

191: Analytic Number Theory

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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THEME 1

ARITHMETIC PROGRESSIONS

1.1 January 18

Here we go.

1.1.1 House-Keeping

We're teaching analytic number theory. Here are some notes.

- We will be referencing [Dav80] mostly, but we will do some things that Davenport does not do. For example, we will discuss the circle method, for which we refer to [Dav05].
- We will assume complex analysis, at the level of Math 185. We will use some Fourier analysis, but we will discuss the relevant parts as we need them. Of course, because this is number theory, we will assume some algebra, such as characters on abelian groups.
- There is a website [here](#), which includes a list of topics. Notably, there is a website for a previous version of the course.
- Grading is still up in there, as is the syllabus. Tentatively, grading will be as follows: by around the middle of the semester, there will be a list of recommended papers to read. Then we will write a 2–6-page report and present it to Professor Zhang. We will not have problem sets.
- Tentatively, office hours will be 90 minutes before lecture on Monday and Wednesday, in Evans 813.
- We should all write an email to Professor Zhang to introduce ourselves; for example, say what you're looking forward to in the course.

1.1.2 Facts about Dirichlet Series

In this first part of the course, we will be moving towards the following result.

Theorem 1.1 (Dirichlet). Fix nonzero integers $a, q \in \mathbb{Z}$ such that $\gcd(a, q) = 1$. Then there exist infinitely many primes p such that $p \equiv a \pmod{q}$.

The statement of Theorem 1.1 is purely elementary, but the standard proof uses complex analysis. The functions we will do analysis on are generalizations of the Riemann ζ function, defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges absolutely for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$. Indeed, we can show this.

Lemma 1.2. Fix some open, connected subset $U \subseteq \mathbb{C}$ and some function $f: U \rightarrow \mathbb{C}$. Given holomorphic functions $f_n: U \rightarrow \mathbb{C}$ for each $n \in \mathbb{N}$, if $f_n \rightarrow f$ uniformly on all compact subsets $D \subseteq U$, then f is holomorphic.

Proof. The point is to use Morera's theorem. Each f_n is continuous, so we see f is continuous as well. Thus, fixing any closed piecewise C^1 path $\gamma: [0, 1] \rightarrow U$, we would like to show

$$\oint_{\gamma} f(z) dz \stackrel{?}{=} 0.$$

Note $\operatorname{im} \gamma$ is compact, so $f_n \rightarrow f$ uniformly on $\operatorname{im} \gamma$. Thus, fixing any $\varepsilon > 0$, we can find some N such that

$$|f(z) - f_n(z)| < \varepsilon$$

for all $n > N$. Fixing any $n > N$, we find

$$\left| \oint_{\gamma} f(z) dz \right| = \left| \oint_{\gamma} f(z) dz - \oint_{\gamma} f_n(z) dz \right| \leq \oint_{\gamma} |f(z) - f_n(z)| dz \leq \varepsilon \ell(\gamma),$$

where $\ell(\gamma)$ is the length of γ . (Note $\ell(\gamma)$ is finite because γ is piecewise C^1 .) Sending $\varepsilon \rightarrow 0^+$ finishes the proof. ■

Proposition 1.3. Let $f: \mathbb{N} \rightarrow \mathbb{C}$ denote a sequence of complex numbers such that $|f(n)| = O(n^{\sigma})$ for some $\sigma \in \mathbb{R}$. Then the series

$$D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

converges absolutely for $s \in \mathbb{C}$ such that $\operatorname{Re} s > \sigma + 1$. Thus, $D(s)$ defines a holomorphic function in this region.

Proof. We are given $|f(n)| \leq Cn^{\sigma}$ for some $C > 0$. Thus, showing the absolute convergence is direct: note

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re}(s) - \sigma}},$$

which converges because $\operatorname{Re}(s) - \sigma > 1$.

We can now convert absolute convergence to uniform convergence of the partial sums $\{D_n\}_{n \in \mathbb{N}}$ of D , from which Lemma 1.2 will finish. Fix some compact subset $D \subseteq U$, and we want to show $D_n \rightarrow D$ uniformly on D . Because D is compact, there exists $s_0 \in D$ with minimal $\operatorname{Re} s_0$; define $\sigma_0 := \operatorname{Re} s_0$. Now, the series

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma_0}}$$

converges by our absolute convergence.

As such, for any $\varepsilon > 0$, select N such that $n_0 > N$ implies

$$\sum_{n > n_0} \frac{|f(n)|}{n^{\sigma_0}} < \varepsilon.$$

Thus, for any $s \in \mathbb{C}$ and $n_0 > N$, we see

$$|D(s) - D_{n_0}(s)| = \left| \sum_{n > n_0} \frac{f(n)}{n^s} \right| \leq \sum_{n > n_0} \frac{|f(n)|}{n^{\operatorname{Re} s}} \leq \sum_{n > n_0} \frac{|f(n)|}{n^{\sigma_0}} < \varepsilon,$$

which is what we wanted. ■

It follows from Proposition 1.3 that $\zeta(s)$ defines a holomorphic function on $\operatorname{Re} s > 1$.

1.1.3 The Euler Product

The following factorization is due to Euler.

Proposition 1.4. Let $f: \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function such that $|f(n)| = O(n^\sigma)$. For any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \sigma + 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right).$$

Proof. Fix $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma + 1$. Roughly speaking, this follows from unique prime factorization in \mathbb{Z} . For and N and M to be fixed later, define

$$P_{N,M} := \prod_{p < N} \left(\sum_{k=0}^{M-1} \frac{f(p^k)}{p^{ks}} \right),$$

and define $P_{N,\infty}$ analogously. Define $A_{N,M}$ to be the set of integers n such that the prime factorization of n includes primes less than N each to a power less than M , and define $A_{N,\infty}$ analogously. Note $A_{N,M}$ is a finite set, so the distributive law implies

$$P_{N,M} = \sum_{n \in A_{N,M}} \frac{f(n)}{n^s}.$$

To begin, we fix N and claim

$$P_{N,\infty} \stackrel{?}{=} \sum_{n \in A_{N,\infty}} \frac{f(n)}{n^s}.$$

Note $P_{N,\infty} = \lim_{M \rightarrow \infty} P_{N,M}$, so we fix some $M > 0$ and compute

$$\left| P_{N,M} - \sum_{n \in A_{N,\infty}} \frac{f(n)}{n^s} \right| = \left| \sum_{n \in A_{N,\infty} \setminus A_{N,M}} \frac{f(n)}{n^s} \right| \leq \sum_{n \notin A_{N,M}} \left| \frac{f(n)}{n^s} \right|.$$

Now, the smallest n such that $n \notin A_{N,M}$ is at least 2^M , so we see

$$\left| P_{N,M} - \sum_{n \in A_{N,\infty}} \frac{f(n)}{n^s} \right| \leq \sum_{n \geq 2^M} \left| \frac{f(n)}{n^s} \right|,$$

which now vanishes as $M \rightarrow \infty$ because $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely by Proposition 1.3. This completes the proof of the claim.

We now send $N \rightarrow \infty$ to finish the proof. For any $N > 0$, we use the claim to note

$$\left| P_{N,\infty} - \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right| = \left| \sum_{n \notin A_{N,\infty}} \frac{f(n)}{n^s} \right| \leq \sum_{n \notin A_{N,\infty}} \left| \frac{f(n)}{n^s} \right|.$$

Now, we note that the smallest $n \notin A_{N,\infty}$ is at least N because any $n < N$ has a prime factor less than N , so

$$\left| P_{N,\infty} - \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right| \leq \sum_{n \geq N} \left| \frac{f(n)}{n^s} \right|,$$

and now we see that the right-hand side goes to 0 as $N \rightarrow \infty$ because $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely by Proposition 1.3. The proposition follows. \blacksquare

Corollary 1.5. We have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Proof. By Proposition 1.4, we see

$$\zeta(s) = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

which is what we wanted. ■

We can now use Corollary 1.5 to give a proof of the infinitude of primes.

Theorem 1.6. There are infinitely many primes. In fact,

$$\sum_{p \text{ prime}} \frac{1}{p} = +\infty.$$

Proof. Throughout the proof, s will be a real number greater than 1. The key estimate is to note

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \int_1^{\infty} x^{-s} dx = -\frac{1}{1-s},$$

which goes to $+\infty$ as $s \rightarrow 1^+$. In particular, $\log \zeta(s) \rightarrow +\infty$ as $s \rightarrow 1^+$.

The last ingredient we need is to bound the Euler product of Corollary 1.5. In particular, we see

$$\log \zeta(s) = \log \left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \right) = \sum_{p \text{ prime}} -\log(1 - p^{-s}).$$

(Formally, one should cap the number of factors and then send the number of factors to infinity.) Using the Taylor expansion of $-\log(1 - x)$, we now see

$$\log \zeta(s) = \sum_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{1}{k p^{ks}} \right) = \left(\sum_{p \text{ prime}} \frac{1}{p^s} \right) + \sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{1}{k p^{ks}} \right).$$

We would like to focus on $\sum_p 1/p^s$, so we quickly show that the other sum converges. All terms are positive, so it suffices to show that it is bounded above, for which we see

$$\sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{1}{k p^{ks}} \right) \leq \sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{1}{p^k} \right) = \sum_{p \text{ prime}} \frac{1/p^2}{1 - 1/p} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1,$$

where we have telescoped in the last equality. Letting the value of this sum be $S(s)$, we see

$$\log \zeta(s) - S(s) = \sum_{p \text{ prime}} \frac{1}{p^s} < \sum_{p \text{ prime}} \frac{1}{p}.$$

Now, as $s \rightarrow 1^+$, we see $\log \zeta(s) - S(s) \rightarrow +\infty$, so the theorem follows. ■

The proof of Theorem 1.1 more or less imitates the argument of Theorem 1.6. Roughly speaking, we will show that

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{q}}} \frac{1}{p} = +\infty,$$

from which our infinitude follows. Finding a way to extract out the equivalence class $a \pmod{q}$ will use a little character theory.

1.1.4 Characters

Throughout, our groups will be finite and abelian, and actually we will be most interested in the abelian groups $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^\times$ for integers n . Formally, here is our definition.

Definition 1.7. Fix a positive integer n . Then we define $(\mathbb{Z}/n\mathbb{Z})^\times$ as the units in $\mathbb{Z}/n\mathbb{Z}$, which is $\{a \pmod n : \gcd(a, n) = 1\}$.

Remark 1.8. It is a fact that $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic for any prime p . This is nontrivial to prove, but we will not show it here.

Notably, given a prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$, there is an isomorphism of rings

$$\mathbb{Z}/n\mathbb{Z} \cong \prod_{p|n} (\mathbb{Z}/p^{\nu_p(n)})$$

and hence also an isomorphism of multiplicative groups, upon taking units.

Having said all that, the theory is most cleanly build working with general finite abelian groups.

Definition 1.9 (dual group). Let G be a group. Then the *dual group* is $\widehat{G} := \text{Hom}(G, \mathbb{C}^\times)$, where the operation is pointwise. Its elements are called *characters*. Note there is a “trivial” character $1: G \rightarrow \mathbb{C}^\times$ sending $g \mapsto 1$, which is the identity. We might call 1 the *principal character*; we might also denote 1 by χ_0 .

Remark 1.10. If G is a finite group, we note that any $\chi \in \widehat{G}$ and $g \in G$ has

$$\chi(g)^{\#G} = \chi(g^{\#G}) = 1,$$

so $\chi(g)$ is a $(\#G)$ th root of unity. In particular, $|\chi(g)| = 1$, so $\overline{\chi(g)} = \chi(g)^{-1} = \chi(g^{-1})$.

It will be helpful to have the following notation.

Notation 1.11. We might write $e: \mathbb{C} \rightarrow \mathbb{C}$ for the function $e(z) := \exp(2\pi iz)$.

We now begin computing \widehat{G} for finite abelian groups.

Lemma 1.12. Suppose G and H are groups. Then $\widehat{G} \times \widehat{H} \cong \widehat{G \times H}$ by sending (χ_G, χ_H) to $(g, h) \mapsto \chi_G(g)\chi_H(h)$.

Proof. We have the following checks. Let e_G and e_H be the identities of G and H , respectively.

- **Well-defined:** given $(\chi_G, \chi_H) \in \widehat{G} \times \widehat{H}$, define $\varphi(\chi_G, \chi_H): G \times H \rightarrow \mathbb{C}^\times$ by $\varphi(\chi_G, \chi_H): (g, h) \mapsto \chi_G(g)\chi_H(h)$. Note $\varphi(\chi_G, \chi_H)$ is a homomorphism: we have

$$\begin{aligned} \varphi(\chi_G, \chi_H)((g, h) \cdot (g', h')) &= \varphi(\chi_G, \chi_H)(gg', hh') \\ &= \chi_G(gg')\chi_H(hh') \\ &= \chi_G(g)\chi_H(h)\chi_G(g')\chi_H(h') \\ &= \varphi(\chi_G, \chi_H)(g, h) \cdot \varphi(\chi_G, \chi_H)(g', h'). \end{aligned}$$

- **Homomorphism:** to show φ is a homomorphism, we have

$$\begin{aligned} \varphi((\chi_G, \chi_H) \cdot (\chi'_G, \chi'_H))(g, h) &= \chi_G(g)\chi'_G(g)\chi_H(h)\chi'_H(h) = \varphi(\chi_G, \chi_H)(g, h) \cdot \varphi(\chi'_G, \chi'_H)(g, h), \\ \text{so } \varphi((\chi_G, \chi_H) \cdot (\chi'_G, \chi'_H)) &= \varphi(\chi_G, \chi_H) \cdot \varphi(\chi'_G, \chi'_H). \end{aligned}$$

- Injective: if $\varphi(\chi_G, \chi_H) = 1$, then

$$\chi_G(g)\chi_H(h) = \varphi(\chi_G, \chi_H)(g, h) = 1$$

for all $g \in G$ and $h \in H$. Setting $g = e_G$ shows that $\chi_H = 1$, and similarly setting $h = e_H$ shows that $\chi_G = 1$. Thus, $(\chi_G, \chi_H) = (1, 1)$.

- Surjective: given a character $\chi: (G \times H) \rightarrow \mathbb{C}^\times$, define $\chi_G(g) := \chi(g, e_H)$ and $\chi_H(h) := \chi(e_G, h)$. Note χ_G is a character because

$$\chi_G(gg') = \chi(gg', e_H) = \chi(g, e_H)\chi(g', e_H) = \chi_G(g)\chi_G(g').$$

Switching the roles of G and H shows that χ_H is also a character. Lastly, we note $\varphi(\chi_G, \chi_H) = \chi$ because

$$\varphi(\chi_G, \chi_H)(g, h) = \chi(g, e_H)\chi(e_G, h) = \chi(g, h).$$

This completes the proof. ■

Lemma 1.13. Suppose $G = \mathbb{Z}/n\mathbb{Z}$ for a positive integer n . Then $\chi_\bullet: \mathbb{Z}/n\mathbb{Z} \cong \widehat{G}$ by sending $[k]$ to the character $\chi_k: [\ell] \mapsto e(k\ell/n)$.

Proof. To begin, note $\chi_k: \mathbb{Z} \rightarrow \mathbb{C}^\times$ defines a homomorphism because

$$\chi_k(\ell + \ell') = e\left(\frac{k(\ell + \ell')}{n}\right) = e\left(\frac{k\ell}{n}\right)e\left(\frac{k\ell'}{n}\right) = \chi_k(\ell)\chi_k(\ell').$$

Further, note $\chi_k(n\ell) = e(k\ell) = 1$ for any $n\ell \in \mathbb{Z}$, so $n\mathbb{Z} \subseteq \ker \chi_k$. It follows that χ_k produces a homomorphism $\chi_k: G \rightarrow \mathbb{C}^\times$.

We now note that $\chi_\bullet: \mathbb{Z} \rightarrow \widehat{G}$ defines a homomorphism: for any $[\ell] \in G$, we see

$$\chi_{k+k'}([\ell]) = e\left(\frac{(k+k')\ell}{n}\right) = e\left(\frac{k\ell}{n}\right)e\left(\frac{k'\ell}{n}\right) = \chi_k([\ell])\chi_{k'}([\ell]).$$

Additionally, $\chi_{nk}([\ell]) = e(k\ell) = 1$, so $\chi_{nk} = 1$, so $nk \in \ker \chi_\bullet$. It follows that χ_\bullet produces a homomorphism $\chi_\bullet: \mathbb{Z}/n\mathbb{Z} \rightarrow \widehat{G}$.

It remains to show that χ_\bullet is a bijection. We have two checks.

- Injective: suppose $\chi_k = 1$ for $k \in \mathbb{Z}$. We must show $k \in n\mathbb{Z}$. Well, we must then have

$$1 = \chi_k([1]) = e(k/n),$$

which forces $n \mid k$.

- Surjective: given some character $\chi: G \rightarrow \mathbb{C}^\times$, we note $\chi([1])^n = \chi([0]) = 1$, so $\chi([1])$ is an n th root of unity. Thus, there exists k such that $\chi([1]) = e(k/n) = \chi_k([1])$. Thus, for any $\ell \in \{0, 1, \dots, n-1\}$, we see

$$\chi([\ell]) = \chi(\underbrace{[1] + \dots + [1]}_\ell) = \underbrace{\chi([1]) \cdot \dots \cdot \chi([1])}_\ell = \underbrace{\chi_k([1]) \cdot \dots \cdot \chi_k([1])}_\ell = \chi_k([\ell]),$$

so $\chi = \chi_k$ follows. ■

Proposition 1.14. Let G be a finite abelian group. Then $G \cong \widehat{G}$.

Proof. By the Fundamental theorem of finitely generated abelian groups, we may write

$$G \cong \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z}$$

for some positive integers n_i . Thus, using Lemma 1.12 and Lemma 1.13, we compute

$$\widehat{G} \cong \left(\prod_{i=1}^n \widehat{\mathbb{Z}/n_i\mathbb{Z}} \right) = \prod_{i=1}^n \widehat{\mathbb{Z}/n_i\mathbb{Z}} \cong \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z} \cong G,$$

which is what we wanted. ■

Proposition 1.14 might look like we now understand dual groups perfectly, but the isomorphism given there is non-canonical because the isomorphism of Lemma 1.13 is non-canonical. In other words, given some $g \in G$, there is in general no good way to produce character $\chi \in \widehat{G}$.

However, there is a natural map $G \rightarrow \widehat{\widehat{G}}$ which is an isomorphism.

Proposition 1.15. Fix a finite abelian group G . Define the map $\text{ev}_\bullet: G \rightarrow \widehat{\widehat{G}}$ by sending $g \in G$ to the map $\text{ev}_g \in \widehat{\widehat{G}}$ defined by $\text{ev}_g: \chi \mapsto \chi(g)$. Then ev_\bullet is an isomorphism.

Proof. We begin by checking that ev_\bullet is a well-defined homomorphism. For each $g \in G$, we see $\text{ev}_g: \widehat{G} \rightarrow \mathbb{C}^\times$ is a homomorphism because

$$\text{ev}_g(\chi\chi') = \chi(g)\chi'(g) = \text{ev}_g(\chi) \text{ev}_g(\chi').$$

Further, ev_\bullet is a homomorphism because

$$\text{ev}_{gg'}(\chi) = \chi(g)\chi(g') = \text{ev}_g(\chi) \text{ev}_{g'}(\chi).$$

It remains to show that ev_\bullet is an isomorphism. We claim that ev_\bullet is injective, which will be enough because $|G| = |\widehat{\widehat{G}}|$ by Proposition 1.14.

For this, we appeal to the following lemma.

Lemma 1.16. Fix a finite abelian group G with identity e . If $g \neq e$, then there exists $\chi \in \widehat{G}$ such that $\chi(g) \neq 1$.

Proof. Using the Fundamental theorem of finitely generated abelian groups, we may write

$$G \cong \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z}$$

for positive integers $n_i \geq 2$. Moving our problem from G to the right-hand side, we are given some $(g_i)_{i=1}^n$ such that $[g_i] \neq [0]$ for at least one i , and we want a character χ such that $\chi((g_i)_{i=1}^n) \neq 1$. Without loss of generality, suppose that $g_1 \neq 0$ and define χ by

$$\chi((k_i)_{i=1}^n) := e(k_1/n_1).$$

Certainly $\chi((g_i)_{i=1}^n) = e(g_1/n_1) \neq 1$, so it remains to show that χ is a character. This technically follows from Lemma 1.12, but we can see it directly by computing

$$\chi((k_i)_{i=1}^n + (k'_i)_{i=1}^n) = e(k_1/n_1)e(k'_1/n_1) = \chi((k_i)_{i=1}^n) \chi((k'_i)_{i=1}^n).$$

This completes the proof. ■

The proof now follows quickly from Lemma 1.16. By contraposition, we see that any $g \in G$ such that $\chi(g) = 1$ for all $\chi \in \widehat{G}$ and must have $g = e$. But this is exactly the statement that $\text{ev}_\bullet: G \rightarrow \widehat{\widehat{G}}$ is injective. ■

1.1.5 Finite Fourier Analysis

We now proceed to essentially do Fourier analysis for finite abelian groups. Here is the idea.



Idea 1.17. We can write general functions $G \rightarrow \mathbb{C}$ as linear combinations of characters.

Remark 1.18. When G is not abelian, one must work with function $G \rightarrow \mathbb{C}$ which are “locally constant” on conjugacy classes of G .

Here is our Fourier transform.

Notation 1.19. Let G be a finite abelian group. Given a function $f: G \rightarrow \mathbb{C}$, we define $\hat{f}: \hat{G} \rightarrow \mathbb{C}$ by

$$\hat{f}(\chi) := \sum_{g \in G} f(g) \overline{\chi(g)}.$$

Recall $\overline{\chi(g)} = \chi(g^{-1})$ by Remark 1.10.

To manifest Idea 1.17 properly, we need the following orthogonality relations.

Proposition 1.20. Let G be a finite abelian group.

- For any fixed $\chi \in \hat{G}$, we have

$$\sum_{g \in G} \chi(g) = \begin{cases} 0 & \text{if } \chi \neq 1, \\ \#G & \text{if } \chi = 1. \end{cases}$$

- For any $g \in G$, we have

$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} 0 & \text{if } g \neq e, \\ \#G & \text{if } g = e. \end{cases}$$

Proof. We show these directly.

- (a) If $\chi = 1$, then the sum is $\sum_{g \in G} 1 = \#G$.

Otherwise, $\chi \neq 1$, so there exists $g_0 \in G$ such that $\chi(g_0) \neq 1$. It follows

$$\chi(g_0) \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(g_0 g) \stackrel{*}{=} \sum_{g \in G} \chi(g),$$

so we must have $\sum_{g \in G} \chi(g) = 0$. Note that we have re-indexed the sum at $*$.

- (b) If $g = e$, then the sum is $\sum_{\chi \in \hat{G}} \chi(g) = \#(\hat{G})$, which is $\#G$ by Proposition 1.14.

Otherwise, $g \neq e$, so by Lemma 1.16, there exists χ_0 such that $\chi_0(g) \neq 1$. Employing the same trick, it follows

$$\chi_0 \sum_{\chi \in \hat{G}} \chi(g) = \sum_{\chi \in \hat{G}} (\chi_0 \chi)(g) \stackrel{*}{=} \sum_{\chi \in \hat{G}} \chi(g),$$

so we must have $\sum_{\chi \in \hat{G}} \chi(g) = 0$. Again, we re-indexed at $*$. ■

Now here is our result.

Theorem 1.21 (Fourier inversion). Let G be a finite abelian group. For any $f: G \rightarrow \mathbb{C}$, we have

$$f(g) = \frac{1}{\#G} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g)$$

for any $g \in G$.

Proof. This is direct computation with Proposition 1.20. Indeed, for any $g_0 \in G$, we see

$$\sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g_0) = \sum_{\chi \in \hat{G}} \sum_{g \in G} f(g) \chi(g^{-1}) \chi(g_0) = \sum_{g \in G} \left(f(g) \sum_{\chi \in \hat{G}} \chi(g^{-1} g_0) \right).$$

Now using Proposition 1.20, given $g \in G$, we see that the inner sum will vanish whenever $g \neq g_0$ and returns $\#G$ when $g = g_0$. In total, it follows

$$\frac{1}{\#G} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g_0) = f(g_0),$$

which is exactly what we wanted. ■

Here is our chief application.

Corollary 1.22. Let G be a finite abelian group. Fixing some $g_0 \in G$, we have

$$1_{g_0}(g) = \frac{1}{\#G} \sum_{\chi \in \hat{G}} \overline{\chi(g_0)} \chi(g)$$

for any $g \in G$.

Proof. Note

$$\hat{1}_{g_0}(\chi) = \sum_{g \in G} 1_{g_0}(g) \overline{\chi(g)} = \overline{\chi(g_0)}$$

because all terms except $g = g_0$ vanish. The result now follows from Theorem 1.21. ■

1.1.6 Dirichlet Characters

We want to extend our characters on $(\mathbb{Z}/q\mathbb{Z})^\times$ to work on all \mathbb{Z} , but this requires some trickery because, for example, 0 is not in general represented in $(\mathbb{Z}/q\mathbb{Z})^\times$. Here is our definition.

Definition 1.23 (Dirichlet character). Let q be a nonzero integer. A *Dirichlet character* $(\bmod q)$ is a function $\chi: \mathbb{Z} \rightarrow \mathbb{C}$ such that there exists a character $\tilde{\chi}: (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ for which

$$\chi(a) = \begin{cases} 0 & \text{if } \gcd(a, q) > 1, \\ \tilde{\chi}([a]) & \text{if } \gcd(a, q) = 1. \end{cases}$$

We might write this situation as $\chi \pmod{q}$. The Dirichlet character corresponding to 1 is denoted χ_0 and still called the *principal character*.

Remark 1.24. Note χ is periodic with period q .

We can finally define our generalization of ζ .

Definition 1.25 (Dirichlet L -function). Fix a Dirichlet character $\chi \pmod{q}$. Then we define the *Dirichlet L -function* as

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

By Proposition 1.3, we have absolute convergence for $\operatorname{Re} s > 1$, and $L(s, \chi)$ defines a holomorphic function there.

Remark 1.26. Continuing in the context of the definition, we note Proposition 1.4 gives

$$L(1, \chi) = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{\chi(p)^k}{p^{ks}} \right) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}}$$

for $\operatorname{Re} s > 1$.

In fact, the summation for $L(s, \chi)$ defines a holomorphic function for $\operatorname{Re} s > 0$, but seeing this requires a little care.

1.2 January 20

A syllabus was posted. There are some extra references posted.

1.2.1 Dirichlet L -Functions

We continue discussing the Dirichlet L -functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

from last class. In particular, we move towards a proof of Theorem 1.1.

Remark 1.27. Note $L(s, \chi) \neq 0$ for $\operatorname{Re} s > 1$ due to the Euler product factorization.

Arguing as in Theorem 1.6, we see that

$$\log L(s, \chi) = \sum_{p \text{ prime}} -\log(1 - \chi(p)p^{-s}) = \sum_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{k p^{ks}} \right)$$

for $\operatorname{Re} s > 1$. By isolating the $k > 1$ terms in the same way, we see

$$\log L(s, \chi) = \left(\sum_{p \text{ prime}} \frac{\chi(p)}{p^s} \right) + O(1)$$

for $\operatorname{Re} s > 1$. Now, the idea is that we can sum over characters χ to get a sum over primes $p \equiv a \pmod{q}$, as described in Corollary 1.22.

Thus, we are motivated to look at how $L(s, \chi)$ behaves as $s \rightarrow^+ 1$. In fact, for non-principal characters, we can easily extend to $\operatorname{Re} s > 0$.

Proposition 1.28. Fix a non-principal Dirichlet character $\chi \pmod{q}$. Then $L(s, \chi)$ has an analytic continuation to $\operatorname{Re} s > 0$.

Proof. The idea is that

$$S(x) := \sum_{n=1}^{\lfloor x \rfloor} \chi(n) = O(1)$$

for any N . As such, we apply summation by parts. Let $L_n(s, \chi)$ denote the n th partial sum

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \int_1^{\infty} x^{-s} dS(x) = s \int_1^{\infty} S(x) x^{-s-1} dx$$

by integrating by parts via Riemann–Stieltjes integration. Thus, because $S(x)$ is bounded, we see that the integral converges absolutely for $\operatorname{Re} s > 0$. ■

Remark 1.29. It follows from the proof that, for any $\sigma_0 > 0$, we see

$$L(s, \chi) < C(\sigma_0) \cdot (|s| + 1)q$$

for $\operatorname{Re} s > \sigma_0$, where $C(\sigma_0) \in \mathbb{R}$ is some constant.

And here is the situation for the principal character. Here, we can quickly reduce to the situation for ζ .

Proposition 1.30. Let $\chi_0 \pmod{q}$ be the principal character. Then

$$L(s, \chi_0) = \left(\prod_{p|q} (1 - p^{-s}) \right) \zeta(s)$$

Proof. Expand with the Euler product. ■

And now let's extend ζ .

Proposition 1.31. We provide a meromorphic continuation for $\zeta(s)$ over $\operatorname{Re} s > 0$, where we have a pole of order 1 and residue 1 at $s = 1$.

Proof. Integrating by parts again, we see

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \int_1^{\infty} x^{-s} d[x] \\ &= x^{-s} [x] \Big|_1^{\infty} + \int_1^{\infty} s x^{-s-1} [x] dx \\ &= 0 + \int_1^{\infty} s x^{-s} dx + \int_1^{\infty} s\{x\} x^{-s-1} dx \\ &= \frac{s}{s-1} + \int_1^{\infty} s\{x\} x^{-s-1} dx. \end{aligned}$$

The left term here is analytic for $\operatorname{Re} s > 0$ with the prescribed pole behavior, and the right term is some bounded analytic function. In fact, we can upper-bound the extra term by $c(\sigma_0)(|s| + 1)$, where $\operatorname{Re} s > \sigma_0 > 0$. ■

Remark 1.32. Doing repeated integration by parts, one can extend the continuations provide above further to the left, but we will not do this. Instead, we will use a functional equation to continue to the left in one fell swoop.

1.2.2 Towards $L(1, \chi)$

We now attack Theorem 1.1 directly. Take $\operatorname{Re} s > 1$. Working in $G := (\mathbb{Z}/q\mathbb{Z})^\times$, Corollary 1.22 tells us

$$1_{[a]}(p) = \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi(a)} \chi(p),$$

so it follows from our preceding discussion that

$$\sum_{p \equiv a \pmod{q}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi(a)} \log L(s, \chi) + O(1). \quad (1.1)$$

Thus, we want to study what happens to $L(s, \chi)$ as $s \rightarrow 1^+$. Philosophically, if $L(s, \chi)$ is not zero or a pole at $s = 1$, then we can legally take $\log L(s, \chi)$ and push the result into the $O(1)$ term.

By Proposition 1.30, we know that $L(s, \chi_0) \rightarrow \infty$ as $s \rightarrow 1^+$ because $\zeta(s) \rightarrow 1^+$. More precisely, we can write

$$\frac{1}{\varphi(q)} \log L(s, \chi_0) = \frac{1}{\varphi(q)} \log \zeta(s) + O_q(1) = -\frac{1}{\varphi(q)} \log(1-s) + O_q(1).$$

It remains to deal with the other characters, which is the meat of the proof. Notably, we know $L(1, \chi)$ is a legitimate complex number. Our goal is to show $L(1, \chi) \neq 0$ for non-principal χ because otherwise we get another pole in Equation (1.1), which could potentially cancel out the contribution of $L(s, \chi_0)$. As an outline, for “complex” characters χ , we are able to compare χ and $\bar{\chi}$ to enforce $L(1, \chi) \neq 0$, but the real characters require more ingenuity.

Let’s see more explicitly what happens with complex characters. Suppose $\chi \neq \bar{\chi}$ so that $\chi(G) \not\subseteq \mathbb{R}$. If $L(1, \chi) = 0$, then we see $L(1, \bar{\chi}) = 0$ as well, so

$$\sum_{p \equiv a \pmod{q}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi(a)} \log L(s, \chi) + O(1),$$

so the right-hand side goes to $-\infty$ as $s \rightarrow 1^+$. But this doesn’t make any sense because the left-hand side is surely positive. It remains to deal with the real characters.

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