174: Category Theory

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Theme 1
Basic Definitions

Category theory is much easier once you realize that it is designed to formalize and abstract things you already know.

—Ravi Vakil

1.1 January 19

Reportedly there is a lot of material that Bryce would like to cover today.

1.1.1 Our Definition

We're doing category theory, so let's define what a category is.

Definition 1.1 (Category)**.** A category $\mathcal{C}$ is a pair of objects and morphisms $(\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C})$ satisfying the following.

- $\operatorname{Ob}\mathcal{C}$ is a collection of *objects*. By abuse of notation, when we write $c \in \mathcal{C}$
- $\operatorname{Mor}\mathcal{C}$ is a collection of *morphisms*. Morphisms might also be called arrows or maps or functions or continuous functions or similar.

  A morphism is written $f : x \to y$ where $x, y \in \operatorname{Ob}\mathcal{C}$. Here, x is the *domain*, and y is the *codomain*.

In the above definition, we have some coherence conditions:

- For each $x \in \mathcal{C}$, there is a morphism $\operatorname{id}_x : x \to x$.
- Given any pair of morphisms $f : x \to y$ and $g : y \to z$, there exists a *composition* $gf : x \to z$. Importantly, the codomain of f is the domain of g.

Additionally, morphisms satisfy the following coherence conditions.

- Associativity: for any morphisms $f : a \to b$ and $g : b \to c$ and $h : c \to d$, we have that $h(gf) = (hg)f$.
- Identity: given any morphism $f : a \to b$, we have $\operatorname{id}_b f = f$ and $f \operatorname{id}_a = f$.

Yes, this is a long definition. For reference, it is on page 3 of Riehl.

The intuition to have here is that we have objects to be thought of as points a bunch of morphisms which are to be thought of arrows between them. Here is an example of some morphisms in a category.

$$\begin{tikzcd}
\bullet \arrow[loop left] \arrow[r] \arrow[d] & \bullet \arrow[loop above] \arrow[dl] \arrow[dr] & \\
\bullet \arrow[loop below] \arrow[rr] & & \bullet \arrow[loop below]
\end{tikzcd}$$

The loops are identity morphisms. As an aside, it is reasonable to think that definition of a category is overly abstract. Most of the time we will be thinking about some concrete category.

Before continuing, we bring in the following definition.

Definition 1.2 (Hom-sets)**.** Fix a category $\mathcal{C}$. Then, given objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ or $\operatorname{Hom}_{\mathcal{C}}(x, y)$ or $\operatorname{Hom}(x, y)$ or $\operatorname{Mor}(x, y)$ for the set of morphisms $f : x \to y$. I personally prefer $\operatorname{Mor}(x, y)$.

Note that two objects need not have a morphism between them. For example, the following is a category even though the two objects have a morphism between them.

$$\begin{tikzcd}
\bullet \arrow[loop left] & \bullet \arrow[loop right]
\end{tikzcd}$$

As a less contrived example, there is no morphism between $\mathbb{F}_2$ and $\mathbb{F}_3$ in the category of fields.

1.1.2 Examples

Let's talk about examples.

Example 1.3. The category Set has objects which are all sets and its morphisms are the functions between sets.

Example 1.4. The category Grp has objects which are all groups and its morphisms are group homomorphisms. Similarly, Ab has abelian groups.

Example 1.5. The category Ring has objects which are all rings (with identity) and its morphisms are group homomorphisms.

Example 1.6. The category Field has objects which are all fields and its morphisms are field/ring homomorphisms.

Example 1.7. The category Vec_k has objects which are all k-vector spaces and its morphisms are k-linear transformations.

Those are the good examples. We like them because they are with familiar objects.

Here are some weirder examples.

Example 1.8 (Walking arrow)**.** The diagram

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

induces a category with a single non-identity morphism.

Note that we will stop writing down all the identity morphisms and all induced morphisms because they're annoying to write out.

Example 1.9 (Walking isomorphism)**.** The diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

induces a category with two non-identity morphisms. We declare that any composition of the two non-identity morphisms is the identity.

There are also such things as a poset category, but for this we should define a poset first.

Definition 1.10 (Poset)**.** A *poset* $(\mathcal{P}, \leq)$ is a set $\mathcal{P}$ and a relation $\leq$ on $\mathcal{P}$ which satisfies the following; let $a, b, c \in \mathcal{P}$.

- Reflexive: $a \leq a$.
- Antisymmetric: $a \leq b$ and $b \leq a$ implies $a = b$.
- Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$.

Now, it turns out that all posets induce a category.

Example 1.11 (Poset category)**.** Given any poset $(\mathcal{P}, \leq)$, we can define the poset category as follows.

- The objects are elements of $\mathcal{P}$.
- For $x, y \in \mathcal{P}$, there is a morphism $x \to y$ if and only if $x \leq y$, and there is only one morphism.

Checking that the poset category is in fact a category is not very interesting. The identity law comes from reflexivity, where id_a witnesses $a \leq a$.

Additionally, transitivity defines our composition: if $a \leq b$ and $b \leq c$, then $a \leq c$, and the morphism representing $a \leq c$ is unambiguous because there is at most one morphism $a \to c$. This uniqueness is in fact crucial for our composition: if $f : a \to b$ and $g : b \to c$ and $h : c \to d$ are morphisms, then $h(gf) = (hg)f$ because they are both morphisms $a \to d$, of which there is at most one.

We continue with our examples. We will not check that these are actually categories formally; perhaps the reader can do the checks on their own time.

Example 1.12 (Groups)**.** Given a group G, we can define the category $\mathrm{B}G$ to have one object $*$ and morphisms $g : * \to *$ given by group elements $g \in G$. Composition in the category is group multiplication; the identity morphism id_* needed is the identity element of G; and the associativity check comes from associativity in G.

Example 1.13 (Pointer sets)**.** We define the category of pointed sets Set_* to consist of objects which are ordered pairs (X, x) where X is a set and $x \in X$ is an element. Then morphism are "based maps" $f : (X, x) \to (Y, y)$ to consist of the data of a function $f : X \to Y$ such that $f(x) = y$.

Example 1.14. Given any set S, we can define a category consisting of objects which are elements of S and morphisms which are only the required identity morphisms.

This last example generalizes.

Definition 1.15 (Discrete, indiscrete)**.** Fix a category $\mathcal{C}$. Then $\mathcal{C}$ is *discrete* if and only if the only morphisms are identity morphisms. Additionally, $\mathcal{C}$ is *indiscrete* if and only if $\mathrm{Mor}(x, y)$ has exactly one element for each pair of objects (x, y).

Warning 1.16. A total order with more than one element is not a category. Namely, if we have distinct objects x and y, then we cannot have both $x \leq y$ and $y \leq x$, so not both $\operatorname{Mor}(x, y)$ and $\operatorname{Mor}(y, x)$ inhabited.

1.1.3 Size Issues

Let's briefly talk about why we are calling $\operatorname{Ob} \mathcal{C}$ and $\operatorname{Mor} \mathcal{C}$ "collections." In short, we cannot have a set that contains all sets, but we would still like a category which contains all categories. There are a few ways around this; here are two.

- Grothendieck inaccessible categories: we essentially upper-bound the size of our sets and then let Set contain all of our sets.
- Proper classes: we add in things called "classes" to foundational mathematics we are allowed to be bigger than sets.

We will avoid doing anything like this in this course, so here is a definition making our avoidance concrete.

Definition 1.17 (Small, locally small)**.** Fix $\mathcal{C}$ a category. Then $\mathcal{C}$ is *small* if and only if $\operatorname{Mor} \mathcal{C}$ is a set. Alternatively, $\mathcal{C}$ is *locally small* if and only if $\operatorname{Mpr}(x, y)$ is a set.

Example 1.18. The category Set is locally small, but it is not small. To see that it is not small, note that $S \mapsto \operatorname{Mor}(\{*\}, S)$ is an injective map, so $\operatorname{Mor} \mathrm{Set}$ must be at least as big as Set.

It turns out that most of our categories will be locally small. It is a very nice property to have.

1.1.4 Isomorphism

In algebra (e.g., group theory), we are interested in when two objects are the same. In category theory, we focus on the morphisms between objects, so we need to be careful how we define this. Here is our definition.

Definition 1.19 (Isomorphism)**.** Fix a category $\mathcal{C}$. Then a morphism $f : x \to y$ is an *isomorphism* if and only if there is a morphism $g : y \to x$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. We call g the *inverse* of f and often notate it f^{-1}.

This is fairly intuitive: isomorphisms are those morphisms with a way to reverse them.

Observe that we called g "the" inverse of f, and we may do so because inverses are unique.

Proposition 1.20. Fix a category $\mathcal{C}$. Inverses of morphisms, if they exist, are unique.

Proof. Fix $f : x \to y$ some isomorphism, and suppose that we have found two inverse morphisms $g, h : y \to x$. Then

$$g = g \, \mathrm{id}_y = g(fh) = (gf)h = \mathrm{id}_x \, h = h,$$

so indeed the inverse morphisms that we found are the same. ■

Anyways, here are some examples.

Example 1.21. In Set, the isomorphisms are the bijective maps. For this we would have to show that bijective maps have inverse maps, which is not too hard to show.

Example 1.22. In Grp, the isomorphisms are group isomorphisms. Similarly, isomorphisms in Ring are ring isomorphisms.

As a warning, we will say now that lots of categories do not have a good categorical notion of injectivity or surjectivity, so we will not be able to say that isomorphisms are merely "bijective" morphisms.

1.2 January 21

By the way, this course is being run by Bryce (interested in category theory, homological algebra, and algebraic topology) and Chris (interested in representation theory and category theory).

1.2.1 Small Correction

Last class we discussed trying to a total order $(\mathcal{P}, \leq)$ into an indiscrete category. One way to do this is to say to give a morphism between two objects $a, b \in \mathcal{P}$ if and only if one of $a < b$ or $b < a$ or $a = a$ is true. Observe that the order does not actually matter here because any two objects have exactly one morphism anyways.

1.2.2 Groupoids

Reportedly, there will usually not be a lecture to begin out our discussion sections, but here is a lecture to begin out our first discussion section.

Last time we left off talking about indiscrete categories. Here is a nice fact.

Proposition 1.23. Fix $\mathcal{C}$ an indiscrete category. Then all maps are isomorphisms.

Proof. Fix any morphism $f : x \to y$. There is also a morphism $g : y \to x$, and we see that $gf \in \mathrm{Mor}(x, x)$. But $\mathrm{id}_x \in \mathrm{Mor}(x, x)$ as well, so we are forced to have $gf = \mathrm{id}_x$ by uniqueness of morphisms. Similar shows that $fg = \mathrm{id}_y$, finishing the proof. ■

Remark 1.24. This statement is also true for discrete categories but only because all identity morphisms are isomorphisms immediately.

The property of the proposition is nice enough to deserve a definition.

Definition 1.25 (Groupoid)**.** A category in which all morphisms are isomorphisms is called a *groupoid*.

Example 1.26. Viewing groups as one-element categories, we see that groups are groupoids because all elements (i.e., morphisms of the one-object set) have inverses and hence are isomorphisms.

Intuitively, a groupoid is a group but more "spread out."

1.2.3 Arrow Words

We close out with some miscellaneous definitions for our morphisms.

Definition 1.27 (Endo-, automorphism)**.** Fix a category $\mathcal{C}$. A morphism $f : x \to y$ is an *endomorphism* if and only if $x = y$. A morphism $f : x \to y$ is an *automorphism* if and only if it is an isomorphism and an endomorphism.

Example 1.28. In the category of abelian groups, the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 is an endomorphism but not an automorphism.

Definition 1.29 (Monic, epic)**.** Fix a category $\mathcal{C}$ and a morphism $f : x \to y$.

- We say f is a *monomorphism* (or is *monic*) if and only if $fg = fh$ implies $g = h$ for any morphisms $g, h : c \to x$. In other words, the map

  $$\operatorname{Mor}(c, x) \stackrel{f\circ -}{\to} \operatorname{Mor}(c, y)$$

  is injective. (This map is called "post-composition.") We might write $f : x \hookrightarrow y$ for emphasis.

- We say f is an *epimorphism* (or is *epic*) if and only if $gf = hf$ implies $g = h$ for any morphisms $g, h : y \to c$. In other words, the map

  $$\operatorname{Mor}(y, c) \stackrel{-\circ f}{\to} \operatorname{Mor}(x, c)$$

  is injective. (This map is called "pre-composition.") We might write $f : x \twoheadrightarrow y$ for emphasis.

Intuitively, the monomorphism condition looks like the injectivity condition (namely, $f(x) = f(y)$ implies $x = y$), so monic is supposed to be a generalization for injective.

Example 1.30. In the category of sets, monic is equivalent to injective, and epic is equivalent to surjective. Then it happens that being monic and epic implies being isomorphic. We will not fill in the details here.

Warning 1.31. It is not always true that being monic and epic implies being isomorphic. It is true in $\mathrm{Set}, \mathrm{Ab}, \mathrm{Grp}$ but not in, say, Ring as the below example shows.

Example 1.32. The inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring is both epic and monic but not an isomorphism. We run some checks.

- We show monic. Suppose $g, h : R \to \mathbb{Z}$ are morphisms with $fg = fh$. We claim $g = h$. Well, for any $r \in R$, we see $g(r) = f(g(r))$ and $h(r) = f(h(r))$ because f is merely an inclusion, so $g(r) = h(r)$ follows.

- We show epic. Suppose $g, h : \mathbb{Q} \to R$ are morphisms with $gf = hf$. We claim $g = h$. We start by noting any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}$ will have

  $$g\left(n/m\right) \cdot g(m) = g(n)$$

  and similar for h. However, $g(m) = g(f(m)) = h(f(m)) = h(m)$ and $g(n) = h(n)$ for the same reason, so $g\left(\frac{n}{m}\right) = g(n)/g(m) = h(n)/h(m) = h\left(\frac{n}{m}\right)$, and we are done because any rational can be expressed as some $\frac{n}{m}$.

- Lastly, f is not an isomorphism because $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. For example, $2x - 1$ has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

And now discussion begins.

1.3 January 24

Chris is giving the lecture today. Reportedly, it might be rough around the edges, but I have full faith in its coherence.

1.3.1 Review

Let's quickly talk about two fun types of categories.

Definition 1.33 (Slice categories)**.** Fix a category $\mathcal{C}$ and an object $c \in \mathcal{C}$.

- We define the *slice category* $\rfloor/\mathcal{C}$ to have objects which are morphisms $f : c \to x$ for objects $x \in \mathcal{C}$. The morphisms from $f : c \to x$ to $g : c \to y$ is a morphism $h : x \to y$ such that $f = gh$. Namely, we require the following triangle to commute.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

- Dual to this is the *slice category* $\mathcal{C}/c$ where we reverse all the arrows. For example, our objects are morphisms $f : x \to c$, and morphisms from $f : x \to c$ to $g : y \to c$ are morphisms $f : x \to y$ such that $g = hf$.

There are also groupoids, which we have defined previously.

1.3.2 Subcategories

We have the following definition.

Definition 1.34 (Subcategory)**.** A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ whose objects and morphisms come from $\mathcal{C}$ and that the composition law is inherited. Explicitly, we require $\mathcal{D}$ to have the identity morphisms and be closed under composition of $\mathcal{C}$ (i.e., if $f : x \to y$ and $g : y \to z$ are morphisms in $\mathcal{D}$, then gf is also a morphism in $\mathcal{D}$.)

We are going to want ways to generate subcategories. Here is one way.

Definition 1.35 (Full subcategory)**.** Fix a category $\mathcal{C}$. Then we define the *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ to be defined by choosing some objects $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{C}$ and then choosing morphisms by taking all of them. Explicitly, for $x, y \in \operatorname{Ob}\mathcal{D}$, we have

$$\operatorname{Mor}_{\mathcal{D}}(x, y) = \operatorname{Mor}_{\mathcal{C}}(x, y).$$

Example 1.36. The category of abelian groups is a full subcategory of the category of groups. Namely, the category of abelian groups is made of the objects which are abelian groups and all arrows are simply all group homomorphisms, so no morphisms have been lost in this restriction.

Example 1.37. The category of finite sets is a full subcategory in the category of sets.

Example 1.38. Given a category $\mathcal{C}$, one can take the *maximal groupoid* of $\mathcal{C}$ to be the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the isomorphisms of $\mathcal{C}$. So as long as $\mathcal{C}$ has morphisms which are not isomorphisms, then the maximal groupoid will not be full.

Example 1.39. The category Rng is a subcategory of Ring, but it is not full. For example, in Ring, the map $\mathbb{Z} \stackrel{\times 2}{\mathbb{Z}}$ is not a morphism even though it is a morphism in Rng.

One has to be a bit careful with this, however.

Non-Example 1.40. The category Grp is not a subcategory of Set because one can endow the same set with different group structures.

1.3.3 Duality

Here is our main character.

Definition 1.41 (Opposite category)**.** Given a category $\mathcal{C}$, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to have objects which are objects of $\mathcal{C}$ and morphisms $f^{\mathrm{op}} : y \to x$ of $\mathcal{C}^{\mathrm{op}}$ are in one-to-one correspondence with morphisms $f : x \to y$ of $\mathcal{C}$. Lastly, composition is defined by, for $f^{\mathrm{op}} : y \to x$ and $g^{\mathrm{op}} : z \to y$, we have

$$f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}.$$

In pictures, the composition law reversed the diagram $x \xrightarrow{f} y \xrightarrow{g} z$ to

$$x \xleftarrow{f^{\mathrm{op}}} y \xleftarrow{g^{\mathrm{op}}} z.$$

Let's see some examples.

Example 1.42. Given a partial order $(\mathcal{P}, \leq)$, the opposite category is by (partial) ordering $\mathcal{P}$ simply by flipping the partial order: $b \leq_{\mathrm{op}} a$ if and only if $a \leq b$. Namely, the opposite category of a partial order remains a partial order.

Example 1.43. Fix a group G and form its category $\mathrm{B}G$. Now, when we reverse the arrows $(\mathrm{B}G)^{\mathrm{op}}$, we get a category corresponding to the group law G^{op} with group law defined by

$$h^{\mathrm{op}} g^{\mathrm{op}} = gh.$$

Namely, the opposite category of a group is still a group.

In fact, we have that $\mathrm{B}G \cong (\mathrm{B}G)^{\mathrm{op}}$ (for whatever $\cong$ means) by taking making our morphisms perform inversion by $\varphi : g \mapsto (g^{\mathrm{op}})^{-1}$. This map is bijective, and we can check the composition by writing

$$\varphi(gh) = ((gh)^{\mathrm{op}})^{-1} = (h^{\mathrm{op}} g^{\mathrm{op}})^{-1} = (g^{\mathrm{op}})^{-1} (h^{\mathrm{op}})^{-1} = \varphi(g)\varphi(h),$$

so everything works.

Example 1.44. Algebraic geometry says that $\mathrm{CRing}^{\mathrm{op}}$ is equivalent to the category of affine schemes AffSch. The point here is that the opposite category is potentially very different from the original category. (Mnemonically, the opposite of algebra is geometry.)

Now, here is the idea of duality.

Idea 1.45. Theorem statements that hold for categories will need to be true for their opposite category as well.

As an example, let's work with monomorphisms and epimorphisms. For example, $f : y \to z$ is monic if and only if the commutativity of the diagram

$$x \underset{h}{\overset{g}{\rightrightarrows}} y \xrightarrow[f]{} z$$

forces $g = h$. Similarly, $f : x \to y$ is epic if and only if the commutativity of the diagram

$$x \xrightarrow[f]{} y \underset{h}{\overset{g}{\rightrightarrows}} z$$

forces $g = h$. But notice that flipping the epic diagram notes that epic condition is equivalent to the commutativity of the diagram

$$x \underset{h^{\text{op}}}{\overset{g^{\text{op}}}{\rightrightarrows}} y \xrightarrow[f^{\text{op}}]{} x$$

forces $g = h$, which is the same thing as $g^{\text{op}} = h^{\text{op}}$. Thus, we have the following lemma.

Lemma 1.46. Fix a category $\mathcal{C}$. Then a morphism f is monic if and only if f^{op} is epic in $\mathcal{C}$.

Proof. This comes from the discussion above. ■

The point is that we can prove theorems about monic and epic maps simultaneously by working with (say) monomorphisms general categories and then dualizing to get the statement about epimorphisms.

Let's see this strategy in action. We have the following definition.

Definition 1.47 (Section, retraction)**.** Suppose that $s : x \to y$ and $r : y \to x$ are morphisms such that $rs = \text{id}_x$; i.e., the composition

$$x \xrightarrow{s} y \xrightarrow{r} x$$

is id_x. Then we say that s is a *section* of r, and r is a *retraction* of s.

Think about these as having a one-sided inverse. We have the following lemma.

Lemma 1.48. A morphism s in $\mathcal{C}$ is a section of some morphism if and only if s^{op} is a retraction in $\mathcal{C}$.

Proof. Fix $s : x \to y$. The condition that there exists r so that $rs = \text{id}_x$ is equivalent to there exists r^{op} such that $s^{\text{op}}r^{\text{op}} = \text{id}_x^{\text{op}}$, which translates into the lemma. ■

And now let's actually see a proof.

Proposition 1.49. A morphism s in $\mathcal{C}$ is a section of some morphism implies that s is a monomorphism.

Proof. Suppose that $s : x \to y$ is a section for the morphism $r : y \to x$ so that $rs = \text{id}_x$. Now, suppose that $sg = sh$ so that we want to show $g = h$. But we see that

$$g = \text{id}_x g = (rs)g = r(sg) = r(sh) = (rs)h = \text{id}_x h = h,$$

so we are done. ■

So here is our dual statement, which we get for free.

Proposition 1.50. A morphism r in $\mathcal{C}$ is a retraction of some morphism implies that r is an epimorphism.

Proof. We note that r is a retraction in $\mathcal{C}$ implies that r^{op} is a section in $\mathcal{C}^{\mathrm{op}}$, so by the above, r^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$. Thus, it follows that r is an epimorphism in $\mathcal{C}$. ■

We've been saying "section of" and "retraction of" a lot, so we optimize out these words in the following definition.

Definition 1.51 (Split monorphism, split epimorphism)**.** We say that a morphism f of $\mathcal{C}$ is a *split monomorphism* if and only if it is a section of some morphism. Similarly, we say that f is a *split epimorphism* if and only if it is the retraction of some morphism.

So the above statements show that split monomorphisms are in fact monomorphisms, and split epimorphisms are in fact epimorphisms.

1.3.4 Yoneda Lite

So far we have said that monic is similar to injective and epic is similar to surjective. We would like to make these sorts of correspondences a little more concrete, so we add more abstraction.

Definition 1.52 (Post- and pre-composition)**.** Fix a morphism $f : x \to y$ of $\mathcal{C}$. Then, given an object $c \in \mathcal{C}$, we define the maps $f_* : \mathrm{Mor}(c, x) \to \mathrm{Mor}(c, y)$ and $f^*(y, c) \to \mathrm{Mor}(x, c)$ by

$$f_*(g) := fg \qquad \text{and} f^*(g) := gf.$$

The map f_* is called *post-composition* because we apply f after; the map f^* is called *pre-composition* because we apply it after.

Note that f_* and f^* are nice because they are all real functions of sets (for locally small categories) with which we can use to understand f. Here are some equivalent conditions.

Proposition 1.53. Fix f a morphism of the category $\mathcal{C}$. Then the following are true.

(a) f is an isomorphism if and only if f_* is bijective if and only if f^* is bijective.

(b) f is monic if and only if f_* is injective.

(c) f is epic if and only if f^* is injective (!).

(d) f is split monic if and only if f^* is surjective.

(e) f is split epic if and only if f_* is surjective.

Proof. We omit most of these; let's show (b). We have two directions. Suppose that f is monic. Then fix an object c, and we show that the map
$$f_* : \mathrm{Mor}(c, x) \to \mathrm{Mor}(c, y)$$
by $f_*(g) := fg$ is injective. But indeed, $f_*(g) = f_*(h)$ implies $fg = fh$ implies $g = h$ by monic, so injectivity follows.

Conversely, suppose f_* is monic. Then suppose that $fg = fh$ for some morphisms $g, h : c \to x$, and we show that $g = h$. But f_* is injective! So

$$f_*(g) = fg = fh = f_*(h)$$

forces $g = h$, and we are done. ■

THEME 2

FUNCTORS AND NATURAL TRANSFORMATIONS

Mathematics is the art of giving the same names to different things

—Henri Poincaré

2.1 January 26

We will start on new things.

2.1.1 Functors

In this class, we will repeatedly talk about the following idea.

Idea 2.1. Everything is a special case of everything else.

In other words, we will want to abstract old ideas from new ones, and this will happen a lot.

The first time we are going to see this is by trying to consider categories of

Remark 2.2. Yes, Russel's paradox prevents a category of all categories. Nevertheless, we will try. One way to get around this is to do size declarations: for example, we can consider the category of all small categories, as we are about to do.

Anyways, we would like to give some categorical structure to (say, small) categories. Well, what will be our morphisms between categories? They will be "functors."

Before defining functors, we should describe what a functor $F : \mathcal{C} \to \mathcal{D}$ should do.

- Viewing $\mathcal{C}$ as consisting of the data of objects and morphisms, an initial requirement might be that F takes objects to objects and morphisms to morphisms.
- We would also like F to preserve the "structure" of our categories, which essentially means we want to preserve composition in our categories. So we will require a "functoriality" condition to preserve this structure.

Let's try to get an intuitive feeling for how functoriality should behave.

Example 2.3. Fix an abelian group A. Then there is a map $\operatorname{Hom}(A,-)$ sending abelian groups Ab to sets Set. In fact, we get a map of morphisms as well, for a morphism $f: X \to Y$ provides a post-composition mapping

$$f_*: \operatorname{Hom}(A,X) \to \operatorname{Hom}(A,Y)$$

by $\varphi \mapsto f\varphi$. This association has some nice properties. For example, we have the following.

- We see $(\mathrm{id}_X)_*: \operatorname{Hom}(A,X) \to \operatorname{Hom}(A,X)$ sends $\varphi \mapsto \varphi$, so $(\mathrm{id}_X)_* = \mathrm{id}_{\operatorname{Hom}(A,X)}$.
- Given $f: X \to Y$ and $g: Y \to Z$, we have $gf: X \to Z$, and we can see that

$$(gf)_*(\varphi) = gf\varphi = g_*(f_*(\varphi)) = (g_*f_*)(\varphi),$$

so we are "preserving composition" in some sense because we composed before and after.

Example 2.4. Given a topological space X, we can create the fundamental group $\pi_1(X)$. This mapping is nice because a continuous map $f: X \to Y$ will induce a map $\pi(f): \pi_1(X) \to \pi_1(Y)$, and in fact we can check that $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X)}$ as well as preserving composition ($f: X \to Y$ and $g: Y \to Z$ gives $\pi_1(gf) = \pi_1(g)\pi_1(f)$).

With the above motivation, we are now ready to give the definition of a functor.

Definition 2.5 (Functor)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a *functor* $F: \mathcal{C} \to \mathcal{D}$ is a pair of "assignments" $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ and $\operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{D}$ satisfying the following coherence laws.

- Morphisms make sense: if $f: x \to y$ a morphism in $\mathcal{C}$, then Ff is a morphism with domain Fx and codomain Fy.
- Identity: given an object $c \in \mathcal{C}$, we require $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Composition: given morphisms $f: x \to y$ and $g: y \to z$ in $\mathcal{C}$, we require that $F(gf) = F(g)F(f)$.

2.1.2 More Examples

Let's do more examples.

Example 2.6 (Forgetful)**.** There is a functor $U: \mathrm{Grp} \to \mathrm{Set}$ which sends a group G to its underlying set G and a group homomorphism to the underlying function. In other words, we are simply forgetting the algebraic structure of the group. Because the composition law in groups is composition of functions, and identities in Grp do nothing like in Set.

Example 2.7 (Forgetful)**.** Here are more forgetful functors.

- $\mathrm{Ring} \to \mathrm{Grp}$ (by $R \mapsto R^\times$)
- $\mathrm{Field} \to \mathrm{Ring}$
- $\mathrm{Ring} \to \mathrm{Ab}$
- $\mathrm{Grp} \to \mathrm{Set}_*$ by sending $G \mapsto (G, e_G)$; namely, we point the set of G by its identity, which must be fixed by group homomorphisms anyways.

With all of our forgetful functors lying around, we have the following definition.

Definition 2.8 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if it has a forgetful functor to Set.

This is not terribly formal because we haven't defined what a forgetful functor means, but hopefully this is sufficiently intuitive: $\mathcal{C}$ should be sets with some extra structure.

Before our next example, we pick up the following example.

Definition 2.9 (Endofunctor)**.** A functor F is an *endofunctor* of its "domain" and "codomain" categories are the same category.

Example 2.10. There is an endofunctor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ sending a set X to its power set $\mathcal{P}(X)$. We send morphisms $f : X \to Y$ to $\mathcal{P}(f)$ by sending subsets $S_X \subseteq X$ in $\mathcal{P}(X)$ to the image $f(S_X) \in \mathcal{P}(Y)$. We will not check the functoriality conditions, but it can be done without too much effort.

And now for more examples.

Example 2.11. There is a functor $\mathrm{Top} \to \mathrm{Htpy}$ by sending a topological space X to the same space up to homotopy. Then we send continuous maps to continuous maps, up to homotopy.

Example 2.12. There is a "free" functor $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$ sending a set S to the abelian group

$$\mathbb{Z}[S] = \bigoplus_{s\in S} \mathbb{Z}s.$$

Essentially, this is the free $\mathbb{Z}$-module generated by S; formally, $\mathbb{Z}[S]$ is made of finite $\mathbb{Z}$-linear combinations of elements of S.

Then we can take a function $f : S \to T$ to a group homomorphism $\mathbb{Z}[S] \to \mathbb{Z}[T]$ because we have described where to send the "basis elements" of S, and hence this f will uniquely determine the full map.

Example 2.13. Fix $\mathcal{C}$ a locally small category, and fix some $x \in \mathcal{C}$. Then there is a functor $\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ by sending

$$y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y) \qquad \text{and} \qquad (f : y \to z) \mapsto f_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z),$$

where $f_* : \varphi \mapsto f\varphi$ is again post-composition.

Example 2.14. There is an endofunctor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ by sending objects and morphisms to themselves.

2.1.3 Categories of Categories

While we're here, we note that we can create new functors from old ones by "composition."

Proposition 2.15. Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ functors. Then the naturally defined map $GF : \mathcal{C} \to \mathcal{E}$ is also a functor.

Proof. We do indeed send objects to objects, and a morphism $f : x \to y$ in $\mathcal{C}$ will be sent to $F(f) : Fx \to Fy$ and then

$$GF(f) : GFx \to GFy.$$

Further, we can check that $GF(\mathrm{id}_x) = G(\mathrm{id}_{Fx}) = \mathrm{id}_{GFx}$, so GF preserves identities. And then, given $f : x \to y$ and $g : y \to z$, we see that

$$GF(gf) = G(F(g)F(f)) = GF(g)GF(f),$$

which finishes the composition check. ■

The point of the above composition law, is that it lets us form a "category."

Definition 2.16. We define Cat to be the category of small categories where morphisms are functors. We define CAT to be the category of locally small categories where morphisms again are functors.

Remark 2.17. Fixing two small categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ can be identified with a function on merely the morphism sets $\operatorname{Mor} \mathcal{C} \to \operatorname{Mor} \mathcal{D}$, which is itself a set. Thus, Cat is a locally small category: $\mathrm{Cat} \in \mathrm{CAT}$.

2.1.4 Subcategories.

To finish out class, we have the following warning.

Warning 2.18. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We check that the naturally defined "image" $F(\mathcal{C})$ need not be a subcategory of $\mathcal{D}$.

Here is an example. Let $\mathcal{C}$ be the following category.

$$a \xrightarrow{f} b$$

$$a' \xrightarrow{f'} b'$$

Then let $\mathcal{D}$ be the following category.

$$0 \xrightarrow{x} 1 \xrightarrow{y} 2$$

Now we define $F : \mathcal{C} \to \mathcal{D}$ by $Ff = x$ and $Ff' = y$, which will make a perfectly fine functor. However, the composition $yx : 0 \to 2$ in $\mathcal{D}$ does not live in the image of F, so this image is not a subcategory.

To fix this problem, one often says something like "given a functor $F : \mathcal{C} \to \mathcal{D}$, consider the full subcategory of $F(\mathcal{C})$" to mean closing up $F(\mathcal{C})$'s potentially unclosed composition.

2.2 January 31

So class is in-person today.

2.2.1 Small Remark

A question was asked in the Discord server about dualizing. In theory, dualizing theorems should be very easy: simply state the theorem in the opposite category, provided we have shown the necessary machinery to make the theorem dualize as necessary.

2.2.2 Contravariance

Today we are talking about contravariance. A functor $F : \mathcal{C} \to \mathcal{D}$ is defined so far as what are called "covariant" functors. We would like to define contravariant functors. There are lots of equivalent ways to do this.

Definition 2.19 (Contravariance, I)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a mapping of objects and morphisms with the following coherence laws.

- If $f : a \to b$ in $\mathcal{C}$, then $Ff : Fb \to Fa$. (Note the reversal of direction!)
- Identity: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for each $c \in \mathcal{C}$.
- Contravariant (!) composition: if $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, then $F(gf) = F(f)F(g)$.

This in fact comes from dualizing.

Definition 2.20 (Contravariance, II)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

To be explicit, if we are given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, then a morphism $f : a \to b$ in $\mathcal{C}$ is first taken to a morphism $f^{\mathrm{op}} : b^{\mathrm{op}} \to a^{\mathrm{op}}$. And if we have another morphism $g : b \to c$ in $\mathcal{C}$, then we see the diagram

$$a \overset{f}{\to} b \overset{g}{\to} c$$

becomes

$$a^{\mathrm{op}} \overset{f^{\mathrm{op}}}{\leftarrow} b^{\mathrm{op}} \overset{g^{\mathrm{op}}}{\leftarrow} c^{\mathrm{op}}$$

becomes

$$Fa^{\mathrm{op}} \overset{Ff^{\mathrm{op}}}{\leftarrow} Fb^{\mathrm{op}} \overset{Fg^{\mathrm{op}}}{\leftarrow} Fc^{\mathrm{op}},$$

which gives our composition law.

We can also dualize in the opposite direction.

Definition 2.21 (Contravariance, III)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Warning 2.22. We will use Definition 2.20 as our definition of contravariance.

Example 2.23. We work with Vec_k the category whose objects are k-vector spaces and morphisms which are linear maps. Then we have a functor

$$-^* : \mathrm{Vec}_k^{\mathrm{op}} \to \mathrm{Vec}_k$$

by taking $V \mapsto V^*$. (Here, $V^* := \mathrm{Hom}_k(V, k)$.) As for morphisms, we need to take $f : V \to W$ to some map $f^* : W^* \to V^*$, which is

$$f^* : \varphi \mapsto \varphi f.$$

Example 2.24. We work with Poset the category whose objects are posets and morphisms which are order-preserving maps. I.e., a map $f : P \to Q$ is order-preserving if and only if $a \le b$ in P implies $f(a) \le f(b)$ in Q. Now we define the contravariant functor $\mathcal{O} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Poset}$ by taking

$$X \mapsto \{U : \text{open } U \subseteq X\},$$

where the order on the right is by inclusion. Then a continuous map $f : X \to Y$ becomes the order-preserving (!) map $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ by

$$\mathcal{O}(f)(U_Y) := f^{-1}(U_Y).$$

Explicitly, open subsets $U_1 \subseteq U_2$ of Y have $f^{-1}(U_1) \subseteq f^{-1}(U_2)$ back in X.

Remark 2.25. We can use the above example to define a presheaf. "Presheaf" can have lots of meanings.

- A "presheaf" can be any contravariant functor.
- A "presheaf" can be any contravariant functor with codomain Set.
- A "presheaf" can be any contravariant functor from $\mathcal{O}(X)^{\mathrm{op}}$. It is Set-valued (respectively, $\mathcal{C}$-valued) if its codomain is Set (respectively, $\mathcal{C}$).

2.2.3 A Lemma

It's a math class, so we should probably prove something today.

Theorem 2.26. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ preserves isomorphisms.

Remark 2.27. By convention, all functors will be covariant, and if we want a contravariant functor, we will write $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. In other words, I will now stop writing "(covariant)."

Proof. Let $f : a \to b$ be an isomorphism in $\mathcal{C}$ with inverse g. We want to show that $F(f)$ is an isomorphism; we claim that $F(g)$ is its inverse. Indeed,

$$F(f)F(g) = F(fg) = F(\mathrm{id}_b) = \mathrm{id}_{F(b)} \qquad \text{and} \qquad F(g)F(f) = F(gf) = F(\mathrm{id}_a) = \mathrm{id}_{F(a)},$$

so indeed, $F(g)$ is an inverse of $F(f)$. So $F(f)$ is an isomorphism, and we are done. ■

This example can do things.

Example 2.28. Fix groups G, H and their one-object categories $\mathrm{B}G, \mathrm{B}H$. We claim that functors $F : \mathrm{B}G \to \mathrm{B}G$ contain exactly the data of a group homomorphism $G \to H$. To see that F induces a group homomorphism, suppose $\sigma, \tau \in G$, we have by funtoriality

$$F(\sigma\tau) = F(\sigma)F(\tau),$$

which is exactly what we need to be a group homomorphism. Conversely, if $f : G \to H$ is a group homomorphism, then f induces a functor: $f(\sigma\tau) = f(\sigma)f(\tau)$ by definition, and $f(\mathrm{id}_G) = \mathrm{id}_H$ is a result of group theory.

Example 2.29. A functor $F : \mathrm{B}G \to \mathcal{C}$ is precisely the data of a G-action of an object $c \in \mathcal{C}$. We send the one object $* \in \mathrm{B}G$ somewhere, say to an object $c \in \mathcal{C}$. Then each $\sigma \in G$ goes to some morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ (which is in fact an isomorphism because σ is an isomorphism $\mathrm{B}G$). So in total we get a map

$$G \to \mathrm{Aut}\, c,$$

which is exactly the data of a group action. This unifies group actions on all sorts of structures.

The above definition is special enough to have a name.

Definition 2.30 (Functorial group action)**.** A *functorial group action* of G on a category $\mathcal{C}$ is a functor $\mathrm{B}G \to \mathcal{C}$.

Remark 2.31. Technically we will be viewing these functors as providing left actions. To get a right action, we want a functor $(\mathrm{B}G)^{\mathrm{op}} \to \mathcal{C}$.

Note, as in the example, the functor contains the same data as a group homomorphism $G \to \mathrm{Aut}\, c$ for some $c \in \mathcal{C}$.

Remark 2.32. Bryce would like to make us aware that writing down $G \to \mathrm{Aut}\, c$ as a group homomorphism is only legal when $\mathcal{C}$ is locally small.

Example 2.33. Given a group G, a G-representation V of G is a functor $\mathrm{B}G \to \mathrm{Vec}_k$ where $* \in \mathrm{B}G$ goes to $V \in \mathrm{Vec}_k$.

2.2.4 The Hom Bifunctor

We have a little time left, so let's do something fun. Given a (locally small) $\mathcal{C}$ and an object $x \in \mathcal{C}$, we get two functors

$$\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set} \qquad \text{and} \qquad \mathrm{Mor}_{\mathcal{C}}(-, x) : \mathcal{C} \to \mathrm{Set}.$$

The former functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $\varphi : y \to z$ to $\varphi_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$ to $\varphi_* : f \mapsto \varphi f$. We can check this functor is covariant because

$$\varphi_* \psi_*(f) = \varphi\psi f = (\varphi\psi)_*(f).$$

Now, the latter functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(y, x)$ and $\varphi : y \to z$ to $\varphi^* : \mathrm{Mor}_{\mathcal{C}}(z, x) \to \mathrm{Mor}_{\mathcal{C}}(y, z)$ by $\varphi^* : f \mapsto \varphi f$. We can check this functor is contravariant because

$$\psi^* \varphi^* f = f\varphi\psi = (\varphi\psi) * f.$$

2.3 February 2

Today we are talking about product categories and the Hom bifunctor.

2.3.1 Hom Bifunctor

Here is our definition.

Definition 2.34 (Product category)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then we define the *product category* $\mathcal{C} \times \mathcal{D}$ as follows.

- We define $\operatorname{Ob} \mathcal{C} \times \mathcal{D}$ to be the collection of ordered pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
- We define $\operatorname{Mor}((c, d), (c', d'))$ to be the collection of ordered pairs (f, g) with $f : c \to c'$ a morphism in $\mathcal{C}$ and $g : d \to d'$ a morphism in $\mathcal{D}$.

Lastly, we define identity to be the identity on each object and composition by composition component-wise.

From yesterday, we have the following functors.

Definition 2.35 (Functors represented by objects)**.** Fix $\mathcal{C}$ a locally small category and $x \in \mathcal{C}$ an object. Then we have the functors

$$\operatorname{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set} \qquad \text{and} \qquad \operatorname{Mor}_{\mathcal{C}}(-, x) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}.$$

The former functor is the *covariant functor represented by* x, and the latter is the *contravariant functor represented by* x.

We would like to codify the structure that having two functors gives us, so we have the following definition.

Definition 2.36 (Bifunctor)**.** A *bifunctor* is a functor whose domain is a product of categories.

In particular, here is our standard example.

Definition 2.37 (Hom bifunntor)**.** Fix $\mathcal{C}$ a locally small category. Then Hom *bifunctor* is the functor given by the functors representing a particular object $x \in \mathcal{C}$. Namely, we have

$$\operatorname{Mor}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$$

by taking $(x, y) \mapsto \operatorname{Mor}_{\mathcal{C}}(x, y)$.

We will not check that this is actually a functor, but it is.

2.3.2 Category Isomorphism

We would like a notion of two categories being the same, but this is somewhat subtle. Here is a first approximation.

Definition 2.38 (Isomorphism)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if and only if there is an inverse functor $G : \mathcal{D} \to \mathcal{C}$ so that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. In this case we say that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*.

Remark 2.39. As usual, isomorphisms are unique and whatnot.

Let's make this definition a little more concrete.

Proposition 2.40. An isomorphism $F : \mathcal{C} \to \mathcal{D}$ descends to a bijective (i.e., injective and surjective) map $\operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D}$.

Remark 2.41. We are attempting to care about set-theoretic issues in our phrasing because Bryce cares about set-theoretic issues.

Proof of Proposition 2.40. Let G be the inverse morphism for F. Then we claim that the induced map $G : \operatorname{Ob}\mathcal{D} \to \operatorname{Ob}\mathcal{C}$ will be the inverse for the induced map for F. This is clear because $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. ■

It turns out that isomorphisms are a little too strong: there are categories we want to be the same but are not actually isomorphic.

Example 2.42. The category

$$\bullet \rightleftarrows \bullet$$

is not isomorphic to

$$\bullet$$

because there are a different number of objects, so there is no bijection.

2.3.3 Natural Transformation

To salvage our notion of categorical isomorphism, we need a notion of naturality. Naturality is more of something that we can feel as mathematicians rather than something we like to formalize.

Example 2.43. Any two trivial groups have a canonical isomorphism between them. In fact, there is only one homomorphism at all.

Non-Example 2.44. There is no "natural" or "canonical" isomorphism $\mathbb{Z}/3\mathbb{Z} \to A_3$, though the groups are isomorphic.

Non-Example 2.45. Given a two-dimensional $\mathbb{R}$-vector space named V, there is no canonical isomorphism $\mathbb{R}^2 \to V$.

We would like maps to preserve all the structure we could want. So here is our notion of naturality for functors.

Definition 2.46 (Natural transformation)**.** Fix functors $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\eta : F \Rightarrow G$ consists of the data of a morphism $\eta_X : Fc \to Gc$ for each $c \in \mathcal{C}$ such that the following diagram always commutes for any morphism $f : c \to c'$ in $\mathcal{C}$.

$$\begin{tikzcd} Fc \arrow[r, "\eta_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\eta_{c'}"'] & Gc' \end{tikzcd}$$

The maps φ_c are called the *components* of φ.

Quote 2.47. Burn this square into your minds. It is the most important square in this class.

As usual, we start with examples.

Exercise 2.48. We work in Vec_k. Then we consider the functor $-^{**} : \mathrm{Vec}_k \to \mathrm{Vec}_k$ by $V \mapsto V^{**}$. Then we claim that there is a natural transformation from $-^{**}$ to id, using the natural transformation

$$\mathrm{ev}_V : V \to V^{**}$$

by $\mathrm{ev}_V(x) := (\lambda \in V^* \mapsto \lambda x)$.

Proof. We need to check that the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\mathrm{ev}_W"'] & W^{**} \end{tikzcd}$$

Very quickly, we recall that $f^{**} : V^{**} \to W^{**}$ is by

$$f(\varphi) = (\lambda \in W^* \mapsto \varphi(\lambda f)).$$

Namely, $\lambda : W \to k$, so $\lambda f : V \to k$ lives in V^*, so $\varphi(\lambda f) \in k$.

Now we check the commutativity of the square. Fix some $x \in V$ and a linear functional $\lambda : W \to k$. Then we can carefully compute, after many tears and groans, that

$$f^{**}(\mathrm{ev}_V(x))(\lambda) = \mathrm{ev}_V(x)(\lambda f) = \lambda f(x) = \mathrm{ev}_W(f(x))(\lambda).$$

Because λ was arbitrary, we see that $f^{**}\,\mathrm{ev}_V(\lambda) = \mathrm{ev}_W\, f(x)$, which then gives us $f^{**}\,\mathrm{ev}_V = \mathrm{ev}_W\, f$. ∎

We have the following definition.

Definition 2.49 (Natural isomorphism)**.** A natural transformation $\eta : F \to C$ is a *natural isomorphism* if and only if its component morphisms are isomorphisms.

Example 2.50. In finVec_k, the above ev is a natural isomorphism because $\mathrm{ev}_V : V \Rightarrow V^{**}$ is an isomorphism when V is finite-dimensional.

Here is a quick proposition.

Proposition 2.51. Let $\varphi : F \Rightarrow G$ be a natural isomorphism. Then the inverse morphisms $\psi_c := \varphi_c^{-1}$ assemble to make a natural transformation $\psi : G \Rightarrow F$.

Proof. We will be brief. Given a morphism $f : x \to y$, we need to check that the following diagram commutes.

$$\begin{tikzcd} Gx \arrow[r, "\psi_x"] \arrow[d, "Gf"'] & Fx \arrow[d, "Ff"] \\ Gy \arrow[r, "\psi_y"'] & Fy \end{tikzcd}$$

In other words, we need to know that $\psi_y Ff = Gf\psi_x$. Well, we already know that

$$\varphi_y Ff = Gf\varphi_x$$

by naturality, so

$$Ff\psi_x = \psi_y\varphi_y Ff\psi_x = \psi_y Gf\varphi_x\psi_x = \psi_y Gf$$

after checking through. ∎

2.4 February 7

2.4.1 Examples of Natural Transformations

We're talking about more natural transformations today. For our first example, consider the covariant power set functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ by $S \mapsto \mathcal{P}(S)$ and $f : S \to T$ to $\mathcal{P}(f)(U) := f(U)$ for $U \subseteq S$.

Exercise 2.52. We define a natural transformation $\eta_\bullet : \mathrm{id}_{\mathrm{Set}} \Rightarrow \mathcal{P}$ a function $\eta_S : S \to \mathcal{P}(S)$ by

$$\eta_S(x) := \{x\}$$

Proof. Fix $f : S \to T$ a morphism in Set. After plugging everything in, we need the following diagram to commute.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\eta_S"'] & T \arrow[d, "\eta_T"] \\ \mathcal{P}(S) \arrow[r, "\mathcal{P}(f)"'] & \mathcal{P}(T) \end{tikzcd}$$

To see this commutes, fix some $x \in S$, and we run it through the diagram as follows.

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\eta_S"'] & f(x) \arrow[d, mapsto, "\eta_T"] \\ \{x\} \arrow[r, mapsto, "\mathcal{P}(f)"'] & \{f(x)\} \end{tikzcd}$$

So indeed, the diagram does commute. ■

Remark 2.53. We may call the second diagram an "internal" diagram because it is looking internally at our objects.

For our next example, recall we defined a functorial G-action on some object $c \in \mathcal{C}$ by a functor $F : \mathrm{B}G \to \mathcal{C}$. Our goal is to define a G-equivariant map between objects.

Exercise 2.54. We track the data between two G-representations $F, G : \mathrm{B}G \to \mathrm{Vec}_k$ by a natural transformation $\eta_\bullet : \mathrm{Vec}_k \Rightarrow \mathrm{Vec}_k$.

Proof. Because $\mathrm{B}G$ has only one object $*$, we set $V := F(*)$ and $W := G(*)$ and need to check the commutativity of the following diagram, for some $g : * \to *$ in G.

$$\begin{tikzcd} V \arrow[r, "Fg"] \arrow[d, "\eta_*"'] & V \arrow[d, "\eta_*"] \\ W \arrow[r, "Fw"'] & W \end{tikzcd}$$

Note that the natural transformation $\eta_\bullet$ really only consists of the map η_*, which is a linear map $V \to W$ which respects the group action: $\eta_*(gv) = g\eta_*(v)$. ■

These G-equivariant maps can be turned into a category.

Definition 2.55 (G-representations)**.** We define the category of G-*representations* to be the category consisting of objects which are functors $F : \mathrm{B}G \to \mathrm{Vec}_k$ and morphisms which are natural transformations between the functors.

Exercise 2.56. We check that there is a category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose

Proof. To define our morphisms, suppose $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\eta_\bullet : F \Rightarrow G$ and $\nu_\bullet : G \Rightarrow H$. Lastly, we define our composition by

$$(\nu\eta)_X := \eta_X \nu_X.$$

To check that $(\eta\nu)_\bullet : F \Rightarrow H$ is in fact a natural transformation, we have the following ladder.

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"'] \arrow[dd, bend right, "(\nu\eta)_x"'] & Fy \arrow[d, "\eta_y"] \arrow[dd, bend left, "(\nu\eta)_y"] \\
Gx \arrow[r, "Gf"] \arrow[d, "\nu_x"'] & Gy \arrow[d, "\nu_y"] \\
Hx \arrow[r, "Hf"'] & Hy
\end{tikzcd}$$

Each square commutes, so the 2×1 rectangle will also commute. We check associativity by drawing a 3×1 rectangle and seeing that it commutes.

To define our identity maps for our category, we take $(\mathrm{id}_F)_X := \mathrm{id}_{F(x)} : Fx \to Fx$. We can check that this works with our composition without too many tears. ■

Definition 2.57 (Functor category)**.** The category of the above exercise is the *functor category*, notated $\mathcal{D}^{\mathcal{C}}$.

Example 2.58. We have that $\mathrm{Rep}_G = \mathrm{Vec}_k^{BG}$.

2.4.2 Yoneda, Contravariant It Is

For the discussion that follows, we fix $\mathcal{C}$ locally small and $f : w \to x$ and $h : y \to z$ some morphisms in $\mathcal{C}$. From this we get the following square.

$$\begin{tikzcd}
\mathrm{Mor}(x, y) \arrow[r, "h-"] \arrow[d, "-f"'] & \mathrm{Mor}(x, z) \arrow[d, "-f"] \\
\mathrm{Mor}(w, y) \arrow[r, "h-"'] & \mathrm{Mor}(w, z)
\end{tikzcd}$$

We can check that this square commutes. Here is the internal square.

$$\begin{tikzcd}
g \arrow[r, mapsto, "h-"] \arrow[d, mapsto, "-f"'] & hg \arrow[d, mapsto, "-f"] \\
gf \arrow[r, mapsto, "h-"'] & hgf
\end{tikzcd}$$

Hooray, it commutes. The point is that $h-$ and $-f$ are going to induce natural transformations of our Mor functors.

- The functors $\mathrm{Mor}(x, -), \mathrm{Mor}(w, -) : \mathcal{C} \to \mathrm{Set}$. Then any morphism $f : w \to x$ induces a natural transformation $-f : \mathrm{Mor}(x, -) \Rightarrow \mathrm{Mor}(w, -)$. The naturality check is the commutativity of the above square.

- Similarly, the functors $\mathrm{Mor}(-, y), \mathrm{Mor}(-, z) : \mathcal{C} \to \mathrm{Set}$. Then any morphism $h : x \to y$ induces a natural transformation $h- : \mathrm{Mor}(-, y) \Rightarrow \mathrm{Mor}(-, w)$. The naturality check is again the commutativity of the above square.

We won't be more explicit about our squares because my head hurts.

Remark 2.59. Later in life we will talk about the Yoneda embedding, which is essentially about the embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}^{\mathcal{C}}$, which takes $x \mapsto \mathrm{Mor}(x, -)$ and $f : x \to y$ to the natural transformation $-f : \mathrm{Mor}(x, -) \Rightarrow \mathrm{Mor}(y, -)$. This will turn out to be a functor and very good. We will not say more for now.

2.4.3 Categorification

The category Set has some nice operations: we can talk about products $A \times B$, disjoint unions $A \sqcup B$, and functions $A^C = \{f : C \to A\}$. Note that these notations are suggestive of multiplication, addition (depending on whom you talk to), and exponentiation. For example,

$$\#(A \times B) = \#A \times \#B, \quad \#(A \sqcup B) = \#A + \#B, \quad \#\left(A^C\right) = \#A^{\#C}.$$

This gives us some notion of a "cardinality functor" $\# : \mathrm{FinSet} \to \mathbb{N}$, which we can check does some things.

This lets us define "categorification." We will not give a formal definition of this, but here are some instructive examples.

Example 2.60. The functor $\# : \mathrm{FinSet} \to \mathbb{N}$ is a decategorification functor. For example, we can categorify $a \times (b + c) = a \times b + a \times c$ in $\mathbb{N}$ to some natural isomorphism

$$A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C).$$

Example 2.61. There is a decategorification functor $\dim : \mathrm{fdRep}_G \to \mathbb{N}$.

2.4.4 Equivalence: Advertisement

Let's close class by defining an equivalence of categories. Recall that we called a functor $F : \mathcal{C} \to \mathcal{D}$ an isomorphism if and only if it has an inverse functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$.

This is a bad notion of saying two categories are the same.

Example 2.62. The categories of k-matrices and k-vector spaces are not isomorphic (they don't have the same), even though we often think about vector spaces as merely being some dimensional space.

Here is the fix

Definition 2.63 (Equivalence)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if and only if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$.

2.5 February 9

2.5.1 Equivalence

We can define a category Mat_k to have objects which are the natural numbers and morphisms which are $\mathrm{Mat}_k(n, m)$ equal to the $m \times n$ matrices with coefficients in k. In linear algebra, we want to think about each natural number n as a k-vector space of dimension n, and we want to think about each matrix $n \to m$ as a linear map. In other words, Mat_k should be "the same" as fdVec_k.

However, fdVec_k and Mat_k do not even have the same number of objects, so they cannot be isomorphic. We still want them to be the same, so we weaken our notion of isomorphism.

Definition 2.64 (Equivalence)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ if and only if $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$. If an equivalence between $\mathcal{C}$ and $\mathcal{D}$ exists, then $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, denoted $\mathcal{C} \simeq \mathcal{D}$.

We should probably start by showing that our notion of equivalence forms what we think of as an equivalence relation.

Remark 2.65 (Bryce)**.** Equivalence does not form an equivalence relation for size reasons.

Lemma 2.66. Fix categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. Then the following hold.

- Reflexive: $\mathcal{C} \simeq \mathcal{C}$.
- Symmetric: $\mathcal{C} \simeq \mathcal{D}$ implies $\mathcal{D} \simeq \mathcal{C}$.
- Transitive: $\mathcal{C} \simeq \mathcal{D}$ and $\mathcal{D} \simeq \mathcal{E}$ implies $\mathcal{C} \simeq \mathcal{E}$.

Proof. We will be brief.

- We have that $\mathrm{id}_{\mathcal{C}}$ provides the needed equivalence.
- If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence with $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$, then G witnesses $\mathcal{D} \simeq \mathcal{C}$.
- Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ witness $C \simeq D$, and fix $F' : \mathcal{D} \to \mathcal{E}$ and $G' : \mathcal{E} \to \mathcal{D}$ witness $D \simeq E$.

  In particular, we are promised natural isomorphisms $\varphi : G \simeq \mathrm{id}_{\mathcal{C}}$ and $\psi : FG \simeq \mathrm{id}_{\mathcal{D}}$ and $\varphi' : G'F' \simeq \mathrm{id}_{\mathcal{D}}$ and $\psi' : F'G' \simeq \mathrm{id}_{\mathcal{E}}$. We would like $GG'F'F \simeq \mathrm{id}_{\mathcal{C}}$, and then $F'FGG' \simeq \mathrm{id}_{\mathcal{E}}$ will follow in a very similar way.

  Well, for an object $c \in \mathcal{C}$, we define our natural transformation $\eta_\bullet$ as having component

  $$\eta_c := \varphi_c \circ G\varphi'_{Fc},$$

  which takes $GG'F'Fc$ to GFc to c. We show naturality directly. Fix some morphism $f : x \to y$ in $\mathcal{C}$. We need the following diagram to commute.

  $$\begin{tikzcd} GG'F'Fx \arrow[r, "\eta_x"] \arrow[d, "GG'F'Ff"'] & x \arrow[d, "f"] \\ GG'F'Fy \arrow[r, "\eta_y"'] & y \end{tikzcd}$$

  To see that this commutes, here is an expanded diagram.

  $$\begin{tikzcd} GG'F'Fx \arrow[r, "G\varphi'_{Fx}"] \arrow[d, "GG'F'Ff"'] \arrow[rr, bend left, "\eta_x"] & GFx \arrow[r, "\varphi_x"] \arrow[d, "GFf"] & x \arrow[d, "f"] \\ GG'F'Fx \arrow[r, "G\varphi'_{Fy}"'] \arrow[rr, bend right, "\eta_y"'] & GFy \arrow[r, "\varphi_y"'] & y \end{tikzcd}$$

By definition of $\eta_\bullet$, it now suffices to show that the left and right squares commute. The right square commutes by naturality of φ_x. To see that the left square commutes, we note that it is what we get after applying G to the naturality square for φ' on the morphism $GFf : GFx \to GFy$.

Lastly, to see that η is a natural isomorphism, we note that each component $\eta_c = \varphi_c \circ G\varphi'_{Fc}$ is the composite of isomorphisms, where we are using that φ and φ' are natural isomorphisms and that functors preserve isomorphisms. ■

This is nice because oftentimes showing that two categories are equivalent is easier by showing a chain of equivalences instead of doing it directly. For example, in our proof that $\mathrm{Mat}_k \simeq \mathrm{fdVec}_k$, we will instead show that both of these categories are equivalent to $\mathrm{fdVec}_k^{\mathrm{basis}}$ of vector spaces with given basis.

2.5.2 Lazy Equivalence

We want to provide a tool for constructing equivalences without having to actually write down a natural transformation. By way of analogy, when showing an "isomorphism of sets" we often show that a given map is both injective and surjective. We will do something similar.

Definition 2.67 (Adjectives for functors). Fix categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $F : \mathcal{C} \to \mathcal{D}$. We consider the map $F^\circ : F : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(Fx, Fy)$. Then

- F is *full* if and only if F° is surjective.
- F is *faithful* if and only if F° is injective.
- F is *fully faithful* if and only if F is full and faithful.
- F is *essentially surjective* on objects if and only if each $d \in \mathcal{D}$ has some $c \in \mathcal{C}$ such that $Fc \cong d$ in $\mathcal{D}$.
- F is an *embedding* if and only if F is faithful and injective on objects.
- F is a *full embedding* if and only if F is an embedding and full.

Remark 2.68. Technically we might want to require that $\mathcal{C}$ and $\mathcal{D}$ be locally small, but there are ways of stating "surjective" and "injective" to note require the underlying domain and codomain to be sets.

Remark 2.69. Being "essentially surjective" will give problems with the axiom of choice later in life because we are not requiring any notion of uniqueness.

We note that a functor being "full" or "faithful" are both local conditions on particular sets of morphisms. For example, if a functor doesn't even hit an object which is outside the image of F, then we can't touch those morphism sets.

Example 2.70. Full and faithful does not imply injective on objects. For example, consider the natural functor F from the left category to the right category, which causes full-on collisions but not locally on the morphism sets.

$$\begin{array}{ccccc} a_1 & & a_2 & & a \\ \downarrow & & \downarrow & \xRightarrow{F} & \downarrow \\ b_1 & & b_2 & & b \end{array}$$

Namely, the maps $\mathrm{Mor}_{\mathcal{C}}(a_\bullet, b_\bullet) \to \mathrm{Mor}_{\mathcal{C}}(a, b)$.

Let's finish class by proving something.

Proposition 2.71. The following are closed under composition.

- Full functors.
- Faithful functors.
- Essentially surjective functors.

Proof. We will be very brief.

- Read the proof of the below and replace all instances of the word "surjective" with "injective."
- Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are faithful functors. Then fix $x, y \in \mathcal{C}$, and we know that the induced maps
$$F^\circ : \operatorname{Mor}_{\mathcal{C}}(x, y) \to \operatorname{Mor}_{\mathcal{D}}(Fx, Fy) \qquad \text{and} \qquad G^\circ : \operatorname{Mor}_{\mathcal{D}}(Fx, Fy) \to \operatorname{Mor}_{\mathcal{D}}(GFx, GFy)$$
are both injective, so their composite is injective. To be explicit, if f and g have $(GF)f = (GF)g$, then $G(Ff) = G(Fg)$, so $Ff = Fg$ by injectivity of G°, so $f = g$ by
- Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are essentially surjective functors. Well, fix any $e \in \mathcal{E}$, and we are promised an object $d \in \mathcal{D}$ such that $Gd \cong e$. But now we are promised an object $c \in \mathcal{C}$ such that $Fc \cong d$, so $GFc \cong Fd \cong e$, which shows that GF is essentially surjective. ■

2.6 February 11

2.6.1 A Better Equivalence

Today we will be talking about the following theorem for our discussion.

Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are true.

(a) If F is an equivalence, then F is fully faithful and essentially surjective.

(b) Assuming a strong form of the axiom of choice, the converse holds.

Remark 2.73. The strong form of the Axiom of choice is for, not sets, but classes/categories depending on how we choose to construct our categories.

Proof of (a) in Theorem 2.72. We will want some lemmas.

Lemma 2.74. Fix a category $\mathcal{C}$. Further, fix a morphism $f : c \to d$ and isomorphisms $\varphi : c \cong c'$ and $\psi : d \cong d'$. Then there is a unique morphism $f' : c' \to d'$ such that one (or equivalently, all) of the following four squares commute.

$$\begin{tikzcd}
c' \arrow[r, shift left] \arrow[d, dashed, "f'"'] & c \arrow[l, shift left, "\varphi"] \arrow[d, "f"] \\
d' \arrow[r, shift left] & d \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

Here, the four squares are achieved by changing the direction of φ and ψ.

Proof. This is on the homework. ■

We now return to the proof of the theorem. In the easier direction, suppose that F is an equivalence with its inverse equivalence $G : \mathcal{D} \to \mathcal{C}$, witnessed by natural isomorphisms $\eta_\bullet : \mathrm{id}_\mathcal{C} \Rightarrow GF$ and $\varepsilon : GF \Rightarrow \mathrm{id}_\mathcal{D}$. We have the following checks.

- We show that F is essentially surjective. Indeed, for any object $d \in \mathcal{D}$, we set $c := Gd$. Then we see $FGd \cong d$ is witnessed by the component isomorphism ε_d.

- We show that F is faithful, for which we have to use Lemma 2.74. Indeed, suppose that we have morphisms $f, g : c \to d$ such that $Ff = Fg$. Then in fact $GFf = Gfg$, so the following diagrams will commute.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "GFf=GFg"'] & GFd \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "g"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "GFf=GFg"'] & GFd \end{tikzcd}$$

  It follows from Lemma 2.74 that there f and g are uniquely determined, so $f = g$.

We quickly remark that, by symmetry, G is also faithful.

- We show that F is full, which will use the lemma as well as the fact that G is faithful (!). Well, suppose that we have some morphism $g : Fc \to Fd$. Passing through to G, we get a morphism $Gg : GFg \to GFg$, so by Lemma 2.74, there is a unique morphism $f : c \to d$ so that the following diagram commutes.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, dashed, "f"'] & GFc \arrow[d, "Gg"] \\ d \arrow[r, "\eta_d"'] & GFd \end{tikzcd}$$

  Now, both GFf and Gg make the following diagram commute.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, dashed, "{Gg,GFf}"] \\ d \arrow[r, "\eta_d"'] & GFd \end{tikzcd}$$

  Thus, by Lemma 2.74, we see $GFf = Gg$, so $Ff = g$ by the faithfulness of G. This finishes. ■

Proof of (b) in Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a fully faithful and essentially surjective functor. We need to construct a $G : \mathcal{D} \to \mathcal{C}$ with some natural isomorphisms. We do this by hand.

- For each $d \in \mathcal{D}$, we callously choose[1] Gd to be any $c \in \mathcal{C}$ together with an isomorphism $\varepsilon_d : GFd \to d$. Indeed, such a d with isomorphism ε_d exists because F is essentially surjective.

- For each $f : d \to d'$ in $\mathcal{D}$, we use Lemma 2.74 to choose h to be the unique morphism making the following diagram commute.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[d, "f"'] & FGd \arrow[d, dashed, "h"] \\ d' \arrow[r, "\varepsilon_{d'}"'] & FGd' \end{tikzcd}$$

  But because F is fully faithful, there will be a unique morphism which we call Gf such that $F(Gf) = h$.

We would like to check that G is in fact our inverse equivalence. However, we don't even know if G is a functor yet.

[1] Note we are using some fuzzy form of the axiom of choice here. We will not say more about this.

- Fix $d \in \mathcal{D}$ and we compute $G(\mathrm{id}_d)$. We run through the definition. Well, we note that id_{FGd} makes the following diagram commute, so it will be the morphism generated by Lemma 2.74.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[d, "\mathrm{id}_d"'] & FGd \arrow[d, dashed, "\mathrm{id}_{FGd}"] \\ d \arrow[r, "\varepsilon_{d'}"'] & FGd' \end{tikzcd}$$

  But now we see that $F(\mathrm{id}_{Gd}) = \mathrm{id}_{FGd}$, so id_{Gd} must be the corresponding morphism promised by the fullness and faithfulness of F. In particular, by definition, $G(\mathrm{id}_d) = \mathrm{id}_{Gd}$.

- Suppose we have $f : d \to d'$ and $g : d' \to d''$. We want to show that $G(gf) = Gg \circ Gf$. For this, we have the following very big diagram.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"'] \arrow[d, "FGf"] \arrow[dd, bend right, "F(Gf \circ Gf)"'] & d \arrow[d, "g"] \arrow[dd, bend left, "gf"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] \arrow[d, "FGg"] & d' \arrow[d, "f"] \\ FGd'' \arrow[r, "\varepsilon_{d''}"'] & d'' \end{tikzcd}$$

  This diagram does commute, from which we see that the left arrow can be either $F(Gg \circ Gf)$ (by funtoriality of F) or $F(G(gf))$. So by Lemma 2.74, we have $F(Gg \circ Gf) = F(G(gf))$, so faithfulness of F implies $Gg \circ Gf = G(gf)$.

Now we construct our natural isomorphisms.

- By construction of the εs, the following diagram commutes.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FGf"'] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$

- For the other direction, we note that if $Fx \cong Fy$ in $\mathcal{D}$, then $x \cong y$, which we will prove on the homework.[2] In particular, to create an isomorphism $\eta_c : c \to GFc$, it suffices to create an isomorphism $Fc \to FGFc$, for which we use $F\eta_c := \varepsilon_{Fc}^{-1}$. For naturality, we suppose we have a morphism $f : c \to c'$, and we note that the following diagram commutes.

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[d, "Ff"'] & FGFc \arrow[r, "\varepsilon_{Fc}"] \arrow[d, "FGFf"] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "F\eta_{c'}"'] & FGFc' \arrow[r, "\varepsilon_{Fc'}"'] & Fc' \end{tikzcd}$$

  Indeed, the outer rectangle commutes by definition of the $\eta_\bullet$s, and the right square commutes by naturality of the $\varepsilon_\bullet$s. Then this forces the left square to commute by an argument by noting

$$\varepsilon_{Fc'} \circ F\eta_{c'} \circ Ff = \varepsilon_{Fc'} \circ FGFf \circ F\eta_c$$

  by the commutativity of the outer diagram, so we get the commutativity by inverting along $\varepsilon_{Fc'}$. ■

2.7 February 14

Here we go.

[2] Yes, I know.

2.7.1 Using Our Equivalence

Last time we proved the following theorem.

Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are true.

(a) If F is an equivalence, then F is fully faithful and essentially surjective.

(b) Assuming a strong form of the axiom of choice, the converse holds.

Let's use this for fun and profit.

Corollary 2.75 (Math 110)**.** The categories Mat_k and fdVec_k are equivalent.

Proof. Fix $\mathcal{C} := \mathrm{fdVec}_k^{\mathrm{basis}}$ to be the category consisting of objects which are ordered pairs $(V, \mathcal{B})$ of vector space equipped with a given ordered basis and morphisms which are linear transformations. I will call these based vector spaces because I can.

Observe that we have a functor $\mathcal{C} \to \mathrm{Mat}_k$ by sending the based vector space $(V, \mathcal{B})$ to $\dim V$ and the linear transformation $T : (V, \mathcal{B}) \to (V', \mathcal{B}')$ to the corresponding matrix representation. We run the following checks.

- The functor F is fully faithful because (based) linear transformations $(V, \mathcal{B}) \to (V', \mathcal{B}')$ are in bijective correspondence with matrices in $k^{\dim V' \times \dim V}$, which is exactly $\mathrm{Mor}_{\mathrm{Mat}_k}$
- This is essentially surjective because it is surjective: the vector space k^n goes to $n \in \mathrm{Mat}_k$.

Thus, F is an equivalence.

To continue, we use the forgetful functor $U : \mathcal{C} \to \mathrm{fdVec}_k$ by simply forgetting the basis. This is fully faithful because look at it, and it is essentially surjective because it is actually surjective. Thus, U witnesses $\mathcal{C} \simeq \mathrm{fdVec}_k$. Applying transitivity, we see

$$\mathrm{Mat}_k \simeq \mathcal{C} \simeq \mathrm{fdVec}_k,$$

which finishes. ■

We have the following definition.

Definition 2.76 (Essential image)**.** The *essential image* of a functor $F : \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ consisting of objects $d \in \mathcal{D}$ such that $d \cong Fc$ for some $c \in \mathcal{C}$.

We are saying "full subcategory" to just throw in all the morphisms, so we don't have to worry about potential composition problems in $\mathcal{D}$.

Corollary 2.77. A fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of $\mathcal{C}$ onto the essential image of F.

Proof. Apply Theorem 2.72, where being essentially surjective follows from the definition of the essential image. ■

2.7.2 Motivating Diagram Chasing

We're going to be talking about diagram-chasing for a little while. This is the technique by which we extract large amounts of information from a commutative diagram. Namely, we will get to formally define what a commutative diagram is and so on. For this, we will want to do a little graph theory.

Definition 2.78 (Path)**.** Fix a category $\mathcal{C}$. Then a *path* in $\mathcal{C}$ is finite sequence of the form

$$(A_1, f_1, A_2, f_2, \ldots, A_n, f_n, A_{n+1}),$$

where $A_1, \ldots, A_{n+1} \in \operatorname{Ob} \mathcal{C}$ and $f_k \in \operatorname{Mor}(A_k, A_{k+1})$ for each k.

Remark 2.79. Equivalently, we could encode this path by the sequence of morphism $f_1, \ldots, f_n$ such that $\operatorname{cod} f_k = \operatorname{dom} f_{k+1}$.

Let's see an example of the power of abstracting diagrams.

Definition 2.80 (Monoid)**.** A *monoid in the category* Set is a set M with morphisms $\mu : M \times M \to M$ and $\eta : \{*\} \to M$ such that the following diagrams commute.

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mu \times \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times \mu"'] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mathrm{id}_M \times \eta"] \arrow[dr, "\mathrm{id}_M"'] & M \times M \arrow[d, "\mu"] & M \arrow[l, "\eta \times \mathrm{id}_M"'] \arrow[dl, "\mathrm{id}_M"] \\ & M & \end{tikzcd}$$

Remark 2.81. Our monoid is made by the binary operation $\cdot_\mu : (a, b) \mapsto \mu(a, b)$ and an identity element $e := \eta(*)$. The left-hand diagram gives associativity in our "monoid" where μ is our binary operation: if $a, b, c \in M$, then we have

$$(a \cdot_\mu b) \cdot_\mu c = a \cdot_\mu (b \cdot_\mu c).$$

The right-hand diagram promises us an identity element $e := \eta(*)$: if $m \in M$, then

$$m \cdot_\mu e = m = e \cdot_\mu m.$$

Remark 2.82. It is not technically necessary for us to use sets M, but if we don't, then we need a good notion of product and one-element set. For example, Top can work instead of Set if we want to keep track of topologies.

Example 2.83. A unital ring R is a monoid in the category of Ab (where our products are tensor products and one-element set is $\mathbb{Z}$). Namely, we have morphisms $\mu : R \otimes R \to R$ and $\eta : \mathbb{Z} \to R$ with the following commutative diagrams.

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu \times \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \times \mu"'] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\mathrm{id}_R \times \eta"] \arrow[dr, "\mathrm{id}_R"'] & R \otimes R \arrow[d, "\mu"] & R \arrow[l, "\eta \times \mathrm{id}_R"'] \arrow[dl, "\mathrm{id}_R"] \\ & R & \end{tikzcd}$$

The left-hand diagram shows that multiplication is an associative bilinear map, and the right-hand diagram promises an identity. We will not be more explicit.

2.7.3 Commutative Diagrams

We should probably define a diagram now.

Definition 2.84 (Diagram)**.** Fix $\mathcal{J}$ and $\mathcal{C}$ categories. A *diagram* in $\mathcal{C}$ indexed by $\mathcal{J}$ is a functor $F : \mathcal{J} \to \mathcal{C}$.

Notably, we are not requiring this functor to be an embedding.

Example 2.85. A diagram of the shape $(0 \to 1)^2$ is a commutative square. To be explicit, our index category is as follows.

$$\begin{tikzcd} (0,0) \arrow[r, "\mathrm{id}\times f"] \arrow[d, "f\times\mathrm{id}"'] \arrow[dr, "f\times f" description] & (0,1) \arrow[d, "f\times\mathrm{id}"] \\ (1,0) \arrow[r, "\mathrm{id}\times f"'] & (1,1) \end{tikzcd}$$

Namely, if we send this to $\mathcal{C}$, we some diagram as follows.

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c' \arrow[d] \\ d \arrow[r] & d' \end{tikzcd}$$

Because we embedded by a functor, we know that $c \to c' \to d'$ is the same as $c \to d \to d'$.

Example 2.86. We can think about triangles as images of squares which collapse a bit, as follows.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, equal] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Alternatively, we could just set the index category to be $\bullet \to \bullet \to \bullet$.

Definition 2.87 (Commutes). A diagram $F : \mathcal{J} \to \mathcal{C}$ *commutes* if and only if, given $k, k' : i \to j$ in $\mathcal{J}$ has $Fk = Fk'$.

The point of this definition is that we don't want composition to matter too much in our index category. For example, if we have morphisms $0 \to 1$ and $1 \to 2$ in $\mathcal{J}$ which go to $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, we want to be sure we have $0 \to 2$ goes to fg without having to look too hard at $\mathcal{J}$.

Example 2.88. Any diagram over a preorder will commute for free because any two i, j has at most one element in $\operatorname{Mor}(i, j)$.

It's a math class, so we should probably prove something today.

Proposition 2.89. Functors preserve commutative diagrams.

Proof. Fix $\mathcal{J}, \mathcal{C}, \mathcal{D}$ all diagrams with a commutative diagram $K : \mathcal{J} \to \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$. Indeed, if $k, k' : i \to j$ in $\mathcal{J}$, then $Kk = Kk'$, so $JKk = JKk'$, so $JK : \mathcal{J} \to \mathcal{D}$ is indeed a commutative diagram. ■

And here is a nice result on commutative diagrams.

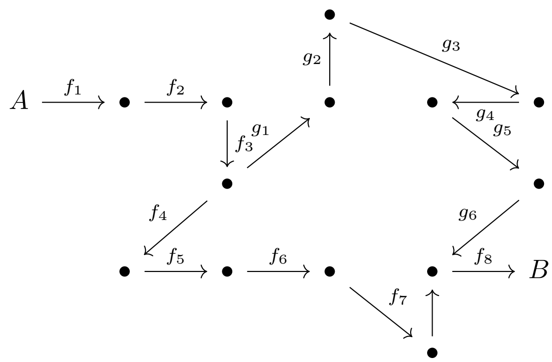
Lemma 2.90. Fix $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are paths in $\mathcal{C}$. Then if we have an equality of composites

$$f_k f_{k-1} \cdots f_{i+1} f_i = g_n g_{n-1} \cdots g_2 g_1,$$

then

$$f_m \cdots f_1 = f_m \cdots f_k g_n \cdots g_1 f_{i-1} \cdots f_1.$$

Here is the image for the above lemma: we are allowed to take either path from A to B, given that the f-parts and g-parts are commuting.



Proof. Look at it. Namely, we have composition is well-defined, so take the given equality and add the required compositions on either end. ■

2.8 February 16

Here we go.

2.8.1 House-Keeping

Let's start with the attendance question from last class because it was a little tricky.

Exercise 2.91. All nonempty indiscrete categories are equivalent.

Proof. The first part of this problem is remembering that indiscrete categories are ones that have all morphism sets are singletons. The second part of the problem is recognizing the following lemma.

Lemma 2.92. Fix $\mathcal{C}$ be a nonempty indiscrete category. Then $\mathcal{C}$ is equivalent to $\mathrm{B}e$, where e is the single-element group.

Proof. We use the functor $F : \mathcal{C} \to \mathrm{B}e$ sending all objects to $*$ and all morphisms to id_*. It is surjective on objects because there is only one object to hit, and $\mathcal{C}$ is nonempty. Further, F is fully faithful because, for any $c, c' \in \mathcal{C}$, the induced map

$$F : \mathrm{Mor}(c, c') \to \mathrm{Mor}(*, *)$$

is a bijection because both of these are singletons. It follows from Theorem 2.72 that F is an equivalence. ■

So transitivity promises that all indiscrete categories are equivalent, finishing the proof. ■

Remark 2.93. In fact, one can use essentially the same proof to show that any functor between indiscrete categories is an equivalence. In particular, the (weak) inverse to the equivalence generated by Lemma 2.92 is not canonical.

2.8.2 Diagram-Chasing Philosophy

We recall that we proved Lemma 2.90 last time, which philosophically means that we should not try to show equalities of morphisms where there is some overlap between the morphisms. For example, to compare all paths in the rectangle

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

above, we merely have to check the commutativity of the squares.

We would like to have some tools to prove that diagrams commute.

Remark 2.94. We remark that the following force commutative diagrams immediately.

- Any diagram indexed by a preorder commutes.
- Any diagram in a preorder commutes because any two morphisms between objects must be equal, so we get the commuting in the image of the index category.

2.8.3 Initial and Final Objects

Let's keep building up our theory.

Definition 2.95 (Initial, final)**.** Fix a category $\mathcal{C}$.

- An object $i \in \mathcal{C}$ is *initial* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(i, c)$.
- The dual notion is that an object $t \in \mathcal{C}$ is *final* or *terminal* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(c, t)$.

Remark 2.96. It is true that initial and final objects are unique up to unique isomorphism. We will not show this here because it might appear on the homework.

And here are many, many examples.

Example 2.97. We work in Set.

- We have $\varnothing$ is initial. Namely, there is only one function $\varnothing \to S$ for any set S by taking all elements of $\varnothing$ to whatever one's heart desires in S, and there is only one way to do this because any two such functions always have the same outputs.
- The singleton set $\{*\}$ is final. Indeed, any set S has a unique function $S \to \{*\}$ by sending all elements of S to $*$.

Example 2.98. In Top, the initial object is $\varnothing$ and the final object is $\{*\}$.

Example 2.99. We work in Set_*, which are ordered pairs (S, s) where $s \in S$. Morphisms $(S, s) \to (T, t)$ are functions $f : S \to T$ such that $f(s) = t$. Singleton sets $\{*\}$ is both initial and final. It's final for the same reason as in Set, and it is initial because any pointed set (S, s) has the unique morphism $* \mapsto s$.

Example 2.100. We work in Ab or Grp. Then the trivial group 0 is the initial and final object by sending identities to identities.

Non-Example 2.101. The object $\mathbb{Z}/2\mathbb{Z}$ is not initial in Ring: there is no morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. Funnily enough, there is at most one morphism from $\mathbb{Z}/2\mathbb{Z}$ to anywhere.

Non-Example 2.102. We work in Ring.

- The object $\mathbb{Z}$ is initial by sending $1 \mapsto 1_R$ (which is forced) for any ring R, and this uniquely determines the rest of the morphism.
- The zero ring 0 is final in Ring because there is only one function $R \to 0$ for any ring R, and it is in fact a ring homomorphism.

Example 2.103. The category Field has no initial or final object. There is no final object because all morphisms are injections, and we cannot embed all fields into one large field. There is no initial object because there are no morphisms between fields of different characteristic. (One can fix this problem by considering the fields of characteristic p, where $\mathbb{F}_p$ is the initial object.)

Quote 2.104. I hate this category, and you should too.

Example 2.105. Let $\mathcal{P}$ be a preorder category.

- We claim that global minimums are equivalent to initial objects. To be explicit, there is surely at most one morphism between any two elements, so the object $m \in \mathcal{P}$ is an initial object if and only if there is a morphism $m \to x$ for each $x \in \mathcal{P}$ if and only if $m \le x$ for each x if and only if m is a global minimum.
- Dually, global maximums are equivalent to final objects.

These new definitions give us a quick criterion for diagram-chasing.

Lemma 2.106. Fix $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ be "parallel" paths in $\mathcal{C}$; i.e., $s := \operatorname{dom} f_1 = \dim g_1$ and $t := \operatorname{cod} f_n = \operatorname{cod} g_m$. If s is initial or t is final, then

$$f_n \cdots f_1 = g_m \cdots g_1.$$

Proof. We have two cases.

- Take s initial. Then $f_n \cdots f_1$ and $g_m \cdots g_1$ are both maps $s \to t$, of which there is a unique map by s being initial, so these are equal.
- Take t final. Then repeat the above sentence using the fact t is final instead of s being initial. ∎

2.8.4 Concrete Categories

We have the following definition.

Definition 2.107 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if there is a fully faithful functor $U : \mathcal{C} \to \mathrm{Set}$. We call U the *forgetful functor*.

For example, this asserts that two morphisms $f, g : x \to y$ in $\mathcal{C}$ are equal if and only if their "restrictions" down in Set are equal, for which we can do an element-wise check on elements of sets.

Lemma 2.108. Fix $U : \mathcal{C} \to \mathcal{D}$ be faithful functors. A diagram in $\mathcal{C}$ commutes if and only if its image through U commutes.

Proof. Fix J our index category with the diagram $K : J \to \mathcal{C}$. We already know that K commuting implies that UK commutes by Proposition 2.89.

In the other direction, suppose that UK commutes. Then pick up $k, k' : i \to j$ in J so that $UKk = UKk'$, but then U being faithful forces

$$Kk = Kk',$$

which is exactly what we need to commute. ■

And here is why we care

Corollary 2.109. Commutativity of a diagram in a concrete category can be checked on "elements."

Proof. Essentially we use the forgetful functor in Lemma 2.108. To be explicit, checking on "elements" is doing the diagram-chase in Set, which we can then pull back to the original concrete category through the forgetful functor via Lemma 2.108. ■

In other words, we can diagram-chase by working everything in set.

2.8.5 Commutative Rectangles

We have the following warning.

Warning 2.110. Consider the following rectangle.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

We know that the squares commuting implies that the rectangle commutes. The converse is not true.

Example 2.111. We work in Ab. The outer rectangle of the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z}
\end{tikzcd}$$

will commute, but the inner squares do not. (The zero map is not the identity map.)

We can salvage Warning 2.110 as follows.

Lemma 2.112. Fix a rectangle as follows.

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[d, "f"'] & b \arrow[r, "k"] \arrow[d, "g"] & c \arrow[d, "h"] \\ a' \arrow[r, "j"'] & b' \arrow[r, "m"'] & c' \end{tikzcd}$$

Suppose the outer rectangle commutes. Then the diagram commutes if

- the right square commutes and m is monic, or
- the left square commutes and e is epic.

Proof. We have separate cases.

- Suppose the right square commutes and m is monic. The right square commutes, so $hk = mg$. Similarly, the outer rectangle commutes, so $hke = mjf$. But then

$$mge = hke = mjf,$$

  so $ge = jf$ because m is monic. This shows the left square commutes, so we are done.

- This holds by running the proof of the above in the opposite category, where the main point is that the left and right squares flip, and m being monic turns into e being epic. ■

2.9 February 18

Apparently I have to take notes today.

2.9.1 Motivating Horizontal Composition

A while ago we discussed vertical composition of natural transformations: if $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, then we can define a natural transformation $(\beta\alpha) : F \Rightarrow H$ by $(\beta\alpha)_c := \beta_c\alpha_c$. To quickly review, the naturality condition can be checked by drawing the following commutative diagram.

$$\begin{tikzcd} Fc \arrow[r, "Ff"] \arrow[d, "\alpha_c"] \arrow[dd, bend right=60, "\beta_c\alpha_c"'] & Fd \arrow[d, "\alpha_d"'] \arrow[dd, bend left=60, "\beta_d\alpha_d"] \\ Gc \arrow[r, "Gf"] \arrow[d, "\beta_c"] & Gd \arrow[d, "\beta_d"'] \\ Hc \arrow[r, "Hf"'] & Hd \end{tikzcd}$$

We are going to discuss horizontal composition because Eckmann–Hamilton would like to know your location. The set-up is as follows: suppose that we have functors $F, G : \mathcal{C} \to \mathcal{D}$ with $\alpha : F \Rightarrow G$ and $F', G' : \mathcal{D} \to \mathcal{E}$ with $\beta : F' \Rightarrow G'$. Here is the diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

Our goal is to define $(\beta * \alpha) : F'F \Rightarrow G'G$.

2.9.2 Whiskering

To define this horizontal composition, we define "whiskering." There are two kinds of whiskering.

- Here is the diagram for left whiskering.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathcal{E} \end{tikzcd}$$

  We would like to define $\alpha H : FH \Rightarrow GH$. Well, we simply define $(\alpha H)_c := \alpha_{Hc}$, which defines a natural transformation by noting the following diagram commutes for a morphism $f : c \to d$ in $\mathcal{C}$ by the naturality of α on $Hf : Hc \to Hd$. This gives the following commutative naturality square.

$$\begin{tikzcd} FHc \arrow[r, "FHf"] \arrow[d, "\alpha_{Hc}"'] & FHd \arrow[d, "\alpha_{Hd}"] \\ GHc \arrow[r, "GHf"'] & GHd \end{tikzcd}$$

- There is also a notion of right whiskering. Here is the diagram.

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{E} \arrow[r, "H'"] & \mathcal{X} \end{tikzcd}$$

  We define $H'\alpha : H'F \Rightarrow H'G$ by $(H'\alpha)_d := H'\alpha_d$. This is a natural transformation because we can pick up some morphism $f : c \to d$ in $\mathcal{D}$ and apply H' to the naturality diagram for α, giving the following commutative naturality square.

$$\begin{tikzcd} H'Fc \arrow[r, "H'Ff"] \arrow[d, "H'\alpha_c"'] & H'Fd \arrow[d, "H'\alpha_d"] \\ H'Gc \arrow[r, "H'Gf"'] & H'Gd \end{tikzcd}$$

2.9.3 Horizontal Composition

From whiskering, there are two ways to define horizontal composition. To review, here is our diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

- We start by whiskering on the left and then whisker on the right. So we start by noting we have $\beta F : F'F \Rightarrow G'F$ induced by whiskering the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

  Then we have $G'\alpha : G'F \Rightarrow G'G$ by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend right, "G'"'] & \mathcal{E} \end{tikzcd}$$

  In total, we see that $(G'\alpha)(\beta F) : F'F \Rightarrow G'G$. Note this is a natural transformation by vertical composition!

- We start by whiskering on the right and then whisker on the left. So we start by noting we have $F'\alpha : F'F \Rightarrow F'G$ by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] & \mathcal{E} \end{tikzcd}$$

Then we have $\beta G : F'G \Rightarrow G'G$ induced by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "G"'] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

In total, we see that $(\beta G)(F'\alpha) : F'F \Rightarrow G'G$, which is a natural transformation by vertical composition.

We now claim that the two horizontal compositions that we just defined are the same. We could just track an element through, or we could simply note that this is the naturality of β applied to the morphism $\alpha_c : Fc \to Gc$. Indeed, we are showing that the following diagram commutes.

$$\begin{tikzcd} F'F \arrow[r, Rightarrow, "F'\alpha"] \arrow[d, Rightarrow, "\beta F"'] & F'G \arrow[d, Rightarrow, "\beta G"] \\ G'F \arrow[r, Rightarrow, "G'\alpha"'] & G'G \end{tikzcd}$$

Now, applying naturality of β to $\alpha_c : Fc \to Gc$, we see that the following diagram commutes.

$$\begin{tikzcd} F'Fc \arrow[r, "F'\alpha_c"] \arrow[d, "\beta_{Fc}"'] & F'Gc \arrow[d, "\beta_{Gc}"] \\ G'Fc \arrow[r, "G'\alpha_c"'] & G'Gc \end{tikzcd}$$

But this diagram is exactly what we wanted, so we are done.

2.9.4 Horizontal and Vertical Composition

For our last note, we show that horizontal composition of vertical compositions is the same as vertical composition of horizontal compositions. Here is our diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \arrow[r, bend left=60, "F'"] \arrow[r, "G'"] \arrow[r, bend right=60, "H'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{E} \end{tikzcd}$$

We claim that

$$(\beta'\alpha') * (\beta\alpha) \stackrel{?}{=} (\beta' * \beta)(\alpha'\alpha).$$

The point is to draw the following giant commuting square. The "morphisms" are induced by various kinds of whiskering in the diagram, and they all commute by uniqueness of horizontal composition.

$$\begin{tikzcd} F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\ G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\ H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H \end{tikzcd}$$

We now follow two paths. Consider the red path below.

$$\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\
G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\
H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H
\end{tikzcd}$$

By definition of horizontal composition, this is $(\beta' * \beta)(\alpha' * \alpha)$. Now consider the different red path below.

$$\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\
G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\
H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H
\end{tikzcd}$$

The top leg is $\beta\alpha$, and the right leg is $\beta'\alpha'$, so this total red path comes out to $(\beta'\alpha')(\beta\alpha)$.

So comparing our two red paths, we see that

$$(\beta'\alpha') * (\beta\alpha) = (\beta' * \beta)(\alpha' * \alpha),$$

which is what we wanted.

Theme 3
Universal Properties

The Yoneda embedding, contravariant it is.

—Mike Stay

3.1 February 23

Today we begin talking about universal properties and associated fun.

Convention 3.1. For today, all of our categories will be locally small. We will not care more about size issues.

3.1.1 A Functorial Initial and Final

We recall the following definition.

Definition 3.2 (Initial, final)**.** Fix a category $\mathcal{C}$.

- An object $i \in \mathcal{C}$ is *initial* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(i, c)$.
- The dual notion is that an object $t \in \mathcal{C}$ is *final* or *terminal* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(c, t)$.

The moral of our story is that being initial and terminal will encode our universal properties.

Here is a nice starting proposition and corollary.

Proposition 3.3. An object $c \in \mathcal{C}$ is initial if and only if $\# \operatorname{Mor}(c, x) = 1$ for each $x \in \mathcal{C}$. Similarly, c is terminal if and only if $\# \operatorname{Mor}(x, c) = 1$ for each $x \in \mathcal{C}$.

Proof. This is a restatement of the definition. For example, $\# \operatorname{Mor}(c, x) = 1$ is asserting there is a unique morphism from c to x for any object x. ■

Corollary 3.4. An object $c \in \mathcal{C}$ is initial if and only if the functor $\mathrm{Mor}(c,-) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ "represented" by c is naturally isomorphic to the (contravariant!) constant functor $\{*\} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ sending everyone in $\mathcal{C}$ to $\{*\}$.

To be explicit, the functor $\{*\} : \mathcal{C} \to \mathrm{Set}$ sends objects $c \in \mathcal{C}$ to $c \mapsto \{*\}$ and sends morphisms $f : c \to d$ to $f \mapsto \mathrm{id}_{\{*\}}$.

Proof. As before, we see that c is initial if and only if $\#\,\mathrm{Mor}(c,x) = 1$ for each x if and only if

$$\mathrm{Mor}(c,x) \cong \{*\}$$

because all singletons form an isomorphism class in Set. We label φ_x to be $\mathrm{Mor}(c,x) \cong \{*\} \cong \{*\}(x)$ to be the unique such isomorphism.

If φ_x assemble to a natural isomorphism, then we get the reverse direction. For the forwards direction, we have to check that the following diagram commutes for naturality: suppose $f : x \to y$ is a morphism in $\mathcal{C}$, and we want

$$\begin{tikzcd} \mathrm{Mor}(c,x) \arrow[r, "\varphi_x"] \arrow[d, "f\circ -"'] & \{*\}(x) \arrow[d, "\mathrm{id}_{\{*\}}"] \\ \mathrm{Mor}(c,y) \arrow[r, "\varphi_y"] & \{*\}(y) \end{tikzcd}$$

to commute. But this commutes for free because $\{*\}(y) = \{*\}$ is a terminal object, so all morphisms to it are the same. ■

Corollary 3.5. An object $c \in \mathcal{C}$ is terminal if and only if the functor $\mathrm{Mor}(-,c) : \mathcal{C} \to \mathrm{Set}$ "represented" by c is naturally isomorphic to the constant functor $\{*\} : \mathcal{C} \to \mathrm{Set}$ sending everyone in $\mathcal{C}$ to $\{*\}$.

Proof. This is dual to the previous corollary. ■

3.1.2 Representability

Here is our central definition.

Definition 3.6 (Representable)**.** Fix a category $\mathcal{C}$.

- A covariant functor $F : \mathcal{C} \to \mathrm{Set}$ is *representable* if and only if there exists some $c \in \mathcal{C}$ such that $F \simeq \mathrm{Mor}(c,-)$.
- A contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is *representable* if and only if there exists some $c \in \mathcal{C}$ such that $F \simeq \mathrm{Mor}(-,c)$.

In either case, we call c together with the promised natural isomorphism the *representation* of F.

Example 3.7. Corollary 3.4 says that c is initial if and only if $\{*\} : \mathcal{C} \to \mathrm{Set}$ is represented by c. Similar holds for the terminal case.

Here is our mantra.

Idea 3.8. A representable functor encodes a universal property of an object.

Remark 3.9. Bryce would like you to repeat Idea 3.8 every day before you go to sleep. He will know if you haven't.

Less formally, Idea 3.8 is saying that a universal property for an object c is a description of $\mathrm{Mor}(c, -)$ or $\mathrm{Mor}(-, c)$.

Let's see some examples.

Exercise 3.10. The identity functor $\mathrm{id}_{\mathrm{Set}} : \mathrm{Set} \to \mathrm{Set}$, is represented by singleton set $\{*\}$.

Proof. To be explicit we would like to show that

$$\mathrm{Mor}(\{*\}, X) \cong X$$

naturally by taking $f \mapsto f(*)$. So we define $\eta_X : \mathrm{Mor}(\{*\}, X) \to X$ by $f \mapsto f(*)$. This is an isomorphism because we have the inverse morphism $\eta_X^{-1} : x \mapsto (* \mapsto x)$. This is natural because, with a morphism $h : X \to Y$, we draw the following diagram.

$$\begin{tikzcd} \mathrm{Mor}(\{*\}, X) \arrow[r, "\eta_X"] \arrow[d, "h \circ -"'] & X \arrow[d, "h"] \\ \mathrm{Mor}(\{*\}, Y) \arrow[r, "\eta_Y"'] & Y \end{tikzcd}$$

This is natural by tracking some $f : \{*\} \to X$ through: along the top, it goes to $h(f(*))$, and along the bottom it goes to $(hf)(*) = h(f(*))$. ■

Exercise 3.11. The forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ is represented by $\mathbb{Z}$.

Proof. The content is to construct an isomorphism

$$\mathrm{Mor}(\mathbb{Z}, G) \cong G$$

for any group G. Well, to see this, we send $f \mapsto f(1)$ and more or less wave our hands to say that a group homomorphism $\mathbb{Z} \to G$ is uniquely determined by where it sends 1 because $f(n) = n \cdot f(1)$, and any such $f(1)$ is legal because we can set $f(n) = n \cdot f(1)$.

So let $\eta_G : \mathrm{Mor}(\mathbb{Z}, G) \to G$ be this isomorphism. For naturality, we need to show that the following diagram commutes for a given group homomorphism $\psi : G \to H$.

$$\begin{tikzcd} \mathrm{Mor}(\mathbb{Z}, G) \arrow[r, "\eta_X"] \arrow[d, "\psi \circ -"'] & G \arrow[d, "U\psi"] \\ \mathrm{Mor}(\mathbb{Z}, H) \arrow[r, "\eta_Y"'] & H \end{tikzcd}$$

Well, along the top, we send f to $f(1)$ to $\psi(f(1))$. Along the bottom, we send f to ψf to $(\psi f)(1)$ to $\psi(f(1))$. ■

Exercise 3.12. The forgetful functor $U : \mathrm{Ring} \to \mathrm{Set}$ is represented by $\mathbb{Z}[x]$.

Proof. The point is that we have an isomorphism

$$\mathrm{Mor}(\mathbb{Z}[x], R) \cong R$$

because the image of $\mathbb{Z}$ is fixed for any morphism $\varphi : \mathbb{Z}[x] \to R$, and where we send x is uniquely determined by a chosen element $r \in R$. ■

Remark 3.13. In some sense, $\mathbb{Z}$ and $\mathbb{Z}[x]$ are the "free" object in their respective categories.

And now for contravariant representable functors.

Exercise 3.14. The functor $\mathcal{P} : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ by sending $X \mapsto \mathcal{P}(X)$ and $f : X \to Y$ to $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is represented by $\Omega = \{0, 1\}$.

Proof. As usual, the content of the proof is our isomorphism

$$\mathrm{Mor}(X, \Omega) \cong \mathcal{P}(X).$$

Namely, we send $f : X \to \Omega$ to $f^{-1}(1)$. Conversely, given a subset $U \subseteq X$, we can track it by the morphism $1_{x \in U} : X \to \Omega$. ∎

Exercise 3.15. Fix sets A and B. Consider the functor $\mathrm{Mor}(- \times A, B) : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$. We claim that this is represented by $\mathrm{Mor}(A, B)$.

Proof. Our isomorphism

$$\mathrm{Mor}(\mathrm{Mor}(A, B), C) \cong \mathrm{Mor}(C \times A, B)$$

is given by currying $f \mapsto ((a, b) \mapsto f(a)(b))$. The inverse mapping is $f \mapsto (a \mapsto b \mapsto f(a, b))$. ∎

Remark 3.16. Many of the above representatives are "nice" in that it seems like they are unique in some sense. This will tie into universal properties.

3.2 February 25

We talk about the Yoneda lemma today.

3.2.1 The Yoneda Lemma

Today we discuss the following question.

Question 3.17. What information "goes into" a natural transformation of a representable functor?

Today we will prove the following theorem.

Theorem 3.18 (Yoneda lemma). Fix $\mathcal{C}$ a locally small category and $F : \mathcal{C} \to \mathrm{Set}$ a functor. Further, fix $c \in \mathcal{C}$. Then there is a "natural" bijection (natural in both c and F)

$$\varphi : \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \cong Fc.$$

Here the outer Mor is in the 2-category, talking about natural transformations $\mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$.

Natural here is in both c and F: if we fix one of them, then the isomorphism is functorial in the other.

Proof. We take this in parts.

- We construct φ. Suppose that $\alpha : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ is a natural transformation. Then we can produce an element of Fc by noting we have a map $\alpha_c : \mathrm{Mor}_{\mathcal{C}}(c, c) \to Fc$, so we can set

$$\boxed{\varphi(\alpha) := \alpha_c(\mathrm{id}_c)} \in Fc.$$

- We construct an inverse $\psi : Fc \to \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c, -), F)$. Well, picking up some $x \in Fc$, then we want a natural transformation $\psi(x) : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$. So given another $d \in \mathcal{C}$, we want a morphism

$$\psi(x)_d : \operatorname{Mor}_{\mathcal{C}}(c, d) \to Fd.$$

To do this, we pick up a morphism $f : c \to d$, and we want an element of Fd. Without any better ideas, we note we have a morphism $Ff : Fc \to Fd$, so we define

$$\boxed{\psi(x)_d(f) := (Ff)(x)}.$$

We now check that $\psi(x)$ is in fact a natural transformation. Well, suppose that we have a map $g : d \to e$, we need the following square to commute.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}}(c, d) \arrow[r, "\psi(x)_d"] \arrow[d, "g\circ-"'] & Fd \arrow[d, "Fg"] \\ \operatorname{Mor}_{\mathcal{C}}(c, e) \arrow[r, "\psi(x)_e"'] & Fe \end{tikzcd}$$

For this, we pick up some morphism $f : c \to d$.

  - Along the top, we go to $(Ff)(x)$ and then $(Fg \circ Ff)(x)$.
  - Along the bottom, we go to gf and then $(F(gf))(x) = (Fg \circ Ff)(x)$.

- We show that $\varphi \circ \psi$ is the identity. Well, pick up some $x \in Fc$. Then

$$\varphi(\psi(x)) = \psi(x)_c(\mathrm{id}_c) = (F\,\mathrm{id}_c)(x) = \mathrm{id}_{Fc}(x) = x.$$

- We show that $\psi \circ \varphi$ is the identity. Well, pick up some natural transformation $\alpha : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ and some object $d \in \mathcal{C}$ and some morphism $f : c \to d$, and we compute

$$\psi(\varphi(\alpha))_d(f) = \psi(\alpha_c(\mathrm{id}_c))_d(f) = (Ff)(\alpha_c(\mathrm{id}_c)) = (Ff \circ \alpha_c)(\mathrm{id}_c).$$

At this point we look stuck, but naturality of $\alpha : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ saves us! We draw the following diagram.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}}(c, c) \arrow[r, "\alpha_c"] \arrow[d, "f\circ-"'] & Fc \arrow[d, "Ff"] \\ \operatorname{Mor}_{\mathcal{C}}(c, d) \arrow[r, "\alpha_d"'] & Fd \end{tikzcd}$$

Thus, we know $Ff \circ \alpha_c = \alpha_d \circ (- \circ f)$, so the above is

$$\psi(\varphi(\alpha))_d(f) = (\alpha_d \circ (f \circ -))(\mathrm{id}_c) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f).$$

So $\psi(\varphi(\alpha))_d$ and α_d match as functions on $\operatorname{Mor}_{\mathcal{C}}(c, d)$, so they are equal. Thus, $\psi(\varphi(\alpha)) = \alpha$ as natural transformations. So we are done.

The above points establish the needed bijection.

It remains to check functoriality.

- We show that φ is functorial in c. We write φ^c for φ given by $c \in \mathcal{C}$. Suppose that we have a morphism $f : c \to c'$, and we want to show that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c, -), F) \arrow[r, "\varphi^c"] \arrow[d] & Fc \arrow[d] \\ \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c', -), F) \arrow[r, "\varphi^{c'}"'] & Fc' \end{tikzcd} \qquad (*)$$

The right arrow is Ff. The left arrow requires some thinking: we pick up some natural transformation $\alpha : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ and want to produce a natural transformation $\operatorname{Mor}_{\mathcal{C}}(c', -) \Rightarrow F$. Visually, the map we want is moving

$$\big((\mathcal{C} \to \mathrm{Set}) \to \mathcal{D}\big) \to \big((\mathcal{C} \to \mathrm{Set}) \to \mathcal{D}\big).$$

Well, given an object $d \in \mathcal{C}$ and morphism $p : c \to d$, we can send $p' : c' \to d$ to

$$\beta_d(p') := \alpha_d(p'f),$$

which we can type-check actually lives in Fd.

We take a moment to verify that β is a natural transformation. For this, we need to check the naturality of the following square, for a morphism $g : d \to e$, that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}}(c', d) \arrow[r, "\beta_d"] \arrow[d, "g\circ-"'] & Fd \arrow[d, "Fg"] \\ \operatorname{Mor}_{\mathcal{C}}(c', e) \arrow[r, "\beta_e"'] & Fe \end{tikzcd}$$

Well, we pick up a morphism $p' : c' \to d$.

- Along the top, we go to $(Fg)(\beta_d(p')) = (Fg)(\alpha_d(p'f)) = (Fg \circ \alpha_d)(p'f)$. By naturality of α, we see $Fg \circ \alpha_d = \alpha_e \circ (g \circ -)$, so we have $\alpha_e(gp'f)$.
- Along the bottom, we go to $\beta_e(gp') = \alpha_e(gp'f)$.

So indeed, β is a natural transformation.

Finally, we check the naturality of $(*)$.

- Along the top, we go to $\varphi^c(\alpha) = \alpha_c(\mathrm{id}_c)$ and then to $(Ff)(\alpha_c(\mathrm{id}_c)) = (Ff \circ \alpha_c)(\mathrm{id}_c)$. By naturality of α, we see $Ff \circ \alpha_c = \alpha_{c'}(f \circ -)$, so we have $(Ff \circ \alpha_c)(\mathrm{id}_c) = \alpha_{c'}(f\,\mathrm{id}_c) = \alpha_{c'}(f)$.
- Along the bottom, we go to

$$\varphi^{c'}(\beta) = \beta_{c'}(\mathrm{id}_{c'}) = \alpha_{c'}(\mathrm{id}_{c'}\, f) = \alpha_{c'}(f).$$

These match, so the diagram commutes.

- We show φ is functorial in F. We write φ^F for φ given by $F : \mathcal{C} \to \mathrm{Set}$. Now, suppose that we have some natural transformation $\eta : F \Rightarrow G$, and we want to show that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c, -), F) \arrow[r, "\varphi^F"] \arrow[d] & Fc \arrow[d] \\ \operatorname{Mor}(\operatorname{Mor}_{\mathcal{C}}(c, -), G) \arrow[r, "\varphi^G"'] & Gc \end{tikzcd}$$

The right arrow is η_c. The left arrow requires some thinking, as before. Fix some natural transformation $\alpha : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$, and we produce a natural transformation $\beta : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow G$. Well, given an object d and morphism $p : c \to d$, we are given an element $\alpha_d(f) \in Fd$, and we want an element in Gd. So we define

$$\beta_d(p) := \eta_d(\alpha_d(p)).$$

We quickly check that this $\beta : \operatorname{Mor}_{\mathcal{C}}(c, -) \Rightarrow G$ actually assembles into a natural transformation. Given $f : d \to e$, we need to check the commutativity of the following diagram.

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{C}}(c, d) \arrow[r, "\beta_d"] \arrow[d, "f\circ-"'] & Gd \arrow[d, "Gf"] \\ \operatorname{Mor}_{\mathcal{C}}(c, e) \arrow[r, "\beta_e"'] & Ge \end{tikzcd} \qquad (**)$$

Well, pick up a morphism $p : c \to d$.

- Along the top, we go to $(Gf)(\beta_d(p)) = (Gf)(\eta_d(\alpha_d(p)))$. By naturality of η, this is $(\eta_e \circ Fe \circ \alpha_d)(p)$. By naturality of α, this is $(\eta_e \circ \alpha_e \circ (f \circ -))(p) = \eta_e(\alpha_e(fp))$.
- Along the bottom, we go to $\beta_e(fp) = \eta_e(\alpha_e(fp))$.

So indeed, β is a natural transformation.

Finally, we check the naturality of $(**)$.

- Along the top, we go to $\varphi^F(\alpha) = \alpha_c(\mathrm{id}_c)$ and then to $\eta_c(\alpha_c(\mathrm{id}_c))$.
- Along the bottom, we go to $\varphi^G(\beta) = \beta_c(\mathrm{id}_c) = \eta_c(\alpha_c(\mathrm{id}_c))$.

These match, so the diagram commute.

Thus, we have checked that φ is functorial in both c and F. I have a headache, so we will call it quits there. ■

3.3 February 28

Chris is back!

3.3.1 Yoneda Lemma Review

Today is more Yoneda lemma. We recall the statement.

Theorem 3.18 (Yoneda lemma)**.** Fix $\mathcal{C}$ a locally small category and $F : \mathcal{C} \to \mathrm{Set}$ a functor. Further, fix $c \in \mathcal{C}$. Then there is a "natural" bijection (natural in both c and F)

$$\varphi : \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \cong Fc.$$

Here the outer Mor is in the 2-category, talking about natural transformations $\mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$.

As seen in the proof, the bijection is by

$$\varphi(\eta) := \eta_c(\mathrm{id}_c) \qquad \text{and} \qquad \varphi^{-1}(x)_d := \big((f \in \mathrm{Mor}(c, d)) \mapsto (Ff)(x)\big).$$

We quickly remark that we can motivate the definition φ^{-1} by drawing the following naturality square with given internal diagram; here $f : c \to d$ is some morphism.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(c,c) \arrow[r, "f\circ -"] \arrow[d, "\varphi^{-1}(x)_c"'] & \mathrm{Mor}_{\mathcal{C}}(c,d) \arrow[d, "\varphi^{-1}(x)_d"] \\
Fc \arrow[r, "Ff"'] & Fd
\end{tikzcd}$$

Because we want $\varphi^{-1}(x)_c(\mathrm{id}_c) = x$, our definition of $\varphi^{-1}(x)_d(f)$ is forced.

As for naturality, we note that we can view Fc as the image of the functor $F : \mathcal{C} \to \mathrm{Set}$ on applying $c \in \mathcal{C}$, or alternatively we could view Fc as the image of the functor $\mathrm{ev}_c : \mathrm{Set}^{\mathcal{C}} \to \mathrm{Set}$ on applying F.

Remark 3.19 (Contravariant Yoneda)**.** There is also a contravariant version of the Yoneda lemma as well, which provides takes a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a functor and some object $c \in \mathcal{C}$. Then there is a "natural" bijection (natural in both c and F)

$$\varphi : \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(-, c), F) \cong Fc.$$

Again, this bijection is natural in F and c.

3.3.2 Yoneda Embeddings

We are going to describe an embedding

$$よ : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

This is defined in essentially exactly the way that we want. We define $よ(c) := \mathrm{Mor}(-, c)$ and send morphisms $f : c \to d$ to the natural transformation $よ(f) : \mathrm{Mor}(-, c) \to \mathrm{Mor}(-, d)$ by $よ(f) : g \mapsto fg$. It is not too hard to see that $よ(f)$ is in fact a natural transformation and composes properly, so we will omit those checks now.

Anyways, here is our theorem.

Theorem 3.20 (Yoneda embedding)**.** Fix $\mathcal{C}$ a category. The Yoneda embedding $よ : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a fully faithful embedding.

Proof. We have the following checks.

- We show that $よ$ is faithful, so suppose we have objects $c, d \in \mathcal{C}$, and we want to show that the map

  $$よ : \mathrm{Mor}_{\mathcal{C}}(c, d) \to \mathrm{Mor}(よ(c), よ(d))$$

  is injective. Namely, if $f, g : c \to d$, then $よ(f) : x \mapsto fx$ and $よ(g) : x \mapsto gx$, so $よ(f) = よ(g)$ forces

  $$f = f\,\mathrm{id}_c = よ(f)(\mathrm{id}_c) = よ(g)(\mathrm{id}_c) = g\,\mathrm{id}_c = g.$$

- We show that $よ$ is full. We will actually use Theorem 3.18. Well, suppose that we have a morphism

  $$\eta : \mathrm{Mor}_{\mathcal{C}}(-, c) \Rightarrow \mathrm{Mor}_{\mathcal{C}}(-, d),$$

  and we want a morphism $f : c \to d$ such that $よ(f) = \eta$. Well, viewing $\mathrm{Mor}(-, d)$ as just some functor $F : \mathcal{C} \to \mathrm{Set}$, we are promised a bijection (by the contravariant version of Theorem 3.18)

  $$\mathrm{Mor}\left(\mathrm{Mor}_{\mathcal{C}}(-, c), \mathrm{Mor}_{\mathcal{C}}(-, d)\right) \to \mathrm{Mor}(c, d).$$

  In particular, η under this bijection goes to some map $f = \eta_c(\mathrm{id}_c) \in \mathrm{Mor}(c, d)$. But in fact $よ(f) = (f \circ -)$ also has $(f \circ -)(\mathrm{id}_c) = f$, so because the above is a bijection, we have $\eta = よ(f)$.

- We show that $よ$ is an embedding. For this, we suppose $c \neq d$ are distinct objects, and we need to show that $\mathrm{Mor}_{\mathcal{C}}(-, c) \neq \mathrm{Mor}_{\mathcal{C}}(-, d)$ as natural transformations. However, morphisms should "remember" their codomain in the data of a morphism, so $\mathrm{Mor}_{\mathcal{C}}(x, c)$ and $\mathrm{Mor}_{\mathcal{C}}(x, d)$ will be distinct automatically.

■

It might feel like cheating that we are forcing our morphisms to remember their codomain, but it is somewhat necessary: in the indiscrete category, we might accidentally try to force our morphisms to be witnessed by the same object, but then the above is not actually an embedding because all morphism sets would be equal.

Just for fun, here is an application.

Corollary 3.21 (Cayley's theorem)**.** Any group G is isomorphic to a subgroup of $\mathrm{Sym}(G) = \mathrm{Aut}_{\mathrm{Set}}(G)$.

Proof. To convert this to category theory, we use the category $\mathrm{B}G$. Well, Theorem 3.20 provides an embedding

$$よ : \mathrm{B}G \to \mathrm{Set}^{\mathrm{B}G^{\mathrm{op}}}.$$

Here, we can think of $\mathrm{Set}^{\mathrm{B}G^{\mathrm{op}}}$ as sets equipped with a right G-action: any such functor $\mathrm{B}G^{\mathrm{op}} \to \mathrm{Set}$ sends the object $* \in \mathrm{B}G$ to a set $S \in Set$ as well as morphisms/elements $g \in G$ to functions $g : S \to S$ satisfying

$$s \cdot (gh) = (s \cdot g) \cdot h$$

for $s \in S$ and $g, h \in G$.

Now, to use よ, we see that $* \in \mathrm{B}G$ goes to the functor よ$(*) = \mathrm{Mor}(-, *) : \mathrm{B}G^{\mathrm{op}} \to \mathrm{Set}$. Fixing any set $\bullet \in \mathrm{B}G$, we see that we really have the data of a morphism $\mathrm{B}G^{\mathrm{op}} \to \mathrm{Mor}(\bullet, *)$. In other words, we are giving $\mathrm{Mor}(\bullet, *)$ a right G-action: each $g \in \mathrm{B}G$ and $x \in \mathrm{Mor}(\bullet, *)$ will have a multiplication given by

$$x \cdot g := g^{\mathrm{op}} x,$$

where we have composition by

$$(x \cdot g) \cdot h = h^{\mathrm{op}} g^{\mathrm{op}} x = (gh)^{\mathrm{op}} x = x \cdot (gh).$$

So indeed, よ$(*)$ is precisely the data of this right G-action on $\mathrm{Mor}(\bullet, *)$. In other words, よ$(*)$ is providing the data of the object $\mathrm{Mor}(\bullet, *)$ in the category of G^{op}-sets.

On the other hand, each morphism $g : * \to *$ of $\mathrm{B}G$ will go to よ$(g) : \mathrm{Mor}(\bullet, *) \Rightarrow \mathrm{Mor}(\bullet, *)$ by よ$(g) : x \mapsto gx$. To be explicit, our multiplication is by

$$\text{よ}(g)(x) := gx.$$

In fact, each element よ(g) is a G^{op}-equivariant map on $\mathrm{Mor}(\bullet, *)$, where this object is thought of as a G^{op}-set. Indeed,

$$\text{よ}(g)(x \cdot h) = g(xh) = (gx)h = (\text{よ}(g)x) \cdot h,$$

which is what we wanted. In particular, よ injects $\mathrm{Mor}_{\mathrm{B}G}(*, *)$ to "G-equivariant" natural transformations $\mathrm{Mor}_{\mathrm{B}G}(-, *) \Rightarrow \mathrm{Mor}_{\mathrm{B}G}(-, *)$.

So to finish up, because よ is a fully faithful functor, we see that we are injective on morphism sets, so we can say

$$G = \mathrm{Mor}_{\mathrm{B}G}(*, *) \overset{\text{よ}}{\hookrightarrow} \mathrm{Mor}_{G^{\mathrm{op}}\text{-set}}\left(\mathrm{Mor}_{\mathrm{B}G}(-, *), \mathrm{Mor}_{\mathrm{B}G}(-, *)\right) \overset{*}{=} \mathrm{Aut}_{G^{\mathrm{op}}\text{-set}}\left(\mathrm{Mor}_{\mathrm{B}G}(-, *), \mathrm{Mor}_{\mathrm{B}G}(-, *)\right),$$

where $\overset{*}{=}$ holds because all elements of G are invertible and hence all the morphisms we are looking at are invertible. Now, we remark that the data of $\mathrm{Mor}(-, *)$ is really only the data of $\mathrm{Mor}(*, *)$ because $\mathrm{B}G$ has only one object, so we actually get to embed

$$G \hookrightarrow \mathrm{Aut}_{G^{\mathrm{op}}\text{-set}}\left(\mathrm{Mor}_{\mathrm{B}G}(*, *), \mathrm{Mor}_{\mathrm{B}G}(*, *)\right) = \mathrm{Aut}_{G^{\mathrm{op}}\text{-set}}(G, G).$$

Lastly, applying the forgetful functor from G-sets to just sets, we have an embedding $G \hookrightarrow \mathrm{Aut}_{\mathrm{Set}}(G)$, so we are done. ■

3.4 March 2

We continue. Chris did some review that the Yoneda embedding よ is full, which can be found in our proof of Theorem 3.20.

3.4.1 Unique Representation

For the attendance question, we have the following.

Proposition 3.22. Fix a category $\mathcal{C}$. Then $x \cong y$ implies that $\mathrm{Mor}(x, -) \simeq \mathrm{Mor}(y, 0)$.

Proof. The point is that the Yoneda embedding よ must induce an isomorphism よ(f) : よ$(x) \cong$ よ(y), which is what we wanted. ■

In fact, the converse is also true. We have the following definition.

Definition 3.23 (Creates, reflects isomorphisms)**.** A functor F *reflects isomorphisms* if and only if Ff being an isomorphism implies that f is an isomorphism. Similarly, a functor F *creates isomorphisms* if and only if $Fx \cong Fy$ forces $x \cong y$.

Example 3.24. The functor from $F : \mathrm{Be} + \mathrm{Be} \to \mathrm{Be}$ sending both points to $*$ will certainly reflect isomorphisms because we can only pull back. This F however does not reflect isomorphisms because $Fe_1 \cong Fe_2$ but e_1 is not isomorphic to e_2; there are no morphisms between them all.

We have the following result.

Proposition 3.25. If a functor F is fully faithful, then F creates and reflects isomorphisms.

Proof. This was on the homework. ■

Remark 3.26. Thus, noting that よ is fully faithful, we see that $\mathrm{Mor}(x, -) \simeq \mathrm{Mor}(y, 0)$ forces $x \cong y$.

The point of all of our discussion is as follows.

Proposition 3.27. Suppose two objects x and y represent a functor $F : \mathcal{C} \to \mathrm{Set}$ by natural isomorphisms $\eta : \mathrm{Mor}(x, -) \Rightarrow F$ and $\mu : \mathrm{Mor}(y, -) \Rightarrow F$. Then there is a canonical isomorphism $x \cong y$ by $\eta^{-1}\mu$.

Proof. Intuitively, $f : x \cong y$ induces $- \circ f : \mathrm{Mor}(y, -) \to \mathrm{Mor}(x, -)$, from we would like the following diagram to commute.

$$\begin{tikzcd}
\mathrm{Mor}(x,-) \arrow[dr, Rightarrow, "\eta"] & \\
& F \\
\mathrm{Mor}(y,-) \arrow[uu, Rightarrow, "-\circ f"] \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}$$

As such, we see that we want f to induce a natural isomorphism $(- \circ f) = \eta^{-1}\mu : \mathrm{Mor}(y, -) \Rightarrow \mathrm{Mor}(x, -)$, which we can in fact pull back to f because of Theorem 3.18. ■

Idea 3.28. Thus, we can say that an object represents a functor is unique up to unique (commuting) isomorphism.

3.4.2 Universal Properties

With our uniqueness in hand, we are ready to talk about universal properties.

Definition 3.29 (Universal property I, element)**.** A *universal property* for an object $c \in \mathcal{C}$ is a representable functor $F : \mathcal{C} \Rightarrow \mathrm{Set}$ along with an element $x \in Fc$ such that x induces (by the Yoneda lemma) a natural isomorphism $\mathrm{Mor}(c, -) \Rightarrow F$. In such a triplet (c, F, x), we call x the *universal element*.

To be explicit, $x \in Fc$ is inducing a natural transformation $\mathrm{Mor}(c, -) \Rightarrow F$ by Theorem 3.18, so the condition we are requiring is that we have a natural isomorphism.

Exercise 3.30. We discuss $\mathbb{Z}[x]$ as the free ring on X.

Proof. We will represent the forgetful functor $U : \mathrm{Ring} \to \mathrm{Set}$. To start, we need to show that $\mathbb{Z}[x]$ does actually represent U, for which we need a natural isomorphism

$$\eta : \mathrm{Mor}_{\mathrm{Ring}}(\mathbb{Z}[x], -) \Rightarrow U.$$

In particular, fixing a ring R, we need an isomorphism $\mathrm{Mor}_{\mathrm{Ring}}(\mathbb{Z}[x], R) \Rightarrow UR$, which we do by sending a morphism $f : \mathbb{Z}[x] \to R$ to $f(x) \in UR$. This is indeed a bijection because we can uniquely determine a morphism $\mathbb{Z}[x] \to R$ by where we send x.

Lastly, our universal element is x. To see this, we track through Theorem 3.18 to compute

$$\eta_{\mathbb{Z}[x]}(\mathrm{id}_{\mathbb{Z}[x]}) = \mathrm{id}_{\mathbb{Z}[x]}(x) = x,$$

which is what we wanted. ■

Remark 3.31. In words, the above proof says that $\mathbb{Z}[x]$ is the universal ring that has a distinguished element x. Being the "universal ring" is usually called being the "free ring."

For our next example, we have the following definition.

Definition 3.32 (Tensor products, I). Fix two k-vector spaces V and W. Then $V \otimes W$ is made of formal sums

$$\sum_{i=1}^{n} v_i \otimes w_i$$

where $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$. Further, $(v, w) \mapsto v \otimes w$ is k-bilinear.

Exercise 3.33. We discuss $V \otimes W$ by universal property.

Proof. The point is to consider the functor

$$\mathrm{Bilin}(V, W, -) : \mathrm{Vec}_k \to \mathrm{Set}$$

taking $U \mapsto \mathrm{Bilin}(V, W, U)$, where $\mathrm{Bilin}(V, W, U)$ consists of the k-bilinear maps $V \times W \to U$. Tensor products are actually intended to represent this functor. So here is a better definition for tensor products.

Definition 3.34 (Tensor products, II). Given vector spaces V and W, we define $V \otimes W$ as the object that represents $\mathrm{Bilin}(V, W, -)$.

We do not actually know if $V \otimes W$ really exists, but we will do so shortly. Our universal element x is intended to live in $\mathrm{Bilin}(V, W, V \otimes W)$, so we are characterizing $V \otimes W$ by the bilinear map $V \times W \to V \otimes W$. In particular, we can now more or less say that $V \otimes W$ is the "universal" vector space with respect to a bilinear map $V \times W \to V \otimes W$.

Unwinding a bit, we will name the map $V \times W \to V \otimes W$ as $\otimes$. In particular, we are hoping that this element induces a natural isomorphism

$$\mathrm{Mor}(V \otimes W, -) \Rightarrow \mathrm{Bilin}(V, W, -).$$

In particular, by the Yoneda lemma, we are hoping that bilinear maps $V \times W \to U$ are in natural bijection with linear maps $V \otimes W \to U$. ■

3.5 March 4

Today we're having both Bryce and Chris lecture today (not in that order). We're in luck.

3.5.1 More on Universal Properties

We recall our definition.

Definition 3.35 (Universal property I, element)**.** A *universal property* for an object $c \in \mathcal{C}$ is a representable functor $F : \mathcal{C} \Rightarrow \mathrm{Set}$ along with an element $x \in Fc$ such that x induces (by the Yoneda lemma) a natural isomorphism $\mathrm{Mor}(c, -) \Rightarrow F$. In such a triplet (c, F, x), we call x the *universal element*.

We continue our discussion of Exercise 3.33. Typically, one would think about the universal property as follows.

Definition 3.36 (Universal property, II)**.** An object $c \in \mathcal{C}$ *satisfies the universal property* given by a functor $F : \mathcal{C} \to \mathrm{Set}$ if we have a universal element x such that (c, F, x) is a universal property.

In particular, last time we talked about having the functor

$$\mathrm{Bilin}(V, W, -) : \mathrm{Vec}_k \to \mathrm{Set}$$

which we claim was represented by some object $V \otimes W$ and our universal element $\otimes \in \mathrm{Bilin}(V, W, -) \to V \otimes W$. By Theorem 3.18, we can unwind this to a natural transformation

$$\eta : \mathrm{Hom}(V \otimes W, -) \Rightarrow \mathrm{Bilin}(V, W, -),$$

which we are claiming is a natural isomorphism to be our universal property. Well, we have our object $\otimes$, so we now track everything through. Here is our diagram to unwind the isomorphism above for a morphism $\overline{f} : V \otimes W \to U$ corresponding to the bilinear map $f : V \times W \to U$.

$$\begin{tikzcd}
\mathrm{Hom}(V \otimes W, V \otimes W) \arrow[r, "\overline{f}\circ -"] \arrow[d, "\eta_{V\otimes W}"'] & \mathrm{Hom}(V \otimes W, U) \arrow[d, "\eta_U"] \\
\mathrm{Bilin}(V, W, V \otimes W) \arrow[r, "\overline{f}\circ -"'] & \mathrm{Bilin}(V, W, U)
\end{tikzcd}$$

So to unwind what η_U means, we plug into $\mathrm{id}_{V\otimes W}$. This makes the following diagram.

$$\begin{tikzcd}
\mathrm{id}_{V\otimes W} \arrow[r, mapsto, "\overline{f}\circ -"] \arrow[d, mapsto, "\eta_{V\otimes W}"'] & \overline{f} \arrow[d, mapsto, "\eta_U"] \\
\otimes \arrow[r, mapsto, "\overline{f}\circ -"'] & f
\end{tikzcd}$$

Namely, we are told that bilinear maps $f : V \times W \to U$ correspond uniquely to a morphism $\overline{f} : V \otimes W \to U$ (by Theorem 2.72) in such a way that the following diagram commutes.

$$\begin{tikzcd}
V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed, "\overline{f}"] \\
& U
\end{tikzcd}$$

So this is our usual universal property for the tensor product.

Remark 3.37. We will actually need to construct $V \otimes W$, which we did not do, in order to show that there exists a way to represent the functor

$$\mathrm{Bilin}(V, W, -) : \mathrm{Vec}_k \to \mathrm{Set}.$$

However, in practice, we only ever want to pay attention to the above universal property.

3.5.2 Category of Elements

Today's discussion will not be discussion. Bryce is, reportedly, sorry. For brevity, we will take the following convention.

Definition 3.38 (Universal)**.** We say that an object $c \in \mathcal{C}$ is *universal* if and only if it is either initial or final.

Of course, $V \otimes W$ will not turn out to be universal in Vec_k, but if we change our category, then it will be, which is nice.

With that said, here is our main character for today.

Definition 3.39 (Category of elements)**.** Fix $F : \mathcal{C} \to \mathrm{Set}$ a functor. Then the *category of elements* of F, denoted $\int F$ is made of the following data.

- Objects are pairs (c, x) where $c \in \mathcal{C}$ and $x \in Fc$. In practice, we should think about the object $x \in Fc$ on its own, but we will have to remember which c it comes from.
- Morphisms $(c, x) \to (d, y)$ made of morphisms $f : c \to d$ which preserve our "base points" as $(Ff)(x) = y$. Importantly, we are keeping track of the arrows in $\mathcal{C}$, not in Set; e.g., F might not be injective on arrows, so we will keep track of these definitions.
- Identities are identities lifted from $\mathcal{C}$.
- Composition is composition in $\mathcal{C}$.

Remark 3.40. There is a natural forgetful functor $\Pi : \int F \to \mathcal{C}$ by

$$\Pi(c, x) := c \qquad \text{and} \qquad \Pi(f) := f.$$

We bring this up because this is roughly why we are keeping track of the morphisms in $\mathcal{C}$ instead of Set.

There is also a contravariant version.

Definition 3.41 (Category of elements, contravariant)**.** Fix $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ a functor. Then the *category of elements* of F, denoted $\int F$ is made of the following data.

- Objects are pairs (c, x) where $c \in \mathcal{C}$ and $x \in Fc$.
- Morphisms $(c, x) \to (d, y)$ made of morphisms $f : c \to d$ which preserve our "base points" as $(Ff)(y) = x$. This flips because F is contravariant.
- Identities are identities lifted from $\mathcal{C}$.
- Composition is composition in $\mathcal{C}$.

Let's see some examples.

Example 3.42. Let $\mathcal{C}$ be a concrete category with faithful (forgetful) functor $U : \mathcal{C} \to \mathrm{Set}$. We work through $\int U$.

- Objects are pairs (c, x) where $x \in Uc$.
- Morphisms are morphisms $f : (c, x) \to (d, y)$ such that $(Uf)(x) = y$.

In other words, $\int U$ is roughly the objects $c \in \mathcal{C}$ with an identified base point. Specifically, $\int(U : \mathrm{Top} \to \mathrm{Set}) = \mathrm{Top}_*$.

Example 3.43. Fix $\mathcal{C}$ a locally small category, which is how you know Bryce is lecturing, which permits a functor $\mathrm{Mor}(c, -) : \mathcal{C} \Rightarrow \mathrm{Set}$. We discuss $\int \mathrm{Mor}(c, -)$.

- Objects are pairs (d, f) where $f \in \mathrm{Hom}(c, d)$. So our objects are morphisms.
- A morphism $\varphi : (d, f) \to (e, g)$ is a morphism $\varphi : d \to e$ in $\mathcal{C}$ such that $\varphi f = (\varphi \circ -)(f) = g$.

In other words, this gives the category under $\mathcal{C}$, denoted $c/\mathcal{C}$. The contravariant version gives $\mathcal{C}/c$.

Exercise 3.44. Fix $F : C^{\mathrm{op}} \to \mathrm{Set}$ a contravariant functor. We recover $\int F$ as a comma category.

Proof. To set up our discussion, we recall that Theorem 3.18 provides us with a sufficiently natural bijection

$$\psi : Fc \cong \mathrm{Mor}(\mathrm{Mor}_{\mathcal{C}}(-, c), F).$$

Now, objects in $\int F$ will naturally be objects $x \in Fc$. We would to track morphisms $f : (c, x) \to (d, y)$ through here as well, which means that we are going to need a morphism $\psi(x) \to \psi(y)$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Roughly speaking, we are going to want the following diagram to commute.

$$\begin{tikzcd} \mathrm{Mor}(-, c) \arrow[r, Rightarrow, "\psi(x)"] \arrow[d, Rightarrow, "(f\circ -)"'] & F \\ \mathrm{Mor}(-, d) \arrow[ur, Rightarrow, "\psi(y)"'] & \end{tikzcd} \tag{$*$}$$

In particular, Theorem 3.20 tells us that all such morphisms between natural transformations take the form $(f \circ -)$ for some morphism f, from which we can track our base point.

The point of all this is that we are going to have a nice correspondence between $\int F$ and the comma category

$$\int F \cong よ \downarrow \widetilde{F},$$

where $\widetilde{F} : \{*\} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the constant functor taking $* \mapsto F$. Indeed, to quickly unwind our definition of the comma category, it is made of triplets $(c \in \mathcal{C}, * \in \{*\}, f : よ(c) \to F(*))$, where morphisms $h : (c, *, f) \to (c', *, f')$ require the following diagram to commute.

$$\begin{tikzcd} よ(c) \arrow[r, "f"] \arrow[d, "Fh"'] & F \arrow[d, equal] \\ よ(c') \arrow[r, "f'"'] & F \end{tikzcd}$$

Notably, we only have to check the $\mathrm{id}_F : F \to F$ morphism because this is the only morphism carried from $\widetilde{F} : \{*\} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. But this diagram above is exactly the one we asked for in ($*$), so we are done. ■

Next time we will discuss the following result.

Proposition 3.45. Fix $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then $\int F$ has an initial object (c, x) if and only if F is representable by c with universal element x.

Proof of the forwards direction. In one direction, take $(c, x) \in \int F$ initial. We would like a natural isomorphism $\eta : \mathrm{Mor}(c, -) \Rightarrow F$. Well, by Theorem 3.18, we get some natural transformation η corresponding to x, where

$$\eta_d(f) := (Ff)(x)$$

by pushing through our definition in Theorem 3.18. For this to be a natural isomorphism, we need the components $\eta_d : \mathrm{Mor}(c, d) \to Fd$ to be isomorphisms. In other words, for each $d \in \mathcal{C}$ and $y \in Fd$, we need some $f : c \to d$ such that

$$(Ff)(x) = \eta_d(f) = y.$$

Equivalently, there is a unique morphism $f : (c, x) \to (d, y)$ in $\int F$, which is what we wanted. ■

Remark 3.46. In the dual case, F will be contravariant, and our initial object becomes final.

3.6 March 7

We continue.

3.6.1 Housekeeping

We begin by discussing a homework problem. Here is a definition.

Definition 3.47 (Divisible)**.** An abelian group A is *divisible* if and only if, for each $a \in A$ and $n \in \mathbb{Z} \setminus \{0\}$.

It happens that the category of divisible abelian groups has non-injective monomorphisms. For example, we have the following.

Exercise 3.48. The map $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Proof. Suppose that we have maps $f, g : A \to \mathbb{Q}$ such that $\pi f = \pi g$. We claim that $f = g$. Indeed, for any $a \in A \setminus \{0\}$, we need to show that $f(a) = g(a)$, for which so far we know that $\pi(f(a)) = \pi(f(g))$, so there is an integer n such that

$$f(a) = g(a) + n.$$

Suppose for the sake of contradiction that $n \neq 0$. Then, because A is divisible, there exists an element $b \in A$ such that $a = 2nb$, so we get to write

$$2nf(b) = f(2nb) = f(a) = g(a) + n = 2ng(b) + n,$$

so $f(b) = g(b) + \frac{1}{2}$. Pushing this though π, we get

$$b = b + \frac{1}{2},$$

so $\frac{1}{2} \in \mathbb{Z}$, which is our contradiction. ■

And here is the attendance question.

Exercise 3.49. We describe $\int F$ where $F : \{*\} \to \mathrm{Set}$ is some functor.

Proof. Set $X := F(*)$. The objects in $\int F$ are pairs $(*, c)$ where $c \in X$, and the morphisms are morphisms $f : * \to *$ such that $Ff(c) = Ff(d)$, but only $f = \mathrm{id}_*$ is permitted. So we have objects which are elements of X and only identities, so this is the discrete category on X. ■

3.6.2 A Representability Test

Last time we were showing the following result.

Proposition 3.45. Fix $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then $\int F$ has an initial object (c, x) if and only if F is representable by c with universal element x.

Last time we showed the forwards direction.

Proof of the backwards direction. Suppose that we have a natural isomorphism $\alpha : \mathrm{Mor}(c, -) \Rightarrow F$, and we need an object to be initial in $\int F$. Without much to do, we set

$$x := \alpha_c(\mathrm{id}_c) \in Fc,$$

and we claim that (c, x) is our desired initial element in $\int F$.

Well, pick up some object (d, y), and we want to show that there is a unique morphism $(c, x) \to (d, y)$. To be explicit, our data consist of $d \in \mathcal{C}$ and $y \in Fd$. The main claim is that, for any morphism $f : c \to d$, we have

$$\alpha_d(f) = (Ff)(f),$$

as we showed in the Yoneda lemma. Here is the relevant naturality diagram.

$$\begin{tikzcd} \mathrm{Mor}(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f\circ -"'] & Fc \arrow[d, "Ff"] \\ \mathrm{Mor}(c,d) \arrow[r, "\alpha_d"] & Fd \end{tikzcd}$$

Tracking through id_c in the diagram gives the result because $\alpha_c(\mathrm{id}_c)$ was defined to be x. It follows that we have a morphism $f : (c, x) \to (d, y)$ if and only if $(Ff)(x) = y$ if and only if $\alpha_d(f) = y$, which we know to be unique because α_d is an isomorphism. ■

From the way we have proven things, we actually have the following result.

Corollary 3.50. In fact, F is represented by c with universal element x if and only if $(c, x) \in \int F$ is initial.

Proof. If $(c, x) \in \int F$ is initial, then we showed last time that c represents our functor, and x is actually our universal property (by staring at our proof). Conversely, if F is represented by c, we conjured our universal element $x := \alpha_c(\mathrm{id}_c)$ to create our initial element (c, x). ■

3.6.3 Unique Representation

Because the Yoneda embedding (Theorem 3.20) creates isomorphisms, if $\mathrm{Mor}(c, -) \simeq \mathrm{Mor}(c', -)$, then $c \cong c'$, so our representing objects are isomorphic. We might hope for something more.

Remark 3.51. There is a technical notion of "evil" that basically says that sometimes in category theory our notion of equality is too strong. For example, isomorphism of categories is too strong, so we had equivalence of categories to fix this.

Example 3.52. "Cardinality" of a category is not preserved by equivalence, so it is evil. For example, any two indiscrete categories are equivalent, but they have different numbers of elements.

Anyways, we have the following result.

Proposition 3.53. For a functor $F : \mathcal{C} \to \mathrm{Set}$, the full subcategory spanned by its representations in $\mathcal{C}$ is either empty or a contractible groupoid.

Wait, contractible groupoid?

Definition 3.54 (Contractible groupoid)**.** A *contractible groupoid* is a category where all morphism sets $\mathrm{Mor}(c, d)$ has exactly one element.

Remark 3.55. The idea is that we can "collapse" our category inwards along unique isomorphisms.

We showed back in Exercise 2.91 that all contractible groupoids are equivalent to Be; here is the idea behind why we are bringing this up.

Idea 3.56. Unique isomorphisms tend to have contractible groupoids in the background.

So the idea behind introducing Proposition 3.53 is that there will be a unique morphism $f : c \to d$ that will also send the corresponding universal elements correctly in that $f : (c, x) \to (d, y)$. It is a good isomorphism.

Before continuing, here is a lemma.

Lemma 3.57. The full subcategory of $\mathcal{C}$ spanned by its final objects is either empty or a contractible groupoid.

Proof. We will be brief. If it is empty, we are done. Otherwise, for any two final objects t_1, t_2, there is exactly one morphism $t_1 \to t_2$ because t_2 is final. So we are done. ■

Remark 3.58. We can dualize the above lemma (by working in $\mathcal{C}^{\mathrm{op}}$) to replace the word "final" with "initial" everywhere.

And now we prove Proposition 3.53.

Proof of Proposition 3.53. If F is not representable, then $\int F$ has no initial objects because initial objects induce representations. Otherwise, $\int F$ will have initial objects, but they form a contractible groupoid by Remark 3.58. ■

3.6.4 Typical Universal Properties

Because we are feeling benevolent today, here are some examples.

Exercise 3.59. Consider the contravariant functor $\mathcal{P} : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$, which sends maps objects by $\mathcal{P} : S \mapsto \mathcal{P}(S)$ and morphisms by taking $f : S \to T$ to $f^{-1} : \mathcal{P}(T) \to \mathcal{P}(S)$. We discuss Proposition 3.45 with this functor.

Proof. Our objects are pairs (X, A) where X is a set and $A \subseteq X$ is a subset. Our morphisms $(X, A) \to (Y, B)$ are maps $f : X \to Y$ such that $f^{-1}(B) = A$.

Now, back in Exercise 3.14, we showed that $\Omega = \{0, 1\}$ represents $\mathcal{P}$ with universal element 1. Accordingly, we claim that $(\Omega, \{1\})$ is final (note $\mathcal{P}$ is contravariant) in $\int F$. Indeed, for any pair (X, A), there is a unique map $f : X \to \Omega$ such that $f^{-1}(\{1\}) = A$ which describes itself. ■

Exercise 3.60. Consider the functor $\operatorname{Bilin}(V, W, -) : \mathrm{Vec}_k$. We discuss Proposition 3.45.

Proof. To start, we note that our objects of $\int \operatorname{Bilin}(V, W, -)$ consists of a vector space U with a bilinear map $f : V \times W \to U$. A morphism $(U, f) \to (U', f')$ is a linear map $g : U \to U'$ such that $gf = f'$; i.e., the following diagram should commute.

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[dr, "f'"'] & U \arrow[d, "g"] \\ & U' \end{tikzcd}$$

Explicitly, we want

$$\operatorname{Bilin}(V, W, -)(g)(f) = f',$$

but $\operatorname{Bilin}(V, W, -)(g) = (g \circ -)$ by definition.

On the other hand, we know that $V \otimes W$ represents $\operatorname{Bilin}(V, W, -)$ with universal element $\otimes : V \times W \to V \otimes W$ by $(v, w) \mapsto v \otimes w$. Noting that this means $(V \otimes W, \otimes)$ ought to be initial, we are told that whenever we have a bilinear map $V \otimes W \to U$, there is a unique map $V \otimes W \to U$ such that the following diagram commutes.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed] \\ & U' \end{tikzcd}$$

This is the typical universal property. ■

Exercise 3.61. Consider the forgetful functor $U : \mathrm{Ring} \to \mathrm{Set}$. We discuss Proposition 3.45.

Proof. Our objects in $\int R$ consists of pairs (R, r) such that $r \in R$. Our morphisms $f : (R, r) \to (S, s)$ is a morphism $f : R \to S$ such that $f(r) = s$.

Now, back in Exercise 3.12, we showed that $\mathbb{Z}[x]$ should represent this functor with universal element x, so we want $(\mathbb{Z}[x], x)$ to be initial in $\int F$. In other words, for any pair (R, r), there is a unique morphism $\mathbb{Z}[x] \to R$ such that $x \mapsto r$. Indeed, this morphism must take $1 \mapsto 1$, so we are sending

$$\sum_{k=0}^{N} a_k x^k \longmapsto \sum_{k=0}^{N} a_k r^k,$$

which finishes. ■

Theme 4
Limits and Colimits

It's true that many pieces of categorical terminology do come from analysis, but maybe all that says is that analysis is an old and venerable subject.

—Tom Leinster

4.1 March 9

The fun continues but now in a different form.

4.1.1 Products

Let's do some examples to start because we are feeling kind today.

Exercise 4.1. We consider products $A \times B$ in Set, defined as

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$$

in a categorical sense.

Remark 4.2. It turns out that products are limits.

Proof. We would like to make this definition more fit for category theory, for which we note that we have projection maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ by $\pi_A : (a, b) \mapsto a$ and $\pi_B : (a, b) \mapsto b$ respectively. In fact, we are universal in the following sense: for any object A and maps $\varphi_A : C \to A$ and $\varphi_B : C \to B$, there is a unique (!) map $\varphi : C \to A \times B$ making the following diagram commute.

$$\begin{tikzcd} & C \arrow[d, dashed, "\varphi"] \arrow[ddl, bend right, "\varphi_A"'] \arrow[ddr, bend left, "\varphi_B"] & \\ & A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A & & B \end{tikzcd}$$

To see uniqueness, we note that we must have

$$\pi_A(\varphi(c)) := \varphi_A(c) \qquad \text{and} \qquad \pi_B(\varphi(c)) := \varphi_B(c),$$

so we must define our map φ as

$$\varphi(c) := (\varphi_A c, \varphi_B c).$$

In fact, we can see that this defined map does have $\pi_A \circ \varphi = \varphi_A$ and $\pi_B \circ \varphi = \varphi_B$, so the diagram does indeed commute. ■

This example should feel similar to universal properties: whenever something occurs in our diagram, we have some unique induced map. To see this more formally, we have the following auxiliary exercise.

Exercise 4.3. We exhibit the universal property for $A \times B$ in the category $\mathcal{C} := \mathrm{Set}$.

Proof. We note that we are being granted a bijection between pairs of maps (φ_A, φ_B) and our maps φ. In other words, there is a bijection

$$\mathrm{Mor}(C, A) \times \mathrm{Mor}(C, B) \to \mathrm{Mor}(C, A \times B).$$

To turn this into a universal property, we consider the functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by

$$F : \mathrm{Mor}(-, A) \times \mathrm{Mor}(-, B).$$

In particular, we send morphisms $f : S \to T$ to $F(f) : \mathrm{Mor}(T, A) \times \mathrm{Mor}(T, B) \to \mathrm{Mor}(S, A) \to \mathrm{MOr}(S, B)$ by

$$F(f) := (- \circ f) \times (- \circ f).$$

We won't check this is a functor, but you can if you like doing that kind of thing.

Now, we to get our universal property, we need to exhibit our natural isomorphism

$$\eta : F \Rightarrow \mathrm{Mor}(-, A \times B).$$

We already gave a bijection of sets in the previous exercise, so we just need to show that it is natural. Well, pick up some morphism $f : T \to S$, and we have the following diagram to check.

$$\begin{tikzcd}
\mathrm{Mor}(S, A) \times \mathrm{Mor}(S, B) \arrow[r, "(-\circ f)\times(-\circ f)"] \arrow[d, "\eta_S"'] & \mathrm{Mor}(T, A) \times \mathrm{Mor}(T, B) \arrow[d, "\eta_T"] \\
\mathrm{Mor}(S, A \times B) \arrow[r, "-\circ f"'] & \mathrm{Mor}(T, A \times B)
\end{tikzcd}$$

So now, pick up some pair $(h, g) \in \mathrm{Mor}(S, A) \times \mathrm{Mor}(S, B)$ and track through. Along the bottom, we go to $h \times g$ which then goes to $hf \times gf$. Along the top, we start with (h, g) then go to (hf, gf) which then goes to $hf \times gf$.

Now let's compute our universal element. For this, we need to find what we are getting out of Theorem 3.18, which is

$$\eta_{A\times B}^{-1}(\mathrm{id}_{A\times B}) = (\pi_A, \pi_B),$$

which is fairly intuitive. In particular, we can track $\eta_{A\times B}((\pi_A, \pi_B))$ through and get $\mathrm{id}_{A\times B}$, which will give what we want. ■

Here are some generalizing remarks.

Remark 4.4. The universal property of products in Exercise 4.3 did not depend on Set, but the construction did. Products are more of an ambient concept that might or might not happen in some given category.

Remark 4.5. We could just have easily defined arbitrary products

$$\prod_{\alpha\in\lambda} S_\alpha$$

for some sets $\{S_\alpha\}_{\alpha\in\lambda}$ by just increasing the number of terms. For example, our functor we want to represent is now

$$F : \prod_{\alpha\in\lambda} \mathrm{Mor}(-, S_\alpha).$$

We can also write out the analogous universal property in terms of Exercise 4.1.

Example 4.6. The product of the one term A equipped with the projection map $\mathrm{id}_A : A \to A$. Indeed, for any map $C \to A$, there is a unique map $C \to A$ making the following diagram commute.

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[dr] & A \arrow[d, equal] \\ & A \end{tikzcd}$$

Example 4.7. The product of no terms at all is the final object X. Indeed, whenever we have no morphisms going anywhere, there is a unique map to X making whatever diagram you want commute.

4.1.2 Coproducts

Next let's discuss coproducts. Let's just give the universal property.

Definition 4.8 (Coproduct)**.** Given two objects $A, B \in \mathcal{C}$, we define the *coproduct* object $A \coprod B$ to be equipped with maps $\iota_A : A \to A \coprod B$ and $\iota_B : B \to A \coprod B$ such that, whenever we have an object Z with maps $\varphi_A : A \to Z$ and $\varphi_B : B \to Z$, there is a unique map $A \coprod B \to Z$ making the following diagram commute.

$$\begin{tikzcd} A \arrow[dr, "\iota_A"] \arrow[ddr, bend right, "\varphi_A"'] & & B \arrow[dl, "\iota_B"'] \arrow[ddl, bend left, "\varphi_B"] \\ & A \coprod B \arrow[d, dashed] & \\ & Z & \end{tikzcd}$$

Example 4.9. The disjoint union $A \sqcup B$ is the coproduct in Set. Indeed, our maps are $\iota_A : a \mapsto (a, 0)$ and $\iota_B : b \mapsto (b, 1)$. To see the universal property, suppose that we have an object C with maps $\varphi_A : A \to C$ and $\varphi_B : B \to C$. To see the uniqueness of $\varphi : A \sqcup B \to C$, we see that we must have

$$\varphi(\iota_A a) = \varphi_A(a) \qquad \text{and} \qquad \varphi(\iota_B b) = \varphi_B(b)$$

which exhausts all possible cases for elements of $A \sqcup B$. It is then not too hard to check that this does satisfy $\varphi \circ \iota_A = \varphi_A$ and $\varphi \circ \iota_B = \varphi_B$ by construction.

Example 4.10. We can generalize to products with multiple terms. If we have one object, the coproduct of A is just A. Similarly, if we have no objects, then the coproduct will be an initial object.

4.1.3 More on Products

Now let's generalize our examples. We begin by making the product even more categorical. At a high level, we might have lots of objects $\{A_\alpha\}_{\alpha\in\lambda}$, we are given maps $\pi_\alpha : \prod A \to A_\alpha$ in some universal way.

$$\begin{tikzcd} & \prod A \arrow[dl, "\pi_\alpha"'] \arrow[dr, "\pi_\beta"] & \\ A_\alpha & \cdots & A_\beta \end{tikzcd}$$

To make this more in terms of category theory, we note that we can formalize the bottom part of the diagram as the image of some functor

$$F : \mathcal{J} \to \mathcal{C}$$

for some discrete category $\mathcal{J}$. Namely, our objects A_α look like $F(\alpha)$ for various $\alpha \in \mathcal{J}$.

To put the product $\prod A$ on the same footing, we will similarly define the constant functor

$$C_x : \mathcal{J} \to \mathcal{C}$$

which sends all objects of $\mathcal{J}$ to x and all morphisms to id_x.

We would like to create arrows between our diagrams, we are asking for an arrow between our functors, so we are more or less asking for a natural transformation $\eta : C_x \Rightarrow F$. Namely, the component morphisms take some $\alpha \in \mathcal{J}$, we are being promised a morphism $\eta_\alpha : x \to A_\alpha$. If we wanted to check that η is a natural transformation, we would pick up a morphism $\mathrm{id}_\alpha : \alpha \to \alpha$ in $\mathcal{J}$, which gives rise to the following diagram.

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "\eta_\alpha"] & x \arrow[d, "\eta_\alpha"] \\ F(\alpha) \arrow[r, "\mathrm{id}_{F(\alpha)}"'] & F(\alpha) \end{tikzcd}$$

Notably, this commutes for free. If we wanted to add more structure to our products, we might want to change $\mathcal{J}$ to be not discrete and have F be a more general diagram. This gives rise to limits.

Definition 4.11 (Cone)**.** Fix an index category $\mathcal{J}$ and a category $\mathcal{C}$ with an object $c \in \mathcal{C}$. Then a *cone* is a natural transformation from the constant functor $C_c \Rightarrow F$, where $F : \mathcal{J} \to \mathcal{C}$ is some diagram.

The limit will be the object $\lim F \in \mathcal{C}$ which is a "universal" cone, in the same way that the product was universal with respect to a "discrete cone." We will not discuss this more formally today, but we will discuss it more next lecture.

4.2 March 11

We do more limits today.

4.2.1 Cones and Cocones

We recall the following definition. For today, we fix $\mathcal{J}$ as our index category.

Definition 4.11 (Cone)**.** Fix an index category $\mathcal{J}$ and a category $\mathcal{C}$ with an object $c \in \mathcal{C}$. Then a *cone* is a natural transformation from the constant functor $C_c \Rightarrow F$, where $F : \mathcal{J} \to \mathcal{C}$ is some diagram.

Definition 4.12 (Apex)**.** A cone over F with *summit* or *apex* c is a natural transformation $\lambda : c \Rightarrow F$ with components $\lambda_j : c \to Fj$. These components are called *legs*.

Informally, λ consists of morphisms $\lambda_j : c \to Fj$ which commute with the morphisms promised by $\mathcal{J}$. Namely, for any morphism $f : i \to j$, we have the following naturality square diagram;

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "\lambda_i"'] & c \arrow[d, "\lambda_j"] \\ Fi \arrow[r, "Ff"'] & Fj \end{tikzcd}$$

Collapsing the top makes this look more like a triangle.

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F_i \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

Of course, we also have a dual notion. We have the following definition.

Definition 4.13 (Nadir)**.** A cone under F (or "cocone") with *nadir* c is a natural transformation $\lambda : F \Rightarrow c$ with components $\lambda_j : Fj \to c$. These components are (still) called *legs*.

Remark 4.14. We use the word nadir because someone wanted to.

This time our picture looks like the following.

$$\begin{tikzcd} Fi \arrow[rr, "Ff"] \arrow[dr, "\lambda_i"'] & & Fj \arrow[dl, "\lambda_j"] \\ & c & \end{tikzcd}$$

Notably, the nadir c is under F this time.

Example 4.15. Fix our index category $\mathcal{J} = \mathbb{Z}$. For a cone $F : \mathcal{J} \to \mathcal{C}$ over c, our diagram looks like the following.

$$\begin{tikzcd} & & c \arrow[dl] \arrow[d] \arrow[dr] & & \\ \cdots \arrow[r] & F(-1) \arrow[r] & F(0) \arrow[r] & F(1) \arrow[r] & \cdots \end{tikzcd}$$

4.2.2 Limits and Colimits

Intuitively, our limits will be the universal apex for a cone. It is the best cone; in some sense, it is the smallest or "closest" apex to the diagram. The diagram looks like the following.

$$\begin{tikzcd} & c' \arrow[ddl, bend right, "\lambda'_i"'] \arrow[ddr, bend left, "\lambda'_j"] & \\ & c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

We have the following definition to induce our desired behavior.

Definition 4.16 (Cone functor)**.** Fix a diagram $F : \mathcal{J} \to \mathcal{C}$.

- We define the functor $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \to \text{Set}$ by

$$c \mapsto \text{Cone}(c, F) := \text{Hom}_{\text{Cat}}(c, F)$$

which are the natural transformations $\lambda : c \Rightarrow F$. Then a morphism $f : c \to d$ goes to the morphism $F(f) : \text{Hom}_{\text{Cat}}(d, F) \to \text{Hom}_{\text{Cat}}(c, F)$ so that a cone $\lambda : d \Rightarrow F$ gives rise to a cone

$$F(f)(\lambda) = \lambda_\bullet \circ f \in \text{Hom}_{\text{Cat}}(c, F).$$

- We define the functor $\text{Cone}(-, F) : \mathcal{C} \to \text{Set}$ by

$$c \mapsto \text{Cone}(F, c) := \text{Hom}_{\text{Cat}}(F, c)$$

which are the natural transformations $\lambda : c \Rightarrow F$. Then a morphism $f : c \to d$ goes to the morphism $F(f) : \text{Hom}_{\text{Cat}}(F, c) \to \text{Hom}_{\text{Cat}}(F, d)$ so that a cone $\lambda : F \Rightarrow c$ gives rise to a cone

$$F(f)(\lambda) = f \circ \lambda_\bullet \in \text{Hom}_{\text{Cat}}(F, d).$$

Here is the image of $\text{Cone}(f)$ creating a cone with apex c from a cone to apex d.

$$\begin{tikzcd}
& d \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\lambda_i \circ f"'] \arrow[ddr, bend left, "\lambda_i \circ f"] & \\
& c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

We will not check the functoriality of this functor, but surely it works: just look at it.

Our functors give the following definitions.

Definition 4.17 (Limit, colimit)**.** A *limit* of a diagram $F : \mathcal{J} \to \mathcal{C}$ is a representation of $\text{Cone}(-, F)$; in other words, it is a natural isomorphism $\mathcal{C}(-, c) \simeq \text{Cone}(-, F)$. (Note $\text{Cone}(-, F)$ is the contravariant.) Dually, a *colimit* is a representation of $\text{Cone}(F, -)$.

We will mostly be talking about limits and leave the discussion of colimits to the curious.

Note that, by Theorem 3.18, we see that a natural transformation

$$\alpha \in \text{Hom}(\mathcal{C}(-, c), \text{Cone}(-, F))$$

corresponds to some literal cone $\text{Cone}(c, F)$. From our discussion of the category of elements, we note that we can also think of a limit in the following way.

Definition 4.18 (Limit, colimit)**.** A *limit* of a diagram $F : \mathcal{J} \to \mathcal{C}$ is a terminal object in $\int \text{Cone}(-, F)$.

To review, our objects of $\int \text{Cone}(-, F)$ look like pairs $(c, \lambda) \in \int \text{Cone}(-, F)$ where $\lambda : c \Rightarrow F$. Then our morphisms $(c, \lambda) \to (c, \mu)$ have the data $f : c \to d$ such that

$$\text{Cone}(f, F)(\mu) = \lambda.$$

In other words, we require $\mu_\bullet \circ f = \lambda_\bullet$, which is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd} & d \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\ & c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

Thus, (c, λ) being terminal in $\int \mathrm{Cone}(-, F)$ means that any pair of objects (d, μ) will have the morphism f be unique.

At this point, we can see that our limits are indeed unique up to unique isomorphism because our terminal objects are unique up to unique isomorphism. Alternatively, we can give the following argument.

Proposition 4.19. The limit of a diagram $F : \mathcal{J} \to \mathcal{C}$ is unique up to unique isomorphism.

Proof. The point is to stack our limits on top of each other. So that (c, λ) and (c', λ') are both limits of F. Then we place them in the following diagram and note that we have unique maps f and g induced by the diagram above.

$$\begin{tikzcd} & c \arrow[d, dashed, "g"] \arrow[dddl, bend right, "\lambda_i"'] \arrow[dddr, bend left, "\lambda_j"] & \\ & c' \arrow[d, dashed, "f"] \arrow[ddl, "\lambda'_i"'] \arrow[ddr, "\lambda'_j"] & \\ & c \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

By uniqueness, we see that $f \circ g$ must be the identity (there is only one such morphism from $c \to c$ making the diagram commute, and id_c works), so f and g are must be inverses by redoing the stacking with f to show $g \circ f = \mathrm{id}_d$. ∎

Notation 4.20. From now on, we will write $\lim F$ for the limit of F and $\operatorname{colim} F$ for F.

Remark 4.21. Not all categories have all their limits and colimits. For example, Field does not have an initial object (we cannot inject into both $\mathbb{F}_2$ and $\mathbb{F}_3$), so Field is missing the limit of the diagram from the empty category.

We close with an example.

Exercise 4.22. We show that product are limits from the discrete category.

Proof. Fix our functor $F : \mathcal{J} \to \mathcal{C}$ with apex (P, π), which means that we have morphisms $\pi_j : P \to Fj$. Note that we have no commutativity among the $j \in \mathcal{J}$ because $\mathcal{J}$ has no non-identity morphisms.

To translate the universal property, we see that whenever we have another apex (c, λ), there is a unique morphism $f : c \to P$ making the following diagram commute.

$$\begin{tikzcd}
 & c \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\lambda_i"'] \arrow[ddr, bend left, "\lambda_j"] & \\
 & P \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

This is what we wrote down in Exercise 4.3. ■

4.3 March 14

The fun, as they say, never stops.

4.3.1 More Examples

Chris is back, so today is just examples.

Exercise 4.23. We discuss the limit of the diagram

$$A \xrightarrow{f} B.$$

Proof. The limit will be an object X with a map $\iota : X \to A$ such that, for any object Y, there is a unique map $Y \to X$ making the following diagram commute.

$$\begin{tikzcd}
Y \arrow[r, dashed] \arrow[d, "\varphi"'] & X \arrow[d] \arrow[dl] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Well, we simply set $X := A$ with $X \to A$ simply as the identity map $\mathrm{id}_A : X \to A$. Then we are forced to have $Y \to X$ be φ by the diagram commuting, which finishes. ■

Exercise 4.24. We exhibit a product where the projection maps are not epimorphisms.

Proof. This is somewhat hard because faithful functors preserve epimorphisms, so concrete categories won't work here. So we consider the following category.

$$\begin{tikzcd}
 & B \arrow[dl] \arrow[dr] \arrow[drr, dashed, bend left] & & \\
A & & C \arrow[r, shift left] \arrow[r, shift right] & D
\end{tikzcd}$$

It is not too hard to see that B is the product of A and C (the only object with map to both A and C is B itself, so it is our only object to check), but the map $B \to C$ is not an epimorphism because of the problems with $C \to D$. ■

4.3.2 Equalizers

Definition 4.25 (Equalizer)**.** An *equalizer* is a limit of the following diagram.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

We denote this by $\operatorname{eq}(f, g)$.

More concretely, we set up our diagram as follows.

$$\begin{tikzcd} E \arrow[d, "e"'] \arrow[dr, "e'"] & \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

By commutativity of the diagram, we want $fe = ge = e'$, so we will ignore the morphism e' entirely: it's induced by the rest of the diagram.

Now, for E to be universal, we are saying that, for any morphism $h : X \to A$, there is a unique morphism $X \to E$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[d, dashed, "h"'] \arrow[dr, "\varphi"] & & \\ E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

This is not an obvious limit; here is an example.

Exercise 4.26. We compute equalizers in Set.

Proof. As a starting example, we note that we do have a "trivial" cone with $X = \varnothing$. This does not use the other information of our limit, so we simply define

$$E := \{a \in A : f(a) = g(a)\}$$

with inclusion morphism $\iota : E \subseteq A$. Certainly $f\iota = g\iota$ by construction.

Now, to show the universal property, any other object X with a morphism $h : X \to A$ such that $fh = gh$, we see that $h(x) \in E$ for each $x \in X$. Thus, h does map into E, so we have our induced map

$$\widetilde{h} : X \to E$$

by simply restricting the codomain. This morphism is unique because any such morphism $\widetilde{h}$ must have $\iota\widetilde{h} = h$, so $\widetilde{h}(x) = h(x)$ for each $x \in X$. ■

Remark 4.27. I think the same construction will work for equalizers in any concrete category.

Exercise 4.28. Working in Ab, we consider the equalizer of the following diagram, where $f : A \to B$ is some morphism.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

In particular, we claim that the equalizer is the kernel.

Proof. By essentially doing the same proof as in Set, the equalizer will be the set

$$E := \{a \in A : f(a) = 0(a) = 0\},$$

which is $\ker f$. ■

Remark 4.29. It follows that $\mathrm{eq}(f, g) = \mathrm{eq}(f - g, 0) = \ker(f - g)$ by tracking through what we need for our diagrams to commute.

Here is a nice result on equalizers.

Proposition 4.30. Given two morphisms $f, g : A \to B$ and an equalizer $e : E \to A$, the map e is always monic

Proof. Fix two maps $h, k : X \to E$ such that $eh = ek$. This has the following diagram.

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Then we see that eh and ek both have $f(eh) = f(ek) = g(eh) = g(ek)$, so there is a unique map $x : X \to E$ such that $ex = eh = ek$. But then we see that h and k both work, so $h = k$ is forced. ■

Remark 4.31. This is notably different from projections failing to be epic because we are really only told that $p_A f = p_B f$ or $p_B f = p_B g$ when looking at just one projection. However, we need both of these for $f = g$.

4.3.3 Coequalizers

Of course, there is also a dual notion of an equalizer.

Definition 4.32 (Coequalizer). A *coequalizer* is a colimit of the following diagram.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

We denote this by $\mathrm{coeq}(f, g)$.

From essentially the same discussion as before, the only data we need for a cocone of the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is an object Q with a morphism $q : B \to Q$. The universal property is saying that any object X with a morphism $\varphi : B \to X$ has a unique induced morphism as follows.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "\varphi"'] & Q \arrow[d, dashed] \\ & & X \end{tikzcd}$$

And now for examples.

Exercise 4.33. We compute coequalizers in Set.

Proof. The "dual" to a subset is a quotient, so we have reason to believe that the coequalizer should be a quotient. Thus, we define the equivalence relation $\sim$ in B generated by $f(a) \sim g(a)$.[1] It will happen that the canonical projection map $B \twoheadrightarrow B/\sim$ is our coequalizer. ■

4.4 March 16

These notes were transcribed from Rhea's notes. Thank you, Rhea!

4.4.1 Limit Review

Let's review the kinds of limits we can do.

- The limit of an empty set is the final object.
- The limit of a discrete category is a product.
- The limit of the arrow
$$A \longrightarrow B$$
is A.
- The limit of the diagram
$$\bullet \rightrightarrows \bullet$$
is the equalizer.

We continue our discussion with diagrams of three points.

Exercise 4.34. We show that limit of the triangle
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$
is A.

Proof. Any apex L for the diagram will consist of maps $\iota_A : L \to A$ and $\iota_B : L \to B$ and $\iota : L \to C$ so that the following diagram commutes.
$$\begin{tikzcd} & L \arrow[dl, "\iota_A"'] \arrow[d, dashed, "\iota_B"] \arrow[dd, bend left, "\iota_C"] \\ A \arrow[r, dashed, "f"] \arrow[dr, "h"'] & B \arrow[d, dashed, "g"] \\ & C \end{tikzcd} \tag{$*$}$$
However, we note that $\iota_B = f\iota_A$ and $\iota_C = h\iota_A$ by the commutativity of the diagram, so in fact, we can make the cone by only specifying ι_A.

And in fact, for any choice $\iota_A : L \to A$, we can induce the above diagram to commute by forcing $\iota_B := f\iota_A$ and $\iota_C : h\iota_A$, which will cause $(*)$ to commute because all the internal triangles commute.

[1] For example, we can take the intersection of all equivalence relations $B \times B$ which contain the requirements $f(a) \sim g(a)$ for each $a \in A$.

We thus claim that A equipped with $\mathrm{id}_A : A \to A$ is our limit. This means that we want a unique induced arrow $\varphi : L \to A$ making the following diagram commute.

$$\begin{tikzcd} L \arrow[r, dashed, "\varphi"] \arrow[d, "\iota_A"'] & A \arrow[dl, "\mathrm{id}_A"] \\ A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Well, any such arrow $\varphi : L \to A$ must satisfy $\varphi = \mathrm{id}_A\,\varphi = \iota_A$, so φ is forced. And indeed, $\varphi = \iota_A$ causes the necessary triangle to commute, we are done. ■

> **Remark 4.35.** At a high level, what is causing this diagram to commute is that we are reducing this limit to a limit on a one-object category, which we know how to do.

4.4.2 Pullbacks

For our next limit, we have the following definition.

> **Definition 4.36** (Cospan)**.** A *cospan* is a diagram of the following form.
>
> $$A \longrightarrow B \longleftarrow C$$
>
> Equivalently, a cospan is a diagram indexed by the following category.
>
> $$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

As with equalizers, we can decrease the number of arrows we have to keep track of in a cone over a cospan. Indeed, an apex L over a cospan is equipped with maps $\varphi_A : L \to A$ and $\varphi_B : L \to B$ and $\varphi_C : L \to C$ such that the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "\varphi_C"] \arrow[d, "\varphi_A"'] \arrow[dr, "\varphi_B"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Now, the commutativity diagram now forces $\varphi_B = f\varphi_A = g\varphi_C$, so we can simply induce φ_B from the rest of the diagram. As such, we decrease the data of a cone over a cospan as merely consisting of the maps $\varphi_A : L \to A$ and $\varphi_C : L \to C$ forcing $f\varphi_A = g\varphi_C$; i.e., we require the following diagram to commute.

$$\begin{tikzcd} L \arrow[r, "\varphi_C"] \arrow[d, "\varphi_A"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Of course, a cone will induce a diagram of the above form by forgetting the morphism φ_B. Conversely, a diagram of the above form makes a cone by setting $\varphi_B := f\varphi_A = g\varphi_C$, which will satisfy the needed commutativity to be a cone by construction.

Anyways, here is our limit.

Definition 4.37 (Pullback)**.** A *pullback* $A \times_B C$ is the limit of a cospan, labeled as follows.

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_C"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The right angle next to $A \times_B C$ is how we diagrammatically notate pullbacks.

Warning 4.38. The pullback $A \times_B C$ also depends on the chosen maps $f : A \to B$ and $g : C \to B$, even though these maps are not included in the notation.

It turns out that pullbacks are actually nontrivial limits, so we will need to fix our category to compute them. Here's an example.

Exercise 4.39. We compute pullbacks in Set.

Proof. Fix our diagram as follows.

$$\begin{tikzcd} L \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

As a first attempt, we might try $L = X \times Y$ with π_X and π_Y being the usual projection. But this does not work because the diagram might not commute: there is no reason to have

$$f(x) = f(\pi_X(x,y)) = (f\pi_X)(x,y) = (g\pi_Y)(x,y) = g(\pi_Y(x,y)) = g(y)$$

for each $x \in X$ and $y \in Y$. However, without much better to do, we force this condition in the rudest way possible: we simply restrict our product to be

$$X \times_Z Y := \{(x,y) \in X \times Y : f(x) = g(y)\},$$

where $\pi_X : (x,y) \mapsto x$ and $\pi_Y : (x,y) \mapsto y$ are the usual projections. This does indeed make a valid cone because any $(x,y) \in X \times Y$ will have

$$(f\pi_X)(x,y) = f(\pi_X(x,y)) = f(x) = g(y) = g(\pi_Y(x,y)) = (g\pi_Y)(x,y),$$

so $f\pi_X = g\pi_Y$.

It remains to show that this $X \times_Z Y$ creates the universal cone. Well, fix a set W with morphisms $\varphi_X : W \to X$ and $\varphi_Y : W \to Y$ so that the following diagram commutes.

$$\begin{tikzcd} W \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\varphi_X"] \arrow[ddr, bend right, "\varphi_Y"'] & & \\ & X \times_Z Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

We need to show that there is a unique arrow φ. To show that it is unique, note that we need

$$\pi_X(\varphi(w)) = (\pi_X\varphi)(w) = \varphi_X(w) \qquad \text{and} \qquad \pi_Y(\varphi(w)) = (\pi_Y\varphi)(w) = \varphi_Y(w)$$

by the commutativity of the diagram. It follows that we are forced to have

$$\varphi(w) := (\varphi_X w, \varphi_Y w).$$

We now show that this works. Note that this φ is well-defined because each $w \in W$ has

$$f(\varphi_X w) = (f\varphi_X)(w) = (g\varphi_Y)(w) = g(\varphi_Y w),$$

so $(\varphi_X w, \varphi_Y w) \in X \times_Z Y$. Then we have $\pi_X\varphi = \varphi_X$ and $\pi_Y\varphi = \varphi_Y$ by construction, forcing the diagram to commute for free. ■

4.4.3 Pullbacks as Equalizers

It is perhaps not too surprising that we ended up with something that looks like a product, with some equalizing condition. In fact, we can realize pullbacks as an equalized product.

Proposition 4.40. Work in some category $\mathcal{C}$. Fix morphisms $f : X \to Z$ and $g : Y \to Z$ in some category. Further, assume that $X \times Y$ exists with the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. If $\operatorname{eq}(f\pi_X, g\pi_Y)$ exists, then it is equal to $X \times_Z Y$.

Proof. Set $E := \operatorname{eq}(f\pi_X, g\pi_Y)$ with equalizing map $e : E \to X \times Y$. Our required map $E \to X$ will be $\pi_X e$; similarly, the required map $E \to Y$ will be $\pi_Y e$. Now, we see that E with $\pi_X e : E \to X$ and $\pi_Y e : E \to Y$ makes the following diagram commute.

$$\begin{tikzcd}
E \arrow[r, "\pi_X e"] \arrow[d, "\pi_Y e"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Indeed, we have

$$f(\pi_X e) = (f\pi_X)e \stackrel{*}{=} (g\pi_Y)e = g(\pi_Y e),$$

where $\stackrel{*}{=}$ is by construction of the equalizer.

It remains to show that E is universal. Well, pick up some object W with maps $\varphi_X : W \to X$ and $\varphi_Y : W \to Y$ such that $f\varphi_X = g\varphi_Y$. We then claim that there is a unique morphism φ causing the following diagram to commute.

$$\begin{tikzcd}
W \arrow[drr, bend left, "\varphi_X"] \arrow[ddr, bend right, "\varphi_Y"'] \arrow[dr, dashed, "\varphi"] & & \\
& E \arrow[r, "\pi_X e"'] \arrow[d, "\pi_Y e"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

We start with the existence of the map φ. For this, we expand the diagram as follows.

$$\begin{tikzcd}
W \arrow[r, dashed, "\varphi"] \arrow[rrrr, bend left, "\varphi_X"] \arrow[drrr, "\varphi_Y"'] & E \arrow[r, "e"] & X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\
& & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Note that the square does not commute anymore. We have two steps.

- We use the universal property of $X \times Y$. The maps φ_X and φ_Y induce a unique map $\psi : W \to X \times Y$ such that $\pi_X\psi = \varphi_X$ and $\pi_Y\psi = \varphi_Y$.

- We use the universal property of E. By construction,

$$(f\pi_X)\psi = f(\pi_X\psi) = f\varphi_X = g\varphi_Y = g(\pi_Y\psi) = (g\pi_Y)\psi,$$

so ψ equalizes $f\pi_X$ and $g\pi_Y$. As such, there is a unique map $\varphi : W \to E$ such that $e\varphi = \psi$. In particular, we see that

$$(\pi_X e)\varphi = \pi_X(e\varphi) = \pi_X\psi = \varphi_X \qquad \text{and} \qquad (\pi_Y e)\varphi = \pi_Y(e\varphi) = \pi_Y\psi = \varphi_Y,$$

so the required diagram commutes.

It remains to show that the map $\varphi : E \to X$ is unique. Suppose that we have two such maps φ_1 and φ_2. We again proceed in two steps.

- We use the universal property of $X \times Y$. Note that

$$\varphi_X = (\pi_X e)\varphi_\bullet = \pi_X(e\varphi_\bullet) \qquad \text{and} \varphi_Y = (\pi_Y e)\varphi_\bullet = \pi_Y(e\varphi_\bullet),$$

so both morphisms $e\varphi_\bullet$ are the needed unique morphism $W \to X \times Y$. So we see $e\varphi_1 = e\varphi_2$.

- We use the universal property of E. Note that

$$(f\pi_X)(e\varphi_\bullet) = f(\pi_X e\varphi_\bullet) = f\varphi_X = g\varphi_Y = g(\pi_Y e\varphi_\bullet) = (g\pi_Y)(e\varphi_\bullet),$$

so the universal property of E forces there to be a unique map φ such $e\varphi = e\varphi_1 = e\varphi_2$. But of course, φ_1 and φ_2 are such maps φ, so $\varphi_1 = \varphi_2$ follows.

This finishes checking that E is universal. ■

Remark 4.41 (Bryce)**.** As Bryce would like to point out, the existence proof might look like it shows that φ is unique immediately—we did use two uniqueness results, after all—but some care is required. Namely, we only know that the morphism φ is the unique morphism commuting with ψ and then happens to make the diagram commute, so φ might not be unique making the diagram commute.

Remark 4.42 (Bryce)**.** It will turn out that all limits can be realized as equalizers of products.

4.4.4 Direct and Inverse Limits

We close lecture with two definitions.

Definition 4.43 (Direct limit)**.** A *direct limit* is a colimit of the poset category $\mathbb{N}$. In other words, a direct limit is a colimit of a diagram of the following form.

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

Intuitively, we can think of direct limits as ascending unions.

Definition 4.44 (Inverse limits)**.** An *inverse limit* is a limit of the poset category $\mathbb{N}^{\text{op}}$. In other words, an inverse limit is a limit of a diagram of the following form.

$$A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow \cdots$$

Dually, we can intuitively think of inverse limits as a descending intersection.

List of Definitions