225A: Model Theory

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

1.1 August 24

It begins.

1.1.1 Logistics

Here are some logistical notes.

- There is a bCourses.
- We will use [Mar02].
- Professor Montalbán and Scanlon will teach the course jointly.
- There will be a midterm (in-class on the 19th of October) and final exam (take-home over three days).
- There are suggested but technically ungraded exercises. They are helpful.
- We will assume basic first-order logic, and examples will be taken from a few other areas of mathematics.
- This is a graduate class. It will be pretty fast.

We are studying model theory, which is the study of models and theories. Our main two theorems are the Compactness theorems and results on admitting types. We will use these results again and again.

1.1.2 Languages and Structures

Let's review chapter 1 of [Mar02]. Here is a language.

Definition 1.1 (language)**.** A *language* $\mathcal{L}$ consists of the sets $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{C}$ of symbols. Here, $\mathcal{F}$ are functions, $\mathcal{R}$ are relations, and $\mathcal{C}$ are constants. Notably, there is an arity function $n\colon (\mathcal{F}\cup\mathcal{R})\to\mathbb{N}$.

Concretely, fix a language $\mathcal{L}=(\mathcal{F},\mathcal{R},\mathcal{C})$. If $f\in\mathcal{F}$ and $n(f)=3$, then we say that f has arity three; the analogous statement holds for relations.

We will often abbreviate a language to just a long tuple. For example, the notation $(\mathbb{N},0,1,+,\leq)$ has the domain $\mathbb{N}$ and constants 0 and 1 and function $+$ and relation $\leq$, even though the notation has not made it obvious what any of these things are.

So far we only have the prototype of data. Here is the data.

Definition 1.2 (structure)**.** Fix a language $\mathcal{L}$. Then an $\mathcal{L}$*-structure* $\mathcal{M}$ consists of the following data.

- Domain: a nonempty set M.
- Functions: for each $f \in \mathcal{F}$, there is a function $f^{\mathcal{M}} \colon M^{n(f)} \to M$.
- Relations: for each $R \in \mathcal{R}$, there is a relation $R^{\mathcal{M}} \subseteq M^{n(r)}$.
- Constants: for each $c \in \mathcal{C}$, there is a constant $c^{\mathcal{M}} \in M$.

The various $(-)^{\mathcal{M}}$ data are called *interpretations*.

Example 1.3. Consider the language $\mathcal{L}$ with the constants 0 and 1 and operations $+$ and $\times$. Then $\mathbb{N}$ is an $\mathcal{L}$-structure, in the obvious way.

In general, algebra provides many examples of languages.

We would like to relate our structures.

Definition 1.4 (homomorphism, embedding, isomorphism)**.** Fix a language $\mathcal{L}$. An $\mathcal{L}$*-homomorphism* $\eta \colon \mathcal{M} \to \mathcal{N}$ of $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ is a one-to-one map $\eta \colon M \to N$ preserving the interpretations, as follows.

- Functions: for each $f \in \mathcal{F}$, we have $\eta \circ f^{\mathcal{M}} = f^{\mathcal{N}} \circ \eta^{n(f)}$.
- Relations: for each $R \in \mathcal{R}$, if $\overline{m} \in R^{\mathcal{M}}$, then $\eta^{n(R)}(m) \in R^{\mathcal{N}}$.
- Constants: for each $c \in \mathcal{C}$, we have $\eta\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$.

If $\eta \colon M \to N$ is one-to-one and the relations condition is an equivalence, then η is an $\mathcal{L}$*-embedding*. If $\eta \colon M \to N$ is the identity $M \subseteq N$, then we say that $\mathcal{M}$ is an $\mathcal{L}$-substructure. In addition, if η is onto, then η is an $\mathcal{L}$*-isomorphism*.

Explicitly, being a substructure means that the functions and relations are restricted appropriately, and the constants remain the same.

Example 1.5. In the language of groups, subgroups make substructures. A similar sentence holds for other algebraic structures.

1.1.3 Formulae

Thus far we have described a vocabulary: the language provides the data for us to manipulate. We now discuss how to "speak" in this language.

Definition 1.6 (term)**.** Let $\mathcal{L}$ be a language. The set of $\mathcal{L}$*-terms* is the smallest set $\mathcal{T}$ satisfying the following.

- Constants: for each $c \in \mathcal{C}$, we have $c \in \mathcal{T}$.
- Variables: $x_i \in \mathcal{T}$ for each $i \in \mathbb{N}$. Notably, we have only countably many variables.
- Functions: if $t_1, \ldots, t_n \in \mathcal{T}$ where $n = n(f)$ for some $f \in \mathcal{F}$, then $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Given an $\mathcal{L}$-structure $\mathcal{M}$ and term $t \in \mathcal{T}$ with variables $x_1, \ldots, x_n$ and elements $a_1, \ldots, a_n \in M$, we define $t^{\mathcal{M}}(\overline{a})$ in the obvious way.

Terms are basically just nouns. We would now like to put them into sentences.

Definition 1.7 (atomic formula)**.** The set of *atomic $\mathcal{L}$-formulae* is the set of expressions of one of the following forms.

- Equality: $t_1 = t_2$ for any $\mathcal{L}$-terms t_1 and t_2.
- Relations: $R(t_1, \ldots, t_n)$ for any n-ary relation R and $\mathcal{L}$-terms $t_1, \ldots, t_n$.

Definition 1.8 (formula)**.** The set of *$\mathcal{L}$-formulae* is the smallest set satisfying the following.

- Any atomic $\mathcal{L}$-formula φ is an $\mathcal{L}$-formula.
- For any $\mathcal{L}$-formulae φ and ψ, then $\neg\varphi$ and $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\mathcal{L}$-formulae.
- For any variable v_i for $i \in \mathbb{N}$, then $\exists v_i \varphi$ is an $\mathcal{L}$-formula.

One can then define the shorthand "$\varphi \to \psi$" for $\neg\varphi \vee \psi$ and "$\forall v_i \varphi$" for $\neg\exists v_i \neg\varphi$.

Now that we can talk about our structure, we would like to know if we are making sense.

Definition 1.9 (free variable)**.** Fix a language $\mathcal{L}$. A variable v in a formula φ is *free* if and only if it is not bound to any quantifier $\exists v$ or $\forall v$. If φ has free variables contained in the variables $x_1, \ldots, x_n$, we write $\varphi(x_1, \ldots, x_n)$.

This definition is vague because we have not said what "bound" means, but it is rather obnoxious to explain what it is rigorously, so we will not bother.

Definition 1.10 (sentence)**.** Fix a language $\mathcal{L}$. An $\mathcal{L}$-formula with no free variables is a *sentence*.

Definition 1.11 (truth)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Further, fix an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and a tuple $\overline{a} \in M^n$. Then we define *truth* as $\mathcal{M} \vDash \varphi(\overline{a})$ to mean that φ is true upon plugging in $\overline{a}$, where our definition is inductive on atomic formulae as follows.

- $\mathcal{M} \vDash (t_1 = t_2)(\overline{a})$ if and only if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.
- $\mathcal{M} \vDash R(t_1, \ldots, t_n)$ if and only if $\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_2^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$.

We define truth inductively on formulae now as follows.

- $\mathcal{M} \vDash (\varphi \wedge \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ and $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash (\varphi \vee \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ or $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash \neg\varphi(\overline{a})$ if and only if we do not have $\mathcal{M} \vDash \varphi(\overline{a})$.
- $\mathcal{M} \vDash \exists v \varphi(\overline{a}, v)$ if and only if there exists $b \in M$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$.

In this case, we say that $\mathcal{M}$ *satisfies, models, etc.* $\varphi(\overline{a})$ and so on.

Here is our first result of substance.

Proposition 1.12. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-embedding $\eta\colon \mathcal{M} \to \mathcal{N}$. Further, fix a quantifier-free formula φ. Then $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi$.

Proof. Induction on φ. Roughly speaking, the point is that the interpretations are the same before and after. ■

Remark 1.13. If we allow variables, the statement is false. For example, $(\mathbb{N}, 0, \leq)$ embeds into $(\mathbb{Z}, 0, \leq)$, but $\forall x(0 \leq x)$ is true in the first formula while false in the second.

In the case of isomorphism, we can say more.

Proposition 1.14. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-isomorphism $\eta\colon \mathcal{M} \to \mathcal{N}$. Further, fix any formula φ with free variables $x_1, \ldots, x_n$ and a tuple $\overline{a} \in M^n$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi(f(\overline{a}))$.

Proof. Induction on φ. The point is that the definition of truth is the same before and after η. ■

1.2 August 29

We continue with the speed run of first-order logic. The goal for today is to state the Compactness theorem.

1.2.1 Theories

Now that we have a notion of truth, it will be helpful to keep track of which sentences exactly we want to be true.

Definition 1.15 (theory)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Then the *theory* $\mathrm{Th}(\mathcal{M})$ of $\mathcal{M}$ is the set of all sentences φ such that $\mathcal{M} \vDash \varphi$.

The theory is essentially all that first-order logic can see.

Definition 1.16 (elementary equivalence)**.** Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. Then we say that $\mathcal{M}$ and $\mathcal{N}$, written $\mathcal{M} \equiv \mathcal{N}$, are *elementarily equivalent* if and only if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Example 1.17. It happens that $(\mathbb{Q}, +) \equiv (\mathbb{R}, +)$ but are not isomorphic because they have different cardinalities.

Example 1.18. Let s denote the successor function. It happens that $(\mathbb{Z}, s) \equiv (\mathbb{Q}, s)$, but one can show that they are not isomorphic.

This notion is different from isomorphism, but it is related.

Lemma 1.19. Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Proof. This is the content of Proposition 1.14 upon unraveling the definitions. ■

Going in the other direction, we might start with some sentences we want to be true and then look for the corresponding models.

Definition 1.20 (theory)**.** Fix a language $\mathcal{L}$. Then an *$\mathcal{L}$-theory T* is a set of $\mathcal{L}$-sentences. For an $\mathcal{L}$-structure $\mathcal{M}$, we say that $\mathcal{M}$ *models T*, written $\mathcal{M} \vDash T$, if and only if $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathcal{M}$. We let $\mathrm{Mod}(T)$ denote the class of all models $\mathcal{M}$ of T, and we call it an *elementary class*.

Example 1.21. The class of all groups arises from the language $\{e, \cdot\}$ with some sentences to make a theory. However, the class of torsion groups is not an elementary class.

We want might want to understand what sentences follow from a given theory.

Definition 1.22. Fix a language $\mathcal{L}$ and theory T. Then we say that T logically implies a sentence φ, written $T \vDash \varphi$, if and only if any $\mathcal{L}$-structure $\mathcal{M}$ modelling T has $\mathcal{M} \vDash \varphi$.

Remark 1.23. Gödel's completeness theorem shows that $T \vDash \varphi$ if and only if there is a "proof" of φ from T. We will not use the notion of proof so much, though its proof is similar to the proof of compactness, which we will show.

1.2.2 Definable Sets

We will want the following notion.

Definition 1.24 (definable)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $B \subseteq M$. Then a subset $X \subseteq M^n$ is B-*definable* if and only if there is a formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_k)$ and tuple $\overline{b} \in B^k$ such that $\overline{a} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{a}, \overline{b})$. The tuple $\overline{b}$ might be called the *parameters*. We may abbreviate M-definable to simply *definable*.

Example 1.25. Any finite set is definable by using the parameters to list out the elements.

Example 1.26. Work with $\mathcal{M} := (\mathbb{Z}, \leq)$. Then $X = \mathbb{N}$ is $\{0\}$-definable by $\varphi(x, 0)$ where $\varphi(x, y)$ is given by $y \leq x$. However, $\mathbb{N}$ is not $\varnothing$-definable, as shown by the following proposition.

Proposition 1.27. Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $A \subseteq M$. Further, suppose $X \subseteq M^n$ is A-definable. For any automorphism $\sigma \colon \mathcal{M} \to \mathcal{M}$ fixing A pointwise must fix X (not necessarily pointwise).

Proof. Suppose $\varphi(\overline{v}, \overline{w})$ defines X with the parameters $\overline{a} \in A^\bullet$. Then $\overline{x} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{x}, \overline{a})$, but then $\mathcal{M} \vDash \varphi(\sigma(\overline{x}), \sigma(\overline{a}))$, so $\mathcal{M} \vDash \varphi(\sigma(\overline{x}), \overline{a})$ so $\sigma(\overline{x}) \in X$. For the converse, use the inverse automorphism σ^{-1}. ■

To further explain Example 1.26, we see that there are automorphisms of $\mathbb{Z}$ (namely, by shifting) which do not fix $\mathbb{N}$, so $\mathbb{N}$ cannot be $\varnothing$-definable.

Example 1.28. Work with $\mathcal{M} := (\{1A, 1B, 2A, 2B\}, \leq)$ with partial ordering given by the number. The set $X := \{1A, 1B\}$ is $\varnothing$-definable by $\varphi(x)$ given by $\exists y((x \neq y) \land (x \leq y))$. However, there is an automorphism of our model swapping $1A$ with $1B$ and $2A$ with $2B$, but this automorphism does not fix X pointwise.

1.2.3 The Compactness Theorem

To state compactness, we want a few definitions.

Definition 1.29 (satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *satisfiable* if and only if it has a model $\mathcal{M}$.

With a notion of proof, one can show that being satisfiable means that there is no proof of $\perp$, but we will avoid a discussion of proofs in this course.

Definition 1.30 (finitely satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *finitely satisfiable* if and only if any finite subset of T is satisfiable.

Of course, being satisfiable implies being finitely satisfiable; the converse will be true but is far from obvious. The following example explains why this is strange.

Example 1.31. Consider the natural numbers $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times, \leq)$ and $\mathcal{N}_c := (\mathbb{N}, 0, 1, +, \times, \leq, c)$, where c is some constant symbol, and set

$$T := \operatorname{Th}(\mathcal{N}) \cup \left\{ c \geq \underbrace{1 + 1 + \cdots + 1}_{n} : n \in \mathbb{N} \right\}.$$

Then T is finitely satisfiable by $\mathcal{N}$ because, for any finite subset of T, the sentences $c \geq 1+1+\cdots+1$ will have to be bounded in length in our finite subset, so we simply find some c large enough in $\mathcal{N}$. However, $\mathcal{N}$ does not model T!

Anyway, here is our statement.

Theorem 1.32 (compactness)**.** Fix a language $\mathcal{L}$ and theory T. If T is finitely satisfiable, then T is satisfiable. Furthermore, T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}| + \aleph_0$.

Remark 1.33. In particular, the theory T of Example 1.31 has a model $\mathcal{N}'$, which is going to look very strange. To begin, there is a segment

$$0 < 1 < 2 < \cdots.$$

But there is now an element c larger than any natural, which produces $c+1, c+2, c+3, \ldots$. But also any nonzero element has a predecessor, so we have elements $c-1, c-2, c-3, \ldots$. Further, any natural number is either odd or even, so there is also either $(c-1)/2$ or $c/2$ sitting between the initial piece of $\mathbb{N}$ and the c piece with $\mathbb{Z}$ added everywhere. In fact, a similar argument holds to produce an element approximately equal to qc for any rational $q \in \mathbb{Q}$.

Remark 1.34. One can of course always make our model larger. For example, suppose we have a theory T with an infinite model. If we want a model with cardinality at least $\mathbb{R}$, we add constants $\{c_r : r \in \mathbb{R}\}$ to our language and add in the sentences

$$\{c_r \neq c_s : \text{distinct } r, s \in \mathbb{R}\}.$$

This remains finitely satisfiable: these constants merely ask for our model to be larger than any finite set. One can even require the new model to be elementarily equivalent to the previous one.

Here are some applications of compactness.

Corollary 1.35. The class of torsion groups is not elementarily definable in the language $\mathcal{L} = \{e, *\}$ of groups.

Notably, it is not okay to write something like

$$\bigvee_{n \in \mathbb{N}} (\forall g \, g^n = e)$$

to encode any torsion because this statement is infinitely long.

Proof. Suppose the class is elementarily definable. Then we have a theory T such that $\operatorname{Mod}(T)$ consists exactly of all torsion groups. Now the trick is to build a model of T which is not actually a torsion group. For this, we expand our language to $\mathcal{L} = \{e, *, c\}$, and let

$$S := T \cup \left\{ \underbrace{c * c * \cdots * c}_{n} \neq e : n \geq 2 \right\}.$$

For any finite subset of S, we can satisfy S by a torsion group containing an element which is not n-torsion for sufficiently large n; for example, $\mathbb{Z}/n\mathbb{Z}$ will do.

Thus, by Theorem 1.32, there is a model $\mathcal{G}$ of S, so in particular, $\mathcal{G}$ has an element $g \in G$ with

$$\underbrace{g * g * \cdots * g}_{n} \neq e$$

for all $n \geq 2$ (namely, g is the interpretation of the constant symbol c), so it follows that G is not torsion. However, $\mathcal{G}$ is also a model of T and thus is supposed to be torsion, so we have a contradiction! This completes the proof. ■

1.3 August 31

Professor Scanlon is back. Let's prove the Compactness theorem. We are going to prove 2.5 times.

1.3.1 Proof of Compactness

Recall the statement.

Theorem 1.32 (compactness)**.** Fix a language $\mathcal{L}$ and theory T. If T is finitely satisfiable, then T is satisfiable. Furthermore, T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}| + \aleph_0$.

Remark 1.36. This result is special to first-order logic: in some sense, Theorem 1.32 combined with a corollary characterizes first-order logic among various logics. Roughly speaking, one wants to formalize what a logic is with its various structures and sentences should do.

Proof with completeness. We can prove this result fairly quickly given the Completeness theorem. Recall that the Completeness theorem says that any theory fails to be satisfiable if and only if there is a proof of contradiction $\perp$; one writes that a theory T proves a sentence φ by $T \vdash \varphi$. We have not discussed how formal proofs work, and we won't discuss it further because this is not a proofs class. Approximately speaking, a formal proof is a list of steps one can use the sentence sin T to produce φ syntactically.

Now, suppose that T fails to be satisfiable. Then there is a proof of $\perp$. But then one can look at the proof, which is necessarily finite in length, and then we pick up any sentence φ occurring in the proof of $\perp$. But then we have a proof of $\perp$ using only finitely many sentences in T, so T fails to be finitely satisfiable! This completes the proof. ■

Anyway, let's present an actual proof.

Definition 1.37 (witness)**.** Fix a theory T of a language $\mathcal{L}$. Then T has *witnesses* (or *Henkin constants*) if and only if each formula $\varphi(x)$ in one free variable x has a constant symbol c such that $\exists x \varphi(x) \to \varphi(c)$ lives in T.

Remark 1.38. If T has witnesses, then $T' \supseteq T$ also has witnesses for any theory T' extending T.

Let's quickly sketch our proof.

1. We will show that if T is finitely satisfiable, then there is an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ and expanded finitely satisfiable $\mathcal{L}'$-theory $T' \supseteq T$ of $\mathcal{L}'$ such that $|\mathcal{L}'| \le |\mathcal{L}| + \aleph_0$, and T' has witnesses (as does any extended theory T'' of T').

2. Next, suppose T is a maximally finitely satisfiable theory (i.e., T is finitely satisfiable, and any proper extension $T' \supseteq T$ fails to be finitely satisfiable[1]). Then we will show T is complete (i.e., each sentence φ has either $\varphi \in T$ or $\neg\varphi \in T$).

3. From here, we want to extend maintaining being complete: if T is finitely satisfiable, then there is an extended language $\mathcal{L}' \supseteq \mathcal{L}$ of size $|\mathcal{L}'| = |\mathcal{L}| + \aleph_0$ and extended theory T' of T which is complete, finitely satisfiable, and has witnesses.

   We can prove this using the previous two steps. Indeed, the first step provides an extended language $\mathcal{L}'$ (of cardinality at most $|\mathcal{L}|+\aleph_0$) and extended theory T' which is finitely satisfiable and has witnesses. Then we use Zorn's lemma to become maximally finitely satisfiable. Let $\mathcal{P}$ denote the set of finitely satisfiable $\mathcal{L}'$-theories T'' extending T' which is finitely satisfiable. Containment shows that $\mathcal{P}$ is a partial order, and it's nonempty because $T' \in \mathcal{P}$. Next up, we note that any nonempty chain $\{T_\alpha\}_{\alpha\in\lambda}$ is upper-bounded by

   $$\bigcup_{\alpha\in\lambda} T_\alpha,$$

   which we can see continues to be finitely satisfiable (any finite set belongs to some fix T_β for β perhaps large) and thus lives in $\mathcal{P}$ and succeeds to upper-bound our chain. Thus, Zorn's lemma provides a maximally finitely satisfiable theory T'' containing T', which will be complete by the previous step. Because T'' contains T', we continue to have witnesses.

4. We are now ready to do our construction. At this point, we may assume that our theory T is finitely satisfiable, complete, and has witnesses. Then we claim that there is a model $\mathcal{M}$ such that $|M| \le |\mathcal{L}|$.

   In fact, the model can be described somewhat explicitly. Take $M := \mathcal{C}/{\sim}$ where $\mathcal{C}$ is our set of constants, and our equivalence relation $\sim$ is given by $c \sim d$ if and only if $(c = d) \in T$. Notably, constants $c \in \mathcal{C}$ are interpreted as $c^{\mathcal{M}} := [c]$. To interpret functions f, we have $f^{\mathcal{M}}([a_1], \ldots [a_n]) = [d]$ if and only if $(f(a_1, \ldots, a_n) = d) \in T$. Lastly, to interpret relations R, we have $R^{\mathcal{M}}([a_1], \ldots, [a_n])$ if and only if $(R(a_1, \ldots, a_n)) \in T$.

Let's start implementing the details.

Remark 1.39. In logic, the answer to a question is often the question. For example, in step 4, we see that T has a model because T says that it has a model.

Here is our first step.

Lemma 1.40. Fix a finitely satisfiable theory T of a language $\mathcal{L}$. Then there is an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ and expanded finitely satisfiable $\mathcal{L}'$-theory $T' \supseteq T$ of $\mathcal{L}'$ such that $|\mathcal{L}'| \le |\mathcal{L}| + \aleph_0$, and T' has witnesses.

Proof. We would like to just set T' to be T together with new constants providing witnesses for all formulae. But these new constants will make new formulae, so we need to do some kind of induction to go upwards.

With this in mind, we will build an increasing sequence of languages

$$\mathcal{L}_0 := \mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots$$

and theories

$$T_0 := T \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

[1] Such a thing exists by some sort of Zorn's lemma argument: note that there is a theory containing T which fails to be finitely satisfiable: take the set of all sentences!

such that T_n is always a finitely satisfiable $\mathcal{L}_n$-theory, and each $\mathcal{L}_n$-formula φ with one free variable has a constant $c \in \mathcal{C}_{\mathcal{L}_n}$ such that $\exists x\varphi(x) \to \varphi(c)$ lives in T_n. We will then set $\mathcal{L}'$ to be the union of the $\mathcal{L}_\bullet$ and T' to be the union of the $T_\bullet$, and this will complete the proof.

We have already built the $n = 0$ stage, as above. Then to build $\mathcal{L}_{n+1}$ from $\mathcal{L}_n$, add in new constant symbols $c_{\varphi(x)}$ for each $\mathcal{L}_n$-formula $\varphi(x)$ with one free variable; all the functions and relations remain the same. Note $\mathcal{L}_{n+1}$ is now the size of the formulae with one free variable in $\mathcal{L}_n$, so $|\mathcal{L}_{n+1}| = |\mathcal{L}_n| + \aleph_0$.

As for our theory, let T_{n+1} be T_n plus the sentences $\exists x\varphi(x) \to \varphi\left(c_{\varphi(x)}\right)$ for each $\mathcal{L}_n$-formula $\varphi(x)$ with one free variable. We are now ready to set

$$\mathcal{L}' \coloneqq \bigcup_{n\in\mathbb{N}} \mathcal{L}_n \qquad \text{and} \qquad T' \coloneqq \bigcup_{n\in\mathbb{N}} T_n.$$

We see that $\mathcal{L}'$ then has the right size, and T' has witnesses: for any $\mathcal{L}'$-formula $\varphi(x)$ with one free variable, note that $\varphi(x)$ has only finitely many symbols, so we can find some fixed level $\mathcal{L}_n$ containing all the symbols used in $\varphi(x)$. But then $\varphi(x)$ has a witness from $T_{n+1} \subseteq T'$, as needed.

It remains to show that T' is finitely satisfiable. It suffices to show that T_n is finitely satisfiable for any $n \in \mathbb{N}$ because any finite set will be contained in some T_n. We show this by induction. For $n = 0$, there is nothing to say. Now suppose T_n is finitely satisfiable, and we show that T_{n+1} is finitely satisfiable.

Fix some finite subset $\Delta \subseteq T_{n+1}$ which we would like to build a model for. Now, Δ will be built by some sentences in T_n plus some new sentences from T_{n+1}. Looking hard at $T_{n+1} \setminus T_n$, we see that we can enumerate $\Delta \setminus T_n$ as some sentences

$$\exists x\psi_k(x) \to \psi_k(c_k)$$

where $\{\psi_k\}_{k=1}^m$ is some $\mathcal{L}_n$-formulae in one free variable.

We now begin building our model. Note $\Delta \cap T_n$ is a finite subset of T_n, so it is satisfiable by some model $\mathcal{M}$. Now, for each k, if there is some $a \in M$ such that $\mathcal{M} \vDash \varphi_k(a)$, set $a \coloneqq a_k$; otherwise, set $a_k \coloneqq m$ for any chosen $m \in M$. (Note structures are nonempty.) We now let $\mathcal{M}'$ be the $\mathcal{L}_{n+1}$-structure with universe M, interpretations of functions and relations the same as in $\mathcal{M}$, and all old constant symbols are also all still interpreted the same way. Then for each new constant symbol, we interpret $c_k^{\mathcal{M}} \coloneqq a_k$, and each other new constant symbol can also go to m. Now $\mathcal{M}'$ is a model for Δ because it models everything in $\Delta \cap T_n$ for free, and it has satisfied $\Delta \setminus T_{n+1}$ by construction, so we are done. ■

To show the second step, we begin with the following lemma.

Lemma 1.41. Fix a finitely satisfiable theory T of a language $\mathcal{L}$. For any $\mathcal{L}$-sentence φ, then either $T\cup\{\varphi\}$ or $T \cup \{\neg\varphi\}$ is finitely satisfiable.

Proof. Suppose that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ both fail to be finitely satisfiable. We will show that T fails to be finitely satisfiable.

Well, we are given finite subsets $\Delta_+ \subseteq T \cup \{\varphi\}$ and $\Delta_- \subseteq T \cup \{\neg\varphi\}$ which are not satisfiable. If Δ_+ fails to contain φ, then Δ_+ is a finite subset of T which is not satisfiable, so T fails to be satisfiable. Thus, we may assume that $\varphi \in \Delta_+$. Analogously, we may assume that $\neg\varphi \in \Delta_-$. Now, $(\Delta_+ \cup \Delta_-) \setminus \{\varphi\}$ and $(\Delta_+ \cup \Delta_-) \setminus \{\neg\varphi\}$ both fail to be satisfiable.

But now suppose for the sake of contradiction that T is finitely satisfiable. Then $(\Delta_+ \cup \Delta_-) \setminus \{\varphi, \neg\varphi\}$ has a model $\mathcal{M}$. But $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \neg\varphi$, so we see that $\mathcal{M}$ will model at least one of $(\Delta_+ \cup \Delta_-) \setminus \{\varphi\}$ or $(\Delta_+ \cup \Delta_-) \setminus \{\neg\varphi\}$, which is the desired contradiction. ■

The second step now follows from a Zorn's lemma argument.

Lemma 1.42. Fix a maximally finitely satisfiable theory T of a language $\mathcal{L}$. Then T is complete.

Proof. Let φ be any $\mathcal{L}$-sentence. Then either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is finitely satisfiable by Lemma 1.41, so by maximality, we may conclude that either $T = T \cup \{\varphi\}$ or $T = T \cup \{\neg\varphi\}$, so either $\varphi \in T$ or $\neg\varphi \in T$, which is what we wanted. ■

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List of Definitions