

as first taking the limit of F over $\mathcal{J}$ (viewing F as a functor from $\mathcal{J}$ to $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ and then taking the limit resulting limit over $\mathcal{I}$.

We start with some motivating exercise.

Exercise 4.106. We draw the diagrams for equalizers commuting with pullbacks.

Proof. Consider the following two categories.

$$\begin{tikzcd}
\mathcal{I} & \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \\
\mathcal{J} & \bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

Then the product can be realized as follows.

$$\begin{tikzcd}
& \bullet \arrow[d] \arrow[dr, shift right] & \\
\bullet \arrow[r] \arrow[dr, shift right] & \bullet \arrow[dr, shift right] & \bullet \arrow[ul, shift right] \arrow[d] \\
& \bullet \arrow[ul, shift right] \arrow[r] & \bullet \arrow[ul, shift right]
\end{tikzcd}$$

There are a few ways to compute this limit.

- We might take the limit over the two copies of $\mathcal{J}$ first (i.e., take the pullbacks first), which with our two limits will end up looking like the following.

  $$\begin{tikzcd}
  \bullet \arrow[dr, shift left] \arrow[dr, shift right] & \\
  & \bullet
  \end{tikzcd}$$

  Then we can compute the limit over this diagram.
- We might take the limit over each of the three copies of $\mathcal{I}$ first (i.e., take the equalizers first), which will give us a diagram that looks like the following.

  $$\begin{tikzcd}
  & \bullet \arrow[d] \\
  \bullet \arrow[r] & \bullet
  \end{tikzcd}$$

  Then we can compute the pullbacks here.

The theorem is saying that the two ways to compute the limit actually coincide. ■

Anyway, here is the proof of our theorem.

Proof of Theorem 4.105. By Theorem 3.19, it suffices to show that

$$\operatorname{Mor}\left(X, \lim_{\mathcal{I}} \lim_{\mathcal{J}} F\right) \cong \operatorname{Mor}\left(X, \lim_{\mathcal{I}\times\mathcal{J}} F\right)$$

naturally. We will show the left isomorphism, and the other one follows by symmetry.

Further, taking the limits out of the morphism sets, we can assume that everything is in set, where we know how to compute limits already. In particular, using the construction in Proposition 4.66, we have

$$\lim_{\mathcal{I}\times\mathcal{J}} F = \operatorname{Cone}(*, F) \qquad \text{and} \qquad \lim_{\mathcal{I}} \lim_{\mathcal{J}} F = \operatorname{Cone}\left(*, \lim_{j\in\mathcal{J}} F(-,j)\right),$$

where the second is by tracking what F means in this case. Explicitly, $\lim_{j\in\mathcal{J}} F(-,j)$ is a functor $\mathcal{I} \to \mathcal{C}$. We construct these isomorphisms by hand.

- We construct a morphism

$$\varphi : \mathrm{Cone}(*, F) \to \mathrm{Cone}\left(*, \lim_{j\in\mathcal{J}} F(-,j)\right).$$

Well, picking up a cone $\lambda : * \Rightarrow F$, we see we are asking for a collection of morphisms

$$\varphi(\lambda)_i : * \to \lim_{j\in\mathcal{J}} F(i,j)$$

for each $i \in \mathcal{I}$. As such, fix some $i \in \mathcal{I}$. Now, the cone λ promises us morphisms $\lambda_{i,j} : * \to F(i,j)$. In particular, $j \mapsto \lambda_{(i,j)}$ gives us a cone which looks like the following.

$$\begin{tikzcd}
* \arrow[r, dashed] \arrow[d, Rightarrow, "\lambda_{i,j}"'] & \lim\limits_{j\in\mathcal{J}} F(i,j) \arrow[dl, Rightarrow] \\
F(i,j) &
\end{tikzcd}$$

Notably, the above diagram has j be variable. We now define $\varphi(\lambda)_i$ to be the induced map in the above diagram.

We now check that $\varphi(\lambda)$ is really a cone. Well, set $f : i \to i'$ to be a morphism in $\mathcal{I}$. Thus, we need to check

$$\lim j \in \mathcal{J} F(f, \mathrm{id}_j) \circ \varphi(\lambda)_i \stackrel{?}{=} \varphi(\lambda)_{i'}.$$

To check this, we draw the following associated diagram.

$$\begin{tikzcd}
* \arrow[r, "\varphi(\lambda_i)"] \arrow[dr, Rightarrow, "\lambda_{i,j}"'] & \lim\limits_{j\in\mathcal{J}} F(i,j) \arrow[r, dashed] \arrow[d, Rightarrow] & \lim\limits_{j\in\mathcal{J}} F(i',j) \arrow[d, Rightarrow] \\
& F(i,j) \arrow[r, Rightarrow, "F(f,\mathrm{id}_j)"'] & F(i',j)
\end{tikzcd}$$

Now, our functoriality is giving the map on the top right as $\lim_{\mathcal{J}} F(f, \mathrm{id}_j)$. The triangle and the square both commute by construction of each, so the full diagram commutes, but then we see that $\varphi(\lambda)_{i'}$ is the only map from $*$ to $\lim_{\mathcal{J}} F(i',j)$ making the diagram commute, so we get

$$\lim j \in \mathcal{J} F(f, \mathrm{id}_j) \circ \varphi(\lambda)_i \stackrel{?}{=} \varphi(\lambda)_{i'}.$$

- We construct a morphism

$$\psi : \mathrm{Cone}\left(*, \lim_{\mathcal{J}} F(-,j)\right) \to \mathrm{Cone}(*, F).$$

This is somewhat hard. Let $\mu : * \Rightarrow \lim_{\mathcal{J}} F(-,j)$ be some cone so that we want a morphism

$$\psi(\mu)_{i,j} : * \to F(i,j).$$

As such, we let $\rho_{i,j} : \lim_{j\in\mathcal{J}} F(i,j) \to F(i,j)$ be the canonical projection. As such, we define

$$\psi(\mu)_{i,j} \coloneqq \rho_{i,j}\mu_i$$

by simply composing our two cones.

It remains to show that we actually have a cone. Well, pick up $(f,g) : (i,j) \to (i',j')$ some morphism. Then we compute

$$F(f,g)\psi(\mu)_{i,j} = F(f, \mathrm{id}_{j'})F(\mathrm{id}_{i'}, g)\rho_{i,j}\mu_i$$

by moving around our definitions. Now, $F(\mathrm{id}_{i'}, g)\rho_{i,j} = \rho_{i,j'}$ because ρ is a cone. So we so far are dealing with

$$F(f, \mathrm{id}_{j'})\rho_{i,j'}\mu_i.$$

For this, we draw the following diagram.

$$\begin{tikzcd}
& * \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_{i'}"] & \\
\lim_{\mathcal{J}} F(i,j) \arrow[rr, "\lim_{\mathcal{J}} F(f,\mathrm{id}_j)"'] \arrow[d, "\rho_{i,j'}"'] & & \lim_{\mathcal{J}} F(i',j) \arrow[d, "\rho_{i',j'}"] \\
F(i,j') \arrow[rr, "F(f,\mathrm{id}_{j'})"'] & & F(i',j')
\end{tikzcd}$$

The top triangle commutes because μ is a cone, and the bottom square commutes by definition of the top map of the square. As such, we see that

$$F(f, \mathrm{id}_{j'})\rho_{i,j'}\mu_i = \rho_{i',j'}\mu_{i'} = \psi(\mu)_{i',j'}$$

by the commutativity of the diagram. This finishes showing that $\psi(\mu)$ assembles into a cone.

It remains to check that φ and ψ are mutually inverse and natural, but we won't bother proving these. ■

4.11 April 8

Today we began with review, which was not recorded.

4.11.1 Group Objects

The main idea is as follows.

Idea 4.107. We want to talk about "groups in a category $\mathcal{C}$."

One way to do this is to simply require a functor $\mathrm{B}G \to \mathcal{C}$, but we would like to have an object in $\mathcal{C}$ into a group, on its own.

As such, here is our definition. Fix a category $\mathcal{C}$ with finite products (and hence a terminal object T). Then a *group object* G in $\mathcal{C}$ has the following data.

- $G \in \mathrm{ob}\,\mathcal{C}$.
- We have a morphism $\nabla : G \times G \to \mathcal{C}$ for our operation.
- We have an identity element $\eta : T \to G$ (via the morphisms-as-elements philosophy).
- Lastly, there is an inverse map $s : G \to G$.

We also require the following diagrams to commute.

- Associativity.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\nabla \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \nabla"'] & G \times G \arrow[d, "\nabla"] \\
G \times G \arrow[r, "\nabla"'] & G
\end{tikzcd} \tag{Ass}$$

This is intended to say that $(ab)c = a(bc)$ for $a, b, c \in G$.

- Left identity.

$$\begin{tikzcd} T \times G \arrow[r, "\eta\times\mathrm{id}_G"] \arrow[dr, "\pi_G"'] & G \times G \arrow[d, "\nabla"] \\ & G \end{tikzcd} \tag{LId}$$

  This is intended to say that $1 \cdot g = g$ for any $g \in G$.

- Right identity.

$$\begin{tikzcd} G \times G \arrow[d, "\nabla"'] & G \times T \arrow[l, "\mathrm{id}_G \times\eta"'] \arrow[dl, "\pi_G"] \\ G \end{tikzcd} \tag{RId}$$

  This is intended to say that $g \cdot 1 = g$ for any $g \in G$.

The data so far assemble into a monoid object. To create a group, we introduce the canonical map $\varepsilon : G \to T$ (the "counit" map) and the diagonal map $\Delta : G \to G \times G$ by $\mathrm{id}_G \times \mathrm{id}_G$. So here is our last diagram.

- Inverse.

$$\begin{tikzcd} G \times G \arrow[rr, "s\times\mathrm{id}_G"] & & G \times G \arrow[d, "\nabla"] \\ G \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\varepsilon"'] & T \arrow[r, "\eta"'] & G \\ G \times G \arrow[rr, "\mathrm{id}_G \times s"'] & & G \times G \arrow[u, "\nabla"'] \end{tikzcd} \tag{Inv}$$

  The top diagram takes $g \in G$ and asserts that $g^{-1} \cdot g$ is equal to the identity element as required by T. The bottom diagram is doing the same for $g \cdot g^{-1}$.

So we now state our definition.

Definition 4.108 (Group object)**.** Fix a category $\mathcal{C}$ with finite products (and hence a terminal object T). Then a *group object* G in $\mathcal{C}$ has the following data.

- $G \in \mathrm{ob}\,\mathcal{C}$.
- We have a morphism $\nabla : G \times G \to \mathcal{C}$ for our operation.
- We have an identity element $\eta : T \to G$ (via the morphisms-as-elements philosophy).
- Lastly, there is an inverse map $s : G \to G$.

We also require (Ass), (LId), (RId), and (Inv) to all commute.

Example 4.109. A group object in Set is a group.

Example 4.110. A group object in Grp is an abelian group. This is by the Eckmann–Hamilton argument because we have made the group G a monoid in two different ways, and we get told that these monoid structures must coincide. However, we have also required that the inverse map $s : G \to G$ to be a group homomorphism. The fact that this inverse map is a group homomorphism requires G to be abelian.

We quickly outline the Eckmann–Hamilton argument. Because $\eta : \{e\} \to G$ is a group homomorphism, we do indeed realize η as the actual identity for G. It remains to show that the ∇ map is correct, so suppose we have a second morphism $\nabla' : G \times G \to G$.

Well, we note that a group homomorphism $f : G \to H$ consists of the data of the following data commuting.

$$\begin{tikzcd} G \times G \arrow[r, "\nabla_G"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\nabla_H"'] & H \end{tikzcd}$$

In particular, $\mathrm{id}_G : G \to G$ will provide the following data.

$$\begin{tikzcd} (G \times G) \times (G \times G) \arrow[r, "\nabla' \times \nabla'"] \arrow[d, "\nabla \times \nabla"'] & G \times G \arrow[d, "\nabla"] \\ G \times G \arrow[r, "\nabla'"'] & G \end{tikzcd}$$

Tracking through $((x, 1), (1, y)) \in G \times G$ shows that $\nabla'(x, y) = xy$, which is what we wanted.

Theme 5
Adjoints

Right foot, let's stomp. Left foot, let's stomp. Cha cha real smooth.

—DJ Casper, [Cas09]

5.1 April 11

We finally begin our discussion of adjoints.

Remark 5.1. Being shocked for thirty minutes after lecturing is strongly cringe.

5.1.1 Introducing Adjunctions

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can ask for the following relations, listed in order of strength.

- We can ask for categories to be equal. This is very strong.
- We can ask for categories to be isomorphic. This is evil.
- We can ask for categories to be equivalent. This is a good notion, but it is also somewhat strong.

For example, we might want to think of Set and Grp to be related. Of course, they are not equivalent (for example, Grp has a zero object $\{e\}$ while Set does not).

The goal for today is to talk about a weaker notion than equivalence (so that we have lots of interesting examples) that is still powerful enough to be useful. Here is our definition.

Definition 5.2 (Adjunction)**.** Fix two categories $\mathcal{C}$ and $\mathcal{D}$. An *adjunction* is a pair of functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ with isomorphisms (of sets)

$$\mathrm{Mor}_{\mathcal{D}}(Fc,d)\cong\mathrm{Mor}_{\mathcal{C}}(c,Gd)$$

natural in both arguments. We will call F a *left adjoint* to G and G a *right adjoint* to F, notated $F\dashv G$.

Notation 5.3. We will write the pair of functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ as $F:\mathcal{C}\rightleftarrows\mathcal{D}:G$.

Remark 5.4. It turns out that adjoints are unique up to some notion of isomorphism, so we may write "the" adjoint. However, we will not prove this uniqueness for a while.

Let's actually write out our naturality square. For c, if $f : c \to c'$ is a morphism, we need the following diagram to commute.

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}}(Fc', d) \arrow[r, phantom, "\cong"] \arrow[d, "- \circ Ff"'] & \mathrm{Mor}_{\mathcal{C}}(c', Gd) \arrow[d, "- \circ f"] \\ \mathrm{Mor}_{\mathcal{D}}(Fc, d) \arrow[r, phantom, "\cong"] & \mathrm{Mor}_{\mathcal{C}}(c, Gd) \end{tikzcd}$$

And here is the other naturality square, for a morphism $g : d \to d'$.

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}}(Fc, d) \arrow[r, phantom, "\cong"] \arrow[d, "g \circ -"'] & \mathrm{Mor}_{\mathcal{C}}(c, Gd) \arrow[d, "Gg \circ -"] \\ \mathrm{Mor}_{\mathcal{D}}(Fc, d') \arrow[r, phantom, "\cong"] & \mathrm{Mor}_{\mathcal{C}}(c, Gd') \end{tikzcd}$$

Here is one last piece of notation.

Notation 5.5. We will notate $f^\sharp : Fc \to d$ being "adjoint" or "transpose" to the morphism $f^\flat : c \to Gd$, by the isomorphism promised by the adjunction.

Let's make the naturality a little faster; it turns out that the naturality is equivalent to having a natural isomorphism as follows.

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\mathcal{D}(F-,-)"] \arrow[r, bend right, "\mathcal{C}(-,G-)"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathrm{Set} \end{tikzcd}$$

Take a moment to verify that the functors actually type-check.

Here is another way to promote the efficiency.

Lemma 5.6. Fix functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ with given isomorphisms $\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$ for $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Then, naturality is equivalent to the following: one of the squares below commutes if and only if the other does, for any morphisms making the diagrams well-defined.

$$\begin{tikzcd} Fc \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & d \arrow[d, "k"] \\ Fc' \arrow[r, "g^\sharp"'] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[d, "h"'] & Gd \arrow[d, "Gk"] \\ c' \arrow[r, "g^\flat"'] & Gd' \end{tikzcd}$$

Proof. This is supposedly on the homework. ∎

In general, the above condition is better to check than naturality.

5.1.2 Examples

We are feeling kind today, so let's see some examples.

Exercise 5.7. The functor $F : \mathrm{Set} \to \mathrm{Grp}$ sending a group X to the free group on X is left adjoint to the forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$.

Proof. The point is that, given a set X and group G, a morphism $FX \to G$ is in some sense the "same data" as a morphism $X \to FG$. For example, given a morphism $f^\flat : X \to UG$, we define $f^\sharp : FX \to G$ by

$$f^\sharp(x_1 x_2 \cdots x_n) = f^\flat(x_1) f^\flat(x_2) \cdots f^\flat(x_n).$$

This turns out to be an isomorphism, and it turns out that we can show naturality everywhere, but we will not bother. ■

Remark 5.8. There are many examples of "free-forgetful" adjunctions.

Remark 5.9. Philosophically, a right adjoint poses a question which the left adjoint answers. For example, $U : \mathrm{Grp} \to \mathrm{Set}$ is asking how to get maps of groups from sets, which the free functor tells us how to do.

Exercise 5.10. The functor $F : \mathrm{Set} \to \mathrm{Top}$ taking a set X to the topological space X equipped with the discrete topology is left adjoint to the forgetful functor $U : \mathrm{Top} \to \mathrm{Set}$.

Similarly, the functor $G : \mathrm{Set} \to \mathrm{Top}$ taking a set X to the topological space X equipped with the indiscrete topology is right adjoint to U.

Proof. For the first claim, we are saying that, given a set X and a topological space T, continuous maps $FX \to T$ have the same data as $X \to UT$. However, all maps (of sets) $FX \to T$ are continuous for free because FX has the discrete topology.

On the other hand, for the second claim, we are saying that, given a set X and a topological space T, maps (of spaces) $UT \to X$ have the same data as continuous maps $T \to GX$. This is true because all maps $T \to GX$ are automatically continuous because we are asking for the preimages of $\varnothing$ and GX to be open, which are true for free. ■

Exercise 5.11. We give the embedding functor between the poset categories $\iota : (\mathbb{Z}, \leq) \to (\mathbb{R}, \leq)$ a right adjoint.

Proof. We claim that the right adjoint is $r \mapsto \lfloor r \rfloor$. This is well-defined and does give a functor. To get a right adjoint, we are saying that $\iota n \leq r$ if and only if $n \leq \lfloor r \rfloor$, which is true by some analysis. Notably, we only care about the existence of morphisms to get our bijections because we are working in poset categories. ■

Remark 5.12. Of course, ι also has a left adjoint as $\lceil \cdot \rceil$.

Exercise 5.13. We give the embedding functor $\iota : \mathrm{Groupoids} \to \mathrm{Cat}$ a left and right adjoint.

Proof. Recall that a groupoid is a category where all morphisms are isomorphisms. There are two ways to do this.

- We can construct the "maximal subgroupoid" $\max \mathcal{C}$ by merely taking $\mathcal{C}$ and throwing out all morphisms which are not isomorphisms. This turns out to be a right adjoint.
- We can force all morphisms to be isomorphisms by adding in all the necessary inverses and quotient out by what we need to remain a category; this process is called "localization," in analogy with localizing a ring. This turns out to be a left adjoint.

We won't actually check that these are adjoints (because the checks are painful), so we will declare that we are done. ■

5.1.3 Units and Counits

We are now done talking about examples. As such, have a lemma.

Lemma 5.14. Fix adjunctions $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$. Then there exists a natural transformation $\eta : \mathrm{id}_\mathcal{C} \Rightarrow GF$, where $\eta_c : c \to GFc$ is defined as the transpose of $\mathrm{id}_{Fc} : Fc \to Fc$ along the isomorphism

$$\mathrm{Mor}_\mathcal{D}(Fc, Fc) \cong \mathrm{Mor}_\mathcal{C}(c, GFc)$$

promised by the adjunction.

Proof. We only have to check naturality. As such, fix some morphism $f : c \to c'$, and we need the following diagram to commute.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, "GFf"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd}$$

Note that id_c is the $f^\flat$ of id_{Fc}, so by Lemma 5.6, it suffices to check that the following diagram commutes.

$$\begin{tikzcd} Fc \arrow[r, "\mathrm{id}_{Fc}"] \arrow[d, "Ff"'] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "\mathrm{id}_{Fc'}"] & Fc' \end{tikzcd}$$

This commutes because look at it. ■

The η in the lemma is special.

Definition 5.15 (Unit)**.** Work in the context of Lemma 5.14. Then η is called the *unit*.

Our lemma also has the following dual.

Lemma 5.16. Fix adjunctions $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$. Then there exists a natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_\mathcal{D}$, where $\varepsilon_d : FGd \to d$ is defined as the transpose of $\mathrm{id}_{Gd} : Gd \to Gd$ along the isomorphism

$$\mathrm{Mor}_\mathcal{D}(FGd, d) \cong \mathrm{Mor}_\mathcal{C}(Gd, Gd)$$

promised by the adjunction.

Proof. Duality. ■

And here is our corresponding word.

Definition 5.17 (Counit)**.** Work in the context of Lemma 5.16. Then ε is called the *counit*.

We close class with an example.

Exercise 5.18. We compute units and counits for the free-forgetful adjoints between Set and Grp.

Proof. Our unit $\eta_X : X \to UFX$ sends an element $x \in X$ to $x \in UFX$, the length-one word. On the other hand, counit $\varepsilon_G : FUG \to G$ sends an element $g_1 \cdots g_n \in FUG$ made of letters of G to its evaluation in G. ■

5.2 April 13

We continue talking about adjoints.

5.2.1 More Examples

We recall the following definition.

Definition 5.2 (Adjunction)**.** Fix two categories $\mathcal{C}$ and $\mathcal{D}$. An *adjunction* is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with isomorphisms (of sets)

$$\mathrm{Mor}_{\mathcal{D}}(Fc, d) \cong \mathrm{Mor}_{\mathcal{C}}(c, Gd)$$

natural in both arguments. We will call F a *left adjoint* to G and G a *right adjoint* to F, notated $F \dashv G$.

From here we were able to define the unit and counit. Another way to view our construction last time is to apply the Yoneda lemma to the natural isomorphism

$$\mathrm{Mor}_{\mathcal{D}}(Fc, -) \cong \mathrm{Mor}_{\mathcal{C}}(c, G-)$$

of functors $\mathcal{D} \to \mathrm{Set}$. In particular, Theorem 3.17 grants us an object $\eta_c : \mathrm{Mor}_{\mathcal{C}}(c, GFc)$ representing this isomorphism, which assembles into our unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$.

Let's see some more examples.

Exercise 5.19. We discuss the product–hom adjunction in Set.

Proof. The point is that there is a natural bijection

$$\mathrm{Mor}(X \times Y, Z) \cong \mathrm{Mor}(X, \mathrm{Mor}(Y, Z)) \qquad (*)$$

by taking the function $f : X \times Y \to Z$ to the function $\widetilde{f}$ defined by $x \mapsto (y \mapsto f(x, y))$. To see that this is a bijection, there is an inverse taking $g \in \mathrm{Mor}(X, \mathrm{Mor}(Y, Z))$ by $\hat{g}(x, y) := g(x)(y)$, and we can check that these are inverses by hand.

This will assemble into an adjunction of the functors $F := - \times Y$ and $G := \mathrm{Mor}(Y, -)$. Thus, $(*)$ turns into a natural isomorphism

$$\mathrm{Mor}(FX, Z) \cong \mathrm{Mor}(X, GY),$$

which is what we need for $F \dashv G$.

We close our discussion by tracking through our unit and counit.

- For the unit, we need to transpose id_{Fc} through $\mathrm{Mor}_{\mathcal{D}}(Fc, Fc) \cong \mathrm{Mor}(c, GFc)$. In particular, we are tracking through $\mathrm{id}_{X \times Y}$ through

$$\mathrm{Mor}(X \times Y, X \times Y) \cong \mathrm{Mor}(X, \mathrm{Mor}(Y, X \times Y)).$$

  Thus, $\eta_X(x)(y)$ should be $\eta_X(x)(y) = \mathrm{id}_{X \times Y}(x, y) := (x, y)$ after moving everything through.

- For the counit, we need to transpose id_{Gd} through $\mathrm{Mor}_{\mathcal{D}}(FGd, d) \cong \mathrm{Mor}(Gd, Gd)$. In particular, we are tracking through $\mathrm{id}_{\mathrm{Mod}(Y,Z)}$ through

$$\mathrm{Mor}(\mathrm{Mod}(Y, Z) \times Y, Z) \cong \mathrm{Mor}(\mathrm{Mod}(Y, Z), \mathrm{Mod}(Y, Z)).$$

  Thus, we can find $\varepsilon_Z(f, y) := f(y)$, which finishes.

This finishes our discussion. ∎

Exercise 5.20. We discuss the hom–tensor adjunction for k-vector spaces.

Proof. It is false that

$$\operatorname{Hom}(V \times W, U) \cong \operatorname{Hom}(V, \operatorname{Hom}(W, U)).$$

This does not make sense because we don't really want to talk about linear maps $V \times W \to U$ but rather being bilinear in both arguments, so we want an isomorphism

$$\operatorname{Bilin}(V \times W, U) \cong \operatorname{Hom}(V, \operatorname{Hom}(W, U))$$

instead. Thus, using the universal property for tensor products, we would have

$$\operatorname{Hom}(V \otimes W, U) \cong \operatorname{Hom}(V, \operatorname{Hom}(W, U)).$$

We can compute that the unit is still $\varepsilon_U(f \otimes w) := f(w)$. Chris does not remember the counit precisely. ■

5.2.2 The Triangle Equations

We should probably prove something today, so let's prove something.

Proposition 5.21. Fix adjoint functors $F \dashv G$ between categories $\mathcal{C}$ and $\mathcal{D}$ with unit η and counit ε. Then we have the triangle equations.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G \end{tikzcd}$$

Proof. Expanding out on objects, our first triangle takes some $c \in \mathcal{C}$ and writes down the following.

$$\begin{tikzcd} Fc \arrow[r, "F(\eta_c)"] \arrow[dr, "\mathrm{id}_{Fc}"'] & FGFc \arrow[d, "\varepsilon_{Fc}"] \\ & Fc \end{tikzcd} \tag{1}$$

Namely, the top arrow is a whiskering. Similarly, the other triangle looks like the following on some $d \in \mathcal{D}$.

$$\begin{tikzcd} Gd \arrow[r, "\eta_{Gd}"] \arrow[dr, "\mathrm{id}_{Gd}"'] & GFGd \arrow[d, "G(\varepsilon_d)"] \\ & Gd \end{tikzcd} \tag{2}$$

To prove the result, we recall the following lemma.

Lemma 5.6. Fix functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ with given isomorphisms $\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$ for $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Then, naturality is equivalent to the following: one of the squares below commutes if and only if the other does, for any morphisms making the diagrams well-defined.

$$\begin{tikzcd} Fc \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & d \arrow[d, "k"] \\ Fc' \arrow[r, "g^\sharp"'] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[d, "h"'] & Gd \arrow[d, "Gk"] \\ c' \arrow[r, "g^\flat"'] & Gd' \end{tikzcd}$$

As such, we start with the following diagram. Note that the following diagram commutes because look at it.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "\eta_c"'] & GFc \arrow[d, "\mathrm{id}_{GFc}"] \\ GFc \arrow[r, "\mathrm{id}_{GFc}"'] & GFc \end{tikzcd}$$

Thus, Lemma 5.6 tells us that the following diagram commutes.

$$\begin{tikzcd} Fc \arrow[r, "\mathrm{id}_{Fc}"] \arrow[d, "F\eta_c"'] & Fc \arrow[d, "\mathrm{id}_{Fc}"] \\ FGFc \arrow[r, "\varepsilon_{Fc}"'] & Fc \end{tikzcd}$$

This is (1).

For the other one, we start with the following square which commutes because look at it.

$$\begin{tikzcd} FGd \arrow[r, "\mathrm{id}_{FGd}"] \arrow[d, "\mathrm{id}_{FGd}"'] & FGd \arrow[d, "\varepsilon_d"] \\ FGd \arrow[r, "\varepsilon_d"'] & d \end{tikzcd}$$

Again applying Lemma 5.6, we get the following commutative diagram.

$$\begin{tikzcd} FGd \arrow[r, "\eta_{Gd}"] \arrow[d, "\mathrm{id}_{Gd}"'] & GFGd \arrow[d, "G(\varepsilon_d)"] \\ FGd \arrow[r, "\mathrm{id}_{Gd}"'] & Gd \end{tikzcd}$$

This is (2). This finishes the proof. ■

As such, we have the following nice result for adjoints.

> **Theorem 5.22.** Fix functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ with natural transformations $\eta : \mathrm{id}_\mathcal{C} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_\mathcal{D}$ satisfying the triangles in Proposition 5.21. Then $F \dashv G$.

Proof. Given some $f^\sharp$ in $\mathcal{C}$, we set $f^\flat$ equal to $Gf^\sharp \circ \eta_c$; conversely, we send $g^\flat$ in $\mathcal{D}$ to $g^\sharp$ equal to $\varepsilon_d \circ Fg^\flat$. The point of showing that these commute is by drawing the following diagram.

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[drr, "\mathrm{id}_{Fc}"'] & FGFc \arrow[r, "FGf^\sharp"] \arrow[dr, "\varepsilon_{Fc}"] & FGd \arrow[r, "\varepsilon_d"] & d \arrow[dl, "f^\sharp"] \\ & & Fc & \end{tikzcd}$$

This diagram commutes by some effort, which will give the inverse conditions.

We can show that these are mutually inverse, and then they are natural in both arguments because of course they are. ■

5.3 April 15

5.3.1 Units and Counits Speed-run

We quickly recall that the transpose of $f : Fc \to d$ is $Gf \circ \eta_c$, and the transpose of $g : c \to Gd$ is $\varepsilon_d \circ Fg$.

We now continue the proof from last class.

Theorem 5.22. Fix functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ with natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ satisfying the triangles in Proposition 5.21. Then $F \dashv G$.

Proof. Suppose that we have functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ with η and ε satisfying the needed triangle equations.

We now define our adjunction by hand. Namely, we define

$$\varphi : \mathrm{Mor}_{\mathcal{D}}(Fc, d) \to \mathrm{Mor}_{\mathfrak{C}}(c, Gd)$$

by sending f to $Gf \circ \eta_c$ by hand. Similarly, we define

$$\psi : \mathrm{Mor}_{\mathcal{C}}(c, Gd) \to \mathrm{Mor}_{\mathcal{D}}(Gc, d)$$

by $\psi : g \mapsto \varepsilon_d \circ Fg$. To check that these are inverses, we see that

$$\psi\varphi(f) = \psi(Gf \circ \eta_c) = \varepsilon_d \circ FGf \circ F\eta_c.$$

Computing, we push ε_d through via the following triangle.

$$\begin{tikzcd} FGFc \arrow[r, "FGf"] \arrow[d, "\varepsilon_{Fc}"'] & FGd \arrow[d, "\varepsilon_d"] \\ Fc \arrow[r, "f"'] & d \end{tikzcd}$$

This commutes by naturality. As such, we find that

$$\psi\varphi(f) = f \circ \varepsilon_{Fc} \circ F\eta_c = f,$$

where in the last equality we have used the triangle equations.

For the other inverse, we compute

$$\varphi\psi(g) = \varphi(\varepsilon_d \circ Fg) = G\varepsilon_d \circ GFGg \circ \eta_c.$$

As such, we draw the following naturality square to push out η.

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[d, "\eta_c"'] & Gd \arrow[d, "\eta_{Gd}"] \\ GFc \arrow[r, "GFg"'] & GFGd \end{tikzcd}$$

Thus, we compute

$$\varphi\psi(g) = G\varepsilon_d \circ \eta_{Gd} \circ g = g,$$

where we once again finished by the triangle equations.

It remains to check naturality. We use Lemma 5.6, of which we show the other direction. Namely, we show that the left square below makes the right square commutes.

$$\begin{tikzcd} Fc \arrow[r, "f"] \arrow[d, "Fh"'] & d \arrow[d, "k"] \\ Fc' \arrow[r, "g"'] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\varphi f"] \arrow[d, "h"'] & Gd \arrow[d, "Gk"] \\ c' \arrow[r, "\varphi g"'] & Gd' \end{tikzcd}$$

The left square commuting gives $kf = gFf$, so

$$Gk \circ Gg \circ \eta_c = Gg \circ GFh \circ \eta_c \qquad (*)$$

by throwing through G and putting η_c on the right. The left-hand side is now $Gk \circ \varphi(f)$. On the other side, we draw the following naturality square.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "h"'] & GFc \arrow[d, "GFh"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd}$$

As such, our right-hand side of $(*)$ becomes

$$Gk \circ \varphi(f) = \varphi(g) \circ h,$$

which gives the commutativity of the desired square. ■

5.3.2 Morphism of Adjunctions

We now begin discussion. We start with the following definition.

Definition 5.23 (Morphism of adjunctions)**.** A *morphism of adjunctions* from $F \dashv G$ to $F' \dashv G'$ is a pair of functors $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to mcD'$ so that the following two diagrams commute.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, "K"'] & \mathcal{D}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{C}' \\ \mathcal{D} \arrow[r, "K"'] \arrow[u, "G"] & \mathcal{D}' \arrow[u, "G'"'] \end{tikzcd}$$

Additionally, we require one of the following (equivalent) conditions.

(a) $H\eta = \eta' H$.

(b) $K\varepsilon = \varepsilon' K$.

(c) The following diagram commutes.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{D}}(Fc, d) \arrow[r] \arrow[d, "K"'] & \operatorname{Mor}_{\mathcal{C}}(c, Gd) \arrow[d, "H"] \\ \operatorname{Mor}_{\mathcal{D}'}(KFc, Kd) \arrow[d, equal] & \operatorname{Mor}_{\mathcal{C}'}(Hc, HGd) \arrow[d, equal] \\ \operatorname{Mor}_{\mathcal{D}'}(F'Hc, Kd) \arrow[r] & \operatorname{Mor}_{\mathcal{C}'}(Hc, G'Kd) \end{tikzcd}$$

Here, ε, η are the unit/counit for $F \dashv G$ and similar for ε', η'.

We start by showing (a) implies (b). For this, we want to show $K\varepsilon_d = \varepsilon'_{Kd}$ for any $d \in \mathcal{D}$. We will show that

$$(K\varepsilon_d)^\sharp \stackrel{?}{=} (\varepsilon'_{Kd})^\sharp = \mathrm{id}_{G'Kd},$$

where we are transposing through $F' \dashv G'$. On the other hand, pushing through the sharp on the left-hand side, we note

$$(K\varepsilon_d)^\sharp = G'(K\varepsilon_d) \circ \eta'_{G'Kd} = G'K\varepsilon_d \circ \eta'_{HGd} \stackrel{*}{=} HG\varepsilon_d \circ H\eta_{Gd} = H(G\varepsilon_d \circ \eta_{Gd}),$$

which is what we want after applying the triangle inequality and using the naturality of H; notably, we used $H\eta = \eta' H$ in $\stackrel{*}{=}$.

We now show (b) implies (c). Suppose $K\varepsilon = \varepsilon' K$, and we want our rectangle to commute. We simply diagram-chase in Set: pick up some $f : Fc \to d$. Along the top, we track the following.

$$\begin{tikzcd} f \arrow[rr] & & Gf \circ \eta_c \arrow[d] \\ & & H(Gf \circ \eta_c) \arrow[d] \\ \varepsilon'_{Kd} \circ F'H(Gf \circ \eta_c) & & H(Gf \circ \eta_c) \arrow[ll] \end{tikzcd}$$

So we would like

$$Kf \stackrel{?}{=} \varepsilon'_{Kd} \circ F'H(Gf \circ \eta_c).$$

Well, we can compute

$$\begin{aligned} \varepsilon'_{Kd} \circ F'H(Gf \circ \eta_c) &= K\varepsilon_d \circ KF(Gf \circ \eta_c) \\ &= K(\varepsilon_d \circ Gf) \circ KF\eta_c \\ &= K(f \circ \varepsilon_{Fc}) \circ KF\eta_c, \end{aligned}$$

where in the last equality we used the naturality of ε as follows.

$$\begin{tikzcd} FGFc \arrow[r, "FGf"] \arrow[d, "\varepsilon_{Fc}"'] & FGd \arrow[d, "\varepsilon_d"] \\ Fc \arrow[r, "f"'] & d \end{tikzcd}$$

Continuing to rearrange, we see that we have

$$Kf \circ K(\varepsilon_{Fc} \circ F\eta_c) = Kf,$$

where we have used the triangle equalities.

5.4 April 18

Bryce's advisor will be giving next week's lectures. So it's time to speed-run adjunctions.

5.4.1 Contravariant Adjoints

Let's discuss trying to use contravariant functors to make adjunctions. We might start with two contravariant functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, but these are not compatible because they don't go both ways properly. As such, we want to turn G into a functor

$$\mathcal{D} \to \mathcal{C}^{\mathrm{op}}.$$

Indeed, functors $\mathcal{A} \to \mathcal{B}$ can become functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ by just reversing all the arrows, so we can indeed view G as a functor $G : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$, as desired.

Now, to make out adjoint, we might just try to require the isomorphism

$$\mathrm{Mor}_{\mathcal{D}}(Fc, d) \cong \mathrm{Mor}_{\mathcal{C}}^{\mathrm{op}}(c, Gd) = \mathrm{Mor}_{\mathcal{C}}(Gd, c),$$

but now both functors are on the left side, so this is a little weird. Nonetheless, we have the following definition.

Definition 5.24 (Mutually adjoint)**.** Two contravariant functors $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ and $G : \mathcal{D}^{\text{op}} \to \mathcal{C}$ (thought of as a functor $\mathcal{D} \to \mathcal{C}^{\text{op}}$) are *mutually left adjoint* if and only if there are natural isomorphisms

$$\operatorname{Mor}_{\mathcal{D}}(Fc, d) \cong \operatorname{Mor}_{\mathcal{C}}(Gd, c).$$

They are *mutually right adjoint* if and only if there are natural isomorphisms

$$\operatorname{Mor}_{\mathcal{D}}(d, Fc) \cong \operatorname{Mor}_{\mathcal{C}}(c, Gd).$$

Exercise 5.25. Fix $\mathcal{P} : \text{Set}^{\text{op}} \to \text{Set}$ sending $A \mapsto \mathcal{P}(A)$ and $f \mapsto f^{-1}$. We claim that $\mathfrak{P}$ is mutually right adjoint with itself.

Proof. We are requiring a natural isomorphism

$$\operatorname{Mor}_{\text{Set}}(Y, \mathcal{P}X) \stackrel{?}{\cong} \operatorname{Mor}_{\text{Set}}(X, \mathcal{P}Y).$$

The main point is that Set is Cartesian-closed, roughly meaning that we can curry as

$$\operatorname{Mor}_{\text{Set}}(X \times Y, Z) \cong \operatorname{Mor}_{\text{Set}}(X, \operatorname{Mor}(Y, Z)).$$

Now, recall that $\mathfrak{P}$ is represented by $\Omega := \{\text{T}, \text{F}\}$, in that $\mathcal{P} \cong \operatorname{Mor}_{\text{Set}}(-, \Omega)$. Thus,

$$\begin{aligned} \operatorname{Mor}_{\text{Set}}(X, \mathcal{P}Y) &\cong \operatorname{Mor}_{\text{Set}}(X, \operatorname{Mor}(Y, \Omega)) \\ &\cong \operatorname{Mor}_{\text{Set}}(X \times Y, \Omega) \\ &\cong \operatorname{Mor}_{\text{Set}}(Y \times X, \Omega) \\ \operatorname{Mor}_{\text{Set}}(Y, \operatorname{Mor}(X, \Omega)) \\ \operatorname{Mor}_{\text{Set}}(Y, \mathcal{P}X), \end{aligned}$$

and everything is natural, so we are done. ■

5.4.2 Uniqueness of Adjoints

We take a moment clean up after ourselves and quickly justify why we have been saying "the adjoint."

Proposition 5.26. Fix functors F, F' which are both left adjoint to G. Then there is a unique natural isomorphism $\theta : F \cong F$ such that the following two triangles commute.

$$\begin{tikzcd} \mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & GF \arrow[d, Rightarrow, "G\theta"] \\ & GF' \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, Rightarrow, "\theta G"'] & \mathrm{id}_{\mathcal{D}} \\ F'G \arrow[ur, Rightarrow, "\varepsilon'"'] & \end{tikzcd}$$

Proof. We start by exhibiting θ, which we do by hand. As such, we define

$$\theta := \varepsilon F' \circ F\eta' \qquad \text{and} \qquad \theta' := \varepsilon' F \circ F'\eta.$$

We show that these are inverse by hand; note that they are natural transformations as composition and whiskering of natural transformations. We can type-check that $F\eta' : F \Rightarrow FGF$ and then $\varepsilon F' : FGF' \Rightarrow F'$ and similar for θ'.

To show these are inverse, we show that

$$\theta'\theta \stackrel{?}{=} \mathrm{id}_F,$$

which we do on components as

$$(\theta'\theta)_c = \mathrm{id}_c$$

for some object c. To do this, we show that the transposes are equal, for which we compute the transpose $(\theta'\theta)_c$ as

$$G(\theta'\theta)\circ\eta_c = G\varepsilon'F\circ GF'\eta\circ G\varepsilon F'\circ GF\eta'\circ\eta,$$

where we have dropped the c out of laziness. We now use naturality. Namely, we see

$$\begin{tikzcd} \mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta'"'] & GF \arrow[d, Rightarrow, "GF\eta'"] \\ GF' \arrow[r, Rightarrow, "\eta GF'"'] & GFGF' \end{tikzcd}$$

gives

$$G\varepsilon'F\circ GF'\eta\circ G\varepsilon F'\circ(GF\eta'\circ\eta) = G\varepsilon'F\circ GF'\eta\circ G\varepsilon F'\circ\eta GF'\circ\eta'.$$

Using the triangle equalities, we note that $G\varepsilon F'\circ\eta GF' = (G\varepsilon\circ\eta G)F'$ vanishes,[1] so we are left with

$$G\varepsilon'F\circ GF'\eta\circ\eta'.$$

Using naturality in the same way as last time but changing the primes around, we get

$$G\varepsilon'F\circ GF'\eta\circ\eta,$$

which again by the triangle equalities simply vanishes into η. So indeed, the transpose of $(\theta'\theta)_c$ is η_c, which is the transpose of id_c.

We now show that our triangles commute as needed. For the left triangle, we compute

$$G\theta\circ\eta = G(\varepsilon F'\circ F\eta')\circ\eta = G\varepsilon F'\circ GF\eta'\circ\eta.$$

By the naturality from before, we have $GF\eta'\circ\eta = \eta GF'\circ\eta'$, which gives

$$G\varepsilon F'\circ\eta GF'\circ\eta' = (G\varepsilon\circ\eta G)\circ\eta' = \eta'$$

where we have used the triangle equalities. The other triangle follows similarly, which we omit.

It remains to show uniqueness of θ. Pick up an object c, and we show that $\theta_c : Fc\to F'c$ is unique. Well, to make the left triangle commute, we need the following triangle to commute.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "\eta'_c"'] & GFc \arrow[d, "G\theta_c"] \\ & GF'c \end{tikzcd}$$

Now, we can compute the transpose of $\theta_c : Fc\to F'c$ (through $F\dashv G$) as $G\theta_c\circ\eta_c$, which is η'_c from the above triangle. As such, θ_c must be the transpose of η'_c, which is exactly how we constructed θ to begin with anyway. ■

5.4.3 Composing Adjunctions

Adjunctions can also create more adjunctions, provided they cohere.

[1] Namely, $G\varepsilon F'\circ\eta G$ is an identity natural transformation, so whiskering with F' on the right gives the morphism $\mathrm{id}_{F'(-)}$, which does indeed vanish

Proposition 5.27. Fix $F \dashv G$ and $F' \dashv G'$ in the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "G'"] \end{tikzcd}$$

Then $F'F \dashv GG'$.

Proof. We compute

$$\begin{aligned} \operatorname{Mor}_{\mathcal{E}}(F'Fc, e) &\cong \operatorname{Mor}_{\mathcal{D}}(Fc, G'e) \\ &\cong \operatorname{Mor}_{\mathcal{C}}(c, GG'e) \end{aligned}$$

by tracking through our various adjunctions. This finishes. ∎

And here is the last thing we will prove today.

Proposition 5.28. Fix an equivalence of categories $F : \mathcal{C} \simeq \mathcal{D} : G$ with its promised natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : \mathrm{id}_{\mathcal{D}} \Rightarrow FG$. We can replace these with an adjoint equivalence by modifying either one of δ or ε (notably not changing F and G!).

Proof. This proof is, reportedly, a little long. Fix η and we modify ε. (For the other result, swap F and G and ε and η.) Now, we fix

$$\gamma := G\varepsilon \circ \eta G$$

which is a natural transformation from $G \Rightarrow G$ (namely, $\eta G : G \Rightarrow GFG$ and $G\varepsilon : GFG \Rightarrow G$). Notably, this is a natural isomorphism because $G\varepsilon$ and ηG are both isomorphisms at each unit.

Our goal is to kill γ to fix our triangle equalities. As such, we set

$$\varepsilon' := \varepsilon \circ F\gamma^{-1},$$

which is now a natural isomorphism $FG \cong \mathrm{id}_{\mathcal{D}}$ by a similar computation to before: we send $FG \Rightarrow FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

We now check our triangle equalities. Note that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[d, Rightarrow, "\gamma^{-1}"'] & GFG \arrow[d, Rightarrow, "GF\gamma^{-1}"] \\ G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "\gamma"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G \end{tikzcd}$$

The bottom triangle commutes by definition of γ, and the square commutes by naturality of ηG—simply chase an element all the way through. As such, we can collapse this diagram down to the following.

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal, "\gamma\circ\gamma^{-1}"'] & GFG \arrow[d, Rightarrow, "G\varepsilon'"] \\ & G \end{tikzcd}$$

This is our first triangle inequality, so we are done.

For the second triangle inequality, we draw the following huge diagram.

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "F\eta"'] & FGF \arrow[r, Rightarrow, "\varepsilon' F"] \arrow[d, Rightarrow, "FGF\eta"] & F \arrow[d, Rightarrow, "F\eta"] \\
FGF \arrow[r, Rightarrow, "F\eta GF"] \arrow[dr, equal] & FGFGF \arrow[r, Rightarrow, "\varepsilon' FGF"] \arrow[d, Rightarrow, "FG\varepsilon' F"] & FGF \arrow[d, Rightarrow, "\varepsilon' F"] \\
& FGF \arrow[r, Rightarrow, "\varepsilon' F"'] & F
\end{tikzcd}$$

We quickly review why this diagram commutes.

- The top-left square commutes by the naturality of η applied to a square of Fs.
- The top-right square commutes by naturality of ε' applied to a square of Fs.
- The bottom-right square commutes by the naturality of ε' applied to a square of Fs.
- The bottom-left triangle commutes by the triangle equality we showed for ε' already.

This diagram now collapses to

$$(\varepsilon' F \circ F\eta)^2 = \varepsilon' F \circ F\eta,$$

which forces $\varepsilon' F \circ F\eta = \mathrm{id}$ because these are isomorphisms, which is what we wanted. ■

5.5 April 20

Chris is back, for the last time.

5.5.1 Adjoints Preserve (Co)limits

Here is a theorem.

Theorem 5.29. Fix a category $\mathcal{C}$.

- $\mathcal{C}$ admits all limits of shape $\mathcal{J}$ if and only if the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ admits a right adjoint. Here, the right adjoint is the $\lim$ functor.
- Dually, $\mathcal{C}$ admits all colimits of shape $\mathcal{J}$ if and only if the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ admits a left adjoint. Here, the left adjoint is the colim functor.

Proof. Here is the image.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta" description] \arrow[r, phantom, shift left=3, "\top"] \arrow[r, phantom, shift right=3, "\bot"] & \mathcal{C}^{\mathcal{J}} \arrow[l, bend right=50, "\operatorname{colim}"'] \arrow[l, bend left=50, "\lim"]
\end{tikzcd}$$

We now recall the following theorem.

Proposition 4.101. Fix a category $\mathcal{C}$ with all limits of shape $\mathcal{J}$. Upon choosing a limit and associated limit cone for each diagram $\mathcal{J} \to \mathcal{C}$ defines the action of objects of a functor

$$\lim_{\mathcal{J}} : \operatorname{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}.$$

As such, we start by proving the forward direction of our theorem. Namely, we are promised a functor

$$\lim\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}.$$

We check that this is the right adjoint of Δ: we merely have to exhibit isomorphisms

$$\mathrm{Mor}_{\mathcal{C}}(c, \lim F) \stackrel{?}{\cong} \mathrm{Mor}_{\mathcal{C}^{\mathcal{J}}}(\Delta c, F).$$

Now, $\mathrm{Mor}_{\mathcal{C}^{\mathcal{J}}}(\Delta c, F)$ denotes natural transformations from c to F, which are just cones over F with apex c. As such, we have

$$\mathrm{Mor}_{\mathcal{C}}(c, \lim F) \stackrel{*}{\cong} \mathrm{Cone}(c, F) \cong \mathrm{Mor}_{\mathcal{C}^{\mathcal{J}}}(\Delta c, F),$$

where $\stackrel{*}{\cong}$ is because $\lim F$ represents the functor $\mathrm{Cone}(-, F)$. This is natural because look at it.

Now work in the other direction. Suppose that $R : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ is right adjoint to Δ. But then

$$\mathrm{Mor}(c, RF) \cong \mathrm{Cone}(c, F)$$

naturally, so RF represents $\mathrm{Cone}(c-, F)$, so RF will be a limit of the diagram F. In fact, we can see that R must now be the limit functor because we showed the limit functor was a right adjoint, and we know limits are unique. ■

Here is our main theorem.

Theorem 5.30. Right adjoints preserve limits.

Proof by diagrams. More rigorously, suppose that $F \dashv G$ are functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. Then, for a diagram $K\colon \mathcal{K} \to \mathcal{D}$, if the limit $\lim K$ exists, then

$$G(\lim K) = \lim GK.$$

We will do a proof by diagrams. To be explicit, we pick our limit cone $\lambda : \lim K \Rightarrow K$. Hitting everything with G, we have the following diagram.

$$\begin{tikzcd} \lim K \arrow[d, Rightarrow] \arrow[r, phantom, "\xrightarrow{G}"] & G \lim k \arrow[d, Rightarrow] \\ K & GK \end{tikzcd}$$

In particular, our legs $\lambda_i : \lim K \to Ki$ become $G\lambda_i : G \lim K \to GKi$, which assembles into a cone $G\lambda : G \lim K \Rightarrow GK$ by whiskering.

We claim that $G\lambda$ is a limit cone. As such, we pick up a cone $\mu\colon c \Rightarrow GK$, and we want a unique map $c \to G \lim K$ commuting with our legs. To use our adjoint, we transpose everything.

Quote 5.31. It's good notation. Fuck you.

In particular, we hope that we have cones as follows.

$$\begin{tikzcd} \lim K \arrow[r, "!"] \arrow[dr, Rightarrow] & Fc \arrow[d, Rightarrow, "\mu_j^\sharp"] \\ & K \end{tikzcd}$$

In particular, we will now check that $\mu^\sharp :: Fc \Rightarrow K$ is actually a cone. To check this, we invoke Lemma 5.6. Indeed, because $G\lambda\colon G \lim K \to GK$ is a cone, the following diagram commutes.

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}_c"] \arrow[d, "\mu_i"'] & c \arrow[d, "\mu_{i'}"] \\ GKi \arrow[r, "GKf"'] & GKi' \end{tikzcd}$$

Transposing, the following diagram commutes.

$$\begin{tikzcd} Fc \arrow[r, "\mathrm{id}_{Fc}"] \arrow[d, "\mu_i^\sharp"'] & Fc \arrow[d, "\mu_{i'}^\sharp"] \\ Ki \arrow[r, "Kf"'] & Ki' \end{tikzcd}$$

So indeed, $\mu^\sharp$ is a cone, so we have a unique map $\tau\colon Fc \to \lim K$ commuting with our cones. So we get a map

$$\tau^\flat c \to\colon G \lim K.$$

Everything commutes because we can transpose triangles back as squares (using Lemma 5.6: do the above diagram argument in reverse), so we have indeed constructed the needed map.

It remains to show that our $\tau^\flat$ is unique. Well, given any $\sigma\colon c \to G\lim K$, we can transpose back to show $\sigma^\sharp$ must match up with τ, which we then bring back to say $\sigma = \tau^\flat$. This finishes. ■

Proof by Yoneda. We can also the Yoneda lemma as follows. Write

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(c, G \lim_{\mathcal{J}} K) &\cong \mathrm{Mor}_{\mathcal{C}}(Fc, \lim_{\mathcal{J}} K) \\ &\overset{*}{\cong} \lim_{\mathcal{J}} \mathrm{Mor}_{\mathcal{D}}(Fc, K) \\ &\cong \lim_{\mathcal{J}} \mathrm{Mor}_{\mathcal{C}}(c, GK) \\ &\cong \mathrm{Mor}_{\mathcal{C}}(c, \lim GK), \end{aligned}$$

which finishes. Notably, $\overset{*}{\cong}$ is the universal property of the limit; this is how limits behave in Set. ■

We also have the following result.

Corollary 5.32. Left adjoints preserve colimits.

Proof. Duality. ■

Quote 5.33. I only lecture properly when I am harassed.

5.5.2 Whiskering

We close class with the following result.

Theorem 5.34. Fix an adjunction $F \dashv G$. If $\mathcal{J}$ is small and $\mathcal{E}$ is locally small, then we have adjunctions as follows.

$$\begin{tikzcd} \mathcal{C}^{\mathcal{J}} \arrow[r, shift left, "F\circ -"] \arrow[r, phantom, "\perp"] & \mathcal{D}^{\mathcal{J}} \arrow[l, shift left, "G\circ -"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\mathcal{C}} \arrow[r, shift left, "-\circ F"] \arrow[r, phantom, "\perp"] & \mathcal{E}^{\mathcal{D}} \arrow[l, shift left, "-\circ G"] \end{tikzcd}$$

Proof. We will show one of these. We start by writing out the triangle identities as follows.

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\varepsilon G"] \arrow[dr, equal] & GFG \arrow[d, "G\eta"] \\ & G \end{tikzcd}$$

As such, we can build units and counits by hand. Indeed, we set $\hat\eta : \mathrm{id}_{\mathcal{D}^{\mathcal{J}}} : (G\circ -)(F\circ -)$ by sending K to ηK. Similarly, we set $\hat\varepsilon : (F\circ -)(G\circ -) \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathcal{J}}}$ by $K \to \varepsilon K$. Drawing the internal diagram for one of the triangle identities shows that they commute. ■

5.6 April 22

It's time to get started, but it is discussion section, for some definition of discussion.

5.6.1

We pick up the following lemma.

Lemma 5.35. Let $F \dashv G$ be an adjunction between categories $\mathcal{C}$ and $\mathcal{D}$, with unit η and counit ε. Then G is faithful/full/both if and only if each $d \in \mathcal{D}$ has ε_d is an epimorphism/split monomorphism/isomorphism.

Proof. We go one at a time.

(i) Fix $d, d' \in \mathcal{D}$ with parallel morphisms $f, g \colon d \to d'$. The main point is to look at

$$f \circ \varepsilon_d, g \circ \varepsilon_d.$$

Passing to the transpose, we go to $(f\varepsilon_d)^\sharp = G(f\varepsilon_d)\eta_{Gd}$ by our discussion of units and things. Distributing this is

$$(f\varepsilon_d)^\sharp = Gf$$

by applying the triangle equalities. Similarly, $(g\varepsilon_d)^\sharp = Gg$, so $f\varepsilon_d = g\varepsilon_d$ if and only if $Gf = Gg$. Thus, if ε_d is an epimorphism, this means $f = g$ if and only if $Gf = Gg$. Similarly, if G is faithful, then $f = g$ if and only if $Gf = Gg$ if and only if $f\varepsilon_d = g\varepsilon_d$, so ε_d is an epimorphism.

(ii) We proceed by force. Suppose G is full. We would like $\varepsilon_d \colon FGd \to d$ to be a split monomorphism, so we need a retraction $r \colon d \to FGd$ so that $r\varepsilon)d = \mathrm{id}_{FGd}$.

As such, we simply pick up $\eta_{Gd} \colon Gd \to GFGd$, but G is full, so we can lift this to some morphism $r \colon d \to FGd$ so that $Gr = \eta_{Gd}$. We now show

$$r\varepsilon_d = \mathrm{id}_{FGd}.$$

Taking the transpose, we are showing

$$(r\varepsilon_d)^\flat = G(r\varepsilon_d)\eta_{Gd} \stackrel{*}{=} Gr = \eta_{Gd}$$

where we are as usual using the triangle inequalities in $\stackrel{*}{=}$. Transposing back finishes.

In the other direction, suppose that each of the ε_d are split monomorphisms. For each $d \in \mathcal{D}$, we are promised a retraction $r_d \colon d \to FGd$ so that $r_d\varepsilon_d = \mathrm{id}_{FGd}$.

We now lift by hand: fix $d, d' \in \mathcal{D}$ with $f \colon Gd \to Gd'$, and we want to lift it by hand. Namely, define

$$g := f^\sharp r_d = \varepsilon_d \circ Ff \circ r_d,$$

which we can see is a map $d \to d'$. We don't want to hit this with G directly because GFf is sad; roughly speaking, we want to put ε_d on the other side of Ff. As such, we compute

$$(Gg)^\sharp = \varepsilon_{d'} \circ FGg = \varepsilon_{d'} \circ FG\varepsilon_{d'} \circ FGFf \circ FGr_d.$$

Now, we use the naturality of the following square.

$$\begin{tikzcd}
FGFGd \arrow[r, "FG\varepsilon d'"] \arrow[d, "\varepsilon FGd'"'] & FGd' \arrow[d, "\varepsilon d'"] \\
FGd' \arrow[r, "\varepsilon d'"'] & d'
\end{tikzcd}$$

This gives us

$$\varepsilon_{d'} \circ \varepsilon_{FGd'} \circ FGFf \circ FGr_d.$$

Next, we use naturality of another square, as follows.

$$\begin{tikzcd} FGFGd \arrow[r, "FGFf"] \arrow[d, "\varepsilon FGd"'] & FGFGd' \arrow[d, "\varepsilon FGd"] \\ FGd \arrow[r, "Ff"'] & FGd' \end{tikzcd}$$

This gives

$$\varepsilon_d \circ Ff \circ \varepsilon_{FGd} \circ FGr_d.$$

Moving things over, we want to put the retraction on the other side, so we draw the following naturality square.

$$\begin{tikzcd} FGd \arrow[r, "FGr_d"] \arrow[d, "\varepsilon_d"'] & FGFGd \arrow[d, "\varepsilon FGd"] \\ d \arrow[r, "r_d"'] & FGd \end{tikzcd}$$

In total, we are left with

$$\varepsilon_{d'} \circ Ff \circ r_d \circ \varepsilon_d$$

which retracts properly to $f^\sharp$

(iii) This follows from adding together (i) and (ii) because isomorphisms are the same as being epic and split monic. ■

Remark 5.36. There is also the following dual statement for F. Namely, F is faithful/full/both if and only if each $d \in \mathcal{D}$ has ε_d is a monomorphism/split epimorphism/isomorphism. In particular, we can pass to the opposite category to change our adjoints.

5.6.2 The Category of Categories

We would like to understand the category of (small) categories. We pick up the following definition.

Definition 5.37. A *reflective* subcategory $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ such that there is a left adjoint $L\colon \mathcal{C} \to \mathcal{D}$ of the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$. This L is called the *reflector* or the *localization*.

The point is that we have a reflector L which gives us a fairly natural way to fix $\mathcal{D}$: the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful, so each of the counit morphisms $\varepsilon_d\colon L_i d \to d$ are all isomorphisms. In other words, for each $d \in \mathcal{D}$, we essentially have

$$Ld \cong \mathrm{id}_{\mathcal{D}}$$

by viewing $d \in \mathcal{C}$ via the embedding.

Example 5.38. The embedding $\iota\colon \mathrm{Ab} \to \mathrm{Grp}$ gives a full subcategory. This is reflective, with its reflector L taking a group G to its abelianization $G/[G,G]$. We won't check that this is a left adjoint, but it follows because mapping into an abelian group is the same thing as mapping from the abelianization.

Example 5.39. The embedding $\mathrm{Sh}(X) \hookrightarrow \mathrm{PSh}(X)$ gives a full subcategory, and its reflector is sheafification.

Exercise 5.40. There is an embedding of $N\colon \mathrm{Cat} \hookrightarrow \mathrm{sSet}$. Here, sSet is the set of presheaves on Δ, where Δ is the "simplex" category with the following data.

- The objects of Δ are sets $[n] \coloneqq \{0, 1, 2, \ldots, n\}$.
- The morphisms of Δ are non-decreasing maps $[n] \to [m]$.

Approximately speaking, the objects of Δ look like n-splices. This embedding makes a reflective subcategory.

Proof. We first describe the embedding. Fix a category $\mathcal{C}$. For notation, given an object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ and let $X_n \coloneqq X([n])$. The way that we are going to embed is by

$$N(\mathcal{C})_n \coloneqq \mathrm{Fun}([n], \mathcal{C}).$$

The point is that we have a left adjoint $h\colon \mathrm{sSet} \to \mathrm{Cat}$ by just restricting down to the 0th component for objects and the 1st component for morphisms. ■

As such, we have the following result.

Proposition 5.41. Suppose that we have an inclusion $\iota\colon \mathcal{D} \to \mathcal{C}$ making $\mathcal{D}$ a reflective subcategory.

(i) The inclusion ι creates limits (that $\mathcal{C}$ admits).

(ii) $\mathcal{D}$ has all colimits that $\mathcal{C}$ admits by applying the reflector.

The point is that understanding $\mathcal{C}$ will give us understanding of $\mathcal{D}$.

Proof. We proceed as follows.

(i) This follows by muttering something about monads.

(ii) We will actually show this. Let $F\colon \mathcal{J} \to \mathcal{D}$ be a diagram such that $\mathcal{C}$ has colimits of shape $\mathcal{J}$. Now, let $\lambda\colon \iota F \Rightarrow c$ be a colimit cone in $\mathcal{C}$, and we note that $L\lambda\colon L\iota F \Rightarrow Lc$ is a colimit cone in $\mathcal{D}$ because left adjoints preserve colimits. However, $L\iota F \cong F$, so we have found our colimit. ■

Corollary 5.42. The category Cat is complete and cocomplete.

Proof. By synthesizing the above discussion, we see that it suffices to show that

$$\mathrm{sSet} = \mathrm{Psh}(\Delta) = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$$

is complete and cocomplete. However, Set is complete and cocomplete, and we know how to compute limits and colimits in functor categories (namely, pointwise) pulling from their codomain. This finishes. ■

THEME 6

KAN EXTENSIONS

I felt profoundly stupid in that moment and he has a PhD in SYNTAX

—Beth Piatote, [Pia]

6.1 April 25

My computer charge is low, so let's see how far we go.

Remark 6.1. Peter Haine took the class when Professor Riehl was writing the book.

6.1.1 Motivation

We are doing an invitation to Kan extensions.

Roughly speaking, fix a functor $K : \mathcal{C} \to \mathcal{D}$ and another category $\mathcal{E}$. Then the precomposition functor

$$(- \circ K)\colon\ \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

preserves all co/limits admitted by $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$. Namely, if $\mathcal{E}$ has all limits and colimits, then $(- \circ K)$ will preserve them. As such, here is our idea.

Idea 6.2. If $\mathcal{E}$ above has all limits and colimits, then we expect the functor $(- \circ K)$ to have both left and right adjoints.

One can often check this by hand, but the point of this week's lecture is to discuss if or when the above is true and to be able to describe the adjoints.

Exercise 6.3. Fix a group G and subgroup $H \subseteq G$; let $\mathrm{B}G$ and $\mathrm{B}H$ be the one-object categories for G and H, respectively. Then we fix a field K and set

$$\mathrm{Rep}_K(G) := \mathrm{Fun}(\mathrm{B}G, \mathrm{Vec}_K)$$

to be the K-representations of the group G; define $\mathrm{Rep}_K(H)$ similarly. We talk through Idea 6.2 here.

Proof. The inclusion $\iota \colon \mathrm{B}H \to \mathrm{B}G$ gives the map

$$\mathrm{res}_H^G \colon \mathrm{Rep}_K(G) \to \mathrm{Rep}_K(H)$$

by precomposition. The adjoints are as follows.

$$\begin{tikzcd} \mathrm{Rep}_K(G) \arrow[r] & \mathrm{Rep}_K(H) \arrow[l, bend right, "\mathrm{Ind}_H^G"'] \arrow[l, bend left, "\mathrm{Coind}_H^G"] \end{tikzcd}$$

These have names from algebra, but that is all that we will say. ■

Anyway, let's say explicitly what we are doing for the rest of the week.

1. We will work through Idea 6.2.
2. We will discuss Kan extensions, which answer Idea 6.2 and in particular give formulae for the inverses.
3. Then we will explain that all concepts are Kan extensions.
4. And lastly, we will get some fun theorems.

For fun, let's give a theorem to whet our appetite.

Theorem 6.4 (Universal property of presheaves)**.** Fix a small category $\mathcal{C}$ so that we can fix the presheaf category $\mathrm{PSh}(\mathcal{C}) := \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$; further, take a category $\mathcal{E}$ with all colimits. Then the restriction of $よ \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ defines an equivalence

$$(- \circ よ) \colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}), \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

Namely, a functor from $\mathcal{C}$ to $\mathcal{E}$ has the same data as a functor from $\mathrm{PSh}(\mathcal{C})$ to $\mathcal{E}$ which preserves colimits.

Intuitively, what is happening is that $\mathrm{PSh}(\mathcal{C})$ is the "free category" preserving colimits. To see this, note that the free-forgetful adjunction from, say, Ab to Set says that

$$\mathrm{Mor}_{\mathrm{Ab}}(\mathbb{Z}[S], G) \cong \mathrm{Mor}_{\mathrm{Set}}(S, G),$$

so now the analogy is a bit clearer.

With all that motivation said, here is what we will do for the rest of the week.

- Today we will introduce Kan extensions.
- On Wednesday, we will talk through the formula for computations involving Kan extensions.
- On Friday, we will explain why all concepts are Kan extensions.

6.1.2 Kan Extensions

So let's talk about Kan extensions. Here is our motivating question.

Question 6.5. Given a functor $K \colon \mathcal{C} \to \mathcal{D}$, how can extend K (covariantly) to $\mathrm{PSh}(\mathcal{C})$?

Here is the image for this.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "K"] \arrow[d, "よ_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "よ_{\mathcal{D}}"] \\ \mathrm{PSh}(\mathcal{C}) \arrow[r, dashed] & \mathrm{PSh}(\mathcal{D}) \end{tikzcd}$$

We would like to draw in the dashed arrow. In general, there need not be a good way to do this, and it need not be unique—in fact, we will have a handedness to our choice of arrow.

As such, here is our definition.

Definition 6.6 (Kan extension)**.** Fix functors $K\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$. A *left Kan extension* of F along K is a functor $\operatorname{Lan}_K F\colon \mathcal{D} \to \mathcal{E}$ and a natural transformation $\eta\colon F \Rightarrow \operatorname{Lan}_K F \circ K$. We also require the data to be "initial" in the following sense: for any other pair $(G\colon \mathcal{D} \to \mathcal{E}, \gamma\colon F \Rightarrow GK)$, we require γ to factor uniquely through η, as follows.

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dd, "K"'] \arrow[dr, phantom, "\Downarrow\eta"] & & \mathcal{E} \\
& {} \arrow[d, phantom, "\overset{!}{\Rightarrow}"] & \\
\mathcal{D} \arrow[uurr, bend left=20, "\operatorname{Lan}_K F"] \arrow[uurr, bend right=40, "G"'] & {} &
\end{tikzcd}
$$

A *right Kan extension* is the same, flipping all the natural transformation arrows: our data are the terminal pair $(\operatorname{Ran}_K F\colon \mathcal{D} \to \mathcal{E}, \eta\colon (\operatorname{Lan} K_F)K \Rightarrow F)$.

Let's see an example.

Exercise 6.7. Fix an object $c \in \mathcal{C}$ to make the pair of functors $c\colon 1 \to \mathcal{C}$ and $*\colon 1 \to \mathrm{Set}$. We describe $\operatorname{Lan}_c *$ using the Yoneda lemma.

Proof. The main point is that, given a functor $G\colon \mathcal{C} \to \mathrm{Set}$, Theorem 3.17 gives us a bijection

$$\psi_\bullet\colon Fc \cong \operatorname{Fun}(\operatorname{Mor}_{\mathcal{C}}(c, -), F).$$

In particular, a natural transformation from $*\colon 1 \to \mathrm{Set}$ to $Fc\colon 1 \to \mathrm{Set}$ consists exactly of the data of a natural transformation from $\operatorname{Mor}_{\mathcal{C}}(c, -)$ to F. In total, we are promised a unique natural transformation ψ_c to make the following diagram commute.

$$
\begin{tikzcd}
1 \arrow[rr, "*"] \arrow[dd, "c"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_c"] & & \mathrm{Set} \\
& {} \arrow[d, phantom, "\overset{\psi_c}{\Rightarrow}"] & \\
\mathcal{C} \arrow[uurr, bend left=20, "\operatorname{Mor}_{\mathcal{C}}(c,-)"] \arrow[uurr, bend right=40, "F"'] & {} &
\end{tikzcd}
$$

Thus, $\operatorname{Lan}_c *$ is $\operatorname{Mor}_{\mathcal{C}}(c, -)$. ■

Anyway, it's a math class, so let's prove something today.

Proposition 6.8. Fix a functor $K\colon \mathcal{C} \to \mathcal{D}$ and category $\mathcal{E}$. If all functors $F\colon \mathcal{C} \to \mathcal{E}$ have Kan extensions, then $\operatorname{Lan}_K(-)$ is a left adjoint of $(- \circ K)$.

$$
\begin{tikzcd}
\operatorname{Fun}(\mathcal{C}, \mathcal{E}) \arrow[r, shift left, "\operatorname{Lan}_K(-)"] \arrow[r, phantom, "\perp"] & \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \arrow[l, shift left, "(-\circ K)"]
\end{tikzcd}
$$

Proof. It suffices to provide the natural bijection

$$\operatorname{Mor}(F, GK) \overset{?}{\cong} \operatorname{Mor}(\operatorname{Lan}_K F, G).$$

Well, by the universal property of $\operatorname{Lan}_K F$, natural transformations $\gamma\colon F \Rightarrow GK$ are in bijection with natural transformations $\gamma'\colon \operatorname{Lan}_K F \Rightarrow G$ such that

$$\gamma = \gamma'\eta,$$

where $\eta\colon F \Rightarrow \operatorname{Lan}_K F$. From this bijection one can turn $\operatorname{Lan}_K(-)$ into a functor by extending the action on objects. ■

Next class we will attempt to discuss when we can remove the seemingly strong condition on having all Kan extensions.

6.2 April 27

We continue our discussion of Kan extensions.

6.2.1 Kan Extensions for Embeddings

Our goal for today is to give a formula for Kan extensions (to be able to determine when they exist), which we saw utility for last class.

Exercise 6.9. There is an order-preserving map/functor between the poset categories

$$2^\bullet\colon \mathbb{Q} \to \mathbb{R}.$$

We extend this functor to all of $\mathbb{R}$ by continuity. Namely, we want to induce the dashed arrow in the following diagram.

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "2^\bullet"] \arrow[d, hook] & \mathbb{R} \\ \mathbb{R} \arrow[ur, dashed] & \end{tikzcd}$$

Proof. Intuitively, we want to write

$$2^x := \lim_{q\to x} 2^q,$$

where the limit is over $q \in \mathbb{Q}$. To make this rigorous, we can define this in two ways.

- On the left, we can write
$$2^x := \sup_{\substack{q\in\mathbb{Q}\\ q\le x}} 2^q.$$

- On the right, we can write
$$2^x := \inf_{\substack{q\in\mathbb{Q}\\ q\ge x}} 2^q.$$

These fit into our context of category theory because the supremum is the colimit in the slice category of objects $q \in \mathbb{Q}$ under x; i.e., we want the objects $q \in \mathbb{Q}$ with a map to x, which is equivalent to $q \le x$. So we can reframe our two stories as follows.

- On the left, we can write
$$\sup_{\substack{q\in\mathbb{Q}\\ q\le x}} 2^q = \operatorname{colim}\left(\mathbb{Q}/x \to \mathbb{Q} \xrightarrow{2^\bullet} \mathbb{R}\right).$$

- On the right, we can write
$$\inf_{\substack{q\in\mathbb{Q}\\ q\geq x}} 2^q = \lim\left(x/\mathbb{Q}\to\mathbb{Q}\overset{2^\bullet}{\to}\mathbb{R}\right).$$

As such, we have a purely categorical extension of $2^\bullet\colon\mathbb{Q}\to\mathbb{R}$ to all of $\mathbb{R}$. It turns out that these are the left and right Kan extensions for $2^\bullet\colon\mathbb{Q}\to\mathbb{R}$, but we will not check this now. ■

More generally, suppose that we have some embedding $K\colon\mathcal{C}\to\mathcal{D}$ with a functor $F\colon\mathcal{C}\to\mathcal{E}$. Then our story of Kan extensions will again be about trying to "extend" F to all of $\mathcal{D}$. Namely, we want to induce the following arrow.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, dashed] & \end{tikzcd}$$

Drawing motivation for example, we would like to define
$$(\operatorname{Lan}_K F)(d) \overset{?}{\cong} \operatorname*{colim}_{c\in\mathcal{C}/d} F(c).$$

Here, $\mathcal{C}/d$ refers to the subcategory of $\mathcal{D}/d$ where we are restricting to $\operatorname{im} K$; i.e., this category consists of objects which are morphisms $Kc\to d$ and morphisms which are triangles commuting under d.

Approximately speaking, we are trying to understand $\operatorname{Lan}_K F$ to be approximating F from below, which is reasonable because defining as a colimit will still be able to have maps out of it for our initial universal property later. Similarly, we might hope
$$(\operatorname{Ran}_K F)(d) \overset{?}{\cong} \lim_{c\in d/\mathcal{C}} F(c).$$

Before continuing, we want to remove the condition that F is an embedding.

6.2.2 Comma Categories, Quickly

We take a moment to talk about comma categories. Suppose that we have the following set-up, where X,Y,Z are categories with F,G functors.

$$\begin{tikzcd} & Y \arrow[d, "G"] \\ X \arrow[r, "F"'] & Z \end{tikzcd}$$

Then we want to define the comma category to be universal with respect to the following diagram.

$$\begin{tikzcd} X\vec{\times}_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "G"] \\ X \arrow[r, "F"'] \arrow[ur, Rightarrow] & Z \end{tikzcd}$$

In particular, we define $X\vec{\times}_Z Y$ as having objects which are morphisms $Fx\to Gy$ (for $x\in X$ and $y\in Y$). Then out morphisms are commuting squares (made of a pair of morphisms $x\to x'$ and $y\to y'$) so that the following diagram commutes.

$$\begin{tikzcd} f(x) \arrow[r] \arrow[d, "Ff"'] & g(y) \arrow[d, "Gg"] \\ f(x') \arrow[r] & g(y') \end{tikzcd}$$

6.2.3 Kan Extensions in General

In particular, with our functor $K\colon \mathcal{C} \to \mathcal{D}$, we can more concretely describe

$$\mathcal{C}\vec{\times}_{\mathcal{D}} d := \mathcal{C}\vec{\times}_Z \{d\}.$$

Namely, its objects has the data of $c \in \mathcal{C}$, the fixed $d \in \mathcal{D}$ and a morphism $Kc \to d$; we may ignore the data of d. Continuing, our morphisms $(c, f) \to (c', f')$ condition is asking for morphisms $k\colon Kc \to d$ and $k'\colon Kc' \to d$ so that the following diagram commutes.

$$\begin{tikzcd} Kc \arrow[r, "k"] \arrow[d, "Kf"'] & d \arrow[d, equal] \\ Kc' \arrow[r, "k'"'] & d \end{tikzcd}$$

Notably, the two vertices on the right collapse, so this is really a commuting triangle "under" d.

This now lets us generalize our Kan extensions.

Theorem 6.10. Fix $K\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$ as usual. If the colimit

$$\ell_K(F)(d) := \operatorname{colim}\left(\mathcal{C}\vec{\times}_{\mathcal{D}} d \to \mathcal{C} \xrightarrow{F} \mathcal{E}\right)$$

exists, then it defines the action of $\operatorname{Lan}_K F$ on objects $d \in \mathcal{D}$; the action on morphisms is induced by action of colim on morphisms along with the functoriality of the construction of $\mathcal{C}\vec{\times}_{\mathcal{D}} d$ in d.

Proof. This is long and therefore omitted. ■

Nonetheless, let's see some corollaries.

Corollary 6.11. Fix a functor $K\colon \mathcal{C} \to \mathcal{D}$ from a small category $\mathcal{C}$ to a locally small category $\mathcal{D}$. If $\mathcal{E}$ is cocomplete, and we have a functor $F\colon \mathcal{C} \to \mathcal{E}$, then $\operatorname{Lan}_K F$ exists by Theorem 6.10.

Proof. Our limit exists because $\mathcal{E}$ has all colimits and $\mathcal{C}$ is a fine index category because it is small by hypothesis. ■

Corollary 6.12. Fix a functor $K\colon \mathcal{C} \to \mathcal{D}$ from a small category $\mathcal{C}$ to a locally small category $\mathcal{D}$. If $\mathcal{E}$ is cocomplete, then the functor

$$(- \circ K)\colon \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$$

has a left adjoint, namely $\operatorname{Lan}_K(-)$.

Proof. The Kan extension exists by Corollary 6.11. ■

6.2.4 Kan Extension Examples

We close class with some examples.

Exercise 6.13. Fix a small category $\mathcal{C}$ and a cocomplete category $\mathcal{E}$ with a functor $F\colon \mathcal{C} \to \mathcal{E}$. We compute the left Kan extension.

Proof. Then Corollary 6.11 gives us a Kan extension from the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\text{よ}"'] & \mathcal{E} \\ \mathrm{PSh}(\mathcal{C}) \arrow[ur, dashed, "\mathrm{Lan}_{\text{よ}}(F)"'] & \end{tikzcd}$$

In particular, Theorem 6.10 promises that we can compute

$$\mathrm{Lan}_K X = \operatorname*{colim}_{\text{よ}c\to X} Fc.$$

As an aside, if $X = \text{よ}c$, then our diagram above is indexed over a category (namely, $\mathcal{E}$) with a terminal object, so we can find its colimit c' in there to be able to say

$$\operatorname*{colim}_{\text{よ}c\to X} Fc = Fc'.$$

So this is nice. ■

As a corollary of the previous exercise, we have proven the following theorem from yesterday.

Theorem 6.4 (Universal property of presheaves)**.** Fix a small category $\mathcal{C}$ so that we can fix the presheaf category $\mathrm{PSh}(\mathcal{C}) := \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$; further, take a category $\mathcal{E}$ with all colimits. Then the restriction of $\text{よ}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ defines an equivalence

$$(-\circ \text{よ})\colon\ \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(\mathcal{C}), \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

Namely, a functor from $\mathcal{C}$ to $\mathcal{E}$ has the same data as a functor from $\mathrm{PSh}(\mathcal{C})$ to $\mathcal{E}$ which preserves colimits.

Namely, the inverse mapping is given by taking $F\colon \mathcal{C} \to \mathcal{E}$ to $\mathrm{Lan}_{\text{よ}} F$.

Exercise 6.14. Fix a group G and subgroup $H \subseteq G$ and a complete and cocomplete category $\mathcal{C}$. Then Corollary 6.11 grants us the following left and right Kan extensions.

$$\begin{tikzcd} \mathrm{Rep}_K(G) \arrow[r] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"'] & \mathrm{Rep}_K(H) \arrow[l, bend right=50, "\mathrm{Ind}_H^G"'] \arrow[l, bend left=50, "\mathrm{Coind}_H^G"] \end{tikzcd}$$

We compute these.

Proof. On the left, we begin by needing to compute

$$\mathrm{B}H \vec{\times}_{\mathrm{B}G} *,$$

which has objects which are morphisms in $\mathrm{B}G$ (namely, we only have to care about our single object $*$ here) and hence in bijection with G. Then the morphisms $g \to g'$ are elements $h \in H$ with $g'h = g$ by tracking through our diagram. Notably, we can forget about g' (and recover it as gh^{-1}).

From these computations, we can actually write out $\mathrm{ind}_H^G(X)$ (for $X\colon \mathrm{B}G \to \mathcal{C}$) as the coequalizer (given at the right) of the following diagram.

$$\coprod_{G\times H} X \rightrightarrows \coprod_G X \longrightarrow \coprod_{G/H} X$$

One has to be a little careful about how our actions of G on the objects are as well as what precisely these morphisms are. There is a similar formula for right Kan extensions. ■

And so we end with the following theorem.

Theorem 6.15 (Frobenius reciprocity). In the previous example, set $\mathcal{C} := \mathrm{Vec}_k$ and assume that $[G : H] < \infty$. Then the functors

$$\mathrm{ind}_H^G, \mathrm{coind}_H^G \colon \mathrm{Rep}_K H \to \mathrm{Rep}_K(G)$$

are naturally isomorphic.

Proof. The point is that we computed these as some products and coproducts, which are equal when finite (which is true because G/H is finite). ∎

6.3 April 29

Welcome to the last day of class.

6.3.1 Ultrafilters

Last time we were able to write down Kan extensions for the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} & \end{tikzcd}$$

by writing down

$$(\mathrm{Lan}_K F)(d) := \mathrm{colim}\left(\mathcal{C} \vec{\times}_{\mathcal{D}} d \to \mathcal{C} \overset{F}{\to} \mathcal{E}\right).$$

Let's see another example: we're going to talk about ultrafilters.

Definition 6.16 (Ultrafilter). Fix a set S. An *ultrafilter* on S is a collection $\mathcal{U} \subseteq \mathcal{P}(S)$ with the following coherence conditions.

(a) $\varnothing \notin \mathcal{U}$.

(b) Upwards closed: if $A, B \in \mathcal{P}(S)$, then $A \in \mathcal{U}$ and $A \subseteq B$ implies $B \in \mathcal{U}$.

(c) Intersection: $A, B \in \mathcal{P}(S)$ implies $A \cap B \in \mathcal{P}(S)$.

(d) For each $A \in \mathcal{P}(S)$, exactly one of $A \in \mathcal{U}$ or $(S \setminus A) \in \mathcal{U}$ is true.

We let $\beta(S)$ denote the set of all ultrafilters on S.

Remark 6.17. Note that, if we threw out the first condition, then $\varnothing \in \mathcal{U}$ would imply $\varnothing \subseteq S \subseteq S$, thus giving $S \in \mathcal{U}$, violating the last condition. So we can actually derive $\varnothing \notin \mathcal{U}$ from the other three.

Ultrafilters are important for point-set topology, for various reasons; it turns out that they are the correct indexing set.

Example 6.18. Given $s \in S$, then we can build

$$\delta_s := \{A \subseteq S : s \in A\},$$

which is an ultrafilter.

It happens that the map $s \mapsto \delta_s$ is a bijection if $S \to \beta(S)$ when S is finite. However, when S is infinite, there might be other ultrafilters, but the axiom of choice is needed in their construction.

Remark 6.19. Here are some use cases for ultrafilters.

- Logic, as in Łoś's theorem.
- Topology, as for limits.
- And more: geometric group theory, dynamics, and so on.

There is also a functoriality of ultrafilters.

Proposition 6.20. Fix a function $f\colon S \to T$, there is a map $f_*\colon \beta(S) \to \beta(T)$ by

$$\mathcal{U} \mapsto \{\beta \subseteq T : f^{-1}(B) \in \mathcal{U}\}$$

Proof. Note that f^{-1} is closed under intersection and unions, so we can just check the axioms one by one. ■

So we have a functoriality and can now do category theory.

Theorem 6.21. The diagram

$$\begin{tikzcd} \mathrm{finSet} \arrow[r, hook] \arrow[d, hook] & \mathrm{Set} \\ \mathrm{Set} \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

makes β a right Kan extension.

Proof. The point is to check

$$\beta(S) = \lim_{S \to F} F,$$

where we have interpreted $S \to F$ in the correct way. ■

Now, having a right Kan extension tells us lots of things by abstract nonsense. For example, by using the functor $\mathrm{id}\colon \mathrm{Set} \to \mathrm{Set}$, the universal property gives us a natural transformation

$$\eta\colon \mathrm{id} \Rightarrow \beta.$$

In the same way, there is also a natural transformation

$$\mu\colon \beta^2 \Rightarrow \beta.$$

Having these two notions lets us write down strange commutative diagrams, like the following.

$$\begin{tikzcd} \beta^3 \arrow[r, Rightarrow, "\mu\beta"] \arrow[d, Rightarrow, "\beta\mu"'] & \beta^2 \arrow[d, Rightarrow, "\mu"] \\ \beta^2 \arrow[r, Rightarrow, "\mu"'] & \beta \end{tikzcd}$$

Observe that this would be essentially impossible to verify by hand (or at least very annoying), but it follows directly from uniqueness of the natural transformation $\beta^3 \Rightarrow \beta$ by universal property.

Remark 6.22. It turns out that η and μ will specify the data of a monad.

6.3.2 Category Theory via Kan Extensions

Today, we will explain why all concepts are Kan extensions.[1] Namely, we are going to reinterpret many concepts in category theory as Kan extensions. But for now, we will talk about ultrafilters.

Let's see an example concept.

Proposition 6.23. Fix a functor $F\colon \mathcal{C} \to \mathcal{D}$.

1. The colimit $\operatorname{colim} F$ exists if and only if the left Kan extension of $!\colon \mathcal{C} \to 1$ exists.
2. Under the hypothesis of (a), $\operatorname{Lan}_! F \cong \operatorname{colim} F$.

Proof. Let's talk about why we might expect (a) to be true. Namely, we are looking for functors $d\colon 1 \Rightarrow \mathcal{D}$ is asking for the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "!"'] & {} \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{D} \\ & 1 \arrow[ur, "d"'] & \end{tikzcd}$$

However, this is equivalent data to a cone $\gamma\colon F \Rightarrow d$. If we have a left Kan extension, then this is again equivalent data to natural transformations $\operatorname{Lan}_! F \Rightarrow d$.

But because we are doing everything over 1, this is the same data as a morphism $\operatorname{Mor}_{\mathcal{D}}(\operatorname{Lan}_! F, d)$. So $\operatorname{Lan}_! F$ will exactly represent $\operatorname{Cone}(F, -)$, meaning that colimits and the left Kan extension have exactly the same data.

This proof technically gives us (b) for free because we showed that we have the same data from both objects, but we can also write down the formula

$$\operatorname{Lan}_! F \cong \operatorname{colim}\left(\mathcal{C}\vec{\times}_1\{*\} \xrightarrow{F} \mathcal{D}\right),$$

but of course $\mathcal{C}\vec{\times}_1\{*\}$ has is the same data as $\mathcal{C}$. ■

Remark 6.24. We can also interpret limits as right Kan extensions, by flipping the arrows.

Next let's see adjoints.

Proposition 6.25. The data of an adjunction $F \dashv G$ with unit $\eta\colon \operatorname{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \operatorname{id}_{\mathcal{D}}$ has the same data of left and right Kan extensions of the following two diagrams, respectively.

$$\begin{tikzcd} \mathcal{C} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "\eta\Downarrow"] & \mathcal{C} \\ \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, equal] \arrow[d, "G"'] \arrow[dr, phantom, "\varepsilon\Uparrow"] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"'] & \end{tikzcd}$$

In particular, the right adjoint G is the left Kan extension $\operatorname{Lan}_F \operatorname{id}_{\mathcal{C}}$, and the left adjoint F is the right Kan extension $\operatorname{Ran}_G \operatorname{id}_{\mathcal{D}}$.

Proof. We will only show one direction; suppose $F \dashv G$. The main point is that we get an adjunction

$$\begin{tikzcd} \operatorname{Fun}(\mathcal{D},\mathcal{C}) \arrow[r, shift right, "-\circ F"'] \arrow[r, phantom, "\perp"] & \operatorname{Fun}(\mathcal{C},\mathcal{C}) \arrow[l, shift right, "G\circ -"'] \end{tikzcd}$$

[1] Alternatively, "all Kancepts are Kan extensions."

from Theorem 5.34. However, the uniqueness of adjoints combined with what we already know about the adjoint of $(- \circ F)$ from Corollary 6.12, we are able to conclude

$$\operatorname{Lan}_F(-) \dashv (- \circ F).$$

In particular, $\operatorname{Lan}_F \operatorname{id}_{\mathcal{C}} \cong G^*(\operatorname{id}_{\mathcal{C}}) = G$, so we are done. ■

6.3.3 The Kan Extension Formula

To set up our discussion, fix $F: \mathcal{C} \to \mathcal{D}$ such that $\mathcal{D}$ has all colimits and that $\mathcal{C}$ is small. Then, by the formula in Corollary 6.11, we know that we can extend F as in the following Kan extension diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, equal] & \mathcal{D} \\ \mathcal{C} \arrow[ur, dashed] \end{tikzcd}$$

However, this extension somewhat clearly should be F, so the formula in Corollary 6.11 tells us

$$Fc \cong \operatorname{colim}\left(\mathcal{C} \vec{\times}_{\mathcal{C}} c \to \mathcal{C} \xrightarrow{F} \mathcal{D}\right)$$

Here are some consequences.

Theorem 6.26 (Co-Yoneda lemma)**.** Fix everything as above. Then we can the diagram

$$\coprod_{y \to x \to c} F(y) \rightrightarrows \coprod_{x \to c} Fx \longrightarrow Fc$$

is a coequalizer diagram; here one of the maps being coequalized is the inclusion from Fy, and the other is the inclusion from Fx.

Theorem 6.27 (Density, I)**.** Fix a functor $F: \mathcal{C} \to \operatorname{Set}$ from a small category $\mathcal{C}$. Then F is the colimit of the diagram

$$\left(\int_{\mathcal{C}} F\right)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \overset{よ}{\hookrightarrow} \operatorname{Fun}(\mathcal{C}, \operatorname{Set}).$$

Dually, a functor $F: \mathcal{C}^{\mathrm{op}} \to \operatorname{Set}$ is the colimit of the diagram

$$\int_{\mathcal{C}^{\mathrm{op}}} F \to \mathcal{C} \overset{よ}{\to} \operatorname{PSh}(\mathcal{C}).$$

Proof. This follows quickly from the Co-Yoneda lemma, with a little elbow grease. ■

Theorem 6.28 (Density, II)**.** The diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "よ"] \arrow[d, hook, "よ"'] & \operatorname{PSh}\mathcal{C} \\ \operatorname{PSh}\mathcal{C} \arrow[ur, equal] \end{tikzcd}$$

gives a left Kan extension.

Proof. Using the formula, we can just compute

$$(\mathrm{Lan}_{よ}\, よ)(F) \cong \operatorname{colim}\left(\mathcal{C} \vec{\times}_{\mathrm{PSh}\,\mathcal{C}} F \to \mathcal{C} \overset{よ}{\hookrightarrow} \mathrm{PSh}\,\mathcal{C}\right).$$

However, $\mathcal{C} \vec{\times}_{\mathrm{PSh}\,\mathcal{C}} F$ is $\int_{\mathcal{C}^{\mathrm{op}}} F$, so the previous density theorem tells us that we get F out of this colimit. ■

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List of Definitions