202A: Introduction to Topology and Analysis

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Contents

Theme 1
Metric Spaces

My personal view on spaces is that every space I ever work with is either metrizable or is the Zariski topology.

—Evan Chen, [Che22]

1.1 August 24

Good morning everyone. This is my first class of the semester.

1.1.1 Administrative Notes

Here are some housekeeping remarks.

- The webpage for this class is `math.berkeley.edu/ rieffel/202AannF22.html`.
- The midterm date is negotiable. We will have a vote on Friday. The possible dates are Friday 14 October, Monday 17 October, or Wednesday 19 October.
- There will be no vote on the final exam. It is on 15 December at 7PM.
- Homework will be due Fridays by midnight, approximately every week.
- There is no particular text for this course, and any given text covers more than we have time for. That said, we will (very) loosely follow [Lan12], but it is helpful to have a number of different expositions around.
- Please wear a mask during lectures and office hours.

Here is a summary of the course.

- We will spend the next couple of lectures talking about metric spaces.
- We will then spend the first half of the course on general topology. The second half of the course will be on measure and integration.
- Throughout we will see a little on functional analysis.

1.1.2 Metric Spaces

Hopefully we remember something about metric spaces. Here's the definition.

Definition 1.1 (Metric)**.** A *metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following rules for any $x, y, z \in X$.

(a) Zero: $d(x,x) = 0$.

(b) Zero: $d(x,y) = 0$ implies $x = y$.

(c) Symmetry: $d(x,y) = d(y,x)$.

(d) Triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$.

We call (X,d) a *metric space*.

We will want some "almost" metrics as well. Here are their names.

Definition 1.2 (Semi-metric)**.** A *semi-metric* d on a set X satisfies (a), (c), and (d) of Definition 1.1. We call (X,d) a *semi-metric space*.

Definition 1.3 (Extended metric)**.** An *extended metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}^{\infty}_{\geq 0}$ satisfying (a)–(d) of Definition 1.1. We call (X,d) an *extended metric space*.

Intuitively, we might want extended metrics if we have points that we never want to be able to get to from other ones.

We can turn spaces with a semi-metric into a space with a metric.

Lemma 1.4. Fix a semi-metric space (X,d), and define the relation $\sim$ on X by $x \sim y$ if and only if $d(x,y) = 0$. Then $\sim$ is an equivalence relation.

Proof. We run these checks by hand. Fix any $x, y, z \in X$.

- Reflexive: $d(x,x) = 0$ means that $x \sim x$.
- Symmetry: if $x \sim y$, then $d(x,y) = 0$, so $d(y,x) = 0$, so $y \sim x$.
- Transitive: if $x \sim y$ and $y \sim z$, then

$$0 \leq d(x,z) \leq d(x,y) + d(y,z) = 0,$$

so $d(x,z) = 0$, so $x \sim z$. ■

As such, given a semi-metric space (X,d), we may look at the set of equivalence classes under $\sim$, which we will denote $X/{\sim}$.[1]

Proposition 1.5. Fix a semi-metric space (X,d) and define $\sim$ as in Lemma 1.4. Then d naturally descends to a metric $\widetilde{d}$ on $X/{\sim}$.

Proof. Let $[x]$ denote the equivalence class of $x \in X$ under $\sim$. We claim that the function

$$\widetilde{d}([x],[y]) := d(x,y)$$

is a well-defined metric. We have the following checks; fix any $x, y, z \in X$.

[1] The notation of $/{\sim}$ is intended to make us think of quotients.

- Well-defined: if $x \sim x'$ and $y \sim y'$, then note that
$$d(x,y) \le d(x,x') + d(x',y) = d(x',y) \le d(x',y') + d(y',y) = d(x',y').$$
By symmetry, we also have $d(x',y') \le d(x,y)$, so equality follows. So d does descent properly to the quotient $X/\sim$.
- Zero: note that $\widetilde{d}([x],[y]) = 0$ if and only if $d(x,y) = 0$ if and only if $x \sim y$ if and only if $[x] = [y]$.
- Symmetry: note that
$$\widetilde{d}([x],[y]) = d(x,y) = d(y,x) = \widetilde{d}([y],[x]).$$
- Triangle inequality: note that
$$\widetilde{d}([x],[z]) = d(x,z) \le d(x,y) + d(y,z) = \widetilde{d}([x],[y]) + \widetilde{d}([y],[z]),$$
which finishes. ■

Here are some examples of metric spaces.

Example 1.6. Given a connected graph $G = (V,E)$ with a weighting function $w\colon E \to \mathbb{R}_{\ge 0}$, we can build a metric as follows: define the "shortest-path" function $d\colon V \times V \to \mathbb{R}_{\ge 0}$ sending two vertices $v, w \in V$ to the length of the shortest path. If the graph G is not connected, we merely have an extended metric.

Example 1.7 (Euclidean metric)**.** The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$
$$d\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) \coloneqq \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$
is a metric.

Observe that it is not completely obvious that Example 1.7 satisfies the triangle inequality, but this will follow from the theory of the next subsections.

1.1.3 Norms on Vector Spaces

Norms provide convenient ways to build metrics.

Definition 1.8 (Norm)**.** Fix a vector space V over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|\colon V \to \mathbb{R}_{\ge 0}$ is a function satisfying the following, for any $r \in \mathbb{R}$ and $v, w \in V$.

(a) Zero: $\|v\| = 0$ if and only if $v = 0$.

(b) Scaling: $\|rv\| = |r| \cdot \|v\|$.

(c) Triangle inequality: $\|v + w\| \le \|v\| + \|w\|$.

Remark 1.9. We can probably work with a more general normed field instead of "merely" $\mathbb{R}$ or $\mathbb{C}$.

And here is our result.

Proposition 1.10. Given a metric space V with a norm $\|\cdot\| : V \to \mathbb{R}_{\ge 0}$, then the function
$$d(v,w) \coloneqq \|v - w\|$$
defines a metric on V.

Proof. We run the checks directly. Let $x, y, z \in V$ be points.

- Zero: note that $d(x, y) = 0$ if and only if $\|x - y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$.
- Symmetry: note that
$$d(x, y) = \|x - y\| = |-1| \cdot \|y - x\| = 1 \cdot \|y - x\| = d(y, x).$$
- Triangle inequality: note that
$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \le \|x - y\| + \|y - z\| = d(x, y) + d(y, z),$$
which finishes the check. ■

Here are the usual examples.

Example 1.11. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then the following are norms on V.

- $\|(x_1, \ldots, x_n)\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.
- $\|(x_1, \ldots, x_n)\|_1 := \sum_{i=1}^n |x_i|$.

Here are some more esotetric examples.

Example 1.12. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then
$$\|(x_1, \ldots, x_n)\|_\infty := \sup\{|x_1|, \ldots, |x_n|\}$$
provides a norm on V.

Example 1.13. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then, given $p \ge 1$,
$$\|(x_1, \ldots, x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$
provides a norm on V.

Remark 1.14. Taking the limit as $p \to \infty$ of $\|f\|_p$ gives $\|f\|_\infty$. This justifies the notation.

Remark 1.15. Despite having lots of examples, all of these norms are equivalent in a topological sense.

These normed vector spaces actually allow us to define a metric on any subset.

Proposition 1.16. Given a metric space (X, d) and a subset $Y \subseteq X$, the restriction of d to $Y \times Y$ is a metric.

Proof. All the requirements for d on $Y \times Y$ are satisfied for any points in X, so we are done by doing no work. ■

Example 1.17. Any subset $X \subseteq \mathbb{R}^n$ has an induced metric by restricting the (say) Euclidean metric.

1.1.4 A Hint of L^p Spaces

Here is a more complicated example of a metric.

Example 1.18. Define $V := C([0,1])$ to be the $\mathbb{R}$-vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0,1]$. The following are norms.

- $\|f\|_\infty := \sup\{|f(x)| : x \in [0,1]\}$.
- $\|f\|_1 := \int_0^1 |f(t)|\,dt$.
- $\|f\|_2 := \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$.
- More generally, given $p \ge 1$
$$\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

These integrals are finite because $[0,1]$ is compact, forcing f to achieve a finite maximum on $[0,1]$.

Remark 1.19. We can tell the same story for $C(X)$, for any measurable compact space X.

Remark 1.20. Note the analogy of Example 1.18 with Example 1.13. To see this more rigorously, set X to be the finite set $\{1,\ldots,n\}$ so that $C(X) = \mathbb{R}^n$.

We should probably justify the claims of this subsection, so here is our result.

Proposition 1.21. Define $V := C([0,1])$ to be the vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0,1]$. Then, given $p \ge 1$, the function $\|\cdot\|_p : C \to \mathbb{R}_{\ge 0}$ by

$$\|f\| := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}$$

is a norm.

Proof. We run the checks directly.

- Zero: if $f = 0$, then of course $\int_0^1 |f(t)|^p\,dt = 0$.
- Zero: suppose that $f \in C([0,1])$ has $f(t_0) \ne 0$ for any $t_0 \in [0,1]$; set $y := f(t_0)$. Then $f^{-1}((y/2, 3y/2))$ is a nonempty open subset of X and hence contains a nonempty open interval (a,b) with $a < b$. As such,
$$\int_X |f(t)|^p\,dt \ge \int_a^b |f(t)|^p\,dt \ge \int_a^b |y/2|^p\,dt > 0,$$
so we are done.
- Scaling: given $f \in C([0,1])$ and a scalar r, we have
$$\|rf\| = \left(\int_0^1 |rf(t)|^p\,dt\right)^{1/p} = \left(|r|^p \int_0^1 |f(t)|^p\,dt\right)^{1/p} = |r| \cdot \|f\|.$$
- Triangle inequality: we borrow from [Tao09]. Given $f, g \in C([0,1])$, for psychological reasons we will assume that f and g are nonzero (else this is clear); then $\|f\|, \|g\| \ne 0$, so we may scale everything so that $\|f\| + \|g\| = 1$. In fact, we may again use scaling to find $a, b \in V$ such that
$$f = (1-\theta)a \qquad \text{and} \qquad g = \theta b$$

where $\theta \in (0,1)$ and $\|a\| = \|b\| = 1$. Now, the triangle inequality translates into showing

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p \, dt = \|(1-\theta)a + \theta b\|_p^p \overset{?}{\le} \left(\|(1-\theta)a\|_p + \|\theta b\|_p\right)^p = 1.$$

Well, because $p \ge 1$, the function $t \mapsto t^p$ is convex, so we get to write

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p \, dt \le (1-\theta)\int_0^1 |a(t)|^p \, dt + \theta \int_0^1 |b(t)|^p \, dt,$$

which is what we wanted.

The above checks complete the proof; note that the proof of the triangle inequality was nontrivial. ■

Remark 1.22. Now, to show Remark 1.20, replace all $\int_0^1$ with $\sum_{i=1}^n$ and adjust all the language accordingly. The point is that "integrating over $[0,1]$" is analogous to "integrating over $\{1,\ldots,n\}$." A more thorough understanding of measure theory will allow us to rigorize this.

Next class we will talk about completeness.

1.2 August 26

Today we're talking about completeness of metric spaces.

1.2.1 Isometries

In mathematics, we are interested in objects not in isolation but as they relate to each other. Namely, we are interested also in the maps between our objects.

The philosophy here comes from category theory, where one is really most interested in the "morphisms" between "objects" instead of the objects themselves. For concreteness, here is a definition of a category.

Definition 1.23 (Category)**.** A *category* $\mathcal{C}$ consists of a class of objects $\operatorname{Ob}\mathcal{C}$ and class of morphisms $\operatorname{Mor}\mathcal{C}$ such that any two objects $A, B \in \operatorname{Ob}\mathcal{C}$ have a morphism class $\operatorname{Mor}(A,B)$. This data satisfy the following properties.

- Composition: given objects $A, B, C \in \operatorname{Ob}\mathcal{C}$, there is a binary composition operation

$$\circ\colon \operatorname{Mor}(B,C) \times \operatorname{Mor}(A,B) \to \operatorname{Mor}(A,C).$$

  Explicitly, given $f \in \operatorname{Mor}(A,B)$ and $g \in \operatorname{Mor}(B,C)$, there is a composition $(g \circ f) \in \operatorname{Mor}(A,C)$.
- Given $A \in \operatorname{Ob}\mathcal{C}$, there is an identity morphism $\operatorname{id}_A \in \operatorname{Mor}(A,A)$.
- Identity: any $f \in \operatorname{Mor}(A,B)$ has $f \circ \operatorname{id}_A = f = \operatorname{id}_B \circ f$.
- Associativity: any $f \in \operatorname{Mor}(A,B)$ and $g \in \operatorname{Mor}(B,C)$ and $h \in \operatorname{Mor}(C,D)$ has $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 1.24. There is a category of groups, where the morphisms are group homomorphisms. The identity function gives the identity morphism, and composition of functions gives the required composition.

For completeness, we check that composition is well-defined: given homomorphisms $f\colon A \to B$ and $g\colon B \to C$, we need $(g \circ f)\colon A \to C$ to be a group homomorphism. Well,

$$(g \circ f)(a \cdot a') = g(f(a \cdot a')) = g(f(a) \cdot f(a')) = g(f(a)) \cdot g(f(a')) = (g \circ f)(a) \cdot (g \circ f)(a').$$

In our discussion of metric spaces, there are many possible kinds of morphisms for us to consider. Here is the strongest type.

Definition 1.25 (Isometry). Given metric spaces (X, d_X) and (Y, d_Y), an *isometry* is a function $f\colon X \to Y$ preserving the metric as

$$d_Y(f(x), f(x')) = d_X(x, x').$$

Example 1.26. The $90°$ rotation $r\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $r(x, y) \mapsto (y, -x)$ is an isometry, where $\mathbb{R}^2$ is given the Euclidean metric. Indeed, any $(x, y), (x', y') \in \mathbb{R}^2$ have

$$\begin{aligned} d\big(r(x,y), r(x',y')\big) &= d\big((y,-x),(y',-x')\big) \\ &= \sqrt{(y-y')^2 + (-x - -x')^2} \\ &= \sqrt{(x-x')^2 + (y-y')^2} \\ &= d\big((x,y),(x',y')\big). \end{aligned}$$

Notation 1.27. Fix two metric spaces (X, d_X) and (Y, d_Y). Given a function $f\colon X \to Y$ with extra structure respecting some aspect of the metric, we might write $f\colon (X, d_X) \to (Y, d_Y)$ to emphasize this.

To show that isometries are valid morphisms, we need to check that the identity function $\mathrm{id}_X\colon X \to X$ is an isometry (which of course it is) and that the composition of two isometries is an isometry. We check this last one in a quick lemma.

Lemma 1.28. Given two isometries $f\colon (X, d_X) \to (Y, d_Y)$ and $g\colon (Y, d_Y) \to (Z, d_Z)$, the composition $g \circ f$ is an isometry.

Proof. Well, any two points $x, x' \in X$ have

$$d_Z(g(f(x)), g(f(x'))) = d_Y(f(x), f(x')) = d_X(x, x'),$$

which is what we wanted. ■

One can restrict further to surjective isometries, where the main point is that (again) the composition of two surjective functions remains surjective. (Note that the identity is of course surjective.) The following is the reason why a surjective isometry is a good notion.

Lemma 1.29. A surjective isometry $f\colon (X, d_X) \to (Y, d_Y)$ is bijective, and its inverse function is also an isometry.

Proof. To see that f is bijective, we only need to know that f is injective. Well, given $x, x' \in X$, note that $f(x) = f(x')$ if and only if $d_Y(f(x), f(x')) = 0$ if and only if $d(x, x') = 0$ if and only if $x = x'$.[2]

Thus, f is indeed bijective; let $g\colon Y \to X$ be its inverse. We now need to show that g is an isometry. Well, given $y, y' \in Y$, we may find $x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$. Then

$$d_X(g(y), g(y')) = d_X((g \circ f)(x), (g \circ f)(x')) = d_X(x, x') \stackrel{*}{=} d_Y(f(x), f(x')) = d_Y(y, y'),$$

where in $\stackrel{*}{=}$ we have used the fact that f is an isometry. ■

[2] In fact, this argument shows that all isometries are injective. We will shortly see that all actually Lipschitz continuous functions are injective.

Remark 1.30. The above result is somewhat subtle in its importance: the inverse function of a bijection is only an inverse in the category of sets. The above result is saying that this inverse morphism in the category of sets is lifting to an inverse morphism in the category of metric spaces with isometries as morphisms. In general, it is not always true that bijective morphisms are invertible, as we shall soon see.

1.2.2 Lipschitz Continuity

Isometries are somewhat restrictive, so we might weaken this as follows.

Definition 1.31 (Lipschitz continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *Lipschitz continuous* if and only if there is a constant $c \in \mathbb{R}$ such that

$$d_Y(f(x), f(x')) \le c \cdot d_X(x, x').$$

Remark 1.32. Equivalently, we are asking for the ratio

$$\frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

to be uniformly bounded above for all $x \neq x'$. Notably, the inequality is trivially satisfied whenever $x = x'$, or equivalently whenever $d(x, x') = 0$.

Example 1.33. Any isometry $f\colon (X, d_X) \to (Y, d_Y)$ is Lipschitz continuous: indeed, set $c := 1$ so that, for any $x, x' \in X$,

$$d_Y(f(x), f(x')) = d_X(x, x') \le 1 \cdot d_X(x, x').$$

Example 1.34. Provide $\mathbb{R}$ and $\mathbb{R}^2$ their usual Euclidean metrics. Then the projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ by $\pi\colon (x, y) \mapsto x$ is Lipschitz continuous: indeed, set $c := 1$ so that, for any $(x, y), (x', y') \in \mathbb{R}^2$, we have

$$d_{\mathbb{R}^2}\big((x, y), (x', y')\big) = \sqrt{(x - x')^2 + (y - y')^2} \ge \sqrt{(x - x')^2} = d_{\mathbb{R}}(x, x') = d_{\mathbb{R}}\big(\pi((x, y)), \pi((x', y'))\big).$$

Again, one can see that the identity function $\mathrm{id}_X\colon (X, d_X) \to (X, d_X)$ is Lipschitz continuous (with $c := 1$), and here is our composition check.

Lemma 1.35. If $f\colon (X, d_X) \to (Y, d_Y)$ and $g\colon (Y, d_Y) \to (Z, d_Z)$ are Lipschitz continuous, then the composition $(g \circ f)\colon (X, d_X) \to (Z, d_Z)$ is also Lipschitz continuous.

Proof. We are given constants c and d such that any $x, x' \in X$ and $y, y' \in Y$ have

$$d_Y(f(x), f(x')) \le c \cdot d_X(x, x') \qquad \text{and} \qquad d_Z(g(y), g(y')) \le d \cdot d_Y(y, y').$$

As such, we use the constant cd to witness our Lipschitz continuity: any $x, x' \in X$ have

$$d_Z(g(f(x)), g(f(x'))) \le d \cdot d_Y(f(x), f(x')) \le cd \cdot d_X(x, x'),$$

which is what we wanted. ■

It will be shortly worth our time to talk about the constant c appearing in Definition 1.31.

Lemma 1.36. Fix a Lipschitz continuous function $f\colon (X,d_X)\to(Y,d_Y)$. Then there exists a constant c_f (possibly $-\infty$) such that any real number $c\ge c_f$ is equivalent to the following property: any $x,x'\in X$ have

$$d_Y(f(x),f(x'))\le c\cdot d_X(x,x').$$

Proof. Let S denote the set of all constants c such that any $x,x'\in X$ have

$$d_Y(f(x),f(x'))\le c\cdot d_X(x,x').$$

Equivalently, using Remark 1.32, S is the set of upper-bounds for

$$R:=\left\{\frac{d_Y(f(x),f(x'))}{d_X(x,x')}:x,x'\in X,x\ne x'\right\}.$$

Now, S is nonempty because f is Lipschitz continuity, so we set $c_f:=\sup R$ to be the least upper bound for R—observe that $c_f=-\infty$ is permissible when X has one point. It is now pretty clear that $S=[c_f,\infty)$. ■

Note that c_f the property stated in the lemma automatically implies that c_f is the least possible constant and is unique. Being least is immediate (by the backwards direction), and being unique follows from being least. So because we have some uniqueness, we get a definition.

Definition 1.37 (Lipschitz constant)**.** Given a Lipschitz continuous function $f\colon (X,d_X)\to(Y,d_Y)$, the *Lipschitz constant* c_f *for* f is the least real number c such that

$$d_Y(f(x),f(x'))\le c\cdot d_X(x,x').$$

We could, as before, look at surjective Lipschitz continuous functions, but these need not be bijective anymore as shown by Example 1.34. What's worse is that, as warned possible in Remark 1.30, bijective Lipschitz continuous functions need not even have a Lipschitz continuous inverse.

Exercise 1.38. We exhibit a function between metric spaces which is bijective and Lipschitz continuous, but its inverse function is not Lipschitz continuous.

Proof. Set $X:=(0,1)$ and $Y:=(1,\infty)$, both metric spaces with the Euclidean (subspace) metric, and set $f\colon(0,\infty)\to(0,\infty)$ by $f\colon x\mapsto 1/x$. Notably, $x\in X$ implies $f(x)\in Y$, and $y\in Y$ implies $f(y)\in X$.

- Note $f|_Y$ is bijective with inverse $f|_X$ because $f(f(x))=f(1/x)=x$ for all $x\in(0,\infty)$.
- Note $f|_Y$ is Lipschitz continuous: set $c:=1$ and note that any $y,y'\in Y$ have

$$|f(y)-f(y')|=\left|\frac{1}{y}-\frac{1}{y'}\right|=\left|\frac{y-y'}{yy'}\right|\le|y-y'|.$$

- But $f|_X$ is not Lipschitz continuous: suppose for contradiction that f_X is Lipschitz continuous, and use Lemma 1.36 to recover the needed constant c_0. Then set $c:=\max\{c_0,4\}$, which must also work as a constant, and set $x:=1/c$ and $x':=1/(3c)$ so that

$$|f(x)-f(x')|=|c-3c|=2c>c\cdot|x-x'|.$$

This is a contradiction, so we are done. ■

Remark 1.39 (Nir)**.** In some sense, the problem here is that the definition of Lipschitz continuity allows $d_Y(f(x),f(x'))$ to be "too small," which permits the inverse function to have distances which blow up.

In light of Exercise 1.38, we introduce a new definition.

Definition 1.40 (Lipschitz isomorphism)**.** Give metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *Lipschitz isomorphism* if and only if f is Lipschitz continuous and has an inverse function which is also Lipschitz continuous.

Remark 1.41. A good reason to care about this notion of continuity (and isomorphism) is that all normed $\mathbb{R}$-vector spaces of some finite dimension n are Lipschitz isomorphic.

1.2.3 Fun with Continuity

Here is yet a weaker notion of morphism.

Definition 1.42 (Uniformly continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *uniformly continuous* if and only if every $\varepsilon > 0$ has some $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

for all $x, x' \in X$.

Example 1.43. Any Lipschitz continuous function $f\colon (X, d_X) \to (Y, d_Y)$ is also uniformly continuous: indeed, for any $\varepsilon > 0$, set $\delta := \max\{c_f, 1\}\varepsilon > 0$ (where c_f is the Lipschitz constant) so that

$$d_X(x, x') < \varepsilon \implies d_Y(f(x), f(x')) \le c_f \cdot d(x, x') < \delta.$$

Example 1.44. Give $[0, 1]$ the Euclidean (subspace) metric, and set $f\colon [0, 1] \to [0, 1]$ by $f(x) := \sqrt{x}$.

- Note f is uniformly continuous because it is continuous on a compact set.
- However, f is not Lipschitz continuous: for any constant $c > 0$, set $x = 1/(c + 1)^2$ and $x' = 0$ so that
$$\left|\frac{f(x) - f(x')}{x - x'}\right| = \left|\frac{1/(c + 1)}{1/(c + 1)^2}\right| = |c + 1| > c,$$
so Remark 1.32 tells us that we are not Lipschitz continuous.

By rearranging quantifiers, we get another useful (but weaker) notion.

Definition 1.45 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous at* $x \in X$ if and only if all $\varepsilon > 0$ have some $\delta_x > 0$ such that

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$

Then f is *continuous* if and only if it is continuous at all $x \in X$.

Example 1.46. All uniformly continuous functions $f\colon (X, d_X) \to (Y, d_Y)$ are continuous. Indeed, at any $x_0 \in X$ with $\varepsilon > 0$, uniform continuity promises $\delta > 0$ so that

$$|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon$$

for all $x, x' \in X$. Setting x' to x_0 recovers continuity.

Example 1.47. Give $\mathbb{R}$ the usual Euclidean metric, and set $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) := x^2$.

- Note $f(x)$ is continuous because it is a polynomial.
- However, $f(x)$ is not uniformly continuous: take $\varepsilon = 1$. Now, for any $\delta > 0$, set $x = 1/\delta$ and $x' = 1/\delta + \delta/2$ so that $|x - x'| < \delta$, but

$$|f(x) - f(x')| = \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2} = 1 + \frac{\delta^2}{4} > \varepsilon.$$

As usual, the identity function is uniformly continuous and continuous (it's an isometry), and these continuities are preserved by composition. We will have a different way to see that continuous functions remain continuous under composition later, so for now we will focus on uniform continuity.

Lemma 1.48. Fix uniformly continuous morphisms $f\colon (X, d_X) \to (Y, d_Y)$ and $g\colon (Y, d_Y) \to (Z, d_Z)$. Then the function $(g \circ f)$ is uniformly continuous.

Proof. For any $\varepsilon > 0$, the uniform continuity of g promises $\delta_g > 0$ such that

$$d_Y(y, y') < \delta_g \implies d_Z(g(y), g(y')) < \varepsilon$$

for any $y, y' \in Y$. Continuing, the uniform continuity of f promises $\delta_f > 0$ such that

$$d_X(x, x') < \delta_X \implies d_Y(f(x), f(x')) < \delta_Y \implies d_Z(g(f(x)), g(f(x'))) < \varepsilon$$

for any $x, x' \in X$, which is what we wanted. ■

Remark 1.49. In some sense, isometries and Lipschitz continuous functions have their definition fundamentally interrelated with the metric. In contrast, the weaker notion of continuity will readily generalize to general topological spaces. Uniform continuity also generalizes to "uniformities," which is a different notion.

1.2.4 Convergence and Completeness

To discuss completeness, we need to talk about convergence.

Definition 1.50 (Converge)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ *converges to* $x \in X$ if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n > N \implies d(x_n, x) < \varepsilon.$$

We might write this as "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$." In this event, we may say that the sequence $\{x_n\}_{n \in \mathbb{N}}$ *converges*, and its limit is x.

Remark 1.51 (Nir)**.** As a sanity check, the limit of a sequence is unique: if $x_n \to x$ and $x_n \to x'$ as $n \to \infty$, then any $\varepsilon > 0$ can find some large n so that $d(x_n, x), d(x_n, x') < \varepsilon/2$. As such,

$$d(x, x') < d(x_n, x) + d(x_n, x') = \varepsilon$$

for any $\varepsilon > 0$, so $d(x, x') = 0$ and thus $x = x'$ is forced.

We have no reason yet to be convinced that any of our morphisms described previously are good notions, so let's start with continuity.

Lemma 1.52. Fix a continuous function between metric spaces $f\colon (X, d_X) \to (Y, d_Y)$. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ converges to $f(x) \in Y$.

Proof. For any $\varepsilon > 0$,the continuity of f implies that we can find $\delta_x > 0$ so that

$$d_X(x_n, x) < \delta_x \implies d_Y(f(x_n), f(x)) < \varepsilon$$

for any x_n. But the fact that $x_n \to x$ as $n \to \infty$ means that there is $N > 0$ so that

$$n > N \implies d_X(x_n, x) < \delta_x \implies d_Y(f(x_n), f(x)) < \varepsilon,$$

so indeed, $f(x_n) \to f(x)$ as $n \to \infty$. ■

We would like a notion of convergence which only uses data internal to the sequence, and this leads to the following definition.

Definition 1.53 (Cauchy)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy sequence* if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n, m > N \implies d(x_n, x_m) < \varepsilon.$$

It would be rude if continuity was always the best kind of morphism, so this time around preserving Cauchy-ness requires something stronger.

Lemma 1.54. Fix a uniformly continuous function between metric spaces $f\colon (X, d_X) \to (Y, d_Y)$. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is Cauchy, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ is also Cauchy.

Proof. For any $\varepsilon > 0$, the uniform continuity of f promises $\delta > 0$ so that

$$d_X(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \varepsilon$$

for any x_n, x_m. However, the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy promises N so that

$$n, m > N \implies d_X(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \varepsilon,$$

which is what we wanted. ■

Example 1.55. Continuous functions do not need to preserve Cauchy sequences: $f\colon (0, \infty) \to (0, \infty)$ by $f(x) := 1/x$ is continuous, and the sequence $\{1/n\}_{n\in\mathbb{N}} \subseteq (0, \infty)$ is Cauchy (it converges to 0 in $\mathbb{R}$) even though $\{f(1/n)\}_{n\in\mathbb{N}} = \{n\}_{n\in\mathbb{N}}$ certainly does not converge.

Anyway, it is quick to check that convergent sequences are Cauchy.

Lemma 1.56. Fix a metric space (X, d). Then all convergent sequences are Cauchy.

Proof. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$. Then, for any $\varepsilon > 0$, find N so that

$$d(x_n, x) < \varepsilon/2$$

for all $n > N$. Then any $n, m > N$ has

$$d(x_n, x_m) \le d(x_n, x) + d(x_m, x) < \varepsilon,$$

so the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. ■

We in general hope that our Cauchy sequences converge. As such, we have the following definition.

Definition 1.57 (Complete). A metric space (X, d) is *complete* if and only if every Cauchy sequence in X converges to a point in X.

We are sad when a metric space is not complete, so we hope to have a way to make it complete. The most natural way to do this is by using the notion of density.

Definition 1.58 (Density). Fix a metric space (X, d). Then $S \subseteq X$ is *dense* if and only if, given any $x \in X$ and $\varepsilon > 0$, we may find $x' \in S$ with $d(x, x') < \varepsilon$.

And here is our completion.

Definition 1.59 (Completion). A *completion* of the metric space (X, d) is a metric space $(\overline{X}, \overline{d})$ equipped with an isometry $\iota \colon X \to \overline{X}$ such that $(\overline{X}, \overline{d})$ is complete and $\operatorname{im} \iota$ is dense in $\overline{X}$.

One can show that any metric space has a completion and that they are all isometric and therefore in some sense the same. We'll do these separately.

1.2.5 Existence of Completions

Let's start with existence.

Theorem 1.60. Any metric space (X, d) has a completion.

Proof. Let $\widetilde{X}$ denote the set of all Cauchy sequences in X. We hope to make $\widetilde{X}$ into our completion, but this requires a little care. To begin, we have the following lemma.

Lemma 1.61. Given a metric space (X, d) with two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, then the sequence

$$\{d(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$$

converges.

Proof. Because $\mathbb{R}$ is a complete metric space, it suffices to show that the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is Cauchy. Well, for any $\varepsilon > 0$, find a sufficiently large N so that

$$n, m > N \implies d(x_n, x_m), d(y_n, y_m) < \varepsilon/2.$$

Then any $n, m > N$ has

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \varepsilon + d(y_m, y_n),$$

and $d(x_m, y_m) < d(x_n, y_n) + \varepsilon$ as well by symmetry. It follows that any $n, m > N$ has

$$\left|d(x_n, y_n) - d(x_m, y_m)\right| < \varepsilon,$$

verifying that our sequence is Cauchy. ∎

Remark 1.62. Here is a quick motivational remark for the definition of our metric below: if (X, d) is a metric space with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then we claim $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. Indeed, for any $\varepsilon > 0$, we can find N large enough so that $d(x_n, x), d(y_n, y) < \varepsilon/2$ for any $n > N$. As such,

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < d(x, y) + \varepsilon.$$

By symmetry, we get $d(x, y) \leq d(x_n, y_n) + \varepsilon$ as well, finishing.

Thus, we define $\widetilde{d}\colon \widetilde{X} \times \widetilde{X} \to \mathbb{R}_{\geq 0}$ by

$$\widetilde{d}\left(\{x_n\},\{y_n\}\right) := \lim_{n\to\infty} d(x_n,y_n).$$

We claim that $\widetilde{d}$ is a semi-metric on $\widetilde{X}$. We have the following checks; fix Cauchy sequences $\{x_n\},\{y_n\},\{z_n\}$.

- Zero: note
$$\widetilde{d}(\{x_n\},\{x_n\}) = \lim_{n\to\infty} d(x_n,x_n) = 0.$$

- Symmetry: note
$$\widetilde{d}(\{x_n\},\{y_n\}) = \lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(y_n,x_n) = \widetilde{d}(\{y_n\},\{x_n\}).$$

- Triangle inequality: note
$$\begin{aligned}\widetilde{d}(\{x_n\},\{y_n\}) + \widetilde{d}(\{y_n\},\{z_n\}) &= \lim_{n\to\infty} d(x_n,y_n) + \lim_{n\to\infty} d(y_n,z_n) \\ &= \lim_{n\to\infty} \left(d(x_n,y_n) + d(y_n,z_n)\right) \\ &\geq \lim_{n\to\infty} d(x_n,z_n) \\ &= \widetilde{d}(x_n,z_n),\end{aligned}$$
where we have implicitly used a number of limit laws.

So because $\widetilde{d}$ is a semi-metric, Proposition 1.5 tells us that $\widetilde{d}$ will descend naturally to a metric $\overline{d}$ on $\overline{X} := \widetilde{X}/{\sim}$, where $\{x_n\} \sim \{y_n\}$ if and only if $\widetilde{d}(\{x_n\},\{y_n\}) = 0$. We will let $[\{x_n\}]$ denote the equivalence class of the Cauchy sequence $\{x_n\} \in \widetilde{X}$ in $\overline{X}$.

We now show that $(\overline{X},\overline{d})$ can be made into a completion for X.

- Given $x \in X$, note that the constant sequence $\{x\}$ is Cauchy (for any $\varepsilon > 0$, set $N = 0$), so we define $\iota\colon X \to \overline{X}$ by
$$\iota(x) := [\{x\}].$$
To see that ι is an isometry, note any $x,x' \in X$ have
$$\overline{d}(\iota(x),\iota(x')) = \widetilde{d}(\{x\},\{y\}) = \lim_{n\to\infty} d(x,y) = d(x,y).$$

- We show that $\operatorname{im}\iota$ is dense in $\overline{X}$. Indeed, fix some $[\{x_n\}] \in \overline{X}$ and $\varepsilon > 0$. Then there is some N so that $n,m > N$ has
$$d(x_n,x_m) < \varepsilon/2.$$
Fixing a particular n_0 with $n_0 > N$, we set $x := x_{n_0}$ so that
$$\overline{d}([\{x_n\}],\iota(x)) = \widetilde{d}(\{x_n\},x_{n_0}) = \lim_{n\to\infty} d(x_n,x_{n_0}).$$
Now, for $n > N$, we have $d(x_n,x_{n_0}) < \varepsilon/2$, so we conclude that this limit must be less than ε.

- We show that $(\overline{X},\overline{d})$ is a complete metric space. Fix a Cauchy sequence $\{\overline{x}_k\}$ in $\overline{X}$. To find the Cauchy sequence we are supposed to converge to, we use our density result: for each $k \in \mathbb{N}$, we can find $y_k \in X$ such that $\overline{d}(\overline{x}_k,\iota(y_k)) < 1/k$.
We claim that $\{y_k\}$ is Cauchy. Indeed, for any $\varepsilon > 0$, we can find N such that $k,\ell > N_0$ has
$$\overline{d}(\overline{x}_k,\overline{x}_\ell) < \varepsilon/3.$$
Then, setting $N := \max\{3/\varepsilon, N_0\}$, we note that $k,\ell > N$ has
$$d(y_k,y_\ell) = \overline{d}(\iota(y_k),\iota(d_\ell)) \leq \overline{d}(\overline{x}_k,\iota(y_k)) + \overline{d}(\overline{x}_\ell,\iota(y_\ell)) + \overline{d}(\overline{x}_k,\overline{x}_\ell) < \varepsilon.$$

Lastly, we claim that $\overline{x}_k \to [\{y_n\}]$ in $\overline{X}$. Indeed, for any $\varepsilon > 0$, find some sufficiently large N so that

$$k, \ell > N \implies d(y_k, y_\ell) < \varepsilon/2.$$

Then $k > \max\{N, 2/\varepsilon\}$ has

$$\overline{d}(\overline{x}_k, [\{y_n\}]) \le \overline{d}(\overline{x}_k, \iota(y_k)) + \overline{d}([\{y_n\}], \iota(y_k)) < \frac{\varepsilon}{2} + \lim_{n\to\infty} d(y_n, y_k).$$

Because $k > N$, we have $d(y_n, y_k) < \varepsilon/2$ for any $n > N$, so the entire right-hand side must be upper-bounded by ε. This finishes.

The above checks complete the proof. ■

Remark 1.63 (Nir)**.** One might complain that we used the completeness of $\mathbb{R}$ in this proof because one common way to construct the real numbers is as the completion of $\mathbb{Q}$ under the Euclidean metric. To remedy this, one ought to define the equivalence relation on Cauchy sequences more directly, saying that two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of real numbers are equivalent under $\sim$ if and only if

$$\lim_{n\to\infty} d_{\mathbb{R}}(x_n, y_n) = 0.$$

1.2.6 Uniqueness of Completions

We now show that any two completions of a metric space (X, d) are isometric, which is our uniqueness result. Here is the main intermediate result.

Lemma 1.64. Fix a metric space (X, d) and a completion $(\overline{X}, \overline{d})$ with its isometry $\iota\colon (X, d) \to (\overline{X}, \overline{d})$. Then, for any complete metric space (Y, d') and isometry $\varphi\colon (X, d) \to (Y, d')$, there is a unique isometry $\psi\colon (\overline{X}, \overline{d}) \to (Y, d')$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \overline{X} \arrow[d, dashed, "\psi"] \\ & Y \end{tikzcd}$$

Proof. We start by showing the uniqueness of ψ. Well, for any $\overline{x} \in \overline{X}$, note that any $n \in \mathbb{N}$ allows us to find $x_n \in X$ with

$$\overline{d}(\overline{x}, \iota(x_n)) < 1/n$$

because $\operatorname{im} \iota$ is dense in $\overline{X}$. Now, we notice that $\iota(x_n) \to \overline{x}$ as $n \to \infty$ because any $\varepsilon > 0$ can set $N = 1/\varepsilon$. As such, we see that Lemma 1.52 applied to any possible $\psi\colon \overline{X} \to Y$ forces

$$\psi(\overline{x}) = \psi\Big(\lim_{n\to\infty} \iota(x_n)\Big) = \lim_{n\to\infty} \psi(\iota(x_n)) = \lim_{n\to\infty} \varphi(x_n).$$

Note that, a priori, we do not know if the sequence $\{\varphi(x_n)\}_{n\in\mathbb{N}}$ converges, but this argument tells us that it must; the limit is unique by Remark 1.51, so $\psi(\overline{x})$ is unique as well.

We now show that ψ exists. As before, any $\overline{x} \in \overline{X}$ can find a sequence $\{x_n\} \subseteq X$ such that $\iota(x_n) \to \overline{x}$ as $n \to \infty$. Thus, we note that $\{\varphi(x_n)\}$ is Cauchy by Lemma 1.54, so the completeness of Y gives it a limit; we set

$$\psi(\overline{x}) \coloneqq \lim_{n\to\infty} \varphi(x_n).$$

We have the following checks on ψ.

- Well-defined: if we have two sequences $\{x_n\}$ and $\{x'_n\}$ such that $\iota(x_n) \to x$ and $\iota(x'_n) \to x$ as $n \to \infty$, we need to show that
$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \varphi(x'_n).$$
For brevity, set y and y' to be the limits of $\{\varphi(x_n)\}$ and $\{\varphi(x'_n)\}$, respectively. Then, for any $\varepsilon > 0$, we note that there is a sufficiently large N such that
$$n > N \implies d_Y(y, \varphi(x_n)), d_Y(y', \varphi(x'_n)) < \varepsilon/4.$$
Further, we can make N even larger so that
$$n > N \implies \overline{d}(\overline{x}, \iota(x_n)), \overline{d}(\overline{x}, \iota(x'_n)) < \varepsilon/4.$$
As such, any $n > N$ has
$$\begin{aligned} d_Y(y, y') &\le d_Y(y, \varphi(x_n)) + d_Y(\varphi(x_n), \varphi(x'_n)) + d_Y(y', \varphi(x'_n)) \\ &< \varepsilon/4 + d_X(x_n, x'_n) + \varepsilon/4 \\ &= \varepsilon/2 + \overline{d}(\iota(x_n), \iota(x'_n)) \\ &\le \varepsilon/2 + \overline{d}(\overline{x}, \iota(x_n)) + \overline{d}(\overline{x}, \iota(x'_n)) \\ &< \varepsilon. \end{aligned}$$
It follows $d_Y(y, y') = 0$, so $y = y'$.
- Isometry: given $\overline{x}, \overline{x}' \in \overline{X}$, find sequences $\{x_n\}$ and $\{x'_n\}$ in X so that $\iota(x_n) \to \overline{x}$ and $\iota(x'_n) \to \overline{x}'$ as $n \to \infty$. Thus,
$$\begin{aligned} d_Y(\psi(\overline{x}), \psi(\overline{x}')) &= d_Y\Big(\lim_{n\to\infty} \varphi(x_n), \lim_{n\to\infty} \varphi(x'_n)\Big) \\ &\overset{*}{=} \lim_{n\to\infty} d_Y(\varphi(x_n), \varphi(x'_n)) \\ &= \lim_{n\to\infty} d(x_n, x'_n) \\ &= \lim_{n\to\infty} \overline{d}(\iota(x_n), \iota(x'_n)) \\ &= \overline{d}\Big(\lim_{n\to\infty} \iota(x_n), \lim_{n\to\infty} \iota(x'_n)\Big) \\ &\overset{*}{=} \overline{d}(\overline{x}, \overline{x}'), \end{aligned}$$
where we have used Remark 1.62 at the $\overset{*}{=}$.
- For any $x \in X$, we see that the (constant) Cauchy sequence $\{\iota(x)\}$ converges to $\iota(x)$, so
$$\psi(\iota(x)) = \lim_{n\to\infty} \varphi(x) = \varphi(x).$$
It follows $\psi \circ \iota = \varphi$.

Thus, we have finished establishing the existence of an isometry $\psi\colon \overline{X} \to Y$ such that $\varphi = \psi \circ \iota$. ■

Remark 1.65. One can also replace all isometries with uniformly continuous functions in the statement.

And here is our uniqueness result.

Theorem 1.66. Fix a metric space (X, d) and two completions $\iota\colon (X, d) \to (\overline{X}, \overline{d})$ and $\iota'\colon (X, d) \to (\overline{X}', \overline{d}')$. Then there is a surjective isometry $\psi\colon (\overline{X}, \overline{d}) \to (\overline{X}', \overline{d}')$.

Proof. Applying Lemma 1.64 twice, we get isometries $\psi\colon (\overline{X}, \overline{d}) \to (\overline{X}', \overline{d}')$ and $\psi'\colon (\overline{X}', \overline{d}') \to (\overline{X}, \overline{d})$ making the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \overline{X} \arrow[d, "\psi"] \\ & \overline{X}' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\iota'"] \arrow[dr, "\iota"'] & \overline{X}' \arrow[d, "\psi'"] \\ & \overline{X} \end{tikzcd}$$

In particular, we see that $\psi' \circ \psi$ makes the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\iota"'] & \overline{X} \arrow[d, "\psi' \circ \psi"] \\ & \overline{X} \end{tikzcd}$$

However, using Lemma 1.64 again, this isometry $\psi' \circ \psi$ is unique to make the diagram commute, and we could of course put the isometry $\mathrm{id}_{\overline{X}}$ here if we wanted to. Thus,

$$\psi' \circ \psi = \mathrm{id}_{\overline{X}}.$$

By symmetry, $\psi \circ \psi' = \mathrm{id}_{\overline{X}'}$, so we do see that ψ and ψ' are inverse isometries. This finishes the proof. ■

1.3 August 29

Good morning everyone.

1.3.1 Some Examples

Let's give some more examples of metric spaces. Let's start with spaces of continuous functions.

Definition 1.67. We denote the $\mathbb{R}$-vector space of $\mathbb{C}$-valued continuous function from a topological space X as $C(X)$.

And here are our two examples. The first is of a complete metric space.

Exercise 1.68. Give $V := C([0, 1])$ the uniform norm

$$\|f\|_\infty := \sup\{|f(t)| : t \in [0, 1]\}.$$

Then V is complete.

Proof. This is merely the statement that a sequence of continuous functions which are uniformly Cauchy will converge uniformly to a continuous function. We will prove this for completeness. Fix a sequence of continuous function $\{f_n\}_{n\in\mathbb{N}}$ which are Cauchy with respect to $\|\cdot\|_\infty$. In other words, for each $\varepsilon > 0$, there exists N_ε so that

$$n, m > N_\varepsilon \implies \|f_n - f_m\|_\infty < \varepsilon,$$

which means that $|f_n(t) - f_m(t)| < \varepsilon$ for all $t \in [0, 1]$.

In particular, for any fixed $t \in [0, 1]$, the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ (using the same N_ε), so we use the completeness of $\mathbb{R}$ to let this sequence converge to $f(t) \in \mathbb{R}$. We have the following checks.

- To see that $f_n \to f$ as $n \to \infty$ (under our metric), select any $\varepsilon > 0$, and then find N so that

$$n, m > N \implies \|f_n - f_m\|_\infty < \varepsilon/3.$$

Further, for any $t \in [0,1]$, we see that we can find a large enough $n_t > N$ so that $|f(t) - f_{n_t}(t)| < \varepsilon/3$. But then $n > N$ has

$$|f_n(t) - f(t)| \le |f_n(t) - f_{n_t}(t)| + |f_{n_t}(t) - f(t)| < 2\varepsilon/3,$$

so $\|f - f_n\|_\infty \le 2\varepsilon/3 < \varepsilon$.

- To see that f is continuous, fix $t \in [0,1]$ so that we want to show f is continuous at t. Well, for any $\varepsilon > 0$, find N large enough so that

$$n, m > N \implies \|f_n - f_m\|_\infty < \varepsilon/4.$$

Now, select $n_t > N$ large enough so that $|f(t) - f_{n_t}(t)| < \varepsilon/4$, and the continuity of f_{n_t} promises us $\delta > 0$ so that

$$|t - t'| < \delta \implies |f_{n_t}(t) - f_{n_t}(t')| < \varepsilon/4.$$

In particular, for any t' with $|t - t'| < \delta$, find $n_{t'} > N$ large enough so that $|f(t') - f_{n_{t'}}(t')| < \varepsilon/4$, and then we see

$$|f(t) - f(t')| \le |f(t) - f_{n_t}(t)| + |f_{n_t}(t) - f_{n_t}(t')| + |f_{n_t}(t') - f_{n_{t'}}(t')| + |f_{n_{t'}}(t') - f(t')| < \varepsilon,$$

which is what we wanted. ■

The second example is the same space, but it is no longer complete.

Example 1.69. Fix $p \ge 1$ finite. Give $V := C([0,1])$ the L^p norm as

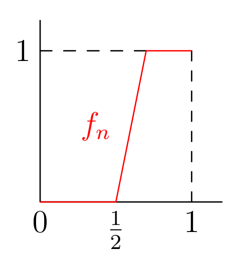
$$\|f\|_p := \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p}.$$

Then V is not complete.

Proof. For each $n \ge 2$, define f_n as the piecewise continuous function

$$f_n(t) := \begin{cases} 0 & 0 \le t \le \frac{1}{2}, \\ n(t - \frac{1}{2}) & \frac{1}{2} \le t \le \frac{1}{2} + \frac{1}{n}, \\ 1 & \frac{1}{2} + \frac{1}{n} \le t \le 1. \end{cases}$$

Here is the image.



The point is that f_n is trying to converge to a discontinuous function. To help us with the proof here, we pick up the following lemma.

Lemma 1.70. Fix $V := C([0,1])$ and some finite $p \ge 1$. If we have a convergent sequence $f_n \to f$ as $n \to \infty$ in the $\|\cdot\|_p$ metric, and $f_n(t) = g(t)$ for all sufficiently large n and $t \in U$ for some open $U \subseteq C([0,1])$, then $f|_U(t) = g(t)$.

Proof. Suppose for the sake of contradiction that we have $t_0 \in U$ with $f(t_0) \neq g(t_0)$; we show that $\{f_n\}$ does not converge to f. Set $\varepsilon := |f(t_0) - g(t_0)|$, which is nonzero. The continuity of $f - g$ now promises that there is $\delta > 0$ for which

$$|t - t_0| < \delta \implies |(f-g)(t_0) - (f-g)(t)| < \varepsilon/2,$$

so in particular $|(f-g)(t)| \geq \varepsilon/2$. It follows that, for sufficiently large n, we have

$$\|f - f_n\|_p^p = \int_0^1 |f(t) - f_n(t)|^p \, dt \geq \int_U |(f-g)(t)| \, dt \geq \int_{U \cap (t_0 - \delta, t_0 + \delta)} \frac{\varepsilon}{2} \, dt.$$

Because $U \cap (t_0 - \delta, t_0 + \delta)$ is open, it has nonzero measure, so this entire right-hand quantity is nonzero, thus violating that $f_n \to f$ as $n \to \infty$. ■

Now suppose for the sake of contradiction that $f_n \to f$ as $n \to \infty$ for some $f \in V$. Then, using $U = (0, 1/2)$, we conclude that $f(t) = 0$ for all $t \in (0, 1/2)$. Similarly, for any n, we set $U_n = (1/2_1/n, 1)$, so $f_m|_{U_n}$ returns 1 always for sufficiently large m; this then implies $f(t) = 1$ for any $t \in U_n$ for any n, so $f(t) = 1$ for any $t \in (1/2, 1)$.

However, the sequences $a_n := \frac{1}{2} - \frac{1}{n}$ and $b_n := \frac{1}{2} + \frac{1}{n}$ (for $n \geq 3$) have $a_n \to \frac{1}{2}$ and $b_n \to \frac{1}{2}$ both as $n \to \infty$ while the continuity of f would require

$$0 = \lim_{n \to \infty} f(a_n) = f(1/2) = \lim_{n \to \infty} f(b_n) = 1,$$

which is a contradiction. ■

Remark 1.71. In an attempt to make this metric space complete, we can try to specify which functions we want to look at, which motivates the theory of measure and integration.

Remark 1.72. The $\|\cdot\|_2$ norm on $C(X)$ for some (say) subset $X \subseteq \mathbb{R}$ with finite measure as coming from an inner product

$$\langle f, g \rangle := \int_X f(t)\overline{g(t)} \, dt.$$

When $\|\cdot\|_2$ is complete, we would then get a Hilbert space, which are very nice normed vector spaces, and we'll see more of them in Math 202B.

Remark 1.73 (Nir)**.** In contrast to the finite case, we see that the $\|\cdot\|_\infty$ norm induces a different (metric) topology on $C([0,1])$ than the $\|\cdot\|_p$ norms with p finite because the former is complete while the latter are not. In fact, all the norms $\|\cdot\|_p$ induce different topologies on $C([0,1])$.

THEME 2

TOPOLOGY

Sets are not doors.

—Munkres

2.1 August 29

We continue lecture by shifting to topology.

2.1.1 Metric Topology

We close our discussion of metric spaces with a taste of topology. Recall the following definition.

Definition 1.45 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous at* $x \in X$ if and only if all $\varepsilon > 0$ have some $\delta_x > 0$ such that

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$

Then f is *continuous* if and only if it is continuous at all $x \in X$.

We are going to want to extend this definition to more general topological spaces. To step in that direction, we will want to talk about open sets, so we start with open balls.

Definition 2.1 (Ball)**.** Fix a metric space (X, d). Then the *open ball of radius* r *centered at* $x_0 \in X$ is

$$B(x_0, r) \coloneqq \{x \in X : d(x, x_0) < r\}.$$

The *closed ball* is $\overline{B(x_0, r)} \coloneqq \{x \in X : d(x, x_0) \le r\}$.

We can now restate continuity as follows.

Definition 2.2 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous* at $x \in X$ if and only if, given any nonempty open ball $B(f(x_0), \varepsilon)$, there exists a nonempty open ball $B(x_0, \delta)$ such that

$$f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon).$$

Namely, we've really only restated our inequalities.

To continue our generalization, we define the pre-image.

Definition 2.3 (Pre-image)**.** Fix a function $f\colon X \to Y$. Then we define the *pre-image* $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

Note that our pre-image notation matches with the notation of an inverse function. In general, no confusion will arise by confusing these two.

As such, let's restate continuity again: observe that $A \subseteq X$ and $B \subseteq Y$ has $f(A) \subseteq B$ if and only if all $a \in A$ have $f(a) \in B$ if and only if all $a \in A$ have $a \in f^{-1}(B)$ if and only if $A \subseteq f^{-1}(B)$.

Definition 2.4 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous* at $x \in X$ if and only if, given any nonempty open ball $B(f(x), \varepsilon)$, there exists a nonempty open ball $B(x, \delta)$ such that

$$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)).$$

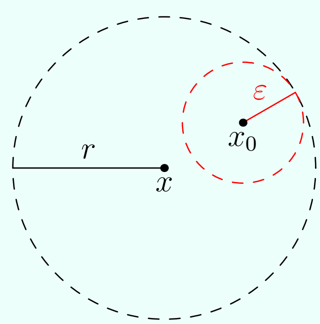
We defined open balls and promised open sets, so now let's define our open sets.

Definition 2.5 (Open set)**.** Fix a metric space (X, d). Then a subset $U \subseteq X$ is *open* if and only if, for each $x \in U$, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. In other words, each point in U has an open ball around it.

Example 2.6. Open balls are open sets. Indeed, given an open ball $B(x, r)$, note that any $x_0 \in B(x, r)$ has $d(x_0, x) < r$, so we take $\varepsilon := r - d(x_0, x)$. To see this works, observe $x' \in B(x_0, \varepsilon)$ will have

$$d(x', x) \leq d(x', x_0) + d(x_0, x) < \varepsilon + (r - \varepsilon) = r,$$

so $B(x_0, \varepsilon) \subseteq B(x, r)$ follows. Here is the image for what just happened.



And here is our definition of corresponding definition of continuity.

Lemma 2.7. Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is continuous at $x \in X$ if and only if, given any open set $U \subseteq Y$ with $f(x) \in U$, there is an open ball $B(x, \delta)$, such that

$$B(x, \delta) \subseteq f^{-1}(U).$$

Proof. Taking f to be continuous, note that we can find $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$ because U is open. Thus, continuity promises $\delta > 0$ such that

$$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)) \subseteq f^{-1}(U).$$

Conversely, if f satisfies the conclusion of the statement, we can take $U = B(f(x), \varepsilon)$ for any $\varepsilon > 0$ by Example 2.6, and the conclusion promises $\delta > 0$ such that

$$B(x,\delta) \subseteq f^{-1}(U) = f^{-1}(B(f(x),\varepsilon)),$$

which is what we wanted. ■

It is cleaner to talk about the entire function being continuous instead of at a point.

Lemma 2.8. Given metric spaces (X, d_X) and (Y, d_Y), a function $f: X \to Y$ is continuous if and only if, given any open set $U \subseteq Y$ with $f(x) \in U$, the pre-image $f^{-1}(U)$ is open.

Proof. This is a matter of rearranging our quantifiers correctly. Lemma 2.7 tells us that, for all $x \in X$, all open $U \subseteq Y$ with $f(x) \in U$ has some $\delta > 0$ such that $B(x,\delta) \subseteq U$. Equivalently, for all open $U \subseteq Y$, any $x \in X$ with $x \in f^{-1}(U)$ has some $\delta > 0$ such that $B(x,\delta) \subseteq U$. But by definition of being open, we're just saying that all open $U \subseteq Y$ has $f^{-1}(U)$ also open. ■

So we have the following definition.

Definition 2.9 (Continuous)**.** A function $f: X \to Y$ between metric spaces is *continuous* if and only if, for any open set $U \subseteq Y$, the pre-image $f^{-1}(U)$ is open.

The philosophy here is to try to understand open sets instead of trying to understand the metrics. This is the idea of topology.

2.1.2 Open Sets

Thus, we are motivated to understand open sets. Here are some basic properties.

Proposition 2.10. Fix a metric space (X, d), and let $\mathcal{T}$ be the collection of open sets.

(a) We have $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union

$$\bigcup_{U \in \mathcal{U}} U$$

is open.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \in \mathcal{T}$, we have

$$\bigcap_{i=1}^{n} U_i$$

is open.

Proof. We go in sequence.

(a) To show $X \in \mathcal{T}$, note that any $x \in X$ has $B(x,1) \subseteq X$ by definition. To show $\varnothing \in \mathcal{T}$, note that any $x \in \varnothing$ has $B(x,1) \subseteq \varnothing$ because there is no $x \in \varnothing$ at all.

(b) For any $x \in \bigcup_{U \in \mathcal{U}} U$, we have $x \in V$ for some particular $V \in \mathcal{U}$. Then the openness of V tells us we can find $\varepsilon > 0$ such that

$$B(x,\varepsilon) \subseteq V \subseteq \bigcup_{U \in \mathcal{U}} U,$$

which finishes.

(c) Fix x in the common intersection. Then, for any i, we have $x \in U_i$, so we have some $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subseteq U$, and so we set

$$\varepsilon := \min_{1 \le i \le n} \varepsilon_i.$$

In particular, $\varepsilon > 0$ because n is finite, and we have

$$B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq U_i$$

for each i, so $B(x, \varepsilon)$ is a subset of our intersection. ■

Remark 2.11. The arbitrary intersection of open sets need not be open: working in $\mathbb{R}$ with the usual metric,

$$\bigcap_{i=1}^{\infty} B(0, 1/n) = \{0\},$$

which is not open. (Namely, no $\varepsilon > 0$ has $B(x, \varepsilon) \subseteq \{0\}$.)

Motivated by Proposition 2.10, we have the following definition.

Definition 2.12 (Topology)**.** Fix a set X. Then a *topology* $\mathcal{T}$ on X is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following.

(a) We have $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union $\bigcup_{U \in \mathcal{U}} U$ lives in $\mathcal{T}$.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $\bigcap_{i=1}^{n} U_i$ lives in $\mathcal{T}$.

We will say that the ordered pair $(X, \mathcal{T})$ is a *topological space*. We say that the sets in $\mathcal{T}$ are *open*.

Example 2.13. By Proposition 2.10, metric spaces with their open sets form a topological space.

Here are some more basic examples.

Definition 2.14 (Discrete topology)**.** Given a set X, the *discrete topology* is the topology $\mathcal{P}(X)$.

Definition 2.15 (Indiscrete topology)**.** Given a set X, the *indiscrete topology* is the topology $\{\varnothing, X\}$.

It is fairly routine to check that the above collections form topologies. In fact, they are closed under both arbitrary union and arbitrary intersection.

Remark 2.16. The discrete topology can be defined by the metric $d \colon X \times X \to \mathbb{R}_{\ge 0}$ by

$$d(x, x') := \begin{cases} 1 & x \ne x', \\ 0 & x = x'. \end{cases}$$

Indeed, for any $x \in X$, we see $B(x, 1/2) = \{x\}$, so any subset $U \subseteq X$ is the open set

$$U = \bigcup_{x \in U} \{x\} = \bigcup_{x \in U} B(x, 1/2).$$

Remark 2.17. If $\#X \geq 2$, the indiscrete topology cannot be given a metric. Indeed, find distinct points $a, b \in X$ and set $r := d(a,b)$, so $a \neq b$ implies $r > 0$. Now, $a \in B(a,r)$, but $b \notin B(a,r)$, so $B(a,r)$ is an open set distinct from both $\varnothing$ and X.

Remark 2.18. One can give topologies a partial order by inclusion. Then the discrete topology is the maximal one (definitionally, any topology is a subset of $\mathcal{P}(X)$), and the indiscrete topology is the minimal one (definitionally, any topology contains $\varnothing$ and X).

And so here is our general definition of continuity.

Definition 2.19 (Continuous)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Then a function $f\colon X \to Y$ is *continuous* if and only if, for any $U_Y \in \mathcal{T}_Y$, we have $f^{-1}(U_Y) \in \mathcal{T}_X$.

2.2 August 31

It is once again the morning.

2.2.1 Intersections of Topologies

We will want to have lots of topologies to work with. Here is a basic way to build them.

Proposition 2.20. Let X be a set, and pick up some collection of topologies $\{\mathcal{T}_\alpha\}_{\alpha\in\lambda}$. Then the intersection

$$\mathcal{T} := \bigcap_{\alpha\in\lambda} \mathcal{T}_\alpha.$$

Proof. This is mostly a matter of writing out the axioms.

(a) Note that $\varnothing, X \in \mathcal{T}_\alpha$ for each α, so $\varnothing, X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, we have $\mathcal{U} \subseteq \mathcal{T}_\alpha$ for each α, so $\bigcup_{U\in\mathcal{U}} U \in \mathcal{T}_\alpha$ for each α, so

$$\bigcup_{U\in\mathcal{U}} U \in \mathcal{T}$$

as well.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, we have $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha$ for each α, so

$$\bigcap_{i=1}^n U_i \in \mathcal{T}$$

follows. ■

Corollary 2.21. Fix a set X. Given a collection $\mathcal{S} \subseteq \mathcal{P}(X)$, there is a smallest topology $\mathcal{T}$ containing $\mathcal{S}$.

Proof. Certainly there is some topology containing $\mathcal{S}$, namely the discrete topology. Thus, we can set our topology to be

$$\bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S} \\ \mathcal{T}\text{ a topology}}} \mathcal{T},$$

which is a topology (by Proposition 2.20) which contains $\mathcal{S}$. ■

To codify this idea, we have the following idea.

Notation 2.22 (Generated topology)**.** Fix a set X. We say that a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ *generates* its smallest topology $\mathcal{T}$.

Remark 2.23. It is fairly clear that this smallest topology $\mathcal{T}$ generated by $\mathcal{S}$ is unique.

2.2.2 Sub-bases and Bases

On the other side of things, we pick up the following definition.

Definition 2.24 (Sub-base)**.** Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{S} \subseteq \mathcal{T}$ is a *sub-base* for $\mathcal{T}$ if and only if the following hold.

(a) $\mathcal{S}$ covers X, in that $X = \bigcup_{U \in \mathcal{S}} U$.

(b) $\mathcal{T}$ is generated by $\mathcal{S}$.

The point is that collections $\mathcal{S}$ are easy to find, so we have therefore found many topologies.

It will be useful to give a more concrete description of the topology generated by a collection $\mathcal{S}$.

Lemma 2.25. Fix a set X and a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with $X = \bigcup_{U \in \mathcal{S}} U$. Then set

$$\mathcal{I}^{\mathcal{S}} := \left\{ \bigcap_{i=1}^{n} U_i : \{U_i\}_{i=1}^{n} \subseteq \mathcal{S} \right\}.$$

Then $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$ and $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection. Further, the topology generated by $\mathcal{I}$ is also the topology generated by $\mathcal{S}$.

Proof. Omitted. ■

Continuing, we have the following.

Lemma 2.26. Fix a set X and a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ closed under finite intersection with $\bigcup_{U \in \mathcal{I}} U = X$. Then the collection of (arbitrary) unions of elements in $\mathcal{I}$, denoted

$$\mathcal{T} := \left\{ \bigcap_{U \in \mathcal{U}} U : \mathcal{U} \subseteq \mathcal{I} \right\},$$

is the smallest topology containing $\mathcal{I}$.

Proof. Certainly any topology containing $\mathcal{I}$ must contain $\mathcal{T}$. Note $\varnothing \in \mathcal{T}$ because it is an empty union. Continuing, $\mathcal{T}$ is closed under arbitrary union because the union of unions is a union.

Lastly, to show that $\mathcal{T}$ is closed under finite intersection, it suffices by induction to show that sets $U, V \in \mathcal{T}$ have $U \cap V \in \mathcal{T}$. However, we can find $\mathcal{U}\mathcal{V} \subseteq \mathcal{T}$ so that

$$U = \bigcup_{U' \in \mathcal{U}} U' \qquad \text{and} \qquad \bigcup_{V' \in \mathcal{V}} V',$$

from which it follows

$$U \cap V = \bigcup_{\substack{U' \in \mathcal{U} \\ V' \in \mathcal{V}}} \underbrace{(U' \cap V')}_{\in \mathcal{I}},$$

which shows $U \cap V \in \mathcal{T}$. ■

Corollary 2.27. Fix a set X and a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with $X = \bigcup_{U \in \mathcal{S}} U$. Letting $\mathcal{I}^{\mathcal{S}}$ be the collection of finite intersections of $\mathcal{S}$ and then $\mathcal{T}$ be the collection of arbitrary unions of $\mathcal{I}^{\mathcal{S}}$, we have that $\mathcal{T}$ is the topology generated by $\mathcal{S}$.

Proof. Combine the previous two lemmas. ■

Remark 2.28. The point of discussing sub-bases is that we will be allowed to check continuity on only a sub-base.

Having defined a sub-base, we should be rightly upset that we have not defined a base.

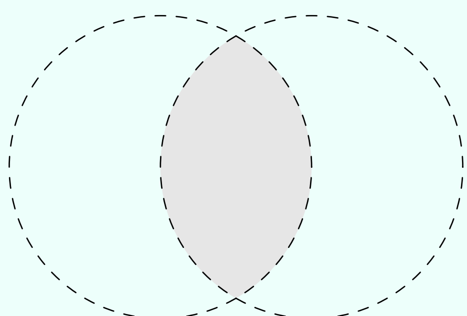
Definition 2.29 (Base)**.** Fix a set X. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *base* (for a topology on X) if and only if the collection of arbitrary unions of $\mathcal{B}$ form a topology on X.

This definition is a little hard to access because we still don't have a good notion of what a topology is.

Example 2.30. Fix a set X. Given any collection $\mathcal{S} \subseteq \mathcal{P}(X)$, the collection of finite intersections $\mathcal{I}^{\mathcal{S}}$ is a base by Lemma 2.26.

However, in general we do not require a base to be closed under finite intersection.

Example 2.31. Fix a metric space (X, d). Then the collection of open balls $\mathcal{B}$ forms a topology by Example 2.13. Notably, the intersection of two open balls need not be an open ball, as follows.



Even though we are not closed under finite intersection, we do have the following.

Lemma 2.32. Fix a set X with a base $\mathcal{B} \subseteq \mathcal{P}(X)$ for a topology $\mathcal{T}$. Then any finite intersection of elements of $\mathcal{B}$ is an arbitrary union of elements of $\mathcal{B}$.

Proof. The finite intersection must live in $\mathcal{T}$ given by $\mathcal{B}$, but any element of $\mathcal{T}$ is an arbitrary union of elements of $\mathcal{B}$. ■

Remark 2.33. Of course, any base is also a sub-base. Notably, sub-bases only require that $X = \bigcup_{U \in \mathcal{S}} U$, which must be satisfied for bases.

Example 2.34. Set $X = \mathbb{R}$ with the usual topology. Then the open intervals (a,b) form a base for the usual topology (these are our open balls), and the collection

$$\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$

forms a sub-base for the usual topology because $(-\infty, b) \cap (a, \infty) = (a, b)$.

Let's go ahead and Remark 2.28.

Proposition 2.35. Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and let $\mathcal{S}$ be a sub-base for $\mathcal{T}_Y$. Then a function $f\colon X \to Y$ is continuous if and only if

$$f^{-1}(U) \in \mathcal{T}_X$$

for all $U \in \mathcal{S}$.

Proof. Certainly if f is continuous then the pre-image of the open set $U \in \mathcal{S}$ must be open. On the other hand, let $\mathcal{T}_Y' \subseteq \mathcal{P}(Y)$ be the collection of subsets U for which $f^{-1}(U) \in \mathcal{T}_X$. We claim that $\mathcal{T}_Y'$ is a topology.

Certainly $f^{-1}(\varnothing) = \varnothing$, so $\varnothing \in \mathcal{T}_Y'$. Also, note $\mathcal{T}_Y'$ is closed under finite intersection: given $U, V \in \mathcal{T}_Y'$, we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \in \mathcal{T}_X.$$

Additionally, $\mathcal{T}_Y'$ is closed under arbitrary union: given $\mathcal{U} \subseteq \mathcal{T}_Y'$, we have

$$f^{-1}\left(\bigcup_{U \in \mathcal{U}} U\right) = \bigcup_{U \in \mathcal{U}} f^{-1}(U) \in \mathcal{T}_X.$$

Thus, $\mathcal{T}_Y'$ is indeed a topology, and it contains $\mathcal{S}$, so it follows that $\mathcal{T}_Y \subseteq \mathcal{T}_Y'$, so $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_Y$. This finishes the check that f is continuous. ■

2.2.3 Induced Topologies

We start with the following motivating example.

Example 2.36. Fix a set X, and give it the discrete topology. Then, for any topological space $(Y, \mathcal{T}_Y)$, any function $f\colon X \to Y$ is continuous because the pre-image of any open subset $U_Y \subseteq Y$ is open in X.

In general, we might have some smallish collection of functions which we want to force to be continuous, so we might ask what topology is forced by their continuity.

Definition 2.37 (Induced topology)**.** Fix a set X and a collection of topologies $\{(Y_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ with some functions $f_\alpha\colon X \to Y_\alpha$ for each $\alpha \in \lambda$. Then

$$\bigcup_{\alpha \in \lambda} \left\{ f_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha \right\}$$

is a sub-base for an *induced topology*.

The one thing to check is that X belongs to the arbitrary unions of our collection, which is clear because $X = f_\alpha^{-1}(Y_\alpha)$.

Definition 2.38 (Relative topology)**.** Fix $(Y, \mathcal{T})$ a topological space. Then the *relative topology* for a subset $X \subseteq Y$ is the topology induced by the natural embedding $\iota\colon X \hookrightarrow Y$.

We have the following more concrete description.

Lemma 2.39. Fix $(Y, \mathcal{T}_Y)$ a topological space. Then the relative topology for a subset $X \subseteq Y$ consists of the subsets
$$\{X \cap U : U \in \mathcal{T}_Y\}.$$

Proof. Let $\iota\colon X \hookrightarrow Y$ be the natural embedding. Then we are given the sub-base
$$\mathcal{S} := \left\{\iota^{-1}(U) : U \in \mathcal{T}_Y\right\}.$$
Now, $\iota^{-1}(U) = X \cap U$, and then we can check directly that this collection gives a topology. ■

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List of Definitions