225A: Model Theory

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Introduction

1.1 August 24

It begins.

1.1.1 Logistics

Here are some logistical notes.

- There is a bCourses.
- We will use [Mar02].
- Professor Montalbán and Scanlon will teach the course jointly.
- There will be a midterm (in-class on the 19th of October) and final exam (take-home over three days).
- There are suggested but technically ungraded exercises. They are helpful.
- We will assume basic first-order logic, and examples will be taken from a few other areas of mathematics.
- This is a graduate class. It will be pretty fast.

We are studying model theory, which is the study of models and theories. Our main two theorems are the Compactness theorems and results on admitting types. We will use these results again and again.

1.1.2 Languages and Structures

Let's review chapter 1 of [Mar02]. Here is a language.

> **Definition 1.1** (language)**.** A *language* $\mathcal{L}$ consists of the sets $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{C}$ of symbols. Here, $\mathcal{F}$ are functions, $\mathcal{R}$ are relations, and $\mathcal{C}$ are constants. Notably, there is an arity function $n\colon (\mathcal{F}\cup\mathcal{R})\to\mathbb{N}$.

Concretely, fix a language $\mathcal{L}=(\mathcal{F},\mathcal{R},\mathcal{C})$. If $f\in\mathcal{F}$ and $n(f)=3$, then we say that f has arity three; the analogous statement holds for relations.

We will often abbreviate a language to just a long tuple. For example, the notation $(\mathbb{N},0,1,+,\leq)$ has the domain $\mathbb{N}$ and constants 0 and 1 and function $+$ and relation $\leq$, even though the notation has not made it obvious what any of these things are.

So far we only have the prototype of data. Here is the data.

Definition 1.2 (structure)**.** Fix a language $\mathcal{L}$. Then an *$\mathcal{L}$-structure* $\mathcal{M}$ consists of the following data.

- Domain: a nonempty set M.
- Functions: for each $f \in \mathcal{F}$, there is a function $f^{\mathcal{M}} \colon M^{n(f)} \to M$.
- Relations: for each $R \in \mathcal{R}$, there is a relation $R^{\mathcal{M}} \subseteq M^{n(r)}$.
- Constants: for each $c \in \mathcal{C}$, there is a constant $c^{\mathcal{M}} \in M$.

The various $(-)^{\mathcal{M}}$ data are called *interpretations*.

Example 1.3. Consider the language $\mathcal{L}$ with the constants 0 and 1 and operations $+$ and $\times$. Then $\mathbb{N}$ is an $\mathcal{L}$-structure, in the obvious way.

In general, algebra provides many examples of languages.

We would like to relate our structures.

Definition 1.4 (homomorphism, embedding, isomorphism)**.** Fix a language $\mathcal{L}$. An *$\mathcal{L}$-homomorphism* $\eta \colon \mathcal{M} \to \mathcal{N}$ of $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ is a one-to-one map $\eta \colon M \to N$ preserving the interpretations, as follows.

- Functions: for each $f \in \mathcal{F}$, we have $\eta \circ f^{\mathcal{M}} = f^{\mathcal{N}} \circ \eta^{n(f)}$.
- Relations: for each $R \in \mathcal{R}$, if $\overline{m} \in R^{\mathcal{M}}$, then $\eta^{n(R)}(m) \in R^{\mathcal{N}}$.
- Constants: for each $c \in \mathcal{C}$, we have $\eta\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$.

If $\eta \colon M \to N$ is one-to-one and the relations condition is an equivalence, then η is an *$\mathcal{L}$-embedding*. If $\eta \colon M \to N$ is the identity $M \subseteq N$, then we say that $\mathcal{M}$ is an $\mathcal{L}$-substructure. In addition, if η is onto, then η is an *$\mathcal{L}$-isomorphism*.

Explicitly, being a substructure means that the functions and relations are restricted appropriately, and the constants remain the same.

Example 1.5. In the language of groups, subgroups make substructures. A similar sentence holds for other algebraic structures.

1.1.3 Formulae

Thus far we have described a vocabulary: the language provides the data for us to manipulate. We now discuss how to "speak" in this language.

Definition 1.6 (term)**.** Let $\mathcal{L}$ be a language. The set of *$\mathcal{L}$-terms* is the smallest set $\mathcal{T}$ satisfying the following.

- Constants: for each $c \in \mathcal{C}$, we have $c \in \mathcal{T}$.
- Variables: $x_i \in \mathcal{T}$ for each $i \in \mathbb{N}$. Notably, we have only countably many variables.
- Functions: if $t_1, \ldots, t_n \in \mathcal{T}$ where $n = n(f)$ for some $f \in \mathcal{F}$, then $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Given an $\mathcal{L}$-structure $\mathcal{M}$ and term $t \in \mathcal{T}$ with variables $x_1, \ldots, x_n$ and elements $a_1, \ldots, a_n \in M$, we define $t^{\mathcal{M}}(\overline{a})$ in the obvious way.

Terms are basically just nouns. We would now like to put them into sentences.

Definition 1.7 (atomic formula)**.** The set of *atomic $\mathcal{L}$-formulae* is the set of expressions of one of the following forms.

- Equality: $t_1 = t_2$ for any $\mathcal{L}$-terms t_1 and t_2.
- Relations: $R(t_1, \ldots, t_n)$ for any n-ary relation R and $\mathcal{L}$-terms $t_1, \ldots, t_n$.

Definition 1.8 (formula)**.** The set of *$\mathcal{L}$-formulae* is the smallest set satisfying the following.

- Any atomic $\mathcal{L}$-formula φ is an $\mathcal{L}$-formula.
- For any $\mathcal{L}$-formulae φ and ψ, then $\neg\varphi$ and $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\mathcal{L}$-formulae.
- For any variable v_i for $i \in \mathbb{N}$, then $\exists v_i \varphi$ is an $\mathcal{L}$-formula.

One can then define the shorthand "$\varphi \to \psi$" for $\neg\varphi \vee \psi$ and "$\forall v_i \varphi$" for $\neg\exists v_i \neg\varphi$.

Now that we can talk about our structure, we would like to know if we are making sense.

Definition 1.9 (free variable)**.** Fix a language $\mathcal{L}$. A variable v in a formula φ is *free* if and only if it is not bound to any quantifier $\exists v$ or $\forall v$. If φ has free variables contained in the variables $x_1, \ldots, x_n$, we write $\varphi(x_1, \ldots, x_n)$.

This definition is vague because we have not said what "bound" means, but it is rather obnoxious to explain what it is rigorously, so we will not bother.

Definition 1.10 (sentence)**.** Fix a language $\mathcal{L}$. An $\mathcal{L}$-formula with no free variables is a *sentence*.

Definition 1.11 (truth)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Further, fix an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and a tuple $\overline{a} \in M^n$. Then we define *truth* as $\mathcal{M} \vDash \varphi(\overline{a})$ to mean that φ is true upon plugging in $\overline{a}$, where our definition is inductive on atomic formulae as follows.

- $\mathcal{M} \vDash (t_1 = t_2)(\overline{a})$ if and only if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.
- $\mathcal{M} \vDash R(t_1, \ldots, t_n)$ if and only if $\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_2^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$.

We define truth inductively on formulae now as follows.

- $\mathcal{M} \vDash (\varphi \wedge \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ and $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash (\varphi \vee \psi)(\overline{a})$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$ or $\mathcal{M} \vDash \psi(\overline{a})$.
- $\mathcal{M} \vDash \neg\varphi(\overline{a})$ if and only if we do not have $\mathcal{M} \vDash \varphi(\overline{a})$.
- $\mathcal{M} \vDash \exists v \varphi(\overline{a}, v)$ if and only if there exists $b \in M$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$.

In this case, we say that $\mathcal{M}$ *satisfies, models, etc.* $\varphi(\overline{a})$ and so on.

Here is our first result of substance.

Proposition 1.12. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-embedding $\eta \colon \mathcal{M} \to \mathcal{N}$. Further, fix a quantifier-free formula φ and $\overline{a} \in M^n$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi(\overline{a})$.

Proof. Induction on φ. Roughly speaking, the point is that the interpretations are the same before and after. ■

Remark 1.13. If we allow variables, the statement is false. For example, $(\mathbb{N}, 0, \leq)$ embeds into $(\mathbb{Z}, 0, \leq)$, but $\forall x(0 \leq x)$ is true in the first formula while false in the second.

In the case of isomorphism, we can say more.

Proposition 1.14. Fix a language $\mathcal{L}$ and an $\mathcal{L}$-isomorphism $\eta\colon \mathcal{M} \to \mathcal{N}$. Further, fix any formula φ with free variables $x_1, \ldots, x_n$ and a tuple $\overline{a} \in M^n$. Then $\mathcal{M} \vDash \varphi(\overline{a})$ if and only if $\mathcal{N} \vDash \varphi(f(\overline{a}))$.

Proof. Induction on φ. The point is that the definition of truth is the same before and after η. ■

1.2 August 29

We continue with the speed run of first-order logic. The goal for today is to state the Compactness theorem.

1.2.1 Theories

Now that we have a notion of truth, it will be helpful to keep track of which sentences exactly we want to be true.

Definition 1.15 (theory)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$. Then the *theory* $\mathrm{Th}(\mathcal{M})$ of $\mathcal{M}$ is the set of all sentences φ such that $\mathcal{M} \vDash \varphi$.

The theory is essentially all that first-order logic can see.

Definition 1.16 (elementary equivalence)**.** Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. Then we say that $\mathcal{M}$ and $\mathcal{N}$, written $\mathcal{M} \equiv \mathcal{N}$, are *elementarily equivalent* if and only if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Example 1.17. It happens that $(\mathbb{Q}, +) \equiv (\mathbb{R}, +)$ but are not isomorphic because they have different cardinalities.

Example 1.18. Let s denote the successor function. It happens that $(\mathbb{Z}, s) \equiv (\mathbb{Q}, s)$, but one can show that they are not isomorphic.

This notion is different from isomorphism, but it is related.

Lemma 1.19. Fix $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Proof. This is the content of Proposition 1.14 upon unraveling the definitions. ■

Going in the other direction, we might start with some sentences we want to be true and then look for the corresponding models.

Definition 1.20 (theory)**.** Fix a language $\mathcal{L}$. Then an *$\mathcal{L}$-theory T* is a set of $\mathcal{L}$-sentences. For an $\mathcal{L}$-structure $\mathcal{M}$, we say that $\mathcal{M}$ *models T*, written $\mathcal{M} \vDash T$, if and only if $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathcal{M}$. We let $\mathrm{Mod}(T)$ denote the class of all models $\mathcal{M}$ of T, and we call it an *elementary class*.

Example 1.21. The class of all groups arises from the language $\{e, \cdot\}$ with some sentences to make a theory. However, the class of torsion groups is not an elementary class.

We want might want to understand what sentences follow from a given theory.

Definition 1.22. Fix a language $\mathcal{L}$ and theory T. Then we say that T logically implies a sentence φ, written $T \vDash \varphi$, if and only if any $\mathcal{L}$-structure $\mathcal{M}$ modelling T has $\mathcal{M} \vDash \varphi$.

Remark 1.23. Gödel's completeness theorem shows that $T \vDash \varphi$ if and only if there is a "proof" of φ from T. We will not use the notion of proof so much, though its proof is similar to the proof of compactness, which we will show.

1.2.2 Definable Sets

We will want the following notion.

Definition 1.24 (definable)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $B \subseteq M$. Then a subset $X \subseteq M^n$ is B-*definable* if and only if there is a formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_k)$ and tuple $\overline{b} \in B^k$ such that $\overline{a} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{a}, \overline{b})$. The tuple $\overline{b}$ might be called the *parameters*. We may abbreviate M-definable to simply *definable*.

Example 1.25. Any finite set is definable by using the parameters to list out the elements.

Example 1.26. Work with $\mathcal{M} \coloneqq (\mathbb{Z}, \leq)$. Then $X = \mathbb{N}$ is $\{0\}$-definable by $\varphi(x, 0)$ where $\varphi(x, y)$ is given by $y \leq x$. However, $\mathbb{N}$ is not $\varnothing$-definable, as shown by the following proposition.

Proposition 1.27. Fix an $\mathcal{L}$-structure $\mathcal{M}$ and subset $A \subseteq M$. Further, suppose $X \subseteq M^n$ is A-definable. For any automorphism $\sigma\colon \mathcal{M} \to \mathcal{M}$ fixing A pointwise must fix X (not necessarily pointwise).

Proof. Suppose $\varphi(\overline{v}, \overline{w})$ defines X with the parameters $\overline{a} \in A^\bullet$. Then $\overline{x} \in X$ if and only if $\mathcal{M} \vDash \varphi(\overline{x}, \overline{a})$, but then $\mathcal{M} \vDash \varphi(\sigma(\overline{x}), \sigma(\overline{a}))$, so $\mathcal{M} \vDash \varphi(\sigma(\overline{x}), \overline{a})$ so $\sigma(\overline{x}) \in X$. For the converse, use the inverse automorphism σ^{-1}. ■

To further explain Example 1.26, we see that there are automorphisms of $\mathbb{Z}$ (namely, by shifting) which do not fix $\mathbb{N}$, so $\mathbb{N}$ cannot be $\varnothing$-definable.

Example 1.28. Work with $\mathcal{M} \coloneqq (\{1A, 1B, 2A, 2B\}, \leq)$ with partial ordering given by the number. The set $X \coloneqq \{1A, 1B\}$ is $\varnothing$-definable by $\varphi(x)$ given by $\exists y((x \neq y) \land (x \leq y))$. However, there is an automorphism of our model swapping $1A$ with $1B$ and $2A$ with $2B$, but this automorphism does not fix X pointwise.

1.2.3 The Compactness Theorem

To state compactness, we want a few definitions.

Definition 1.29 (satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *satisfiable* if and only if it has a model $\mathcal{M}$.

With a notion of proof, one can show that being satisfiable means that there is no proof of $\bot$, but we will avoid a discussion of proofs in this course.

Definition 1.30 (finitely satisfiable)**.** Fix a language $\mathcal{L}$ and theory T. Then T is *finitely satisfiable* if and only if any finite subset of T is satisfiable.

Of course, being satisfiable implies being finitely satisfiable; the converse will be true but is far from obvious. The following example explains why this is strange.

Example 1.31. Consider the natural numbers $\mathcal{N} = (\mathbb{N}, 0, 1, +, \times, \leq)$ and $\mathcal{N}_c := (\mathbb{N}, 0, 1, +, \times, \leq, c)$, where c is some constant symbol, and set

$$T := \operatorname{Th}(\mathcal{N}) \cup \left\{ c \geq \underbrace{1 + 1 + \cdots + 1}_{n} : n \in \mathbb{N} \right\}.$$

Then T is finitely satisfiable by $\mathcal{N}$ because, for any finite subset of T, the sentences $c \geq 1+1+\cdots+1$ will have to be bounded in length in our finite subset, so we simply find some c large enough in $\mathcal{N}$. However, $\mathcal{N}$ does not model T!

Anyway, here is our statement.

Theorem 1.32 (compactness)**.** Fix a language $\mathcal{L}$ and theory T. If T is finitely satisfiable, then T is satisfiable. Furthermore, T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}| + \aleph_0$.

Remark 1.33. In particular, the theory T of Example 1.31 has a model $\mathcal{N}'$, which is going to look very strange. To begin, there is a segment

$$0 < 1 < 2 < \cdots.$$

But there is now an element c larger than any natural, which produces $c+1, c+2, c+3, \ldots$. But also any nonzero element has a predecessor, so we have elements $c-1, c-2, c-3, \ldots$. Further, any natural number is either odd or even, so there is also either $(c-1)/2$ or $c/2$ sitting between the initial piece of $\mathbb{N}$ and the c piece with $\mathbb{Z}$ added everywhere. In fact, a similar argument holds to produce an element approximately equal to qc for any rational $q \in \mathbb{Q}$.

Remark 1.34. One can of course always make our model larger. For example, suppose we have a theory T with an infinite model. If we want a model with cardinality at least $\mathbb{R}$, we add constants $\{c_r : r \in \mathbb{R}\}$ to our language and add in the sentences

$$\{c_r \neq c_s : \text{distinct } r, s \in \mathbb{R}\}.$$

This remains finitely satisfiable: these constants merely ask for our model to be larger than any finite set. One can even require the new model to be elementarily equivalent to the previous one.

Here are some applications of compactness.

Corollary 1.35. The class of torsion groups is not elementarily definable in the language $\mathcal{L} = \{e, *\}$ of groups.

Notably, it is not okay to write something like

$$\bigvee_{n \in \mathbb{N}} (\forall g\, g^n = e)$$

to encode any torsion because this statement is infinitely long.

Proof. Suppose the class is elementarily definable. Then we have a theory T such that $\operatorname{Mod}(T)$ consists exactly of all torsion groups. Now the trick is to build a model of T which is not actually a torsion group. For this, we expand our language to $\mathcal{L} = \{e, *, c\}$, and let

$$S := T \cup \left\{ \underbrace{c * c * \cdots * c}_{n} \neq e : n \geq 2 \right\}.$$

For any finite subset of S, we can satisfy S by a torsion group containing an element which is not n-torsion for sufficiently large n; for example, $\mathbb{Z}/n\mathbb{Z}$ will do.

Thus, by Theorem 1.32, there is a model $\mathcal{G}$ of S, so in particular, $\mathcal{G}$ has an element $g \in G$ with

$$\underbrace{g * g * \cdots * g}_{n} \neq e$$

for all $n \geq 2$ (namely, g is the interpretation of the constant symbol c), so it follows that G is not torsion. However, $\mathcal{G}$ is also a model of T and thus is supposed to be torsion, so we have a contradiction! This completes the proof. ■

1.3 August 31

Professor Scanlon is back. Let's prove the Compactness theorem. We are going to prove 2.5 times.

1.3.1 Proof of Compactness

Recall the statement.

Theorem 1.32 (compactness)**.** Fix a language $\mathcal{L}$ and theory T. If T is finitely satisfiable, then T is satisfiable. Furthermore, T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}| + \aleph_0$.

Remark 1.36. This result is special to first-order logic: in some sense, Theorem 1.32 combined with a corollary characterizes first-order logic among various logics. Roughly speaking, one wants to formalize what a logic is with its various structures and sentences should do.

Proof with completeness. We can prove this result fairly quickly given the Completeness theorem. Recall that the Completeness theorem says that any theory fails to be satisfiable if and only if there is a proof of contradiction $\perp$; one writes that a theory T proves a sentence φ by $T \vdash \varphi$. We have not discussed how formal proofs work, and we won't discuss it further because this is not a proofs class. Approximately speaking, a formal proof is a list of steps one can use the sentence sin T to produce φ syntactically.

Now, suppose that T fails to be satisfiable. Then there is a proof of $\perp$. But then one can look at the proof, which is necessarily finite in length, and then we pick up any sentence φ occurring in the proof of $\perp$. But then we have a proof of $\perp$ using only finitely many sentences in T, so T fails to be finitely satisfiable! This completes the proof. ■

Anyway, let's present an actual proof.

Definition 1.37 (witness)**.** Fix a theory T of a language $\mathcal{L}$. Then T has *witnesses* (or *Henkin constants*) if and only if each formula $\varphi(x)$ in one free variable x has a constant symbol c such that $\exists x \varphi(x) \to \varphi(c)$ lives in T.

Remark 1.38. If T has witnesses, then $T' \supseteq T$ also has witnesses for any theory T' extending T.

Let's quickly sketch our proof.

1. We will show that if T is finitely satisfiable, then there is an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ and expanded finitely satisfiable $\mathcal{L}'$-theory $T' \supseteq T$ of $\mathcal{L}'$ such that $|\mathcal{L}'| \le |\mathcal{L}| + \aleph_0$, and T' has witnesses (as does any extended theory T'' of T').

2. Next, suppose T is a maximally finitely satisfiable theory (i.e., T is finitely satisfiable, and any proper extension $T' \supseteq T$ fails to be finitely satisfiable[1]). Then we will show T is complete (i.e., each sentence φ has either $\varphi \in T$ or $\neg\varphi \in T$).

3. From here, we want to extend maintaining being complete: if T is finitely satisfiable, then there is an extended language $\mathcal{L}' \supseteq \mathcal{L}$ of size $|\mathcal{L}'| = |\mathcal{L}| + \aleph_0$ and extended theory T' of T which is complete, finitely satisfiable, and has witnesses. This essentially follows from a Zorn's lemma argument.

4. We are now ready to do our construction. At this point, we may assume that our theory T is finitely satisfiable, complete, and has witnesses. Then we claim that there is a model $\mathcal{M}$ such that $|M| \le |\mathcal{L}|$.

   In fact, the model can be described somewhat explicitly. Take $M := \mathcal{C}/\sim$ where $\mathcal{C}$ is our set of constants, and our equivalence relation $\sim$ is given by $c \sim d$ if and only if $(c = d) \in T$. Notably, constants $c \in \mathcal{C}$ are interpreted as $c^{\mathcal{M}} := [c]$. To interpret functions f, we have $f^{\mathcal{M}}([a_1], \ldots [a_n]) = [d]$ if and only if $(f(a_1, \ldots, a_n) = d) \in T$. Lastly, to interpret relations R, we have $R^{\mathcal{M}}([a_1], \ldots, [a_n])$ if and only if $(R(a_1, \ldots, a_n)) \in T$.

Let's start implementing the details.

Remark 1.39. In logic, the answer to a question is often the question. For example, in step 4, we see that T has a model because T says that it has a model.

Here is our first step.

Lemma 1.40. Fix a finitely satisfiable theory T of a language $\mathcal{L}$. Then there is an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ and expanded finitely satisfiable $\mathcal{L}'$-theory $T' \supseteq T$ of $\mathcal{L}'$ such that $|\mathcal{L}'| \le |\mathcal{L}| + \aleph_0$, and T' has witnesses.

Proof. We would like to just set T' to be T together with new constants providing witnesses for all formulae. But these new constants will make new formulae, so we need to do some kind of induction to go upwards.

With this in mind, we will build an increasing sequence of languages

$$\mathcal{L}_0 := \mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots$$

and theories

$$T_0 := T \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

such that T_n is always a finitely satisfiable $\mathcal{L}_n$-theory, and each $\mathcal{L}_n$-formula φ with one free variable has a constant $c \in \mathcal{C}_{\mathcal{L}_n}$ such that $\exists x\varphi(x) \to \varphi(c)$ lives in T_n. We will then set $\mathcal{L}'$ to be the union of the $\mathcal{L}_\bullet$ and T' to be the union of the $T_\bullet$, and this will complete the proof.

We have already built the $n = 0$ stage, as above. Then to build $\mathcal{L}_{n+1}$ from $\mathcal{L}_n$, add in new constant symbols $c_{\varphi(x)}$ for each $\mathcal{L}_n$-formula $\varphi(x)$ with one free variable; all the functions and relations remain the same. Note $\mathcal{L}_{n+1}$ is now the size of the formulae with one free variable in $\mathcal{L}_n$, so $|\mathcal{L}_{n+1}| = |\mathcal{L}_n| + \aleph_0$.

As for our theory, let T_{n+1} be T_n plus the sentences $\exists x\varphi(x) \to \varphi\left(c_{\varphi(x)}\right)$ for each $\mathcal{L}_n$-formula $\varphi(x)$ with one free variable. We are now ready to set

$$\mathcal{L}' := \bigcup_{n \in \mathbb{N}} \mathcal{L}_n \qquad \text{and} \qquad T' := \bigcup_{n \in \mathbb{N}} T_n.$$

We see that $\mathcal{L}'$ then has the right size, and T' has witnesses: for any $\mathcal{L}'$-formula $\varphi(x)$ with one free variable, note that $\varphi(x)$ has only finitely many symbols, so we can find some fixed level $\mathcal{L}_n$ containing all the symbols used in $\varphi(x)$. But then $\varphi(x)$ has a witness from $T_{n+1} \subseteq T'$, as needed.

[1] Such a thing exists by some sort of Zorn's lemma argument: note that there is a theory containing T which fails to be finitely satisfiable: take the set of all sentences!

It remains to show that T' is finitely satisfiable. It suffices to show that T_n is finitely satisfiable for any $n \in \mathbb{N}$ because any finite set will be contained in some T_n. We show this by induction. For $n = 0$, there is nothing to say. Now suppose T_n is finitely satisfiable, and we show that T_{n+1} is finitely satisfiable.

Fix some finite subset $\Delta \subseteq T_{n+1}$ which we would like to build a model for. Now, Δ will be built by some sentences in T_n plus some new sentences from T_{n+1}. Looking hard at $T_{n+1} \setminus T_n$, we see that we can enumerate $\Delta \setminus T_n$ as some sentences

$$\exists x \psi_k(x) \to \psi_k(c_k)$$

where $\{\psi_k\}_{k=1}^m$ is some $\mathcal{L}_n$-formulae in one free variable.

We now begin building our model. Note $\Delta \cap T_n$ is a finite subset of T_n, so it is satisfiable by some model $\mathcal{M}$. Now, for each k, if there is some $a \in M$ such that $\mathcal{M} \vDash \varphi_k(a)$, set $a := a_k$; otherwise, set $a_k := m$ for any chosen $m \in M$. (Note structures are nonempty.) We now let $\mathcal{M}'$ be the $\mathcal{L}_{n+1}$-structure with universe M, interpretations of functions and relations the same as in $\mathcal{M}$, and all old constant symbols are also all still interpreted the same way. Then for each new constant symbol, we interpret $c_k^{\mathcal{M}} := a_k$, and each other new constant symbol can also go to m. Now $\mathcal{M}'$ is a model for Δ because it models everything in $\Delta \cap T_n$ for free, and it has satisfied $\Delta \setminus T_{n+1}$ by construction, so we are done. ■

To show the second step, we begin with the following lemma.

Lemma 1.41. Fix a finitely satisfiable theory T of a language $\mathcal{L}$. For any $\mathcal{L}$-sentence φ, then either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is finitely satisfiable.

Proof. Suppose that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ both fail to be finitely satisfiable. We will show that T fails to be finitely satisfiable.

Well, we are given finite subsets $\Delta_+ \subseteq T \cup \{\varphi\}$ and $\Delta_- \subseteq T \cup \{\neg\varphi\}$ which are not satisfiable. If Δ_+ fails to contain φ, then Δ_+ is a finite subset of T which is not satisfiable, so T fails to be satisfiable. Thus, we may assume that $\varphi \in \Delta_+$. Analogously, we may assume that $\neg\varphi \in \Delta_-$. Now, $(\Delta_+ \cup \Delta_-) \setminus \{\varphi\}$ and $(\Delta_+ \cup \Delta_-) \setminus \{\neg\varphi\}$ both fail to be satisfiable.

But now suppose for the sake of contradiction that T is finitely satisfiable. Then $(\Delta_+ \cup \Delta_-) \setminus \{\varphi, \neg\varphi\}$ has a model $\mathcal{M}$. But $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \neg\varphi$, so we see that $\mathcal{M}$ will model at least one of $(\Delta_+ \cup \Delta_-) \setminus \{\varphi\}$ or $(\Delta_+ \cup \Delta_-) \setminus \{\neg\varphi\}$, which is the desired contradiction. ■

The second step now follows from a Zorn's lemma argument.

Lemma 1.42. Fix a maximally finitely satisfiable theory T of a language $\mathcal{L}$. Then T is complete.

Proof. Let φ be any $\mathcal{L}$-sentence. Then either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is finitely satisfiable by Lemma 1.41, so by maximality, we may conclude that either $T = T \cup \{\varphi\}$ or $T = T \cup \{\neg\varphi\}$, so either $\varphi \in T$ or $\neg\varphi \in T$, which is what we wanted. ■

Combining the work so far completes the third step.

Lemma 1.43. Fix a finitely satisfiable theory T of a language $\mathcal{L}$. Then there is an extended language $\mathcal{L}' \supseteq \mathcal{L}$ of size $|\mathcal{L}'| \leq |\mathcal{L}| + \aleph_0$ and extended theory T' of T which is complete, finitely satisfiable, and has witnesses.

Proof. We can prove this using the previous two steps.

1. Lemma 1.40 provides an extended language $\mathcal{L}'$ (of cardinality at most $|\mathcal{L}| + \aleph_0$) and extended theory T' which is finitely satisfiable and has witnesses.

2. We use Zorn's lemma to become maximally finitely satisfiable. Let $\mathcal{P}$ denote the set of finitely satisfiable $\mathcal{L}'$-theories T'' extending T' which is finitely satisfiable. Containment shows that $\mathcal{P}$ is a partial order, and it's nonempty because $T' \in \mathcal{P}$. Next up, we note that any nonempty chain $\{T_\alpha\}_{\alpha\in\lambda}$ is upper-bounded by
$$\bigcup_{\alpha\in\lambda} T_\alpha,$$
which we can see continues to be finitely satisfiable (any finite set belongs to some fix T_β for β perhaps large) and thus lives in $\mathcal{P}$ and succeeds to upper-bound our chain. Thus, Zorn's lemma provides a maximally finitely satisfiable theory T'' containing T', which will be complete by Lemma 1.42. Because T'' contains T', we continue to have witnesses. ■

1.4 September 5

In this lecture, we will complete our proof of Theorem 1.32.

1.4.1 Completing the proof of Theorem 1.32

Last class, we left off having shown Lemma 1.43, which was the third step of our outline. The last step of the proof is the following lemma.

Lemma 1.44. Fix a language $\mathcal{L}$ with a theory T which is finitely satisfiable, complete, and has witnesses. Then T has a model $\mathcal{M}$ with cardinality at most $|\mathcal{L}|$.

Proof. As we did last class, we go ahead and explicitly describe our model and then show that it makes sense. Take $M := \mathcal{C}/{\sim}$ where $\mathcal{C}$ is our set of constants, and our equivalence relation $\sim$ is given by $c \sim d$ if and only if $(c = d) \in T$. Notably, constants $c \in \mathcal{C}$ are interpreted as $c^{\mathcal{M}} := [c]$. To interpret functions f, we have $f^{\mathcal{M}}([a_1], \ldots [a_n]) = [d]$ if and only if $(f(a_1, \ldots, a_n) = d) \in T$. Lastly, to interpret relations R, we have $R^{\mathcal{M}}([a_1], \ldots, [a_n])$ if and only if $(R(a_1, \ldots, a_n)) \in T$.

We now check that this makes sense. Note that in the following checks, we are a bit sloppy in differentiating between constants and their equivalence classes in C when there is unlikely to be problems from doing so.

1. We show that $\sim$ is in fact an equivalence relation on $\mathcal{C}$. There are the following checks.
   - Reflexive: we must show $c = c$ is a sentence in T. Because T is complete, one of $c = c$ or $\neg(c = c)$ is in T. But T is finitely satisfiable, and the sentence $\neg(c = c)$ has no model, so it cannot live in T. So instead $c = c$ lives in T.
   - Symmetric: suppose $c \sim c'$ so that $c = c'$ is a sentence in T; we want to show that $c' = c$ is a sentence in T. Well, by completeness one of $c' = c$ or $\neg(c' = c)$ lives in T. But if we have $\neg(c' = c)$, then the finite theory $\{\neg(c' = c), c = c'\}$ will have no model (symmetry of equality will hold in the model), violating that T is finitely satisfiable. So we must have $c' = c$ instead.
   - Transitive: suppose $c \sim c'$ and $c' \sim c''$ so that $c = c'$ and $c' = c''$ are sentences in T. We want to show that $c \sim c''$, or equivalently that $c = c''$ lives in T. Well, by completeness, one of $c = c''$ or $\neg(c = c'')$ lives in T. However, if $\neg(c = c'')$ lives in T, then we note that $\{c = c', c' = c'', \neg(c = c'')\}$ is a subset of T with no model, which is a contradiction. So instead $c = c''$ lives in T.
2. We show that our interpretation of functions makes sense. Fix an n-ary function f. We need to show that $f(a_1, \ldots, a_n)$ has a unique interpretation in $\mathcal{M}$.
   - Existence: for constants $a_1, \ldots, a_n$, we show that there is a constant b such that $f(a_1, \ldots, a_n) = b$ in T. This holds by having witnesses: let $\varphi(x)$ be the formula $f(a_1, \ldots, a_n) = x$, and having witnesses tells us that T contains the sentence
$$\exists x\, \varphi(x) \to \varphi(b)$$

for some constant b. We show that T contains the sentence $\varphi(b)$. Otherwise, because T is complete, T will have the sentence $\neg\varphi(b)$, but being finitely satisfiable means that

$$\{\exists x\varphi(x) \to \varphi(b), \neg\varphi(b)\}$$

must have a model; this is an issue because all models satisfy $\exists x\, f(a_1, \ldots, a_n) = x$ and therefore must satisfy $\varphi(b)$, which is a contradiction to satisfying $\neg\varphi(b)$.

- Uniqueness: for constants $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ and b and b' such that $a_i \sim a'_i$ for all i and both $f(a_1, \ldots, a_n) = b$ and $f(a'_1, \ldots, a'_n) = b'$, we must show that actually $b \sim b'$.
  Well, by completeness, if $b \sim b'$ is not true, then $\neg(b = b')$ lives in T. Then the theory

$$\{a_1 = a'_1, \ldots, a_n = a'_n, f(a_1, \ldots, a_n) = b, f(a'_1, \ldots, a'_n) = b', \neg(b = b')\}$$

  is a subset of T but is not satisfiable (because of how functions work in set theory), which is a contradiction.

3. We show that our interpretation of relations makes sense. Fix an n-ary relation R. Essentially, if we have constants $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ such that $a_i \sim a'_i$ for each i, then we will have $R(a_1, \ldots, a_n) \in T$ if and only if $R(a'_1, \ldots, a'_n) \in T$. Because $\sim$ is symmetric as shown above, it suffices to show that $R(a_1, \ldots, a_n) \in T$ implies $R(a'_1, \ldots, a'_n) \in T$.

   Well, T is complete, so if T fails to contain $R(a'_1, \ldots, a'_n)$, then it must contain $\neg R(a'_1, \ldots, a'_n)$ instead. But then
$$\{a_1 = a'_1, \ldots, a_n = a'_n, R(a_1, \ldots, a_n), \neg R(a'_1, \ldots, a'_n)\}$$
   is a finite subset of T with no model because of how relations work in set theory; this is a contradiction.

4. As an intermediate step, before going on to show that $\mathcal{M} \vDash T$, we show that terms behave: suppose $t(x_1, \ldots, x_n)$ is a term. For constants $c_1, \ldots, c_n, c'$, we show that $t(c_1, \ldots, c_n) = d$ is in T if and only if $t^{\mathcal{M}}([c_1], \ldots, [c_n]) = [d]$.

   Let T' be the subset of T with this property. Note that T' contains constants by our first check above. To show that $T' = T$, we suppose that $t_1, \ldots, t_m \in T'$ and that f is an m-ary function, and we want to show that $f(t_1, \ldots, t_m)$ is in T'. Fix enough constants $c_1, \ldots, c_n$ (namely, more than the number of free variables of each $t_\bullet$). Then we note $t_i^{\mathcal{M}}([c_1], \ldots, [c_n]) = [d_i]$ for some $[d] \in \mathcal{M}$, which then implies that

$$t_i(c_1, \ldots, c_n) = d_i$$

   is a sentence in T for each t_i. Now, $f^{\mathcal{M}}\left(t_1^{\mathcal{M}}, \ldots, t_m^{\mathcal{M}}\right)(\overline{c})$ is certainly equal to some constant $[d]$, which is now equivalent to having
$$f(d_1, \ldots, d_m) = d$$
   in T by the functions check above. Now, the finite satisfiable and completeness of T imply that having the above sentence in T is equivalent to having the sentence

$$f(t_1, \ldots, t_m)(\overline{c}) = d$$

   in T because T already contains $t_i(\overline{c}) = d_i$ for each i. For example, if T fails to contain $f(t_1, \ldots, t_m)(\overline{c})$, then it will contain $\neg f(t_1, \ldots, t_m)(\overline{c}) = d$ by completeness, but this contradicts $f(d_1, \ldots, d_m) = d$ and $t_i(\overline{c}) = d_i$ for each i and therefore the finite subset with all these sentences is not satisfiable. The reverse implication is similar.

5. We show that $\mathcal{M}$ actually satisfies all sentences in T; in fact, we will show $T \vDash \varphi(\overline{a})$ for any φ and $\overline{a}$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$. We proceed by induction, starting with atomic formulae.

   - Our most basic cases are sentences of the form $c_1 = c_2$ and $R(c_1, \ldots, c_n)$ where R is some n-ary relation and $c_1, \ldots, c_n$ are constants. These are satisfied by $\mathcal{M}$ basically by construction: the definition of $\sim$ establishes from $c_1 = c_2$ that $c_1 \sim c_2$ and thus $c_1^{\mathcal{M}} = [c_1] = [c_2] = c_2^{\mathcal{M}}$. And $R^{\mathcal{M}}\left(c_1^{\mathcal{M}}, \ldots, c_n^{\mathcal{M}}\right)$ is equivalent to $R(c_1, \ldots, c_n) \in T$.

- For any terms t and s and enough constants $\overline{a}$ and $\overline{b}$, we claim that having $(t = s)(\overline{a}, \overline{b})$ in T implies $\mathcal{M} \vDash (t = s)(\overline{a}, \overline{b})$. The previous step promises constants c and d such that $t(\overline{a}) = c$ and $s(\overline{b}) = d$ are in T and that this is equivalent to $t^{\mathcal{M}}(\overline{a}) = [c]$ and $s^{\mathcal{M}}(\overline{b}) = [d]$.

  Now, $(t = s)(\overline{a}, \overline{b})$ being in T is thus equivalent to having $c = d$ in T by the usual argument using the completeness and finite satisfiability of T. Then having $c = d$ is equivalent to $[c] = [d]$, which is equivalent to $t^{\mathcal{M}}(\overline{a}) = s^{\mathcal{M}}(\overline{b})$, which is equivalent to $\mathcal{M} \vDash (t = s)(\overline{a}, \overline{b})$.
- For any n-ary relation R and terms $t_1, \ldots, t_n$ and enough constants $\overline{a}$, we claim $R(t_1, \ldots, t_n)(\overline{a})$ being in T implies $\mathcal{M} \vDash R(t_1, \ldots, t_n)(\overline{a})$. Well, for each term t_i, the previous step promises us a constant c_i such that $t_i(\overline{a}) = c_i$ is in T and has $t_i^{\mathcal{M}}(\overline{a}) = [c_i]$.

  Now, having the sentences $t_i(\overline{a}) = c_i$ for each i implies that $R(t_1, \ldots, t_n)(\overline{a})$ lives in T if and only if $R(c_1, \ldots, c_n)$ lives in T by the usual argument using the completeness and finite satisfiability of T. But by our relations check, we know that $R(c_1, \ldots, c_n)$ lives in T if and only if $R^{\mathcal{M}}([c_1], \ldots, [c_n])$ is true, which is equivalent to $R^{\mathcal{M}}\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_n^{\mathcal{M}}(\overline{a})\right)$.

We now build up from atomic formulae. Let F' be the subset of formulae such that $\varphi(\overline{a})$ being in T for some constants $\overline{a}$ if and only if $\mathcal{M} \vDash \varphi(\overline{a})$. The above checks show that F' contains atomic formulae.

- Suppose $\varphi \in F'$. We show $\neg\varphi \in F'$. Well, $\neg\varphi(\overline{a})$ fails to live in T if and only if $\varphi(\overline{a})$ lives in T (by completeness), which is equivalent to $\mathcal{M} \vDash \varphi(\overline{a})$, which is equivalent to $\mathcal{M}$ not satisfying $\neg\varphi(\overline{a})$.
- Suppose $\varphi, \psi \in F'$. We show that $\varphi \wedge \psi$. Well, $(\varphi \wedge \psi)(\overline{a})$ lives in T if and only if both $\varphi(\overline{a})$ and $\psi(\overline{a})$ live in T (using the usual argument with the completeness and finite satisfiability of T), which is equivalent to $\mathcal{M} \vDash \varphi(\overline{a})$ and $\mathcal{M} \vDash \psi(\overline{a})$, which is equivalent to $\mathcal{M} \vDash (\varphi \wedge \psi)(\overline{a})$.
- Suppose $\varphi(x) \in F'$. We show that $\exists x\, \varphi(x) \in F'$. Well, $\mathcal{M} \vDash (\exists x\, \varphi(x))(\overline{a})$ if and only if there is $[b] \in M$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$. By hypothesis, this is equivalent to having some constant b such that $\varphi(\overline{a}, b)$ is in T.

  Now, if $\varphi(\overline{a}, b)$ is in T for some constant b, then the usual argument with completeness and finite satisfiability requires $(\exists x\, \varphi(x))(\overline{a})$ to be in T. Conversely, if $(\exists x\, \varphi(x))(\overline{a})$ is in T, then the fact that T has witnesses implies that there is a constant c such that $\varphi(\overline{a}, b)$ is in T from the usual argument. In particular, the sentence $\exists x\, \varphi(\overline{a})(x) \to \varphi(\overline{a})(b)$ belongs to T for some constant b.

The above checks complete the induction on formulae. ■

Theorem 1.32 now follows from combining Lemmas 1.43 and 1.44.

1.5 September 7

In this lecture, we will provide another proof of Theorem 1.32, using ultrafilters.

1.5.1 Ultrafilters

Unsurprisingly, the main character of our story will be ultrafilters.

Definition 1.45 (filter)**.** Fix a set I. Then a *filter* $\mathcal{F}$ on I is a subset of $\mathcal{P}(I)$ satisfying the following.

(a) $I \in \mathcal{F}$.

(b) Finite intersection: for $X, Y \in \mathcal{F}$, we have $X \cap Y \in \mathcal{F}$.

(c) Containment: if $X \in \mathcal{F}$ and $Y \subseteq I$ contains X, then $Y \in \mathcal{F}$ also.

The intuition here is that filters contain "large" subsets of I.

Example 1.46. Fix a set I. Then $\{I\}$ is a filter.

Example 1.47. Fix a set I and a filter $\mathcal{F}$ on I. If $\varnothing \in \mathcal{F}$, then we see that any subset $X \subseteq I$ contains $\varnothing$ and thus must live in $\mathcal{F}$. Thus, $\mathcal{F} = \mathcal{P}(I)$, which is in fact a filter. We call $\mathcal{P}(I)$ the "trivial filter."

Example 1.48. More generally, fix any subset $X \subseteq I$. Then $\mathcal{F}_X \coloneqq \{Y \subseteq I : X \subseteq Y\}$ is a filter.

(a) Note $X \subseteq I$, so $I \in \mathcal{F}_X$.

(b) Intersection: if $Y, Z \in \mathcal{F}_X$, then $X \subseteq Y$ and $X \subseteq Z$, so $X \subseteq Y \cap Z$, so $Y \cap Z \in \mathcal{F}_X$.

(c) Containment: if $Y \in \mathcal{F}_a$, and $Z \subseteq I$ contains Y, then $X \subseteq Y \subseteq Z$, so $Z \in \mathcal{F}_a$.

Example 1.49. Fix a set I, and define $\mathcal{F} \subseteq \mathcal{P}(I)$ by $X \in \mathcal{F}$ if and only if $I \setminus X$ is finite. We check that $\mathcal{F}$ is a filter.

(a) Note $I \in \mathcal{F}$ because $I \setminus I = \varnothing$ is finite.

(b) Intersection: if $X, Y \in \mathcal{F}$, then $I \setminus (X \cap Y) = (I \setminus X) \cup (I \setminus Y)$ is finite and thus $X \cap Y \in \mathcal{F}$.

(c) Containment: if $X \in \mathcal{F}$ and $Y \subseteq I$ contains X, then $I \setminus Y \subseteq I \setminus X$ is finite, so $Y \in \mathcal{F}$.

Ultrafilters are the largest filters.

Definition 1.50 (ultrafilter)**.** Fix a set I. Then an *ultrafilter* $\mathcal{F}$ on I is a nontrivial filter on I such that each subset $X \subseteq I$ has one of $X \in \mathcal{F}$ or $I \setminus X \in \mathcal{F}$.

Example 1.51. Fix a set I and element $a \in I$. Define the "principal ultrafilter"

$$\mathcal{F}_a \coloneqq \{X \subseteq I : a \in X\}.$$

We show that $\mathcal{F}_a$ is an ultrafilter. Note $\mathcal{F}_a$ is already a filter by Example 1.48. To be ultrafilter, for each $X \subseteq I$, either $a \in X$ or $a \in I \setminus X$, which imply $X \in \mathcal{F}_a$ or $I \setminus X \in \mathcal{F}_a$ respectively.

The following result rigorizes the notion that ultrafilters are the largest filters.

Lemma 1.52. Fix a set I and a filter $\mathcal{U}$ on I. The following are equivalent.

(a) $\mathcal{U}$ is an ultrafilter.

(b) $\mathcal{U}$ is maximal among the partially ordered set of nontrivial filters on I, ordered by inclusion.

Proof. We have two implications to show.

- We show (a) implies (b). Suppose $\mathcal{U}'$ is a filter properly containing $\mathcal{U}$, and we want to show that $\mathcal{U}' = \mathcal{P}(I)$. Well, $\mathcal{U}'$ properly contains $\mathcal{U}$, so there is some $X \in \mathcal{U}' \setminus \mathcal{U}$. But $X \notin \mathcal{U}$ requires $I \setminus X \in \mathcal{U}$, so $I \setminus X \in \mathcal{U}'$ too, but then
$$\varnothing = X \cap (I \setminus X)$$
lives in $\mathcal{U}'$. It follows that $\mathcal{U}' = \mathcal{P}(I)$ by Example 1.47.

- We show (b) implies (a). Certainly $\mathcal{U}$ is nontrivial. Now, fix any subset $X \subseteq I$. Suppose $I \setminus X \notin \mathcal{U}$, and we want to show that $X \in \mathcal{U}$. Indeed, consider the filter
$$\mathcal{U}' \coloneqq \{Y \subseteq I : Y \supseteq X \cap X' \text{ for some } X' \in \mathcal{U}\}.$$
Quickly, we check that $\mathcal{U}'$ is a nontrivial filter containing $\mathcal{U}$.

  - Note $I \supseteq X \cap I$, so $I \in \mathcal{U}'$.
  - Intersection: if $Y_1, Y_2 \in \mathcal{U}'$, then find $X_1, X_2 \in \mathcal{U}$ such that $Y_i \supseteq X \cap X_i$ for each i, so $X_1 \cap X_2 \in \mathcal{U}$ implies $Y_1 \cap Y_2 \supseteq X \cap (X_1 \cap X_2)$ and so $Y_1 \cap Y_2 \in \mathcal{U}'$.
  - Containment: if $Y \in \mathcal{U}'$ and $Z \subseteq I$ contains Y, then find $X' \in \mathcal{U}$ such that $Y \supseteq X \cap X'$, so $Z \supseteq X \cap X'$, so $Z \in \mathcal{U}'$.
  - Contains $\mathcal{U}$: for each $X' \in \mathcal{U}$, note $X' \supseteq X \cap X'$, so $X' \in \mathcal{U}'$.
  - Nontrivial: having $\varnothing \in \mathcal{U}'$ would imply $\varnothing \supseteq X \cap X'$ for some $X' \in \mathcal{U}$, which is equivalent to $X' \subseteq I \setminus X$, so it would follow that $I \setminus X \in \mathcal{U}$, which is a contradiction.

  We conclude that $\mathcal{U} = \mathcal{U}'$ by maximality of $\mathcal{U}$. However, $X \supseteq I \cap X$ forces $X \in \mathcal{U}' = \mathcal{U}$, so we are done. ■

It is important to know that it is relatively easy to build ultrafilters.

Proposition 1.53. Fix a nontrivial filter $\mathcal{F}$ on a set I. Then there exists an ultrafilter $\mathcal{U}$ containing $\mathcal{F}$.

Proof. Let $\mathcal{P}$ be the set of nontrivial filters containing $\mathcal{F}$, which we turn into a partially ordered by set by inclusion; note $\mathcal{F} \in \mathcal{P}$, so $\mathcal{P}$ is nonempty. Using Lemma 1.52, we would like to show that $\mathcal{P}$ has a maximal element, for which we use Zorn's lemma. Fix a nonempty chain $\mathcal{C} \subseteq \mathcal{P}$, which we must upper-bound. We claim that

$$\mathcal{F}_u \coloneqq \bigcup_{\mathcal{F}' \in \mathcal{C}} \mathcal{F}'$$

is a filter containing $\mathcal{F}$ upper-bounding $\mathcal{C}$, which will complete the proof. Here are our checks.

- Upper-bounds: for any $\mathcal{F}' \in \mathcal{C}$, we see that $\mathcal{F}' \subseteq \mathcal{F}_u$ by construction.
- Any $\mathcal{F}' \in \mathcal{C}$ contains I, so $I \in \mathcal{F}_u$.
- Intersection: if $X, Y \in \mathcal{F}_u$, then we can find $\mathcal{F}'_X, \mathcal{F}'_Y \in \mathcal{C}$ containing X and Y, respectively. Because $\mathcal{C}$ is a chain, we may find $\mathcal{F}' \in \mathcal{C}$ containing both $\mathcal{F}'_X$ and $\mathcal{F}'_Y$. Then $X, Y \in \mathcal{F}'$, so $X \cap Y \in \mathcal{F}' \subseteq \mathcal{F}_u$ because $\mathcal{F}'$ is a filter.
- Containment: if $X \in \mathcal{F}_u$ and we have a subset $Y \subseteq I$ containing X, then we find $\mathcal{F}' \in \mathcal{C}$ containing X and find that $Y \in \mathcal{F}' \subseteq \mathcal{F}_u$ because $\mathcal{F}'$ is a filter. ■

1.5.2 Compactness via Ultraproducts

For our application, we will want the notion of an ultraproduct.

Lemma 1.54. Fix a language $\mathcal{L}$ and some $\mathcal{L}$-structures $\{\mathcal{M}_\alpha\}_{\alpha \in I}$. Now, define an $\mathcal{L}$-structure $\mathcal{M}$ as follows.

- The universe M is $\prod_{\alpha \in I} M_\alpha$ modded out by the equivalence relation $\sim$ given by $(a_\alpha) \sim (b_\alpha)$ if and only if
$$\{\alpha \in I : a_\alpha = b_\alpha\} \in \mathcal{U}.$$
- Functions are interpreted component-wise.
- For an n-ary relation R, $R^{\mathcal{M}}((a_{1\alpha}), \ldots, (a_{n\alpha}))$ if and only if the set of α such that $R^{M_\alpha}(a_{1\alpha}, \ldots, a_{n\alpha})$ is in $\mathcal{U}$.

Then $\mathcal{M}$ is a well-defined $\mathcal{L}$-structure.

Proof. Here are our various checks.

- We check that $\sim$ is an equivalence relation.
  - Reflexive: note $(a_\alpha) \sim (a_\alpha)$ because $\{\alpha \in I : a_\alpha = a_\alpha\} = I$ lives in $\mathcal{U}$.
  - Symmetric: if $(a_\alpha) \sim (b_\alpha)$, then
    $$\{\alpha \in I : b_\alpha = a_\alpha\} = \{\alpha \in I : a_\alpha = b_\alpha\},$$
    which is in $\mathcal{U}$ by hypothesis.
  - Transitive: if $(a_\alpha) \sim (b_\alpha)$ and $(b_\alpha) \sim (c_\alpha)$, then $\{\alpha \in I : a_\alpha = c_\alpha\}$ contains the set
    $$\{\alpha \in I : a_\alpha = b_\alpha = c_\alpha\} = \{\alpha \in I : a_\alpha = b_\alpha\} \cap \{\alpha \in I : a_\alpha = c_\alpha\},$$
    which lives in $\mathcal{U}$ because $\mathcal{U}$ is a filter.
- We check that interpretation of functions makes sense. Fix an n-ary function f and some elements $(a_{1\alpha}), \ldots, (a_{n\alpha})$ and $(b_{1\alpha}), \ldots, (b_{n\alpha})$. We must show
  $$\left(f^{\mathcal{M}}(a_{1\alpha}, \ldots, a_{n\alpha})\right) \sim \left(f^{\mathcal{M}}(b_{1\alpha}, \ldots, b_{n\alpha})\right).$$
  Well, we note $\left\{\alpha \in I : f^{\mathcal{M}}(a_{1\alpha}, \ldots, a_{n\alpha}) = f^{\mathcal{M}}(b_{1\alpha}, \ldots, b_{n\alpha})\right\}$ contains the set
  $$\bigcap_{i=1}^{n} \{\alpha \in I : a_{i\alpha} = b_{i\alpha}\},$$
  which lives in $\mathcal{U}$ because $\mathcal{U}$ is a filter.
- We check that interpretation of relations makes sense. Fix an n-ary function R and some elements $(a_{1\alpha}), \ldots, (a_{n\alpha})$ and $(b_{1\alpha}), \ldots, (b_{n\alpha})$. We must show
  $$R((a_{1\alpha}), \ldots, (a_{n\alpha})) \iff R((b_{1\alpha}), \ldots, (b_{n\alpha})).$$
  Unwrapping the definition of $R^{\mathcal{M}}$, this is equivalent to
  $$\left\{\alpha \in I : R^{M_\alpha}(a_{1\alpha}, \ldots, a_{n\alpha})\right\} \in \mathcal{U} \iff \left\{\alpha \in I : R^{M_\alpha}(b_{1\alpha}, \ldots, b_{n\alpha})\right\} \in \mathcal{U}.$$
  By symmetry, it's enough to show the forward direction, for which we note that the right-hand set contains
  $$\left\{\alpha \in I : R^{M_\alpha}(a_{1\alpha}, \ldots, a_{n\alpha})\right\} \cap \bigcap_{i=1}^{n} \{\alpha \in I : a_{i\alpha} = b_{i\alpha}\},$$
  which lives in $\mathcal{U}$ because $\mathcal{U}$ is a filter. ■

Definition 1.55 (ultraproduct)**.** Fix a language $\mathcal{L}$ and some $\mathcal{L}$-structures $\{\mathcal{M}_\alpha\}_{\alpha \in I}$. The *ultraproduct* is the $\mathcal{L}$-structure defined in Lemma 1.54, denoted $\prod_{\alpha \in I} M_\alpha / \mathcal{U}$ or $\prod_{\mathcal{U}} M_\alpha$.

We are now ready to begin our proof of Theorem 1.32. We want the following definition.

Definition 1.56 (expansion)**.** Fix a language $\mathcal{L}$ and structure $\mathcal{M}$. Given a subset $A \subseteq M$, we define the *expansion* $\mathcal{L}_A$ as having the same constants in addition to the constants in A but the same functions and relations.

Remark 1.57. Fix a language $\mathcal{L}$ and structure $\mathcal{M}$ and subset $A \subseteq M$. Then $\mathcal{M}$ is in fact an $\mathcal{L}_A$-structure, where we interpret the new constants $a \in A$ by $a^{\mathcal{M}} := a$.

Compactness will follow from the result.

Theorem 1.58 (Łoś). Fix a language $\mathcal{L}$ and $\mathcal{L}$-structures $\{\mathcal{M}_\alpha\}_{\alpha\in I}$. Expand $\mathcal{L}$ to the language $\mathcal{L}' := \mathcal{L}_{\prod_{\alpha\in I} M_\alpha}$. Now, let $\mathcal{U}$ be an ultrafilter on I so that $\mathcal{M} := \prod_{\mathcal{U}} M_\alpha$ is an $\mathcal{L}'$-structure. Then for any $\mathcal{L}$-formula $\varphi(x_1,\ldots,x_n)$ has $\mathcal{M}\vDash\varphi\left(a_1^{\mathcal{M}},\ldots,a_n^{\mathcal{M}}\right)$ if and only if

$$\{\alpha\in I:\mathcal{M}_\alpha\vDash\varphi(a_1,\ldots,a_n)\}\in\mathcal{U}.$$

Proof. To see that $\mathcal{M}$ is in fact an $\mathcal{L}'$-structure, note $\mathcal{M}$ is already an $\mathcal{L}$-structure, and we may interpret the constant (a_α) of $\mathcal{L}'$ by the corresponding equivalence class in $\mathcal{M}$. Anyway, the content of the proof is to induct on φ.

- Let c_1 and c_2 be constants. Then $\mathcal{M}\vDash(c_1=c_2)$ if and only if $c_1^{\mathcal{M}}=c_2^{\mathcal{M}}$ if and only if the set of α such that $c_1^{M_\alpha}=c_2^{M_\alpha}$ is in $\mathcal{U}$.
- Let $t(x_1,\ldots,x_n)$ be a term and c be a constant. We claim that $\mathcal{M}\vDash(t=c)(a_1,\ldots,a_n)$ if and only if

  $$\{\alpha\in I:\mathcal{M}_\alpha\vDash(t=c)(a_1,\ldots,a_n)\}\in\mathcal{U}.$$

  This is done by induction on the term t. If t is a constant there is nothing to say. Otherwise, suppose that f is an m-ary function, and we have terms $t_1(x_1,\ldots,x_n),\ldots,t_m(x_1,\ldots,x_n)$. Now, $\mathcal{M}\vDash(f(t_1,\ldots,t_m)=c)(a_1,\ldots,a_n)$ if and only if $f^{\mathcal{M}}\left(t_1^{\mathcal{M}}(\overline{a}),\ldots,t_m^{\mathcal{M}}(\overline{a})\right)=c^{\mathcal{M}}$, which after taking enough intersection is equivalent to having $f^{\mathcal{M}}\left(c_1^{\mathcal{M}},\ldots,c_m^{\mathcal{M}}\right)=c^{\mathcal{M}}$ for suitable constants $c_\bullet$ coming from the inductive hypothesis. One can then continue the argument backwards to complete.
- Let $t_1(x_1,\ldots,x_n)$ and $t_2(x_1,\ldots,x_n)$ be terms. Then $\mathcal{M}\vDash(t_1=t_2)\left(a_1^{\mathcal{M}},\ldots,a_n^{\mathcal{M}}\right)$ if and only if the set of α such that

  $$t_1^{\mathcal{M}_\alpha}\left((a_1^{M_\alpha}),\ldots,(a_2^{M_\alpha})\right)=t_2^{\mathcal{M}_\alpha}\left((a_1^{M_\alpha}),\ldots,(a_2^{M_\alpha})\right)$$

  is contained in $\mathcal{U}$. Choosing constants c_1 and c_2 suitably as above and using the filter property, this is equivalent to having $c_1^{\mathcal{M}}=c_2^{\mathcal{M}}$, from which we can go backwards to complete the argument.
- The same argument holds for atomic formulae of the form $R(t_1,\ldots,t_n)$ where R is an n-ary relation.

We now begin inducting on formulae. Let $\mathcal{F}'$ be the set of desired $\mathcal{L}'$-formulae. The above checks show that $\mathcal{F}'$ contains atomic formulae.

- Suppose $\varphi,\psi\in\mathcal{F}'$. Then $\mathcal{M}\vDash(\varphi\wedge\psi)(\overline{a})$ if and only if $\mathcal{M}\vDash\varphi(\overline{a})$ and $\mathcal{M}\vDash\psi(\overline{a})$ if and only if

  $$\{\alpha\in I:\mathcal{M}_\alpha\vDash\varphi(\overline{a})\}\cap\{\alpha\in I:\mathcal{M}_\alpha\vDash\psi(\overline{a})\}$$

  lives in $\mathcal{U}$, which is equivalent to

  $$\{\alpha\in I:\mathcal{M}_\alpha\vDash(\varphi\wedge\psi)(\overline{a})\}$$

  by the intersection property of $\mathcal{U}$.
- Suppose $\varphi\in\mathcal{F}'$. Then $\mathcal{M}\vDash(\neg\varphi)(\overline{a})$ is false if and only if $\mathcal{M}\vDash\varphi(\overline{a})$ if and only if

  $$\{\alpha\in I:\mathcal{M}_\alpha\vDash\varphi(\overline{a})\}\in\mathcal{U},$$

  which because $\mathcal{U}$ is an ultrafilter is equivalent to

  $$I\setminus\{\alpha\in I:\mathcal{M}_\alpha\vDash\varphi(\overline{a})\}\notin\mathcal{U},$$

  from which we can work backwards to complete the argument. (To see the last equivalence, note that each $X\subseteq I$ has exactly one of $X\in\mathcal{U}$ or $I\setminus X\in\mathcal{U}$: at least one is true because $\mathcal{U}$ is an ultrafilter, and at most one is true because both being true requires $\varnothing\in\mathcal{U}$, making $\mathcal{U}$ the trivial filter.)
- Suppose $\varphi(x,\overline{a})\in\mathcal{F}'$. Then $\mathcal{M}\vDash(\exists x\,\varphi(x))(\overline{a})$ if and only if there is some $b\in M$ (i.e., b a constant because we expanded our language) such that $\mathcal{M}\vDash\varphi(b,\overline{a})$, which is equivalent to

  $$\{\alpha\in I:\mathcal{M}_a\vDash\varphi(b,\overline{a})\}\in\mathcal{U}$$

  for some constant b. ■

Corollary 1.59. Let T be a finitely satisfiable $\mathcal{L}$-theory. Then T is satisfiable.

Proof. We follow [Mar02, Exercise 2.5.20]. We may suppose that T is nonempty. Let I be the set of finite subsets of T, and for each $\Delta \in I$, let $\mathcal{M}_\Delta$ be a model for Δ. We have two steps.

1. We define a filter. For each $\varphi \in T$, let $X_\varphi := \{\Delta \in I : \mathcal{M}_\Delta \vDash \varphi\}$. Then we define
$$D := \{A \in I : A \supseteq X_\varphi \text{ for some } \varphi \in T\}.$$
We show that D is a nontrivial filter on I.
    - Note that $\varnothing \notin D$ because this would require that $\varnothing \supseteq X_\varphi$ for some $\varphi \in T$, which is bad because $\mathcal{M}_{\{\varphi\}} \vDash \varphi$ shows X_φ is nonempty.
    - Note any $\varphi \in T$ has $X_\varphi \subseteq I$, so $I \in D$.
    - Intersection: if $A, B \in D$, then find $\varphi, \psi \in T$ such that $X_\varphi \subseteq A$ and $X_\psi \subseteq B$. Then $A \cap B$ contains $X_\varphi \cap X_\psi$, but $X_\varphi \cap X_\psi$ consists of Δ such that $\mathcal{M}_\Delta$ models both φ and ψ, which is equivalent to $\mathcal{M} \vDash \varphi \wedge \psi$, so $X_\varphi \cap X_\psi = X_{\varphi\wedge\psi}$.
    - Containment: if $A \in D$ is contained in $B \subseteq I$, then find $\varphi \in T$ with $A \supseteq X_\varphi$ so that $B \supseteq X_\varphi$ as well.
2. Let $\mathcal{U}$ be an ultrafilter containing D, and let $\mathcal{M}$ be $\prod_\mathcal{U} \mathcal{M}_\Delta$. Then for each $\varphi \in T$, we see by Theorem 1.58 that $\mathcal{M} \vDash \varphi$ if and only if
$$\{\Delta \in I : \mathcal{M}_\Delta \vDash \varphi\} \in \mathcal{U},$$
which is true by construction of $\mathcal{U}$. ■

Remark 1.60. Theorem 1.32 was able to bound the size of the model, but the above proof does not. Indeed, the models $\mathcal{M}_\Delta$ are potentially large, and $\mathcal{M}$ is approximately the size of all of them multiplied together.

1.6 September 12

We started class by showing that Theorem 1.58 implies the compactness theorem. Professor Scanlon's proof is distinct from the one in my notes, but I have not bothered to record his proof.

1.6.1 Elementary Equivalence

The following notion will be helpful.

Definition 1.61 (theory)**.** Fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$. Then the *theory* $\operatorname{Th}_\mathcal{L}(\mathcal{M})$ is the set of sentences φ such that $\mathcal{M} \vDash \varphi$.

The following notions are also sometimes helpful.

Definition 1.62 (diagram)**.** Fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$. The *diagram* $\operatorname{Diag}(\mathcal{M})$ is the set φ of atomic $\mathcal{L}_M$-sentences (in the expanded language $\mathcal{L}_M$) or negations of atomic sentences such that $\mathcal{M} \vDash \varphi$. The *elementary diagram* is the theory $\operatorname{Th}_{\mathcal{L}_M}(\mathcal{M}_M)$.

The theory is in some sense everything that a structure can see. As such, we make the following definition.

Definition 1.63 (elementarily equivalent)**.** Fix a language $\mathcal{L}$. Then two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, written $\mathcal{M} \equiv \mathcal{N}$ if and only if $\operatorname{Th}_{\mathcal{L}}(\mathcal{M}) = \operatorname{Th}_{\mathcal{L}}(\mathcal{N})$.

Remark 1.64. In fact, it is enough to merely have $\operatorname{Th}_{\mathcal{L}}(\mathcal{M}) \supseteq \operatorname{Th}_{\mathcal{L}}(\mathcal{N})$. Indeed, suppose for the sake of contradiction that $\operatorname{Th}_{\mathcal{L}}(\mathcal{M}) \supsetneq \operatorname{Th}_{\mathcal{L}}(\mathcal{N})$. Then there is a sentence φ with $\mathcal{M} \vDash \varphi$ but $\mathcal{N}$ does not satisfy φ. But then $\mathcal{N} \vDash \neg\varphi$, so $\mathcal{M} \vDash \neg\varphi$ too! But this does not make sense because $\mathcal{M}$ cannot satisfy both φ and $\neg\varphi$.

Proposition 1.65. Fix a language $\mathcal{L}$ and isomorphic $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$. Then $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent.

Proof. We show this by induction. Fix an isomorphism $f\colon \mathcal{M} \to \mathcal{N}$. We will actually show that $\mathcal{M}_M \equiv \mathcal{N}_M$, where $\mathcal{N}_M$ means $\mathcal{M}$ viewed as an $\mathcal{L}_M$-structure where the constants $a \in M$ are interpreted as $a^{\mathcal{N}} := f(a)$.

Anyway, we induct on φ.

- Suppose that φ is atomic of the form $t_1(\overline{a}) = t_2(\overline{a})$. If $\mathcal{M}_M \vDash (t_1(\overline{a}) = t_2(\overline{a}))$, then an induction on terms t shows that
$$t^{\mathcal{N}}(\overline{a}) = f\left(t^{\mathcal{M}}(\overline{a})\right).$$
Indeed, if t is a constant term, then this follows directly from f being an isomorphism. Otherwise, t takes the form $g(t_1, \ldots, t_n)$ for a function symbol g, and the interpretation of g is also respected by f because it is an isomorphism.

  Now, $\mathcal{M}_M \vDash (t_1(\overline{a}) = t_2(\overline{a}))$ if and only if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$, which is equivalent to $\mathcal{N}_M \vDash (t_1(\overline{a}) = t_2(\overline{a}))$ by passing through f as above.

- Suppose that φ is atomic of the form $R(t_1(\overline{a}), \ldots, t_n(\overline{a}))$. Well, $\mathcal{M}_M \vDash R(t_1(\overline{a}), \ldots, t_n(\overline{a}))$ if and only if $\left(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_n^{\mathcal{M}}(\overline{a})\right) \in R^{\mathcal{M}}$, and then passing everything through f shows that this is equivalent to
$$\left(t_1^{\mathcal{N}}(\overline{a}), \ldots, t_n^{\mathcal{N}}(\overline{a})\right) \in R^{\mathcal{N}},$$
which is $\mathcal{N}_M \vDash R(t_1(\overline{a}), \ldots, t_n(\overline{a}))$.

- Suppose that φ takes the form $\neg\psi$. Then the usual semantic argument takes care of us.

- Suppose that φ takes the form $\psi \land \theta$. Then the usual semantic argument takes care of us.

- Suppose that φ takes the form $\exists x\, \psi(x)$. Then $\mathcal{M}_M$ models this if and only if there is some $a \in M$ such that $\mathcal{M}_M \vDash \psi(a)$, but $\psi(a)$ is a perfectly valid sentence in our language because we expanded our constants, so this is equivalent to $\mathcal{N}_M \vDash \psi(a)$ for some $a \in M$. This last assertion is equivalent to $\mathcal{N}_M \vDash \exists x\, \psi(x)$ (the forward direction is clear, and the backward direction is because any $b \in \mathcal{N}$ witnessing takes the form $f(a)$ for some $a \in \mathcal{M}$ because f is a bijection on the universe).

The above induction completes the argument. ■

Proposition 1.65 is a nice result. We might hope for a converse, but it is false in general. There is a converse for finite structures.

Proposition 1.66. Fix a finite language $\mathcal{L}$ and a finite structure $\mathcal{M}$. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M} \cong \mathcal{N}$.

Proof. Say that $\mathcal{M}$ has n elements. Then we build a sentence which asserts that there are exactly n elements $x_1, \ldots, x_n$, and then add on conditions for each m-ary function symbol f what $f(x_{i_1}, \ldots, x_{i_m})$ should equal, for each m-ary function symbol R whether $R(x_{i_1}, \ldots, x_{i_m})$ should be, and so on.

Let's write this out. The start of this sentence

$$\exists x_1 \cdots \exists x_n \left(\left(\left(\bigwedge_{i \neq j} \neg(x_i \neq x_j) \right) \wedge \left(\forall y \bigvee_{i=1}^{n} (y = x_i) \right) \wedge \cdots \right)$$

dictates that any model satisfying this sentence has exactly n elements. (Namely, the first part asserts that the model has at least n elements, and the second bit says that any element equals one of the given n elements.) Next we write in function symbols. Enumerate $\mathcal{M}$ as $a_1, \ldots, a_n$. For each m-ary function symbol f in the language $\mathcal{L}$, and m elements $a_{i_1}, \ldots, a_{i_m}$ of M, we note that $f^{\mathcal{M}}(a_{i_1}, \ldots, a_{i_m})$ is some element of M, which by abuse of notation we will write as $a_{\overline{f}(i_1,\ldots,i_m)}$. As such, we next tack on the sentence

$$\bigwedge_{m\text{-ary } f} \bigwedge_{1 \le i_1, \ldots, i_m \le n} \left(f(x_{i_1}, \ldots, x_{i_m}) = x_{\overline{f}(i_1,\ldots,i_m)} \right).$$

Next up, we interpret constant symbols: by abuse of notation, let $c^{\mathcal{M}}$ be $a_{\overline{c}}$, so we add on the sentence

$$\bigwedge_{c \text{ constant}} (c = x_{\overline{c}}).$$

Lastly, we interpret relations: we need the sentence

$$\bigwedge_{m\text{-ary } R} \bigwedge_{\substack{1 \le i_1, \ldots, i_m \le n \\ R(a_{i_1}, \ldots, a_{i_m})}} R(x_{i_1}, \ldots, x_{i_m}).$$

In total, our sentence looks like

$$\begin{aligned} \exists x_1 \cdots \exists x_n \Bigg(\left(\left(\bigwedge_{i \neq j} \neg(x_i \neq x_j) \right) \wedge \left(\forall y \bigvee_{i=1}^{n} (y = x_i) \right) \right. \\ \wedge \bigwedge_{m\text{-ary } f} \bigwedge_{1 \le i_1, \ldots, i_m \le n} \left(f(x_{i_1}, \ldots, x_{i_m}) = x_{\overline{f}(i_1,\ldots,i_m)} \right) \\ \wedge \bigwedge_{c \text{ constant}} (c = x_{\overline{c}}) \\ \wedge \bigwedge_{m\text{-ary } R} \bigwedge_{\substack{1 \le i_1, \ldots, i_m \le n \\ R(a_{i_1}, \ldots, a_{i_m})}} R(x_{i_1}, \ldots, x_{i_m}) \\ \wedge \bigwedge_{m\text{-ary } R} \bigwedge_{\substack{1 \le i_1, \ldots, i_m \le n \\ \neg R(a_{i_1}, \ldots, a_{i_m})}} \neg R(x_{i_1}, \ldots, x_{i_m}) \Bigg). \end{aligned}$$

Let's quickly explain why this works. Notably, $\mathcal{M}$ satisfies the above sentence by taking x_i to be a_i. On the other hand, for any $\mathcal{N}$ which is an $\mathcal{L}$-structure satisfying the above sentence, the first line dictates that $\mathcal{N}$ must have exactly n elements $b_1, \ldots, b_n$. The second line dictates what $f^{\mathcal{N}}(b_{i_1}, \ldots, b_{i_m})$ must equal for each m-ary function symbol f. The third line dictates what $c^{\mathcal{N}}$ for each constant symbol c. Lastly, the last two lines dictate what $R^{\mathcal{N}}(b_{i_1}, \ldots, b_{i_m})$ for each m-ary relation symbol R. Thus, we see that we have an isomorphism $\rho\colon \mathcal{M} \to \mathcal{N}$ by $a_i \mapsto b_i$.

Writing this out a bit, let's check that ρ preserves function symbols. The other checks are no harder. By construction, we see that

$$\begin{aligned} \rho\left(f^{\mathcal{M}}(a_{i_1}, \ldots, a_{i_m})\right) &= \rho\left(a_{\overline{f}(i_1,\ldots,i_m)}\right) \\ &= b_{\overline{f}(i_1,\ldots,i_m)} \\ &= f^{\mathcal{N}}(b_{i_1}, \ldots, b_{i_m}), \end{aligned}$$

which is what we wanted. Notably, the last equality holds because it was required by our sentence. ■

Remark 1.67. The infinite language case might be an interesting question for the midterm exam. The proof should be quite similar.

Let's verify that infinite structures are not determined by their theories.

Proposition 1.68. Fix a language $\mathcal{L}$ and infinite $\mathcal{L}$-structure $\mathcal{M}$. Then there exists an $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{M} \not\cong \mathcal{N}$ but $\mathcal{M} \equiv \mathcal{N}$.

Proof. We will choose $\mathcal{N}$ to simply be larger than $\mathcal{M}$. Choose a cardinal κ strictly larger than $|M|$, and let $\mathcal{L}'$ be an expanded language with κ new constants c_α for each $\alpha \in \kappa$.

We now use compactness to construct $\mathcal{N}$. Choose the theory T to be

$$\operatorname{Th}_{\mathcal{L}}(\mathcal{M}) \sqcup \{c_\alpha \neq c_\beta : \alpha \neq \beta \text{ for } \alpha, \beta \in \kappa\}.$$

We claim that T is finitely satisfiable. Indeed, for any finite subset Δ, we claim that $\mathcal{M}$ can be made into a model for Δ. Well, $\mathcal{M}$ certainly satisfies $T \cap \Delta \subseteq \operatorname{Th}_{\mathcal{L}}(\mathcal{M})$, and then $\Delta \setminus \operatorname{Th}_{\mathcal{L}}(\mathcal{M})$ is just asserting that $\mathcal{M}$ has some finite number of distinct elements, which is true

More explicitly, let $\lambda \subseteq \kappa$ be a finite subset such that any c_α appearing in a sentence of Δ has $\alpha \in \lambda$. Then choose some element $a_0 \in \mathcal{M}$ and then $|\lambda|$ distinct elements a_α for each $\alpha \in \lambda$. We interpret c_α as a_α for each $\alpha \in \lambda$ and interpret each c_β as a_0 for each $\beta \notin \lambda$. We can see that this new model $\mathcal{M}'$ models Δ, so we are safe.

Anyway, Theorem 1.32 now provides us with a model $\mathcal{N}'$ of T. Notably, $\mathcal{N}'$ can be restricted to an $\mathcal{L}$-structure by simply forgetting how to interpret the κ new constants, and we see that $\operatorname{Th}_{\mathcal{L}}(\mathcal{N}) \supseteq \operatorname{Th}_{\mathcal{L}}(\mathcal{M})$, so $\mathcal{M} \equiv \mathcal{N}$ follows by Remark 1.64. However, $|\mathcal{N}| \geq \kappa > |\mathcal{M}|$ requires that $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic. ■

Here are some follow-up questions. Fix a language $\mathcal{L}$.

1. If we have $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}|$, can we construct an example with $\mathcal{M} \not\cong \mathcal{N}$? This is true for some theories $\operatorname{Th}_{\mathcal{L}}(\mathcal{M})$ where this is true but not always. For example, for countable models, this is (roughly speaking) the theory of types.
2. If $\mathcal{M} \equiv \mathcal{N}$, can we find a nonempty index set I and an ultrafilter $\mathcal{U}$ such that $\mathcal{M}^I/\mathcal{U} \cong \mathcal{N}^I/\mathcal{U}$? The converse is certainly true by Theorem 1.58. This forward direction turns out to be yes and is Keisler–Shelah. By the end of the course, we will be able to show this under some assumptions (countable languages, countable structures, and assuming the continuum hypothesis).

1.7 September 14

Today we will prove the Löwenheim–Skolem Theorem.

1.7.1 The Löwenheim–Skolem Theorem

We will want the following definition.

Definition 1.69 (elementary substructure)**.** Fix a language $\mathcal{L}$ and two structures $\mathcal{M}$ and $\mathcal{N}$. Then we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, written $\mathcal{M} \preceq \mathcal{N}$ if and only if $\mathcal{M}$ is a substructure of $\mathcal{N}$ and $\mathcal{M}_M \equiv \mathcal{N}_M$.

Remark 1.70. It is not enough to have $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \equiv \mathcal{N}$. For example, take the language $\mathcal{L} = \{<\}$ and let $\mathcal{M} = (\mathbb{N}, <)$ and $\mathcal{N} = (\mathbb{Z}^+, <)$. Then $\mathcal{M} \subseteq \mathcal{N}$, and $\mathcal{M} \equiv \mathcal{N}$. To see that $\mathcal{M} \equiv \mathcal{N}$ because $\mathcal{M} \cong \mathcal{N}$ (subtracting one is an isomorphism $\mathbb{Z}^+ \to \mathbb{N}$), which is enough by Proposition 1.65. However, $\mathcal{M} \not\preceq \mathcal{N}$: the sentence $\forall x\, 1 \leq x$ is true in $\mathcal{M}$ but not in $\mathcal{N}$.

Here is the result we are going to show.

Theorem 1.71. Fix a language $\mathcal{L}$ and infinite structure $\mathcal{M}$. For all subsets $A \subseteq M$, there exists an elementary substructure $\mathcal{N} \leq \mathcal{M}$ containing A with $|N| = |A| + |\mathcal{L}| + \aleph_0$.

Proof. We essentially do a more careful version of the Henkin construction. Set $T := \mathrm{Th}(\mathcal{M}_A)$. Let $\mathcal{L}'$ and T' be the language and theory extending $\mathcal{L}$ and T (respectively) obtained from the construction in Lemma 1.40 by adding witnessing constants. Quickly, we recall that T' and $\mathcal{L}'$ are constructed inductively as follows.

- Set $T_0 := T$ and $\mathcal{L}_0 := \mathcal{L}$.
- Set $\mathcal{L}_{n+1}$ to be $\mathcal{L}_n$ with a constant c_φ for each $\mathcal{L}_n$-formula φ with a variable x, and then we add $\exists\varphi(x) \to \varphi(c_\varphi)$ to T'. The function and relation symbols are the same between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$.
- Lastly, $\mathcal{L}'$ is the union of the $\mathcal{L}_n$s, and T' is the union of the T_ns.

We now expand $\mathcal{M}$ to be a model $\mathcal{M}'$ of T'. One only has to deal with the constants added by $\mathcal{L}'$. We will do this inductively.

- Set $\mathcal{M}_0 := \mathcal{M}_A$, and we construct $\mathcal{M}_n$ to model T_n.
- Given $\mathcal{M}_n \vDash T_n$, we construct $\mathcal{M}_{n+1}$ to be an $\mathcal{L}_{n+1}$-structure as follows. Well, we only need to worry about interpreting the new constants c_φ where φ is an $\mathcal{L}_n$-formula with free variable x, and we interpret $c_\varphi^{\mathcal{M}_{n+1}}$ as some $a_\varphi \in \mathcal{M}_n$ if $\mathcal{M}_n \vDash \varphi(a_\varphi)$ if such some a_φ exists, and we set $c_\varphi^{\mathcal{M}_{n+1}}$ to be any element of $\mathcal{M}_n$ if no such a_φ exists.

  Then $\mathcal{M}_{n+1}$ certainly satisfies everything in T_n (by inductive hypothesis), and it satisfies every one of the new sentences $\exists x\varphi(x) \to \varphi(c_\varphi)$ by construction of $c_\varphi^{\mathcal{M}_{n+1}}$, so we conclude $\mathcal{M}_{n+1} \vDash T_{n+1}$, as needed.
- Lastly, we define $\mathcal{M}'$ to be the union of the $\mathcal{M}_n$, and we conclude our construction. One can see that $\mathcal{M}' \vDash T'$ directly from the construction of the previous step because any $\varphi \in T'$ belongs to some T_n for finite n.

To continue the proof, we want the following result to check that we have built an elementary substructure.

Lemma 1.72 (Tarski–Vaught test)**.** Fix an $\mathcal{L}$-structure $\mathcal{M}$ and a subset $A \subseteq M$. Call A "realizable" if and only if any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y)$ and n-tuple $\overline{a} \in A^n$ has $\mathcal{M} \vDash (\exists y\varphi(\overline{x}, y))(\overline{a})$ if and only if there is some $b \in A$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$. Then A is realizable if and only if there is an elementary substructure $\mathcal{A} \leq \mathcal{M}$ with universe A.

Proof. There is some content here because the assertion $\mathcal{M}_A \equiv \mathcal{A}_A$ does not even make sense without having constructed $\mathcal{A}$. Anyway, we have two implications to show.

- Suppose that A is the universe of an elementary substructure $\mathcal{A} \leq \mathcal{M}$. We want to show that $\mathcal{A}$ is realizable. Well, let $\varphi(x_1, \ldots, x_n, y)$ be an $\mathcal{L}$-formula, and choose some $\overline{a} \in A^n$. Now, $\mathcal{M} \vDash (\exists y\varphi(\overline{x}, y))(\overline{a})$ if and only if $\mathcal{M}_A \vDash \exists y\varphi(\overline{a}, y)$. Now, because $\mathcal{A} \leq \mathcal{M}$, this is equivalent to $\mathcal{A}_A \vDash \exists y\varphi(\overline{a}, y)$, which his equivalent to having some $b \in A$ such that $\mathcal{A}_A \vDash \varphi(\overline{a}, b)$, which is equivalent to $\mathcal{M}_A \vDash \varphi(\overline{a}, b)$, which means there is $b \in A$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$.
- Suppose A is realizable. The main content here is to check that A is the universe of an $\mathcal{L}$-substructure of $\mathcal{M}$. We have the following checks.
  - Certainly $A \subseteq M$.
  - For each constant symbol c, we need $c^{\mathcal{M}} \in A$. Well, look at the formula $\varphi(y)$ given by $y = c$. Then $\mathcal{M} \vDash \exists y\varphi(y)$ by $c^{\mathcal{M}}$, so being realizable grants some $b \in A$ such that $\mathcal{M} \vDash \varphi(b)$, which means $c^{\mathcal{M}} = b \in A$, as needed.

- For each n-ary function symbol $f(x_1, \ldots, x_n)$ and $\overline{a} \in A$, we need to check $f^{\mathcal{M}}(\overline{a}) \in A$. Well, look at the formula $\varphi(x_1, \ldots, x_n, y)$ which is $y = f(x_1, \ldots, x_n, y)$. Then $\mathcal{M} \vDash \exists y \varphi(\overline{a})$, so being realizable promises some $b \in A$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$, which is asserting $f(a_1, \ldots, a_n) = b$.

We now need to show $\mathcal{M}_A \equiv \mathcal{A}_A$. We induct to show that an $\mathcal{L}_A$-sentence ψ has $\mathcal{M}_A \vDash \psi$ if and only if $\mathcal{A}_A \vDash \psi$. Let $\mathcal{F}'$ be the set of such $\mathcal{L}_A$-sentences.

- For atomic formulae, we use Proposition 1.12 so that we don't have to do any more work.
- The usual arguments tell us that $\varphi, \psi \in \mathcal{F}'$ implies that $\neg\varphi \in \mathcal{F}'$ and $\varphi \wedge \psi \in \mathcal{F}'$. We won't write this out.
- Lastly, suppose ψ is of the form $\exists y \varphi(y)$. Because $\exists y \varphi$ is an $\mathcal{L}_A$-sentence, we can write $\varphi(y)$ as $\varphi'(\overline{a}, y)$ where $\varphi'(x_1, \ldots, x_n, y)$ is some $\mathcal{L}$-formula and $\overline{a} \in A^n$.
  Now, in one direction, $\mathcal{A}_A \vDash \psi$ if and only if some $b \in A$ such that $\mathcal{A}_A \vDash \varphi(a)$, so by induction $\mathcal{M}_A \vDash \psi(b)$, which is implies $\mathcal{M} \vDash \psi$, as needed.
  To go the other direction, we need to pull a witness down from $\mathcal{M}$ to $\mathcal{A}$, which is harder. Suppose $\mathcal{M}_A \vDash \psi$. Then $\mathcal{M}_A \vDash (\exists y \varphi'(x, y))(\overline{a})$, from which being realizable grants $b \in A$ such that $\mathcal{M}_A \vDash \varphi'(\overline{a}, b)$. This sentence is simpler, so by induction we get $\mathcal{A} \vDash \varphi'(\overline{a}, b)$, which is equivalent to $\mathcal{A} \vDash \exists y \varphi(y)$, as needed. ■

Remark 1.73. There is not really anything to do when checking the reverse direction of being realizable: having $b \in A$ such that $\mathcal{M} \vDash \varphi(\overline{a}, b)$ of course implies that $\mathcal{M} \vDash (\exists y \varphi(\overline{x}, y))(\overline{a})$ by choosing y to be this $b \in A$. The content is the reverse direction where we pull down the witness from $\mathcal{M}$ to $\mathcal{A}$.

Now, let the set N be the set of interpretations of constant symbols $c^{\mathcal{M}'}$ for each constant symbol c of $\mathcal{L}'$. Notably, $A \subseteq \mathcal{L}'$, and $a^{\mathcal{M}'} = a$, so $a \in N$, so $A \subseteq N$. We would like to turn N into an elementary substructure, for which we use Lemma 1.72.

It suffices to check that N is realizable. Let $\varphi(x_1, \ldots, x_n, y)$ be an $\mathcal{L}$-formula and $(a_1, \ldots, a_n) \in N^n$. Suppose $\mathcal{M} \vDash (\exists y \varphi(\overline{x}, y))(\overline{a})$. Then $\mathcal{M}' \vDash (\exists y \varphi(\overline{x}, y))(\overline{a})$ by choosing the same y, which means $\mathcal{M}' \vDash \varphi(\overline{a}, y)$, but $\mathcal{M}' \vDash \exists y \varphi(\overline{a}, y) \to \varphi(\overline{a}, c)$ for some constant symbol c of $\mathcal{L}'$. Combining, we get $\mathcal{M}' \vDash \varphi(\overline{a}, c)$. But then setting $d := c^{\mathcal{M}'}$ (which lives in N!), we achieve $\mathcal{M}' \vDash \varphi(\overline{a}, d)$.

Thus, N is the universe of some elementary substructure $\mathcal{N} \preceq \mathcal{M}$. We saw that N contains A, and we see $|N|$ is at most the size of the constants of $\mathcal{L}'$, which has size $|\mathcal{L}| + \aleph_0 + |A|$. This completes the proof. ■

One can also go up, which was essentially Proposition 1.68.

Proposition 1.74. Fix an infinite $\mathcal{L}$-structure $\mathcal{M}$. For any cardinal $\kappa \geq |M| + |\mathcal{L}|$, there exists an $\mathcal{L}$-structure $\mathcal{N}$ with cardinality κ and $\mathcal{M} \preceq \mathcal{N}$.

Proof. As in Proposition 1.68, let $\mathcal{L}'$ be the language $\mathcal{L}$ where we add constants c_α for each $\alpha \in \kappa$, and then we let T' be

$$\operatorname{Th}(\mathcal{M}_M) \sqcup \{c_\alpha \neq c_\beta : \alpha \neq \beta \text{ for } \alpha, \beta \in \kappa\}.$$

We showed in Proposition 1.68 that T' is finitely satisfiable, so we produce a model $\mathcal{N}_0$ of T'. Now, let A be the set of interpretations of constants $c^{\mathcal{N}_0}$ for each constant c in $\mathcal{L}'$. Notably, A contains M, and the map $\kappa \to A$ given by $a \mapsto c_\alpha^{\mathcal{N}_0}$ is one-to-one, so $|A| \geq \kappa$. On the other hand, $|A|$ has size bounded by the constants of $\mathcal{L}'$, which has size $\kappa + |\mathcal{M}| + |\mathcal{L}|$, which is κ, so $|A|$ has size exactly κ.

Now, by Theorem 1.71, we produce an elementary substructure $\mathcal{N} \preceq \mathcal{N}_0$ containing A. Because $\mathcal{M} \subseteq \mathcal{N} \preceq \mathcal{N}_0$ and $\mathcal{M} \preceq \mathcal{N}_0$ (by construction of $\mathcal{N}_0$), so we conclude $\mathcal{M} \preceq \mathcal{N}$ by chasing our formulae around. ■

1.8 September 19

Here we go.

1.8.1 An Example of the Back-and-Forth Method

For our example, let $\mathcal{L}$ be a language with one binary relation E, which will be considered to be an equivalence relation. Consider the structure $\mathcal{M}_0$ with universe $(x, y) \in \mathbb{N}^2$ where $x < y$, where $(x, y)E(x', y')$ if and only if $y = y'$.

We claim that there is another countable model with the same theory. For example, we consider $\mathcal{M}_\omega$ which is $\mathcal{M}_0$ with a disjoint copy of $\mathbb{N}^2 \times \{0\}$ where $(x, y, 0)E(x', y', 0)$ if and only if $y = y'$. Let's check that the theory of $\mathcal{M}_0$ has the same theory of $\mathcal{M}_\omega$. This essentially follows from compactness (Theorem 1.32) and Theorem 1.71 to the theory T consisting of the elementary diagram of $\mathcal{M}_0$ plus the sentences

$$\{c_{xy} \neq c_{x'y'} : \text{ for } (x, y) \neq (x', y')\} \cup \{c_{xy} E c_{x'y} : x, x', y \in \mathbb{N}\} \cup \{c_{xy} E c_{x'y'} : x, x', y, y' \in \mathbb{N} \text{ where } y \neq y'\},$$

where we have introduced these new constants c_{xy} to an extended language $\mathcal{L}'$. Namely, Theorem 1.32 permits us to find a countable model of this above theory: to see that the above set of sentences is satisfiable, we note that $\mathcal{M}_0$ is able to model any finite subset of the above theory is only asking for arbitrary many arbitrarily large equivalence classes, which $\mathcal{M}_0$ provides.

So we produce a countable model $\mathcal{M}'$ of T. We claim that $\mathcal{M}' \cong \mathcal{M}_\omega$ in the language $\mathcal{L}$. This will use the back-and-forth method.

Lemma 1.75. Fix everything as above. Then $\mathcal{M}' \cong \mathcal{M}_\omega$, where $\mathcal{M}'$ is considered as an $\mathcal{L}$-structure.

Proof. We build our isomorphism via approximations $f_i : X_i \to Y_i$ for $i \in \mathbb{N}$, where $X_i \subseteq \mathcal{M}'$ and $Y_i \subseteq \mathcal{M}_\omega$. We require that $i \leq j$ means $X_i \subseteq X_j$ and $Y_i \subseteq Y_j$ and then $f_j|_{X_i} = f_i$, and we also want f_i to be an isomorphism of $\mathcal{L}$-strucutres for $i > 0$. By the end of this process, we will want $\bigcup_{i\in\mathbb{N}} X_i = \mathcal{M}'$ and $\bigcup_{i\in\mathbb{N}} Y_i = \mathcal{M}_\omega$ so that we have a well-defined isomorphism $f : \mathcal{M}' \to \mathcal{M}_\omega$ at the end. This last bit is going to be a little tricky. For this, we enumerate $M' = \{a_i\}_{i=0}^\infty$ and $M_\omega = \{b_i\}_{i=0}^\infty$, and we will ask that each n have $\{a_j : j < n\} \subseteq X_{2n}$ and $\{b_j : j < n\} \subseteq \operatorname{im} f_{2n+1}$.

Alright, let's get started. Take f_0 to be the unique function $X_0 \to Y_0$ where $X_0 = Y_0 = \varnothing$. One can check that this trivially works for all of our hypotheses. We now induct in two cases.

- Suppose we have $f_{2n} : X_{2n} \to Y_{2n}$, and we want to produce $f_{2n+1} : X_{2n+1} \to Y_{2n+1}$. The point is that b_n now needs to appear in the range of f_{2n+1}. We have the following cases.
  - If b_n is already in the range, do nothing. In the following cases, we suppose that b_n is not in the range of f_{2n} already.
  - Suppose that b_n is not equivalent to some element of $\operatorname{im} f_{2n}$. If b_n is in a finite equivalence class, map it to the corresponding unique equivalence class in $\mathcal{M}'$, which cannot have been chosen so far because f_{2n} is an isomorphism. If b_n lives in an infinite equivalence class, then go find an unused infinite equivalence class in $\mathcal{M}'$, which is possible because f_{2n} has finite domain currently.
  - Suppose that b_n is equivalent to some element $b' \in \operatorname{im} f_{2n}$. By the nature of $f_\bullet$ being an isomorphism, we are arranging so that the size of the equivalence class of a and $f_\bullet(a)$ are always the same. So the size of the equivalence class of $f_{2n}^{-1}(b')$ must have space (even if finite!) because the element of b_n not being hit so far requires us to have space in the equivalence class of $f_{2n}^{-1}(b')$.
- Going forward the argument is essentially the same just talking in reverse.

Assembling the $f_\bullet$ together produces the desired result. ■

We now conclude by remarking that $\operatorname{Th}_{\mathcal{L}_{\mathcal{M}_0}}(\mathcal{M}_0) = \operatorname{Th}_{\mathcal{L}_{\mathcal{M}_0}}(\mathcal{M}_\omega)$, so $\mathcal{M}_0 \preceq \mathcal{M}_\omega$.

Remark 1.76. We can now define $\mathcal{M}_n := \mathcal{M}_0 \sqcup \mathbb{N} \times \{0, 1, \ldots, n-1\} \times \{0\}$ as a substructure of $\mathcal{M}_\omega$. One can repeat the above argument with $\mathcal{M}_0$ replaced by $\mathcal{M}_n$ to conclude that $\mathcal{M}_n \preceq \mathcal{M}_\omega$ again. We conclude that $\mathcal{M}_0 \equiv \mathcal{M}_n$ for each n. In total, we have produced countably many non-isomorphic models. It turns out that these are all the countable ones.

One might now go back and ask for the number of models of $\mathrm{Th}_{\mathcal{L}_{\mathcal{M}_0}}(\mathcal{M}_0)$ of cardinality $\aleph_1$. It turns out that there are again countably many. The point is that a model $\mathcal{M}$ of cardinality $\aleph_1$ can be attached the two invariants

$$\kappa_0(\mathcal{M}) := \#\{[x] \in M/E : [x]_E \text{ has size } \aleph_0\},$$
$$\kappa_1(\mathcal{M}) := \#\{[x] \in M/E : [x]_E \text{ has size } \aleph_1\}.$$

One can show that $\mathcal{M}_1 \cong \mathcal{M}_2$ if and only if $\kappa_0(\mathcal{M}_1) = \kappa_0(\mathcal{M}_2)$ and $\kappa_1(\mathcal{M}_1) \cong \kappa_1(\mathcal{M}_2)$ by using some set theory, and then one can produce a model with given invariants κ_0 and κ_1 arbitrarily provided that $\aleph_0\kappa_0 + \aleph_1\kappa_1 = \aleph_1$.

1.8.2 Dense Linear Orders Without Endpoints

Let's see another example.

Proposition 1.77. Fix a language $\mathcal{L}$ with a single binary relation $<$. Then $\mathrm{Th}_{\mathcal{L}}(\mathbb{Q}, <)$ is $\aleph_0$-categorical.

We should perhaps define $\aleph_0$-categorical.

Definition 1.78 (κ-categorical)**.** A theory T of a language $\mathcal{L}$ is *κ-categorical* if and only if T has exactly one isomorphism class of models of cardinality κ.

In fact, we will show the following.

Proposition 1.79. Fix a language $\mathcal{L}$ with a single binary relation $<$, and let DLO be the following theory, of dense linear orders without endpoints.

- $<$ is a total ordering.
- Dense: $\forall x \forall y (x < y \to \exists z (x < z \wedge z < y))$.
- Without endpoints: $\forall x \exists y (y < x)$ and $\forall x \exists y (x < y)$.

Then DLO is $\aleph_0$-categorical.

Note that $\mathbb{Q}$ models DLO, so Proposition 1.77 will follow. Anyway, let's show Proposition 1.79.

Proof of Proposition 1.79. Let $\mathcal{A}$ and $\mathcal{B}$ be models of DLO. Enumerate $\mathcal{A} = \{a_i\}_{i=0}^{\infty}$ and $\mathcal{B} = \{b_i\}_{i=0}^{\infty}$, and we will work in the same set-up as the back-and-forth argument previously described. Namely, we describe a sequence of compatible isomorphisms $f_i \colon X_i \to Y_i$ where X_{2n} contains $\{a_1, \ldots, a_{n-1}\}$ and Y_{2n+1} contains $\{b_1, \ldots, b_{n-1}\}$. Take f_0 to be the unique function $\varnothing \to \varnothing$.

- Suppose we have f_{2n-1}, and we want to build f_{2n}. If a_n is already in the domain of f_{2n-1}, do nothing. We have three cases.
  - If $a_n < x$ for all $x \in X_{2n-1}$, use that $\mathcal{B}$ has no endpoints to find $f(a_n)$ less than everyone in Y_{2n-1}.
  - If $a_n > x$ for all $x \in X_{2n-1}$, make a similar argument as the previous case.
  - Otherwise, find $x, y \in X_{2n-1}$ so that $x < a_n < y$, and nothing in X_{2n-1} lives between x and y; this is possible because $<$ is a total ordering. Then use the density of $\mathcal{B}$ to find some $f(a_n)$ strictly between x and y to complete.
- To extend f_{2n} to f_{2n+1}, repeat the above argument in reverse.

Now, assembling our $f_\bullet$ produces our isomorphism. ■

Remark 1.80. We now might ask how many models DLO has of cardinality $\aleph_1$. There are apparently $2^{\aleph_1}$ many up to isomorphism. Of course, this is an upper bound on the number of models because an ordering is asking for a subset of $\aleph_1 \times \aleph_1$. So the name of the game now is to produce enough models; one cannot really hope to precisely describe all the models.

To wrap us up, let's pick up the following result.

Proposition 1.81. Fix an $\mathcal{L}$-theory T which is κ-categorical for cardinality κ. If T has only infinite models, then T is complete; i.e., any $\mathcal{L}$-sentence φ has either $T \vDash \varphi$ or $T \vDash \neg\varphi$.

Proof. Let $\mathcal{M}$ be a model of T of cardinality κ. Now, for any sentence φ, if $T \vDash \varphi$ and $T \vDash \neg\varphi$, then there is a model $\mathcal{M}_+$ and $\mathcal{M}_-$ satisfying $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$, respectively. By Theorem 1.71, we may bring $\mathcal{M}_+$ and $\mathcal{M}_-$ to have cardinality κ, so being κ-categorical requires $\mathcal{M}_+ \cong \mathcal{M}_-$, which is a contradiction because then $\mathcal{M}_+ \equiv \mathcal{M}_-$. ∎

Example 1.82. Thus, Proposition 1.79 requires that DLO is complete. As such, the theory DLO must complete to exactly $\mathrm{Th}(\mathbb{Q}, <)$.

1.9 September 21

Today, we will go on to some more nontrivial examples.

1.9.1 Algebraically Closed Fields

Consider the language $\mathcal{L}$ with binary operations $+$ and $\cdot$, a unary operation $-$, and constants 0 and 1. The theory of fields has the sentences given by the ones in a standard algebra class.

- $\forall x \forall y((x + y = y + x) \land (x \cdot y = y \cdot x))$.
- $\forall x \forall y \forall z(((x + y) + z = x + (y + z)) \land ((x \cdot y) \cdot z = x \cdot (y \cdot z)))$.
- $\forall x((x + (-x) = 0) \land ((-x) + x = 0))$.
- $\forall x \exists y(x \cdot y = 1)$.
- $\forall x \forall y \forall z(x \cdot (y + z) = x \cdot y + x \cdot z)$.
- $\forall x((x + 0 = x) \land (x \cdot 1 = x))$.
- $\neg(0 = 1)$.

To make this algebraically closed, we want every monic polynomial to have a root. For this, we should go degree-by-degree. For example, for degree d which is a positive integer, we write the sentence φ_d to be

$$\forall a_1 \cdots \forall a_{d-1} \exists x \left(x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0\right).$$

Call this theory ACF. Notably, we then have used infinitely many axioms.

As an aside, we note there is no finite set of sentences characterizing algebraically closed fields. Let's show this.

Lemma 1.83. Suppose a satisfiable theory T is finitely axiomatizable: there is a finite set of sentences $\varphi_1, \ldots, \varphi_n$ such that $\mathcal{M} \vDash T$ for a structure $\mathcal{M}$ if and only if $\mathcal{M} \vDash \varphi_\bullet$ for each $\varphi_\bullet$. Then there is a finite subset $T_0 \subseteq T$ such that $\mathcal{M} \vDash T$ if and only if $\mathcal{M} \vDash T_0$.

Proof. The reverse direction is clear by just taking T_0 to be our finite set of axioms.

In the other direction, suppose that $\varphi := \varphi_1 \wedge \cdots \wedge \varphi_n$ axiomatizes T. We now apply compactness $\Sigma := T \cup \{\neg\varphi\}$. Note Σ is not satisfiable because $\mathcal{M} \vDash T$ if and only if $\mathcal{M} \vDash \varphi$. Thus, by Theorem 1.32, we see that Σ cannot be finitely satisfiable. But T is finitely satisfiable, so there is some finite subset of the form $T_0 \cup \{\neg\varphi\}$ which is not satisfiable.

We now check that T_0 does the trick. However, this means that any structure $\mathcal{M}$ such that $\mathcal{M} \vDash T_0$ requires $\mathcal{M} \vDash \varphi$, and conversely, $\mathcal{M} \vDash \varphi$ implies $\mathcal{M} \vDash T$ implies $\mathcal{M} \vDash T_0$. Thus, T_0 is the needed subset. ■

Let's apply this lemma to ACF. Let T_0 be some finite subset of ACF, and we show that T_0 is not equivalent to ACF. Add in any of the field axioms necessary, and we know there is some upper bound N such that T_0 is then contained in the field axioms plus $\{\varphi_1, \ldots, \varphi_d\}$. To show that T_0 is not equivalent to ACF, we construct a field $K/\mathbb{Q}$ which models T_0 but not ACF. Well, construct K by a tower

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots,$$

where K_{n+1} consists of all numbers which are roots of polynomials in K_n of degree at most N. Then set $K := \bigcup_{n=0}^{\infty} K_n$, and we see $K \vDash T_0$.

Well, for a piece of algebra, we note that the polynomial $f_p(x) := x^p - 2 \in \mathbb{Q}[x]$ is irreducible for any prime p. Choosing $p > N$, we then claim that $f_p(x) \in K[x]$ has no root. Indeed, any root would need to live in some $K_{n+1}[x]$, which means that $x^p - 2$ has some root shared with a polynomial of degree at most N whose coefficients live in K_n. However, extracting out the necessary coefficients into a field L, we see that $L/\mathbb{Q}$ has degree coprime to p (it's constructed using roots of polynomials of degrees at most N, and $p > N$ is prime), but then $\mathbb{Q}[x]/(x^p - 2) \subseteq L$ has degree p, so it cannot possibly be a subfield.

Remark 1.84. The same argument shows that one can finitely axiomatize fields of characteristic 0. We produce the theory of characteristic-0 fields by adding in the sentences

$$\underbrace{1 + \cdots + 1}_{p} = 0$$

for each positive prime p. But then no finite subset of these axioms will do because there are fields of arbitrarily large (but still finite) characteristic.

Anyway, here is our theorem.

Theorem 1.85. The completion of ACF are the theories ACF_p where p is a prime or zero, where ACF_p adds in the condition of being characteristic p (via the sentence $1 + \cdots + 1 = 0$ for nonzero p and $1 + \cdots + 1 \neq 0$ for all lengths when $p = 0$).

In fact, we will show the following stronger result.

Theorem 1.86. Fix p to be prime or zero. Then ACF_p is κ-categorical for any $\kappa > \aleph_0$.

This will be enough to prove Theorem 1.85 by Proposition 1.81 because ACF_p certainly has models of size $\kappa > \aleph_0$ by taking $\overline{k(\kappa)}$ where κ is being used as a transcendence basis. Notably, $\overline{k(\kappa)}$ has size $\kappa + \aleph_0 = \kappa$.

Anyway, let's prove Theorem 1.86.

Proof with algebra. Let k be the smallest field of that characteristic (the finite field when $p > 0$ and $\mathbb{Q}$ when $p = 0$).

Now, suppose we have two fields K_1 and K_2 which satisfy ACF_p of cardinality κ. Now, let $X_i \subseteq K_i$ be a transcendence basis for each i, meaning that X_i is a maximal algebraically independent set of elements. As such, K_i is algebraic over $\mathbb{F}_p(X_i)$. Now, $|k(X_i)| = |X_i| + \aleph_0$, so taking algebraic closure has $\kappa = |K_i| = |k(X_i)| + \aleph_0 = |X_i| + \aleph_0$, so $\kappa = |X_i|$. Thus, $k(X_1) \cong k(X_2)$, so taking algebraic closure enforces $K_1 \cong K_2$ by taking algebraic closure. ■

Corollary 1.87. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathcal{P}$. Then we have a field isomorphism

$$\mathbb{C} \cong \prod_{\mathcal{U}} \overline{\mathbb{F}_p}.$$

Proof. By Theorem 1.58, we see that $\prod_{\mathcal{U}} \overline{\mathbb{F}_p}$ is algebraically closed because being algebraically closed field is held in each factor of the ultraproduct. It remains to compute the characteristic. Well, the sentence $1 + \cdots + 1 = 0$ for any length p fails to hold in all but finitely many of these factors, so we see that the sentence

$$\underbrace{1 + \cdots + 1}_{p} \neq 0$$

holds in all but finitely many of the factors of our ultrafilter. Thus, the ultraproduct has characteristic 0 by Theorem 1.58 again, and we see that $\mathbb{C}$ has the same cardinality as our ultrafilter, so the result follows by Theorem 1.86. To compute this cardinality, we note that

$$\left|\prod_{\mathcal{U}} \overline{\mathbb{F}_p}\right| \le \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

One can then embed this ultraproduct into a tree; one uses Theorem 1.58. More generally, one we will be able to show that $|X_i| \ge 2^i$ for some collection $\{X_i\}_{i \in \mathbb{N}}$ has $\prod_{\mathcal{U}} X_i$ of cardinality $2^{\aleph_0}$. ■

Let's improve our proof of Theorem 1.86. We will show the following stronger result.

Theorem 1.88. The theory ACF eliminates quantifiers. In other words, for any formula $\varphi(x_1, \ldots, x_n)$, there is a quantifier-free formula $\psi(x_1, \ldots, x_n)$ such that $\mathrm{ACF} \vDash \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.

Remark 1.89. The theory of Peano arithmetic does not eliminate quantifiers: there are very complicated sets that one can define.

There is a proof in Tarski's RAND paper. We are not going to follow it. We are going to do a back-and-forth argument. To begin, we have the following step.

Proposition 1.90. Fix two algebraically closed fields K_1 and K_2 of cardinality $\kappa > \aleph_0$. Suppose, we have an isomorphism $f \colon L_1 \to L_2$ of subfields $L_1 \subseteq K_1$ and $L_2 \subseteq K_2$ where L_1 and L_2 are subfields of cardinality less than κ. Then f extends to an isomorphism $K_1 \to K_2$.

Proof. We construct this isomorphism using a back-and-forth argument. Treat κ as an ordinal, and enumerate $K_1 = \{a_\alpha : \alpha \in \kappa\}$ and $K_2 = \{b_\alpha : \alpha \in \kappa\}$. We will build a sequence of isomorphisms $g_\alpha \colon L_\alpha^1 \to L_\alpha^2$ for each $\alpha \in \kappa$ so that g_β extends g_α whenever $\alpha \le \beta$. We will also arrange so that $g_0 := f$ and $a_\beta \in L_\alpha^1$ and $b_\beta \in L_\alpha^2$ for each $\beta \in \alpha$; it will also help to have $L_\alpha^\bullet$ always have cardinality less than κ. If we can do this, we simply define $g \colon K_1 \to K_2$ by taking the union of all these isomorphisms.

For g_0, there is nothing to do. If α is a limit ordinal, then take g_α to be the union of the g_β for $\beta < \alpha$. Notably, the domain and codomain are the unions of the domains and codomains; of course, this is still an isomorphism, and it satisfies our necessary property because any $\beta < \alpha$ has a_β and b_β in the domain and codomain of $g_{\beta+1}$, respectively. Lastly, the domain and codomain is an ascending union of sets of cardinality less than κ, which is typically less than κ.[2]

In our last case, take $\alpha := \beta + 1$. Then we need to tell g_β where to send a_β. If a_β is already in the domain, do nothing. Otherwise, there are two cases.

- Suppose that a_β is algebraic over L^1_β with monic irreducible polynomial $P(x)$. Passing through g_β, we see that $g_\beta(P(x)) \in L^2_\beta[x]$ will fully factor in K_2, and one of the roots cannot have been hit by g_β because then their pre-images in L^1_β includes a_β already. So send a_β to a root not hit yet.

- Suppose that a_β is transcendental over L^1_β. Now, $\left|\overline{L^2_\beta}\right| = \left|L^2_\beta\right| + \aleph_0 < \kappa$, so there is a transcendental element of K_2 not in L^2_β. Send a_β to such a transcendental element.

For b_β to go backwards, do the same argument in reverse. ■

Corollary 1.91. Fix algebraically closed fields K_1 and K_2, and fix tuples $\overline{a} \in K_1^n$ and $\overline{b} \in K_2^n$. Then the following are equivalent.

(a) The structures $(K_1, \overline{a})$ and $(K_2, \overline{b})$ are equivalent in an expanded language.

(b) $k_1(\overline{a}) = k_2(\overline{\beta})$ where $k_1 \subseteq K_1$ and $k_2 \subseteq K_2$ are the prime subfields.

(c) For any quantifier-free formulae θ, we have $K_1 \vDash \theta(\overline{a})$ if and only if $K_2 \vDash \theta(\overline{b})$.

1.10 September 26

Today, we will give a structural way to look at quantifier elimination.

1.10.1 A Taste of Types

We split our discussion of quantifier elimination into the following lemmas.

Lemma 1.92. Fix $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$. Further, fix $\overline{a} \in \mathcal{A}^n$ and $\overline{b} \in \mathcal{B}^n$ with $n \geq 1$. Then the following are equivalent.

(a) For any quantifier-free $\mathcal{L}$-formula φ, we have $\mathcal{A} \vDash \varphi(\overline{a})$ if and only if $\mathcal{B} \vDash \varphi(\overline{b})$.

(b) There is an isomorphism of substructures $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$ containing $\overline{a}$ and $\overline{b}$ respectively, and the isomorphism sends $\overline{a}$ to $\overline{b}$.

We will prove this in a moment, but we quickly note that it motivates the following definition.

Definition 1.93 (quantifier-free type)**.** Fix an $\mathcal{L}$-structure $\mathcal{A}$ and some $\overline{a} \in A^n$. Then the *quantifier-free type of* $\overline{a}$, denoted $\operatorname{qftp}^{\mathcal{A}}(\overline{a})$, is the set of quantifier free formulae φ such that $\mathcal{A} \vDash \varphi(\overline{a})$.

Anyway, here is our proof of Lemma 1.92.

Proof of Lemma 1.92. We have two implications to show.

- We show (b) implies (a). Suppose we have an isomorphism $f \colon \mathcal{A}' \to \mathcal{B}'$ as described. Now, suppose $\varphi(\overline{x})$ is a quantifier-free $\mathcal{L}$-formula with n free variables. Then $\mathcal{A}' \vDash \varphi(\overline{a})$ if and only if $\mathcal{B}' \vDash \varphi(\overline{b})$ by the nature of our isomorphism (see Proposition 1.14). Then this comes down to the substructure because φ is quantifier-free by Proposition 1.12.

[2] One needs to do something here in the case that κ is a singular.

- We show (a) implies (b). Define $A' \subseteq A$ to be the set of terms t evaluated on $\overline{a}$ as $t^{\mathcal{A}}(\overline{a})$, and define $B' \subseteq B$ similarly. We do need to check that A' is the universe of an $\mathcal{L}$-substructure of $\mathcal{A}$, and the check for B' will be similar. Well, we interpret constants (which are terms) exactly as they were interpreted in $\mathcal{A}$. We interpret functions exactly as they were interpreted in $\mathcal{A}$ because terms are closed under applying functions. Lastly, relations are defined by intersection with A, which is what is needed to provide a substructure.

  We now define $\mathcal{A}' \to \mathcal{B}'$ by sending the term $t^{\mathcal{A}}(a_1, \ldots, a_n)$ to $t^{\mathcal{B}}(b_1, \ldots, b_n)$. We have the following checks.

  - Well-defined and injective: if s and t are terms with $s^{\mathcal{A}}(\overline{a}) = t^{\mathcal{A}}(\overline{a})$, then this is equivalent to $\mathcal{A} \vDash (s(\overline{x}) = t(\overline{x}))(\overline{a})$, which is equivalent to $\mathcal{B} \vDash (s(\overline{x}) = t(\overline{x}))(\overline{b})$ by hypothesis, which at the end is equivalent to $s^{\mathcal{B}}(\overline{b}) = t^{\mathcal{B}}(\overline{b})$.
  - Surjective: any element of $\mathcal{B}'$ takes the form $t^{\mathcal{B}}(\overline{b})$ for some term t, which is hit by $t^{\mathcal{A}}(\overline{a})$.
  - Isomorphism: this has many checks in itself. For any constant symbol c, we see $f\left(c^{\mathcal{A}'}\right) = c^{\mathcal{B}'}$ by viewing c as a term which does not care about the input $\overline{a}$. Now suppose F is an m-ary function symbol, then

  $$\begin{aligned} f\left(F^{\mathcal{A}'}(t_1^{\overline{A}'}(\overline{a}), \ldots, t_m^{\overline{A}'}(\overline{a}))\right) &= f\big(\underbrace{F(t_1(\overline{x}), \ldots, t_m(\overline{x}))}_{\text{some term!}}(\overline{a})\big) \\ &= F^{\mathcal{B}}(t_1^{\mathcal{B}}(\overline{b}), \ldots, t_m^{\mathcal{B}}(\overline{b})) \\ &= F^{\mathcal{B}}\left(f(t_1^{\mathcal{A}}(\overline{a}), \ldots, t_m^{\mathcal{A}}(\overline{a}))\right). \end{aligned}$$

  Lastly, let R be an m-ary relation symbol. Then $\left(t_1^{\mathcal{A}}(\overline{a}), \ldots, t_m^{\mathcal{A}}(\overline{a})\right) \in R^{\mathcal{A}'}$ if and only if $\mathcal{A}' \vDash R(t_1, \ldots, t_m)(\overline{a})$ if and only if $\mathcal{A} \vDash R(t_1, \ldots, t_m)(\overline{a})$ by Proposition 1.12, which is now equivalent to $\mathcal{B} \vDash R(t_1, \ldots, t_m)(\overline{b})$ and then equivalent to $\mathcal{B}' \vDash R(t_1, \ldots, t_m)(\overline{b})$. ■

Remark 1.94. The $\mathcal{A}'$ given in the proof above is the smallest substructure of $\mathcal{A}$ containing $\overline{a}$.

More generally, we might be interested in types.

Definition 1.95 (type)**.** Fix an $\mathcal{L}$-structure $\mathcal{A}$. Further, fix an n-tuple $\overline{a} \in A^n$. Then the *type*, denoted $\operatorname{tp}^{\mathcal{A}}(\overline{a})$ is the set of $\mathcal{L}$-formulae $\varphi(\overline{x})$ such that $\mathcal{A} \vDash \varphi(\overline{a})$.

Here is the corresponding result.

Lemma 1.96. Fix $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, and further fix $a \in A^n$ and $b \in B^n$. Suppose that there are elementary extensions $\mathcal{A}' \geq \mathcal{A}$ and $\mathcal{B}' \geq \mathcal{B}$ with an isomorphism $f\colon \mathcal{A}' \to \mathcal{B}'$ sending $\overline{a}$ to $\overline{b}$. Then $\operatorname{tp}^{\mathcal{A}}(\overline{a}) = \operatorname{tp}^{\mathcal{B}}(\overline{b})$.

Proof. Note that we have elementary expansions $\mathcal{A}_{\overline{a}} \leq \mathcal{A}'_{\overline{a}}$ and $\mathcal{B}_{\overline{b}} \leq \mathcal{B}'_{\overline{b}}$. By hypothesis, the isomorphism $\mathcal{A}' \cong \mathcal{B}'$ sends $\overline{a}$ to $\overline{b}$, so in fact $\mathcal{A}'_{\overline{a}}$ is isomorphic to $\mathcal{B}_{\overline{b}}$. Tracking everything through, we see $\mathcal{A} \vDash \varphi(\overline{a})$ if and only if $\mathcal{A}_{\overline{a}} \vDash \varphi(\overline{a})$ if and only if $\mathcal{A}'_{\overline{a}} \vDash \varphi(\overline{a})$ if and only if $\mathcal{B}'_{\overline{b}} \vDash \varphi(\overline{b})$ if and only if $\mathcal{B}_{\overline{b}} \vDash \varphi(\overline{b})$ if and only if $\mathcal{B} \vDash \varphi(\overline{b})$. ■

Remark 1.97. The converse of this result is true, and we will prove it later in this class.

1.10.2 Back to Algebraically Closed Fields

Let's return to our discussion of algebraically closed fields.

Definition 1.98 (eliminates quantifiers). An $\mathcal{L}$-theory T *eliminates quantifiers* if and only if any formula $\varphi(\overline{x})$ has some quantifier-free formula $\overline{\psi}(\overline{x})$ such that $T \vDash \forall x(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.

Theorem 1.99. Say that an $\mathcal{L}$-theory T is "isomorphism-extendable" if and only if it has the following property: for any models $\mathcal{A}, \mathcal{B} \vDash T$ with fixed n-tuples $a \in A^n$ and $b \in B^n$ equipped with an isomorphism $f\colon \mathcal{A}' \to \mathcal{B}'$ of substructures containing $\overline{a}$ and $\overline{b}$ (respectively) which sends $\overline{a}$ to $\overline{b}$, then any elementary superstructures $\mathcal{A}^* \geq \mathcal{A}$ and $\mathcal{B}^* \geq \mathcal{B}$ have an isomorphism extending f. Then if T is isomorphism-extendable, then T eliminates quantifiers.

Proof. Fix a formula $\varphi(\overline{x})$. Observe that being isomorphism-extendable implies that $\overline{a}$ and $\overline{b}$ having the same quantifier-free type implies that they have the same type by combining Lemmas 1.92 and 1.96.

For technical reasons, we extend the language to $\mathcal{L}^*$ to have some new constants c_1 and c_2 for each of the old constants c. Our functions are the same, and we add in one more unary relation U. The point of introducing $\mathcal{L}^*$ is to be able to talk about two $\mathcal{L}$-structures of the same type.

Explicitly, given an $\mathcal{L}^*$-structure where U contains the c_1s and the complement contains the c_2s (and these are nonempty), then we can restrict to U and its complement to provide two $\mathcal{L}$-structures. Conversely, given $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, we build an $\mathcal{L}^*$-structure with universe $A \sqcup B$ as follows: interpret the constants c_1 and c_2 as in $\mathcal{A}$ and $\mathcal{B}$, respectively. Interpret the values $f(\overline{a})$ and $f(\overline{b})$ for $\overline{a} \in A^\bullet$ and $\overline{b} \in B^\bullet$ as in $\mathcal{A}$ and $\mathcal{B}$, respectively, and interpret $f(\overline{e})$ for any other $\overline{e}$ however we wish. One does something similar for the relations. Notably, the $\mathcal{L}^*$-structure, which we call $\mathcal{A}$, is not exactly the same data as two $\mathcal{L}$-structures because one has to say what happens on the function and relation symbols when we have not been told by $\mathcal{A}$ and $\mathcal{B}$ alone.

Anyway, let $\varphi(\overline{x})$ be an $\mathcal{L}$-formula, and we expand $\mathcal{L}^*$ to add in some new constant symbols $\overline{a}$ and $\overline{b}$. We now relative to build a new theory. The observation is that, using the construction of the previous paragraph, there is a function $\widehat{\cdot}^{\mathcal{A}}$ such that $\mathcal{A} \vDash \varphi(\overline{a})$ if and only if $\mathcal{C} \vDash \varphi^{\mathcal{A}}(\overline{a})$. As such, we adjust T to the theory Σ be an $\mathcal{L}^*$-theory by adjusting cs to c_1s and c_2s in the natural way, and we also add in the sentences $U(a_\bullet)$ and $\neg U(b_\bullet)$. Further, we add in the sentences

$$\{\varphi^{\mathcal{A}}(\overline{a}) \leftrightarrow \psi^{\mathcal{B}}(\overline{b})\}$$

as well as $\widehat{\varphi}^{\mathcal{A}}(\overline{a}) \leftrightarrow \varphi^{\mathcal{B}}(\overline{b})$. This theory is inconsistent by the type discussion at the very beginning of this proof: we are being promised that $\overline{a}$ and $\overline{b}$ have the same type, but then they disagree on φ!

Thus, by compactness, there is a finite set Ψ of quantifier-free formulae with the following property for any models $\mathcal{A}, \mathcal{B} \vDash T$ with $\overline{a} \in A^n$ and $b \in B^n$: if $\mathcal{A} \vDash \psi(\overline{a})$ is equivalent to $\mathcal{B} \vDash \psi(\overline{b})$ for each $\psi \in \Psi$, then we must have $\mathcal{A} \vDash \varphi(\overline{a})$ is equivalent to $\mathcal{B} \vDash \varphi(\overline{b})$. We now construct our quantifier-free formula: for each $X \subseteq \Psi$, we define

$$\theta_X \coloneqq \bigwedge_{\psi \in X} \psi \wedge \bigwedge_{\psi \in \Psi \setminus X} \neg\psi,$$

and we let G be the set of subsets such that there is a model $\mathcal{A} \vDash T$ with $\mathcal{A} \vDash \exists \overline{x}(\varphi(\overline{x}) \wedge \theta_X(\overline{x}))$. Then we set $\eta(\overline{x})$ to be the disjunction over all the θ_X where $X \in G$. Note that $\eta(\overline{x})$ is quantifier-free.

We now claim that $T \vDash \forall \overline{x}(\eta(\overline{x}) \leftrightarrow \varphi(\overline{x}))$. Suppose $\mathcal{A} \vDash T$ and we have some $\overline{a} \in \mathcal{A}$ with $\mathcal{A} \vDash \varphi(\overline{a})$. Then we consider the subset X of Ψ such that $\mathcal{A} \vDash \psi(\overline{a})$ if and only if $\psi \in X$. Then $\mathcal{A}$ is in fact modelling $\varphi(\overline{a})$ along with the sentences $\psi(\overline{a})$ for each $\psi \in X$ and then $\neg\psi(\overline{a})$ for each $\psi \notin X$. Thus, $\mathcal{A} \vDash \theta_X(\overline{a}) \wedge \varphi(\overline{a})$, so $X \in G$, and $T \vDash \forall \overline{x}(\varphi(\overline{x}) \to \eta(\overline{x}))$ follows.

We now go in the other direction. Suppose $\mathcal{A} \vDash T$ is a model, and suppose we have $\overline{a} \in A^n$ and $\mathcal{A} \vDash \eta(\overline{a})$. Then there is some $X \in G$ such that $\mathcal{A} \vDash \theta_X(\overline{a})$, but being in G promises us a model $\mathcal{B} \vDash T$ and $\overline{b} \in B^n$ with $\mathcal{B} \vDash \varphi(\overline{b}) \wedge \theta_X(\overline{b})$. But then any $\psi \in \Psi$ has $\mathcal{A} \vDash \psi(\overline{a})$ if and only if $\mathcal{B} \vDash \psi(\overline{b})$ by definition of θ_X, so $\mathcal{A}$ and $\mathcal{B}$ must agree on $\varphi(\overline{b})$. In other words, we conclude $\mathcal{A} \vDash \varphi(\overline{a})$, and we are done. ■

Corollary 1.100. The theory ACF eliminates quantifiers.

Proof. We show the hypothesis of the above theorem. Given two algebraically closed fields K and L with an isomorphism $f\colon K' \to L'$ where $K' \subseteq K$ and $L' \subseteq L$ are algebraically closed subfields, we need an isomorphism $f^*\colon K^* \to L^*$ extending f. As long as K and L have the same cardinality, we can simply do this with $K = K^*$ and $L = L^*$. In general, with $|K| \leq |L|$, we might need to use a transcendence basis to expand K and take an algebraic closure, and this is an elementary extension because ACF is κ-categorical. ■

Corollary 1.101. The theory of dense linear order without endpoints eliminates quantifiers.

Proof. Use the theorem. ■

Non-Example 1.102. The theory of an equivalence relation with exactly one equivalence class of size each positive integer does not eliminate quantifiers. To see this, consider the sentence which says that a free variable x is in an equivalence class of size 2.

1.11 September 28

Let's talk about some game.

1.11.1 Ehrenfeucht–Fraïssé Games

For today's lecture, let's discuss Ehrenfeucht–Fraïssé Games. Recall the following definition.

Definition 1.103 (unnested)**.** An atomic $\mathcal{L}$-formula φ is *unnested* if and only if it takes one of the following forms.

- Equalities: $t_i = t_j$ or $x_i = c$ where the $t_\bullet$ are variables or constants.
- Relations: $R(t_1, \ldots, t_n)$ where the $t_\bullet$ are variables or constants.
- Functions: $f(t_1, \ldots, t_n) = t_{n+1}$ where the $t_\bullet$ are variables or constants.

For our discussion today, we let U_0 denote the set of finite boolean combinations of unnested atomic formulae, up to provable equivalence (e.g., we don't want to include $\varphi \wedge \varphi$ from φ), and we inductively set U_{n+1} to be finite boolean combinations (again, up to provable equivalence) of formulae of the form $\exists x \psi$ where $\psi \in U_n$ and x is a variable.

Proposition 1.104. Fix a finite language $\mathcal{L}$. Then for each n and m, there are only finitely many formulae in U_n with the variables $x_1, \ldots, x_m$ (up to provable equivalence).

Proof. Fix m, and we induct on n. We start with $n = 0$. For number unnested atomic formulae is finite because the problem is just combinatorics to count sentences of each type. As for the boolean combinations, we note that the boolean algebra generated by a finite set is finite, so there are only finitely many classes up to provable equivalence. Then to go up, we place $\exists x_\bullet$ or not in front of each formula, so there continue to be only finitely many formulae, and the boolean algebra generated continues to be finite, so we are okay. ■

Our observation, now, is that every $\mathcal{L}$-formula is equivalent to some formula in one of the U_n.

Proposition 1.105. Fix a language $\mathcal{L}$. Then any $\mathcal{L}$-formula φ is equivalent to some $\psi \in U_n$ for some n.

Proof. It suffices to check this for atomic formulae; all other formulae follow by adding enough quantifiers and taking boolean combinations. Here are our cases.

- Take sentences of the form $t_1 = t_2$. We now have to induct on the complexity of the terms. If we have an equality of variables $x_i = x_j$ or an equality $x_i = c$ for constant c, there is nothing to say. If we have $c = x_i$, then this is equivalent to the unnested formula $x_i = c$. Lastly, $c = d$ is equivalent to the sentence $\exists x(x = c \wedge x = d)$.

  Now if we have something of the type $t_1 = f(s_1, \ldots, s_n)$, then by induction, we can achieve any of the formulae $x_{n+1} = t_1$ and $x_i = s_i$ for each i where the $x_\bullet$ are variables. So $t_1 = f(s_1, \ldots, s_n)$ is equivalent to
  $$\exists x_1 \cdots \exists x_n \left(\bigwedge_{i=1}^{n} x_i = s_i \wedge x_{n+1} = t_1 \wedge x_{n+1} = f(x_1, \ldots, x_n) \right).$$
  This induction completes this case.

- For relations, one does essentially the same trick. If we have $R(t_1, \ldots, t_n)$, we simply look at the sentences $x_i = t_i$ combined with $R(x_1, \ldots, x_n)$, reducing to the previous case. ■

Now let's play a game. Fix a language $\mathcal{L}$ with two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, and we fix a natural number n. The game $EF_n(\mathcal{A}, \mathcal{B})$ of length n is played as follows.

- Player I picks $\mathcal{A}$ or $\mathcal{B}$ and chooses some $a_1 \in A$ or $b_1 \in B$. Then Player II chooses an element $b_1 \in B$ or $a_1 \in A$ from the opposite universe to the one Player I chose.
- Then the above move is repeated until we have two n-tuples $(a_1, \ldots, a_n)$ or $(b_1, \ldots, b_n)$.
- Player II wins if, for any unnested atomic formula $\psi(x_1, \ldots, x_n)$, we have $\mathcal{A} \vDash \psi(\overline{a})$ is equivalent to $\mathcal{B} \vDash \psi(\overline{b})$. Otherwise, Player I wins.

Roughly speaking, Player I wants to make $\mathcal{A}$ and $\mathcal{B}$ look different, and Player II wants them to look similar. We write $\mathcal{A} \equiv^n \mathcal{B}$ to mean that Player II can win the EF_n game.

Example 1.106. Fix the language $\mathcal{L} = \{<\}$, and take $\mathcal{A}$ to be $\omega + \omega^*$, where the ω^* means we have concatenated ω on top of ω^* but in reverse (so that 0^* is the largest element). We then let $\mathcal{B}$ be the set $\{0, 1, 2, \ldots, 6\}$ for some natural m, and we play the game. Player I can win the game in four moves, but Player II can win in three moves.

Here is our result.

Proposition 1.107. Fix a finite language $\mathcal{L}$. For each n and structures $\mathcal{A}$ and $\mathcal{B}$, Player II has a winning strategy in the $EF_n(\mathcal{A}, \mathcal{B})$ game if and only if $\mathcal{A} \vDash \psi$ is equivalent to $\mathcal{B} \vDash \psi$ for all sentences $\psi \in U_n$.

Proof. We prove this by induction on n, but the inductive hypothesis will allow $\mathcal{A}$ and $\mathcal{B}$ to vary. At $n = 0$, we are asking for $\mathcal{A} \vDash \varphi$ if and only if $\mathcal{B} \vDash \varphi$ where φ is an unnested atomic formula, so Player II wins if and only if this is satisfied.

For our induction, suppose n, and we get $n + 1$. There are two implications to show.

- In one direction, suppose Player II has a winning strategy. Suppose Player I has picked $a_1 \in A$ (without loss of generality). Then Player II responds with some $b_1 \in B$ according to the winning strategy. Now, the rest of the game is a length-n game in the language $\mathcal{L}'$ expanded by a constant symbol c with the structures $\mathcal{A}'$ and $\mathcal{B}'$ have $c^{\mathcal{A}'} = a_1$ and $c^{\mathcal{B}'} = b_1$. So we are now playing $EF_n(\mathcal{A}', \mathcal{B}')$. So Player II has a winning strategy in $EF_{n+1}(\mathcal{A}, \mathcal{B})$ if and only if, for all $a \in A_1$, there exists $b_1 \in B$ such that Player II has a winning strategy in $EF_n(\mathcal{A}', \mathcal{B}')$. Anyway, by the induction, we get $\mathcal{A}' \equiv^n \mathcal{B}'$ in $\mathcal{L}'$.

We now show that $\mathcal{A} \equiv^{n+1} \mathcal{B}$. Thus far we are given that $\mathcal{A}' \vDash \psi$ if and only if $\mathcal{B}' \vDash \psi$ for any $\mathcal{L}'$-sentence $\psi \in U_n$. We now do our check. Fix a sentence $\theta \in U_{n+1}$ of the form $\exists x_1 \varphi$ where $\varphi \in U_n$. Then $\mathcal{A} \vDash \theta$ is equivalent to having some $a_1 \in A$ such that $\mathcal{A} \vDash \varphi(a_1)$. Let b_1 be the resulting choice of Player II. But now using our hypothesis at the beginning of the paragraph, we achieve $\mathcal{A}' \vDash \varphi(c)$, so $\mathcal{B}' \vDash \varphi(c)$, so $\mathcal{B} \vDash \varphi(b_1)$. The reverse implication is similar.

- Conversely, suppose that $\mathcal{A} \vDash \psi$ is equivalent to $\mathcal{B} \vDash \psi$ for all sentences $\psi \in U_n$. We give a winning strategy for Player II. Let's say $a_1 \in A$ is chosen by Player I. Let Ψ be the set of formulae $\psi(x_1) \in U_n$ with at most $(n+1)$ variables such that $\mathcal{A} \vDash \psi(a_1)$, which is a finite set up to provable equivalence by Proposition 1.104. It is important that Ψ is finite because now

$$\mathcal{A} \vDash \exists x_1 \bigwedge_{\psi \in \Psi} \psi(x_1).$$

This formula lives in U_{n+1}, so by hypothesis, we get

$$\mathcal{B} \vDash \exists x_1 \bigwedge_{\psi \in \Psi} \psi(x_1),$$

so we get $b_1 \in B$ satisfying all $\mathcal{B} \vDash \psi(b_1)$ for $\psi \in \Psi$.

Now build $\mathcal{L}'$ and structures $\mathcal{A}'$ and $\mathcal{B}'$ as before. We claim that $\mathcal{A}' \vDash \varphi$ if and only if $\mathcal{B}' \vDash \varphi$ for all $\mathcal{L}'$-sentences $\varphi \in U_n$. Indeed, simply view φ as an $\mathcal{L}$-formula $\widetilde{\varphi}(x)$ by extracting out the constant c and replacing it with c, and we see $\mathcal{A}' \vDash \varphi$ is equivalent to $\mathcal{A} \vDash \widetilde{\varphi}(a_1)$, which is indeed equivalent to $\mathcal{B} \vDash \widetilde{\varphi}(b_1)$.

Now by induction, Player II has a winning strategy in the game $EF_n(\mathcal{A}', \mathcal{B}')$, which is equivalent to winning the original game, as discussed in the previous implication. ■

Corollary 1.108. Fix a language $\mathcal{L}$. Then $\mathcal{A} \equiv \mathcal{B}$ if and only if, for all finite language $\mathcal{L}' \subseteq \mathcal{L}$, we have $\mathcal{A}|_{\mathcal{L}'} \equiv \mathcal{B}|_{\mathcal{L}'}$.

Proof. Play the above game. Note $\mathcal{A} \equiv \mathcal{B}$ if and only if they satisfy the same formulae, which is equivalent to having $\mathcal{A}|_{\mathcal{L}'} \equiv \mathcal{B}|_{\mathcal{L}'}$ for all finite $\mathcal{L}' \subseteq \mathcal{L}$ because any formula will only contain finitely many symbols. Then this is in fact equivalent to satisfying the same $\mathcal{L}'$-sentences in U_n for all n, which finishes by Proposition 1.107. ■

Remark 1.109. Here is a challenge problem: for which m and n does Player II win the game of length n between the groups $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$? There does exist some n such that Player I will always win this game. Approximately speaking, one needs a sentence true in $\mathbb{Z}$ which is false in the $\mathbb{Z}/m\mathbb{Z}$s.

1.12 October 3

Let's play the game to start off the class.

Example 1.110. We work with ordinals in the language $\mathcal{L} = \{<\}$.

- We play with $\varepsilon_0 = \sup\{\omega, \omega^\omega, \ldots\}$ and 2. Then Player II loses after, say, 2 moves: Player I selects anything, Player II selects (say) 0, and then Player I chooses something smaller than what they chose in ε_0.
- We play with ε_0 and ω_1. Then Player II can always win. The point is that there is some kind of finite-length back-and-forth argument

1.12.1 Real Closed Fields

Let's discuss real closed fields because they, in some sense, will tell us that Euclidean geometry is decidable (approximately speaking). Our language will be the language $\mathcal{L} = \{+, -, \cdot, <, 0, 1\}$ of ordered rings. The theory of ordered fields OrdFld is axiomatized by writing the axioms for fields, for a total order, and requiring that addition and multiplication respect this ordering. We won't bother writing down the first two lists of axioms, but the third list is given as follows.

- $\forall a \forall b \forall c((a < b) \to (a + c < b + c))$.
- $\forall a \forall b \forall c(((a < b) \land (c > 0)) \to (a \cdot c < b \cdot c))$.

So we have a finitely axiomatized our theory OrdFld.

Example 1.111. Any subfield of $\mathbb{R}$ will do is a model.

Example 1.112. We can use compactness to provide a model of $\mathbb{R}$ with an element larger than any other element but the same cardinality.

We will be actually be interested in the theory RCF of real closed fields, which is the theory OrdFld plus the intermediate value theorem for polynomials. This is an infinite list of axioms, approximately saying that, for any model $\mathcal{R}$ with universe R, and polynomial $f \in R[x]$ with inputs $a, b \in R$ such that $f(a) < 0 < f(b)$ has some $c \in R$ such that $f(c) = 0$.

To write this out, we choose a degree of n and write down the sentence

$$\begin{aligned} \forall a_0 \cdots \forall a_n \forall a \forall b \big(\, ((a < b) \land \big(a_0 a^0 + \cdots + a_n a^n < 0 < a_0 b^0 + \cdots + a_n b^n\big)\big) \\ \to \exists c \left(a < c < b \land a_0 c^0 + \cdots + a_n c^n = 0 \right) \big). \end{aligned}$$

We cannot finitely axiomatize these sentences using an argument like Lemma 1.83.

Remark 1.113. Any ordered field $(\mathcal{R}, +, -, 0, 1, <)$ has $(\mathcal{R}, <)$ satisfying DLO. We know that we are a linear order, we have no endpoints because $x + -1 < x < x + 1$ for ant $x \in R$, and we are dense because $x < \frac{x+y}{2} < y$ for any $x, t \in R$. Note that checking $x < x + 1$ (for example) requires knowing that $0 < 1$, which is a nontrivial fact on its own (one should use trichotomy and rule our $0 = 1$ by fields and rule our $0 > 1$ because this would imply $-1 > 0$ and then $1 > 0$ by squaring). There are lots of these nontrivial facts (e.g., we also want to know $0 < 1/2 < 1$), but we won't bother to show this.

For ordered fields, there is an order topology, and one can show that various functions like $+$ and $\cdot$ and polynomials are all continuous.

We will define the function $|\cdot| : R \to R$ given by

$$|x| := \begin{cases} +x & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases}$$

Now, if $x < 0$, then $-x > 0$ by subtraction, so we see that $|x| > 0$ for all $x \neq 0$. The standard casework is also able to prove the triangle inequality $|x + y| \leq |x| + |y|$ by some casework. If both nonpositive or nonnegative, then we have equality, and if they have different signs (say, $x > 0 > y$ without loss of generality and $|x| \geq |y|$), then we are looking at $x + y \leq x - y$, which is true.

For notation, we will also want the function sgn given by

$$\operatorname{sgn}(x) := \begin{cases} +1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Now, one is able to check the following, which tells us that polynomials "go off to infinity."

Proposition 1.114. Fix an ordered field $\mathcal{R}$ and a polynomial $f(x) \in R[x]$ of positive degree d, written

$$f(x) = \sum_{i=0}^{d} c_d x^d$$

where $c_d \neq 0$. If

$$x > 1 + \frac{1}{|c_d|} \sum_{i=0}^{d-1} |c_i| \, ,$$

then $\operatorname{sgn} f(x) = \operatorname{sgn} c_d$.

Proof. Boring bounding. Note

$$\operatorname{sgn} f(x) = \operatorname{sgn} c_d \cdot \operatorname{sgn} \left(x^d + \sum_{i=0}^{d-1} \frac{c_i}{c_d} x^d \right),$$

so by scaling down, it is enough to consider the case where $c_d = 1$.

As an aside, we note that any $x \geq 1$ and nonnegative integer n will have $x^n \geq x$, which is true by induction because $x^{n+1} \geq x^n$, where our base case is $x^1 = x \geq 1 = x^0$. With this in mind, we see that x satisfying the desired inequality will have

$$x^d = x \cdot x^{d-1} > \sum_{i=0}^{d-1} |c_i| \, x^{d-1} \geq \sum_{i=0}^{d-1} |c_i| \, x^i \geq \sum_{i=0}^{d-1} -c_i x^i,$$

so $f(x) > 0$ follows. ■

Corollary 1.115. If $\mathcal{R}$ is a real closed field and $a \geq 0$, then there exists $b \geq 0$ such that $b^2 = a$.

Proof. If $a = 0$, set $b = 0$. Otherwise, consider the polynomial $f(x) := x^2 - a$. Note $f(0) < 0$, and Proposition 1.114 tells us that $f(1 + a) > 0$, so the intermediate value theorem for polynomials tells us that there is some b such that $f(b) = 0$, so $b^2 = a$. ■

Corollary 1.116. If $\mathcal{R}$ is a real closed field, then any polynomial $f(x)$ of odd degree has a root.

Proof. Write

$$f(x) = \sum_{i=0}^{d} c_i x^i$$

where d is odd and $c_d \neq 0$, and let $N := 2 + \frac{1}{|c_d|} \sum_{i=0}^{d-1} |c_i|$. By Proposition 1.114, we have N such that $\operatorname{sgn} f(N) = \operatorname{sgn} c_d$, and we see similarly that the polynomial $f(-x)$ will now have $\operatorname{sgn} f(-N) = \operatorname{sgn}(-c_d)$. Thus, $f(N)$ and $f(-N)$ have different signs, so the intermediate value theorem for polynomials grants f a root. ■

The above two corollaries turn out to characterize real closed fields.

Remark 1.117. We can now remove the ordering from our real closed fields by declaring that squares are exactly the nonnegative elements. It is in general an interesting question when we can give a field an order; for example, -1 cannot be a sum of squares because $-1 < 0$. This turns out to be good enough to make a field orderable!

BIBLIOGRAPHY

[Mar02] D. Marker. *Model Theory : An Introduction*. Graduate Texts in Mathematics. Springer New York, 2002. ISBN: 9780387987606. URL: https://books.google.com/books?id=QieAHk--GCcC.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions