18.708: Topics in Algebra

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

de Rham Cohomology in Mixed Characteristic

These talks were given by Alexander Petrov.

1.1 February 2

Here we go.

1.1.1 Algebraic de Rham Cohomology

Let's begin by describing what we mean by de Rham cohomology. We will consider a smooth variety X over an algebraically closed field F.

> **Definition 1.1** (smooth). We say that a variety X over a field F is *smooth* if and only if $\Omega_{X/F}$ is a vector bundle of rank $\dim X$ on each connected component. Here, on an affine open subset $U \subseteq X$, recall that $\Omega_{X/F}(U)$ is spanned by symbols of the form $f \, dg$, where the symbol d is (as usual) F-linear and satisfies the Leibniz rule.

> **Definition 1.2** (algebraic de Rham cohomology). Fix a smooth variety X over a field F. Then one can iterate the F-linear map $d \colon \mathcal{O}_X \to \Omega_{X/F}$ to a map $d \colon \Omega^i_{X/F} \to \Omega^{i+1}_{X/F}$ for each i, where $\Omega^i_{X/F} := \wedge^i \Omega_{X/F}$. We now define the *de Rham complex* to the complex
> $$\Omega^\bullet_{X/F} \colon 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/F} \xrightarrow{d} \cdots,$$
> and we define the *de Rham cohomology* $\mathrm{H}^n_{\mathrm{dR}}(X/F)$ to be the nth hypercohomology of $\Omega^\bullet_{X/F}$. Here, hypercohomology means the total cohomology of some produced acyclic double complex which resolves the complex (e.g., a Čech resolution). Note that this hypercohomology is merely a vector space over F.

> **Example 1.3.** The map $d \colon \Omega^1_{X/F} \to \Omega^2_{X/F}$ is given by $d(f \, dg) = df \wedge dg$.

Example 1.4. Suppose that X is affine. Then vector bundles are already acyclic, so the hypercohomology does nothing. Thus,
$$\mathrm{H}^n_{\mathrm{dR}}(X/F) = \mathrm{H}^n\left(X; 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/F} \xrightarrow{d} \cdots\right).$$
As usual, this is $\ker(d|_{\Omega^n}) / \operatorname{im}(d|_{\Omega^{n-1}})$.

Remark 1.5. If X is affine and $i > \dim X$, then $\Omega^i_{X/F}$ vanishes, so the algebraic de Rham cohomology also vanishes.

Remark 1.6. A different definition is required for non-smooth X. Roughly speaking, one should embed into a smooth variety and take cohomology there.

Here is one way to convince ourselves that this is a reasonable cohomology theory.

Theorem 1.7 (Grothendieck). Suppose that X is a smooth variety over $\mathbb{C}$. Then there is a canonical isomorphism
$$\mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C}); \mathbb{C}) \to \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}).$$
Here, the left-hand side is Betti cohomology (also called singular cohomology).

Sketch. We argue in the case that X is affine. Then $X(\mathbb{C})$ already has a notion of $\Omega^{i,\mathrm{an}}_{X/\mathbb{C}}$ given by the holomorphic forms. Algebraic forms embed into holomorphic ones, which produces a morphism

$$
\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{O}_X(X) & \longrightarrow & \Omega^1_{X/\mathbb{C}}(X) & \longrightarrow & \Omega^2_{X/\mathbb{C}}(X) & \longrightarrow \cdots \\
& \Big\uparrow & & \Big\downarrow & & \Big\downarrow & \\
0 \longrightarrow & \mathcal{O}^{\mathrm{an}}_X(X) & \longrightarrow & \Omega^{1,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow & \Omega^{2,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow \cdots
\end{array}
$$

of complexes. It then turns out that this is an isomorphism on cohomology, so we reduce to comparing analytic de Rham cohomology with singular cohomology.

This is now a problem of analysis. One can pass from holomorphic differentials to smooth differentials via a similar process, which produces another morphism

$$
\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{O}^{\mathrm{an}}_X(X) & \longrightarrow & \Omega^{1,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow & \Omega^{2,\mathrm{an}}_{X/\mathbb{C}}(X) & \longrightarrow \cdots \\
& \Big\downarrow & & \Big\downarrow & & \Big\downarrow & \\
0 \longrightarrow & C^\infty(X(\mathbb{C}), \mathbb{C}) & \longrightarrow & \Omega^1_{C^\infty}(X(\mathbb{C})) & \longrightarrow & \Omega^2_{C^\infty}(X(\mathbb{C})) & \longrightarrow \cdots
\end{array}
$$

of complexes, which is also an isomorphism on complexes. We are now reduced to the setting of de Rham's theorem for real manifolds. ∎

Example 1.8. Consider $X := \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = \operatorname{Spec} k[t, 1/t]$.

- Our differential map $d\colon \mathbb{C}[t, 1/t] \to \mathbb{C}[t, 1/t]\, dt$ sends t^n to $nt^{n-1}\, dt$. Thus, $\mathrm{H}^0_{\mathrm{dR}}(X)$ is one-dimensional given by the constants, and $\mathrm{H}^1_{\mathrm{dR}}(X)$ is one-dimensional spanned by dt/t.

- The point above works also for holomorphic differentials. The interesting bit is in degree 1, where the point is that there is no global antiderivative for dx/x.

- On the other hand, $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle, so we expect its singular cohomology to be supported in degrees 0 and 1, where it should be one-dimensional.

> **Corollary 1.9** (Artin vanishing). If X is an affine algebraic complex smooth variety, then $\mathrm{H}^n(X(\mathbb{C}); \mathbb{C}) = 0$ for $n > \dim X$.

Proof. The algebraic de Rham cohomology complex vanishes above $\dim X$. ∎

> **Corollary 1.10.** Fix a smooth variety X over $\mathbb{C}$. Then $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C})$ is finite-dimensional.

Proof. Pass to singular cohomology. ∎

> **Remark 1.11.** This corollary still admits algebraic proofs in characteristic zero by working with holonomic $\mathcal{D}$-modules. Pavel Etingof claims that there is an algebraic proof using the fact that the direct image of a holonomic $\mathcal{D}$-module is a holonomic $\mathcal{D}$-module.

We would like to point out that our de Rham cohomology is algebraic but still interesting.

> **Remark 1.12.** Suppose that X is smooth over $\mathbb{Q}$. Base-changing by a field is exact, so
>
> $$\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{Q})_{\mathbb{C}} \cong \mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}).$$
>
> However, Theorem 1.7 grants an isomorphism to $\mathrm{H}^n(X(\mathbb{C}); \mathbb{C}) \cong \mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C}); \mathbb{Z})_{\mathbb{C}}$. Notably, we then find a lattice and a rational structure over in some complex vector space, but the comparison between the two is quite interesting mathematically (and amounts to the study of periods).

> **Example 1.13.** In the case that $X = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\}$, the comparison between $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})_{\mathbb{C}}$ and $\mathrm{H}^1_{\mathrm{B}}(X(\mathbb{C}); \mathbb{Z})$ is mediated by a constant $2\pi i$. Indeed, once unwinds the de Rham theorem, this amounts to the statement that a contour integral of dx/x going once around the origin is $2\pi i$.

1.1.2 Frobenius Structure

We now pass to positive characteristic. Let k be a perfect field of positive characteristic p, and we may still consider a smooth variety X.

> **Remark 1.14.** If k is perfect, then $\Omega^1_{X/k} = \Omega^1_{X/\mathbb{F}_p}$ by doing some thinking about inseparable extensions. The moral is that
>
> $$y^{1/p}\, dy = d\left((y^{1/p})^p\right),$$
>
> so the coefficients can be brought down when everything is a pth power.

This cohomology is rather strangely behaved.

> **Example 1.15.** Take $X := \mathbb{A}^1_k$. The de Rham cohomology still lives in degrees zero and one, so we would like to study the kernel and cokernel of the k-linear map $d\colon k[t] \to k[t]\, dt$ given by $t^n \mapsto nt^{n-1}$.
>
> - We see that $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \ker d$ is spanned by t^{pi} for each i.
>
> - We see that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \operatorname{im} d$ is infinite-dimensional because the differentials $t^{mp-1}\, dt$ fail to be in the image. In fact, these classes form a basis.

Let's try to view these infinite-dimensional groups as a feature instead of a bug. Indeed, it turns out that the de Rham complex has some extra structure. The de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots$$

is merely made of sheaves of k-vector spaces over X. In characteristic zero, this is all the structure present, but in characteristic p, we have more structure.

> **Notation 1.16.** Fix a variety X over a field k of characteristic p. For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we define
>
> $$\mathcal{F}^p := \{f^p : f \in \mathcal{O}_X\}$$
>
> to locally be given by the pth powers.

The moral is that $d(f^p) = 0$ always, so the de Rham complex is in fact $\mathcal{O}_X^p$-linear! Let's attempt to codify this.

> **Definition 1.17** (relative Frobenius). Fix a scheme X over a field k of characteristic p. Then there is an *absolute Frobenius* $F_{\mathrm{abs}} \colon X \to X$ which is the identity on topological spaces and the pth power on sheaves. This is a morphism of schemes but not of k-schemes (in general). The *relative Frobenius* $F \colon X \to X^{(1)}$ is the morphism fitting into the following diagram.
>
>

> **Remark 1.18.** Note that $X^{(p)}$ is isomorphic to X as a scheme but not as a k-scheme! However, we now benefit because the relative Frobenius F is morphism of k-schemes.

> **Remark 1.19.** The relative Frobenius $F \colon X \to X^{(1)}$ is finite flat of degree $p^{\dim X}$

> **Example 1.20.** If $X = \operatorname{Spec} k[t_1, \ldots, t_n]$, then $X^{(1)} = \operatorname{Spec} k[t_1^p, \ldots, t_n^p]$. Thus, we see that the embedding
>
> $$k[t_1^p, \ldots, t_n^p] \subseteq k[t_1, \ldots, t_n]$$
>
> is indeed finite flat of degree p^n.

We now see that

$$0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/F} \to \cdots$$

is a complex of quasicoherent sheaves on $X^{(1)}$. In fact, because F is finite flat, these are all vector bundles: $F_*\mathcal{O}_X$ has rank $p^{\dim X}$ and $F_*\Omega^i_{X/k}$ has rank $p^{\dim X}\binom{\dim X}{i}$. Because $\mathcal{O}_{X^{(1)}} = (F_*\mathcal{O}_X)^p$, we see that this complex is in fact $\mathcal{O}_{X^{(1)}}$-linear.

> **Example 1.21.** Take $X = \operatorname{Spec} k[t]$. Then $X^{(1)} := \operatorname{Spec} k[t^p]$, and $d \colon k[t] \to k[t]\,dt$ is $k[t^p]$-linear! Thus, $\mathrm{H}^i_{\mathrm{dR}}(X/k)$ was required to be given by $k[t^p]$-modules, which explains why we received vector spaces of infinite dimension.

Note that passing through F_* is not going to adjust the underlying k-vector spaces, so

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \mathbb{H}^n_{\mathrm{Zar}}\left(X^{(1)}; 0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/k} \xrightarrow{d} \cdots\right).$$

To see why this has globalized the $\mathcal{O}^p_X$-linearity, we need the Cartier isomorphism.

Theorem 1.22 (Cartier isomorphism)**.** Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism
$$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$
Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Remark 1.23. This is a reason why characteristic p may be more convenient than characteristic 0: one could still try to understand $\mathcal{H}^i(\Omega^\bullet_{X/k})$ when $\mathrm{char}\, k = 0$, but this has no easy answer.

Example 1.24. Consider $X = \mathbb{A}^1_k$. Then $\mathcal{H}^1$ is given by the module

$$\frac{k[t]\, dt}{d(k[t])},$$

which our formalism now remembers is a $k\,[t^p]$-module. And indeed, we can show that this is isomorphic to $k\,[t^p] \cdot t^{p-1}\, dt$. Setting $s := t^p$, we know that $\Omega^1_{X^{(1)}/k}$ is given by the module $k[s]\, ds$, so our isomorphism of modules is given by sending ds to $t^{p-1}\, dt$. One can even check that this isomorphism is canonical in the sense that it will not change under automorphisms of $\mathbb{A}^1$.

We will prove Theorem 1.22 later after a detour.

1.1.3 Crystalline Cohomology

We continue with our perfect field k of positive characteristic p. Our story so far has taken a variety X over a field k, and then we have produced some (total) complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$ in the derived category $D(\mathrm{Vec}_k)$. Crystalline cohomology will allow us to produce an answer in characteristic 0 instead of characteristic p. The idea is to "choose" a lift to characteristic p and then check that the answer is independent of the lift.

The correct formalism for this lifting is that of a "formal scheme."

Definition 1.25 (Witt ring)**.** Fix a perfect field k of characteristic p. Then there is a ring $W(k)$ satisfying that

- $W(k)$ is p-torsion-free,

- $W(k)/p \cong k$, and

- $W(k)$ is the limit of the $W(k)/p^n$ as $n \to \infty$.

This ring $W(k)$ turns out to be unique up to unique isomorphism. We may write $W_n(k) := W(k)/p^n$.

Example 1.26. One can see that $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W(\overline{\mathbb{F}}_p)$ is its unramified closure.

Remark 1.27. There is a completely explicit construction of $W(k)$, but it is rather involved: given a p-torsion-free ring R, we identify $W(R) \coloneqq R^{\mathbb{N}}$ but with ring structure chosen so that

$$(a_0, a_1, a_2, \dots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$$

is a ring homomorphism $W(R) \to R$. It turns out that this ring structure is given by some polynomials (called "ghost coordinates"), so we are allowed to define $W(k)$. From a higher level, it turns out that $W(k)$ is the unique deformation of k, which exists because $\Omega^1_{k/\mathbb{F}_p} = 0$.

Definition 1.28 (formal scheme)**.** Fix a perfect field k of characteristic p. A p-*adic formal scheme* X is a collection of schemes X_n over $W_n(k)$ equipped with isomorphisms

$$X_{n+1} \times_{W_{n+1}(k)} W_n(k) \to X_n.$$

The structure sheaf $\widehat{\mathcal{O}}_X$ is the inverse limit of the $\mathcal{O}_{X_n}$s.

Example 1.29. Given a scheme Y over $W(k)$, we can produce a formal scheme $\widehat{Y}$ with $\widehat{Y}_n \coloneqq Y \times_{W(k)} W_n(k)$ and the induced internal isomorphisms.

Remark 1.30. If X_1 is affine, then all the nilpotent thickenings are affine, so we may say that the full formal scheme is affine.

Remark 1.31. We can even define $\widehat{\Omega} X^1_{\widetilde{X}}$.

We can now describe crystalline cohomology.

Theorem 1.32. Fix a perfect field k of positive characteristic p. Then there is a functor sending smooth k-varieties X to a complex $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ in the derived category $D(\mathrm{Mod}_{W(k)})$ satisfying the following.

(a) There is a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes^{\mathbb{L}}_{W(k)} k \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$.

(b) If $\widetilde{X}$ is a smooth formal scheme over $W(k)$ (meaning that $\widetilde{X}_n$ is smooth over $W_n(k)$ for all n), then the scheme $X \coloneqq \widetilde{X} \times_{W(k)} k$ has

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \cong \mathrm{R}\Gamma_{\mathrm{dR}}\left(X; \widehat{\mathcal{O}}_{\widetilde{X}} \xrightarrow{d} \widehat{\Omega}^1_{\widetilde{X}} \xrightarrow{d} \cdots\right).$$

This right-hand side can be thought of as $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/W(k))$.

Remark 1.33. Here, (a) immediately tells us that the cohomology of $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$ is not expected to be finitely generated.

Remark 1.34. There is something remarkable here, which is that choosing two different lifts of X to a smooth formal scheme produces the same cohomology!

Remark 1.35. It turns out that flatness is equivalent to smoothness in this context.

1.2 February 4

Today we continue our discussion of crystalline cohomology.

1.2.1 Example of Crystalline Cohomology

We continue working over a perfect field k of characteristic p. Let's run an example.

> **Remark 1.36.** If our formal scheme $\widetilde{X}$ is smooth and affine, then $\mathrm{R}\Gamma_{\mathrm{dR}}(X/W(k))$ can be computed directly as the complex
> $$0 \to \mathcal{O}(\widetilde{X}) \xrightarrow{d} \Omega^1(\widetilde{X}) \xrightarrow{d} \Omega^2(\widetilde{X}) \xrightarrow{d} \cdots.$$

> **Example 1.37.** Take $X = \mathbb{A}_k^1$, which we can deform to a formal scheme $\widetilde{X} = \widehat{\mathbb{A}}_k^1$. Then
> $$\mathcal{O}(\widetilde{X}) = \lim W_n[t],$$
> which we may refer to as $W\langle t\rangle$. Here, $W\langle t\rangle$ contains those power series whose terms are more and more divisible by p. Similarly, we can compute that $\Omega^1(\widetilde{X}) = W\langle t\rangle \, dt$. For example, we now see that W has characteristic zero, so the kernel of the differential $d\colon W\langle t\rangle \to W\langle t\rangle \, dt$ is exactly the constants. On the other hand, Theorem 1.32 tells us that $\mathrm{H}_{\mathrm{dR}}^1(\widetilde{X}/W(k))$ is required to be quite interesting because the de Rham complex of $\mathbb{A}_k^1$ is interesting.

> **Remark 1.38.** One can extend Theorem 1.32 slightly: even if some k-scheme X can only be deformed to a smooth scheme X_n over $W_n(k)$, then we still have
> $$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes_{W(k)}^{\mathbb{L}} W_n(k) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X_n/W_n(k)).$$

The above example falls under the paradigm where the formal scheme comes from a genuine scheme. Here are some motivational remarks about this case.

> **Example 1.39.** Let's consider the case where the formal scheme comes from a genuine scheme. Given a smooth scheme Y over $W(k)$ with formal completion $\widehat{Y}$, then $\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k))$ is the derived p-adic completion of $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$. Explicitly, one takes a complex representing $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ and takes a p-completion of each module. (This turns out to be a well-defined operation on the derived category.)

> **Example 1.40.** We continue the previous example. Suppose further that Y is smooth and proper. Then it turns out that $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ can be represented by finitely generated projective $W(k)$-modules, so the p-completion does nothing, so
> $$\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k)) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k)).$$

1.2.2 The Mysterious Functor

Crystalline cohomology produces the following strange functoriality.

> **Corollary 1.41.** Fix a smooth formal scheme $\widetilde{X}$ over $W(k)$. Given an endomorphism f of $\widetilde{X}$ such that f is the identity $(\bmod\ p)$, then f^* is the identity on the cohomology groups $\mathrm{H}_{\mathrm{dR}}^n(\widetilde{X}/W(k))$.

Proof using Theorem 1.32. By Theorem 1.32, one may pass to crystalline cohomology of the reduction $X :=$ $X \times_k W(k)$, where the result has no content. ∎

Theorem 1.32 admits the following version in characteristic zero, where we heuristically replace the deformation $W(k) \to k$ with $\mathbb{C}[[t]] \to \mathbb{C}$.

> **Proposition 1.42.** Given a smooth formal scheme $\widetilde{X}$ over $\mathbb{C}[[t]]$ with reduction X over $\mathbb{C}$. Then
>
> $$\mathrm{R\Gamma_{dR}}(\widetilde{X}/\mathbb{C}[[t]]) \cong \mathrm{R\Gamma_{dR}}(X/\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[t]].$$

Sketch. The idea of the proof is to pass through the larger de Rham cohomology $\mathrm{R\Gamma_{dR}}(\widetilde{X}/\mathbb{C})$; by definition, this is the inverse limit of the de Rham cohomology of the reductions $\widetilde{X}_n$. By the Poincaré lemma, it turns out that $\mathrm{R\Gamma_{dR}}(\widetilde{X}/\mathbb{C})$ is quasi-isomorphic to $\mathrm{R\Gamma_{dR}}(X/\mathbb{C})$. (Approximately speaking, this is saying that a tubular neighborhood of X is homotopic to X.) The result now follows by extending scalars. ∎

> **Remark 1.43.** This proof does not work in our mixed characteristic situation $W(k) \to k$ because there is no direct way to geometrically link $\widetilde{X}$ and X via a Poincaré lemma.

To prove Theorem 1.32, it will turn out that the key ideas can be used to instead prove Corollary 1.41.

Proof of Corollary 1.41 when $p > 2$. We attempt to give a version of Poincaré's lemma. The idea there is to use Cartan's formula, which arises by viewing differentials as living in flows of vector fields.

1. Our analog of this will be

   $$\log f := \sum_{n \geq 1} (-1)^{n+1} \frac{(f - \mathrm{id})^{\circ n}}{n},$$

   viewed as a W-linear endomorphism of $\mathcal{O}_{\widetilde{X}}$. To see that this makes sense, note that $f - \mathrm{id} = pA$ for some endomorphism A, so it is at least true that the terms of the series are divisible by larger and larger powers of p. Thus, the series descends to a compatible sequence of endomorphisms for each $\mathcal{O}_{\widetilde{X}_n}$ and hence an endomorphism of $\mathcal{O}_{\widetilde{X}}$.

2. It further turns out that $\log f$ is a derivation of $\mathcal{O}_{\widetilde{X}}$, meaning that

   $$(\log f)(ab) = (\log f)(a)b + a(\log f)(b).$$

   This is a formal consequence of the fact that f is a ring map, and $\log f$ takes products to sums. (Indeed, one can already find this in characteristic zero.)

3. To continue, we take $p > 2$ for simplicity. Indeed, then one can see $f = \exp(\log f)$, meaning that

   $$f \stackrel{?}{=} \sum_{n \geq 0} \frac{(\log f)^{\circ n}}{n!}.$$

   The right-hand sum makes sense because $\log f$ is divisible by p, which then implies that the sum converges. Thus, we may write $\log f = pD$ for some derivation D. The equality now follows from some formal calculation of the series.

4. We now pass to the de Rham complex. We would like to check that the endomorphism $f^* \colon \Omega^{\bullet}_{\widetilde{X}} \to \Omega^{\bullet}_{\widetilde{X}}$ is a quasi-isomorphism. Well, there is a "Lie derivative" L_D for which f^* acts on $\Omega^i_{\widetilde{X}}$ by

   $$\sum_{n \geq 0} \frac{p^n}{n!} L_D^{\circ n}.$$

Let's explain what this L_D is, which we may define affine-locally. For any $W(k)$-algebra A, a derivation $W \colon A \to A$ induces an endomorphism L_D on differentials as follows. Indeed, $\mathrm{id} + \varepsilon D$ is an automorphism of the thickening $A[\varepsilon]/\left(\varepsilon^2\right)$, so $\mathrm{id} + \varepsilon D$ acts on $\Omega^i_{A[\varepsilon]/(\varepsilon^2)}$. Certainly it must reduce to identity modulo ε, so we conclude that $(\mathrm{id} + \varepsilon D)^* = \mathrm{id} + \varepsilon L_D$ for some endomorphism L_D on differentials!

5. The previous step now tells us that $f^* = \mathrm{id} + L_D \circ G$ for some G. We will be done as soon as we can check that the endomorphism L_D of the de Rham complex is homotopic to zero. In other words, we are on the hunt for maps $\iota_D \colon \Omega^i_{\widetilde{X}} \to \Omega^{i-1}_{\widetilde{X}}$ fitting into the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\widetilde{X}} & \longrightarrow & \Omega^1_{\widetilde{X}} & \longrightarrow & \Omega^2_{\widetilde{X}} & \longrightarrow & \cdots \\
& & \Big\downarrow{\scriptstyle L_D} & {\scriptstyle \iota_D}\nearrow & \Big\downarrow{\scriptstyle L_D} & {\scriptstyle \iota_D}\nearrow & \Big\downarrow{\scriptstyle L_D} & & \\
0 & \longrightarrow & \mathcal{O}_{\widetilde{X}} & \longrightarrow & \Omega^1_{\widetilde{X}} & \longrightarrow & \Omega^2_{\widetilde{X}} & \longrightarrow & \cdots
\end{array}
$$

so that $L_D = d\iota_D + \iota_D D$. This is precisely Cartan's formula, which admits an algebraic proof. (Namely, one takes ι_D to be a contraction whose vector field is D.) To show this, one can show directly that $d\iota_D + \iota_D D$ is a derivation, and then one can check that there is an equality on functions and 1-forms. ∎

> **Remark 1.44.** The construction of L_D may appear ad-hoc. The intuition here is that our objects are not only functorial under endomorphisms but also under derivations.

It turns out that all the interesting algebra to prove Theorem 1.32 is already contained in the proof of Corollary 1.41, though this should not be obvious yet.

1.2.3 The Cartier Isomorphism

Recall that we wanted to prove Theorem 1.22.

> **Theorem 1.22** (Cartier isomorphism). Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism
> $$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$
> Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Proof. Let's start with $i = 0$. Note that there is a map

$$\mathcal{O}_{X^{(1)}} \to \ker\left(d \colon F_*\mathcal{O}_X \to F_*\Omega^1_X\right)$$

given by sending a function f to f^p. We would like to check that this is an isomorphism. Well, it is enough to check this on an open cover, but because X is smooth, we may pass étale-locally along some $f \colon U \to \mathbb{A}^n_k$ (where $U \subseteq X$ is some open subset) because the de Rham complex is immune to such deformations: indeed,

$$\left(F_*\Omega^0_X\right)\big|_{U^{(1)}} \cong f^{(1)*}\left(F_*\Omega^0_{\mathbb{A}^n_k}\right).$$

Thus, we may pass to $X = \mathbb{A}^n$.

We now work in general. By working locally, we may assume that X is affine and equal to some $\operatorname{Spec} A$. Then we would like an isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(X/k) \overset{?}{\cong} \Omega^i_{A^{(1)}/k}$$

of A-modules. Well, consider $\mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k)) := \mathrm{R}\Gamma_{\mathrm{cris}}(A/W(k)) \otimes_{W(k)} W_2(k)$, which we know to be isomorphic to $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/W_2(k))$ for any deformation $\widetilde{X}$ of X to $W_2(k)$. Accordingly, we receive a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{dR}}(A/k) \to \mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k)) \to \mathrm{R}\Gamma_{\mathrm{dR}}(A/k)$$

induced by the lifting $k \to W_2(k) \to k$, where the map $k \to W_2(k)$ is multiplication by p. Thus, we receive a long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathrm{dR}}(A/k) \to \mathrm{H}^i_{\mathrm{dR}}(A/k) \xrightarrow{\beta_i} \mathrm{H}^{i+1}_{\mathrm{dR}}(A/k) \to \cdots .$$

It turns out that the graded map $\bigoplus_i \beta_i$ on the de Rham cohomology ring $\bigoplus_{i>0} \mathrm{H}^i_{\mathrm{dR}}(X/k)$ is a derivation; this is just some exercise in the homological algebra. Explicitly, one has to check that

$$\beta_{i+j}(x_i \wedge x_j) = \beta_i(x_i) \wedge x_j + (-1)^i x_i \wedge \beta_j(x_j)$$

for any x_i and x_j of degrees i and j, respectively. It further turns out that $\beta_{i+1} \circ \beta_i = 0$.

Now that we have found a derivation, we can make differentials appear. The universal property of differentials produce morphisms

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{A^{(1)}} & \xrightarrow{d} & \Omega^1_{A^{(1)}} & \xrightarrow{d} & \Omega^2_{A^{(1)}} & \xrightarrow{d} & \cdots \\
& & \downarrow{\scriptstyle c_0} & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_2} & & \\
0 & \longrightarrow & \mathrm{H}^0_{\mathrm{dR}}(A/k) & \xrightarrow{\beta} & \mathrm{H}^1_{\mathrm{dR}}(A/k) & \xrightarrow{\beta} & \mathrm{H}^2_{\mathrm{dR}}(A/k) & \xrightarrow{\beta} & \cdots
\end{array}
$$

where f_0 is induced by the degree-zero argument, and all the relevant morphisms intertwine β and d and produce maps of graded rings. For example, one finds that we need $c_1(df) = \beta_0(c_0(f))$ for functions f. One now checks that this in fact an isomorphism on the level of modules, which is checked étale-locally and then passed to affine spaces. ∎

> **Remark 1.45.** We can afford to be a little more explicit about our construction of the maps $c_\bullet$. Fix a lift $\widetilde{A}$ over $W_2(k)$ as well as a lift of the Frobenius $\widetilde{F}$. Given $\omega \in \Omega^i_{A^{(1)}/k}$, we pull it back along $\widetilde{F}$ to get a differential divisible by p in $\mathrm{H}^i_{\mathrm{dR}}(\widetilde{A}/W_2(k))$. Then one can check that the class $\widetilde{F}^*(\omega)/p$ is well-defined in $\mathrm{H}^i_{\mathrm{dR}}(A/k)$.

> **Remark 1.46.** Crystalline cohomology is not technically necessary because we could choose a lift $\widetilde{X}$ by hand using Remark 1.45. However, one needs to check that the constructed map is independent of the lift. This is not impossible (such lifts are well-understood in some cohomology group by deformation theory), but it is a little difficult.

1.2.4 $\mathcal{D}$-modules

We close our class by saying something about $\mathcal{D}$-modules.

> **Definition 1.47** (D-module). Fix a smooth scheme X over a ring R. Then we define the quasicoherent sheaf $\mathcal{D}_{X/R}$ of associative algebras on X defined explicitly as
>
> $$\mathcal{O}_X\{\partial_v : v \in T_X\},$$
>
> where the relations ∂_v satisfies the relations $\partial_{v_1+v_2} = \partial_{v_1} + \partial_{v_2}$, $\partial_{fv} = f\partial_v$, $[\partial_v, f] = L_v(f)$, and $[\partial_v, \partial_w] = \partial_{[v,w]}$. Here, T_X is the tangent bundle. A $\mathcal{D}$-*module* is a quasicoherent sheaf which is a module for $\mathcal{D}_{X/R}$.

> **Remark 1.48.** Precisely, we have given a definition of an associative ring on affine opens, which then glue together on affines as a sheaf on a base.

Bibliography

[Shu16]   Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions