250B: Commutative Algebra

For the Morbidly Curious

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Contents

Theme 1
Working in Chains

But this is like trying to scale a glacier. It's hard to get your footing, and your fingertips get all red and frozen and torn up.

—Anne Lamott

1.1 March 8

Let's begin.

1.1.1 A Criterion for Flatness

Let's review the following result.

Theorem 1.1. An R-module M is flat if and only if $\operatorname{Tor}_1^R(R/I, M) = 0$ for all finitely generated ideals $I \subseteq R$. Equivalently, $\operatorname{Tor}_1^R(R/I, M) = 0$ if and only if the natural map $I \otimes M \to M$ is injective.

Proof. We proceed with the following steps.

1. From the long exact sequence of Tor, the injectivity of $I \otimes M \to M$ is equivalent to
$$\operatorname{Tor}_1^R(R/I, M) = 0$$
for all finitely generated ideals I. In particular, we use the short exact sequence
$$0 \to I \to R \to R/I \to 0$$
and consider the long exact sequence we get upon applying $- \otimes_R M$.

2. We extend to all ideals $I = (x_\alpha)_{\alpha \in \lambda}$. Well, if we have a nontrivial element $x \otimes m$ of the kernel of $I \otimes M \to M$, then we can write
$$x = \sum_{i=1}^{n} r_i x_{\alpha_i} \otimes m,$$
because x only uses finitely many of the x_α, so we see that the finitely generated (!) ideal $(x_{\alpha_1}, \ldots, x_{\alpha_n})$ will also have a kernel.

3. We reduce checking that, for all submodule $N' \hookrightarrow N$, we have that $M \otimes_R N' \hookrightarrow M \otimes_R N$, to the case where N is finitely generated. Well, if there is a nontrivial kernel, then we write a nontrivial element of the kernel as
$$\sum_{i=1}^{n} m_i \otimes n_i' \mapsto 0,$$
so the same trick lets us assume that both M and N' are finitely generated.

4. We would like to check that $\operatorname{Tor}_1^R(M,N) = 0$ for finitely generated N. Well, let n be the minimal number of generators for N. For $n = 0$, we have $N = 0$ and are done. For $n = 1$, we use step 1. Then for our induction, we write
$$0 \to N'' \to N \to N' \to 0$$
where N'' and N' have fewer than n generators. Then the long exact sequence tells us that
$$\operatorname{Tor}_1^R(M,N'') \to \operatorname{Tor}_1^R(M,N) \to \operatorname{Tor}_1^R(M,N)$$
is exact, so the induction forces the left and right terms to vanish, so $\operatorname{Tor}_1^R(M,N) = 0$. ■

Let's see some example.

Exercise 1.2. Set $R := k[x]/\left(x^2\right)$. We show that an R-module M is flat if and only if M is free.

Proof. We use Theorem 1.1. Certainly if $I = R$ or $I = 0$ we are done. The only other ideal to check is (x), so we just need to verify that
$$(x)M \to M$$
is injective. In particular, we need to check that $\ker(x)/\operatorname{im}(x) = 0$, which amounts to verifying that M is free because x is the only element that could provide us with a kernel. ■

Remark 1.3 (Serganova)**.** In fact, we can show that any R-module M can be written as $M_0 \oplus F$ where $M_0 \cong \ker(x)/\operatorname{im}(x)$ and F is free.

Exercise 1.4. Fix R a principal ideal domain. Then an R-module M is flat if and only if M is torsion-free.

Proof. All ideals take the form (a), but $(a) \cong R$ because R is a principal ideal domain, so we are merely verifying that the map $R \otimes_R M \to M$ is injective, which is true if and only if M is torsion free. ■

Example 1.5. If a $\mathbb{Z}$-module M is finitely generated and torsion free, then M must actually be free. Also, $\mathbb{Q}$ is torsion free and hence flat.

1.1.2 Flatness Locally

We note the following.

Lemma 1.6. Fix R a ring and $\mathfrak{p}$ a prime. If M is a flat R-module, then $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module.

Proof. Use the Tor condition for flatness and note that $R_\mathfrak{p}$ being flat allows us to simply tensor in the projective resolution for M to give a projective resolution for $M_\mathfrak{p}$. Alternatively, simply note that $R_\mathfrak{p} \otimes_R M$ is the tensor product of two flat modules. ■

We might hope the converse holds. Indeed, it does.

Proposition 1.7. Fix R a ring and M an R-module. If $M_\mathfrak{p}$ is flat for all primes $\mathfrak{p}$, then M is also flat.

Proof. Well, fix some inclusion $N \subseteq N'$ so that we want to show that

$$M \otimes_R N' \to M \otimes_R N$$

is also an inclusion. Well, we know that, upon localization, we have an inclusion

$$M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N'_\mathfrak{p} \to M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p},$$

so because this is an inclusion locally, it becomes an inclusion globally as we showed a while ago. ■

So we are motivated to study how flat modules behave under localization.

Proposition 1.8. Fix R a local ring with maximal ideal $\mathfrak{p}$. Further, let M be a finitely presented R-module. If M is flat, then M is free.

Proof. The idea is to use Nakayama's lemma. Because M is finitely presented, we can build a short exact sequence

$$0 \to N \to F \to M \to 0,$$

where N is finitely generated, and F is free. Upon tensoring with $R/\mathfrak{p}$, we get the right-exact sequence

$$N/\mathfrak{p}N \to F/\mathfrak{p}F \to M/\mathfrak{p}M \to 0.$$

Now, choose F such that $\dim F/\mathfrak{p}F = \dim M/\mathfrak{p}M$, for otherwise we could use Nakayama's lemma to generate F by fewer (namely, $\dim F/\mathfrak{p}F$ many) elements.

It follows that $N/\mathfrak{p}N$ must vanish, so because N is finitely generated, Nakayama's lemma promises that $N = 0$. Thus, $F = M$, so M is free, so we are done. ■

Remark 1.9. In fact, any projective module over a local ring is free. The case for finitely generated modules is on the homework. The homework also includes some examples of flatness checks.

1.1.3 Completion

The point is to give certain very nice rings some very nice topologies. Here is our definition.

Definition 1.10 (Completion, rings)**.** Fix R with a filtration $\mathcal{J}$ given by

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots.$$

Then we define the *completion* $\hat{R}_\mathcal{J}$ as a subring of $\prod_s R$ by

$$\hat{R}_\mathcal{J} = \{(r_0, r_1, \ldots) : r_i \equiv r_j \pmod{I_j} \text{ for } i > j\}.$$

Most specifically, the most interesting filtrations for us will be the I-adic filtrations.

Let's see some examples.

Example 1.11. Fix $R := k[x]$ with $\mathfrak{m} := (x)$ a maximal ideal. Then $\hat{R}_\mathfrak{m}$ consists of sequences of polynomials $\{p_n(x)\}_{n\in\mathbb{N}}$ such that

$$p_n \equiv p_m \pmod{x^m}$$

for $n > m$. These sequences just define formal power series, where the last n terms of the power series are determined by p_n, and the coherence above guarantees that this is well-defined.

Example 1.12. Fix $R := \mathbb{Z}$ with $\mathfrak{m} := (p)$. Then $\hat{R}_\mathfrak{m}$ consists of sequences $\{b_n\}_{n\in\mathbb{N}}$ which behave as "formal power series" in p as

$$a_0 + a_1 p + a_2 p^2 + \cdots,$$

where $a_\bullet \in \mathbb{Z}/p\mathbb{Z}$ with $b_n \equiv \sum_{k<n} a_k p^k$. These are the p-adic integers.

Example 1.13. The 2-adic integer $u \in \mathbb{Z}_2$ given by

$$u := 1 + 2 + 2^2 + 2^3 + \cdots$$

is actually just -1. Indeed, if we multiply $(1-2)u$, then we get 1 after the mass cancellation, so we get $(1-2)u = 1$, so $u = -1$.

We quickly note that, as in the above examples, we have a natural inclusion

$$\iota : R \to \hat{R}_\mathcal{J}$$

by simply taking $r \mapsto (r, r, r, \ldots)$. This gives rise to the following definition.

Definition 1.14 (Complete)**.** Fix a filtration $\mathcal{J}$ and a ring R. Then the ring R is *complete with respect to* $\mathcal{J}$ if and only if $\hat{R}_\mathcal{J} = R$, in that the natural map $\iota : R \to \hat{R}_\mathcal{J}$ is an isomorphism.

We have the following check to justify the name completion.

Lemma 1.15. Fix R a ring and $\mathcal{J}$ a filtration. Then $\hat{R}_\mathcal{J}$ is a complete ring with respect to the induced filtration $\hat{\mathcal{J}}$ given by

$$\hat{R} = I_0\hat{R} \supseteq I_1\hat{R} \supseteq I_2\hat{R} \supseteq \cdots.$$

Proof. We omit this proof. ∎

Let's have a little fun with our completions.

Definition 1.16 (Limit)**.** Fix a completion $\hat{R}_\mathcal{J}$. Then we say that an element $a \in \hat{R}_\mathcal{J}$ is the *limit* of a sequence $\{a_k\}_{k\in\mathbb{N}}$ if and only if, for each n, there exists N such that $a_i - a \in I_n$ for each $i > N$.

Definition 1.17 (Cauchy sequence)**.** Fix a completion $\hat{R}_\mathcal{J}$. Then a sequence $\{a_k\}_{k\in\mathbb{N}}$ is *Cauchy* if and only if, for each n, there exists N such that $a_i - a_j \in I_n$ for each $i, j > N$.

So the following is why we used the term "complete."

Lemma 1.18. A ring R is complete with respect to the filtration $\mathcal{J}$ if and only if every Cauchy sequence in $\hat{R}$ converges.

Proof. In one direction, suppose that ∎

As an aside, we note that, if we have a sequence $a := \{a_k\}_{k\in\mathbb{N}} \in \hat{R}_\mathcal{J}$, then we have $a_{i+1} - a_i \in \mathfrak{I}_n$, so we can write

$$a = \sum_{j=1}^{\infty} (a_{j+1} - a_j)$$

to be an infinite convergent series; in particular, the partial sums converge.

1.1.4 Properties Preserved by Completion

Taking the completion also tends to look like localization.

Proposition 1.19. Fix R complete with respect to an I-adic filtration. Then any element $1-a$ with $a \notin I$ is a unit.

Proof. We see that

$$(1-a)^{-1} = 1 + a + a^2 + \cdots$$

will work, essentially by just summing an infinite geometric series. ■

Corollary 1.20. Fix R a ring and $\mathfrak{m}$ a maximal ideal. Then $\hat{R}_\mathfrak{m}$ is a local ring with maximal ideal $\mathfrak{m}\hat{R}_\mathfrak{m}$.

Proof. We need to check that $a \notin \mathfrak{m}\hat{R}_\mathfrak{m}$ implies that a is a unit. Well, we can write out

$$a = \sum_{k=0}^{n} b_k$$

where $b_k \in \mathfrak{m}^k$. We can select b so that $ab \equiv 1 \pmod{\mathfrak{m}}$ because $R/\mathfrak{m}$ is a field, so we see that $ab = 1 + c$ for some $c \in \mathfrak{m}$. But now Proposition 1.19 assures us that $1 + c$ is a unit, so a must be a unit also. ■

In fact, we can see how similar completion looks like localization by how it focuses on $\mathfrak{m}$.

Lemma 1.21. We have that $R/\mathfrak{m}^n \cong \hat{R}/\mathfrak{m}\hat{R}_\mathfrak{m}^n$.

Proof. This is by definition of $\mathfrak{m}\hat{R}_\mathfrak{m}^n$. ■

And now we can glue these isomorphisms together.

Corollary 1.22. We have that $\operatorname{gr}_\mathfrak{m} R \cong \operatorname{gr}_{\mathfrak{m}\hat{R}_\mathfrak{m}} \hat{R}_\mathfrak{m}$.

Proof. Glue together the isomorphisms of the preceding result. The multiplicative structure matches. ■

We can also say the following.

Theorem 1.23. Fix R a Noetherian ring and $\mathfrak{m} \subseteq R$ an ideal. Then $\hat{R}_\mathfrak{m}$ is Noetherian.

Proof. We need the following lemma.

Lemma 1.24. Fix R a ring complete with respect to the filtration $\mathcal{J}$ denoted by

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots.$$

Further, if we have $I = (a_1, \ldots, a_n)$, then $(\operatorname{in} I) = (\operatorname{in} a_1, \ldots, \operatorname{in} a_n)$.

Proof. Suppose for the sake of contradiction that we have some element $f \notin (\operatorname{in} a_1, \ldots, \operatorname{in} a_n)$. But now, we can write

$$f = \sum_{i=1}^{n} a_i g_i$$

for some elements g_i, so if $f \in I^d \setminus I^{d+1}$, then we have

$$\operatorname{in} f = \left[\sum_{i=1}^{n} a_k g_i \right]_{\mathfrak{m}^{d+1}}.$$

Now, lifting the elements g_i to the completion, we hit our contradiction because the above should be an equality. ■

We now attack the proof of the theorem. Well, fix some ideal $I \subseteq \hat{R}_\mathfrak{m}$. Then we project I into $\operatorname{gr}_\mathfrak{m} R$, which is equal to $\operatorname{gr}_{\mathfrak{m}\hat{R}_\mathfrak{m}} \hat{R}_\mathfrak{m}$, and here we see that I must be finitely generated because $\operatorname{gr}_\mathfrak{m} R$ needs to be Noetherian as lifted from R, by the lemma. So we are done. ■

We close by stating the following theorem.

Definition 1.25 (Completion, modules)**.** Fix R a ring with filtration $\mathcal{J}$ denoted by

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots.$$

Further, given an R-module M, there is an induced filtration

$$M = I_0 M \supseteq I_1 M \supseteq I_2 M \supseteq \cdots.$$

Doing the same construction as for $\hat{R}$ gives rise to $\hat{R}$.

We can check that $\hat{M}$ is an $\hat{R}$-module.

We have the following, which says that completion is really looking like localization.

Theorem 1.26. Fix R a Noetherian ring with ideal I. Given a finitely generated module M, we have

$$\hat{M}_\mathfrak{m} \cong \hat{R}_\mathfrak{m} \otimes_R M.$$

In fact, $\hat{R}_\mathfrak{m}$ is a flat R-module.

Proof. We will show this next time. ■

List of Definitions