737: Weil II for Curves

Nir Elber

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

REVIEW OF ÉTALE THEORY

1.1 January 23

This is a pretty small class, so it will be rather informal. This course is going to assume some basic étale theory, roughly speaking up to the construction of the derived functors and some of their fundamental properties. We will also freely black-box the difficult theorems of the theory, most notably the Grothendieck–Lefschetz trace formula.

In this course, we are interested in proving the Weil conjectures, but we will be modest and focus on curves. Historically, the proof of the Weil conjectures for curves is much older than Weil II, but part of our goal will be to introduce the important relative techniques. For example, there should be a notion of weights attached to sheaves on a variety, known already from Hodge theory. However, we will require a way to see this purely from algebraic geometry; in fact, one expects the notion of weight to be motivic.

1.1.1 The Zeta Function

Let's begin by setting some notation which will be in place for the entire course. We take k to be a finite field $\mathbb{F}_q$ of characteristic p, embedded in a fixed algebraic closure $\overline{k} = \overline{\mathbb{F}_p}$; we write $q = p^n$. For brevity, we may write $k_m = \mathbb{F}_{q^m}$ for each $m \geq 1$. Then we let X be a smooth, projective, geometrically connected variety over the field k; we set $d \coloneqq \dim X$.

Definition 1.1 (zeta function)**.** Let X be a variety over $\mathbb{F}_q$. Then we define the *zeta function* as the generating function

$$\zeta_X(T) \coloneqq \exp\left(\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| \frac{T^m}{m}\right).$$

In order to do algebraic geometry to $\zeta_X(T)$, we would like to have a different description for $X(\mathbb{F}_{q^m})$. For this, we need to discuss closed points.

Definition 1.2 (closed point)**.** Let X be a variety over k. Then a point $x \in X$ is *closed* if and only if $\dim \overline{\{x\}} = 0$. Its *degree* $\deg x$ is the degree $[k(x) : k]$, where $k(x)$ is the minimal field of definition.

We now see that

$$X(\mathbb{F}_{q^m}) = \operatorname{Mor}_{\mathbb{F}_q}(\operatorname{Spec} \mathbb{F}_{q^m}, X).$$

For example, we see that this consists of the collection of closed points $x \in X$ of degree dividing m, counted with a certain multiplicity.

Now, to read off fields of definition, we introduce some Frobenius morphisms.

Definition 1.3. Fix a scheme X over $k = \mathbb{F}_q$. Then there is a *Frobenius morphism* $\operatorname{Frob}_X \colon X \to X$ defined as being an identity on the underlying topological space and the q-power map on $\mathcal{O}_X$. We may write $\operatorname{Frob}_{X,q}$ for Frob_X if we want to remember the power. We may also extend scalars and write $\operatorname{Frob}_{X_{\overline{k}},q} = \operatorname{Frob}_{X,q} \times \operatorname{id}_{\overline{k}}$, which we note is a morphism of schemes over $\overline{k}$ by its construction.

Remark 1.4. Fix a morphism $f \colon X \to Y$ of schemes over $\mathbb{F}_q$. Then we see $\operatorname{Frob}_Y \circ f = f \circ \operatorname{Frob}_X$, which can be checked directly: both sides are f on the topological spaces, and both sides are the same on the level of sheaves.

Example 1.5. On $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, our Frobenius map may be defined as the k-algebra endomorphism of $k[x_1, \ldots, x_n]$ which sends $x_i \mapsto x_i^q$. Thus, on points, we see that $(p_1, \ldots, p_n) \in \mathbb{A}^n_k(\overline{k})$ has

$$F_{\mathbb{A}^n_k}(p_1, \ldots, p_n) = \left(\operatorname{Frob}_q^{-1} p_1, \ldots, \operatorname{Frob}_q^{-1} p_n\right).$$

Remark 1.6. We now see that we can think about $X(\mathbb{F}_q)$ as the subset of $X(\overline{k})$ fixed by F_{X,q^m}. Thus, we note that one can realize $X(k_m)$ as the set of closed points of the scheme $(\Gamma_{F_{X,q^m}} \cap \Delta)$, where $\Delta \colon X \times X \to X$ is the diagonal map.

Definition 1.7 (arithmetic Frobenius)**.** The *arithmetic Frobenius* Frob_k is the q-power automorphism of $\overline{k}$.

Definition 1.8 (geometric Frobenius)**.** Let X be a scheme over k. Then we define the *geometric Frobenius* of $X_{\overline{k}}$ as $F_X \coloneqq \operatorname{id}_{X_{\overline{k}}} \times \operatorname{Frob}_{\overline{k}}^{-1}$. It fits in the following commutative diagram.

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r] \arrow[d] \arrow[dr, dashed, "F_X"] & X \arrow[dr, equal] & \\
\operatorname{Spec} \overline{k} \arrow[dr] & X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} \overline{k} \arrow[r, "\operatorname{Frob}_{\overline{k}}^{-1}"'] & \operatorname{Spec} k
\end{tikzcd}$$

Definition 1.9 (absolute Frobenius)**.** Let X be a scheme over k. One can check that F_X commutes with $\operatorname{Frob}_{X,q}$. We then define the *absolute Frobenius* as the composite $F_X \circ \operatorname{Frob}_{X_{\overline{k}},q}$.

Remark 1.10. It turns out that the absolute Frobenius is the identity on the level of étale cohomology.

We now return to our zeta function. To be able to undo the exponential, we note

$$\log\left(\frac{1}{1-T^d}\right) = \sum_{m\geq 1} d \cdot \frac{T^{nd}}{dn}.$$

Thus,

$$\sum_{m\geq 1} |X(\mathbb{F}_{q^m})| \frac{T^m}{m} = \sum_{\text{closed } x\in X} \log\left(\frac{1}{1-T^{\deg x}}\right),$$

so taking the exponential reveals

$$\zeta_X(T) = \prod_{\text{closed } x \in X} \frac{1}{1 - T^{\deg x}},$$

and now this Euler product appears similar to the usual Euler products we expect.

1.2 January 28

Today we do something with cohomology.

1.2.1 The Rationality Conjecture

We would like to relate our zeta function to cohomology. It turns out that the key input is the following result.

Theorem 1.11 (Grothendieck–Lefschetz trace formula)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$. Then

$$X\left(\mathbb{F}_{q^m}\right) = \sum_{i \geq 0} (-1)^i \operatorname{tr}\left(\operatorname{Frob}^m_{X_{\overline{k}}}; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

Here, recall that

$$\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \varprojlim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}/\ell^\bullet\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Namely, this is our Weil cohomology (over the field $\mathbb{Q}_\ell$) produced by étale cohomology.

Remark 1.12. It is the goal of Weil II (and thus of the course) to be able to work more general local systems than the "constant" sheaf $\mathbb{Q}_\ell$.

To relate this to ζ_X, we recall the following result from linear algebra.

Lemma 1.13. Fix an endomorphism φ of a finite-dimensional vector space V (over a field K). Then we have an equality of power series

$$\exp\left(\sum_{m \geq 1} \operatorname{tr}\left(\varphi^m; V\right) \frac{T^m}{m}\right) = \det\left(1 - \varphi T; V\right)^{-1}.$$

Proof. It is enough to check the equality after base-changing to the algebraic closure, so we may assume that K is algebraically closed. Then we may give V a basis so that φ is upper-triangular.

Let $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of φ. Then we are tasked with showing

$$\exp\left(\sum_{m \geq 0} \sum_{i=1}^{d} \lambda_i^m \cdot \frac{T^m}{m}\right) \stackrel{?}{=} \prod_{i=1}^{d} \frac{1}{1 - \lambda_i T}.$$

Well, we may move the sum on the left-hand side outside so that we see we are interested in showing

$$\exp\left(\sum_{m \geq 1} \frac{(\lambda T)^m}{m}\right) = \frac{1}{1 - \lambda T}$$

for any eigenvalue λ of φ. The result now follows by considering the Taylor expansion $-\log(1 - x) = \sum_{m \geq 1} x^m/m$. ∎

Here is the punchline: we are able to prove the rationality conjecture.

Proposition 1.14 (Rationality)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Then there are polynomials $P_0, \ldots, P_{2d} \in \mathbb{Q}_\ell[T]$ such that

$$Z_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}.$$

Proof. By Theorem 1.11, we see that

$$Z_X(T) = \prod_{i=0}^{2d} \exp\left(\sum_{m \geq 1} \operatorname{tr}\left(\operatorname{Frob}_{X_{\overline{k}}}^m ; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) \right) \frac{T^m}{m} \right)^{(-1)^i}.$$

We now define

$$P_i(T) := \det\left(1 - \operatorname{Frob}_{X_{\overline{k}}} T ; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

The result now follows from Lemma 1.13. ■

Remark 1.15. In fact, we see that $P_i(T)$ has degree $\dim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$. This fact can be combined with the comparison theorem to Betti cohomology.

Remark 1.16. We thus see that $Z_X(T) \in \mathbb{Q}_\ell(T)$, so because we already know $Z_X(T) \in \mathbb{Q}[[T]]$, we see $Z_X(T) \in \mathbb{Q}(T)$.

Remark 1.17. It turns out that $P_i(T) \in \mathbb{Z}[T]$ and is independent of ℓ, but the proof above does not show this.

Example 1.18. At $i = 0$, we see that the Frobenius acts trivially on $\mathrm{H}^0_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, so $P_0(T) = 1 - T$. Using Poincaré duality, we can similarly compute $P_{2d}(T) = 1 - q^d T$.

Remark 1.19. There is also a functional equation for $Z_X(T)$, which is purely formal from the above expression for Z_X when combined with Poincaré duality for étale cohomology.

1.2.2 The Riemann Hypothesis

This course will be interested in the following conjecture.

Conjecture 1.20 (Riemann hypothesis)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Fix an index $i \in \{0, \ldots, 2d\}$.

(a) The eigenvalues of $\operatorname{Frob}_{X_{\overline{k}}}$ on $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ are algebraic integers of magnitude $q^{i/2}$.

(b) $P_i(T) \in \mathbb{Z}[T]$ and is independent of ℓ.

The condition in (a) is interesting enough to deserve a name.

Definition 1.21 (q-Weil)**.** An algebraic integer $\alpha \in \overline{\mathbb{Q}}$ is *q-Weil of weight i* if and only if $|\iota(\alpha)| = q^{i/2}$ for all embeddings $\iota \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Example 1.22. The number $\sqrt{2}$ is a 2-Weil number. The number $1+\sqrt{2}$ is not a q-Weil number for any q.

In general, we find that the eigenvalues of a Frobenius action on a local system will still be q-Weil numbers of prescribed weight.

To be precise, the goal of this course will be to prove the following generalization of the above Riemann hypothesis.

Theorem 1.23 (Deligne)**.** Let $f\colon X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X that is mixed of weights at most n. Then $R^i f_! \mathcal{F}$ is also mixed of weights at most $w+i$.

We will define the notion of weights shortly. The idea intuitively comes from Hodge theory: the cohomology groups on a complex Kähler manifold naturally have a weight filtration, which then lifts to sheaves by taking a suitable compactification and studying differential forms suitably. Weights in our context will come from reading off q-Weil numbers.

Remark 1.24. Issues with compactification explain why we are forced to merely deal with mixed weights instead of upgrading this result to one on pure weights. Already this can be seen in Hodge theory.

This course will not prove Theorem 1.23 in full. Instead, we will focus on the case where f has fibers of dimension 1; it turns out that the general case follows from this from some argument involving fibering by curves and using the Leray spectral sequence.

Corollary 1.25. Let X be a scheme of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X.

(a) If $\mathcal{F}$ is mixed of weights at most n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at most $n+i$.

(b) If $\mathcal{F}$ is mixed of weights at least n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at least $n+i$.

(c) Assume that X is smooth and that $\mathcal{F}$ is pure of weight n. Then the image of the canonical map $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

(d) Assume that X is smooth and proper and that $\mathcal{F}$ is pure of weight n. Then $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

Proof. Here, (a) is direct from Theorem 1.23. Then (b) will follow from (a) via Poincaré duality as soon as we know that the duality given by Poincaré duality inverts the weights. Now, (c) follows from combining (a) and (b), and (d) follows from (c). ■

Remark 1.26. One can then prove the result for sheaves over $\mathbb{Q}_\ell$ by base-changing up to the algebraic closure.

The moral of the story is that we are going to use weights to significant profit in this course. Next class we will define weights.

BIBLIOGRAPHY

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions