256B: Algebraic Geometry

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Curves

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

1.1 January 18

Here we go.

1.1.1 House-Keeping

Here are some notes on the course.

- We will continue to use [Har77]. Note that [Vak17] is also popular, as is [SP].
- Office hours will probably be after class on Wednesday and Friday.
- There is a `bCourses`.
- In the course, we plan to cover curves, some coherent cohomology (and maybe on Zariski sheaves), and some surfaces if we have time.
- Grading will be homework and a term paper. Homework will be challenging, so collaboration is encouraged.

In this course, we will discuss coherent cohomology, but we will begin by talking about curves.

1.1.2 Serre Duality Primer

For the next few weeks, we will focus on non-singular curves over an algebraically closed field. Here is our definition.

> **Definition 1.1** (curve)**.** Fix a field k. A k-*curve* is an integral, proper, normal scheme of dimension 1. Note that being normal is equivalent to being smooth, so we are requiring our curves to be smooth!

We will want to talk about genus a little. Here is a working definition.

Definition 1.2 (arithmetic genus)**.** Fix a projective k-variety X. Then the *arithmetic genus* $p_a(X)$ is defined

Definition 1.3 (geometric genus)**.** Fix an irreducible k-variety X. Then the *geometric genus* is $p_g(X) \coloneqq \dim_k \Gamma(X, \omega_X)$, where ω_X is the canonical sheaf. Explicitly, ω_X is the top exterior power of the sheaf of differential forms on X.

In general, the above notions are not the same, but they will be for curves.

Proposition 1.4. Fix a k-curve X. Then $p_g(X) = p_a(X)$. We denote this genus by $g(X)$ or g when the curve is clear.

We would like to actually compute some genera, but this is a bit difficult. One goal of the class is to build a cohomology theory $H^i(X, \mathcal{F})$ for coherent sheaves $\mathcal{F}$ on X, and it turns out we can use these cohomology groups to compute the genus of X. Roughly speaking, we will derive (on the right) the left-exact functor $\Gamma(X, \cdot)$, so the cohomology will in some sense measure the difference between global sections and local sections. For example, flasque sheaves will have trivial cohomology.

For now, we will black-box various things. Here is an example of something we will prove.

Proposition 1.5. Fix a projective k-variety X, and let $\mathcal{F}$ be a coherent sheaf. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$, and $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces for all $i \geq 0$.

To show the Riemann–Roch theorem, we will black-box Serre duality, which we will prove much later. In the case of curves, it says the following.

Theorem 1.6 (Serre duality)**.** Fix a k-curve X. Then, for any vector bundle $\mathcal{L}$ on X, there is a duality

$$H^i(X, \mathcal{L}^\vee \otimes \omega_X) \otimes_k H^{1-i}(X, \mathcal{L}) \to k,$$

where $i \in \{0, 1\}$.

Remark 1.7. Notably, we see $p_g(X) = \dim_k \Gamma(X, \omega_X) = \dim_k H^0(X, \omega_X) = \dim_k H^1(X, \mathcal{O}_X)$.

We will also want the following fact.

Proposition 1.8. Fix a closed embedding $i\colon X \to Y$ of schemes. Given a sheaf $\mathcal{F}$ of abelian groups on Y, then

$$H^i(X, i_*\mathcal{F}) = H^i(Y, \mathcal{F}).$$

1.1.3 Divisors Refresher

We also want to recall a few facts about divisors. We begin with Weil divisors.

Definition 1.9 (Weil divisor)**.** Fix an irreducible k-variety X. A *Weil divisor* $\operatorname{Div}(X)$ are $\mathbb{Z}$-linear combinations of codimension-1 irreducible closed subschemes. Then the *principal divisors* are the image of the map $\operatorname{div}\colon K(X) \to \operatorname{Div}(X)$, where div takes rational functions to poles. The *class group* $\operatorname{Cl} X$ is the quotient.

More generally, we have Cartier divisors.

Definition 1.10 (Cartier divisor)**.** Fix a scheme X. A *Cartier divisor* in $\operatorname{CaDiv} X$ is a global section of $\Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$, where $K^\times$ is the sheafification of the presheaf $U \mapsto \operatorname{Frac} \mathcal{O}_X(U)$. The *principal divisors* are the image of $\Gamma(X, \mathcal{K}^\times)$, and the *class group* $\operatorname{CaCl} X$ is the quotient.

Notably, if X is an integral sheaf, then $\mathcal{K}$ is the constant sheaf $K(X)$. Then a global section is given by the pair $(\{U_i\}, \{f_i\})$ where the U_i cover X, and $f_i \in K(X)^\times$ so that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^\times$. (The coherence condition allows the Cartier divisors to glue.) Notably, each $f \in K(X)$ grants a principal divisor $(\{X\}, \{f\})$, which are exactly the principal divisors.

Here is the main result on these divisors.

Proposition 1.11. If X is an integral, separated, Noetherian, and locally factorial (notably, regular in codimension 1), then Weil divisors are in canonical isomorphism with Cartier divisors. Further, the principal divisors are in correspondence, and so the class groups are also isomorphic.

Example 1.12. Non-singular k-curves have all the required adjectives. Namely, codimension-1 means we are looking at points, and being smooth implies being regular, so all the local rings are dimension-1 regular local rings, which are discrete valuation rings. Notably, discrete valuation rings are

Yet another connection to divisors comes from invertible sheaves. Namely, for integral schemes X, the group of invertible sheaves $\operatorname{Pic} X$ is isomorphic to $\operatorname{CaCl} X$. The point here is that invertible shaves can be embedded into $\mathcal{K}^\times$ when X is integral.

We will be interested in some special divisors.

Definition 1.13 (effective)**.** Fix a k-curve X. Then an *effective Weil divisor* is a $\mathbb{Z}_{\geq 0}$ linear combination of closed points of X. Note that the collection of effective Weil divisors forms a submonoid of $\operatorname{Div} X$. We might be interested in knowing how many effective divisors are equivalent to some given divisor; the set of these is denoted $|D|$.

When our schemes X have enough adjectives, we note that the above correspondences tell us that there is a way to send a Cartier divisor $(\{U_i\}, \{f_i\})$ to a line bundle $\mathcal{L}$ embedded in $\mathcal{K}^\times$. Explicitly, we build $\mathcal{L}(D)$ by $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_X|_{U_i} \subseteq \mathcal{K}$, where the last isomorphism is by sending $1 \mapsto f_i^{-1}$. Notably, if D is effective, then the global section 1 of $\mathcal{K}^\times$ can be pulled back along to a nonzero global section on $\mathcal{L}(D)$ which is f_i on each U_i.

1.2 January 20

We continue moving towards Riemann–Roch.

1.2.1 Linear Systems

Let's discuss linear systems. Let X be a non-singular projective irreducible variety over a field k, and let D be a divisor of X.

Recall that a Cartier divisor $D = \{(U_i, f_i)\}$ on X is associated to the line bundle $\mathcal{L}(D)$ which is locally trivial on each U_i, given as $f_i^{-1}\mathcal{O}_X|_{U_i}$. Conversely, suppose that $\mathcal{L}$ is a line bundle on X. Then we pick up some nonzero global section $\Gamma(X, \mathcal{L})$. Give $\mathcal{L}$ a trivializing open cover $\{U_i\}$, where we are given isomorphisms $\varphi_i \colon \mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$. Setting $f_i \coloneqq \varphi_i(s)$ recovers an (effective) Cartier divisor $\{(U_i, f_i)\}$ on X. We call this line bundle $\operatorname{div}(\mathcal{L}, s)$.

This thinking gives the following result.

Proposition 1.14. Let X be a non-singular projective integral variety over a field k. Given a Cartier divisor D_0, and let $\mathcal{L} := \mathcal{L}(D_0)$ be the corresponding line bundle.

(a) For each nonzero section $s \in \Gamma(X, \mathcal{L})$, the divisor $\operatorname{div}(\mathcal{L}, s)$ is an effective divisor linearly equivalent to D_0.

(b) Every effective divisor linearly equivalent tot D_0 is obtained in this way.

(c) If k is algebraically closed, we have $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s')$ if and only if s and s' differ by a scalar in $k^\times$.

The above result essentially says that we can study $\Gamma(X, \mathcal{L})$ as a k-vector space instead of trying to understand linear equivalence of divisors. For example, if $\Gamma(X, \mathcal{L}) = 0$, then D is not equivalent to any effective divisor!

Proof. We go one at a time.

(a) Embed $\mathcal{L} \subseteq \mathcal{K}_X$ as usual. Then $s \in \Gamma(X, \mathcal{L})$ becomes a rational function in $K(X)$. By the construction of $\mathcal{L}$, we have an open cover $\{U_i\}$ and some f_i so that $\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_X|_U$. Because we have a global section, we may write $\varphi_i(s) = f_i f$ for some fixed f, and then tracking through our Cartier divisor, we get
$$\operatorname{div}(\mathcal{L}, s) = D_0 + \operatorname{div}(f),$$
as needed.

(b) Suppose D is an effective divisor with $D = D_0 + \operatorname{div}(f)$. Then we see $(f) \geq -D_0$, so f determines a nonzero global section of $\mathcal{L}\mathcal{L}(D_0)$ by tracking through the above constructions: namely, set $s|_{U_i} = f_i^{-1}f$ and glue. (In particular, $(f) \geq -D_0$ means $f/f_i \in \mathcal{O}_X(U_i)$ for each i.) So we see $D = \operatorname{div}(\mathcal{L}, s)$.

(c) One can see directly that $s = cs'$ for $c \in k^\times$ will have $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s)$. Conversely, if $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s')$, then under the embedding $\mathcal{L} \subseteq \mathcal{K}_X$, we may correspond s and s' to $f, f' \in K(X)^\times$. Thus, $f/f' \in \Gamma(X, \mathcal{O}_X^\times)$. But because k is algebraically closed and X is proper over k, we have $\Gamma(X, \mathcal{O}_X) = k$, so we are done. ∎

Remark 1.15. More generally, we have the following: let k be a field, and let X be a proper, geometrically reduced scheme over k. Then $\Gamma(X, \mathcal{O}_X) = k$ if and only if X is geometrically reduced.

So we have the following.

Corollary 1.16. Let X be a non-singular projective integral variety over a field k. The set $|D_0|$ of effective divisors linearly equivalent to a given divisor D_0 is in natural bijection with $(\Gamma(X, \mathcal{L}(D)) \setminus \{0\})/k^\times$.

With this in mind, we set the following notation.

Notation 1.17. Let X be a non-singular projective integral variety over a field k. Given a divisor D_0 of X, we define $\ell(D_0) := \dim_k \Gamma(X, \mathcal{L}(D))$ and $\dim D_0 := \ell(D_0) - 1$.

The Riemann–Roch theorem is interested in the values of $\ell(D_0)$. Here is a quick lemma.

Lemma 1.18. Let X be a non-singular projective integral variety over a field k. Fix a divisor D of X.

(a) If $\ell(D) \neq 0$, then $\deg D \geq 0$.

(b) If $\ell(D) \neq 0$ and $\deg D = 0$, then D is linearly equivalent to 0.

Proof. Note $\ell(D) \neq 0$ enforces $D \sim D_0$ for some effective divisor D, so $\deg D = \deg D_0 \geq 0$, which shows (a). Then for (b), we note $\deg D_0 = 0$ forces $D_0 = 0$. ∎

1.2.2 Riemann–Roch for Curves

We now force $\dim X = 1$, meaning that X is a curve. Let $\Omega_{X/k}$ denote the sheaf of differentials, which is equal to the canonical sheaf $\omega_X = \bigwedge^{\dim X} \Omega_{X/k}$. Any divisor linearly equivalent to $\Omega_{X/k}$ will be denoted K and is called the "canonical divisor." Note that the canonical divisor is really a canonical divisor class.

Theorem 1.19 (Riemann–Roch)**.** Let D be a divisor on a k-curve X, and let g be the genus of X. Further, suppose k is algebraically closed. Then

$$\ell(D) - \ell(K - D) = \deg D + 1 - g.$$

Proof. Set $\mathcal{L} := \mathcal{L}(D)$ for brevity. Note $\mathcal{L}(K-D) \cong \omega_X \otimes \mathcal{L}^\vee$, so Serre duality implies

$$\ell(K-D) = \dim_k \Gamma(\omega_X \otimes \mathcal{L}^\vee) = \dim_k H^1(X, \mathcal{L}).$$

Thus, our left-hand side is $\chi(\mathcal{L}) := \dim H^0(X,\mathcal{L}) - \dim H^1(X,\mathcal{L})$.[1] Quickly, note $D = 0$ can be seen directly by

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = \dim k - g = 1 - g,$$

which is what we wanted.

We now perturb D by a point. We show the formula holds for D if and only if the formula holds for $D+p$, where $p \in X$ is some closed point. Note we have a short exact sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{O}_X \to k(p) \to 0,$$

where $k(p)$ refers to the skyscraper sheaf which is the structure sheaf about p. Tensoring with $\mathcal{L}(D+p)$, we get

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D+p) \to k(p) \to 0.$$

Now, χ is additive in short exact sequences by using the long exact sequence in cohomology, so

$$\chi(\mathcal{L}(D)) = \chi(\mathcal{L}(D+p)) + \chi(k(P)),$$

but $\chi(k(p)) = \dim_k \Gamma(X, k(p)) = \dim_k k = 1$ because k is algebraically closed. The conclusion now follows because $\deg(D+p) = \deg D + 1$. ■

1.3 January 23

Today we apply the Riemann–Roch theorem.

Remark 1.20. Here is a quick hint for the homework: fix a Weil divisor $D = \sum_P n_P P$ on a k-curve X, where k is algebraically closed. Then $\Gamma(X, \mathcal{O}_X(D))$ can be described as space of rational functions f on X such that $D + \operatorname{div}(f)$ is effective. In other words, for each point $P \in X$, we see f has a pole of order at worse n_P at P.

1.3.1 Applications of Riemann–Roch

Let's give a few applications of Theorem 1.19.

Example 1.21. Fix a k-curve X, where k is algebraically closed. Further, let g be the genus of X and K the canonical divisor. We can compute $\deg K$ as follows: plugging into Theorem 1.19, we see

$$g - 1 = \ell(K) - \ell(0) = \deg K - 1 + g,$$

so $\deg K = 2g - 2$.

[1] This is the Euler characteristic of $\mathcal{L}(D)$ because our higher cohomology groups vanish.

Remark 1.22. More generally, we can see that plugging in $K - D$ into Theorem 1.19 is only able to deduce $\deg K = 2g - 2$.

Example 1.23. Let D be a divisor on a k-curve X, where k is algebraically closed. Further, let g be the genus of X and K the canonical divisor. We would like to study $\dim|nD| = \ell(nD) - 1$ for $n \in \mathbb{Z}^+$. We have the following cases.

- If $\deg D < 0$, then $\deg(nD) < 0$ still, so $\ell(nD) = 0$, so $\dim|nD| = -1$ always.
- If $\deg D = 0$, then there are two possibilities. Namely, if nD is linearly equivalent to 0, then $\ell(nD) = 1$, so $\dim|nD| = 0$; otherwise, D will not be linearly equivalent to any effective divisor (the only effective divisor with degree 0 is 0), so $\dim|nD| = -1$.
- If $\deg D > 0$, then for n large enough, we see $\deg(K - nD) < 0$, so $\ell(K - nD) = 0$, so Theorem 1.19 implies $\ell(nD) = n\deg D + 1 - g$, so $\dim|nD| = n\deg D - g$. Here, "n large enough" is just $n > \deg K/\deg D$.

Here is a more interesting corollary.

Lemma 1.24. Let X be a k-curve, where k is algebraically closed. Suppose that two distinct closed points P and Q produce linearly equivalent Weil divisors. Then $X \cong \mathbb{P}^1_k$.

Proof. We are given that $\operatorname{div}(f) = P - Q$ for some $f \in K(X)$. Thus, we induce a map $k(t) \to K(X)$ given by $t \mapsto f$, where we view $k(t)$ as the fraction field of $\mathbb{P}^1_k$. Notably, t has a zero at 0 and a pole at ∞, and f has a zero at Q and a pole at P. This will induce a finite map $g\colon X \to \mathbb{P}^1$, which we can compute to have degree 1 by the following discussion (notably, the pull-back of the divisor $[0]$ is $[P]$), so g is a birational map and hence an isomorphism.

Now, for any finite map of curves $g\colon X \to Y$, recall there is a map on divisors $g^*\colon \operatorname{Cl}(Y) \to \operatorname{Cl}(X)$ as follows: given point $P \in Y$ inside an affine open subscheme $V \subseteq Y$, we can take the pre-image to X to produce a Weil divisor.[2] More formally, we send P to

$$g^*(P) \coloneqq \sum_{Q \in g^{-1}(\{P\})} v_Q(t)Q,$$

What? where t is a uniformizer parameter for $Q_{X,P}$, and $v_Q(t)$ is its valuation at the local ring $\mathcal{O}_{X,Q}$.

In fact, we showed the following last semester, which we used in the proof above.

Proposition 1.25. Let $g\colon X \to Y$ be a finite map of k-curves. For any divisor D on Y, we have $\deg g^*D = (\deg g)(\deg D)$.

Proof. Let's recall the proof: it suffices to show this in the case where $D = P$ is a point. Plugging into the definition of g^*, we are showing that

$$\sum_{Q \in g^{-1}(\{P\})} v_Q(t) = \deg g^*P \stackrel{?}{=} \deg g.$$

This statement is local at P, so we may assume that $Y = \operatorname{Spec} B$, whereupon taking the pre-image along g enforces $X = \operatorname{Spec} A$ for some A. For dimension-theory reasons, we see that g is dominant, so the induced map $B \to A$ is injective.

Localizing, we set $A' \coloneqq A \otimes_B \mathcal{O}_{Y,P}$, so we are really interested in the map $\mathcal{O}_{Y,P} \to A'$, which is still injective. It follows that A' is a finite (by g) torsion-free (by this injectivity argument) module over $\mathcal{O}_{Y,P}$. But

[2] Alternatively, one can view this operation as the pullback $g^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$ and then recall that each element of the class group corresponds to an isomorphism class in Pic.

$\mathcal{O}_{Y,P}$ is a principal ideal domain, so we may appeal to the structure theorem. Namely, we want to compute the rank of A' over $\mathcal{O}_{Y,P}$, for which it suffices to take fraction fields everywhere and instead compute

$$\operatorname{rank}_{\mathcal{O}_{Y,P}} A' = [\operatorname{Frac} A' : \operatorname{Frac} \mathcal{O}_{Y,P}] = [\operatorname{Frac} A : \operatorname{Frac} B].$$

On the other hand, given uniformizer t of $\mathcal{O}_{X,P}$, we can compute the corresponding rank of A'/tA' over $k = k(P)$ is $\deg g$. However, $\operatorname{Spec} A'/tA'$ is the pre-image of P, so we go ahead and note A'/tA' is a product of local Artinian rings which are quotients corresponding to points in $g^{-1}(\{P\})$. In particular, for each $Q \in g^{-1}(P)$, we see $v_Q(t) = m$ means that $\mathcal{O}_{X,Q}$ appears in A'/tA' as $\mathcal{O}_Q/(\varpi_Q)^m$, where ϖ_Q is a uniformizer at Q. So we can write

$$A'/tA' = \prod_{Q \in g^{-1}(\{P\})} \mathcal{O}_{X,Q}/(\varpi_Q)^{v_Q(t)}.$$

But the k-rank of this is $\deg g^*(\{P\})$ by definition of g^*, which must equal the $\mathcal{O}_{Y,P}$-rank of A', so we are done. ■

This proposition finishes justifying the first paragraph of the proof. ■

Corollary 1.26. Let X be a k-curve of genus 0, where k is algebraically closed. Then X is isomorphic to $\mathbb{P}^1_k$.

Proof. As an aside, we note that $\mathbb{P}^1_k$ is certainly a k-curve of genus 0.

Quickly, choose any two points P and Q on X. As such, we take $D := P - Q$ so that $\deg(K - D) = -2 < 0$, so $\ell(K - D) = 0$. Thus, Theorem 1.19 implies $\ell(D) = 1$. Thus, D is linearly equivalent to an effective divisor, but the only effective divisor with degree 0 is 0 itself, so we see that $P - Q$ is linearly equivalent to 0. This is enough to conclude that $X \cong \mathbb{P}^1_k$ by Lemma 1.24; note that X being a curve requires X to have infinitely many points and thus distinct points. ■

Lastly, let's give a corollary for elliptic curves.

Definition 1.27 (elliptic)**.** A (proper) k-curve X is *elliptic* if and only if X has genus 1.

Corollary 1.28. Let X be an elliptic k-curve, where k is algebraically closed. We give $X(k)$ a group law arising from $\operatorname{Pic} X$.

Proof. Let K be a canonical divisor for X, and we see $\deg K = 0$ by Example 1.21. However, $\ell(K) = 1$ is the genus, so K is linearly equivalent to some effective divisor, so as usual we note that K is linearly equivalent to 0.

Quickly, we note that the group structure on the Picard group $\operatorname{Pic} X$ of isomorphism classes of line bundles on $X(k)$ induces a group law on X. Indeed, fix some k-point $P_0 \in X$. We now claim that the map $X(k) \to \operatorname{Pic}^0 X$ given by

$$P \mapsto \mathcal{O}_X(P - P_0)$$

is a bijection. (Here, $\operatorname{Pic}^0 X$ is the subgroup of degree-0 line bundles.) This will give $X(k)$ a group law by stealing it from $\operatorname{Pic} X$.

Because we already know that $\operatorname{Pic} X$ is in bijection with divisors more generally, it's enough to show that any divisor D of degree 0 is linearly equivalent to a divisor of the form $P - P_0$ for $P \in X(k)$. Well, we use Theorem 1.19 with $D + P_0$, which yields

$$\ell(D + P_0) - \ell(K - D - P_0) = 1 + 1 - g = 1,$$

but $K - D - P_0$ has degree -1 and so $\ell(K - D - P_0) = 0$. Thus, $\dim |D + P_0| = 0$, so there is a unique effective divisor of degree 1 linearly equivalent to $D + P_0$. However, an effective divisor of degree 1 is just a point P, so we are done. ■

1.4 January 25

Today we discuss ramification.

1.4.1 Ramification

Given a finite morphism $f\colon X \to Y$ of k-curves, there are some numbers we can attach to f. For example, we have the degree $\deg f := [K(X) : K(Y)]$. Additionally, for each $y \in Y$, we have a map

$$\mathcal{O}_{Y,y} \to \prod_{x \in f^{-1}(\{y\})} \mathcal{O}_{X,x}.$$

Each of these discrete valuation rings have residue field k, so when k is algebraically closed, this is pretty simple to understand. Indeed, this gives rise to "ramification" information.

Definition 1.29 (ramification)**.** Let $f\colon X \to Y$ be a morphism of k-curves. Then for $x \in X$, we define the *ramification index* as $e_x := v_x(t_y)$, where $t_y \in \mathcal{O}_{Y,y}$ is a uniformizer for $\mathcal{O}_{Y,y}$, and $v_x(t_y)$ refers to the valuation of t_y embedded in $\mathcal{O}_{X,x}$.

Note $e_x > 0$ because $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a map of local rings.

When k is not algebraically closed, things get a little more complicated, and we will also want to keep track of the degree of the corresponding residue field extension.

Definition 1.30 (ramification)**.** Let $\varphi\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a map of discrete valuation rings. Fix a uniformizer $\varpi \in \mathfrak{m}$, and define $e := v_B(\varphi(\varpi))$ and $f := [B/\mathfrak{n} : A/\mathfrak{m}]$ and $p := \operatorname{char}(A/\mathfrak{m})$.

- φ is *unramified* if and only if $e = 1$ and $B/\mathfrak{n}$ is separable over $A/\mathfrak{m}$.
- φ is *tamely ramified* if and only if $p \nmid e$ and $B/\mathfrak{n}$ is separable over $A/\mathfrak{m}$.
- Otherwise, φ is *wildly ramified.*

Remark 1.31. Algebraic number theory has a lot to say about how the above process works for local fields (or even just number rings).

When k is algebraically closed, we see that the extension $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k$, so this extension is of course separable, so Definition 1.30 simplifies somewhat in our situation.

Definition 1.32. Let $f\colon X \to Y$ be a finite morphism of k-curves. Then we say that $x \in X$ is *unramified/tamely ramified/wildly ramified* if and only if the corresponding map $f^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is unramified/tamely ramified/wildly ramified.

Notably, last class we recalled that

$$f^*[y] = \sum_{x \in f^{-1}(\{y\})} e_x[x].$$

Now, even in our algebraically closed situation, we will want to care about separable extensions.

Definition 1.33 (separable)**.** Let $f\colon X \to Y$ be a finite morphism of k-curves. Then f is *separable* if and only if the extension $K(Y) \subseteq K(X)$ is separable.

Now, we would like to keep track of our ramification information all at once.

Lemma 1.34. Let $f\colon X \to Y$ be a finite separable morphism of k-curves. Then

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

is an exact sequence of line bundles on X.

Proof. We know from last semester that this map is exact on the right, so we need the map $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. Well, we may check exactness of quasicoherent sheaves on affine open subschemes, so we may assume that $X = \operatorname{Spec} A$, where the map looks like $A \to A$. Letting I denote the kernel of this map, we get an embedding $A/I \subseteq A$, but if I is nontrivial, then this means that the map $A \to A$ is zero at the generic point.

Thus, we want to check that the map $A \to A$ is nonzero at the generic point. Everything is compatible with localization, so we are now looking at

$$f^*\Omega_{K(Y)/k} \to \Omega_{K(X)/k} \to \Omega_{K(X)/K(Y)} \to 0.$$

Thus, to show that the map on the left is nonzero, it suffices to show that $\Omega_{K(X)/K(Y)} = 0$. We now use the fact that $K(X)/K(Y)$ is separable, for which the statement is true. ■

The point here is that $\Omega_{X/Y}$ precisely measures the "difference" between $\Omega_{Y/k}$ and $\Omega_{X/k}$.

In what follows, we fix the following notation. Let $f\colon X \to Y$ be a finite morphism of k-curves. Given $x \in X$ and $y := f(x)$, let ϖ_x be a uniformizer for $\mathcal{O}_{X,x}$ and ϖ_y be a uniformizer for $\Omega_{Y,y}$. Then we note $d\varpi_x$ generates $(\Omega_{X/k})_x$, and $d\varpi_y$ generates $(\Omega_{Y/k})_y$, and we have a map

$$(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x \simeq \mathcal{O}_{X,x}$$

by sending $f^*\colon d\varpi_y \mapsto d\varpi_y/d\varpi_x \cdot d\varpi_x$, where $d\varpi_y/d\varpi_x$ is an element of $\mathcal{O}_{X,x}$. (This is the definition of $d\varpi_y/d\varpi_x$.)

Proposition 1.35. Let $f\colon X \to Y$ be a finite separable morphism of k-curves.

(a) $\Omega_{X/Y}$ is supported on exactly the set of ramification points of f, so the set of ramified points is finite.

(b) For each $x \in X$, we have $(\Omega_{Y/X})_x$ is a principal $\mathcal{O}_{X,x}$-module of length $v_x(d\varpi_y/d\varpi_x)$.

(c) If f is tamely ramified at x, then the length of $(\Omega_{Y/X})_x$ is $e_x - 1$; if it's wildly ramified, then the length is larger.

Proof. We show these one at a time. As a warning, all uniformizers might be swapped here.

(a) Recall that $\Omega_{Y/X}$ is generically zero. Now, $(\Omega_{Y/X})_p = 0$ if and only if the map

$$(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x$$

is an isomorphism, which means that a uniformizer for $\mathcal{O}_{Y,f(x)}$ is going to a uniformizer of $\mathcal{O}_{X,x}$, which is equivalent to f being unramified at x. The point here is that the set of ramified points correspond to some dimension-zero subset and is therefore finite.

What?

(b) The length of $(\Omega_{Y/X})_p$ is its k-dimension, which we can compute as

$$\operatorname{length}(\Omega_{X/k})_x - \operatorname{length}(f^*\Omega_{Y/K})_x,$$

which we can compute is e_x by hand. Notably, this has to do with how we identify $(\Omega_{X/k})_x$ with $\mathcal{O}_{X,x}$.

(c) Letting e denote our ramification index, we may set $\varpi_y = a\varpi_x^e$ where $a \in \mathcal{O}_{X,x}^\times$, which upon taking differentials reveals
$$d\varpi_y = ea\varpi_x^{e-1}d\varpi_x + \varpi_x^e da.$$
Now, if f is tamely ramified at x, we see $\operatorname{char} k \nmid e$, so we see that the valuation here is in fact $e_x - 1$. The statement for wild ramification follows similarly. ■

Remark 1.36. The length of the modules here coincides with the dimension as a k-vector space. This is because $\mathcal{O}_{X,x}$ is a discrete valuation ring with residue field k.

1.5 January 27

There is homework due tonight. Today we keep talking towards the Riemann–Hurwitz formula.

1.5.1 The Riemann–Hurwitz Formula

Throughout, $f\colon X \to Y$ is a finite separable morphism of k-curves, where k is algebraically closed.

Definition 1.37 (ramification divisor)**.** Let $f\colon X \to Y$ be a finite separable morphism of k-curves, where k is algebraically closed. Then the *ramification divisor* is the divisor
$$R(f) := \sum_{x\in X} \operatorname{length}(\Omega_{Y/X})_x x.$$
Note that there are only finitely many ramified points, so this is indeed a divisor.

In particular, in tame ramification, this length is in fact exactly our ramification.

Lemma 1.38. Let $f\colon X \to Y$ be a finite separable morphism of k-curves, where k is algebraically closed. Further, let K_X and K_Y denote the canonical divisors. Then K_X is linearly equivalent to $f^*K_Y + R$.

Proof. We can see by hand that the structure sheaf $\mathcal{O}_R$ of R as a closed subscheme is exactly $\Omega_{Y/X}$, so the exact sequence
$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{Y/X} \to 0$$
What? can be tensored with $\Omega_{X/k}^{-1}$ to give
$$0 \to f^*\Omega_{Y/k} \otimes \Omega_{X/k}^{-1} \to \mathcal{O}_X \to \mathcal{O}_R \to 0.$$
Now, we see $\mathcal{O}_R = \mathcal{O}_X(-R)$ by unwinding definitions, so $f^*\Omega_{Y/k} \otimes \Omega_{X/k}^{-1} = \mathcal{O}_X(-R)$, which is what we wanted. ■

Theorem 1.39 (Hurwitz)**.** Let $f\colon X \to Y$ be a finite separable map of k-curves, where k is algebraically closed. Letting $n := \deg f$, we have
$$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R(f).$$

Proof. Take degrees of Lemma 1.38. ■

Let's derive some corollaries.

Definition 1.40 (étale)**.** A morphism locally of finite presentation $f\colon X \to Y$ of schemes is *étale* at $x \in X$ if and only if the following conditions hold.

- f is flat at x. In other words, $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module.
- $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a field and separable over $k(f(x))$. Equivalently, we are requiring $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and for the residue field extension to be separable.

Then f is étale if and only if f is étale at all points.

Remark 1.41. The locus of points for which a morphism is étale is open essentially because this is true for both conditions individually.

Remark 1.42. If the morphism of k-curves $f\colon X \to Y$ is étale, then it is unramified because we are essentially saying $\operatorname{length}\Omega_{X/Y} = 0$. In fact, unramified morphisms are étale, which should roughly be our intuition. Flatness is a bit mysterious, but such is life.

Remark 1.43. Equivalently, a morphism $f\colon X \to Y$ is étale if and only if f is a smooth morphism of relative dimension 0. For example, if both X and Y are varieties, then we are asking for the varieties to have the same sheaf of differentials.

Geometrically, we imagine finite étale morphisms as covering space maps. This motivates the following definition.

Definition 1.44 (simply connected)**.** A finite étale morphism $f\colon X \to Y$ is *trivial* if and only if X is a disjoint union of copies of Y, and f is a disjoint union of automorphisms. Then a curve Y is *simply connected* if and only if all finite étale morphisms $X \to Y$ are trivial.

Roughly speaking, we are trying to say that the fundamental group is trivial.

Proposition 1.45. Let k be an algebraically closed field. Then $\mathbb{P}^1_k$ is simply connected.

Example 1.46. We can see that the sphere $\mathbb{P}^1_{\mathbb{C}}$ is simply connected.

Proof. Fix some finite étale morphism $f\colon X \to Y$. By breaking this morphism into connected components, we may assume that X is connected. We show that $X = \mathbb{P}^1_k$. Because f is smooth, we see that the structure map $X \to Y \to k$ is smooth, so X is a (smooth) curve. (In particular, we can also see that X is irreducible.) Now, because f is étale, it is étale at the generic point, so f is also separable.

We are now ready to apply Theorem 1.39. Here, $R(f) = 0$ and $g(Y) = 0$, so we are left with

$$2g(X) - 2 = (\deg f)(0-2) = -2\deg f.$$

However, $g(X) \geq 0$ and $\deg f \geq 1$, so we must have $g(X) = 0$ and $\deg f = 1$, so f is in fact an isomorphism $X \cong \mathbb{P}^1_k$. ∎

Remark 1.47. In characteristic 0, we will have $\mathbb{A}^1_k$ is simply connected. However, in positive characteristic, this is no longer true; indeed, $\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{F}_p})$ is infinite.

1.6 January 30

Homework was assigned and still due on Friday, sadly.

Remark 1.48. Let k be an algebraically closed field of positive characteristic p. It turns out $G := \pi_1^{\text{ét}}(\mathbb{A}_k^1)$ is profinite but not topologically finitely generated—it's very large. In fact, one can show that any finite p-group arises as a quotient of $\pi_1^{\text{ét}}(\mathbb{A}_k^1)$. More generally, any finite quasi-p-group is a quotient, where a quasi-p-group is a finite group generated by its Sylow p-subgroups cover G.

1.6.1 Everything Is Frobenius

Thus far we roughly understand finite separable morphisms of curves. We now investigate the purely inseparable case. In particular, today k will be an algebraically closed field of positive characteristic p. Note there is a canonical embedding $\mathbb{F}_p \hookrightarrow k$, which gives rise to the Frobenius automorphism as follows.

Definition 1.49 (Frobenius)**.** Let k be a field of characteristic $p > 0$. Given a k-scheme X, we define the *Frobenius automorphism* $F\colon X \to X$ as being the identity on topological spaces and being the pth-power map $F_U^\sharp : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ for all open $U \subseteq X$.

We can see that this map takes units of $\mathcal{O}_{X,x}$ to units of $\mathcal{O}_{X,x}$ for any $x \in X$, so we have defined a morphism of locally ringed spaces.

Example 1.50. Let $X = \operatorname{Spec} A$ be an $\mathbb{F}_p$-scheme. Then the ring homomorphism $F\colon A \to A$ given by the pth power map is the Frobenius $F\colon X \to X$. To show the map is the identity on the topological space, we note $a^p \in \mathfrak{p}$ is equivalent to $a \in \mathfrak{p}$ for $a \in A$, where we are using the primality of $\mathfrak{p}$. Thus, $F^{-1}(\mathfrak{p}) = \mathfrak{p}$ for any prime $\mathfrak{p} \in \operatorname{Spec} A$.

The Frobenius map defined above is not k-linear because it is the pth power map on k too. To make this k-linear, we essentially cheat.

Definition 1.51. Fix a scheme X over a field k of characteristic p. Then we define the k-scheme X_p which is equal to X as a scheme but whose structure morphism to $\operatorname{Spec} k$ is given by

$$X_p \to \operatorname{Spec} k \xrightarrow{F} \operatorname{Spec} k.$$

The point is that the diagram

$$\begin{tikzcd} X_p \arrow[r, "F"] \arrow[d] \arrow[dr] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, "F"] & \operatorname{Spec} k \end{tikzcd} \tag{1.1}$$

commutes, so we do genuinely have a Frobenius morphism $F\colon X_p \to X$.

Remark 1.52. Explicitly, for each affine open $\operatorname{Spec} A \subseteq X$, we get an identical affine open $\operatorname{Spec} A_p \subseteq X_p$, but if the k-algebra structure on A is given by $g\colon k \to A$, then the k-algebra structure on A is given by $g_p(x) \cdot \alpha := g\left(x^p\right) \cdot \alpha$.

Namely, with everything being contravariant on functions, we see that

$$\begin{tikzcd} A & A \arrow[l, "F"'] \\ k \arrow[u] & k \arrow[l, "F"'] \arrow[u] \end{tikzcd}$$

commutes.

Remark 1.53. Certainly $X_p \cong X$ as schemes because they are literally the same data. If k is perfect, then $F\colon X_p \to X$ is an isomorphism of k-schemes because the pth-power map on k is an isomorphism. Explicitly, (1.1) has an isomorphism at the bottom row but is also a pullback square (which can be checked by hand), so the top row is also an isomorphism.

Example 1.54. Let k be a field of characteristic $p > 0$, and let X be an integral k-scheme. Then $F\colon X_p \to X$ is given by a morphism $K(X) \to K(X_p)$ by $\alpha \mapsto \alpha^p$. (Yes, this is k-linear because the k-action on $K(X_p)$ is by pth powers.) As such, we have defined an embedding of $K(X)$ into an algebraic extension $K(X_p)$: namely, every $\alpha \in K(X_p)$ is a root of the polynomial $t^p - \alpha^p = 0$ in $K(X)[t]$.

Conversely, $t^p - \beta \in K(X)[t]$ always has a root in $K(X_p)$ because $\beta \in K(X)$ embeds into $K(X_p)$ as β^p, so this polynomial "looks like" $t^p - \beta^p$ in $K(X_p)$, where β is our root. Thus,

$$K(X_p) = K(X)^{1/p}.$$

The point is that a k-curve X will have the Frobenius morphism $X_p \to X$ induced by the embedding $K(X) \to K(X)^{1/p}$.

Now, here is our main result.

Theorem 1.55. Let k be an algebraically closed field of characteristic $p > 0$. Further, let $f\colon X \to Y$ be a finite map of k-curves which induces a purely inseparable extension $f^\sharp\colon K(Y) \to K(X)$. Then f is some iterate of the Frobenius morphism. In particular, X and Y are isomorphic as schemes.

Proof. Note $\deg f = [K(X) : K(Y)]$ is a power of p, which we call p^ν. Being purely inseparable then enforces $K(X) \subseteq K(Y)^{1/p^\nu}$; namely, the minimal polynomial if all elements $\alpha \in K(X)$ must be the minimal polynomial of the form $x^{p^r} - \beta = 0$, for otherwise there is a separable subextension, violating our pure inseparability.

Now, consider iterated Frobenius morphisms

$$Y_{p^\nu} \to Y_{p^{\nu-1}} \to \cdots \to Y_p \to Y,$$

which corresponds to the inclusion of fields

$$K(Y) \subseteq K(Y)^{1/p} \subseteq \cdots \subseteq K(Y)^{1/p^{\nu-1}} \subseteq K(Y)^{1/p^\nu},$$

where the inclusions have reversed.

Thus, to conclude, we would like to show $K(X) = K(Y)^{1/p^\nu}$. By degree arguments, it's enough to conclude $\left[K(Y)^{1/p^\nu} : K(Y)\right] = p^\nu$. By induction, it's enough to show $\left[K(Y)^{1/p} : K(Y)\right] = p$. We now must use the fact that Y is a smooth k-curve, so we will push its proof into the following lemma.

Lemma 1.56. Let k be an algebraically closed field of characteristic $p > 0$. If Y is a smooth k-curve, then $\left[K(Y)^{1/p} : K(Y)\right] = p$.

Proof. Equivalently, we would like to show that $[K(Y) : K(Y)^p] = p$. Note that $\Omega_{K(Y)/k}$ is a $K(Y)$-vector space of dimension 1; we refer to Theorem II.8.6.A, where the point is that k being perfect tells us $K(Y)/k$ is separably generated, so $\dim_k \Omega_{K(Y)/k}$ is the transcendence degree of $K(Y)$ over k, which is 1.

We now note that dx generates $\Omega_{K(Y)/k}$ if and only if $x \in K(Y)$ yields a power basis $\{1, x, \ldots, x^{p-1}\}$ of $K(Y)$ over $K(Y)^p$, which completes the proof. ■

Okay.

The above lemma completes the proof. ■

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List of Definitions