225A: Model Theory

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

1.1 August 24

It begins.

1.1.1 Logistics

Here are some logistical notes.

- There is a bCourses.
- We will use [Mar02].
- Professor Montalbán and Scanlon will teach the course jointly.
- There will be a midterm (in-class on the 19th of October) and final exam (take-home over three days).
- There are suggested but technically ungraded exercises. They are helpful.
- We will assume basic first-order logic, and examples will be taken from a few other areas of mathematics.
- This is a graduate class. It will be pretty fast.

We are studying model theory, which is the study of models and theories. Our main two theorems are the Compactness theorems and results on admitting types. We will use these results again and again.

1.1.2 Languages and Structures

Let's review chapter 1 of [Mar02]. Here is a language.

Definition 1.1 (language)**.** A *language* $\mathcal{L}$ consists of the sets $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{C}$ of symbols. Here, $\mathcal{F}$ are functions, $\mathcal{R}$ are relations, and $\mathcal{C}$ are constants. Notably, there is an arity function $n\colon (\mathcal{F}\cup\mathcal{R})\to\mathbb{N}$.

Concretely, fix a language $\mathcal{L}=(\mathcal{F},\mathcal{R},\mathcal{C})$. If $f\in\mathcal{F}$ and $n(f)=3$, then we say that f has arity three; the analogous statement holds for relations.

We will often abbreviate a language to just a long tuple. For example, the notation $(\mathbb{N},0,1,+,\leq)$ has the domain $\mathbb{N}$ and constants 0 and 1 and function $+$ and relation $\leq$, even though the notation has not made it obvious what any of these things are.

So far we only have the prototype of data. Here is the data.

Definition 1.2 (structure)**.** Fix a language $\mathcal{L}$. Then an $\mathcal{L}$-*structure* $\mathcal{M}$ consists of the following data.

- Domain: a set M.
- Functions: for each $f \in \mathcal{F}$, there is a function $f^{\mathcal{M}}: M^{n(f)} \to M$.
- Relations: for each $R \in \mathcal{R}$, there is a relation $R^{\mathcal{M}} \subseteq M^{n(r)}$.
- Constants: for each $c \in \mathcal{C}$, there is a constant $c^{\mathcal{M}} \in M$.

The various $(-)^{\mathcal{M}}$ data are called *interpretations*.

Example 1.3. Consider the language $\mathcal{L}$ with the constants 0 and 1 and operations $+$ and $\times$. Then $\mathbb{N}$ is an $\mathcal{L}$-structure, in the obvious way.

In general, algebra provides many examples of languages.

We would like to relate our structures.

Definition 1.4 (homomorphism, embedding, isomorphism)**.** Fix a language $\mathcal{L}$. An $\mathcal{L}$-*homomorphism* $\eta: \mathcal{M} \to \mathcal{N}$ of $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ is a one-to-one map $\eta: M \to N$ preserving the interpretations, as follows.

- Functions: for each $f \in \mathcal{F}$, we have $\eta \circ f^{\mathcal{M}} = f^{\mathcal{N}} \circ \eta^{n(f)}$.
- Relations: for each $R \in \mathcal{R}$, if $\overline{m} \in R^{\mathcal{M}}$, then $\eta^{n(R)}(m) \in R^{\mathcal{N}}$.
- Constants: for each $c \in \mathcal{C}$, we have $\eta\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$.

If $\eta: M \to N$ is one-to-one and the relations condition is an equivalence, then η is an $\mathcal{L}$-*embedding*. If $\eta: M \to N$ is the identity $M \subseteq N$, then we say that $\mathcal{M}$ is an $\mathcal{L}$-substructure. In addition, if η is onto, then η is an $\mathcal{L}$-*isomorphism*.

Explicitly, being a substructure means that the functions and relations are restricted appropriately, and the constants remain the same.

Example 1.5. In the language of groups, subgroups make substructures. A similar sentence holds for other algebraic structures.

1.1.3 Formulae

Thus far we have described a vocabulary: the language provides the data for us to manipulate. We now discuss how to "speak" in this language.

Definition 1.6 (term)**.** Let $\mathcal{L}$ be a language. The set of $\mathcal{L}$-*terms* is the smallest set $\mathcal{T}$ satisfying the following.

- Constants: for each $c \in \mathcal{C}$, we have $c \in \mathcal{T}$.
- Variables: $x_i \in \mathcal{T}$ for each $i \in \mathbb{N}$. Notably, we have only countably many variables.
- Functions: if $t_1, \ldots, t_n \in \mathcal{T}$ where $n = n(f)$ for some $f \in \mathcal{F}$, then $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Given an $\mathcal{L}$-structure $\mathcal{M}$ and term $t \in \mathcal{T}$ with variables $x_1, \ldots, x_n$ and elements $a_1, \ldots, a_n \in M$, we define $t^{\mathcal{M}}(\overline{a})$ in the obvious way.

Terms are basically just nouns. We would now like to put them into sentences.

Definition 1.7 (atomic formula)**.** The set of *atomic $\mathcal{L}$-formulae* is the set of expressions of one of the following forms.

- Equality: $t_1 = t_2$ for any $\mathcal{L}$-terms t_1 and t_2.
- Relations: $R(t_1, \ldots, t_n)$ for any n-ary relation R and $\mathcal{L}$-terms $t_1, \ldots, t_n$.

Definition 1.8 (formula)**.** The set of *$\mathcal{L}$-formulae* is the smallest set satisfying the following.

- Any atomic $\mathcal{L}$-formula φ is an $\mathcal{L}$-formula.
- For any $\mathcal{L}$-formulae φ and ψ, then $\neg\varphi$ and $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\mathcal{L}$-formulae.
- For any variable v_i for $i \in \mathbb{N}$, then $\exists v_i \varphi$ is an $\mathcal{L}$-formula.

One can then define the shorthand "$\varphi \to \psi$" for $\neg\varphi \vee \psi$ and "$\forall v_i \varphi$" for $\neg\exists v_i \neg\varphi$.

Now that we can talk about our structure, we would like to know if we are making sense.

Definition 1.9 (free variable)**.** Fix a language $\mathcal{L}$. A variable v in a formula φ is *free* if and only if it is not bound to any quantifier $\exists v$ or $\forall v$. If φ has free variables contained in the variables $x_1, \ldots, x_n$, we write $\varphi(x_1, \ldots, x_n)$.

This definition is vague because we have not said what "bound" means, but it is rather obnoxious to explain what it is rigorously, so we will not bother.

Definition 1.10 (sentence)**.** Fix a language $\mathcal{L}$. An $\mathcal{L}$-formula with no free variables is a *sentence*.

Definition 1.11 (truth)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Further, fix an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and a tuple $\overline{a} \in M^n$. Then we define *truth* as $\mathcal{M} \vDash \varphi(\overline{a})$ to mean that φ is true upon plugging in $\overline{a}$, where our definition is inductive on atomic formulae as follows.

- $\mathcal{M} \vDash (t_1 = t_2)(\overline{a})$ if and only if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.
- $\mathcal{M} \vDash R(t_1, \ldots, t_n)$ if and only if $\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_2^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$.

We define truth inductively on formulae now as follows.

- $\mathcal{M} \vDash (\varphi \wedge \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ and $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash (\varphi \vee \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ or $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash \neg\varphi(\overline{a})$ if and only if we do not have $\mathcal{M} \vDash \varphi(\overline{a})$.
- $\mathcal{M} \vDash \exists v \varphi(\overline{a}, v)$ if and only if there exists $b \in M$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$.

In this case, we say that $\mathcal{M}$ *satisfies, models, etc.* $\varphi(\overline{a})$ and so on.

Here is our first result of substance.

Proposition 1.12. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-embedding $\eta\colon \mathcal{M} \to \mathcal{N}$. Further, fix a quantifier-free formula φ. Then $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi$.

Proof. Induction on φ. Roughly speaking, the point is that the interpretations are the same before and after. ■

Remark 1.13. If we allow variables, the statement is false. For example, $(\mathbb{N}, 0, \leq)$ embeds into $(\mathbb{Z}, 0, \leq)$, but $\forall x(0 \leq x)$ is true in the first formula while false in the second.

In the case of isomorphism, we can say more.

Proposition 1.14. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-isomorphism $\eta\colon \mathcal{M} \to \mathcal{N}$. Further, fix any formula φ with free variables $x_1, \ldots, x_n$ and a tuple $\overline{a} \in M^n$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi(f(\overline{a}))$.

Proof. Induction on φ. The point is that the definition of truth is the same before and after η. ■

1.2 August 29

We continue with the speed run of first-order logic. The goal for today is to state the Compactness theorem.

1.2.1 Theories

Now that we have a notion of truth, it will be helpful to keep track of which sentences exactly we want to be true.

Definition 1.15 (theory)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Then the *theory* $\operatorname{Th}(\mathcal{M})$ of $\mathcal{M}$ is the set of all sentences φ such that $\mathcal{M} \vDash \varphi$.

The theory is essentially all that first-order logic can see.

Definition 1.16 (elementary equivalence)**.** Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. Then we say that $\mathcal{M}$ and $\mathcal{N}$, written $\mathcal{M} \equiv \mathcal{N}$, are *elementarily equivalent* if and only if $\operatorname{Th}(\mathcal{M}) = \operatorname{Th}(\mathcal{N})$.

Example 1.17. It happens that $(\mathbb{Q}, +) \equiv (\mathbb{R}, +)$ but are not isomorphic because they have different cardinalities.

Example 1.18. Let s denote the successor function. It happens that $(\mathbb{Z}, s) \equiv (\mathbb{Q}, s)$, but one can show that they are not isomorphic.

This notion is different from isomorphism, but it is related.

Lemma 1.19. Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Proof. This is the content of Proposition 1.14 upon unraveling the definitions. ■

Going in the other direction, we might start with some sentences we want to be true and then look for the corresponding models.

Definition 1.20 (theory)**.** Fix a language $\mathcal{L}$. Then an *$\mathcal{L}$-theory* T is a set of $\mathcal{L}$-sentences. For an $\mathcal{L}$-structure $\mathcal{M}$, we say that $\mathcal{M}$ *models* T, written $\mathcal{M} \vDash T$, if and only if $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathcal{M}$. We let $\operatorname{Mod}(T)$ denote the class of all models $\mathcal{M}$ of T, and we call it an *elementary class*.

Example 1.21. The class of all groups arises from the language $\{e, \cdot\}$ with some sentences to make a theory. However, the class of torsion groups is not an elementary class.

We want might want to understand what sentences follow from a given theory.

Definition 1.22. Fix a language $\mathcal{L}$ and theory T. Then we say that T logically implies a sentence φ, written $T \vDash \varphi$, if and only if any $\mathcal{L}$-structure $\mathcal{M}$ modelling T has $\mathcal{M} \vDash \varphi$.

Remark 1.23. Gödel's completeness theorem shows that $T \vDash \varphi$ if and only if there is a "proof" of φ from T. We will not use the notion of proof so much, though its proof is similar to the proof of compactness, which we will show.

1.2.2 Definable Sets

We will want the following notion.

Definition 1.24 (definable)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $B \subseteq M$. Then a subset $X \subseteq M^n$ is B-*definable* if and only if there is a formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_k)$ and tuple $\overline{b} \in B^k$ such that $\overline{a} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{a}, \overline{b})$. The tuple $\overline{b}$ might be called the *parameters*. We may abbreviate M-definable to simply *definable*.

Example 1.25. Any finite set is definable by using the parameters to list out the elements.

Example 1.26. Work with $\mathcal{M} := (\mathbb{Z}, \leq)$. Then $X = \mathbb{N}$ is $\{0\}$-definable by $\varphi(x, 0)$ where $\varphi(x, y)$ is given by $y \leq x$. However, $\mathbb{N}$ is not $\varnothing$-definable, as shown by the following proposition.

Proposition 1.27. Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $A \subseteq M$. Further, suppose $X \subseteq M^n$ is A-definable. For any automorphism $\sigma \colon \mathcal{M} \to \mathcal{M}$ fixing A pointwise must fix X (not necessarily pointwise).

Proof. Suppose $\varphi(\overline{v}, \overline{w})$ defines X with the parameters $\overline{a} \in A^\bullet$. Then $\overline{x} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{x}, \overline{a})$, but then $\mathcal{M} \vDash \varphi(\sigma(\overline{x}), \sigma(\overline{a}))$, so $\mathcal{M} \vDash \varphi(\sigma(\overline{x}), \overline{a})$ so $\sigma(\overline{x}) \in X$. For the converse, use the inverse automorphism σ^{-1}. ■

To further explain Example 1.26, we see that there are automorphisms of $\mathbb{Z}$ (namely, by shifting) which do not fix $\mathbb{N}$, so $\mathbb{N}$ cannot be $\varnothing$-definable.

Example 1.28. Work with $\mathcal{M} := (\{1A, 1B, 2A, 2B\}, \leq)$ with partial ordering given by the number. The set $X := \{1A, 1B\}$ is $\varnothing$-definable by $\varphi(x)$ given by $\exists y((x \neq y) \wedge (x \leq y))$. However, there is an automorphism of our model swapping $1A$ with $1B$ and $2A$ with $2B$, but this automorphism does not fix X pointwise.

1.2.3 The Compactness Theorem

To state compactness, we want a few definitions.

Definition 1.29 (satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *satisfiable* if and only if it has a model $\mathcal{M}$.

With a notion of proof, one can show that being satisfiable means that there is no proof of $\perp$, but we will avoid a discussion of proofs in this course.

Definition 1.30 (finitely satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *finitely satisfiable* if and only if any finite subset of T is satisfiable.

Of course, being satisfiable implies being finitely satisfiable; the converse will be true but is far from obvious. The following example explains why this is strange.

Example 1.31. Consider the natural numbers $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times, \leq)$ and $\mathcal{N}_c := (\mathbb{N}, 0, 1, +, \times, \leq, c)$, where c is some constant symbol, and set

$$T := \operatorname{Th}(\mathcal{N}) \cup \left\{ c \geq \underbrace{1 + 1 + \cdots + 1}_{n} : n \in \mathbb{N} \right\}.$$

Then T is finitely satisfiable by $\mathcal{N}$ because, for any finite subset of T, the sentences $c \geq 1+1+\cdots+1$ will have to be bounded in length in our finite subset, so we simply find some c large enough in $\mathcal{N}$. However, $\mathcal{N}$ does not model T!

Anyway, here is our statement.

Theorem 1.32 (compactness)**.** Fix a language $\mathcal{L}$ and theory T. If T is finitely satisfiable, then T is satisfiable. Furthermore, T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}| + \aleph_0$.

Remark 1.33. In particular, the theory T of Example 1.31 has a model $\mathcal{N}'$, which is going to look very strange. To begin, there is a segment

$$0 < 1 < 2 < \cdots.$$

But there is now an element c larger than any natural, which produces $c+1, c+2, c+3, \ldots$. But also any nonzero element has a predecessor, so we have elements $c-1, c-2, c-3, \ldots$. Further, any natural number is either odd or even, so there is also either $(c-1)/2$ or $c/2$ sitting between the initial piece of $\mathbb{N}$ and the c piece with $\mathbb{Z}$ added everywhere. In fact, a similar argument holds to produce an element approximately equal to qc for any rational $q \in \mathbb{Q}$.

Remark 1.34. One can of course always make our model larger. For example, suppose we have a theory T with an infinite model. If we want a model with cardinality at least $\mathbb{R}$, we add constants $\{c_r : r \in \mathbb{R}\}$ to our language and add in the sentences

$$\{c_r \neq c_s : \text{distinct } r, s \in \mathbb{R}\}.$$

This remains finitely satisfiable: these constants merely ask for our model to be larger than any finite set. One can even require the new model to be elementarily equivalent to the previous one.

Here are some applications of compactness.

Corollary 1.35. The class of torsion groups is not elementarily definable in the language $\mathcal{L} = \{e, *\}$ of groups.

Notably, it is not okay to write something like

$$\bigvee_{n \in \mathbb{N}} (\forall g\, g^n = e)$$

to encode any torsion because this statement is infinitely long.

Proof. Suppose the class is elementarily definable. Then we have a theory T such that $\mathrm{Mod}(T)$ consists exactly of all torsion groups. Now the trick is to build a model of T which is not actually a torsion group. For this, we expand our language to $\mathcal{L} = \{e, *, c\}$, and let

$$S \coloneqq T \cup \left\{ \underbrace{c * c * \cdots * c}_{n} \neq e : n \geq 2 \right\}.$$

For any finite subset of S, we can satisfy S by a torsion group containing an element which is not n-torsion for sufficiently large n; for example, $\mathbb{Z}/n\mathbb{Z}$ will do.

Thus, by Theorem 1.32, there is a model $\mathcal{G}$ of S, so in particular, $\mathcal{G}$ has an element $g \in G$ with

$$\underbrace{g * g * \cdots * g}_{n} \neq e$$

for all $n \geq 2$ (namely, g is the interpretation of the constant symbol c), so it follows that G is not torsion. However, $\mathcal{G}$ is also a model of T and thus is supposed to be torsion, so we have a contradiction! This completes the proof. ■

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List of Definitions