185: Introduction to Complex Analysis

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THEME 1

INTRODUCTION

Our reality isn't about what's real, it's about what we pay attention to.

—Hank Green

1.1 January 19

It is reportedly close enough to start.

1.1.1 Logistics

We are online for the first two weeks, as with the rest of Berkeley. We will be using `bCourses` a lot, so check it frequently. There is also a website. There is a homework due on Friday, but do not worry about it.

Here are some syllabus things.

- Our main text is *Complex Variables and Applications, 8th Edition* because it is the version that Professor Morrow used. There is a free copy online.
- The homework consists of readings (for each course day) and weekly problem sets. Late homework is never accepted.
- Lowest two homework scores are dropped.
- There are two midterms and a final. The final is cumulative, as usual. The final can replace one midterm if the score is higher.
- Regrade requests can be made in GradeScope within one week of being graded.
- The class is curved but usually only curved at the end. The average on exams is expected to be 80%–83%.

1.1.2 Complex Numbers

Welcome to complex analysis. What does that mean?

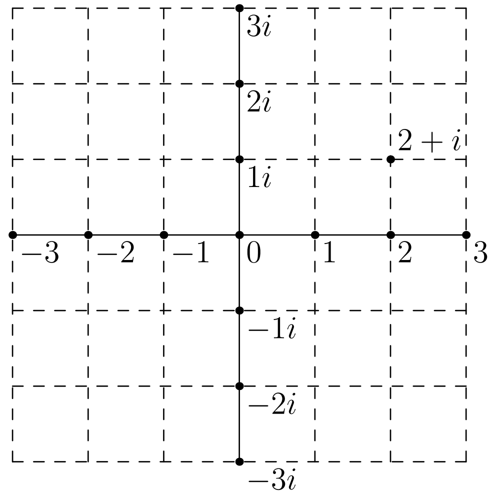
Idea 1.1. In complex analysis, we study functions $f : \mathbb{C} \to \mathbb{C}$, usually analytic to some extent.

There are two pieces here: we should study $\mathbb{C}$ in themselves, and then we will study the functions.

Definition 1.2 (Complex numbers)**.** The set of complex numbers $\mathbb{C}$ is $\{a + bi : a, b \in \mathbb{R}\}$, where $i^2 = -1$.

Hopefully $\mathbb{R}$ is familiar from real analysis. As an aside, we see $\mathbb{R} \subseteq \mathbb{C}$ because $a = a + 0i \in \mathbb{C}$ for each $a \in \mathbb{R}$.

The complex numbers have an inherent geometry as a two-dimensional plane.



The point is that $\mathbb{C}$ looks like the real plane $\mathbb{R}^2$. More precisely, $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-vector space, where our basis is $\{1, i\}$.

We would like to understand $\mathbb{C}$ geometrically, "as a space." The first step here is to create a notion of size.

Definition 1.3 (Norm on $\mathbb{C}$)**.** We define the *norm map* $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by $|z| := \sqrt{z\overline{z}}$. In other words,

$$|a + bi| := \sqrt{a^2 + b^2}.$$

Note that this agrees with the absolute value on $\mathbb{R}$: for $a \in \mathbb{R}$, we have $\sqrt{a^2} = |a|$.

Norm functions, as in the real case, give us a notion of distance.

Definition 1.4 (Metric on $\mathbb{C}$)**.** We define the *metric on* $\mathbb{C}$ to be $d_{\mathbb{C}}(z_1, z_2) := |z_1 - z_2|$.

One can check that this is in fact a metric, but we will not do so here.

Remark 1.5. The distance in $\mathbb{C}$ is defined to match the distance in $\mathbb{R}^2$ under the basis $\{1, i\}$.

Again as we discussed in real analysis, having a metric gives us a metric topology by open balls. Lastly it is this topology that our geometry will follow from: we have turned $\mathbb{C}$ into a topological space.

1.1.3 Complex Functions

There are lots of functions on $\mathbb{C}$, and lots of them are terrible. So we would like to focus on functions with some structure. We'll start with *continuous functions*, which are more or less the functions that respect topology.

Then from continuous functions, we will be able to define *holomorphic functions*, which are complex differentiable. This intended to be similar to being real differentiable, but complex differentiable turns out to be a very strong condition. Nevertheless, everyone's favorite functions are holomorphic.

Example 1.6. Polynomials, $\exp$, $\sin$, and $\cos$ are all holomorphic.

To make concrete that complex differentiability is stronger than real differentiability, the Cauchy–Riemann equations which provides a partial differential equation to test complex differentiability.

From here we define *analytic* functions, which essentially are defined as taking the form

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Analytic functions are super nice in that we have an ability to physically write them down, so the following theorem is amazing.

Theorem 1.7. Holomorphic functions on $\mathbb{C}$ are analytic.

To prove this, we will need the following result, which is what Professor Morrow calls the most fundamental result in complex analysis, the *Cauchy integral formula*.

In short, the Cauchy integral formula lets us talk about the value of holomorphic functions (and derivatives) at a point in terms of integrals around the point. This will essentially let us build the power series for a holomorphic function by hand. But as described, we will need a notion of complex (path) integration to even be able to talk about the Cauchy integral formula.

The Cauchy integral formula has lots of applications; for example, *Liouville's theorem* on holomorphic functions and the *Fundamental theorem of algebra*.

Remark 1.8. It is very hard to spell Liouville.

Additionally, we remark that our study of holomorphic functions, via the Cauchy integral formula, will boil down to a study of complex path integrals. So we will finish out our story with the *Residue theorem*, which provides a very convenient way to compute such integrals.

Then as a fun addendum, we talk about automorphisms of the complex numbers.

Definition 1.9 (Automorphisms of $\mathbb{C}$)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is an *automorphism of* $\mathbb{C}$ if it is bijective and both f and f^{-1} are holomorphic.

What is amazing is that all of these functions have a concrete description in terms of *Möbius transformations*.

1.1.4 Why Care?

Whenever taking a class, it is appropriate to ask why one should care. Here are some reasons to care.

- Algebraic geometry in its study of complex analytic spaces uses complex analysis.
- Analytic number theory (e.g., the Prime number theorem) makes heavy use of complex analysis.
- Combinatorics via generating functions can use complex analysis.
- Physics uses complex analysis.

The first two Professor Morrow is more familiar with, the last two less so.

1.2 January 21

We're reviewing set theory today.

1.2.1 Set Theory Notation

We have the following definitions.

- $\varnothing$ means the empty set.
- $a \in X$ means that a is an element of the set X.
- $A \subseteq B$ means that A is a subset of B.
- $A \subsetneq B$ means that A is a proper subset of B.
- $A \cup B$ consists of the elements which are in at least one of A or B.
- $A \cap B$ consists of the elements which are in both A and B.
- $A \setminus B$ consists of the elements of A which are not in B.
- Two sets A and B are *disjoint* if and only if $A \cap B = \varnothing$.
- Given a set X, we define $\mathcal{P}(X)$ to be the set of all subsets of X.
- $|X| = \#X$ is the cardinality of X, or (roughly speaking) the number of elements of X.

As an example of unwinding notation, we have the following.

Proposition 1.10 (De Morgan's Laws)**.** Fix $\mathcal{S} \subseteq \mathcal{P}(X)$ a collection of subsets of a set X. Then

$$X \Big\backslash \bigcap_{S\in\mathcal{S}} S = \bigcup_{S\in\mathcal{S}} (X \setminus S) \qquad \text{and} \qquad X \setminus \bigcup_{S\in\mathcal{S}} S = \bigcap_{S\in\mathcal{S}} (X \setminus S).$$

Proof. We take these one at a time.

- Note $a \in X \setminus \bigcap \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcap \mathcal{S}$. However, $a \notin \bigcap \mathcal{S}$ is merely saying that a is not in all the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for one of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for some $S \in \mathcal{S}$, which is equivalent to $a \in \bigcup_{S\in\mathcal{S}} (X \setminus S)$.
- Note $a \in X \setminus \bigcup \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcup \mathcal{S}$. However, $a \notin \bigcup \mathcal{S}$ is merely saying that a is not in any of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for each of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for each $S \in \mathcal{S}$, which is equivalent to $a \in \bigcap_{S\in\mathcal{S}} (X \setminus S)$. ■

1.2.2 Some Conventions

In this class, we take the following names of standard sets.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Importantly, $0 \in \mathbb{N}$.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}$ and $q \neq\}$ is the set of rationals.
- $\mathbb{R}$ is the set of real numbers. We will not specify a construction here; see any real analysis class.
- $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq\}$ is the nonzero real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ is the positive real numbers.

- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ is the nonnegative real numbers.
- $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$ is the nonpositive real numbers.
- $\mathbb{C}$ is the complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$ is the set of nonzero complex numbers.

1.2.3 Relations

Let's review some set theory definitions.

Definition 1.11 (Cartesian product)**.** Given two sets A and B, we define the *Cartesian product* $A \times B$ to be the set of ordered pairs (a, b) such that $a \in A$ and $b \in B$.

Definition 1.12 (Binary relation)**.** A *binary relation* on A is any subset $R \subseteq A^2 := A \times A$. We may sometimes notate $(x, y) \in R$ by xRy, read as "x is related to y."

Example 1.13. Equality is a binary relation on any set A; namely, it is the subset $\{(a, a) : a \in A\}$.

The best relations are equivalence relations.

Definition 1.14 (Equivalence relation)**.** An *equivalence relation* on A is a binary relation R satisfying the following three conditions.

- Reflexive: each $x \in A$ has $(x, x) \in R$.
- Symmetric: each $x, y \in A$ has $(x, y) \in R$ implies $(y, x) \in R$.
- Transitive: each $x, y, z \in A$ has $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Equivalence relations are nice because they allow us to partition the set into "equivalence classes."

Definition 1.15 (Equivalence class)**.** Fix A a set and $R \subseteq A^2$ an equivalence relation. Then, for given $x \in A$, we define

$$[x]_R := \{y \in A : (x, y) \in R\}$$

to be the *equivalence class* of x.

The hope is that equivalence classes partition the set. What is a partition?

Definition 1.16 (Parition)**.** A *partition* of a set A is a collection of nonempty subsets $\mathcal{S} \subseteq \mathcal{P}(A)$ of A such that any two distinct $S_1, S_2 \in \mathcal{S}$ are disjoint while $A = \bigcup_{S \in \mathcal{S}} S$.

And now let's manifest our hope.

Lemma 1.17. Equivalence relations are in one-to-one correspondence with partitions of A.

Proof. Given an equivalence relation R, we define the collection

$$\mathcal{S}(R) = \{[x]_R : x \in A\}.$$

We claim that $R \mapsto \mathcal{S}(R)$ is our needed bijection. We have the following checks.

- Well-defined: observe that $\mathcal{S}(R)$ does partition A: if we have $[x]_R, [y]_R \in \mathcal{S}$, then $[x]_R \cap [y]_R \neq \varnothing$ implies there is some z with $(x,z) \in R$ and $(z,y) \in R$, so $x \in [y]_R$ and then $[x]_R \subseteq [y]_R$ follows. So by symmetry, $[y]_R \subseteq [x]_R$ as well, so we finish the disjointness check.

  Further, we see that
  $$A = \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x\in A} [x]_R \subseteq A$$
  because $x \in [x]_R$, so indeed the equivalence classes cover A.

- Injective: suppose R_1 and R_2 have $\mathcal{S}(R_1) = \mathcal{S}(R_2)$. We show that $R_1 \subseteq R_2$, and $R_2 \subseteq R_1$ will follow by symmetry, finishing.

  We notice that, for any $\mathcal{S}$ partitioning A, being a partition, will have exactly one subset which contains x. But for $\mathcal{S}(R)$ for an equivalence relation R, we see $x \in [x]_R \in \mathcal{S}(R)$, so this equivalence class must be the one.

  So because $[x]_{R_1}$ and $[x]_{R_2}$ are the only subsets of $\mathcal{S}(R_1)$ and $\mathcal{S}(R_2)$ containing x (respectively), we must have $[x]_{R_1} = [x]_{R_2}$. Thus, $(x,y) \in R_1$ implies $y \in [x]_{R_1} = [x]_{R_2}$ implies $(x,y) \in R_2$.

- Surjective: suppose that $\mathcal{S}$ is a partition of A. As noted above, each $x \in A$ is a member of exactly one set $S \in \mathcal{S}$, which we call $[x]$. Then we define $R \subseteq A^2$ by $(x,y) \in R$ if and only if $y \in [x]$. One can check that this is an equivalence relation, which we will not do here in detail.[1]

  The point is that
  $$[x]_R = \{y : (x,y) \in R\} = \{y : y \in [x]\} = [x],$$
  so $\mathcal{S}(R) = \mathcal{S}$. So our mapping is surjective. ■

We continue our discussion.

Definition 1.18 (Quotient set)**.** Given an equivalence relation $R \subseteq A^2$, we define the *quotient set* A/R is the set of equivalence classes of R. In other words,
$$A/R = \{[x]_R : x \in A\}.$$

Intuitively, the quotient set is the set where we have gone ahead and identified the elements which are "similar" or "related."

We would like a more concrete way to talk about equivalence classes, for which we have the following.

Definition 1.19 (Representatives)**.** Given an equivalence relation $R \subseteq A^2$, we say that $C \subseteq A$ is a *set of representatives of R-equivalence classes of A* if and only if C consists of exactly one element from each equivalence class in A/R.

1.2.4 Functions

To finish off, we discuss functions.

Definition 1.20 (Functions)**.** A *function* $f : X \to Y$ is a relation $f \subseteq X \times Y$ satisfying the following.

- For each $x \in X$, there is some $y \in Y$ such that $(x,y) \in f$. Intuitively, each $x \in X$ goes somewhere.
- For each $x \in X$ and given some $y_1, y_2 \in Y$ such that $(x,y_1), (x,y_2) \in f$, then $y_1 = y_2$. Intuitively, each $x \in X$ goes to at most one place.

We will write $f(x) = y$ as notational sugar for $(x,y) \in f$. Note this equality is legal because the value y with $(x,y) \in f$ is uniquely given.

[1] Note $x \in [x]$ by definition of $[x]$. If $y \in [x]$, then note $y \in [y]$ as well, so $[x] = [y]$ is forced by uniqueness, so $x \in [y]$. If $y \in [x]$ and $z \in [y]$, then again by uniqueness $[x] = [y] = [z]$, so $z \in [x]$ follows.

We would like to create new functions from old. Here are two ways to do this.

Definition 1.21 (Restriction)**.** Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define

$$f|_A = \{(x, y) \in f : x \in A\} \subseteq A \times Y$$

to be a function $f|_A : A \to Y$.

We will not check that $f|_A$ is actually a function; it is, roughly speaking inherited from f.

Definition 1.22. Given two functions $f : X \to Y$ and $g : Y \to Z$, we define the *composition* of f and g to be some function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) := g(f(x)).$$

Again, we will not check that this makes a function; it is.

Functions can also help create new sets.

Definition 1.23 (Image)**.** Given a function $f : X \to Y$, we define the *image* of f to be

$$\operatorname{im} f = f(X) := \{y \in Y : \text{there is } x \in X \text{ such that } f(x)y\}.$$

Namely, $\operatorname{im} f$ consists of all elements hit by someone in X hit by f.

Definition 1.24 (Fiber, pre-image)**.** Given a function $f : X \to Y$ and some $y \in Y$, we define the *fiber* of f over y to be

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

In general, we define the *pre-image* of a subset $A \subseteq X$ to be

$$f^{-1}(A) := \{x \in A : f(x) \in A\} = \bigcup_{a \in A} \{x \in A : f(x) = a\} = \bigcup_{a \in A} f^{-1}(a).$$

Some functions have nicer properties than others.

Definition 1.25 (Inj-, sur-, bijective)**.** Fix a function $f : X \to Y$. We have the following.

- Then f is *injective* or *one-to-one* if and only if, given $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- Then f is *surjective* or *onto* if and only if $\operatorname{im} f = Y$. In other words, for each $y \in Y$, there exists $x \in X$ with $f(x) = y$.
- Then f is *bijective* if and only if it is both injective and surjective.

Here is an example.

Definition 1.26 (Identity)**.** For a given set X, the function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) := x$ is called the *identity function*.

For completeness, here are the checks that id_X is bijective.

- Injective: given $x_1, x_2 \in X$, we see $\mathrm{id}_X(x_1) = \mathrm{id}_X(x_2)$ implies $x_1 = \mathrm{id}_X(x_1) = \mathrm{id}_X(x_2) = x_2$.
- Surjective: given $x \in X$, we see that $x \in \operatorname{im} \mathrm{id}_X$ because $x = \mathrm{id}_X$.

We leave with some lemmas, to be proven once in one's life.

Lemma 1.27. Fix finite sets X and Y such that $\#X = \#Y$. Then a function $f : X \to Y$ is bijective if and only if it is injective or surjective.

Proof. Certainly if f is bijective, then it is both injective and surjective, so there is nothing to say.

The reverse direction is harder. We proceed by induction on $\#X = \#Y$. If $\#X = \#Y = 0$, then $X = Y = \varnothing$, and all functions $f : \varnothing \to \varnothing$ are vacuously bijective: for injective, note that any $x_1, x_2 \in \varnothing$ have $x_1 = x_2$; for surjective, note that any $x \in \varnothing$ has $f(x) = x$.

Otherwise, $\#X = \#Y > 0$. We have two cases.

- Take f injective; we show f is surjective. In this case, $\#X > 0$, so choose some $a \in X$. Note that $x \in X$ with $x \neq a$ will have $f(x) \neq f(a)$ by injectivity, so we may define the restriction
$$f|_{X\setminus\{a\}} : X \setminus \{a\} \to Y \setminus \{f(a)\}.$$
Observe that $f|_{X\setminus\{a\}}$ is injective because f is: if $x_1, x_2 \in X \setminus \{a\}$ have
$$f(x_1) = f|_{X\setminus\{a\}}(x_1) = f|_{X\setminus\{a\}}(x_2) = f(x_2),$$
then $x_1 = x_2$ follows.
Now, $\#(X \setminus \{a\}) = \#(Y \setminus \{f(a)\}) = \#X - 1$, so by induction $f|_{X\setminus\{a\}}$ will be bijective because it is injective. In particular, f by way of $f|_{X\setminus\{a\}}$ fully hits $Y \setminus \{f(a)\}$ in its image, so because $f(a) \in \operatorname{im} f$ as well, we conclude $\operatorname{im} f = Y$. So f is surjective.

- Take f surjective; we show f is injective. Define a function $g : Y \to X$ as follows: for each $y \in Y$, the surjectivity of f promises some $x \in X$ such that $f(x) = y$, so choose any such x and define $g(y) := x$.[2] Observe that $f(g(y)) = y$ by construction.
Now, we notice that g is injective: if $y_1, y_2 \in Y$ have $g(y_1) = g(y_2)$, then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$. So the previous case tells us that g is in fact bijective.
So now choose any $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. The surjectivity of f promises some $y_1, y_2 \in Y$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$, so we see that
$$x_1 = g(y_1) = g(f(g(y_1))) = g(f(x_1)) = g(f(x_2)) = g(f(g(y_2))) = g(y_2) = x_2,$$
proving our injectivity. ■

Lemma 1.28. Fix $f : X \to Y$ a bijective function. Then there is a unique function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Proof. We show existence and uniqueness separately.

- We show existence. Note that, because $f : X \to Y$ is surjective, each $y \in Y$ has some $x \in X$ such that $f(x) = y$. In fact, this $x \in X$ is uniquely defined because $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so we may define $g(y)$ as the value x for which $f(x) = y$.
By construction, $f(g(y)) = y$, so $f \circ g = \mathrm{id}_Y$. Additionally, we note that, given any $x \in X$, the value x_0 for which $f(x) = f(x_0)$ is $x = x_0$ by the injectivity, so $g(f(x)) = x$. Thus, $g \circ f = \mathrm{id}_X$, as claimed.

- We show uniqueness. Suppose that we have two functions $g_1, g_2 : Y \to X$ which satisfy
$$f \circ g_1 = f \circ g_2 = \mathrm{id}_Y \qquad \text{and} \qquad g_1 \circ f = g_2 \circ f = \mathrm{id}_X .$$
Then we see that
$$g_1 = g_1 \circ \mathrm{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathrm{id}_X \circ g_2 = g_2,$$
where we have used the fact that function composition associates. This finishes. ■

[2] Technically we are using the Axiom of Choice here. One can remove this with an induction because all sets are finite, but I won't bother.

THEME 2

COMPLEX NUMBERS AND THEIR TOPOLOGY

This somewhat laborious proof could have been avoided if one had defined a complex analytic structure

—Jean-Pierre Serre

2.1 January 24

Good morning everyone.

2.1.1 Algebraic Structure

Today we are reviewing the complex numbers (reportedly, "some basics"). Or at least it is hopefully mostly review. Here is our main character this semester.

Definition 2.1 (Complex numbers)**.** The set $\mathbb{C}$ of *complex numbers* is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Here i is some symbol such that $i^2 = -1$ formally.

In particular, two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

The complex numbers also have some algebraic structure.

Definition 2.2 ($+$ and $\times$ in $\mathbb{C}$)**.** Given complex numbers $a_1 + b_1 i, a_2 + b_2 i \in \mathbb{C}$, we define

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i,$$

and

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i,$$

defined essentially by direct expansion, upon recalling $i^2 = -1$.

Here is the corresponding algebraic structure.

Proposition 2.3. The set $\mathbb{C}$ with the above operations is a two-dimensional $\mathbb{R}$-vector space with basis $\{1, i\}$.

Proof. The elements $\{1, i\}$ span $\mathbb{C}$ because all complex numbers in $\mathbb{C}$ can be written as $a + bi = a \cdot 1 + b \cdot i$ by definition.

To see that these elements are linearly independent, suppose $a + bi = 0$. If $b = 0$, then $a = 0$ follows, and we are done. Otherwise, take $b \neq 0$, but then we see $(-a/b) = i$, so

$$(-a/b)^2 = -1 < 0,$$

which does not make sense for real numbers. This finishes. ■

Proposition 2.4. The set $\mathbb{C}$ with the above operations is a field.

Proof. We have the following checks.

- The element $0 + 0i$ is our additive identity. Indeed, one can check that $(0 + 0i) + (a + bi) = (a + bi) + (0 + 0i) = a + bi$.
- The element $1 + 0i$ is our multiplicative identity. Indeed, one can check that $(1 + 0i)(a + bi) = (a + bi)(1 + 0i) = a + bi$.
- Commutativity of addition and multiplication follow from by expansion.
- The distributive laws can again be checked by expansion.
- The additive inverse of $a + bi$ is $(-a) + (-b)i$.
- The multiplicative inverse of $a + bi$ can be found by wishing really hard and writing
$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i.$$
Then one can check this works. ■

Sometimes we would like to extract our coefficients from our basis.

Definition 2.5 (Re and Im)**.** Given $z := a + bi \in \mathbb{C}$, we define the operations

$$\operatorname{Re} z := a \qquad \text{and} \qquad \operatorname{Im} z := b.$$

Importantly, $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$.

Because we are merely doing basis extraction, it makes sense that these operations will preserve some (additive) structure.

Proposition 2.6. Fix $z = a + bi$ and $w = c + di$. Then the following.

(a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$.

(b) $\operatorname{Im}(z + w) = \operatorname{Im} z + \operatorname{Im} w$.

Proof. We proceed by direct expansion. Observe

$$\operatorname{Re}(z + w) = \operatorname{Re}((a + c) + (b + d)i) = a + c = \operatorname{Re} z + \operatorname{Re} w,$$

and

$$\operatorname{Im}(z + w) = \operatorname{Im}((a + c) + (b + d)i) = b + d = \operatorname{Im} z + \operatorname{Im} w.$$

This finishes. ■

It also turns out that the complex numbers have a very special transformation.

Definition 2.7 (Conjugate)**.** Given $z := a+bi \in \mathbb{C}$, we define the *complex conjugate* to be $\overline{z} := a-bi \in \mathbb{C}$.

We promised conjugation would be special, so here are some special things.

Proposition 2.8. Fix $z = a + bi \in \mathbb{C}$. Then the following.

(a) $z + \overline{z} = 2\operatorname{Re} z$.

(b) $z - \overline{z} = 2i\operatorname{Im} z$.

(c) $\overline{\overline{z}} = z$.

Proof. We take these one at a time.

(a) Write $a + bi + \overline{a + bi} = a + bi + a - bi = 2a$.

(b) Write $a + bi - \overline{a + bi} = a + bi - (a - bi) = 2bi$.

(c) Write $\overline{\overline{a + bi}} = \overline{a - bi} = a + bi$. ■

In fact, more is true.

Proposition 2.9. Fix $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$. Then the following.

(a) $\overline{z + w} = \overline{z} + \overline{w}$.

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$.

Proof. We take these one at a time.

- Write
$$\overline{z + w} = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z} + \overline{w}.$$
- Write
$$\begin{aligned}\overline{z} \cdot \overline{w} &= (a - bi)(c - di) \\ &= (ac - bd) - (ad + bc)i \\ &= \overline{(ac - bd) + (ad + bc)i} \\ &= \overline{zw}.\end{aligned}$$

This finishes. ■

2.1.2 Defining Distance

Complex conjugation actually gives rise to a notion of size.

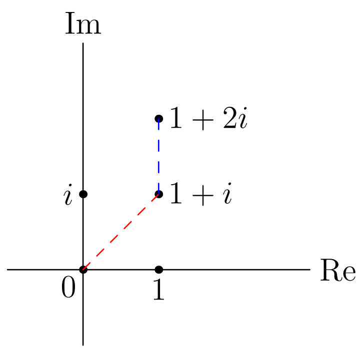
Definition 2.10 (Norm on $\mathbb{C}$)**.** Given $z := a + bi$, we define the *norm function on* $\mathbb{C}$ by
$$|z| := \sqrt{a^2 + b^2}.$$

Size actually gives distance.

Definition 2.11 (Distance on $\mathbb{C}$)**.** Given complex numbers $z = a + bi$ and $w = c + di$, we define the *distance* between z and w to be

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2}.$$

Here are some examples.



One can ask what is the distance between $0 + 0i$ and $1 + 1i$, and we can compute directly that this is $\sqrt{1+1} = \sqrt{2}$. Similarly, the distance between $1 + 2i$ and $1 + i$ is $|(1 + 2i) - (1 + i)| = |i| = 1$. It should agree with our geometric intuition.

We mentioned complex conjugation is involved here, so we have the following lemma.

Lemma 2.12. Fix $z, w \in \mathbb{C}$. The following are true.

(a) $|z|^2 = z\overline{z}$.

(b) $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.

(c) $|z| = |\overline{z}| = |-z|$.

(d) $|z| = 0$ if and only if $z = 0$.

(e) $|zw| = |z| \cdot |w|$.

Proof. We take these one at a time. Set $z = a + bi$.

(a) We have

$$|z|^2 = a^2 + b^2 = (a + bi)(a - bi) = z\overline{z}.$$

Here we have used subtraction of two squares, which one can see when writing $a^2 + b^2 = a^2 - (ib)^2$.

(b) We have $a^2 \le a^2 + b^2$ and $b^2 \le a^2 + b^2$ by the Trivial inequality, so

$$|\operatorname{Re} z| = |a| \le \sqrt{a^2 + b^2} = |z|,$$

and similarly,

$$|\operatorname{Im} z| = |b| \le \sqrt{a^2 + b^2} = |z|.$$

(c) Note

$$|\overline{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|,$$

and

$$|-z| = |-a - bi| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(d) From (b), we know that $|\operatorname{Re} z|, |\operatorname{Im} z| \le |z|$, but $|z| = 0$ then forces $\operatorname{Re} z = \operatorname{Im} z = 0$, so $z = 0$.

(e) From (a), we can write $|zw|^2 = zw \cdot \overline{zw}$, which will expand out into

$$z \cdot w \cdot \overline{z} \cdot \overline{w}.$$

We can collect this into $z\overline{z} \cdot w\overline{w} = |z|^2|w|^2$. Thus, by (a) again, $|zw|^2 = |z|^2|w|^2$. But because all norms must be nonnegative real numbers, we may take square roots to conclude $|zw| = |z| \cdot |w|$. ■

Remark 2.13. Norms are actually more general constructions. For example, the requirement $|zw| = |z| \cdot |w|$ makes $| \cdot |$ into a "multiplicative" norm.

To finish off, we actually show that our distance function is good: we show the triangle inequality.

Lemma 2.14 (Triangle inequality)**.** For every $x, y, z \in \mathbb{C}$, we claim

$$|z - x| \le |z - y| + |y - z|.$$

This claim should be familiar from real analysis. Intuitively, it means that travelling between z and x cannot be made into a shorter trip by taking a detour to some other point y first.

Proof. Let $a := z - y$ and $b := y - z$ so that $a + b = z - x$. Thus, we are showing that

$$|a + b| \stackrel{?}{\le} |a| + |b|,$$

which is nicer because it only has two letters. For this, because everything is a nonnegative real numbers, it suffices to show the square of this requirement; i.e., we show

$$(|a| + |b|)^2 - |a + b|^2 \stackrel{?}{\ge} 0.$$

Fully expanding, it suffices to show

$$|a|^2 + |b|^2 + 2|a| \cdot |b| - |a + b|^2 \stackrel{?}{\ge} 0.$$

Expanding out $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$, we are showing

$$a\overline{a} + b\overline{b} + 2|a| \cdot |b| - (a + b)(\overline{a} + \overline{b}) \stackrel{?}{\ge} 0.$$

This is nice because the expansion of the rightmost term will induce some cancellation: it expands into $a\overline{a} + a\overline{b} + \overline{a}b + b\overline{b}$, so we are left with showing

$$2|a| \cdot |b| - (a\overline{b} + b\overline{a}) \stackrel{?}{\ge} 0.$$

Note that $\overline{a}b = \overline{a\overline{b}}$, so we can collect the final term as $2\operatorname{Re}(a\overline{b})$. Similarly, we can write $|a| \cdot |b| = |a| \cdot |\overline{b}| = |a\overline{b}|$, so we are showing

$$2|a\overline{b}| - 2\operatorname{Re}(a\overline{b}) \ge 0,$$

which is true because the real part does exceed the norm. This finishes. ■

2.2 January 26

In-person class should start on Monday. Homework #2 will be released on Friday.

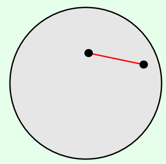
2.2.1 Geometry on $\mathbb{C}$

So let's try to build a topology on $\mathbb{C}$ today. We pick up the following definition.

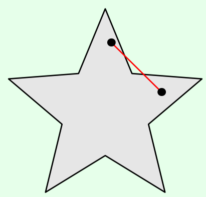
Definition 2.15 (Convex)**.** A subset $X \subseteq \mathbb{C}$ is *convex* if and only if, for every $z, w \in X$ and $t \in [0,1]$, we have that $w + t(z-w) \in X$.

Intuitively, "convex" means that X contains the line segment of any two points in X.

Example 2.16. The circle is convex: any line with endpoints in the circle lives in the circle.



Non-Example 2.17. The star-shape is not convex: the given line goes outside the star.



To define our open sets, we will define balls first.

Definition 2.18 (Open ball)**.** Given some $z_0 \in \mathbb{C}$, then *open ball* centered at z_0 with radius $r > 0$ is

$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Observe $z_0 \in B(z_0, r)$.

Open balls let us define all sorts of properties.

Definition 2.19 (Isolated)**.** Fix $X \subseteq \mathbb{C}$. A point $z \in X$ is isolated in X if and only if there exists $r > 0$ such that

$$B(z, r) \cap X = \{z\}.$$

Definition 2.20 (Discrete)**.** A subset $X \subseteq \mathbb{C}$ is *discrete* if and only if every point is isolated.

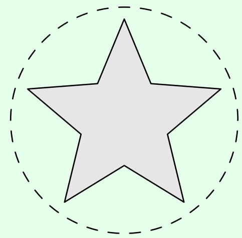
Example 2.21. Any finite subset of $X \subseteq \mathbb{C}$ is discrete. Namely, any point $z \in X$ can take

$$r = \frac{1}{2} \min_{w \in X \setminus \{z\}} |z - x|.$$

Example 2.22. The subset $\mathbb{Z} \subseteq \mathbb{C}$ is isolated. Namely, take $r = \frac{1}{2}$ for any given point.

Definition 2.23 (Bounded)**.** A subset $X \subseteq \mathbb{C}$ is *bounded* if and only if there is an M such that $X \subseteq B(0, M)$.

Example 2.24. The star from earlier fits into a large circle and is therefore bounded.



And here is our fundamental definition for our topology.

Definition 2.25 (Open)**.** A subset $X \subseteq \mathbb{C}$ is *open* if and only if, for each $z \in X$, there exists $r > 0$ such that $B(z, r) \subseteq X$.

Remark 2.26 (Nir)**.** We should probably show that open balls are open; let $B(z, r)$ be an open ball. Well, for any $w \in B(z, r)$, set $r_w := r - |z - w|$, which is positive because $w \in B(z, r)$ requires $|z - w| < r$. Now, $w' \in B(w, r_w)$ implies that $|w - w'| < r - |z - w|$, so by the triangle inequality,

$$|z - w'| \leq |z - w| + |w - w'| < r,$$

so $w' \in B(z, r)$ follows. So indeed, each $w \in B(z, r)$ has $B(w, r_w) \subseteq B(z, r)$.

Open lets us define closed.

Definition 2.27 (Closed)**.** A subset $X \subseteq \mathbb{C}$ is *closed* if and only if $\mathbb{C} \setminus X$ is open.

Warning 2.28. Sets are not doors: a set can be both open and closed.

2.2.2 Unions and Intersections

Here are some basic properties of our topology.

Lemma 2.29. The subsets $\varnothing$ and $\mathbb{C}$ are open and closed in $\mathbb{C}$.

Proof. It suffices to show that $\varnothing$ and $\mathbb{C}$ are both open, by definition of closed. That $\varnothing$ is open holds vacuously because one cannot find any $z \in \varnothing$ anyways. That $\mathbb{C}$ is open holds because open balls are subsets of $\mathbb{C}$, so any $z \in \mathbb{C}$ can take $r = 1$ so that

$$B(z, r) \subseteq \mathbb{C}.$$

So we are done. ■

Lemma 2.30. Fixing some $z \in \mathbb{C}$, the set $\{z\}$ is closed.

Proof. We show that $U := \mathbb{C} \setminus \{z\}$ is open. Well, fix any $w \in U$, and because $w \neq z$, we note $|z - w| > 0$, so we set $r := \frac{1}{2}|z - w|$. It follows that

$$z \notin B(w, r)$$

because $|z - w| > r$. But this is equivalent to $B(w, r) \subseteq \mathbb{C} \setminus \{x\} = U$, so we are done. ■

We would like to make new open and closed subsets from old ones. Here is one way to do so.

Lemma 2.31. The following are true.

(a) Arbitrary union: if $\mathcal{U}$ is any collection of open subsets of $\mathbb{C}$, then the union $\bigcup_{U\in\mathcal{U}} U$ is also open.

(b) Arbitrary intersection: if $\mathcal{V}$ is any collection of closed subsets of $\mathbb{C}$, then intersection $\bigcap_{V\in\mathcal{V}} V$ is also closed.

Proof. We take these one at a time.

(a) Fix $z \in \bigcup_{U\in\mathcal{U}} U$. We need to show there is some $r > 0$ such that

$$B(z,r) \stackrel{?}{\subseteq} \bigcup_{U\in\mathcal{U}} U.$$

Well, we know there must be some $U_z \in \mathcal{U}$ such that $z \in U_z$ by definition of the union. But now U_z is open, and therefore we are promised an $r > 0$ such that

$$B(z,r) \subseteq U_z \subseteq \bigcup_{U\in\mathcal{U}} U,$$

so we are done.

(b) Fix $\mathcal{V}$ a collection of closed subsets of $\mathbb{C}$. We want to show that

$$\mathbb{C} \Bigg\backslash \bigcap_{V\in\mathcal{V}} V$$

is open, which by de Morgan's law is equivalent to

$$\bigcup_{V\in\mathcal{V}} (\mathbb{C}\setminus V)$$

being open. However, each $V \in \mathcal{V}$ is closed, so $\mathbb{C}\setminus V$ will be open, so we are done by (a). ■

Lemma 2.32. The following are true.

(a) Finite intersection: if $\{U_k\}_{k=1}^n$ is a finite collection of open subsets of $\mathbb{C}$, then the intersection $\bigcap_{k=1}^n U_k$ is also open.

(b) Finite union: if $\{V_k\}_{k=1}^n$ is a finite collection of closed subsets of $\mathbb{C}$, then $\bigcup_{k=1}^n V_k$ is also closed.

Proof. We take these one at a time.

(a) Fix $z \in \bigcap_{k=1}^n U_k$ so that we need to find $r > 0$ such that

$$B(z,r) \overset{\subseteq}{\bigcup}{}^{n}_{k=1} U_k.$$

Well, $z \in U_k$ for each k, and each U_k is open, so there is an $r_k > 0$ such that $B(z,r_k) \subseteq U_k$. Thus, we set $r := \min_k\{r_k\}$; because there are only finitely many r_k, we are assured that $r > 0$. Now, we observe that

$$B(z,r) \subseteq B(z,r_k) \subseteq U_k.$$

(Explicitly, $|w-z| < r$ implies $|w-z| < r_k$ because $r \le r_k$.) Thus, it follows that

$$B(z,r) \subseteq \bigcap_{k=1}^n U_k,$$

as desired.

(b) We use de Morgan's laws. We want to show that

$$\mathbb{C} \Big\backslash \bigcup_{k=1}^{n} V_k$$

is open, which by de Morgan's laws is the same thing as showing that

$$\bigcap_{k=1}^{n} (\mathbb{C} \setminus V_k)$$

is open. However, each $\mathbb{C} \setminus V_k$ is open by hypothesis on the V_k, so the full intersection is open by (a). This finishes. ■

Remark 2.33. The finiteness is in fact necessary. For example,

$$\bigcap_{n\in\mathbb{N}} B(0, 1/n) = \{0\}.$$

Then one can check that each open ball is open while singletons in $\mathbb{C}$ are not.

2.2.3 Interior, Closure

Let's see more definitions.

Definition 2.34 (Interior)**.** Given a subset $X \subseteq \mathbb{C}$, we define the *interior* X° of X to be the union of all open sets contained in X (which will be open by Lemma 2.31).

Remark 2.35. In fact, X° is the largest open subset of X, for any open subset $U_0 \subseteq \mathbb{C}$ contained in X will have

$$U_0 \subseteq \bigcup_{\text{open } U \subseteq X} U = X^\circ.$$

It follows X is open if and only if $X = X^\circ$: if $X = X^\circ$, then X is open because X° is open; if X is open, then X is the largest open subset of $\mathbb{C}$ contained in X, so $X = X^\circ$.

Definition 2.36 (Closure)**.** Given a subset $X \subseteq \mathbb{C}$, we define the *closure* $\overline{X}$ of X to be the intersection of all closed sets containing X (which will be closed by Lemma 2.31).

Remark 2.37. In fact, X° is the smallest closed set containing X, for any closed subset $V_0 \subseteq \mathbb{C}$ containing X will have

$$V_0 \supseteq \bigcap_{\text{open } V \supseteq X} V = \overline{X}.$$

It follows X is closed if and only if $X = \overline{X}$: if $X = \overline{X}$, then X is open because $\overline{X}$ is closed; if X is closed, then X is the smallest closed subset of $\mathbb{C}$ containing X, so $X = \overline{X}$.

By the above definitions, it is not too hard to see that $X^\circ \subseteq X \subseteq \overline{X}$.

The interior and closure also let us define the boundary.

Definition 2.38 (Frontier, boundary)**.** Given a subset $X \subseteq \mathbb{C}$, we define the *frontier* or *boundary* ∂X of X to be $\overline{X} \setminus X^\circ$.

2.2.4 Connectivity

Definition 2.39 (Disconnected)**.** A subset $X \subseteq \mathbb{C}$ is *disconnected* if and only if there exists nonempty disjoint open subsets U_1 and U_2 such that $X \subseteq U_1 \cup U_2$ and $X \cap U_1, X_\cap U_2 \neq \varnothing$. (In other words, the subspace of $X \subseteq \mathbb{C}$ is (topologically) disconnected.) In this case, we say that U_1 and U_2 *disconnect* X.

Lastly, we say X is *connected* if and only if it is not disconnected.

Example 2.40. The set $\varnothing$ is connected because it is impossible for $U \cap \varnothing \neq \varnothing$ for any open set U of $\mathbb{C}$.

Example 2.41. Any singleton $\{z\}$ is connected. In fact, one cannot decompose $\{x\}$ into two disjoint sets at all, much less into disjoint sets of the form $U \cap \{x\}$ with U open.

Example 2.42. Any open ball $B(z,r)$ is connected. This is surprisingly annoying to check.

Example 2.43. The set $\{1,2\}$ is disconnected by $U_1 = B(1,1/2)$ and $U_2 = B(2,1/2)$.

Connectivity plays nicely with the rest of our definitions as well.

Lemma 2.44. A given subset $X \subseteq \mathbb{C}$ is connected if and only if the only subsets of X which are both open and closed (in the subspace topology) are $\varnothing$ and X.

Proof. We take the directions independently. For the forwards direction, take X connected, and suppose that $U \subseteq X$ is open and closed. In the subspace topology, we get that $X \setminus U$ will also be open, and then the subsets U and $X \setminus U$ are both open, disjoint and have

$$X = U \cup (X \setminus U).$$

Thus, we require $U = \varnothing$ or $X \setminus U = \varnothing$, so $U \in \{\varnothing, X\}$.

We leave the reverse direction as an exercise. Suppose that X is disconnected, and we show that there is a nonempty proper closed and open subset of X. Well, because X is disconnected, we have disjoint open sets U_1 and U_2 of $\mathbb{C}$ such that $X \cap U_1, X \cap U_2 \neq \varnothing$ and $X \subseteq U_1 \cup U_2$. It follows that

$$X = (U_1 \cap X) \cup (U_2 \cap X). \tag{$*$}$$

However, now consider the open subset $U := U_1 \cap X$ of X. We note that $(U_1 \cap X) \cap (U_2 \cap X) = \varnothing$, so by $(*)$ we see that $U_1 \cap X = X \setminus (U_2 \cap X)$, so $U_1 \cap X$ is closed as well.

To finish, we note that $U \neq \varnothing$ is nonempty, and its complement is $X \setminus U = U_2 \cap X$ is also nonempty, so $U \neq X$ is proper. Thus, U is a proper nonempty closed and open subset of X. This finishes. ■

Remark 2.45 (Nir)**.** It is actually important that the open subsets in the above lemma are in the subspace topology and are not required to be $\mathbb{C}$-open. For example, $X = \{1,2\}$ is disconnected, but it has no nonempty $\mathbb{C}$-open subsets to witness this.

Lemma 2.46. Fix $\mathcal{S}$ a collection of connected subsets of $\mathbb{C}$. If $\bigcap_{S \in \mathcal{S}} S$ is nonempty, then $\bigcup_{S \in \mathcal{S}} S$ will be connected.

Proof. Suppose $\bigcup_{S\in\mathcal{S}} S$ is contained in the disjoint open subsets U_1 and U_2 of $\mathbb{C}$; we claim $U_1 \cap \left(\bigcup_{S\in\mathcal{S}} S\right) = \varnothing$ or $U_2 \cap \left(\bigcup_{S\in\mathcal{S}} S\right) = \varnothing$, which will finish.

Pick up some

$$z \in \bigcap_{S\in\mathcal{S}} S,$$

which exists because the intersection is nonempty. Without loss of generality, we may assume that $z \in U_1$.

Now, $z \in S$ for each $S \in \mathcal{S}$, so we see $U_1 \cap S \neq \varnothing$, so because $(U_1 \cap S) \cup (U_2 \cap S) = S$, we see that $U_2 \cap S = \varnothing$ by hypothesis on S's connectivity. Thus, taking the union over the $U_2 \cap S = \varnothing$,

$$U_2 \cap \left(\bigcup_{S\in\mathcal{S}} S\right) = \varnothing,$$

which finishes the proof. ■

Remark 2.47. The condition with nonempty intersection is necessary: $\{0\}$ and $\{1\}$ are connected, but $\{0\} \cup \{1\}$ is not.

2.3 January 28

Hopefully we'll be in-person on Monday. Homework 2 will be released later today, due next Friday.

2.3.1 Sequences

Today we're talking about sequences, building towards a theory of sequences and series. Next week we will begin studying holomorphic functions and actually doing complex analysis.

Anyways, here is a series of definitions.

Definition 2.48 (Sequence)**.** A *sequence* of complex numbers is a function $f : \mathbb{N} \to \mathbb{C}$. Often we will notate this by $\{z_n\}_{n\in\mathbb{N}}$ where $z_n := f(n)$.

By convention, all of our sequences will be sequences of complex numbers unless otherwise stated.

Definition 2.49 (Subsequence)**.** A sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a *subsequence* of a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ if and only if there is some strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{g(n)}$.

Definition 2.50 (Bounded)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is *bounded* if and only if there exists a positive real number $M > 0$ such that

$$\{z_n\}_{n\in\mathbb{N}} \subseteq B(0, M).$$

In other words, $|z_n| < M$ for each $n \in \mathbb{N}$.

We are in particular interested in convergence in analysis.

Definition 2.51 (Converges)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *converges* to some $z \in \mathbb{C}$ if and only if, for each $\varepsilon > 0$, there exists some N such that $n > N$ implies

$$|z - z_n| < \varepsilon.$$

We will notate this by $z_n \to z$ or $\lim_{n\to\infty} z_n = z$.

Note that the definition of the limit above says that

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} |z_n - z| = 0.$$

Intuitively, the distance between the z_n and the z has to "narrow in" on z.

We would like some real-analytic tools for our complex analysis. Here is a convergence lemma.

Lemma 2.52. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a sequence. Then, letting $z_n := x_n + y_n i$, we have that $z_n \to z$ where $z = x + yi$ if and only if $x_n \to x$ and $y_n \to y$.

Proof. This is essentially by definition of the metric on $\mathbb{C}$. We take the directions one at a time.

- Suppose that $z_n \to z$ in $\mathbb{C}$. Then we claim that $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z_n$ in $\mathbb{R}$. Indeed, for any $\varepsilon > 0$, there is N such that
$$n > N \implies |z - z_n| < \varepsilon.$$
But now we see that $|\operatorname{Re} z_n - \operatorname{Re} z|, |\operatorname{Im} z_n - \operatorname{Im} z| \le \sqrt{(\operatorname{Re} z_n - \operatorname{Re} z)^2 + (\operatorname{Im} z_n - \operatorname{Im} z)^2}$, so it follows
$$n > N \implies |\operatorname{Re} z_n - \operatorname{Re} z|, |\operatorname{Im} z_n - \operatorname{Im} z| < \varepsilon,$$
finishing.

- Suppose that $\operatorname{Re} z_n \to x$ and $\operatorname{Im} z_n \to y$. We claim that $z_n \to x + yi$. Indeed, for any $\varepsilon > 0$, there exists N_x such that
$$n > N_x \implies |\operatorname{Re} z_n - x| < \varepsilon/2$$
and N_y such that
$$n > N_y \implies |\operatorname{Im} z_n - y| < \varepsilon/2.$$
It follows that
$$n > \max\{N_x, N_y\} \implies |z_n - (x + yi)| = \sqrt{|\operatorname{Re} z_n - x|^2 + |\operatorname{Im} z_n - y|^2} \le \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2} < \varepsilon.$$
This finishes. ∎

Essentially, this means that checking convergence of complex numbers is the same as checking real and imaginary parts individually, so we can turn convergence questions into ones from real analysis.

We also have the following basic properties about convergence.

Proposition 2.53. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a convergent sequence. The following are true.

(a) $\{z_n\}_{n\in\mathbb{N}}$ is bounded.

(b) The limit of $\{z_n\}_{n\in\mathbb{N}}$ is unique.

(c) Every subsequence of $\{z_n\}_{n\in\mathbb{N}}$ converges to z.

Proof. We take the claims one at a time. Let $z \in \mathbb{C}$ be so that $z_n \to z$.

(a) Fix $\varepsilon = 1$ so that there exists N so that $n > N$ implies $|z_n - z| < 1$. Now set
$$M := \max(\{|z_n| + 1 : n \le N\} \cup \{|z| + 1\}).$$
We claim that $|z_n| < M$ for each $n \in \mathbb{N}$. We have two cases.

- If $n \le N$, then $|z_n| < |z_n| + 1 \le M$.

- Otherwise, $n > N$ so that
$$|z_n| \le |z_n - z| + |z| < |z| + 1 \le M,$$
so we are done.

(b) Suppose that $z_n \to z'$ for some $z' \in \mathbb{C}$, and we show $z = z'$. Indeed, if $z = z'$, then we are done, so suppose that $z \neq z'$ so that $|z - z'| \neq 0$. Then we set $\varepsilon := \frac{1}{2}|z - z'| > 0$, and we are promised some N, N' such that
$$n > N \implies |z - z_n| < \frac{\varepsilon}{2} \quad \text{and} \quad n > N' \implies |z' - z_n| < \frac{\varepsilon}{2}.$$
In particular, we see that, for $n > \max\{N, N'\}$, we have
$$|z - z'| \le |z - z_n| + |z_n - z'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \frac{1}{2}|z - z'|.$$
But because $0 \le |z - z'|$, we see that this forces $|z - z'| = 0$, so $z = z'$ follows. (Technically we have hit contradiction, but we do not need to use this.)

(c) Note that subsequences can be characterized by choosing a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ so that we want to show $z_{f(n)} \to z$. Indeed, for any $\varepsilon > 0$, we are promised some N so that
$$n > N \implies |z - z_n| < \varepsilon.$$
Now, for each $n \in \mathbb{N}$, we have[1] $f(n) \ge n$, so we see that
$$n > N \implies f(n) > N \implies |z - z_{f(n)}| < \varepsilon,$$
which finishes. ■

Sequences themselves have an arithmetic.

Proposition 2.54. Fix $\{z_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ sequences such that $z_n \to z$ and $w_n \to w$. Then the following hold.

(a) $z_n + w_n \to z + w$.

(b) $z_n w_n \to zw$.

(c) If $w \neq 0$ and $w_n \neq 0$ for each $n \in \mathbb{N}$, then $\frac{1}{w_n} \to \frac{1}{w}$.

Proof. We take these one at a time, essentially borrowing the proof from metric spaces.

(a) Fix some $\varepsilon > 0$. We can find some N_z such that
$$n > N_z \implies |z - z_n| < \varepsilon/2$$
and some N_w such that
$$n > N_w \implies |w - w_n| < \varepsilon/2.$$
Now, taking $N := \max\{N_z, N_w\}$ so that the triangle inequality gives
$$n > N \implies |(z + w) - (z_n + w_n)| \le |z - z_n| + |w - w_n| < \varepsilon,$$
which finishes.

[1] We can show this by induction on n, for $f(0) \ge 0$ and $f(n+1) > f(n)$ forces $f(n+1) \ge f(n) + 1$.

(b) We have to use the fact that the sequences are bounded here. Because $w_n \to w$, the sequence is bounded, so there is an M so that $|w_n| < M$ for each $n \in \mathbb{N}$. Now, the key inequality is that

$$|z_n w_n - zw| \le |z_n w_n - z w_n| + |z w_n - zw| \le M|z_n - z| + |z| \cdot |w_n - w|. \qquad (*)$$

So with this in mind, fix any $\varepsilon > 0$, and we see that we are promised N_z such that

$$n > N_z \implies |z_n - z| < \frac{\varepsilon}{2M}$$

and some N_w such that

$$n > N_w \implies |w_n - w| < \frac{\varepsilon}{2|z|}$$

so that $(*)$ implies

$$n > \max\{N_x, N_w\} \implies |z_n w_n - zw| < \varepsilon,$$

finishing.

(c) We begin with some motivating arithmetic. Observe that

$$\left|\frac{1}{w} - \frac{1}{w_n}\right| = \frac{|w_n - w|}{|w w_n|}.$$

We can upper-bound the numerator without tears, so we see the main difficulty is lower-bounding the denominator. Well, because $w \neq 0$, we can set $\varepsilon = |w|/2$ so that there exists N_0 such that

$$n > N_0 \implies |w_n - w| < |w|/2.$$

In particular, it follows that $|w_n| \ge |w| - |w - w_n| = |w|/2$ for $n > N_0$.

With this in mind, fix any $\varepsilon > 0$. Then we are promised some N_1 such that

$$n > N_1 \implies |w_n - w| < |w|^2 \varepsilon/2$$

so that we see

$$n > \max\{N_0, N_1\} \implies \left|\frac{1}{w} - \frac{1}{w_n}\right| = \frac{|w_n - w|}{|w| \cdot |w_n|} \le \frac{|w|^2\varepsilon/2}{|w| \cdot |w|/2} = \varepsilon,$$

finishing. ■

2.3.2 Limit Points

Here is our main character.

Definition 2.55 (Limit point). Fix $X \subseteq \mathbb{C}$ and some $z \in \mathbb{C}$. Then we say that z is a *limit point* if and only if there exists some sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq X$ such that $z_n \to z$.

Definition 2.56 (Accumulation point). Fix $X \subseteq \mathbb{C}$ and some $z \in \mathbb{C}$. Then we say that z is an *accumulation point* if and only if there exists some sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq X \setminus \{z\}$ such that $z_n \to z$.

Essentially accumulation points do not allow isolated points while limit points do.

The above essentially gives a more directly topological definition of "closed set." It also gives us a more directly topological definition of the closure.

Lemma 2.57. Fix $X \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. The following are equivalent.

(a) We have that $z \in \overline{X}$.

(b) For all $\varepsilon > 0$, we have $B(z, \varepsilon) \cap X \neq \varnothing$.

(c) There exists $\{z_n\}_{n \in \mathbb{N}} \subseteq X$ such that $z_n \to z$.

Proof. We show our directions one at a time.

- We show (a) implies (b). Suppose $z \in \overline{X}$, and for the sake of contradiction suppose we have $\varepsilon > 0$ such that $B(z, \varepsilon) \cap X = \varnothing$. In particular, $z \notin X$.

  Now, $z \in \overline{X}$ implies that z is contained in every closed set containing X by definition of $\overline{X}$. But because $B(z, \varepsilon)$ is open and is disjoint from X, we see

  $$z \in \overline{X} \subseteq \mathbb{C} \setminus B(z, \varepsilon),$$

  which is a contradiction.

- We show (b) implies (c). For each $n \in \mathbb{N}$, we know that $B(z, 1/n) \cap X \neq \varnothing$, so we find some $z_n \in B(z, 1/n)$. Now, for any $\varepsilon > 0$, choose $N := 1/\varepsilon$ so that

  $$n > N \implies |z_n - z| < \frac{1}{n} < \frac{1}{N} = \varepsilon,$$

  so indeed $z_n \to z$.

- We show (b) implies (a). We proceed by contraposition. Suppose that $z \notin \overline{X}$. It follows that $z \in \mathbb{C} \setminus \overline{X}$, which is open, so there exists an $r > 0$ such that

  $$B(z, r) \subseteq \mathbb{C} \setminus \overline{X} \subseteq \mathbb{C} \setminus X.$$

  It follows that $B(z, r) \cap X = \varnothing$.

- We show (c) implies (b). Suppose $\{z_n\}_{n \in \mathbb{N}} \subseteq X$ has $z_n \to z$ for some $z \in \mathbb{C}$. For any $\varepsilon > 0$, there exists N such that

  $$n > N \implies |z_n - z| < \varepsilon,$$

  so in particular, choosing any $n := \lceil N \rceil + 1$ has $z_n \in B(z, \varepsilon) \cap X$, so $B(z, \varepsilon) \cap X \neq \varnothing$. ■

The above discussion can give us a more directly topological definition of "closed."

Lemma 2.58. A subset $X \subseteq \mathbb{C}$ is closed in $\mathbb{C}$ if and only if X contains all of its limit points.

Proof. By the previous lemma, we see that $z \in \overline{X}$ if and only if z is a limit point of X, so $\overline{X}$ is the set of limit points of X. Now, X is closed if and only if $X = \overline{X}$, so X is closed if and only if all limit points of X are in fact points of X. (Note that all points of X are automatically limit points essentially because $X \subseteq \overline{X}$ for free.) ■

While we're here, we can pick up a nice topological result.

Lemma 2.59. Fix $X \subseteq \mathbb{C}$ a connected subset. Then $\overline{X}$ is also connected.

Proof. This argument is purely topological. We proceed by contraposition: suppose $\overline{X}$ is disconnected by $U_1, U_2 \subseteq \mathbb{C}$. We claim that U_1, U_2 disconnect X. Well, we already know that $A \subseteq \overline{A} \subseteq U_1 \cup U_2$, and we already know that U_1 and U_2 are disjoint.

We claim that, for $U \subseteq \mathbb{C}$ an open subset, if $U \cap \overline{X} \neq \varnothing$, then $U \cap X \neq \varnothing$ as well. Indeed, we proceed by contraposition: if $U \cap X = \varnothing$, then $X \subseteq \mathbb{C} \setminus U$, but $\mathbb{C} \setminus U$ is closed, so

$$\overline{X} \subseteq \mathbb{C} \setminus U,$$

so $\overline{X} \cap U = \varnothing$.

Thus, it follows from $U_1 \cap \overline{X}, U_2 \cap \overline{X} \neq \varnothing$ that $U_1 \cap X, U_2 \cap X \neq \varnothing$. This finishes the proof that U_1 and U_2 disconnect X. Indeed, ■

2.3.3 Cauchy Sequences

Just like in real analysis, we will be interested in Cauchy sequences.

Definition 2.60 (Cauchy sequence)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a *Cauchy sequence* if and only if, for each $\varepsilon > 0$, there exists an N such that

$$n, m > N \implies |z_n - z_m| < \varepsilon.$$

We have the following results on Cauchy sequences.

Proposition 2.61. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a sequence. If $\{z_n\}_{n\in\mathbb{N}}$ converges, it is Cauchy.

Proof. This proof uses no special properties of $\mathbb{C}$. If $z_n \to z$, then for a given $\varepsilon > 0$, there exists N such that

$$n > N \implies |z_n - z| < \varepsilon/2.$$

It follows that

$$n, m > N \implies |z_n - z_m| < |z_n - z| + |z_m - z| < \varepsilon,$$

finishing. ■

Proposition 2.62. Every Cauchy sequence in $\mathbb{C}$ converges.

Proof. If $\{z_n\}_{n\in\mathbb{N}}$ is Cauchy, then we claim $\{\operatorname{Re} z_n\}_{n\in\mathbb{N}}$ and $\{\operatorname{Im} z_n\}_{n\in\mathbb{N}}$ are Cauchy sequences. Indeed, for any $\varepsilon > 0$, there exists N so that

$$n, m > N \implies |z_n - z_m| < \varepsilon,$$

but then $|\operatorname{Re} z_n - \operatorname{Re} z_m| < |z_n - z_m|$ and $|\operatorname{Im} z_n - \operatorname{Im} z_m| < |z_n - z_m|$, so the same N witnesses that $\{\operatorname{Re} z_n\}_{n\in\mathbb{N}}$ and $\{\operatorname{Im} z_n\}_{n\in\mathbb{N}}$ are Cauchy in $\mathbb{R}$.

Now, Cauchy sequences in $\mathbb{R}$ converge, so there are reals $x, y \in \mathbb{R}$ such that $\operatorname{Re} z_n \to x$ and $\operatorname{Im} z_n \to w$. It follows that $z_n \to x + yi$, finishing. ■

2.3.4 A Little More Topology

We close with one more topological definition.

Definition 2.63 (Sequentially compact)**.** A subset $X \subseteq \mathbb{C}$ is *sequentially compact* if and only if every $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ has a convergent subsequence which converges in X.

Remark 2.64. This happens to be equivalent to X is compact because $\mathbb{C} \cong \mathbb{R}^2$ satisfies the fact that all compact sets are closed and bounded.

Example 2.65. Every finite set is compact.

And here is a last definition.

Definition 2.66 (Tends to infinity)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *tends to infinity* (notated $z_n \to \infty$) if and only if each $M > 0$ has some $N \in \mathbb{N}$ such that

$$n > N \implies |z_n| > M.$$

Essentially the points of $\{z_n\}_{n\in\mathbb{N}}$ wander infinitely away.

2.4 January 31

So we are lecturing in-person today. Good morning everyone.

Quote 2.67. If I don't fall off the stage, I will consider it a major accomplishment.

Homework 2 is due Friday, the 4th of February. Office hours will occur at the usual times, but they will now occur in-person at Evans 749.

2.4.1 Series

Today we're mostly talking about series, and on Friday we'll talk about continuous functions.

Definition 2.68 (Series)**.** An *infinite series* over $\mathbb{C}$ is an infinite sum

$$S := \sum_{n=1}^{\infty} z_n$$

where $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a sequence of complex numbers.

With respect to series, we really want to know when various series converge so that the series is well-defined.

Definition 2.69 (Converge, diverge)**.** Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers, we define the mth partial sum to be

$$S_m := \sum_{n=0}^{m} z_m.$$

Then we say that the infinite series *converges* if and only if the sequence $\{S_m\}$ of partial sums converges. Otherwise, we say that S is *divergent*.

As usual, we start with some basic examples.

Exercise 2.70. Fix some $z \in \mathbb{C}$ with $|z| < 1$, we define $z_n := z^n$. Then we have

$$S = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Proof. Fix some partial sum

$$S_N := \sum_{k=0}^{N} z^k = 1 + z + z^2 + \cdots + z^N.$$

Multiplying by z, we see that

$$zS_n = z + z^2 + \cdots + z^N + z^{N+1}.$$

It follows that

$$S_N - zS_N = 1 - z^{N+1}.$$

Because $|z| < 1$, we have $z \neq 1$, so we may write

$$S_N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}.$$

However, we may note that as $N \to \infty$, the bad term z^{N+1} will have size

$$\left|z^{N+1}\right| = |z|^{N+1},$$

which goes to 0 (because $|z| < 1$).[2] It follows that

$$\lim_{N\to\infty} S_N = \frac{1}{1-z},$$

which is what we wanted. ■

Anyways, here are some basic lemmas.

Lemma 2.71 (Divergence test)**.** Suppose that $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers such that $\sum z_n$ converges. Then $z_n \to 0$ as $n \to \infty$.

Proof. Let S_n be the nth partial sum so that we are given $S_n \to L$ for some $L \in \mathbb{C}$. But now we see that

$$z_{n+1} = \left(\sum_{k=0}^{N+1} z_k\right) - \left(\sum_{k=0}^{N} z_k\right) = S_{n+1} - S_n.$$

Using limit laws, we see that

$$\lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = L - L = 0.$$

Shifting the indices back gives $z_n \to 0$ as $N \to \infty$. ■

Here is an important example of a divergent series.

Exercise 2.72. We claim that

$$S = \sum_{k=1}^{\infty} \frac{1}{k}$$

does not converge.

[2] This is surprisingly annoying to rigorize with an ε-δ proof, so we won't do so here. The interested can try to use induction to manually bound $|z|^n$ by $\frac{c}{n}$ for some c.

Proof. We will show that the sequence of partial sums $\{S_n\}_{n=1}^\infty$ is not Cauchy, which will show that the series diverges. Well, observe that

$$S_{2^{n+1}} - S_{2^n} = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}$$

after cancelling out all of our terms. However, each term in the sum is at least $\frac{1}{2^{n+1}}$, so we bound

$$S_{2^{n+1}} - S_{2^n} \geq \frac{1}{2^{n+1}} \left(2^{n+1} - 2^n\right) = \frac{1}{2}.$$

We now show that the partial sums are not Cauchy. Fix ε. Supposing for the sake of contradiction that the sequence is Cauchy, there exists N so that $n, m > N$ has

$$|S_n - S_m| < \frac{1}{2}.$$

However, we can find some power of 2 named 2^r which exceeds N, in which case we find $2^{r+1}, 2^r > N$ and

$$|S_{2^{r+1}} - S_{2^r}| \geq \frac{1}{2},$$

which is our contradiction. ■

Remark 2.73. Because a sequence will converge if and only if its real and imaginary parts do, we can turn a convergence test into a real-number test by taking the real and imaginary parts of the sum.

2.4.2 The Comparison Test

Recall the comparison test in $\mathbb{R}$.

Theorem 2.74 (Comparison test)**.** Fix $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ sequences of real numbers. Further, suppose that we there exists a positive constant $c > 0$ such that $0 \leq x_n \leq cy_n$. Then the following hold.

- If $\sum y_n$ converges, then $\sum x_n$ converges as well.
- If $\sum x_n$ diverges, then $\sum y_n$ diverges as well.

Proof. We appeal to real analysis. The interested can see Theorem 2.1.21 in Eterović. The main point is to use the Monotone sequence theorem. ■

Here is an example.

Exercise 2.75. Fix $s > 1$ an integer. Then the series

$$S = \sum_{k=1}^\infty \frac{1}{k^s}$$

converges.

Proof. Because s is an integer, we have $s \geq 2$. Namely, $\frac{1}{k^s} \leq \frac{1}{k^2}$, so by the comparison test it suffices to just show the convergence of

$$S' := \sum_{k=1}^\infty \frac{1}{k^2}.$$

For this, we apply some trickery. In particular, for $k > 1$, we bound

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

In particular,

$$S' = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} < 1 + \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right).$$

Thus, by the comparison test, it suffices to show the convergence of

$$T := \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right).$$

But the nth partial sum will telescope, giving

$$T_n := \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 - \frac{1}{n},$$

so $T_n \to 1$ as $n \to \infty$, and $T = 1$. It follows that S' is upper-bounded by $1 + T \le 2$. ■

2.4.3 Absolute Convergence

The following kind of convergence is nontrivially stronger, but that makes it better.

Definition 2.76 (Absolute convergence)**.** Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers. Then the sum $S := \sum z_n$ converges *absolutely* if and only if the series

$$\sum_{n=0}^{\infty} |z_n|$$

also converges. In other words, the partial sums of the above series converges.

We have the following quick lemma to justify naming this "convergence."

Lemma 2.77. If a series converges absolutely, then the series also converges.

Proof. The idea is to use the triangle inequality. Fix our series

$$S := \sum_{n=0}^{\infty} z_n$$

for which

$$T := \sum_{n=0}^{\infty} |z_n|$$

converges. Let S_n be the nth partial sum of S and T_n the n the partial sum of T.

Our goal is to show that $\{S_n\}_{n\in\mathbb{N}}$ is Cauchy. Observe $\{T_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers because $|z| \ge 0$ always. To start off our arithmetic, we note that, for $n, m \in \mathbb{N}$ with $n > mn$, we have

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} z_k \right|,$$

which by the triangle inequality can be bounded by

$$|S_n - S_m| \leq \sum_{k=m+1}^{n} |z_k| = T_m - T_n.$$

But now we can use the fact that $\{T_n\}_{n\in\mathbb{N}}$ must be Cauchy to finish: for any $\varepsilon > 0$, there exists some N such that $n > m > N$ implies $T_m - T_n < \varepsilon$. But then this same N promises $n > m > N$ implies

$$|S_n - S_m| < T_m - T_n < \varepsilon,$$

which is what we wanted. ■

Here is a surprise tool that will help us later.

Lemma 2.78. Fix a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of nonzero complex numbers. Further, suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ tends to infinity (i.e., $|a_n| \to \infty$ as $n \to \infty$), then for any positive real number $r \in \mathbb{R}^+$, the series

$$\sum_{k=0}^{\infty} \left(\frac{r}{|a_k|}\right)^k$$

converges.

Proof. We need the a_n to be nonzero in order to allow division, so the real puzzle is to determine how to use the fact $|a_n| \to \infty$. Well, there exists some N such that $n > N$ has

$$|a_n| > 2r.$$

But then $\frac{r}{|a_n|} < \frac{1}{2}$ for each $n > N$, so we can use the comparison test as follows: observe that

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

will converge, and there will exist some $c > 1$ so that

$$\frac{r}{|a_k|} < \frac{c}{2^k}$$

for $0 \leq k \leq N$; and then for $n > N$, we get the above inequality anyways as discussed earlier (observe we took $c > 1$). ■

Quote 2.79. I can't break math on the first day of class. I can do it later on.

Lemma 2.80. Suppose that we have two series $S := \sum_{k\in\mathbb{N}} z_k$ and $T := \sum_{k\in\mathbb{N}} w_k$ are both absolutely convergent. Then the sum

$$P := \sum_{k=0}^{\infty} \left(\sum_{i+j=k} z_i w_j \right)$$

is absolutely convergent as well. In fact, P will converge to ST.

Proof. We sketch the result, and the remaining details are in Eterović. As usual, consider the partial sums

$$A_n := \sum_{k=0}^{n} |z_k| \qquad \text{and} \qquad B_n = \sum_{k=0}^{n} |w_k|,$$

both of which will converge as $n \to \infty$. Brazenly multiplying these together, we see that

$$A_n B_n = \sum_{i,j=0}^{n} |z_i w_j| = \sum_{k=0}^{n} \sum_{\substack{i+j=k \\ 0 \le i,j \le n}} |z_i w_j| + \sum_{k>n} \sum_{\substack{i+j=k \\ 0 \le i,j \le n}} |z_i w_j|.$$

In the first sum, observe that any time $i+j=k$, we will automatically have $i, j \le k \le n$. It follows that

$$A_n B_n = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_i w_j \right) + \underbrace{\sum_{\substack{i+j>n \\ 0 \le i,j \le n}} |z_i w_j|}_{R_n}.$$

The key claim is that $R_n \to 0$. The main idea is that $i+j>n$ implies that $i \ge n/2$ or $j \ge n/2$, so we can write

$$|R_n| \le \sum_{i=0}^{n} \sum_{j=n/2}^{n} |z_i w_j| + \sum_{i=n/2}^{n} \sum_{j=0}^{n} |z_i w_j| = \left(\sum_{i=0}^{n} |z_i| \right) \left(\sum_{j=n/2}^{n} |w_j| \right) + \left(\sum_{i=n/2}^{n} |z_i| \right) \left(\sum_{j=0}^{n} |w_j| \right).$$

Now, fix any $\varepsilon > 0$, and we show there exists X so that $n > X$ has $|R_n| < \varepsilon$. Note $A := \sum |z_k|$ and $B := \sum |w_k|$ both converge and hence have Cauchy partial sums. Because the partial sums are increasing, we bound

$$|R_n| \le A \left(\sum_{j=n/2}^{n} |w_j| \right) + B \left(\sum_{i=n/2}^{n} |z_i| \right)$$

So there exists N such that $n > m > N$ has

$$\sum_{i=m+1}^{n} |z_i| < \frac{\varepsilon}{2B}$$

Similarly there exists M so that $n > m > M$ has

$$\sum_{j=m+1}^{n} |w_j| < \frac{\varepsilon}{2A},$$

from which it follows that $n > n/2 > \max\{N, M\}$ will have

$$|R_n| \le A \cdot \frac{\varepsilon}{2A} + B \cdot \frac{\varepsilon}{2B} = \varepsilon,$$

which finishes.

Now, because $R_n \to 0$, we see

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left(\sum_{i+j=k} |z_i| \cdot |w_j| \right) = \lim_{n\to\infty} A_n B_n - \lim_{n\to\infty} R_n,$$

which does indeed converge, so indeed the series

$$\sum_{k=0}^{\infty} \left(\sum_{i+j=k} |z_i| \cdot |w_j| \right)$$

will converge. By the comparison test (using the triangle inequality), it follows that

$$P = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} z_i w_j \right)$$

will also absolutely converge.

To show that P converges to ST, we observe that the difference of the nth partial sum is

$$P_n - S_nT_n = \sum_{k=0}^{n}\left(\sum_{i+j=k} z_iw_j\right) - \sum_{i,j=0}^{n} z_iw_j = \sum_{k=0}^{n}\left(\sum_{i+j=k} z_iw_j\right) - \sum_{k=0}^{n}\left(\sum_{i+j=k}^{n} z_iw_j\right) + \sum_{\substack{0\le i,j\le n\\ i+j<n}} z_iw_j,$$

so

$$P_n - S_nT_n = \sum_{\substack{0\le i,j\le n\\ i+j<n}} z_iw_j.$$

But by the triangle inequality, we see $|P_n - S_nT_n| \le R_n$, so $P_n - S_nT_n \to 0$ as $n \to \infty$. It follows P_n and S_nT_n have the same limit as $n \to \infty$ (which exists because S_n and T_n have a limit). So indeed, $P = ST$. ■

2.5 February 2

Good morning everyone. Here is some house-keeping.

- Homework #2 is due on Friday at 11:59, on GradeScope. The assignment has just been added.
- There are office hours to talk about the homework. Please come if you have questions.

2.5.1 Summation Review

Today we finish our discussion of series, for now. We quickly recall the definitions.

Definition 2.81 (Series)**.** An *infinite series* over $\mathbb{C}$ is an infinite sum

$$S := \sum_{n=1}^{\infty} z_n$$

where $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a sequence of complex numbers.

Definition 2.82 (Converge, diverge)**.** Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers, we define the mth partial sum to be

$$S_m := \sum_{n=0}^{m} z_m.$$

Then we say that the infinite series *converges* if and only if the sequence $\{S_m\}$ of partial sums converges. Otherwise, we say that S is *divergent*.

Today we are building towards proving Dirichlet's convergence theorem. We pick up the following lemmas.

Lemma 2.83. Fix sequences $\{z_{k,\ell}\}_{k,\ell\in\mathbb{N}}$ a collection of complex numbers satisfying the following conditions.

- Fixing any k, the sum $\sum_{\ell=0}^{\infty}|z_{k,\ell}|$ converges.
- The sum $\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|$ converges.

Then the following are true.

(a) Fix any ℓ, the sum $\sum_{k=0}^{\infty}|z_{k,\ell}|$ converges; i.e., the terms in the left sum below are well-defined.

(b) We have that
$$\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}z_{k,\ell}=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}z_{k,\ell},$$
and both sums converge.

Intuitively, the first condition is requiring that the series horizontally does not grow too fast. The second condition is requiring an absolute convergence condition. Then the first conclusion says we can sum vertically instead, and the second conclusion says that we can move around the order of summation.

Proof. We will sketch this proof; we prove (a) and (b) more or less simultaneously. To turn the infinite double sum into something we can consider finite partial sums of, we set, for each natural N,
$$S_n:=\sum_{k=0}^{n}\sum_{\ell=0}^{n}|z_{k,\ell}|.$$
The main claim is that
$$\lim_{n\to\infty}S_n\overset{?}{=}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|.$$
Indeed, fix any $\varepsilon>0$. Because the latter sum converges, there exists some natural A such that
$$\sum_{k>A}\sum_{\ell=0}^{\infty}|z_{k,\ell}|<\frac{\varepsilon}{2}.$$
Further, there exists some natural B_k such that
$$\sum_{\ell>B_k}|z_{k,\ell}|<\frac{\varepsilon}{2A}$$
for each $k\in\mathbb{N}$. Take $B:=\max_{0\le k<A}B_k$. Now, we set $N:=\max\{A,B\}$. To start off our inequalities, we note that
$$S_n=\sum_{k=0}^{n}\sum_{\ell=0}^{n}|z_{k,\ell}|\le\sum_{k=0}^{n}\sum_{\ell=0}^{\infty}|z_{k,\ell}|\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|,$$
so we know the sign of our difference. In particular, for any $n>N$, we see that
$$S_n=\sum_{k=0}^{N}\sum_{\ell=0}^{N}|z_{k,\ell}|\ge\sum_{k=0}^{K}\sum_{\ell=0}^{L}|z_{k,\ell}|.$$
Thus,
$$0\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|-S_n\le\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}|-\sum_{k=0}^{K}\sum_{\ell=0}^{L}|z_{k,\ell}|=\sum_{k>K}\sum_{\ell=0}^{\infty}|z_{k,\ell}|+\sum_{k=0}^{K}\sum_{\ell>L}|z_{k,\ell}|$$
after some cancellation. But we can upper-bound the last quantity by $\frac{\varepsilon}{2}+K\cdot\frac{\varepsilon}{2K}=\varepsilon$, so we are done.

The main point of the above lemma is that we are told each $\varepsilon > 0$ has some N so that $n > N$ implies

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}| - S_n = \sum_{\substack{(k,\ell)\in\mathbb{Z}^2 \\ k>n \text{ or } \ell>n}} |z_{k,\ell}| < \varepsilon.$$

We now take the two parts in sequence.

(a) Fix an index ℓ'; we show absolute convergence by showing that the partial sums of $\sum_{k=0}^{\infty}|z_{k,\ell'}|$ are Cauchy. Indeed, fix some $\varepsilon > 0$, and we know there exists N so that each $n > N$ has

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}| - S_n < \varepsilon.$$

Now, we see that any $n > m > N$ will have

$$\sum_{k=m+1}^{n}|z_{k,\ell'}| \le \sum_{k=m+1}^{N}\sum_{\ell=0}^{\infty}|z_{k,\ell}| \le \sum_{k=N+1}^{\infty}\sum_{\ell=0}^{\infty}|z_{k,\ell}| + \sum_{k=0}^{N}\sum_{\ell=N+1}^{\infty}|z_{k,\ell}| < \varepsilon,$$

so we are done.

(b) As above, fix some $\varepsilon > 0$, and we are promised N so that

$$\sum_{\substack{(k,\ell)\in\mathbb{Z}^2 \\ k>N \text{ or } \ell>N}} |z_{k,\ell}| < \varepsilon/2.$$

Observe, for $K, L > N$, we have by the triangle inequality that

$$\left|\sum_{\ell=0}^{L}\sum_{k=0}^{K} z_{k,\ell} - S_N\right| < \varepsilon/2.$$

This bounds holds for any K, so we can send K arbitrarily large; that inner sum will converge, so in fact we can send K to ∞ without ill effect. (Formally, the inner term is an increasing sequence bounded above, so it will converge as $K \to \infty$.) This gives

$$\left|\sum_{\ell=0}^{L}\sum_{k=0}^{\infty} z_{k,\ell} - S_N\right| \le \varepsilon/2.$$

Again, the inner term is an increasing sequence as $L \to \infty$ but still bounded above as $\varepsilon/2$, so the inner sum will converge as $L \to \infty$ and still give the inequality

$$\left|\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} z_{k,\ell} - S_N\right| < \varepsilon.$$

Now as we send $\varepsilon \to 0$, we see that $\lim_{N\to\infty} S_N = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} z_{k,\ell}$, which finishes. ■

2.5.2 Dirichlet Test

We now go directly for the Dirichlet test for convergence.

Lemma 2.84 (Summation by parts)**.** Fix sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ sequences of complex numbers. Then we define

$$B_n := \sum_{k=0}^{N} b_n,$$

and $B_{-1} = 0$ to be the empty sum. It follows that, for any $n, m \in \mathbb{N}$ with $n > m$,

$$\sum_{k=m}^{n} a_k b_k = a_n B_n - a_m B_{m-1} + \sum_{k=m}^{n-1} B_k(a_k - a_{k+1}).$$

Proof. This comes down to a direct computation. The main point is that $b_k = B_k - B_{k-1}$, which even works with $k = 0$. Indeed,

$$\begin{aligned}\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} a_k (B_k - B_{k-1}) \\ &= \sum_{k=m}^{n} a_k B_k - \sum_{k=m}^{n} a_k B_{k-1} \\ &\overset{*}{=} a_n B_n + \sum_{k=m}^{n-1} a_k B_k - a_m B_{m-1} - \sum_{k=m}^{n} a_{k+1} B_k \\ &= a_n B_n - a_m B_{m-1} + \sum_{k=m}^{n-1} B_k (a_k - a_{k+1}),\end{aligned}$$

which is what we wanted. The important step to pay attention to is the rearrangement we did in $\overset{*}{=}$ in order to message the sums together. ■

And here is our theorem.

Theorem 2.85 (Dirichlet's test)**.** Fix $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ a sequence of real numbers and $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a sequence of complex numbers satisfying the following conditions.

- The sequence $\{a_n\}_{n\in\mathbb{N}}$ is decreasing.
- We have $a_n \to 0$ as $n \to \infty$.
- Bounded partial sums: there exists a positive real number M such that

  $$\left| \sum_{k=0}^{n} b_k \right| < M$$

  for each $n \in \mathbb{N}$.

Then we claim that

$$\sum_{k=0}^{\infty} a_k b_k$$

converges.

Proof. As usual, fix our partial sums

$$S_n := \sum_{k=0}^{n} a_k b_k \qquad \text{and} \qquad B_n := \sum_{k=0}^{n} b_k.$$

We are given that the B_k are bounded, so we are going to want to use Lemma 2.84, which tells us that

$$S_n = a_n B_n + \sum_{k=0}^{n-1} B_k (a_k - a_{k+1}).$$

We examine the convergence of these terms individually.

- For the sum, we will show that it absolutely converges. We are given that the partial sums B_n are bounded by M, so we note $|B_k(a_k - a_{k+1})| < M|a_k - a_{k+1}|$, so it suffices to show that

  $$M \sum_{k=0}^{n-1} |a_k - a_{k+1}|$$

converges as $n \to \infty$. It would be great if this would telescope, and indeed it does! Because the a_k are decreasing,

$$\sum_{k=0}^{\infty} |a_k - a_{k+1}| = \sum_{k=0}^{\infty} (a_k - a_{k+1}) = a_0 - a_{n+1}.$$

Because $a_n \to 0$ as $n \to \infty$, we see that this sum will converge to a_0. It follows that

$$\sum_{k=0}^{\infty} |B_k(a_k - a_{k+1})|$$

will converge by the Comparison test, so

$$\sum_{k=0}^{\infty} B_k(a_k - a_{k+1})$$

converges by absolute convergence.

- Note that the B_n are bounded in norm by M, so $|a_n B_n| \le M|a_n|$, but $|a_n| \to 0$ as $n \to \infty$, so $|a_n B_n| \to 0$. ■

Eterović has lots of different convergence tests in his notes, but we don't care about most of them. Here is one that we do care about.

Theorem 2.86 (Integral test)**.** Fix a decreasing function $f : [1, \infty) \to \mathbb{R}^+$ and for which

$$\int_k^{k+1} f(x)\,dx$$

always exists. Then the sequence of integrals $I_n := \int_1^n f(x)\,dx$ converges if and only if the summation

$$\sum_{k=1}^{\infty} f(k)$$

converges.

Proof. We omit this proof; it's a reasonably standard real-analytic test. ■

2.6 February 4

Today we are talking about continuity.

Warning 2.87. The first half of this lecture was transcribed from Professor Morrow's notes because I had to miss class for a job interview

2.6.1 Limits

Before defining continuity, we have the following definitions.

Definition 2.88 (Limit)**.** Fix $f : X \to \mathbb{C}$ a function and $z_0 \in \overline{X}$. Then we say *the limit of $f(z)$ as z approaches z_0 equals w*, denoted

$$\lim_{z\to z_0} f(z) = w,$$

if and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - w| < \varepsilon$$

for $z \in X$.

This is the standard ε-δ definition.

We also pick up the following convention as a surprise tool that may help us later.

Definition 2.89 (Infinite limits)**.** Fix $f : X \to \mathbb{C}$ a function. Then we say *the limit of $f(z)$ as z tends to infinity equals w*, denoted

$$\lim_{z\to\infty} f(z) = w,$$

if and only if, for each $\varepsilon > 0$, there exists $N > 0$ such that

$$|z| > N \implies |f(z) - w| < \varepsilon$$

for $z \in X$.

As in real analysis, the ε-δ definition of a limit can be translated to a statement about sequences.

Proposition 2.90. Fix $\alpha \in \overline{X}$. Then $\lim_{z\to\alpha} f(z) = w$ if and only if, for each $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to \alpha$ as $n \to \infty$, we have $f(z_n) \to w$ as $n \to \infty$.

Proof. In the forwards direction, fix $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to \alpha$, and we show that $f(z_n) \to w$. Well, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon,$$

where $z \in X$. But for this $\delta > 0$, there exists N such that

$$n > N \implies |z_n - \alpha| < \delta \implies |f(z_n) - f(\alpha)| < \varepsilon.$$

So indeed, $f(z_n) \to f(\alpha)$.

In the reverse direction, suppose that $f(z)$ does not approach w as $z \to \alpha$. Then, there exists $\varepsilon_0 > 0$ such that, for any $\delta > 0$, there is $z \in X$ such that $|z - \alpha| < \delta$ while $|f(z) - w| > \varepsilon_0$. Well, for any $n \in \mathbb{N}$, taking $\delta = 1/(n+1)$, we are promised $z_n \in X$ such that

$$|z_n - \alpha| < \frac{1}{n+1} qquad \text{and} \qquad |f(z_n) - w| > \varepsilon_0.$$

So to finish, we claim that $z_n \to \alpha$ as $n \to \infty$, but $f(z_n)$ does not approach w as $n \to \infty$.

- For any $\varepsilon > 0$, we note that $N := 1/\varepsilon$ has $n > N$ implies

  $$|z_n - \alpha| < \frac{1}{n+1} < \frac{1}{N+1} < \frac{1}{N} = \varepsilon,$$

  so indeed $z_n \to \alpha$ as $N \to \infty$.
- We note that $\varepsilon_0 > 0$ satisfies that

  $$|f(z_n) - w| > \varepsilon_0$$

  for any $n \in \mathbb{N}$, so no N will have $n > N$ implies $|f(z_n) - w| < \varepsilon_0$. Thus, $f(z_n)$ does not approach w as $n \to \infty$.

The sequence $\{z_n\}_{n\in\mathbb{N}}$ now completes the proof by showing the reverse direction by contraposition. ■

While we're here, we pick up the following definitions.

Definition 2.91 (Bounded)**.** A function $f : X \to \mathbb{C}$ is *bounded* if there exists $R > 0$ such that $\operatorname{im} f \subseteq B(0, R)$.

Definition 2.92 (Bounded near)**.** Fix a nonempty open subset $\Omega \subseteq \mathbb{C}$ and $z_0 \in \Omega$. Then $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is *bounded near* z_0 if and only if

$$\lim_{z\to z_0}(z - z_0)f(z) = 0.$$

2.6.2 Continuity

And here is our central definition for today.

Definition 2.93 (Continuous)**.** A function $f : X \to \mathbb{C}$ is *continuous at* $z_0 \in X$ if and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon,$$

where $z \in X$. Further, f is *continuous on* X if and only if f is continuous at each $z_0 \in X$.

We have the following lemma of equivalent definitions.

Lemma 2.94. Suppose that $f : X \to \mathbb{C}$.

(a) Then f is continuous at w if and only if every sequence $\{z_n\} \subseteq X$ such that $z_n \to z$ implies $f(z_n) \to f(z)$.

(b) We have that f is continuous on X if and only if every open set $U \subseteq \mathbb{C}$ has $f^{-1}(U)$ open in X.

(c) We have that f is continuous on X if and only if each closed set $V \subseteq X$ has $f^{-1}(V)$ closed in X.

(d) Lastly, we have that f is continuous at if and only if, for each $\varepsilon > 0$ and $z \in \mathbb{C}$, we have that $f^{-1}(B(z, \varepsilon))$ is open in X.

Proof. We take the parts one at a time.

(a) We could use Proposition 2.90, but we will just do this by hand. For the forwards direction, suppose that $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ converges to some w. Then let $\varepsilon > 0$. By assumption, there exists some $\delta > 0$ such that

$$|z - w| < \delta \implies |f(x) - f(w)| < \varepsilon.$$

It follows from $z_n \to w$ that there exists some N such that

$$n > N \implies |z_n - w| < \delta \implies |f(z_n) - f(z)| < \varepsilon,$$

so it follows that $f(z_n) \to f(z)$.

In the reverse direction, take f not continuous at w, so there exists $\varepsilon > 0$ so that for all $n \in \mathbb{N}$, there exists some chosen z_n with

$$|z_n - w| < \delta \implies |f(z_n) - f(w)| > \varepsilon.$$

But as $z_n \to w$, we see that $f(z_n)$ does not approach $f(w)$, so we are done.

(b) In the forwards direction, suppose that $U \subseteq \mathbb{C}$ is open, and we show that $f^{-1}(U)$ is open in X. Well, suppose that $z \in f^{-1}(U)$, and we will find $\delta > 0$ such that $B(z, \delta) \subseteq f^{-1}(U)$.

Well, $f(z) \in U$, so there exists $\varepsilon > 0$ such that $B(f(z), \varepsilon) \subseteq U$. Thus, continuity of f requires some $\delta > 0$ such that

$$|w - z| < \delta \implies |f(w) - f(z)| < \varepsilon,$$

which implies $f(w) \in B(f(z), \varepsilon) \subseteq U$ implies $w \in f^{-1}(U)$. So indeed, $B(z, \delta) \subseteq f^{-1}(U)$.

In the reverse direction, suppose that each open $U \subseteq \mathbb{C}$ has $f^{-1}(U)$ is open. Now fix any $z \in X$ and $\varepsilon > 0$. The set $B(f(z), \varepsilon)$ is open, so

$$f^{-1}(B(f(z), \varepsilon))$$

is open. But $z \in f^{-1}(B(f(z), \varepsilon))$, so we can find $\delta > 0$ such that $B(z, \delta) \subseteq f^{-1}(B(f(z), \varepsilon))$. Thus, $|w - z| < \delta$ implies $w \in f^{-1}(B(f(z)), \varepsilon)$ implies $f(w) \in B(f(z), \varepsilon)$ implies $|f(w) - f(z)| < \varepsilon$, finishing.

(c) In the forwards direction, suppose f is continuous so that $U \subseteq \mathbb{C}$ open implies $f^{-1}(U) \subseteq X$ is open. But then, if $V \subseteq \mathbb{C}$ is closed, then $\mathbb{C} \setminus V$ is open, so[3]

$$f^{-1}(\mathbb{C} \setminus V) = f^{-1}(\mathbb{C}) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$$

is open, so $f^{-1}(V)$ is closed.

In the backwards direction, suppose f^{-1} preserves closed sets. Then, if $U \subseteq \mathbb{C}$ is open, $\mathbb{C} \setminus U$ is closed, so

$$f^{-1}(\mathbb{C} \setminus U) = f^{-1}(\mathbb{C}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$$

is closed, so $f^{-1}(U)$ is open. Thus, f^{-1} preserves open sets, so f must be continuous.

(d) In the forwards direction, fix $\varepsilon > 0$ and $z \in \mathbb{C}$, so $B(z, \varepsilon)$ is open, so $f^{-1}(B(z, \varepsilon))$ is open in X, finishing.

In the other direction fix $\varepsilon > 0$ and $z \in \mathbb{C}$ to consider $B(f(z), \varepsilon) \subseteq U$. Thus, continuity of f requires some $\delta > 0$ such that

$$|w - z| < \delta \implies |f(w) - f(z)| < \varepsilon,$$

which implies $f(w) \in B(f(z), \varepsilon) \subseteq U$ implies $w \in f^{-1}(U)$. So indeed, $B(z, \delta) \subseteq f^{-1}(U)$.

In the reverse direction, fix U open, and we show that $f^{-1}(U)$ is open. Well, each $z \in U$ has some ε_z such that $B(z, \varepsilon_z) \subseteq U$. But $f^{-1}(B(z, \varepsilon_z))$ is open by hypothesis, so

$$f^{-1}(U) = f^{-1}\left(\bigcup_{z \in U} B(z, \varepsilon_z)\right) = \bigcup_{z \in U} f^{-1}(B(z, \varepsilon_z))$$

is an arbitrary union of open sets and hence open. ■

And here are some special examples.

Example 2.95. Fix some $z_0 \in \mathbb{C}$. Then $f(z) := |z - z_0|$ is continuous on $\mathbb{C}$. Indeed, fix any $w \in \mathbb{C}$. Then for any $\varepsilon > 0$, we set $\delta := \varepsilon$ so that $|z - w| < \delta$ implies

$$|f(z) - f(w)| = \big||z - z_0| - |w - z_0|\big| \le |z - w| < \delta = \varepsilon.$$

Example 2.96. The functions Re and Im is continuous. Indeed, fix any $w \in \mathbb{C}$. Then, for any $\varepsilon > 0$, take $\delta := \varepsilon$ so that $|z - w| < \delta$ implies

$$|\operatorname{Re} z - \operatorname{Re} w| = |\operatorname{Re}(z - w)| \le |z - w| < \delta = \varepsilon,$$

and similarly,

$$|\operatorname{Im} z - \operatorname{Im} w| = |\operatorname{Im}(z - w)| \le |z - w| < \delta = \varepsilon,$$

Continuous functions also have some arithmetic.

[3] To see $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, note that $x \in f^{-1}(A \setminus B)$ if and only if $f(x) \in A \setminus B$ if and only if $f(x) \in A$ but $f(x) \notin B$ if and only if $x \in f^{-1}(A)$ but $x \notin f^{-1}(B)$.

Proposition 2.97. Fix $f, g : X \to \mathbb{C}$ to functions continuous at $z_0 \in X$. Then $f + g, f \cdot g$ are both continuous at $z_0 \in X$, and f/g is continuous at z_0 provided $g(z_0) \neq 0$.

Proof. The point is to appeal to the corresponding results on convergence of sequences. In particular, we use the idea that f is continuous at z_0 if and only if each sequence $z_n \to z_0$ in X has $f(z_n) \to f(z_0)$. We omit the details because they are essentially the same as in a real analysis class. ■

Corollary 2.98. Every polynomial in one variable is a continuous function $X \to \mathbb{C}$ for any $X \subseteq \mathbb{C}$.

Proof. Note that $x \mapsto x$ is continuous, so by induction $x \mapsto x^n$ is continuous for each $n \in \mathbb{N}$. Taking a $\mathbb{C}$-linear combination gives arbitrary polynomials. ■

Here is another sort of arithmetic.

Lemma 2.99. The composition of two continuous functions is continuous.

Proof. Omitted. ■

2.6.3 Connectedness

We want to build towards a particular type of continuous function.

Proposition 2.100. Fix $X \subseteq \mathbb{C}$ a connected subset. Then a continuous function $f : X \to \mathbb{C}$ has connected image $f(X)$.

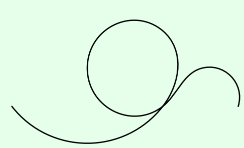
Proof. The main point is to use the topological characterization of continuity. In particular, suppose that $f(X)$ is disconnected, and we show that X is disconnected. In particular, suppose that U_1 and U_2 disconnect $f(X)$, and we have that $f^{-1}(U_1)$ and $f^{-1}(U_2)$ disconnect X. We will not run all the checks here; the main point is that $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open because f is continuous. ■

Definition 2.101 (Path)**.** A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$ where $a < b$ are real numbers.

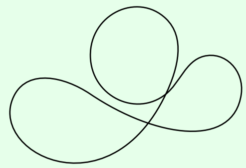
Definition 2.102. We say that a path γ is *closed* if and only if $\gamma(a) = \gamma(b)$. We say that γ is *simple* if and only if $\gamma : (a, b) \to \mathbb{C}$ is injective.

Remark 2.103. The point of restricting γ to the open interval at the end so that closed, simple paths are allowed to exist.

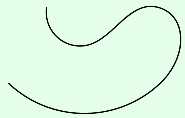
Example 2.104. Here is a path.



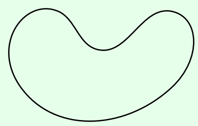
Example 2.105. Here is a closed path.



Example 2.106. Here is a simple path.



Example 2.107. Here is a closed, simple path, also called a loop.

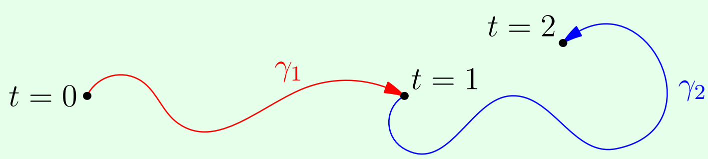


Definition 2.108 (Concatenation)**.** Fix $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d]$ paths in $\mathbb{C}$ such that $\gamma_1(b) = \gamma_2(c)$. Then we define the *concatenation* of γ_1 and γ_2 to be

$$(\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a, b], \\ \gamma_2(t) & t \in [b, d + c - b]. \end{cases}$$

The main point is that we are doing one path after the other.

Example 2.109. The following shows an example concatenation of $\gamma_1 * \gamma_2$, where $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$.



The entire path is $\gamma_1 * \gamma_2$.

Paths give us the following notion.

Definition 2.110 (Path-connected)**.** A subset $X \subseteq \mathbb{C}$ is *path connected* if and only if, for any two $x_0, x_1 \in X$, there exists a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Lemma 2.111. The open ball $B(z, r)$ and closed ball $\overline{B(z, r)}$ are both path-connected.

Proof. The point is that $B(z, r)$ and $\overline{B(z, r)}$ are both convex, so the path

$$\gamma(t) := z_0 + t(z_1 - z_0)$$

will work. ■

Here is the basic result.

Proposition 2.112. A space X is path-connected implies that X is connected. If X is open and connected, then X is path-connected.

Proof. We will show this next class. ■

2.7 February 7

Good morning everyone. A few announcements.

- Homework #3 is due on Friday.
- There will be no in-person class on Wednesday or Friday.
- Office hours this week are today (1:00PM–2:30PM) and tomorrow (2:00PM–3:30PM).

2.7.1 Connectedness

Today we're going to talk more about continuous functions.

Last time we ended with the following proposition.

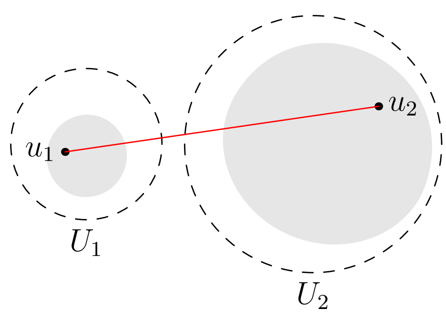
Proposition 2.113. A space X is path-connected implies that X is connected. If X is open and connected, then X is path-connected.

Proof. We do these separately.

- Suppose that $X = U_1 \sqcup U_2$ is disconnected, and we show that X is not path-connected. Namely, we have $U_1, U_2 \subseteq X$ open subsets (in X) which are disjoint and nonempty. Because U_1 and U_2 are nonempty, find $x_1 \in U_1$ and $x_2 \in U_2$.

  However, we claim there is no continuous path $\gamma : [0,1] \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Indeed, the image of $\gamma([0,1])$ must be connected, but then we can disconnect $\gamma([0,1])$ by U_1 and U_2: $\gamma([0,1]) \subseteq U_1 \cup U_2$ and $x_\bullet \in U_\bullet \cap \gamma([0,1])$ and $U_1 \cap U_2 = \varnothing$.

  At a high level, here is the image that a disconnected X cannot have a path between any two pair points: there is no possible red path below which stays in the gray region.

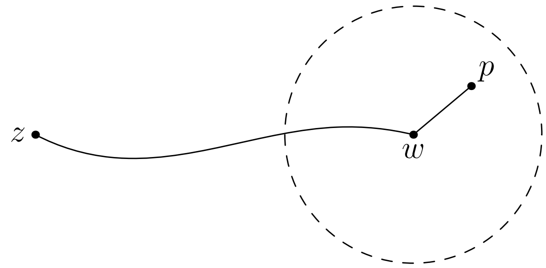


- Suppose we have a point $z \in X$, and we set

$$C(z) := \{w \in X : \text{there is a path from } z \text{ to } w\}.$$

  We claim that $C(z)$ is closed and open in X, which will force $C(z) = X$ because X is connected and $C(z)$ is nonempty ($z \in C(z)$ by the trivial path $\gamma : t \mapsto z$).

  We start by showing $C(z)$ is open: because X is open, there exists $r > 0$ such that $B(w, r) \subseteq X$. But with $w \in C(z)$, there will be a path between any point in $p \in B(z, w)$ and w, so there is a path from z to w to p. Here is the image.



Now we show that $C(z)$ is closed. Suppose that $w \in X \setminus C(z)$, and we have to show that there is an open ball around w in $X \setminus C(z)$. To see this, fix an open ball $B(w,r) \subseteq X$ for $r > 0$, but now there can be no path from z to anywhere in $B(w,r)$, for then we could just run the above argument again to show that $w \in C(z)$. ■

Remark 2.114. The proof for the second part merely needs X to be locally path-connected, not a metric space.

Corollary 2.115. We have that $\mathbb{C}$ is path-connected and therefore connected.

Proof. Given any two points $z, w \in \mathbb{C}$, we choose the path $\gamma : [0,1] \to \mathbb{C}$ by

$$\gamma(t) = tz + (1-t)w.$$

Indeed, $\gamma(0) = w$ and $\gamma(1) = z$, and γ is somewhat clearly continuous by, say, checking component-wise. ■

2.7.2 Compactness

Let's do compactness better this time.

Lemma 2.116. Fix $X \subseteq \mathbb{C}$ (sequentially) compact. Then X is both closed and bounded.

Proof. We start by showing X is closed. For this, we show that X contains all of its limit points.

Well, suppose that $z \in X$ is a limit point so that we have a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to z$. But by the (sequential) compactness of X, this sequence has a convergent subsequence $\{z_{\sigma n}\}_{n\in\mathbb{N}}$ which does converge in X. But any subsequence will converge to the same limit (!), so $z_{\sigma n} \to z$ as well, so $z \in X$ is forced.

We now show that X is bounded. We proceed by contraposition: if X is unbounded, then for any $n \in \mathbb{N}$, then we can find some $z_n \in X \setminus B(0,n)$. But then we can check that $\{z_n\}_{n\in\mathbb{N}}$ has no convergent subsequence, essentially because it tends off to infinity. ■

Our goal for the rest of class is to prove the following two results.

Proposition 2.117. A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if it is closed and bounded.

Theorem 2.118. [Heine–Borel] A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if every open cover of X has a finite subcover.

On the homework, we showed the backward direction of Theorem 2.118.

Remark 2.119. Our hope is to have lots of equivalent characterizations of compactness so that we can have easier proofs of statements about compact sets.

To start off, here are some lemmas we will need.

Lemma 2.120. Fix $X \subseteq \mathbb{C}$ (sequentially) compact. For any $\varepsilon > 0$, there exist only finitely many points $z_1, \ldots, z_n \in X$ such that

$$X \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon).$$

Proof. The point is to build some inductive argument: one fixes an $\varepsilon > 0$ and then continues choosing random points out of X until we cover X. Indeed, if the process does not terminate, then the sequence we generate has no convergent subsequence.

Rigorously, if X is empty, then just choose no points at all and be done. Otherwise, we can find some $z_1 \in X$. Inductively, suppose we have a sequence $\{z_1, \ldots, z_m\}$. If

$$X \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon),$$

then we are done. Otherwise, we can find $z_{m+1} \in X \setminus \bigcup_{k=1}^{n} B(z_k, \varepsilon)$.

If the above inductive process terminates, then we get the result. Otherwise, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$z_{n+1} \in X \Big\backslash \bigcup_{k=1}^{n} B(z_k, \varepsilon).$$

We claim that $\{z_n\}_{n\in\mathbb{N}}$ has subsequence converging in X. Indeed, suppose for the sake of contradiction that $z_{\sigma n} \to z$ for some strictly increasing σ and $z \in X$. Then there exists N such that $n > N$ implies

$$|z_{\sigma n} - z| < \varepsilon/2.$$

But then, finding some $n+1, n > N$, we have

$$|z_{\sigma(n+1)} - z_{\sigma n}| < |z_{\sigma(n+1)} - z| + |z_{\sigma n} - z| < \varepsilon,$$

so

$$z_{\sigma(n+1)} \in \bigcup_{k=1}^{\sigma(n+1)-1} B(z_k, \varepsilon),$$

which is our contradiction to the construction of $z_\bullet$. ■

Lemma 2.121. Fix $X \subseteq \mathbb{C}$ (sequentially) compact with some open cover $\mathcal{U}$ of X. Then there is an $\varepsilon > 0$ such that, for every $z \in X$, there is $U \in \mathcal{U}$ such that $B(z, \varepsilon) \subseteq U$.

Proof. Suppose that, for all $\varepsilon > 0$, there exists some $z \in X$ such that no $U \in \mathcal{U}$ has $B(z, \varepsilon) \subseteq U$. We construct a sequence in X with no subsequence converging in X. Indeed, for any $n \in \mathbb{N}$, we find $z_n \in X$ such that no $U \in \mathcal{U}$ has $B(z_n, 1/n) \subseteq U$. We claim that $\{z_n\}_{n\in\mathbb{N}}$ has no subsequence converging in X.

Indeed, suppose that we have $z \in X$ and strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_{\sigma n} \to z$. We will then be able to find some z_n such that $B(z_n, 1/n) \subseteq U$ for some $U \in \mathcal{U}$, which will be a contradiction. Indeed, $z \in X$, and $\mathcal{U}$ covers z, so there is some $U \in \mathcal{U}$ with $z \in U$. In fact, U is open, so there is an $\varepsilon > 0$ such that

$$B(z, \varepsilon) \subseteq U.$$

Now, there is N such that for $n > N$, we can guarantee that $|z - z_n| < \varepsilon/2$. Further, for $n > 2/\varepsilon$, we have $1/n < \varepsilon/2$. So $n > \max\{N, 2/\varepsilon\}$ will have $\sigma n > \max\{N, 2/\varepsilon\}$, implying

$$|w - z_n| < 1/n < \varepsilon/2 \implies |w - z| < |w - z_n| + |z - z_n| = \varepsilon \implies w \in B(z, \varepsilon) \subseteq U,$$

so $B(z_n, 1/n) \subseteq U$. This contradiction finishes. ■

This is saying that there is a universal ε that we can find for our open cover.

Lemma 2.122. Fix X a bounded set. Then, for any $\varepsilon > 0$, there exist finitely many points $z_1, \ldots, z_n$ such that

$$X \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon).$$

Proof. The point is to reduce this to the case of $[-M, M]^2$ which can cover X because X is bounded, and then we can create the cover for X by hand. ■

Now let's attack one of our equivalent conditions for compactness.

Proposition 2.117. A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if it is closed and bounded.

Proof. The forwards direction we have already done.

In the backwards direction, suppose that $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ is some sequence. Our main goal is to construct a convergent subsequence. Because X is bounded, we can choose $w_{1,1}, \ldots, w_{1,\ell_1}$ such that

$$X \subseteq \bigcup_{k=1}^{\ell_1} B(w_{1,k}, 1/2).$$

Now, because $\{z_n\}_{n\in\mathbb{N}}$ is infinite, there must be some index $w_1 := w_{1,k_1}$ such that

$$L_1 = \{n \in \mathbb{N} : z_n \in B(w_{1,k_1}, 1/2)\}$$

is infinite. The important point is that $\{z_n\}_{n\in L_1}$ has formed a subsequence which lives inside a ball of radius $1/2$. We can continue this process: again using our bounded condition, we can find some $w_{2,1}, \ldots, w_{2,\ell_2} \in B(w_{1,k_1}, 1/2)$ such that

$$B(w_{1,k_1}, 1/2) \subseteq \bigcup_{k=1}^{\ell_2} B(w_{2,k}, 1/4).$$

Then we can choose L_2 from here by choosing one of the $w_{2,k}$ with infinitely many indices. Continuing this process forces our sequence to converge.

To more explicitly appeal to choice, we note that we can always find some sequence $\{w_{k,i}\} \subseteq X$ such that

$$X \subseteq \bigcup_{k=1}^{\ell_n} B(w_{k,i}, 1/2^k),$$

but L_{i-1} is infinite, so there is a specific $w_k := w_{k,i}$ such that

$$L_i := \{n \in L_{i-1} : |z_n - w_k| < 2^{-k}\}$$

is infinite. To actually construct our sequence from these infinite subsets, we define a choice function over our indices: define $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi(n+1)$ is the smallest number exceeding $\varphi(n)$ with $\varphi(n+1) \in L_{n+1}$. Then we know that

$$|z_{\varphi(n)} - w_k| < 2^{-n}$$

for each $1 \le k \le n$. Thus, for $n \ge m > N$, we have

$$|z_{\varphi(n)} - z_{\varphi(m)}| \le |z_{\varphi(n)} - w_m| + |z_{\varphi(n)} - w_m| < 2 \cdot 2^{-m} < 2^{-N+1},$$

so for any $\varepsilon > 0$, we can choose $N := 1 - \log_2 \varepsilon$ sufficiently large so that $n, m > N$ implies

$$|z_{\varphi(n)} - z_{\varphi(m)}| < 2^{-N+1} = \varepsilon.$$

It follows that the subsequence defined by φ is Cauchy and hence converges. But because X is closed, any convergent sequence in X will be in X, so our sequence in X has a convergent subsequence. ■

2.8 February 9

2.8.1 More Compactness

To wrap up from last class, we show the following.

Theorem 2.118. [Heine–Borel] A subset $X \subseteq \mathbb{C}$ is (sequentially) compact if and only if every open cover of X has a finite subcover.

Proof. The direction that sequentially compact implies closed and bounded was done on the homework.

We focus on the other direction. Fix $\mathcal{U}$ an open cover of X. By Lemma 2.121, we know there exists $\varepsilon > 0$ such that, for each $z \in X$, there is some $U \in \mathcal{U}$ such that $B(z, \varepsilon) \subseteq U$. But in fact, with this $\varepsilon > 0$, Lemma 2.122 tells us that there exists finitely many points $z_1, \ldots, z_\ell$ such that

$$X \subseteq \bigcup_{k=1}^{\ell} B(z_k, \varepsilon).$$

But now, finding U_k such that $B(z_k, \varepsilon) \subseteq U_k$ (possible by construction of ε), we see that $\{U_k\}_{k=1}^{\ell}$ will be our finite subcover. ■

Remark 2.123. The conclusion of the above theorem is the usual notion of compactness, so I will stop writing "(sequentially)" whenever I say "compact."

Let's see a use for our definitions of compactness.

Corollary 2.124. Let $X \subseteq \mathbb{C}$ be a compact space and $f : X \to \mathbb{C}$ continuous. Then $f(X)$ is compact.

Proof. Give $f(X)$ some open cover $\mathcal{U}$. Because f is continuous, we see that

$$\left\{f^{-1}(U)\right\}_{U \in \mathcal{U}}$$

is an open cover for X. But X is compact, so we can find some finite subcover $\{U_k\}_{k=1}^{n} \subseteq \mathcal{U}$ so that $\left\{f^{-1}(U_k)\right\}_{k=1}^{n}$ covers X. But then the $\{U_k\}_{k=1}^{n}$ will cover X by taking the union over our open subcover. ■

2.8.2 Uniform Continuity

The point of uniform convergence is to make fewer choices in our notion of continuity.

Definition 2.125 (Uniform continuity)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset. Then a function $f : X \to \mathbb{C}$ is *uniformly continuous* if and only if, for each $\varepsilon > 0$, there exists a single $\delta > 0$ so that $z, w \in X$ have

$$|z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Importantly, this definition has δ not depend on either z nor w, where continuity would allow δ to depend on one of them.

Example 2.126. The functions $\mathrm{id}_{\mathbb{C}}$ and $z \mapsto \overline{z}$ are both uniformly continuous on $\mathbb{C}$. Letting f be either of these functions, we see that, for any $\varepsilon > 0$, we may take $|z - w| < \varepsilon$ to imply

$$|f(z) - f(w)| = |z - w| < \varepsilon.$$

Here is a nice result.

Proposition 2.127. Fix X a nonempty, compact subset. Then any continuous function $f : X \to \mathbb{C}$ is uniformly continuous.

Proof. The point is to let $\delta \to 0$ until we can fit some prescribed ε bound. Choose $\delta = 1/n$ as n varies over positive integers, and we imagine fixing sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ such that

$$|z_n - w_n| < 1/n.$$

Now we use the sequential compactness of X, which forces $\{z_n\}_{n=1}^\infty$ to have a convergent subsequence, so we conjure $\alpha \in X$ and a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_{\sigma n} \to \alpha$ as $n \to \infty$.

We now claim that $w_{\sigma n} \to \alpha$ as well. In particular, for any $\delta > 0$, there is some N_1 so that $n > N_1$ implies

$$|z_{\sigma n} - \alpha| < \delta/2.$$

Choosing N to be larger than N_1 and $2/\delta$, we see that $n > N$ will have

$$|w_{\sigma n} - \alpha| \le |z_{\sigma n} - w_{\sigma n}| + |z_{\sigma n} - \alpha| < \frac{1}{\sigma n} + \frac{\delta}{2} \le \frac{1}{n} + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

so indeed $w_{\sigma n} \to \alpha$ as $n \to \infty$.

Only now we suppose for the sake of contradiction we have some $\varepsilon > 0$ such that any $\delta > 0$ has some z and w such that $|z - w| < \delta$ actually has $|f(z) - f(w)| \ge \varepsilon$. Taking $\delta := 1/n$, we are promised some sequences $\{z_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ so that

$$|z_n - w_n| < \delta \qquad \text{and} \qquad |f(z_n) - f(w_n)| \ge \varepsilon.$$

Using the above machinery, we see that $z_{\sigma n} \to \alpha$ and $w_{\sigma n} \to \alpha$ force $f(z_{\sigma n}) \to f(\alpha)$ and $f(w_{\sigma n}) \to f(\alpha)$ by continuity of f, but the sequences $f(z_{\sigma n})$ and $f(w_{\sigma n})$ are supposed to be ε far apart! Explicitly, we can find sufficiently large N_1 and N_2 such that

$$\begin{aligned} n > N_1 &\implies |f(z_{\sigma n}) - \alpha| < \varepsilon/4, \\ n > N_2 &\implies |f(w_{\sigma n}) - \alpha| < \varepsilon/4, \end{aligned}$$

which by the triangle inequality means that any $n > \max\{N_1, N_2\}$ will give

$$|f(z_{\sigma n}) - f(w_{\sigma n})| \le |f(z_{\sigma n}) - \alpha| + |f(w_{\sigma n}) - \alpha| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon,$$

which is a contradiction to the construction of $z_{\sigma n}$ and $w_{\sigma n}$. ■

2.8.3 Uniform Convergence

We next talk about uniform convergence for functions. Here is our starter pack.

Definition 2.128 (Sequence of functions)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset. Then a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is a function $\varphi : \mathbb{N} \to (X \to \mathbb{C})$. Namely, for each $n \in \mathbb{N}$, we are given a function $\varphi(n) : X \to \mathbb{C}$.

Definition 2.129 (Pointwise convergence)**.** Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then $\{f_n\}$ *converges to some* $g : X \to \mathbb{C}$ *pointwise* if and only if, for each $z \in X$, we have $f_n(z) \to g(z)$ as $n \to \infty$. We write this as $f_n \to g$.

This is called pointwise convergence because we are only checking convergence at each individual point $z \in X$, without caring about the larger structure of the function. This will cause problems later but not now.

Definition 2.130 (Bounded)**.** We say that a function $f : X \to \mathbb{C}$ is *bounded* if and only if $f(X) \subseteq \mathbb{C}$ is bounded. In other words, there is some $M > 0$ so that $f(X) \subseteq B(0, M)$.

Definition 2.131 (Uniform convergence)**.** Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then $\{f_n\}$ *converges to some* $g : X \to \mathbb{C}$ *pointwise* if and only if, for each $\varepsilon > 0$, there is some N so that

$$n > N \implies |f_n(z) - g(z)| < \varepsilon$$

for each $z \in X$.

The uniformity here is that the value of N is no longer allowed to depend on z. Here is an alternate definition.

Proposition 2.132. Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ some function. Then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to g if and only if

$$\lim_{n\to\infty} \sup_{z\in X}\{|f_n(z) - g(z)|\} = 0.$$

Proof. We take the directions independently.

- In the forward direction, we know that there is an N_1 so that $n > N_1$ implies each $z \in X$ has

$$|f_n(z) - g(z)| < 1.$$

  In particular, for $n > N_1$, the set $\{|f_n(z) - g(z)| : z \in X\}$ is bounded above by 1, so the supremum will exist; set $y_n := \sup\{|f_n(z) - g(z)| : z \in X\}$ so that we want to show $y_n \to 0$ as $n \to \infty$.

  More generally, for any $\varepsilon > 0$, there exists some N so that $n > N_0$ implies

$$|f_n(z) - g(z)| < \varepsilon/2.$$

  So $n > \max\{N_0, N_1\}$, we will have that $y_n = \sup\{|f_n(z) - g(z)| : z \in X\}$ both exists and has $y_n \le \varepsilon/2 < \varepsilon$. So we do get $y_n \to 0$ as $n \to \infty$.

- In the reverse direction, set $y_n := \sup\{|f_n(x) - g(x)| : x \in X\}$ so that $y_n \to 0$ as $n \to \infty$. Namely, for each $\varepsilon > 0$, there exists some N so that $n > N$ has $y_n < \varepsilon$. In particular, we see $n > N$ has

$$|f_n(x_0) - g(x_0)| \le \sup\{|f_n(x) - g(x)| : x \in X\} = y_n < \varepsilon$$

  for each $x_0 \in X$. So indeed, f_n converges to g uniformly. ■

2.8.4 Distances Between Functions

Later in life it will be nice to view functions as forming a metric under $d(f, g) := \sup\{|f(x) - g(x)|\}$. However, this supremum need not only exist; here is one condition in which it does.

Lemma 2.133. Fix $f, g : X \to \mathbb{C}$ bounded functions. Then $\sup\{|f(x) - g(x)| : x \in X\}$ exists.

Proof. Because f is bounded, there exists M_f so that each $x \in X$ has $|f(x)| < M_f$. Similarly, because g is bounded, there exists M_g so that each $x \in X$ has $|g(x)| < M_g$. It follows that, for each $x \in X$,

$$|f(x) - g(x)| \le |f(x)| + |g(x)| \le M_f + M_g,$$

so the set $\{|f(x) - g(x)| : x \in X\}$ is bounded above and in particular has a supremum. ■

Proposition 2.134. Fix $f, g, h : X \to \mathbb{C}$ all bounded functions. Then

$$\sup_{x \in X}\{|f(x) - h(x)|\} \le \sup_{x \in X}\{|f(x) - g(x)|\} + \sup_{x \in X}\{|g(x) - h(x)|\}.$$

Note that all the suprema above exist by Lemma 2.133

Proof. The point is to reduce to the typical triangle inequality. Indeed, for any $x \in X$, we see that

$$|f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)|.$$

Thus,

$$\begin{aligned}\sup_{x \in X}\{|f(x) - h(x)|\} &\le \sup_{x \in X}\{|f(x) - g(x)| + |g(x) - h(x)|\} \\ &\le \sup_{x \in X}\{|f(x) - g(x)|\} + \sup_{x \in X}\{|g(x) - h(x)|\},\end{aligned}$$

which is what we wanted. We have used the fact that $\sup(A + B) \le \sup A + \sup B$ for $A, B \subseteq \mathbb{R}$, which is not hard to show: if $a + b \in A + B$, then $a \le \sup A$ and $b \le \sup B$, so $a + b \le \sup A + \sup B$; thus, $\sup(A + B) \le \sup A + \sup B$.[4] ■

Remark 2.135 (Nir)**.** Viewing Lemma 2.133 as providing a distance metric, the above proposition proves the triangle inequality for this metric.

We can also build a Cauchy criterion for uniform convergence.

Proposition 2.136. A sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then $\{f_n\}_{n \in \mathbb{N}}$ converges to some function uniformly if and only if the quantity $\sup_{x \in X}\{f_n(x) - f_m(x)\}$ exists and, for any $\varepsilon > 0$, there exists some N so that $n, m > N$ implies

$$\sup_{x \in X}\{|f_n(x) - f_m(x)|\} < \varepsilon$$

for any $x \in X$.

We note that the hypothesis that the supremum exists can be removed if the functions are presupposed to be bounded.

Proof. We again take the directions independently.

[4] In fact, $\sup A + \sup B \le \sup(A+B)$ as well. We show $\sup A \le \sup(A+B) - \sup B$. Fixing $a \in A$, we need $a \le \sup(A+B) - \sup B$, so we show $\sup B \le \sup(A + B) - a$. Fixing $b \in B$, we need $b \le \sup(A + B) - a$, which is clear.

- Suppose that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges to a function g uniformly. Then, for any $\varepsilon > 0$, we are promised some N so that $n > N$ will have

  $$|f_n(x) - g(x)| < \varepsilon/4$$

  for any $x \in X$. In particular, for any $n, m > N$, we see

  $$|f_n(x) - f_m(x)| < |f_n(x) - g(x)| + |f_m(x) - g(x)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

  so

  $$\sup_{x\in X}\{|f_n(x) - f_m(x)|\} \le \frac{\varepsilon}{2} < \varepsilon,$$

  which is what we wanted.

- There are two steps.
  - We begin by constructing g. Well, for each $x \in X$, we note that any $\varepsilon > 0$ will have some N so that $n, m > N$ implies

    $$|f_n(x) - f_m(x)| \le \sup_{x\in X}\{|f_n(x) - f_m(x)|\} < \varepsilon,$$

    so the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges in $\mathbb{C}$. We define $g(x)$ to be the limit of $f_n(x)$ as $n \to \infty$.
  - Next we show the uniform convergence. Fix some $\varepsilon > 0$. Then we are promised some N so that $n, m > N$ has

    $$\sup_{x\in X}\{|f_n(x) - f_m(x)|\} < \varepsilon.$$

    In particular, for any $x \in X$

    $$f_n(x) - g(x) = \lim_{m\to\infty}(f_n(x) - f_m(x)),$$

    so because $z \mapsto |z|$ is continuous, any $n > N$ will have

    $$|f_n(x) - g(x)| = \lim_{m\to\infty}|f_n(x) - f_m(x)| \le \lim_{m\to\infty}\sup_{z\in X}\{|f_n(x) - f_m(x)|\} < \lim_{m\to\infty}\varepsilon = \varepsilon,$$

    where the last inequality holds by taking m sufficiently large (that is, $m > N$). So we have been provided uniform convergence. ■

Remark 2.137 (Nir)**.** In the language of metric topology, the above proposition asserts that the space of (bounded) functions is metrically complete. For this, one must technically show that $\{f_n\}_{n\in\mathbb{N}}$ being bounded implies that the convergent g is bounded, but this is not hard: there is N so that $n > N$ has $|f_n(x) - g(x)| < 1$ for each $x \in X$.

Remark 2.138. In lecture, Professor Morrow asserted that we require these functions to be bounded. I do not think this is the case; indeed, the above proof never uses this hypothesis.

We close with one result which shows that uniform continuity is nice.

Proposition 2.139. Fix $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ all continuous at some $x \in X$. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to some function $g : X \to \mathbb{C}$, then g is also continuous.

Proof. The idea is to well-approximate g by f_ns. Fix any $\varepsilon > 0$. By the uniform convergence, there will be some N so that

$$|f_n(z) - g(z)| < \varepsilon/3$$

for any $n > N$ and $z \in X$; fix some $n > N$. Because f_n is continuous, we are promised some $\delta > 0$ (allowed to vary with our chosen $x \in X$) so that

$$|z - x| < \delta \implies |f_n(z) - f_n(x)| < \varepsilon/3$$

for any $z \in X$. Well, if $|z - x| < \delta$, then the triangle inequality gives

$$|g(z) - g(x)| \le |g(z) - f_n(z)| + |f_n(z) - f_n(x)| + |f_n(x) - g(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which is what we needed. ■

Remark 2.140 (Nir)**.** In fact, if the $\{f_n\}_{n\in\mathbb{N}}$ are uniformly continuous, then g will also be uniformly continuous. The argument is similar.

THEME 3

HOLOMORPHIC AND ANALYTIC FUNCTIONS

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.

—Charles Hermite

3.1 February 11

The wheel of time marches on. Today, we actually start talking about complex analysis.

3.1.1 Differentiability

We are going to talk about holomorphic functions.

Convention 3.1. We set Ω to be some open subset of $\mathbb{C}$.

This gives the following definition.

Definition 3.2 (Differentiable)**.** Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a function. Then f is *complex differentiable at* $z_0 \in \Omega$ *with derivative* $\alpha \in \mathbb{C}$ if and only if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \alpha.$$

We write this as $f'(z_0) = \alpha$.

If f' is itself a differentiable function, then f would be "twice" differentiable, and we denote this function by f''. In general, if f can be differentiated n times, we denote the corresponding function by $f^{(n)}$.

Warning 3.3. In the definition of complex differentiability, we are taking the limit with $h \in \mathbb{C}$, not $h \in \mathbb{R}$. This will make complex differentiability significantly more structured.

Differentiability gives rise to the following definition.

Definition 3.4 (Holomorphic, entire)**.** Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a function. Then f is *holomorphic on* Ω if and only if f is complex differentiable at each $z_0 \in \mathbb{C}$. If $\Omega = \mathbb{C}$, then we say f is *entire*.

Here is a small usual lemma.

Lemma 3.5. Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a function. Then if f is differentiable at $z_0 \in \Omega$, then f is continuous at $z_0 \in \mathbb{C}$.

Proof. We compute that

$$\begin{aligned}\lim_{z\to z_0}\left(f(z)-f(z_0)\right) &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\lim_{z\to z_0}(z-z_0)\\ &\overset{*}{=} \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\cdot\lim_{z\to z_0}(z-z_0)\\ &= f'(z_0)\cdot 0\\ &= 0.\end{aligned}$$

It follows by rearrangement that $\lim_{z\to z_0} f(z) = f(z_0)$, which is what we wanted. Notably, $\overset{*}{=}$ sets $h := z - z_0$. ■

3.1.2 Basic Properties

As usual, differentiable functions have an arithmetic.

Proposition 3.6. Fix an open subset $\Omega \subseteq \mathbb{C}$ and $f, g : \Omega \to \mathbb{C}$ functions differentiable at $z_0 \in \mathbb{C}$.

(a) We have that $(af + bg)'(z_0) = af'(z_0) + bg'(z_0)$, where $a, b \in \mathbb{C}$.

(b) We have that $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.

(c) If $g'(z_0) \neq 0$, then

$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. We copy the proofs from real analysis.

(a) We check that

$$\begin{aligned}\lim_{h\to 0}\frac{(af+bg)(z_0+h)+(af+bg)(z_0)}{h} &= a\cdot\lim_{h\to 0}\frac{f(z_0+h)+f(z_0)}{h}+b\cdot\lim_{h\to 0}\frac{g(z_0+h)-g(z_0)}{h}\\ &= a\cdot f'(z_0)+b\cdot g'(z_0),\end{aligned}$$

which is what we wanted.

(b) The key idea is to add and subtract $f(z_0)g(z_0+h)$. Indeed, we see

$$\begin{aligned}\lim_{h\to 0}\frac{(fg)(z_0+h)-(fg)(z_0)}{h} &= \lim_{h\to 0}\frac{f(z_0+h)g(z_0+h)-f(z_0)g(z_0+h)}{h}\\ &\quad+\lim_{h\to 0}\frac{f(z_0)g(z_0+h)-f(z_0)g(z_0)}{h}\\ &= \left(\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\right)\left(\lim_{h\to 0}g(z_0+h)\right)\\ &\quad+f(z_0)\left(\lim_{h\to 0}\frac{g(z_0+h)-g(z_0)}{h}\right)\\ &= f'(z_0)g(z_0)+f(z_0)g'(z_0),\end{aligned}$$

which is what we wanted.

(c) This will follow from applying the product rule to $f \cdot \frac{1}{g}$, where we can compute the derivative of $\frac{1}{g}$ by the chain rule. We refer to Eterović's notes for the details. ■

Remark 3.7 (Nir)**.** Technically part (c) will require us to compute the derivative of $f(z) := \frac{1}{z}$ for $z \neq 0$ to finish the proof. Well, for any $z \in \mathbb{C} \setminus \{0\}$, we see that

$$\frac{f(z+h)-f(z)}{h} = \frac{\frac{1}{z+h} - \frac{1}{z}}{h} = \frac{z-(z+h)}{hz(z+h)} = -\frac{1}{z(z+h)}.$$

Taking $h \to 0$ reveals that the derivative is in fact $f'(z) = -\frac{1}{z^2}$.

Let's give some examples of entire functions.

Exercise 3.8. Fix n some positive integer. We show that the function $f(z) := z^n$ is entire with derivative $f'(z) := nz^{n-1}$.

Proof. We employ the usual proof involving the binomial theorem. Note that

$$f(z+h) = (z+h)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k,$$

so

$$\frac{f(z+h)-f(z)}{h} = \sum_{k=1}^{n} \binom{n}{k} z^{n-k} h^{k-1},$$

where notably the $k = 0$ term was killed by the $-f(z)$. Thus,

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_{k=1}^{n} \binom{n}{k} z^{n-k} \left(\lim_{h\to 0} h^{k-1} \right),$$

but all terms except $k = 1$ will now vanish as $h \to 0$, so we are left with nz^{n-1} as our limit. ■

Remark 3.9 (Nir)**.** One could also show this by induction, using the product rule.

Corollary 3.10. Any polynomial function is entire.

Proof. Polynomials are (finite) linear combinations of the monomials $f_n(z) := z^n$, so this follows from combining the above two results. ■

3.1.3 Advanced Properties

We also have a notion of L'Hôpital's rule.

Proposition 3.11 (L'Hôpital's rule)**.** Fix $\Omega \subseteq \mathbb{C}$ an open subset with $f, g : \Omega \to \mathbb{C}$ holomorphic functions. Then, given $z_0 \in \Omega$ with $f(z_0) = g(z_0) = 0$ while $g'(z_0) \neq 0$, we have that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof. Note that, because $f(z_0) = g(z_0) = 0$, we see that

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)}{z - z_0} \qquad \text{and} \qquad g'(z_0) = \lim_{z\to z_0} \frac{g(z)}{z - z_0}.$$

Dividing, we see that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} = \lim_{z\to z_0} \frac{f'(z_0)}{g'(z_0)} = \frac{f'(z_0)}{g'(z_0)},$$

which is what we wanted. ■

Remark 3.12 (Nir)**.** The above proof technically does not work because we have not ruled out the possibility that g might vanish arbitrarily close to z_0, thus making the limits not actually make sense. We will not fix this problem, but we will remark that a holomorphic function will only have finitely many zeroes on a compact set, which we could use to create a neighborhood for z_0 on which g doesn't vanish.

And here is our chain rule.

Proposition 3.13 (Chain rule)**.** Fix Ω and U open subsets of $\mathbb{C}$ with functions $f : \Omega \to U$ and $g : U \to \mathbb{C}$. Further, suppose that f is differentiable at $z_0 \in \Omega$ and that g is differentiable at $f(z_0) \in U$. Then $(g \circ f)$ is differentiable at z_0 with derivative

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. This proof is long, so we will try to be brief. The main idea is to consider the auxiliary function $r : U \setminus \{f(z_0)\} \to \mathbb{C}$ defined by

$$r(w) := \frac{g(w) - g(f(z_0))}{w - f(z_0)} - g'(f(z_0)).$$

We extend r to $f(z_0)$ by setting $r(f(z_0)) := 0$. Now, the differentiability of g at $f(z_0)$ implies that

$$\lim_{z\to z_0} \frac{g(z) - g(f(z_0))}{z - z_0} = g'(f(z_0)),$$

so in particular rearranging implies that r is continuous on at $f(z_0) \in U$.

The reason we used the letter r is that we should think of r as a remainder term. Indeed, we see

$$g(w) - g(f(z_0)) = g'(f(z_0))(w - f(z_0)) + r(w)(w - f(z_0)).$$

Plugging in $w = f(z)$, we get

$$g(f(z)) - g(f(z_0)) = g'(f(z_0))(f(z) - f(z_0)) + r(f(z))(f(z) - f(z_0)),$$

so

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = g'(f(z_0)) \cdot \frac{f(z) - f(z_0)}{z - z_0} + r(f(z)) \cdot \frac{f(z) - f(z_0)}{z - z_0}.$$

Sending $z \to z_0$ makes the rightmost term vanish by continuity because $r(f(z_0)) = 0$ and the limit is $f'(z_0)$, so we are left with

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0),$$

which is what we wanted. ■

Remark 3.14 (Nir)**.** Let's complete the proof of quotient rule. Note that the derivative of $\frac{1}{g(z)}$ will be, by the chain rule, $-\frac{1}{g(z)^2} \cdot g'(z)$. Thus, the derivative of $\frac{f(z)}{g(z)} = f(z) \cdot \frac{1}{g(z)}$ will be

$$f'(z) \cdot \frac{1}{g(z)} - f(z) \cdot \frac{g'(z)}{g(z)^2} = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

And we finish with a result which is less common in real analysis, essentially saying that differentiable functions are "approximately" linear.

Proposition 3.15 (Carathéodory)**.** Fix $\Omega \subseteq \mathbb{C}$ an open subset with a function $f : \Omega \to \mathbb{C}$ and point $z_0 \in \Omega$. Then f is differentiable at z_0 if and only if there exists a function $h : \Omega \to \mathbb{C}$ which is continuous at z_0 such that

$$f(z) - f(z_0) = h(z)(z - z_0).$$

In particular, $h(z_0) = f'(z_0)$.

Proof. We show the directions independently.

- Suppose f is differentiable at z_0. We construct the function h manually. We define

$$h(z) := \begin{cases} (f(z) - f(z_0))/(z - z_0) & z \in \Omega \setminus \{z_0\}, \\ f'(z_0) & z = z_0. \end{cases}$$

  In particular, we note that h is continuous at z_0 because $h(z) \to f'(z_0)$ as $z \to z_0$ by differentiability of f.

- Suppose h is such a function. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} h(z) = h(z_0)$$

  by continuity. Formally, the first equality is holding for the limit in $\Omega \setminus \{z_0\}$, and the second equality is continuity for $h|_{\Omega \setminus \{z_0\}}$.

To finish, we note that the second part shows that $h(z_0) = f'(z_0)$. ■

3.2 February 14

Happy Valentine's Day, I suppose. Homework #4 is due on Sunday. Homework #5 will be released on Friday.

3.2.1 Motivating Cauchy–Riemann Equations

Today we're talking about the Cauchy–Riemann equations.

Idea 3.16. The Cauchy–Riemann equations are necessary conditions for a function to be holomorphic.

In fact, they will be sufficient as well, but we will only see this next class.

Throughout today's class, we will fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset. We recall that a function $f : \Omega \to \mathbb{C}$ is "differentiable" at some $z_0 \in \Omega$ if and only if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If it exists, we denoted it by $f'(z_0)$, though we will not assume it exists yet. If we fix $\Delta z := z - z_0$, then we can write the above as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{\Delta z}.$$

Now, to motivate our discussion, we recall that under the isomorphism $\mathbb{C} \cong \mathbb{R}^2$ with basis $\{1, i\}$, we can define $u(x,y) := \operatorname{Re} f(x+yi)$ and $v(x,y) := \operatorname{Im} f(x+yi)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$ so that

$$f(x+yi) = u(x,y) + iv(x,y).$$

The point of this is to encode some geometry directly into our set-up.

> **Example 3.17.** Given $f(z) = z^2$, we can plug in
>
> $$f(x+yi) = (x+yi)^2 = \underbrace{x^2 - y^2}_{u} + i \cdot \underbrace{2xy}_{v}.$$

Now that we're moving things to $\mathbb{R}^2$, we will fix $z_0 := x_0 + y_0 i$ for $x_0, y_0 \in \mathbb{R}$ with $z = x + yi$ so that $\Delta z = (x - x_0) + (y - y_0)i = \Delta x + i\Delta y$. And for a little more convenience, we fix $\Delta w := f(z_0 + z) - f(z_0)$ so that

$$f'(z_0) \stackrel{?}{=} \lim_{z \to z_0} \frac{\Delta w}{\Delta z},$$

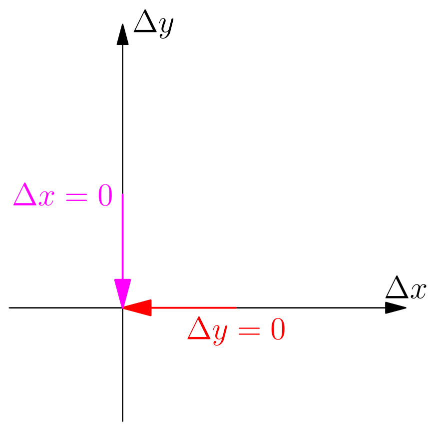
if the limit exists. Expanding out f into real and imaginary parts, we find

$$\Delta w := \big(u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\big) + i\big(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\big).$$

Now assume that f is inf act differentiable at z_0 so that $f'(z_0)$ will actually exist. Our key idea to continue is to split up the limit into real and imaginary parts because it will exist if and only if the limits of the real and imaginary parts exist. So we note

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to 0} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x, \Delta y) \to 0} \operatorname{Im}\left(\frac{\Delta w}{\Delta z}\right) \end{aligned} \tag{$*$}$$

We will now compute this limit in two ways to get the Cauchy–Riemann equations, as follows.



These are probably the easiest two limits that we could think of, so it's nice that they will be so useful. Anyways, here is our working out.

- We set $\Delta y = 0$ so that $\Delta z = \Delta x$. This gives

$$\frac{\Delta w}{\Delta x} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \cdot \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

On one hand, we can use (∗) to show the real part comes out to

$$\operatorname{Re} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Re}\left(\frac{\Delta w}{\Delta x}\right) = \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x}.$$

This limit must exist because f is differentiable at z_0, and when this limit exists, the rightmost limit is called the partial derivative $u_x(x_0, y_0)$.

On the other hand, the imaginary part comes out to

$$\operatorname{Im} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Im}\left(\frac{\Delta w}{\Delta x}\right) = \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x},$$

which comes out to $v_x(x_0, y_0)$ because we know that the limit exists.

So in total, we see $f'(z_0) = u_x(x_0, y_0) + i \cdot v_x(x_0, y_0)$.

- We set $\Delta x = 0$ so that $\Delta z = i\Delta y$. Be warned that an unexpected sign is about to appear from this i. This time we get

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i\cdot\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}.$$

To "rationalize" the deminators, we write

$$\frac{\Delta w}{\Delta z} = \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\Delta y} - i\cdot\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y},$$

where we are using $1/i = -i$. Note that the us and vs have swapped from the last computation!

We now compute our limits. On one hand,

$$\operatorname{Re} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\Delta y},$$

which is $v_y(x_0, y_0)$ because the limit exists. On the other hand,

$$\operatorname{Im} f'(z_0) = \lim_{(\Delta x,\Delta y)\to 0} \operatorname{Im}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y\to 0} -\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y},$$

which is $-u_y(x_0, y_0)$ because the limit exists.

So in total, we see $f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

Remark 3.18. Either equation itself is pretty useful to actually compute formulae for the derivatives.

Synthesizing, we see

$$f'(z_0) = u_x(x_0, y_0) + i\cdot v_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Comparing real and imaginary parts, we get the following.

Theorem 3.19 (Cauchy–Riemann)**.** Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f : \Omega \to \mathbb{C}$ a function differentiable at some $z_0 = x_0 + y_0 i \in \mathbb{C}$. If we write $f(x + yi) = u(x, y) + i(x, y)$, then

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0), \\ v_x(x_0, y_0) = -u_y(x_0, y_0). \end{cases}$$

In fact, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

Proof. This follows from the above discussion. ■

3.2.2 Examples

Let's see some examples to be convinced of the utility of Theorem 3.19. Let's start by checking that something is differentiable.

Example 3.20. Take $f(z) = z^2$ so that

$$(x + yi) = (x + yi)^2 = \left(x^2 - y^2\right) + i(2xy)$$

so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ has $f(x + yi) = u(x, y) + iv(x, y)$. We know that f is entire (it's impossible), so picking up any $z = x + yi \in \mathbb{C}$, we compute

$$u_x(x, y) = 2x = v_y(x, y) \qquad \text{and} \qquad v_x(x, y) = 2y = -(-2y) = -u_y(x_0, y_0),$$

verifying Theorem 3.19. In fact, we can see that $f'(z) = u_x(x, y) + v_x(x, y) = 2x + 2yi = 2z$.

And now let's see something which isn't differentiable.

Example 3.21. Take $f(z) = |z|^2$ so that

$$f(x + yi) = |x + yi|^2 = (x + yi)(x - yi) = x^2 + y^2,$$

which only has a real part! Namely, we have $u(x, y) = x^2+y^2$ and $v(x, y) = 0$ to make $f(x+yi) = u(x, y)+ iv(x, y)$. Now suppose for the sake of contradiction that f were differentiable at some $z = x + yi \in \mathbb{C}$. Then we are forced into

$$2x = u_x(x, y) = v_x(x, y) = 0 \qquad \text{and} \qquad 0 = v_x(x, y) = -u_y(x, y) = -2y,$$

which means $x = y = 0$. So f is differentiable at nowhere outside $\mathbb{C} \setminus \{0\}$.

Observe that the above example does not show that f is differentiable at $0 \in \mathbb{C}$, though this is true. To be explicit, Theorem 3.19 does not tell us that satisfying the Cauchy–Riemann equations implies differentiability.

Remark 3.22. Extending Example 3.21, we can show that the only entire real-valued function is constant.

Let's also close with an application of Theorem 3.19.

Corollary 3.23. Fix $\Omega \subseteq \mathbb{C}$ a connected nonempty open subset and $f : \Omega \to \mathbb{C}$ a function differentiable on all of Ω so that $f'(z) = 0$ for all $z \in \Omega$. Then f is constant.

Proof. By Theorem 3.19, we see that, for any $z = x + yi$, we see

$$u_x(x, y) = v_y(x, y) = \operatorname{Re} f'(z) = 0 \qquad \text{and} \qquad v_x(x, y) = -u_y(x, y) = \operatorname{Im} f'(z) = 0.$$

In particular, for some function $g : C \to \mathbb{R}$ for some $C \subseteq \mathbb{R}^2$ connected and open, having $g_x = 0$ forces g to be constant as a function of x on any connected horizontal line, and $g_y = 0$ forces g to be constant as a function of y.

Now, because any path between two points in an open subset can be approximated by vertical and horizontal line segments contained in neighborhoods of points, we see that the endpoints of any path in C must have the same value.[1] But C is open and connected and hence path-connected, so C any two points can be connected by path, so g must be constant on all of C.

Returning to f, we see that u and v will be constant on the embedding of Ω into $\mathbb{R}^2$ (recall that $\mathbb{C} \cong \mathbb{R}^2$ topologically, so $\Omega \subseteq \mathbb{R}^2$ remains open and connected), so f is constant on Ω. This is what we wanted. ■

[1] Please don't ask me to rigorize this.

Remark 3.24. We do need the connected hypothesis: we could take $\Omega = \mathbb{C} \setminus \mathbb{R}$ and with $f(z) = 1_{\operatorname{Re} z > 0}$.

3.3 February 16

We talk more about the Cauchy–Riemann equations today. For our announcements, Homework #4 is due on Sunday. There is a midterm next Friday; we will get a review sheet and some practice problems in the next few days. There will be no homework, and there will be extra office hours.

3.3.1 Introducing Sufficient Conditions

The slogan for today as follows.

Idea 3.25. The Cauchy–Riemann equations provide a sufficient condition for differentiability.

Recall our theorem.

Theorem 3.19 (Cauchy–Riemann). Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f : \Omega \to \mathbb{C}$ a function differentiable at some $z_0 = x_0 + y_0 i \in \mathbb{C}$. If we write $f(x + yi) = u(x, y) + i(x, y)$, then

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0), \\ v_x(x_0, y_0) = -u_y(x_0, y_0). \end{cases}$$

In fact, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$.

These are sufficient conditions for differentiability. Today we are discussing necessary conditions for differentiability.

Theorem 3.26. Fix $\Omega \subseteq \mathbb{C}$ a nonempty open subset and $f : \Omega \to \mathbb{C}$ a function. Writing $f(x + yi) = u(x, y) + iv(x, y)$ and fixing some $z_0 := x_0 + y_0 i$, then suppose we have the following.

- We have u_x, u_y, v_x, v_y all exist and are continuous (!).
- We have
$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0), \\ v_x(x_0, y_0) = -u_y(x_0, y_0). \end{cases}$$

Then f is differentiable at z_0.

Remark 3.27. It is possible to construct functions which are differentiable at z_0 but do not have continuous first partial derivatives.

Let's do some examples of Theorem 3.26 to see its utility.

Example 3.28. Fix $f(x + yi) = x^2 + y + i\left(y^2 - x\right)$. Here, $u(x, y) = x^2 + y$ and $v(x, y) = y^2 - x$, so we see

$$u_x(x, y) = 2x, \quad u_y(x, y) = 1, \quad v_x(x, y) = -1, \quad \text{and} \quad v_y(x, y) = 2y.$$

So all first partial derivatives are continuous. To satisfy the Cauchy–Riemann equations, we see that we need $u_x = v_y$ and $u_y = -v_x$, which is equivalent to $2x = 2y$ and $1 = --1$. It follows from Theorem 3.26 that f is differentiable on the line $y = x$, and f is not differentiable anywhere else by Theorem 3.19.

Remark 3.29. Another type of question is to be given $u(x, y)$ and be asked for what $v(x, y)$ might be.

3.3.2 Proving Sufficient Conditions

Let's go ahead and prove Theorem 3.26.

Proof of Theorem 3.26. As with last time, we fix $\Delta z := z - z_0$ and $\Delta x = x - x_0$ and $\Delta y = y - y_0$ so that our difference quotient is

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \underbrace{\frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z}}_{\Delta u/\Delta z :=} + i \cdot \underbrace{\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta z}}_{\Delta v/\Delta z :=}.$$

So our goal is to show that

$$\lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \cdot \frac{\Delta v}{\Delta y} \right)$$

exists and is equal to $u_x(x_0, y_0) + iv_x(x_0, y_0)$. So we need to force our first partial derivatives into the limit.

We start with $\Delta u/\Delta z$. To make our partial derivatives appear, we write

$$\begin{aligned} \frac{\Delta u}{\Delta z} &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z} \\ &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)}{\Delta z} + \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z}. \end{aligned}$$

To get our partial derivatives, we apply the Mean value theorem (!): define

$$F(x) := u(x, y_0 + \Delta y) \qquad \text{and} \qquad F(y) := u(x_0, y).$$

We do our applications one at a time.

- Note that $F(x)$ is differentiable everywhere from x_0 to $x_0 + \Delta x$, so the Mean value theorem provides some x_0^* between x_0 and $x_0 + \Delta x$ such that

$$F(x_0 + \Delta x) - F(x_0) = F'(x_0^*)\Delta x.$$

- Similarly, $F(y)$ is differentiable everywhere from y_0 to $y_0 + \Delta y$, so the Mean value theorem provides some y_0^* between y_0 and $y_0 + \Delta y$ such that

$$F(y_0 + \Delta x) - F(y_0) = F'(y_0^*)\Delta y.$$

Synthesizing and plugging in, we get

$$\frac{\Delta u}{\Delta z} = \frac{u_x(x_0^*, y_0)\Delta x}{\Delta z} + \frac{u_y(x_0, y_0^*)\Delta y}{\Delta z}.$$

We now use continuity of our first partial derivative. Our hope is that sending $\Delta z \to 0$ will send $u_x(x_0^*, y_0) \to u_x(x_0, y_0)$ and $u - y(x_0, y_0^*) \to u_y(x_0, y_0)$. To show this, we show the difference will be small. We write

$$\frac{\Delta u}{\Delta z} = \frac{u_x(x_0, y_0)\Delta x}{\Delta z} + \frac{u_y(x_0, y_0)\Delta y}{\Delta z} + E_{ux} + E_{uy},$$

where

$$E_{ux} = \big(u_x(x_0^*, y_0) - u_x(x_0, y_0)\big)\frac{\Delta x}{\Delta z} \quad \text{and} \quad E_{uy} = \big(u_y(x_0, y_0^*) - u_y(x_0, y_0)\big)\frac{\Delta y}{\Delta z}.$$

We now remark that we can repeat the entire above argument for $\frac{\Delta v}{\Delta z}$. Namely, running the above machinery lets us write

$$\frac{\Delta v}{\Delta z} = \frac{v_x(x_0, y_0)\Delta x}{\Delta z} + \frac{v_y(x_0, y_0)\Delta y}{\Delta z} + E_{vx} + E_{vy},$$

where

$$E_{vx} = \left(u_x(x_0^{**}, y_0) - u_x(x_0, y_0)\right)\frac{\Delta x}{\Delta z} \quad \text{and} \quad E_{vy} = \left(u_y(x_0, y_0^{**}) - u_y(x_0, y_0)\right)\frac{\Delta y}{\Delta z}.$$

We now show that the various $E_\bullet$ terms vanish as $\Delta z \to 0$. Note that, as $\Delta z \to 0$, the following happen.

- Because x_0^* and x_0^{**} are bounded between x_0 and $x_0 + \Delta x$, they will approach x_0.
- Because y_0^* and y_0^{**} are bounded between y_0 and $y_0 + \Delta y$, they will approach y_0.
- We will have $\left|\frac{\Delta x}{\Delta z}\right| \le 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \le 1$ by direct expansion of the norm because $\operatorname{Re} \Delta z = \Delta x$ and $\operatorname{Im} \Delta z = \Delta y$.

It follows that each of the $E_\bullet$ do indeed vanish as $\Delta z \to 0$. For example,

$$\left|\left(u_x(x_0^*, y_0) - u_x(x_0, y_0)\right)\frac{\Delta x}{\Delta z}\right| \le \left|u_x(x_0^*, y_0) - u_x(x_0, y_0)\right|$$

will go to 0 as $\Delta z \to 0$ by the continuity of u_x at (x_0, y_0).

Now we return to our difference quotient. We see

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \cdot \frac{\Delta v}{\Delta z}\right) \\
&= \lim_{\Delta z \to 0} \left(\frac{u_x(x_0, y_0)\Delta x}{\Delta z} + \frac{u_y(x_0, y_0)\Delta y}{\Delta z} + i \cdot \frac{v_x(x_0, y_0)\Delta x}{\Delta z} + i \cdot \frac{v_y(x_0, y_0)\Delta y}{\Delta z}\right) \\
&\qquad + \lim_{\Delta z \to 0} E_{ux} + \lim_{\Delta z \to 0} E_{uy} + \lim_{\Delta z \to 0} E_{vx} + \lim_{\Delta z \to 0} E_{vy} \\
&= \lim_{\Delta z \to 0} \left(\frac{u_x(x_0, y_0)\Delta x}{\Delta z} + \frac{u_y(x_0, y_0)\Delta y}{\Delta z} + i \cdot \frac{v_x(x_0, y_0)\Delta x}{\Delta z} + i \cdot \frac{v_y(x_0, y_0)\Delta y}{\Delta z}\right),
\end{aligned}$$

using the fact that our error terms all vanish. At this point we use the Cauchy–Riemann equations. We see

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \left(\frac{u_x(x_0, y_0)\Delta x}{\Delta z} - \frac{v_x(x_0, y_0)\Delta y}{\Delta z} + i \cdot \frac{v_x(x_0, y_0)\Delta x}{\Delta z} + i \cdot \frac{u_x(x_0, y_0)\Delta y}{\Delta z}\right) \\
&= \lim_{\Delta z \to 0} \left(u_x(x_0, y_0) \cdot \frac{\Delta x + i\Delta y}{\Delta z}\right) + i \cdot \lim_{\Delta z \to 0} \left(v_x(x_0, y_0) \cdot \frac{\Delta x + i\Delta y}{\Delta z}\right),
\end{aligned}$$

which finishes after evaluating our first partial derivatives. ■

3.4 February 18

Good morning everyone. Here are some announcements.

- Homework #4 is due on Sunday.
- Next Friday is our midterm. A review sheet has been posted. Some practice problems and a practice midterm will be released today or tomorrow.
- Next week will have office hours every day.
- Next Wednesday will be a review class.

3.4.1 Power Series

Today we are building towards a discussion of analytic functions. We won't actually define what "analytic" means, but it will be important, so we will spend today setting up the definitions and results.

Definition 3.30 (Complex power series)**.** A *complex power series* is a formal expression of the form

$$S(z) := \sum_{k=0}^{\infty} a_k x^k$$

where $\{a_k\}_{k\in\mathbb{N}} \subseteq \mathbb{C}$ and z is a (formal) variable taking complex values.

definition Our main goal for today is to be able to answer the following question.

Question 3.31. For which z will $S(z)$ converge?

The answer to this question is essentially the same as for real analysis: it's the radius of convergence.

Definition 3.32 (Radius of convergence)**.** The *radius of convergence* of a complex power series $S(z) = \sum_{k=0}^{\infty} a_k z^k$ is defined to be equal to the radius of convergence of the real power series

$$T(x) = \sum_{k=0}^{n} |a_k| x^k.$$

Concretely, we define

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

We should probably check convergence in the radius of convergence.

Proposition 3.33. Fix a complex power series $S(z) = \sum_{k=0}^{\infty} a_k z^k$ with radius of convergence R. Then the following hold.

(a) The sequence of partial sums $\sum_{k=0}^{n} \left|a_k z^k\right|$ converge for any z with $|z| < R$. In other words, $S(z)$ converges absolutely.

(b) The series $S(z)$ will diverge for z with $|z| > R$.

Proof. We take these one at a time. The point is to imitate the proofs from real analysis.

(a) We note that, if $R = 0$, there is nothing to prove here. Otherwise, fix z with $|z| < R$ so that there exists some $\rho \in \mathbb{R}$ with $|z| < \rho < R$. For example, $\rho := \frac{|z|+R}{2}$ will do.

Now, because $\rho < R$, we see that $\frac{1}{\rho} > \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ (this is legal because $R \neq 0$), so there exists some N for which any fixed $n > N$ has

$$\sup_{k\geq n} \sqrt[k]{|a_k|} < \frac{1}{\rho}.$$

In particular, each $k > N$ will have $\sqrt[k]{|a_k|} < 1/\rho$, so $|a_k|\rho^k < 1$. So, setting

$$M := \max\left(\{1\} \cup \{|a_k| : k \leq N\}\right),$$

we see that $|a_k|\rho^k \leq M$ for each $k \in \mathbb{N}$.

But because $|z| < \rho$, we note that $|z|/\rho < 1$, so we bound

$$\left|a_k z^k\right| = \left|a_k \rho^k\right| \cdot \left|\frac{z^n}{\rho^n}\right| \leq M \left|\frac{z}{\rho}\right|^n.$$

However, $|z/\rho| < 1$, so the series $\sum_{k=0}^{\infty} |z/\rho|^k$ will converge as a geometric series, so we are done by the comparison test.

(b) We proceed by contraposition. Suppose that $S(z)$ converges, so by Lemma 2.71, $a_k z^k \to 0$ as $k \to \infty$. We show that $|z| \le R$. If $z = 0$, there is nothing to say; otherwise, it will suffice to show that

$$\frac{1}{|z|} \overset{?}{\ge} \limsup_{k\to\infty} \sqrt[k]{|a_k|}.$$

For this, fix $\varepsilon = 1$, so we are granted some N for which $k > N$ has

$$\left|a_k z^k\right| < 1.$$

In particular, this rearranges into $1/|z| > \sqrt[k]{|a_k|}$. So for each $n > N$, we see $1/|z| > \sqrt[k]{|a_k|}$ for $k > n$, so $1/|z| \ge \sup\{\sqrt[k]{|a_k|} : k > n\}$, so

$$\frac{1}{|z|} \ge \lim_{n\to\infty} \sup\left\{\sqrt[k]{|a_k|} : k > n\right\} = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

which is what we wanted. ■

Remark 3.34 (Nir)**.** The proof of (b) might feel weird because we are not using the full power of $S(z)$ converging, just that its terms go to 0. However, a power series will "essentially" converge whenever its terms go to 0 (aside from boundary cases), so it is not too surprising that this is all that we need.

We have the following warning.

Warning 3.35. Proposition 3.33 is agnostic to the case of $|z| = R$.

In general, the behavior need not be uniform, as with $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$.

3.4.2 Series of Functions

We will be interested in series of functions, which generalize power series.

Definition 3.36 (Series of functions)**.** Fix $X \subseteq \mathbb{C}$ a nonempty set and $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$. Then we define the *series of functions*

$$S(z) = \sum_{k=0}^{\infty} f_k(z)$$

for each $z \in \mathbb{C}$.

Observe that the partial sums of some $S(z) = \sum_{k=0}^{m} f_k(z)$ will look like some finite sum

$$S_m(z) = \sum_{k=0}^{m} f_k(z),$$

which defines a sequence of functions $\{S_m\}_{m\in\mathbb{N}}$ where $S_m : X \to \mathbb{C}$. We are interested in the convergence of S as a function.

Definition 3.37 (Uniform convergence)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset and $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ defining a series of functions $S(z) = \sum_{k=0}^{\infty} f_k(z)$. Then $S(z)$ *converges uniformly* if and only if the sequence of partial sums converges uniformly to a function on X.

Uniform convergence will be nice because (say) it will preserve continuity, but before talking about utility, we discuss a way to check uniform convergence.

Theorem 3.38 (Weierstrass M-test)**.** Fix $X \subseteq \mathbb{C}$ a nonempty subset and $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions $X \to \mathbb{C}$ defining a series of functions $S(z) = \sum_{k=0}^{\infty} f_k(z)$. Further, suppose that, for each $k \in \mathbb{N}$, there exists some M_k such that

$$|f_k(z)| \le M_k$$

for each $z \in X$. If $\sum_{k=0}^{\infty} M_k$ converges, then $S(z)$ converges uniformly.

In other words, we can determine uniform convergence of a series of functions by bounding the functions individually.

Proof of Theorem 3.38. This is not as hard as it looks. Let S_m denote the mth partial sum. By Proposition 2.136, it suffices to show that, for each $\varepsilon > 0$, there exists some N such that $n \ge m > N$ implies

$$\sup_{x\in X}\{|S_n(z) - S_m(z)|\} < \varepsilon.$$

Well, we know that the series $\sum_{k=0}^{\infty} M_k$ converges, so its partial sums are Cauchy, so there exists some N such that $n \ge m > N$ implies

$$\sum_{k=m+1}^{n} M_k < \varepsilon,$$

where the left-hand side is the difference between the nth and mth partial sums. So now we bound

$$|S_n(z) - S_m(z)| = \left|\sum_{k=m+1}^{n} f_k(z)\right| \le \sum_{k=m+1}^{n} |f_k(z)| \le \sum_{k=m+1}^{n} M_k,$$

for any $z \in X$. Thus,

$$\sup_{z\in X}\{|S_n(z) - S_m(z)|\} \le \sum_{k=m+1}^{n} M_k < \varepsilon.$$

This finishes the proof. ■

And now let's apply the Weierstrass M-test to power series.

Corollary 3.39. Fix $S(z) = \sum_{k=0}^{\infty} a_k z^k$ a power series with positive radius of convergence $R > 0$. We have the following.

(a) For any r such that $0 < r < R$, the power series $S(z)$ converges uniformly in $\overline{B(0,r)}$.

(b) The power series $S(z)$ is continuous on $B(0,r)$.

Proof. Most of our work will be done in (a), which comes from the Weierstrass M-test.

(a) Fix some r with $0 < r < R$. Note that $S(r)$ converges absolutely by Proposition 3.33. To use the Weierstrass M-test, we set $f_k(z) := a_k z^k$ with $M_k := |a_k| r^k$ so that $|z| \le r$ implies

$$|f_k(z)| = \left|a_k z^k\right| = |a_k| \cdot |z|^k \le |a_k| r^k.$$

But we know that $S(r)$ converges absolutely, so

$$\sum_{k=0}^{\infty} \left|a_k r^k\right| = \sum_{k=0}^{\infty} M_k$$

converges, so now Theorem 3.38 promises that $S(z)$ will converge uniformly for each $z \in \overline{B(0,r)}$.

(b) Note that, for every k, the function $f_k(z) = a_k z^k$ is a polynomial and hence entire and hence continuous on $B(0,R)$.

The trick is to apply (a) by starting with a fixed $z_0 \in B(0,R)$ with r such that $|z_0| < r < R$. In particular, by restriction, it suffices to show that $S|_{B(0,r)}$ is continuous at z_0. (For example, $r = \frac{|z_0|+R}{2}$ will work.) So now we note that the continuous partial sums of $S(z)$ converge uniformly to $S(z)$ on $B(0,r)$ by (a), so Proposition 2.139 forces $S(z)$ itself to be continuous on $B(0,r)$. This finishes. ■

We remark that the restriction to $S|_{B(0,r)}$ only works because $B(0,r)$ is an open set. Here is the exact lemma we just used.

Lemma 3.40. Fix $f : X \to \mathbb{C}$ a function and $U \subseteq \mathbb{C}$ an open subset X with $x \in U \cap X$. Then f is continuous at x if and only if the restriction $f|_{U\cap X} : U \cap X \to \mathbb{C}$ is continuous at x.

An alternate way to give the hypothesis on U is that $U \cap X$ is an open subset of X.

Proof. We show the directions independently.

- Suppose that f is continuous at x. We show that $f|_{U\cap X}$ is continuous at x. Well, for any $\varepsilon > 0$, we are promised some $\delta > 0$ so that any $z \in X$ has
$$|z - x| < \delta \implies |f(z) - f(x)| < \varepsilon.$$
In particular, any $z \in X \cap U$ has
$$|z - x| < \delta \implies \left|f|_{U\cap X}(z) - f|_{U\cap X}(x)\right| = |f(z) - f(x)| < \varepsilon.$$

- Suppose that $f|_{U\cap X}$ is continuous at x. Fix any $\varepsilon > 0$. Because $x \in U$ and U is open, there exists $r > 0$ such that $B(x,r) \subseteq U$. Because $f|_{U\cap X}$ is continuous at x, there exists some $\delta_0 > 0$ such that
$$|z - x| < \delta_0 \implies |f(z) - f(x)| = |f_{U\cap X}(z) - f_{U\cap X}(z)| < \varepsilon$$
for $z \in U \cap X$. However, taking $\delta := \min\{r, \delta\}$, we see that any $z \in X$ with $|x - z| < \delta$ will have $z \in B(x,\delta) \subseteq U$, so $z \in U \cap X$ automatically. So $|z - x| < \delta$ will still imply
$$|f(z) - f(x)| < \varepsilon,$$
and we are done. ■

Remark 3.41 (Nir)**.** More generally, if we have a sequence of continuous functions $f_k : X \to \mathbb{C}$ such that the series $S(z) := \sum_{k=0}^{\infty} f_k(z)$ converges uniformly on X, then S is a continuous function on X. Indeed, fix some $z_0 \in X$ and $\varepsilon > 0$. We have the following.

- There is N so that $n > N$ has $|S(z) - f_n(z)| < \varepsilon/3$ for $z \in X$. Fix some $n > N$.
- There is $\delta > 0$ so that $|z - z_0| < \delta$ has $|f_n(z) - f_n(z_0)| < \varepsilon/3$.

Thus, $|z - z_0| < \delta$ will have
$$|S(z) - S(z_0)| < |S(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - S(z_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

3.5 February 23

Good morning everyone. We are doing review today.

3.5.1 Review Highlights

Here are some of the answers to questions asked in class.

- The midterm will be like the practice midterm.
- You do not need to be stressed about the midterm.
- Things proven in class we will not be asked to prove on the midterm.
- We will probably will not be asked to do anything involving summation by parts.
- Professor Morrow will not curve downward.
- Please write more on the exam for the sake of partial credit.
- For words with multiple definitions (like continuity and compactness), the first definition is preferred, though other definitions will likely be accepted.
- We may cite facts from real analysis, which is a requirement for this class; e.g., $[0,1]$ is compact.
- Lemmas elided from class we will not be responsible for. Essentially, please know the things on the review.
- Please know the definitions of things is important. They will be graded fairly harshly because these are critical to know to going forwards.

Let's do a practice problem.

Exercise 3.42. Find the possible functions $v(x,y):\mathbb{R}^2\to\mathbb{R}$ such that

$$f(z)=f(x+iy)=x^2-y^2+iv(x,y)$$

is entire and $f(0)=0$.

Proof 1. The point is to use the Cauchy–Riemann equations. We set $u(x,y):=x^2-y^2$ so that $f(x+yi)=u(x,y)+iv(x,y)$. If we want this to be differentiable, we want

$$u_x(x,y)=2x=v_y(x,y)$$

by Theorem 3.19. This means that $v(x,y)=2xy+h(x)$ for some function $h(x):\mathbb{R}\to\mathbb{R}$. Again, we note

$$u_y(x,y)=-2y=-v_x(x,y)=-2y-h'(x),$$

so we want $h'(x)=0$. So h is a constant function, so we set $h(x)=c$ for some $c\in\mathbb{R}$.

It remains to determine c. Well, so far the story is that

$$f(x+iy)=x^2-y^2+i(2xy+c).$$

Plugging in $x=y=0$ forces $c=0$, so we see that we get $\boxed{f(x+iy)=x^2-y^2-i\cdot 2xy}$. ■

Remark 3.43. The current form of the answer is fine: we do not have to simplify in terms of z or something. More generally, we will not have to spend large amounts of time simplifying on the exam.

Let's present another proof.

Proof 2. The point is to use the x information to fully piece together $f'(z)$. As before, set, $u(x,y) = x^2 - y^2$. Namely, the Cauchy–Riemann equations promise

$$f'(z) = f'(x+yi) = u_x(x,y) + iv_x(x,y) = u_x(x,y) - iu_y(x,y).$$

Taking partial derivatives of u implies that

$$f(z) = 2x - i(-2y) = 2x + i \cdot 2y = 2(x+yi) = 2z.$$

So from here, we can take the "antiderivative" (i.e., guess) that $f(z) = z^2 + c$. Lastly, plugging in $f(0) = 0$, we get $c = 0$, so $\boxed{f(z) = z^2}$. ■

Remark 3.44. We can rigorize that this is the only possible solution because any other solution $g(z)$ must have $g(z) - z^2$ with constant derivative 0, from which we can argue that $g(z) - z^2$ is constant using the Cauchy–Riemann equations and the fact that $\mathbb{C}$ is path-connected. To be explicit, we are using Corollary 3.23.

3.6 February 25

There was no lecture today because we had a midterm.

3.7 February 28

Good morning, everyone. Here are some announcements.

- Midterm grades will be posted today or tomorrow, on `bCourses`.
- Class on Wednesday will be a recording. Professor Morrow will be giving a talk, at 9AM as decided by the powers that be.
- There is no homework due Friday because we haven't covered anything since the midterm.

3.7.1 Holomorphic Power Series

Today we actually talk about analytic functions. Professor Morrow promises that it is actually complex analysis today, and once we talk about analytic functions and path integration, we will prove the Cauchy integral formula, which is one of the major results of the course.

We recall the following definition.

Definition 3.30 (Complex power series)**.** A *complex power series* is a formal expression of the form

$$S(z) := \sum_{k=0}^{\infty} a_k x^k$$

where $\{a_k\}_{k\in\mathbb{N}} \subseteq \mathbb{C}$ and z is a (formal) variable taking complex values.

So far we've talked about the radius of convergence of a power series as well as some properties of series of functions in general (e.g., the Weierstrass M-test).

Today we are showing the following result.

Proposition 3.45. Fix $S(z) = \sum_{k=0}^\infty a_k z^k$ a (complex) power series with radius of convergence $R > 0$. Then $S(z)$ is holomorphic on $B(0, R)$ with derivative

$$S'(z) = \sum_{k=1}^\infty k a_k z^{k-1}.$$

Further, $S'(z)$ also has radius of convergence R.

Note that this derivative is essentially the "term-wise" derivative of $S(z)$, so it is more or less the best thing that we could want.

Proof. We will symbolically define

$$S'(z) := \sum_{k=1}^\infty k a_k z^{k-1}$$

and show that it is equal to the requested derivative. We start by noting the radius of convergence of S' is

$$\frac{1}{\lim_{k\to\infty} \sqrt[k]{|(k+1)a_k|}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} \cdot \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} = 1 \cdot R = R,$$

so at the very least our radius of convergence matches, as claimed.

Fix $0 < r < R$ a real number (i.e., we don't want to deal with $R = +\infty$), so that it suffices to show S is holomorphic with the given derivative on $B(0, r)$. (Namely, for a given $w \in B(0, R)$, choose any r with $|w| < r < R$.)

Indeed, given $w \in B(0, r)$, it suffices to show that S is differentiable at w with the requested derivative, for which we claim

$$\left(\lim_{z\to w} \frac{S(z) - S(w)}{z - w}\right) - S'(w) \stackrel{?}{=} 0,$$

where $S'(z)$ is the claimed derivative. To set up our computation, we fix a positive integer m and work with the mth partial sum, computing

$$\begin{aligned}
\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) &= \sum_{k=0}^m \frac{a_k z^k - a_k w^k}{z - w} - \sum_{k=1}^m k a_k w^{k-1} \\
&= (a_0 - a_0) + \sum_{k=1}^m a_k \left(\frac{z^k - w^k}{z - w} - k w^{k-1}\right) \\
&= \sum_{k=1}^m a_k \left(\sum_{a+b=k-1} z^b w^a - \sum_{a+b=k-1} w^{k-1}\right) \\
&= \sum_{k=1}^m a_k \left(\sum_{a+b=k-1} \left(z^b w^a - w^{k-1}\right)\right) \\
&= \sum_{k=1}^m a_k \left(\sum_{a+b=k-1} w^a \left(z^b - w^b\right)\right).
\end{aligned}$$

With this in mind, we set

$$h_k(z) = \sum_{a+b=k-1} w^a \left(z^b - w^b\right),$$

which we note is a polynomial in $z \in B(0, r)$ because we fixed w to be constant. In particular, we have

$$\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) = \sum_{k=1}^m a_k h_k(z).$$

We now show that this series converges uniformly as $m \to \infty$; we will use Theorem 3.38. For this, we bound

$$|h_k(z)| = \left| \sum_{a+b=k-1} w^a \left(z^b - w^b\right) \right| \le \sum_{a+b=k-1} |w|^a \left(|z|^b + |w|^b\right) < \sum_{a+b=k-1} r^a \left(r^b + r^b\right) = 2(k-1)r^{k-1},$$

so we bound $|a_k h_k(z)| < |a_k| \cdot 2(k-1)r^{k-1}$. Namely, by Theorem 3.38, it suffices to show that the series

$$\sum_{k=1}^{\infty} 2(k-1)|a_k|r^{k-1}$$

converges. Well, $\sum_{k=1}^{\infty} 2(k-1)|a_k|x^{k-1}$ is a power series with radius of convergence

$$\frac{1}{\lim_{k\to\infty}\left(\sqrt[k]{2k} \cdot \sqrt[k]{|a_{k+1}|}\right)} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{2k}} \cdot \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_{k+1}|}} = R,$$

so indeed the power series $\sum_{k=1}^{\infty} 2(k-1)|a_k|x^{k-1}$ converges at $x = r < R$.

So in total, we see that the series of functions

$$\sum_{k=1}^{\infty} a_k h_k(z)$$

uniformly converges as $m \to \infty$. Because each component function $a_k h_k(z)$ is continuous, we see that the entire series will converge to a continuous function by Remark 3.41. In other words, we can evaluate

$$\lim_{z\to w} \lim_{m\to\infty} \left(\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) \right) = \lim_{z\to w} \sum_{k=1}^{\infty} a_k h_k(z) = \sum_{k=1}^{\infty} a_k h_k(w).$$

But now we notice that $h_k(w) = 0$ for each h_k, so this sum does indeed vanish.

We are now essentially done. We compute

$$\begin{aligned}
\lim_{z\to w} \frac{S(z) - S(w)}{z - w} &= \lim_{z\to w} \left(\frac{S(z) - S(w)}{z - w} - S'(w) \right) + \lim_{z\to w} S'(w) \\
&= \lim_{z\to w} \left(\frac{\lim_{m\to\infty} S_m(z) - \lim_{m\to\infty} S_m(w)}{z - w} - \lim_{m\to\infty} S_m'(w) \right) + S'(w) \\
&= \lim_{z\to w} \lim_{m\to\infty} \left(\frac{S_m(z) - S_m(w)}{z - w} - S_m'(w) \right) + S'(w) \\
&= S'(w),
\end{aligned}$$

so we are done. ■

So indeed, power series are holomorphic. Here is nice application of this fact.

Corollary 3.46. Fix

$$S(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad T(z) = \sum_{k=0}^{\infty} b_k z^k$$

two complex power series with radius of convergence $R > 0$. If $S(z) = T(z)$ for all $z \in B(0, R)$, then $a_k = b_k$ for each k.

Proof. We proceed inductively, in spirit. For example $a_0 = S(0) = T(0) = b_0$, so these are equal as our base case. Further, we could take one derivative to see that

$$S'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \text{and} \qquad T'(z) = \sum_{k=1}^{\infty} k b_k z^{k-1},$$

so $a_1 = S'(0) = T'(0) = b_1$. More generally, setting $S^{(m)}$ to be the mth derivative, we can see that

$$S^{(m)}(z) = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)a_k z^{k-m} \qquad \text{and} \qquad T^{(m)} = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)a_k z^{k-m},$$

and both of these have the same radius of convergence. So now $a_m = \frac{1}{m!}S^{(m)}(0) = \frac{1}{m!}T^{(m)}(0) = b_m$. ■

3.7.2 Analytic Functions

To define analytic, we need one more definition.

Definition 3.47 (Power series expansion)**.** Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f : X \to \mathbb{C}$ a function. We say that f has a *power series expansion centered at* $z_0 \in X$ if and only if there is a positive real number r such that $B(z_0, r) \subseteq X$ and further there is a power series defined by $\{a_k\}_{k\in\mathbb{N}}$ which has

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

for each $z \in B(z_0, r)$.

And here is our definition.

Definition 3.48 (Analytic)**.** Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f : X \to \mathbb{C}$ a function. Then f is *analytic* at $z_0 \in \mathbb{C}$ if and only if f has a power series expansion at z_0. Explicitly, there is a power series $S(z) = \sum_{k=0}^{\infty} a_k z^k$ and positive real number $r > 0$ (less than the radius of convergence) such that

$$f(z) = S(z-z_0) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

for any $z \in B(z_0, r)$. Then f is *analytic* if and only if it is analytic at each $z_0 \in \mathbb{C}$.

Here is the idea.

Idea 3.49. Analytic functions are locally power series.

Being analytic is a very nice condition. For example, we have the following.

Proposition 3.50. Analytic functions are holomorphic on their domain.

Proof. Fix $f : X \to \mathbb{C}$ an analytic function. For each $x \in X$, we note that f is locally equal to a power series at x (i.e., $f|_{B(x,r)}$ is a power series), which is holomorphic by Proposition 3.45. Because f is locally differentiable at each point, f will be actually differentiable at each point. ■

Remark 3.51. It will turn out that the converse is also true, but this is a pretty deep result. We will prove it from the Cauchy integral formula. The main obstacle is how we should construct the power series, which the Cauchy integral formula will tell us how to do.

Anyways, let's prove something of substance.

Lemma 3.52. Fix $X \subseteq \mathbb{C}$ a nonempty open subset and $f : X \to \mathbb{C}$ an analytic function. Then f' is also analytic.

Proof. Fix $z_0 \in X$. Because f is analytic, there is a positive real number $r > 0$ and power series $S(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ (with radius of convergence at least r) such that

$$f(z) = S(z-z_0) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

for each $z \in B(z_0, r)$. By Proposition 3.45, we see that

$$f'(z) = S'(z-z_0) = \sum_{k=1}^{\infty} ka_k(z-z_0)^{k-1}$$

for each $z \in B(0, r)$. So we see that f' has a power series expansion at our arbitrarily chosen $z_0 \in X$, so f' is analytic at each $z_0 \in X$, so f' is analytic. ■

Remark 3.53. We can iterate the above lemma to show that an analytic function is infinitely differentiable.

Remark 3.54. In fact, because analytic will turn out to be equivalent to holomorphic, we will see that being once differentiable implies being analytic implies being infinitely differentiable. This is pretty nice.

Next class we will start talking about the exponential function, a very important analytic function.

3.8 March 2

This lecture was recorded.

3.8.1 Definition of the Exponential

For the next couple lectures we will be discussing the very special functions $\exp$ and $\log$. For now, we will focus on $\exp$, defined as follows.

Definition 3.55 ($\exp$)**.** We define the *complex exponential* $\exp : \mathbb{C} \to \mathbb{C}$ by the power series

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

In particular, we are going to be building our exponentiation from scratch. Nevertheless, we promise that it will work fine.

As such, we have the following checks.

Lemma 3.56. We have that $\exp$ is analytic and entire with derivative $\exp'(z) = \exp(z)$.

Proof. Very quickly, we note that the radius of convergence of $\exp$ is lower-bounded by

$$\left(\lim_{n\to\infty} \sqrt[n]{|1/n!|}\right)^{-1} \geq \left(\lim_{n\to\infty} \sqrt[n]{n^{-n/2}}\right)^{-1} = \left(\lim_{n\to\infty} n^{-1/2}\right)^{-1} = \infty,$$

so our radius of convergence is actually ∞. As such Proposition 3.45 tells us that $\exp$ is holomorphic on $B(0, \infty) = \mathbb{C}$ (i.e., entire) with derivative

$$\exp'(z) = \sum_{k=1}^{\infty} \frac{k}{k!} z^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} z^{k-1} = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

where we have shifted indices in the last step. So indeed, $\exp'(z) = \exp(z)$.

Lastly, to show that $\exp$ is analytic, we need to show that $\exp$ can be locally expanded as a power series. For this, we appeal to the following lemma.

Lemma 3.57. Fix $S(z) := \sum_{k=0}^\infty a_k z^k$ a power series with radius of convergence $R > 0$. Then $S(z)$ is analytic on $B(0, R)$.

Proof. There is actually something to show here: given $z_0 \in \mathbb{C}$, we need to expand $S(z)$ locally at a power series at z_0. In particular, we need to be able to write

$$S(z) = \sum_{k=0}^\infty b_k (z - z_0)^k,$$

where the series on the right converges for any $z \in B(z_0, r)$ for some $r > 0$. For this, we expand

$$S(z + z_0) = \sum_{n=0}^\infty a_k (z + z_0)^n,$$

under the assumption $z, z_0, z+z_0, |z|+|z_0| \in B(0, R)$. (We will discuss how to ensure these conditions later.)

The short version of what we are about to do is that we will expand out this power series in terms of z and then collect terms of the same degree. Making this rigorous requires some care to the uniform convergence, but everything is okay because we converge absolutely.

Heuristically, we have

$$\sum_{n=0}^\infty a_n (z + z_0)^n = \sum_{n=0}^\infty \left(\sum_{k+\ell=n} \binom{n}{k} a_n z^k z_0^\ell \right) \stackrel{*}{=} \sum_{k=0}^\infty \left(\sum_{\ell=0}^\infty \binom{n}{k} a_n z_0^\ell \right) z^k,$$

where $\stackrel{*}{=}$ is the equality which requires attention. To rigorize $\stackrel{*}{=}$, we use Lemma 2.83.[2] Indeed, to make the application clearer, we set

$$a_{n,k} := \begin{cases} \binom{n}{k} a_n z^k z_0^{n-k} & k \le n, \\ 0 & k > n \end{cases}$$

so that we are interested in exchanging the order of the summation

$$\sum_{n=0}^\infty \left(\sum_{k+\ell=n} \binom{n}{k} a_n z^k z_0^\ell \right) = \sum_{n=0}^\infty \sum_{k=0}^\infty a_{n,k}.$$

Well, for fixed n, we see that $\sum_{k=0}^\infty |a_{n,k}|$ is a finite sum and hence converges. And further, we see that

$$\sum_{n=0}^\infty \sum_{k=0}^\infty |a_{n,k}| = \sum_{n=0}^\infty \left(\sum_{k=0}^n \binom{n}{k} a_n |z|^k |z_0|^{n-k} \right) = \sum_{n=0}^\infty a_n (|z| + |z_0|)^n = S(|z| + |z_0|),$$

which converges because $|z| + |z_0| \in B(0, R)$. As such, Lemma 2.83 tells us that

$$S(z + z_0) = \sum_{n=0}^\infty \sum_{k=0}^\infty a_{k,\ell} = \sum_{k=0}^\infty \sum_{n=0}^\infty a_{k,\ell} = \sum_{k=0}^\infty \sum_{n=k}^\infty \binom{n}{k} a_n z^k z_0^{n-k}.$$

The inner sums we may simplify as $\sum_{n=k}^\infty \binom{n}{k} a_n z^k z_0^{n-k} = z^k \sum_{\ell=0}^\infty \binom{n}{k} a_n z_0^\ell$, so we do indeed find that

$$S(z + z_0) = \sum_{k=0}^\infty \left(\sum_{\ell=0}^\infty \binom{n}{k} a_n z_0^\ell \right) z^k,$$

[2] Yes, I, too, am impressed that this lemma is seeing use.

for any $z \in \mathbb{C}$. In particular, plugging in $z - z_0$ tells us that

$$S(z) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{\infty} \binom{n}{k} a_n z_0^{\ell} \right) (z - z_0)^k,$$

which gives us our power series expansion at z_0.

It remains to show the power series expansion will hold in some neighborhood $B(z_0, r)$. Translating back, we need to know that the power series expansion for $S(z + z_0)$ will hold in some neighborhood $S(0, r)$. To review, our hypotheses were that

$$z, z_0, z + z_0, |z| + |z_0| \in B(0, R).$$

Recalling that $z_0 \in B(0, R)$ automatically, we set $r := R - |z_0| > 0$. Then $r < R$, so $z \in B(0, R)$. Similarly,

$$|z + z_0| \le |z| + |z_0| < r + |z_0| = R,$$

so we get $z + z_0, |z| + |z_0| \in B(0, R)$ as well. So we have constructed our neighborhood and have verified that $S(z)$ is analytic at z_0. ■

Thus, because we defined $\exp$ as a power series with infinite radius of convergence, we see that $\exp$ is analytic everywhere on $\mathbb{C}$. ■

3.8.2 Basic Properties of the Exponential

Now that we know $\exp'(z) = \exp(z)$, we can begin actually building some theory. We pick up the following nice properties of $\exp$.

Proposition 3.58. Fix $z, w \in \mathbb{C}$.

(a) We have that $\exp(z + w) = \exp(z)\exp(w)$.

(b) We have that $\exp(z) \neq 0$.

(c) We have that $\exp(-z) = 1/\exp(z)$.

Proof. Parts (b) and (c) will follow from (a), so we will focus our attention on (a). Fixing some $\alpha \in \mathbb{C}$, the trick is to consider

$$f(z) = \exp(z)\exp(\alpha - z).$$

Observe that $z \mapsto z$ and so $\alpha - z$ are entire, so the chain rule promises each factor of f is entire, so f is entire by the product rule. Tracking all this through, we can compute the derivative as

$$\begin{aligned} f'(z) &= \exp'(z)\exp(\alpha - z) + \exp(z)\exp'(\alpha - z) \cdot (-1) \\ &= \exp(z)\exp(\alpha - z) - \exp(z)\exp(\alpha - z) \\ &= 0. \end{aligned}$$

Thus, f' is constantly 0 everywhere (and $\mathbb{C}$ is connected by Corollary 2.115), so f is constant on $\mathbb{C}$ by Corollary 3.23. However, we can plug in $z = \alpha$ into f to see that

$$f(\alpha) = \exp(\alpha) \cdot \exp(0) = \exp(\alpha),$$

where $\exp(0) = 1$ by construction of $\exp$. In particular, we see that

$$\exp(z)\exp(\alpha - z) = \exp(\alpha)$$

for any $z, \alpha \in \mathbb{C}$. Setting $\alpha := w + z$ recovers $\exp(z + w) = \exp(z)\exp(w)$ part (a).

We now show (b) and (c). Setting $z = -w \in \mathbb{C}$ in (a), we see that

$$1 = \exp(0) = \exp(z + -z) = \exp(z)\exp(-z).$$

Thus, because $\mathbb{C}$ is an integral domain, we see that $\exp(z) \neq 0$ automatically, which is (b). So, using the field structure of $\mathbb{C}$ to divide by $\exp(z)$, we conclude that

$$\exp(-z) = 1/\exp(z),$$

which proves (c). ■

Remark 3.59 (Nir)**.** In other words, $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a homomorphism: $\exp$ does map to $\mathbb{C}^\times$ by (c) of the proposition, and $\exp$ satisfies the needed homomorphism property by (a).

In fact, $\exp$ will behave with our complex analytic structure.

Lemma 3.60. Fix any $z \in \mathbb{C}$. Then

$$\overline{\exp(z)} = \exp(\overline{z}).$$

Proof. The main point is that $z \mapsto \overline{z}$ is continuous on $\mathbb{C}$, say by Example 2.126. Thus, we compute

$$\overline{\exp(z)} = \overline{\lim_{n\to\infty} \sum_{k=0}^{n} \frac{z^k}{k!}} \stackrel{*}{=} \lim_{n\to\infty} \overline{\sum_{k=0}^{n} \frac{z^k}{k!}} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\overline{z}^k}{k!} = \exp(\overline{z}),$$

where we have used the continuity of $z \mapsto \overline{z}$ in $\stackrel{*}{=}$. In particular, the point is that the sequence of partial sums $S_n := \sum_{k=0}^{n} \frac{z^k}{k!}$ approach $\exp(z)$, so by continuity, $\overline{S_n}$ (which goes to $\overline{\exp(z)}$ definitionally) must approach $\exp(\overline{z})$. ■

Our next goal is to study certain outputs of $\exp$. Like a good algebraist, we will particularly be interested in the "kernel" of $\exp$ (as a homomorphism). For now, we will avoid saying the word "kernel" and instead simply solve for the output 1.

Lemma 3.61. Fix any $t \in \mathbb{R}$. Then $|\exp(it)| = 1$.

Proof. Note that

$$\overline{\exp(it)} = \exp(\overline{it}) = \exp(-it) = 1/\exp(it),$$

where we have used Lemma 3.60 followed by Proposition 3.58. Thus,

$$|\exp(it)|^2 = \exp(it) \cdot \overline{\exp(it)} = 1,$$

so $|\exp(it)| = 1$ follows because the norm is always a positive real number. ■

In fact, we can do better than the above.

Corollary 3.62. Fix any $z \in \mathbb{C}$. Then $|\exp(z)| = 1$ if and only if $\operatorname{Re}(z) = 0$.

Proof. We show our implications separately.

- Suppose that $\operatorname{Re}(z) = 0$. Then we can write $z = it$ for some $t \in \mathbb{R}$, from which Lemma 3.61 tells us that $|\exp(z)| = |\exp(it)| = 1$ for free.

- Suppose that $|\exp(z)| = 1$. Writing $z = x + yi$ with $x, y \in \mathbb{R}$, we compute
$$\exp(z) = \exp(x)\exp(iy) = \exp(x),$$
where we have used Proposition 3.58 and Lemma 3.61. Now, taking norms, we see that $|\exp(x)| = |\exp(z)| = 1$.
However, $\exp|_{\mathbb{R}}$ is a strictly increasing function: it is differentiable with continuous nonzero derivative (using Proposition 3.58), so the Intermediate value theorem implies that the derivative must stay the same sign for all $x_0 \in \mathbb{R}$. So noting $\exp(0) = 1$ is enough to conclude $\exp'(x_0) > 0$ for any $x_0 \in \mathbb{R}$, so $\exp$ is strictly increasing from a Mean value theorem argument.[3]
Thus, if $x < 0$, then $|\exp(x)| = \exp(x) < 1$, and if $0 < x$, then $1 < \exp(x) = |\exp(x)|$. So we see that $x = 0$ with $|\exp(x)| = 1$ is our only way to hit 1, so $\operatorname{Re} z = x = 0$ follows. ■

So far we understand $|\exp(z)|$ pretty well. It is time to turn to $\exp$.

Definition 3.63 (Kernel of $\exp$)**.** We define the *kernel of* $\exp$ as
$$\ker \exp := \{z \in \mathbb{C} : \exp(z) = 1\}.$$

Remark 3.64. This is intended to align with abstract algebra: viewing $\exp : \mathbb{C} \to \mathbb{C}^\times$ as a homomorphism, we see that we are asking for the values of $z \in \mathbb{C}$ which go to the identity of $\mathbb{C}^\times$, which is 1.

Example 3.65. We have that $\exp(0) = 1$, so $0 \in \ker \exp$.

To better access the kernel, we will want to talk about the real and imaginary parts of $\exp(it)$.

Definition 3.66 ($\sin$ and $\cos$)**.** Given $z \in \mathbb{C}$, we define the (complex) $\sin$ and $\cos$ functions as
$$\cos z := \frac{\exp(iz) + \exp(-iz)}{2} \qquad \text{and} \qquad \sin z := \frac{\exp(iz) - \exp(-iz)}{2i}.$$

We can see pretty directly that
$$\cos z + i \sin z = \frac{\exp(iz) + \exp(-iz)}{2} - \frac{\exp(iz) - \exp(-iz)}{2} = \exp(iz).$$

In the case where z is real, we get to say a little more.

Remark 3.67. Using Proposition 2.8 with Lemma 3.60, we see that, for when $t \in \mathbb{R}$,
$$\cos t = \frac{\exp(it) + \exp(-it)}{2} = \frac{\exp(it) + \overline{\exp(it)}}{2} = \operatorname{Re}\exp(it),$$
and
$$\sin t = \frac{\exp(it) - \exp(-it)}{2i} = \frac{\exp(it) - \overline{\exp(it)}}{2i} = \operatorname{Im}\exp(it).$$
In particular $\exp(it) = \cos t + i \sin t$ is our decomposition into real and imaginary parts.

3.8.3 Some Trigonometry

Before we go any further, we do some trigonometry. We want to establish that $\exp(it)$ is periodic, but this requires a little effort; we follow sx63102.

[3] If $a < b$, then use the Mean value theorem to find $x \in (a, b)$ with $f(b) - f(a) = (b - a)f'(x) > 0$, so $f(a) < f(b)$.

Lemma 3.68. For each $z \in \mathbb{C}$, we have $\cos^2 z + \sin^2 z = 1$.

Proof. We directly compute

$$\cos^2 z + \sin^2 z = \frac{\exp(iz)^2 + 2\exp(iz)\exp(-iz) + \exp(-iz)^2}{4} + \frac{\exp(iz)^2 - 2\exp(iz)\exp(-iz) + \exp(-iz)^2}{-4}.$$

After the dust settles, we are left with

$$\cos^2 z + \sin^2 z = \exp(iz)\exp(-iz),$$

which is 1 by Proposition 3.58. ■

More or less by just staring at $\cos$ and $\sin$, we can see that they are entire.

Lemma 3.69. For each $z \in \mathbb{C}$, we have $\frac{d}{dz}\cos z = -\sin z$ and $\frac{d}{dt}\sin z = \cos z$.

Proof. We directly compute

$$\frac{d}{dz}\frac{\exp(iz) + \exp(-iz)}{2} = \frac{i\exp(iz) - i\exp(iz)}{2} = -\frac{\exp(iz) - \exp(-iz)}{2i} = -\sin z,$$

and

$$\frac{d}{dz}\frac{\exp(iz) - \exp(-iz)}{2i} = \frac{i\exp(iz) + i\exp(iz)}{2} = \frac{\exp(iz) + \exp(-iz)}{2} = \cos z,$$

which is what we wanted. ■

Lemma 3.70. For $z \in \mathbb{C}$, we have

$$\cos z = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}z^{2k} \qquad \text{and} \qquad \sin z = \sum_{k=0}^\infty \frac{(-1)^{k-1}}{(2k+1)!}z^{2k+1}.$$

Proof. We directly compute, for any $z \in \mathbb{C}$, we have

$$\cos z = \frac{1}{2}(\exp(iz) + \exp(-iz)) = \frac{1}{2}\left(\sum_{k=0}^\infty \frac{i^k}{k!}z^k + \sum_{k=0}^\infty \frac{(-i)^k}{k!}z^k\right) = \frac{1}{2}\sum_{k=0}^\infty \frac{i^k + (-i)^k}{k!}z^k.$$

Here, we were allowed to merge the two sums because they are just limits which converge. Now, we note that

$$i^k + (-i)^k = \begin{cases} 2 & k \equiv 0 \pmod 4, \\ 0 & k \equiv 1 \pmod 2, \\ -2 & k \equiv 2 \pmod 4, \end{cases}$$

so all the odd terms vanish, leaving us with

$$\cos z = \frac{1}{2}\sum_{k=0}^\infty \frac{2(-1)^k}{(2k)!}z^{2k} = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}z^{2k},$$

which is what we wanted.

On the other hand, we note that $\cos z$ is an entire function, and its power series will converge everywhere because the power series for $\exp$ also converges everywhere. In particular, Proposition 3.45 tells us that

$$\sin z = -\frac{d}{dz}\cos z = \sum_{k=1}^\infty \frac{(-1)^k \cdot 2k}{(2k)!}z^{2k-1} = \sum_{k=1}^\infty \frac{(-1)^k}{(2k-1)!}z^{2k-1},$$

which gives the power series for $\sin$ after shifting over our indices. Notably, Proposition 3.45 assures us that this also has infinite radius of convergence. ■

To continue, we have to do a little real analysis.

Lemma 3.71. There exists the smallest positive real number θ such that $\cos\theta = 0$.

Proof. On one hand, note $\cos 0 = 1$. On the other hand, using the Alternating series bound, we note

$$\cos 2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot 2^{2k} \le 1 - \frac{4}{2} + \frac{16}{24} = -\frac{1}{3} < 0.$$

Thus, there certainly exists some $t \in [0,2]$ such that $\cos t = 0$, so we define

$$\theta := \inf\{t > 0 : \cos t = 0\}.$$

Because $\cos$ is continuous, we note that the set $\{t : \cos t = 0\}$ will be closed and hence contain all of its limit points, so we do have $\cos\theta = 0$.

Further, $\cos 0 = 1$ implies there is some δ such that $|t| < \delta$ has $|\cos t - 1| < 1$, meaning there is an open neighborhood around 0 for which $\cos t \ne 0$. In particular, we must have $\theta \ge \delta > 0$, so θ is a positive real number. So lastly, we note that any $t > 0$ for which $\cos t = 0$ must have $t \ge \theta$ by construction, so θ is indeed the smallest positive real number with $\cos\theta = 0$. ■

And now we get to define π.

Definition 3.72 (π)**.** We define $\pi \in \mathbb{R}$ so that $\pi/2$ is the smallest positive real number such that $\cos \pi/2 = 0$.

And now let's show our periodicity.

Lemma 3.73. We have that $\exp(z + 2\pi i) = \exp(z)$ for any $z \in \mathbb{C}$. In fact, 2π is the smallest positive real number θ such that $\exp(i\theta) = 1 = \exp 0$.

Proof. We start with the second sentence. We are given that $\cos \pi/2 = 0$ already, and $\pi/2$ is the smallest such positive real number. From Lemma 3.68, we see that this requires $\sin \pi/2 \in \{\pm 1\}$. However,

$$\frac{d}{dt} \sin t = \cos t$$

must be positive in the interval $(0, \pi/2)$ because $\cos 0 = 1 > 0$ and $\cos$ is nonzero on $(0, \pi/2)$. In particular, a Mean value theorem argument tells us that $\sin$ is strictly increasing on $(0, \pi/2)$, so we have

$$\sin \pi/2 > \sin 0 = 0,$$

so $\sin \pi/2 = 1$. Plugging into Remark 3.67, we get that $\exp(i\pi/2) = i$, so

$$\exp(2\pi i) = \exp(4 \cdot i\pi/2) = \exp(i\pi/2)^4 = i^4 = 1.$$

It remains to show that 2π is the smallest such positive real number. Well, suppose that $\theta > 0$ has $\exp(\theta i) = 1$ and is the smallest such positive real number; we get for free that $\theta \le 2\pi$ by the above. On the other hand, we compute

$$\exp(\theta/4 \cdot i)^4 = \exp(\theta i) = 1,$$

but we can factor $z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$, so $\exp(\theta/4 \cdot i) \in \{\pm 1, \pm i\}$. Certainly if $\exp(\theta/4 \cdot i) \in \{\pm 1\}$, then $\exp(\theta/2 \cdot i) = \exp(\theta/4 \cdot i)^2 = 1$, but $\theta/2 < \theta/4$, so this cannot be. So instead, we have that

$$\exp(\theta/4 \cdot i) = \pm i,$$

so in particular, Remark 3.67 tells us that $\cos(\theta/4) = \operatorname{Re} \exp(\theta/4 \cdot i) = 0$. Thus, $\theta/4 \ge \pi/2$ by the definition of π, so $\theta \ge 2\pi$. It follows $\theta = 2\pi$.

We now show the first sentence. By Proposition 3.58, we merely have to compute

$$\exp(z+2\pi i)=\exp(z)\exp(2\pi i)=\exp z,$$

so we are done. ■

While we're here, we note that also get access to the kernel from our work.

Proposition 3.74. We have that $\ker\exp=\{2\pi i n : n\in\mathbb{Z}\}$.

Proof. In one direction, certainly

$$\exp(2\pi in)=\exp(2\pi i)^n=1$$

by Lemma 3.73. In the other direction, suppose $\exp z=1$. Then Corollary 3.62 forces $\operatorname{Re} z=0$, so we can write $z=it$. By the division algorithm, we can write

$$t=2\pi q+r,$$

where $q\in\mathbb{Z}$ and $r\in[0,2\pi)$, from which we see

$$1=\exp z=\exp(it)=\exp(2\pi iq+ir)=\exp(2\pi iq)\exp(ir)=\exp(ir).$$

However, $r<2\pi$ is smaller than the smallest positive real number for which $\exp(ir)=1$, so r cannot be a positive real number at all. But we do know $r\geq 0$, so $r=0$ is forced. Thus, $t=2\pi iq$, as needed. ■

Remark 3.75 (Nir)**.** As a last remark, it would be a crime to note say that $\exp(i\pi)=-1$. Indeed,

$$\exp(i\pi)^2=\exp(2\pi i)=1,$$

but we can factor $z^2-1=(z+1)(z-1)$, s $\exp(i\pi)\in\{\pm1\}$. But $\pi<2\pi$, so we cannot have $\exp(i\pi)=1$, so $\exp(i\pi)=-1$ is forced.

3.8.4 Polar Coordinates

We would like to talk about polar coordinates, so for this we would like to access the arctangent function. This requires a little care.

Lemma 3.76. We have that $\cos(-z)=\cos z$ and $\sin(-z)=-\sin z$ for any $z\in\mathbb{C}$.

Proof. This comes down to computing

$$\cos(-z)=\frac{\exp(i(-z))+\exp(-i(-z))}{2}=\frac{\exp(iz)+\exp(-iz)}{2}=\cos z.$$

Similarly,

$$\sin(-z)==\frac{\exp(i(-z))-\exp(-i(-z))}{2i}=-\frac{\exp(iz)-\exp(-iz)}{2i}=-\sin z,$$

which is what we wanted. ■

So now we note that $\cos$ is, by definition of $\pi/2$, nonzero on $[0,\pi/2)$. The above lemma lets us extend this nonzero region to $(-\pi/2,\pi/2)$, permitting the following definition.

Definition 3.77. Given a real number $t\in(-\pi/2,\pi/2)$, we define $\tan t:=\frac{\sin t}{\cos t}$. Note that this definition is legal because $\cos t\neq 0$ for $(-\pi/2,\pi/2)$.

Lemma 3.78. The function $\tan$ is real differentiable and strictly increasing.

Proof. That $\tan$ is real differentiable follows from the quotient rule, which applies because the denominator $\cos$ is nonzero on all of $(-\pi/2, \pi/2)$.[4] In fact, we can compute the derivative as

$$\frac{d}{dt}\tan t = \frac{d}{dt}\frac{\sin t}{\cos t} = \frac{(\cos t)(\cos t) - (\sin t)(-\sin t)}{(\cos t)^2},$$

where we have used Lemma 3.69. So from Lemma 3.68, we see that $\frac{d}{dt}\tan t = \frac{1}{(\cos t)^2}$, which is positive for real numbers t. Thus, $\tan t$ is in fact strictly increasing. ■

We would like to show that $\tan$ surjects onto $\mathbb{R}$. To start, we note $\tan 0 = \sin 0/\cos 0 = 0/1 = 0$.

Lemma 3.79. For $t \in (-\pi/2, \pi/2)$, we have that $\tan(-z) = -\tan z$.

Proof. By brute force, Lemma 3.76 tells us that

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = \tan t,$$

which is what we wanted. ■

Lemma 3.80. The function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection.

Proof. We already know that $\tan$ is injective because it is strictly increasing by Lemma 3.78, so we have left to show the surjection. Additionally, Lemma 3.79 implies that we merely have to show that $\tan$ surjects onto $\mathbb{R}_{\geq 0}$, and because $\tan 0 = 0$, we merely have to show that $\tan$ surjects onto $\mathbb{R}^+$.

Now, $\tan$ is continuous (by Lemma 3.78), so the Intermediate value theorem means that we merely need to show $\tan$ takes on arbitrarily large values in $\mathbb{R}^+$. For this, we claim that

$$\lim_{t \to \pi/2} \tan t = \infty,$$

which will be enough. So fix any $M > 0$. Well, because $\cos$ is continuous, we see that

$$\lim_{t \to \pi/2} \cos t = \cos \pi/2 = 0.$$

Thus, for $\varepsilon = 1/(2M)$, there exists some $\delta_1 > 0$ so that $\pi/2 - \delta_1 < t < \pi/2$ will have $\cos t < \varepsilon$. Because $\cos$ must be positive for $t < \pi/2$, we actually have $0 < \cos t < \varepsilon$. Additionally, because $\sin$ is continuous, we see that

$$\lim_{t \to \pi/2} \sin t = \sin \pi/2 = 1.$$

Thus, there exists some $\delta_2 > 0$ so that $\pi/2 - \delta_2 < t < \pi/2$ will have $\sin t > 1/2$. In particular, setting $\delta := \min\{\delta_1, \delta_2\}$, we see $\pi/2 - \delta < t < \pi/2$ implies that

$$\tan t = \frac{\sin t}{\cos t} > \frac{1/2}{\varepsilon} = \frac{1}{2\varepsilon} = M.$$

This finishes. ■

The above check permits the following definition.

[4] Technically, we should extend $\tan$ to a small open strip around $(-\pi/2, \pi/2)$ in order to make the complex quotient rule work and then restrict $\tan$ afterwards. We will settle for merely saying that we should do this instead of actually doing it.

Definition 3.81 $(\arctan)$**.** We define $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ to be the inverse function of $\tan$.

Note that the above definition makes sense because $\tan$ is a bijection $(-\pi/2, \pi/2) \to \mathbb{R}$. In fact, the proof of Lemma 3.80 lets us say

$$\lim_{t\to\infty} \arctan t = \frac{\pi}{2}.$$

In fact, we see $\tan(-t) \to -\pi/2$ as $t \to \infty$, so

$$\lim_{t\to-\infty} \arctan t = -\frac{\pi}{2}.$$

We are now ready to give polar form.

Proposition 3.82 (Polar form)**.** For any $z \in \mathbb{C}^\times$, there exist unique real numbers $r > 0$ and $\theta \in [-\pi, \pi)$ such that $z = r\exp(i\theta)$.

Proof. We start by showing uniqueness because it is easier: if $r_1 \exp(i\theta_1) = r_2 \exp(i\theta_2)$, then taking magnitudes tells us that

$$|r_1| = |r_1 \exp(i\theta_1)| = |r_2 \exp(i\theta_2)| = |r_2|,$$

where we have used Corollary 3.62. Because r_1 and r_2 are positive real numbers, we conclude $r_1 = r_2$. So now

$$\exp(i(\theta_1 - \theta_2)) = \exp(i\theta_1)/\exp(i\theta_2) = 1$$

using Proposition 3.58. By Proposition 3.74, this forces $\theta_1 - \theta_2 \in 2\pi i\mathbb{Z}$. However, $-\pi \le \theta_1, \theta_2 < \pi$ implies that

$$-2\pi < \theta_1 - \theta_2 < 2\pi,$$

so $\theta_1 - \theta_2 = 0$ is forced, so $\theta_1 = \theta_2$.

We now show that the r and θ actually exist for any $z \in \mathbb{C}^\times$. As above, we take $r = |z|$, so we need to set θ. Well, we see that Remark 3.67 gives

$$r\exp(i\theta) = r\cos\theta + ir\sin\theta.$$

So we want a value $\theta \in [-\pi, \pi)$ such that $\operatorname{Re} z = r\cos\theta$ and $\operatorname{Im} z = r\sin\theta$. Noting that $z \ne 0$ implies $r \ne 0$, we want to choose θ such that

$$(\cos\theta, \sin\theta) \stackrel{?}{=} (\operatorname{Re} z/r, \operatorname{Im} z/r).$$

In particular, we set $a := \operatorname{Re} z/r$ and $b := \operatorname{Im} z/r$ so that $a^2 + b^2 = \frac{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}{r^2} 1$. So, given $(a, b) \in \mathbb{R}^2$ such that $a^2 + b^2 = 1$, we need to find θ such that

$$(\cos\theta, \sin\theta) \stackrel{?}{=} (a, b).$$

We set θ by hand. We do casework.

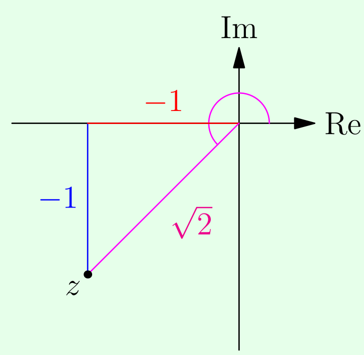
- If $a = 0$, then $\cos\theta = 0$ and $b = \pm 1$. Well, for $b = \pm 1$, we set $\theta = \pm\frac{\pi}{2}$ so that $\cos \pm\frac{\pi}{2} = \cos\frac{\pi}{2}$ and $\sin \pm\frac{\pi}{2} = \pm\sin\frac{\pi}{2} = \pm 1$ by Lemma 3.76.
- If $a > 0$, then we choose $\theta = \arctan(b/a) \in (-\pi/2, \pi/2)$. In particular, we see that $\tan\theta = \frac{b}{a}$, so we have the system of equations

  $$\frac{\sin\theta}{\cos\theta} = \frac{b}{a} \qquad \text{and} \qquad (\cos\theta)^2 + (\sin\theta)^2 = 1.$$

  In particular, $\sin\theta = \frac{b}{a}\cos\theta$, so $\left(1 + \frac{b^2}{a^2}\right)(\cos\theta)^2 = 1$, so $\cos\theta = \pm a$ because $a^2 + b^2$. But $\cos\theta$ is positive on $(-\pi/2, \pi/2)$, so we see that $\cos\theta = a$, from which we can read $\sin\theta = \frac{b}{a} \cdot a = b$.

- If $a < 0$, we note $-a > 0$, so we use the above argument to choose $\gamma = \arctan(b/-a) \in (-\pi/2, \pi/2)$ so that
$$\cos\gamma = -a \qquad \text{and} \qquad \sin\gamma = b.$$
In particular, we see that $-\gamma$ has
$$\exp(i(-\gamma)) = \overline{\exp(i\gamma)} = \overline{-a+bi} = -a-bi.$$
In particular, multiplying this through by $\exp(i\pi) = -1$, we see that $\exp(i(\pi-\gamma)) = a+bi$, giving $\cos(\pi-\gamma) = a$ and $\sin(\pi-\gamma) = b$.
It remains to force $\pi-\gamma$ into $[-\pi,\pi)$. However, $\exp(it)$ is periodic with period 2π, so we can callously shift $2\pi-\gamma$ into $[0,2\pi)$ via the division algorithm and then subtract γ to get a representative of $\pi-\gamma$ in $[-\pi,\pi)$. This finishes. ■

Example 3.83. We take $z = -1-i$. Here, $|z| = \sqrt{1+1} = \sqrt{2}$; further $\operatorname{Re} z < 0$, so we compute
$$\pi - \arctan(-1/-(-1)) = \pi - -\frac{\pi}{4} = \frac{5\pi}{4},$$
so we take $\theta = -3\pi/4$ after shifting. So the above argument assures us that $z = \sqrt{2}\exp(-i\cdot 3\pi/4)$. Here is the image.



3.9 March 4

Good morning everyone. Today's lecture was not recorded.

- Homework #5 will be uploaded today, due next Friday.
- The class average on the midterm was a 74; it might have been a little long. There will probably be something approximately equal to a 6-point curve.

3.9.1 Arguments

Today we talk more about the exponential function. Last time we proved the following.

Proposition 3.82 (Polar form). For any $z \in \mathbb{C}^\times$, there exist unique real numbers $r > 0$ and $\theta \in [-\pi,\pi)$ such that $z = r\exp(i\theta)$.

As a brief review, we recall that we took $r = |z|$, and we computed θ in terms of some $\arctan$s. Essentially, this means that we can effectively compute polar form without tears.

Remark 3.84. The interval $[-\pi, \pi)$ is somewhat arbitrary; we can choose any set of representatives for $\mathbb{R}/2\pi\mathbb{Z}$. To see this, we note that the unique $\theta \in [-\pi, \pi)$ will have a unique representative in any set of representatives for $\mathbb{R}/2\pi\mathbb{Z}$ and vice versa. For example, any half-open interval of length 2π (such as $[0, 2\pi)$) will do the trick. To see this,

We can in fact use polar form to talk about the exponential map.

Corollary 3.85. For any $z \in \mathbb{C}^\times$, there exists some $w \in \mathbb{C}$ such that $\exp(w) = z$.

Proof. To start, we know that we can write $z = r\exp(i\theta)$. So, using real analysis, we set

$$w := \log r + i\theta,$$

where $\log : \mathbb{R}^+ \to \mathbb{R}$ is the real logarithm. Thus,

$$z = r\exp(i\theta) = \exp(\log r)\exp(i\theta) = \exp(\log r + i\theta) = \exp(w),$$

which is what we wanted. ■

Continuing to talk about polar form, we have the following definition.

Definition 3.86 (Argument)**.** Given a complex number $z \in \mathbb{C}^\times$, we define the *principal argument* $\arg z \in [-\pi, \pi)$ by writing $z := r\exp(i\theta)$ and taking $\arg z := \theta$. More generally, for any $\eta \in \mathbb{R}$, we define

$$\arg_\eta : \mathbb{C}^\times \to [\eta, \eta + 2\pi)$$

by $\arg_\eta(z) := \arg z + \pi + \eta$.

Example 3.87. We have that $\arg_{-\pi} = \arg$.

3.9.2 The Complex Logarithm

The logarithm is somewhat subtle, so we have to be careful. We take the following definition.

Definition 3.88 (Branch of the logarithm)**.** Fix $\Omega \subseteq \mathbb{C} \setminus \{0\}$ an open, connected subset. A *branch of the logarithm* is a continuous function $f : \Omega \to \mathbb{C}$ such that

$$\exp(f(z)) = z.$$

Intuitively, f will "look like" an inverse for $\exp$.

Nevertheless, there is a fairly standard choice of branch.

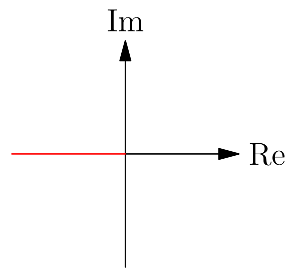
Definition 3.89. Taking $\Omega := \mathbb{C} \setminus \mathbb{R}_{\le 0}$, we define the *principal branch of the logarithm* as $\operatorname{Log} : \Omega \to \mathbb{C}$ by

$$z \mapsto \log|z| + i\arg z.$$

Remark 3.90. Again, $\log : \mathbb{R}^+ \to \mathbb{R}$ here is the real logarithm, legal because $z \ne 0$ so that $|z| > 0$.

In particular, we are essentially using the construction from back in Corollary 3.85.

As some brief geometric commentary, we are calling these "branches" our open sets Ω are typically $\mathbb{C}$ minus a single line, and the subtlety of why we have to do this is to make the logarithm continuous. For example, in the principal branch, we deleted $\mathbb{R}_{\le 0}$, which has the following image.



We should probably check that Log is actually well-formed; namely, it turns out that we had some choice in our construction of Log.

Lemma 3.91. Fix $z, w \in \mathbb{C}$ such that $\exp(z) \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$ and $\operatorname{Log}\exp(z) = w$. Then there is a $k \in \mathbb{Z}$ such that $z = w + 2\pi i k$.

Proof. Write $z = x + iy$ so that $\exp z = \exp(x)\exp(iy)$. Now, we know that $\exp(\alpha) = 0$ if and only if $\alpha \in 2\pi i\mathbb{Z}$, so for example, we can write

$$\exp(yi + 2\pi i n) = \exp(iy)$$

for any $n \in \mathbb{Z}$. So, by the division algorithm, we choose a $k \in \mathbb{Z}$ so that

$$\widetilde{y} := y + 2\pi k$$

has $\widetilde{y} \in [-\pi, \pi)$. But now, because $\exp(z) \notin \mathbb{R}_{\ge 0}$, we see that we cannot have $\widetilde{y} = -\pi$ because this would make $\exp(iy) = -1$ and therefore $\exp z = -\exp(x) \in \mathbb{R}_{\le 0}$.

The point of choosing this $\widetilde{y}$ is that we still have $\exp(z) = \exp(x)\exp(iy) = \exp(x) = \exp(i\widetilde{y})$, but now $\widetilde{y} \in (-\pi, \pi)$, so we are assured

$$\arg\exp(z) = \widetilde{y}.$$

At this point, we just write out

$$w = \operatorname{Log}\exp(z) = \operatorname{Log}\exp(x + iy) = \log(|\exp(x)\exp(it)|) + i\arg\exp(z) = x + i\widetilde{y}.$$

So now we can write $w = x + iy - 2\pi i k$, which is what we wanted. ■

Let's return to our discussion of branches. There are a few reasons why we want "branches" for Log. Roughly speaking, here is the reasoning.

- The function $\exp$ is not injective: it has kernel $\ker\exp = 2\pi i\mathbb{Z}$. In particular, if we wanted to define Log on $1 \in \mathbb{C}$, then we need to make a choice among the representatives in $2\pi i\mathbb{Z}$.
- In order to avoid having to make a choice, we chose Log to have imaginary part in $[-\pi, \pi)$ always (in fact, $-\pi$ is illegal because Log doesn't take inputs in $\mathbb{R}_{\le 0}$).
- But making this choice makes Log not continuous at values in $\mathbb{R}_{\le 0}$ because (notably!) $\arg z$ is not continuous on $\mathbb{R}_{\le 0}$. In particular, $z \to -1$ from above gives $\arg z \to \pi$ while $z \to -1$ from below gives $\arg z \to -\pi$.
- So the point of introducing the branch is to simply throw out the $\mathbb{R}_{\le 0}$ and recover our continuity.

In particular, we do indeed have that $\arg$ is continuous where we want it to be.

Lemma 3.92. The restricted argument function $\arg : \mathbb{C} \setminus \mathbb{R}_{\le 0} \to [-\pi, \pi)$ is continuous.

Proof. We defined $\arctan$ piece-wise, so it is not hard to check that $\arg$ is continuous on each quadrant. So the problematic regions are the imaginary axes, for which we refer to Eterović's notes for the details. ■

Corollary 3.93. The function $\operatorname{Log} : \mathbb{C} \setminus \mathbb{R}_{\le 0} \to \mathbb{C}$ is continuous.

Proof. Well, we write

$$\operatorname{Log} z = \log|z| + i \arg z,$$

and we now know that each component is continuous, so the total function is continuous. ■

In fact, we get that Log is holomorphic, essentially inherited from $\exp$.

Lemma 3.94. Fix $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ connected and open subsets. Further, suppose we have a continuous function $f : \Omega_1 \to \Omega_2$ and a holomorphic function $g : \Omega_2 \to \Omega_1$ such that $g(f(z)) = z$ for each $z \in \Omega_1$. Then f is holomorphic on Ω_1 with derivative

$$f'(z) = \frac{1}{g'(f(z))}.$$

Proof. Omitted. ■

Proposition 3.95. The function Log is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$.

Proof. We simply apply Lemma 3.94 with $\Omega_1 = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ and $\Omega_2 = \mathbb{C}$ and $f = \operatorname{Log}$ and $g = \exp$. The hypotheses are satisfied. ■