

735: Automorphic Representations

Nir Elber

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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THEME 1

INTRODUCTION

1.1 January 20

I'm only here for a few weeks, but here we go anyway.

1.1.1 Overview

There will be three parts to the course, approximately going around the trace formula.

1. Representations of real and p -adic groups: we will not cover structure theory of reductive groups, but we will try to say some interesting things.

For p -adic groups, we will mainly follow Bernstein's Harvard notes from 1993; Casselman's notes are also good but are more elementary. Here are some topics: we will talk about the Bernstein center following some notes of Deligne, and we will talk about the Plancherel formula approximately following the statements of Sakellaridis–Venkatesh. Note that the Plancherel formula gains importance because it more or less computes $L^2(G)$.

For real groups, there is a "lifting" problem of topological representations compared to algebraic representations (i.e., (\mathfrak{g}, K) -modules). We may also say something about the construction of representations (focusing on discrete series and inductions) and cohomology of representations. In particular, (\mathfrak{g}, K) -cohomology has an application to locally symmetric spaces. Time-permitting, we can say something about localization theory.

2. The (relative) trace formula: automorphic representations are found in $L^2([G])$, where $[G]$ is some adelic quotient. Then the trace formula classically computes the functional $f \mapsto \text{tr } Rf$, where f is some element of a Hecke algebra, and R is a representation. When $[G]$ is compact, this automatically makes sense and is classical; however, when $[G]$ is not compact, this trace need not be finite. Our goal, then, is to explain how to make sense of such traces. For example, we will discuss Arthur's truncation.
3. Endoscopy and stabilization: we want to say something about how the trace formula can be used to prove Langlands's functoriality conjecture for the standard representations ${}^L G \rightarrow \text{GL}_n$, where G is a classical group. It turns out that this is done via a comparison of trace formulae: on one hand, there is the stable trace formula for G ; on the other hand, we can compare this to some part of the twisted trace formula for GL_n . The comparison between these trace formulae is done via a "stabilization," which we will only discuss in the non-twisted case.

Stabilization starts with the following observation: for a local field $F \in \{\mathbb{R}, \mathbb{Q}_p\}$, there are elements of $\text{SL}_2(F)$ which are not conjugate but become conjugate over \overline{F} . (This is not true for GL_n !) Conjugacy over \overline{F} is known as stable conjugacy. It was discovered that certain representations of $\text{SL}_2(F)$

come in pairs $\{\pi_+, \pi_-\}$ such that the difference of the characters looks like a character of a torus (e.g., supported on the image of a torus in $\mathrm{SL}_2(F)$). This is the most basic instance of endoscopy.

There will be notes embedded into the automorphic project.

1.1.2 Reciprocity

There's not much better to do today, so let's say something about reciprocity. Fix a global field K .

- There ought to be a reciprocity map sending Galois representations $\mathrm{Gal}(\bar{K}/K) \rightarrow \mathrm{GL}_n(\mathbb{C})$ to automorphic representations of GL_n . In fact, these automorphic representations ought to be the ones with vanishing infinitesimal character.
- There ought to be a reciprocity map sending representations of the motivic Galois group to automorphic representations. In fact, these automorphic representations ought to be the ones with integral infinitesimal characters.
- Lastly, there ought to be some conjectural Γ for which its representations correspond to all automorphic representations.

1.2 January 22

Today we start discussing the complex representation theory of p -adic groups. Specifically, we are hoping to say something about the Bernstein center over the next few lectures. The original paper

Remark 1.1. Everything we say will also be fine over base fields of characteristic 0, not necessarily algebraically closed.

1.2.1 Smooth Representations

For now, G will be a totally disconnected, locally compact, second countable topological group.

Definition 1.2 (td group). A td group is a totally disconnected, locally compact, second countable topological group.

Example 1.3. For any affine algebraic group G over a local field K , the topological group $G(K)$ is totally disconnected, locally compact, and second countable.

Example 1.4. The group $G = \mathbb{Z}$ is td.

We are currently interested in representation theory of td groups.

Definition 1.5 (smooth). Fix a td group G . Then a (complex) representation V of G is *smooth* if and only if the stabilizer of any $v \in V$ is open in G . The category of smooth representations is denoted by $\mathcal{M}(G)$.

Remark 1.6. To say a little about where we are going, the theory of the Bernstein center will allow us to decompose the category $\mathcal{M}(G)$ into subcategories indexed by pairs of a Levi subgroup and a supercuspidal representation (up to equivalence). (Such a pair is frequently referred to as a cuspidal support.)

It will be worthwhile to be able to view the category of (smooth) representations of G as a module category. This is the purpose of the Hecke algebra.

Notation 1.7. Fix a td group G . Then $S(G)$ denotes the collection $C_c^\infty(G)$ of compactly supported, locally compact functions on G . The collection $C^\infty(G)$ consists of the uniformly smooth functions, which are those which are locally constant and smooth as a representation of $G \times G$.

Definition 1.8 (Hecke algebra). Fix a td group G . Then the Hecke algebra $\mathcal{H}(G)$ is the convolution algebra of complex, compactly supported, smooth Borel measures on G (meaning that they are invariant under translation by an open subgroup of G).

Remark 1.9. One can show that $\mathcal{H}(G) = S(G) dg$, where dg is some choice of Haar measure. Note that the definition of $\mathcal{H}(G)$ descends canonically to \mathbb{Q} upon choosing dg to have integral measure on some chosen open subgroup of G .

Remark 1.10. The algebra $\mathcal{H}(G)$ is an “idempotent algebra,” meaning that any finite collection $\mathcal{F} \subseteq \mathcal{H}(G)$ admits an idempotent e for which

$$ef = fe = f$$

for all $f \in \mathcal{F}$. For example, one can take e to be the characteristic measure of some small compact open subgroup built from the f s. It follows that

$$\mathcal{H}(G) = \lim_{K \subseteq G} e_K \mathcal{H}(G) e_K.$$

Definition 1.11. Fix a td group G . Then an $\mathcal{H}(G)$ -module M is *non-degenerate* if and only if the natural map

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \rightarrow M$$

is surjective.

Remark 1.12. It looks like the natural map should always be an isomorphism, but this is not the case because $\mathcal{H}(G)$ is not necessarily unital!

Remark 1.13. It turns out that the surjectivity of this map is equivalent to being an isomorphism.

Proposition 1.14. Fix a td group G . The natural functor from $\mathcal{M}(G)$ to the category of non-degenerate $\mathcal{H}(G)$ -modules is an equivalence. The functor sends a representation V of G to the Hecke module V with Hecke action given by

$$(f dg) \cdot v := \int_G f(g)(gv) dg.$$

Proof. We will say nothing of content, but let's describe the inverse functor. Given the isomorphism

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} V \rightarrow V,$$

one can construct a G -action on V by figuring out how δ_g should act on V , which is done by taking a limit back in $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} V$. ■

Corollary 1.15. Fix a td group G . Then $\mathcal{M}(G)$ is abelian and has enough projectives.

Proof. The category of $\mathcal{H}(G)$ -modules has these properties, and one can check that they descend to the subcategory of non-degenerate modules. The difficult point is to check that there are enough projectives, so let's explain this. Fix a non-degenerate module M . Then each $m \in M$ admits an idempotent $e_m \in \mathcal{H}(G)$ with $e_m m = m$. Then there is a natural surjection

$$\bigoplus_{m \in M} \mathcal{H}(G)e_m \rightarrow M$$

where $\mathcal{H}e_m$ goes to $e_m m = m$. Thus, it is enough to note that $\mathcal{H}(G)e_m$ is projective, which is true because the functor $\text{Hom}_{\mathcal{H}(G)}(\mathcal{H}(G)e_m, -)$ is simply the functor which takes e_m -invariants. ■

Example 1.16. In an abelian category like $\mathcal{M}(G)$, a morphism which is monic and epic is automatically an isomorphism. But we do not expect this from our representation theoretic categories in general: for example, in the category of Banach spaces, it is quite possible to find vector spaces with dense subspaces.

Remark 1.17. When the td group G is the \mathbb{Q}_p -points of an algebraic group, one can further find that $\mathcal{M}(G)$ admits enough injectives.

While we're here, we note that there is a Schur's lemma.

Lemma 1.18 (Schur). Fix a smooth irreducible representation V of a td group G . Then

$$\text{End}_G V = \mathbb{C}.$$

Proof. This requires the assumption that G is second countable. We proceed in steps.

1. Note that $\dim V$ is at most countable: smoothness implies that

$$V = \bigcup_{K \subseteq G} V^K.$$

This union can be made countable because G is second countable! Now, upon choosing a vector $v \in V$, we see that V^K is $e_K \mathcal{H}(G) e_K \cdot v$, which is at most countable dimensional because $e_K \mathcal{H}(G) e_K$ is.

2. Now, choose some G -invariant operator T of V . We will show that T admits an eigenvalue λ , which means that $\ker(T - \lambda \text{id}_V)$. Well, if T has no eigenvalue, then $T - \lambda \text{id}_V$ is invertible for all $\lambda \in \mathbb{C}$, so we have an uncountable family of operators

$$\left\{ \frac{1}{T - \lambda \text{id}_V} : \lambda \in \mathbb{C} \right\}.$$

However, $\text{End}_G V$ is countable, so these operators admit a relation. Clearing denominators implies that T is the root of a polynomial of finite degree, which is a contradiction to admitting no eigenvalues. ■

Remark 1.19. This proof works fine as long as the base field is uncountable. If we want to work over $\overline{\mathbb{Q}}$ or number fields, then there is a counterexample: take $G = \overline{\mathbb{Q}}^\times$ and $V = \overline{\mathbb{Q}}$. This problem will be fixed when we pass to p -adic groups.

1.2.2 Finiteness Conditions

It will be worthwhile to have more finiteness criteria on our representations.

Definition 1.20 (admissible). Fix a td group G . Then a representation V of G is *admissible* if and only if

$$\dim V^K < \infty$$

for all open subgroups $K \subseteq G$.

Example 1.21. Finite-dimensional representations of \mathbb{Z} are admissible, and these are the only ones because $\{0\}$ is a compact open subgroup. For example, the regular representation fails to be admissible, which indicates that the category admissible representations is too small!

Here is an application of admissibility.

Definition 1.22 (contragredient). Fix a td group G . Then the *contragredient* \tilde{V} of a representation V of G consists of the smooth vectors of the dual of V .

Remark 1.23. If V is smooth, then there is a canonical embedding $V \hookrightarrow \tilde{V}$ given by $v \mapsto (\varphi \mapsto \varphi(v))$. This embedding is an isomorphism if and only if V is admissible. To see this last claim, note that \tilde{V} is smooth, so

$$\tilde{V} = \bigcup_{K \subseteq G} \tilde{V}^K.$$

Now, some functional $\ell \in \tilde{V}^K$ factors through the K -coinvariants of V . However, by fixing a Haar measure, coinvariants are identified with invariants (send a vector v in the coinvariants to its K -average), so we find that \tilde{V}^K is identified with $(V^K)^\vee$. Thus, \tilde{V} is identified with $\bigcup_K (V^*)^K$, and the claim follows from usual representation theory.

Here is another finiteness condition.

Definition 1.24 (compact). A smooth representation V of a td group G is *compact* if and only if its matrix coefficients are compactly supported. Similarly, V is compactly supported modulo the center if and only if its matrix coefficients are compactly supported modulo the center. Here, a matrix coefficient is a function in the image of the natural evaluation map

$$\begin{aligned} \tilde{V} \otimes V &\rightarrow C^\infty(G) \\ \ell \otimes v &\mapsto (g \mapsto \ell(gv)) \end{aligned}$$

outputting uniformly smooth functions.

Remark 1.25. It turns out that V is compact if and only if, for each open compact subgroup $K \subseteq G$ and vector $v \in V$, the functional

$$g \mapsto e_K gv$$

has compact support. (Here, e_K is the idempotent for $K \subseteq G$ in the Hecke algebra.)

Remark 1.26. One can translate everything into Hecke modules as follows: one may identify $C^\infty(G)$ with $\widehat{\mathcal{H}(G)}$ because functions pair with measures to produce scalars. Then the matrix coefficient map sends $\ell \otimes v$ to $(T \mapsto \ell(Tv))$.

Now is as good a time as any to introduce the following idea.



Idea 1.27. Harmonic analysis of a group G has categorical interpretations.

Here's an example, which uses the notion of compactness: compact representations split $\mathcal{M}(G)$.

Proposition 1.28. Fix a unimodular td group G , and choose a compact representation V .

(a) There is a splitting

$$\mathcal{H}(G) = \mathcal{H}(G)_V \oplus \mathcal{H}(G)_{\neg V}$$

of $\mathcal{H}(G)$ into two-sided ideals.

(b) There is a splitting of categories

$$\mathcal{M}(G) = \mathcal{M}(G)_V \oplus \mathcal{M}(G)_{\neg V}.$$

Proof. Note (b) follows from (a) by taking module categories from the two ideals: send $M \in \mathcal{M}(G)$ to the decomposition $\mathcal{H}(G)_V M \oplus \mathcal{H}(G)_{\neg V} M$. Let m be the matrix coefficient map for V . It remains to show (a). Here, $\mathcal{H}(G)_V$ is defined to be the image of $m dg$, which we can see is a two-sided ideal. The complement $\mathcal{H}(G)_{\neg V}$ is defined by embedding $\mathcal{H}(G)$ into $L^2(G)$ (by choosing a Haar measure) and then the inner product on $L^2(G)$ yields the desired complement. ■

Remark 1.29. If V is irreducible (and Schur's lemma holds), there is an "algebraic" argument which does not require us to embed $\mathcal{H}(G)$ into $L^2(G)$. Note that there is a composite

$$\tilde{V} \otimes V \xrightarrow{m} S(G) \xrightarrow{dg} \mathcal{H}(G) \rightarrow \text{End}(V)^\infty = \tilde{V} \otimes V.$$

Here, $(-)^{\infty}$ denotes smooth endomorphisms. As soon as we choose a compact open subgroup $K \subseteq G$, we can take invariants for $K \times K$ everywhere, which provides an identity of $\text{End}(V)^\infty$. Then $\mathcal{H}(G)_V$ is taken to be the projection on this identity, and $\mathcal{H}(G)_{\neg V}$ is the kernel of this identity. (As a technical point, we note that the irreducibility of V is used to show that the splitting has the required properties: the map $\mathcal{H}(G) \rightarrow \text{End}(V)^{K \times K}$ is surjective.)

Our two finiteness conditions are related as follows.

Lemma 1.30. Fix a td group G . Then finitely generated compact representations of G are admissible.

Proof. Fix such a representation V . For simplicity, we assume that V has only a single generator v . It follows that V^K is spanned by

$$\{e_K g v : g \in G\}.$$

However, the map $g \mapsto e_K g v$ is compactly supported by Remark 1.25, so the set in the previous sentence is finite, so V^K is finite-dimensional. ■

1.3 January 29

Last class was cancelled due to snow!

1.3.1 Inductions and Restrictions

Today, we will take G to be some reductive group over a nonarchimedean local field F . Today, we will classify the compact representations of $G(F)$. This will imply that $\mathcal{M}(G(F))$ splits into a cuspidal part and parabolically induced part. This eventually produces the theory of the Bernstein center, which is a finer decomposition of $\mathcal{M}(G(F))$.

Quickly, recall that (maximal) parabolic subgroups $P \subseteq G$ can be described as those for which G/P is compact. They can be parameterized by cocharacters $\lambda: \mathbb{G}_m \rightarrow G$: the character λ induces a grading on the Lie algebra \mathfrak{g} ; the Lie algebra of the parabolic is then given by the Lie subalgebra of the nonnegatively graded pieces. Accordingly, we let P_λ be the associated parabolic and U_λ be the associated unipotent radical. (It turns out that $\text{Lie } U_\lambda$ consists of the subspace of \mathfrak{g} with positive degree.) There is also a Levi $L_\lambda \subseteq G$ for which $P_\lambda = U_\lambda L_\lambda$.

To motivate the notion of a cuspidal representation, we recall Frobenius reciprocity.

Definition 1.31 (induction, restriction). Fix a topological subgroup H of a td group G . Then the natural restriction functor $\text{Res}_H^G: \mathcal{M}(G) \rightarrow \mathcal{M}(H)$ admits a right adjoint $\text{Ind}_H^G: \mathcal{M}(H) \rightarrow \mathcal{M}(G)$, called *induction*. If H is open, then there is a left adjoint, denoted ind_H^G or cInd_H^G , and it is called *compact induction*.

Remark 1.32. One can construct $\text{Ind}_H^G W$ as the smooth sections of the equivariant sheaf $W \times^H G$ on $H \backslash G$. Concretely, these are the smooth functions $f: G \rightarrow W$ which are H -invariant.

Remark 1.33. It turns out that $\text{cInd}_H^G W$ admits the same description, with the extra condition that $\text{supp } f/H$ is compact.

Remark 1.34. If H is not open, then one can still define a functor cInd_H^G , but it is no longer adjoint; one should also change the definition of cInd_H^G as measures on $H \backslash G$ valued in the sheaf $W \times^H G$.

Remark 1.35. The isomorphism

$$\text{Hom}_G(V, \text{Ind}_H^G W) \rightarrow \text{Hom}_H(\text{Res}_H^G V, W)$$

is given by evaluating the function in $\text{Ind}_H^G W$ at the identity.

Definition 1.36 (parabolic induction). Fix a reductive group G over a nonarchimedean local field F , and choose a parabolic $P \subseteq G$ with Levi subgroup L . Then we define *parabolic induction* I_P^G as the composite functor

$$\mathcal{M}(L(F)) \xrightarrow{\text{Inf}} \mathcal{M}(P(F)) \xrightarrow{\text{Ind}} \mathcal{M}(G(F)).$$

Definition 1.37 (Jacquet). Fix a reductive group G over a nonarchimedean local field F , and choose a parabolic $P \subseteq G$. Then we define the *Jacquet functor* R_P^G to be the left adjoint of I_P^G .

Remark 1.38. It turns out that R_P^G admits an explicit description as taking the U -coinvariants, where $U \subseteq P$ is the unipotent radical.

It turns out that taking coinvariants always right exact, for any subgroup $H \subseteq G$. In the unipotent case, one can do better.

Theorem 1.39. Fix a reductive group G over a nonarchimedean local field F , and choose a parabolic $P \subseteq G$. Then R_P^G is exact.

Proof. We need to show that taking coinvariants by a unipotent subgroup U is exact. The main point is that a unipotent group U admits a filtration $\{U_n\}_n$ by compact unipotent subgroups.

Example 1.40. We will not prove this latter claim, but we will note that it is not so hard to see for GL_n , where the unipotent radical is simply given by the upper-triangular matrices. Then one can take the subgroup to be the matrices generated by ones coming from coefficients in $p^{-\bullet}\mathcal{O}$ just above the diagonal. One can upgrade this to work in general because unipotent groups are in general filtered by \mathbb{G}_a (in characteristic 0).

But there is a natural isomorphism $(-)_{\bar{H}} \rightarrow (-)^H$ for any compact group H given by integration: send a vector v to $\int_H hv dh$. The result now follows by using the filtration.

Let's explain the last deduction. We need to show that an embedding $A \hookrightarrow B$ induces an injection $A_U \hookrightarrow B_U$. Then we want to show that the kernel of the composite

$$A \rightarrow B \rightarrow B_U$$

is exactly the vectors of the form $\{ua - a : a \in A\}$. Well, choose some $a \in A$ in the kernel, so we are granted $b \in B$ and $u \in U_n$ (for n large) so that $a = b - ub$. This means that a is in the kernel of the averaging map by U_n , so the result follows. ■

Remark 1.41. It may occasionally be convenient to normalize I_P^G by a factor of $\delta_P^{1/2}$, where δ_P is the modular character. This is convenient because it sends us to half-densities instead of sections of the line bundle. Explicitly, this normalized parabolic induction preserves unitarity (namely, one can consider square integrals). On the other hand, it is occasionally inconvenient to be forced to choose a square root.

1.3.2 Cuspidal Representations

Here is the main definition for this part of the course.

Definition 1.42 (quasicuspidal, cuspidal). Fix a reductive group G over a nonarchimedean local field F . A smooth representation W of $G(F)$ is *quasicuspidal* if and only if $R_P^G W = 0$ for all proper parabolic subgroups $P \subseteq G$. We say that W is *cuspidal* if and only if W is quasicuspidal and finitely generated. Lastly, W is *supercuspidal* if and only if W is quasicuspidal and irreducible.

Example 1.43. Consider the group $G = \mathbb{G}_m$. Then the compact induction $c\mathrm{Ind}_{\mathcal{O}^\times}^{F^\times} \mathbb{C}$ is cuspidal but not finite length. This same example works for any torus.

Theorem 1.44. Fix a reductive group G over a nonarchimedean local field F . Then a representation W of $G(F)$ is compact modulo center if and only if it is quasicuspidal.

The proof is based on a “ U_p -operator.”

Example 1.45. Consider the group $G = \mathrm{GL}_2$, which admits Iwahori subgroup

$$J = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{bmatrix}.$$

Then there is an operator U_p (in the Iwahori Hecke algebra) which is the characteristic function on

$$J \mathrm{diag}(p, 1) J.$$

It turns out that U_p^n is (up to scalar) $J \mathrm{diag}(p^n, 1) J$. These operators turn out to provide a subalgebra of the Iwahori Hecke algebra $\mathcal{H}(J \backslash G / J)$ isomorphic to $\mathbb{C}[\mathbb{N}]$.

To generalize this, we recall the following facts about reductive groups.

Theorem 1.46 (Weak Cartan decomposition). Fix a reductive group G over a nonarchimedean local field F . Choose a minimal parabolic $P_0 \subseteq G$ with Levi subgroup L_0 , and let $A_0 \subseteq L_0$ be the maximal split (central) torus. Then $\Lambda := X_*(A_0)$ is a free abelian group, and there is a dominant subset $\Lambda^+ := \Lambda$ consisting of those elements acting nonnegatively on the weights of P_0 . Then there is a compact subset $K \subseteq G$ for which

$$G = \bigcup_{\lambda \in \Lambda^+} K \lambda(\varpi) K.$$

Example 1.47. For $G = \mathrm{GL}_n$, the usual Cartan decomposition reads

$$\mathrm{GL}_n(F) = \bigsqcup_{\lambda \in \Lambda^+} K \lambda(\varpi) K,$$

where $K = G(\mathcal{O})$, and Λ^+ consists of the dominant cocharacters.

Theorem 1.48. Fix a reductive group G over a nonarchimedean local field F with minimal parabolic P_0 with Levi decomposition $P_0 = L_0 U_0$. Then $G(F)$ admits a basis of open compact subgroups J_n with Iwahori factorization

$$J_n = J_{n+} J_{n0} J_{n-}$$

with respect to P_0 , where $J_{n0} := J_n \cap L_0$ and $J_{n+} := J_n \cap U_0$ and $J_{n-} := J_n \cap U_0^-$.

Remark 1.49. One can interchange the subgroups in the Iwahori factorization.

Remark 1.50. It turns out that any such J_n admits an embedding $\mathbb{C}[\Lambda^+] \hookrightarrow \mathcal{H}(J_n \backslash G / J_n)$ given by (up to some scalar)

$$\lambda \mapsto 1_{J_n \lambda(\varpi) J_n}.$$

Indeed, up to some scalar, it is enough to check that

$$J_n \lambda(\varpi) J_n \cdot J_n \lambda'(\varpi) J_n \stackrel{?}{=} J_n \lambda \lambda'(\varpi) J_n.$$

The point is that the 0 parts commute with the $\lambda(\varpi)$ and $\lambda'(\varpi)$, and the positive and negative parts can be moved around with some explicit factors.

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