18.917: The Chromatic Splitting Conjecture

Nir Elber

Spring 2026

CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

INTRODUCTION

1.1 February 2

Here we go.

1.1.1 Idempotent Algebras

The goal of this class is to understand some topics related to the chromatic splitting conjecture. Thus, the first half of the class will try to understand the statement, and the second half of the class will explain how it relates to other problems in algebra.

> **Warning 1.1.** All categories in this course are ∞-categories.

> **Example 1.2.** Given a ring R, we have a stable, symmetric monoidal ∞-category $D(R)$ of chain complexes of R-modules, considered up to quasi-isomorphism. Notably, the symmetric monoidal structure is given by the derived tensor product.

We begin our story with idempotent algebras.

> **Definition 1.3** (idempotent algebra)**.** Fix a ring R. An *idempotent algebra* is an object $E \in D(R)$ equipped with a unit map $R \to E$ such that the composite
>
> $$E = E \otimes_R R \to E \otimes_R E$$
>
> is an equivalence.

> **Remark 1.4.** Such an object E grants E a multiplication structure $E \otimes_R E \to E$, and E gains the structure of a differentially graded algebra.

> **Example 1.5.** Consider $R = \mathbb{Z}$. Then for each prime p, the algebra $\mathbb{Z}_{(p)}$ is idempotent: localizing $\mathbb{Z}_{(p)}$ further at (p) does nothing!

> **Non-Example 1.6.** The $\mathbb{Z}$-algebra $\mathbb{F}_p$ is not idempotent because the tensor product we are considering is derived. Indeed, we computed $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p$ last semester.

Here is a quick reason why one might care about idempotent algebras.

> **Theorem 1.7** (Neeman)**.** Fix a Noetherian ring R. Then the lattice of idempotent algebras is equivalent to the data of $\operatorname{Spec} R$ as a topological space.

> **Example 1.8.** For $R = \mathbb{Z}$, it turns out that the idempotent algebras are either $\mathbb{Z}_{(p)}$ or $\mathbb{Q}$, and the maps between them look like the specializations of $\operatorname{Spec} R$.

Of course, we are homotopy theorists, so we have less reason to care about $\mathbb{Z}$. Recall that $\mathbb{Z}$ is obtained from $\mathbb{N}$ by formally adding inverses. But $\mathbb{N}$ is basically isomorphism classes of FinSet; if we had instead formally added inverses directly to FinSet (instead of taking isomorphism classes first), we would have found the sphere spectrum $\mathbb{S}$. In particular, we will be interested in the category $D(\mathbb{S})$ of $\mathbb{S}$-modules, also called spectra.

We now no longer have access to algebraic geometry directly on $\mathbb{S}$. Instead, Theorem 1.7 motivates us to look for the idempotent algebras for $\mathbb{S}$.

> **Remark 1.9.** For any $x \in \pi_* \mathbb{S}$, there is an idempotent algebra $\mathbb{S}\left[x^{-1}\right]$. For example, $\pi_0 \mathbb{S} = \mathbb{Z}$, so there is an idempotent algebra $\mathbb{S}_{(p)}$.

Here is our first main theorem.

> **Theorem 1.10** (Nishida)**.** Fix some $x \in \pi_* \mathbb{S}$ of positive degree. Then x is nilpotent.

Thus, the idempotent algebras $\mathbb{S}\left[x^{-1}\right]$ do not look genuinely "new." To get other idempotent algebras, we need more tools.

1.1.2   The Adams–Novikov Spectral Sequence

Recall the $\mathbb{S}$-algebra MU defined as the colimit of the embedding $\operatorname{BU} \to \operatorname{BLG}(\mathbb{S}) \subseteq \operatorname{Mod}(\mathbb{S})$. Let's compute its homotopy groups.

> **Definition 1.11** (formal group law)**.** Fix a commutative ring R. Then a *commutative formal group law* over R is a power series $f(x, y) \in R[[x, y]]$ satisfying
>
> (a) $f(x, 0) = x$ and $f(0, y) = y$,
>
> (b) $f(x, y) = f(y, x)$, and
>
> (c) $f(x, f(y, z)) = f(x, f(y, z))$.

> **Definition 1.12** (Lazard ring)**.** The *Lazard ring* is the ring L which is exactly the quotient of $\mathbb{Z}[\{a_{ij}\}_{ij}]$ by the relations dictating that
> $$f(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j$$
> is a commutative formal group law.

> **Remark 1.13.** In other words, L represents the collection formal group laws, in the sense that the data of a formal group law for a ring R amounts to the data of a ring homomorphism $L \to R$.

Remark 1.14. By definition, there is a "universal" formal group law f_L in L given exactly by

$$f_L(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j.$$

Theorem 1.15 (Quillen). The ring $\pi_* \mathrm{MU}$ is exactly the Lazard ring.

Remark 1.16. Quillen also computed $\pi_*(\mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU})$ as well as the two natural maps $\pi_* \mathrm{MU} \to \pi_*(\mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU})$. It turns out that this is more or less related to some notion of isomorphism of the formal group laws.

The use of MU is that it produces a spectral sequence with which we can understand $\pi_* \mathbb{S}$. By Čech descent along the map $\mathbb{S} \to \mathrm{MU}$, we see that $\mathbb{S}$ is the limit of the diagram

$$\mathrm{MU} \rightrightarrows \mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU} \qquad \cdots$$

which we can then truncate as $\mathrm{fil}^n \mathbb{S}$ in order to get a descending filtration to $\mathrm{fil}^0 \mathbb{S}$. Computing homotopy along this filtration produces the desired spectral sequence, as soon as we compute homotopy groups of the various tensor powers of the MUs and so on.

Theorem 1.17 (Adams−Novikov). Let $\mathcal{M}_{\mathrm{fg}}$ be the moduli space of formal groups. Then there is a spectral sequence

$$E_2 = \mathrm{H}^s(\mathcal{M}_{\mathrm{fg}}; \omega^{\otimes t}) \Rightarrow \pi_{2t-s} \mathbb{S}.$$

Remark 1.18. It turns out that the spectral sequence is concentrated in the region $s \leq 2t - s$.

Example 1.19. Along the line $s = 2t - s$, there is some h_1 at $(s, 2t - s) = (1, 1)$, and then we can take powers of it to go up the line. It turns out that h_1 survives the spectral sequence, and it goes to the "Hopf map" $\eta \in \pi_1 \mathbb{S}$; however, $\eta^4 = 0$, though the Adams−Novikov spectral sequence cannot see it!

Thus, we see that the Adams−Novikov spectral sequence is not an amazing approximation: the E_2 page sees many classes which we know abstractly must vanish! Life is better if we pass to E_∞ instead; the following is our first main theorem.

Theorem 1.20 (Devinatz−Hopkins−Smith). The E_∞ page of the Adams−Novikov spectral sequence lies under a curve which grows more slowly than any line.

Note that this immediately implies Theorem 1.10. On the other hand, we will see that the topological input of Theorem 1.10 plus some algebraic facts about formal group laws will prove the above big theorem.

Remark 1.21. The curve is known to be faster than logarithmic, but not much else is known. Our proof will not help us much because our proof of Theorem 1.10 will be ineffective.

1.1.3   Back to Idempotent Algebras

Let's return to trying to find some idempotent algebras.

Notation 1.22. Define the power series $[n] \in L[[x]]$ to be adding with f a total of n times.

Example 1.23. We see that $[2](x) = f(x,x)$ and $[5](x) = f(f(f(f(x,x),x),x),x)$.

Notation 1.24. Fix a prime p. Then we define the class $v_n \in \pi_* \mathrm{MU}$ to be the coefficient of x^{p^n} in the power series $[p](x)$.

Now, because localization is exact, we see that $\mathbb{S}_{(p)}$ is the limit of the nerve

$$\mathrm{MU}_{(p)} \rightrightarrows \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \qquad \cdots$$

so it is not unreasonable to consider the following limit.

Notation 1.25. Fix a prime p and some $n \geq 0$. Then we define $L_n \mathbb{S}_{(p)}$ as the limit of the following diagram.

$$\mathrm{MU}_{(p)}\left[v_n^{-1}\right] \rightrightarrows \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right]$$

We may abbreviate $L_n \mathbb{S}_{(p)}$ to $L_n \mathbb{S}$ if there is no possibility of confusion.

Remark 1.26. It turns out that there are natural maps $L_{n+1}\mathbb{S} \to L_n\mathbb{S}$.

These spectra $L_n\mathbb{S}$ give us new idempotent algebras, more or less granting us further understanding of the "spectrum" of $\mathbb{S}$.

Theorem 1.27 (Hopkins–Ravenel)**.** Fix a prime p and some $n \geq 0$. Then $L_n\mathbb{S}_{(p)}$ is an idempotent algebra.

Remark 1.28. Ravenel has conjectured that if E is a nonzero idempotent algebra under $\mathbb{S}_{(p)}$, then E is either $\mathbb{Q}$ or one of the $L_n\mathbb{S}$s. This was recently disproved. It is current work to attempt a classification.

Nonetheless, $\mathbb{S}_{(p)}$ can be understood well from the $L_n\mathbb{S}$s.

Theorem 1.29 (Hopkins–Ravenel)**.** Fix a prime p. Then $\mathbb{S}_{(p)}$ is the limit of the diagram

$$\cdots \to L_3\mathbb{S}_{(p)} \to L_2\mathbb{S}_{(p)} \to L_1\mathbb{S}_{(p)}.$$

1.1.4 Completion

Continue with our fixed prime p. For motivation, we return to abelian groups.

Remark 1.30. For any $M \in D(\mathbb{Z})$, the p-localization sits in a pullback square

$$\begin{array}{ccc} M_{(p)} & \longrightarrow & M_p^{\wedge} \\ \downarrow & & \downarrow \\ M \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & M_p^{\wedge} \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

more or less corresponding to finding the "lattice" $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$.

Analogously, there is a completion of $L_n \mathbb{S}$ which fits into a diagram

$$
\begin{array}{ccc}
L_n E & \longrightarrow & L_{K(n)} E \\
\downarrow & & \downarrow \\
L_{n-1} E & \longrightarrow & L_{n-1} L_{K(n)} E
\end{array}
\tag{1.1}
$$

where $L_n E \coloneqq L_n \mathbb{S} \otimes_{\mathbb{S}} E$. We are now ready to state the chromatic splitting conjecture.

Conjecture 1.31 (Chromatic splitting). For any $n \geq 2$, the inclusion

$$
L_{K(n)} \mathbb{S} \to L_{n-1} L_{K(n)} \mathbb{S}
$$

is an inclusion of a direct summand.

Remark 1.32. This implies that the natural map

$$
\mathbb{S}_p^\wedge \to \prod_{n \geq 1} L_{K(n)} \mathbb{S}
$$

is the inclusion of a direct summand. The point is that the squares (1.1) are rather degenerate, which would let us compute the homotopy groups of $L_n \mathbb{S}$ from the completions.

Remark 1.33. Conjecture 1.31 is known at $n = 2$ and all primes, by work of many people.

The goal of the present class is to review the homotopy theory required to understand the statement of Conjecture 1.31 formally, and then we will discuss why perfectoid geometry may be useful to prove it.

Let's see why passing to $L_{K(n)} \mathbb{S}$ is genuinely easier.

Example 1.34. For $p > 2$, we can define $L_{K(1)} \mathbb{S}$ as the homotopy fiber of the endomorphism $\psi^g - 1$ of $\mathrm{KU}_p^\wedge$, where g is a choice of topological generator of $\mathbb{Z}_p^\times$, and ψ is some action of $\mathbb{Z}_p^\times$ on $\mathrm{KU}_p^\wedge$.

Theorem 1.35 (Goerss–Hopkins–Miller, Rogres). Fix a prime p. For each $n \geq 1$, there is an $\mathbb{S}$-algebra E_n and a profinite group $\mathbb{G}_n$ for which

$$
L_{K(n)} \mathbb{S} = (E_n)^{\mathbb{G}_n}.
$$

In fact, E_n is a Galois extension of $L_{K(n)} \mathbb{S}$.

Remark 1.36. We will only be able to keep track of this sort of "infinite Galois theory" with condensed mathematics.

Remark 1.37. The profinite group $\mathbb{G}_n$ is some subgroup of automorphisms of formal group laws.

Remark 1.38. For any spectrum X, there is some "Galois descent"

$$
L_{K(n)} X = \left(L_{K(n)} (E_n \otimes X) \right)^{\mathbb{G}_n}.
$$

This generalizes to a spectral sequence

$$
\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n; \pi_*(L_{K(n)}(E_n \otimes X))) \Rightarrow \pi_* L_{K(n)} X.
$$

The previous remark produces a spectral sequence

$$\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n; \pi_* E_n) \Rightarrow \pi_* L_{K(n)}\mathbb{S}.$$

If p is large compared to n, then it turns out that the spectral sequence collapses for degree reasons, so we are reduced to a pure algebra problem.

The end of the course will be interested $L_{K(n-1)}L_{K(n)}\mathbb{S}_{(p)}$ for general n but p very large. Conjecture 1.31 tells us that this should be fairly easy to understand, so we can view the end of the course as trying to provide some evidence for the conjecture. For example, work in progress by many people has recently culminated in the following strategy.

> **Notation 1.39.** Fix $\mathbb{B} := E_{n-1} \otimes_{\mathbb{S}} L_{K(n-1)}E_n$.

> **Remark 1.40.** It turns out that $\mathbb{B}$ is Galois over $L_{K(n-1)}L_{K(n)}\mathbb{S}$ with Galois group $\mathbb{G}_{n-1} \times \mathbb{G}_n$. Thus, we can hope to be able to use some Galois descent spectral sequence to understand $L_{K(n-1)}L_{K(n)}\mathbb{S}$, as in Remark 1.38.

Now, $\pi_*\mathbb{B}$ is a local ring, so one becomes motivated to consider a perfection $\widehat{\mathbb{B}}$. In particular, it turns out that there is a $\mathbb{G}_n \times \mathbb{G}_{n-1}$-equivariant map $\mathbb{B} \to \widehat{\mathbb{B}}$, so taking fixed points produces a map out of $L_{K(n-1)}L_{K(n)}\mathbb{S}$. This is the sort of thing that Conjecture 1.31 asks us to do! Of course, the target is related to the perfection $\widehat{\mathbb{B}}$, which we now want to understand.

> **Theorem 1.41.** The groups $\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n \times \mathbb{G}_{n-1}; \pi_*\widehat{\mathbb{B}})$ is the same as the cohomology of the structure sheaf of some diamond related to the Fargues–Fontaine curve.

Let's explain the application to Conjecture 1.31: this calculation tells us that $(\widehat{\mathbb{B}})^{\mathbb{G}_n \times \mathbb{G}_{n-1}}$ is $L_{K(n-1)}\mathbb{S} \oplus \Sigma L_{K(n-1)}\mathbb{S}$, from which our small piece of Conjecture 1.31 follows!

1.2 February 4

Our next goal is to prove Theorem 1.10.

1.2.1 Spectra

Let's set some notation. We are interested in the category Spaces of spaces, also called anima.

> **Definition 1.42** (pointed space)**.** The category of pointed spaces is denoted Space_*.

> **Remark 1.43.** There is a functor $(-)_+ \colon \mathrm{Spaces} \to \mathrm{Spaces}_*$ which simply adds a basepoint to any topological space.

Our next category will be spectra.

> **Definition 1.44** (spectra)**.** A *spectrum* is an infinite tuple $(X_0, X_1, \ldots)$ of spaces, equipped with isomorphisms $X_0 \cong \Omega X_1 \cong \Omega^2 X_1 \cong \cdots$. The category of spectra is, unsurprisingly, denoted $\mathrm{Spectra}$.

> **Remark 1.45.** The functor $\Omega \colon \mathrm{Spectra} \to \mathrm{Spectra}$, given by shifting all the spaces down, is an auto-equivalence. Its inverse functor is Σ. Thus, $\mathrm{Spectra}$ is a stable category.

Remark 1.46. The functor $\Omega^\infty\colon \mathrm{Spectra} \to \mathrm{Spaces}$ given by sending a spectrum X to X_0 admits an adjoint Σ^∞. We may denote the composite functor $X \mapsto \Sigma^\infty(X_+)$ by Σ^∞_+.

Remark 1.47. The category $\mathrm{Spectra}$ admits products and coproducts, which are the same and denoted $\oplus$.

Example 1.48. We let $\mathbb{S}$ denote the sphere spectrum $\Sigma^\infty S^0$. We can also think about this as $\Sigma^\infty_+ \mathrm{pt}$.

Remark 1.49. The category $\mathrm{Spectra}$ admits internal Homs which has a left adjoint $\otimes_\mathbb{S}$. This gives a symmetric monoidal structure to $\mathrm{Spectra}$.

Remark 1.50. For any space X, the constant map $X \to \mathrm{pt}$ induces a map $\Sigma^\infty_+ X \to \mathbb{S}$. On the other hand, for any pointed space X, the canonical map $\mathrm{pt} \to X$ induces a map $\mathbb{S} \to \Sigma^\infty X$. Thus, when X is pointed, we find

$$\Sigma^\infty_+ X = \Sigma^\infty X \oplus \mathbb{S}.$$

1.2.2 Module Categories

We now pass to other rings.

Definition 1.51 ($\mathbb{E}_\infty$-ring)**.** An $\mathbb{E}_\infty$-*ring* R is a spectrum R equipped with a multiplication $R \otimes_\mathbb{S} R \to R$ and a unit $\mathbb{S} \to R$, as well as much other data (e.g., many other operations all required to cohere with each other).

One can define modules in an expected way.

Example 1.52. For any $\mathbb{E}_\infty$-ring R, the category $\mathrm{Mod}(R)$ of R-modules continues to be a stable, symmetric monoidal category, where the symmetric monoidal structure is given by some tensor product $\otimes_R$.

Remark 1.53. Conversely, given a symmetric monoidal stable category $\mathcal{C}$, we can recover $\mathcal{C}$ as a module category over the ring $R := \mathrm{End}_\mathcal{C} 1$, where 1 is a tensor unit. Indeed, if $\mathcal{C} = \mathrm{Mod}_R$, we see that

$$\mathrm{Hom}_\mathcal{C}(R, R) = \mathrm{Hom}_\mathbb{S}(\mathcal{S}, R)$$

by the tensor–hom adjunction. This is then $\mathrm{Hom}_\mathbb{S}(\Sigma^\infty_+ \mathrm{pt}, R)$, which is just $\Omega^\infty R$ because Σ^∞ is left adjoint to Ω^∞.

Here are some constructions with modules.

Definition 1.54. Fix an $\mathbb{E}_\infty$-ring R. For any $x \in \pi_i R$, we receive a spectrum map $x\colon \Sigma^i \mathbb{S} \to R$ and hence an R-module map $\Sigma^i R \to R$. We now define $R\left[x^{-1}\right]$ to be the R-module

$$\mathrm{colim}\left(R \xrightarrow{x} \Sigma^{-i} R \xrightarrow{x} \Sigma^{-2i} R \to \cdots \right).$$

Remark 1.55. By construction, there is a unit $R \to R\left[x^{-1}\right]$. Additionally, by understanding maps from $\mathbb{S}$ to the colimit, we see that $\pi_* R\left[x^{-1}\right] = (\pi_* R)\left[x^{-1}\right]$.

Note that we have only defined $R\left[x^{-1}\right]$ as a module. One may hope to upgrade it to a ring.

Remark 1.56. One can check that the canonical map

$$R \otimes_R R\left[x^{-1}\right] \to R\left[x^{-1}\right] \otimes_R R\left[x^{-1}\right]$$

is an equivalence: on homotopy groups, both sides are $(\pi_* R)\left[x^{-1}\right]$. It follows that $R\left[x^{-1}\right]$ is an idempotent algebra, defined by taking the inverse of the equivalence.

Remark 1.57. Here is another way to find that $R\left[x^{-1}\right]$ is a ring. Let $\mathcal{C}$ be the full subcategory of R-modules M such that the action of x on $\pi_* M$ is invertible. Then $\mathcal{C}$ is closed under the tensor product, so it is a stable symmetric monoidal category, and then Remark 1.53 allows us to find $R\left[x^{-1}\right]$ as the corresponding ring.

1.2.3 Some Free Constructions

Here is a way to construct an $\mathbb{E}_\infty$-ring.

Remark 1.58. Given a spectrum R, note that $X \otimes_{\mathbb{S}} X$ admits a natural action by the symmetric group Σ_2. This produces a functor $\mathrm{B}\Sigma_2 \to \mathrm{Spectra}$ which sends the one object to $X \otimes_{\mathbb{S}} X$ and the nontrivial morphism to the swapping action. We may let $(X \otimes_{\mathbb{S}} X)_{h\Sigma_2}$ denote the "quotient." In general, one can form the "quotient" $(R^{\otimes_{\mathbb{S}} n})_{h\Sigma_n}$.

Remark 1.59. If R is an $\mathbb{E}_\infty$-ring, then there is a canonical map $(R \otimes_{\mathbb{S}} R)_{h\Sigma_2} \to R$ given by the commutativity. The associativity and commutativity produce a map

$$\left(R^{\otimes_{\mathbb{S}} n}\right)_{h\Sigma_n} \to R.$$

Note that the existence of this map requires us to remember the higher coherences present in the definition of the $\mathbb{E}_\infty$-ring R.

Definition 1.60 (free ring)**.** Fix a spectrum X. Then we define the *free* $\mathbb{E}_\infty$*-ring* $\mathrm{Free}_{\mathbb{E}_\infty} X$ to be

$$\mathrm{Free}_{\mathbb{E}_\infty} X := \mathbb{S} \oplus X \oplus (X \otimes_{\mathbb{S}} X)_{h\Sigma_2} \oplus \cdots.$$

The operation is induced by the construction.

Remark 1.61. The free rings have a grading, but the general ones do not.

Remark 1.62. There is an analogous notion of a free $\mathbb{E}_\infty$-space of a space X, which is

$$\mathrm{Free}_{\mathbb{E}_\infty} X := \mathrm{pt} \sqcup X \sqcup (X \times X)_{h\Sigma_2} \sqcup \cdots.$$

Notably, Σ_+^∞ preserves the symmetric monoidal structure and is a left adjoint, so one knows by pure nonsense that

$$\mathrm{Free}_{\mathbb{E}_\infty} \Sigma_+^\infty X = \Sigma_+^\infty \mathrm{Free}_{\mathbb{E}_\infty} X.$$

Recall that $\mathbb{E}_\infty$-spaces are interesting because they allow us to access many spectra.

Remark 1.63. If X is an $\mathbb{E}_\infty$-space, then $\pi_0 X$ is a commutative monoid.

Definition 1.64 (group-like). An $\mathbb{E}_\infty$-space is *group-like* if and only if $\pi_0 X$ is a group.

Remark 1.65. The embedding from group-like $\mathbb{E}_\infty$-spaces to $\mathbb{E}_\infty$-spaces admits a right adjoint $\mathrm{B}\Omega$.

Theorem 1.66 (May). Consider the category of group-like $\mathbb{E}_\infty$-spaces.

(a) There is a fully faithful embedding from the category of group-like $\mathbb{E}_\infty$-spaces to the category of spectra.

(b) Its essential image consists of those spectra with no homotopy groups in negative degrees.

(c) The inverse functor to the fully faithful embedding is Ω^∞.

Note that we currently have two ways to construct an $\mathbb{E}_\infty$-space: there is the free construction, but we could also take $\Omega^\infty \Sigma_+^\infty$ because Ω^∞ automatically outputs $\mathbb{E}_\infty$-spaces. These constructions are related, but a modification is required because Ω^∞ outputs group-like $\mathbb{E}_\infty$-spaces.

Theorem 1.67. For any space X, there is an isomorphism

$$\Omega \mathrm{B} \, \mathrm{Free}_{\mathbb{E}_\infty} X \cong \Omega^\infty \Sigma_+^\infty X$$

of group-like $\mathbb{E}_\infty$-spaces.

Proof. We use the Yoneda lemma: for any group-like $\mathbb{E}_\infty$-space A, which we may naturally view as a spectrum, we see

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{E}_\infty}(\Omega^\infty \Sigma_+^\infty X, A) &= \mathrm{Hom}_{\mathbb{S}}(\Sigma_+^\infty X, A) \\
&= \mathrm{Hom}_{\mathrm{Spaces}}(X, A) \\
&= \mathrm{Hom}_{\mathbb{E}_\infty}(\mathrm{Free}_{\mathbb{E}_\infty} X, A) \\
&= \mathrm{Hom}_{\mathbb{E}_\infty}(\Omega \mathrm{B} \, \mathrm{Free}_{\mathbb{E}_\infty} X, A),
\end{aligned}$$

where the last equality holds because $\Omega^\infty A$ is group-like. ∎

Example 1.68. Note that $\Omega \mathrm{B} \, \mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt} = \Omega^\infty \Sigma_+^\infty \mathrm{pt}$, which is $\Omega^\infty \mathbb{S}$. On the other hand, $\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt}$ by definition(!) is

$$\mathrm{pt} \sqcup \mathrm{pt} \sqcup (\mathrm{pt} \times \mathrm{pt})_{h\Sigma_2} \sqcup \cdots,$$

which is $\mathrm{B}\Sigma_0 \sqcup \mathrm{B}\Sigma_1 \sqcup \mathrm{B}\Sigma_2 \sqcup \cdots$, also known as the category of finite sets up to isomorphism. Thus, we see that the "group completion" of the category of finite sets (up to isomorphism) is $\Omega^\infty \mathbb{S}$, indicating that the group completion of the category of finite sets ought to be $\mathbb{S}$.

Remark 1.69 (telescope). There is a space sitting between $\mathrm{FinSet}^{\cong}$ and its group completion $\Omega^\infty \mathbb{S}$. It is the colimit of the diagram

$$\mathrm{FinSet}^{\cong} \to \mathrm{FinSet}^{\cong} \to \cdots,$$

where the individual maps "add 1" in the sense that they are $X \mapsto X \sqcup \mathrm{pt}$. As such, this space turns out to be isomorphic to $\mathbb{Z} \times \mathrm{B}\Sigma_\infty$. The induced map $\mathbb{Z} \times \mathrm{B}\Sigma_\infty \to \Omega^\infty \mathbb{S}$ is not an isomorphism, but it becomes an isomorphism after Σ_+^∞.

Remark 1.70. The free $\mathbb{E}_\infty$-ring on $\mathbb{S}$ turns out to be

$$\Sigma_+^\infty \mathrm{FinSet}^\cong = \bigoplus_{n \geq 0} \Sigma_+^\infty B\Sigma_n.$$

In particular, we see that taking π_0 gives $\mathbb{Z}[x_0]$. Notably, taking a further group completion will invert by this x_0.

We can also take some free $\mathbb{E}_\infty$-spaces of pointed spaces.

Definition 1.71. Fix a pointed space X. Then $\mathrm{Free}_{\mathbb{E}_\infty,*} X$ is the pushout of the following diagram.

$$\begin{array}{ccc}
\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt} & \longrightarrow & \mathrm{Free}_{\mathbb{E}_\infty} X \\
\downarrow & & \downarrow \\
\mathrm{pt} & \longrightarrow & \mathrm{Free}_{\mathbb{E}_\infty,*} X
\end{array}$$

Theorem 1.72 (May)**.** If X is a pointed space, then

$$\Omega B \, \mathrm{Free}_{\mathbb{E}_\infty,*} X = \Omega^\infty \Sigma^\infty X.$$

Example 1.73. If X is a connected pointed space, then one can compute that $\mathrm{Free}_{\mathbb{E}_\infty,*} X$ is connected (intuitively, the connected components of $\mathrm{Free}_{\mathbb{E}_\infty} X$ have collapsed). Thus, $\mathrm{Free}_{\mathbb{E}_\infty,*} X$ is already group-like (it's π_0 is just a point!), so $\mathrm{Free}_{\mathbb{E}_\infty,*} X = \Omega^\infty \Sigma^\infty X$. Thus, we may view $\Omega^\infty \Sigma^\infty X$ as the "group completion" of X.

Here is an application of our "free" constructions which does not use the word "free."

Theorem 1.74 (Snaith, Jones)**.** Fix a connected pointed space X. Then

$$\Sigma_+^\infty \Omega^\infty \Sigma^\infty X = \mathbb{S} \oplus \Sigma^\infty X \oplus (\Sigma^\infty X)_{h\Sigma_2}^{\otimes 2} \oplus \cdots .$$

Proof. Recall that we have a pushout

$$\begin{array}{ccc}
\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt} & \longrightarrow & \mathrm{Free}_{\mathbb{E}_\infty} X \\
\downarrow & & \downarrow \\
\mathrm{pt} & \longrightarrow & \mathrm{Free}_{\mathbb{E}_\infty,*} X
\end{array}$$

of $\mathbb{E}_\infty$-spaces. Note that the bottom-right is $\Omega^\infty \Sigma^\infty X$ by Theorem 1.72. Hitting this with the left adjoint Σ_+^∞ produces a pushout

$$\begin{array}{ccc}
\mathrm{Free}_{\mathbb{E}_\infty} \mathbb{S} & \longrightarrow & \mathrm{Free}_{\mathbb{E}_\infty} \Sigma_+^\infty X \\
\downarrow & & \downarrow \\
\mathbb{S} & \longrightarrow & \Sigma_+^\infty \Omega^\infty \Sigma^\infty X
\end{array}$$

of $\mathbb{E}_\infty$-rings, where the top-right is has $\Sigma_+^\infty X = \Sigma^\infty X \oplus \mathbb{S}$ and so

$$\begin{aligned}
\mathrm{Free}_{\mathbb{E}_\infty} \Sigma_+^\infty X &= \mathrm{Free}_{\mathbb{E}_\infty} (\Sigma^\infty X \oplus \mathbb{S}) \\
&= \mathrm{Free}_{\mathbb{E}_\infty} \Sigma^\infty X \otimes_{\mathbb{S}} \mathrm{Free}_{\mathbb{E}_\infty} \mathbb{S} \\
&= \mathrm{Free}_{\mathbb{E}_\infty} \Sigma^\infty X.
\end{aligned}$$

The result now follows by computing our pushout.   ∎

Remark 1.75. This is a remarkable theorem! Basically, it tells us that taking Σ_+^∞ of some group-like $\mathbb{E}_\infty$-space should have homology which splits in a very natural way.

Remark 1.76. It turns out that $\Sigma_+^\infty \Omega^\infty \mathbb{S} = \mathrm{Free}_{\mathbb{E}_\infty} \mathbb{S}\left[x^{-1}\right]$ for some class x arising from the class of a point (in degree one). Note that Theorem 1.74 does not apply here because the space S^0 is not connected! The general statement is that

$$\Sigma_+^\infty \Omega^\infty \Sigma^\infty X = \mathrm{Free}_{\mathbb{E}_\infty}(X)\left[I^{-1}\right],$$

where I is the set of elements in the image of $\pi_0 X \to \pi_0 \Sigma^\infty X$. The proof of this statement amounts to "forcing" π_0 to be a group.

1.2.4 The Transfer

Here is a more complicated free example.

1.3 February 9

Here we go.

Remark 1.77. The functor Σ_+^∞ taking $\mathbb{E}_\infty$-spaces to $\mathbb{E}_\infty$-rings admits a left adjoint known as Ω^∞. Note that $\Omega^\infty R$ is merely a (multiplicative) $\mathbb{E}_\infty$-space, which need not be grouplike.

1.3.1 Fun with Σ_2

Let's do a little combinatorics with $\mathrm{FinSet}(\Sigma_2)$.

Example 1.78. There is a single point set, which has automorphism group Σ_2.

Example 1.79. The two-elements sets are $\mathrm{pt} \sqcup \mathrm{pt}$, whose stabilizer is Σ_2, and Σ_2, whose stabilizer is also Σ_2.

Example 1.80. There is a three-element set $\mathrm{pt}^{\sqcup 3}$, whose stabilizer is Σ_3. There is also the three-element set $\mathrm{pt} \sqcup \Sigma_2$, whose stabilizer is Σ_2.

Example 1.81. Consider the four-element set $\Sigma_2 \sqcup \Sigma_2$. Then the automorphism group has eight elements (one can swap within the Σ_2s or also swap the two copies of Σ_2), and one can draw a square with each Σ_2 on the diagonals to show that the stabilizer is D_8.

Remark 1.82. There is a fiber sequence

$$\mathrm{B}C_2 \times \mathrm{B}C_2 \to \mathrm{B}D_8 \to \mathrm{B}C_2$$

induced by the short exact sequence of groups. It follows that $\mathrm{B}D_8 = (\mathrm{B}C_2 \times \mathrm{B}C_2)_{h\Sigma_2}$. In particular, $\mathrm{B}D_8$ lives in $\mathrm{Free}_{\mathbb{E}_\infty}(\mathrm{B}C_2)$.

Example 1.83. Consider the full category of finite Σ_2-sets, considered up to isomorphism. Such a set is a union of some points and some two-element sets with a swapping action. Enumerating all such finite sets, we find that this category is $\mathrm{Free}_{\mathbb{E}_\infty}(\mathrm{pt} \sqcup B\Sigma_2)$. By Theorem 1.67, we thus see that

$$\Omega B\,\mathrm{Free}_{\mathbb{E}_\infty}(\mathrm{pt} \sqcup B\Sigma_2) = \Omega^\infty\left(\mathbb{S} \oplus \Sigma_+^\infty B\Sigma_2\right).$$

Here are some linear maps with $\mathrm{FinSet}^{\cong}$.

- There is a forgetful functor $\mathrm{FinSet}(\Sigma_2) \to \mathrm{FinSet}$ which descends to isomorphism classes. It also sends disjoint unions to disjoint unions, so by taking group completion, one receives a map

$$\Sigma_+^\infty B\Sigma_2 \oplus \mathbb{S} \to \mathbb{S},$$

  so there is a "transfer" map $\mathrm{tr}\colon \Sigma_+^\infty B\Sigma_2 \to \mathbb{S}$. (Note that we have silently removed our Ω^∞, which we can do because the forgetful functor preserves disjoint unions, so Ω^∞ can be removed by Theorem 1.66 because these things are group-like.) It turns out that this map is given by $(\mathrm{tr}, 1)$.

- There is a map $\mathrm{FinSet} \to \mathrm{FinSet}(\Sigma_2)$ giving a finite set the trivial Σ_2-action. It sends disjoint unions to disjoint unions, so one can argue as above to show that we receive a map

$$\Omega^\infty \mathbb{S} \to \Omega^\infty \mathbb{S} \times \Omega^\infty \Sigma_+^\infty B\Sigma_2$$

  on group completions. It is the inclusion into the first factor, as one can see by tracking around our isomorphisms.

- There is a map $\mathrm{FinSet}(\Sigma_2) \to \mathrm{FinSet}$ which takes the quotient by the Σ_2-action. This time, the corresponding map on group completion

$$\Sigma_+^\infty B\Sigma_2 \oplus \mathbb{S} \to \mathbb{S},$$

  and it takes the form $(\varepsilon, 1)$.

Remark 1.84. It turns out that the map ε is an unstable map, coming from the natural projection $\mathbb{RP}^\infty \to \mathrm{pt}$. The map tr is stable.

Remark 1.85 (Siegel's conjecture)**.** It turns out that all maps $\Sigma_+^\infty B\Sigma_2 \to \mathbb{S}$ can be obtained from tr and ε.

1.3.2   The Kahn–Priddy Theorem

Here are some nonlinear maps.

Example 1.86 (squaring)**.** There is a squaring map $x \mapsto x^2$ on $\Omega^\infty \mathbb{S}$. This squares on π_0, so it is not linear. Nonetheless, it sends 0 to 0, so it is at least pointed if 0 is the basepoint. We claim that it vanishes on higher homotopy groups.

Proof. An element in $\pi_i \Omega^\infty \mathbb{S}$ amounts to the data of a map $f\colon S^i \to \Omega^\infty \mathbb{S}$, and we can see that squaring is the map

$$S^i \xrightarrow{\Delta} S^i \times S^i \xrightarrow{(f,f)} \Omega^\infty \mathbb{S} \times \Omega^\infty \mathbb{S} \xrightarrow{m} \Omega^\infty \mathbb{S}.$$

For $i > 0$, it turns out that the map $S^i \to S^i \times S^i$ factors (up to homotopy) with $S^i \vee S^i$, which we can see by pinching some corners. Now, mapping out of $S^i \vee S^i$ is determined by its map on each of the factors, which we can see is nullhomotopic in the composite. It follows that the squaring map $x \mapsto x^2$ is trivial on higher homotopy groups. ∎

Example 1.87 (Norm)**.** There is a canonical factorization

$$\Omega^\infty(\Sigma_+^\infty B\Sigma_2 \oplus \mathbb{S})$$

$$\Omega^\infty\mathbb{S} \xrightarrow[x \mapsto x^2]{} \Omega^\infty\mathbb{S}$$

with the map Norm (dashed) into the top object and a vertical map $(\text{tr},1)$.

which defines the norm.

Proof. Combinatorially, the point is that there is a composite

$$\text{FinSet}^{\cong} \to \text{FinSet}(\Sigma_2)^{\cong} \to \text{FinSet}$$

where the left map is squaring, and the right map is forgetful. However, squaring does not preserve disjoint unions, so we cannot just take group completion. However, it turns out that we can move down to the telescope (as in Remark 1.69)

$$\mathbb{Z} \times B\Sigma_\infty \to \text{Telescope}(\text{FinSet}(\Sigma_2)) \to \mathbb{Z} \times B\Sigma_\infty,$$

where the middle object is the colimit of $\text{FinSet}(\Sigma_2)$ where we are adding in $\text{pt} \sqcup B\Sigma_2$ at all levels. We are interested in getting this composite to map to $\Omega^\infty\mathbb{S}$, but by the adjunction, we can instead attempt to get the composite

$$\Sigma_+^\infty(\mathbb{Z} \times B\Sigma_\infty) \to \Sigma_+^\infty \text{Telescope}(\text{FinSet}(\Sigma_2)) \to \Sigma_+^\infty(\mathbb{Z} \times B\Sigma_\infty)$$

to $\mathbb{S}$. However, the above composite is made of group completions—we are using the "Group completion theorem," whose proof is some messing around with the Yoneda lemma. ∎

We will want the following result.

Theorem 1.88 (Kahn−Priddy)**.** The map $\pi_* \text{tr}\colon \pi_*\Sigma_+^\infty B\Sigma_2 \to \pi_*\mathbb{S}$ is surjective in positive degrees.

Proof. Consider the map $\Omega^\infty\mathbb{S} \to \Omega^\infty\Sigma_+^\infty B\Sigma_2$ by $X \mapsto X^2 \setminus X$, where we are thinking about this as coming from finite sets (mapping to free Σ_2-sets), which upgrades to the group completion (at least as a map to $\Omega^\infty\mathbb{S}$) by the above telescoping trick. The total composite is now $x \mapsto x^2 - x$ on π_0 and so an isomorphism on higher homotopy groups because we showed earlier in Example 1.86 that squaring vanishes on higher homotopy. ∎

1.3.3   Nishida Nilpotence

Before we jump into Nishida nilpotence, we will need the D_2 construction.

Definition 1.89 (D_2 construction)**.** Suppose R is an $\mathbb{E}_\infty$-ring, and choose an element $x \in \pi_0 R$, which provides the data of a map $x\colon \mathbb{S} \to R$. We can then form a composite

$$\left(\mathbb{S}^{\otimes 2}\right)_{h\Sigma_2} \to \left(R^{\otimes_{\mathbb{S}}2}\right)_{h\Sigma_2} \to R,$$

where the left map is $x \otimes x$, and the right map is multiplication. Note that the left object is $\text{colim}_{B\Sigma_2} \mathbb{S} = \Sigma_+^\infty \text{colim}_{B\Sigma_2} \text{pt} = \Sigma_+^\infty B\Sigma_2$, so we see that we have found an element $D_2(x) \in R^0(\mathbb{RP}^\infty)$.

Example 1.90. Take $R = \mathbb{S}$. Then $D_2(1) = \varepsilon$. Indeed, the multiplication map $\mathbb{S}_{h\Sigma_2}^{\otimes 2} \to \mathbb{S}$ is Σ_+^∞ of the projection $B\Sigma_2 \to \text{pt}$, so we can see that the total composite is in fact ε.

Example 1.91. Take $R = \mathbb{S}$. To compute $D_2(2)$, we note that

$$(\mathbb{S} \oplus \mathbb{S})^{\otimes 2} = \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{S},$$

and the canonical swapping map on the left fixes two factors and swaps the other two. (Indeed, the isomorphism is basically given by $(x + y)^2 = x^2 + xy + yx + y^2$.) Taking $(-)_{h\Sigma_2}$, it turns out that we get

$$\Sigma_+^\infty \mathbb{RP}^\infty \to \Sigma_+^\infty \mathbb{RP}^\infty \oplus \mathbb{S} \oplus \Sigma_+^\infty \mathbb{RP}^\infty \to \Sigma_+^\infty \mathbb{RP}^\infty.$$

In particular, $(\mathbb{S} \oplus \mathbb{S})_{h\Sigma_2}$ can be seen to be $\mathbb{S}$ again by considering a left Kan extension of the map $\mathbb{S}\colon \mathrm{pt} \to \mathrm{Spectra}$ to $B\Sigma_2$. Now, it turns out that the left map is $(1, \mathrm{tr}, 1)$ (one can see that the middle map is transfer by returning to combinatorics of finite sets), so by mapping back to $\mathbb{S}$, we see that $D_2(2) = 2\varepsilon + \mathrm{tr}$.

Remark 1.92. The method of Example 1.91 can show that $D_2(a + b) = D_2(a) + D_2(b) + ab\,\mathrm{tr}$ for any $a, b \in \pi_0 \mathbb{S}$. For example, $D_2(4) = r\varepsilon + 6\,\mathrm{tr}$.

And here is our theorem.

Theorem 1.93 (Nishida nilpotence). If $x \in \pi_i \mathbb{S}$ for $i > 0$, then x is nilpotent.

Proof. It suffices to show that $\mathbb{S}\left[x^{-1}\right]$ is the trivial ring, meaning that we want to show that $1 = 0$ in $\pi_0 \mathbb{S}\left[x^{-1}\right]$. Now, every higher homotopy group is torsion, so there is at least some positive integer N for which $N = 0$ in $\pi_0 \mathbb{S}\left[x^{-1}\right]$. If N admits no prime factors, then we are done. Otherwise, choose a prime p dividing N, and we would like to replace N with N/p, for which it is enough to check that $\mathbb{S}_{(p)}\left[x^{-1}\right] = 0$.

    We will give the proof in an example. For example, let's work with $p = 2$ so that we want to show $\mathbb{S}_{(2)}\left[x^{-1}\right] = 0$. By the discussion of the previous paragraph, we may even assume that some power of 2 vanishes. Further specializing with loss of generality, let's say that 4 vanishes. (The same idea turns out to work in general.) This means that the composite

$$\mathbb{S} \xrightarrow{4} \mathbb{S} \to \mathbb{S}_{(2)}\left[x^{-1}\right]$$

is nullhomotopic. Squaring, we can form the diagram

$$
\begin{array}{ccc}
\mathbb{S}_{h\Sigma_2}^{\otimes 2} & \xrightarrow{\ 4\ } \mathbb{S}_{h\Sigma_2}^{\otimes 2} \longrightarrow & \mathbb{S}_{(2)}\left[x^{-1}\right]^{\otimes 2} \\
\| & & \downarrow{\scriptstyle m} \\
\mathbb{S} & \longrightarrow & \mathbb{S}_{(2)}\left[x^{-1}\right]
\end{array}
$$

where the top composite is nullhomotopic. But the zigzag path is $D_2(4)$, so we conclude that the composite

$$\mathbb{S}_{h\Sigma_2}^{\otimes 2} \xrightarrow{D_2(4)} \mathbb{S} \to \mathbb{S}_{(2)}\left[x^{-1}\right]$$

is nullhomotopic. Now, $D_2(4) = 4 + 6\,\mathrm{tr}$, so because $4 = 0$, we further see that the composite

$$\mathbb{S}_{h\Sigma_2}^{\otimes 2} \xrightarrow{2\,\mathrm{tr}} \mathbb{S} \to \mathbb{S}_{(2)}\left[x^{-1}\right]$$

is nullhomotopic. But now Theorem 1.88 grants us a map $\mathbb{S}^i \to \Sigma_+^\infty \mathbb{RP}^\infty$ which goes back to x under $\pi_* \mathrm{tr}$, so it follows that

$$\mathbb{S}^i \xrightarrow{2x} \mathbb{S} \to \mathbb{S}_{(2)}\left[x^{-1}\right]$$

is nullhomotopic. Thus, $2x = 0$ in $\mathbb{S}_{(2)}\left[x^{-1}\right]$, so $2 = 0$.

    Let's at least say a sentence or two about why this should work in general. Continuing with $p = 2$, the point is that $D_2(2a) = 2D_2(a) + a^2\,\mathrm{tr}$, and it turns out that the trace term again has lower 2-adic valuation. Thus, one can expect to be able to induct and handle general powers of 2. For other primes p, we need a D_p construction, and again, the point is that the multinomial coefficients in an expansion $(a_1 + \cdots + a_p)^p$ will have smaller p-adic valuation. ∎

Remark 1.94. There is a specific element in $\kappa \in \pi_{24}\mathbb{S}$ for which we do not know if $\kappa^{11} = 0$ or if merely $\kappa^{12} = 0$.

Remark 1.95. One can upgrade this argument to show the following: for an $\mathbb{E}_\infty$-ring R, one has $R = 0$ if and only if $\mathbb{Z} \otimes_{\mathbb{S}} R = 0$. (This truly uses the $\mathbb{E}_\infty$-structure: it is false if one tries to work with $\mathbb{E}_n$-rings for any finite n!) In particular, plugging in $\mathbb{S}\left[x^{-1}\right]$ for R recovers Theorem 1.93.

1.4 February 11

Today, we say a little bit about MU. There are many existing sources: there is Adams's red book, Ravenel's green book, Wilson's *An Introduction and Sampler*, Lurie's chromatic homotopy notes, Pstragowki's book, and Peterson's book.

1.4.1 Formal Group Laws

Let's start with some pure algebra.

Definition 1.96 (formal group law). Fix a commutative ring R. Then a *formal group law* over R is a power series $f \in R[[x, y]]$ such that

(a) $f(x, 0) = f(0, x) = x$,

(b) $f(x, y) = f(y, x)$, and

(c) $f(f(x, y), z) = f(x, f(y, z))$.

We may write $x +_f y$ for $f(x, y)$.

Remark 1.97. One is supposed to think of f as providing the formula for an actual group law, under the assumption that f always converges. More rigorously, these are supposed to be additions on $\widehat{\mathbb{A}}^1$.

Definition 1.98 (isomorphism). Fix a commutative ring R. An *isomorphism* $g \colon f_1 \to f_2$ of formal group laws is a power series $g \in R[[x]]$ for which $g(0) = 0$, $g'(0) \in R^\times$ and

$$g(x +_{f_1} y) = x +_{f_2} y.$$

We say that g is *strict* if and only if $g'(0) = 1$.

Example 1.99. There is the additive formal group law $f(x, y) := x + y$.

Example 1.100. There is the multiplicative formal group law $f(x, y) = x + y + xy$.

There is an important "universal" formal group law.

Definition 1.101 (Lazard ring)**.** The *Lazard ring* L is defined to co-represent the functor sending a ring R of all formal group laws over R. Explicitly, we can realize it as a quotient of the ring $\mathbb{Z}[\{a_{ij}\}_{ij}]$ by the relations required to make the power series

$$\sum_{i,j \geq 0} a_{ij} x^i y^j$$

into a formal group law. We give L a grading by $\deg a_{ij} := 2(i + j - 1)$.

Definition 1.102. We define $\mathbb{Z}[b_1, b_2, \ldots]$ to be the universal ring equipped with a formal group law and an isomorphism of this formal group law. Explicitly, it is a ring $\mathbb{Z}[b_1, b_2, \ldots]$ equipped with a map $\varphi : L \to \mathbb{Z}[b_1, b_2, \ldots]$ to classify $g\left(g^{-1}(x) + g^{-1}(y)\right)$, where $g(x) := x + b_1 x^2 + b_2 x^3$. We give this ring a grading by $\deg b_i = 2i$, so the map φ is graded.

Remark 1.103. One can see that $L[b_1, b_2, \ldots]$ classifies two formal group laws and a choice of strict isomorphism between them. Indeed, L provides the formal group law, and the $b_\bullet$s provide the strict isomorphism.

Theorem 1.104 (Lazard)**.** For all $n > 0$, the map $\varphi_{2n} : L_{2n} \to \mathbb{Z}[b_1, b_2, \ldots]_{2n}$ is an injection after taking a quotient by the decomposable elements. Explicitly, the target is isomorphic to $\mathbb{Z}b_{2n}$, and the image is full if $n + 1$ is not a prime power; otherwise, the image is $p\mathbb{Z}$ if $n + 1$ is a power of p.

Idea. The hard part is to show that L_{2n} modulo decomposable elements is isomorphic to $\mathbb{Z}$. Accordingly, let this quotient be Q_{2n}. Then one needs to exhibit a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}}(Q_{2n}, M) \cong M$, but one finds that maps $Q_{2n} \to M$ are given by formal group laws over the square zero extension $\mathbb{Z} \oplus M$ of the form

$$x + y + \sum_{i+j=n+1} x^i y^j.$$

Then one needs to classify such things, which is an algebra problem with binomial coefficients. ∎

Corollary 1.105. For each i, choose $x_i \in L$ of degree $2i$ which generates L_{2i} modulo decomposable elements. Then the induced map

$$\mathbb{Z}[x_1, x_2, \ldots] \to L$$

is an isomorphism of rings.

Proof. This follows from Theorem 1.104: indeed, Theorem 1.104 shows that we have produced an isomorphism after passing to the associated graded ring for the relevant filtration, which is enough to show that we have an isomorphism. ∎

Thus, L is just a polynomial ring! However, this presentation is not very helpful because it requires infinitely many choices. Alternatively, the map φ is an isomorphism after taking tensor product with $\mathbb{Q}$ by Theorem 1.104, but taking this tensor product may lose information we care about.

1.4.2   The Spectrum MU

We now return to algebraic topology. We would like to make sense of the following theorem.

Theorem 1.106 (Quillen)**.** There is an $\mathbb{E}_\infty$-ring MU satisfying the following.

(a) $\pi_* \mathrm{MU} \cong L$.

(b) The induced map $\pi_* \mathrm{MU} \to \pi_*(\mathbb{Z} \otimes_{\mathbb{S}} \mathrm{MU})$ is isomorphic to φ.

(c) There are isomorphisms $\pi_*(\mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU}) \cong \pi_* \mathrm{MU}[b_1, b_2, \dots] \cong L[b_1, b_2, \dots]$ such that the two induced maps $\pi_* \mathrm{MU} \to \pi_*(\mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU})$ correspond to the source and target of the universal formal group laws of $L[b_1, b_2, \dots]$.

Of course, we will have to start by constructing MU.

Remark 1.107. There is a functor $S \colon \mathrm{Vec}_{\mathbb{C}} \to \mathrm{Top}_*$ of ordinary category given by taking the one-point compactification. It satisfies $S(V \oplus W) = SV \wedge SW$, so S is symmetric monoidal, and we get a map

$$\bigsqcup_{n \geq 0} \mathrm{BU}(n) \to \mathrm{Top}_* \to \mathrm{Spaces}_*$$

sending V to $\Sigma^\infty S^V$.

Definition 1.108 (Picard group)**.** Fix an $\mathbb{E}_\infty$-ring R. Then $\mathrm{Pic}\, R$ is the full subcategory of $\mathrm{Mod}(R)$ of $\otimes$-invertible objects, only equipped with the isomorphisms.

Example 1.109. The only invertible objects in $\mathrm{Pic}\, \mathbb{S}$ are powers of $\mathbb{S}$. Here is a sketch: any invertible object remains invertible after $- \otimes_{\mathbb{S}} \mathbb{Z}$, so it becomes an invertible object in $\mathrm{Mod}(\mathbb{Z})$. But the only invertible chain complexes are shifts of $\mathbb{Z}$, so we know that any invertible object has homology supported in one degree. By the Hurewicz theorem, it therefore has homotopy starting in one degree (and zero below), so we receive a map from a sphere, and one can check that it is an isomorphism by a variant of Whitehead's theorem.

Remark 1.110. Note that $\mathrm{Pic}\, R$ is in fact a group-like $\mathbb{E}_\infty$-space, where the operation is given by $\otimes$.

Remark 1.111. It thus turns out that the map $\bigsqcup_{n \geq 0} \mathrm{BU}(n) \to \mathrm{Spectra}$ given by $V \mapsto \Sigma^\infty SV$ factors through $\mathrm{Pic}\, \mathbb{S}$ because it only produces powers of spheres.

Definition 1.112 (Thom spectrum)**.** Fix an $\mathbb{E}_\infty$-ring R. If $f \colon X \to \mathrm{Pic}\, R$ is a map of groupoids, then we define the *Thom spectrum* $\mathrm{Thom}(f)$ to be the colimit of the diagram

$$X \to \mathrm{Pic}\, R \to \mathrm{Mod}(R).$$

Remark 1.113. Intuitively, the map $f \colon X \to \mathrm{Pic}\, R$ is a bundle of invertible spectra on R, so we are recovering a classical notion.

Example 1.114. We define the map $\bigsqcup_{n \geq 0} \mathrm{BU}(n) \to \mathrm{Pic}\, \mathbb{S}$ to have Thom spectrum $\bigoplus_{n \geq 0} \mathrm{MU}(n)$.

Example 1.115. We claim that $\mathrm{MU}(1)$ is $\Sigma^\infty \mathbb{CP}^\infty$.

Proof. We are looking at the colimit which $\mathrm{BU}(1) \to \mathrm{Spectra}$ which sends a line bundle $\mathcal{L} \in \mathrm{B}S^1$ to the spectrum $\Sigma^\infty S\mathcal{L}$. Let $S^1\mathcal{L}$ be the unit sphere in $\mathcal{L}$, and then there is a homotopy fiber sequence

$$S^1\mathcal{L}_+ \to \mathrm{pt}_+ \to S\mathcal{L}.$$

Intuitively, this homotopy fiber sequence is gluing some part of $S\mathcal{L}$ against a point to produce a sphere with a point.

It follows that $\mathrm{MU}(1)$ can be computed as

$$\begin{aligned}
\operatorname*{colim}_{\mathrm{B}S^1} \Sigma^\infty S\mathcal{L} &= \Sigma^\infty \operatorname*{colim}_{\mathrm{B}S^1} S\mathcal{L} \\
&= \Sigma^\infty \operatorname*{colim}_{\mathrm{B}S^1} \mathrm{cofiber}\left(S^1\mathcal{L}_+ \to \mathrm{pt}_+\right) \\
&= \mathrm{cofiber}\left(\operatorname*{colim}_{\mathrm{B}S^1} \Sigma^\infty_+ S^1\mathcal{L} \to \operatorname*{colim}_{\mathrm{B}S^1} \Sigma^\infty_+ \mathrm{pt}\right).
\end{aligned}$$

Now, the right colimit has Σ^∞_+ come out, so it is $\Sigma^\infty_+ \mathbb{CP}^\infty$. The left colimit is the colimit of the group S^1 acting on itself, which is the sphere $\mathbb{S}$. Now, the map $\mathbb{S} \to \Sigma^\infty_+ \mathbb{CP}^\infty$ just gets rid of the point, so the cofiber is $\Sigma^\infty \mathbb{CP}^\infty$. ∎

Now, we recognize that we have a map $\bigsqcup_{n\geq 0} \mathrm{BU}(n) \to \mathrm{Pic}\,\mathbb{S}$ which has target equal to a grouplike $\mathbb{E}_\infty$-space. Thus, we can take a group completion to produce a map

$$\Omega\mathrm{B}\bigsqcup_{n\geq 0} \mathrm{BU}(n) \to \mathrm{Pic}\,\mathbb{S}.$$

The left-hand side is just $\mathrm{BU} \times \mathbb{Z}$ (here, BU is the colimit BU_∞), and it has no negative homotopy groups, so we can realize it as some $\Omega^\infty \mathrm{ku}$ for some ku.

> **Definition 1.116.** Fix everything as above. Then MU is the Thom spectrum of the sequence
> $$\mathrm{BU} \to \mathrm{BU} \times \mathbb{Z} \to \mathrm{Pic}\,\mathbb{S}.$$

> **Remark 1.117.** One can realize the left map as Ω^∞ applied to the sequence $\tau_{\geq 2}\mathrm{ku} \to \mathrm{ku} \to \mathrm{pic}\,\mathbb{S}$, where $\mathrm{pic}\,\mathbb{S}$ is the space for which $\Omega^\infty \mathrm{pic}\,\mathbb{S} = \mathrm{Pic}\,\mathbb{S}$.

> **Remark 1.118.** It was important to mention Pic so that we could take group completion.

> **Remark 1.119.** Our definition of MU automatically gives it the structure of a group-like $\mathbb{E}_\infty$-space. It is already a colimit valued in $\mathbb{S}$-modules, so MU becomes an $\mathbb{E}_\infty$-ring!

> **Remark 1.120.** The Thom spectrum of the map
> $$\Omega\mathrm{B}\bigsqcup_{n\geq 0} \mathrm{BU}(n) \to \mathrm{Pic}\,\mathbb{S}$$
> produces a space MUP, which is isomorphic to $\bigoplus_{k\in\mathbb{Z}} \Sigma^k \mathrm{MU}$. Its ring structure is not unique! It is an open question if the ring structure on MU is unique.

1.4.3 Complex Orientations

Let's find some structure on MU.

Remark 1.121. By taking Thom spectra of the map

$$\mathbb{CP}^\infty \to \bigsqcup_{n \geq 0} \mathrm{BU}(n) \to \mathrm{BU} \times \mathbb{Z} \to \mathrm{Pic}\,\mathbb{S}$$

produces a map $\mathrm{MU}(1) \to \mathrm{MUP}$. Note that the copy of $\mathbb{Z}$ here keeps track of the dimension of the ambient vector bundle, so our copy of $\mathrm{BU}(1)$ will land in the first component of $\mathrm{BU} \times \mathbb{Z}$. Expanding out MUP, we see that we have induced a map $\Sigma^\infty \mathbb{CP}^\infty \to \Sigma^2 \mathrm{MU}$.

The sort of gadget in Remark 1.121 is important enough to deserve a name.

Definition 1.122 (complex orientation). Fix an $\mathbb{E}_\infty$-ring R. A *complex orientation* is a map $\Sigma^\infty \mathbb{CP}^\infty \to \Sigma^2 R$ for which the composite

$$\Sigma^\infty \mathbb{CP}^1 \to \Sigma^\infty \mathbb{CP}^\infty \to \Sigma^2 R$$

is the unit map $\Sigma^2 \mathbb{S} \to \Sigma^2 R$.

Complex orientations give rise to formal group laws. Let's explain how this is done.

Remark 1.123. A choice of complex orientation yields an isomorphism

$$R^* \mathbb{CP}^\infty \cong \pi_{-*} R[[t]],$$

where $t \in R^2 \mathbb{CP}^\infty$ is the class induced by the complex orientation. Indeed, this is some spectral sequence calculation. The point is that the Atiyah–Hirzebruch spectral sequence tells us that $R^* \mathbb{CP}^\infty$ certainly looks like $\pi_{-*} R[[t]]$ on E_2, and there cannot be differentials making this smaller because t is a genuine cohomology class.

Remark 1.124. One can similarly show that a complex orientation on R produces an isomorphism

$$R^* (\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_{-*} R[[x, y]].$$

Remark 1.125. The map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ given by the tensor product of lines is the same data as a map $\Sigma^2 \mathbb{Z} \oplus \Sigma^2 \mathbb{Z} \to \Sigma^2 \mathbb{Z}$ after Ω^∞. This then induces a map $\pi_{-*} R[[t]] \to \pi_{-*} R[[x, y]]$. The image of t turns out to be a formal group law, which one can see all the back from the line bundle maps.

Thus, we have a formal group law on $\pi_* \mathrm{MU}$, so we receive a map $L \to \pi_* \mathrm{MU}$. Theorem 1.106 will tell us that this map is an isomorphism. One next proceeds by producing maps

$$\pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbb{Z}) \to \mathrm{H}_*(\mathrm{MU}; \mathbb{Q}).$$

Thus, it is profitable to understand the target rings. To this end, note that we have a square

$$
\begin{array}{ccc}
\mathrm{Pic}\,\mathbb{S} & \longrightarrow & \mathrm{Mod}(\mathbb{S}) \\
\downarrow & & \downarrow{\scriptstyle -\otimes \mathbb{Z}} \\
\mathrm{Pic}\,\mathbb{Z} & \longrightarrow & \mathrm{Mod}(\mathbb{Z})
\end{array}
$$

where the vertical maps are given by base-change, and the horizontal maps are inclusions. Observe that $\pi_0 \mathrm{Pic}\,\mathbb{Z} = \mathbb{Z}$ because the invertible chain complexes are all just $\mathbb{Z}$ shifted by some degree. It then turns out that $\tau_{\geq 1} \mathrm{Pic}\,\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ embeds into homomorphisms $\mathbb{Z} \to \mathbb{Z}$, which is $\Omega^\infty \mathbb{Z}$.

Now, to compute $\mathrm{H}_*(\mathrm{MU}; \mathbb{Z}) = \pi_*(\mathrm{MU} \otimes_\mathbb{S} \mathbb{Z})$, we are interested in making BU map all the way to $\mathrm{Mod}(\mathbb{Z})$. But $\mathrm{Pic}\,\mathbb{Z}$ only has homotopy groups below 1, and BU is simply connected, so the induced map $\mathrm{BU} \to \mathrm{Mod}(\mathbb{Z})$ is nullhomotopic. It follows that

$$\mathrm{MU} \otimes_\mathbb{S} \mathbb{Z} = \Sigma^\infty_+ \mathrm{BU} \otimes_\mathbb{S} \mathbb{Z},$$

so $\pi_*(\mathrm{MU} \otimes_\mathbb{S} \mathbb{Z}) = \mathrm{H}_*(\mathrm{BU}; \mathbb{Z})$. Then one can make some arguments with spectral sequences and so on.

1.5 February 17

Today we continue.

1.5.1 Maps from MU

Let's try to better understand MU. For example, we can try to understand its functor of points.

> **Remark 1.126.** Choose an $\mathbb{E}_\infty$-ring R. If the composite
>
> $$\mathrm{BU} \to \operatorname{Pic} \mathbb{S} \to \operatorname{Pic} R$$
>
> is nullhomotopic, then taking the colimit to $\operatorname{Mod}(R)$ shows that $R \otimes_{\mathbb{S}} \mathrm{MU}$ is a constant colimit
>
> $$\operatorname*{colim}_{\mathrm{BU}} R = R \otimes \operatorname*{colim}_{\mathrm{BU}} \mathbb{S} = R \otimes \Sigma^\infty_+ \mathrm{BU}.$$
>
> But note BU is already pointed, so $\Sigma^\infty_+ \mathrm{BU} = \Sigma^\infty \mathrm{BU} \oplus \mathbb{S}$, so we receive an R-module map $R \otimes_{\mathbb{S}} \mathrm{MU} \to R$, which is the same data as an $\mathbb{S}$-module map $\mathrm{MU} \to R$.

> **Remark 1.127.** This discussion of nullhomotopies has a precursor: the data of a complex orientation on E is equivalent to the data of a nullhomotopy of the composite $\mathbb{CP}^\infty \to \operatorname{Pic} \mathbb{S} \to \operatorname{Pic} E$. Indeed, simply apply the adjunction between Σ^2 and Ω^2.

> **Remark 1.128.** One can upgrade Remark 1.126 to show that the space of unital maps $\mathrm{MU} \to R$ is equivalent to the data of a nullhomotopy of the composite
>
> $$\mathrm{BU} \to \operatorname{Pic} \mathbb{S} \to \operatorname{Pic} R$$
>
> in Spaces_*. (Indeed, providing a map out of MU is a map out of a certain colimit, and unwinding the data finds the required nullhomotopy.) Furthermore, the space of $\mathbb{E}_n$-ring maps $\mathrm{MU} \to R$ is equivalent to the space of nullhomotopies of the same composite in the category of n-fold loop spaces.

We can work out the above remark in some controlled situations. Here are the "$\mathbb{Z}$-points" of MU.

> **Example 1.129.** The map $\mathrm{BU} \to \operatorname{Pic} \mathbb{S} \to \operatorname{Pic} \mathbb{Z}$ is nullhomotopic in the category of infinite loop spaces because $\operatorname{Pic} \mathbb{Z}$ only has homotopy in two degrees (namely 0 and 1), and BU has no homology there. Thus, there is an isomorphism $\mathbb{Z} \otimes_{\mathbb{S}} \Sigma^\infty_+ \mathrm{BU} \cong \mathbb{Z} \otimes_{\mathbb{S}} \mathrm{MU}$ of $\mathbb{E}_\infty$-rings, so one can compute
>
> $$\mathrm{H}_*(\mathrm{MU}; \mathbb{Z}) \cong \mathrm{H}_*(\mathrm{BU}; \mathbb{Z}),$$
>
> and then one can calculate $\mathrm{H}^*(\mathrm{BU}; \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \ldots]$.

> **Remark 1.130.** Let's explain where the b_is may come from. The natural map $\mathrm{MU}(1) \to \Sigma^2 \mathrm{MU}$ (placing $\mathrm{MU}(1)$ in the correct degree) is the same as the complex orientation $\Sigma^\infty \mathbb{CP}^\infty \to \Sigma^2 \mathrm{MU}$. This then induces a map
>
> $$\mathbb{Z} \otimes_{\mathbb{S}} \Sigma^\infty \mathbb{CP}^\infty \to \mathbb{Z} \otimes_{\mathbb{S}} \Sigma^2 \mathrm{MU}.$$
>
> But now the homology of $\mathbb{CP}^\infty$ is the free $\mathbb{Z}$-module on the generators $b_1, b_2, \ldots$.

Here are the MU-points of MU.

Example 1.131. There is a canonical identity $\mathrm{MU} \to \mathrm{MU}$, which thus induces a nullhomotopy $\mathrm{BU} \to \mathrm{Pic}\,\mathbb{S} \to \mathrm{Pic}\,\mathrm{MU}$ in the category of infinite loop spaces. Thus, we see that

$$\mathrm{MU}_*\mathrm{MU} = \mathrm{MU}_*\Sigma_+^\infty \mathrm{BU}.$$

Now, the Atiyah–Hirzebruch spectral sequence degenerates, so this is $\pi_*\mathrm{MU}[b_1, b_2, \ldots]$. One way to see it degenerates is to know that $\pi_*\mathrm{MU}$ is concentrated in even degrees (via Theorem 1.106). There is also an easier way to see this, basically by directly showing that the b_is are permanent cycles, which by the Leibniz rule is equivalent to showing that b_1 is permanent, which can be checked more directly from $\mathrm{MU}(1)$.

The moral of the previous example is that $\mathrm{MU}_*\mathrm{MU}$ is the universal ring for a pair of formal group laws and a choice of strict isomorphism between them.

Proposition 1.132. Fix an even $\mathbb{E}_\infty$-ring E. Then a complex orientation of E extends to an $\mathbb{E}_2$-map $\mathrm{MU} \to E$.

Proof. Note that there is a canonical line bundle $\mathcal{L} \colon \mathbb{CP}^\infty \to \mathrm{BU} \times \mathbb{Z}$, so there is a map $(\mathcal{L} - 1) \colon \mathbb{CP}^\infty \to \mathrm{BU} \times \mathbb{Z}$, which then maps to $\mathrm{Pic}\,\mathbb{S}$ as before. As discussed last class, we see the Thom spectrum of $(\mathcal{L} - 1)$ is $\Sigma^{-2}\mathrm{MU}(1)$. The moral is that we have some composite

$$\mathbb{CP}^\infty \to \mathrm{BU} \to \mathrm{Pic}\,\mathbb{S} \to \mathrm{Pic}\,E.$$

Taking Σ^2 yields

$$\Sigma^2\mathbb{CP}^\infty \to \mathrm{BSU} \to \mathrm{B}^2\,\mathrm{Pic}\,\mathbb{S} \to \mathrm{B}^2\,\mathrm{Pic}\,E.$$

A nullhomotopy of the full sequence is a complex orientation on E, but a choice of nullhomotopy $\mathrm{BSU} \to \mathrm{B}^2\,\mathrm{Pic}\,E$ on its own is an $\mathbb{E}_2$-map $\mathrm{MU} \to E$. Thus, we want to show that the former maps always extend to the latter maps. By splitting up $\mathrm{B}^2\,\mathrm{Pic}\,E = \Omega^\infty\Sigma^2\,\mathrm{pic}\,E$ into its Postnikov tower, it is enough to check that the maps

$$\mathrm{H}^*(\mathrm{BSU}; -) \to \mathrm{H}^*\left(\Sigma^2\mathbb{CP}^\infty; -\right)$$

are surjective, which can be checked by calculations with the spectral sequence. ∎

Remark 1.133. As a consequence, we note that an even $\mathbb{E}_\infty$-ring E makes $E^*\mathbb{CP}^\infty$ into a formal group. Choosing coordinates means choosing an isomorphism $E^*\mathbb{CP}^\infty \cong \pi_{-*}E[[t]]$, which means choosing a formal group law up to isomorphism and thus induces a map $L \to \pi_*E$. Our discussion is intended to motivate the fact that choosing a coordinate alternatively should come from a choice of complex orientation and therefore the map $L \to \pi_*E$ comes from an $\mathbb{E}_2$-ring map $\mathrm{MU} \to E$. It is an interesting (open) question when one can upgrade this $\mathbb{E}_2$-ring map into an $\mathbb{E}_\infty$-ring map.

1.5.2 Localization

Let's try to apply some of this discussion.

Notation 1.134. Given a formal group law f, we define the power series $[n]_f(x)$ as $x +_f \cdots +_f x$, where x is added to itself n times. We will write $[n]$ for $[n]_f$ if there is no possibility of confusion.

Example 1.135. In the Lazard ring $L_{(p)}$, one finds that

$$[3](x) = 3x - 8x_2x^3 + 72x_2^2x^5 - 840x_2^3x^7 + \cdots,$$

where the $x_\bullet$s are the Hazewinkel generators of $L_{(p)} \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$. (Here, x_i has degree $2i$, and x has degree -2, so the above polynomial is homogeneous.)

Remark 1.136. With the Hazewinkel generators, one can show that the universal formal group law f over $L_{(p)}$ is isomorphic to one that only involves the coefficients $x_{p^\bullet - 1}$, and in fact, up to units, $x_{p^i - 1}$ is the coefficient of x^{p^i}. Thus, setting $v_i := x_{p^i - 1}$, we see that the ring

$$\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$$

carries the universal formal p-typical group law up isomorphism! (Here, p-typical means that our coefficients x_j vanish for j which is not one less than a power of p.) The point is that we only have to care about these coefficients v_i to define such a formal group law.

Example 1.137. There is an idempotent ring endomorphism $f : L_{(p)} \to L_{(p)}$ sending x_i to itself if i is one less than a power of p and sending x_i to zero otherwise. Note that the colimit of the diagram

$$L_{(p)} \xrightarrow{f} L_{(p)} \xrightarrow{f} \cdots$$

is just $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$.

The above remarks prove the following.

Proposition 1.138. There is an $\mathbb{E}_2$-ring map $\mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ whose telescope (in the category of $\mathbb{E}_2$-rings) is an $\mathbb{E}_2$-ring BP with

$$\pi_* \mathrm{BP} \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots],$$

where v_i lives in degree $2p^i - 2$.

Proof. Let's explain how to produce the map $\mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$. Indeed, the algebraic map $L \to \pi_* \mathrm{MU}_{(p)}$ (given by the change of coordinates just described) lifts to an $\mathbb{E}_2$-map $\mathrm{MU} \to \mathrm{MU}_{(p)}$ (because this is how one maps out of MU), which is then further localized. ∎

The moral of BP is that we have removed all of our "junk" generators.

Remark 1.139. It is known that BP is not $\mathbb{E}_\infty$, but it is $\mathbb{E}_4$. Exactly the highest $\mathbb{E}_n$ which works is currently open.

Remark 1.140. As before, one can compute $\mathrm{BP}_* \mathrm{BP}$ as $\mathbb{Z}_{(p)}[v_1, v_2, \ldots][t_1, t_2, \ldots]$, which classifies two formal p-local formal group laws and a choice of p-typical strict isomorphism between them. Namely, the isomorphism is given by $x \mapsto x +_f t_1 x^p +_f t_2 x^{p^2} +_f \cdots$.

1.6   February 18

Let's start using MU to understand $\mathbb{S}$.

1.6.1   Filtrations for Fun and Profit

Here is how we will use MU to get to $\mathbb{S}$.

Lemma 1.141. The spectrum $\mathbb{S}$ is the limit of the following diagram.

$$\mathrm{MU} \rightrightarrows \mathrm{MU}^{\otimes 2} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{MU}^{\otimes 3} \qquad \cdots$$

Proof. This follows from the following general fact: if $A \to B$ is a map of $\mathbb{E}_\infty$-rings whose fiber has homotopy groups only in positive degree, then A is the limit of the Čech nerve of B over A (which is the given colimit). This fact is checked by considering the partial limits and showing that they induce isomorphisms on higher and higher homotopy groups, so we get an isomorphism in the limit. Anyway, it is enough to check that $\mathbb{S} \to \mathrm{MU}$ is an isomorphism in degree zero, which is true by Theorem 1.106. ∎

Notation 1.142. We let $\mathrm{fil}^n \mathbb{S}$ be the limit of the following diagram.

$$\tau_{\geq 2n}\mathrm{MU} \rightrightarrows \tau_{\geq 2n}\mathrm{MU}^{\otimes 2} \rightrightarrows \tau_{\geq 2n}\mathrm{MU}^{\otimes 3} \qquad \cdots$$

Using $2n$ instead of n is a convention, motivated by Theorem 1.106.

Example 1.143. Note that MU has only homotopy groups in nonnegative degrees, we see that $\mathrm{fil}^n \mathbb{S} = \mathbb{S}$ for $n \leq 0$. It follows that the colimit of the diagram

$$\cdots \to \mathrm{fil}^3 \mathbb{S} \to \mathrm{fil}^2 \mathbb{S} \to \mathrm{fil}^1 \mathbb{S} \to \mathrm{fil}^0 \mathbb{S}$$

is $\mathbb{S}$. In fact, the limit of this diagram is 0, which more or less follows from Lemma 1.141.

We are thus motivated to say something about filtrations of spectra.

Definition 1.144. A *filtered spectrum* X_* is a diagram of the form

$$\cdots \to X_{-2} \to X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots .$$

A morphism of such filtered spectra is a morphism of the underlying diagrams, including homotopies for all the relevant squares.

Remark 1.145. It turns out that the category of filtered spectra is then a stable ∞-category. It is further symmetric monoidal, with $\otimes$ given by

$$(X_* \otimes Y_*)_n := \operatorname*{colim}_{i+j \geq n}(X_i \otimes Y_j).$$

For example, if the filtered spectrum is in fact given from a "graded spectrum" (so that the internal maps between the X_*s and Y_*s are all split embeddings), then the right-hand side is $\bigoplus_{i+j=n}(X_i \otimes Y_j)$. This formula shows that the object $\cdots 0 \to 0 \to \mathbb{S} \to \mathbb{S} \to \cdots$ is the identity.

Example 1.146. If E is an $\mathbb{E}_\infty$-ring, then $\{\tau_{\geq 2*}E\}_n$ is an $\mathbb{E}_\infty$-algebra valued in filtered complexes.

Example 1.147. It now makes sense to say that $\mathrm{fil}^* \mathbb{S}$ is the limit of the Čech nerve

$$\tau_{\geq 2*}\mathrm{MU} \to \tau_{\geq 2*}\mathrm{MU}^{\otimes 2} \to \cdots$$

in the category of filtered $\mathbb{E}_\infty$-rings. It follows that $\mathrm{fil}^* \mathbb{S}$ gains the structure of an $\mathbb{E}_\infty$-algebra valued in the category of filtered spectra.

The benefit of doing category theory is that we can make categories.

Definition 1.148 (synthetic spectrum)**.** A *synthetic spectrum* is a module of $\mathrm{fil}^* \mathbb{S}$ in the category of filtered spectra. The (stable ∞-)category of synthetic spectra is denoted $\mathrm{SynSpectra}$.

Example 1.149. There is a functor ν which takes a spectrum X and produces the synthetic spectrum

$$\lim \tau_{\geq 2*} \left(X \otimes \mathrm{MU}^{\otimes \bullet + 1} \right).$$

Here is the point of working with synthetic spectra.

Proposition 1.150. If X is spectrum with homotopy groups bounded below, then $\nu(X)$ is a filtered spectrum $F_\bullet$, and its limit is 0 and colimit is X. There is then a spectral sequence

$$E_2 \colon \pi_* C_* \Rightarrow \pi_* X,$$

where C_i is defined as the cofiber of $F_{i-1} \to F_i$.

Remark 1.151. There are also higher coherences present in this spectral sequence arising because $\nu(E)$ has some higher coherences in its construction. For example, the fact that $\mathrm{fil}^* \, \mathbb{S}$ is a ring produces the Leibniz rule.

1.6.2   The Adams–Novikov Spectral Sequence

Later in life, we will want to localize at a prime p.

Lemma 1.152. For each prime p,

$$\lim \tau_{\geq 2*} \mathrm{MU}_{(p)}^{\otimes \bullet} = \lim \tau_{\geq 2*} \mathrm{BP}_{(p)}^{\otimes \bullet + 1}.$$

Proof. Certainly there is a natural map $\mathrm{BP} \to \mathrm{MU}$, so we just have to check that it induces an isomorphism of the filtered spectra. By some argument with the Five lemma, it is enough to check that it gives an isomorphism on the associated graded spectra. Well, taking this grading passes through the limit, so it becomes a natural map

$$\Sigma^{2i} \lim \pi_{2i} \mathrm{BP}_{(p)}^{\otimes \bullet + 1} \to \Sigma^{2i} \lim \pi_{2i} \mathrm{MU}_{(p)}^{\otimes \bullet + 1},$$

which is now just a morphism taking place in $\mathrm{Mod}(\mathbb{Z})$. We can now use the Dold–Kan correspondence to pass to chain complexes. It turns out that the homology on the left computes cohomology of $\mathcal{M}_{\mathrm{fg},(p)}^{\mathrm{strict}}$ (which is the moduli stack of formal group laws up to strict isomorphism, and then we invert everything away from p), and the homology on the right computes moduli of p-typical formal group laws up to strict isomorphism. ∎

Remark 1.153. In fact, the homology of the aforementioned complex is exactly the starting page of the spectral sequence.

Example 1.154. At $p = 2$, our MU complex looks like

$$\mathbb{Z}_{(2)}[v_1, \ldots] \to \mathbb{Z}_{(2)}[v_1, \ldots][t_1, \ldots] \to \mathbb{Z}_{(2)}[v_1, \ldots][t_1 \otimes t_1, \ldots].$$

For example, one can compute that $d(v_1) = \pm 2 t_1$, which because $\mathbb{Z}_{(2)}$ is $\mathbb{Z}$-torsion-free, we see $d(t_1) = 0$. (Formulae for the differentials have been computed for many degrees.) One then sees that t_1 represents a class in the first homology which is 2-torsion. Unravelling the spectral sequence, one sees that an ith homology class in degree $2j$ (of the polynomial ring) maps to $\pi_{2j-i} \mathbb{S}$. For example, t_1 goes to the Hopf map in $\pi_1 \mathbb{S}$.

> **Remark 1.155.** Even though BP is not an $\mathbb{E}_\infty$-ring, it turns out that $\mathrm{fil}^* \, \mathbb{S}_{(p)} = \lim \tau_{\geq 2*} \mathrm{BP}^{\otimes \bullet + 1}$ is a filtered $\mathbb{E}_\infty$-ring, as we just showed. There are some reasons why $\mathrm{fil}^* \, \mathbb{S}_{(p)}$ may be "more canonical" than $\mathbb{S}$.
>
> - One can find $\mathrm{fil}^* \, \mathbb{S}$ is the limit of the filtered spectra $\tau_{\geq 2*} E$ as E varies over all even $\mathbb{E}_\infty$-rings.
>
> - The p-completed category $(\mathrm{SynSp})_p^\wedge$ embeds fully faithfully into the p-completed category of $\mathbb{C}$-motivic spectra.

One will frequently write down the E_2 page of our associated spectral sequence for $\lim \tau_{\geq 2*} \mathrm{BP}^{\otimes \bullet + 1}$ by graphing along axes i and $2j - i$, where i is the homological degree, and $2j$ is the top degree of the relevant polynomial in the homotopy groups of MU. (Recall that a class in homological degree i and polynomial degree $2j$ maps to $\pi_{2j-i}\mathbb{S}$.) For example, the E_2 page for $p = 2$ looks something like the following.

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & t_1 \otimes t_1 \otimes t_1 & \cdots \\
0 & t_1 \otimes t_1 & ? & \cdots \\
t_1 & ? & ? & \cdots
\end{array}
$$

> **Remark 1.156.** It turns out that multiplication by t_1 is an isomorphism on the E_2 page, above a line of slope $1/5$. But $t_1^4 = 0$ (even though E_2 does not show this), it follows that the E_n page of the Adams–Novikov spectral sequence for large n (which can see $t_1^4 = 0$ vanishes) above some line of slope $1/5$.

Our next target is the following.

> **Theorem 1.157.** For any positive slope, there is a page E_n which vanishes above a line of that slope.

One key idea in the proof will be to not merely view the spectral sequence as some formal algebraic gadget but as an actual object in the category of filtered spectra.

1.6.3 Towards Hopf Algebroids

We now see that we are interested in understanding the Čech nerve

$$\pi_* \mathrm{BP} \to \pi_* \mathrm{BP}^{\otimes 2} \to \pi_* \mathrm{BP}^{\otimes 3} \to \cdots .$$

This has the data of a "Hopf algebroid." Indeed, maps to R will get some kind of Agroup structure.

- A map $\pi_* \mathrm{BP} \to R$ is the data of a p-typical formal group law.

- The data of a map from
$$\pi_* \mathrm{BP}^{\otimes 2} = \pi_* \mathrm{BP} \otimes_{\pi_* \mathrm{BP}} \pi_* \mathrm{BP}$$

  (note $\mathrm{BP} \otimes \mathrm{BP}$ is free as a module over $\pi_* \mathrm{BP}$) classifies two formal group laws with an isomorphism between them. For example, the canonical map from $\pi_* \mathrm{BP}$ explains what the identity is.

- The data of a map from $\pi_* \mathrm{BP}^{\otimes 3}$ similarly explains how to compose the two formal group laws from $\pi_* \mathrm{BP}^{\otimes 2}$.

In general, given any $\mathbb{E}_\infty$-ring R, if $R \otimes R$ is free over R, then one gets a Hopf algebroid.

1.7 February 25

Here we go.

1.7.1 Hopf Algebroids

Given an $\mathbb{E}_\infty$-ring R, one has an ∞-category of R-modules. If R is a discrete ring, then this is the derived category of R-modules. One may want such an explicit description of the R-modules in more general situations.

> **Definition 1.158** (totalization). The *totalization* of a cosimplicial discrete ring $\Delta^{\mathrm{op}} \to \mathrm{Ring}$ is the limit of this functor.

> **Example 1.159.** One can define a ring R as the limit of the Čech nerve
> $$\mathrm{BP}_* \to \mathrm{BP}_*\mathrm{BP} \to \cdots .$$

> **Remark 1.160.** Before totalization, a cosimplicial discrete ring is some "affine stack," meaning that it is a functor which takes a commutative ring to simplicial sets.

It will sometimes be true that such an affine stack will actually output to classical groupoids instead of arbitrary simplicial sets. In this situation, the cosimplicial ring is a Hopf algebroid.

> **Definition 1.161** (Hopf algebroid). A *Hopf algebroid* is a pair of commutative rings (A, Γ) equipped with maps $\eta_L, \eta_R \colon A \to \Gamma$, a comultiplication $\Delta \colon \Gamma \to \Gamma \otimes_A \Gamma$, a counit $\varepsilon \colon \Gamma \to A$, and an inversion $\iota \colon \Gamma \to \Gamma$, all of which satisfy the following.
>
> - η_L makes Γ into a flat A-module.
>
> - There are various other identities so that $\mathrm{Hom}(A, R)$ is the set of objects and $\mathrm{Hom}(\Gamma, R)$ is the set of morphisms of a groupoid.
>
> When no confusion is possible, we will simply call Γ the Hopf algebroid.

> **Remark 1.162.** A Hopf algebroid (A, Γ) produces a cosimplicial ring as follows.
> $$A \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Gamma \Rrightarrow \Gamma \times_A \Gamma \qquad \cdots$$
> The totalization is an $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra.

> **Example 1.163.** Any Hopf algebra Γ over a field k produces a Hopf algebroid by taking $\eta_L = \eta_R$ to both be the same identity map. For example, one can consider the Hopf algebra $\alpha_p := \mathbb{F}_p[x]/(x^p)$ over $\mathbb{F}_p$, which is a subscheme of $\mathbb{G}_{a,\mathbb{F}_p}$.

> **Example 1.164.** If we take $A = \pi_*\mathrm{BP}$ and $\Gamma = \pi_*(\mathrm{BP}_*\mathrm{BP})$, then the Hopf algebroid (co)represents the moduli space of p-typical formal groups up to strict isomorphism.

We thus will satisfy ourselves with trying to understand the category of modules for such a Hopf algebroid. The point is to have some module for A and Γ which communicate with each other in some way.

Definition 1.165 (module)**.** Fix a Hopf algebroid Γ. Then a *discrete left Γ-comodule* is a discrete left A-module M equipped with an A-module map

$$\psi \colon M \to \Gamma \otimes_A M,$$

which is counital and coassociative.

Remark 1.166. The idea is that one can always present a module for the totalization as a sequence of modules, each for the discrete ring A and then Γ and then $\Gamma \otimes_A \Gamma$, and so on, always with some homotopies for coherence. But one can think of any module (indeed, any spectrum) by filtering it with its Postnikov tower, which in the case of $\mathbb{Z}$-modules allows one to recover chain complexes. The same thing works for Γ, thereby granting us our description of $\mathrm{Mod}(R)$, which are now the inductive systems of the bounded chain complexes. (Technically speaking, bounded chain complexes are those R-modules which are finite limits of just R, and we need to close this category under filtered colimits to get all of $\mathrm{Mod}(R)$.)

Remark 1.167. The (discrete) A-linear dual $\Gamma^\vee$ has the natural structure of a Hopf algebroid. Then the A-linear dual of a comodule M is simply a right $\Gamma^\vee$-module.

Remark 1.168 (Hahn)**.** Let's not pretend we care about homological algebra. Homological algebra is some fiddly way of pointing out modules.

1.7.2 Vanishing Lines

Let's return to α_p.

Example 1.169. Take $(A, \Gamma) = (\mathbb{F}_p, \mathbb{F}_p[x]/(x^p))$, and let R be the totalization. Then a perfect R-module is a bounded chain complex of finite-dimensional $\mathbb{F}_p$-modules M equipped with the structure map $\psi \colon M \to \mathbb{F}_p[x]/(x^p) \otimes_{\mathbb{F}_p} M$. However, such a map is the same as giving M the structure of an $\mathbb{F}_p[x]/(x^p)$-module because Γ is its own self-dual (indeed, $\alpha_p = \alpha_p^\vee$).

Example 1.170. We continue from the previous example. The homotopy groups of the totalization R of this Hopf algebroid can be computed to be $\mathrm{Ext}_\Gamma(\mathbb{F}_p, \mathbb{F}_p)$ via the Dold–Kan correspondence. This turns out to be $\mathbb{F}_p[x] \otimes \Lambda(h)$, where h is in degree 2 and β is in degree approximately p. Letting x have some grading (e.g., a $\mathbb{Z}/p\mathbb{Z}$-grading), then $\pi_* R$ admits a bigrading, so we see that $\pi_* R$ is supported on two lines of positive slope in the bigrading.

Warning 1.171. By convention, in all of our bigradings, homological degree is the vertical degree.

One can use the above examples to show the following.

Proposition 1.172. Let R be the totalization of the Hopf algebroid $\mathbb{F}_p[x]$, and choose a perfect R-module M. Then either

 (a) $\pi_* M$ has a horizontal vanishing line in its bigraded homotopy groups, or

 (b) there is a map $\Theta \colon \Sigma^{x,y} M \to M$ of R-modules which is not nilpotent and of the same slope as β. In this case, $M/\Theta \cong \mathrm{cofiber}\,\Theta$.

Proof. Simply classify all R-modules, as above. Roughly speaking, by the Dold–Kan correspondence, the homotopy groups of a discrete R-module M are computed as $\mathrm{Ext}_{\mathbb{F}_p[x]/(x^p)}(M, \mathbb{F}_p)$. Thus, everything is built from the following two simple discrete modules.

- For the module $\mathbb{F}_p$, we see that its homotopy groups have the second property.

- For the module $\mathbb{F}_p[x]/(x^p)$, its homotopy groups are finitely supported.   ∎

One can even pass to higher powers.

> **Proposition 1.173.** Let R be the totalization of the Hopf algebroid $\mathbb{F}_p[x]/\left(x^{p^2}\right)$. For every perfect R-module M, the homotopy groups $\pi_* M$ admit a vanishing line with minimal slope and an endomorphism Θ with slope parallel to that vanishing line. The cofiber of Θ will have a smaller slope vanishing line.

Proof. Note that R is an extension of the totalization R_0 of $\mathbb{F}_p[x]/(x^p)$ by itself. The point is that this sort of statement is immune to taking extensions in some formal way.   ∎

Here is another example.

> **Example 1.174.** Let $\mathcal{P}_* := \pi_{2*}(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p)$ be the "even" Steenrod algebra, which is a free polynomial ring over $\mathbb{F}_p$ with generators ξ_i in degree $2p^i - 2$. It turns out that one has a Hopf algebra coming from $(A, \Gamma) := (\pi_{2*}\mathbb{F}_p, \mathcal{P}_*)$. It is difficult to explicitly write down the coproduct $\mathcal{P}_* \to \mathcal{P}_* \otimes_{\mathbb{F}_p} \mathcal{P}_*$, which come from the Adem relations. Nonetheless, we can define this map by taking homotopy groups of the composite
> $$\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p = \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{S} \otimes_{\mathbb{S}} \mathbb{F}_p \to \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p = (\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p) \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p).$$
> In particular, because $\mathbb{F}_p$ is a field, taking homotopy groups of the target produces $\mathcal{P}_* \otimes_{\mathbb{F}_p} \mathcal{P}_*$.

> **Remark 1.175.** It turns out that the Hopf algebra $\mathcal{P}_*$ (co)represents the additive group with its automorphisms.

> **Remark 1.176.** It turns out that the Hopf algebra $\mathcal{P}_*$ is an iterated extension of Hopf algebras, and this process gives us the associated graded which is a tensor product of $\mathbb{F}_p\left[\xi_i^{p^j}\right]/\left(\xi_i^{p^{j+1}}\right)$s.

One can now use our ideas from α_p and $\mathbb{F}_p[x]/\left(x^{p^2}\right)$ to show the following.

> **Theorem 1.177** (Palmieri)**.** Fix a perfect module M over the graded $\mathbb{E}_\infty$-ring R which is the totalization of the Hopf algebra $\pi_2(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p)$. Then the homotopy groups $\pi_* M$ admit a vanishing line of some slope contained in the set
> $$\left\{ \frac{1}{p^{j+1}(p^i - 1) - 1} : i > j \geq 0 \right\} \cup \{\infty\}.$$
> Further, there is an endomorphism Θ of M with this slope which is not nilpotent, and the cofiber of Θ is an R-module with smaller slope.

> **Example 1.178.** If X is a finite CW-complex, then
> $$\lim \pi_{2*}\left(\Sigma_+^\infty X \otimes_{\mathbb{S}} \mathbb{F}_p^{\otimes(i+1)}\right).$$

Bibliography

[Shu16]   Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions