18.708: Topics in Algebra

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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Theme 1
de Rham Cohomology in Mixed Characteristic

These talks were given by Alexander Petrov.

1.1 February 2

Here we go.

1.1.1 Algebraic de Rham Cohomology

Let's begin by describing what we mean by de Rham cohomology. We will consider a smooth variety X over an algebraically closed field F.

Definition 1.1 (smooth)**.** We say that a variety X over a field F is *smooth* if and only if $\Omega_{X/F}$ is a vector bundle of rank $\dim X$ on each connected component. Here, on an affine open subset $U \subseteq X$, recall that $\Omega_{X/F}(U)$ is spanned by symbols of the form $f\,dg$, where the symbol d is (as usual) F-linear and satisfies the Leibniz rule.

Definition 1.2 (algebraic de Rham cohomology)**.** Fix a smooth variety X over a field F. Then one can iterate the F-linear map $d\colon \mathcal{O}_X \to \Omega_{X/F}$ to a map $d\colon \Omega^i_{X/F} \to \Omega^{i+1}_{X/F}$ for each i, where $\Omega^i_{X/F} := \wedge^i \Omega_{X/F}$. We now define the *de Rham complex* to the complex

$$\Omega^\bullet_{X/F}\colon 0 \to \mathcal{O}_X \stackrel{d}{\to} \Omega^1_{X/F} \stackrel{d}{\to} \cdots,$$

and we define the *de Rham cohomology* $\mathrm{H}^n_{\mathrm{dR}}(X/F)$ to be the nth hypercohomology of $\Omega^\bullet_{X/F}$. Here, hypercohomology means the total cohomology of some produced acyclic double complex which resolves the complex (e.g., a Čech resolution). Note that this hypercohomology is merely a vector space over F.

Example 1.3. The map $d\colon \Omega^1_{X/F} \to \Omega^2_{X/F}$ is given by $d(f\,dg) = df \wedge dg$.

Example 1.4. Suppose that X is affine. Then vector bundles are already acyclic, so the hypercohomology does nothing. Thus,

$$\mathrm{H}^n_{\mathrm{dR}}(X/F) = \mathrm{H}^n\left(X; 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/F} \xrightarrow{d} \cdots\right).$$

As usual, this is $\ker\left(d|_{\Omega^n}\right)/\operatorname{im}\left(d|_{\Omega^{n-1}}\right)$.

Remark 1.5. If X is affine and $i > \dim X$, then $\Omega^i_{X/F}$ vanishes, so the algebraic de Rham cohomology also vanishes.

Remark 1.6. A different definition is required for non-smooth X. Roughly speaking, one should embed into a smooth variety and take cohomology there.

Here is one way to convince ourselves that this is a reasonable cohomology theory.

Theorem 1.7 (Grothendieck)**.** Suppose that X is a smooth variety over $\mathbb{C}$. Then there is a canonical isomorphism

$$\mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C}); \mathbb{C}) \to \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}).$$

Here, the left-hand side is Betti cohomology (also called singular cohomology).

Sketch. We argue in the case that X is affine. Then $X(\mathbb{C})$ already has a notion of $\Omega^{i,\mathrm{an}}_{X/\mathbb{C}}$ given by the holomorphic forms. Algebraic forms embed into holomorphic ones, which produces a morphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(X) \arrow[r] \arrow[d, hook] & \Omega^1_{X/\mathbb{C}}(X) \arrow[r] \arrow[d, hook] & \Omega^2_{X/\mathbb{C}}(X) \arrow[r] \arrow[d, hook] & \cdots \\
0 \arrow[r] & \mathcal{O}^{\mathrm{an}}_X(X) \arrow[r] & \Omega^{1,\mathrm{an}}_{X/\mathbb{C}}(X) \arrow[r] & \Omega^{2,\mathrm{an}}_{X/\mathbb{C}}(X) \arrow[r] & \cdots
\end{tikzcd}$$

of complexes. It then turns out that this is an isomorphism on cohomology, so we reduce to comparing analytic de Rham cohomology with singular cohomology.

This is now a problem of analysis. One can pass from holomorphic differentials to smooth differentials via a similar process, which produces another morphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}^{\mathrm{an}}_X(X) \arrow[r] \arrow[d, hook] & \Omega^{1,\mathrm{an}}_{X/\mathbb{C}}(X) \arrow[r] \arrow[d, hook] & \Omega^{2,\mathrm{an}}_{X/\mathbb{C}}(X) \arrow[r] \arrow[d, hook] & \cdots \\
0 \arrow[r] & C^\infty(X(\mathbb{C}), \mathbb{C}) \arrow[r] & \Omega^1_{C^\infty}(X(\mathbb{C})) \arrow[r] & \Omega^2_{C^\infty}(X(\mathbb{C})) \arrow[r] & \cdots
\end{tikzcd}$$

of complexes, which is also an isomorphism on complexes. We are now reduced to the setting of de Rham's theorem for real manifolds. ■

Example 1.8. Consider $X := \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = \operatorname{Spec} k[t, 1/t]$.

- Our differential map $d\colon \mathbb{C}[t, 1/t] \to \mathbb{C}[t, 1/t]\, dt$ sends t^n to $nt^{n-1}\, dt$. Thus, $\mathrm{H}^0_{\mathrm{dR}}(X)$ is one-dimensional given by the constants, and $\mathrm{H}^1_{\mathrm{dR}}(X)$ is one-dimensional spanned by dt/t.
- The point above works also for holomorphic differentials. The interesting bit is in degree 1, where the point is that there is no global antiderivative for dx/x.
- On the other hand, $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle, so we expect its singular cohomology to be supported in degrees 0 and 1, where it should be one-dimensional.

Corollary 1.9 (Artin vanishing)**.** If X is an affine algebraic complex smooth variety, then $\mathrm{H}^n(X(\mathbb{C});\mathbb{C}) = 0$ for $n > \dim X$.

Proof. The algebraic de Rham cohomology complex vanishes above $\dim X$. ∎

Corollary 1.10. Fix a smooth variety X over $\mathbb{C}$. Then $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C})$ is finite-dimensional.

Proof. Pass to singular cohomology. ∎

Remark 1.11. This corollary still admits algebraic proofs in characteristic zero by working with holonomic $\mathcal{D}$-modules. Pavel Etingof claims that there is an algebraic proof using the fact that the direct image of a holonomic $\mathcal{D}$-module is a holonomic $\mathcal{D}$-module.

We would like to point out that our de Rham cohomology is algebraic but still interesting.

Remark 1.12. Suppose that X is smooth over $\mathbb{Q}$. Base-changing by a field is exact, so

$$\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{Q})_\mathbb{C} \cong \mathrm{H}^n_{\mathrm{dR}}(X_\mathbb{C}/\mathbb{C}).$$

However, Theorem 1.7 grants an isomorphism to $\mathrm{H}^n(X(\mathbb{C});\mathbb{C}) \cong \mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C});\mathbb{Z})_\mathbb{C}$. Notably, we then find a lattice and a rational structure over in some complex vector space, but the comparison between the two is quite interesting mathematically (and amounts to the study of periods).

Example 1.13. In the case that $X = \mathbb{A}^1_\mathbb{Q} \setminus \{0\}$, the comparison between $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})_\mathbb{C}$ and $\mathrm{H}^1_{\mathrm{B}}(X(\mathbb{C});\mathbb{Z})$ is mediated by a constant $2\pi i$. Indeed, once unwinds the de Rham theorem, this amounts to the statement that a contour integral of dx/x going once around the origin is $2\pi i$.

1.1.2 Frobenius Structure

We now pass to positive characteristic. Let k be a perfect field of positive characteristic p, and we may still consider a smooth variety X.

Remark 1.14. If k is perfect, then $\Omega^1_{X/k} = \Omega^1_{X/\mathbb{F}_p}$ by doing some thinking about inseparable extensions. The moral is that

$$y^{1/p}\,dy = d\left((y^{1/p})^p\right),$$

so the coefficients can be brought down when everything is a pth power.

This cohomology is rather strangely behaved.

Example 1.15. Take $X := \mathbb{A}^1_k$. The de Rham cohomology still lives in degrees zero and one, so we would like to study the kernel and cokernel of the k-linear map $d\colon k[t] \to k[t]\,dt$ given by $t^n \mapsto nt^{n-1}$.

- We see that $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \ker d$ is spanned by t^{pi} for each i.
- We see that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \operatorname{im} d$ is infinite-dimensional because the differentials $t^{mp-1}\,dt$ fail to be in the image. In fact, these classes form a basis.

Let's try to view these infinite-dimensional groups as a feature instead of a bug. Indeed, it turns out that the de Rham complex has some extra structure. The de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots$$

is merely made of sheaves of k-vector spaces over X. In characteristic zero, this is all the structure present, but in characteristic p, we have more structure.

Notation 1.16. Fix a variety X over a field k of characteristic p. For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we define

$$\mathcal{F}^p \coloneqq \{f^p : f \in \mathcal{O}_X\}$$

to locally be given by the pth powers.

The moral is that $d(f^p) = 0$ always, so the de Rham complex is in fact $\mathcal{O}_X^p$-linear! Let's attempt to codify this.

Definition 1.17 (relative Frobenius)**.** Fix a scheme X over a field k of characteristic p. Then there is an *absolute Frobenius* $F_{\mathrm{abs}} \colon X \to X$ which is the identity on topological spaces and the pth power on sheaves. This is a morphism of schemes but not of k-schemes (in general). The *relative Frobenius* $F \colon X \to X^{(1)}$ is the morphism fitting into the following diagram.

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "F_{\mathrm{abs}}"] \arrow[ddr, bend right] & & \\
& X^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& k \arrow[r, "F_{\mathrm{abs}}"] & k
\end{tikzcd}$$

Remark 1.18. Note that $X^{(p)}$ is isomorphic to X as a scheme but not as a k-scheme! However, we now benefit because the relative Frobenius F is morphism of k-schemes.

Remark 1.19. The relative Frobenius $F \colon X \to X^{(1)}$ is finite flat of degree $p^{\dim X}$

Example 1.20. If $X = \operatorname{Spec} k[t_1, \ldots, t_n]$, then $X^{(1)} = \operatorname{Spec} k\left[t_1^p, \ldots, t_n^p\right]$. Thus, we see that the embedding

$$k\left[t_1^p, \ldots, t_n^p\right] \subseteq k[t_1, \ldots, t_n]$$

is indeed finite flat of degree p^n.

We now see that

$$0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/F} \to \cdots$$

is a complex of quasicoherent sheaves on $X^{(1)}$. In fact, because F is finite flat, these are all vector bundles: $F_*\mathcal{O}_X$ has rank $p^{\dim X}$ and $F_*\Omega^i_{X/k}$ has rank $p^{\dim X}\binom{\dim X}{i}$. Because $\mathcal{O}_{X^{(1)}} = (F_*\mathcal{O}_X)^p$, we see that this complex is in fact $\mathcal{O}_{X^{(1)}}$-linear.

Example 1.21. Take $X = \operatorname{Spec} k[t]$. Then $X^{(1)} \coloneqq \operatorname{Spec} k\left[t^p\right]$, and $d \colon k[t] \to k[t]\,dt$ is $k\left[t^p\right]$-linear! Thus, $\mathrm{H}^i_{\mathrm{dR}}(X/k)$ was required to be given by $k\left[t^p\right]$-modules, which explains why we received vector spaces of infinite dimension.

Note that passing through F_* is not going to adjust the underlying k-vector spaces, so

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \mathbb{H}^n_{\mathrm{Zar}}\left(X^{(1)}; 0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/k} \xrightarrow{d} \cdots\right).$$

To see why this has globalized the $\mathcal{O}_X^p$-linearity, we need the Cartier isomorphism.

Theorem 1.22 (Cartier isomorphism)**.** Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism

$$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$

Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Remark 1.23. This is a reason why characteristic p may be more convenient than characteristic 0: one could still try to understand $\mathcal{H}^i(\Omega^\bullet_{X/k})$ when $\operatorname{char} k = 0$, but this has no easy answer.

Example 1.24. Consider $X = \mathbb{A}^1_k$. Then $\mathcal{H}^1$ is given by the module

$$\frac{k[t]\,dt}{d(k[t])},$$

which our formalism now remembers is a $k\left[t^p\right]$-module. And indeed, we can show that this is isomorphic to $k\left[t^p\right] \cdot t^{p-1}\,dt$. Setting $s := t^p$, we know that $\Omega^1_{X^{(1)}/k}$ is given by the module $k[s]\,ds$, so our isomorphism of modules is given by sending ds to $t^{p-1}\,dt$. One can even check that this isomorphism is canonical in the sense that it will not change under automorphisms of $\mathbb{A}^1$.

We will prove Theorem 1.22 later after a detour.

1.1.3 Crystalline Cohomology

We continue with our perfect field k of positive characteristic p. Our story so far has taken a variety X over a field k, and then we have produced some (total) complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$ in the derived category $D(\mathrm{Vec}_k)$. Crystalline cohomology will allow us to produce an answer in characteristic 0 instead of characteristic p. The idea is to "choose" a lift to characteristic p and then check that the answer is independent of the lift.

The correct formalism for this lifting is that of a "formal scheme."

Definition 1.25 (Witt ring)**.** Fix a perfect field k of characteristic p. Then there is a ring $W(k)$ satisfying that

- $W(k)$ is p-torsion-free,
- $W(k)/p \cong k$, and
- $W(k)$ is the limit of the $W(k)/p^n$ as $n \to \infty$.

This ring $W(k)$ turns out to be unique up to unique isomorphism. We may write $W_n(k) := W(k)/p^n$.

Example 1.26. One can see that $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W(\overline{\mathbb{F}}_p)$ is its unramified closure.

Remark 1.27. There is a completely explicit construction of $W(k)$, but it is rather involved: given a p-torsion-free ring R, we identify $W(R) \coloneqq R^{\mathbb{N}}$ but with ring structure chosen so that

$$(a_0, a_1, a_2, \ldots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$$

is a ring homomorphism $W(R) \to R$. It turns out that this ring structure is given by some polynomials (called "ghost coordinates"), so we are allowed to define $W(k)$. From a higher level, it turns out that $W(k)$ is the unique deformation of k, which exists because $\Omega^1_{k/\mathbb{F}_p} = 0$.

Definition 1.28 (formal scheme)**.** Fix a perfect field k of characteristic p. A *p-adic formal scheme* X is a collection of schemes X_n over $W_n(k)$ equipped with isomorphisms

$$X_{n+1} \times_{W_{n+1}(k)} W_n(k) \to X_n.$$

The structure sheaf $\widehat{\mathcal{O}}_X$ is the inverse limit of the $\mathcal{O}_{X_n}$s.

Example 1.29. Given a scheme Y over $W(k)$, we can produce a formal scheme $\widehat{Y}$ with $\widehat{Y}_n \coloneqq Y \times_{W(k)} W_n(k)$ and the induced internal isomorphisms.

Remark 1.30. If X_1 is affine, then all the nilpotent thickenings are affine, so we may say that the full formal scheme is affine.

Remark 1.31. We can even define $\widehat{\Omega} X^1_{\widetilde{X}}$.

We can now describe crystalline cohomology.

Theorem 1.32. Fix a perfect field k of positive characteristic p. Then there is a functor sending smooth k-varieties X to a complex $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ in the derived category $D(\mathrm{Mod}_{W(k)})$ satisfying the following.

(a) There is a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes^{\mathbb{L}}_{W(k)} k \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$.

(b) If $\widetilde{X}$ is a smooth formal scheme over $W(k)$ (meaning that $\widetilde{X}_n$ is smooth over $W_n(k)$ for all n), then the scheme $X \coloneqq \widetilde{X} \times_{W(k)} k$ has

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \cong \mathrm{R}\Gamma_{\mathrm{dR}}\left(X; \widehat{\mathcal{O}}_{\widetilde{X}} \xrightarrow{d} \widehat{\Omega}^1_{\widetilde{X}} \xrightarrow{d} \cdots\right).$$

This right-hand side can be thought of as $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/W(k))$.

Remark 1.33. Here, (a) immediately tells us that the cohomology of $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ is not expected to be finitely generated.

Remark 1.34. There is something remarkable here, which is that choosing two different lifts of X to a smooth formal scheme produces the same cohomology!

Remark 1.35. It turns out that flatness is equivalent to smoothness in this context.

1.2 February 4

Today we continue our discussion of crystalline cohomology.

1.2.1 Example of Crystalline Cohomology

We continue working over a perfect field k of characteristic p. Let's run an example.

Remark 1.36. If our formal scheme $\widetilde{X}$ is smooth and affine, then $\mathrm{R}\Gamma_{\mathrm{dR}}(X/W(k))$ can be computed directly as the complex

$$0 \to \mathcal{O}(\widetilde{X}) \xrightarrow{d} \Omega^1(\widetilde{X}) \xrightarrow{d} \Omega^2(\widetilde{X}) \xrightarrow{d} \cdots .$$

Example 1.37. Take $X = \mathbb{A}^1_k$, which we can deform to a formal scheme $\widetilde{X} = \widehat{\mathbb{A}}^1_k$. Then

$$\mathcal{O}(\widetilde{X}) = \lim W_n[t],$$

which we may refer to as $W\langle t\rangle$. Here, $W\langle t\rangle$ contains those power series whose terms are more and more divisible by p. Similarly, we can compute that $\Omega^1(\widetilde{X}) = W\langle t\rangle\, dt$. For example, we now see that W has characteristic zero, so the kernel of the differential $d\colon W\langle t\rangle \to W\langle t\rangle\, dt$ is exactly the constants. On the other hand, Theorem 1.32 tells us that $\mathrm{H}^1_{\mathrm{dR}}(\widetilde{X}/W(k))$ is required to be quite interesting because the de Rham complex of $\mathbb{A}^1_k$ is interesting.

Remark 1.38. One can extend Theorem 1.32 slightly: even if some k-scheme X can only be deformed to a smooth scheme X_n over $W_n(k)$, then we still have

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes^{\mathbb{L}}_{W(k)} W_n(k) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X_n/W_n(k)).$$

The above example falls under the paradigm where the formal scheme comes from a genuine scheme. Here are some motivational remarks about this case.

Example 1.39. Let's consider the case where the formal scheme comes from a genuine scheme. Given a smooth scheme Y over $W(k)$ with formal completion $\widehat{Y}$, then $\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k))$ is the derived p-adic completion of $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$. Explicitly, one takes a complex representing $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ and takes a p-completion of each module. (This turns out to be a well-defined operation on the derived category.)

Example 1.40. We continue the previous example. Suppose further that Y is smooth and proper. Then it turns out that $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ can be represented by finitely generated projective $W(k)$-modules, so the p-completion does nothing, so

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k)) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k)).$$

1.2.2 The Mysterious Functor

Crystalline cohomology produces the following strange functoriality.

Corollary 1.41. Fix a smooth formal scheme $\widetilde{X}$ over $W(k)$. Given an endomorphism f of $\widetilde{X}$ such that f is the identity $\pmod p$, then f^* is the identity on the cohomology groups $\mathrm{H}^n_{\mathrm{dR}}(\widetilde{X}/W(k))$.

Proof using Theorem 1.32. By Theorem 1.32, one may pass to crystalline cohomology of the reduction $X :=$ $X \times_k W(k)$, where the result has no content. ∎

Theorem 1.32 admits the following version in characteristic zero, where we heuristically replace the deformation $W(k) \to k$ with $\mathbb{C}[[t]] \to \mathbb{C}$.

Proposition 1.42. Given a smooth formal scheme $\widetilde{X}$ over $\mathbb{C}[[t]]$ with reduction X over $\mathbb{C}$. Then

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/\mathbb{C}[[t]]) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[t]].$$

Sketch. The idea of the proof is to pass through the larger de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/\mathbb{C})$; by definition, this is the inverse limit of the de Rham cohomology of the reductions $\widetilde{X}_n$. By the Poincaré lemma, it turns out that $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/\mathbb{C})$ is quasi-isomorphic to $\mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{C})$. (Approximately speaking, this is saying that a tubular neighborhood of X is homotopic to X.) The result now follows by extending scalars. ∎

Remark 1.43. This proof does not work in our mixed characteristic situation $W(k) \to k$ because there is no direct way to geometrically link $\widetilde{X}$ and X via a Poincaré lemma.

To prove Theorem 1.32, it will turn out that the key ideas can be used to instead prove Corollary 1.41.

Proof of Corollary 1.41 when $p > 2$. We attempt to give a version of Poincaré's lemma. The idea there is to use Cartan's formula, which arises by viewing differentials as living in flows of vector fields.

1. Our analog of this will be
$$\log f := \sum_{n \geq 1} (-1)^{n+1} \frac{(f - \mathrm{id})^{\circ n}}{n},$$
viewed as a W-linear endomorphism of $\mathcal{O}_{\widetilde{X}}$. To see that this makes sense, note that $f - \mathrm{id} = pA$ for some endomorphism A, so it is at least true that the terms of the series are divisible by larger and larger powers of p. Thus, the series descends to a compatible sequence of endomorphisms for each $\mathcal{O}_{\widetilde{X}_n}$ and hence an endomorphism of $\mathcal{O}_{\widetilde{X}}$.
2. It further turns out that $\log f$ is a derivation of $\mathcal{O}_{\widetilde{X}}$, meaning that
$$(\log f)(ab) = (\log f)(a)b + a(\log f)(b).$$
This is a formal consequence of the fact that f is a ring map, and $\log f$ takes products to sums. (Indeed, one can already find this in characteristic zero.)
3. To continue, we take $p > 2$ for simplicity. Indeed, then one can see $f = \exp(\log f)$, meaning that
$$f \stackrel{?}{=} \sum_{n \geq 0} \frac{(\log f)^{\circ n}}{n!}.$$
The right-hand sum makes sense because $\log f$ is divisible by p, which then implies that the sum converges. Thus, we may write $\log f = pD$ for some derivation D. The equality now follows from some formal calculation of the series.
4. We now pass to the de Rham complex. We would like to check that the endomorphism $f^* \colon \Omega^\bullet_{\widetilde{X}} \to \Omega^\bullet_{\widetilde{X}}$ is a quasi-isomorphism. Well, there is a "Lie derivative" L_D for which f^* acts on $\Omega^i_{\widetilde{X}}$ by
$$\sum_{n \geq 0} \frac{p^n}{n!} L_D^{\circ n}.$$

Let's explain what this L_D is, which we may define affine-locally. For any $W(k)$-algebra A, a derivation $W: A \to A$ induces an endomorphism L_D on differentials as follows. Indeed, $\mathrm{id} + \varepsilon D$ is an automorphism of the thickening $A[\varepsilon]/\left(\varepsilon^2\right)$, so $\mathrm{id} + \varepsilon D$ acts on $\Omega^i_{A[\varepsilon]/(\varepsilon^2)}$. Certainly it must reduce to identity modulo ε, so we conclude that $(\mathrm{id} + \varepsilon D)^* = \mathrm{id} + \varepsilon L_D$ for some endomorphism L_D on differentials!

5. The previous step now tells us that $f^* = \mathrm{id} + L_D \circ G$ for some G. We will be done as soon as we can check that the endomorphism L_D of the de Rham complex is homotopic to zero. In other words, we are on the hunt for maps $\iota_D: \Omega^i_{\widetilde{X}} \to \Omega^{i-1}_{\widetilde{X}}$ fitting into the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\widetilde{X}} \arrow[r] \arrow[d, "L_D"] & \Omega^1_{\widetilde{X}} \arrow[r] \arrow[d, "L_D"] \arrow[dl, "\iota_D"'] & \Omega^2_{\widetilde{X}} \arrow[r] \arrow[d, "L_D"] \arrow[dl, "\iota_D"'] & \cdots \\ 0 \arrow[r] & \mathcal{O}_{\widetilde{X}} \arrow[r] & \Omega^1_{\widetilde{X}} \arrow[r] & \Omega^2_{\widetilde{X}} \arrow[r] & \cdots \end{tikzcd}$$

so that $L_D = d\iota_D + \iota_D D$. This is precisely Cartan's formula, which admits an algebraic proof. (Namely, one takes ι_D to be a contraction whose vector field is D.) To show this, one can show directly that $d\iota_D + \iota_D D$ is a derivation, and then one can check that there is an equality on functions and 1-forms. ■

Remark 1.44. The construction of L_D may appear ad-hoc. The intuition here is that our objects are not only functorial under endomorphisms but also under derivations.

It turns out that all the interesting algebra to prove Theorem 1.32 is already contained in the proof of Corollary 1.41, though this should not be obvious yet.

1.2.3 The Cartier Isomorphism

Recall that we wanted to prove Theorem 1.22.

Theorem 1.22 (Cartier isomorphism)**.** Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism

$$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$

Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Proof. Let's start with $i = 0$. Note that there is a map

$$\mathcal{O}_{X^{(1)}} \to \ker\left(d: F_*\mathcal{O}_X \to F_*\Omega^1_X\right)$$

given by sending a function f to f^p. We would like to check that this is an isomorphism. Well, it is enough to check this on an open cover, but because X is smooth, we may pass étale-locally along some $f: U \to \mathbb{A}^n_k$ (where $U \subseteq X$ is some open subset) because the de Rham complex is immune to such deformations: indeed,

$$\left(F_*\Omega^0_X\right)|_{U^{(1)}} \cong f^{(1)*}\left(F_*\Omega^0_{\mathbb{A}^n_k}\right).$$

Thus, we may pass to $X = \mathbb{A}^n$.

We now work in general. By working locally, we may assume that X is affine and equal to some $\operatorname{Spec} A$. Then we would like an isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(X/k) \overset{?}{\cong} \Omega^i_{A^{(1)}/k}$$

of A-modules. Well, consider $\mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k)) := \mathrm{R}\Gamma_{\mathrm{cris}}(A/W(k)) \otimes_{W(k)} W_2(k)$, which we know to be isomorphic to $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/W_2(k))$ for any deformation $\widetilde{X}$ of X to $W_2(k)$. Accordingly, we receive a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{dR}}(A/k) \to \mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k)) \to \mathrm{R}\Gamma_{\mathrm{dR}}(A/k)$$

induced by the lifting $k \to W_2(k) \to k$, where the map $k \to W_2(k)$ is multiplication by p. Thus, we receive a long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathrm{dR}}(A/k) \to \mathrm{H}^i_{\mathrm{dR}}(A/k) \xrightarrow{\beta_i} \mathrm{H}^{i+1}_{\mathrm{dR}}(A/k) \to \cdots .$$

It turns out that the graded map $\bigoplus_i \beta_i$ on the de Rham cohomology ring $\bigoplus_{i\geq 0} \mathrm{H}^i_{\mathrm{dR}}(X/k)$ is a derivation; this is just some exercise in the homological algebra. Explicitly, one has to check that

$$\beta_{i+j}(x_i \wedge x_j) = \beta_i(x_i) \wedge x_j + (-1)^i x_i \wedge \beta_j(x_j)$$

for any x_i and x_j of degrees i and j, respectively. It further turns out that $\beta_{i+1} \circ \beta_i = 0$.

Now that we have found a derivation, we can make differentials appear. The universal property of differentials produce morphisms

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{A^{(1)}} \arrow[r, "d"] \arrow[d, "c_0"] & \Omega^1_{A^{(1)}} \arrow[r, "d"] \arrow[d, "c_1"] & \Omega^2_{A^{(1)}} \arrow[r, "d"] \arrow[d, "c_2"] & \cdots \\
0 \arrow[r] & \mathrm{H}^0_{\mathrm{dR}}(A/k) \arrow[r, "\beta"] & \mathrm{H}^1_{\mathrm{dR}}(A/k) \arrow[r, "\beta"] & \mathrm{H}^2_{\mathrm{dR}}(A/k) \arrow[r, "\beta"] & \cdots
\end{tikzcd}$$

where f_0 is induced by the degree-zero argument, and all the relevant morphisms intertwine β and d and produce maps of graded rings. For example, one finds that we need $c_1(df) = \beta_0(c_0(f))$ for functions f. One now checks that this in fact an isomorphism on the level of modules, which is checked étale-locally and then passed to affine spaces. ■

Remark 1.45. We can afford to be a little more explicit about our construction of the maps $c_\bullet$. Fix a lift $\widetilde{A}$ over $W_2(k)$ as well as a lift of the Frobenius $\widetilde{F}$. Given $\omega \in \Omega^i_{A^{(1)}/k}$, we pull it back along $\widetilde{F}$ to get a differential divisible by p in $\mathrm{H}^i_{\mathrm{dR}}(\widetilde{A}/W_2(k))$. Then one can check that the class $\widetilde{F}^*(\omega)/p$ is well-defined in $\mathrm{H}^i_{\mathrm{dR}}(A/k)$.

Remark 1.46. Crystalline cohomology is not technically necessary because we could choose a lift $\widetilde{X}$ by hand using Remark 1.45. However, one needs to check that the constructed map is independent of the lift. This is not impossible (such lifts are well-understood in some cohomology group by deformation theory), but it is a little difficult.

1.3 February 9

Today we say more about the construction of crystalline cohomology.

1.3.1 $\mathcal{D}$-modules

We are going to use $\mathcal{D}$-modules to glue our de Rham complexes together.

Definition 1.47 ($\mathcal{D}$-module)**.** Fix a smooth scheme X over a ring R. Then we define the quasicoherent sheaf $\mathcal{D}_{X/R}$ of associative algebras on X defined explicitly as

$$\mathcal{O}_X\{\partial_v : v \in T_X\},$$

where the relations ∂_v satisfies the relations $\partial_{v_1+v_2} = \partial_{v_1} + \partial_{v_2}$, $\partial_{fv} = f\partial_v$, $[\partial_v, f] = L_v(f)$, and $[\partial_v, \partial_w] = \partial_{[v,w]}$. Here, T_X is the tangent bundle, which is the dual of $\Omega^1_{X/R}$ (equivalently, the sheaf of derivations $\mathcal{O}_X \to \mathcal{O}_X$). A $\mathcal{D}$*-module* is a quasicoherent sheaf for which the $\mathcal{O}_X$-action extends to the action of a module for $\mathcal{D}_{X/R}$.

Remark 1.48. Precisely, we have given a definition of an associative ring on affine opens, which then glue together on affines as a sheaf on a base.

Remark 1.49. There is a natural action of $\mathcal{D}_{X/R}$ on $\mathcal{O}_X$, where functions act by multiplication, and the derivations act by taking derivatives. If R contains $\mathbb{Q}$, then this natural map $\mathcal{D}_{X/R} \to \operatorname{End}_R(\mathcal{O}_X)$ is injective.

Example 1.50. In positive characteristic p, the action map $\mathcal{D}_{X/R} \to \operatorname{End}_R(\mathcal{O}_X)$ need not be injective. Indeed, with $X = \mathbb{A}^1_{\mathbb{F}_p}$ over $\mathbb{F}_p$, we see that the iterated derivation ∂_t^p kills all polynomials (because $p! = 0$).

Remark 1.51. The various relations make it so that a $\mathcal{D}$-module is exactly a vector bundle with a flat connection $\nabla\colon M \to M \otimes \Omega^1_{X/R}$.

Remark 1.52. The algebra $\mathcal{D}_{X/R}$ admits a filtration by the subgroups $\mathcal{D}^{\le n}_{X/R}$ of differentials at most n. It turns out that $\operatorname{gr}\mathcal{D}_{X/R} = \operatorname{Sym}^\bullet T_X$, which is probably a variant of the PBW theorem.

Here is why $\mathcal{D}$-modules are relevant to us.

Lemma 1.53. Fix a smooth scheme X over a ring R. Then the de Rham complex $\Omega^\bullet_{X/R}$ is quasi-isomorphic to $\operatorname{RHom}_{\mathcal{D}_{X/R}}(\mathcal{O}_X, \mathcal{O}_X)$ in $D(\operatorname{Sh}(X))$.

Proof. We are going to find a locally projective resolution for $\mathcal{O}_X$ as a $\mathcal{D}_{X/R}$-module. To start, note that there is a projection

$$\mathcal{D}_{X/R} \to \mathcal{O}_X$$

given by sending $\delta \mapsto \delta(1)$. This is surjective because functions act by left multiplication. By definition, all the derivations ∂_v are in the kernel, and these are exactly what goes to zero, so we continue our sequence by writing

$$\mathcal{D}_{X/R} \otimes_{\mathcal{O}_X} T_{X/R} \to \mathcal{D}_{X/R} \to \mathcal{O}_X \to 0,$$

where the left map sends $\delta \otimes v \mapsto \delta\partial_v$. Then $\partial_v \otimes w - \partial_w \otimes v$ is in the kernel, and we can find that we should extend our complex by

$$\mathcal{D}_{X/R} \otimes \mathcal{O}_X \wedge^2 T_{X/R} \to \mathcal{D}_{X/R} \otimes_{\mathcal{O}_X} T_{X/R} \to \mathcal{D}_{X/R} \to \mathcal{O}_X \to 0,$$

where the left map sends $\delta \otimes (v_1 \wedge v_2)$ to $\delta\partial_{v_1} \otimes v_2 - \delta\partial_{v_2} \otimes \partial_{v_1} - \delta \otimes [v_1, v_2]$. This suggests that we are looking at a Koszul complex

$$\cdots \to \mathcal{D}_{X/R} \otimes \mathcal{O}_X \wedge^2 T_{X/R} \to \mathcal{D}_{X/R} \otimes_{\mathcal{O}_X} T_{X/R} \to \mathcal{D}_{X/R} \to \mathcal{O}_X \to 0,$$

and one can check that it is acyclic by passing to the associated graded everywhere.

We now compute RHom by hitting our complex with $\operatorname{Hom}_{\mathcal{D}_X}(-, \mathcal{O}_X)$. Then the point is that there is an isomorphism $\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes E, \mathcal{O}_X) = E^\vee$ in the obvious way, so taking $\operatorname{Hom}_{\mathcal{D}_X}(-, \mathcal{O}_X)$ produces the complex

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/R} \xrightarrow{d} \Omega^2_{X/R} \xrightarrow{d} \cdots,$$

which is exactly the de Rham complex. ■

Remark 1.54. One can replace the target $\mathcal{O}_X$ with any quasicoherent sheaf to obtain a similar result, which allows us to compute de Rham cohomology of an arbitrary quasicoherent sheaf.

1.3.2 The Category of Crystals

We are now ready to construct crystalline cohomology.

Theorem 1.55. Suppose $p > 2$. Fix any smooth variety X over a perfect field k of characteristic p. For each $n \geq 1$, then there is an abelian tensor category $\operatorname{Cris}(X/W_n(k))$ of crystals with unit object $\mathcal{O}_X^{\mathrm{cris}}$, with the following property: for any lift X_n of X to $W_n(k)$, there is an isomorphism

$$\operatorname{Cris}(X/W_n(k)) \cong \operatorname{Mod}(\mathcal{D}_{X_n/W_n})$$

sending $\mathcal{O}_X^{\mathrm{cris}}$ to $\mathcal{O}_X$.

Remark 1.56. This certainly implies Theorem 1.32 simply by defining

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W_n(k)) \coloneqq \operatorname{RHom}_{\operatorname{Cris}}(X/W_n(k))(\mathcal{O}_X^{\mathrm{cris}}, \mathcal{O}_X^{\mathrm{cris}}).$$

Namely, we may view the target as living in $\mathcal{D}_{X_n/W_n}$-modules for any lift X_n, which is then related to de Rham cohomology via Lemma 1.53. One can construct $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ by taking a suitable limit over n.

Remark 1.57. The category quasicoherent sheaves on X_n (as a tensor category) recovers X_n as a tensor category, so it is remarkable that the category of $\mathcal{D}$-modules does not!

Remark 1.58. The sheaf of algebras $\mathcal{D}_{X_n/W_n}$ does depend on the lift to X_n, so it is remarkable that we are able to construct $\operatorname{Cris}(X/W_n(k))$ at all!

We have been using a bit of deformation theory (for motivation) throughout, but let's state what we need more explicitly.

Theorem 1.59 (Deformation). Fix a commutative ring A and a nilpotent thickening $A \to A/I$ where $I^2 = 0$, and choose a smooth scheme Y over A/I.

(a) There is a (natural) class $\mathrm{ob}_{Y,A} \in \mathrm{H}^2(Y; T_{Y/(A/I)} \otimes_{A/I} I)$ which vanishes if and only if there is a smooth lift $\widetilde{Y}$ over A for which $\widetilde{Y}_{A/I} \cong Y$.

(b) Given a choice of lift $\widetilde{Y}$ with a choice of isomorphism $\rho\colon \widetilde{Y}_{A/I} \to Y$, then $\mathrm{H}^1(Y; T_{Y/(A/I)} \otimes I)$ parameterizes all lifts.

(c) Given a choice of lift $\widetilde{Y}$ with a choice of isomorphism $\rho\colon \widetilde{Y}_{A/I} \to Y$, then the group of automorphisms $f\colon \widetilde{Y} \to \widetilde{Y}$ which reduce to the identity modulo I is isomorphic to $\mathrm{H}^0(Y; T_{Y/(A/I)} \otimes_{A/I} I)$.

Example 1.60. By (a), we see that one can always lift smooth curves.

Remark 1.61. One can directly prove the commutativity of the group of automorphisms (roughly) as follows: f amounts to a deformation which looks like $\mathrm{id} + \varepsilon$ (e.g., on the sheaf of rings), which all commute with each other because $I^2 = 0$.

Let's explain how we use Theorem 1.59: if X is a smooth variety over a field k, then X is separated, so it admits an affine open cover $\{U_i\}_{i \in \Lambda}$ so that each intersection $U_i \cap U_j$ is still affine. By the first two parts of Theorem 1.59 (applied iteratively), we are granted a unique smooth lift $\widetilde{U}_\alpha$ over $W_n(k)$ (because higher cohomology of affine schemes will vanish). By the uniqueness, we are granted isomorphisms

$$f_{ij}\colon \mathcal{O}_{\widetilde{U}_i}|_{U_{ij}} \cong \mathcal{O}_{\widetilde{U}_j}|_{U_{ij}},$$

but the third part of Theorem 1.59 tells us that there are many choices for such an automorphism. Nonetheless, for any triple (i, i', i''), we see that the composite $f_{i''i}f_{i'i''}f_{ii'}$ is the identity $\pmod p$. But this does not mean that our sheaves glue! This explains why we cannot lift X.

However, we will be able to lift $\mathcal{D}$-modules to a category of $\mathcal{D}$-modules with W_n-coefficients without ever having to lift X to X_n. The key is the following, which asserts that such "automorphisms which are the identity $\pmod p$" are inner for $\mathcal{D}$-modules.

Lemma 1.62. Suppose $p > 2$, and fix a smooth affine scheme Y_n over W_n. Then for any automorphism f of Y_n which is the identity $\pmod p$, there is a $D_f \in \mathcal{D}_{Y_n/W_n}(Y_n)^\times$ which is $1 \pmod p$, and

$$f^*\delta = D_f \delta D_f^{-1}$$

for any $\delta \in \mathcal{D}_{Y_n/W_n}(Y_n)$.

Proof. We may find Δ_f to satisfy

$$p\Delta_f = \log f^*,$$

where $\log f^*$ refers to the power series $\sum_{r\geq 1}(-1)^{r+1}(f-\mathrm{id})^{\circ r}/r$. (This exists because $p > 2$.) This logarithm is some endomorphism on $\mathcal{O}_{Y_n}$, and it is even a derivation of $\mathcal{O}_{Y_{n-1}}$, which one can check directly. We now define $D_f := \exp(p\partial_{\Delta_f})$, and one can check the identity directly from the identity $f^* = \exp(p\Delta_f)$. ■

Remark 1.63. The moral of Lemma 1.62 is that any $\mathcal{D}_{Y_n}$-module M admits a natural isomorphism $M \cong (f^*)_* M$, where the right-hand module means that $\mathcal{D}_{Y_n}$ acts on M through the automorphism f^*. Indeed, simply send m to $D_f m$.

Remark 1.64. Lemma 1.62 even has a formulation in characteristic zero, which admits basically the same proof.

We are now ready to prove Theorem 1.55.

Proof of Theorem 1.55. Fix an affine open cover $\{U_i\}_{i\in\Lambda}$ on X with f_{ij}s as before. The category of $\mathcal{D}_{X/k}$-modules can be thought of as the category of big tuples $\{M_i\}_i$, where M_i is a module for $\mathcal{D}_{X/k}(U_i)$, and we are equipped with isomorphisms between $M_i|_{U_{ij}}$ and $M_j|_{U_{ij}}$ that satisfy some cocycle condition. In light of the discussion preceding Lemma 1.62, our problem is that we do not know how to fix the cocycle condition.

We now imitate this definition. Define our category $\mathrm{Cris}(X/W_n(k))$ of crystals as being the category of tuples $\{M_i\}_i$ of modules, where M_i is a module over $\mathcal{D}_{\widetilde{U}_i/W_n}(\widetilde{U}_i)$, equipped with isomorphisms

$$\alpha_{ij} \colon f_{ij}^* M_i|_{U_{ij}} \to M_j|_{U_{ij}}$$

such that the composites $\alpha_{i''i}\alpha_{i'i''}\alpha_{ii'}$ produces an automorphism $f_{i''i}^* f_{i'i''}^* f_{ii'}^* M_i \to M_i$, where this is an automorphism where we have identified the left module with M_i via Lemma 1.62.

It may appear that our construction depends rather poorly on the choice of cover. However, this is not the case: one can simply show that the uniqueness of our constructions means that our construction does not depend on refinements of open covers, so it does not actually depend on the choice of open cover. (It may look like this is so because the identifications depend on the choice of D_f in Lemma 1.62, but this construction was in fact canonical in the proof.) Professor Petrov seems to think that one will need to check some cocycle condition on intersections of four open subsets.

We now run our checks.

- For example, we note that the $\mathcal{O}_{\widetilde{U}_i}$s glue to an object $\mathcal{O}_X^{\mathrm{cris}}$ because all data is only every constructed $\pmod p$, and the cocycle condition is satisfied by the particular construction of D_f.
- Given a choice of lift X_n, we may actually lift everything to X_n, which shows that our crystals agree with $\mathcal{D}$-modules on X_n. ■

Remark 1.65. One can relax the hypothesis that X is separated by further covering the U_{ij}s by affines.

Remark 1.66. These are called "crystals" because they are rigid and they grow. Namely, we constructed them locally, but they were rigid enough to be able to be glued to a global category.

Remark 1.67. Additionally, morphisms of varieties are finite type and separated, so it is not too hard to construct functoriality for our category of crystals. Explicitly, a morphism $f\colon X \to Y$ produces a morphism

$$f^*\colon \mathrm{R}\Gamma_{\text{cris}}(Y/W(k)) \to \mathrm{R}\Gamma(X/W(k)).$$

1.4 February 11

Today, we give some applications of crystalline cohomology.

1.4.1 The Ax–Katz Theorem

Crystalline cohomology comes with a natural Frobenius.

Example 1.68. For any smooth scheme X over k, there is a Frobenius $F\colon X \to X^{(1)}$, which produces an endomorphism

$$F^*\colon \mathrm{R}\Gamma_{\text{cris}}\left(X^{(1)}/W(k)\right) \to \mathrm{R}\Gamma_{\text{cris}}(X/W(k)).$$

For example, if X is defined over $\mathbb{F}_{q^r}$, then $X = X^{(r)}$, then we have an endomorphism $(F^*)^r$ of the complex $\mathrm{R}\Gamma_{\text{cris}}(X/W(k))$.

Remark 1.69. If X is smooth and proper over $\mathbb{F}_q$ where $q = p^r$, then $\mathrm{H}^n_{\text{cris}}(X/W(\mathbb{F}_q))$ is finitely generated over $W(\mathbb{F}_q)$, and it turns out that

$$\#X(\mathbb{F}_q) = \sum_{i\geq 0}(-1)^i \operatorname{tr}\left((F^*)^r; \mathrm{H}^i_{\text{cris}}(X/W(\mathbb{F}_q))\left[\frac{1}{p}\right]\right).$$

One has to check that crystalline cohomology is a Weil cohomology theory, which is true. It is not too difficult to check this formula directly when X is zero-dimensional, meaning that it is a finite disjoint union of points.

Theorem 1.70 (Ax–Katz)**.** Let X be a smooth proper geometrically connected variety over $\mathbb{F}_q$ for which $\mathrm{H}^i(X; \mathcal{O}_X) = 0$ for all $i > 0$. Then

$$\#X(\mathbb{F}_q) \equiv 1 \pmod{q}.$$

Proof. The idea is to use Remark 1.69. Note that $\mathrm{H}^0_{\text{cris}}(X/W(\mathbb{F}_q))$ is the one-dimensional space with trivial Frobenius action, so it is enough to show that

$$\operatorname{tr}\left((F^*)^r; \mathrm{H}^i_{\text{cris}}(X/W(\mathbb{F}_q))\left[\frac{1}{p}\right]\right) \equiv 0 \pmod{q}$$

for all $i \geq 1$. With an inversion of p, one will not be able to do better than $\pmod{p}$ information, so we see that we have to use the integral structure.

For a given module M over $W(\mathbb{F}_q)$, we let M_{tf} denotes the quotient by torsion so that M_{tf} is torsion-free. It is now enough to show that the map

$$F^*\colon \mathrm{H}^i_{\text{cris}}(X^{(1)}/W(\mathbb{F}_q))_{\text{tf}} \to \mathrm{H}^i_{\text{cris}}(X/W(\mathbb{F}_q))_{\text{tf}}$$

is divisible by p. Indeed, the trace after $(-)[1/p]$ is the same as the trace on the torsion-free part (because then inverting p does nothing), so we can just iterate this divisibility r times to prove the claim. We now reduce $\pmod p$, for which we recall that

$$\mathrm{R}\Gamma_{\text{cris}}(X/W(\mathbb{F}_q)) \otimes^{\mathbb{L}}_{W(\mathbb{F}_q)} \mathbb{F}_q \cong \mathrm{R}\Gamma_{\text{dR}}(X/\mathbb{F}_q).$$

Now, the same argument as in Universal coefficient theorem yields an exact sequence

$$0 \to \mathrm{H}^n_{\text{cris}}(X/W(\mathbb{F}_q))/p \to \mathrm{H}^n_{\text{dR}}(X/\mathbb{F}_q) \to \mathrm{H}^{n+1}_{\text{cris}}(X/W(\mathbb{F}_q))[p] \to 0.$$

By functoriality of the short exact sequence, if we could show that Frobenius was zero on de Rham cohomology (i.e., the map $F^*\colon \mathrm{H}^n_{\text{dR}}(X^{(1)}/\mathbb{F}_q) \to \mathrm{H}^n_{\text{dR}}(X/\mathbb{F}_q)$ vanishes), then it would vanish on $\mathrm{H}^n_{\text{cris}}(X/W(\mathbb{F}_q))/p$ and thus vanish on $\mathrm{H}^n_{\text{cris}}(X/W(\mathbb{F}_q))_{\text{tf}}/p$, which was the required claim.

We are thus reduced to computing some Frobenius action on the de Rham complex. Let's recall how this Frobenius action is defined: by pulling back along F^*, there are natural maps

$$\begin{tikzcd}
F^{-1}\mathcal{O}_{X^{(1)}} \arrow[r] \arrow[d, "F^*"'] & F^{-1}\Omega^1_{X^{(1)}} \arrow[r] \arrow[d, "F^*"'] & F^{-1}\Omega^2_{X^{(1)}} \arrow[r] \arrow[d, "F^*"'] & \cdots \\
\mathcal{O}_X \arrow[r] & \Omega^1_X \arrow[r] & \Omega^2_X \arrow[r] & \cdots
\end{tikzcd}$$

which then descend to cohomology. But note that $F^*(f\,dg) = f^p\,d(g^p) = 0$, so F^* vanishes on all the higher terms. Thus, the above morphism F^* of complexes factors through the complex $\mathcal{O}_X \to 0 \to 0 \to \cdots$. Thus, after taking hypercohomology, we see that F^* factors through $\mathrm{H}^n(X;\mathcal{O}_X)$, which vanishes by assumption! ■

Remark 1.71. We can remove the properness assumption, and one still finds that F^* finds on de Rham cohomology.

Remark 1.72 (Chevalley–Warning)**.** If $X \subseteq \mathbb{P}^N_{\mathbb{F}_q}$ is a hypersurface of degree at most N, then the hypothesis is satisfied. It is a classical result of Chevalley–Warning that $\#X(\mathbb{F}_q) \equiv 1 \pmod p$. A cohomological argument could proceed by using the Atiyah–Bott formula

$$\#X(\mathbb{F}_q) = \sum_{i\geq 0}(-1)^i \operatorname{tr}\left((F^*)^r; \mathrm{H}^n(X;\mathcal{O}_X)\right) \pmod p$$

and then argue as above. To access $\pmod q$ information, we need a different cohomology theory.

Remark 1.73. If one wants to prove that the Frobenius acting on crystalline cohomology was instead divisible by p^2, then there starts to be contributions of the cohomology of Ω_X, via some discussion of Hodge–Tate weights.

1.4.2 A Weak Hodge Decomposition

Here is another application.

Notation 1.74. Given a formal scheme $\widetilde{X}$ over $W(k)$, we define

$$\widetilde{X}^{(1)} \coloneqq \widetilde{X} \times_{W(k)} W(k),$$

where the internal map $W(k) \to W(k)$ is the natural Frobenius on $W(k)$.

Proposition 1.75. Fix a smooth scheme X over a perfect field k of characteristic p. Suppose that we can lift X to a formal scheme $\widetilde{X}$ over $W(k)$ as well as a lift $\widetilde{F}\colon \widetilde{X} \to \widetilde{X}^{(1)}$ of F. Then the de Rham complex

$$\mathcal{O}_{\widetilde{X}} \stackrel{d}{\to} \Omega^1_{\widetilde{X}} \stackrel{d}{\to} \Omega^2_{\widetilde{X}} \stackrel{d}{\to} \cdots$$

is quasi-isomorphic to

$$\mathcal{O}_{\widetilde{X}^{(1)}} \stackrel{pd}{\to} \Omega^1_{\widetilde{X}^{(1)}} \stackrel{pd}{\to} \Omega^2_{\widetilde{X}^{(1)}} \stackrel{pd}{\to} \cdots.$$

Remark 1.76. If X is affine, then such lifts $\widetilde{X}$ and $\widetilde{F}$ always exist.

As an application, we get a version of the Hodge decomposition.

Corollary 1.77. Fix a smooth scheme X over a perfect field k of characteristic p. Suppose that we can lift X to a formal scheme $\widetilde{X}$ over $W(k)$ as well as a lift $\widetilde{F}\colon \widetilde{X} \to \widetilde{X}^{(1)}$ of F. Then the de Rham complex

$$\mathcal{O}_X \stackrel{d}{\to} \Omega^1_X \stackrel{d}{\to} \Omega^2_X \to \cdots$$

is quasi-isomorphic to

$$\mathcal{O}_{X^{(1)}} \stackrel{0}{\to} \Omega^1_{X^{(1)}} \to \Omega^2_{X^{(2)}} \to \cdots.$$

Thus,

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \bigoplus_{i+j=n}^{n} \mathrm{H}^i\left(X^{(1)}; \Omega^j_{X^{(1)}}\right).$$

Proof. To show the quasi-isomorphism, merely reduce Proposition 1.75 modulo p, and then we know that crystalline cohomology becomes de Rham cohomology. To prove the last claim, note that the hypercohomology of the first complex is the de Rham cohomology, and hypercohomology of the second complex produces the right-hand side. Indeed, to take hypercohomology, one needs to resolve the second complex, but the "columns" in our resolution do not need to communicate with each other, so

$$\mathbb{H}^n\left(\mathcal{O}_{X^{(1)}} \stackrel{0}{\to} \Omega^1_{X^{(1)}} \to \Omega^2_{X^{(2)}} \to \cdots\right) = \bigoplus_{i+j=n}^{n} \mathrm{H}^i\left(X^{(1)}; \Omega^j_{X^{(1)}}\right)$$

follows after we keep track of our degrees. ■

Proof of Proposition 1.75. We are interested in filling in the following diagram.

$$\begin{tikzcd}
\mathcal{O}_{\widetilde{X}} \arrow[r] & \Omega^1_{\widetilde{X}} \arrow[r] & \Omega^2_{\widetilde{X}} \arrow[r] & \cdots \\
\mathcal{O}_{\widetilde{X}^{(1)}} \arrow[r] \arrow[u, "\widetilde{F}^*"] & \Omega^1_{\widetilde{X}^{(1)}} \arrow[r] \arrow[u, dashed] & \Omega^2_{\widetilde{X}^{(1)}} \arrow[r] \arrow[u, dashed] & \cdots
\end{tikzcd}$$

Staring at the diagram, we see that the map $\Omega^i_{\widetilde{X}^{(1)}} \to \Omega^i_{\widetilde{X}}$ had better be $p^{-i}F^*$. The proof of Theorem 1.70 checked that $p^{-1}F^*$ makes sense for $i > 0$, and by discussing Frobenius on higher wedges of differentials finds that even $p^{-i}F^*$ will always make sense.

We have thus provided our morphism of complexes, which we want to be a quasi-isomorphism. Because the terms are p-adically complete, it turns out that it is enough to check that it is a quasi-isomorphism $\pmod{p}$.[1] It turns out that one can proceed as in the proof of Theorem 1.22. ■

This decomposition remains true with weaker assumptions, which will be our next goal.

Theorem 1.78 (Berthelot–Ogus)**.** Fix a smooth scheme X over a perfect field k which can be lifted to a formal scheme $\widetilde{X}$ over $W(k)$. Suppose further that $\dim X < p$. Then

$$\mathcal{O}_{\widetilde{X}} \xrightarrow{d} \Omega^1_{\widetilde{X}} \xrightarrow{d} \Omega^2_{\widetilde{X}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim X}_{\widetilde{X}}$$

is quasi-isomorphic to

$$\mathcal{O}_{\widetilde{X}^{(1)}} \xrightarrow{pd} \Omega^1_{\widetilde{X}^{(1)}} \xrightarrow{pd} \Omega^2_{\widetilde{X}^{(1)}} \xrightarrow{pd} \cdots \xrightarrow{pd} \Omega^{\dim X}_{\widetilde{X}^{(1)}}.$$

Remark 1.79. The same argument as in Corollary 1.77 is able to show that

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) \approx \bigoplus_{i+j=n} \mathrm{H}^i(X^{(1)}; \Omega^j_{X^{(1)}}).$$

(We will not be precise about what $\approx$ means now.) It turns out that this decomposition remains true if X can merely be lifted to $W_2(k)$, and it even depends on this choice of lift.

More precisely, our goal will be to prove the above remark.

1.4.3 $\mathcal{D}$-modules by Crystals

In order to access our Hodge decomposition, we are going to realize $\mathcal{D}_X$-modules as quasicoherent sheaves on some stack X^{dR}; if X is affine, then X^{dR} will be able to be realized as the quotient of some formal scheme by a formal group scheme. Notably, quasicoherent sheaves can then just be thought of as certain equivariant sheaves on a formal scheme.

To understand where X^{dR} may come from, we return to characteristic 0.

Lemma 1.80. Fix a smooth scheme F over a field F of characteristic zero. For a $\mathcal{D}_X$-module M and a point $x \in X(F)$, there is a natural isomorphism

$$M|_{\widehat{\mathcal{O}}_{X,x}} \cong M_x \widehat{\otimes}_F \widehat{\mathcal{O}}_{X,x}.$$

Here, the left-hand side is $\lim M \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/\mathfrak{m}_x^\bullet$, and $M_x = M \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Remark 1.81. The moral is that there is a canonical trivialization "M_x" of M in a formal neighborhood of a point.

Proof. We will produce an isomorphism $M_x \to \left(M|_{\widehat{\mathcal{O}}_{X,x}}\right)^{\nabla=0}$, where the target means that we are looking for sections with flat connection. Accordingly, choose $s \in M_x$, which can be lifted to some $\widetilde{s} \in M(U)$. Now, $\widetilde{s}$ has no reason to be flat, so we have to fix it. Accordingly, we use Taylor's formula to expand out U in local coordinates $(t_1, \ldots, t_d)$ and then define

$$\sum_{i_1,\ldots,i_d \geq 0} (-1)^{i_1+\cdots+i_d} \frac{t_1^{i_1} \cdots t_d^{i_d}}{i_1! \cdots i_d!} \nabla^{i_1}_{\partial_{t_1}} \circ \cdots \circ \nabla^{i_d}_{\partial_{t_d}}(\widetilde{s}),$$

[1] This is a variant of Nakayama's lemma. For example, a sample statement is that a morphism $f\colon M \to N$ of torsion-free p-adically complete $\mathbb{Z}_p$-modules (meaning that $M \cong \lim M/p^\bullet$) can be checked to be an isomorphism $\pmod{p}$. After deriving, it turns out that we don't have to check that it is torsion-free.

and one can hit this with ∇ to check that it is a flat section. Further, note that $i_1 = \cdots = i_d = 0$ reproduces s, so this section continues to lift s. We have thus defined our map, and one can check that it produces the required isomorphism by Nakayama's lemma. ■

Example 1.82. Let's write this out for $d = 1$. Then in the first degree, we have replaced $\widetilde{s}$ with $t - \nabla_{\partial_t}\widetilde{s}$; hitting this with ∇_{∂_t}, we receive $-t\nabla^2_{\partial_t}(\widetilde{s})$. This is still nonzero, and we then further correct by $\frac{1}{2}t^2\nabla^2_{\partial_t}(\widetilde{s})$ and continue.

We may want to extend these canonical trivializations into a full trivialization of M, but then we have to explain how to relate the various points.

Notation 1.83. Fix a smooth scheme X over a field F of characteristic zero. Then we define $(X \times_F X)^\wedge_\Delta$ to be the union of the closed subschemes of $X \times X$ cut out by the powers of the quasicoherent ideal sheaf $\mathcal{I}_\Delta$ (of the diagonal). We can define other powers similarly.

Remark 1.84. Intuitively, $(X \times_F X)^\wedge_\Delta$ is the union of all nilpotent thickenings of Δ.

Theorem 1.85. Fix a smooth scheme X over a field F of characteristic zero. Define the formal scheme $(X \times_F X)^\wedge_\Delta$ to be the formal completion at the diagonal, which is the union of the nilpotent thickenings of the diagonal Δ. Then the data of a $\mathcal{D}_X$-module M is equivalent to the data of a quasicoherent sheaf M on X and an isomorphism $\alpha\colon \operatorname{pr}_1^* M \to \operatorname{pr}_2^* M$ on $(X \times_F X)^\wedge_\Delta$ making the diagram of sheaves on $(X \times X \times X)^\wedge_\Delta$

$$\begin{tikzcd} \operatorname{pr}_1^* M \arrow[r, "\operatorname{pr}_{12}^* \alpha"] \arrow[dr, "\operatorname{pr}_{13}^* \alpha"'] & \operatorname{pr}_2^* M \arrow[d, "\operatorname{pr}_{23}^* \alpha"] \\ & \operatorname{pr}_3^* M \end{tikzcd}$$

commute.

Sketch. We describe how to turn a $\mathcal{D}_X$-module M into the desired data. Namely, we need to provide the data of α. For example, to define α along the fiber $\operatorname{pr}_1^{-1}(x) \cong \operatorname{Spf}\widehat{\mathcal{O}}_{X,x}$ needs to provide an isomorphism

$$\operatorname{pr}_1^* M|_{\operatorname{pr}_1^{-1} x} \to \operatorname{pr}_2^* M|_{\operatorname{pr}_1^{-1} x}.$$

The left-hand side is just $M_x \otimes_F \widehat{\mathcal{O}}_{X,x}$ because we start at the point x (which gives M_x) which is then expanded along the formal neighborhood. On the other hand, the right-hand side is $M|_{\widehat{\mathcal{O}}_{X,x}}$ because the pullback is able to remember some horizontal information.[2] It turns out that the maps of Lemma 1.80 now suitably glue over all points. ■

Remark 1.86. The moral of α is that it lets us identify infinitesimally nearby fibers. Once one has identified "nearby" fibers, then one can expect to be able to take derivatives, which gives the $\mathcal{D}_X$-module structure.

Remark 1.87. The data (M, α) is sometimes called a "crystal."

[2] This discussion is a little confusing. It becomes easier to think about in the case of $X = \mathbb{A}^1$, where we can think of Δ as the genuine diagonal of a square.

Example 1.88. Consider $X = \mathbb{A}^1_F$ (or any other algebraic group). Then we find that $\mathcal{D}_{\mathbb{A}^1}$-modules can be seen to be the same as quasicoherent sheaves on $\mathbb{A}^1$ with $\widehat{\mathbb{G}}_a$-equivariant structure. Well, we are asking for a suitable identification along the two projections

$$\left(\mathbb{A}^1 \times \mathbb{A}^1\right)^\wedge_\Delta \rightrightarrows \mathbb{A}^1.$$

However, the left-hand formal scheme is $\mathbb{A}^1 \times \widehat{\mathbb{G}}_a$ (given by sending (x, y) to $(x, x-y)$), and one can check that the cocycle condition corresponds to an equivariant structure. It follows that we are looking at quasicoherent sheaves on $\mathbb{A}^1/\widehat{\mathbb{G}}_a$. Unwinding, we remark that a $\mathcal{D}_{\mathbb{A}^1}$-module M goes to the same $R[t]$-module M with $\widehat{\mathbb{G}}_a$-(co)action $M \to M\widehat{\otimes}\mathcal{O}(\widehat{\mathbb{G}}_a)$ given by

$$m \mapsto \sum_{n \geq 0} \partial_t^n m \otimes \frac{t^n}{n!}.$$

1.5 February 17

Today, we discuss the Hodge decomposition.

1.5.1 Divided Powers

We are interested in sketching the following result, fulfilling our promise in Remark 1.79.

Theorem 1.89 (Drinfeld, Bhatt–Lurie)**.** Fix a smooth variety X over a perfect field k of positive characteristic p. Given a smooth lift $\widetilde{X}$ over $W_2(k)$, there is a natural $\mathbb{Z}/p\mathbb{Z}$-grading

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} \mathrm{H}^n_{\mathrm{dR}}(X/k)_i$$

such that $\mathrm{H}^n_{\mathrm{dR}}(X/k)_i \cong \mathrm{H}^{n-i}\left(X^{(1)}; \Omega^i_{X^{(1)}}\right)$ for $n < p$ and $i \in \{0, 1, \ldots, p-1\}$.

Remark 1.90. This is not exactly a Hodge decomposition because this is only a $\mathbb{Z}/p\mathbb{Z}$-grading. Additionally, the Hodge decomposition usually only exists when X is further proper.

Remark 1.91. It is possible to find a smooth proper surface without the lift to $W_2(k)$ and also so that

$$\dim \mathrm{H}^n_{\mathrm{dR}}(X/k) \neq \sum_{0 \leq i < p} \mathrm{H}^{n-i}\left(X^{(1)}; \Omega_{X^{(1)}}\right).$$

Such varieties even appear in nature.

It is difficult to find a grading directly on de Rham cohomology. Instead, we will pass through the de Rham stack. We may want to retell some of our story from last class with crystals, but there are problems with this construction in positive characteristic. For example, the power series defining the coaction in Example 1.88 have large denominators.

To fix this problem, we simply add in the required symbols by hand.

Definition 1.92. We define the affine group scheme $\mathbb{G}_a^\sharp$ over $\mathbb{F}_p$ to be given by the ring

$$\mathbb{Z}\left[t, t/1, t^2/2, t^3/3!, \ldots\right] \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

The group structure $\mathbb{G}_a^\sharp \times \mathbb{G}_a^\sharp \to \mathbb{G}_a^\sharp$ is given by sending $t_1 \otimes t_2$ to $\frac{1}{n!}(t_1+t_2)^n = \sum_{i+j=n} \frac{t_1^i}{i!} \cdot \frac{t_2^j}{j!}$.

Remark 1.93. Explicitly, the ring is given by

$$\frac{\mathbb{F}_p[t, u_1, u_2, u_3 \ldots]}{(u_1 - t, 2tu_2 - u_1, 3tu_3 - u_2, \ldots)}.$$

It turns out to be enough to merely add in the divided powers coming from powers of p.

Remark 1.94. One can also take the formal group scheme $\widehat{\mathbb{G}}_a^\sharp$ over $\mathbb{F}_p$ to be given by the union of the affine group schemes given by the quotient rings

$$\frac{\mathbb{Z}\left[t, t/1, t^2/2, t^3/3!, \ldots\right]}{(t^m/m!)_{m \geq n}} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Alternatively, this is the formal scheme attached to the completed ring $\mathbb{F}_p[t][[t^n/n!]]_{n \geq 0}$.

Remark 1.95. The previous remark provides a natural map $\mathbb{G}_a^\sharp \to \mathbb{G}_a$. This is some map of group schemes, and we see that its kernel is given by

$$\frac{\mathbb{F}_p\left[t, t^p/p!, t^{p^2}/(p^2)!, \ldots\right]}{(t)},$$

which continues to be some infinite-dimensional group scheme. Similarly, the kernel of the natural map $\widehat{\mathbb{G}}_a^\sharp \to \mathbb{G}_a$ is the formal scheme attached to the ring $\mathcal{O}(\widehat{\mathbb{G}}_a^\sharp)/(t)$.

Here is our analog of Example 1.88.

Lemma 1.96. Fix a ring R. Then the category of $\mathcal{D}_{\mathbb{A}^1_R/R}$-modules is equivalent to the category of quasi-coherent sheaves on $\mathbb{A}^1_R$ with an equivariant action by $\widehat{\mathbb{G}}_a^\sharp$.

Proof. The functor in the forward direction was already given by Example 1.88. We will not check that it is an equivalence. ∎

Example 1.97. Because the natural map $\widehat{\mathbb{G}}_a^\sharp \to \mathbb{G}_a$ admits a kernel, we see that its kernel will act by endomorphisms on a $\mathcal{D}$-module M. (Indeed, this kernel does not interact with the rest of the $\mathbb{A}^1$-action.)

1.5.2 The p-curvature

It is remarkable that we have found so many additional endomorphisms. Let's explain this. We have already seen something like this before: recall that the natural map $\mathcal{D}_X \to \mathcal{E}nd_k(\mathcal{O}_X)$ admitted a kernel in positive characteristic.

Lemma 1.98. Fix a smooth scheme X over a field k of positive characteristic p. For any derivation $v\colon \mathcal{O}_X \to \mathcal{O}_X$, the p-fold composition $v^{\circ p}$ is also a derivation.

Proof. Direct calculation. ■

Notation 1.99. Fix a scheme X over a field k of positive characteristic p. Given a vector field v on some open subset $U \subseteq X$, we let $v^{[p]}$ denote the vector field corresponding to the p-fold derivation $v^{\circ p}$.

Example 1.100. With $X = \mathbb{A}^1$, we see that $(\partial_t)^{[p]} = 0$ because hitting any t^n with p derivatives will kill it. Similarly, we see that $(t\partial_t)^{[p]} = t\partial_t$ because hitting t^n with $t\partial_t$ a total of p times goes to $n^p t^n = nt^n$.

The point is that the element $\partial_v^{\circ p} - \partial_{v^{[p]}}$ acts by zero as a derivation on $\mathcal{O}_X$ because $\partial_v^{\circ p}$ and $\partial_{v^{[p]}}$ admit the same action on $\mathcal{O}_X$ (by construction of $v^{[p]}$). In particular, it lives in the kernel of the natural map $\mathcal{D}_X \to \mathcal{E}nd_k(\mathcal{O}_X)$! It turns out that these elements generate.

Proposition 1.101. Fix a smooth variety X over a perfect field k of positive characteristic p.

(a) Then $F_*\mathcal{D}_X$ is a sheaf of $\mathcal{O}_{X^{(1)}}$-algebras, and its center is the symmetric algebra generated by the elements
$$\partial_v^p - \partial_{v^{[p]}}$$
as v varies over $T_{X^{(1)}}$.

(b) The kernel of $\mathcal{D}_X \to \mathcal{E}nd_k(\mathcal{O}_X)$ is generated by the homogeneous elements generated by $\partial_v^p - \partial_{v^{[p]}}$ in positive degree.

Proof. Omitted. ■

Thus, we have found many endomorphisms.

Definition 1.102 (p-curvature)**.** Fix a smooth variety X over a perfect field k of positive characteristic p. For a vector field v and a $\mathcal{D}$-module M, the *p-curvature* of M is the action of
$$\psi_v \coloneqq \partial_v^{\circ p} - \partial_v.$$

Remark 1.103. The action of ψ_v is an endomorphism as a $\mathcal{D}_X$-module, which more or less follows from part (a) of Proposition 1.101. (Note that the F_* does not actually do much to the commutativity condition we are trying to check.) This action turns out to be the same as the one found in Example 1.97.

1.5.3 Quasi-nilpotent Modules

We are now allowed to make the following definition, which will let us remove some formal schemes from view.

Definition 1.104 (quasi-nilpotent)**.** Fix a smooth variety X over a perfect field k of positive characteristic p. Then a $\mathcal{D}_X$-module M is *quasi-nilpotent* if and only if, for all vector fields v on $U \subseteq X$, the action of ψ_v on $M(U)$ is locally nilpotent: for any $m \in M(U)$, there is N for which $\psi_v^{\circ N}(m) = 0$. More generally, if X is a smooth scheme over a $\mathbb{Z}/p^n\mathbb{Z}$-algebra R, then a $\mathcal{D}$-module M is *quasi-nilpotent* if and only if $M_{R/p}$ is nilpotent as a $\mathcal{D}$-module over $X_{R/p}$.

Example 1.105. Because the action by the ψ_vs is zero on $\mathcal{O}_X$, we see that $\mathcal{O}_X$ is quasi-nilpotent.

Non-Example 1.106. The action of $\mathcal{D}_X$ on $\mathcal{D}_X$ is faithful, so $\mathcal{D}_X$ is not quasi-nilpotent.

Let's now upgrade Lemma 1.96.

Lemma 1.107. The category of quasi-nilpotent $\mathcal{D}$-modules on $\mathbb{A}^1_{\mathbb{F}_p}$ is equivalent to the category of quasicoherent sheaves on $\mathbb{A}^1_{\mathbb{F}_p}$ with an equivariant action by $\mathbb{G}_a^\sharp$.

Proof. Via Lemma 1.96, we only have to characterize the image of the constructed functor. Indeed, the functor defined by Example 1.88 finds that the sum

$$\sum_{n\geq 0} \partial_t^n m \otimes \frac{t^n}{n!}$$

is always finite because M is quasi-nilpotent. ∎

Remark 1.108. For later use, it is worth noting that the de Rham complex continues to be computed by quasi-nilpotent sheaves: as in Lemma 1.53, we find that

$$\Omega^\bullet_{X/R} \cong \mathrm{RHom}_{\mathrm{Mod}(\mathcal{D}_X)^{\mathrm{quasi-nilpotent}}}(\mathcal{O}_X, \mathcal{O}_X).$$

We additionally have a notion of crystals!

Definition 1.109. Fix a closed, regular embedding $Y \hookrightarrow X$ of smooth varieties over a field k. Then we define the *de Rham stack* $X_Y^\sharp$ as the "divided power envelope"

$$X_Y^\sharp = \mathrm{Spec}_X\, \mathcal{O}_X \left[\frac{f^n}{n!}\right]_{n\geq 1, f\in \mathcal{I}_Y},$$

where $\mathcal{I}_Y$ is the ideal sheaf.

Remark 1.110. Explicitly, if Y is cut out by a regular sequence $\{f_1, \ldots, f_r\}$, then the ring can be described as the tensor product of $\mathcal{O}_X$ with the free divided power algebra on the letters $x_1, \ldots, x_r$, where x_i corresponds to f_i. It is not obvious that this ring does not depend on the choice of regular sequence!

As promised, here is our analog of Theorem 1.85.

Theorem 1.111. Fix a smooth variety X over a perfect field k of positive characteristic p. Then the category of quasi-nilpotent $\mathcal{D}_X$-modules is equivalent to the category of quasicoherent sheaves M on X equipped with an isomorphism $\mathrm{pr}_1^* M \cong \mathrm{pr}_2^* M$ (and satisfying a suitable cocycle condition), where pr_1 and pr_2 are the canonical projections $(X \times X)^\sharp_\Delta \to X$.

Proof. Omitted. ∎

1.5.4 The de Rham Stack

We are now ready to define the de Rham stack.

Definition 1.112 (quotient stack)**.** A *quotient stack*, denoted $[X/G]$, is a pair (G, X), where G is a group scheme acting on a scheme X. Two quotient stacks $[X/G]$ and $[X'/G']$ are *strongly equivalent* if and only if there is a flat surjection $G' \to G$ of group schemes and a map $X' \to X$ so that $X' \to X$ is a torsor for $\ker f$. Two quotient stacks $[X/G]$ and $[X'/G']$ are *equivalent* if and only if they can be connected by a finite sequence of strong equivalences.

Notation 1.113. Given a quotient stack $[X/G]$, we define $\operatorname{QCoh}([X/G])$ to consist of the G-equivariant sheaves on X.

Remark 1.114. One can check that $\operatorname{QCoh}([X/G])$ does not depend on the equivalence class of $[X/G]$.

Here is a first definition of our de Rham stack, for affine schemes.

Definition 1.115 (de Rham stack)**.** Fix an affine scheme $X \subseteq \mathbb{A}_k^n$ over a perfect field k of positive characteristic p. Then we define the *de Rham stack* X^{dR} to be the quotient stack

$$\left[(\mathbb{A}^n)_X^\sharp \,/\, \left(\mathbb{G}_a^\sharp\right)^n\right].$$

It turns out that this construction does not depend on the choice of embedding to $\mathbb{A}_k^n$.

Remark 1.116. It turns out that the quasicoherent sheaves on X^{dR} correspond exactly to crystals and thus to quasi-nilpotent $\mathcal{D}_X$-modules. We have already seen this for $\mathbb{A}^1$ in Example 1.88.

To make a definition in general, note that one can locally fit X into a pullback square as follows.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathbb{A}^n \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

(One may not be able to do this globally if X is not a complete intersection.) But now we may as well base-change by $(\mathbb{A}^m)_0^\sharp$, which produces the following pullback square.

$$\begin{tikzcd} X^{\mathrm{dR}} \arrow[r] \arrow[d] & (\mathbb{A}^n)^{\mathrm{dR}} \arrow[d] \\ \operatorname{Spec} k \arrow[r] & (\mathbb{A}^m)^{\mathrm{dR}} \end{tikzcd}$$

Here, note $(\operatorname{Spec} k)^{\mathrm{dR}} = \operatorname{Spec} k$ because $\mathcal{D}$-modules on a point are just vector spaces. But now we see that $(\mathbb{A}^n)^{\mathrm{dR}} = \left(\mathbb{A}^1\right)^{\mathrm{dR},n}$, so it becomes interesting to understand what the polynomial map $\mathbb{A}^n \to \mathbb{A}^m$ becomes after these isomorphisms.

Remark 1.117. Note that $(\mathbb{A}^1)^{\mathrm{dR}}$ is a ring stack: one can just base-change the addition and multiplication up from $\mathbb{A}^1$ to the de Rham stack. In fact, this is a k-algebra stack, meaning that each $a \in k$ produces a multiplication $\mu_a \colon (\mathbb{A}^1)^{\mathrm{dR}} \to (\mathbb{A}^1)^{\mathrm{dR}}$.

The moral is that the functor $X \mapsto X^{\mathrm{dR}}$ for every X is understood just from gluing and taking closed subschemes from the k-algebra stack $\left(\mathbb{A}^1\right)^{\mathrm{dR}}$. Indeed, one can make sense of the action of "polynomials," which is enough to cut out X. Thus, we could recover X^{dR} for affine schemes by only understanding the k-algebra stack structure on $\left(\mathbb{A}^1\right)^{\mathrm{dR}}$.

We end our class by giving a coordinate-free definition.

Definition 1.118 (de Rham stack)**.** Fix a smooth variety X over a perfect field k. Then we define the *de Rham stack* to be the quotient $X^{\mathrm{dR}} := X/(X \times X)^\sharp_\Delta$.

Remark 1.119. It turns out that $\operatorname{QCoh} X^{\mathrm{dR}}$ is the category of quasi-nilpotent $\mathcal{D}_X$-module.

1.6 February 18

Here we go.

1.6.1 How to Grade

We are going to prove Theorem 1.89 by finding new structure on X^{dR}.

Notation 1.120. We define $\mathbb{G}_m^\sharp$ to be the group scheme with ring given by

$$\operatorname{Spec} k\left[t, 1/t, \{(t-1)^n/n!\}_n\right].$$

Remark 1.121. There is a canonical map $k[t,1/t] \to k[t,1/t,\{(t-1)^n/n!\}_n]$, but note that it factors through

$$\frac{k[t,1/t]}{(t-1)^p}$$

because $(t-1)^p$ is divisible by p in the target. Thus, the canonical map $\mathbb{G}_m^\sharp \to \mathbb{G}_m$ factors through μ_p.

Remark 1.122. There is a map $\log\colon \mathbb{G}_m^\sharp \to \mathbb{G}_a$ given by

$$\log t := \sum_{n\geq 1}(-1)^{n+1}\frac{(t-1)^n}{n},$$

and one can calculate that $\ker\log = \mu_p$. (Note that this sum is finite because $(t-1)^n/n = (t-1)^n/n! \cdot (n-1)!$, and $(n-1)!$ is eventually divisible by p.)

Theorem 1.123. Fix a smooth variety X over a perfect field k of positive characteristic p. Given a smooth lift $\widetilde{X}$ over $W_2(k)$, there is a natural action by $\mathbb{G}_m^\sharp$ on X^{dR}.

Proof that Theorem 1.123 implies Theorem 1.89. We will only explain how to produce the grading; the requirement in low degrees follows from the construction of the action. Now, an action of $\mathbb{G}_m^\sharp$ on X^{dR} amounts to an action of $\mathbb{G}_m^\sharp$ on

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X/k) = \operatorname{Ext}^\bullet_{\operatorname{QCoh}(X^{\mathrm{dR}})}(\mathcal{O}_{X^{\mathrm{dR}}}, \mathcal{O}_{X^{\mathrm{dR}}})$$

because $\operatorname{QCoh}(X^{\mathrm{dR}})$ is the category of quasi-nilpotent $\mathcal{D}$-modules by Remark 1.108.

Now, a $\mathbb{G}_m$-action on a vector space is the data of a $\mathbb{Z}$-grading (by taking the eigenvalues of the $\mathbb{G}_m$-action). Thus, we may hope that our $\mathbb{G}_m^\sharp$-action can produce a similar grading. Thus, there is a canonical copy of μ_p in $\mathbb{G}_m^\sharp$, which then acts on $\mathrm{H}^\bullet_{\mathrm{dR}}(X/k)$. But now an action of μ_p on V corresponds to a $\mathbb{Z}/p\mathbb{Z}$-grading given by taking the eigenspaces $V_i := \{v \in V : tv = t^v \text{ for } t \in \mu_p\}$. We have thus exhibited the grading! ■

Remark 1.124. It turns out that $\mathbb{G}_m^\sharp \cong \mu_p \times \mathbb{G}_a^\sharp$. Thus, there is an interesting action by $\mathbb{G}_a^\sharp$ on the degree pieces.

For motivation, note that Theorem 1.123 can be refined if we give ourselves more lifting.

Proposition 1.125. Fix a smooth variety X over a perfect field k of positive characteristic p. Given a smooth lift $\widetilde{X}$ over $W_2(k)$ and a lift $\widetilde{F}\colon \widetilde{X} \to \widetilde{X}^{(1)}$ of the Frobenius, there is a natural action of $\mathbb{G}_m$ on X^{dR}.

Remark 1.126. As before, a natural action of $\mathbb{G}_m$ on X^{dR} induces a $\mathbb{Z}$-grading on the de Rham cohomology.

One may hope to use Proposition 1.125 by working with $\mathbb{A}^1$ and then passing to X using the k-algebra stack structure. This is not possible because there is no Frobenius lift of $\mathbb{A}^1$ on $W_2(k)$ which commutes with the structure as an additive group scheme: indeed, even for $k = \mathbb{F}_p$, there is no polynomial $f(x) \in \mathbb{Z}/p^2\mathbb{Z}[x]$ such that $f(x+y) = f(x) + f(y)$ and $f(x) \equiv x^p \pmod p$. Thus, we cannot hope to use some "canonical" Frobenius lifts and also reduce everything to the ring stack structure of $\mathbb{A}^1$: one needs to use the data of the lift to $W_2(k)$ to produce the Frobenius lift.

Anyway, let's prove Theorem 1.123 in the case of $\mathbb{A}^1$.

Lemma 1.127. The kernel of the projection $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ is given by $\mathbb{G}_a^{\sharp,(1)}$.

Proof. The idea is that the map $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ factors through α_p because $t^p \in k[t]$ is divisible by p in the divided power ring, which is the kernel of $F\colon \mathbb{G}_a \to \mathbb{G}_a^{(1)}$. ■

Example 1.128. We exhibit a $\mathbb{G}_m^\sharp$-action on $\mathbb{A}^{1,\mathrm{dR}} = \mathbb{A}^1/\mathbb{G}_a^\sharp$. Indeed, $\mathbb{G}_m^\sharp$ acts on $\mathbb{G}_a^{\sharp,(1)}$ simply by scaling $\lambda\colon t^p/p \mapsto \lambda t^p/p$. This then embeds into $\mathbb{G}_a^\sharp$ as the kernel down to $\mathbb{G}_a$. The action follows.

1.6.2 Witt Vectors

We now move to the general case. Motivated by Proposition 1.125, we try to find a "universal" algebra with a Frobenius lift, which is the Witt vectors.

Lemma 1.129. There is a unique flat $\mathbb{Z}_p$-algebra $\mathcal{O}(W)$ with a Frobenius lift $F\colon \mathcal{O}(W) \to \mathcal{O}(W)$ such that $F(y) \equiv y^p \pmod p$ for all y satisfying the following universal property: for every map $\mathbb{Z}_p[x] \to R$ with R flat and admitting a Frobenius lift factors uniquely through $\mathcal{O}(W)$.

Proof. As one does in commutative algebra, we simply define

$$\mathcal{O}(W) \coloneqq \mathbb{Z}_p\left[x, F(x), \frac{F(x)-x^p}{p}, F\left(\frac{F(x)-x^p}{p}\right), \ldots\right],$$

where $F(x)$ is viewed as a formal variable, and then one explicitly finds some generators and relations. There is some work to show that W is flat, and it satisfies the universal property. ■

Definition 1.130 (Witt vectors)**.** We define the *Witt vector scheme* W to be the scheme $\operatorname{Spec} \mathcal{O}(W)$.

Proposition 1.131. There is a unique ring scheme structure on W such that the map $W \to \mathbb{A}^1$ is a ring map. In fact, $\mathcal{O}(W)$ admits a presentation

$$\mathcal{O}(W) = \mathbb{Z}_p[a_0, a_1, \ldots]$$

so that $F^n(a_0) = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$.

Proof. Omitted. ■

Example 1.132. One has that $W(\mathbb{F}_p) = \mathbb{Z}_p$.

Remark 1.133. Here is how to work formally with $W(R)$: as a set, we know that $W(R) = \prod_{m\geq 0} R$, and we must have that the "ghost component" map

$$\prod_{m\geq 0} R \to R$$

given by $\mapsto a_0^{p^n} + \cdots + p^n a_n$ is a ring map. For example, $(a+b)_0 = a_0 + b_0$, and $(a+b)_1 = a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0+b_0)^p}{p}$.

Remark 1.134. The Witt vectors W admit a Frobenius lift, which continues to be a ring map.

We are now ready to give some kind of Frobenius lift to $\mathbb{A}^{1,\mathrm{dR}}$.

Lemma 1.135. The de Rham stack $\mathbb{A}^{1,\mathrm{dR}}$ is isomorphic to $[W/pW]$, where pW means that W is acting additively on W by multiplication by p.

Proof. It turns out that the kernel of $F\colon W \to W$ is $\mathbb{G}_a^\sharp$. For example, $(Fa)_0 = a_0^p + pa_1$, so a_0 has pth power divisible by p, which explains why divided powers may appear (and allows us to define a map $\mathbb{G}_a^\sharp \to \ker F$). ■

Proof of Theorem 1.123. We would like a ring stack action of $\mathbb{G}_m^\sharp$ on $\mathbb{A}^{1,\mathrm{dR}}$; this then formally goes to all X^{dR}s. By Lemma 1.135, we know that $\mathbb{A}^{1,\mathrm{dR}} = [W/pW]$. Taking units, we know that $\mathbb{G}_m^\sharp$ is isomorphic to the kernel of $F\colon W^\times \to W^\times$. Thus, $\mathbb{G}_m^\sharp$ is able to rescale W, and it intertwines the multiplication-by-p map, where the target W has trivial action by $\mathbb{G}_m^\sharp$.

Let's quickly explain why one may need a smooth lift of X. Suppose in fact that X admits a lift $\widetilde{X}$ fully to $W(k)$. Then, working locally on X, we may assume that $\widetilde{X}$ is a locally complete intersection (e.g., the fiber of some $f\colon \mathbb{A}^n \to \mathbb{A}^m$ even over $W(k)$), and then the action on X^{dR} is found by pullback on ring stacks $f^{\mathrm{dR}}\colon \mathbb{A}^{n,\mathrm{dR}} \to \mathbb{A}^{m,\mathrm{dR}}$. ■

Remark 1.136. If we merely have a smooth lift to $W_2(k)$, then one has to redo the argument replacing Ws with W_2s everywhere.

THEME 2
HITCHIN SYSTEMS

These talks were given by Pavel Etingof.

2.1 February 18

Here we go.

2.1.1 Principal Bundles

Let X be a smooth irreducible projective curve over a field k, and let G be a split, connected reductive group over k.

Example 2.1. We will usually take $k = \mathbb{C}$ and G to be some kind of linear group.

Definition 2.2 (principal bundle)**.** A *principal G-bundle* is a variety P over k equipped with a map $\pi \colon P \to X$ and a right G-action which preserves π and locally on X is isomorphic to $G \times U \to U$.

Remark 2.3. There was some disagreement about what "locally" means. In general, we should work étale-locally, but it is enough to work Zariski-locally if $k = \mathbb{C}$.

Remark 2.4. Concretely, we may define a G-bundle by choosing an (étale) atlas $\{U_i\}$ on X and some "clutching maps" $g_{ij} \colon U_{ij} \to G$ satisfying the cocycle conditions $g_{ii} = 1$ and $g_{ij}g_{ji} = 1$ and $g_{ij}g_{jk}g_{ki} = 1$. Indeed, given the variety P, we take the atlas $\{U_i\}$ to be a trivializing cover for P, and g_{ij} is induced by the composite

$$G \times U_i \cong \pi^{-1}(U_i) \supseteq \pi^{-1}(U_ij) \subseteq \pi^{-1}(U_j) \cong G \times U_j.$$

Conversely, given the atlas $\{U_i\}$, we define P by taking a disjoint union of the varieties $\{G \times U_i\}_i$ and then gluing along the g_{ij}s; the resulting P is in fact a variety by some faithfully flat descent.

Remark 2.5. Let $\mathcal{O}_{X,G}$ be the sheaf of functions on X valued in G. The previous remark explains that the data of a principal G-bundle comes from some Čech 1-cocycle valued in $\mathcal{O}_{X,G}$, and one can check adjusting such a cocycle by a coboundary does nothing to the isomorphism class of the actual bundle. In fact, one finds that cocycle classes are in bijection with isomorphism classes of bundles, so $\mathrm{H}^1_{\text{ét}}(X; \mathcal{O}_{X,G})$ classifies our bundles. Note that this $\mathrm{H}^1_{\text{ét}}(X; \mathcal{O}_{X,G})$ is not even a group in general!

Our bundles can be classified by stacks.

Definition 2.6. There is a stack $\mathrm{Bun}_G(X)$ which classifies G-bundles on X: namely, given a test scheme T, maps $T \to \mathrm{Bun}_G(X)$ are exactly given by G-bundles on $X \times T$.

Example 2.7. If $G = \mathrm{GL}_1$, then principal bundles are line bundles, so $\mathrm{Bun}_{\mathrm{GL}_1}(X) = \mathrm{Pic}\, X$.

Remark 2.8. It turns out that $\mathrm{Bun}_G(X)$ is a smooth algebraic stack.

Example 2.9. Here is an example of a stack: given an algebraic group H acting on a scheme Y (not necessarily freely), the quotient stack $[Y/H]$ is defined by its functor of points. Indeed, maps from a test scheme T to $[Y/H]$ are given by a pair (E, φ) of a principal H-bundle on S and a map $\varphi\colon E \to Y$ which commutes with the H-action. The point is that a test scheme T should fit into a commutative diagram as follows.

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ S \arrow[r] & {[Y/H]} \end{tikzcd}$$

However, it is possible for $\mathrm{Bun}_G(X)$ to have stabilizers of arbitrary dimension, so it is not a quotient stack. Instead, it turns out to be a nested union of quotient stacks.

2.2 February 23

Here we go.

2.2.1 Bundles on $\mathbb{P}^1$

We continue letting X be a curve over $\mathbb{C}$.

Example 2.10. The simplest stack is $\mathrm{B}G$ is the quotient stack $[\mathrm{pt}/G]$. Thus, maps $S \to \mathrm{B}G$ are the principal G-bundles on S.

Example 2.11. The stack $\mathrm{Pic}\, X := \mathrm{Bun}_{\mathrm{GL}_1}(X)$ classifies line bundles on X. (Indeed, one is simply trying to glue scalars.) For example, when X is a curve, then $\mathrm{Pic}\, X$ admits a surjective degree map, so it has infinitely many automorphisms. Even the degree-zero component $\mathrm{Pic}^0 X$ may not quite be a variety because line bundles may have automorphisms.

Example 2.12. With $G = \mathrm{GL}_n$, we find that $\mathrm{Bun}_G(X)$ classifies vector bundles on X of rank n.

Example 2.13. With $G = \mathrm{SL}_n$, we find that $\mathrm{Bun}_G(X)$ classifies vector bundles with trivial determinant. Similarly, if $G = \mathrm{PGL}_n$, then we are classifying vector bundles up to equivalence by tensoring with line bundles.

Example 2.14. We classify GL_2-bundles on the complex curve $\mathbb{P}^1$ as direct sums of two line bundles.

Proof. Note that any bundle E admits some meromorphic section on $X = \mathbb{P}^1$, so $\operatorname{Hom}(\mathcal{O}(m), E) \neq 0$ for sufficiently small m, and this map is an inclusion for generic such maps. Thus, we get a short exact sequence

$$0 \to \mathcal{O}(m) \to E \to \mathcal{O}(n) \to 0,$$

where m is small, and $\mathcal{O}(n)$ is the line bundle representing the quotient. (The quotient is a line bundle for generic inclusions $\mathcal{O}(m) \to E$.) Recall that the transition function for $\mathcal{O}(m)$ is given by z^m (used to transition between charts). Thus, the short exact sequence given above shows that the transition function for E is given by

$$\begin{bmatrix} z^m & f(z) \\ 0 & z^n \end{bmatrix},$$

where $f(z) \in k[z, 1/z]$. By changing the choice of trivialization, one can multiply this matrix on the left or right by some upper-triangular matrices, and one can use such moves to force $f(z)$ to have monomials concentrated in degrees between m and n.

One can now study the actual isomorphism class of taking each affine open for E. In particular, letting r be the maximal integer which admits an inclusion $\mathcal{O}(r) \to E$, which we see lives between m and n. By massaging with these transition maps, one is able to show that $r \leq (m+n)/2$ with some degree arguments. Eventually, one finds that the corresponding short exact sequence

$$0 \to \mathcal{O}(r) \to E \to \mathcal{O}(r') \to 0,$$

with $r' := m+n-r$, which by using a similar argument to before forces this short exact sequence to genuinely split. While we're here, we remark that these r and r' have been chosen canonically from E, and so the decomposition of E into $\mathcal{O}(r) \oplus \mathcal{O}(r')$ has unique r and r'. ■

Remark 2.15. Here is another way to see the class of short exact sequence of E: the short exact sequence is some class in

$$\operatorname{Ext}^1(\mathcal{O}(n), \mathcal{O}(m)) \cong \mathrm{H}^1(X; \mathcal{O}(m-n)),$$

which by Serre duality is

$$\mathrm{H}^0(X; \mathcal{O}(n-m-2))$$

because the canonical bundle is $\mathcal{O}(-2)$. But this space gives the homogeneous polynomials of degree $n-m-2$, which has dimension $\max\{n-m-1, 0\}$.

Remark 2.16. One can use this to find connected components of $\mathrm{Bun}_{\mathrm{GL}_2}(\mathbb{P}^1)$, which are labeled by their degree $m+n$ where the bundles are $\mathcal{O}(m) \oplus \mathcal{O}(n)$. Further, the points inside a connected component can be labeled by the invariant $n-m$ when $n \geq m$, so the points are $P_0, P_1, \ldots, P_{n-m}$.

It turns out that P_i specializes to P_j whenever $j \geq i$, which one sees by suitably choosing degenerations of families of bundles. For example, consider the family of bundles E_t with the clutching map $\begin{bmatrix} z^{-1} & t \\ 0 & z \end{bmatrix}$. Then $E_t \cong \mathcal{O} \oplus \mathcal{O}$ for $t \neq 0$ (by explicitly finding a global trivialization), but $E_0 \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$

One can upgrade this discussion as follows.

Theorem 2.17 (Grothendieck)**.** A vector bundle of rank N on $\mathbb{P}^1$ can be uniquely written as

$$\bigoplus_{i=1}^{N} \mathcal{O}(m_i),$$

where $\{m_i\}$ is a decreasing sequence of integers.

Proof. We proceed by induction. The case of $N = 1$ is classical, so we focus on the inductive step. Let E be a vector bundle of rank $N + 1$. As before, choose maximal m_0 so that there is an embedding $\mathcal{O}(m_0) \to E$, and generic such choices makes the quotient into another vector bundle E'. Thus, by the induction, we have a short exact sequence

$$0 \to \mathcal{O}(m_0) \to E \to \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_N) \to 0.$$

Now, for each i, we can let E' be the kernel of E projecting onto the sum of the $\mathcal{O}(m_j)$s with $j \neq i$ and $j > 0$, so there is a short exact sequence

$$0 \to \mathcal{O}(m_0) \to E' \to \mathcal{O}(m_i) \to 0.$$

The argument on transition functions from before on $N = 2$ shows that $m_0 \geq (m_0 + m_i)/2$, so $m_0 \geq m_i$ for all i follows. A calculation with Ext groups as before shows that the short exact sequence for E must split, so we are done! (As before, to show uniqueness of the sequence $\{m_i\}$, we simply note that the maximal term m_0 is uniquely constructed above.) ■

Remark 2.18. This argument is more or less due to Birkhoff, though reinterpreted by Grothendieck.

More generally, one has the following.

Theorem 2.19. Fix a reductive group G with maximal torus T. Then a G-bundle on $\mathbb{P}^1$ admits a T-structure, and there is a bijection given by induction

$$[\operatorname{Bun}_T/W] \to \operatorname{Bun}_G$$

Remark 2.20. The moral is that bundles can be labeled by dominant integral weights for the dual group.

2.2.2 Double Quotients

We will want the following motivational tool.

Theorem 2.21 (Harder). Fix a reductive group G over an algebraically closed field k. Then any G-bundle on an affine curve is trivial.

Proof. Omitted. This is pretty hard! ■

The moral is that one expects to be able to glue bundles over just two charts on a projective curve. For us, we will use the affine chart $X \setminus \{x\}$ and the formal disk D_x around x, which have intersection $D_x^\times := D_x \setminus \{x\}$.

Notably, the transition function on $D_x^\times$ is some element $g \in G(K)$, where $K = k[D_x^\times]$; note that K is isomorphic to $k((t))$, where t is a choice of uniformizer.[1] Thus, elements $g \in G(K)$ parameterize bundles; note that there is no required cocycle condition because there are only two charts.

However, two G-bundles will be isomorphic if we decide to take a quotient on the right side by $G(\mathcal{O})$ and on the left side by $G(R)$, where $R = k[X \setminus \{x\}]$. It follows that

$$\operatorname{Bun}_G(X) = G(R)\backslash G(K)/G(\mathcal{O}).$$

We are now motivated to make the following definition.

[1] Technically, any such g spreads out to an open neighborhood of x, which then allows us to glue.

Definition 2.22 (affine Grassmannian)**.** Fix an affine algebraic group G. Then we define the *affine Grassmannian* to be

$$\mathrm{Gr}_G := G(K)/G(\mathcal{O}),$$

which we note is just $G(k((t)))/G(k[[t]])$. In particular, Gr_G does not depend on the choice of curve.

We can tell the same story for any finite set of points $S \subseteq X$, from which we find that

$$\mathrm{Bun}_G(X) = G(R)\backslash \prod_{x\in S} G(K_x)/\prod_{x\in S} G(\mathcal{O}_x).$$

This does not quite work when G is no longer assumed to be semisimple because G does not trivialize on $X \setminus \{x\}$. In fact, G has no reason to trivialize on any given set of points: even line bundles have no reason to trivialize away from a given set of points.

The right thing to do is to simply take the colimit over all points in $X(k)$. Then we still know that any G-bundle can be trivialized away from finite set of points in $X(k)$, and we get the following.

Proposition 2.23. Fix a curve X over an algebraically closed field k, and let G be an affine algebraic group over k. Then

$$\mathrm{Bun}_G(X)(k) = G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_{\mathbb{A}}),$$

where $F = k(X)$, $\mathbb{A} = \prod_{x\in X(k)}(K_x, \mathcal{O}_x)$, and $\mathcal{O}_{\mathbb{A}} = \prod_{x\in X(k)} \mathcal{O}_x$.

Remark 2.24. The restricted direct product appears as a colimit of those "meromorphic power series" which are regular almost everywhere.

Remark 2.25. The same discussion works even if k is not algebraically closed. In this case, the restricted product should take place over the Galois orbits of $X(\overline{k})$.

Remark 2.26. This realizes $\mathrm{Bun}_G(X)(k)$ as an arithmetic quotient. Namely, let F be a number field or field of functions of a curve. Then let $V(F)$ be its set of valuations (constant on the base field when F is over a curve), and let F_v be the completion for each $v \in V(F)$, and let $\mathcal{O}_v$ be the ring of integers in F_v when v is nonarchimedean. Then the arithmetic quotient is

$$G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_{\mathbb{A}}).$$

The case where F is a number field is of interest to number theorists. When $F = \mathbb{Q}$, one finds that this double quotient is $G(\mathbb{Z})\backslash G(\mathbb{R})$ when one has weak approximation, meaning that $\prod_p (G(\mathbb{Q}_p), G(\mathbb{Z}_p)) = G(\mathbb{Q})\prod_p G(\mathbb{Z}_p)$. For example, modular forms arise from the case of $G = \mathrm{SL}_2$.

2.2.3 Higgs Fields

Let's use the double quotient to write down some tangent spaces. Let E_γ be the line bundle defined by $\gamma \in G(k((t)))$. If we are in the case that $k = \mathbb{C}$ so that G is a Lie group with Lie algebra $\mathfrak{g}$, then taking the quotient shows that

$$T_{E_\gamma} \mathrm{Bun}_G(X) = \frac{\mathfrak{g}((t))}{\left(\mathrm{Ad}_\gamma^{-1} \mathfrak{g}(X \setminus \{x\}) + \mathfrak{g}[[t]]\right)}.$$

This is the space of Čech 1-cocycles for the "adjoint" vector bundle $\operatorname{ad} E_\gamma$ modulo coboundaries for our given open cover, so we see that $E := E_\gamma$ ought to have

$$T_E \mathrm{Bun}_G(X) \cong \mathrm{H}^1(X; \operatorname{ad} E),$$

which by Serre duality is $\mathrm{H}^0(X; K_X \otimes (\operatorname{ad} E)^*)^*$. If one fixes an invariant inner product on $\mathfrak{g}$, then we may identify $\operatorname{ad} E$ with its dual. Removing the dual on the outside produces Higgs fields.

Definition 2.27 (Higgs field)**.** Fix a complex curve X, and let G be an affine algebraic group. Then the space of *Higgs fields* is

$$\mathrm{H}^0(X; K_X \otimes \operatorname{ad} E).$$

We let $\mathcal{M}_G(X)$ be the *Hitchin stack* classifying pairs (E, φ), where E is a principal G-bundle, and φ is a Higgs field.

Remark 2.28. This argument also works over an algebraically closed field k of sufficiently large characteristic.

The above discussion motivates us to study 1-forms on X valued in $\operatorname{ad} E$. For simplicity, we assume that a generic bundle has automorphism group $Z(G)$, which is true as long as G is semisimple and the genus of X exceeds 1.

Notation 2.29. Fix a complex curve X, and let G be an affine algebraic group. Let $\operatorname{Bun}_G^0(X)$ be the open subset of regularly stable G-bundles.

Remark 2.30. Any regularly stable bundle has automorphism group $Z(G)$.

Example 2.31. If $G = \mathrm{GL}_n$ (or PGL_n or SL_n), then "stable" means that for all sub-bundles $F \subseteq E$, we have

$$\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}.$$

Remark 2.32. Take G semisimple. For a regular stable E, one finds that $\mathrm{H}^0(X; \operatorname{ad} E) = \operatorname{Lie} \operatorname{Aut} E$, which vanishes because $Z(G)$ is finite. Thus, Serre duality implies $\mathrm{H}^1(X; K_X \otimes \operatorname{ad} E) = 0$, so

$$\dim \mathrm{H}^0(X; K_X \otimes \operatorname{ad} E) = \chi(K_X \otimes \operatorname{ad} E).$$

Because $\operatorname{ad} E$ is some degree-zero bundle, one finds that the right-hand side is $\chi(K_X) \dim \mathfrak{g}$. (Indeed, one can reduce to the case of line bundles by producing a filtration of $\operatorname{ad} E$ by degree-zero vector bundles.) Thus, our dimension is $\dim \mathfrak{g}(g(X) - 1)$.

Thus, we see that $\operatorname{Bun}_G^0(X)$ is a smooth variety of dimension $\dim \mathfrak{g}(g(X) - 1)$.

Remark 2.33. If G is reductive, then a similar calculation shows that

$$\dim \operatorname{Bun}_G^0(X) = \dim \mathfrak{g}(g(X) - 1) + \dim Z(\mathfrak{g}).$$

For example, $G = \mathrm{GL}_n$ gives $n^2(g-1)+1$.

This has application to our Hitchin stack $\mathcal{M}_G(X)$ because $T^* \operatorname{Bun}_G^0(X)$ is an open subscheme of the Hitchin stack $\mathcal{M}_G(X)$. In fact, $T^* \operatorname{Bun}_G^0(X)$ is a symplectic variety because cotangent spaces have a canonical symplectic structure.

2.2.4 Hamiltonian Reduction

Let's recall something about Hamiltonian reduction. If a group H with Lie algebra $\mathfrak{h}$ acts on a variety Y, then H acts no T^*Y in a Hamiltonian way. There is also a moment map $\mu\colon T^*Y \to \mathfrak{h}^*$ which is dual to the canonical map $\mathfrak{h} \to \operatorname{Vect} Y$ given by the infinitesimal action. (Namely, the Lie algebra should act infinitesimally by vector fields.)

Now, if H acts freely, then

$$\mu^{-1}(0)/H \cong T^*(Y/H).$$

In fact, H-stable functions f on T^*Y automatically descend to $T^*(Y/H)$. The left-hand object is called the Hamiltonian reduction. In this way, one sees that $T^* \operatorname{Bun}_G(X)$ is the Hamiltonian reduction of $T^*G(K)$ by $G(R) \times G(\mathcal{O})$.

Remark 2.34. Here is something called the Poisson bracket: given a nondegenerate symplectic form ω on M, meaning that $d\omega = 0$, then ω^{-1} defines a non-degenerate symplectic form on $\wedge^2 TM$. Accordingly, we may define

$$\{f, g\} \coloneqq \left\langle \omega^{-1}, df \otimes dg \right\rangle,$$

where f and g live on $\mathcal{O}(M)$. For example, on $M = k^{2n}$ with coordinates $\{x_i, p_i\}$, there is an explicit way to write this out in terms of some derivatives.

2.3 February 25

Today we talk about the classical Hitchin integrable system.

2.3.1 The Classical Hitchin Integrable Systems

For today, we continue to let X be an irreducible smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$, and G is a connected reductive group over $\mathbb{C}$.

Definition 2.35 (Hitchin base)**.** The *Hitchin base* of X for PGL_n is the vector space

$$\mathcal{B} \coloneqq \bigoplus_{i=1}^{n-1} \mathrm{H}^0\left(X; K_X^{\otimes(i+1)}\right).$$

Remark 2.36. If $m \geq 2$, then

$$\dim \mathrm{H}^0\left(X; K_X^{\otimes m}\right) = (2m-1)(g-1)$$

by the Riemann–Roch theorem, so

$$\dim \mathcal{B} = \sum_{m=2}^{n} (2m-1)(g-1) = \left(n^2 - 1\right)(g-1).$$

Thus, $\mathcal{B}$ has the same dimension as $\operatorname{Bun}_{\mathrm{PGL}_n}(X)$.

Definition 2.37 (Hithin map)**.** Suppose $G = \mathrm{PGL}_n$. The *Hitchin map* is the map $p\colon T^* \operatorname{Bun}_G^0(X) \to \mathcal{B}$ is the map

$$\rho(E, \varphi) \coloneqq \left(\operatorname{tr} \wedge^2 \varphi, -\operatorname{tr} \wedge^3 \varphi, \ldots, (-1)^n \operatorname{tr} \wedge^n \varphi\right).$$

To make sense of this, we note that $\varphi \in \mathrm{H}^0(X; K_X \otimes \mathfrak{sl}(E))$ has trace zero. Notably, taking each further wedge power increases the level of the differentials.

Theorem 2.38 (Hitchin)**.** Suppose $G = \mathrm{PGL}_n$. Then the map p is generically a Lagrangian fibration, meaning that the generic fibers are Lagrangian.

Proof. We will do this later! ■

Remark 2.39. For example, a dimension argument is able to show that p is dominant.

Remark 2.40. Concretely, this means that coordinate functions on $\mathcal{B}$ will pull back to algebraically independent functions, and their Poisson brackets vanish. Explicitly, by choosing a basis $\{b_j\}$ of $\mathcal{B}$ (compatible with the direct sum decomposition of $\mathcal{B}$), we see that

$$p(E,\varphi)=\sum_j H_j(E,\varphi)b_j,$$

where $H_j\colon T^*\operatorname{Bun}_G^0(X)\to\mathbb{C}$ are some algebraically independent coordinate functions, and $\{H_i,H_j\}=0$ for all i and j. This is exactly the data of an integrable system.

Remark 2.41. For example, we can see that the fiber over 0 is the nilpotent cone. Thus, p is not smooth! However, p turns out to be flat.

One can define the Hitchin map in other situations. For example, GL_n should add in the trace.

Definition 2.42 (Hitchin map)**.** The *Hitchin base* of X for GL_n is the vector space

$$\mathcal{B}\coloneqq\bigoplus_{i=0}^{n-1}\mathrm{H}^0\left(X;K_X^{\otimes(i+1)}\right).$$

In this case, the *Hitchin map* $p\colon T^*\operatorname{Bun}_G^0(X)\to\mathcal{B}$ is the map

$$\rho(E,\varphi)\coloneqq\left(-\operatorname{tr}\varphi,\operatorname{tr}\wedge^2\varphi,-\operatorname{tr}\wedge^3\varphi,\ldots,(-1)^n\operatorname{tr}\wedge^n\varphi\right).$$

Remark 2.43. Here, we find that $\dim\mathcal{B}=\left(n^2-1\right)(g-1)=n^2(g-1)+1$, which is $\dim\operatorname{Bun}_G^0(X)$.

We turn to general groups. To define our Hitchin base, we must generalize our functions $\operatorname{tr}\wedge^\bullet\varphi$. The importance of these functions is that they are functions on $\mathfrak{g}$ which are G-invariant (with the adjoint action). This is desirable because we want to define a function on pairs (E,φ), but changing charts may adjust the local coordinates of E by some conjugation by elements of G. Thus, whatever function on φ we choose needs to be invariant!

We are thus motivated to study $\mathbb{C}[\mathfrak{g}]^G$, for which we recall the following result.

Theorem 2.44 (Chevalley)**.** Fix a connected reductive group G over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and rank r. Then there are homogeneous polynomials $\{_1,\ldots,Q_r\}$ for which

$$\mathbb{C}[\mathfrak{g}]^G=\mathbb{C}[Q_1,\ldots,Q_r].$$

In fact, the tuple of degrees $\{\deg Q_i\}_i$ are independent of the precise choices of $Q_\bullet$s.

Example 2.45. For $G=\mathrm{GL}_n$, the degrees are $\{1,\ldots,n\}$, and one can take $Q_i\coloneqq(-1)^i\operatorname{tr}\wedge^i\varphi$ for each $\varphi\in\mathfrak{g}$.

Example 2.46. For $G=\mathrm{SL}_n$, the degrees are $\{2,\ldots,n\}$. One can take $Q_i\coloneqq(-1)^i\operatorname{tr}\wedge^i\varphi$ again, where $i\in\{2,\ldots,n\}$.

Remark 2.47. It turns out that the degrees $\{d_i\}$ of the group G arise via the decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{r} L_{2(d_i-1)}$$

as a representation of the principal $\mathfrak{sl}_2$-triple. (The principal $\mathfrak{sl}_2$-triple is given by setting e to be the sum of the simple e_is, which is a regular nilpotent element.)

Definition 2.48 (Hitchin map)**.** Let the degrees of G be $\{d_i\}_{i=1}^r$. The *Hitchin base* for G is

$$\mathcal{B}_{G,X} := \bigoplus_{i=1}^{r} \mathrm{H}^0\left(X; K_X^{\otimes d_i}\right).$$

When no confusion is possible, we write $\mathcal{B}$ for $\mathcal{B}_{G,X}$. We define the *Hitchin map* $p\colon T^*\operatorname{Bun}_G^0(X) \to \mathcal{B}$ given by

$$p(E,\varphi) := (Q_1(\varphi), \ldots, Q_r(\varphi)).$$

Remark 2.49. The space $\mathcal{B}_{G,X}$ admits a "canonical" definition as the space of sections of the weighted vector bundle $\mathfrak{h}/W \otimes K_X$.

Remark 2.50. By the same Riemann–Roch argument, one finds that

$$\dim \mathcal{B} = \sum_{i=1}^{r} (2d_i - 1)(g-1) + \#\{i : d_i = 1\}.$$

The latter term is the rank $\dim Z(G)$, so the entire dimension is $\dim \operatorname{Bun}_G^0(X)$.

We are now ready to restate Theorem 2.38.

Theorem 2.51. Fix an irreducible smooth projective curve X over $\mathbb{C}$ of genus $g \geq 2$, and let G be a connected reductive group over $\mathbb{C}$. Then the Hitchin map $p\colon T^*\operatorname{Bun}_G^0(X) \to \mathcal{B}$ is generically a Lagrangian fibration.

We will prove that the relevant $H_\bullet$s are Poisson-commutative for all semisimple G, but we will only show the independence for $G = \mathrm{GL}_n$.

Proof of commutativity. For the Poisson commutativity, we use Hamiltonian reduction. Choose a local coordinate t at some $x \in X$. Then our discussion of double quotients granted

$$\operatorname{Bun}_G(X) = G(R)\backslash G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]),$$

where $R = \mathbb{C}[X \setminus \{x\}]$. (We have identified $\mathbb{C}[[t]]$ with $\widehat{\mathcal{O}}_x$.) Then the pre-image of $\operatorname{Bun}_G^0(X)$ is some $G(\mathbb{C}((t)))^0$.

Now, the group $H := G(R) \times G(\mathbb{C}[[t]])/Z(G)$ acts freely on $Y := G(\mathbb{C}[[t]])^0$, and the quotient is $\operatorname{Bun}_G^0(X)$. Because this action is free, we conclude that

$$T^*(Y/H) = T^*Y/H,$$

where the latter is the Hamiltonian reduction $\mu^{-1}(\{0\})$, where $\mu\colon T^* \to \mathfrak{h}^*$ is the moment map. Now, if $\{F_i\}$ are H-invariant Poisson-commutative functions on T^*Y, then they descend to $\mu^{-1}(\{0\})$ by restriction and then to $\mu^{-1}(\{0\})/H$ by the invariance, so they give functions in $T^*(Y/H)$. In fact, these descended functions continue to be Poisson-commutative. The moral of this reduction is that it is comparatively easier to find functions on T^*Y.

We are thus on the hunt for Poisson-commutative functions on

$$T^*G(\mathbb{C}((t)))^0 = \mathfrak{g}((t))^* \times G((t))^0.$$

The idea is to use left translation to produce invariant functions. Let's start by computing this dual. Note that $\mathfrak{g}((t)) = \mathfrak{g}\otimes\mathbb{C}((t))$, and there is a canonical invariant pairing $\mathbb{C}((t))\times\mathbb{C}((t))\,dt \to \mathbb{C}$ given by the residue, so $\mathfrak{g}((t))^* = \mathfrak{g}\otimes\mathbb{C}((t))\,dt$.

Let's now define our functions. By definition, a point in $T^*G(\mathbb{C}((t)))$ is represented by a pair (g,φ), where $g\in G((t))$, and $\varphi\in\mathfrak{g}\otimes\mathbb{C}((t))\,dt$. But this φ is a Higgs field in the punctured formal disk $D_x^\times$, so we can use our prior discussion to get well-defined maps

$$\varphi\mapsto (Q_1(\varphi),\ldots,Q_r(\varphi)),$$

where $Q_i(\varphi)\in\mathbb{C}((t))\,(dt)^{\otimes d_i}$. Taking Laurent coefficients of Q_i produces many functions $Q_{i,n}$ on $T^*G((t))$. They are conjugation-invariant (because Q_i is), and they are invariant by left translation (because of how we parallelized our cotangent space), so they are invariant under the two-sided action by $G(R)\times G(\mathbb{C}[[t]])$.

Let's check that the $Q_{i,n}$s are Poisson-commutative. In general, for any Lie group A with Lie algebra $\mathfrak{a}$, we are finding a Poisson structure on $\mathfrak{a}^*$ given by the Lie bracket: choosing a basis $\{a_i\}$ of $\mathfrak{a}$ so that $[a_i,a_j]=\sum_k c_{ij}^k a_k$, one defines

$$\{f,g\} \coloneqq \sum_{i,j,k} c_{ij}^k \frac{\partial f}{\partial a_i}\frac{\partial g}{\partial a_j} a_k$$

for any $f,g\colon \mathfrak{a}^*\to\mathbb{C}$. In particular, if f is invariant, then $\{a_j,f\}=0$ for all j by a direct calculation, so $\{-,f\}$ vanishes in general. The same sort of argument works in our situation because the functions $Q_{i,n}$ are suitable invariant.

Thus, we receive functions which descend to Y/H on $\operatorname{Bun}_G^0(X)$. Poisson-commutativity will now follow as soon as we show that these functions descend to the required Hitchin map p. Well, Hamiltonian reduction produces a diagram

$$\begin{tikzcd}
\mu^{-1}(\{0\}) \arrow[r] \arrow[d] & T^*G(\mathbb{C}((t)))^0 \arrow[r, "Q_1,\ldots,Q_r"] & \bigoplus_{i=1}^r \mathbb{C}((t))\,dt^{\otimes d_i} \\
T^*\operatorname{Bun}_G^0(X) \arrow[rr] & & \bigoplus_{i=1}^r \mathrm{H}^0\left(X;K_X^{\otimes d_i}\right) \arrow[u]
\end{tikzcd}$$

where the right vertical map is given by expanding the global differentials at x. It is not too hard to check that this diagram commutes, so the p^*b_js are linear combinations of the $Q_{i,n}$s, so they are Poisson commutative. ■

Remark 2.52. The expansion of p^*b_j in terms of the $Q_{i,n}$s is far from unique!

2.3.2 Spectral Curve

It remains to show the algebraic independence in Theorem 2.51, which we only do for $G=\mathrm{GL}_n$. For this, we will use the notion of the "spectral curve."

Definition 2.53 (spectral curve)**.** Take $G = \mathrm{GL}_n$. For $b \in \mathcal{B}$, write $b = (b_1, \ldots, b_n)$, where we have $b_i \in \mathrm{H}^0(X; K_X^{\otimes i})$ for each i, and consider the factorized polynomial

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_n = \prod_{i=1}^{n}(\lambda - \lambda_i).$$

Thus, for each $x \in X$, we receive a set of roots $\{\lambda_1, \ldots, \lambda_n\}$in $T_x^* X = K_{X,x}$. As $x \in X$ varies, this cuts out the *spectral curve* C of X. Equivalently, $C \subseteq T^*X$ cut out by the above polynomial (which notably outputs to $(T^*X)^{\otimes n}$).

Remark 2.54. The curve C has no reason to be connected or regular, but it is projective.

Remark 2.55. This is called the spectral curve because if $b = p(E, \varphi)$, then it turns out that C gives the spectrum of φ.

Here are some facts about this spectral curve.

Theorem 2.56 (Hitchin)**.** For generic $b \in \mathcal{B}$, the fiber C_b is smooth and irreducible.

Proof. The fiber being smooth and irreducible is an open condition, so it is enough to exhibit a single $b \in \mathcal{B}$. We will want the following result.

Lemma 2.57. Fix a smooth irreducible curve X over $\mathbb{C}$ of genus $g \geq 2$.

(a) If $\mathcal{L}$ is a line bundle on X of degree at least $2g$, hen a generic sectino of $\mathcal{L}$ has only simple zeroes.

(b) A generic section of $\mathcal{K}_X^{\otimes n}$ has only simple zeroes for $n \geq 2$.

Proof. Quickly, note that (a) implies (b) by a degree calculation. It remains to show (a). Well, let $Y \subseteq \mathrm{H}^0(X; \mathcal{L})$ be the subset of sections which have a double root, so we want to show that Y is positive codimension. By the Riemann–Roch theorem, we know that any line bundle $\mathcal{M}$ of degree $m \geq 2g - 2$ has

$$\dim \mathrm{H}^0(X; \mathcal{M}) = \begin{cases} m - g + 1 & \text{if } \mathcal{M} \neq K_X, \\ g & \text{if } \mathcal{M} = K_X. \end{cases}$$

Now, for each $s \in Y$ with a double root at $z \in X$, we can view s as a section of $\mathcal{L} \otimes \mathcal{O}(-2z)$. But $\deg(\mathcal{L} \otimes \mathcal{O}(-2z)) \geq 2g - 2$. Thus, if $\mathcal{L} \otimes \mathcal{O}(-2z) \neq \mathcal{K}_X$ (which happens for all but finitely many z because this requires $\mathcal{O}(2z)$ to equal a fixed line bundle, which is impossible for divisor reasons), then the space of global sections has dimension $\deg \mathcal{L} - g - 1$. Looping over all zs shows that

$$\dim Y \leq d - g < \dim \mathrm{H}^0(X; \mathcal{L}),$$

so we are done. ■

We work with b of the form $(0, \ldots, 0, s)$, where $s \in \mathrm{H}^0\left(X; \mathcal{K}_X^{\otimes n}\right)$ is chosen generically to have no simple zeroes. Thus, our spectral curve is defined by the equation

$$\lambda^n = s(x),$$

which one can check is smooth and irreducible. Indeed, it is irreducible because it is ■

Remark 2.58. In fact, this argument can also show that the fibers C_b generically have genus $n^2(g-1)+1$. Indeed, one can calculate this genus via the Hurwitz formula with the projection $C_b \to X$. Indeed, the section s admits $2n(g-1)$ simple zeroes, so

$$\chi(C_b) = n\chi(X) - (n-1) \cdot 2n(g-1),$$

which rearranges into the required claim.

Bibliography

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