174: Category Theory

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Contents

THEME 1: BASIC DEFINITIONS

Category theory is much easier once you realize that it is designed to formalize and abstract things you already know.

—Ravi Vakil

1.1 January 19

Reportedly there is a lot of material that Bryce would like to cover today.

1.1.1 Our Definition

We're doing category theory, so let's define what a category is.

Category

Definition 1.1 (Category)**.** A category $\mathcal{C}$ is a pair of objects and morphisms $(\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C})$ satisfying the following.

- $\operatorname{Ob}\mathcal{C}$ is a collection of *objects*. By abuse of notation, when we write $c \in \mathcal{C}$
- $\operatorname{Mor}\mathcal{C}$ is a collection of *morphisms*. Morphisms might also be called arrows or maps or functions or continuous functions or similar.

  A morphism is written $f : x \to y$ where $x, y \in \operatorname{Ob}\mathcal{C}$. Here, x is the *domain*, and y is the *codomain*.

These morphisms have a little extra structure.

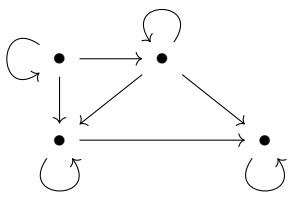
- For each $x \in \mathcal{C}$, there is a morphism $\operatorname{id}_x : x \to x$.
- Given any pair of morphisms $f : x \to y$ and $g : y \to z$, there exists a *composition* $gf : x \to z$. Importantly, the codomain of f is the domain of g.

Additionally, morphisms satisfy the following coherence conditions.

- Associativity: for any morphisms $f : a \to b$ and $g : b \to c$ and $h : c \to d$, we have that $h(gf) = (hg)f$.
- Identity: given any morphism $f : a \to b$, we have $\operatorname{id}_b f = f$ and $f \operatorname{id}_a = f$.

Yes, this is a long definition. For reference, it is on page 3 of Riehl.

The intuition to have here is that we have objects to be thought of as points a whole bunch of morphisms which are to be thought of arrows between them. Here is an example of some morphisms in a category.



The loops are identity morphisms. As an aside, it is reasonable to think that definition of a category is overly abstract. Most of the time we will be thinking about some concrete category.

Before continuing, we bring in the following definition.

Hom-sets

Definition 1.2 (Hom-sets)**.** Fix a category $\mathcal{C}$. Then, given objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ or $\mathrm{Hom}_{\mathcal{C}}(x, y)$ or $\mathrm{Hom}(x, y)$ or $\mathrm{Mor}(x, y)$ for the set of morphisms $f : x \to y$. I personally prefer $\mathrm{Mor}(x, y)$.

Note that two objects need not have a morphism between them. For example, the following is a category even though the two objects have a morphism between them.

As a less contrived example, there is no morphism between $\mathbb{F}_2$ and $\mathbb{F}_3$ in the category of fields.

1.1.2 Examples

Let's talk about examples.

Example 1.3. The category Set has objects which are all sets and its morphisms are the functions between sets.

Example 1.4. The category Grp has objects which are all groups and its morphisms are group homomorphisms. Similarly, Ab has abelian groups.

Example 1.5. The category Ring has objects which are all rings (with identity) and its morphisms are group homomorphisms.

Example 1.6. The category Field has objects which are all fields and its morphisms are field/ring homomorphisms.

Example 1.7. The category Vec_k has objects which are all k-vector spaces and its morphisms are k-linear transformations.

Those are the good examples. We like them because they are with familiar objects.

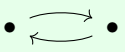
Here are some weirder examples.

Example 1.8 (Walking arrow)**.** The diagram

indues a category with a single non-identity morphism.

Note that we will stop writing down all of the identity morphisms and all induced morphisms because they're annoying to write out.

Example 1.9 (Walking isomorphism)**.** The diagram



induces a category with two non-identity morphisms. We declare that any composition of the two non-identity morphisms is the identity.

There are also such things as a poset category, but for this we should define a poset first.

Poset

Definition 1.10 (Poset)**.** A *poset* $(\mathcal{P}, \leq)$ is a set $\mathcal{P}$ and a relation $\leq$ on $\mathcal{P}$ which satisfies the following; let $a, b, c \in \mathcal{P}$.

- Reflexive: $a \leq a$.
- Antisymmetric: $a \leq b$ and $b \leq a$ implies $a = b$.
- Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$.

Now, it turns out that all posets induce a category.

Example 1.11 (Poset category)**.** Given any poset $(\mathcal{P}, \leq)$, we can define the poset category as follows.

- The objects are elements of $\mathcal{P}$.
- For $x, y \in \mathcal{P}$, there is a morphism $x \to y$ if and only if $x \leq y$, and there is only one morphism.

Checking that the poset category is in fact a category is not very interesting. The identity law comes from reflexivity, where id_a witnesses $a \leq a$.

Additionally, transitivity defines our composition: if $a \leq b$ and $b \leq c$, then $a \leq c$, and the morphism representing $a \leq c$ is unambiguous because there is at most one moprhism $a \to c$. This uniqueness is in fact crucial for our composition: if $f : a \to b$ and $g : b \to c$ and $h : c \to d$ are morphisms, then $h(gf) = (hg)f$ because they are both morphisms $a \to d$, of which there is at most one.

We continue with our examples. We will not check that these are actually categories formally; perhaps the reader can do the checks on their own time.

Example 1.12 (Groups)**.** Given a group G, we can define the category bG to have one object $*$ and morphisms $g : * \to *$ given by group elements $g \in G$. Composition in the category is group multiplication; the identity morphism id_* needed is the identity element of G; and the associativity check comes from associativity in G.

Example 1.13 (Pointer sets)**.** We define the category of pointed sets Set_* to consist of objects which are orderd pairs (X, x) where X is a set and $x \in X$ is an element. Then morphism are "based maps" $f : (X, x) \to (Y, y)$ to consist of the data of a function $f : X \to Y$ such that $f(x) = y$.

Example 1.14. Given any set S, we can define a category consisting of objects which are elements of S and morphisms which are only the required identity morphisms.

This last example generalizes.

Discrete, indiscrete

Definition 1.15 (Discrete, indiscrete)**.** Fix a category $\mathcal{C}$. Then $\mathcal{C}$ is *discrete* if and only if the only morphisms are identity morphisms. Additionally, $\mathcal{C}$ is *indiscrete* if and only if $\mathrm{Mor}(x, y)$ has exactly one element for each pair of objects (x, y).

Warning 1.16. A total order with more than one element is not a category. Namely, if we have distinct objects x and y, then we cannot have both $x \leq y$ and $y \leq x$, so not both $\mathrm{Mor}(x, y)$ and $\mathrm{Mor}(y, x)$ inhabited.

1.1.3 Size Issues

Let's briefly talk about why we are calling $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ "collections." In short, we cannot have a set that contains all sets, but we would still like a category which contiains all categories. There are a few ways around this; here are two.

- Grothendeik inaccessible categories: we essentially upper-bound the size of our sets and then let Set contain all of our sets.
- Proper classes: we add in things called "classes" to foundational mathematics we are allowed to be bigger than sets.

We will avoid doing anything like this in this course, so here is a definition making our avoidance concrete.

Small, locally small

Definition 1.17 (Small, locally small)**.** Fix $\mathcal{C}$ a category. Then $\mathcal{C}$ is *small* if and only if $\mathrm{Mor}\,\mathcal{C}$ is a set. Alternatively, $\mathcal{C}$ is *locally small* if and only if $\mathrm{Mpr}(x, y)$ is a set.

Example 1.18. The category Set is locally small, but it is not small. To see that it is not small, note that $S \mapsto \mathrm{Mor}(\{*\}, S)$ is an injective map, so $\mathrm{Mor}\,\mathrm{Set}$ must be at least as big as Set.

It turns out that most of our categories will be locally small. It is a very nice property to have.

1.1.4 Isomorphism

In algebra (e.g., group theory), we are interested in when two objects are the same. In category theory, we focus on the morphisms between objects, so we need to be careful how we define this. Here is our definition.

Isomorphism

Definition 1.19 (Isomorphism)**.** Fix a category $\mathcal{C}$. Then a morphism $f : x \to y$ is an *isomorphism* if and only if there is a morphism $g : y \to x$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. We call g the *inverse* of f and often notate it f^{-1}.

This is fairly intuitive: isomorphisms are those morphisms with a way to reverse them.

Observe that we called g "the" inverse of f, and we may do so because inverses are unique.

Proposition 1.20. Fix a category $\mathcal{C}$. Inverses of morphisms, if they exist, are unique.

Proof. Fix $f : x \to y$ some isomorphism, and suppose that we have found two inverse morphisms $g, h : y \to x$. Then

$$g = g\,\mathrm{id}_y = g(fh) = (gf)h = \mathrm{id}_x\,h = h,$$

so indeed the inverse morphisms that we found are the same. ∎

Anyways, here are some examples.

Example 1.21. In Set, the isomorphisms are the bijective maps. For this we would have to show that bijective maps have inverse maps, which is not too hard to show.

Example 1.22. In Grp, the isomorphisms are group isomorphisms. Similarly, isomorphisms in Ring are ring isomorphisms.

As a warning, we will say now that lots of categories do not have a good categorial notion of injectivity or surjectivity, so we will not be able to say that isomorphisms are merely "bijective" morphisms.

1.2 January 21

By the way, this course is being run by Bryce (interested in category theory, homological algebra, and algebraic topology) and Chris (interested in representation theory and category theory).

1.2.1 Small Correction

Last class we discussed trying to a total order $(\mathcal{P}, \leq)$ into an indiscrete category. One way to do this is to say to give a morphism between two objects $a, b \in \mathcal{P}$ if and only if one of $a < b$ or $b < a$ or $a = a$ is true. Observe that the order does not actually matter here because any two objects have exactly one morphism anyways.

1.2.2 Groupoids

Reportedly, there will usually not be a lecture to begin out our discussion sections, but here is a lecture to begin out our first discussion section.

Last time we left off talking about indiscrete categories. Here is a nice fact.

Proposition 1.23. Fix $\mathcal{C}$ an indiscrete category. Then all maps are isomorphisms.

Proof. Fix any morphism $f : x \to y$. There is also a morphism $g : y \to x$, and we see that $gf \in \mathrm{Mor}(x, x)$. But $\mathrm{id}_x \in \mathrm{Mor}(x, x)$ as well, so we are forced to have $gf = \mathrm{id}_x$ by uniqueness of morphisms. Similar shows that $fg = \mathrm{id}_y$, finishing the proof. ■

Remark 1.24. This statement is also true for discrete categories but only because all identity morphisms are isomorphisms immediately.

The property of the proposition is nice enough to deserve a definition.

Groupoid

Definition 1.25 (Groupoid)**.** A category in which all morphisms are isomorphisms is called a *groupoid*.

Example 1.26. Viewing groups as one-element categories, we see that groups are groupoids because all elements (i.e., morphisms of the one-object set) have inverses and hence are isomorphisms.

Intuitively, a groupoid is a group but more "spread out."

1.2.3 Arrow Words

We close out with some miscellaneous definitions for our morphisms.

Endo-, automorphism

Definition 1.27 (Endo-, automorphism)**.** Fix a category $\mathcal{C}$. A morphism $f : x \to y$ is an *endomorphism* if and only if $x = y$. A morphism $f : x \to y$ is an *autormophism* if and only if it is an isomorphism and an endomorphism.

Example 1.28. In the category of abelian groups, the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 is an endomorphism but not an automorphism.

Monic, epic

Definition 1.29 (Monic, epic)**.** Fix a category $\mathcal{C}$ and a morphism $f : x \to y$.

- We say f is a *monomorphism* (or is *monic*) if and only if $fg = fh$ implies $g = h$ for any morphisms $g, h : c \to x$. In other words, the map
$$\operatorname{Mor}(c, x) \xrightarrow{f\circ -} \operatorname{Mor}(c, y)$$
is injective. (This map is called "post-composition.") We might write $f : x \hookrightarrow y$ for emphasis.
- We say f is an *epimorphism* (or is *epic*) if and only if $gf = hf$ implies $g = h$ for any morphisms $g, h : y \to c$. In other words, the map
$$\operatorname{Mor}(y, c) \xrightarrow{-\circ f} \operatorname{Mor}(x, c)$$
is injective. (This map is called "pre-composition.") We might write $f : x \twoheadrightarrow y$ for emphasis.

Intuitively, the monomorphism condition looks like the injectivity condition (namely, $f(x) = f(y)$ implies $x = y$), so monic is supposed to be a generalization for injective.

Example 1.30. In the category of sets, monic is equivalent to injective, and epic is equivalent to surjective. Then it happens that being monic and epic implies being an isomorphic. We will not fill in the details here.

Warning 1.31. It is not always true that being monic and epic implies being isomorphic. It is true in $\mathrm{Set}, \mathrm{Ab}, \mathrm{Grp}$ but not in, say, Ring as the below example shows.

Example 1.32. The inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring is both epic and monic but not an isomorphism. We run some of the checks.

- We show monic. Suppose $g, h : R \to \mathbb{Z}$ are morphisms with $fg = fh$. We claim $g = h$. Well, for any $r \in R$, we see $g(r) = f(g(r))$ and $h(r) = f(h(r))$ because f is merely an inclusion, so $g(r) = h(r)$ follows.
- We show epic. Suppose $g, h : \mathbb{Q} \to R$ are morphisms with $gf = hf$. We claim $g = h$. We start by noting any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}$ will have
$$g\left(n/m\right) \cdot g(m) = g(n)$$
and similar for h. However, $g(m) = g(f(m)) = h(f(m)) = h(m)$ and $g(n) = h(n)$ for the same reason, so $g\left(\frac{n}{m}\right) = g(n)/g(m) = h(n)/h(m) = h\left(\frac{n}{m}\right)$, and we are done because any rational can be expressed as some $\frac{n}{m}$.
- Lastly, f is not an isomorphism because $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. For example, $2x - 1$ has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

And now discussion begins.

1.3 January 24

Chris is giving the lecture today. Reportedly, it might be rough around the edges, but I have full faith in its coherence.

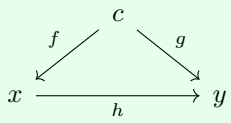
1.3.1 Review

Let's quickly talk about two fun types of categories.

Slice categories

Definition 1.33 (Slice categories)**.** Fix a category $\mathcal{C}$ and an object $c \in \mathcal{C}$.

- We define the *slice category* $\rfloor/\mathcal{C}$ to have objects which are morphisms $f : c \to x$ for objects $x \in \mathcal{C}$. The morphisms from $f : c \to x$ to $g : c \to y$ is a morphism $h : x \to y$ such that $f = gh$. Namely, we require the following triangle to commute.



- Dual to this is the *slice category* $\mathcal{C}/c$ where we reverse all the arrows. For example, our objects are morphisms $f : x \to c$, and morphisms from $f : x \to c$ to $g : y \to c$ are morphisms $f : x \to y$ such that $g = hf$.

There are also groupoids, which we ahvae defined previously.

1.3.2 Subcategories

We have the following definition.

Subcategory

Definition 1.34 (Subcategory)**.** A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ whose objects and morphisms come from $\mathcal{C}$ and that the composition law is inherited. Explicitly, we require $\mathcal{D}$ to have the identity morphisms and be closed under composition of $\mathcal{C}$ (i.e., if $f : x \to y$ and $g : y \to z$ are morphisms in $\mathcal{D}$, then gf is also a morphism in $\mathcal{D}$.)

We are going to want ways to generate subcategories. Here is one way.

Full subcategory

Definition 1.35 (Full subcategory)**.** Fix a category $\mathcal{C}$. Then we define the *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ to be defined by choosing some objects $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{C}$ and then choosing morphisms by taking all of them. Explicitly, for $x, y \in \operatorname{Ob}\mathcal{D}$, we have

$$\operatorname{Mor}_{\mathcal{D}}(x, y) = \operatorname{Mor}_{\mathcal{C}}(x, y).$$

Example 1.36. The category of abelian groups is a full subcategory of the category of groups. Namely, the category of abelian groups is made of the objects which are abelian groups and all arrows are simply all group homomorphisms, so no morphisms have been lost in this restriction.

Example 1.37. The category of finite sets is a full subcategory in the category of sets.

Example 1.38. Given a category $\mathcal{C}$, one can take the *maximal groupoid* of $\mathcal{C}$ to be the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the isomorphisms of $\mathcal{C}$. So as long as $\mathcal{C}$ has morphisms which are not isomorphisms, then the maximal groupoid will not be full.

Example 1.39. The category Rng is a subcategory of Ring, but it is not full. For example, in Ring, the map $\mathbb{Z} \overset{\times 2}{\mathbb{Z}}$ is not a morphism even though it is a morphism in Rng.

One has to be a bit careful with this, however.

Non-Example 1.40. The category Grp is not a subcategory of Set because one can endow the same set with different group structures.

1.3.3 Duality

Here is our main character.

Opposite category

Definition 1.41 (Opposite category)**.** Given a category $\mathcal{C}$, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to have objects which are objects of $\mathcal{C}$ and morphisms $f^{\mathrm{op}} : y \to x$ of $\mathcal{C}^{\mathrm{op}}$ are in one-to-one correspondence with morphisms $f : x \to y$ of $\mathcal{C}$. Lastly, composition is defined by, for $f^{\mathrm{op}} : y \to x$ and $g^{\mathrm{op}} : z \to y$, we have

$$f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}.$$

In pictures, the composition law reversed the diagram $x \xrightarrow{f} y \xrightarrow{g} z$ to

$$x \xleftarrow{f^{\mathrm{op}}} y \xleftarrow{g^{\mathrm{op}}} z.$$

Let's see some examples.

Example 1.42. Given a partial order $(\mathcal{P}, \leq)$, the opposite category is by (partial) ordering $\mathcal{P}$ simply by flipping the partial order: $b \leq_{\mathrm{op}} a$ if and only if $a \leq b$. Namely, the opposite category of a partial order remains a partial order.

Example 1.43. Fix a group G and form its category $\mathrm{B}G$. Now, when we reverse the arrows $(\mathrm{B}G)^{\mathrm{op}}$, we get a category corresponding to the group law G^{op} with group law defined by

$$h^{\mathrm{op}} g^{\mathrm{op}} = gh.$$

Namely, the opposite category of a group is still a group.

In fact, we have that $\mathrm{B}G \cong (\mathrm{B}G)^{\mathrm{op}}$ (for whatever $\cong$ means) by taking making our morphisms perform inversion by $\varphi : g \mapsto (g^{\mathrm{op}})^{-1}$. This map is bijective, and we can check the composition by writing

$$\varphi(gh) = ((gh)^{\mathrm{op}})^{-1} = (h^{\mathrm{op}} g^{\mathrm{op}})^{-1} = (g^{\mathrm{op}})^{-1} (h^{\mathrm{op}})^{-1} = \varphi(g)\varphi(h),$$

so everything works.

Example 1.44. Algebraic geometry says that $\mathrm{CRing}^{\mathrm{op}}$ is equivalent to the category of affine schemes AffSch. The point here is that the opposite category is potentially very different from the original category. (Mnemonically, the opposite of algebra is geometry.)

Now, here is the idea of duality.

Idea 1.45. Theorem statements that hold for categories will need to be true for their opposite category as well.

As an example, let's work with monomorphisms and epimorphisms. For example, $f : y \to z$ is monic if and only if the commutativity of the diagram

$$x \underset{h}{\overset{g}{\rightrightarrows}} y \xrightarrow[f]{} z$$

forces $g = h$. Similarly, $f : x \to y$ is epic if and only if the commutativity of the diagram

$$x \xrightarrow[f]{} y \underset{h}{\overset{g}{\rightrightarrows}} z$$

forces $g = h$. But notice that flipping the epic diagram notes that epic condition is equivalent to the commutativity of the diagram

$$x \underset{h^{\mathrm{op}}}{\overset{g^{\mathrm{op}}}{\rightrightarrows}} y \xrightarrow[f^{\mathrm{op}}]{} x$$

forces $g = h$, which is the same thing as $g^{\mathrm{op}} = h^{\mathrm{op}}$. Thus, we have the following lemma.

Lemma 1.46. Fix a category $\mathcal{C}$. Then a morphism f is monic if and only if f^{op} is epic in $\mathcal{C}$.

Proof. This comes from the discussion above. ∎

The point is that we can prove theorems about monic and epic maps simultaneously by working with (say) monomorphisms general categories and then dualizing to get the statement about epimorphisms.

Let's see this strategy in action. We have the following definition.

Section, retraction

Definition 1.47 (Section, retraction)**.** Suppose that $s : x \to y$ and $r : y \to x$ are morphisms such that $rs = \mathrm{id}_x$; i.e., the composition

$$x \xrightarrow{s} y \xrightarrow{r} x$$

is id_x. Then we say that s is a *section* of r, and r is a *retraction* of s.

Think about these as having a one-sided inverse. We have the following lemma.

Lemma 1.48. A morphism s in $\mathcal{C}$ is a section of some morphism if and only if s^{op} is a retraction in $\mathcal{C}$.

Proof. Fix $s : x \to y$. The condition that there exists r so that $rs = \mathrm{id}_x$ is equivalent to there exists r^{op} such that $s^{\mathrm{op}} r^{\mathrm{op}} = \mathrm{id}_x^{\mathrm{op}}$, which translates into the lemma. ∎

And now let's actually see a proof.

Proposition 1.49. A morphism s in $\mathcal{C}$ is a section of some morphism implies that s is a monomorphism.

Proof. Suppose that $s : x \to y$ is a section for the morphism $r : y \to x$ so that $rs = \mathrm{id}_x$. Now, suppose that $sg = sh$ so that we want to show $g = h$. But we see that

$$g = \mathrm{id}_x \, g = (rs)g = r(sg) = r(sh) = (rs)h = \mathrm{id}_x \, h = h,$$

so we are done. ∎

So here is our dual statement, which we get for free.

Proposition 1.50. A morphism r in $\mathcal{C}$ is a retraction of some morphism inplies that r is an epimorphism.

Proof. We note that r is a retraction in $\mathcal{C}$ implies that r^{op} is a section in $\mathcal{C}^{\mathrm{op}}$, so by the above, r^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$. Thus it follows that r is an epimorphism in $\mathcal{C}$. ∎

We've been saying "section of" and "retraction of" a lot, so we optimize out these words in the following definition.

Split mono-, split epi-morphism

Definition 1.51 (Split mono-, split epi-morphism)**.** We say that a morphism f of $\mathcal{C}$ is a *split monomorphism* if and only if it is a section of some morphism. Similarly, we say that f is a *split epimorphism* if and only if it is the retraction of some morphism.

So the above statements show that split monomorphisms are in fact monomorphisms, and split epimorphisms are in fact epimorphisms.

1.3.4 Yoneda Lite

So far we have said that monic is similar to injective and epic is similar to surjective. We would like to make these sorts of correspondences a little more concrete, so we add more abstraction.

Post- and pre-composition

Definition 1.52 (Post- and pre-composition)**.** Fix a morphism $f : x \to y$ of $\mathcal{C}$. Then, given an object $c \in \mathcal{C}$, we define the maps $f_* : \mathrm{Mor}(c, x) \to \mathrm{Mor}(c, y)$ and $f^*(y, c) \to \mathrm{Mor}(x, c)$ by

$$f_*(g) := fg \qquad \text{and} f^*(g) := gf.$$

The map f_* is called *post-composition* because we apply f after; the map f^* is called *pre-composition* because we apply it after.

Note that f_* and f^* are nice because they are all real functions of sets (for locally small categories) with which we can use to understand f. Here are some equivalent conditions.

Proposition 1.53. Fix f a morphism of the category $\mathcal{C}$. Then the following are true.

(a) f is an isomorphism if and only if f_* is bijective if and only if f^* is bijective.

(b) f is monic if and only if f_* is injective.

(c) f is epic if and only if f^* is injective (!).

(d) f is split monic if and only if f^* is surjective.

(e) f is split epic if and only if f_* is surjective.

Proof. We omit most of these; let's show (b). We have two directions. Suppose that f is monic. Then fix an object c, and we show that the map

$$f_* : \mathrm{Mor}(c, x) \to \mathrm{Mor}(c, y)$$

by $f_*(g) := fg$ is injective. But indeed, $f_*(g) = f_*(h)$ implies $fg = fh$ implies $g = h$ by monic, so injectivity follows.

Conversely, suppose f_* is monic. Then suppose that $fg = fh$ for some morphisms $g, h : c \to x$, and we show that $g = h$. But f_* is injective! So

$$f_*(g) = fg = fh = f_*(h)$$

forces $g = h$, and we are done. ∎

1.4 January 26

We will start on new things.

1.4.1 Functors

In this class, we will repeatedly talk about the following idea.

Idea 1.54. Everything is a special case of everything else.

In other words, we will want to abstract old ideas from new ones, and this will happen a lot.

The first time we are going to see this is by trying to consider categories of

Remark 1.55. Yes, Russel's paradox prevents a category of all categories. Nevertheless, we will try. One way to get around this is to do size declarations: for example, we can consider the category of all small categories, as we are about to do.

Anyways, we would like to give some categorial structure to (say, small) categories. Well, what will be our morphisms between categories? They will be "functors."

Before defining functors, we should describe what a functor $F : \mathcal{C} \to \mathcal{D}$ should do.

- Viewing $\mathcal{C}$ as consisting of the data of objects and morphisms, an initial requirement might be that F takes objects to objects and morphisms to morphisms.
- We would also like F to preserve the "structure" of our categories, which essentially means we want to preserve composition in our categories. So we will require a "functoriality" condition to preserve this structure.

Let's try to get an intuitive feeling for how functoriality should behave.

Example 1.56. Fix an abelian group A. Then there is a map $\mathrm{Hom}(A, -)$ sending abelian groups Ab to sets Set. In fact, we get a map of morphisms as well, for a morphism $f : X \to Y$ provides a post-composition mapping

$$f_* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$$

by $\varphi \mapsto f\varphi$. This association has some nice properties. For example, we have the following.

- We see $(\mathrm{id}_X)_* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, X)$ sends $\varphi \mapsto \varphi$, so $(\mathrm{id}_X)_* = \mathrm{id}_{\mathrm{Hom}(A,X)}$.
- Given $f : X \to Y$ and $g : Y \to Z$, we have $gf : X \to Z$, and we can see that

$$(gf)_*(\varphi) = gf\varphi = g_*(f_*(\varphi)) = (g_*f_*)(\varphi),$$

so we are "preserving composition" in some sense because we composed before and after.

Example 1.57. Given a topological space X, we can create the fundamental group $\pi_1(X)$. This mapping is nice because a continuous map $f : X \to Y$ will induce a map $\pi(f) : \pi_1(X) \to \pi_1(Y)$, and in fact we can check that $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X)}$ as well as preserving composition ($f : X \to Y$ and $g : Y \to Z$ gives $\pi_1(gf) = \pi_1(g)\pi_1(f)$).

With the above motivation, we are now ready to give the definition of a functor.

Functor

Definition 1.58 (Functor)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of "assignments" $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ satisfying the following coherence laws.

- Morphisms make sense: if $f : x \to y$ a morphism in $\mathcal{C}$, then Ff is a morphism with domain Fx and codomain Fy.
- Identity: given an object $c \in \mathcal{C}$, we require $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Composition: given morphisms $f : x \to y$ and $g : y \to z$ in $\mathcal{C}$, we require that $F(gf) = F(g)F(f)$.

1.4.2 More Examples

Let's do more examples.

Example 1.59 (Forgetful)**.** There is a functor $U : \mathrm{Grp} \to \mathrm{Set}$ which sends a group G to its underlying set G and a group homomorphism to the underlying function. In other words, we are simply forgetting the algebraic structure of the group. Because the composition law in groups is composition of functions, and identites in Grp do nothing like in Set.

Example 1.60 (Forgetful)**.** Here are more forgetful functors.

- $\mathrm{Ring} \to \mathrm{Grp}$ (by $R \mapsto R^\times$)
- $\mathrm{Field} \to \mathrm{Ring}$
- $\mathrm{Ring} \to \mathrm{Ab}$
- $\mathrm{Grp} \to \mathrm{Set}_*$ by sending $G \mapsto (G, e_G)$; namely, we point the set of G by its identity, which must be fixed by group homomorphisms anyways.

With all of our forgetful functors lying around, we have the following definition.

Concrete

Definition 1.61 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if it has a forgetful functor to Set.

This is not terribly formal because we haven't defined what a forgetful functor means, but hopefully this is sufficiently intuitive: $\mathcal{C}$ should be sets with some extra structure.

Before our next example, we pick up the following example.

Endofunctor

Definition 1.62 (Endofunctor)**.** A functor F is an *endofunctor* of its "domain" and "codomain" categories are the same category.

Example 1.63. There is an endofunctor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ sending a set X to its power set $\mathcal{P}(X)$. We send morphisms $f : X \to Y$ to $\mathcal{P}(f)$ by sending subsets $S_X \subseteq X$ in $\mathcal{P}(X)$ to the image $f(S_X) \in \mathcal{P}(Y)$. We will not check the functoriality conditions, but it can be done without too much effort.

And now for more examples.

Example 1.64. There is a functor $\mathrm{Top} \to \mathrm{Htpy}$ by sending a topological space X to the same space up to homotopy. Then we send continuous maps to continuous maps, up to homotopy.

Example 1.65. There is a "free" functor $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$ sending a set S to the abelian group

$$\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z}s.$$

Essentially, this is the free $\mathbb{Z}$-module generated by S; formally, $\mathbb{Z}[S]$ is made of finite $\mathbb{Z}$-linear combinations of elements of S.

Then we can take a function $f : S \to T$ to a group homomorphism $\mathbb{Z}[S] \to \mathbb{Z}[T]$ because we have described where to send the "basis elements" of S, and hence this f will uniquely determine the full map.

Example 1.66. Fix $\mathcal{C}$ a locally small category, and fix some $x \in \mathcal{C}$. Then there is a functor $\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ by sending

$$y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y) \qquad \text{and} \qquad (f : y \to z) \mapsto f_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z),$$

where $f_* : \varphi \mapsto f\varphi$ is again post-composition.

Example 1.67. There is an endofunctor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ by sending objects and morphisms to themselves.

1.4.3 Categories of Categories

While we're here, we note that we can create new functors from old ones by "composition."

Proposition 1.68. Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ functors. Then the naturally defined map $GF : \mathcal{C} \to \mathcal{D}$ is also a functor.

Proof. We do indeed send objects to objects, and a morphism $f : x \to y$ in $\mathcal{C}$ will be sent to $F(f) : Fx \to Fy$ and then

$$GF(f) : GFx \to GFy.$$

Further, we can check that $GF(\mathrm{id}_x) = G(\mathrm{id}_{Fx}) = \mathrm{id}_{GFx}$, so GF preseres identities. And then, given $f : x \to y$ and $g : y \to z$, we see that

$$GF(gf) = G(F(g)F(f)) = GF(g)GF(f),$$

which finishes the composition check. ∎

The point of the above composition law, is that it lets us form a "category."

Definition 1.69. We define Cat to be the category of small categories where morphisms are functors. We define CAT to be the category of locally small categories where morphisms again are functors.

Remark 1.70. Fixing two small categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ can be identified with a function on merely the morphism sets $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$, which is itself a set. Thus, Cat is a locally small category: $\mathrm{Cat} \in \mathrm{CAT}$.

1.4.4 Subcategories.

To finish out class, we have the following warning.

Warning 1.71. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We check that the naturally defined "image" $F(\mathcal{C})$ is need not be a subcategory of $\mathcal{D}$.

Here is an example. Let $\mathcal{C}$ be the following category.

$$a \xrightarrow{f} b$$

$$a' \xrightarrow{f'} b'$$

Then let $\mathcal{D}$ be the following category.

$$0 \xrightarrow{x} 1 \xrightarrow{y} 2$$

Now we define $F : \mathcal{C} \to \mathcal{D}$ by $Ff = x$ and $Ff' = y$, which will make a perfectly fine functor. However, the composition $yx : 0 \to 2$ in $\mathcal{D}$ does not live in the image of F, so this image is not a subcategory.

To fix this problem, one often says something like "given a functor $F : \mathcal{C} \to \mathcal{D}$, consider the full subcategory of $F(\mathcal{C})$" to mean closing up $F(\mathcal{C})$'s potentially unclosed composition.

1.5 January 31

So class is in-person today.

1.5.1 Small Remark

A question was asked in the Discord server about dualizing. In theory, dualizing theorems should be very easy: simply state the theorem in the opposite category, provided we have shown the necessary machinery to make the theorem dualize as necessary.

1.5.2 Contravariance

Today we are talking about contravariance. A functor $F : \mathcal{C} \to \mathcal{D}$ is defined so far as what are called "covariant" functors. We would like to define contravariant functors. There are lots of equivalent ways to do this.

Contravariance, I

Definition 1.72 (Contravariance, I)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a mapping of objects and morphisms with the following coherence laws.

- If $f : a \to b$ in $\mathcal{C}$, then $Ff : Fb \to Fa$. (Note the reversal of direction!)
- Identity: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for each $c \in \mathcal{C}$.
- Contravariant (!) composition: if $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, then $F(gf) = F(f)F(g)$.

This in fact comes from dualizing.

Contravariance, II

Definition 1.73 (Contravariance, II)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

To be explicit, if we are given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, then a morphism $f : a \to b$ in $\mathcal{C}$ is first taken to a morphism $f^{\mathrm{op}} : b^{\mathrm{op}} \to a^{\mathrm{op}}$. And if we have another morphism $g : b \to c$ in $\mathcal{C}$, then we see the diagram

$$a \xrightarrow{f} b \xrightarrow{g} c$$

becomes

$$a^{\mathrm{op}} \xleftarrow{f^{\mathrm{op}}} b^{\mathrm{op}} \xleftarrow{g^{\mathrm{op}}} c^{\mathrm{op}}$$

becomes

$$Fa^{\mathrm{op}} \xleftarrow{Ff^{\mathrm{op}}} Fb^{\mathrm{op}} \xleftarrow{Fg^{\mathrm{op}}} Fc^{\mathrm{op}},$$

which gives our composition law.

We can also dualize in the opposite direction.

Contravariance, III

Definition 1.74 (Contravariance, III)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Warning 1.75. We will use Definition 1.73 as our definition of contravariance.

Example 1.76. We work with Vec_k the category whose objects are k-vector spaces and morphisms which are linear maps. Then we have a functor

$$-^* : \mathrm{Vec}_k^{\mathrm{op}} \to \mathrm{Vec}_k$$

by taking $V \mapsto V^*$. (Here, $V^* := \mathrm{Hom}_k(V, k)$.) As for morphisms, we need to take $f : V \to W$ to some map $f^* : W^* \to V^*$, which is

$$f^* : \varphi \mapsto \varphi f.$$

Example 1.77. We work with Poset the category whose objects are posets and morphisms which are order-preserving maps. I.e., a map $f : P \to Q$ is order-preserving if and only if $a \leq b$ in P implies $f(a) \leq f(b)$ in Q. Now we define the contravariant functor $\mathcal{O} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Poset}$ by taking

$$X \mapsto \{U : \text{open } U \subseteq X\},$$

where the order on the right is by inclusion. Then a continuous map $f : X \to Y$ becomes the order-preserving (!) map $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ by

$$\mathcal{O}(f)(U_Y) := f^{-1}(U_Y).$$

Explicitly, open subsets $U_1 \subseteq U_2$ of Y have $f^{-1}(U_1) \subseteq f^{-1}(U_2)$ back in X.

Remark 1.78. We can use the above example to define a presheaf. "Presheaf" can have lots of meanings.

- A "presheaf" can be any contravariant functor.
- A "presheaf" can be any contravariant functor with codomain Set.
- A "presheaf" can be any contravariant functor from $\mathcal{O}(X)^{\mathrm{op}}$. It is Set-valued (respectively, $\mathcal{C}$-valued) if its codomain is Set (respectively, $\mathcal{C}$).

1.5.3 A Lemma

It's a math class, so we should probably prove something today.

Theorem 1.79. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ preserves isomorphisms.

Remark 1.80. By convention, all functors will be covariant, and if we want a contravariant functor, we will write $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. In other words, I will now stop writing "(covariant)."

Proof. Let $f : a \to b$ be an isomorphism in $\mathcal{C}$ with inverse g. We want to show that $F(f)$ is an isomorphism; we claim that $F(g)$ is its inverse. Indeed,

$$F(f)F(g) = F(fg) = F(\mathrm{id}_b) = \mathrm{id}_{F(b)} \qquad \text{and} \qquad F(g)F(f) = F(gf) = F(\mathrm{id}_a) = \mathrm{id}_{F(a)},$$

so indeed, $F(g)$ is an inverse of $F(f)$. So $F(f)$ is an isomorphism, and we are done. ∎

This example can do things.

Example 1.81. Fix groups G, H and their one-object categories $\mathrm{B}G, \mathrm{B}H$. We claim that functors $F : \mathrm{B}G \to \mathrm{B}G$ contain exactly the data of a group homomorphism $G \to H$. To see that F induces a group homomorphism, suppose $\sigma, \tau \in G$, we have by funtoriality

$$F(\sigma\tau) = F(\sigma)F(\tau),$$

which is exactly what we need to be a group homomorphism. Conversely, if $f : G \to H$ is a group homomorphism, then f induces a functor: $f(\sigma\tau) = f(\sigma)f(\tau)$ by definition, and $f(\mathrm{id}_G) = \mathrm{id}_H$ is a result of group theory.

Example 1.82. A functor $F : \mathrm{B}G \to \mathcal{C}$ is precisely the data of a G-action of an object $c \in \mathcal{C}$. We send the one object $* \in \mathrm{B}G$ somewhere, say to an object $c \in \mathcal{C}$. Then each $\sigma \in G$ goes to some morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ (which is in fact an isomorphism because σ is an isomorphism $\mathrm{B}G$). So in total we get a map

$$G \to \mathrm{Aut}\, c,$$

which is exactly the data of a group action. This unifies group actions on all sorts of structures.

The above definition is special enough to have a name.

Functorial group action

Definition 1.83 (Functorial group action)**.** A *functorial group action* of G on a category $\mathcal{C}$ is a functor $\mathrm{B}G \to \mathcal{C}$.

Remark 1.84. Technically we will be viewing these functors as providing left actions. To get a right action, we want a functor $(\mathrm{B}G)^{\mathrm{op}} \to \mathcal{C}$.

Note, as in the example, the functor contains the same data as a group homomorphism $G \to \mathrm{Aut}\, c$ for some $c \in \mathcal{C}$.

Remark 1.85. Bryce would like to make us aware that writing down $G \to \mathrm{Aut}\, c$ as a group homomorphism is only legal when $\mathcal{C}$ is locally small.

Example 1.86. Given a group G, a G-representation V of G is a functor $\mathrm{B}G \to \mathrm{Vec}_k$ where $* \in \mathrm{B}G$ goes to $V \in \mathrm{Vec}_k$.

1.5.4 The Hom Bifucntor

We have a little time left, so let's do something fun. Given a (locally small) $\mathcal{C}$ and an object $x \in \mathcal{C}$, we get two functors

$$\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set} \qquad \text{and} \qquad \mathrm{Mor}_{\mathcal{C}}(-, x) : \mathcal{C} \to \mathrm{Set}.$$

The former functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $\varphi : y \to z$ to $\varphi_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$ to $\varphi_* : f \mapsto \varphi f$. We can check this functor is covariant because

$$\varphi_*\psi_*(f) = \varphi\psi f = (\varphi\psi)_*(f).$$

Now, the latter functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(y, x)$ and $\varphi : y \to z$ to $\varphi^* : \mathrm{Mor}_{\mathcal{C}}(z, x) \to \mathrm{Mor}_{\mathcal{C}}(y, z)$ by $\varphi^* : f \mapsto \varphi f$. We can check this functor is contravariant because

$$\psi^*\varphi^* f = f\varphi\psi = (\varphi\psi) * f.$$

1.6 February 2

Today we are talking about produt categories and the Hom bifunctor.

1.6.1 Hom Bifunctor

Here is our definition.

Product category

Definition 1.87 (Product category)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then we define the *product category* $\mathcal{C} \times \mathcal{D}$ as follows.

- We define $\mathrm{Ob}\,\mathcal{C} \times \mathcal{D}$ to be the collection of ordered pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
- We define $\mathrm{Mor}((c, d), (c', d'))$ to be the collection of ordered pairs (f, g) with $f : c \to c'$ a morphism in $\mathcal{C}$ and $g : d \to d'$ a morphism in $\mathcal{D}$.

We define identity to be the identity on each object and composition by composition componentwise.

From yesterday, we have the following functors.

Functors represented by objects

Definition 1.88 (Functors represented by objects)**.** Fix $\mathcal{C}$ a locally small category and $x \in \mathcal{C}$ an object. Then we have the functors

$$\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set} \qquad \text{and} \qquad \mathrm{Mor}_{\mathcal{C}}(-, x) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}.$$

The former functor is the *covariant functor represented by* x, and the latter is the *contravariant functor represented by* x.

We would like to codify the structure that having two functors gives us, so we have the following definition.

Bifunctor

Definition 1.89 (Bifunctor)**.** A *bifunctor* is a functor whose domain is a product of categories.

In particular, here is our standard example.

Hom bifunntor

Definition 1.90 (Hom bifunntor)**.** Fix $\mathcal{C}$ a locally small category. Then Hom *bifunctor* is the functor given by the functors representing a particular object $x \in \mathcal{C}$. Namely, we have

$$\mathrm{Mor}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$$

by taking $(x, y) \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y)$.

We will not check that this is actually a functor, but it is.

1.6.2 Category Isomorphism

We would like a notion of two categories being the same, but this is somewhat subtle. Here is a first approximation.

Isomorphism

Definition 1.91 (Isomorphism)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if and only if there is an inverse functor $G : \mathcal{D} \to \mathcal{C}$ so that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. In this case we say that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*.

Remark 1.92. As usual, isomorphisms are unique and whatnot.

Let's make this definition a little more concrete.

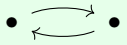
Proposition 1.93. An isomorphism $F : \mathcal{C} \to \mathcal{D}$ descends to a bijective (i.e., injective and surjective) map $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$.

Remark 1.94. We are attempting to care about set-theoretic issues in our phrasing because Bryce cares about set-theoretic issues.

Proof of Proposition 1.93. Let G be the inverse morphism for F. Then we claim that the induced map $G : \operatorname{Ob}\mathcal{D} \to \operatorname{Ob}\mathcal{C}$ will be the inverse for the induced map for F. This is clear because $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. ■

It turns out that isomorphisms are a little too strong: there are categories we want to be the same but are not actually isomorphic.

Example 1.95. The category



is not isomorphic to

because there are a different number of objects, so there is no bijection.

1.6.3 Natural Transformation

To salvage our notion of categorical isomorphism, we need a notion of naturality. Naturality is more of something that we can feel as mathematicians rather than something we like to formalize.

Example 1.96. Any two trivial groups have a canonical isomorphism between them. In fact, there is only one homomorphism at all.

Non-Example 1.97. There is no "natural" or "canonical" isomorphism $\mathbb{Z}/3\mathbb{Z} \to A_3$, though the groups are isomorphic.

Non-Example 1.98. Given a two-dimensional $\mathbb{R}$-vector space named V, there is no canonical isomorphism $\mathbb{R}^2 \to V$.

We would like maps to preserve all of the structure we could want. So here is our notion of naturality for functors.

Natural transformation

Definition 1.99 (Natural transformation)**.** Fix functors $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\eta : F \Rightarrow G$ consists of the data of a morphism $\eta_X : Fc \to Gc$ for each $c \in \mathcal{C}$ such that the following diagram always commutes for any morphism $f : c \to c'$ in $\mathcal{C}$.

$$\begin{array}{ccc} Fc & \xrightarrow{\eta_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow[\eta_{c'}]{} & Gc' \end{array}$$

The maps φ_c are called the *components* of φ.

Quote 1.100. Burn this square into your minds. It is the most important square in this class.

As usual, we start with examples.

Exercise 1.101. We work in Vec_k. Then we consider the functor $-^{**} : \mathrm{Vec}_k \to \mathrm{Vec}_k$ by $V \mapsto V^{**}$. Then we claim that there is a natural transformation from $-^{**}$ to id, using the the natural transformation

$$\mathrm{ev}_V : V \to V^{**}$$

by $\mathrm{ev}_V(x) := (\lambda \in V^* \mapsto \lambda x)$.

Proof. We need to check that the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{ev}_V} & V^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ W & \xrightarrow[\mathrm{ev}_W]{} & W^{**} \end{array}$$

Very quickly, we recall that $f^{**} : V^{**} \to W^{**}$ is by

$$f(\varphi) = (\lambda \in W^* \mapsto \varphi(\lambda f)).$$

Namely, $\lambda : W \to k$, so $\lambda f : V \to k$ lives in V^*, so $\varphi(\lambda f) \in k$.

Now we check the commutativity of the square. Fix some $x \in V$ and a linear functional $\lambda : W \to k$. Then we can carefully compute, after many tears and groans, that

$$f^{**}(\mathrm{ev}_V(x))(\lambda) = \mathrm{ev}_V(x)(\lambda f) = \lambda f(x) = \mathrm{ev}_W(f(x))(\lambda).$$

because λ was arbitrary, we see that $f^{**}\,\mathrm{ev}_V(\lambda) = \mathrm{ev}_W\, f(x)$, which then gives us $f^{**}\,\mathrm{ev}_V = \mathrm{ev}_W\, f$. ∎

We have the following definition.

Natural isomorphism

Definition 1.102 (Natural isomorphism)**.** A natural transformation $\eta : F \to C$ is a *natural isomorphism* if and only if its component morphisms are isomorphisms.

Example 1.103. In finVec_k, the above ev is a natural isomorphism because $\mathrm{ev}_V : V \Rightarrow V^{**}$ is an isomorphism when V is finite-dimensional.

Here is a quick proposition.

Proposition 1.104. Let $\varphi : F \Rightarrow G$ be a natural isomorphism. Then the inverse morphisms $\psi_c := \varphi_c^{-1}$ assemble to make a natural transformation $\psi : G \Rightarrow F$.

Proof. We will be brief. Given a morphism $f : x \to y$, we need to check that the following diagram commutes.

$$\begin{array}{ccc} Gx & \xrightarrow{\psi_x} & Fx \\ {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ Gy & \xrightarrow[\psi_y]{} & Fy \end{array}$$

In other words, we need to know that $\psi_y F f = G f \psi_x$. Well, we already know that

$$\varphi_y F f = G f \varphi_x$$

by naturality, so

$$F f \psi_x = \psi_y \varphi_y F f \psi_x = \psi_y G f \varphi_x \psi_x = \psi_y G f$$

after checking through. ∎