256A: Algebraic Geometry

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Theme 1
Sheaf Theory

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 24

A feeling of impending doom overtakes your soul.

1.1.1 Administrative Notes

Here are housekeeping notes.

- Here are some housekeeping notes. There is a syllabus on `bCourses`.
- We hope to cover Chapter II of [Har77], mostly, supplemented with examples from curves.
- There are lots of books.
  - We use [Har77] because it is short.
  - There is also [Vak17], which has more words.
  - The book [Liu06] has notes on curves.
  - There are more books in the syllabus. Professor Tang has some opinions on these.
- Some proofs will be skipped in lecture. Not all of these will appear on homework.
- Some examples will say lots of words, some of which we won't have good definitions for until later. Do not be afraid of words.

Here are assignment notes.

- Homework is 70% of the class.
- Homework is due on noon on Fridays. There will be 6–8 problems, which means it is pretty heavy. The lowest homework score will be dropped.

- Office hours exist. Professor Tang also answers emails.
- The term paper covers the last 30% of the grade. They are intended to be extra but interesting topics we don't cover in this class.

1.1.2 Motivation

We're going to talk about schemes. Why should we care about schemes? The point is that schemes are "correct."

Example 1.1. In algebraic topology, there is a cup product map in homology, which is intended to algebraically measure intersections. However, intersections are hard to quantify when we aren't dealing with, say manifolds.

Here is an example of algebraic geometry helping us with this rigorization.

Theorem 1.2 (Bézout)**.** Let C_1 and C_2 be curves in $\mathbb{P}^2(k)$, for some algebraically closed k, where C_1 and C_2 are defined by homogeneous polynomials f_1 and f_2. Then the "intersection number" between the curves C_1 and C_2 is $(\deg f_1)(\deg f_2)$.

This is a nice result, for example because it automatically accounts for multiplicities, which would be difficult to deal with directly using (say) geometric techniques. Schemes will help us with this.

Example 1.3. Moduli spaces are intended to be geometric objects which represent a family of geometric objects of interest. For example, we might be interested in the moduli space of some class of elliptic curves.

It turns out that the correct way to define these objects is using schemes as a functor; we will see this in this class.

Remark 1.4. One might be interested in when a functor is a scheme. We will not cover this question in this class in full, but it is an interesting question, and we will talk about this in special cases.

1.1.3 Elliptic Curves

For the last piece of motivation, let's talk about elliptic curves, over a field k.

Definition 1.5 (Elliptic curve)**.** An *elliptic curve* over k is a smooth projective curve of genus 1, with a marked k-rational point.

Remember that we said that we not to be afraid of words. However, we should have some notion of what these words mean: being a curve means that we are one-dimensional, being smooth is intuitive, and having genus 1 roughly means that base-changing to a complex manifold has one hole. Lastly, the k-rational point requires defining a scheme as a functor.

Here's another (more concrete) definition of an elliptic curve.

Definition 1.6 (Elliptic curve)**.** An *elliptic curve* over k is an affine variety in $\mathbb{A}^2(k)$ cut out be a polynomial of the form

$$y^2 + a_1xy + a_3y^2 = x^3 + a_2x^2 + a_4x + a_6$$

with nonzero discriminant plus a point $\mathcal{O}$ at infinity.

Remark 1.7. Why are these equivalent? Well, the Riemann–Roch theorem approximately lets us take a smooth projective curve of genus 1 and then write it as an equation; the marked point goes to $\mathcal{O}$. In the reverse direction, one merely needs to embed our affine curve into projective space and verify its smoothness and genus.

Instead of working with affine varieties, we can also give a concrete description of an elliptic curve using projective varieties.

Definition 1.8 (Elliptic curve)**.** An *elliptic curve* over k is a projective variety in $\mathbb{P}^2(k)$ cut out be a polynomial of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with nonzero discriminant.

We get the equivalence of the previous two definitions via the embedding $\mathbb{A}_2(k) \hookrightarrow \mathbb{P}^2(k)$ by $(x, y) \mapsto [x : y : 1]$; the point at infinity $\mathcal{O}$ is $[0 : 1 : 0]$.

1.1.4 Crackpot Varieties

In order to motivate schemes, we should probably mention varieties, so we will spend some time in class discussing affine and projective varieties. For convenience, we work over an algebraically closed field k.

Definition 1.9 (Affine space)**.** Given a field k, we define *affine n-space over k*, denoted $\mathbb{A}^n(k)$. An *affine variety* is a subset $Y \subseteq \mathbb{A}^n(k)$ of the form

$$Y = V(S) := \{p \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } f \in S\},$$

where $S \subseteq k[x_1, \ldots, x_n]$.

Remark 1.10. The set $S \subseteq k[x_1, \ldots, x_n]$ in the above definition need not be finite or countable. In certain cases, we can enforce this condition; for example, if $n = 1$, then $k[x]$ is a principal ideal domain, so we may force $\#S = 1$.

Note that we have defined vanishing sets $V(S)$ from subsets $S \subseteq k[x_1, \ldots, x_n]$. We can also go from vanishing sets to subsets.

Definition 1.11. Fix a field k and subset $Y \subseteq \mathbb{A}^n(k)$. Then we define the ideal

$$I(Y) := \{f \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } p \in Y\}.$$

Remark 1.12. One should check that this is an ideal, but we won't bother.

So we've defined some geometry. But we're in an algebraic geometry class; where's the algebra?

Theorem 1.13 (Hilbert's Nullstellensatz)**.** Fix an algebraically closed field k and ideal $J \subseteq k[x_1, \ldots, x_n]$. Then

$$I(V(J)) = \operatorname{rad} I,$$

where $\operatorname{rad} I$ is the radical of I.

Remark 1.14. The Nullstellensatz has no particularly easy proof.

The point of this result is that it ends up giving us a contravariant equivalence of posets of radical ideals and affine varieties.

Why do we care? In some sense, we prefer to work with ideals because it "remembers" more information than merely the points on the variety. To see this, note that elements $f \in k[x_1, \ldots, x_n]$ we are viewing as giving functions on $\mathbb{A}^n(k)$. However, when we work on a variety $Y \subseteq \mathbb{A}^n(k)$, then sometimes two functions will end up being identical on Y. So the correct ring of functions on Y is

$$k[x_1, \ldots, x_n]/I(Y),$$

so indeed keeping track of the (algebraic) ideal $V(Y)$ gets us some extra (geometric) information.

We will use this discussion as a jumping-off point to discuss affine schemes and then schemes. Affine schemes will have the following data.

- A commutative ring A, which we should think of as the ring of functions on a variety.
- A topological space $\operatorname{Spec} A$, which has more information than merely points on the variety.
- A structure sheaf of functions on $\operatorname{Spec} A$.

Remark 1.15. Our topological space $\operatorname{Spec} A$ will contain more points than just the points on the variety. In some sense, these extra points make the topology more apparent.

Remark 1.16. Going forward, one might hope to remove requirements that the field k is algebraically closed (e.g., to work with a general ring) or talk about ideals which are not radical. This is the point of scheme theory.

1.2 August 26

Let's finish up talking varieties, and then we'll move on to affine schemes.

1.2.1 Projective Varieties

We're going to briefly talk about projective varieties. Let's start with projective space.

Definition 1.17 (Projective space)**.** Given a field k, we define *projective n-space over k*, denoted $\mathbb{P}^n(k)$ as

$$\frac{k^{n+1} \setminus \{(0, \ldots, 0\}}{\sim},$$

where $\sim$ assigns two points being equivalent if and only if they span the same 1-dimensional subspace of k^{n+1}. We will denote the equivalence class of a point $(a_0, \ldots, a_n)$ by $[a_0 : \ldots : a_n]$.

To work with varieties, we don't quite cut out by general polynomials but rather by homogeneous polynomials.

Definition 1.18 (Projective variety)**.** Given a field k and a set of some homogeneous polynomials $T \subseteq k[x_1, \ldots, x_n]$, we define the *projective variety* cut out by T as

$$V(T) := \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in T\}.$$

Example 1.19. The elliptic curve corresponding to the affine algebraic variety in $\mathbb{A}^2(k)$ cut out by $y^2 - x^3 - 1$ becomes the projective variety in $\mathbb{P}^2(k)$ cut out by

$$Y^2Z - X^3 - Z^3 = 0.$$

Remark 1.20. One can give projective varieties some Zariski topology as well, which we will define later in the class.

What to remember about projective varieties is that we can cover $\mathbb{P}^2(k)$ (say) by affine spaces as

$$\begin{aligned}\mathbb{P}^2(k) &= \{[X:Y:Z] : X,Y,Z \in k \text{ not all } 0\} \\ &= \{[X:Y:Z] : X,Y,Z \in k \text{ and } X \neq 0\} \\ &\quad \cup \{[X:Y:Z] : X,Y,Z \in k \text{ and } Y \neq 0\} \\ &= \{[1:y:z] : y,z \in k\} \\ &\quad \cup \{[x:1:z] : x,z \in k\} \\ &\simeq \mathbb{A}^2(k) \cup \mathbb{A}^2(k).\end{aligned}$$

The point is that we can decompose $\mathbb{P}^2(k)$ into an affine cover.

Example 1.21. Continuing from Example 1.19, we can decompose $Z\left(Y^2Z - X^3 - Z^3\right)$ into having an affine open cover by

$$\underbrace{\left\{(x,y) : y^2 - x^3 - 1 = 0\right\}}_{z\neq 0} \cup \underbrace{\left\{(x,z) : z - x^3 - z^3 = 0\right\}}_{y\neq 0} \cup \underbrace{\left\{(y,z) : y^2z - 1 - z^3 = 0\right\}}_{x\neq 0}.$$

Notably, we get almost everything from just one of the affine chunks, and we get the point at infinity by taking one of the other chunks.

Remark 1.22. It is a general fact that we only need two affine chunks to cover our projective curve.

1.2.2 The Spectrum

The definition of a(n affine) scheme requires a topological space and its ring of functions. We will postpone talking about the ring of functions until we discuss sheaves, so for now we will focus on the space.

Definition 1.23 (Spectrum)**.** Given a ring A, we define the *spectrum*

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}.$$

Example 1.24. Fix a field k. Then $\operatorname{Spec} k = \{(0)\}$. Namely, non-isomorphic rings can have homeomorphic spectra.

Exercise 1.25. Fix a field k. We show that

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(\pi) : \pi \text{ is monic, irred.}, \deg \pi > 0\}.$$

Proof. To begin, note that (0) is prime, and (π) is prime for irreducible non-constant polynomials π because irreducible elements are prime in principal ideal domains. Additionally, we note that all the given primes are distinct: of course (0) is distinct from any prime of the form (π), but further, given monic non-constant irreducible polynomials α and β, having

$$(\alpha) = (\beta)$$

forces $\alpha = c\beta$ for some $c \in k[x]^\times$. But $k[x]^\times = k^\times$, so $c \in k^\times$, so $c = 1$ is forced by comparing the leading coefficients of α and β.

It remains to show that all prime ideals $\mathfrak{p} \subseteq k[x]$ take the desired form. Well, $k[x]$ is a principal ideal domain, so we may write $\mathfrak{p} = (\pi)$ for some $\pi \in k[x]$. If $\pi = 0$, then we are done. Otherwise, $\deg \pi \geq 0$, but $\deg \pi > 0$ because $\deg \pi = 0$ implies $\pi \in k[x]^\times$. By adjusting by a unit, we may also assume that π is monic. And lastly, note that (π) is prime means that π is prime, so π is irreducible. ■

Example 1.26. If k is an algebraically closed field, then the only nonconstant irreducible polynomials are linear (because all nonconstant polynomials have a root and hence a linear factor), and of course any linear polynomial is irreducible. Thus,

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(x-\alpha) : \alpha \in k\}.$$

Set $\mathfrak{m}_\alpha := (x-\alpha)$ so that $\alpha \mapsto \mathfrak{m}_\alpha$ provides a natural map from $\mathbb{A}^1_k$ to $\operatorname{Spec} k[x]$. In this way we can think of $\operatorname{Spec} k[x]$ as $\mathbb{A}^1_k$ with an extra point (0).

Remark 1.27. Continuing from Example 1.26, observe that we can also recover function evaluation at a point $\alpha \in \mathbb{A}^1_k$: given $f \in k[x]$, the value of $f(\alpha)$ is the image of f under the canonical map

$$k[x] \twoheadrightarrow \frac{k[x]}{\mathfrak{m}_\alpha} \cong k,$$

where the last map is the forced $x \mapsto \alpha$. Observe running this construction at the point $(0) \in \operatorname{Spec} k[x]$ makes the "evaluation" map just the identity.

Example 1.28. Similar to $k[x]$, we can classify $\operatorname{Spec} \mathbb{Z}$: all ideals are principal, so our primes look like (p) where $p = 0$ or is a rational prime. Namely, essentially the same proof gives

$$\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) : p \text{ prime}, p > 0\}.$$

The condition $p > 0$ is to ensure that all the points on the right-hand side are distinct; certainly we can write all nonzero primes $(p) \subseteq \mathbb{Z}$ for some nonzero (p), and we can adjust p by a unit to ensure $p > 0$. Conversely, $(p) = (q)$ with $p, q > 0$ forces $p \mid q$ and $q \mid p$ and so $p = q$.

We might hope to have a way to view $\operatorname{Spec} k[x]$ as points even when k is not algebraically closed.

Example 1.29. Set $k = \mathbb{Q}$. There is a map sending a nonconstant monic irreducible polynomial $\pi \in \mathbb{Q}[x]$ to its roots in $\overline{\mathbb{Q}}$, and note that this map is injective because one can recover a polynomial from its roots. Further, all the roots of π are Galois conjugate because π is irreducible, and a Galois orbit S_α of a root α corresponds to the polynomial

$$\pi(x) = \prod_{\beta \in S_\alpha} (x - \beta),$$

where $\pi(x) \in \mathbb{Q}[x]$ because its coefficients are preserved the Galois action. Thus, there is a bijection between the nonconstant monic irreducible polynomials $\pi \in \mathbb{Q}[x]$ and Galois orbits of elements in $\overline{\mathbb{Q}}$.

So far, all of our examples have been "dimension 0" (namely, a field k) or "dimension 1" (namely, $\mathbb{Z}$ and $k[x]$). Here is an example in dimension 2.

Exercise 1.30. Let k be algebraically closed. Any $\mathfrak{p} \in \operatorname{Spec} k[x,y]$ is one of the following types of prime.

- Dimension 2: $\mathfrak{p} = (0)$.
- Dimension 1: $\mathfrak{p} = (f(x,y))$ where f is nonconstant and irreducible.
- Dimension 0: $\mathfrak{p} = (x-\alpha, y-\beta)$, where $\alpha, \beta \in k$.

Proof. We follow [Vak17, Exercise 3.2.E]. If $\mathfrak{p} = (0)$, then we are done. If $\mathfrak{p}$ is principal, then we can write $\mathfrak{p} = (f)$ where $f \in k[x, y]$ is a prime element and hence irreducible. Observe that if f is irreducible, then f is also a prime element because $k[x, y]$ is a unique factorization domain.

Lastly, we suppose that $\mathfrak{p}$ is not principal. We start by finding $f, g \in \mathfrak{p}$ with no nonconstant common factors. Because $\mathfrak{p} \neq 0$, we can find $f_0 \in \mathfrak{p} \setminus \{0\}$, and assume that (f_0) is maximal with respect to this (namely, $f_0 \notin (f_0')$ for any $f_0' \in \mathfrak{p}$). Because $\mathfrak{p}$ is not principal, we can find $g_0 \in \mathfrak{p} \setminus (f_0)$. Now, we can use unique prime factorization of f_0 and g_0 to find some $d \in k[x, y]$ such that

$$f_0 = fd \qquad \text{and} \qquad g_0 = gd$$

where f and g share no common factors. (Namely, $\nu_\pi(d) = \min\{\nu_\pi(f_0), \nu_\pi(g_0)\}$ for all irreducible factors $\pi \in k[x, y]$.) Note $d \notin \mathfrak{p}$ by the maximality of f_0, so $f, g \in \mathfrak{p}$ is forced.

Continuing, embedding f and g into $k(x)[y]$ and using the Euclidean algorithm there, we can write

$$af + bg = 1$$

where $a, b \in k(x)[y]$, because f and g have no common factors in $k(x)[y]$. (Any common factor would lift to a common factor in $k[x, y]$.[1]) Clearing denominators, we see that we can find $h(x) \in k[x] \cap \mathfrak{p}$, but by factoring $h(x)$ using the fact that k is algebraically closed, we see that we can actually enforce $(x - \alpha) \in \mathfrak{p}$ for some $\alpha \in k$.

By symmetry, we can force $(y - \beta) \in \mathfrak{p}$ for some $\beta \in \mathfrak{p}$ as well, so $(x - \alpha, y - \beta) \subseteq \mathfrak{p}$. However, we see that $(x - \alpha, y - \beta)$ is maximal because of the isomorphism

$$\frac{k[x, y]}{(x - \alpha, y - \beta)} \to k$$

by $x \mapsto \alpha$ and $y \mapsto \beta$. Thus, $\mathfrak{p} = (x - \alpha, y - \beta)$ follows. ■

Remark 1.31. The intuition behind Exercise 1.30 is that the prime ideal $(x - \alpha, y - \beta)$ "cuts out" the zero-dimensional point $(\alpha, \beta) \in \mathbb{A}_k^2$. Then the prime ideal (f) cuts out some one-dimensional curve in $\mathbb{A}_k^2$, and the prime ideal (0) cuts out the entire two-dimensional plane. We have not defined dimension rigorously, but hopefully the idea is clear.

Remark 1.32. It is remarkable that the number of equations we need to cut out a variety of dimension d is $2 - d$. This is not always true.

The point is that we seem to have recovered $\mathbb{A}_k^1$ by looking at $\operatorname{Spec} k[x]$ and $\mathbb{A}_k^2$ by looking at $\operatorname{Spec} k[x, y]$, so we can generalize this to arbitrary rings cleanly, realizing some part of Remark 1.16.

Definition 1.33 (Affine space)**.** Given a ring R, we define *affine n-space over R* as

$$\mathbb{A}_R^n \coloneqq \operatorname{Spec} R[x_1, \ldots, x_n].$$

So far all the rings we've looked at so far have been integral domains, but it's worth pointing out that working with general rings allows more interesting information.

Example 1.34. We classify $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$. Notably, all prime ideals here must correspond to prime ideals of $k[\varepsilon]$ containing $\left(\varepsilon^2\right)$ and hence contain $\operatorname{rad}\left(\varepsilon^2\right) = (\varepsilon)$, which allows only the prime (ε). (We will make this correspondence precise later.) So $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ has a single point.

[1] If $d(x, y)/e(x)$ divides both f and g in $k(x)[y]$, where d and e share no common factors, then $d \mid fe, ge$ in $k[x, y]$. Unique prime factorization now forces $d \mid f, g$ in $k[x, y]$.

Remark 1.35. In some sense, $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ will be able to let us talk about differential information algebraically: ε is some very small nonzero element such that $\varepsilon^2 = 0$. So we can study a "function" $f \in k[x]$ locally at a point p by studying $f(p+\varepsilon)$. Rigorously, $f(x) = \sum_{i=0}^d a_i x^i$ has

$$f(x+\varepsilon) = \sum_{i=0}^d a_i (x+\varepsilon)^i = \sum_{i=0}^d a_i x^i + \sum_{i=1}^d i a_i x^{i-1}\varepsilon = f(x) + f'(x)\varepsilon.$$

One can recover more differential information by looking at $k[\varepsilon]/\left(\varepsilon^n\right)$ for larger n.

1.2.3 The Zariski Topology

Thus far we've defined our space. Here's our topology.

Definition 1.36 (Zariski topology). Fix a ring A. Then, for $S \subseteq A$, we define the *vanishing set*

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\}$$

Then the *Zariski topology* on $\operatorname{Spec} A$ is the topology whose closed sets are the $V(S)$.

Intuitively, we are declaring A as the (continuous) functions on $\operatorname{Spec} A$, and the evaluation of the function $f \in A$ at the point $\mathfrak{p} \in \operatorname{Spec} A$ is $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27). Then the vanishing sets of a continuous function must be closed, and without easy access to any other functions on $\operatorname{Spec} A$, we will simply declare that these are all of our closed sets.

In the affine case, we can be a little more rigorous.

Example 1.37. Set $A := k[x_1, \ldots, x_n]$, where k is algebraically closed. Then, given $f \in k[x_1, \ldots, x_n]$, we want to be convinced that $V(\{f\})$ matches up with the affine k-points $(a_1, \ldots, a_n)$ which vanish on f. Well, $(a_1, \ldots, a_n)$ corresponds to the prime ideal $(x_1 - a_1, \ldots, x_n - a_n) \in \operatorname{Spec} A$, and

$$\{f\} \subseteq (x_1 - a_1, \ldots, x_n - a_n)$$

is equivalent to f vanishing in the evaluation map

$$k[x_1, \ldots, x_n] \twoheadrightarrow \frac{k[x_1, \ldots, x_n]}{(x_1 - a_1, \ldots, x_n - a_n)} \to k,$$

which is equivalent to $f(a_1, \ldots, a_n) = 0$. So indeed, f vanishes on $(a_1, \ldots, a_n)$ if and only if the corresponding maximal ideal is in $V(\{f\})$.

With intuition out of the way, we should probably check that the sets $V(S)$ make a legitimate topology. To begin, here are some basic properties.

Lemma 1.38. Fix a ring A.

(a) If subsets $S, T \subseteq A$ have $S \subseteq T$, then $V(T) \subseteq V(S)$.

(b) A subset $S \subseteq A$ has $V(S) = V((S))$.

(c) An ideal $\mathfrak{a} \subseteq A$ has $V(\mathfrak{a}) = V(\operatorname{rad} \mathfrak{a})$.

Proof. We go in sequence.

(a) Note $\mathfrak{p} \in V(T)$ implies that $T \subseteq \mathfrak{p}$, which implies $S \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(S)$.

(b) Surely $S \subseteq (S)$, so $V((S)) \subseteq V(S)$. Conversely, if $\mathfrak{p} \in V(S)$, then $S \subseteq \mathfrak{p}$, but then the generated ideal (S) must also be contained in $\mathfrak{p}$, so $\mathfrak{p} \in V((S))$.

(c) Surely $\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a}$, so $V(\operatorname{rad} \mathfrak{a}) \subseteq V(I)$. Conversely, if $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{p} \subseteq \mathfrak{a}$, so

$$\mathfrak{p} \subseteq \bigcap_{\mathfrak{q} \supseteq \mathfrak{a}} \mathfrak{q} = \operatorname{rad} \mathfrak{a},$$

so $\mathfrak{p} \in V(\operatorname{rad} \mathfrak{a})$. ■

Remark 1.39. In light of (b) and (c) of Lemma 1.38, we can actually write all closed subsets of $\operatorname{Spec} A$ as $V(\mathfrak{a})$ for a radical ideal $\mathfrak{a}$. We will use this fact freely.

And here are our checks.

Lemma 1.40. Fix a ring A.

(a) $V(A) = \varnothing$ and $V((0)) = \operatorname{Spec} A$.

(b) Given ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$, then $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(c) Given a collection of ideals $\mathcal{I} \subseteq \mathcal{P}(A)$, we have

$$\bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a}) = V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right).$$

Proof. We go in sequence.

(a) All primes are proper, so no prime $\mathfrak{p}$ has $A \subseteq \mathfrak{p}$, so $V(A) = \varnothing$. Also, 0 is an element of all ideals, so all $\mathfrak{p} \in \operatorname{Spec} A$ have $(0) \subseteq \mathfrak{p}$, so $V((0)) = \operatorname{Spec} A$.

(b) Note $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$, so $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ follows. Conversely, take $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, and suppose $\mathfrak{p} \notin V(\mathfrak{a})$ so that we need $\mathfrak{p} \in V(\mathfrak{b})$. Well, $\mathfrak{p} \notin V(\mathfrak{a})$ implies $\mathfrak{a} \not\subseteq \mathfrak{p}$, so we can find $a \in \mathfrak{a} \setminus \mathfrak{p}$. Now, for any $b \in \mathfrak{b}$, we see

$$ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p},$$

so $a \notin \mathfrak{p}$ forces $b \in \mathfrak{p}$. Thus, $\mathfrak{b} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{b})$.

(c) Certainly any $\mathfrak{b} \in \mathcal{I}$ has $\mathfrak{b} \subseteq \sum_{\mathfrak{a} \in \mathcal{I}} I$, so $V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right) \subseteq \bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a})$ follows.

Conversely, suppose $\mathfrak{p} \in \bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a})$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{a} \in \mathcal{I}$, so $\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a} \subseteq \mathfrak{p}$ follows. Thus, $\mathfrak{p} \in V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right)$. ■

Remark 1.41. For ideals $I, J \subseteq A$, note that $IJ \subseteq I \cap J$. Additionally, $I \cap J \subseteq \operatorname{rad}(IJ)$: if $f \in I \cap J$, then $f^2 \in (I \cap J)^2 \subseteq IJ$. It follows from Lemma 1.38 that

$$V(IJ) \supseteq V(I \cap J) \supseteq V(\operatorname{rad}(IJ)) = V(IJ),$$

so $V(I) \cup V(J) = V(IJ) = V(I \cap J)$. So V does respect some poset structure.

It follows that the collection of vanishing sets is closed under finite union and arbitrary intersection, so they do indeed specify the closed sets of a topology.

1.2.4 Easy Nullstellensatz

While we're here, let's also generalize Definition 1.11 in the paradigm that $\operatorname{Spec} A$ is the analogue for affine space.

Definition 1.42. Fix a ring A. Then, given a subset $Y \subseteq \operatorname{Spec} A$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Remark 1.43. To see that this is the correct definition, note we want $f \in I(Y)$ if and only if f vanishes at all points $\mathfrak{p} \in Y$. We said earlier that the value of f at $\mathfrak{p}$ should be $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27), so f vanishes at $\mathfrak{p}$ if and only if $f \in \mathfrak{p}$. So we want

$$I(Y) = \{f \in A : f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in Y\} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

As before, we'll write in a few basic properties of I.

Lemma 1.44. Fix a ring A, and fix subsets $X, Y \subseteq \operatorname{Spec} A$.

(a) If $X \subseteq Y$, then $I(Y) \subseteq I(X)$.

(b) The ideal $I(X)$ is radical.

Proof. We go in sequence.

(a) Note
$$I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = I(X).$$

(b) Suppose that $f^n \in I(X)$ for some positive integer n, and we need to show $f \in I(X)$. Then $f^n \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, so $f \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, so $f \in I(X)$. ■

And here is our nice version of Theorem 1.13.

Proposition 1.45. Fix a ring A.

(a) Given an ideal $\mathfrak{a} \subseteq A$, we have $I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$.

(b) Given a subset $X \subseteq \operatorname{Spec} A$, we have $V(I(X)) = \overline{X}$.

(c) The functions V and I provide an inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. We go in sequence.

(a) Observe
$$I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} = \operatorname{rad} \mathfrak{a}.$$

(b) Using Lemma 1.40, we find
$$\overline{X} = \bigcap_{V(\mathfrak{a}) \supseteq X} V(\mathfrak{a}) = V\left(\sum_{V(\mathfrak{a}) \supseteq X} \mathfrak{a}\right).$$

Now, $X \subseteq V(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in X$, which is equivalent to $\mathfrak{a} \subseteq I(X)$. Thus,

$$\overline{X} = V\left(\sum_{\mathfrak{a}\subseteq I(X)} \mathfrak{a}\right) = V(I(X)).$$

(c) Note that V sends (radical) ideals to closed subsets of $\operatorname{Spec} A$ by the definition of the Zariski topology. Also, I sends (closed) subsets of $\operatorname{Spec} A$ to radical ideals by Lemma 1.44. Additionally, for a closed subset $X \subseteq \operatorname{Spec} A$, we have

$$V(I(X)) = \overline{X} = X,$$

and for a radical ideal $\mathfrak{a}$, we have

$$I(V(\mathfrak{a})) = \operatorname{rad}\mathfrak{a} = \mathfrak{a},$$

so I and V are in fact mutually inverse. ■

Remark 1.46. Given $X \subseteq \operatorname{Spec} A$, we claim $I(X) = I(\overline{X})$. Well, these are both radical ideals, so it suffices by Proposition 1.45 (c) to show $V(I(X)) = V(I(\overline{X}))$, which is clear because these are both $\overline{X}$.

Remark 1.47. Intuitively, what makes proving Proposition 1.45 so much easier than Theorem 1.13 is that we've added extra points to our space in order to track varieties better.

1.2.5 Some Continuous Maps

As a general rule, we will make continuous maps between our spectra by using ring homomorphisms. Here is the statement.

Lemma 1.48. Given a ring homomorphism $\varphi\colon A \to B$, the pre-image function $\varphi^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$ induces a continuous function $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$.

Proof. We begin by showing $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is well-defined: given a prime $\mathfrak{q} \subseteq \operatorname{Spec} B$, we claim that $\varphi^{-1}\mathfrak{q}$ is a prime in $\operatorname{Spec} A$. Well, if $ab \in \varphi^{-1}\mathfrak{q}$, then $\varphi(a)\varphi(b) \in \mathfrak{q}$, so $\varphi(a) \in \mathfrak{q}$ or $\varphi(b) \in \mathfrak{q}$. So indeed, $\varphi^{-1}\mathfrak{q}$ is prime.

We now show that $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous. It suffices to show that the pre-image of a closed set $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ under φ^{-1} is a closed set. For concreteness, we will make $\operatorname{Spec}\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ our pre-image map so that we want to show $(\operatorname{Spec}\varphi)^{-1}(V(\mathfrak{a}))$ is closed. Well,

$$\begin{aligned}(\operatorname{Spec}\varphi)^{-1}(V(\mathfrak{a})) &= \{\mathfrak{q} \in \operatorname{Spec} B : (\operatorname{Spec}\varphi)(\mathfrak{q}) \in V(\mathfrak{a})\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{a} \subseteq (\operatorname{Spec}\varphi)(\mathfrak{q})\} \\ &= \left\{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}\right\}.\end{aligned}$$

Now, if $\mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}$, then any $a \in \mathfrak{a}$ has $\varphi(a) \in \mathfrak{q}$, so $\varphi(\mathfrak{a}) \subseteq \mathfrak{q}$. Conversely, if $\varphi(\mathfrak{a}) \subseteq \mathfrak{q}$, then any $a \in \mathfrak{a}$ has $\varphi(a) \in \mathfrak{q}$ and hence $a \in \varphi^{-1}\mathfrak{q}$, so $\mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}$ follows. In total, we see

$$(\operatorname{Spec}\varphi)^{-1}(V(\mathfrak{a})) = \{\mathfrak{q} \in \operatorname{Spec} B : \varphi(\mathfrak{a}) \subseteq \mathfrak{q}\} = V(\varphi(\mathfrak{a})),$$

which is closed. ■

In fact, we have defined a (contravariant) functor.

Proposition 1.49. The mapping Spec sending rings A to topological spaces $\operatorname{Spec} A$ and ring homomorphisms $\varphi\colon A \to B$ to continuous maps $\operatorname{Spec}\varphi = \varphi^{-1}$ assembles into a functor $\operatorname{Spec}\colon \operatorname{Ring}^{\mathrm{op}} \to \operatorname{Top}$.

Proof. Thus far our data is sending objects to objects and morphisms to (flipped) morphisms, so we just need to run the functoriality checks.

- Identity: note that $\operatorname{Spec}\operatorname{id}_A$ sends a prime $\mathfrak{p} \in \operatorname{Spec} A$ to
$$(\operatorname{Spec}\operatorname{id}_A)(\mathfrak{p}) = \operatorname{id}_A^{-1}(\mathfrak{p}) = \{a \in A : \operatorname{id}_A a \in \mathfrak{p}\} = \mathfrak{p},$$
so indeed, $\operatorname{Spec}\operatorname{id}_A = \operatorname{id}_{\operatorname{Spec} A}$.
- Functoriality: given morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, as well as a prime $\mathfrak{r} \in \operatorname{Spec} C$, we compute
$$\begin{aligned}(\operatorname{Spec}(\psi\circ\varphi))(\mathfrak{r}) &= (\psi\circ\varphi)^{-1}(\mathfrak{r}) \\ &= \{a \in A : \psi(\varphi(a)) \in \mathfrak{r}\} \\ &= \{a \in A : \varphi(a) \in (\operatorname{Spec}\psi)(\mathfrak{r})\} \\ &= \{a \in A : a \in (\operatorname{Spec}\varphi)((\operatorname{Spec}\psi)(\mathfrak{r}))\} \\ &= (\operatorname{Spec}\varphi \circ \operatorname{Spec}\psi)(\mathfrak{r}).\end{aligned}$$
So indeed, $\operatorname{Spec}(\psi\circ\varphi) = \operatorname{Spec}\varphi\circ\operatorname{Spec}\psi$. ■

Here is a quick example.

Definition 1.50 (k-points)**.** Given a ring A and field k, a *k-point* of $\operatorname{Spec} A$ is a ring homomorphism $\iota\colon A \to k$.

Remark 1.51. To see that Definition 1.50 does indeed cut out a single point, note $\iota\colon A \to k$ induces $\operatorname{Spec}\iota\colon \operatorname{Spec} k \to \operatorname{Spec} A$ and therefore picks out a single point of $\operatorname{Spec} A$ because $\operatorname{Spec} k = \{(0)\}$.

Remark 1.52. To see that Definition 1.50 is reasonable, let $A = k[x_1, \ldots, x_n]$ so that $\operatorname{Spec} A = \mathbb{A}_k^n$. Then a map $\iota\colon A \to k$ is determined by $a_i := \iota(x_i)$, so we expect this ι to correspond to the point $(a_1, \ldots, a_n)$. Indeed, Remark 1.51 says we should compute
$$(\operatorname{Spec}\iota)((0)) = \iota^{-1}((0)) = \ker\iota = (x_1 - a_1, \ldots, x_n - a_n),$$
which does indeed correspond to the point $(a_1, \ldots, a_n)$.

Here is a more elaborate example: closed subsets can be realized as spectra themselves!

Exercise 1.53. Fix a ring A and ideal $\mathfrak{a} \subseteq A$. Letting $\pi\colon A \twoheadrightarrow A/\mathfrak{a}$ be the natural projection, we have that
$$\operatorname{Spec}\pi\colon \operatorname{Spec} A/\mathfrak{a} \to V(\mathfrak{a})$$
is a homeomorphism.

Proof. To be more explicit, we claim that the maps
$$\begin{aligned}\operatorname{Spec} A/\mathfrak{a} &\cong V(\mathfrak{a}) \\ \mathfrak{q} &\mapsto \pi^{-1}\mathfrak{q} \\ \pi(\mathfrak{p}) &\leftarrow\!\shortmid \mathfrak{p}\end{aligned}$$
are continuous inverses. Here are our well-definedness and continuity checks.

- That $\mathfrak{q} \mapsto \pi^{-1}\mathfrak{q}$ is continuous follows from Lemma 1.48. Note $\pi^{-1}\mathfrak{q}$ contains $\mathfrak{a}$ because any $a \in \mathfrak{q}$ has $\pi(a) = [0]_\mathfrak{a} \in \mathfrak{q}$.
- For any $\mathfrak{p}$ containing $\mathfrak{a}$, we need to show that $\pi(\mathfrak{p})$ is prime. Of course, if $\mathfrak{p}$ is proper, then $\pi(\mathfrak{p})$ is proper as well. For the primality check, note $[a]_\mathfrak{a} \cdot [b]_\mathfrak{a} \in \pi(\mathfrak{p})$ implies $ab \in \mathfrak{p} + \mathfrak{a} = \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, so $[a]_\mathfrak{a} \in \mathfrak{p}$ or $[b]_\mathfrak{a} \in \mathfrak{p}$.
- To show that $\mathfrak{p} \mapsto \pi\mathfrak{p}$ is continuous, note that a closed set $V(\overline{S}) \subseteq \operatorname{Spec} A/\mathfrak{a}$ has pre-image
$$\pi^{-1}(V(\overline{S})) = \{\mathfrak{p} : \pi\mathfrak{p} \supseteq \overline{S}\}.$$
Now, set $S = \pi^{-1}(\overline{S})$. Now $\pi\mathfrak{p} \supseteq \overline{S}$ if and only if each $a \in S$ has $\pi(a) \in \pi\mathfrak{p}$, which is equivalent to $a \in \mathfrak{a} + \mathfrak{p} = \mathfrak{p}$. Thus,
$$\pi^{-1}(V(\overline{S})) = V(S),$$
which is closed.

Here are our inverse checks.

- Given $\mathfrak{p} \in V(\mathfrak{a})$, note
$$\pi^{-1}(\pi\mathfrak{p}) = \{a \in A : \pi(a) \in \pi\mathfrak{p}\} = \{a \in A : a \in \mathfrak{a} + \mathfrak{p}\} = \mathfrak{a} + \mathfrak{p} = \mathfrak{p}.$$
- Given $\mathfrak{q} \in \operatorname{Spec} A/\mathfrak{a}$, note
$$\pi\left(\pi^{-1}\mathfrak{q}\right) = \pi\left(\{a \in A : \pi(a) \in \mathfrak{q}\}\right).$$
Because $\pi\colon A \twoheadrightarrow A/\mathfrak{a}$ is surjective, the output here is just $\mathfrak{q}$. ■

A similar story exists for open sets, but we must be more careful. Here are our open sets.

Definition 1.54 (Distinguished open sets). Given a ring A and element $f \in A$, we define the *distinguished open set*
$$D(f) := (\operatorname{Spec} A) \setminus V(\{f\}) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}.$$

Intuitively, these are the points on which f does not vanish.

Remark 1.55. In fact, the distinguished open sets form a base: any open set takes the form $(\operatorname{Spec} A) \setminus V(S)$ for some $S \subseteq A$, so we write
$$(\operatorname{Spec} A) \setminus V(S) = \{\mathfrak{p} : S \not\subseteq \mathfrak{p}\} = \bigcup_{f \in S} \{\mathfrak{p} : f \notin \mathfrak{p}\} = \bigcup_{f \in S} D(f).$$

Remark 1.56. The distinguished open base is good in that $D(f) \cap D(g) = \{\mathfrak{p} : f \notin \mathfrak{p}, g \notin \mathfrak{p}\} = \{\mathfrak{p} : fg \notin \mathfrak{p}\} = D(fg)$.

Here is our statement.

Exercise 1.57. Fix a ring A and element $f \in A$. Letting $\iota\colon A \to A_f$ be the localization map,
$$\operatorname{Spec} \iota\colon \operatorname{Spec} A_f \to D(f)$$
is a homeomorphism.

Proof. The arguments here are analogous to Exercise 1.53. To be explicit, we will say that our maps are
$$\begin{array}{rrcl} & \operatorname{Spec} A_f & \cong & D(f) \\ \varphi\colon & \mathfrak{P} & \mapsto & \iota^{-1}\mathfrak{P} \\ \psi\colon & \mathfrak{p}A_f & \leftarrow\!\shortmid & \mathfrak{p} \end{array}$$
for which it remains to run the various checks.

- We show φ is well-defined. Namely, we need to show that $\iota^{-1}\mathfrak{P}$ does not contain f for any $\mathfrak{P} \in \operatorname{Spec} A_f$. Well, if $f \in \iota^{-1}\mathfrak{P}$, then $f/1 \in \mathfrak{P}$, but $f/1 \in A_f^\times$, so this would violate $\mathfrak{P}$ being a proper ideal.
- We show ψ is well-defined. More formally, we have
$$\mathfrak{p}A_f = \{a/f^n : a \in \mathfrak{p}, n \in \mathbb{N}\}.$$
Quickly, if $a/f^k \cdot b/f^\ell \in \mathfrak{p}$, then $(ab)/f^{k+\ell} \in \mathfrak{p}$, so there exists $c \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that
$$\frac{ab}{f^{k+\ell}} = \frac{c}{f^n}.$$
Clearing denominators, there is some $N, M \in \mathbb{N}$ such that $f^N ab = f^M c \in \mathfrak{p}$, but $f \notin \mathfrak{p}$ forces $ab \in \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. It follows $a/f^k \in \mathfrak{p}A_f$ or $b/f^\ell \in \mathfrak{p}A_f$.
We should also check that $\mathfrak{p}A_f$ is proper. Indeed, if $1/1 \in \mathfrak{p}A_f$, there exists $a \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that $a/f^n = 1/1$, so there exists $N, M \in \mathbb{N}$ such that $f^N a = f^M$, which is a contradiction because $f^N a \in \mathfrak{p}$ while $f^N \notin \mathfrak{p}$.
- Note φ is continuous by Lemma 1.48.
- We show ψ is continuous. It suffices to check this on the distinguished base; for $a/f^m \in A_f$, we need to compute $\psi^{-1}(D(a/f^m))$. Well,
$$\psi^{-1}(D(a/f^m)) = \{\mathfrak{p} \in D(f) : a/f^m \notin \mathfrak{p}A_f\}.$$
Now, $a/f^m \in \mathfrak{p}A_f$ means there is $b \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that $a/f^m = b/f^n$, so clearing denominators promises $N, M \in \mathbb{N}$ such that
$$f^N a = f^M b \in \mathfrak{p},$$
so $a \in \mathfrak{p}$ follows. Conversely, $a \in \mathfrak{p}$ of course $a/f^m \in \mathfrak{p}A_F$, so we see that
$$\psi^{-1}(D(a/f^m)) = \{\mathfrak{p} \in D(f) : a/f^m \notin \mathfrak{p}A_f\} = \{\mathfrak{p} \in D(f) : a \notin \mathfrak{p}\} = D(f) \cap D(a)$$
is certainly open in $D(f) \subseteq \operatorname{Spec} A$.

And here our are inverse checks.

- We show $\psi \circ \varphi$ is the identity. Namely, given $\mathfrak{P} \in \operatorname{Spec} A_f$, we have to show that $(\iota^{-1}\mathfrak{P})A_f = \mathfrak{P}$.
In one direction, elements in $(\iota^{-1}\mathfrak{P})A_f$ take the form a/f^n where $a \in \iota^{-1}\mathfrak{P}$, which is equivalent to being in the form a/f^n where $a/1 \in \mathfrak{P}$, from which $a/f^n \in \mathfrak{P}$ certainly follows.
In the other direction, pick up some $a/f^n \in \mathfrak{P}$. Then $a/1 \in \mathfrak{P}$, so $a \in \iota^{-1}\mathfrak{P}$, so $a/f^n \in (\iota^{-1}\mathfrak{P})A_f$.
- We show $\varphi \circ \psi$ is the identity. Namely, given $\mathfrak{p} \in D(f)$, we have to show that $\iota^{-1}(\mathfrak{p}A_f) = \mathfrak{p}$. In one direction, if $a \in \mathfrak{p}$, then $a/1 \in \mathfrak{p}A_f$, so $a \in \iota^{-1}(\mathfrak{p}A_f)$.
In the other direction, if $a \in \iota^{-1}(\mathfrak{p}A_f)$, then $a/1 \in \mathfrak{p}A_f$. Then there exists $b \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that $a/1 = b/f^n$, so clearing denominators promises $N, M \in \mathbb{N}$ such that
$$f^N a = f^M b \in \mathfrak{p},$$
so $f \notin \mathfrak{p}$ forces $a \in \mathfrak{p}$. ■

Remark 1.58. Not every open set is a distinguished open set. For example, taking k algebraically closed,
$$\mathbb{A}_k^2 \setminus \{(0,0)\} \subseteq \mathbb{A}_k^2$$
is an open set not in the form $D(f)$; equivalently, we need to show $V(\{f\}) \neq \{(x,y)\}$ for any $f \in k[x,y]$. Intuitively, this is impossible because a curve cuts out a one-dimensional variety of $\mathbb{A}_k^2$, not a zero-dimensional point.

Rigorously, we are requiring $f \in k[x,y]$ to have $f \in \mathfrak{p}$ if and only if $\mathfrak{p} = (x,y)$. However, f is certainly nonzero and nonconstant, so f has an irreducible factor π, which means that $f \in (\pi)$, where (π) is prime because $k[x,y]$ is a unique factorization domain.

1.3 August 29

Today we talk about the structure sheaf. To review, so far we have defined the spectrum $\operatorname{Spec} A$ of a ring A and given it a topology. The goal for today is to define its structure sheaf. Here is a motivating example.

Example 1.59. Set $A := \mathbb{C}[x_1, \ldots, x_n]$ so that $\operatorname{Spec} A = \mathbb{A}^n_k$. Recall that $\{D(f)\}_{f \in A}$ is a base for the Zariski topology, and we would like the functions on this ring to be A_f, the rational polynomials which allow some f in the denominator. In other words, these are rational functions on $\mathbb{C}^n$ whose poles are allowed on $V(\{f\})$ only.

1.3.1 Sheaves

Sheaves are largely a topological object, so we will forget that we are interested in the Zariski topology for now. Throughout, X will be a topological space.

Notation 1.60. Given a topological space X, we let $\operatorname{Op} X$ denote the poset (category) of its open sets.

Namely, the objects of $\operatorname{Ob} X$ are open sets, and

$$\operatorname{Mor}(V, U) = \begin{cases} \{*\} & V \subseteq U, \\ \varnothing & \text{else}. \end{cases}$$

Here is our definition.

Definition 1.61 (Presheaf)**.** A *presheaf* $\mathcal{F}$ on a topological space X valued in a category $\mathcal{C}$ is a contravariant functor $\mathcal{F} \colon (\operatorname{Ob} X)^{\mathrm{op}} \to \mathcal{C}$. More concretely, $\mathcal{F}$ has the following data.

- Given an open set $U \subseteq X$, we have $\mathcal{F}(U) \in \mathcal{C}$.
- Given open sets $V \subseteq U \subseteq X$, we have a restriction map $\operatorname{res}_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$.

This data satisfies the following coherence conditions.

- Identity: given an open set $U \subseteq X$, $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(U)}$.
- Functoriality: given open sets $W \subseteq V \subseteq U$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\operatorname{res}_{U,V}"] \arrow[dr, "\operatorname{res}_{U,W}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,W}"] \\ & \mathcal{F}(W) \end{tikzcd}$$

Notation 1.62. We might call an element $f \in \mathcal{F}(U)$ a *section over* U.

As suggested by our language and notation, we should think about (pre)sheaves as mostly being "sheaves of functions." We will see a few examples shortly.

Notation 1.63. Given $f \in \mathcal{F}(U)$, we might write $f|_V := \operatorname{res}_{U,V} f$.

Remark 1.64. In principle, one can have any target category $\mathcal{C}$ for our presheaf. However, we will only work Set, Ab, Ring, Mod_R in this class. In particular, we will readily assume that $\mathcal{C}$ is a concrete category.

Now that we've defined an algebraic object, we should discuss its morphisms.

Definition 1.65 (Presheaf morphism)**.** Fix a topological space X. A *presheaf morphism* between $\mathcal{F}$ and $\mathcal{G}$ is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$. In other words, for each open set $U \subseteq X$, we have a morphism $\eta_U\colon \mathcal{F}(U) \to \mathcal{F}(V)$; these morphisms make the following diagram commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"'] & \mathcal{G}(V) \end{tikzcd}$$

We've talked about presheaves a lot; where are sheaves?

Definition 1.66 (Sheaf)**.** Fix a topological space X. A presheaf $\mathcal{F}\colon (\operatorname{Ob} X)^{\mathrm{op}} \to \mathcal{C}$ is a *sheaf* if and only if it satisfies the following for any open set $U \subseteq X$ with an open cover $\mathcal{U}$.

- Identity: if $f_1, f_2 \in \mathcal{F}(U)$ have $f_1|_V = f_2|_V$ for all $V \in \mathcal{U}$, then $f_1 = f_2$.
- Glubility: if we have $f_V \in \mathcal{F}(V)$ for all $V \in \mathcal{U}$ such that
$$f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$$
for all $V_1, V_2 \in \mathcal{U}$, then there is $f \in \mathcal{F}(U)$ such that $f|_V = f_V$ for all $V \in \mathcal{U}$.

Ok, so we've defined the sheaf as an algebraic object, so here are its morphisms.

Definition 1.67 (Sheaf morphism)**.** A *sheaf morphism* is a morphism of the (underlying) presheaves.

Because there is an identity natural transformation and because the composition of natural transformations is a natural transformation, we see that we have the necessary data for a category PreSh_X of presheaves on X and a category Sh_X of sheaves on X.

As an aside, we note that we can succinctly write the sheaf conditions in an exact sequence.

Lemma 1.68. Fix a topological space X and presheaf $\mathcal{F}\colon (\operatorname{Ob} X)^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{C}$ is an abelian category or Grp. Then $\mathcal{F}$ is a sheaf if and only if the sequence

$$\begin{array}{ccccc} 0 \to \mathcal{F}(U) & \to & \prod_{V\in\mathcal{U}} \mathcal{F}(V) & \to & \prod_{V_1,V_2\in\mathcal{U}} \mathcal{F}(V_1\cap V_2) \\ f & \mapsto & (f|_V)_{V\in\mathcal{U}} & & \\ & & (f_V)_{V\in\mathcal{U}} & \mapsto & (f_{V_1}|_{V_1\cap V_2} - f_{V_2}|_{V_1\cap V_2})_{V_1,V_2} \end{array} \tag{1.1}$$

is exact.

Proof. In one direction, suppose that $\mathcal{F}$ is a sheaf, and we will show that (1.1) is exact for any open cover $\mathcal{U}$ of an open set U.

- Exact at $\mathcal{F}(U)$: suppose $f_1, f_2 \in \mathcal{F}(U)$ have the same image in $\prod_{V\in\mathcal{U}} \mathcal{F}(V)$. This means that
$$f_1|_V = f_2|_V$$
for all $V \in \mathcal{U}$, so the identity axiom tells us that $f_1 = f_2$.
- Exact at $\prod_{V\in\mathcal{U}} \mathcal{F}(V)$: of course any $f \in \mathcal{F}(U)$ goes to $(f|_V)_{V\in\mathcal{U}}$, which goes to
$$f|_{V_1}|_{V_1\cap V_2} - f|_{V_2}|_{V_1\cap V_2} = f|_{V_1\cap V_2} - f|_{V_1\cap V_2} = 0 \in \prod_{V_1,V_2\in\mathcal{U}} \mathcal{F}(V_1\cap V_2)$$

and therefore lives in the kernel. Conversely, suppose $(f_V)_{V\in\mathcal{U}}$ vanishes in $\prod_{V_1,V_2}\mathcal{F}(V_1\cap V_2)$. Rearranging, this means that

$$f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2},$$

so the gluability axiom tells us that we can find $f\in\mathcal{F}(U)$ such that $f|_V = f_V$. This finishes.

Conversely, suppose that $\mathcal{F}$ makes (1.1) always exact, and we will show that $\mathcal{F}$ is a sheaf. Fix an open cover $\mathcal{U}$ of an open set U.

- Identity: suppose that $f_1, f_2\in\mathcal{F}(U)$ have $f_1|_V = f_2|_V$ for any $V\in\mathcal{U}$. This means that f_1 and f_2 have the same image in $\prod_{V\in\mathcal{U}}\mathcal{F}(V)$, so the exactness of (1.1) at $\mathcal{F}(U)$ enforces $f_1 = f_2$.
- Gluability: suppose that we have $f_V\in\mathcal{F}(V)$ for each $V\in\mathcal{U}$ in such a way that $f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$ for all $V_1, V_2\in\mathcal{U}$. Then the image of $(f_V)_{V\in\mathcal{U}}$ in $\prod_{V_1,V_2\in\mathcal{U}}\mathcal{F}(V_1\cap V_2)$ is

$$(f_{V_1}|_{V_1\cap V_2} - f_{V_2}|_{V_1\cap V_2})_{V_1,V_2} = (0)_{V_1,V_2},$$

so exactness of (1.1) forces there to be $f\in\mathcal{F}(U)$ such that $f|_V = f_V$ for each $V\in\mathcal{U}$. This finishes. ■

Remark 1.69. One might want to continue this left-exact sequence. To see this, we will have to talk about cohomology, which is a task for later in life.

1.3.2 Examples of Sheaves

Sheaves of functions will be our key example here. Intuitively, any type of function which can be determined "locally" will form a sheaf; for example, here are continuous functions.

Remark 1.70. For most of our examples, the identity axiom is easily satisfied: intuitively, the identity axiom says that two sections are equal if and only if they agree locally. However, gluability is usually the tricky one: it requires us to build a function from local behavior.

Exercise 1.71. Fix topological spaces X and Y. For each $U\subseteq X$, let $\mathcal{F}(U)$ denote the set of continuous functions $f\colon U\to Y$, and equip these sets with the natural restriction maps. Then $\mathcal{F}$ is a sheaf.

Proof. To begin, here are the functoriality checks.

- Identity: for any $f\in\mathcal{F}(U)$, we have $f|_U = f$.
- Functoriality: if $W\subseteq V\subseteq U$, any $f\in\mathcal{F}(U)$ will have $(f|_V|_W)(w) = f(w) = (f|_W)(w)$ for any $w\in W$, so $f|_V|_W = f|_W$ follows.

Here are sheaf checks. Fix an open cover $\mathcal{U}$ of an open set $U\subseteq X$.

- Identity: suppose $f_1, f_2\in\mathcal{U}$ have $f_1|_V = f_2|_V$ for all $V\in\mathcal{U}$. Now, for all $x\in U$, we see $x\in U_x$ for some $U_x\in\mathcal{U}$, so

$$f_1(x) = (f_1|_{U_x})(x) = (f_2|_{U_x})(x) = f_2(x),$$

so $f_1 = f_2$ follows.
- Gluability: suppose we have $f_V\in\mathcal{F}(V)$ for each $V\in\mathcal{U}$ such that $f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$ for each $V_1, V_2\in\mathcal{U}$. Now, for each $x\in\mathcal{U}$, find $U_x\in\mathcal{U}$ with $x\in U_x$ and set

$$f(x) := f_{U_x}(x).$$

Note this is well-defined: if $x \in U_x$ and $x \in U_{x'}$, then $f_{U_x}(x) = f_{U_x}|_{U_x \cap U'_x}(x) = f_{U'_x}|_{U_x \cap U'_x}(x) = f_{U'_x}(x)$. Additionally, we see that, for each $V \in \mathcal{U}$ and $x \in V$, we have

$$f|_V(x) = f(x) = f_V(x)$$

by construction, so we are done.

Lastly, we need to check that f is continuous. Well, for any open set $V_0 \subseteq Y$, we can compute

$$f^{-1}(V_0) = \{x \in U : f(x) \in V_0\} = \bigcup_{V \in \mathcal{U}} \{x \in V : f(x) \in V_0\} = \bigcup_{V \in \mathcal{U}} \{x \in V : f_V(x) \in V_0\} = \bigcup_{V \in \mathcal{U}} f_V^{-1}(V_0),$$

which is open as the arbitrary union of open sets because $f_V : V \to Y$ is a continuous function. ■

Another key geometric example going forward will be the following.

Exercise 1.72. Set $X := \mathbb{C}$. For each open $U \subseteq X$, let $\mathcal{O}_X(U)$ denote the set of holomorphic functions $U \to \mathbb{C}$, and equip these sets with the natural restriction maps. Then $\mathcal{O}_X$ is a sheaf.

Proof. Again, the point here is that being differentiable can be checked locally. Anyway, we note that our presheaf checks are exactly the same as in Exercise 1.71, as is the check of the sheaf identity axiom.

The gluability axiom is also mostly the same. Given an open cover $\mathcal{U}$ of an open set $U \subseteq X$, pick up $f_V \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that

$$f_{V_1}|_{V_1 \cap V_2} = f_{V_2}|_{V_1 \cap V_2}.$$

As before, we note that each $x \in U$ has some $U_x \in \mathcal{U}$ containing x, so we may define $f : U \to \mathbb{C}$ by $f(x) := f_{U_x}(x)$. The arguments of Exercise 1.71 tell us that this function f is well-defined and has $f|_V = f_V$ for each $V \in \mathcal{U}$.

It remains to check that f is actually holomorphic. This requires that, for each $x \in X$, the limit

$$\lim_{x' \to x} \frac{f(x) - f(x')}{x - x'}.$$

However, this limit can be computed locally for $x' \in U_x$ because U_x contains an open neighborhood around x. As such, it suffices to show that the limit

$$\lim_{x' \to x} \frac{f|_{U_x}(x) - f|_{U_x}(x')}{x - x'} = \lim_{x' \to x} \frac{f_{U_x}(x) - f_{U_x}(x')}{x - x'}$$

exists, which is true because $f_{U_x} \in \mathcal{F}(U_x)$ is holomorphic. ■

In contrast, sheaves have trouble keeping track of "global" information.

Example 1.73. For each $U \subseteq \mathbb{R}$, let $\mathcal{F}(\mathbb{R})$ denote the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, and equip these sets with the natural restriction maps. Then $\mathcal{F}$ is not a sheaf: for each open set $(n-1, n+1)$ for $n \in \mathbb{N}$, the function $f_{(n-1,n+1)} := \mathrm{id}_{(n-1,n=1)}$ is bounded and continuous, but the glued function $f = \mathrm{id}_{\mathbb{R}}$ is not bounded on all of $\mathbb{R}$. (We glued using Exercise 1.71, which does force the definition of f.)

1.3.3 Sheaf on a Base

In light of our sheaf language, we are trying to define a "structure" sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$, and we wanted to have

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f.$$

We aren't going to be able to specify a presheaf with this data, but we can specify a sheaf. In some sense, the presheaf is unable to build up locally in the way that a sheaf can, so having the data on a base like $\{D(f)\}_{f \in A}$ need not be sufficient to define the full presheaf.

But as alluded to, we can do this for sheaves. We begin by defining a sheaf on a base.

Definition 1.74 (Sheaf on a base)**.** Fix a topological space X and a base $\mathcal{B}$ for its topology. Then a *sheaf on a base valued in* $\mathcal{C}$ is a contravariant functor $F\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ satisfying the following identity and gluability axioms: for any $B \in \mathcal{B}$ with a basic cover $\{B_i\}_{i\in I}$, we have the following.

- Identity: if we have $f_1, f_2 \in F(B)$ such that $f_1|_{B_i} = f_2|_{B_i}$ for all B_i, then $f_1 = f_2$.
- Gluability: if we have $f_i \in F(B_i)$ for each i such that $f_i|_B = f_j|_B$ for each $B \subseteq B_i \cap B_j$, then there is $f \in F(B)$ such that $f|_{B_i} = f_i$ for each i.

Example 1.75. Given a topological space X and a base $\mathcal{B}$, any sheaf $\mathcal{F}\colon (\operatorname{Op} X)^{\mathrm{op}} \to \mathcal{C}$ "restricts" to a sheaf on a base $\mathcal{F}_\mathcal{B}$ by setting $\mathcal{F}_\mathcal{B}(B) := \mathcal{F}(B)$ for all $B \in \mathcal{B}$ and reusing the same restriction maps. The identity and gluability axioms follow from their (stronger) sheaf counterparts; checking this amounts writing down the axioms.

Morphisms are constructed in the obvious way.

Definition 1.76 (Sheaf on a base morphisms)**.** Fix a topological space X and a base $\mathcal{B}$ for its topology. Then a *morphism* between two sheaves F and G on the base $\mathcal{B}$ is a natural transformation of the (underlying) contravariant functors.

Example 1.77. Given a topological space X and a base $\mathcal{B}$, any sheaf morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ restricts in the obvious way to a morphism $\eta_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$ (namely, $(\eta_\mathcal{B})_B = \eta_B$) on the corresponding sheaves on a base. Checking this amounts to saying out loud that the diagram on the left commutes for any $B' \subseteq B$ because it is the same as the diagram on the right.

$$\begin{tikzcd} \mathcal{F}_\mathcal{B}(B) \arrow[r, "(\eta_\mathcal{B})_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{G}_\mathcal{B}(B) \arrow[d, "\mathrm{res}_{B,B'}"] \\ \mathcal{F}_\mathcal{B}(B') \arrow[r, "(\eta_\mathcal{B})_{B'}"'] & \mathcal{G}_\mathcal{B}(B') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(B) \arrow[r, "\eta_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{G}(B) \arrow[d, "\mathrm{res}_{B,B'}"] \\ \mathcal{F}(B') \arrow[r, "\eta_{B'}"'] & \mathcal{G}(B') \end{tikzcd}$$

Remark 1.78. Example 1.75 and Example 1.77 combine into the data of a forgetful functor $(-)_\mathcal{B}$ from sheaves on X to sheaves on a base $\mathcal{B}$. Here are the last two checks.

- Identity: given a sheaf $\mathcal{F}$ on X, note $(\mathrm{id}_\mathcal{F})_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{F}_\mathcal{B}$ sends $s \in \mathcal{F}_\mathcal{B}(B) = \mathcal{F}(B)$ to itself, so $(\mathrm{id}_\mathcal{F})_\mathcal{B} = \mathrm{id}\,\mathcal{F}_\mathcal{B}$.
- Functoriality: given sheaf morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ and some $B \in \mathcal{B}$, we see

$$(\varphi \circ \psi)_\mathcal{B}(B) = (\varphi \circ \psi)_B = \varphi_B \circ \psi_B = (\varphi_\mathcal{B} \circ \psi_\mathcal{B})(B).$$

We are interested in showing that we can build a sheaf from a sheaf on a base uniquely, but it will turn out to be fruitful to spend a moment to discuss how this behaves on morphisms first for the uniqueness part of this statement.

Lemma 1.79. Fix a topological space X with a base $\mathcal{B}$ for its topology. Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X with values in $\mathcal{C}$ and a morphism of the (underlying) sheaves on a base $\eta_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$, there is a unique sheaf morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ such that $(\eta_\mathcal{B})_B = \eta_B$ for each $B \in \mathcal{B}$.

Proof. We show uniqueness before existence.

- Uniqueness: fix any open $U \subseteq X$, and we will try to solve for $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$. Well, fix a basic open cover $\mathcal{U}$ of U; then, for any $B \in \mathcal{U}$, we need the following diagram to commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\mathrm{res}_{U,B}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}_{U,B}"] \\ \mathcal{F}(B) \arrow[r, "\eta_B = (\eta_{\mathcal{B}})_B"'] & \mathcal{G}(B) \end{tikzcd}$$

  In particular, for any $f \in \mathcal{F}(U)$, we need $\eta_U(f)|_B = (\eta_{\mathcal{B}})_B(f|_B)$. Thus, $\eta_U(f)|_B$ is fully specified by the data provided by $\eta_{\mathcal{B}}$, so the identity axiom for $\mathcal{G}$ forces $\eta_U(f)$ to be unique.

- Existence: to begin, fix any open $U \subseteq X$, and we will define $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$. As alluded to above, we let $\mathcal{U}$ be the set of basis elements which are contained in U so that $\mathcal{U}$ is a (large) basic cover of U.

  Then, picking up $f \in \mathcal{F}(U)$, we will try to use the gluability axiom by setting $g_B := (\eta_{\mathcal{B}})_B(f|_B)$ for each $B \in \mathcal{U}$. In particular, for any $B, B' \in \mathcal{U}$, any basic $B_0 \subseteq B \cap B'$ has

$$(g_B|_{B\cap B'})|_{B_0} = g_B|_{B_0} = (\eta_{\mathcal{B}})_B(f|_B)|_{B_0} = (\eta_{\mathcal{B}})_{B_0}(f|_B|_{B_0}) = \eta_{B_0}(f|_{B_0}) = g_{B_0},$$

  which is also $(g_{B'}|_{B\cap B'})_{B_0}$ by symmetry, so the identity axiom applied to $B \cap B'$ implies $g_B|_{B\cap B'} = g_{B'}|_{B\cap B'}$. Thus, the gluability axiom applied to U gives us a unique $g \in \mathcal{G}(U)$ such that

$$g|_B = (\eta_{\mathcal{B}})_B(f|_B)$$

  for each basic set $B \subseteq U$. We define $\eta_U(f) := g$.

  It remains to show that η does in fact assemble into a sheaf morphism. Fix open sets $V \subseteq U$, and we need the following diagram to commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"'] & \mathcal{G}(V) \end{tikzcd}$$

  Well, pick up any $f \in \mathcal{F}(U)$. Then, for any basic $B \subseteq V \subseteq U$, we see that

$$\eta_U(f)|_B = (\eta_{\mathcal{B}})_B(f|_B) = (\eta_{\mathcal{B}})_B(f|_V|_B),$$

  so the uniqueness of $\eta_V(f|_V)$ forces $\eta_V(f|_V) = \eta_U(f)|_U$. This finishes. ■

1.3.4 Extending a Sheaf on a Base

We dedicate this subsection to the following result, describing how to extend a sheaf on a base to a full sheaf.

Proposition 1.80. Fix a topological space X with a base $\mathcal{B}$ for its topology. Given a sheaf on a base $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$, there is a sheaf $\mathcal{F}$ and isomorphism (of sheaves on a base) $\iota : F \to \mathcal{F}_{\mathcal{B}}$ satisfying the following universal property: any sheaf $\mathcal{G}$ with a morphism (of sheaves on a base) $\varphi : F \to \mathcal{G}_{\mathcal{B}}$ has a unique sheaf morphism $\psi : \mathcal{F} \to \mathcal{G}$ making the following diagram commute.

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{F}_{\mathcal{B}} \arrow[d, "\psi_{\mathcal{B}}"] \\ & \mathcal{G}_{\mathcal{B}} \end{tikzcd} \tag{1.2}$$

Proof. We begin by providing a construction of $\mathcal{F}$. For each open set $U \subseteq X$, define

$$\mathcal{F}(U) := \varprojlim_{B \subseteq U} F(B) = \left\{ (f_B)_{B\subseteq U} \in \prod_{B\subseteq U} F(B) : f_B|_{B'} = f_{B'} \text{ for each } B' \subseteq B \subseteq U \right\}.$$

(Namely, we are implicitly assuming that our target category has limits.) Observe that, when $V \subseteq U$, the natural surjection

$$\prod_{B\subseteq U} \mathcal{F}(B) \to \prod_{B\subseteq V} \mathcal{F}(B)$$

induces a map $\mathcal{F}(U) \to \mathcal{F}(V)$. Indeed, an element $(f_B)_{B\subseteq U} \in \mathcal{F}(U)$ gets sent to $(f_B)_{B\subseteq V}$, and it is still the case that $B' \subseteq B \subseteq V$ implies $f_B|_{B'} = f_{B'}$ because actually $B' \subseteq B \subseteq U$. Thus, $(f_B)_{B\subseteq V} \in \mathcal{F}(V)$, so we have a well-defined map

$$\begin{array}{rccc} \operatorname{res}_{U,V}\colon & \mathcal{F}(U) & \to & \mathcal{F}(V) \\ & (f_B)_{B\subseteq U} & \mapsto & (f_B)_{B\subseteq V} \end{array}$$

which will serve as our restrictions. We start by checking that these data assemble into a presheaf.

- When $U = V$, we are sending $(f_B)_{B\subseteq U} \in \mathcal{F}(U)$ to itself, so $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(I)}$.
- Given $W \subseteq V \subseteq U$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\operatorname{res}_{U,V}"] \arrow[dr, "\operatorname{res}_{U,W}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,W}"] \\ & \mathcal{F}(W) \end{tikzcd} \qquad \begin{tikzcd} (f_B)_{B\subseteq U} \arrow[r, mapsto] \arrow[dr, mapsto] & (f_B)_{B\subseteq V} \arrow[d, mapsto] \\ & (f_B)_{B\subseteq W} \end{tikzcd}$$

  commutes, which is our functoriality check.

We now show that these data make a sheaf. Fix an open set $U \subseteq X$ with an open cover $\mathcal{U}$. To help our constructions, given any open subset $V \subseteq X$, let $\mathcal{B}_V$ denote the collection of basis elements B contained in V; notably $\mathcal{B}_V$ is a basic cover for V. Then, for any open $U' \subseteq U$, we let

$$\mathcal{S}_{U'} := \bigcup_{V\subseteq U} \mathcal{S}_{U'\cap V}.$$

Notably, $\mathcal{S}_{U'}$ is a basic cover for U' such that any $B \in \mathcal{S}_{U'}$ is contained in some element of $\mathcal{U}$.

- Identity: suppose that $(f_B)_{B\subseteq U}, (g_B)_{B\subseteq U} \in \mathcal{F}(U)$ restrict to the same element on any $V \in \mathcal{U}$. Now, fix any $B_0 \subseteq U$, and we will show $f_{B_0} = g_{B_0}$.

  Now consider $\mathcal{S}_{B_0}$: for each $B' \in \mathcal{S}$, we can find $V \in \mathcal{U}$ so that $B' \subseteq V$, for which we know

  $$(f_B)_{B\subseteq V} = (g_B)_{B\subseteq V}.$$

  In particular $f_{B_0}|_{B'} = f_{B'} = g_{B'} = g_{B_0}|_{B'}$ for any $B \in \mathcal{S}$, so the identity axiom for the sheaf on a base F forces $f_{B_0} = g_{B_0}$.

- Gluability: suppose we are given some $(f_{V,B})_{B\subseteq V} \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that

  $$(f_{V,B})_{B\subseteq V\cap V'} = (f_{V,B})_{B\subseteq V}|_{V\cap V'} = (f_{V',B})_{B\subseteq V}|_{V\cap V'} = (f_{V',B})_{B\subseteq V\cap V'}$$

  for any $V, V' \in \mathcal{U}$. In other words, for any basic $B \subseteq V \cap V'$, we have $f_{V,B} = f_{V',B}$.

  Now, for any basic $B_0 \subseteq U$, we will solve for f_{B_0}. Using $\mathcal{S}_{B_0}$, note that any $B \in \mathcal{S}_{B_0}$ has some $V_B \in \mathcal{U}$ such that $B \subseteq V_B$, so we will use $f_{V_B,B}$ at this point. Note that if $B \subseteq V_B'$ as well, then $f_{V_B,B} = f_{V_B',B}$, so our $f_{V_B,B}$ is independent of V_B. Continuing, if we have $B \subseteq B_1 \cap B_2$, then

  $$f_{V_{B_1},B_1}|_B = f_{V_{B_1},B} = f_{V_{B_2},B} = f_{V_{B_2},B_2}|_B,$$

so gluability applied to our sheaf F on a base promises us a unique f_{B_0} such that $f_{B_0}|_B = f_{V_B,B}$ for any $B \in \mathcal{S}_{B_0}$.

We now need to show that the $(f_B)_{B\subseteq U}$ assemble into an element of $\mathcal{F}(U)$. Namely, if we have $B_0' \subseteq B_0$, we need to show that $f_{B_0}|_{B_0'} = f_{B_0'}$. Well, for any $B \in \mathcal{S}_{B_0'}$, we compute

$$f_{B_0}|_{B_0'}|_B = f_{B_0}|_B = f_{V_B,B} = f_{B_0}|_B,$$

so the uniqueness of f_{B_0} gives the equality.

For our next step, we define $\iota_{B_0}\colon F(B) \to \mathcal{F}_\mathcal{B}(B_0)$ by

$$\iota_{B_0}(f) := (f|_B)_{B\subseteq B_0}.$$

Here are the checks on ι.

- Well-defined: note $\iota_{B_0}(f)$ is an element of $\mathcal{F}_\mathcal{B}(B_0)$ because $B' \subseteq B \subseteq B_0$ will have $f|_B|_{B'} = f|_{B'}$.
- Natural: if $B \subseteq B'$, then note that the diagrams

$$\begin{tikzcd} F(B_0) \arrow[r, "\iota_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{F}_\mathcal{B}(B_0) \arrow[d, "\mathrm{res}_{B,B'}"] \\ F(B_0') \arrow[r, "\iota_{B'}"'] & \mathcal{F}_\mathcal{B}(B_0') \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & (f|_B)_{B\subseteq B_0} \arrow[d, mapsto] \\ f|_{B_0'} \arrow[r, mapsto] & (f|_B)_{B\subseteq B_0'} \end{tikzcd}$$

  commute, finishing.
- Injective: suppose that $f, g \in F(B_0)$ have the same image in $\mathcal{F}_\mathcal{B}(B_0)$. This means that $(f|_B)_{B\subseteq B_0} = (g|_B)_{B\subseteq B_0}$, so $f = f|_{B_0} = g|_{B_0} = g$, so we are done.
- Surjective: fix some $(f_B)_{B\subseteq B_0} \in \mathcal{F}_\mathcal{B}(B_0)$. Notably, for any basic $B_1, B_2 \subseteq B_0$ with some basic $B \subseteq B_1 \cap B_2$, we have

$$f_{B_1}|_B = f_B = f_{B_2}|_B,$$

  so gluability applied to F promises $f \in F(B_0)$ such that $f|_B = f_B$ for all basic $B \subseteq B_0$. So $\iota_{B_0}(f) = (f_B)_{B\subseteq B_0}$.

We now begin showing that $\mathcal{F}$ satisfies the universal property. Fix some sheaf $\mathcal{G}$ on X with a morphism $\varphi\colon F \to \mathcal{G}_\mathcal{B}$.

In light of Lemma 1.79, it suffices to show the existence and uniqueness of a morphism $\psi_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$ on the base $\mathcal{B}$ making (1.2) commute. Namely, the existence of $\psi_\mathcal{B}$ promises a full sheaf morphism $\psi\colon \mathcal{F} \to \mathcal{G}$ extending via Lemma 1.79; for uniqueness, two possible $\psi, \psi'\colon \mathcal{F} \to \mathcal{G}$ with $\psi_\mathcal{B}$ and $\psi'_\mathcal{B}$ both commuting will enforce $\psi_\mathcal{B} = \psi'_\mathcal{B}$ and then $\psi = \psi'$ by the uniqueness of Lemma 1.79.

Continuing with the proof, we note that the fact that ι is an isomorphism means that the commutativity of (1.2) is equivalent to the diagram

$$\begin{tikzcd} F & \mathcal{F}_\mathcal{B} \arrow[l, "\iota^{-1}"'] \arrow[d, "\psi_\mathcal{B}"] \\ & \mathcal{G}_\mathcal{B} \arrow[from=1-1, "\varphi"'] \end{tikzcd}$$

commuting. However, the commutativity of this diagram is equivalent to setting $\psi_\mathcal{B} := \varphi \circ \iota^{-1}$. Thus, uniqueness of $\psi_\mathcal{B}$ is immediate, and existence of $\psi_\mathcal{B}$ amounts to noting the composition of natural transformations remains a natural transformation. ■

> **Remark 1.81.** One can also define $\mathcal{F}(U)$ as compatible systems of stalks, but we have not defined stalks yet.

Remark 1.82. The universal property implies that the pair $(\mathcal{F}, \iota)$ is unique up to unique isomorphism, for a suitable notion of unique isomorphism. Namely, the usual abstract nonsense arguments with universal properties is able to show that if we have another sheaf $\mathcal{F}'$ with isomorphism $\iota'\colon F \to \mathcal{F}'_{\mathcal{B}}$ satisfying the universal property, then $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic. (This isomorphism $\eta\colon \mathcal{F} \cong \mathcal{F}'$ is unique if we ask for the corresponding diagram

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \mathcal{F}_{\mathcal{B}} \arrow[d, "\eta_{\mathcal{B}}"] \\ & \mathcal{F}'_{\mathcal{B}} \end{tikzcd}$$

to commute.)

The universal property actually gives a functor from sheaves F on a base $\mathcal{B}$ to sheaves $\mathcal{F}_B$ on X.

Lemma 1.83. Fix a topological space X with a base $\mathcal{B}$ for its topology. Then the map sending a sheaf F on a base to its sheaf $\mathcal{E}(F)$ describes the action of a functor on objects.

Proof. Given a sheaf on a base F, let $\iota_F\colon F \to \mathcal{E}(F)_{\mathcal{B}}$ be the inclusion. Now, given a morphism $\varphi\colon F \to G$ of sheaves on a base, note that there is a unique morphism $\mathcal{E}(\varphi)$ making the diagram

$$\begin{tikzcd} F \arrow[r, "\iota_F"] \arrow[d, "\varphi"'] & \mathcal{E}(F)_{\mathcal{B}} \arrow[d, dashed, "\mathcal{E}(\varphi)_{\mathcal{B}}"] \\ G \arrow[r, "\iota_G"] & \mathcal{E}(G)_{\mathcal{B}} \end{tikzcd}$$

commute by Lemma 1.79. We now need to show that this data assembles into a functor.

- Identity: given a sheaf on a base F, note that id_F induces the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\iota_F"] \arrow[d, "\mathrm{id}_F"'] & \mathcal{E}(F)_{\mathcal{B}} \arrow[d, dashed, "(\mathrm{id}_{\mathcal{E}(F)})_{\mathcal{B}}"] \\ F \arrow[r, "\iota_F"] & \mathcal{E}(F)_{\mathcal{B}} \end{tikzcd}$$

  which makes us conclude $\mathcal{E}(\mathrm{id}_F) = \mathrm{id}_{\mathcal{E}(F)}$.

- Functoriality: given morphisms $\varphi\colon F \to G$ and $\psi\colon G \to H$ of sheaves on a base, we note that the diagram

$$\begin{tikzcd} F \arrow[r, "\iota_F"] \arrow[d, "\varphi"'] & \mathcal{E}(F)_{\mathcal{B}} \arrow[d, "\mathcal{E}(\varphi)_{\mathcal{B}}"'] \arrow[dd, dashed, bend left, "\mathcal{E}(\psi\circ\varphi)_{\mathcal{B}}"] \\ G \arrow[r, "\iota_G"] \arrow[d, "\psi"'] & \mathcal{E}(G)_{\mathcal{B}} \arrow[d, "\mathcal{E}(\psi)_{\mathcal{B}}"'] \\ H \arrow[r, "\iota_H"] & \mathcal{E}(H)_{\mathcal{B}} \end{tikzcd}$$

  commutes, so the uniqueness of the arrow $\mathcal{E}(\psi \circ \varphi)_{\mathcal{B}}$ forces $\mathcal{E}(\psi \circ \varphi) = \mathcal{E}(\psi) \circ \mathcal{E}(\varphi)$. ■

Remark 1.84. In fact, the functor $\mathcal{E}$ is the right adjoint to the forgetful functor $(-)_{\mathcal{B}}$ from sheaves on a base $\mathcal{B}$ to sheaves on X, which also essentially follows from the universal property. We will not bother showing this.

1.4 August 31

We finish defining the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ of an affine scheme today.

Remark 1.85. One complaint about sheaves on a base is that we have to choose a base. To be more canonical, we will discuss stalks today, which treats all points the same.

1.4.1 The Structure Sheaf

We are now ready to define the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ of a ring A, which we will define a sheaf on a base. Recall from Remark 1.55 that $\{D(f)\}_{f\in A}$ forms a base of the Zariski topology of $\operatorname{Spec} A$, so it will suffice to set

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) \coloneqq A_{S(D(f))},$$

where

$$S(D(f)) \coloneqq \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f)\}.$$

In other words, $S(D(f))$ consists of the set of functions in A which only vanish outside $D(f)$ so that we can invert them on $D(f)$.

Remark 1.86. In essence, $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ is supposed to be the functions on $D(f)$, which is why we want to be able to invert functions which only vanish on $(\operatorname{Spec} A) \setminus D(f)$.

Remark 1.87. The subset $S(D(f))$ only depends on $D(f)$, not f, so $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ is well-defined. With that said, we note that $f \in S(D(f))$ gives a natural localization map $A_f \to A_{S(D(f))}$ induced by id_A. Similarly, any $g \in S(D(f))$ has $V((g)) \subseteq V((f))$ and so Proposition 1.45 tells us that

$$\operatorname{rad}(f) = I(V((f))) \subseteq I(V((g))) = \operatorname{rad}(g),$$

so $f \in \operatorname{rad}(g)$, so $f^n = ag$ for some positive integer n and $a \in A$; this means $g \in A_f^\times$, so actually $S(D(f)) \subseteq A_f^\times$, allowing another natural localization map $A_{S(D(f))} \to A_f$ induced by id_A. These natural localization maps are inverse (their compositions are induced by id_A), so $\mathcal{O}_{\operatorname{Spec} A}(D(f)) \cong A_f$.

Remark 1.88 (Nir)**.** In class, Professor Tang defined the structure sheaf on a base by $\mathcal{O}_{\operatorname{Spec} A}(D(f)) \coloneqq A_f$. I have chosen to follow [Vak17] here because I don't like $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ to depend on $f \in A$ when it should only depend on $D(f)$.

To define our (pre)sheaf on a base, we also need to provide restriction maps. Well, for $f, f' \in A$ with $D(f') \subseteq D(f)$, we see that

$$S(D(f)) = \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f)\} \subseteq \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f')\} = S(D(f')),$$

so there is a natural localization map

$$\operatorname{res}_{D(f),D(f')} \colon A_{S(D(f))} \to A_{S(D(f'))}$$

induced by id_A. These data give all the data we need to define a sheaf on a base. We will throw the remaining checks into the following lemma.

Lemma 1.89. Fix a ring A. The above data define a sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on the base $\{D(f)\}_{f\in A}$.

Proof. We begin by showing that the data gives a presheaf.

- Identity: if $D(f) = D(f')$, then $S(D(f)) = S(D(f'))$, so the localization map

$$\operatorname{res}_{D(f),D(f)} \colon A_{S(D(f))} \to S_{D(f)}$$

  is simply $\operatorname{id}_{A_{S(D(f))}}$.
- Functoriality: suppose $D(f'') \subseteq D(f') \subseteq D(f)$. Then we note that the diagram

$$\begin{tikzcd} A_{S(D(f))} \arrow[r, "\operatorname{res}"] \arrow[dr, "\operatorname{res}"'] & A_{S(D(f'))} \arrow[d, "\operatorname{res}"] \\ & A_{S(D(f''))} \end{tikzcd} \qquad \begin{tikzcd} a/g \arrow[r, mapsto] \arrow[dr, mapsto] & a/g \arrow[d, mapsto] \\ & a/g \end{tikzcd}$$

  commutes because everything is induced by id_A, so we are done.

It remains to check the identity and gluability axioms. For this, we will need a basis set $D(f)$ and a basic cover $\{D(f_\alpha)\}_{\alpha\in\lambda}$. To access this cover, we have the following lemma.

Lemma 1.90. Fix a ring A. Then, given $f \in A$ and $\{f_\alpha\}_{\alpha\in\lambda} \subseteq A$, the following are equivalent.

(a) $D(f) \subseteq \bigcup_{\alpha\in\lambda} D(f_\alpha)$.

(b) $f \in \operatorname{rad}(f_\alpha)_{\alpha\in\lambda}$.

Proof. Note that

$$\bigcup_{\alpha\in\lambda} D(f_\alpha) = \operatorname{Spec} A \setminus \bigcap_{\alpha\in\lambda} V((f_\alpha)) = \operatorname{Spec} A \setminus V((f_\alpha)_{\alpha\in\lambda}),$$

so (a) is equivalent to $V((f_\alpha)_{\alpha\in\lambda}) \subseteq V((f))$. Now, Proposition 1.45 tells us that (a) implies

$$\operatorname{rad}(f) = I(V((f))) \subseteq I(V((f_\alpha)_{\alpha\in\lambda})) = \operatorname{rad}(f_\alpha)_{\alpha\in\lambda},$$

from which (b) follows. Conversely, if (b) holds, then $\operatorname{rad}(f) \subseteq \operatorname{rad}(f_\alpha)_{\alpha\in\lambda}$ by taking radicals, so Proposition 1.45 again promises

$$V((f_\alpha)_{\alpha\in\lambda}) = V(\operatorname{rad}(f_\alpha)_{\alpha\in\lambda}) \subseteq V(\operatorname{rad}(f)) = V((f)),$$

which we showed is equivalent to (a). ■

Corollary 1.91. Fix a ring A. Then any cover $\{D(f_\alpha)\}_{\alpha\in\lambda}$ of $D(f)$ has a finite subcover.

Proof. Note Lemma 1.90 tells us that $f \in \operatorname{rad}(f_\alpha)_{\alpha\in\lambda}$, so there is a positive integer n and finite subset $\lambda \subseteq \lambda'$ so that

$$f^n = \sum_{\alpha\in\lambda'} a_\alpha f_\alpha,$$

but then $f \in \operatorname{rad}(f_\alpha)_{\alpha\in\lambda'}$, so $D(f)$ is covered by the (finite) cover $\{D(f_\alpha)\}_{\alpha\in\lambda'}$. ■

We now show the identity and gluability axioms separately.

- Identity: note Corollary 1.91 promises us some $\lambda' \subseteq \lambda$ such that the $\{D(f_\alpha)\}_{\alpha\in\lambda'}$ still covers $D(f)$. We will now forget about λ entirely and deal with the finite λ' instead.

  For identity, we suppose that we have $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ such that $s|_{D(f_\alpha)} = 0$ for all $\alpha \in \lambda'$, and we want to show that $s = 0$. Under the (canonical) isomorphism $\mathcal{O}_{\operatorname{Spec} A}(D(f_\alpha)) \simeq A_{f_\alpha}$, we see that we must have

$$f_\alpha^{d_\alpha} s = 0$$

for some d_α, for each α. Now, $D(f_\alpha) = D\left(f_\alpha^{d_\alpha}\right)$, so the $D\left(f_\alpha^{d_\alpha}\right)$ still cover $D(f)$; it follows from Lemma 1.90 that there is some d for which

$$f^d = \sum_{\alpha\in\lambda'} c_\alpha f_\alpha^{d_\alpha}.$$

Multiplying both sides by s (after embedding in $A_{S(D(f))}$) tells us that $f^d s = 0$ in $A_{S(D(f))}$, so $s = 0$ because $f \in A_{S(D(f))}^\times$.

- Finite gluability: fix sections $s_\alpha \in \mathcal{O}_{\operatorname{Spec} A}(D(f_\alpha))$ such that

$$s_\alpha|_{D(f_\alpha)\cap D(f_\beta)} = s_\beta|_{D(f_\alpha)\cap D(f_\beta)}.$$

For concreteness, use $\mathcal{O}_{\operatorname{Spec} A}(D(f)) \simeq A_f$ to write $s_\alpha := a_\alpha/f_\alpha^n$, where n is the maximum of all the possibly needed denominators.

Noting that $D(f_\alpha)\cap D(f_\beta) = D(f_\alpha f_\beta)$, our coherence is equivalent to asking for

$$(f_\alpha f_\beta)^m \left(f_\beta^n a_\alpha - f_\alpha^n a_\beta\right) = 0,$$

where again m is chosen to be large enough among the finitely many possibilities for α and β. We now notice that

$$s_\alpha = \frac{a_\alpha}{f_\alpha^n} = \frac{f_\alpha^m a_\alpha}{f_\alpha^{n+m}},$$

so we set $b_\alpha := f_\alpha^m a_\alpha$ and $g_\alpha := f_\alpha^{n+m}$, which means

$$g_\beta b_\alpha = g_\alpha b_\beta$$

for all α, β. Notably, $\operatorname{rad}(f_\alpha) = \operatorname{rad}(g_\alpha)$, so $D(f_\alpha) = D(g_\alpha)$, so the $\{D(g_\alpha)\}_{\alpha\in\lambda}$ still cover $D(f)$, so Lemma 1.90 tells us that we can write

$$f^n = \sum_{\alpha\in\lambda} c_\alpha g_\alpha$$

for some positive integer n. In particular, we set $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f)) \simeq A_f$ by

$$s := \frac{1}{f^n}\sum_{\alpha\in\lambda} c_\alpha b_\alpha.$$

In particular, for any $\beta\in\lambda$, we see

$$g_\beta s = \frac{1}{f^n}\sum_{\alpha\in\lambda} c_\alpha g_\beta b_\alpha = \frac{1}{f^n}\sum_{\alpha\in\lambda} c_\alpha g_\alpha b_\beta = b_\beta$$

in A_f, so our restriction is $s|_{D(g_\beta)} = b_\beta/g_\beta = s_\beta$, which is what we wanted.

- Gluability: we show general gluability from finite gluability. Fix sections $s_\alpha \in \mathcal{O}_{\operatorname{Spec} A}(D(f_\alpha))$ such that

$$s_\alpha|_{D(f_\alpha)\cap D(f_\beta)} = s_\beta|_{D(f_\alpha)\cap D(f_\beta)} \tag{1.3}$$

for each $\alpha, \beta \in \lambda$. Using Corollary 1.91, we can find a finite subcover using $\lambda' \subseteq \lambda$, and the sections $\{s_\alpha\}_{\alpha\in\lambda'}$ still satisfy (1.3), so finite gluability (and identity!) gives a unique $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ with

$$s|_{D(f_\alpha)} = s_\alpha.$$

We claim that actually $s|_{D(f_\alpha)} = s_\alpha$ for all $\alpha\in\lambda$. Well, for any $\beta\in\lambda$, apply finite gluability to $\lambda'\cup\{\beta\}$ to find $s' \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ such that $s'|_{D(f_\alpha)} = s_\alpha$ for all $\alpha \in \lambda'\cup\{\beta\}$.

It follows from the identity axiom that on the open cover $\{D(f_\alpha)\}_{\alpha\in\lambda'}$ that $s = s'$, so we conclude

$$s|_{D(f_\beta)} = s'|_{D(f_\beta)} = s_\beta$$

for any $\beta\in\lambda$. ■

Having finished the last of our checks, we see that our data make a sheaf on a base, so Proposition 1.80 promises a unique sheaf extending this sheaf on a base. This is the (affine) structure sheaf, and it finishes our definition of an affine scheme.

Definition 1.92 (Affine scheme)**.** Fix a ring A. An *affine scheme* is the topological space $\operatorname{Spec} A$ (given the Zariski topology) together with the sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ such that

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_{S(D(f))}$$

for each $f \in A$; here $S(D(f)) = \{g \in A : D(f) \subseteq D(g)\}$.

Note that we are somewhat sloppily identifying the outputs of the structure sheaf with its outputs on the base.

1.4.2 Stalks

To define a morphism of schemes, we will want to discuss stalks.

Remark 1.93. We might expect a morphism of (affine) schemes to be merely a continuous map together with a natural transformation of the structure sheaves (perhaps with some coherence conditions). However, this will not be enough data. Namely, we want all of our morphisms of affine schemes to be induced by ring homomorphisms, and this will require exploiting a little more data.

The extra data in those morphisms will come from stalks.

Definition 1.94 (Stalk)**.** Fix a presheaf $\mathcal{F}$ on a topological space X. For a point $p \in X$, we define the *stalk of $\mathcal{F}$ at $p \in X$* to be the direct limit

$$\mathcal{F}_p := \varinjlim_{U \ni p} \mathcal{F}(U).$$

Concretely, elements of $\mathcal{F}_p$ are ordered pairs (U, s) where $s \in \mathcal{F}(U)$ with $p \in U$, modded out by an equivalence relation $\sim$; here, $(U, s) \sim (U', s')$ if and only if there is $W \subseteq U \cap U'$ such that $s|_W = s'|_W$.

Remark 1.95 (Nir)**.** In some sense, the stalk is intended to encode "local information" at the point $p \in X$ in a particularly violent way: whenever two functions $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ (where $p \in U_1 \cap U_2$) are equal locally on some open set U containing p, then we identify s_1 and s_2. As such, $\mathcal{F}_p$ can really study functions locally at p.

Remark 1.96. We go ahead and check that $\sim$ forms an equivalence relation. Fix (U_i, s_i) with $s_i \in \mathcal{F}(U_i)$ for $i \in \{1, 2, 3\}$.

- Reflexive: note $U_1 \subseteq U_1$ and $s_1|_{U_1} = s_1 = s_1|_{U_1}$, so $(U_1, s_1) \sim (U_1, s_1)$.
- Symmetry: if $(U_1, s_1) \sim (U_2, s_2)$, we can find an open $V \subseteq U_1 \cap U_2$ with $s_1|_V = s_2|_V$, which implies $s_2|_V = s_1|_V$, so $(U_2, s_2) \sim (U_1, s_1)$.
- Transitive: if $(U_1, s_1) \sim (U_2, s_2)$ and $(U_2, s_2) \sim (U_3, s_3)$, we can find open $V_1 \subseteq U_1 \cap U_2$ and $V_2 \subseteq U_2 \cap U_3$ such that $s_1|_{V_1} = s_2|_{V_1}$ and $s_2|_{V_2} = s_3|_{V_2}$. Then $V_1 \cap V_2 \subseteq U_1 \cap U_3$, and we can see

$$s_1|_{V_1 \cap V_2} = s_1|_{V_1}|_{V_1 \cap V_2} = s_2|_{V_1}|_{V_1 \cap V_2} = s_2|_{V_1 \cap V_2} = s_2|_{V_2}|_{V_1 \cap V_2} = s_3|_{V_2}|_{V_1 \cap V_2} = s_3|_{V_1 \cap V_2}.$$

Definition 1.97 (Germ)**.** Fix a presheaf $\mathcal{F}$ on a topological space X. For a point $p \in X$ and section $s \in \mathcal{F}(U)$ with $p \in U$, the *germ of s at p* is the element

$$[(U, s)] \in \mathcal{F}_p.$$

Notation 1.98. I will write the germ of $f \in \mathcal{F}(U)$ at $p \in U$ as $f|_p$. This notation is not standard, but I like it because I think of taking the germ of a section at p as analogous to "restricting" to the point p.

As a warning, later on, we will want to consider tuples of sections $(f_p)_p$, and we will want to distinguish the notation for an element of this tuple as f_p with the notation for the corresponding germ $f|_p$.

Remark 1.99. As justification for my notation, if $f \in \mathcal{F}(U)$ while $p \in V \subseteq U$, then

$$f|_p = f|_V|_p$$

because $[(U, f)] = [(V, f|_V)]$ can be witnessed by $f|_V = f|_V|_V$.

Here are some examples of stalks.

Lemma 1.100. Fix a presheaf $\mathcal{F}$ on a topological space X, and give the topology on X a base $\mathcal{B}$. For a point $p \in X$, we have the isomorphism

$$\begin{aligned} \varphi\colon \varinjlim_{B \ni p} \mathcal{F}(B) &\simeq \quad \mathcal{F}_p \\ [(B, s)] &\mapsto [(B, s)] \end{aligned}$$

where the colimit is taken over $B \in \mathcal{B}$ such that $p \in B$.

Proof. The main point to show that φ is well-defined is that the system of maps $\mathcal{F}(B) \to \mathcal{F}_p$ for each $B \in \mathcal{B}$ containing p induce the map φ by the universal property. Concretely, if $(B_1, s_1) \sim (B_2, s_2)$, then we can find $B \subseteq B_1 \cap B_2$ such that $s_1|_B = s_2|_B$, which means that $[(B_1, s_1)] = [(B_2, s_2)]$ in $\varinjlim_{B \ni p} \mathcal{F}(B)$ implies the equality in $\mathcal{F}_p$. Now, any structure that φ needs to preserve (e.g., being a homomorphism of some kind) will be immediately preserved.

We now exhibit the map in the reverse direction. Note that any $U \subseteq X$ containing p can find some basis element $B \in \mathcal{B}$ such that $p \in B \subseteq U$. As such, we define $\psi\colon \mathcal{F}_p \to \varinjlim_{B \ni p} \mathcal{F}(B)$ by

$$\psi\colon [(U, s)] \mapsto [(B, s|_B)].$$

To show that this map is well-defined, first note that ψ does not depend on B: if we have basis sets B_1 and B_2 inside U containing p, we can find basis sets $B \subseteq B_1 \cap B_2$ giving $s|_{B_1}|_B = s|_B = s|_{B_2}|_B$, so

$$[(B_1, s|_{B_1})] = [(B_2, s|_{B_2})].$$

Second, note that ψ does not depend on the representative of $[(U, s)]$. Indeed, if $(U_1, s_1) \sim (U_2, s_2)$, then we are promised $U \subseteq U_1 \cap U_2$ such that $s_1|_U = s_2|_U$. Now, find B contained in U containing p, so we see $s_1|_B = s_2|_B$, so

$$[(B, s_1|_B)] = [(B, s_2|_B)].$$

So we have a well-defined map ψ.

We now show that ψ and φ are inverse. In one direction, given some $[(B, s)]$, we note we can write

$$\psi(\varphi([(B, s)])) = \psi([(B, s)]) = [(B, s)],$$

where the last equality is legal because B is a basis set containing p which is contained in B. In the other direction, given some $[(U, s)]$, find a basis set $B \subseteq U$ containing p so that

$$\varphi(\psi([(U, s)])) = \varphi([(B, s|_B)]) = [(B, s|_B)],$$

and we note that $[(B, s|_B)] = [(U, s)]$ because $B \subseteq U$ has $s|_B|_B = s|_B$. ∎

Lemma 1.101. Fix a ring A. Then, for any prime $\mathfrak{p}$, $A_\mathfrak{p} \simeq \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ induced by $a \mapsto a|_\mathfrak{p}$.

Proof. The point here is that $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ permits denominators from anyone in $A \setminus \mathfrak{p}$. In one direction, note that $A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$, so there is a canonical map

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) & \to & \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} \\ s & \mapsto & s|_\mathfrak{p} \end{array}$$

because $\mathfrak{p} \in \operatorname{Spec} A$. Call this map φ. Note, for any $f \in A \setminus \mathfrak{p}$, we see that $\mathfrak{p} \in D(f)$, so the canonical map

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) \to \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$$

permits us to write

$$[(\operatorname{Spec} A, f)] \cdot [(D(f), 1/f)] = [(D(f), f)] \cdot [(D(f), 1/f)] = [(D(f), 1)]$$

is the unit element of $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$. Thus, $\varphi(f) \in \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}^\times$ for each $f \in A \setminus \mathfrak{p}$, so φ induces a natural map $\varphi\colon A_\mathfrak{p} \to \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ sending a/f to $[(D(f), a/f)]$.

In the other direction, we can directly pick up any $[(D(f), a/f^n)] \in \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$, where we are thinking about the colimit as happening over the distinguished base according to Lemma 1.100. Now, $\mathfrak{p} \in D(f)$ is equivalent to $f \notin \mathfrak{p}$, so $f \in A_\mathfrak{p}^\times$, so we can define $\psi\colon \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} \to A_\mathfrak{p}$ by

$$\psi\colon\ [(D(f), a/f^n)] \mapsto a/f^n.$$

To see that ψ is well-defined, note $(D(f_1), a_1/f_1^{n_1}) \sim (D(f_2), a_2/f_2^{n_2})$ means we can find $D(f) \subseteq D(f_1) \cap D(f_2)$ containing $\mathfrak{p}$ with

$$f^n \left(f_2^{n_2} a_1 - f_1^{n_1} a_2\right) = 0$$

in A. Rearranging, it follows that $a_1/f_1^{n_1} = a_2/f_2^{n_2}$ in $A_\mathfrak{p}$.

We won't bother checking that ψ is a ring map; just look at it. However, we will check that ψ and φ are inverses (which tells us that ψ is a ring map automatically). Well, given $a/f \in A_\mathfrak{p}$, we see

$$\psi(\varphi(a/f)) = \psi([(D(f), a/f)]) = a/f.$$

On the other hand, given $[(D(f), a/f^n)] \in \mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$, write

$$\varphi(\psi([(D(f), a/f^n)])) = \varphi\left(a/f^n\right) = [(D(f^n), a/f^n)] = [(D(f), a/f)],$$

where the last equality holds because $D(f^n) = (\operatorname{Spec} A) \setminus V((f^n)) = (\operatorname{Spec} A) \setminus V(\operatorname{rad}(f^n)) = (\operatorname{Spec} A) \setminus V((f)) = D(f)$. ∎

Remark 1.102. Notably, $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ is always a local ring, and the maximal ideal $\mathfrak{p}A_\mathfrak{p}$ corresponds to germs $[(D(f), a/f)]$ such that $a/f \in \mathfrak{p}A_\mathfrak{p}$, or equivalently, such that $a \in \mathfrak{p}$. Namely, the maximal ideal consists of our germs which vanish at $\mathfrak{p}$.

Example 1.103. Continuing from Exercise 1.72, set $X := \mathbb{C}$ and $\mathcal{O}_X$ to be the sheaf of holomorphic functions. Then, for any $z_0 \in X$, we have

$$\mathcal{O}_{X,z_0} = \left\{ \sum_{n=0}^\infty a_n(z - z_0)^n \text{ with positive radius of convergence} \right\}.$$

Indeed, any germ $[(U, f)]$ with f holomorphic actually has f analytic, so f is equal to a (unique) power series of the given form in some small enough neighborhood. And of course, each power series with positive radius of convergence gives rise to a germ.

Remark 1.104 (Nir). As in Remark 1.102, we note that $\mathcal{O}_{X,z_0}$ is a local ring with maximal ideal

$$\mathfrak{m}_{X,z_0} = \left\{ \sum_{n=1}^{\infty} a_n(z-z_0)^n \text{ with positive radius of convergence} \right\}.$$

Of course $\mathfrak{m}_{X,z_0} \subseteq \mathcal{O}_{X,z_0}$ is an ideal. Conversely, one can see that any germ $[(f,U)]$ with $f(z_0) \neq 0$ is nonzero in some neighborhood around z_0 (by continuity) and therefore is invertible in $\mathcal{O}_{X,z_0}$, so $\mathcal{O}_{X,z_0} \setminus \mathfrak{m}_{X,z_0} = \mathcal{O}_{X,z_0}^\times$.

1.4.3 Stalk Memory

Here is why we care about stalks.

Idea 1.105. Stalks remember everything about a sheaf.

Again, the reason why we expect Idea 1.105 to be true is that the stalk is able to remember local information, so having all the local information should be able to recover the original sheaf. Here is a rigorization.

Proposition 1.106. Fix a sheaf $\mathcal{F}$ and a presheaf $\mathcal{G}$ on X. Also, fix an open subset $U \subseteq X$.

(a) The natural embedding

$$\begin{aligned} \iota\colon \mathcal{F}(U) &\to \prod_{p\in U} \mathcal{F}_p \\ f &\mapsto (f|_p)_{p\in U} \end{aligned}$$

is injective.

(b) A tuple $(f_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p$ is in $\operatorname{im}\iota$ if and only if, for each $p \in U$, there is an open set U_p containing p such that we can find $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that all $q \in U_p$ have $f_q = \widetilde{f}_p|_q$.

Remark 1.107. Intuitively, part (b) is saying that all stalks in a small neighborhood come from a single section.

Proof. Here we go.

(a) We use the identity axiom on $\mathcal{F}$. Suppose that $f, g \in \mathcal{F}(U)$ have $f|_p = g|_p$ for all $p \in U$. Thus, for each $p \in U$, we can find $U_p \subseteq U$ containing p such that $f|_{U_p} = g|_{U_p}$.

Now, $U \subseteq \bigcup_{p\in U} U_p \subseteq U$, so $\{U_p\}_{p\in U}$ is an open cover for U, so the identity axiom on $\mathcal{F}$ forces $f = g$.

(b) We use the gluability axiom on $\mathcal{F}$. In one direction, suppose $\iota(f) = (f_p)_{p\in U}$ so that $f|_p = f_p$ for each $p \in U$. This means that, for each $p \in U$, we can set $U_p := U$ and $\widetilde{f}_p := f \in \mathcal{F}(U_p)$ so that any $q \in U_p$ have

$$f_q = f|_q = \widetilde{f}_p|_q.$$

In the other direction, suppose we have germs $(f_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p$ such that any $p \in U$ has an open set U_p and a section $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $f_q = \widetilde{f}_p|_q$ for any $q \in U_p$. We claim that

$$\widetilde{f}_p|_{U_p\cap U_q} \stackrel{?}{=} \widetilde{f}_q|_{U_p\cap U_q}. \tag{1.4}$$

Well, for any $r \in U_p \cap U_q$, we know that $\widetilde{f}_p|_{U_p\cap U_q}|_r = f_r = \widetilde{f}_q|_{U_p\cap U_q}|_r$, so there is an open set $V_r \subseteq U_p \cap U_q$ containing r such that

$$\widetilde{f}_p|_{U_p\cap U_q}|_{V_r} = \widetilde{f}_q|_{U_p\cap U_q}|_{V_r}.$$

Now, applying the identity axiom of $\mathcal{F}$ on the open cover $\{V_r\}_{r\in U_p\cap U_q}$ forces (1.4). Thus, the gluability axioms grants $f \in \mathcal{F}(U)$ such that $f|_{U_p} = \widetilde{f}_p$ for each $p \in U$, so it follows that

$$f|_p = f|_{U_p}|_p = f_p$$

for each $p \in U$. ■

We are going to want a name for the condition in Proposition 1.106 (b).

Definition 1.108 (Compatible germ)**.** Fix a sheaf $\mathcal{F}$ on a topological space X. Then, given a subset $U \subseteq X$, a *system of compatible germs* is a tuple $(f_p)_{p\in U}$ such that, for each $p \in U$, there is an open set U_p containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that all $q \in U_p$ have $f_q = \widetilde{f}_p|_q$.

As a quick sanity check, we can see by hand that morphisms preserve compatibility.

Lemma 1.109. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. If $(f_p)_{p\in U}$ is a system of compatible germs for $\mathcal{F}(U)$, then $(\varphi_p f_p)_{p\in U}$ is a system of compatible germs for $\mathcal{G}(U)$.

Proof. For each $p \in U$, we can find $U_p \subseteq U$ containing p and a lift $\widetilde{f}_p$ so that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. Thus, for each p, we set $\widetilde{g}_p := \varphi_{U_p}(\widetilde{f}_p)$ so that any $q \in U_p$ has

$$\widetilde{g}_p|_q = \varphi_{U_p}(\widetilde{f}_p)|_q = [(U_p, \varphi_{U_p}(\widetilde{f}_p))] = \varphi_q([(U_p, \widetilde{f}_p)]) = \varphi_q(\widetilde{f}_p|_q) = \varphi_q(f_q),$$

which finishes our check. ■

In addition to sections, stalks also remember morphisms.

Proposition 1.110. Fix presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X with a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$.

(a) For any $p \in X$, there is a natural map $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$.

(b) Suppose $\mathcal{G}$ is a sheaf. Given presheaf morphisms $\varphi, \varphi'\colon \mathcal{F} \to \mathcal{G}$ such that $\varphi_p = \varphi'_p$ for all $p \in X$, we have $\varphi = \varphi'$.

Proof. We go in sequence.

(a) It is possible to induce this map from abstract nonsense. Alternatively, we can write this explicitly as being induced by
$$\varphi_p\colon [(U, s)] \mapsto [(U, \varphi_U(s))].$$
To see that φ_p is well-defined, suppose $(U_1, s_1) \sim (U_2, s_2)$ so that we have some $U \subseteq U_1 \cap U_2$ with $s_1|_U = s_2|_U$. Then
$$\varphi_{U_1}(s_1)|_U = \varphi_U(s_1|_U) = \varphi_U(s_2|_U) = \varphi_{U_2}(s_2)|_U,$$
so $(U_1, \varphi_U(s_1)) \sim (U_2, \varphi_U(s_2))$. Now, φ_p will preserve whatever extra structure we need it to because it is essentially induced by the φ_U.

(b) Fix an open set $U \subseteq X$ so that we need $(\varphi_1)_U = (\varphi_2)_U$. Now, the point is that any $\psi\colon \mathcal{F} \to \mathcal{G}$ will make the diagram
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\psi_U"'] & \prod\limits_{p\in U} \mathcal{F}_p \arrow[d, "\prod \psi_p"] \\ \mathcal{G}(U) \arrow[r, hook] & \prod\limits_{p\in U} \mathcal{G}_p \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & (f|_p)_{p\in U} \arrow[d, mapsto] \\ \psi_U f \arrow[r, mapsto] & ((\psi_U f)|_p)_{p\in U} \end{tikzcd}$$
commute. In particular, if $\varphi_p = \varphi'_p$ for all $p \in X$, then we see that $\varphi_U(f)|_p = \varphi_p(f) = \varphi'_p(f) = \varphi_U(f)|_p$. Thus, the injectivity of the map $\mathcal{G}(U) \to \prod_{p\in U} \mathcal{G}_p$ of Proposition 1.106 forces $\varphi_U(f) = \varphi_U(f)$. ■

Remark 1.111. It is not hard to see that $(-)_p\colon \operatorname{PreSh}_X \to \mathcal{C}$ is a functor, where $\mathcal{C}$ is the target category for our sheaves. We can see this because we're just computing limits, but we can also see this concretely. We have already described the action on (pre)sheaves and morphisms, so it remains to check functoriality. Fix $p \in X$.

- Identity: note that $[(U,f))] \in \mathcal{F}_p$ has $(\mathrm{id}_{\mathcal{F}})_p\colon [(U,f)] \mapsto [(U,f)]$.
- Functoriality: given $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ as well as $[(U,f)] \in \mathcal{F}_p$, we have

$$\psi_p(\varphi_p([(U,f)])) = \psi_p([U,\varphi_U f]) = [(U,(\psi\circ\varphi)_U f)] = (\psi\circ\varphi)_p([(U,f)]).$$

1.4.4 The Category of Sheaves Is Additive

We are going to want to do category theory on sheaves, so let's begin. Our end goal is to show that the category of (pre)sheaves over a topological space X valued in an abelian category is itself abelian. Throughout, our target category for our sheaves will be abelian (and concrete). Explicitly, the target category will essentially be a subcategory of Mod_R always.

To begin, we need to show that we can give morphisms of sheaves an abelian group structure.

Lemma 1.112. Fix presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X. Then, given morphisms $\varphi,\psi\colon \mathcal{F} \to \mathcal{G}$, we can define

$$(\varphi+\psi)_U \coloneqq \varphi_U + \psi_U$$

for each $U \subseteq X$. Then $(\varphi+\psi)\colon \mathcal{F} \to \mathcal{G}$ is a presheaf morphism. This operation $+$ makes $\operatorname{Mor}(\mathcal{F},\mathcal{G})$ an abelian group, and composition of morphisms distributes over addition.

Proof. To check that $\varphi+\psi$ is a presheaf morphism, pick up a containment of open sets $V \subseteq U$, and we need to check that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{(\varphi+\psi)_U} & \mathcal{G}(U) \\ {\scriptstyle \operatorname{res}_{U,V}}\downarrow & & \downarrow{\scriptstyle \operatorname{res}_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{(\varphi+\psi)_V} & \mathcal{G}(B) \end{array}$$

commutes. Well, for any $s \in \mathcal{F}(U)$, we note

$$(\varphi+\psi)_U(s)|_V = (\varphi_U s + \psi_U s)|_V = \varphi_U(s)|_V + \psi_U(s)|_V \overset{*}{=} \varphi_V(s|_V) + \psi_V(s|_V) = (\varphi+\psi)_V(s|_V),$$

where we have used the fact that φ and ψ are presheaf morphisms in $\overset{*}{=}$.

To check that $\operatorname{Mor}(\mathcal{F},\mathcal{G})$ is an abelian group under $+$, we note that

$$\operatorname{Mor}(\mathcal{F},\mathcal{G}) \subseteq \prod_{U\subseteq X} \operatorname{Mor}(\mathcal{F}(U),\mathcal{G}(U)),$$

where the latter is a product group under the same addition operation. We have already established that $\operatorname{Mor}(\mathcal{F},\mathcal{G})$ is closed under the addition operation. So we have two more checks to establish that we have a subgroup.

- Zero: the zero element $0 \in \operatorname{Mor}(\mathcal{F},\mathcal{G})$ is then made of the zero morphisms $0_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ sending all elements to zero. The uniqueness of zero morphisms ensures that $0\colon \mathcal{F} \to \mathcal{G}$ is a presheaf morphism. Namely, any $V \subseteq U$ and $s \in \mathcal{F}(U)$ gives $0_U(s)|_V = 0 = 0_V(s|_V)$.
- Inverses: given a sheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, we define $(-\varphi)_U \coloneqq -\varphi_U$ for each $U \subseteq X$. The $(-\varphi)$ assembles into a presheaf morphism: for any $V \subseteq U$ and $s \in \mathcal{F}(U)$, we see that $(-\varphi)_U(s)|_V = -\varphi_U(s)|_V = -\varphi_V(s|_V) = (-\varphi)_V(s|_V)$.

It remains to check distributivity. Let $\varphi_1, \varphi_1 \colon \mathcal{F} \to \mathcal{G}$ and $\psi_1, \psi_2 \colon \mathcal{G} \to \mathcal{H}$ be presheaf morphisms. Then, for any $U \subseteq X$ and $s \in \mathcal{F}(U)$, we compute

$$\begin{aligned}((\psi_1+\psi_2)\circ(\varphi_1+\varphi_2))_U(s) &= (\psi_1+\psi_2)_U\big((\varphi_1+\varphi_2)_U(s)\big)\\ &= ((\psi_1)_U+(\psi_2)_U)\big((\varphi_1)_U(s)+(\varphi_2)_U(s)\big)\\ &= ((\psi_1)_U+(\psi_2)_U)\big((\varphi_1)_U(s)\big)+((\psi_1)_U+(\psi_2)_U)\big((\varphi_2)_U(s)\big)\\ &= (\psi_1\circ\varphi_1)_U(s)+(\psi_2\circ\varphi_1)_U(s)+(\psi_1\circ\varphi_2)_U(s)+(\psi_2\circ\varphi_2)_U(s)\\ &= (\psi_1\circ\varphi_1+\psi_2\circ\varphi_1+\psi_1\circ\varphi_2+\psi_2\circ\varphi_2)_U(s),\end{aligned}$$

so $(\psi_1+\psi_2)\circ(\varphi_1+\varphi_2) = \psi_1\circ\varphi_1+\psi_2\circ\varphi_1+\psi_1\circ\varphi_2+\psi_2\circ\varphi_2$ follows. ■

Remark 1.113. Of course, replacing all presheaves with sheaves in Lemma 1.112 makes the statement still true because sheaf morphisms are just presheaf morphisms. This will be a recurring theme.

Continuing, we should define a zero presheaf.

Definition 1.114 (Zero presheaf)**.** Given a topological space X, the *zero presheaf* on X is the presheaf $\mathcal{Z}$ such that $\mathcal{Z}(U) = 0$ for all open $U \subseteq X$.

Lemma 1.115. The zero presheaf $\mathcal{Z}$ on X is the zero object in the category PreSh_X.

Proof. The restriction maps for $\mathcal{Z}$ are all zero maps; the functoriality checks are all immediate because zero maps are unique (namely, $\mathrm{id}_0 = 0$ and $0 \circ 0 = 0$). Now, given any presheaf $\mathcal{F}$, we need to exhibit unique presheaf morphisms to and from $\mathcal{Z}$.

- Initial: we show there is a unique sheaf morphism $\varphi\colon \mathcal{Z} \to \mathcal{F}$. For uniqueness, note that any $U \subseteq X$ needs a map
$$\varphi_U \colon \mathcal{Z}(U) \to \mathcal{F}(U),$$
so because $\mathcal{Z}(U) = 0$ is initial, there is a unique possible map. To check that this data actually assembles into a presheaf morphism, we need to check that any containment of open sets $V \subseteq U$ causes the diagram
$$\begin{tikzcd} \mathcal{Z}(U) \arrow[r, "0"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ \mathcal{Z}(V) \arrow[r, "0"] & \mathcal{F}(V) \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "0"] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ 0 \arrow[r, "0"] & \mathcal{F}(V) \end{tikzcd}$$
commutes, which is clear by the uniqueness of our zero maps. Namely, the map $0 \to 0 \to \mathcal{F}(V)$ and $0 \to \mathcal{F}(U) \to \mathcal{F}(V)$ must both just be the map $0 \to \mathcal{F}(V)$.
- Terminal: one merely has to reverse all the arrows in the previous argument. Notably, the zero object 0 in the target category of $\mathcal{Z}$ is terminal in addition to being initial. ■

As in Remark 1.113, we can quickly move the zero presheaf to being the zero sheaf.

Corollary 1.116. The zero presheaf $\mathcal{Z}$ on a topological space X is a sheaf and hence the zero object in the category Sh_X.

Proof. The main point here is that the zero presheaf $\mathcal{Z}$ is in fact a sheaf. This is easy to check: fix an open cover $\mathcal{U}$ of an open set $U \subseteq V$. If we are given sections $f, g \in \mathcal{Z}(U)$, then we don't even need any other conditions to know that
$$f = g \in \mathcal{Z}(U) = 0$$

because there is only one element in the zero object. Similarly, given sections $f_V \in \mathcal{Z}(V)$ for each $V \in \mathcal{U}$, we note that $f_V = 0$ everywhere, so we can set $f_U = 0 \in \mathcal{Z}(U)$ so that $f|_V = f_V$; this proves the gluability axiom.

We now check the universal property. Given any sheaf $\mathcal{F}$, we know from Lemma 1.115 that there are unique presheaf morphisms $\mathcal{F} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{F}$. Because sheaf morphisms are presheaf morphisms, it follows that there are unique sheaf morphisms as well. ■

To show that our category of (pre)sheaves is additive, it remains to exhibit (finite) products.

Definition 1.117 (Product presheaf)**.** Given presheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on a topological space X, the *product presheaf* $\mathcal{F}_1 \times \mathcal{F}_2$ by

$$(\mathcal{F}_1 \times \mathcal{F}_2)(U) := \mathcal{F}_1(U) \times \mathcal{F}_2(U)$$

with the restriction maps induced by $\mathcal{F}_1$ and $\mathcal{F}_2$.

Lemma 1.118. Given presheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X, the product presheaf $\mathcal{F}_1 \times \mathcal{F}_2$ is the categorical product in PreSh_X.

Proof. We begin by showing that $\mathcal{F}_1 \times \mathcal{F}_2$ is in fact a presheaf. To be explicit, our restriction maps for opens $V \subseteq U \subseteq X$ are

$$\begin{aligned} \operatorname{res}_{U,V} \colon (\mathcal{F}_1 \times \mathcal{F}_2)(U) &\to (\mathcal{F}_1 \times \mathcal{F}_2)(V) \\ (f_1, f_2) &\mapsto (f_1|_V, f_2|_V). \end{aligned}$$

Here are our presheaf checks.

- Identity: with an open $U \subseteq X$ and $(f_1, f_2) \in (\mathcal{F}_1 \times \mathcal{F}_2)(U)$, we have $(f_1, f_2)|_U = (f_1|_U, f_2|U) = (f_1, f_2)$.
- Functoriality: with opens $W \subseteq V \subseteq U$ and $(f_1, f_2) \in (\mathcal{F}_1 \times \mathcal{F}_2)(U)$, we have

$$(f_1, f_2)|_V|_W = (f_1|_V|_W, f_2|_V|_W) = (f_1|_W, f_2|_W) = (f_1, f_2)|_W.$$

It remains to show our universal property for products. Given an open $U \subseteq X$, define $(\pi_1)_U \colon (\mathcal{F}_1 \times \mathcal{F}_2)(U) \to \mathcal{F}_1(U)$ by projection onto the first coordinate. To show that π_1 assembles into a presheaf morphism, pick up opens $V \subseteq U \subseteq X$ and $(f_1, f_2) \in (\mathcal{F}_1 \times \mathcal{F}_2)(U)$ and check

$$(\pi_1)_U((f_1, f_2))|_V = f_1|_V = (\pi_1)_V((f_1|_V, f_2|_V)).$$

We can define the presheaf morphism $\pi_2 \colon (\mathcal{F}_1 \times \mathcal{F}_2) \to \mathcal{F}_2$ by projection onto the second coordinate, which is a presheaf morphism by symmetry.

For our presheaf morphism, suppose that we have a presheaf $\mathcal{G}$ with maps $\varphi_1 \colon \mathcal{G} \to \mathcal{F}_1$ and $\varphi_2 \colon \mathcal{G} \to \mathcal{F}_2$. We need a unique presheaf morphism $\varphi \colon \mathcal{G} \to (\mathcal{F}_1 \times \mathcal{F}_2)$ making the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\varphi_2"] \arrow[ddr, bend right, "\varphi_1"'] & & \\ & \mathcal{F}_1 \times \mathcal{F}_2 \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & \mathcal{F}_2 \\ & \mathcal{F}_1 & \end{tikzcd} \tag{1.5}$$

commute. We show uniqueness and existence separately.

- Uniqueness: if $\varphi \colon \mathcal{G} \to (\mathcal{F}_1 \times \mathcal{F}_2)$ makes (1.5) commute, at any given open $U \subseteq X$ and $g \in \mathcal{G}(U)$, we must have

$$(\pi_1)_U(\varphi_U g) = (\varphi_1)_U(g) \qquad \text{and} \qquad (\pi_2)_U(\varphi_U g) = (\varphi_2)_U(g),$$

so $\varphi_U(g) := ((\varphi_1)_U g, (\varphi_2)_U g)$ is forced.

- Existence: as above, given an open $U \subseteq X$ and $g \in \mathcal{G}(U)$, define
$$\varphi_U(g) \coloneqq ((\varphi_1)_U g, (\varphi_2)_U g).$$
We can see, as above, that $(\pi_1)_U \circ \varphi_U = (\varphi_1)_U$ and similar for π_2, so (1.5) will commute as long as φ actually assembles into a presheaf morphism.
Well, given $V \subseteq U \subseteq X$ and $g \in \mathcal{G}(U)$, note
$$\varphi_U(g)|_V = ((\varphi_1)_U g, (\varphi_2)_U g)|_V = ((\varphi_1)_U(g)|_V, (\varphi_2)_U(g)|_V) = ((\varphi_1)_V(g|_V), (\varphi_2)_V(g|_V)) = \varphi_V(g|_V),$$
which finishes. ■

Corollary 1.119. Given sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X, the product presheaf $\mathcal{F}_1 \times \mathcal{F}_2$ is a sheaf and hence the categorical product in Sh_X.

Proof. As in Corollary 1.116, the main point is to show that $\mathcal{F}_1 \times \mathcal{F}_2$ is in fact a sheaf. Fix an open cover $\mathcal{U}$ of U.

- Identity: given $(f_1, f_2) \in (\mathcal{F}_1 \times \mathcal{F}_2)(U)$ with $(f_1, f_2)|_V = 0$ for all $V \in \mathcal{U}$, we see $f_1|_V = f_2|_V = 0$ is forced for all $V \in \mathcal{U}$, so the identity axiom on $\mathcal{F}_1$ and $\mathcal{F}_2$ forces $f_1 = f_2 = 0$. Thus, $(f_1, f_2) = 0$.
- Gluability: pick up sections $(f_{1,V}, f_{2,V})(\mathcal{F}_1 \times \mathcal{F}_2)(V)$ for each $V \in \mathcal{U}$ such that any $V, V' \in \mathcal{U}$ have
$$(f_{1,V}|_{V\cap V'}, f_{2,V}|_{V\cap V'}) = (f_{1,V}, f_{2,V})|_{V\cap V'} = (f_{1,V'}, f_{2,V'})|_{V\cap V'} = (f_{1,V'}|_{V\cap V'}, f_{2,V'}|_{V\cap V'}).$$
Thus, the gluability axiom on $\mathcal{F}_1$ and $\mathcal{F}_2$ promises $f_1 \in \mathcal{F}_1(U)$ and $f_2 \in \mathcal{F}_2(U)$ such that $f_1|_V = f_{1,V}$ and $f_2|_V = f_{2,V}$ for each $V \in \mathcal{U}$. Thus, $(f_1, f_2)|_V = (f_1|_V, f_2|_V) = (f_{1,V}, f_{2,V})$ for each $V \in \mathcal{U}$, as needed.

We now discuss the universal property. This immediately follows from the corresponding statement in the category of presheaves, but for completeness, we will say out loud what's going on. Let $\pi_1 \colon (\mathcal{F}_1 \times \mathcal{F}_2) \to \mathcal{F}_1$ and $\pi_2 \colon (\mathcal{F}_1 \times \mathcal{F}_2) \to \mathcal{F}_2$ be the projection (pre)sheaf morphisms.

Suppose we have a sheaf $\mathcal{G}$ with sheaf morphisms $\varphi_1 \colon \mathcal{G} \to \mathcal{F}_1$ and $\varphi_2 \colon \mathcal{G} \to \mathcal{F}_2$. Then we are promised a unique presheaf morphism $\varphi \colon \mathcal{G} \to (\mathcal{F}_1 \times \mathcal{F}_2)$ such that $\varphi_1 = \pi_1 \circ \varphi$ and $\varphi_2 = \pi_2 \circ \varphi$. Thus, there is also a unique sheaf morphism φ satisfying the same constraint because sheaf morphisms are just presheaf morphisms. ■

Remark 1.120. The above discussion immediately generalizes to arbitrary products, but we will not need these.

Corollary 1.121. The category Sh_X of sheaves on a topological space X valued in a (concrete) abelian category $\mathcal{C}$ is additive.

Proof. Combine Lemma 1.112, Corollary 1.116, and Corollary 1.119. ■

1.4.5 Sheaf Kernels

We continue working with (pre)sheaves valued in a concrete abelian category. The next step to show that the category is abelian is to exhibit kernels and cokernels. Cokernels will turn out to be difficult, so we begin with kernels.

Definition 1.122 (Presheaf kernel)**.** Given a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X, we define the *presheaf kernel* as

$$(\ker\varphi)(U) := \ker\varphi_U$$

for each $U \subseteq X$, where restriction maps are induced by $\mathcal{F}$. Then $\ker\varphi$ is our *presheaf kernel.*

Lemma 1.123. Given a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X, the presheaf kernel $\ker\varphi$ is a categorical kernel.

Proof. We haven't actually defined the restriction maps for the presheaf kernel, so we do this now: for each open $U \subseteq X$ with $V \subseteq U$, note $\ker\varphi_U \subseteq \mathcal{F}(U)$, so we can restrict the map $\operatorname{res}_{U,V}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ to a map

$$\ker\varphi_U \to \mathcal{F}(V).$$

Now, for any $s \in \ker\varphi_U$, we note that actually $\varphi_V(s|_V) = \varphi_U(s)|_V = 0$, so our restriction map restricts to $\operatorname{res}_{U,V}\colon \ker\varphi_U \to \ker\varphi_V$ as needed. The presheaf checks on $\ker\varphi$ of identity and functoriality checks are inherited from $\mathcal{F}$.

It remains to check the universal property: we need $\ker\varphi$ to be the limit of the following diagram.

$$\begin{tikzcd} & \mathcal{Z} \arrow[d] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

There is an inclusion $\ker\varphi_U \subseteq \mathcal{F}(U)$ for each open $U \subseteq X$, which induces maps $\iota_U\colon (\ker\varphi)(U) \to \mathcal{F}(U)$. To see that ι_U assembles into a presheaf morphism, pick up a containment $V \subseteq U$ and $s \in \mathcal{F}(U)$, and we check $\iota_U(s)|_V = s|_V = \iota_V(s|_V)$. Additionally, there is a canonical 0 map $0\colon (\ker\varphi) \to \mathcal{Z}$, so we claim that the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"] & \mathcal{Z} \arrow[d] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

commutes. Well, for any $U \subseteq X$ and $f \in (\ker\varphi)(U)$, note $\varphi_U(\iota_U(f)) = 0$, so the presheaf morphism $\varphi \circ \iota$ is just the zero morphism, as needed.

We are now ready to show the universal property. Fix a presheaf $\mathcal{H}$ with a map $\psi\colon \mathcal{H} \to \mathcal{F}$ such that $\varphi \circ \psi = 0$. Then we claim that there is a unique map $\overline{\psi}$ making the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[dr, dashed, "\overline{\psi}"] \arrow[drr, bend left] \arrow[ddr, bend right, "\psi"'] & & \\ & \ker\varphi \arrow[r] \arrow[d, "\iota"] & \mathcal{Z} \arrow[d] \\ & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd} \tag{1.6}$$

commute. We show uniqueness and existence separately.

- Uniqueness: for any subset $U \subseteq X$ and $h \in \mathcal{H}(U)$, (1.6) forces

$$\iota_U(\overline{\psi}_U(h)) = \psi_U(h).$$

However, ι_U is just an inclusion (of sets, say), so we must have $\overline{\psi}_U(h) = \iota_U^{-1}(\psi_U(h))$. As such, $\overline{\psi}$ is uniquely determined.

- Existence: for any subset $U \subseteq X$ and $h \in \mathcal{H}(U)$, (1.6) forces $\varphi_U(\psi_U(h)) = 0$, so $\psi_U(h) \in \ker \varphi_U$. So we can restrict the image of ψ_U to define a map

$$\overline{\psi}_U(h) := \psi_U(h).$$

Of course, $\iota_U(\overline{\psi}_U(h)) = \psi_U(h)$, so (1.6) will commute as long as $\overline{\psi}$ assembles into a presheaf morphism. Well, for a containment $V \subseteq U$ and $h \in \mathcal{H}(U)$, we see

$$\overline{\psi}_U(h)|_V = \psi_U(h)|_V = \psi_V(h|_V) = \overline{\psi}_V(h|_V),$$

as needed. ■

What makes the presheaf kernel nice is that is actually the sheaf kernel.

Lemma 1.124. Fix a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$. Then $\ker \varphi$ is a sheaf and hence the categorical kernel.

Proof. As usual, the main point is to show that $\ker \varphi$ is a sheaf. For clarity, label the (canonical) inclusion $\iota\colon (\ker \varphi) \to \mathcal{F}$; note ι_U is injective at each open $U \subseteq X$. Now, fix an open cover $\mathcal{U}$ for an open set $U \subseteq X$.

- Identity: fix $f, g \in (\ker \varphi)(U)$ such that $f|_V = g|_V$ for all $V \in \mathcal{U}$. However, all of this is embedded in $\mathcal{F}$ by ι, so we really have $\iota f, \iota g \in \mathcal{F}(U)$ with $(\iota_U f)|_V = \iota_V(f|_V) = \iota_V(g|_V) = (\iota_U g)|_V$ for all $V \in \mathcal{U}$, so the identity axiom promises that $\iota_U f = \iota_U g$. Thus, $f = g$ follows.

- Gluability: fix sections $f_V \in (\ker \varphi)(V)$ for each $V \in \mathcal{F}(V)$ such that

$$f_V|_{V \cap V'} = f_{V'}|_{V \cap V'}$$

for each $V, V' \in \mathcal{U}$. Embedding everything in $\mathcal{F}$, we see

$$(\iota_V f_V)|_{V \cap V'} = \iota_{V \cap V'}(f_V|_{V \cap V'}) = \iota_{V \cap V'}(f_{V'}|_{V \cap V'}) = (\iota_{V'} f_{V'})|_{V \cap V'},$$

so the gluability axiom on $\mathcal{F}(U)$ tells us there is $f \in \mathcal{F}(U)$ with $f|_V = \iota_V(f_V)$ for each $V \in \mathcal{U}$.

We now need to show $f \in (\ker \varphi)(U)$. Well, for each $V \in \mathcal{U}$, we see

$$\varphi_U(f)|_V = \varphi_V(f|_V) = \varphi_V(f_V) = 0,$$

where the last equality is because $f_V \in (\ker \varphi)(V)$. Thus, the identity axiom on $\mathcal{G}$ tells us $f \in \ker \varphi_U$, so we can pull f back to an element $f \in (\ker \varphi)(U)$ such that $f|_V = f_V$ for each $V \in \mathcal{U}$.

Checking the universal property is a matter of stating it and noting that working in the category PreSh_X immediately forces the universal property to work in the subcategory Sh_X. We showed what this looks like in the last paragraph of Corollary 1.119. ■

Now, having a kernel gives us a definition.

Definition 1.125 (Injective morphism)**.** A morphism of (pre)sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if and only if the kernel (pre)sheaf $\ker \varphi$ is identically zero. Equivalently, we are asking for φ_U to be injective everywhere.

We briefly convince ourselves that this is the correct definition.

Lemma 1.126. Let $\mathcal{C}$ be a category with a zero object and kernels, and fix a morphism $\varphi\colon A \to B$. Then φ is monic if and only if $\ker \varphi$ vanishes.

Proof. This is purely categorical; let $\iota\colon (\ker\varphi) \to A$ be the kernel map. In one direction, suppose that $\ker\varphi$ vanishes. To show φ is monic, write down

$$C \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} A \xrightarrow{\varphi} B$$

with $\varphi\circ\psi_1 = \varphi\circ\psi_2$, we need to show $\psi_1 = \psi_2$. Well, $\psi := \psi_1 - \psi_2$ has $\varphi\circ\psi = 0$, so our kernel promises a unique map $\overline{\psi}\colon C \to (\ker\varphi)$ with $\psi = \iota\circ\overline{\psi}$. However, $\ker\varphi$ is the zero object, so we conclude $\psi = 0$.

In the other direction, suppose φ is monic, and we show that the zero object Z satisfies the universal property of the kernel. Well, fix an object C with a map $\psi\colon C \to A$ such that $\varphi\circ\psi = 0$. Then we need a unique map $\overline{\psi}$ making

$$\begin{tikzcd} C \arrow[r, dashed, "\overline{\psi}"] \arrow[d, "\psi"'] & Z \arrow[dl] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

commute. Well, the map $C \to Z$ is certainly unique because Z is terminal. Additionally, we note that the zero map $C \to Z$ does indeed make the diagram commute: $\varphi\circ\psi = 0 = \varphi\circ 0$ forces $\psi = 0$, so ψ is the zero map. ■

1.4.6 Injectivity at Stalks

In our stalk philosophy, we might hope we can detect injectivity at stalks. Indeed, we can.

Lemma 1.127. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then, for any $p \in X$, the inclusion $(\ker\varphi) \to \mathcal{F}$ induces an isomorphism

$$(\ker\varphi)_p \simeq \ker\varphi_p.$$

Proof. Let $\iota\colon (\ker\varphi) \to \mathcal{F}$ denote the inclusion. Then Proposition 1.110 grants us a map $\iota_p\colon (\ker\varphi)_p \to \mathcal{F}_p$. Now, for any $[(U,f)] \in (\ker\varphi)_p$, we have

$$\varphi_p(\iota_p([(U,f)])) = [(U,\varphi_U(\iota_U(f)))] = [(U,0)] = 0$$

by how these maps are defined in Proposition 1.110. Thus, we can restrict the image of ι_p to $\ker\varphi_p \subseteq \mathcal{F}_p$.

In the other direction, suppose that we have a germ $[(U,f)] \in \ker\varphi_p$ so that $[(U,\varphi_U(f))] = 0$, which means there is $V \subseteq U$ containing p such that $\varphi_V(f|_V) = \varphi_U(f)|_V = 0$. In particular, $f|_V \in \ker\varphi_V$, so we have $[(V,f|_V)] \in (\ker\varphi)_p$. Thus, we define the map $\pi\colon \ker\varphi_p \to (\ker\varphi)_p$ by

$$\pi\colon [(U,f)] \mapsto [(V,f|_V)].$$

Note that π does not depend on the choice of $V \subseteq U$: if $V' \subseteq U$ also have $\varphi_{V'}(f|_{V'}) = 0$, then we note $(V,f|_V) \sim (V',f|_{V'})$ because $f|_V|_{V\cap V'} = f|_{V'}|_{V\cap V'}$. Additionally, π does not depend on the choice of representative for $[(U,f)]$: if $(U,f) \sim (U',f')$ in $\ker\varphi_p$, then find $V \subseteq U \cap U'$ small enough so that $f|_V = f'|_V$ and $\varphi_V(f|_V) = \varphi_V(f'|_V) = 0$ so that $\pi([(U,f)]) = [(V,f|_V)] = [(V,f'|_V)] = \pi([(U,f')])$.

Lastly, we check ι_p and π are inverse. In one direction, given $[(U,f)] \in (\ker\varphi)_p$, we note $\varphi_U(f) = 0$, so

$$\pi(\iota_p([(U,f)])) = \pi([(U,f)]) = [(U,f)].$$

In the other direction, given $[(U,f)] \in \ker\varphi_p$, find $V \subseteq U$ small enough so that $\varphi_V(f|_V) = 0$. Then

$$\iota_p(\pi([(U,f)])) = \iota([(V,f|_V)]) = [(V,f|_V)] = [(U,f)],$$

finishing. ■

Lemma 1.128. Fix a sheaf $\mathcal{F}$ on a topological space X. The following are equivalent.

(a) $\mathcal{F}$ is the zero sheaf.

(b) $\mathcal{F}(U) \simeq 0$ for each open $U \subseteq X$.

(c) $\mathcal{F}_p \simeq 0$ for each $p \in X$.

Proof. Our construction of the zero presheaf tells us that (a) implies (c): any germ $[(U, f)] \in \mathcal{Z}_p$ has $f \in \mathcal{Z}(U) = 0$, so $[(U, f)] = 0$. Note we are using the fact that isomorphic sheaves have isomorphic stalks. To show that (c) implies (b), we note that $\mathcal{F}$ being a sheaf grants us the inclusion

$$\mathcal{F}(U) \hookrightarrow \prod_{p \in U} \mathcal{F}_p$$

by Proposition 1.106. However, the right-hand side is 0, so the left-hand side must also be 0.

Lastly, we show that (b) implies (a). Well, note that the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for an inclusion $V \subseteq U$ are forced because zero morphisms are unique. Similarly, letting $\mathcal{Z}$ denote the zero sheaf, we have isomorphisms $\varphi_U : \mathcal{F}(U) \simeq \mathcal{Z}(U)$ induced by these zero maps for all $U \subseteq X$, and we thus assemble into a natural isomorphism $\varphi : \mathcal{F} \to \mathcal{Z}$ because the uniqueness of zero maps makes the naturality square commute. ∎

Proposition 1.129. Fix a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$. The following are equivalent.

(a) φ is monic.

(b) φ_U is monic for each open $U \subseteq X$.

(c) φ_p is monic for each $p \in X$.

Proof. By Lemma 1.126, these are equivalent to the following.

(a') $\ker \varphi$ vanishes.

(b') $(\ker \varphi)(U)$ vanishes for each open $U \subseteq X$.

(c') $\ker \varphi_p$ vanishes for each $p \in X$. By Lemma 1.127, this is equivalent to $(\ker \varphi)_p$ vanishing for each $p \in X$.

These are equivalent by Lemma 1.128. ∎

Remark 1.130. Technically, we only need to know that $\mathcal{F}$ is a sheaf for Proposition 1.129.

Being careful, one can extend Proposition 1.129 as follows.

Proposition 1.131. Fix a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$. The following are equivalent.

(a) φ is an isomorphism.

(b) φ_U is an isomorphism for each open $U \subseteq X$.

(c) φ_p is an isomorphism for each $p \in X$.

Proof. To begin, (a) and (b) are equivalent by category theory: natural isomorphisms are just natural transformations whose component morphisms are isomorphisms. The main check here is that the inverse morphisms $\varphi^{-1}(U)\colon \mathcal{G}(U) \to \mathcal{F}(U)$ cohere into a bona fide natural transformation, which is true because, for any containment $V \subseteq U$, the commutativity of the left diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(U) \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[l, "\varphi_U^{-1}"'] \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) & \mathcal{G}(V) \arrow[l, "\varphi_V^{-1}"] \end{tikzcd}$$

is equivalent to the commutativity of the right diagram.

Additionally, it is also fairly easy that (a) implies (c); fix some $p \in X$. Give φ an inverse morphism ψ, and we claim that φ_p is the inverse of ψ_p. Indeed, for any $[(U, f)] \in \mathcal{F}_p$, we see

$$\psi_p(\varphi_p([(U,f)])) = \psi_p([(U,\varphi_U(f))]) = [(U,\psi_U\varphi_U(f))] = [(U,f)].$$

By symmetry, we see $\varphi_p \circ \psi_p = \mathrm{id}_{\mathcal{G}_p}$ as well, finishing.

Thus, the hard direction is showing that φ_p being an isomorphism for all $p \in X$ promises that φ_U is an isomorphism for each $U \subseteq X$. Proposition 1.129 already tells us that φ_U is injective, so we focus on showing φ_U is surjective. Well, for any $g \in \mathcal{G}(U)$, we get a system of compatible germs $(g|_p)_{p\in U}$ (by Proposition 1.129), so because φ_p is an isomorphism, we may set

$$f_p \coloneqq \varphi_p^{-1}(g|_p).$$

We claim that f_p is a set of compatible germs, which gives rise to a section $f \in \mathcal{F}(U)$ by Proposition 1.129. Well, giving f_p some representative $[(V_p, s_p)]$, we see $\varphi_p(f_p) = [(V_p, \varphi_{V_p}(s_p))]$. Thus, by appropriately restricting V_p, we see $\varphi_p(f_p) = g|_p$ means that we can find some open $U_p \subseteq U$ containing p and a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that

$$\varphi_{U_p}(\widetilde{f}_p) = g|_{U_p}.$$

In particular, for all $q \in U_p$, we see that

$$\varphi_q(\widetilde{f}_p|_q) = g|_q,$$

so we see that $\widetilde{f}_p|_q = \varphi_q^{-1}(g|_q) = f_q$. This finishes the compatibility check.

Thus, $(f_p)_{p\in U}$ is a system of compatible germs and therefore lifts to some $f \in \mathcal{F}(U)$ with $f|_p = f_p$ everywhere. So $\varphi_U(f)|_p = \varphi_p(f|_p) = \varphi_p(f_p) = g|_p$ for each $p \in X$, so Proposition 1.129 gives $\varphi_U(f) = g$. ■

Remark 1.132. We are avoiding surjectivity for the moment because it is a little trickier. In particular, a morphism φ will be able to be epic without being each φ_U being epic. However, surjectivity will still be equivalent to surjectivity on the stalks.

Remark 1.133. It is possible for sheaves to isomorphic stalks but to not be isomorphic. At a high level, any line bundle over S^1 has stalks isomorphic to $\mathbb{R}$, but not all line bundles are homeomorphic (e.g., the Möbius strip and the trivial line bundle are not homeomorphic). The issue here is that there need not even be a candidate isomorphism between line bundles over S^1 at all!

1.5 September 2

It is another day.

Remark 1.134. Facts used on the homework from Vakil which are in Vakil without proof should be proven on the homework.

We begin lecture by providing an example which we don't quite have the language to describe yet, but we will elaborate on it more later.

elaborate

Example 1.135. Fix $X = \mathbb{C}$ with the usual topology, and give it the sheaf $\mathcal{O}_X$ of holomorphic functions. There is a constant sheaf $\underline{\mathbb{Z}}$ returning $\mathbb{Z}$ at its stalks. Then there is an exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \overset{2\pi i}{\to} \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^\times \to 1 \tag{1.7}$$

even though the last map is not always surjective for any $U \subseteq \mathbb{C}$; for example, take $U = \mathbb{C} \setminus \{0\}$. (However, if U is simply connected, then the map will be surjective.)

Remark 1.136. Cohomology applied to (1.7) (with X some smooth projective curve) shows a special case of the Hodge conjecture.

The point here is that surjectivity cannot be checked on open sets the way that injectivity can. At some level, the issue here is that the cokernel presheaf is not a sheaf, so we have to apply a sheafification operation to fix this.

Remark 1.137. Setting

$$\mathcal{F}(U) := \operatorname{im} \exp(U)$$

makes $\mathcal{F}$ a presheaf but does not give a sheaf.

1.5.1 Sheafification

We introduce sheafification by its universal property.

Definition 1.138 (Sheafification)**.** Fix a presheaf $\mathcal{F}$ on X valued in a (concrete) category $\mathcal{C}$. The *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^{\text{sh}}, \text{sh})$ where $\text{sh}\colon \mathcal{F} \to \mathcal{F}^{\text{sh}}$ satisfies the following universal property: any sheaf $\mathcal{G}$ with a presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ has a unique sheaf morphism $\overline{\varphi}\colon \mathcal{F}^{\text{sh}} \to \mathcal{G}$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\text{sh}"] \arrow[dr, "\varphi"'] & \mathcal{F}^{\text{sh}} \arrow[d, "\overline{\varphi}"] \\ & \mathcal{G} \end{tikzcd}$$

Of course, there are some checks we should do before using this object.

Lemma 1.139. The sheafification of a presheaf $\mathcal{F}$ on X exists and is unique up to (a suitable notion of) unique isomorphism.

Proof. The idea of the construction is to set $\mathcal{F}^{\text{sh}}(U)$ to be systems of compatible germs; precisely,

$$\mathcal{F}^{\text{sh}}(U) := \left\{ (f_p)_{p \in U} \in \prod_{p \in U} \mathcal{F}_p : (f_p)_{p\in U} \text{ is a compatible system of germs} \right\}.$$

Given open sets $V \subseteq U$, we define the restriction map

$$\begin{aligned} \operatorname{res}_{U,V}\colon \mathcal{F}^{\text{sh}}(U) &\to \mathcal{F}^{\text{sh}}(V) \\ (f_p)_{p\in U} &\mapsto (f_p)_{p \in V} \end{aligned}$$

though we do have to check this is well-defined: to show $(f_p)_{p\in V} \in \mathcal{F}^{\mathrm{sh}}(V)$, we note $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$ promises that each $p \in U$ has $U_p \subseteq U$ containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ so that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. As such, each $p \in V$ has $V_p \coloneqq U_p \cap V$ containing p with a lift $\widetilde{f}_p|_{U_p\cap V} \in \mathcal{F}(U_p \cap V)$ so that $\widetilde{f}_p|_{U_p\cap V}|_q = \widetilde{f}_p|_q = f_q$ for each $q \in U_p \cap V$. Thus, $(f_p)_{p\in V}$ is indeed a system of compatible germs.

We now check that $\mathcal{F}^{\mathrm{sh}}$ is a presheaf.

- Identity: given $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, we see $(f_p)_{p\in U}|_U = (f_p)_{p\in U}$.
- Functoriality: given open sets $W \subseteq V \subseteq U$, we see $(f_p)_{p\in U}|_V|_W = (f_p)_{p\in V}|_W = (f_p)_{p\in W} = (f_p)_{p\in U}|_W$.

Next up, we check that $\mathcal{F}^{\mathrm{sh}}$ is a sheaf. Fix an open cover $\mathcal{U}$ of an open set $U \subseteq X$.

- Identity: suppose that $(f_p)_{p\in U}, (g_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$ have $(f_p)_{p\in U}|_V = (g_p)_{p\in U}|_V$ for each $V \in \mathcal{U}$. Now, for each $q \in U$, there is some $V \in \mathcal{U}$ containing q, so we note

  $$(f_p)_{p\in V} = (f_p)_{p\in U}|_V = (g_p)_{p\in U}|_V = (g_p)_{p\in V}$$

  forces $f_q = g_q$. Thus, $(f_p)_{p\in U} = (g_p)_{p\in U}$.

- Gluability: suppose we have $(f_{V,p})_{p\in V} \in \mathcal{F}^{\mathrm{sh}}(V)$ for each $V \in \mathcal{U}$ so that

  $$(f_{V,p})_{p\in V\cap V'} = (f_{V,p})_{p\in V}|_{V\cap V'} = (f_{V',p})_{p\in V'}|_{V\cap V'} = (f_{V',p})_{p\in V\cap V'}.$$

  Now, for each $q \in U$, find any $V \in \mathcal{U}$ containing q, and set $f_q \coloneqq f_{V,q}$. Note that this is independent of the choice of V: if we have $q \in V \cap V'$ with $V, V' \in \mathcal{U}$, then $(f_{V,p})_{p\in V\cap V'} = (f_{V',p})_{p\in V\cap V'}$ tells us that $f_{V,q} = f_{V',q}$. Further, we note that $(f_p)_{p\in U}|_V = (f_p)_{p\in V} = (f_{V,p})_{p\in V}$ for any $V \in \mathcal{U}$.

  So it remains to show that $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$. Well, for each $p \in U$, find some $V \in \mathcal{U}$ containing p. Then $(f_{V,p})_{p\in V}$ is a system of compatible germs, so we can find $U_p \subseteq V$ containing p and a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that

  $$\widetilde{f}_p|_q = f_{V,q} = f_q$$

  for each $q \in U_p$. This finishes checking that $(f_p)_{p\in U}$ is a compatible system of germs.

We now begin showing the universal property. The sheafification map is defined as

$$\begin{aligned} \mathrm{sh}_U \colon \mathcal{F}(U) &\to \mathcal{F}^{\mathrm{sh}}(U) \\ f &\mapsto (f|_p)_{p\in U} \end{aligned}$$

for any open set $U \subseteq X$. Note $f \in \mathcal{F}(U)$ does indeed give $(f|_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$ because each $p \in U$ can choose $U_p \coloneqq U$ (which contains p) with lift $\widetilde{f}_p \coloneqq f$ so that $\widetilde{f}_p|_q = f|_q$ for each $q \in U_p$.

Additionally, it is fairly quick to check that sh is actually a presheaf morphism: given open sets $V \subseteq U$ and $f \in \mathcal{F}(U)$, we compute

$$\mathrm{sh}_U(f)|_V = (f|_p)_{p\in U}|_V = (f|_p)_{p\in V} = (f|_V|_p)_{p\in V} = \mathrm{sh}_V(f|_V).$$

We are now ready to prove the universal property. Fix any sheaf $\mathcal{G}$ with a presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$. We need to show there is a unique sheaf morphism $\overline{\varphi}\colon \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ such that $\varphi = \overline{\varphi} \circ \mathrm{sh}$. We show these separately.

- Uniqueness: fix an open set $U \subseteq X$ and $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, and we will solve for $\overline{\varphi}_U((f_p)_{p\in U})$. Well, each $p \in U$ has some $U_p \subseteq U$ containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. As such, for each $q \in U$,

  $$\overline{\varphi}_U((f_p)_{p\in U})|_{U_q} = \overline{\varphi}_{U_q}((f_p)_{p\in U}|_{U_q}) = \overline{\varphi}_{U_q}((f_p)_{p\in U_q}) = \overline{\varphi}_{U_q}\big((\widetilde{f}_q|_p)_{p\in U_q}\big) = \overline{\varphi}_{U_q}(\mathrm{sh}_{U_q}\widetilde{f}_q) = \varphi_{U_q}(\widetilde{f}_q).$$

  Thus, restrictions $\overline{\varphi}_U((f_p)_{p\in U})|_{U_q}$ are fixed by φ, so the identity axiom on $\mathcal{G}$ makes $\overline{\varphi}_U((f_p)_{p\in U})$ unique.

- Existence: fix an open set $U \subseteq X$ and $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, and we will define $\overline{\varphi}_U((f_p)_{p\in U})$. Well, $(\varphi_p f_p)_{p\in U}$ is a system of compatible germs in $\mathcal{G}(U)$ by Lemma 1.109, so there is a unique $g \in \mathcal{G}(U)$ such that $g|_p = \varphi_p(f_p)$ for each $p \in U$. (Uniqueness is by Proposition 1.106.) Thus, we set $\overline{\varphi}_U((f_p)_{p\in U}) \coloneqq g$ so that $\overline{\varphi}_U((f_p)_{p\in U})$ is the unique section in $\mathcal{G}(U)$ such that

  $$\overline{\varphi}_U((f_p)_{p\in U})|_q = \varphi_q(f_q)$$

  for each $q \in U$. Note any section $f \in \mathcal{F}(U)$ has

  $$(\overline{\varphi} \circ \mathrm{sh})_U(f)|_q = \overline{\varphi}_U((f|_p)_{p\in U})|_q = \varphi_q(f|_q) = \varphi_U(f)|_q$$

  for any $q \in U$, so Proposition 1.106 applied to the sheaf $\mathcal{G}$ forces equality, implying $\overline{\varphi} \circ \mathrm{sh} = \varphi$.

  So we will be done as soon as we can show $\overline{\varphi}_U$ is a (pre)sheaf morphism. Well, given open sets $V \subseteq U$ and some $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, we note any $q \in V$ has

  $$\overline{\varphi}_U((f_p)_{p\in U})|_V|_q = \overline{\varphi}_U((f_p)_{p\in U})|_q = \varphi_q(f_q),$$

  so the uniqueness of $\overline{\varphi}_V((f_p)_{p\in V})$ forces $\overline{\varphi}_U((f_p)_{p\in U})|_V = \overline{\varphi}_V((f_p)_{p\in U}|_V)$, as desired. ■

Here are some basic properties.

Proposition 1.140. Fix a presheaf $\mathcal{F}$ on X with a sheafification $\mathrm{sh}\colon \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$. For given $p \in X$, the induced map $\mathrm{sh}_p\colon \mathcal{F}_p \to (\mathcal{F}^{\mathrm{sh}})_p$ on stalks is an isomorphism.

Proof. We use the explicit description of the sheafification. To be explicit, our map $\mathrm{sh}_p\colon \mathcal{F}_p \to (\mathcal{F}^{\mathrm{sh}})_p$ sends $[(U, f)]$ to $[(U, (f|_q)_{q\in U})]$.

For the inverse morphism $\pi_p\colon (\mathcal{F}^{\mathrm{sh}})_p \to \mathcal{F}_p$, we simply send

$$\pi_p\colon [(U, (f_q)_{q\in U})] \mapsto f_p.$$

Notably, this is well-defined: $[(U, (f_q)_{q\in U})] = [(U', (f'_q)_{q\in U})]$, then there is $V \subseteq U \cap U'$ such that $(f_q)_{q\in U})|_V = (f'_q)_{q\in U'}|_V$, which implies $f_p = f'_p$.

It remains to show that these are inverse. Well, for $[(U, f)] \in \mathcal{F}_p$, we see

$$\pi_p(\mathrm{sh}_p([(U, f)])) = \pi_p([(U, (f|_q)_{q\in U})]) = f|_p.$$

And for $[(U, (f_q)_{q\in U})] \in (\mathcal{F}^{\mathrm{sH}})_p$, we see

$$\mathrm{sh}_p(\pi_p([(U, (f_q)_{q\in U})])) = \mathrm{sh}_p(f_p).$$

Now, because $(f_q)_{q\in U}$ is a compatible system of germs, we may find $U_p \subseteq U$ containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. It follows

$$\mathrm{sh}_p(f_p) = \mathrm{sh}_p(\widetilde{f}_p|_p) = [(U_p, (\widetilde{f}_p|_q)_{q\in U_p})] = [(U_p, (f_q)_{q\in U})] = [(U, (f_q)_{q\in U})],$$

finishing this check. ■

Remark 1.141. If $\mathcal{F}$ is itself a sheaf, then we can see fairly directly that $\mathcal{F}$ satisfies the universal property for $\mathcal{F}^{\mathrm{sh}}$. Alternatively, the sheafification map $\mathrm{sh}\colon \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ is a sheaf morphism which is an isomorphism on stalks by Proposition 1.140 and thus an isomorphism of sheaves by Proposition 1.131.

Proposition 1.142. Sheafification $\mathcal{F} \mapsto \mathcal{F}^{\text{sh}}$ defines a functor $(-)^{\text{sh}} : \text{PreSh}_X \to \text{Sh}_X$ which is left adjoint to the forgetful functor $U : \text{Sh}_X \to \text{PreSh}_X$.

Proof. We begin by describing the functor $(-)^{\text{sh}}$. We know its behavior on objects, so we still need to know its behavior on morphisms $\eta : \mathcal{F} \to \mathcal{G}$. Well, note that we have a composite map $\mathcal{F} \to \mathcal{G} \to \mathcal{G}^{\text{sh}}$, and $\mathcal{G}^{\text{sh}}$ is a sheaf, so the universal property of $\mathcal{F}^{\text{sh}}$ induces a unique map $\eta^{\text{sh}} : \mathcal{F}^{\text{sh}} \to \mathcal{G}^{\text{sh}}$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\eta"'] & \mathcal{F}^{\text{sh}} \arrow[d, dashed, "\eta^{\text{sh}}"] \\ \mathcal{G} \arrow[r] & \mathcal{G}^{\text{sh}} \end{tikzcd}$$

commute. We quickly check functoriality.

- Identity: note $\text{id}_{\mathcal{F}^{\text{sh}}}$ makes the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\text{id}_{\mathcal{F}}"'] & \mathcal{F}^{\text{sh}} \arrow[d, dashed, "\text{id}_{\mathcal{F}^{\text{sh}}}"] \\ \mathcal{F} \arrow[r] & \mathcal{F}^{\text{sh}} \end{tikzcd}$$

  commute, so by definition, we see $(\text{id}_{\mathcal{F}})^{\text{sh}} = \text{id}_{\mathcal{F}^{\text{sh}}}$.

- Functoriality: given presheaf morphisms $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$, we note that $\psi^{\text{sh}} \circ \varphi^{\text{sh}}$ makes the outer rectangle of

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\varphi"] \arrow[dd, bend right, "\psi\circ\varphi"'] & \mathcal{F}^{\text{sh}} \arrow[d, "\varphi^{\text{sh}}"'] \arrow[dd, dashed, bend left, "\psi^{\text{sh}}\circ\varphi^{\text{sh}}"] \\ \mathcal{G} \arrow[r] \arrow[d, "\psi"] & \mathcal{G}^{\text{sh}} \arrow[d, "\psi^{\text{sh}}"'] \\ \mathcal{H} \arrow[r] & \mathcal{H}^{\text{sh}} \end{tikzcd}$$

  commute, so by definition, we see $\psi^{\text{sh}} \circ \varphi^{\text{sh}} = (\psi \circ \varphi)^{\text{sh}}$.

We will not check that the forgetful functor U is a functor; the main point is that it does nothing to morphisms. Also, we will not formally check the adjoint pair, but we will say that it requires exhibit a natural isomorphism

$$\text{Mor}_{\text{Sh}_X}(F^{\text{sh}}, \mathcal{G}) \simeq \text{Mor}_{\text{PreSh}_X}(F, U\mathcal{G})$$

where $F \in \text{PreSh}_X$ and $\mathcal{G} \in \text{Sh}_X$. And we will describe this isomorphism: if $\text{sh}: F \to F^{\text{sh}}$ is the sheafification map, the isomorphism is given by

$$\begin{array}{ccc} \text{Mor}_{\text{Sh}_X}(F^{\text{sh}}, \mathcal{G}) & \simeq & \text{Mor}_{\text{PreSh}_X}(F, U\mathcal{G}) \\ \varphi & \mapsto & \varphi \circ \text{sh} \\ \overline{\psi} & \leftarrow\!\shortmid & \psi \end{array}$$

where $\overline{\psi}$ is the morphism induced by the universal property of sheafification applied to the presheaf morphism $\psi : F \to \mathcal{G}$. That this is an isomorphism follows from the universal property, and the naturality checks for the adjoint pair are a matter of writing down the squares and checking them. ■

Remark 1.143. Sheafification being a left adjoint means that it preserves limits. Kernels and limits, so we see that the sheafification of the presheaf kernel is just the presheaf kernel again. The point here is that we don't need to sheafify the kernel, which is why we could talk about them before sheafification, but we will not be so lucky with cokernels.

1.5.2 Sheaf Cokernels

Now that we have sheafification, we may continue showing that the category sheaves valued in an abelian category is abelian. For this, we need to understand cokernels.

Definition 1.144 (Sheaf cokernel)**.** Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then the *presheaf cokernel* $\operatorname{coker}\varphi$ is the sheaf found by setting

$$(\operatorname{coker}^{\mathrm{pre}}\varphi)(U) := \operatorname{coker}\varphi_U = \mathcal{G}(U)/\operatorname{im}\varphi_U.$$

We define the *sheaf kernel* as the sheafification of the presheaf $\operatorname{coker}^{\mathrm{pre}}\varphi$.

We begin by running our checks on the presheaf cokernel.

Lemma 1.145. Fix a presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$. Then $\operatorname{coker}^{\mathrm{pre}}\varphi$ is a presheaf, and it is the cokernel of φ in the category PreSh_X.

Proof. To begin, we must exhibit our restriction maps. Given open sets $V \subseteq U$ and $[g] \in (\operatorname{coker}^{\mathrm{pre}}\varphi)(U) = \operatorname{coker}\varphi_U$, we define

$$\operatorname{res}_{U,V}([g]) := [g|_V].$$

Note this is well-defined: if $[g] = [g']$, then $g - g' \in \operatorname{im}\varphi_U$, so write $g - g' = \varphi_U(f)$ for $f \in \mathcal{F}(U)$. Thus, $g|_V - g'|_V = (g - g')|_V = \varphi_U(f)|_V = \varphi_V(f|_V)$ is in the image of φ_V, so $[g|_V] = [g'|_V]$.

We quickly check that this data assembles into a presheaf.

- Identity: given $g \in (\operatorname{coker}^{\mathrm{pre}}\varphi)(U)$, note $[g]|_U = [g|_U] = [g]$.
- Functoriality: given open sets $W \subseteq V \subseteq U$ and some $g \in (\operatorname{coker}^{\mathrm{pre}}\varphi)(U)$, we see $[g]|_V|_W = [g|_V]|_W = [g|_V|_W] = [g|_W] = [g]|_W$.

It remains to check the universal property: we need $\operatorname{coker}^{\mathrm{pre}}\varphi$ to be the colimit of the following diagram.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \mathcal{Z} & \end{tikzcd}$$

To begin, we define a morphism $\pi\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}}\varphi$. Well, for each open $U \subseteq X$, there is a natural projection $\pi_U\colon \mathcal{G}(U) \twoheadrightarrow \operatorname{coker}\varphi_U$ by $\pi_U\colon g \mapsto [g]$, which we need to assemble into a natural transformation. Indeed, given open sets $V \subseteq U$ and a section $g \in \mathcal{G}(U)$, we compute

$$\pi_U(g)|_V = [g]|_V = [g|_V] = \pi_V(g|_V).$$

This map $\pi\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}}\varphi$ induces the other needed maps $\mathcal{F} \to \operatorname{coker}^{\mathrm{pre}}\varphi$ (as $\pi \circ \varphi$) and $\mathcal{Z} \to \operatorname{coker}^{\mathrm{pre}}\varphi$ (which is the zero map). Further, note that any open $U \subseteq X$ has $(\pi \circ \varphi)_U = \pi_U \circ \varphi_U = 0$ because π_U returns 0 on $\operatorname{im}\varphi_U$; thus, $\pi \circ \varphi = 0$. Thus, the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r] & \operatorname{coker}^{\mathrm{pre}}\varphi \end{tikzcd}$$

commutes.

We are now ready to show the universal property. Fix a presheaf $\mathcal{H}$ with a map $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $\psi \circ \varphi = 0$. Then we need a unique map $\overline{\psi}\colon (\operatorname{coker}^{\text{pre}} \varphi) \to \mathcal{H}$ such that $\psi = \overline{\psi} \circ \pi$; i.e., such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \arrow[ddr, bend left, "\psi"] & \\ \mathcal{Z} \arrow[r] \arrow[drr, bend right] & \operatorname{coker}^{\text{pre}} \varphi \arrow[dr, dashed, "\overline{\psi}"] & \\ & & \mathcal{H} \end{tikzcd}$$

commutes. We show uniqueness and existence of $\overline{\psi}$ separately.

- Uniqueness: given an open set $U \subseteq X$ and some $[g] \in (\operatorname{coker}^{\text{pre}})(U)$, we must have
$$\overline{\psi}_U([g]) = \overline{\psi}_U(\pi_U g) = \psi_U(g),$$
so $\overline{\psi}_U$ is uniquely determined.
- Existence: given an open set $U \subseteq X$ and some $[g] \in (\operatorname{coker}^{\text{pre}} \varphi)(U)$, we simply define
$$\overline{\psi}_U([g]) \coloneqq \psi_U(g).$$
Note this is well-defined: if $[g] = [g']$, then $g - g' \in \operatorname{im} \varphi_U$, so write $g - g' = \varphi_U(f)$. Then $\psi_U(g) - \psi_U(g') = \psi_U(\varphi_U f) = 0$, so $\psi_U(g) = \psi_U(g')$.
Additionally, we note that any $g \in \mathcal{G}(U)$ will have $\overline{\psi}_U(\pi_U g) = \overline{\psi}_U([g]) = \psi_U(g)$, so we conclude $\overline{\psi} \circ \pi = \psi$. It remains to show that $\overline{\psi}$ is actually a presheaf morphism. Well, any open sets $V \subseteq U$ and $[g] \in (\operatorname{coker}^{\text{pre}} \varphi)(U)$ has
$$\overline{\psi}_U([g])|_V = \psi_U(g)|_V = \psi_V(g|_V) = \overline{\psi}_V([g|_V]) = \overline{\psi}_V([g]|_V),$$
finishing. ■

And now we run the checks on the sheaf kernel.

Lemma 1.146. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X. Then $\operatorname{coker} \varphi$ is the cokernel in the category Sh_X.

Proof. Let $\pi^{\text{pre}}\colon \mathcal{G} \to \operatorname{coker}^{\text{pre}} \varphi$ be the projection map of Lemma 1.145 and $\operatorname{sh}\colon \operatorname{coker}^{\text{pre}} \varphi \to \operatorname{coker} \varphi$ be the sheafification map. Then we define $\pi \coloneqq \operatorname{sh} \circ \pi^{\text{pre}}$, so we claim that this map makes $\operatorname{coker} \varphi$ the colimit of the following diagram.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \mathcal{Z} & \end{tikzcd}$$

Notably, we have $\pi \circ \varphi = \operatorname{sh} \circ \pi^{\text{pre}} \circ \varphi = \operatorname{sh} \circ 0 = 0$, so π at least works as a candidate morphism.

To show the universal property, fix a sheaf $\mathcal{H}$ with a map $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $\psi \circ \varphi = 0$. Then we need a unique map $\overline{\psi}\colon \operatorname{coker} \varphi \to \mathcal{H}$ such that $\psi = \overline{\psi} \circ \pi$, or equivalently, making

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \arrow[ddr, bend left, "\psi"] & \\ \mathcal{Z} \arrow[r] \arrow[drr, bend right] & \operatorname{coker} \varphi \arrow[dr, dashed, "\overline{\psi}"] & \\ & & \mathcal{H} \end{tikzcd}$$

commute. We show existence and uniqueness separately.

- Existence: working in PreSh_X for a moment, the fact that $\psi \circ \varphi = 0$ promises a map $\overline{\psi}^{\mathrm{pre}} \colon \operatorname{coker}^{\mathrm{pre}} \varphi \to \mathcal{H}$ such that $\overline{\psi}^{\mathrm{pre}} \circ \pi^{\mathrm{sh}} = \psi$. Now, from the definition of sheafification, we get a map $\overline{\psi} \colon \operatorname{coker} \varphi \to \mathcal{H}$ such that
$$\overline{\psi} \circ \mathrm{sh} = \overline{\psi}^{\mathrm{pre}}.$$
Thus, $\overline{\psi} \circ \pi = \overline{\psi} \circ \mathrm{sh} \circ \pi^{\mathrm{pre}} = \overline{\psi}^{\mathrm{pre}} \circ \pi^{\mathrm{pre}} = \psi$, as needed.
- Uniqueness: suppose $\overline{\psi}_1, \overline{\psi}_2 \colon \operatorname{coker} \varphi \to \mathcal{H}$ have $\psi = \overline{\psi}_1 \circ \pi = \overline{\psi}_2 \circ \pi$. Then we see that actually
$$\psi = (\overline{\psi}_1 \circ \mathrm{sh}) \circ \pi^{\mathrm{pre}} = (\overline{\psi}_2 \circ \mathrm{sh}) \circ \pi^{\mathrm{pre}},$$
but the universal property of $\operatorname{coker}^{\mathrm{pre}} \varphi$ has a uniqueness forcing $\overline{\psi}_1 \circ \mathrm{sh} = \overline{\psi}_2 \circ \mathrm{sh}$. But then the universal property of sheafification says there is a unique map $\overline{\psi} \colon \operatorname{coker} \varphi \to \mathcal{H}$ such that
$$\overline{\psi} \circ \mathrm{sh} = \overline{\psi}_1 \circ \mathrm{sh} = \overline{\psi}_2 \circ \mathrm{sh},$$
so $\overline{\psi} = \overline{\psi}_1 = \overline{\psi}_2$ follows. ■

As before, we take a moment to verify that vanishing cokernel does indeed mean epic.

Lemma 1.147. Let $\mathcal{C}$ be a category with a zero object and cokernels. Then a morphism $\varphi \colon A \to B$ is epic if and only if $\operatorname{coker} \varphi$ vanishes.

Proof. Reverse all the arrows in Lemma 1.126. Notably, the dual of the kernel is the cokernel, the dual of a monic map is an epic map, and the dual of the zero object is still the zero object. ■

1.5.3 Surjectivity at Stalks

We are now ready to fix our surjectivity. Just like injectivity, we can check surjectivity at stalks.

Lemma 1.148. Fix a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then, for any p, the projection $\mathcal{G} \to \operatorname{coker}^{\mathrm{pre}} \varphi$ induces an isomorphism
$$\operatorname{coker} \varphi_p \to (\operatorname{coker}^{\mathrm{pre}} \varphi)_p.$$
Thus, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the projection $\mathcal{G} \to \operatorname{coker} \varphi$ induces an isomorphism $\operatorname{coker} \varphi_p \simeq (\operatorname{coker} \varphi)_p$.

Proof. Let $\pi^{\mathrm{pre}} \colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}} \varphi$ be the natural projection witnessing that $\operatorname{coker}^{\mathrm{pre}} \varphi$ is the presheaf cokernel.

To show the second sentence note π^{pre} induces a map $\mathcal{G}_p \to (\operatorname{coker}^{\mathrm{pre}} \varphi)_p$ as
$$\pi_p^{\mathrm{pre}} \colon [(U, g)] \mapsto [(U, \pi_U^{\mathrm{pre}} g)].$$
Note that, if $[(U, g)] \in \operatorname{im} \varphi_p$, then we can write $[(U, g)] = [(V, \varphi_V f)]$ for some $f \in \mathcal{F}(V)$, so
$$\pi_p^{\mathrm{pre}}(f|_p) = (\pi_V^{\mathrm{pre}} f)|_p = 0|_p = 0,$$
so $\operatorname{im} \varphi_p \subseteq \ker \pi_p^{\mathrm{pre}}$, so we have actually induced a map $\operatorname{coker} \varphi_p \to (\operatorname{coker}^{\mathrm{pre}} \varphi)_p$.

In the other direction, we define $\varphi_p \colon (\operatorname{coker}^{\mathrm{pre}} \varphi)_p \to \operatorname{coker} \varphi_p$ by
$$\varphi_p \colon [(U, [g])] \mapsto (g|_p + \operatorname{im} \varphi_p).$$
We do need to check that this is well-defined: if $(U, [g]) \sim (U', [g'])$, then we can find $V \subseteq U \cap U'$ such that $[(g - g')|_V] = [g]|_V - [g']|_V = 0$, so there is $f \in \mathcal{F}(V)$ such that $(g - g')|_V = \varphi_V(f)$. Thus, $g|_p - g'|_p = (g - g')|_p = (g - g')|_V|_p = \varphi_V(f)|_p$ is in $\operatorname{im} \varphi_p$.

Lastly, we need to check that π_p^{pre} and φ_p are inverse. Given $[(U,g)]+\operatorname{im}\varphi_p\in\operatorname{coker}\varphi_p$, we note

$$\varphi_p\left(\pi^{\mathrm{pre}}([(U,g)]+\operatorname{im}\varphi_p)\right)=\varphi_p([(U,[g])])=g|_p+\operatorname{im}\varphi_p.$$

Conversely, given $[(U,[g])]\in(\operatorname{coker}^{\mathrm{pre}}\varphi)_p$, we note

$$\pi_p^{\mathrm{pre}}(\varphi_p([(U,[g])]))=\pi^{\mathrm{pre}}([(U,g)]+\operatorname{im}\varphi_p)=[(U,[g])],$$

finishing.

We now show the last sentence. Let $\mathrm{sh}\colon\operatorname{coker}^{\mathrm{pre}}\varphi\to\operatorname{coker}\varphi$ be the sheafification map. Then $\pi_p=(\mathrm{sh}\circ\pi^{\mathrm{pre}})_p$ we can check to be $\mathrm{sh}_p\circ\pi_p^{\mathrm{pre}}$ (by, say, Remark 1.111). Stringing these isomorphisms together, we see

$$\operatorname{coker}\varphi_p\to(\operatorname{coker}^{\mathrm{pre}}\varphi)_p\simeq(\operatorname{coker}\varphi)_p,$$

which is what we wanted. ■

And here is our result.

Proposition 1.149. Fix a morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ of sheaves on X. The following are equivalent.

(a) φ is epic.

(b) $(\operatorname{coker}\varphi)(U)$ vanishes for all open $U\subseteq X$.

(c) φ_p is epic for all $p\in X$.

Proof. By Lemma 1.147, these are equivalent to the following.

(a') $\operatorname{coker}\varphi$ vanishes.

(b') $(\operatorname{coker}\varphi)(U)$ vanishes for all open $U\subseteq X$.

(c') $\operatorname{coker}\varphi_p$ vanishes for all $p\in X$. By Lemma 1.148, this is equivalent to $(\operatorname{coker}\varphi)_p$ vanishing for all $p\in X$.

These are equivalent by Lemma 1.128. ■

1.5.4 The Category of Sheaves Is Abelian

Now that our category of sheaves (valued in an abelian category) has kernels and cokernels for our morphisms, we have two more conditions to check.

Lemma 1.150. Fix a monic morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ of sheaves on X. Then actually $\varphi\colon\mathcal{F}\to\mathcal{G}$ makes $\mathcal{F}$ the kernel of the cokernel $\pi\colon\mathcal{G}\to\operatorname{coker}\varphi$.

Proof. We need to show that $\mathcal{F}$ is the limit of the following diagram.

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

To begin, note that $\varphi\colon\mathcal{F}\to\mathcal{G}$ makes the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

commute by the construction of $\pi\colon \mathcal{G} \to \operatorname{coker}\varphi$. We now show that $\mathcal{F}$ satisfies the universal property. Fix a sheaf morphism $\psi\colon \mathcal{H} \to \mathcal{G}$ such that $\pi \circ \psi = 0$. Then we need a unique map $\overline{\psi}\colon \mathcal{H} \to \mathcal{F}$ making the diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[drr, bend left, "\psi"] \arrow[dr, dashed, "\overline{\psi}"'] \arrow[ddr, bend right] & & \\
& \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"] \\
& \mathcal{Z} \arrow[r] & \operatorname{coker}\varphi
\end{tikzcd}$$

commute; i.e., we need $\psi = \varphi \circ \overline{\psi}$. We show uniqueness and existence separately.

- Uniqueness: this follows because φ is monic. Indeed, if $\overline{\psi}_1, \overline{\psi}_2$ have $\varphi \circ \overline{\psi}_1 = \psi = \varphi \circ \overline{\psi}_2$, then $\overline{\psi}_1 = \overline{\psi}_2$ because φ is monic.
- Existence: this is trickier. Let $\pi^{\text{pre}}\colon \mathcal{G} \to \operatorname{coker}^{\text{pre}}\varphi$ be the natural projection, and let $\operatorname{sh}\colon \operatorname{coker}^{\text{pre}}\varphi \to \operatorname{coker}\varphi$ be the sheafification map.

  Now, given $U \subseteq X$ and $h \in \mathcal{H}(U)$, set $g := \psi_U(h)$ for brevity. Notably, we have $\pi_U \circ \psi_U = 0$, so $\pi_U(g) = 0$. It follows $\pi_p(g|_p) = \pi_U(g)|_p = 0$ for each $p \in U$, so $g|_p \in \ker \pi_p$ for each $p \in U$. Now, for each $p \in U$, by Lemma 1.148, $\ker \pi_p = \operatorname{im} \varphi_p$, and by Proposition 1.129, φ_p is monic, so is a unique $f_p \in \mathcal{F}_p$ such that
  $$\varphi_p(f_p) = g|_p.$$
  We claim that $(f_p)_{p \in U}$ is a system of compatible germs. To begin, choose some representative $f_p = [(U_p', \widetilde{f}_p')]$ and note that we have
  $$[(U, g)] = g|_p = \varphi_p(f_p) = [(U_p', \varphi_{U_p'}(\widetilde{f}_p'))],$$
  so we can find $U_p \subseteq U_p'$ containing p with $\widetilde{f}_p = \widetilde{f}_p'|_{U_p}$ small enough so that $g|_{U_p} = \varphi_{U_p}(\widetilde{f}_p)$. As such, any $q \in U_p$ has
  $$\varphi_q(\widetilde{f}_p|_q) = [(U_p, \varphi_{U_p}(\widetilde{f}_p))] = [(U_p, g|_{U_p})] = g|_p,$$
  so $\widetilde{f}_p|_q = f_q$ follows.

  Thus, Proposition 1.106 promises a unique $f \in \mathcal{F}(U)$ such that $f|_p = f_p$ for each $p \in U$. So we define $\overline{\psi}_U(h) := f$ to be the unique element such that
  $$\varphi_p(\overline{\psi}_U(h)|_p) = \psi_U(h)|_p$$
  for all $p \in U$.

  It remains to show that $\overline{\psi}$ assembles into a presheaf morphism. Well, for open sets $V \subseteq U$ and $h \in \mathcal{H}(U)$, we see that any $p \in V$ will have
  $$\varphi_p(\overline{\psi}_U(h)|_V|_p) = \varphi_p(\overline{\psi}_U(h)|_p) = \psi_U(h)|_p = \psi_V(h|_V)|_p,$$
  so the uniqueness of $\psi_V(h|_V)$ forces $\overline{\psi}_U(h)|_V = \psi_V(h|_V)$. ■

Lemma 1.151. Fix an epic morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X. Then actually $\varphi\colon \mathcal{F} \to \mathcal{G}$ makes $\mathcal{G}$ the cokernel of the kernel $\iota\colon \ker\varphi \to \mathcal{F}$.

Proof. We need to show that $\mathcal{G}$ is the colimit of the following diagram.

$$\begin{tikzcd}
\ker\varphi \arrow[r] \arrow[d, "\iota"'] & \mathcal{Z} \\
\mathcal{F} &
\end{tikzcd}$$

To begin, note that $\varphi\colon \mathcal{F}\to\mathcal{G}$ makes the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"'] & \mathcal{Z} \arrow[d] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

commute by the construction of $\iota\colon \ker\varphi\to\mathcal{G}$. We are now ready to show that $\mathcal{G}$ satisfies the universal property. Fix a sheaf $\mathcal{H}$ with a morphism $\psi\colon\mathcal{F}\to\mathcal{H}$ such that $\psi\circ\iota=0$. We need a unique map $\overline{\psi}\colon\mathcal{G}\to\mathcal{H}$ such that $\psi=\overline{\psi}\circ\varphi$, or equivalently, making the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"'] & \mathcal{Z} \arrow[d] \arrow[ddr, bend left] & \\ \mathcal{F} \arrow[r, "\varphi"] \arrow[drr, bend right, "\psi"'] & \mathcal{G} \arrow[dr, dashed, "\overline{\psi}"] & \\ & & \mathcal{H} \end{tikzcd}$$

commute. We show uniqueness and existence separately.

- Uniqueness: this follows because φ is epic. Indeed, if $\overline{\psi}_1,\overline{\psi}_2\colon\mathcal{G}\to\mathcal{H}$ have $\overline{\psi}_1\circ\varphi=\psi=\overline{\psi}_2\circ\varphi$, then $\overline{\psi}_1=\overline{\psi}_2$ because φ is epic.

- Existence: given $U\subseteq X$ and $g\in\mathcal{G}(U)$, we define $\overline{\psi}_U(g)$ by hand. By Proposition 1.149, we see that φ being epic means that φ_p is surjective for each $p\in U$, we can find $f_p\in\mathcal{F}_p$ with $\varphi_p(f_p)=g|_p$ for each p. We now set
$$h_p:=\psi_p(f_p).$$
We claim that h_p is independent of the choice for f_p. Indeed, if we have $[(U,f)]$ and $[(U',f')]$ in $\mathcal{F}_p$ with $[(U,\varphi_U f)]=[(U',\varphi_{U'}f')]=g|_p$, then there is an open $V\subseteq U\cap U'$ such that $\varphi_V(f|_V-f'|_V)=0$. Thus, $f-f'\in\ker\varphi_V=(\ker\varphi)(V)$, so it follows $\psi_V((f-f')|_V)=0$. Thus, so
$$\psi_p([(U,f)])-\psi_p([(U',f')])=\psi_p([(V,\psi_V((f-f')|_V))])=\psi_p([(V,0)])=0.$$
Next, we claim that the $(h_p)_{p\in U}$ forms a compatible system of germs. Well, for each $p\in U$, we can find a sufficiently small open set U_p with a lift $\widetilde{f}_p\in\mathcal{F}(U_p)$ such that $\varphi_{U_p}(\widetilde{f}_p)=g|_{U_p}$. Set $\widetilde{h}_p:=\psi_{U_p}(\widetilde{f}_p)$ so that for each $q\in U_p$ has $\varphi_q(\widetilde{f}_p|_q)=\varphi_{U_p}(\widetilde{f}_p)|_q=g|_q$ and thus
$$h_q=\psi_q(\widetilde{f}_p|_q)=\psi_{U_p}(\widetilde{f}_p)|_q.$$
It follows Proposition 1.106 that we have a unique $h\in\mathcal{H}(U)$ such that $h|_p=h_p$ for each $p\in U$, so we define $\overline{\psi}_U(g):=h$. Explicitly, $\overline{\psi}_U(g)$ is the unique element of $\mathcal{H}(U)$ such that
$$\overline{\psi}_U(g)|_p=\psi_p\left(\varphi_p^{-1}(g|_p)\right)$$
for each $p\in U$.
We now run checks on $\overline{\psi}$. To see that we have a morphism $\mathcal{G}\to\mathcal{H}$, note that any opens $V\subseteq U$ and $g\in\mathcal{G}(U)$ will have, for each $p\in U$,
$$\overline{\psi}_U(g)|_V|_p=\overline{\psi}_U(g)|_p=\psi_p\left(\varphi_p^{-1}(g|_p)\right)=\psi_p\left(\varphi_p^{-1}(g|_V|_p)\right),$$
so the uniqueness of $\overline{\psi}_V(g|_V)$ forces $\overline{\psi}_U(g)|_V=\overline{\psi}_V(g|_V)$.
Lastly, we note that $\psi=\overline{\psi}\circ\varphi$: for any open $U\subseteq X$ and $f\in\mathcal{F}(U)$, all points $p\in U$ give
$$\overline{\psi}_U(\varphi_U(f))|_p=\psi_p\left(\varphi_p^{-1}(\varphi_U(f)|_p)\right)=\psi_p\left(\varphi_p^{-1}(\varphi_p(f|_p))\right)=\psi_p(f|_p)=\psi_U(f)|_p,$$
so the injectivity of Proposition 1.106 forces our equality. ■

And here is our result.

Theorem 1.152. The category Sh_X of sheaves on a topological space X valued in a (concrete) abelian category $\mathcal{C}$ is additive.

Proof. The category is additive by Corollary 1.121. Kernels exist by Lemma 1.124, and cokernels exist by Lemma 1.146. The last conditions to check are Lemma 1.150 and Lemma 1.151. ■

1.5.5 Exactness via Stalks

It is a general philosophy, well-exhibited by Theorem 1.152, that we can prove (categorical) facts about sheaves by passing to stalks. Here is an example.

Proposition 1.153. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X valued in an abelian category. Then $\operatorname{coker}\varphi \simeq \operatorname{coker}\varphi^{\text{sh}}$.

Proof. We merely need to exhibit a candidate isomorphism and then check that it is an isomorphism on stalks. Here is our diagram.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\text{sh}"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\text{sh}"] & \operatorname{coker}^{\text{pre}}\varphi \arrow[r, "\text{sh}"] & \operatorname{coker}\varphi \\
\mathcal{F}^{\text{sh}} \arrow[r, "\varphi^{\text{sh}}"] & \mathcal{G}^{\text{sh}} \arrow[r, "\pi'"] & \operatorname{coker}^{\text{pre}}\varphi^{\text{sh}} \arrow[r, "\text{sh}"] & \operatorname{coker}\varphi^{\text{sh}}
\end{tikzcd}$$

Note that the composite $\mathcal{F} \to \mathcal{G} \to \mathcal{G}^{\text{sh}} \to \operatorname{coker}^{\text{pre}}\varphi \to \operatorname{coker}\varphi^{\text{sh}}$ is the zero map because it is the same as the same as

$$\mathcal{F} \to \underbrace{\mathcal{F}^{\text{sh}} \to \mathcal{G}^{\text{sh}} \to \operatorname{coker}^{\text{pre}}\varphi^{\text{sh}}}_{0} \to \operatorname{coker}\varphi^{\text{sh}}.$$

Thus, the universal property of $\operatorname{coker}\varphi$ induces a unique sheaf morphism $\psi\colon \operatorname{coker}^{\text{pre}}\varphi \to \operatorname{coker}\varphi^{\text{sh}}$ making

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\text{sh}"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\text{sh}"] & \operatorname{coker}^{\text{pre}}\varphi \arrow[r, "\text{sh}"] \arrow[d, dashed, "\psi"] & \operatorname{coker}\varphi \\
\mathcal{F}^{\text{sh}} \arrow[r, "\varphi^{\text{sh}}"] & \mathcal{G}^{\text{sh}} \arrow[r, "\pi'"] & \operatorname{coker}^{\text{pre}}\varphi^{\text{sh}} \arrow[r, "\text{sh}"] & \operatorname{coker}\varphi^{\text{sh}}
\end{tikzcd}$$

commute. Now, sheafification promises a unique map ψ^{sh} making

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\text{sh}"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\text{sh}"] & \operatorname{coker}^{\text{pre}}\varphi \arrow[r, "\text{sh}"] \arrow[d, "\psi"] & \operatorname{coker}\varphi \arrow[d, dashed, "\psi^{\text{sh}}"] \\
\mathcal{F}^{\text{sh}} \arrow[r, "\varphi^{\text{sh}}"] & \mathcal{G}^{\text{sh}} \arrow[r, "\pi'"] & \operatorname{coker}^{\text{pre}}\varphi^{\text{sh}} \arrow[r, "\text{sh}"] & \operatorname{coker}\varphi^{\text{sh}}
\end{tikzcd}$$

commute. We claim that ψ^{sh} is the desired isomorphism, for which it suffices by Proposition 1.131 to take stalks at $p \in X$ everywhere. This gives the following diagram.

$$\begin{tikzcd}
\mathcal{F}_p \arrow[r, "\varphi_p"] \arrow[d, "\text{sh}_p"'] & \mathcal{G}_p \arrow[r, "\pi_p"] \arrow[d, "\text{sh}_p"] & (\operatorname{coker}^{\text{pre}}\varphi)_p \arrow[r, "\text{sh}_p"] \arrow[d, "\psi_p"] & (\operatorname{coker}\varphi)_p \arrow[d, "\psi_p^{\text{sh}}"] \\
\mathcal{F}_p^{\text{sh}} \arrow[r, "\varphi_p^{\text{sh}}"] & \mathcal{G}_p^{\text{sh}} \arrow[r, "\pi_p'"] & (\operatorname{coker}^{\text{pre}}\varphi^{\text{sh}})_p \arrow[r, "\text{sh}_p"] & (\operatorname{coker}\varphi^{\text{sh}})_p
\end{tikzcd}$$

All the sh_p morphisms are isomorphisms by Proposition 1.140, so to show ψ_p^{sh} is an isomorphism, it suffices to show that ψ_p is an isomorphism. Now, by Lemma 1.148, we see that $\operatorname{im}\varphi_p$ lives in the kernel of $\mathcal{G}_p \to$

$(\operatorname{coker}^{\text{pre}}\varphi)_p$ and analogously for the bottom row. So the fact that the sh_ps are isomorphisms induces the diagram

$$\begin{tikzcd} \mathcal{G}_p/\operatorname{im}\varphi_p \arrow[r, "\overline{\pi}_p"] \arrow[d, "\mathrm{sh}_p"] & (\operatorname{coker}^{\text{pre}}\varphi)_p \arrow[d, "\psi_p"] \\ \mathcal{G}_p^{\text{sh}}/\operatorname{im}\varphi_p \arrow[r, "\overline{\pi}'_p"] & (\operatorname{coker}^{\text{pre}}\varphi^{\text{sh}})_p \end{tikzcd}$$

where sh_p is still an isomorphism because it was an isomorphism before. However, Lemma 1.148 actually tells us that this map π_p from $\mathcal{G}_p/\operatorname{im}\varphi_p = \operatorname{coker}\varphi_p$ to $(\operatorname{coker}^{\text{pre}}\varphi)_p$ is an isomorphism, and analogous holds for the bottom row, so it follows that ψ_p is an isomorphism. This finishes. ■

Remark 1.154. Thinking about cokernels as quotients, Proposition 1.153 roughly says that $(\mathcal{F}/\mathcal{G})^{\text{sh}} \simeq (\mathcal{F}^{\text{sh}}/\mathcal{G}^{\text{sh}})^{\text{sh}}$, where the "embedding" $\varphi\colon \mathcal{F}\to\mathcal{G}$ has been made implicit.

As an example application, we define the sheaf image.

Definition 1.155 (Sheaf image)**.** Fix a morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ of presheaves on X. Then the *sheaf image* $\operatorname{im}\varphi$ of φ is the sheafification of the presheaf image

$$(\operatorname{im}^{\text{pre}}\varphi)(U) = \operatorname{im}\varphi_U.$$

We go ahead and check that we have an image presheaf very quickly.

Lemma 1.156. Fix a morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ of presheaves on X. Then $\operatorname{im}^{\text{pre}}\varphi$ is a presheaf on X.

Proof. We quickly define restriction maps in the obvious way. Given a containment $U\subseteq V$, we define $\operatorname{res}_{U,V}\colon \operatorname{im}\varphi_U\to\operatorname{im}\varphi_V$ by restricting $\operatorname{res}_{U,V}\colon\mathcal{G}(U)\to\mathcal{G}(V)$. This is well-defined: if $g\in(\operatorname{im}^{\text{pre}}\varphi)(U) = \operatorname{im}\varphi_U$, then we write $g=\varphi_U(f)$ for some $f\in\mathcal{F}(U)$, so $g|_V=\varphi_U(f)|_V=\varphi_V(f|_V)\in\operatorname{im}\varphi_V$.

Now, here are our presheaf checks.

- Identity: note $g\in\operatorname{im}\varphi_U$ has $g|_U=g$.
- Functoriality: given open sets $W\subseteq V\subseteq U$ and $g\in\operatorname{im}\varphi_U$, we have $g|_V|_W=g|_W$. ■

Remark 1.157. Note there is an obvious inclusion $\iota_U^{\text{pre}}\colon(\operatorname{im}^{\text{pre}}\varphi)(U)\to\mathcal{G}(U)$ by $g\mapsto g$. This assembles into a presheaf morphism: given open sets $V\subseteq U$ and $g\in(\operatorname{im}^{\text{pre}}\varphi)(U)$, we have

$$\iota_U^{\text{pre}}(g)|_V = g|_V = \iota_V^{\text{pre}}(g|_V).$$

Thus, when $\mathcal{G}$ is a sheaf, sheafification induces a unique sheaf morphism $\iota\colon\operatorname{im}\varphi\to\mathcal{G}$.

We quickly check that our sheaf image is the categorical image.

Proposition 1.158. Fix a morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ of sheaves on X, and let $\pi\colon\mathcal{G}\to\operatorname{coker}\varphi$ be the canonical projection. Then

$$\operatorname{im}\varphi\simeq\ker\pi.$$

In other words, the canonical inclusion $\iota\colon\operatorname{im}\varphi\to\mathcal{G}$ is a kernel for π.

Proof. Pass to stalks. ■

Having defined an image sheaf, we may deal with exactness.

Definition 1.159 (Exact sequence)**.** Fix an abelian category $\mathcal{C}$. Then a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* B if and only if $\operatorname{im} f \simeq \ker g$. More precisely, this is asking for the image of f, thought of as $\iota\colon \operatorname{im} f \to B$, to be a kernel of g.

And here is our main result.

Proposition 1.160. A sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

of sheaves on X is exact (at $\mathcal{G}$) if and only if it is exact at all stalks.

Proof. Unsurprisingly, pass to stalks. ■

1.5.6 The Direct Image Sheaf

We now discuss how to build some new sheaves from old.

Definition 1.161 (Direct image sheaf)**.** Fix a continuous map $f\colon X \to Y$ of topological spaces. Given a (pre)sheaf $\mathcal{F}$ on X, we define the *direct image (pre)sheaf* on Y to be

$$f_*\mathcal{F}(U) := \mathcal{F}\left(f^{-1}(U)\right).$$

Here are our checks on the direct image sheaf.

Lemma 1.162. Fix a continuous map $f\colon X \to Y$.

(a) If $\mathcal{F}$ is a presheaf on X, then $f_*\mathcal{F}$ defines a presheaf on Y.

(b) If $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ defines a sheaf on Y.

Proof. We do these one at a time.

(a) We begin by defining our restriction maps. Well, if we have open sets $V \subseteq U \subseteq Y$, then $f^{-1}(V) \subseteq f^{-1}(U) \subseteq X$, so there is a restriction map

$$\operatorname{res}_{f^{-1}(U),f^{-1}(V)}\colon \mathcal{F}\left(f^{-1}(U)\right) \to \mathcal{F}\left(f^{-1}(V)\right).$$

Thus, we set our restriction map $\operatorname{res}_{U,V}\colon f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ as $\operatorname{res}_{U,V} := \operatorname{res}_{f^{-1}(U),f^{-1}(V)}$.

Here are our presheaf checks.

- Identity: given $s \in f_*\mathcal{F}(U) = \mathcal{F}\left(f^{-1}(U)\right)$, note $s|_U = s|_{f^{-1}(U)} = s$.
- Functoriality: given open sets $W \subseteq V \subseteq U$ and some $s \in f_*\mathcal{F}(U)$, we compute

$$f|_V|_W = f|_{f^{-1}(V)}|_{f^{-1}(W)} = f|_{f^{-1}(W)} = f|_W.$$

(b) Suppose $\mathcal{F}$ is a sheaf. We now run our sheaf checks. Fix an open cover $\mathcal{U}$ for an open set $U \subseteq Y$. Then define $V := f^{-1}(U)$ and $\mathcal{V} :- \left\{f^{-1}(U_0) : U_0 \in \mathcal{U}\right\}$; notably, $\mathcal{U}$ being an open cover of $U \subseteq Y$ promises that $\mathcal{V}$ is an open cover for V.

- Identity: suppose $s_1, s_2 \in f_*\mathcal{F}(U) = \mathcal{F}(V)$ has

$$s_1|_{U_0} = s_2|_{U_0}$$

for each $U_0 \in \mathcal{U}$. Then, moving back to X, we have $s_1|_{V_0} = s_2|_{V_0}$ for each $V_0 \in \mathcal{V}$, so it follows $s_1 = s_2$ as sections in $\mathcal{F}(V) = f_*\mathcal{F}(U)$ by the identity axiom of $\mathcal{F}$.

- Gluability: suppose we have sections $s_{U_0} \in f_*\mathcal{F}(U_0) = \mathcal{F}\left(f^{-1}(U_0)\right)$ for each $U_0 \in \mathcal{U}$ such that
$$s_{U_0}|_{U_0 \cap U_0'} = s_{U_0'}|_{U_0 \cap U_0'}.$$
Moving back to X, we have sections $t_{f^{-1}(U_0)} := s_{U_0}$ such that
$$t_{f^{-1}(U_0)}|_{f^{-1}(U_0)\cap f^{-1}(U_0')} = t_{f^{-1}(U_0')}|_{f^{-1}(U_0)\cap f^{-1}(U_0')}.$$
As such, the gluability axiom of $\mathcal{F}$ applied to the open cover $\mathcal{V}$ promises $s \in \mathcal{F}(V) = f_*\mathcal{F}(U)$ such that $s|_{U_0} = s|_{f^{-1}(U_0)} = t_{f^{-1}(U_0)} = s_{U_0}$ for each $U_0 \in \mathcal{U}$. This finishes. ■

In fact, we can build a functor out of this.

Lemma 1.163. Fix a continuous map $f\colon X \to Y$. Given a morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ of (pre)sheaves on X, there is an induced morphism $f_*\eta\colon f_*\mathcal{F} \to f_*\mathcal{G}$ of (pre)sheaves on Y. This makes $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$ into a functor.

Proof. For open $U \subseteq Y$, define $f_*\eta_U\colon f_*\mathcal{F}(U) \to f_*\mathcal{G}(U)$ by $f_*\eta_U := \eta_{f^{-1}(U)}$. Note this makes sense because
$$\eta_{f^{-1}(U)}\colon \mathcal{F}\left(f^{-1}(U)\right) \to \mathcal{G}\left(f^{-1}(U)\right).$$
Observe quickly that $f_*\eta$ is indeed a morphism of (pre)sheaves: given open sets $U' \subseteq U$ and some $s \in f_*\mathcal{F}(U)$, we have
$$f_*\eta_U(s)|_{U'} = \eta_{f^{-1}(U)}(s)|_{f^{-1}(U')} = \eta_{f^{-1}(U')}(s|_{f^{-1}(U')}) = f_*\eta_{U'}(s|_{U'}).$$
We now run functoriality checks on the functor $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$.

- Identity: given a (pre)sheaf $\mathcal{F}$ on X, an open set $U \subseteq Y$, and a section $s \in f_*\mathcal{F}(U)$, we compute
$$(f_*\mathrm{id}_{\mathcal{F}})_U(s) = (\mathrm{id}_{\mathcal{F}})_{f^{-1}(U)}(s) = s = (\mathrm{id}_{f_*\mathcal{F}})_U(s).$$
- Functoriality: given morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of (pre)sheaves on X, pick up an open set $U \subseteq Y$ and compute
$$f_*(\psi \circ \varphi)_U = (\psi \circ \varphi)_{f^{-1}(U)} = \varphi_{f^{-1}(U)} \circ \psi_{f^{-1}(U)} = f_*\psi \circ f_*\varphi,$$
which is what we wanted. ■

Remark 1.164. Given continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$, we have
$$(g \circ f)_* = g_* \circ f_*$$
as functors $\mathrm{Sh}_X \to \mathrm{Sh}_Z$. To see this, we have two checks. Fix any $U \subseteq Z$ and morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X.

- On objects, we see $(g \circ f)_*\mathcal{F}(U) = \mathcal{F}\left((g \circ f)^{-1}(U)\right) = \mathcal{F}\left(f^{-1}(g^{-1}(U))\right) = g_*(f_*\mathcal{F})(U)$. Additionally, given $U' \subseteq U$, the restriction map for $(g \circ f)_*\mathcal{F}$ is $\mathrm{res}_{(g\circ f)^{-1}(U),(g\circ f)^{-1}(U')}$ of $\mathcal{F}$. This matches the restriction map for $g_*f_*\mathcal{F}$.
- On morphisms, we see $(g \circ f)_*\varphi_U = \varphi_{((g\circ f)^{-1}(U))} = \varphi_{f^{-1}(g^{-1}(U))} = g_*(f_*\varphi)_U$.

Philosophically, we see that the point of the direct image sheaf is to use a continuous map $f\colon X \to Y$ to take a (pre)sheaf on X to a (pre)sheaf on Y. Under our stalk philosophy, we might want something like $(f_*\mathcal{F})_{f(x)} = \mathcal{F}_x$, but this need not be the case; essentially, $(f_*\mathcal{F})_{f(x)}$ is a colimit over all open sets containing $f(x)$, but we want to only consider the ones of the form $f^{-1}(U)$ where $x \in U$.

Nonetheless, there is a canonical map.

Lemma 1.165. Fix a continuous map $f\colon X\to Y$ and a (pre)sheaf $\mathcal{F}$ on X. Then, at any $x\in X$, there is a canonical map

$$(f_*\mathcal{F})_{f(x)}\to\mathcal{F}_x.$$

Proof. A germ in $(f_*\mathcal{F})_{f(x)}$ looks like $[(U,s)]$ where $f(x)\in U$ and $s\in f_*\mathcal{F}(U)=\mathcal{F}\left(f^{-1}(U)\right)$. As such, we will callously define

$$\begin{array}{rccc}\varphi\colon & (f_*\mathcal{F})_{f(x)} & \to & \mathcal{F}_x \\ & [(U,s)] & \mapsto & \left[(f^{-1}(U),s)\right]\end{array}$$

which we will only have to verify is well-defined. Well, suppose $[(s,U)]=[(s',U')]$ in $(f_*\mathcal{F})_{f(x)}$ so that we can find an open $V\subseteq U\cap U'$ such that $s|_V=s'|_V$. Moving back to $\mathcal{F}$, this translates to

$$s|_{f^{-1}(V)}=s'|_{f^{-1}(V)},$$

so $\left[(f^{-1}(U),s)\right]=\left[(f^{-1}(U'),s')\right]$ follows. ■

Remark 1.166. If f is a homeomorphism (with inverse $g\colon Y\to X$), then this canonical map is an isomorphism. Indeed, we can see that the maps

$$\begin{array}{ccc}(f_*\mathcal{F})_{f(x)} & \to & \mathcal{F}_x \\ [(U,s)] & \mapsto & \left[(f^{-1}(U),s)\right] \\ \left[g^{-1}(V),s\right] & \leftarrow\!\shortmid & [(V,s)]\end{array}$$

are well-defined, essentially by the above argument, and they are inverse because $g^{-1}(f^{-1}(U))=f(f^{-1}(U))=U$ and similar on the other side.

1.5.7 The Inverse Image Sheaf

Given a continuous map $f\colon X\to Y$, the direct image sheaf tells us how to take a sheaf on X to a sheaf on Y. We can also define an inverse image sheaf.

Definition 1.167 (Inverse image sheaf)**.** Fix a continuous map $f\colon X\to Y$ of topological spaces. Given a (pre)sheaf $\mathcal{G}$ on Y, we define the *inverse image sheaf* $f^{-1}\mathcal{G}$ on X to be the sheafification of the presheaf

$$f^{-1,\mathrm{pre}}\mathcal{G}(U):=\varinjlim_{V\supseteq f(U)}\mathcal{G}(V)=\left\{(V,s):s\in\mathcal{F}(V)\text{ and }V\supseteq f(U)\right\}/\sim,$$

where $(V,s)\sim(V',s')$ if and only if there is some $V''\subseteq V\cap V'$ containing $f(U)$ such that $s|_{V''}=s'|_{V''}$.

As usual, here are the checks on the inverse image sheaf.

Lemma 1.168. Fix a continuous map $f\colon X\to Y$.

(a) If $\mathcal{G}$ is a presheaf on Y, then $f^{-1,\mathrm{pre}}\mathcal{G}$ defines a presheaf on X.

(b) If $\mathcal{G}$ is a sheaf on Y, then $f^{-1}\mathcal{G}$ defines a sheaf on X.

Proof. Note that (b) is immediate from (a) because $f^{-1}\mathcal{G}$ is the sheafification of $f^{-1,\mathrm{pre}}\mathcal{G}$. So we will focus on showing (a).

To begin, we define our restriction maps for open sets $U'\subseteq U$ as

$$\begin{array}{rccc}\mathrm{res}_{U,U'}\colon & f^{-1,\mathrm{pre}}\mathcal{G}(U) & \to & f^{-1,\mathrm{pre}}\mathcal{G}(U') \\ & [(V,s)] & \mapsto & [(V,s)]\end{array}$$

which at least makes sense because $V \supseteq f(U) \supseteq f(U')$. To see that this is well-defined, suppose $[(V,s)] = [(V',s')]$ as elements of $f^{-1,\text{pre}}\mathcal{G}(U)$. Then there is $V'' \subseteq V \cap V'$ with $V'' \supseteq f(U)$ such that $s|_{V''} = s'|_{V''}$. As such, $V'' \supseteq f(U')$ as well while $s|_{V''} = s'|_{V''}$, so $[(V,s)] = [(V',s')]$ as elements of $f^{-1,\text{pre}}\mathcal{G}(U')$.

We now check our presheaf conditions.

- Identity: observe that $[(V,s)] \in f^{-1,\text{pre}}\mathcal{G}(U)$ has $[(V,s)]|_U = [(V,s)]$.
- Functoriality: fix open sets $U'' \subseteq U' \subseteq U$ and some $[(V,s)] \in f^{-1,\text{pre}}\mathcal{G}(U)$. Then

$$[(V,s)]|_{U'}|_{U''} = [(V,s)]|_{U''} = [(V,s)] = [(V,s)]|_{U''},$$

finishing. ■

As before, we actually have a functor.

Lemma 1.169. Fix a continuous map $f\colon X \to Y$. Given a morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ of sheaves on Y, there is an induced morphism $f^{-1}\eta\colon f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$ of sheaves on X. This makes $f^{-1}\colon \mathrm{Sh}_Y \to \mathrm{Sh}_X$ into a functor.

Proof. For open $U \subseteq X$, define

$$\begin{aligned} f^{-1,\text{pre}}\eta_U\colon f^{-1,\text{pre}}\mathcal{F}(U) &\to f^{-1,\text{pre}}\mathcal{G}(U) \\ [(V,s)] &\mapsto [(V,\eta_V(s))] \end{aligned}$$

where $[(V,\eta_V(s))] \in f^{-1,\text{pre}}\mathcal{G}(U)$ again at least makes sense because $V \supseteq f(U)$. To see that this is well-defined, note $[(V,s)] = [(V',s')]$ as elements of $f^{-1,\text{pre}}\mathcal{F}(U)$ promises some $V'' \subseteq V \cap V'$ containing $f(U)$ such that $s|_{V''} = s'|_{V''}$. Then

$$\eta_V(s)|_{V''} = \eta_{V''}(s|_{V''}) = \eta_{V''}(s'|_{V''}) = \eta_{V'}(s')|_{V''},$$

so we conclude $[(V,\eta_V(s))] = [(V',\eta_{V'}(s'))]$ as elements of $f^{-1,\text{pre}}\mathcal{G}(U)$.

Additionally, we see that $f^{-1,\text{pre}}\eta$ assembles into a presheaf morphism: given open sets $U' \subseteq U$, note that the diagram

$$\begin{tikzcd} f^{-1,\text{pre}}\mathcal{F}(U) \arrow[r, "f^{-1,\text{pre}}\eta_U"] \arrow[d, "\text{res}_{U,U'}"'] & f^{-1,\text{pre}}\mathcal{G}(U) \arrow[d, "\text{res}_{U,U'}"] \\ f^{-1,\text{pre}}\mathcal{F}(U') \arrow[r, "f^{-1,\text{pre}}\eta_{U'}"] & f^{-1,\text{pre}}\mathcal{G}(U') \end{tikzcd} \qquad \begin{tikzcd} {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V,\eta_V(s))]} \arrow[d, mapsto] \\ {[(V,s)]} \arrow[r, mapsto] & {[(V,\eta_V(s))]} \end{tikzcd}$$

commutes. We now run the functoriality checks on $f^{-1,\text{pre}}$.

- Identity: given a (pre)sheaf $\mathcal{F}$ on X, we see

$$(f^{-1,\text{pre}}\mathrm{id}_{\mathcal{F}})_U\big([(V,s)]\big) = [(V, \mathrm{id}_{\mathcal{F}(V)}\, s)] = [(V,s)].$$

- Functoriality: given morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of (pre)sheaves on X, pick up an open set $U \subseteq X$ and some $[(V,s)] \in f^{-1,\text{pre}}\mathcal{F}(U)$. Then we see

$$f^{-1,\text{pre}}(\psi\circ\varphi)\big([(V,s)]\big) = [(V,\psi_V\varphi_V(s))] = (f^{-1,\text{pre}}\psi \circ f^{-1,\text{pre}}\varphi)_U\big([(V,s)]\big).$$

To finish, we define $f^{-1}\eta := \left(f^{-1,\text{pre}}\eta\right)^{\text{sh}}$ to be a map $f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$. Here are the functoriality checks.

- Identity: given a sheaf $\mathcal{F}$ on X, we see $f^{-1}\mathrm{id}_{\mathcal{F}} = (f^{-1,\text{pre}}\,\mathrm{id}_{\mathcal{F}})^{\text{sh}} = \mathrm{id}^{\text{sh}}_{f^{-1,\text{pre}}\mathcal{F}} = \mathrm{id}_{f^{-1}\mathcal{F}}$.
- Functoriality: given morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of sheaves on X, we see

$$f^{-1}(\varphi\circ\psi) = \left(f^{-1,\text{pre}}(\varphi\circ\psi)\right)^{\text{sh}} = \left(f^{-1,\text{pre}}\varphi \circ f^{-1,\text{pre}}\psi\right)^{\text{sh}} = f^{-1}\varphi \circ f^{-1}\psi,$$

finishing. ■

Remark 1.170. As in Remark 1.164, continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ give $(f\circ g)^{-1} = g^{-1}\circ f^{-1}$ as functors $\mathrm{Sh}_Z \to \mathrm{Sh}_X$. Dealing with the implicit intermediate sheafification is not something that fits into a remark, so we will omit showing this. I suspect that we will not use this fact.

Here is, approximately, the reason that we like the inverse image sheaf.

Lemma 1.171. Fix a continuous map $f\colon X \to Y$ and a sheaf $\mathcal{G}$ on Y. Then, for any $x \in X$, we have $\left(f^{-1}\mathcal{G}\right)_x \simeq \mathcal{G}_{f(x)}$.

Proof. This is somewhat annoying because $\left(f^{-1}\mathcal{G}\right)_x$ involves equivalence classes of equivalence classes. In particular, an element looks like $[(U,[(V,s)])]$ where $[(V,s)] \in f^{-1}\mathcal{G}(U)$, meaning $s \in \mathcal{F}(V)$ while $V \supseteq f(U)$. Thus, we see $x \in U$ gives $f(x) \in V$, so we define our map as

$$\begin{array}{rccc}\varphi\colon & \left(f^{-1}\mathcal{G}\right)_x & \to & \mathcal{G}_{f(x)} \\ & [(U,[(V,s)])] & \mapsto & [(V,s)]\end{array}$$

which again makes sense because $s \in \mathcal{F}(V)$ and $f(x) \in V$. We have the following checks on φ.

- Well-defined: if $[(U,[(V,s)])] = [(U',[(V',s')])]$, then there is an open set $U'' \subseteq U\cap U'$ containing $f(x)$ such that $[(V,s)] = [(V',s')]$ as elements of $f^{-1}\mathcal{G}(U'')$. Thus, we are promised $V'' \subseteq V\cap V'$ containing $f(U'')$ and thus $f(x)$ such that $s|_{V''} = s'|_{V''}$. It follows $[(V,s)] = [(V',s')]$ as elements of $\mathcal{G}_{f(x)}$.
- Injective: suppose that $[(U,[(V,s)])]$ and $[(U',[(V',s')])]$ have $[(V,s)] = [(V',s')]$ as elements of $\mathcal{G}_{f(x)}$. This then promises some open $V'' \subseteq V\cap V'$ containing $f(x)$ such that $s|_{V''} = s'|_{V''}$. As such, set
$$U'' := f^{-1}(V'')\cap U \cap U'.$$
We automatically see $U'' \subseteq U\cap U'$ and $V'' \supseteq f(U'')$, so we note $[(V,s)] = [(V',s')]$ as elements of $f^{-1}\mathcal{G}(U'')$. Thus,
$$[(U,[(V,s)])] = [(U'',[(V,s)]|_{U''})] = [(U'',[(V,s)])] = [(U'',[(V',s')])] = [(U',[(V',s')])].$$
- Surjective: pick up some $[(V,s)] \in \mathcal{G}_{f(x)}$ we would like to hit with φ. Well, set $U := f^{-1}(V)$ so that $V \supseteq f(U)$ and $x \in U$, meaning that $[(U,[(V,s)])]$ is a valid element of $\left(f^{-1}\mathcal{G}\right)_x$, which we can fairly directly check goes to $[(V,s)]$ under φ. ■

1.5.8 A Sheaf Adjunction

The two sheaves we just introduced are intertwined, as follows.

Proposition 1.172. There is a natural bijection

$$\mathrm{Mor}_{\mathrm{Sh}_X}\left(f^{-1}\mathcal{G},\mathcal{F}\right) \simeq \mathrm{Mor}_{\mathrm{Sh}_Y}\left(\mathcal{G},f_*\mathcal{F}\right).$$

In other words, we have a pair of adjoint functors.

Proof. We proceed with in steps. The main point is to define a unit and counit map.

1. We define the natural map $\varepsilon\colon f^{-1}f_*\mathcal{F} \to \mathcal{F}$ given a sheaf $\mathcal{F}$ on X. Well, for any open set $U \subseteq X$, we compute
$$\left(f^{-1,\mathrm{pre}}f_*\mathcal{F}\right)(U) = \varinjlim_{V\supseteq f(U)} f_*\mathcal{F}(V) = \varinjlim_{V\supseteq f(U)} \mathcal{F}\left(f^{-1}(V)\right).$$
Notably, $V \supseteq f(U)$ implies $U_X \subseteq f^{-1}(U)$, so we can take some $[(V,s)]$ with $s \in \mathcal{F}\left(f^{-1}(V)\right)$ to $s|_U \in \mathcal{F}(U)$. As such, we define
$$\begin{array}{rccc}\varepsilon_U^{\mathrm{pre}}\colon & \left(f^{-1,\mathrm{pre}}f_*\mathcal{F}\right)(U) & \to & \mathcal{F}(U_X) \\ & [(V,s)] & \mapsto & s|_U\end{array}$$
for which we have the following checks.

- Well-defined: if $(V,s)\sim(V',s')$, then there is some open set $V''\subseteq V\cap V'$ with $V\supseteq f(U_X)$ such that $s|_{f^{-1}(V'')}=s'|_{f^{-1}(V'')}$. As such, we find

$$s|_U=s|_{f^{-1}(V'')}|_U=s'|_{f^{-1}(V'')}|_U=s'|_U,$$

so $\varepsilon_U^{\text{pre}}([(V,s)])$ is in fact well-defined.

- Natural: we verify that ε^{pre} is a (pre)sheaf morphism. Fix open sets $U'\subseteq U\subseteq X$. Then we see that the diagram

$$\begin{tikzcd} \left(f^{-1,\text{pre}}f_*\mathcal{F}\right)(U) \arrow[r, "\varepsilon_U^{\text{pre}}"] \arrow[d, "\text{res}_{U,U'}"'] & \mathcal{F}(U) \arrow[d, "\text{res}_{U,U'}"] & {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & s|_U \arrow[d, mapsto] \\ \left(f^{-1,\text{pre}}f_*\mathcal{F}\right)(U') \arrow[r, "\varepsilon_{U'}^{\text{pre}}"] & \mathcal{F}(U') & {[(V,s)]} \arrow[r, mapsto] & s|_V \end{tikzcd}$$

commutes.

- Natural: we verify that ε^{pre} assembles into a natural transformation $f^{-1,\text{pre}}f_*\Rightarrow\text{id}_{\text{PreSh}_X}$. Indeed, given a presheaf morphism $\varphi\colon\mathcal{F}\to\mathcal{F}'$, observe that the left diagram

$$\begin{tikzcd} f^{-1,\text{pre}}f_*\mathcal{F} \arrow[r, "f^{-1,\text{pre}}f_*\varphi"] \arrow[d, "\varepsilon^{\text{pre}}"'] & f^{-1,\text{pre}}f_*\mathcal{F}' \arrow[d, "\varepsilon^{\text{pre}}"] & {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V,\varphi_V(s))]} \arrow[d, mapsto] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{F}' & s|_U \arrow[r, mapsto] & \varphi_V(s)|_U \end{tikzcd}$$

commutes at each open set $U\subseteq X$, as shown in the right diagram.

The universal property of sheafification tells us that there is a unique sheaf morphism $\varepsilon\colon f^{-1}f_*\mathcal{F}\to\mathcal{F}$ making the diagram

$$\begin{tikzcd} f^{-1,\text{pre}}f_*\mathcal{F} \arrow[r, "\text{sh}"] \arrow[dr, "\varepsilon^{\text{pre}}"'] & f^{-1}f_*\mathcal{F} \arrow[d, dashed, "\varepsilon"] \\ & \mathcal{F} \end{tikzcd}$$

commute. We quickly check the naturality of $\varepsilon\colon f^{-1}f_*\Rightarrow\text{id}_{\text{Sh}_X}$: given a sheaf morphism $\varphi\colon\mathcal{F}\to\mathcal{F}'$, the outer square of

$$\begin{tikzcd} f^{-1,\text{pre}}f_*\mathcal{F} \arrow[rrr, "f^{-1,\text{pre}}f_*\varphi"] \arrow[dr, "\text{sh}"] \arrow[ddd, "\varepsilon_{\mathcal{F}}^{\text{pre}}"'] & & & f^{-1,\text{pre}}f_*\mathcal{F}' \arrow[dl, "\text{sh}"'] \arrow[ddd, "\varepsilon_{\mathcal{F}'}^{\text{pre}}"] \\ & f^{-1}f_*\mathcal{F} \arrow[r, "f^{-1}f_*\varphi"] \arrow[ddl, "\varepsilon_{\mathcal{F}}"] & f^{-1}f_*\mathcal{F}' \arrow[ddr, "\varepsilon_{\mathcal{F}'}"'] & \\ & & & \\ \mathcal{F} \arrow[rrr, "\varphi"] & & & \mathcal{F}' \end{tikzcd}$$

commutes by our previous naturality check. Additionally, the triangles and top square commutes by sheafification. We want the bottom square to commute. Well, the path

$$f^{-1,\text{pre}}f_*\mathcal{F}\to\mathcal{F}\to\mathcal{F}'$$

by sheafification induces a unique morphism $f^{-1}f_*\mathcal{F}\to\mathcal{F}'$ making the diagram commute. Comparing our two candidates, we see that $\varphi\circ\varepsilon_{\mathcal{F}}=\varepsilon_{\mathcal{F}'}\circ f^{-1}f_*\varphi$. This finishes our check.

2. We define the natural map $\eta\colon\mathcal{G}\to f_*f^{-1}\mathcal{G}$ given a sheaf $\mathcal{G}$ on Y. Well, for any open set $U\subseteq Y$, we compute

$$\left(f_*f^{-1,\text{pre}}\mathcal{G}\right)(U)=f^{-1,\text{pre}}\mathcal{G}\left(f^{-1}(U)\right)=\varinjlim_{V\supseteq f(f^{-1}(U))}\mathcal{G}(V).$$

As such, there is a natural map

$$\begin{aligned} \eta_U^{\text{pre}} \colon \mathcal{G}(U) &\to \left(f_*f^{-1,\text{pre}}\mathcal{G}\right)(U) \\ s &\mapsto [(U,s)] \end{aligned}$$

which makes sense because $U \supseteq f\left(f^{-1}(U)\right)$. We have the following naturality checks on η_U^{pre}.

- Natural: we verify that ε^{pre} assembles into a (pre)sheaf morphism. Indeed, given open sets $U' \subseteq U \subseteq Y$, the diagram

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, "\eta_U^{\text{pre}}"] \arrow[d, "\text{res}_{U,U'}"'] & f_*f^{-1,\text{pre}}\mathcal{G}(U) \arrow[d, "\text{res}_{U,U'}"] \\ \mathcal{G}(U') \arrow[r, "\eta_{U'}^{\text{pre}}"] & f_*f^{-1,\text{pre}}\mathcal{G}(U') \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & {[(U,s)]} \arrow[d, mapsto] \\ s|_{U'} \arrow[r, mapsto] & {[(U,s)]} \end{tikzcd}$$

  commutes because $s|_{U'}|_{U'} = s|_{U'}$ verifies $[(U', s|_{U'})] = [(U,s)]$.

- Natural: we verify that ε^{pre} assembles into a natural transformation $\text{id}_{\text{PreSh}_Y} \Rightarrow f_*f^{-1}$. Indeed, given a presheaf morphism $\varphi\colon \mathcal{G} \to \mathcal{G}'$, observe the left diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\varphi"] \arrow[d, "\eta_{\mathcal{G}}^{\text{pre}}"'] & \mathcal{G}' \arrow[d, "\eta_{\mathcal{G}'}^{\text{pre}}"] \\ f_*f^{-1,\text{pre}}\mathcal{G} \arrow[r, "f_*f^{-1,\text{pre}}\varphi"] & f_*f^{-1,\text{pre}}\mathcal{G}' \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_U(s) \arrow[d, mapsto] \\ {[(U,s)]} \arrow[r, mapsto] & {[(U,\varphi_U(s))]} \end{tikzcd}$$

  commutes at each open set $U \subseteq X$ as shown in the right diagram.

Denoting our sheafification map by $\text{sh}\colon f^{-1,\text{pre}}\mathcal{G} \to \mathcal{G}$, we define $\eta \coloneqq f_*\,\text{sh} \circ \eta^{\text{pre}}$. We automatically know that η is always a sheaf morphism, but to see that a natural transformation $\eta\colon \text{id}_{\text{Sh}_Y} \Rightarrow f_*f^{-1}$, observe that a morphism $\varphi\colon \mathcal{G} \to \mathcal{G}'$ makes the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\varphi"] \arrow[d, "\eta_{\mathcal{G}'}"'] & \mathcal{G}' \arrow[d, "\eta_{\mathcal{G}'}"] \\ f_*f^{-1,\text{pre}}\mathcal{G} \arrow[r, "f_*f^{-1,\text{pre}}\varphi"] \arrow[d, "f_*\,\text{sh}"'] & f_*f^{-1,\text{pre}}\mathcal{G}' \arrow[d, "f_*\,\text{sh}"] \\ f_*f^{-1}\mathcal{G} \arrow[r, "f_*f^{-1}\varphi"] & f_*f^{-1}\mathcal{G}' \end{tikzcd}$$

commute: the top square commutes as shown above, and the bottom square by applying f_* to the usual sheafification square. As such, the outer rectangle commutes, finishing.

3. We verify the triangle identities.

- Given a sheaf $\mathcal{F}$ on X, we verify that the diagram

$$\begin{tikzcd} f_*\mathcal{F} \arrow[r, "\eta_{f_*\mathcal{F}}"] \arrow[dr, equal] & f_*f^{-1}f_*\mathcal{F} \arrow[d, "f_*\varepsilon_{\mathcal{F}}"] \\ & f_*\mathcal{F} \end{tikzcd}$$

  commutes. Indeed, for any open set $U \subseteq Y$ and $s \in \mathcal{F}\left(f^{-1}(U)\right)$, we see

$$\begin{tikzcd} f_*\mathcal{F}(U) \arrow[r, "\eta_{f_*\mathcal{F}}^{\text{pre}}(U)"] \arrow[dr, equal] & f_*f^{-1,\text{pre}}f_*\mathcal{F}(U) \arrow[d, "f_*\varepsilon_{\mathcal{F}}^{\text{pre}}(U)"] \\ & f_*\mathcal{F}(U) \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[dr, mapsto] & {[(U,s)]} \arrow[d, mapsto] \\ & s|_U = s|_{f^{-1}(U)} = s \end{tikzcd}$$

commutes. As such, the outer triangle of

$$\begin{tikzcd}
f_*\mathcal{F} \arrow[dr, "\eta^{\mathrm{pre}}_{f_*\mathcal{F}}"'] \arrow[drr, "\eta_{f_*\mathcal{F}}"] \arrow[dd, equal] & & \\
& f_*f^{-1,\mathrm{pre}}f_*\mathcal{F} \arrow[r, dashed, "\mathrm{sh}"] \arrow[dl, "f_*\varepsilon^{\mathrm{pre}}_{\mathcal{F}}"'] & f_*f^{-1}f_*\mathcal{F} \arrow[dll, "f_*\varepsilon_{\mathcal{F}}"] \\
f_*\mathcal{F} & &
\end{tikzcd}$$

commutes, making the inner triangle commute by definition of those morphisms.

- Give a sheaf $\mathcal{G}$ on Y, we verify that the diagram

$$\begin{tikzcd}
f^{-1}\mathcal{G} \arrow[r, "f^{-1}\eta_{\mathcal{G}}"] \arrow[dr, equal] & f^{-1}f_*f^{-1}\mathcal{G} \arrow[d, "\varepsilon_{f^{-1}\mathcal{G}}"] \\
& f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes. Indeed, for any open set $U \subseteq X$ and $[(V,s)] \in f^{-1}\mathcal{G}(U)$, we see

$$\begin{tikzcd}
f^{-1,\mathrm{pre}}\mathcal{G}(U) \arrow[r, "f^{-1,\mathrm{pre}}\eta^{\mathrm{pre}}_{f^{-1}\mathcal{G}}(U)"] \arrow[dr, equal] & f^{-1,\mathrm{pre}}f_*f^{-1,\mathrm{pre}}\mathcal{G}(U) \arrow[d, "\varepsilon^{\mathrm{pre}}_{f^{-1,\mathrm{pre}}\mathcal{G}}(U)"] \\
& f^{-1,\mathrm{pre}}\mathcal{G}(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(V,s)]} \arrow[r, mapsto] \arrow[dr, mapsto] & {[(V,\eta_{\mathcal{G}}(s))] = [(V,[(V,s)])]} \arrow[d, mapsto] \\
& {[(V,s)]|_U = [(V,s)]}
\end{tikzcd}$$

commutes; here we have extended η^{pre} and $\varepsilon^{\mathrm{pre}}$ in the natural way to all presheaves. Thus, we claim that the diagram

$$\begin{tikzcd}
f^{-1}\mathcal{G} \arrow[r, "f^{-1}\eta^{\mathrm{pre}}_{\mathcal{G}}"] \arrow[rr, bend left, "f^{-1}\eta_{\mathcal{G}}"] & f^{-1}f_*f^{-1,\mathrm{pre}}\mathcal{G} \arrow[r, "f^{-1}f_*\,\mathrm{sh}"] & f^{-1}f_*f^{-1}\mathcal{G} \arrow[dd, bend left, "\varepsilon_{f^{-1}\mathcal{G}}"] \\
f^{-1,\mathrm{pre}}\mathcal{G} \arrow[u, "\mathrm{sh}"] \arrow[r, "f^{-1,\mathrm{pre}}\eta^{\mathrm{pre}}_{\mathcal{G}}"] \arrow[dr, equal] \arrow[ur, phantom, "(1)"] & f^{-1,\mathrm{pre}}f_*f^{-1,\mathrm{pre}}\mathcal{G} \arrow[u, "\mathrm{sh}"] \arrow[r, "f^{-1,\mathrm{pre}}f_*\,\mathrm{sh}"] \arrow[d, "\varepsilon^{\mathrm{pre}}_{f^{-1,\mathrm{pre}}\mathcal{G}}"] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & f^{-1,\mathrm{pre}}f_*f^{-1}\mathcal{G} \arrow[u, "\mathrm{sh}"] \arrow[d, "\varepsilon^{\mathrm{pre}}_{f^{-1}\mathcal{G}}"] \\
& f^{-1,\mathrm{pre}}\mathcal{G} \arrow[r, "\mathrm{sh}"] & f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes. The triangle commutes as checked above; (1) and (2) both commute because f^{-1} is sheafification applied to the functor $f^{-1,\mathrm{pre}}$. Lastly, (3) is a naturality square for $\varepsilon^{\mathrm{pre}}$ applied to $\mathrm{sh}\colon f^{-1,\mathrm{pre}}\mathcal{G} \to \mathcal{G}$. Collapsing the above diagram, we conclude that

$$\begin{tikzcd}
f^{-1,\mathrm{pre}}\mathcal{G} \arrow[r, equal] \arrow[d, "\mathrm{sh}"'] & f^{-1,\mathrm{pre}}\mathcal{G} \arrow[d, "\mathrm{sh}"] \\
f^{-1}\mathcal{G} \arrow[r, "\varepsilon_{f^{-1}\mathcal{G}} \circ f^{-1}\eta_{\mathcal{G}}"'] & f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes, but because sheafification is a functor, we are forced to have $\varepsilon_{f^{-1}\mathcal{G}} \circ f^{-1}\eta_{\mathcal{G}} = \mathrm{id}_{f^{-1}\mathcal{G}}$, which finishes this check.

4. We now exhibit our natural bijection as follows; fix sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y.

$$\begin{array}{ccc} \mathrm{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G},\mathcal{F}) & \simeq & \mathrm{Mor}_{\mathrm{Sh}_Y}(\mathcal{G},f_*\mathcal{F}) \\ \varphi & \mapsto & f_*\varphi\circ\eta_\mathcal{G} \\ \varepsilon_\mathcal{F}\circ f^{-1}\psi & \leftarrow\!\shortmid & \psi \end{array}$$

We have the following checks.

- Bijective: starting with $\varphi\colon f^{-1}\mathcal{G}\to\mathcal{F}$, we get mapped to

$$\begin{aligned} \varepsilon_\mathcal{F}\circ f^{-1}\left(f_*\varphi\circ\eta_\mathcal{G}\right) &= \varepsilon_\mathcal{F}\circ f^{-1}f_*\varphi\circ f^{-1}\eta_\mathcal{G} \\ &\overset{*}{=} \varphi\circ\varepsilon_{f^{-1}\mathcal{G}}\circ f^{-1}f_*\varphi\circ f^{-1}\eta_\mathcal{G} \\ &= \varphi, \end{aligned}$$

  where in $\overset{*}{=}$ we used the naturality of ε, and the last equality used the triangle equalities. Similarly, starting with $\psi\colon\mathcal{G}\to f_*\mathcal{F}$, we get mapped to

$$\begin{aligned} f_*\left(\varepsilon_\mathcal{F}\circ f^{-1}\psi\right)\circ\eta_\mathcal{G} &= f_*\varepsilon_\mathcal{F}\circ f_*f^{-1}\psi\circ\eta_\mathcal{G} \\ &\overset{*}{=} f_*\varepsilon_\mathcal{F}\circ\eta_{f_*\mathcal{F}}\circ\psi \\ &= \psi, \end{aligned}$$

  where in $\overset{*}{=}$ we used the naturality of η, and the last equality used the triangle equalities.
- Natural: given a morphism $\alpha\colon\mathcal{F}\to\mathcal{F}'$ of sheaves on X, the square

$$\begin{tikzcd} \mathrm{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G},\mathcal{F}) \arrow[d, "\alpha\circ-"'] \arrow[r, phantom, "\simeq"] & \mathrm{Mor}_{\mathrm{Sh}_Y}(\mathcal{G},f_*\mathcal{F}) \arrow[d, "f_*\alpha\circ-"] & \varphi \arrow[r, mapsto] \arrow[d, mapsto] & f_*\varphi\circ\eta_\mathcal{G} \arrow[d, mapsto] \\ \mathrm{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G},\mathcal{F}') \arrow[r, phantom, "\simeq"] & \mathrm{Mor}_{\mathrm{Sh}_Y}(\mathcal{G},f_*\mathcal{F}') & \alpha\circ\varphi \arrow[r, mapsto] & f_*(\alpha\circ\varphi)\circ\eta_\mathcal{G} \end{tikzcd}$$

  commutes. Similarly, given a morphism $\beta\colon\mathcal{G}\to\mathcal{G}'$ of sheaves on Y, the square

$$\begin{tikzcd} \mathrm{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G}',\mathcal{F}) \arrow[d, "-\circ f^{-1}\beta"'] \arrow[r, phantom, "\simeq"] & \mathrm{Mor}_{\mathrm{Sh}_Y}(\mathcal{G}',f_*\mathcal{F}) \arrow[d, "-\circ\beta"] & \varepsilon_\mathcal{F}\circ f^{-1}\psi \arrow[d, mapsto] & \psi \arrow[l, mapsto] \arrow[d, mapsto] \\ \mathrm{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G},\mathcal{F}) \arrow[r, phantom, "\simeq"] & \mathrm{Mor}_{\mathrm{Sh}_Y}(\mathcal{G},f_*\mathcal{F}) & \varepsilon_\mathcal{F}\circ f^{-1}(\psi\circ\beta) & \psi\circ\beta \arrow[l, mapsto] \end{tikzcd}$$

  commutes.

The above checks finish the proof. ■

1.5.9 The Restriction Sheaf

One particular example of the inverse image sheaf is for an embedding.

Definition 1.173 (Restriction sheaf)**.** Fix a topological space X and a subset $S\subseteq X$; let $\iota\colon S\to X$ be the embedding. Then a sheaf $\mathcal{F}$ on X restricts to a sheaf $\mathcal{F}|_S:=\iota^{-1}\mathcal{F}$ on S.

For example, our computation of stalks for the inverse image sheaf tells us that any $p\in S$ has

$$(\mathfrak{F}|_S)_p=\left(\iota^{-1}\mathcal{F}\right)_{\iota(p)}\simeq\mathcal{F}_p$$

by Lemma 1.171.

A special case of this embedding will be of interest.

Lemma 1.174. Fix a topological space X and an open subset $U \subseteq X$; let $\iota\colon U \to X$ be the embedding. Then a sheaf $\mathcal{F}$ on X actually restricts to a sheaf

$$\iota^{-1,\text{pre}}\mathcal{F}(V) := \mathcal{F}(V)$$

on U.

Proof. We already know that $\iota^{-1,\text{pre}}\mathcal{F}$ is a presheaf by Lemma 1.168, where the restriction map for $V' \subseteq V \subseteq U$ going $\iota^{-1,\text{pre}}\mathcal{F}(V') \to \iota^{-1,\text{pre}}\mathcal{F}(V)$ is just $\mathcal{F}(V') \to \mathcal{F}(V)$. It remains to show the sheaf axioms. Fix an open cover $\mathcal{V}$ of an open set $V \subseteq U$.

- Identity: if $f_1, f_2 \in \iota^{-1,\text{pre}}\mathcal{F}(V)$ have $f_1|_W = f_2|_W$ for each $W \in \mathcal{V}$, then we are actually saying that $f_1, f_2 \in \mathcal{F}(V)$ have
$$f_1|_W = f_2|_W$$
for all $W \in \mathcal{V}$, so $f_1 = f_2$ follows from the identity axiom of $\mathcal{F}$.

- Gluability: suppose we have $f_W \in \iota^{-1,\text{pre}}\mathcal{F}(W)$ for each $W \in \mathcal{V}$ such that $f_W|_{W\cap W'} = f_{W'}|_{W\cap W'}$ for each $W, W' \in \mathcal{V}$. This actually translates into $f_W \in \mathcal{F}(W)$ and
$$f_W|_{W\cap W'} = f_{W'}|_{W\cap W'}$$
for each $W, W' \in \mathcal{V}$, from which it follows we can find $f \in \mathcal{F}(V) = \iota^{-1,\text{pre}}\mathcal{F}(V)$ such that $f|_W = f_W$ for each $W \in \mathcal{V}$. ■

Remark 1.175. Note that there is a natural isomorphism

$$\begin{array}{rcl} \varinjlim_{W\supseteq V} \mathcal{F}(W) & \simeq & \mathcal{F}(V) \\ (W, s) & \mapsto & s|_W \\ (V, s) & \leftarrow\!\shortmid & s \end{array}$$

which motivates makes our definition of $\iota^{-1,\text{pre}}\mathcal{F}$ above make sense.

With the above in mind, in order to avoid a level of sheafification in this special case, we will sloppily set the following notation.

Notation 1.176. Fix a topological space X and an open subset $U \subseteq X$. Then, given a sheaf $\mathcal{F}$ we will set the restriction sheaf $\mathcal{F}|_U$ to actually be $\iota^{-1,\text{pre}}\mathcal{F}$, where $\iota\colon U \to X$ is the embedding.

Notably, because $\mathcal{F}|_U$ is already a sheaf, the isomorphism class remains well-defined among our notation.

1.5.10 More Sheaves

Let's see a few more examples, for fun.

Definition 1.177 (Constant sheaf)**.** Fix a set S and a topological space X. Then the *constant sheaf* is

$$\underline{S}(U) := \operatorname{Mor}_{\text{Top}}(U, S),$$

where S has been turned into a topological space by giving it the discrete topology.

Remark 1.178. Intuitively, one should think of $\underline{S}(U)$ as$S^{\oplus \pi_0(U)}$ where $\pi_0(U)$ is the number of connected components in U. We have chosen not to do this because this definition is hard to work with for proofs.

Remark 1.179. All the stalks of $\underline{S}$ are S.

Definition 1.180 (Skyscraper sheaf)**.** Fix a topological space Y and a set S. For $y \in Y$, set $X := \{y\}$ so that there is a continuous map $\iota\colon X \hookrightarrow Y$. Then we define the *skyscraper sheaf* as

$$\iota_* S(U) := \begin{cases} S & y \in U, \\ \{*\} & y \notin U. \end{cases}$$

Remark 1.181. For $z \in Y$, we can compute the stalk of the skyscraper sheaf as

$$(\iota_* S)_z = \begin{cases} S & z \in \overline{\{y\}}, \\ \{*\} & z \notin \overline{\{y\}}. \end{cases}$$

For another remark, we pick up the following definition.

Definition 1.182 (Support)**.** Fix a sheaf $\mathcal{F}$ on a topological space x. Then we define the *support* of $\mathcal{F}$ to be

$$\operatorname{supp} \mathcal{F} := \{x \in X : \#\mathcal{F}_x \text{ is not terminal}\}.$$

Remark 1.183. The support of $\iota_* S$ is $\overline{\{y\}}$.

Here is another result, which explains why we care about the skyscraper sheaf.

Proposition 1.184. There is a natural bijection

$$\operatorname{Mor}_{\{y\}}(\mathcal{F}_y, \mathcal{G}) \simeq \operatorname{Mor}_Y(\mathcal{F}, \iota_* \mathcal{G}).$$

In other words, understanding maps from stalks is roughly the same as understanding maps to the corresponding skyscraper sheaf.

THEME 2
INTRODUCING SCHEMES

when it is right, the things you reach for in life, the things you deeply hope for, they will reach back.

—Bianca Sparacino, [Spa18]

2.1 September 7

Today we define schemes.

2.1.1 Locally Ringed Spaces

Schemes will be a special kind of locally ringed space, so we take a moment to define these.

Definition 2.1 (Locally ringed space)**.** A *locally ringed space* is an ordered pair $(X, \mathcal{O}_X)$ of a topological space X and sheaf of rings $\mathcal{O}_X$ such that all stalks are local rings.

Example 2.2. Affine schemes are locally ringed spaces by Lemma 1.101.

Example 2.3. Fix a locally ringed space $(X, \mathcal{O}_X)$. For any open subset $U \subseteq X$, we see that $(U, \mathcal{O}_X|_U)$ is a locally ringed space as well. Namely, $\mathcal{O}_X|_U$ is certainly a sheaf of rings on U, and by Lemma 1.171 tells us that any $x \in U$ makes

$$(\mathcal{O}_X|_U)_x = \mathcal{O}_{X,x}$$

a local ring, so all stalks are indeed local rings.

Having been introduced to a new algebraic object, one should ask how to define a morphism. This is somewhat subtle. We begin by just giving the definition.

Definition 2.4 (Morphism of locally ringed spaces)**.** Given locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ locally ringed spaces, a *morphism* is a pair $(\varphi, \varphi^\sharp)$ of a continuous map $\varphi\colon X \to Y$ and a sheaf morphism $\varphi^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Further, we require that, at each $x \in X$, the induced map

$$\begin{array}{ccccc} \mathcal{O}_{Y,\varphi(x)} & \overset{\varphi^\sharp_{\varphi(x)}}{\to} & (\varphi_*\mathcal{O}_X)_{\varphi(x)} & \to & \mathcal{O}_{X,x} \\ [(V,s)] & \mapsto & \left[(V, \varphi^\sharp_V(s))\right] & \mapsto & \varphi^\sharp_V(s)|_x \end{array}$$

is a morphism of local rings; i.e., the image of $\mathfrak{m}_{Y,\varphi(x)}$ is contained in $\mathfrak{m}_{X,x}$, or equivalently the pre-image of $\mathfrak{m}_{X,x}$ is $\mathfrak{m}_{Y,\varphi(x)}$.

Notably, the last map $(f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ above is the canonical map of Lemma 1.165.

Remark 2.5. Using the inverse image sheaf instead of the direct image sheaf, we can use Proposition 1.172 to think about $f^\sharp$ as

$$f^\flat\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

One might want to do this because the stalks of $f^{-1}\mathcal{O}_Y$ are nicely behaved by Lemma 1.171.

We take a moment to provide two ways to motivate Definition 2.4.

1. On the algebraic side, it will turn out that this definition makes the only morphisms of affine schemes (which are locally ringed spaces) come from ring homomorphisms, so we can "check" that this definition is the correct one.

   To help see why Definition 2.4 looks the way that it does a ring homomorphism $f\colon A \to B$ gives rise to a continuous map $\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$, but the function data still goes to $A \to B$. This explains why $\varphi^\sharp$ should go $\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$.

   Lastly, we can view the local ring condition as checking that we cohere with the "local" part of a locally ringed space.

2. On the geometric side, we should imagine that a morphism of locally ringed spaces is like a map $\varphi\colon X \to Y$ of manifolds, where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the sheaf of holomorphic functions on each. Then the sheaf morphism

   $$\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$$

   is saying that a holomorphic function $f\colon V \to \mathbb{C}$ (for some open $V \subseteq Y$) should pull back through φ to a differential function

   $$\varphi^{-1}(V) \overset{\varphi}{\to} V \overset{f}{\to} \mathbb{C} \tag{2.1}$$

   which is simply true. Importantly, there isn't really a way to take a holomorphic function $X \to \mathbb{C}$ and "push it" through φ to a holomorphic function $Y \to \mathbb{C}$.

   Lastly, the local ring condition is saying that a germ $f \in \mathcal{O}_{Y,\varphi(x)}$ will vanish at $\varphi(x)$ will pull back via (2.1) to a germ in $\mathcal{O}_{X,x}$ which vanishes at x. Again, this is simply true.

Here are some quick checks on locally ringed spaces.

Lemma 2.6. All locally ringed spaces equipped with the defined morphisms makes a category.

Proof. Here is the extra data we need to define.

- Identity: given a locally ringed space $(X, \mathcal{O}_X)$, we define $\mathrm{id}_{(X,\mathcal{O}_X)}$ as given by the continuous map $\mathrm{id}_X\colon X \to X$ and sheaf morphism $\mathrm{id}_{\mathcal{O}_X}\colon \mathcal{O}_X \to \mathcal{O}_X$. (Notably, $(\mathrm{id}_X)_*\mathcal{O}_X$ is the same as $\mathcal{O}_X$ by Lemma 1.163.) Checking stalks, we see that any $x \in X$ has

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & \overset{\mathrm{id}_{\mathcal{O}_X,x}}{\to} & ((\mathrm{id}_{\mathcal{O}_X})_*\mathcal{O}_X)_x & \to & \mathcal{O}_{X,x} \\ [(U,s)] & \mapsto & [(U,s)] & \mapsto & [(U,s)] \end{array}$$

is the identity and hence a map of local rings.

- Composition: given two morphisms $(\varphi, \varphi^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(\psi, \psi^\sharp) \colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, we define the composition as having the continuous map $\psi \circ \varphi \colon X \to Y$ and sheaf morphism

$$\mathcal{O}_Z \overset{\psi^\sharp}{\to} \psi_*\mathcal{O}_Y \overset{\psi_*\varphi^\sharp}{\to} \psi_*\varphi_*\mathcal{O}_X.$$

Notably, $\psi_*\varphi_*\mathcal{O}_X = (\psi \circ \varphi)_*\mathcal{O}_X$ by Remark 1.164, so at least all of our data look correct.

Checking stalks, fix $x \in X$ and $[(U, s)] \in \mathfrak{m}_{Z,\psi(\varphi(z))}$. Because $(\varphi, \varphi^\sharp)$ and $(\psi, \psi^\sharp)$ are morphisms of locally ringed spaces, we see that $[(\psi^{-1}U, \psi^\sharp_U z)] \in \mathfrak{m}_{Y,\varphi(z)}$, so

$$\left[((\psi \circ \varphi)^{-1}U, (\psi_*\varphi^\sharp \circ \psi^\sharp)_U s)\right] = \left[(\varphi^{-1}\psi^{-1}U, \varphi^\sharp_{\psi^{-1}U}\psi^\sharp_U s)\right] \in \mathfrak{m}_{X,x},$$

which finishes the check.

We have the following coherence checks.

- Identity: given a morphism $(\varphi, \varphi^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, we compute

$$(\varphi, \varphi^\sharp) \circ \mathrm{id}_{(X,\mathcal{O}_X)} = \left(\varphi \circ \mathrm{id}_X, \varphi_*\mathrm{id}_{\mathcal{O}_X} \circ \varphi^\sharp\right) = \left(\varphi \circ \mathrm{id}_X, \mathrm{id}_{\varphi_*\mathcal{O}_X} \circ \varphi^\sharp\right) = (\varphi, \varphi^\sharp),$$

and

$$\mathrm{id}_{Y,\mathcal{O}_Y} \circ (\varphi, \varphi^\sharp) = \left(\mathrm{id}_Y \circ \varphi, (\mathrm{id}_Y)_*\varphi^\sharp \circ \mathrm{id}_{\mathcal{O}_Y}\right) = \left(\mathrm{id}_Y \circ \varphi, \varphi^\sharp \circ \mathrm{id}_{\mathcal{O}_Y}\right) = (\varphi, \varphi^\sharp).$$

- Associativity: given morphisms $(\alpha, \alpha^\sharp) \colon (A, \mathcal{O}_A) \to (B, \mathcal{O}_B)$ and $(\beta, \beta^\sharp) \colon (B, \mathcal{O}_B) \to (C, \mathcal{O}_C)$ and $(\gamma, \gamma^\sharp) \colon (C, \mathcal{O}_C) \to (D, \mathcal{O}_D)$, we compute

$$\begin{aligned}(\gamma, \gamma^\sharp) \circ \left((\beta, \beta^\sharp) \circ (\alpha, \alpha^\sharp)\right) &= (\gamma, \gamma^\sharp) \circ (\beta \circ \alpha, \beta_*\alpha^\sharp \circ \beta^\sharp) \\ &= (\gamma \circ \beta \circ \alpha, \gamma_*\beta_*\alpha^\sharp \circ \gamma_*\beta^\sharp \circ \gamma^\sharp) \\ &= (\gamma \circ \beta, \gamma_*\beta^\sharp \circ \gamma^\sharp) \circ (\alpha, \alpha^\sharp) \\ &= \left((\gamma, \gamma^\sharp) \circ (\beta, \beta^\sharp)\right) \circ (\alpha, \alpha^\sharp),\end{aligned}$$

finishing. ■

Thus, an isomorphism of locally ringed spaces is, of course, an isomorphism in the category. This carries a lot of data, so it will be helpful to have a shorter version of the data to carry around.

Lemma 2.7. A morphism of ringed spaces $(\varphi, \varphi^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if and only if φ is a homeomorphism and $\varphi^\sharp$ is an isomorphism of sheaves.

Proof. Note that $(\varphi, \varphi^\sharp)$ is a morphism of locally ringed spaces already because any $x \in X$ has

$$\mathcal{O}_{Y,\varphi(x)} \overset{\varphi^\sharp_{\varphi(x)}}{\to} (\varphi_*\mathcal{O}_X)_{\varphi(x)} \to \mathcal{O}_{X,x}$$

is a string of isomorphisms: the former is an isomorphism because $\varphi^\sharp$ is, and the last is an isomorphism by Remark 1.166. Thus, this is a map of local rings for free.

We now construct the inverse for $(\varphi, \varphi^\sharp)$. Let $\psi \colon Y \to X$ be the inverse continuous map for φ. Also, for each $V \subseteq \mathcal{O}_Y$, define the morphism $\psi^\sharp_V \colon \mathcal{O}_Y(V) \to \psi_*\mathcal{O}_X(V)$ as the inverse of the morphism

$$\varphi^\sharp_{\varphi(V)} \colon \mathcal{O}_X(\varphi(V)) \to \varphi_*\mathcal{O}_Y(\varphi(V))$$

which makes sense because $\psi^{-1}(V)=\varphi(V)$ and $\varphi^{-1}(\varphi(V))=V$. To check that $\psi^\sharp$ assembles into a sheaf morphism, we note that open subsets $V'\subseteq V\subseteq Y$ make the left diagram below

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\psi^\sharp_V"] \arrow[d, "\mathrm{res}_{V,V'}"'] & \psi_*\mathcal{O}_X(V) \arrow[d, "\mathrm{res}_{V,V'}"] \\ \mathcal{O}_Y(V') \arrow[r, "\psi^\sharp_{V'}"] & \psi_*\mathcal{O}_X(V') \end{tikzcd} \qquad \begin{tikzcd} \varphi_*\mathcal{O}_Y(\varphi(V)) \arrow[d, "\mathrm{res}_{\varphi(V),\varphi(V')}"'] & \mathcal{O}_X(\varphi(V)) \arrow[l, "\varphi^\sharp_{\varphi(V)}"'] \arrow[d, "\mathrm{res}_{\varphi(V),\varphi(V')}"] \\ \varphi_*\mathcal{O}_Y(\varphi(V')) & \mathcal{O}_X(\varphi(V')) \arrow[l, "\varphi^\sharp_{\varphi(V')}"'] \end{tikzcd}$$

commute because it is the same as the one on the right. Additionally, we can quickly check that we have a morphism of locally ringed spaces; by Remark 1.166, we are actually given that any $x\in X$ has

$$\mathcal{O}_{Y,\varphi(x)}\to(\varphi_*\mathcal{O}_X)_{\varphi(x)}\simeq\mathcal{O}_{X,x}$$

is a map of local rings. Inverting this map, we see that any $y\in Y$ has

$$\mathcal{O}_{X,\psi(y)}\to(\psi_*\mathcal{O}_Y)_{\psi(y)}\simeq\mathcal{O}_{Y,y}$$

is also a map of local rings.

It remains to see that $(\psi,\psi^\sharp)$ is actually the inverse of $(\varphi,\varphi^\sharp)$. On one side, we see that

$$(\varphi,\varphi^\sharp)\circ(\psi,\psi^\sharp)=(\varphi\circ\psi,\varphi_*\psi^\sharp\circ\varphi^\sharp).$$

Now, $\varphi\circ\psi=\mathrm{id}_Y$ by definition of ψ, and for any $U\subseteq X$, we note $\psi^\sharp_{\varphi^{-1}(U)}=(\varphi^\sharp_U)^{-1}$ by definition of $\psi^\sharp$. So the above is indeed $\mathrm{id}_{(Y,\mathcal{O}_Y)}$. The other side inverse check is entirely symmetric. ■

For schemes, we will be very interested in special (open) subsets of the underlying topological space. The following lemma will be of use.

Lemma 2.8. Fix a morphism $(\varphi,\varphi^\sharp)\colon(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ of locally ringed spaces. Then, for any open subset $U\subseteq Y$, φ will restrict to a morphism of locally ringed spaces

$$(\varphi,\varphi^\sharp)|_U\colon\left(\varphi^{-1}(U),\mathcal{O}_X|_{\varphi^{-1}U}\right)\to(U,\mathcal{O}_Y|_U).$$

In particular, if $(\varphi,\varphi^\sharp)$ is an isomorphism, then $(\varphi,\varphi^\sharp)|_U$ is an isomorphism.

Proof. We will define $(\varphi,\varphi^\sharp)_U$ by hand. We set $\psi\colon\varphi^{-1}(U)\to U$ to just be $\varphi|_{\varphi^{-1}(U)}$, which is continuous by restriction. Additionally, for any open subset $V\subseteq U$, we define

$$\psi^\sharp_V\colon\underbrace{\mathcal{O}_Y|_U(V)}_{\mathcal{O}_Y(V)}\to\underbrace{\psi_*(\mathcal{O}_X|_{\varphi^{-1}U})(V)}_{\mathcal{O}_X(\psi^{-1}(V))}$$

as just $\varphi^\sharp_V$, which makes sense because $\mathcal{O}_X(\psi^{-1}(V))=\mathcal{O}_X(\varphi^{-1}(V))=\varphi_*\mathcal{O}_X(V)$. To see that $\psi^\sharp$ assembles into a morphism of sheaves, we see that any $V'\subseteq V\subseteq U$ makes the left diagram of

$$\begin{tikzcd} \mathcal{O}_Y|_U(V) \arrow[r, "\psi^\sharp_V"] \arrow[d, "\mathrm{res}_{V,V'}"'] & \psi_*(\mathcal{O}_X|_{\varphi^{-1}U})(V) \arrow[d, "\mathrm{res}_{V,V'}"] \\ \mathcal{O}_Y|_U(V') \arrow[r, "\psi^\sharp_{V'}"] & \psi_*(\mathcal{O}_X|_{\varphi^{-1}U})(V') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\varphi^\sharp_V"] \arrow[d, "\mathrm{res}_{V,V'}"'] & \varphi_*\mathcal{O}_X(V) \arrow[d, "\mathrm{res}_{V,V'}"] \\ \mathcal{O}_Y(V') \arrow[r, "\varphi^\sharp_{V'}"] & \varphi_*\mathcal{O}_X(V') \end{tikzcd}$$

commutes because it is the same as the right diagram. Continuing, $(\psi,\psi^\sharp)$ is a morphism of locally ringed spaces because any $x\in\varphi^{-1}(U)$ makes the diagram

$$\begin{tikzcd} (\mathcal{O}_Y|_U)_{\psi(x)} \arrow[r, "\psi^\sharp_{\psi(x)}"] \arrow[d] & \left(\psi_*(\mathcal{O}_X|_{\varphi^{-1}(U)})\right)_{\psi(x)} \arrow[r] & (\mathcal{O}_X|_{\varphi^{-1}(U)})_x \arrow[d] \\ \mathcal{O}_{Y,\varphi(x)} \arrow[r, "\varphi^\sharp_{\varphi(x)}"] & (\varphi_*\mathcal{O}_X)_{\varphi(x)} \arrow[r] & \mathcal{O}_{X,x} \end{tikzcd} \qquad \begin{tikzcd} {[(V,s)]} \arrow[r, mapsto] \arrow[d, equal] & \psi^\sharp_V(s)|_x \arrow[d, equal] \\ {[(V,s)]} \arrow[r, mapsto] & \varphi^\sharp_V(s)|_x \end{tikzcd}$$

commute, where the vertical morphisms are the isomorphisms of Lemma 1.171. In particular, the top composite is a map of local rings because the bottom one is.

It remains to show that $(\varphi, \varphi^\sharp)$ being an isomorphism forces $(\psi, \psi^\sharp)$ is an isomorphism. Well, to see that $\psi\colon \varphi^{-1}(U) \to U$ is a homeomorphism, note that ψ is an injective, continuous, open map as inherited from φ, and ψ is surjective onto U by construction. Additionally, $\psi^\sharp$ is an isomorphism because its components morphisms come from $\varphi^\sharp$, which are all isomorphisms. Thus, we are done by Lemma 2.7. ■

2.1.2 K-points

The morphism of a locally ringed space contains a lot of data, so it will be helpful to see all this data go to use. Here's an example.

Definition 2.9 (Residue field)**.** Fix a locally ringed space $(X, \mathcal{O}_X)$. Given a point $x \in X$, define $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ to be the unique maximal ideal of $\mathcal{O}_{X,x}$. Then the *residue field* of x is $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Exercise 2.10. Fix a locally ringed space $(X, \mathcal{O}_X)$ and a field K. Then the data of a morphism of locally ringed spaces $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (X, \mathcal{O}_X)$ can be equivalently presented as a point $p \in X$ equipped with an inclusion $\iota\colon k(p) \to K$.

Intuitively, we are saying that morphisms from the affine scheme over K correspond to "K-points of X," for a suitable definition of K-points.

Proof. Let M be the set of morphisms $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (X, \mathcal{O})$, and let P be the set of ordered pairs (p, ι) where $p \in X$ is a point and $\iota\colon k(p) \hookrightarrow K$ is an embedding. We exhibit a bijection between M and P. Here are the maps.

- We exhibit a map $\alpha\colon M \to P$. Well, given a morphism $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (X, \mathcal{O})$, we have an underlying continuous map $\varphi\colon \operatorname{Spec} K \to X$ and sheaf morphism $\varphi^\sharp\colon \mathcal{O} \to \varphi_*\mathcal{O}_{\operatorname{Spec} K}$.

  Now, $(0) \in \operatorname{Spec} K$, so we set $p := \varphi((0))$. Then $\varphi^\sharp$ will provide a map

  $$\mathcal{O}_p \stackrel{\varphi_p^\sharp}{\to} (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p \to \mathcal{O}_{\operatorname{Spec} K,(0)} \cong K_{(0)} = K.$$

  This is supposed to be a map of local rings, so the pre-image of the maximal ideal $(0) \subseteq K$ is supposed to equal $\mathfrak{m}_p$, so we actually induce an embedding $\iota\colon \mathcal{O}_p/\mathfrak{m}_p \hookrightarrow K$. Thus, we set $\alpha((\varphi, \varphi^\sharp)) = (p, \iota)$.

- We exhibit a map $\beta\colon P \to M$. We are provided with a point $p \in X$ and an inclusion $\mathcal{O}_p/\mathfrak{m}_p \to K$. Here is the defining data.

  - Define $\varphi\colon \operatorname{Spec} K \to X$ by $\varphi((0)) := p$. To see that this continuous, note any open subset $U \subseteq X$ containing p have $\varphi^{-1}(U) = \{(0)\} = \operatorname{Spec} K$, which is open. Otherwise, the open subset $U \subseteq X$ does not contain p, so $\varphi^{-1}(U) = \varnothing$, which is still open.
  - Given an open subset $U \subseteq X$, we define $\varphi_U^\sharp\colon \mathcal{O}(U) \to \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U)$. If U does not contain p, then

    $$\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) = \mathcal{O}_{\operatorname{Spec} K}\left(\varphi^{-1}(U)\right) = \mathcal{O}_{\operatorname{Spec} K}(\varnothing) = 0,$$

    so we set $\varphi_U^\sharp$ to be the zero map. Otherwise, when U contains p, we see

    $$\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) = \mathcal{O}_{\operatorname{Spec} K}\left(\varphi^{-1}(U)\right) = \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K) = K,$$

    so we need to exhibit a map $\varphi_U^\sharp\colon \mathcal{O}(U) \to K$. For this, we use the composite map

    $$\begin{array}{ccccccc} \mathcal{O}(U) & \to & \mathcal{O}_p & \to & \mathcal{O}_p/\mathfrak{m}_p & \stackrel{\iota}{\to} & K \\ s & \mapsto & s|_p & \mapsto & (s|_p + \mathfrak{m}_p) & \mapsto & \iota(s|_p + \mathfrak{m}_p) \end{array}$$

    as our $\varphi_U^\sharp$.

We quickly check that $\varphi^\sharp$ assembles into a map of sheaves. Fix open sets $U' \subseteq U$, and we want the diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, "\varphi^\sharp_U"] \arrow[d, "\operatorname{res}_{U,U'}"'] & \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) \arrow[d, "\operatorname{res}_{U,U'}"] \\ \mathcal{O}(U') \arrow[r, "\varphi^\sharp_{U'}"] & \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U') \end{tikzcd} \tag{2.2}$$

to commute. We have two cases.

- If $p \notin U'$, then $\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U') = 0$, so (2.2) commutes for free.
- If $p \in U'$, then $p \in U$ as well, so (2.2) becomes the diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, "\varphi^\sharp_U"] \arrow[d, "\operatorname{res}_{U,U'}"'] & K \arrow[d, "\operatorname{res}_{U,U'}"] \\ \mathcal{O}(U') \arrow[r, "\varphi^\sharp_{U'}"] & K \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & \iota(s|_p + \mathfrak{m}_p) \arrow[d, mapsto] \\ s|_{U'} \arrow[r, mapsto] & \iota(s|_p + \mathfrak{m}_p) \end{tikzcd}$$

  which does indeed commute.

Next we check that $(\varphi, \varphi^\sharp)$ assembles into a morphism of locally ringed spaces. For this we have to check that, for any $\mathfrak{p} \in \operatorname{Spec} K$, the composite

$$\mathcal{O}_{\varphi(\mathfrak{p})} \stackrel{\varphi^\sharp_{\mathfrak{p}}}{\to} (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_{\varphi(\mathfrak{p})} \to (\mathcal{O}_{\operatorname{Spec} K})_{\mathfrak{p}}$$

is a map of local rings. Notably, the only point we have to check this is on $\mathfrak{p} = (0)$ because $\operatorname{Spec} K = \{(0)\}$, and $\varphi((0)) = p$, so we are checking that

$$\begin{array}{ccccc} \mathcal{O}_p & \stackrel{\varphi^\sharp_p}{\to} & (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p & \to & (\mathcal{O}_{\operatorname{Spec} K})_0 \\ [(U,s)] & \mapsto & \left[(U, \iota(s|_p + \mathfrak{m}_p))\right] & \mapsto & \left[(\varphi^{-1}(U), \iota(s|_p + \mathfrak{m}_p))\right] \end{array}$$

is a map of local rings. Notably, $\varphi^{-1}(U) = \{(0)\} = \operatorname{Spec} K = D(1)$, so we can chain the above composite with the isomorphism $(\mathcal{O}_{\operatorname{Spec} K})_0 \cong K_{(0)} = K$, which will send $[(D(1), \overline{s})]$ to $\overline{s}$. So we are showing that

$$\begin{array}{ccc} \mathcal{O}_p & \to & K \\ [(U,s)] & \mapsto & \iota(s|_p + \mathfrak{m}_p) \end{array}$$

is a map of local rings. (Namely, isomorphisms are maps of local rings, so we can "unchain" the above map with the previous isomorphisms to recover the needed map of local rings.) Well, the pre-image of the maximal ideal $(0) \subseteq K$ consists of sections $s_p \in \mathcal{O}_p$ such that $\iota(s_p + \mathfrak{m}_p) = 0$; because ι is injective, we see that this is equivalent to $s_p \in \mathfrak{m}_p$.

So indeed, the pre-image of the maximal ideal (0) is $\mathfrak{m}_p$, verifying that we have a map of local rings. As such, we may define $\beta((p, \iota)) := (\varphi, \varphi^\sharp)$.

We now have to show that α and β are inverses.

- Fix some $(p, \iota) \in P$. We show that $(\alpha \circ \beta)((p, \iota)) = (p, \iota)$. Set $(\varphi, \varphi^\sharp) := \beta((p, \iota))$, and we need to compute $\alpha((\varphi, \varphi^\sharp))$. To start, by construction, we see

$$\varphi((0)) = p,$$

  as it should be. To solve for ι, we note that above we tracked through the map

$$\begin{array}{ccccccc} \mathcal{O}_p & \stackrel{\varphi^\sharp_p}{\to} & (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p & \to & (\mathcal{O}_{\operatorname{Spec} K})_0 & \cong K_{(0)} = & K \\ [(U,s)] & \mapsto & \left[(U, \iota(s|_p + \mathfrak{m}_p))\right] & \mapsto & \left[(\varphi^{-1}(U), \iota(s|_p + \mathfrak{m}_p))\right] & \mapsto & \iota(s|_p + \mathfrak{m}_p) \end{array}$$

  as having kernel $\mathfrak{m}_p$, so the induced map $\mathcal{O}_p/\mathfrak{m}_p \to K$ is just $(s_p + \mathfrak{m}_p) \mapsto \iota(s_p + \mathfrak{m}_p)$. Thus, this map induced by $\alpha((\varphi, \varphi^\sharp))$ is exactly ι, as needed.

- Fix some $(\varphi, \varphi^\sharp) \in M$. We show that $(\beta \circ \alpha)((\varphi, \varphi^\sharp)) = (\varphi, \varphi^\sharp)$. Set $(p, \iota) := \alpha((\varphi, \varphi^\sharp))$ and $(\psi, \psi^\sharp) := \beta((p, \iota))$, and we will show $(\psi, \psi^\sharp) = (\varphi, \varphi^\sharp)$. By construction, we see $\psi((0)) = p$, so we get $\psi = \varphi$ immediately.

  To show $\psi^\sharp = \varphi^\sharp$, we need to show that $\psi^\sharp_U = \varphi^\sharp_U$ as functions $\mathcal{O}(U) \to \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U)$ for each open $U \subseteq X$. We have two cases.

  - If $p \notin U$, then $\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) = \mathcal{O}_{\operatorname{Spec} K}\left(\varphi^{-1}(U)\right) = \mathcal{O}_{\operatorname{Spec} K}(\varnothing) = 0$, so $\psi^\sharp_U$ and $\varphi^\sharp_U$ must both be the zero map because 0 is terminal.
  - Otherwise, we have $p \in U$; note that $\varphi^\sharp_U, \psi^\sharp_U \colon \mathcal{O}(U) \to K$ now. By definition, $\psi^\sharp_U$ sends a section $s \in \mathcal{O}(U)$ to $\iota(s|_p + \mathfrak{m}_p)$; by definition, ι sends some $s|_p + \mathfrak{m}_p$ down the composite

$$\begin{array}{ccccccc} \mathcal{O}_p & \overset{\varphi^\sharp_p}{\to} & (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p & \to & \mathcal{O}_{\operatorname{Spec} K,(0)} & \cong K_{(0)} = & K \\ [(U,s)] & \mapsto & \left[(U, \varphi^\sharp_U(s))\right] & \mapsto & \left[(\varphi^{-1}(U), \varphi^\sharp_U(s))\right] & \mapsto & \varphi^\sharp_U(s) \end{array}$$

    which verifies that $\psi^\sharp_U$ is sending $s \in \mathcal{O}(U)$ all the way to $\varphi^\sharp_U(s)$.

  From the above, it follows that $\psi^\sharp_U = \varphi^\sharp_U$, which finishes this last check. ∎

Remark 2.11. There is a similar story one can tell for $K[\varepsilon]/(\varepsilon^2)$, where we can see that we will also want to keep track of some differential information from the ε.

2.1.3 Schemes

We finally arrive at the definition of a scheme.

Definition 2.12 (Scheme)**.** A *scheme* is a ringed space $(X, \mathcal{O}_X)$ such that, for each $x \in X$, there is an open set $U \subseteq X$ containing x such that the restriction

$$(U, \mathcal{O}_X|_U)$$

is isomorphic (as a locally ringed space) to an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. We call the $(U, \mathcal{O}_X|_U)$ an *affine open subscheme* of $(X, \mathcal{O}_X)$.

Here are some quick facts about this definition.

Lemma 2.13. Fix a ring A and a distinguished open set $D(f) \subseteq \operatorname{Spec} A$. Then

$$(A_f, \mathcal{O}_{\operatorname{Spec} A_f}) \cong (D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}).$$

Proof. The underlying homeomorphism is provided by Exercise 1.57; namely, set $\psi := D(f) \to \operatorname{Spec} A_f$ by $\psi \colon \mathfrak{p} \mapsto \mathfrak{p}A_f$, which we showed is a homeomorphism. By Lemma 2.7, it suffices to exhibit a sheaf isomorphism

$$\psi^\sharp \colon \mathcal{O}_{\operatorname{Spec} A_f} \to \psi_*\mathcal{O}_{\operatorname{Spec} A}|_{D(f)}.$$

By Lemma 1.83, it suffices provide a sheaf isomorphism on the distinguished base. Well, given a distinguished basis set $D(a/f^m) \subseteq \operatorname{Spec} A_f$, we showed in Exercise 1.57 that $\psi^{-1}(D(a/f^m)) = D(a) \subseteq D(f)$, so we define

$$\psi^\sharp_{D(a/f^m)} \colon \mathcal{O}_{\operatorname{Spec} A_f}(D(a/f^m)) \to \mathcal{O}_{\operatorname{Spec} A}\left(\psi^{-1}(D(a))\right)$$

as the string of isomorphisms

$$\mathcal{O}_{\operatorname{Spec} A_f}(D(a/f^m)) = \mathcal{O}_{\operatorname{Spec} A_f}(D(a)) \simeq A_{f,a} \simeq A_a \simeq \mathcal{O}_{\operatorname{Spec} A}(D(a)),$$

where $A_{f,a} \simeq A_a$ because $D(a) \subseteq D(f)$. In particular, this assembles into a morphism of sheaves on a base because, for $D(a/f^m) \subseteq D(b/f^n)$, the diagram

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A_f}(D(a/f^m)) \arrow[d, "\text{res}"'] \arrow[r, phantom, "="] & \mathcal{O}_{\operatorname{Spec} A_f}(D(a)) \arrow[d, "\text{res}"'] \arrow[r, phantom, "\simeq"] & A_{f,a} \arrow[r, phantom, "\simeq"] & A_a \arrow[r, phantom, "\simeq"] & \mathcal{O}_{\operatorname{Spec} A}(D(a)) \arrow[d, "\text{res}"'] \\
\mathcal{O}_{\operatorname{Spec} A_f}(D(b/f^n)) \arrow[r, phantom, "="] & \mathcal{O}_{\operatorname{Spec} A_f}(D(b)) \arrow[r, phantom, "\simeq"] & A_{f,b} \arrow[r, phantom, "\simeq"] & A_b \arrow[r, phantom, "\simeq"] & \mathcal{O}_{\operatorname{Spec} A}(D(b))
\end{tikzcd}$$

because all the maps are just localization maps in various orders. In particular, all elements in the top row can be written in the form s/a^m for some $s \in A$ and $m \in \mathbb{N}$, which makes all maps on the top the identity. Then transporting this element to the bottom row, all elements become exactly s/a^m, so the diagram is indeed commuting. ■

> **Corollary 2.14.** Fix a scheme $(X, \mathcal{O}_X)$. Then any open subset $U \subseteq X$ induces an open subscheme $(U, \mathcal{O}_X|_U)$.

Proof. The affine case follows from Lemma 2.13. The general case follows by reducing to an affine open cover.

To see this explicitly, fix some $p \in U$. We need to find an open subset $U_p \subseteq U$ such that $(U_p, \mathcal{O}_X|_U|_{U_p})$ is an affine open subscheme of X; quickly, note we will have $\mathcal{O}_X|_U|_{U_p} = \mathcal{O}_X|_{U_p}$, which is clear on the level of open sets, and the restriction maps are just induced.

Because X is a scheme, we can find some affine open subset $V_p \subseteq X$ containing p, so find our isomorphism

$$(\varphi, \varphi^\sharp) \colon (V_p, \mathcal{O}_X|_{V_p}) \cong (\operatorname{Spec} B_p, \mathcal{O}_{\operatorname{Spec} B_p}).$$

Now, $U \cap V_p \subseteq V_p$ is an open subset, so $\varphi(U) \subseteq \operatorname{Spec} B_p$ is still an open subset. Thus, we can find an element $D(f) \subseteq \varphi(U)$ of the distinguished base containing $\varphi(p)$. Setting $U_p := \varphi^{-1}(D(f))$ (which is still open), we have the chain of isomorphisms

$$(U_p, \mathcal{O}_X|_{U_p}) \overset{(\varphi,\varphi^\sharp)|_{D(f)}}{\cong} (D(f), \mathcal{O}_{\operatorname{Spec} B_p}|_{D(f)}) \cong (\operatorname{Spec}(B_p)_f, \mathcal{O}_{\operatorname{Spec}(B_p)_f}).$$

Namely, the first isomorphism is from Lemma 2.8, and the second isomorphism is from Lemma 2.13. ■

2.1.4 Geometry Is Opposite Algebra

Here is the fun part of our definition of morphisms for locally ringed spaces.

> **Proposition 2.15.** The functors
>
> $$\begin{array}{rcl} \text{Rings}^{\text{op}} & \simeq & \text{AffSch} \\ A & \mapsto & (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\ \mathcal{O}_X(X) & \leftarrow\!\shortmid & (X, \mathcal{O}_X) \end{array}$$
>
> define an equivalence of categories.

Proof. For brevity, let $\mathcal{O}_A$ denote the structure sheaf of a ring A. Now, the leftward map is essentially surjective by definition of an affine scheme, so the main point is that we have to show

$$\operatorname{Mor}_{\text{AffSch}}((\operatorname{Spec} B, \mathcal{O}_B), (\operatorname{Spec} A, \mathcal{O}_A)) \cong \operatorname{Hom}_{\text{Ring}}(A, B).$$

We define the left and right maps separately.

- In one direction, suppose we have a ring homomorphism $f^\sharp\colon A \to B$, and we need to recover a morphism of affine schemes. We already have a continuous map $f := \operatorname{Spec} f^\sharp$ going $\operatorname{Spec} B \to \operatorname{Spec} A$.

  Additionally, we can extend $f^\sharp$ to be a sheaf morphism $f^\sharp\colon \mathcal{O}_A \to f_*\mathcal{O}_B$. It is enough to define this morphism on a base: pick up some open $D(s) \subseteq \operatorname{Spec} A$ so that $\mathcal{O}_A(D(s)) = A_s$. It follows

  $$f_*\mathcal{O}_B(D(s)) = \mathcal{O}_B\left(f^{-1}(D(s))\right) = \mathcal{O}_B(D(f^\sharp s)) = B_{f^\sharp(s)},$$

  so there is a natural map $f^\sharp(D(s))\colon A_s \to B_{f^\sharp(s)}$. From here, it's not hard to check that this gives a morphism of sheaves on a base.

  Lastly, we need to check that we actually have a morphism of locally ringed spaces. Well, given $\mathfrak{p} \in \operatorname{Spec} B$, the stalk map turns out to be

  $$f^\sharp_{f(\mathfrak{p})}\colon A_{f(\mathfrak{p})} \to B_{\mathfrak{p}},$$

  which we can see to be a local ring homomorphism by passing through the maximal ideal $\mathfrak{m}_{A,f(\mathfrak{p})}$ by hand.

  For notation, we define $\operatorname{Spec} f^\sharp$ to be this morphism of local rings $(f, f^\sharp)$.

- In the other direction, suppose we have a morphism of affine schemes $(f, f^\sharp)$. Then $f^\sharp$ as a morphism of locally ringed spaces can take global sections to recover a ring homomorphism.

To finish the proof, we have to show that the composition of our two maps is the identity.

- Starting with a ring homomorphism, extending it to a morphism of affine schemes, and then restricting it back to a ring homomorphism will overall unsurprisingly do nothing.

- Suppose we start with $(f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ as a morphism of affine schemes. Taking global sections gives

  $$f^\sharp_{\operatorname{Spec} A}\colon A \to B.$$

  Define $\varphi := f^\sharp_{\operatorname{Spec} A}$. We want to show that $\operatorname{Spec} \varphi = (f, f^\sharp)$. Note we are starting with

  $$\varphi\colon \mathcal{O}_A(\operatorname{Spec} A) \to \mathcal{O}_B(\operatorname{Spec} B).$$

  Now, fix $\mathfrak{p} \in \operatorname{Spec} B$ going to some $f(\mathfrak{p}) \in \operatorname{Spec} A$. Taking stalks everywhere, we see that the diagram

  $$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^\sharp_{f(\mathfrak{p})}"] & B_{\mathfrak{p}} \end{tikzcd}$$

  commutes. In particular, we see that $\varphi^{-1}(\mathfrak{p}) \subseteq f(\mathfrak{p})$ by tracking units through the bottom map. On the other hand, $f^\sharp_{f(\mathfrak{p})}$ was assumed to be a local ring homomorphism, so actually $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ follows. Thus, $\operatorname{Spec} \varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ matches with our continuous map f!

  It remains to show that $f^\sharp$ agrees with $\operatorname{Spec} \varphi$ as a morphism of sheaves. Well, it suffices to check that these agree on stalks by Proposition 1.110. To begin, we note that φ on stalks looks like $\varphi^\sharp_{f(\mathfrak{p})}$ making the bottom arrow of the diagram

  $$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, dashed] & B_{\mathfrak{p}} \end{tikzcd}$$

  commute. But we can also put $f^\sharp_{f(\mathfrak{p})}$ here even though this arrow is unique by the universal property of localization. This finishes this check. ■

Remark 2.16. In some sense, Proposition 2.15 is intended to be fact-checking: at the end of the day, we really just want the categorical equivalence and don't care much for its proof.

We will quickly provide an example that says that we really do need to pay attention to morphisms of locally ringed spaces.

Non-Example 2.17. Consider ring homomorphism $\mathbb{Z}_p \to \mathbb{Q}_p$. However, $\operatorname{Spec}\mathbb{Z}_p = \{(0),(p)\}$ while $\operatorname{Spec}\mathbb{Q}_p = \{(0)\}$. From the natural embedding $\iota\colon \mathbb{Z}_p \to \mathbb{Q}_p$, we get a map sending $(0) \mapsto (0)$, and it will not be possible to get a ring homomorphism to send (0) to (p) because this forces $\mathbb{Q}_p$ to have torsion. Nonetheless, one can upgrade sending $(0) \mapsto (p)$ to a full morphism of sheaves even though it will not be a morphism of locally ringed spaces.

2.1.5 Extending Geometry Is Opposite Algebra

It turns out that we can extend Proposition 2.15 to work with general locally ringed spaces. We will state this as an adjunction of two functors. Here are our two functors.

Lemma 2.18. The mapping $(X,\mathcal{O}_X) \mapsto \mathcal{O}_X(X)$ defines the action of a functor

$$\Gamma\colon \mathrm{LocRingSpace} \to \mathrm{Ring}^{\mathrm{op}}$$

on objects.

Proof. Given a morphism $(\varphi,\varphi^\sharp)\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$, we induce a morphism

$$\mathcal{O}_Y(Y) \overset{\varphi^\sharp_Y}{\to} \varphi_*\mathcal{O}_X(Y) = \mathcal{O}_X(X),$$

so we define $\Gamma((\varphi,\varphi^\sharp)) \coloneqq \varphi^\sharp_Y$. Here are our functoriality checks.

- Identity: note that $\mathrm{id}_{(X,\mathcal{O}_X)} = (\mathrm{id}_X, \mathrm{id}_{\mathcal{O}_X})$, so this will induce the morphism $\Gamma(\mathrm{id}_{(X,\mathcal{O}_X)}) = (\mathrm{id}_{\mathcal{O}_X})_X = \mathrm{id}_{\mathcal{O}_X(X)}$.
- Functoriality: given morphisms $(\varphi,\varphi^\sharp)\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ and $(\psi,\psi^\sharp)\colon (Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z)$, we note the composite $(\psi,\psi^\sharp)\circ(\varphi,\varphi^\sharp)$ taken through Γ acts as

$$\mathcal{O}_Z(Z) \overset{\psi^\sharp_Z}{\to} \psi_*\mathcal{O}_Y(Z) \overset{(\psi_*\varphi^\sharp)_Z}{\to} \psi_*\varphi_*\mathcal{O}_X(Z) = \mathcal{O}_X(X)$$

  on global sections. Now, we can see that this composite is $\varphi^\sharp_Y \circ \psi^\sharp_Z = \Gamma((\varphi,\varphi^\sharp))\circ\Gamma((\psi,\psi^\sharp))$ on global sections. ■

Lemma 2.19. The mapping $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ defines the action of a functor

$$\operatorname{Spec}\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{LocRingSpace}$$

on objects.

Proof. Given a ring homomorphism $f\colon A \to B$, we need to induce a morphism $(\varphi,\varphi^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. For brevity, given a ring R, we will define $\mathcal{O}_R \coloneqq \mathcal{O}_{\operatorname{Spec} R}$.

- On topological spaces, we define $\varphi \coloneqq f^{-1}$ to be our continuous map $\operatorname{Spec} B \to \operatorname{Spec} A$. This is continuous by Lemma 1.48 and functorial by Lemma 1.48.

- On sheaves, it suffices to induce the morphism $\varphi^\sharp\colon \mathcal{O}_A \to \varphi_*\mathcal{O}_B$ on the distinguished base $\{D(a)\}_{a\in A}$ by Lemma 1.48. Well, given $a \in A$, we can compute

$$\varphi^{-1}(D(a)) = \{\mathfrak{q} \in \operatorname{Spec} B : a \notin f^{-1}\mathfrak{q}\} = \{\mathfrak{p} \in \operatorname{Spec} A : f(a) \notin \mathfrak{q}\} = D(f(a)),$$

so $\varphi^\sharp_{D(a)}$ is a map $A_a \to B_{f(a)}$. However, this is induced directly from the localization map $A \to B \to B_{f(a)}$ upon noting that $a \in A$ goes to a unit $f(a) \in B_{f(a)}$.

To finish, we do need to check that this is a morphism of sheaves on a base. Suppose $D(a') \subseteq D(a)$, which means that there is a canonical localization map $A_a \simeq A_{S(D(a))} \to A_{S(D(a'))} \simeq A_{a'}$. (Namely, $a \in A_{a'}^\times$.) Then we see that the left diagram of

$$\begin{array}{ccc} \mathcal{O}_A(D(a)) & \xrightarrow{\varphi^\sharp_{D(a)}} & \varphi_*\mathcal{O}_B(D(a)) \\ {\scriptstyle \operatorname{res}_{D(a),D(a')}}\downarrow & & \downarrow{\scriptstyle \operatorname{res}_{D(a),D(a')}} \\ \mathcal{O}_A(D(a')) & \xrightarrow{\varphi^\sharp_{D(a')}} & \varphi_*\mathcal{O}_B(D(a')) \end{array} \qquad \begin{array}{ccc} A_a & \xrightarrow{f} & B_{f(a)} \\ \downarrow & & \downarrow \\ A_{a'} & \xrightarrow{f} & B_{f(a')} \end{array} \qquad \begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x/1 & \longmapsto & f(x)/1 \end{array}$$

commutes because it is the same as the middle diagram.

To finish our construction, we need to know that $(\varphi,\varphi^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ assembles into a map of locally ringed spaces. Namely, we need to verify that, for each $\mathfrak{q} \in \operatorname{Spec} B$, the map

$$\begin{array}{ccccc} \mathcal{O}_{A,\varphi(\mathfrak{q})} & \overset{\varphi^\sharp_{\mathfrak{q}}}{\to} & (\varphi_*\mathcal{O}_B)_{\mathfrak{q}} & \to & \mathcal{O}_{B,\mathfrak{q}} \\ {}[(D(a), s/a^m)] & \mapsto & [(D(a), f(s)/f(a)^m)] & \mapsto & [(D(f(a)), f(s)/f(a)^m)] \end{array}$$

is a map of local rings; notably, we are using Lemma 1.100 to define the stalk on the distinguished base. Well, given $[(D(a), s/a^m)] \in \mathfrak{m}_{\varphi(\mathfrak{q})}$ implies that $s \in \varphi(\mathfrak{q}) = f^{-1}(\mathfrak{q})$, so $f(s) \in \mathfrak{q}$, so $[(D(f(a)), f(s)/f(a)^m)] \in \mathfrak{m}_{\mathfrak{q}}$.

Thus, we define $\operatorname{Spec} f := (\varphi, \varphi^\sharp)$. It remains to run our functoriality checks.

- Identity: note that $f = \mathrm{id}_A$ makes $\varphi\colon \operatorname{Spec} A \to \operatorname{Spec} A$ the identity Lemma 1.48, and each $a \in A$ induces the localization map $A_a \to A_{f(a)}$, which we can see visually is just the identity map. Thus, $\varphi^\sharp_{D(a)}$ is the identity for each distinguished base element $D(a)$, so $\varphi^\sharp\colon \mathcal{O}_A \to \varphi_*\mathcal{O}_A$ is just the identity by Lemma 1.79.

  Thus, $(\varphi, \varphi^\sharp)$ is the identity morphism on the locally ringed space $(\operatorname{Spec} A, \mathcal{O}_A)$.

- Functoriality: fix ring morphisms $f\colon A \to B$ and $g\colon B \to C$ yielding morphisms of locally ringed spaces $(\varphi,\varphi^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ and $(\psi,\psi^\sharp)\colon (\operatorname{Spec} C, \mathcal{O}_C) \to (\operatorname{Spec} B, \mathcal{O}_B)$. For brevity, we will also define $(\gamma, \gamma^\sharp) = \operatorname{Spec}(g \circ f)$ to be the morphism induced by the composite. We need to show $(\gamma,\gamma^\sharp) = (\varphi,\varphi^\sharp) \circ (\psi,\psi^\sharp)$.

  We already know we are functorial on the level of topological spaces (i.e., $\gamma = \varphi \circ \psi$) by Lemma 1.48. Then on sheaves, we need to check that $\gamma^\sharp = \varphi_*\psi^\sharp \circ \varphi^\sharp$. Well, for some distinguished open set $D(a) \subseteq \operatorname{Spec} A$, our composite is

$$(\varphi_*\psi^\sharp \circ \varphi^\sharp)_{D(a)} = \psi^\sharp_{\varphi^{-1}(D(a))} \circ \varphi^\sharp_{D(a)} = \psi^\sharp_{D(f(a))} \circ \varphi^\sharp_{D(a)}$$

  using computations above. Unwrapping our construction, we see that this composite is the composite of the localized maps

$$A_a \xrightarrow{f} B_{f(a)} \xrightarrow{g} C_{g(f(a))},$$

  which of course is just the single map $A_a \to C_{g(f(a))}$ induced by localizing $g \circ f$. So we do indeed match with $\gamma^\sharp$ on the base, so we have an equality of sheaves on the base by Lemma 1.79. ■

We now exhibit the first of our natural maps.

Lemma 2.20. We exhibit a map $\varepsilon_\bullet\colon \operatorname{id}_{\mathrm{Ring}} \Rightarrow \Gamma\operatorname{Spec}$.

Proof. Fix an object A. To begin, we compute

$$\Gamma((\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A,$$

so we set $\varepsilon_A := \operatorname{id}_A$. ■

Going in the other direction is a little more subtle because we need to construct a morphism of locally ringed spaces from a morphism of just the global sections. For example, we will need to construct a continuous map, so we will need access to some open sets. Here are the ones we will need.

Lemma 2.21. Fix a locally ringed space $(X, \mathcal{O}_X)$. Then, for some $f \in \mathcal{O}_X(X)$, the subset

$$X_f := \left\{p \in X : f|_p \in \mathcal{O}_{X,p}^\times\right\}$$

is open in X.

Proof. For each $p \in X_f$, we need to provide an open neighborhood $U_p \subseteq X_f$ containing p. Well, we are given $f|_p \in \mathcal{O}_{X,p}^\times$, so there is some germ g_p such that

$$g_p \cdot f|_p = 1.$$

In particular, giving g_p a sufficiently restricted representative, we can find an open subset U_p containing p and some $g \in \mathcal{O}_X(U_p)$ such that

$$g \cdot f|_{U_p} = 1.$$

In particular, any $q \in U_p$ will thus have $g|_q \cdot f|_q = 1$, so $f|_q \in \mathcal{O}_{X,q}^\times$, so $q \in X_f$. Thus, $U_p \subseteq X_f$ does the trick. ■

Here is another quick fact we will want.

Lemma 2.22. Fix a locally ringed space $(X, \mathcal{O}_X)$. For some $f \in \mathcal{O}_X(X)$, consider the open set X_f of Lemma 2.21. Then $f \in \mathcal{O}_X(X_f)^\times$.

Proof. For each $p \in X_f$, we know that $f|_p \in \mathcal{O}_{X,p}$ is a unit, so find $g_p \in \mathcal{O}_{X,p}$ with $f|_p \cdot g_p = 1$. We claim that $(g_p)_{p \in X_f}$ is a compatible system of germs. Well, for each $p \in U$, the equation

$$f|_p \cdot g_p = 1$$

promises an open set $U_p \subseteq X_f$ containing p and a lift $\widetilde{g}_p \in \mathcal{O}_X(U_p)$ such that $f|_{U_p} \cdot \widetilde{g}_p = 1$. Thus, for any $q \in U_p$, we see

$$f|_q \cdot \widetilde{g}_p|_q = 1,$$

so uniqueness of multiplicative inverses forces $\widetilde{g}_p|_q = g_q$. This finishes the compatibility check.

Thus, we are granted $g \in \mathcal{O}_X(X_f)$ such that $(fg)|_p = f|_p \cdot g|_p = 1$ for each $p \in X_f$. It follows that $fg = 1$ by Proposition 1.106, so we have witnessed $f \in \mathcal{O}_X(X_f)^\times$. ■

And here is the result.

Lemma 2.23. We exhibit a map $\eta_\bullet\colon \mathrm{id}_{\mathrm{LocRingSpace}} \Rightarrow \operatorname{Spec}\Gamma$.

Proof. Fix a locally ringed space $(X, \mathcal{O}_X)$ so that we need to exhibit a map

$$\varepsilon_X\colon (X, \mathcal{O}_X) \to (\operatorname{Spec}\mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}).$$

We define this map in pieces.

- We need a continuous map $\varphi\colon X \to \operatorname{Spec}\mathcal{O}_X(X)$. Well, given $p \in X$, we define $\varphi(p) \in \operatorname{Spec}\mathcal{O}_X(X)$ as the kernel of the composite

  $$\mathcal{O}_X(X) \to \mathcal{O}_{X,p} \twoheadrightarrow \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}.$$

  This kernel makes a prime ideal because modding out by it induces a subring of $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$, which must be an integral domain.

  To see that φ is continuous, it suffices to check on the distinguished open base. Well, for some $f \in \mathcal{O}_X(X)$, we see that

  $$\varphi^{-1}(D(f)) = \{p \in X : \varphi(p) \in D(f)\} = \{p \in X : f \notin \varphi(p)\} = \{p \in X : f|_p \notin \mathfrak{m}_{X,p}\}.$$

  However, this last condition is equivalent to $f|_p \in \mathcal{O}_{X,p}^\times$, so our pre-image is the open set X_f by Lemma 2.21.

- Next we need a sheaf morphism $\varphi^\sharp\colon \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)} \to \varphi_*\mathcal{O}_X$. By Lemma 1.79, it suffices to exhibit $\varphi^\sharp$ on the distinguished base of $\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}$. Well, for some $f \in \mathcal{O}_X(X)$, we need a map

  $$\mathcal{O}_X(X)_f \simeq \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(f)) \to \mathcal{O}_X\left(\varphi^{-1}(D(f))\right) = \mathcal{O}_X(X_f).$$

  Now, there is the obvious restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_f)$, and $f \in \mathcal{O}_X(X_f)^\times$ by Lemma 2.22 will exhibit the required map $\varphi^\sharp_{D(f)}\colon \mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$.

  We now check that we have built a morphism of sheaves on the distinguished base. Well, given that $D(f') \subseteq D(f)$ for $f, f' \in \mathcal{O}_X(X)$, we see that the diagram

  $$\begin{array}{ccccc} \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(f)) & \simeq & \mathcal{O}_X(X)_f & \longrightarrow & \mathcal{O}_X(X_f) \\ \big\downarrow{\scriptstyle \mathrm{res}_{D(f),D(f')}} & & & & {\scriptstyle \mathrm{res}_{X_f,X_{f'}}}\big\downarrow \\ \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(f')) & \simeq & \mathcal{O}_X(X)_{f'} & \longrightarrow & \mathcal{O}_X(X_{f'}) \end{array} \qquad \begin{array}{ccc} a \cdot f^{-n} & \longmapsto & a|_{X_f} \cdot (f|_{X_f})^{-n} \\ \big\downarrow & & \big\downarrow \\ a \cdot f^{-n} & \longmapsto & a|_{X_{f'}} \cdot (f|_{X_{f'}})^{-n} \end{array}$$

  commutes, so we have uniquely induced $\varphi^\sharp$ by Lemma 1.79.

We should also verify that $(\varphi, \varphi^\sharp)$ assembles into a morphism of locally ringed spaces. Fixing a point $p \in X$, we need to show

$$\begin{array}{ccccc} (\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)})_{\varphi(p)} & \overset{\varphi^\sharp_{\varphi(p)}}{\to} & (\varphi_*\mathcal{O}_X)_{\varphi(p)} & \to & \mathcal{O}_{X,p} \\ {[(D(f), s/f^n)]} & \mapsto & [(D(f), s|_{X_f} \cdot (f|_{X_f})^{-n})] & \mapsto & [(X_f, s|_{X_f} \cdot (f|_{X_f})^{-n})] \end{array}$$

is a map of local rings; notably, we are using Lemma 1.100 to define the stalk on the distinguished base. Well, $[(D(f), s/f^n)] \in \mathfrak{m}_{\varphi(p)}$ implies $s \in \varphi(p)$, so $s|_p \in \mathfrak{m}_p$, so $\left[(X_f, s|_{X_f} \cdot (f|_{X_f})^{-n})\right] \in \mathfrak{m}_p$. Thus, we have indeed defined a morphism $\eta_{X,\mathcal{O}_X} := (\varphi, \varphi^\sharp)$. ■

We are now ready to show the main result.

Theorem 2.24. Given a locally ringed space $(X, \mathcal{O}_X)$ and a ring A, there is a natural bijection

$$\begin{array}{rcl} \operatorname{Hom}((X,\mathcal{O}_X),(\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A})) & \simeq & \operatorname{Hom}(A,\mathcal{O}_X(X)) \\ (\varphi,\varphi^\sharp) & \mapsto & A = \mathcal{O}_{\operatorname{Spec} A}(A) \to \mathcal{O}_X(X) \\ (X,\mathcal{O}_X) \to (\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}) \stackrel{\operatorname{Spec} f}{\to} (\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A}) & \leftarrow\!\shortmid & f \end{array}$$

where we are using the natural maps constructed in Lemma 2.20 and Lemma 2.23.

Proof. Naturality will follow easily from what we've already done as soon as we show that these are inverses. We have two checks.

- Beginning with a ring homomorphism $f\colon A \to \mathcal{O}_X(X)$, for brevity set $(\varphi,\varphi^\sharp) := \operatorname{Spec} f$ and $\eta_X = (\psi,\psi^\sharp)$. Now, we are studying the composite
$$(X,\mathcal{O}_X) \stackrel{(\psi,\psi^\sharp)}{\to} (\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}) \stackrel{(\varphi,\varphi^\sharp)}{\to} (\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A}).$$
We are interested in what this composite looks like on global sections. On sheaves, we are looking at
$$\mathcal{O}_{\operatorname{Spec} A} \stackrel{\varphi^\sharp}{\to} \varphi_*\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)} \stackrel{\varphi_*\psi^\sharp}{\to} \varphi_*\psi_*\mathcal{O}_X.$$
However, on global sections, this simplifies to
$$\underbrace{\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)}_{A} \stackrel{\varphi^\sharp_{D(1)}}{\to} \underbrace{\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(\operatorname{Spec}\mathcal{O}_X(X))}_{\mathcal{O}_X(X)} \stackrel{\psi^\sharp_{D(1)}}{\to} \mathcal{O}_X(X).$$
Expanding out our definitions, $\varphi^\sharp_{D(1)}$ is supposed to be f localized at 1, but this is just f; also, $\psi^\sharp_{D(1)}$ is supposed to be some localized restriction map, but it is just the identity. So indeed, our composite is f.
- Begin with a morphism of locally ringed spaces $(\pi,\pi^\sharp)\colon (X,\mathcal{O}_X) \to (\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A})$. Under Γ, this gives rise to the ring homomorphism $\pi^\sharp_{\operatorname{Spec} A}\colon A \to \mathcal{O}_X(X)$; set $f := \pi^\sharp_{\operatorname{Spec} A}$ and $(\varphi,\varphi^\sharp) = \operatorname{Spec} f$. Going backward, we label our natural map by $\eta_{(X,\mathcal{O}_X)}$ by $(\alpha,\alpha^\sharp)\colon (X,\mathcal{O}_X) \to (\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)})$, and we want to show that $(\pi,\pi^\sharp)$ agrees with the composite
$$(X,\mathcal{O}_X) \stackrel{(\alpha,\alpha^\sharp)}{\to} (\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}) \stackrel{(\varphi,\varphi^\sharp)}{\to} (\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A}).$$
As before, there are two checks.
  - On topological spaces, we want the diagram
$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\pi"'] & \operatorname{Spec}\mathcal{O}_X(X) \arrow[d, "\varphi"] \\ & \operatorname{Spec} A \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, mapsto] \arrow[drr] & \ker(\mathcal{O}_X(X) \to \mathfrak{m}_p) \\ & & \pi(p) \end{tikzcd}$$
to commute. As such, recalling φ is given by f^{-1}, we compute
$$\begin{aligned} f^{-1}(\ker(\mathcal{O}_X(X) \to \mathfrak{m}_p)) &= f^{-1}(\{r \in \mathcal{O}_X(X) : r|_p \in \mathfrak{m}_p\}) \\ &= \{a \in A : f(a)|_p \in \mathfrak{m}_p\} \\ &= \{a \in A : \pi^\sharp_{\operatorname{Spec} A}(a)|_p \in \mathfrak{m}_p\}. \end{aligned}$$
However, $(\pi,\pi^\sharp)$ being a morphism of locally ringed spaces says that $\pi^\sharp_{\operatorname{Spec} A}(a)|_p \in \mathfrak{m}_p$ is equivalent to $a|_{\pi(p)} \in \mathfrak{m}_{\pi(p)}$.
Now, for a prime $\mathfrak{p} \in \operatorname{Spec} A$, we have $a|_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ if and only if $fa \in \mathfrak{p}$ for some $f \notin \mathfrak{p}$, which is equivalent to $a \in \mathfrak{p}$. Thus, we do indeed get out $\pi(p)$ from the above computation, as needed.

– On sheaves, we want the diagram

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} A} \arrow[r, "\varphi^\sharp"] \arrow[dr, "\pi^\sharp"'] & \varphi_*\mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)} \arrow[d, "\varphi_*\alpha^\sharp"] \\ & \pi_*\mathcal{O}_X \end{tikzcd}$$

to commute, which at least see makes sense from the above topological check. By Lemma 1.79, it suffices to check this on the distinguished base of $\operatorname{Spec} A$, so fix some $a \in A$. Plugging in $D(a)$, we want the diagram

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} A}(D(a)) \arrow[r, "\varphi^\sharp_{D(a)}"] \arrow[dr, "\pi^\sharp_{D(a)}"'] & \varphi_*\mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}(D(a)) \arrow[d, "(\varphi_*\alpha^\sharp)_{D(a)}"] \\ & \pi_*\mathcal{O}_X(D(a)) \end{tikzcd}$$

to commute. Recalling the computations from Lemma 2.19, we see that $\varphi^{-1}(D(a)) = D(f(a))$, and the map $\varphi^\sharp_{D(a)}\colon A_a \to \mathcal{O}_X(X)_{f(a)}$ is induced by localizing f. Similarly, the definition of $\alpha^\sharp$ says that $(\varphi_*\alpha^\sharp)_{D(a)} = \alpha^\sharp_{D(f(a))}$ is the localization of the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_{f(a)})$. So we are staring at the diagram

$$\begin{tikzcd} A_a \arrow[r, "f"] \arrow[dr, "\pi^\sharp_{D(a)}"'] & \mathcal{O}_X(X)_{f(a)} \arrow[d, "\operatorname{res}_{X,X_{f(a)}}"] \\ & \mathcal{O}_X(X_{f(a)}) \end{tikzcd}$$

which commutes because $f = \pi^\sharp_{\operatorname{Spec} A}$. Indeed, this triangle is the localization of the diagram

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \arrow[r, "\pi^\sharp_{\operatorname{Spec} A}"] \arrow[d, "\operatorname{res}_{\operatorname{Spec} A, D(a)}"'] & \mathcal{O}_X(X) \arrow[d, "\operatorname{res}_{X, X_{\pi^\sharp_{\operatorname{Spec} A}(a)}}"] \\ \mathcal{O}_{\operatorname{Spec} A}(D(a)) \arrow[r, "\pi^\sharp_{D(a)}"] & \mathcal{O}_X(X_{\pi^\sharp_{\operatorname{Spec} A}(a)}) \end{tikzcd}$$

at $a \in A$.

The above checks show that we have defined inverse morphisms. There are two naturality checks.

- Given a ring homomorphism $h\colon B \to A$, we see that the diagram

$$\begin{tikzcd} \operatorname{Hom}((X,\mathcal{O}_X),(\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A})) \arrow[r, phantom, "\simeq"] \arrow[d, "\operatorname{Spec} h \circ -"] & \operatorname{Hom}(A,\mathcal{O}_X(X)) \arrow[d, "- \circ h"] & \operatorname{Spec} f \circ \eta_{(X,\mathcal{O}_X)} \arrow[d, mapsto] & f \arrow[l, mapsto] \arrow[d, mapsto] \\ \operatorname{Hom}((X,\mathcal{O}_X),(\operatorname{Spec} B,\mathcal{O}_{\operatorname{Spec} B})) \arrow[r, phantom, "\simeq"] & \operatorname{Hom}(B,\mathcal{O}_X(X)) & \operatorname{Spec}(f \circ h) \circ \eta_{(X,\mathcal{O}_X)} & f \circ h \arrow[l, mapsto] \end{tikzcd}$$

  commutes.
- Given a scheme morphism $(\pi,\pi^\sharp)\colon (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$, we see that the diagram

$$\begin{tikzcd} \operatorname{Hom}((X,\mathcal{O}_X),(\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A})) \arrow[r, phantom, "\simeq"] \arrow[d, "- \circ (\pi,\pi^\sharp)"] & \operatorname{Hom}(A,\mathcal{O}_X(X)) \arrow[d, "\Gamma((\pi,\pi^\sharp)) \circ -"] & (\varphi,\varphi^\sharp) \arrow[r, mapsto] \arrow[d, mapsto] & \Gamma((\varphi,\varphi^\sharp)) \arrow[d, mapsto] \\ \operatorname{Hom}((Y,\mathcal{O}_Y),(\operatorname{Spec} A,\mathcal{O}_{\operatorname{Spec} A})) \arrow[r, phantom, "\simeq"] & \operatorname{Hom}(A,\mathcal{O}_X(Y)) & (\varphi,\varphi^\sharp) \circ (\pi,\pi^\sharp) \arrow[r, mapsto] & \Gamma((\varphi,\varphi^\sharp) \circ (\pi,\pi^\sharp)) \end{tikzcd}$$

  commutes.

The above checks complete the proof. ■

As a nice consequence, we get a pretty nice check for a scheme to be affine.

Corollary 2.25. If $(X, \mathcal{O}_X)$ is an affine scheme, then the map $\varepsilon_X : (X, \mathcal{O}_X) \to (\mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)})$ of Lemma 2.23 is an isomorphism.

Proof. Because $(X, \mathcal{O}_X)$ is affine, there is some ring A with an isomorphism $(\varphi, \varphi^\sharp) : (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong (X, \mathcal{O}_X)$. However, on global sections, we see are granted an isomorphism

$$\Gamma((\varphi, \varphi^\sharp)) : \mathcal{O}_X(X) \to A.$$

Set $f := \Gamma((\varphi, \varphi^\sharp))$, so applying Spec gives us the composite

$$(X, \mathcal{O}_X)(\varphi, \varphi^\sharp)^{-1} \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \overset{\operatorname{Spec} f}{\cong} (\operatorname{Spec} \mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}),$$

which on global sections behaves as

$$\mathcal{O}_X(X) \overset{(\varphi^\sharp_{\operatorname{Spec} A})^{-1}}{\cong} \mathcal{O}_{\operatorname{Spec} A}(A) = A \overset{f}{\cong} \mathcal{O}_X(X),$$

where this composite is just the identity by definition.

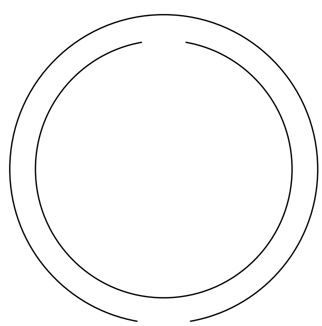
Thus, we have induced an isomorphism $(X, \mathcal{O}_X) \cong (\operatorname{Spec} \mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)})$ which is the identity on global sections. Applying Theorem 2.24, we see that Spec of the identity is still the identity, so this morphism is just given by ε_X. ■

2.1.6 Scheme Examples

Schemes have a lot of data. Let's try to make it more concrete; we'll be satisfied with just two examples today. We won't be very rigorous because we haven't defined gluing yet.

Remark 2.26. Today, we are only gluing two things together at a time because we don't want to worry about the "cocycle condition" for gluing.

Our first example is the projective line. Here is the image of our affine cover.



Here is the rigorization of our affine cover.

Example 2.27 (Projective line)**.** Let R be a ring. Then we can glue two copies of $\mathbb{A}^1_R$ (which is $\operatorname{Spec} R[x]$) as subrings of $\operatorname{Spec} R[x, x^{-1}]$. Then we can identify our copies $\operatorname{Spec} R[x, x^{-1}]$ and $\operatorname{Spec} R[y, y^{-1}]$ by sending $x \mapsto y^{-1}$.

To be rigorous, one should also define our full sheaf on this topological space; this comes from the homework problem explaining how to glue together sheaves.

Here is the image for our next example.

Here is the rigorization.

Example 2.28 (Doubled origin)**.** Let R be a ring. Then we can glue two copies of $\mathbb{A}^1_R$ (which is $\operatorname{Spec} R[x]$) as subrings of $\operatorname{Spec} R[x, x^{-1}]$. Then we can identify our copies $\operatorname{Spec} R[x, x^{-1}]$ and $\operatorname{Spec} R[y, y^{-1}]$ by sending $x \mapsto y$.

Remark 2.29. Later on, we will add certain adjectives (namely, "separated") which disallow the above scheme.

For our last example, we return to elliptic curves.

Example 2.30. We build the elliptic curve carved out by $Y^2Z = X^3 - Z^3$. Our two affine patches are

$$\operatorname{Spec} \frac{k[x,y]}{(y^2 - x^3 + 1)} \qquad \text{and} \qquad \operatorname{Spec} \frac{k[x,z]}{(z - x^3 + z^3)}.$$

To glue these together, we identify

$$\operatorname{Spec} \frac{k[x,y,y^{-1}]}{(y^2 - x^3 + 1)} \qquad \text{and} \qquad \operatorname{Spec} \frac{k[x,z,z^{-1}]}{(z - x^3 + z^3)}$$

by sending $x \mapsto x/z$ and $y \mapsto z^{-1}$.

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The fun continues.

2.2.1 Modules

We are not going to need modules for quite some time, but we will go ahead and define them now.

Definition 2.31 (Module)**.** Fix a scheme $(X, \mathcal{O}_X)$. Then an *$\mathcal{O}_X$-module* is a sheaf $\mathcal{F}$ on X with sheaf morphisms for addition $+ \colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and a scalar multiplication $\cdot \colon \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$.

Remark 2.32. In particular, for each $U \subseteq X$, we see $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

We will remark that one can define kernels, cokernels, and images as we did for sheaves values in general abelian categories, so the category of $\mathcal{O}_X$-modules is again an abelian category. We refer to [Vak17, §2.6.3].

2.2.2 Gluing Schemes

Here is the main idea.

Definition 2.33 (Open subscheme)**.** Fix a scheme $(X, \mathcal{O}_X)$ and an open subset $U \subseteq X$. Then we define the scheme $(U, \mathcal{O}_X|_U)$ to be an *open subscheme*.

Checking that we in fact have a scheme is annoying but not particularly hard; one merely has to restrict the affine open cover to U.

Example 2.34. Given an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we note that taking $U := D(f)$ has

$$(U, \mathcal{O}_{\operatorname{Spec} A}|_U) = (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f}),$$

where we are using Exercise 1.57.

Before we define gluing, we should pick up the following.

Proposition 2.35. Fix a scheme $(X, \mathcal{O}_X)$. Given an affine scheme $(\operatorname{Spec} A, \mathcal{O}_A)$, we can define

$$\operatorname{Mor}_{\operatorname{Sch}_X}((X, \mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_A)) \simeq \operatorname{Hom}_{\operatorname{Ring}}(A, \mathcal{O}_X(X)).$$

This is intended to generalize Proposition 2.15.

Remark 2.36. Reversing the arguments in Proposition 2.35 is no longer true.

Proposition 2.35 will follow from the following, where we let the open cover $\mathcal{U}$ below comes from the affine open cover.

Proposition 2.37. Fix schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Let $\mathcal{U}$ be an open cover for X; then, given scheme morphisms $\varphi_U : U \to Y$ of such that

$$\varphi_U|_{U\cap U'} = \varphi_{U'}|_{U\cap U'},$$

there is a unique scheme morphism $\varphi\colon X \to Y$ such that $\varphi|_U = \varphi_U$.

Proof. Apply force. ■

So far we've glued together morphisms. It remains to glue together schemes. This will be similar to the way that we glue together sheaves.

Proposition 2.38. Fix schemes $(X_i, \mathcal{O}_i)$ for each $i \in I$, with an open subset $U_{ij} \subseteq X_i$ for each i, where $X_{ii} = X_i$; let X_{ij} be the induced open subscheme. Further, pick up some isomorphisms $f_{ji}\colon X_{ij} \to X_{ji}$ satisfying the cocycle condition

$$f_{ki} = f_{kj} \circ f_{ji},$$

on $X_{ij} \cap X_{ik}$, where we implicitly assume that $f_{ji}(X_{ij} \cap X_{ik}) \subseteq X_{ji}$. Then there is a unique scheme X covered by open subschemes $U_i \subseteq X$, equipped with isomorphisms $\varphi_i\colon X_i \to U_i$ such that $\varphi_i|_{X_{ij}} = \varphi_j \circ f_{ji}$ and $\varphi_i(X_i) \cap \varphi_j(X_j) = \varphi_i(X_{ij}) = \varphi_J(X_{ji})$.

Proof. Glue the topological space first using the cocycle condition. Second, glue the sheaves together as described earlier. Lastly, X is a scheme by using the affine open covers of the various X_i. The uniqueness up to unique isomorphism of X follows by keeping track of all the data. ■

2.2.3 Projective Space by Gluing

Fix a ring R. Let's define $\mathbb{P}^n_R$ by gluing $n+1$ different affine sets $\mathbb{A}^n_R$. Intuitively, we want to define projective space to have the topological space of homogeneous coordinates

$$[X_0 : X_1 : \ldots : X_n],$$

and we would like the ith affine piece of this space to be given by

$$\left(\frac{X_0}{X_i}, \frac{X_1}{X_i}, \ldots, \frac{X_n}{X_i}\right).$$

Notably, this has killed a coordinate with $X_i/X_i = 1$.

As such, to glue properly, we define the ith affine piece to be

$$X_i \coloneqq \operatorname{Spec} R\left[x_{0/i}, x_{1/i}, \ldots, x_{(i-1)/i}, x_{(i+1)/i}, \ldots, x_{n/i}\right].$$

To glue this X_i piece to the X_j piece, we need to force $x_{j/i}$ to be nonzero (namely, to invert it), so we look at the open subscheme

$$X_{ij} \coloneqq \operatorname{Spec} R\left[x_{0/i}, x_{1/i}, \ldots, x_{(i-1)/i}, x_{(i+1)/i}, \ldots, x_{n/i}, x_{j/i}^{-1}\right].$$

To glue these open subschemes directly, we remember that $x_{i/j}$ is supposed to mean X_i/X_j as a quotient not always defined, so we define our isomorphism as

$$\begin{aligned} f_{ji} \colon X_{ij} &\to X_{ji} \\ x_{k/i} &\mapsto x_{k/j}/x_{i/j} \end{aligned}$$

from which we can pretty directly check the cocycle condition. (The f_{ji} is an isomorphism because we can see its inverse is f_{ij}.) This gives us our definition.

Definition 2.39 (Projective space)**.** Fix a ring R. Then we define *projective n-space over R*, denoted $\mathbb{P}^n_R$ to be the scheme obtained from the above gluing data.

Remark 2.40. One can see that

$$\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R) = R.$$

Indeed, any global section $s \in \mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R)$ must restrict to each affine open set X_i; however, looking at our gluing data X_i and X_j tells us that we cannot use a non-constant polynomial because having any positive degree (in, say $x_{i/j}$), would induce a denominator when pushing to X_i. Thus, $\mathbb{P}^n_R$ is not an affine scheme unless $n = 0$, for we would be asserting that $\mathbb{P}^n_R$ is the affine scheme $\operatorname{Spec} R$.

2.2.4 Graded Rings

Another way to look at projective schemes is to approach them from graded rings.

Definition 2.41 (Graded rings)**.** Fix a commutative monoid $(M, +)$. An *M-graded ring S* is a ring S equipped with a decomposition of abelian groups

$$S = \bigoplus_{d \in M} S_d$$

such that $S_k \cdot S_k \subseteq S_{k+\ell}$ for any $k, \ell \in M$. By convention, a *graded ring* will be an $\mathbb{N}$-graded ring.

Remark 2.42. If S is an M-graded ring, then $S_0 \subseteq S$ is a subring. Here are our checks.

- Note $0 \in S_0$ and that S_0 is closed under addition and subtraction because S_0 is an abelian group.
- We check $1 \in S_0$. Well, suppose $1 = \sum_{d \in M}^N s_d$. Observe that $s_0 s_d \in S_0 S_d \subseteq S_d$ for each $d \in M$, so by comparing degrees, we are forced to have $s_0 s_d = s_d$. But then

  $$s_0 = s_0 \cdot 1 = s_0 \sum_{d \in M} s_d = \sum_{d \in M} s_d = 1,$$

  so $1 = s_0 \in S_0$ follows.
- For $s, s' \in S_0$, we see $ss' \in S_0 S_0 \subseteq S_0$.

Remark 2.43. Certainly, if S is an $\mathbb{N}$-graded ring, then S is a $\mathbb{Z}$-graded ring by just setting $S_d = 0$ for $d < 0$.

Example 2.44. Take $S := R[x_0, \ldots, x_n]$ graded by degree; namely, S_k is the set of homogeneous polynomials of degree k with 0. Because $\deg(fg) = \deg f + \deg g$, we do indeed have $S_k S_\ell \subseteq S_{k+\ell}$.

Example 2.45. If S is a graded ring, and $f \in S_n$, then S_f is a $\mathbb{Z}$-graded ring, where we are allowing negative degrees coming from $1/f$.

We will want our ideals to keep track of the grading, so we have the following definition.

Definition 2.46 (Homogeneous element)**.** Fix an M-graded ring S. Then an element $f \in S$ is *homogeneous* if and only if $f \in S_d$ for some $d \in M$. If $s \in S_d \setminus \{0\}$ is nonzero and homogeneous, we set $\deg s := S_d$.

Definition 2.47 (Homogeneous ideal)**.** Fix an M-graded ring S. An ideal $I \subseteq S$ is *homogeneous* if and only if I is generated by homogeneous elements.

Remark 2.48. Directly from the definition, we can see that the (arbitrary) sum of homogeneous ideals is homogeneous by just taking the union of the homogeneous generators. Also, if $I = (r_\alpha)_{\alpha\in\lambda}$ and $J = (s_\beta)_{\beta\in\kappa}$ are homogeneous ideals, we see

$$IJ = (r_\alpha s_\beta)_{(\alpha,\beta)\in\lambda\times\kappa},$$

so IJ is homogeneous as well; namely, $r_\alpha s_\beta \in S_{\deg r_\alpha} S_{\deg s_\beta} = S_{\deg r_\alpha + \deg s_\beta}$.

This definition of a homogeneous ideal is easy to think about, but it is not yet clear why it "respects the grading."

Lemma 2.49. Fix an M-graded ring S and ideal $I \subseteq S$. The following are equivalent.

(a) I is generated by homogeneous elements.

(b) If $s = \sum_{d\in M} s_d$ lives in I, then $s_d \in I$ for each $d \in M$.

Proof. To see that (b) implies (a), note that I is generated by

$$I = \left(\sum_{d\in M} s_d : \sum_{d\in M} s_d \in I\right) \subseteq \left(s_d : \sum_{d\in M} s_d \in I\right).$$

However, $\sum_{d\in M} s_d \in I$ implies $s_d \in I$ for each $d \in M$, so in fact

$$\left(s_d : \sum_{d\in M} s_d \in I\right) \subseteq I,$$

giving the needed equality. Thus, we have shown I to be generated by homogeneous elements.

We now show that (a) implies (b). Suppose I is generated by the homogeneous elements $\{s_\alpha\}_{\alpha\in\lambda}$, where the degree of s_α is d_α. Now, for any $s \in I$, write $s = \sum_{d\in M} s_d$ for $s_d \in S_d$. Of course, we can also write

$$\sum_{d\in M} s_d = s = \sum_{\alpha\in\lambda} r_\alpha s_\alpha$$

for some $r_\alpha \in S$. Writing $r_\alpha = \sum_{d\in M} r_{\alpha,d}$, we have

$$\sum_{d\in M} s_d = \sum_{\alpha\in\lambda}\sum_{d\in M} r_{\alpha,d}s_\alpha.$$

Comparing the dth degree on both sides, we see that

$$s_d = \sum_{\alpha\in\lambda} r_{\alpha,d_\alpha-d}s_d,$$

which is indeed an element of I. This finishes. ■

Corollary 2.50. Fix an M-graded ring S and homogeneous ideal $I \subseteq S$. Then, setting $I_d \coloneqq I \cap S_d$, we see S/I is an M-graded ring by $(S/I)_d \simeq S_d/I_d$ for each $d \in M$.

Proof. Note we have the surjection

$$\begin{aligned} S &\simeq \bigoplus_{d\in M} S_d \twoheadrightarrow \bigoplus_{d\in M} S_d/I_d \\ \sum_{d\in M} s_d &\mapsto (s_d)_{d\in M} \mapsto (s_d + I_d)_{d\in M} \end{aligned}$$

which is indeed a surjection because some $(s_d + I_d)_{d\in M} \in \bigoplus_{d\in M} S_d/I_d$ will just lift right back to $(s_d)_{d\in M} \in \bigoplus_{d\in M} S_d$, where $s_d = 0$ if $s_d + I_d = I_d$ (which occurs all but finitely often). Additionally, an element $\sum_{d\in M} s_d \in S$ lives in the kernel of this map if and only if $s_d \in I_d$ for each $d \in M$, which by Lemma 2.49 is equivalent to $\sum_{d\in M} s_d \in I$. So we actually have the isomorphism

$$\begin{aligned} S/I &\simeq \bigoplus_{d\in M} S_d/I_d \\ \sum_{d\in M} s_d + I &\mapsto (s_d + I_d)_{d\in M} \end{aligned}$$

which becomes a grading upon noting that $k, \ell \in M$ with $s_k + I \in (S/I)_k \simeq S_k/I_k$ and $s_\ell + I \in (S/I)_\ell \simeq S_\ell/I_\ell$ will have $s_ks_\ell + I \in (S/I)_{k+\ell} \simeq S_{k+\ell}/I_{k+\ell}$. ■

Here are some other quick facts about homogeneous ideals.

Corollary 2.51. Fix an M-graded ring S and homogeneous ideals $\{I_\alpha\}_{\alpha\in\lambda}$. Then $\bigcap_{\alpha\in\lambda} I_\alpha$ is also a homogeneous ideal.

Proof. Set $I \coloneqq \bigcap_{\alpha\in\lambda} I_\alpha$. We use Lemma 2.49. Indeed, if $s = \sum_{d\in M} s_d$ lives in I, then $s \in I_\alpha$ for each $\alpha \in \lambda$, so each $d \in M$ has $s_d \in I_\alpha$ for each $\alpha \in \lambda$. Thus, $s_d \in I$ for each $d \in M$. ■

Lemma 2.52. Fix an M-graded ring S and homogeneous ideal I. Then I is prime if and only if, for any homogeneous elements $ab \in I$, we have $a, b \in I$.

Proof. Certainly if I is prime, then the conclusion holds. Conversely, we need to show that I is prime. Well, suppose $a = \sum_{d\in M} a_d$ and $b = \sum_{d\in M} b_d$ have $ab \notin I$. Expanding,

$$ab = \sum_{d\in M}\left(\sum_{k+\ell=d} a_kb_\ell\right) \notin I,$$

so there is some term $a_kb_\ell \notin I$. Using the hypothesis, we see $a_k \notin I$ and $b_\ell \notin I$, so because I is homogeneous, we conclude $a \notin I$ and $b \notin I$ by Lemma 2.49. ■

Lemma 2.53. Fix an M-graded ring S and homogeneous ideal I. Then

$$\operatorname{rad} I = \bigcap_{\substack{\mathfrak{p}\supseteq I \\ \mathfrak{p}\text{ homogeneous}}} \mathfrak{p}.$$

In particular, $\operatorname{rad} I$ is homogeneous.

Proof. We follow [Kid12]. The main claim is the first one; that $\operatorname{rad} I$ is homogeneous will follow by Corollary 2.51. Now, for any prime ideal $\mathfrak{p}$ containing I, let $\mathfrak{p}'$ denote the ideal generated by the homogeneous elements of $\mathfrak{p}$. We collect the following facts.

- By definition, $\mathfrak{p}'$ is homogeneous, and $\mathfrak{p}' \subseteq \mathfrak{p}$.
- Note $\mathfrak{p}'$ is prime by Lemma 2.52: given homogeneous elements a, b with $ab \in \mathfrak{p}'$, we see $ab \in \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, so $a \in \mathfrak{p}'$ or $b \in \mathfrak{p}'$ by definition of $\mathfrak{p}'$.
- If $s = \sum_{d\in M} s_d$ lives in I, then $s_d \in I \subseteq \mathfrak{p}$ for each $d \in M$, so $s_d \in \mathfrak{p}'$ for each $d \in M$, so $s \in \mathfrak{p}'$. Thus, $I \subseteq \mathfrak{p}'$.

From the above, we see

$$\operatorname{rad} I = \bigcap_{\mathfrak{p}\supseteq I} \mathfrak{p} \supseteq \bigcap_{\mathfrak{p}\supseteq I} \mathfrak{p}' \supseteq \bigcap_{\substack{\mathfrak{p}\supseteq I \\ \mathfrak{p}\text{ homogeneous}}} \mathfrak{p} \supseteq \bigcap_{\mathfrak{p}\supseteq I} \mathfrak{p},$$

which is what we wanted. ■

It turns out that some ideals do not carry geometric information.

Definition 2.54 (Irrelevant ideal)**.** Fix a graded ring S. Then the *irrelevant ideal* S_+ is the ideal of S generated by the homogeneous elements of positive degree.

We will see why this ideal is called the irrelevant ideal shortly. For now, note that S_+ is a homogeneous ideal, and because

$$(S_+)_d := S_+ \cap S_d = \begin{cases} 0 & d = 0, \\ S_d & d > 0, \end{cases}$$

we see that

$$S/S_+ \simeq \bigoplus_{d\in\mathbb{N}} (S_d/(S_+)_d) = S_0 \oplus \bigoplus_{d\in\mathbb{N}} 0 \simeq S_0.$$

2.2.5 The Topological Space Proj

Fix a graded ring S. We now construct $\operatorname{Proj} S$. Intuitively, we want to have $\operatorname{Proj} R[x_0, \ldots, x_n] = \mathbb{P}^n_R$ and $\operatorname{Proj} S[x_0, \ldots, x_n]/I = V(I)$ when I is a homogeneous ideal. Rigorously, we are going to retell the affine story but add the word homogeneous everywhere.

Let's speak a little non-rigorously for a moment. In some sense, the point $p = [\lambda_0 : \lambda_1 : \ldots : \lambda_n] \in \mathbb{P}^n_R$ should correspond to the ideal of $R[x_0, \ldots, x_n]$ which cuts out this line. Supposing $\lambda_0 \neq 0$ without loss of generality, we can see that the correct ideal is

$$\mathfrak{m}_p = (\lambda_0 x_1 - \lambda_1 x_0, \lambda_0 x_2 - \lambda_2 x_0, \ldots, \lambda_0 x_n - \lambda_n x_0).$$

In particular, $x_i \in \mathfrak{m}_p$ if and only if $\lambda_i = 0$, so we can encode the condition that $\lambda_i \neq 0$ for some i by requiring $\mathfrak{p} \not\supseteq R[x_0, \ldots, x_n]_+$—namely, our irrelevant ideal $R[x_0, \ldots, x_n]_+$ carves out no points.[1] This gives our definition.

[1] This is why S_+ is called the irrelevant ideal.

Definition 2.55 (Proj)**.** Given a graded ring S, we define

$$\operatorname{Proj} S := \{\mathfrak{p} \in \operatorname{Spec} S : \mathfrak{p} \text{ homogeneous}, \mathfrak{p} \not\supseteq S_+\}.$$

Having defined a version of our spectrum, we should give it a Zariski topology.

Definition 2.56 (Zariski topology)**.** Fix a graded ring S. Given a homogeneous ideal $\mathfrak{a} \subseteq S$, define

$$V_+(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a}\}.$$

In other words, $V_+(\mathfrak{a}) = V(\mathfrak{a}) \cap \operatorname{Proj} S$.

Remark 2.57. As before, we see homogeneous ideals $\mathfrak{a} \subseteq \mathfrak{b}$ give

$$V_+(\mathfrak{b}) = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{b}\} \subseteq \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a}\} = V_+(\mathfrak{a}).$$

Remark 2.58. In light of Lemma 2.53, we may say

$$V_+(\mathfrak{a}) = V(\mathfrak{a}) \cap \operatorname{Proj} S = V(\operatorname{rad} \mathfrak{a}) \cap \operatorname{Proj} S = V_+(\mathfrak{a}).$$

Here is the check that we have defined a topology.

Lemma 2.59. Fix a graded ring S. Then the subsets $\{V_+(\mathfrak{a})\}$ define a topology of closed sets on $\operatorname{Proj} S$. In particular, we have the following.

(a) $V_+(S_+) = \varnothing$ and $V_+((0)) = \operatorname{Proj} S$.

(b) Arbitrary intersection: homogeneous ideals $\{\mathfrak{a}_\alpha\}_{\alpha\in\lambda}$ give $\bigcap_{\alpha\in\lambda} V_+(\mathfrak{a}_\alpha) = V_+\left(\sum_{\alpha\in\lambda} \mathfrak{a}_\alpha\right)$.

(c) Finite union: homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ give $V_+(\mathfrak{a}\mathfrak{b}) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b})$.

Proof. This largely follows straight from Lemma 1.40.

(a) Note there is no $\mathfrak{p} \in \operatorname{Proj} S$ with $\mathfrak{p} \supseteq S_+$ by construction, so $V_+(S_+) = \varnothing$. Also, all ideals contain (0), so $V_+((0)) = \operatorname{Proj} S$. We also note that S_+ and (0) are both homogeneous ideals.

(b) Using Lemma 1.40, we see

$$V_+\left(\sum_{\alpha\in\lambda} \mathfrak{a}_\alpha\right) = V\left(\sum_{\alpha\in\lambda} \mathfrak{a}_\alpha\right) \cap \operatorname{Proj} S = \left(\bigcap_{\alpha\in\lambda} V(\mathfrak{a}_\alpha)\right) \cap \operatorname{Proj} S = \bigcap_{\alpha\in\lambda} \underbrace{(V(\mathfrak{a}_\alpha) \cap \operatorname{Proj} S)}_{V_+(\mathfrak{a}_\alpha)}.$$

We close by noting that $\sum_{\alpha\in\lambda} \mathfrak{a}_\alpha$ is a homogeneous ideal by Remark 2.48.

(c) Again using Lemma 1.40, we see

$$V_+(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) \cap \operatorname{Proj} S = (V(\mathfrak{a}) \cup V(\mathfrak{b})) \cap \operatorname{Proj} S = \underbrace{(V(\mathfrak{a}) \cap \operatorname{Proj} S)}_{V_+(\mathfrak{a})} \cup \underbrace{(V(\mathfrak{b}) \cap \operatorname{Proj} S)}_{V_+(\mathfrak{b})}.$$

We close by noting that $\mathfrak{a}\mathfrak{b}$ is a homogeneous ideal by Remark 2.48. ■

As before, we will have a distinguished base, but we will be a little more careful.

Definition 2.60 (Distinguished open sets)**.** Fix a graded ring S. For a homogeneous element $f \in S_+$, we define

$$D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}.$$

As before, we see $D_+(f) = D(f) \cap \operatorname{Proj} S$.

Here is the analogue of Remark 1.55.

Lemma 2.61. Fix a graded ring S. The open sets $\{D_+(f)\}_{f \in S_+}$ form a base of the Zariski topology on $\operatorname{Proj} S$.

Proof. Given any open subset $(\operatorname{Proj} S) \setminus V_+(\mathfrak{a})$ and point $\mathfrak{p} \in (\operatorname{Proj} S) \setminus V_+(\mathfrak{a})$, we need to find $f \in S_+$ such that $D_+(f)$ contains $\mathfrak{p}$ and $D_+(f) \subseteq (\operatorname{Proj} S) \setminus V_+(\mathfrak{a})$. In other words, we need $f \notin \mathfrak{p}$ while $V_+(\mathfrak{a}) \subseteq V_+((f))$. As such, it will suffice to find $f \notin \mathfrak{p}$ with $f \in \mathfrak{a}$ by Remark 2.57.

Note that $\mathfrak{a}$ is generated by homogeneous elements, so there certainly must exist some homogeneous element in $\mathfrak{a}$ which is not in $\mathfrak{p}$. If this element has positive degree, we are done immediately. Otherwise, suppose for contradiction the only homogeneous elements $f \in \mathfrak{a} \setminus \mathfrak{p}$ have degree zero. Then any homogeneous $s \in S_+$ of positive degree will have

$$fs \in \mathfrak{a}$$

while fs has positive degree, but then we forced ourselves into having $s \in \mathfrak{p}$. Thus, $\mathfrak{p}$ contains all homogeneous elements of S_+, so $\mathfrak{p} \supseteq S_+$ because S_+ is homogeneous (!), which contradicts $\mathfrak{p} \in \operatorname{Proj} S$. ■

2.2.6 Easy Nullstellensatz for Proj

For fun, we take a moment to establish the analogue for Proposition 1.45.

Definition 2.62. Fix a graded ring S. Then, given a subset $Y \subseteq \operatorname{Proj} S$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Remark 2.63. Identically as in Lemma 1.44, we have $X \subseteq Y \subseteq \operatorname{Proj} S$ implies $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = I(X)$.

Remark 2.64. Because the intersection of homogeneous radical ideals is homogeneous (Corollary 2.51) and radical, we see that $I(Y)$ is a homogeneous radical ideal for any $Y \subseteq \operatorname{Proj} S$.

And here is our analogue.

Proposition 2.65. Fix a graded ring S.

(a) Given a homogeneous ideal $\mathfrak{a} \subseteq A$, we have $I(V_+(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$.

(b) Given a subset $X \subseteq \operatorname{Proj} S$, we have $V_+(I(X)) = \overline{X}$.

(c) The functions V_+ and I provide an inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. The proof are essentially analogous to Proposition 1.45; we record them for completeness.

(a) Note
$$I(V_+(\mathfrak{a})) = \bigcap_{\mathfrak{p}\in V_+(\mathfrak{a})} \mathfrak{a} = \bigcap_{\substack{\mathfrak{p}\supseteq\mathfrak{a} \\ \mathfrak{p}\text{ homogeneous}}} \mathfrak{p} = \operatorname{rad}\mathfrak{a},$$
where the last equality follows from Lemma 2.53.

(b) Using Lemma 2.59, we see
$$\overline{X} = \bigcap_{V_+(\mathfrak{a})\supseteq X} V_+(\mathfrak{a}) = V_+\left(\sum_{V_+(\mathfrak{a})\supseteq X} \mathfrak{a}\right).$$
In particular, $X \subseteq V_+(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in X$, which means $\mathfrak{a} \subseteq I(X)$. Thus, $\overline{X} = V\left(\sum_{\mathfrak{a}\subseteq I(X)} \mathfrak{a}\right) = V(I(X))$.

(c) As before, V_+ sends radical homogeneous ideals to closed subsets of $\operatorname{Proj} S$ (by definition of the topology), and I sends closed subsets of $\operatorname{Proj} S$ to radical homogeneous ideals by Remark 2.64. These mappings are inclusion-reversing by Remark 2.57 and Remark 2.63. Lastly, (a) and (b) show that these mappings compose to the identity. ■

2.2.7 The Structure Sheaf for Proj

One can again check that this makes a topology. In fact, given $f \in S$, we can define
$$D_+(f) := (\operatorname{Proj} S) \setminus V((f))$$
and then check that this makes a basis for our topology, essentially for the same reason.

Remark 2.66. One can check that the map
$$\begin{aligned} D_+(f) &\simeq \operatorname{Spec}(S_f)_0 \\ \mathfrak{p} &\mapsto (\mathfrak{p}S_f) \cap (S_f)_0 \end{aligned}$$
is a homeomorphism.

As such, we give the open set $D_+(f)$ the structure sheaf $\mathcal{O}_{\operatorname{Spec}((S_f)_0)}$. To glue these together, we choose the affine open subset
$$\operatorname{Spec}((S_f)_0)_{g^{\deg f}/f^{\deg g}} \subseteq \operatorname{Spec}(S_f)_0$$
and identify them with $\operatorname{Spec}(S_{fg})_0$.

2.3 September 12

The classroom is emptier than usual.

2.3.1 Projective Schemes from Proj

We quickly finish our definition of a projective scheme.

Definition 2.67 (Projective scheme)**.** Fix a ring R. A scheme $(X, \mathcal{O}_X)$ is a *projective scheme over* R if and only if $(X, \mathcal{O}_X)$ is isomorphic (as schemes) to some
$$\operatorname{Proj} R[x_0, \ldots, x_n]/I$$
for a homogeneous ideal $I \subseteq R[x_0, \ldots, x_n]$. Equivalently, $(X, \mathcal{O}_X)$ is isomorphic to some $\operatorname{Proj} S$, where S is a finitely generated graded R-algebra.

Intuitively, the ring map

$$R[x_0,\ldots,x_n] \twoheadrightarrow R[x_0,\ldots,x_n]/I$$

will induce an embedding from $(X,\mathcal{O}_X)$ into $\mathbb{P}^n_R$. So a projective scheme is really just one which has an embedding into projective space.

Remark 2.68. It is not totally trivial that we may allow S to be finitely generated from elements outside S_1. See [Vak17, Section 7.4.4].

Here is another equivalent definition.

Definition 2.69 (Projective scheme)**.** Fix a ring R. A scheme $(X,\mathcal{O}_X)$ is a *projective scheme over* R if and only if there is a "closed embedding" $X \hookrightarrow \mathbb{P}^n_R$ of schemes.

We haven't defined a closed embedding yet, but we will do this soon.

2.3.2 Topological Adjectives

We start by describing a scheme by focusing on its topological space.

Definition 2.70 (Connected)**.** A scheme $(X,\mathcal{O}_X)$ is *connected* if and only if X is connected as a topological space. In other words, if $X = V_1 \sqcup V_2$ for closed subsets $V_1, V_2 \subseteq X$, then one of the $V_1 = X$ or $V_2 = X$.

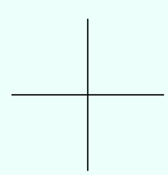
Definition 2.71 (Irreducible)**.** A scheme $(X,\mathcal{O}_X)$ is *irreducible* if and only if X is irreducible as a topological space. In other words, we require X to be nonempty, and if $X = V_1 \cup V_2$ for closed subsets $V_1, V_2 \subseteq X$, then one of the $V_1 = X$ or $V_2 = X$.

Example 2.72. Let X be a topological space. Then, for any $x \in X$, the subset $\overline{\{x\}} \subseteq X$ is irreducible: if closed sets $V_1, V_2 \subseteq X$ have $\overline{\{x\}} \in V_1 \cup V_2$, then $x \in V_1$ or $x \in V_2$, so $\overline{\{x\}} \subseteq V_1$ or $\overline{\{x\}} \subseteq V_2$.

What.

Example 2.73. Projective space $\mathbb{P}^n_R$ is irreducible.

Non-Example 2.74. The scheme $\operatorname{Spec} k[x,y]/(xy)$ is connected but not irreducible. The picture is as follows.



We will explain this example in more detail shortly in Remark 2.85.

We like compact topological spaces, so here is the scheme analogue.

Definition 2.75 (Quasicompact)**.** A scheme $(X,\mathcal{O}_X)$ is *quasicompact* if and only if any open cover of the topological space X has a finite subcover.

Example 2.76. The scheme $\operatorname{Spec} A$ is quasicompact.

Explain with gluing.

Non-Example 2.77. The infinite disjoint union $\bigsqcup_{i=1}^{\infty} \operatorname{Spec} \mathbb{Z}$ is not quasicompact.

Non-Example 2.78. The scheme $\operatorname{Proj} k[x_1, x_2, \ldots]$ is not quasicompact.

2.3.3 Components

Having discussed the entire topological space, we might be interested in studying some interesting subspaces.

Definition 2.79 (Connected component)**.** Fix a topological space X. A *connected component* is a maximal connected subset of X.

Definition 2.80 (Irreducible component)**.** Fix a topological space X. An *irreducible component* is a maximal irreducible subset of X.

Here are some quick facts.

Lemma 2.81. Fix a topological space X.

(a) If a subset $V \subseteq X$ is irreducible, then V is connected.

(b) If a subset $V \subseteq X$ is irreducible (respectively, connected), then so is $\overline{V}$.

(c) All points $x \in X$ are contained in an irreducible component. Also, all points of $x \in X$ are contained in a connected component.

Proof. We go one at a time.

(a) Suppose that $V \subseteq V_1 \sqcup V_2$ where $V_1, V_2 \subseteq X$ are closed subsets. Being irreducible forces $V \subseteq V_1$ or $V \subseteq V_2$, so connectivity of V follows.

(b) We have two claims to show.

- Take V irreducible so that we want to show $\overline{V}$ is irreducible. Suppose $\overline{V} \subseteq V_1 \cup V_2$ where $V_1, V_2 \subseteq X$ are closed. Then $V \subseteq V_1 \cup V_2$, so $V \subseteq V_1$ or $V \subseteq V_2$, so properties of the closure promise $\overline{V} \subseteq V_1$ or $\overline{V} \subseteq V_2$.
- Take V connected so that we want to show $\overline{V}$ is connected. Well, replace the $\cup$ in the previous proof with a $\sqcup$, and the proof goes through verbatim.

(c) Observe that $\{x\}$ is irreducible: if $\{x\} \subseteq V_1 \cup V_2$ where $V_1, V_2 \subseteq X$ are closed, then $x \in V_1$ or $x \in V_2$, so $\{x\} \subseteq V_1$ or $\{x\} \subseteq V_2$.

We now apply Zorn's lemma twice.

- Let $\mathcal{I}_x$ denote the set of irreducible subsets of X containing x. We need to show that $\mathcal{I}_x$ has a maximal element, which will finish because any maximal element of $\mathcal{I}_x$ will be maximal among all irreducible subsets.

  Note $\{x\} \in \mathcal{I}_x$ means that $\mathcal{I}_x$ is nonempty. We now show that $\mathcal{I}_x$ satisfies the ascending chain condition: given a totally ordered set λ and nonempty ascending chain $\{V_\alpha\}_{\alpha\in\lambda} \subseteq \mathcal{I}_x$, we claim that

$$V := \bigcup_{\alpha\in\lambda} V_\alpha$$

and contains x. That $x \in V$ is clear because x lives in any of the V_α. To see irreducibility, suppose that $V \subseteq V_1 \cup V_2$.
If $V \subseteq V_1$, then we are done, so suppose that we can find $p \in V \setminus V_1$. This means that $p \in V_\beta \setminus V_1$ for some $\beta \in \lambda$, so $p \in V_\beta \setminus V_1$ for all $\alpha \geq \beta$. However, $V_\alpha \subseteq V_1 \cup V_2$ still even though $V_1 \not\subseteq V_1$, so we must instead have

$$V_\alpha \subseteq V_2$$

for all $\alpha \geq \beta$. It follows $V = \bigcup_{\alpha \geq \beta} V_\alpha \subseteq V_2$.

- Let $\mathcal{C}_x$ denote the set of connected subsets of X containing x. We actually claim that

$$V := \bigcup_{C \in \mathcal{C}_x} C$$

is connected. This will finish because V is a connected component containing x: if $V \subseteq V'$ with V' connected, then $x \in V'$, so $V' \in \mathcal{C}_x$, so $V' \subseteq V$.
We now check V is connected. Suppose $V \subseteq V_1 \sqcup V_2$ for closed subsets $V_1, V_2 \subseteq X$.
The main point is that $x \in V_1$ or $x \in V_2$. Without loss of generality, take $x \in V_1$ so that $x \notin V_2$. Now, any $C \in \mathcal{C}_x$ has $C \subseteq V_1 \sqcup V_2$, so $C \subseteq V_1$ or $C \subseteq V_2$. However, $x \in C \setminus V_2$, so we must have $C \subseteq V_1$ instead, meaning that actually $V \subseteq C_1$. ■

Remark 2.82. It follows from the above proof that any connected subset C of x is contained in the connected component of x.

Here is another nice result.

Proposition 2.83. If X is an irreducible topological space, then all nonempty open subsets $U \subseteq X$ have U irreducible and $\overline{U}$ dense in X.

Proof. We have two claims to show.

- We show U is irreducible. Suppose $U \subseteq V_1 \cup V_2$ for closed subsets $V_1, V_2 \subseteq X$. It follows that

$$X \subseteq \big((X \setminus U) \cup V_1\big) \cup V_2$$

has covered X by closed subsets. It follows that either $V_2 = X$ (and hence covers U) or $(X \setminus U) \cup V_1 = X$ (and so $V_1 \supseteq U$).

- We show $\overline{U} = X$. Indeed, we can cover X by closed sets as

$$X = (X \setminus U) \cup \overline{U},$$

so either $X \setminus U = X$, which is impossible because U is nonempty, or $\overline{U} = X$, which finishes. ■

Even though irreducible components are a little weird in typical point-set topology, they are of interest in scheme theory.

Lemma 2.84. Fix a ring A.

(a) Given an ideal $I \subseteq A$, the subset $V(I) \subseteq \operatorname{Spec} A$ is irreducible if and only if $\operatorname{rad} I$ is prime.

(b) The irreducible components of X are

$$\{V(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A \text{ is a minimal prime}\}.$$

Proof. As usual, we go in sequence.

(a) We have two claims to show.

- Suppose that $\mathfrak{p} := \operatorname{rad} I$ is prime. Then Proposition 1.45 tells us

$$V(I) = V(\operatorname{rad} I) = V(\mathfrak{p}) = V(I(\{\mathfrak{p}\})) = \overline{\{\mathfrak{p}\}},$$

  which is irreducible by Example 2.72.

- Suppose that $V(I)$ is irreducible; by replacing I with $\operatorname{rad} I$, we may assume that I is radical. We want to show that I is prime. Well, if $a, b \in A$ have $ab \in I$, we want to show $a \notin I$ and $b \notin I$. Well, $ab \in I$ means any $\mathfrak{p}$ containing $V(I)$ contains (ab), so

$$V(I) \subseteq V(ab) = V((a)) \cup V((b))$$

  using Lemma 1.40. Because $V(I)$ is irreducible, we conclude that $V(I) \subseteq V((a))$ without loss of generality. Thus, $\operatorname{rad}((a)) \subseteq \operatorname{rad} I = I$ by Proposition 1.45, so $a \in I$.

(b) The inclusion reversing-bijection of Proposition 1.45 takes prime ideals $\mathfrak{p} \subseteq A$ to $V(\mathfrak{p})$, which we have seen is an irreducible closed subset; and it takes closed subsets $X \subseteq \operatorname{Spec} A$, which we know must take the form $V(\mathfrak{p})$ for a prime $\mathfrak{p}$, to a prime ideal $I(X) = I(V(\mathfrak{p})) = \operatorname{rad} \mathfrak{p} = \mathfrak{p}$.

Thus, the inclusion-reversing bijection restricts to an inclusion-reversing bijection between prime ideals of A and irreducible closed subsets of $\operatorname{Spec} A$. Thus, the maximal irreducible closed subsets of $\operatorname{Spec} A$ correspond (under this bijection) to minimal prime ideals of A.

The claim follows, upon remarking that irreducible components are equal to their closure and hence closed (by Lemma 2.81 (b)), so maximal irreducible closed subsets are just maximal irreducible subsets. ■

Remark 2.85. We are now ready to explain Non-Example 2.74.

- Not irreducible: note that any prime $\mathfrak{p} \in \operatorname{Spec} k[x,y]/(xy)$ contains $xy = 0 \in \mathfrak{p}$, so $(x) \subseteq \mathfrak{p}$ or $(y) \subseteq \mathfrak{p}$. On the other hand, (x) and (y) are primes (with quotients isomorphic to $k[t]$), so (x) and (y) are our minimal primes. Thus, Lemma 2.84 tells us that $V((x))$ and $V((y))$ are our irreducible components; in particular, the entire space is not irreducible.

- Connected: note $(x,y) \in V((x))$ and $(x,y) \in V((y))$, so the connected component of (x,y) contains $V((x)) \cup V((y))$, which is the entire space because we have taken the union of our irreducible components (and all points live in some irreducible component by Lemma 2.81). So the full space is connected.

2.3.4 Closed and Generic Points

We like our topological spaces to be Hausdorff, but we have seen that this need not happen in our schemes. So let's keep track of the good points we try to be Hausdorff

Definition 2.86 (Closed point)**.** Fix a topological space X. Then a point $x \in X$ is a *closed point* if and only if $\overline{\{x\}} = \{x\}$.

In the variety setting, we are more interested in counting closed points, which correspond to the "actual" points on our variety. As such, we might hope that we have "lots" of closed points in our schemes, and under suitable smallness conditions, we do.

Lemma 2.87. Let $(X, \mathcal{O}_X)$ be a quasicompact scheme. Then any nonempty closed subset $V \subseteq X$ contains a closed point.

Proof. Note that this is essentially ring theory for affine schemes: for an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we see that a closed subset $V(I) \subseteq \operatorname{Spec} A$ being nonempty forces I to be proper, so I is contained in some maximal ideal $\mathfrak{m}$. So $\mathfrak{m} \in V(I)$ while Proposition 1.45 says

$$\overline{\{\mathfrak{m}\}} = V(I(\{\mathfrak{m}\})) = V(\mathfrak{m}) = \{\mathfrak{m}\}$$

because $\mathfrak{m}$ is maximal.

We now attack the general case. Because X has an affine open cover, the quasicompactness condition gives V (which is closed in a quasicompact space and hence quasicompact) a finite affine open cover $\{U_i\}_{i=1}^n$ so that we have rings A_i such that

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$$

for each i. We may assume that none of the $\{V \cap U_i\}_{i=1}^n$ is contained in the union of the other, for otherwise we could remove the offending U_i.[2] Now,

$$U_1 \cap \left(V \cap \bigcap_{i=1}^n (X \setminus U_i) \right)$$

is a closed subset of U_1, so because $(U_1, \mathcal{O}_X|_{U_1})$ is an affine scheme, it will have a closed point $p \in U_1$.

Notably, $p \in V$ by construction, so it remains to show that $\{p\} \subseteq X$ is closed. Well, $\{p\} \subseteq U_1$ is closed, so $U_1 \setminus \{p\} \subseteq U_1$ is open, so there is some open set $U' \subseteq X$ such that $U' \cap U_1 = U_1 \setminus \{p\}$. It follows that

$$X \setminus \{p\} = (U_1 \setminus \{p\}) \cup \bigcup_{i=2}^n U_i$$

because $p \notin U_i$ for any $i \neq 1$; in particular, $X \setminus \{p\} \subseteq X$ is open, finishing. ■

Remark 2.88. Sadly, there are examples of schemes with no closed points.

Having kept track of our closed points, we don't want to shame our "unclosed points," so we give them a name as well.

Definition 2.89 (Generic point)**.** Fix a topological space X. Then a point $x \in X$ is a *generic point* of an irreducible subset $V \subseteq X$ if and only if $V = \overline{\{x\}}$.

Example 2.90. Given a ring A, the point $\mathfrak{p} \in \operatorname{Spec} A$ is the (unique!) generic point of $V(\mathfrak{p})$. Indeed, certainly $V(\mathfrak{p}) = V(I(\{\mathfrak{p}\})) = \overline{\{\mathfrak{p}\}}$. On the other hand, if some $\mathfrak{q}$ has

$$V(\mathfrak{p}) = \overline{\{\mathfrak{q}\}} = V(\mathfrak{q}),$$

then $\mathfrak{p} \supseteq \mathfrak{q}$ and $\mathfrak{q} \supseteq \mathfrak{p}$, so $\mathfrak{p} = \mathfrak{q}$.

Example 2.91. Fix an integral domain A. Then (0) is the generic point for $\operatorname{Spec} A$. Notably, $\overline{\{(0)\}} = V((0)) = \operatorname{Spec} A$.

The relationship between generic points will be important to keep track of.

Definition 2.92 (Specialization, generalization)**.** Fix a topological space X and two points $x, y \in X$. We say that x is a *specialization* of y (or equivalently, y is a *generalization* of x) if and only if $x \in \overline{\{y\}}$.

[2] Here we implicitly use the fact that there are only finitely many U_i.

Example 2.93. Given a ring A, we see that $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ if and only if $\mathfrak{q} \supseteq \mathfrak{p}$.

This provides a sort of ordering on our space. Closed points are the most "specific," so let's keep track of the most generic.

Definition 2.94 (Generic point)**.** Fix a topological space X. A point $p \in X$ is a *generic point* if and only if the only point specializing to $\overline{\{p\}}$ is p.

We saw in Example 2.90 that in fact every irreducible closed subset had a unique generic point. This can be extended so schemes.

Lemma 2.95. Fix a scheme $(X, \mathcal{O}_X)$. Then any nonempty irreducible closed subset $Z \subseteq X$ has a unique generic point $p \in X$. In other words, we can write any nonempty irreducible closed subset $Z \subseteq X$ as $Z = \overline{\{p\}}$ for some $p \in X$.

Proof. Give X an affine open cover $\mathcal{U}$ so that each $U \in \mathcal{U}$ has $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A_U, \mathcal{O}_{\operatorname{Spec} A_U})$ for some ring A_U. Now, let $\mathcal{V}$ be the open sets $U \in \mathcal{U}$ such that $Z \cap U \neq \varnothing$. As such,

$$Z = \bigcup_{V \in \mathcal{V}} (Z \cap V).$$

Now, $(V, \mathcal{O}_X|_V)$ is an affine scheme for each $V \in \mathcal{V}$. Further, $Z \cap V \subseteq V$ is a closed subset by the induced topology, and it is irreducible because $Z \cap V \subseteq (V_1 \cap V) \cup (V_2 \cap V)$ for closed subsets $V_1, V_2 \subseteq V$ tells us that

$$Z \subseteq (X \setminus V) \cup V_1 \cup V_2,$$

so irreducibility of Z and $Z \cap V \neq \varnothing$ forces $Z \subseteq V_1$ or $Z \subseteq V_2$.

Thus, Example 2.90 promises a unique point $p_V \in Z \cap V$ such that $Z \cap V = \overline{\{p_V\}} \cap V$ for each $V \in \mathcal{V}$. Fixing some p_V, we claim that $Z \cap W = \overline{\{p_V\}} \cap W$ for each $W \in \mathcal{V}$. Indeed, note

$$Z \cap W \subseteq (Z \cap V \cap W) \cup (W \setminus V) \subseteq (\overline{\{p_V\}} \cap W) \cup (W \setminus V).$$

The right-hand side exhibits $Z \cap W$ as the union of two closed subsets of W, so the irreducibility of $Z \cap W$ tells us $Z \cap W \subseteq (\overline{\{p_V\}} \cap W)$ or $Z \cap W \subseteq (W \setminus V)$. The second case would imply $Z \subseteq (X \setminus V) \cup (X \setminus W)$, which by irreducibility of Z forces $Z \cap V = \varnothing$ or $Z \cap W = \varnothing$, which is assumed false. So instead, we have

$$Z \cap W \subseteq \overline{\{p_V\}} \cap W,$$

but of course $p_V \in Z$ forces the other inclusion.

It follows that

$$Z = \bigcup_{W \in \mathcal{V}} (Z \cap W) = \bigcup_{W \in \mathcal{V}} (\overline{\{p_V\}} \cap W) \subseteq \overline{\{p_V\}}.$$

But $p_V \in Z$ and the fact that Z is closed gives the other inclusion, so we conclude $Z = \overline{\{p_V\}}$. So we have indeed given Z some generic point.

It remains to show that this generic point is unique. Well, suppose $p, q \in X$ have $\overline{\{p\}} = \overline{\{q\}}$. The affine open cover $\mathcal{U}$ from earlier grants us some open set U containing q. Note that $p \notin U$ would imply that $\{p\} \subseteq X \setminus U$ and so $\overline{\{p\}} \subseteq X \setminus U$, meaning that $q \notin \overline{\{p\}}$, which is assumed false. So we must have $p \in U$ as well. But then

$$\overline{\{p\}} \cap U = \overline{\{q\}} \cap U$$

tells us that $p = q$ by the uniqueness of generic points of irreducible closed subsets in affine schemes, from Example 2.90. ■

Remark 2.96. It is not in general case that nonempty irreducible closed subsets of a topological space X can be uniquely written as $\overline{\{x\}}$ for some $x \in X$. Here are two examples.

- If $X = \mathbb{R}$ has the indiscrete topology, the closure of any point is the full space X.
- If $X = \mathbb{R}$ has the cofinite topology, the full space X is irreducible (because all proper closed subsets are finite, so the finite union of proper closed subsets cannot cover X), but X is not the closure of any point (because all points are closed).

2.3.5 Noetherian Conditions

Noetherian rings are good, so we will want to push this to our schemes as well.

Definition 2.97 (Locally Noetherian). A scheme $(X, \mathcal{O}_X)$ is *locally Noetherian* if and only if X has an open cover $\mathcal{U}$ where each $U \in \mathcal{U}$ has $(U, \mathcal{O}_X|_U)$ isomorphic to the affine scheme of a Noetherian ring.

Noetherian is about making infinite things finite, so we want to add a quasicompact condition this.

Definition 2.98 (Noetherian). A scheme $(X, \mathcal{O}_X)$ is *Noetherian* if and only if X is quasicompact and locally Noetherian.

Example 2.99. The scheme from Non-Example 2.77 is locally Noetherian (it's the infinite disjoint union of $\operatorname{Spec} \mathbb{Z}$s, and $\mathbb{Z}$ is Noetherian), but it is not quasicompact and hence not Noetherian.

There is also a separate notion of a Noetherian topological space.

Definition 2.100 (Noetherian). A topological space X is *Noetherian* if and only if the subsets of X satisfy the descending chain condition.

Example 2.101. If A is a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space: given a totally ordered set λ, any descending chain of closed subsets $\{V_\alpha\}_{\alpha\in\lambda}$ gives an ascending chain of A-ideals $\{I(V_\alpha)\}_{\alpha\in\lambda}$. Because A is Noetherian, $\{I(V_\alpha)\}_{\alpha\in\lambda}$ must stabilize past some β, so for $\alpha \geq \beta$, we have $I(V_\alpha) = I(V_\beta)$, so

$$V_\alpha = \overline{V_\alpha} = V(I(V_\alpha)) = V(I(V_\beta)) = \overline{V_\beta} = V_\beta,$$

so the descending chain of closed subsets stabilizes past β.

Non-Example 2.102. Fix a field k and ring $A := k[x_1, x_2, x_3, \ldots]/\left(x_1, x_2^2, x_3^3, \ldots\right)$. Then $\operatorname{Spec} A$ a Noetherian topological space even though A is not a Noetherian ring.

- Observe that any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ must contain $x_k^k = 0$ for each $k \geq 1$, so $x_k \in \mathfrak{p}$. Thus, $\mathfrak{m} := (x_1, x_2, x_3, \ldots)$ is contained in $\mathfrak{p}$, but $\mathfrak{m}$ is maximal in A because $A/\mathfrak{m} \cong k$. So we conclude that $\mathfrak{p} = \mathfrak{m}$, meaning that $\operatorname{Spec} A = \{\mathfrak{m}\}$ has a single point and so is Noetherian as a topological space.
- The ascending chain

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$$

shows that A is not a Noetherian ring.

Having seen that Noetherian rings give Noetherian spaces, we might hope that a similar result holds for schemes. As in Lemma 2.87, we must add a quasicompactness hypothesis.

Lemma 2.103. If $(X, \mathcal{O}_X)$ is a Noetherian scheme, then X is a Noetherian topological space.

Proof. As usual, give X an affine open cover $\mathcal{U}$ where the affine schemes are from Noetherian rings; we may make $\mathcal{U}$ finite because X is quasicompact. Now, let λ be a totally ordered set, and pick up some descending chain $\{V_\alpha\}_{\alpha\in\lambda}$ of closed subsets of X.

Now, for each $U \in \mathcal{U}$, so we see that $\{U \cap V_\alpha\}_{\alpha\in\lambda}$ is a descending chain of closed subsets of the affine open set U, so because $(U, \mathcal{O}_X|_U)$ is the affine scheme of a Noetherian ring, Example 2.101 tells us that $\{U \cap V_\alpha\}$ will stabilize after some β_U. Thus, we set

$$\beta := \max\{\beta_U : U \in \mathcal{U}\},$$

which exists because $\mathcal{U}$ is finite. Then any $\alpha \geq \beta$ and $U \in \mathcal{U}$ will have

$$V_\alpha \cap U = V_{\beta_U} \cap U = V_\beta \cap U,$$

so $V_\alpha = V_\beta$ after taking the union over $\mathcal{U}$. So indeed, our descending chain stabilized after β. ■

Remark 2.104. More generally, we have shown that a finite union of Noetherian topological spaces is a Noetherian topological space.

Remark 2.105. There are some nice philosophical remarks in [Vak17, Section 3.6.21] about when we might care about non-Noetherian things.

As a benefit to keeping things finite, we have the following.

Lemma 2.106. Fix a Noetherian topological space X. Then any open subset $U \subseteq X$ is quasicompact.

Proof. We proceed by contraposition. Suppose that we can find an open cover $\mathcal{V}$ of U with no finite subcover. In other words, any finite subset of $\mathcal{V}$ cannot cover U, so we can build some strictly ascending chain of open subsets

$$V_1 \subsetneq V_1 \cup V_2 \subsetneq V_1 \cup V_2 \cup V_3 \subsetneq \cdots$$

by choosing each $V_n \in \mathcal{V}$ inductively to not be contained in $\bigcup_{k<n} V_k$. (If no such V_n existed, then we would have $U = \bigcup_{V\in\mathcal{V}} V \subseteq \bigcup_{k<n} V_k$, granting a finite open subcover.) For brevity, define

$$V_n' := \bigcup_{k\leq n} V_k$$

so that $\{V_n'\}_{n\geq1}$ is a strictly ascending chain of open subsets. Taking complements, we see that $\{X \setminus V_n'\}_{n\geq1}$ is a strictly descending chain of closed subsets, which means that our space X is not Noetherian. ■

Here are some applications to affine open subschemes.

Lemma 2.107. Fix a scheme $(X, \mathcal{O}_X)$.

(a) If $(X, \mathcal{O}_X)$ is locally Noetherian, then any open subset $U \subseteq X$ makes a locally Noetherian subscheme $(U, \mathcal{O}_X|_U)$.

(b) If $(X, \mathcal{O}_X)$ is Noetherian, then any open subset $U \subseteq X$ makes a Noetherian subscheme $(U, \mathcal{O}_X|_U)$.

(c) All stalks $\mathcal{O}_{X,p}$ of a locally Noetherian scheme $(X, \mathcal{O}_X)$ are Noetherian rings.

Proof. We go in sequence.

(a) As usual, begin by giving X an affine open cover $\mathcal{U}$, and use the locally Noetherian condition to promise that each $V \in \mathcal{U}$ has $\varphi^V : (V, \mathcal{O}_X|_V) \cong (\operatorname{Spec} A_V, \mathcal{O}_{\operatorname{Spec} A_V})$ for a Noetherian ring A_U.

Now, pick up some $V \in \mathcal{U}$ and focus on $U \cap V$; it suffices to cover $U \cap V$ with affine open subschemes of Noetherian rings. Notably, φ^V makes $U \cap V$ an open subset of $(\operatorname{Spec} A_V, \mathcal{O}_{\operatorname{Spec} A_V})$. In particular, using the distinguished open base of $\operatorname{Spec} A_V$, we can write

$$\varphi^V(U \cap V) = \bigcup_{f \in F_V} D(f),$$

where $F_V \subseteq A_V$ is some subset. Now let $U_f \subseteq U \cap V$ be the pre-image of $D(f)$ under the homeomorphism φ^V, and we can use our isomorphism to give the scheme isomorphisms

$$(U \cap V, \mathcal{O}_X|_U|_{U \cap V}) \cong (D(f), \mathcal{O}_{\operatorname{Spec} A_U}|_{D(f)}) \cong (\operatorname{Spec} A_{U,f}, \mathcal{O}_{\operatorname{Spec}_{A_{U,f}}}),$$

where we have used Lemma 2.8 for the first isomorphism and Lemma 2.13 for the second isomorphism. Now, $A_{U,f}$ is the localization of the Noetherian ring and hence Noetherian. This is what we wanted.

(b) This follows directly from (a) and Lemma 2.106: by (a), we see that $(U, \mathcal{O}_X|_U)$ is locally Noetherian, and because X is a Noetherian topological space by Lemma 2.103, Lemma 2.106 tells us that U is quasicompact as a topological space. It follows $(U, \mathcal{O}_X|_U)$ is a Noetherian scheme.

(c) For our point $p \in X$, the locally Noetherian condition promises an open set $U \subseteq X$ containing p and a Noetherian ring A such that

$$(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

Now, Lemma 1.171 tells us that

$$\mathcal{O}_{X,p} \simeq (\mathcal{O}_X|_U)_p = \mathcal{O}_{\operatorname{Spec} A,p},$$

which is $A_\mathfrak{p}$ by Lemma 1.101. This ring $A_\mathfrak{p}$ is Noetherian because A is Noetherian. ■

2.4 September 14

We continue to study things which go bump in the night.

2.4.1 A Better Noetherian

One annoying thing about our locally Noetherian definition is that we are being forced to choose a very special affine open cover. However, we can remove this stress because it turns out all affine open subsets will be Noetherian.

Proposition 2.108. Fix a locally Noetherian scheme $(X, \mathcal{O}_X)$. Then any affine open subset $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong \operatorname{Spec} A$ for a ring A has A a Noetherian ring.

Proof. Fix an affine open cover $\mathcal{U}$ for U making everything Noetherian. By intersecting $\operatorname{Spec} A$ with this open cover, we can pick up a finite subcover contained in particular distinguished open sets $D(f_i)$ for various $f_i \in A$. As such, we see that the A_{f_i} are Noetherian because they are contained form our open cover.

We now upgrade to A. Suppose $I \subseteq A$ is an ideal. Then each IA_{f_i} is finitely generated, so find finitely many elements in I which generate IA_{f_i} as an A_{f_i}-ideal (by clearing denominators as necessary). All these elements combine to give a finitely generated ideal J, which we want to show equals I. Namely, we have $J \subseteq I$ of course, and

$$JA_{f_i} = IA_{f_i}$$

for each i. Intuitively, what's going on here is that J and I are "sheaves" with a morphism $J \subseteq I$ which is an isomorphism on our open cover, so they should be the same ideal.

Let's see this rigorously. In particular, by taking the direct limit over our localizations, we see

$$IA_\mathfrak{p} = JA_\mathfrak{p}$$

by essentially taking stalks. Now, suppose $x \in I \setminus J$, and set

$$\mathfrak{a} := \{a \in A : ax \in J\}.$$

We can check $\mathfrak{a} \subseteq A$ is an ideal, and $\mathfrak{a}$ is proper because $1 \notin \mathfrak{a}$; as such, put $\mathfrak{a}$ inside a maximal ideal $\mathfrak{m}$. However,

$$IA_\mathfrak{m} = JA_\mathfrak{m},$$

which is going to give us a contradiction: indeed, we get $s \notin \mathfrak{m}$ with $sx \in J$ because $x \in IA_\mathfrak{m} = JA_\mathfrak{m}$, which is a contradiction to the construction of $\mathfrak{m}$. ■

Corollary 2.109. Fix a ring A. Then $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is locally Noetherian if and only if A is Noetherian.

Proof. Note $\operatorname{Spec} A \subseteq \operatorname{Spec} A$ makes $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ an affine open subset of itself. ■

Remark 2.110. The above result says that being "locally Noetherian" can be checked locally. It will turn out that this is a general notion, but we will not discuss this generalization for some time.

2.4.2 Reduced Schemes

Here is the definition.

Definition 2.111 (Reduced)**.** A scheme $(X, \mathcal{O}_X)$ is *reduced* if and only if each $p \in X$ give a reduced local ring $\mathcal{O}_{X,p}$; i.e., we are asking for $\operatorname{nilrad} \mathcal{O}_{X,p} = (0)$.

Remark 2.112. On the homework, we will show that being reduced is equivalent to showing that $\mathcal{O}_X(U)$ is reduced for all open $U \subseteq X$.

Example 2.113. The affine ring $\mathbb{P}^n_k$ is reduced because its affine open patches are reduced.

Remark 2.114. In particular, it turns out that an affine scheme $\operatorname{Spec} A$ is reduced if and only if A is reduced. This is also on the homework.

Non-Example 2.115. The scheme $\operatorname{Spec} k[x,y]/(x^2)$ is not reduced because of the stalk at (x).

Being reduced is a nice property, so we might want to force schemes to be reduced.

Definition 2.116 (Reduced scheme associted)**.** Given a scheme $(X, \mathcal{O}_X)$, the *reduced scheme associated* to $(X, \mathcal{O}_X)$ is the scheme $(X, \mathcal{O}_X/\mathcal{N})$, where $\mathcal{N}(U) := \{s \in \mathcal{O}_X(U) : s|_x \in \mathcal{O}_{X,x} \text{ is nilpotent}\}$ for each $U \subseteq X$.

This satisfies the following universal property.

Lemma 2.117. Fix a scheme $(X, \mathcal{O}_X)$, and let $(X^{\text{red}}, \mathcal{O}_X^{\text{red}})$ be the reduced scheme. Then, for any reduced scheme $(Y, \mathcal{O}_Y)$ and map $\varphi\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, there is a unique map $\overline{\varphi}\colon (Y, \mathcal{O}_Y) \to (X^{\text{red}}, \mathcal{O}_X^{\text{red}})$ making the following diagram commute.

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) \arrow[r, "\varphi"] \arrow[dr, dashed, "\overline{\varphi}"'] & (X, \mathcal{O}_X) \\
& (X^{\text{red}}, \mathcal{O}_X^{\text{red}}) \arrow[u]
\end{tikzcd}$$

Proof. Omitted. ■

Example 2.118. The reduced scheme associated to $\operatorname{Spec} k[x, y]/(x^2)$ just becomes $\operatorname{Spec} k[y]$. Intuitively, we are "deleting" all of our differential information.

2.4.3 Integral Schemes

Here is the definition.

Definition 2.119 (Integral)**.** A scheme $(X, \mathcal{O}_X)$ is *integral* if and only if all open subsets $U \subseteq X$ give an integral domain $\mathcal{O}_X(U)$.

Remark 2.120. Note that X being integral will imply that each $\mathcal{O}_{X,x}$ is an integral domain by taking the direct limit, but the converse does not hold.

Proposition 2.121. A scheme $(X, \mathcal{O}_X)$ is integral if and only if $(X, \mathcal{O}_X)$ is reduced and irreducible.

Proof. In the forward direction, note that $(X, \mathcal{O}_X)$ is easily reduced. Further, if $(X, \mathcal{O}_X)$ is not irreducible, then we have two proper closed subsets $V_1, V_2 \subseteq X$ covering X. This grants nonempty open subsets $U_1, U_2 \subseteq X$, and we can force these to be disjoint. It follows

$$\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$

is not integral.

In the other direction, suppose $(X, \mathcal{O}_X)$ is irreducible and integral. Well, we have an open subset $U \subseteq X$ and nonzero $f, g \in \mathcal{O}_X(U)$ with $fg = 0$. We now define

$$V(a) := \{x \in U : g_x \in \mathfrak{m}_x\},$$

and we can check that $V(f)$ and $V(g)$ are going to be closed subsets of U. Because X is irreducible, U is as well, so $U = V(g) \cup V(f)$ forces U to be contained in one of these closed subsets, so without loss of generality take $U = V(f)$. Going down to an affine open patch makes $V(f)$ equal to a full spectrum $\operatorname{Spec} A$, so we conclude

$$f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \operatorname{nilrad} A.$$

But being reduced now forces $f = 0$, finishing. ■

Example 2.122. A reduced scheme whose stalks are integral domains even though X is not irreducible will not make X in total integral. Somehow being integral is a more global property.

Here is a good property of integral schemes.

Proposition 2.123. An integral scheme $(X, \mathcal{O}_X)$ has a unique generic point ξ for X. Then

$$\mathcal{O}_{X,\xi} = \operatorname{Frac} \mathcal{O}_X(U)$$

for any $U \subseteq X$.

Proof. The existence of ξ comes from the fact that irreducible components (which here is only X) have a unique generic point. The second claim follows by taking the direct limit over our U, trying to get down to the generic point. ■

So we get the following nice definition.

Definition 2.124 (Function field)**.** An integral scheme $(X, \mathcal{O}_X)$ with generic point ξ has *function field*

$$\mathcal{O}_{X,\xi} := \operatorname{Frac} \mathcal{O}_X(X)$$

The point here is that we can retrieve the field out from some $\operatorname{Spec} k[x, y]$, say. This also allows us to define regular functions.

Definition 2.125 (Regular)**.** Fix an integral scheme $(X, \mathcal{O}_X)$ with generic point ξ. Then $f \in \mathcal{O}_{X,\xi}$ is *regular* at a point $x \in X$ if and only if f lifts to $\mathcal{O}_{X,x}$.

2.4.4 Closed Subschemes

Open subschemes had natural subscheme structure by just taking restriction. Closed subschemes are a little harder.

Example 2.126. Set $X := \operatorname{Spec} k[x, y]$. Then the closed subset $V(x)$ will have lots of natural homeomorphisms

$$V(x) \cong \operatorname{Spec} \frac{k[x, y]}{(x^n)},$$

for any $n \geq 1$, so there is no canonical way to set the structure sheaf.

The idea to define a closed subscheme is to instead keep track of the morphism which does the embedding.

Definition 2.127 (Closed immersion)**.** A scheme morphism $(f, f^\sharp) \colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a *closed immersion* if and only if the following two conditions hold.

- The map $f \colon Z \to X$ is a homeomorphism from Z onto a closed subset of X.
- The map $f^\sharp \colon \mathcal{O}_X \to f_* \mathcal{O}_Z$ is epic.

Germs should pull back to germs.

If in fact $Z \subseteq X$ is a closed subset, then we will say Z is a closed subscheme.

The main point here is that we would like our closed immersions in the affine case to be induced by $A \twoheadrightarrow A/I$ as in Exercise 1.53.

Proposition 2.128. Fix an affine scheme $(X, \mathcal{O}_X) := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

(a) Each ideal $I \subseteq A$ induces a closed immersion

$$\operatorname{Spec} A/I \to \operatorname{Spec} A$$

from the projection map $A \twoheadrightarrow A/I$. In particular, this gives $V(I) \subseteq \operatorname{Spec} A$ the structure of a closed subscheme.

(b) The map of (a) provides a bijection between ideals of A and closed subschemes of $\operatorname{Spec} A$.

Proof. Here we go.

(a) From Exercise 1.53, we already have the natural homeomorphism $\operatorname{Spec} A/I \cong V(I)$. On the level of sheaves, we only need to check surjectivity at stalks, for which we look at the distinguished open base. Namely, at some $D(f)$, we are studying the map

$$A_f = \mathcal{O}_X(D(f)) \to \mathcal{O}_Z(D(f+I)) = (A/I)_f \simeq A_f/I_f,$$

which we can see is surjective here. Taking the direct limit shows that we remain surjective on the level of stalks.

(b) This proof will be able to simplified later in life when we have talked about coherent sheaves. Fix a closed subscheme $\iota\colon Z \to X$ with $\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_Z$. Now, define

$$I_Z := \ker(\iota^\sharp_X).$$

Then one can show that I_Z is the ideal we want, providing the inverse to (a). In particular, one can show that Z is identified with $\operatorname{Spec} A/I_Z$ as schemes. Notably, there is an embedding $Z \hookrightarrow Y$ by first looking on the level of topological spaces and then carrying over to schemes. It remains to show that this is an isomorphism.

- We show that we have a homeomorphism of our topological spaces. Note that Z is quasicompact, so we can write it as a finite union of affine open subschemes. Now, for $Z \subseteq V(s)$, we will show that $V(s) = Y$. We see $s \in B := A/I_Z$, so note that $s \in \mathcal{O}_X(U_i)$ will be nilpotent because of its definition. Looping over entire affine open cover forces $s^n = 0$, so the injectivity of the open cover

$$A/I_Z \hookrightarrow \mathcal{O}_Z(Z)$$

forces $s^N = 0$ in A/I_Y, meaning $Y \subseteq V(s)$.

- We now need to show $\mathcal{O}_Y \to \iota_*\mathcal{O}_Z$. Surjectivity follows from the construction looking at the global sections. The injectivity follows by looking at stalks and making an argument similar to the above. ■

2.5 September 16

We continue.

2.5.1 Schemes over a scheme

Here is our definition.

Definition 2.129 (Schmes over a scheme)**.** Fix a scheme $(S, \mathcal{O}_S)$. An S-scheme is a scheme $(X, \mathcal{O}_X)$ equipped with a morphism $(\pi, \pi^\sharp)\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ of schemes. We might write X/S.

Example 2.130. All schemes are naturally a scheme over $\operatorname{Spec}\mathbb{Z}$.

A morphism of two S-schemes $\pi\colon X \to S$ and $\pi'\colon X' \to S$ is a morphism $\varphi\colon X \to X'$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

commute.

Remark 2.131. Fix $S := \operatorname{Spec} A$. Then we can define a projective scheme as a scheme X over S with a closed embedding into $\mathbb{P}^n_S$. We will show the equivalence of this definition later; the point is that Proposition 2.128 should have an analogue to graded rings.

2.5.2 Reduced Schemes

Given a closed subscheme $Z \subseteq X$, there might be many natural scheme structures from the topology. However, if we ask for it to be reduced, then it is unique.

Proposition 2.132. Fix a scheme X and a closed subset $Z \subseteq X$. Then there is a unique reduced, closed subscheme $Z \subseteq X$ whose topological space agrees with Z.

Proof. We start by showing uniqueness. Note that we may replace X with its reduced scheme without headaches, so we assume that X is a reduced scheme. Fix an affine open subscheme $U \subseteq X$, and write $U = \operatorname{Spec} A$ for some ring A. Now, $U \cap Z$ is a reduced, closed subscheme of U, so by Proposition 2.128 tells us that

$$(U \cap Z, \mathcal{O}_Z|_U)$$

comes from a radical ideal I, and we see that this is unique.

This also tells us how to construct Z. Namely, each affine open subscheme $U \subseteq X$ with $U = \operatorname{Spec} A$ can take some $I = I(U \cap Z) \subseteq \operatorname{Spec} A$. As such, we can give $U \cap Z \subseteq U$ a reduced scheme structure from $\mathcal{O}_{\operatorname{Spec} A/I}$, and we finish the construction by gluing these subschemes together. Notably, the gluing is possible because the uniqueness forces the cocycle condition. ■

Remark 2.133. Given a scheme morphism $f\colon X \to Y$, we might want to think about the image of f. The correct way to think about this is to say that there is a unique closed subscheme $Z \subseteq Y$ such that f factors through Z, with the following universal property: for all closed subschemes $Z' \subseteq Y$ factoring through f, we have $Z \subseteq Z'$.

At a high level, in some nice cases one takes Z to be the kernel of $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Then we will look at $\mathcal{O}_Y \to \mathcal{O}_Y/\ker f^\sharp$, at least when X is reduced.

2.5.3 Quasiprojective Schemes

We might want to talk about affine and projective schemes at the same time. Here is how we do this.

Definition 2.134 (Quasiprojective)**.** Fix an affine scheme $S := \operatorname{Spec} A$. Then a scheme X/S is *quasiprojective* if and only if X/S is a quasicompact open S-subscheme of some projective S-scheme.

Example 2.135. Affine k-varieties are quasiprojective.

Here is a related definition.

Definition 2.136 (Locally closed embedding)**.** A scheme morphism $\pi\colon X \to Y$ is a *locally closed embedding* if and only if we can factor π into

$$X \hookrightarrow Z \hookrightarrow Y$$

where $X \hookrightarrow Z$ is a closed embedding, and $Y \hookrightarrow Z$ is an open embedding.

The reason that this is called a "locally closed embedding" is because it becomes a closed embedding on a sufficiently small open subset.

Remark 2.137. For a locally closed embedding $\pi\colon X \to Y$, then under suitable smallness conditions (i.e., for π to be quasicompact), we can find Z' so that π factors as

$$X \hookrightarrow Z' \hookrightarrow Y$$

where $X \hookrightarrow Z'$ is open, and $Z' \hookrightarrow Y$ is closed. The idea here is that we want to generalize the notion of "constructible subsets" which are intersections of open and closed subsets. This finiteness result is telling us that these are approximately the same notion.

2.5.4 Dimension

The last topological property we will talk about is dimension.

Definition 2.138 (Dimension)**.** Fix a topological space X. Then the *dimension* of X is the longest possible length n of a chain of closed irreducible subsets

$$Z_0 \subsetneq Z_1 \subsetneq Z_3 \subsetneq \cdots \subsetneq Z_n \subseteq X.$$

Example 2.139. If $X = \operatorname{Spec} A$, then $\dim X$ is the Krull dimension $\dim A$.

Example 2.140. We see $\dim \mathbb{A}_k^n = \dim \mathbb{P}_k^n = n$.

Sometimes a topological space might have components of larger dimension than others, which is undesirable.

Definition 2.141 (Pure dimension)**.** Fix a topological space X. Then X is of *pure dimension* n if and only if all irreducible components of X have dimension n.

Having defined a notion of dimension, we can now define codimension.

Definition 2.142 (Codimension)**.** Fix a topological space X and an irreducible closed subset $Z \subseteq X$. Then the *codimension* $\operatorname{codim}_Z X$ is the supremum of the length n of a chain of irreducible closed subsets

$$Z = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n \subseteq X.$$

Remark 2.143. Of course, $\dim Z + \operatorname{codim}_X Z \leq \dim Z$, but equality is not true in general because we lack pure dimension.

2.5.5 The Functor of Points

Here is our definition.

Definition 2.144 (Functor of points)**.** Fix an S-scheme X. Then the functor of points of X is the functor defined as follows.

$$\begin{aligned} h_X \colon (\mathrm{Sch}_S)^{\mathrm{op}} &\to \mathrm{Set} \\ Y &\mapsto \mathrm{Mor}_{\mathrm{Sch}_S}(Y, X) \end{aligned}$$

This provides the correct intuitive definition, say when $S = k$ is some field. In particular, the A-points of X are made of the scheme morphisms

$$\operatorname{Spec} A \to X,$$

so we are taking a general scheme Y to its "Y-points." Notably, a morphism of schemes $X \to X'$ will induce a natural transformation $h_X \Rightarrow h_{X'}$.

A little category theory will be informative.

Theorem 2.145 (Yoneda's lemma)**.** Fix a category $\mathcal{C}$, and define the functor よ$\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ taking objects X to h_X.

(a) Natural transformations from h_X to a functor $\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ are canonically isomorphic to $\mathcal{F}(X)$.

(b) The functor $X \mapsto h_X$ is fully faithful.

We will be interested in the following special case.

Corollary 2.146. Fix $\mathcal{C}$ to be the category of schemes over $S := \operatorname{Spec} R$. Then the functor $h_\bullet$ taking X to h_X^{aff} (where h_X maps R-algbras A to $\mathrm{Mor}_{\mathrm{Sch}_{\operatorname{Spec} R}}(\operatorname{Spec} A, X)$) is fully faithful.

Proof. For a given S-scheme Y, reduce to the case of affine open subsets, and then in the affine case we get to appeal directly to Yoneda's lemma.

Namely, cover X with some affine open cover $\mathcal{U}$. Then, given a natural transformation $\varphi \colon h_X^{\mathrm{aff}} \to h_{X'}^{\mathrm{aff}}$, we need to construct a (unique) morphism $X \to X'$. Well, we simply go down to each affine piece $U \in \mathcal{U}$, use the affine case which provides some

$$\varphi(A_U) \colon \ \mathrm{Mor}_R(U, X) \to \mathrm{Mor}_R(U, X')$$

for each $U \in \mathcal{U}$. Passing the inclusion $U \hookrightarrow X$ through this proof, we get a bunch of morphisms $U \hookrightarrow X'$, which we then glue to a morphism. ■

2.6 September 19

Bump, bump, bump.

2.6.1 Fiber Products

Here is today's main result.

Theorem 2.147. Fix two S-schemes X and Y. Then the fiber product $X \times_S Y$ exists.

Remark 2.148. Even if X and Y are Noetherian, it does not necessarily follow that $X \times_S Y$ is Noetherian. For example, taking $\operatorname{Spec} \overline{\mathbb{Q}}$ and $\operatorname{Spec} \overline{\mathbb{Q}}$ are both Noetherian, but the fiber product turns out to be

$$\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

In particular, this is zero-dimensional but has infinitely many points and is therefore not Noetherian.

The fiber product is a purely categorical construct, which is the limit of the diagram as follows.

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

In other words, there are canonical projection maps $X \times_S Y \to X$ and $X \times_S Y \to Y$ with a suitable universal property. As usual, the universal property means that this is unique up to unique isomorphism.

Remark 2.149. Even without knowing that $X \times_S Y$ exists, we may note that we have a natural isomorphism

$$\operatorname{Mor}_S(Z, X \times_S Y) \cong \operatorname{Mor}_S(Z, X) \times_{\operatorname{Mor}_S(Z,S)} \operatorname{Mor}_S(Z, Y)$$

coming straight from the universal property. In other words, the Z-points of an S-fiber product is just going to be the product of the two S-points. In this way, we can view the below proof as asking for a particular functor (on the right here) to be representable, which will be enlightening after some thought.

Schemes are made of affine schemes, so we will begin with affine schemes, with the hope of patching these together later.

Lemma 2.150. Fix affine schemes X, Y, and S, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Then

$$X \times_S Y = \operatorname{Spec} A \otimes_R B.$$

Proof. Notably, the canonical maps $X \to S$ and $Y \to S$ makes A and B into R-algebras. To see this, we simply compute, for some scheme S,

$$\begin{aligned} \operatorname{Mor}_S(Z, \operatorname{Spec} A \otimes_R S) &\simeq \operatorname{Hom}_R(A \otimes_R S, \mathcal{O}_Z(Z)) \\ &\simeq \operatorname{Hom}_R(A, \mathcal{O}_Z(Z)) \times_{\operatorname{Hom}_R(R, \mathcal{O}_Z(Z))} \operatorname{Hom}_R(B, \mathcal{O}_Z(Z)), \end{aligned}$$

which unravels into the correct thing. This finishes by the Yoneda lemma. One can write out all of this as some diagram-chase, but it is equivalent to the above computation. ∎

Remark 2.151. One can view the above proof as basically preserving the fact that $A \otimes_R B$ is the fiber product of A and B as R-algebras.

We now begin our gluing. Here is the key case to glue.

Lemma 2.152. Fix affine schemes X and S and a scheme Y, where X and Y are S-schemes. Then the fiber product exists.

We hope that a gluing will be able to give $X \times_S Y$ for S affine and X and Y arbitrary, from which the general case will follow by gluing S.

To prove Lemma 2.152, we will want to glue to get up to Y, so we have the following smaller case.

Lemma 2.153. Fix affine schemes X and S and a scheme Y, where X and Y are S-schemes. Given an open embedding $Y \hookrightarrow Y'$, if $X \times_S Y'$ exists, then $X \times_S Y$ also exists as

$$X \times_S Y = \pi_{Y'}^{-1}(Y),$$

where $\pi_{Y'} \colon X \times_S Y' \to Y'$ is the canonical map.

Sketch. We use the universal property a bunch of times. Then

$$\begin{tikzcd} \pi_{Y'}^{-1}(Y) \arrow[r, "\pi_Y"] \arrow[d] & Y \arrow[d] \\ X \times_S Y' \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y' \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

we can show has the top and bottom squares are both pullback squares, so the full rectangle is a pullback square, which finishes. ■

We can now prove Lemma 2.152.

Proof of Lemma 2.152. Give Y the affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$, writing $Y_{\alpha\beta} \coloneqq Y_\alpha \cap Y_\beta$ as a subset of Y_α to prepare for our gluing. Now, we see that Lemma 2.153 grants us schemes

$$X \times_S Y_{\alpha\beta} \subseteq X \times_S Y_\alpha \qquad \text{and} \qquad X \times_S Y_{\beta\alpha} \subseteq X \times_S Y_\beta,$$

which must be identified because $Y_{\alpha\beta} = Y_{\beta\alpha}$. These identified schemes are going to satisfy the cocycle condition because these fiber products are unique up to unique isomorphism.

To see the cocycle condition more concretely, we can just say out loud that we have the scheme

$$X \times_S Y_{\alpha\beta\gamma} = (X \times_S Y_{\alpha\beta}) \cap (X \times_S Y_{\alpha\gamma}) \subseteq X \times_S Y_\alpha.$$

On the other side, we have a similar computation for Y_{jik} and Y_{kij}, so chaining these diagrams together will identify the needed isomorphisms by the uniqueness of our various isomorphisms.

At this point, we have been glued together a (unique) scheme W with "compatible" open embeddings $X \times_S Y_\alpha \hookrightarrow W$. It remains to verify that W is our fiber product $X \times_S Y$. Our map $\pi_X \colon W \to X$ is induced by projecting down to X in all the glued components; our map $\pi_Y \colon W \to Y$ is made by gluing the various maps $\pi_\alpha \colon W \to Y_\alpha$ (which appear from the projections $X \times_S Y_\alpha \to Y_\alpha$).

We now verify the universal property. Fix morphisms $\varphi_X \colon Z \to X$ and $\varphi_Y \colon Y \to Z$ making the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\ & W \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

commute, and we want to induce a unique arrow where the dashed arrow is. The big diagram commuting gives an internal commuting diagram

$$\begin{tikzcd} Z_\alpha \arrow[dr, dashed, "\varphi_\alpha"] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\ & X \times_S Y_\alpha \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y_\alpha \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

with $Z_\alpha \coloneqq \varphi_Y^{-1}(Y_\alpha)$, where we have already used the universal property to construct the unique morphism φ_α making the diagram commute. This gives maps

$$Z_\alpha \to X \times_S Y_\alpha \subseteq W,$$

and we just show that these various maps will be compatible to give us a unique map $Z \to W$. One can track through all the various universal properties to show that we have in fact induced a unique map making the original diagram commute.

We now show that we can in fact glue the morphisms $Z_\alpha \to W$. Well, set $Z_{\alpha\beta} \coloneqq \varphi_Y^{-1}(Y_{\alpha\beta})$ to be $Z_\alpha \cap Z_\beta$, and we can see that the diagram

$$\begin{tikzcd} Z_{\alpha\beta} \arrow[dr] \arrow[drr, bend left, "\varphi_\alpha"] \arrow[ddr, bend right, "\varphi_\beta"'] & & \\ & X \times_S Y_{\alpha\beta} \arrow[r] \arrow[d] & X \times_S Y_\alpha \\ & X \times_S Y_\beta & \end{tikzcd}$$

commutes, which finishes. ■

Note that the same proof will show that we can build the fiber product for $X \times_S Y$ when S is affine and X and Y are arbitrary; namely, nowhere did we use in the proof that X was affine (only that the fiber products $X \times_S Y_\alpha$ exist), so we can just apply the same gluing process.

Lastly, here is the general case.

Proof of Theorem 2.147. Give S an affine open cover $\{S_\alpha\}_{\alpha\in\lambda}$, with our natural maps $f_X \colon X \to S$ and $f_Y \colon Y \to S$. Then we set $X_\alpha \coloneqq f_X^{-1}(S_\alpha)$ and similar for Y_α. By previous work, we have the fiber products $X_\alpha \times_{S_\alpha} Y_\alpha$, so we can just glue these together as usual. ■

A little more rigorously, the following result aids the above gluing.

Lemma 2.154. Fix schemes X and Y over a scheme S. If $S \subseteq S'$ is an affine open embedding of schemes such that $X \times_{S'} Y$ exists, we have

$$f_X^{-1}(S) \times_S f_Y^{-1}(S) = (f_X \circ \pi_X)^{-1}(S') = (f_Y \circ \pi_Y)^{-1}(S').$$

The intuition is that the above lemma should be pulling back the required fiber product along the various legs of the following diagram.

$$\begin{tikzcd} X \times_{S'} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S' \end{tikzcd}$$

This will give the intersection data in the gluing for Theorem 2.147, which will finish the gluing.

Remark 2.155. We will provide a more categorical viewpoint of this construction next class. This categorical viewpoint will be helpful for when we want to define the Grassmannian.

2.7 September 21

Today we return to fiber products and discuss some applications.

2.7.1 Representability

We start with a few definitions.

Definition 2.156 (Zariski sheaf)**.** A functor $F\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ is a *Zariski sheaf* if and only if F is a sheaf when the scheme is viewed as merely a topological space. Namely, each scheme T has

$$0 \to F(T) \to \prod_i F(T_i) \to \prod_{i,j} F(T_i \cap T_j)$$

exact for any open cover $\{T_i\}$ of T.

Definition 2.157 (Open subfunctor)**.** Fix functors $F, F'\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$. Then $F' \subseteq F$ is an *open subfunctor* if and only if each scheme T, every natural transformation $\psi\colon h_T \Rightarrow F$ yielding a pullback square

$$\begin{tikzcd} F_{i,\psi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & h_T \arrow[d] \\ F_i \arrow[r] & F \end{tikzcd}$$

already has each $F_{i,\psi}$ represented by a scheme T_i with the natural transformation $F_{i,\psi} \hookrightarrow h_T$ given by an open embedding $T_i \hookrightarrow T$.

Here is an abstract lemma.

Lemma 2.158. A functor $F\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ is representable if and only if the following conditions are satisfied.

- F is a Zariski sheaf.
- Locally representable: there are representable subfunctors $F_i \subseteq F$ for each i, and $F_i \subseteq F$ is an open subfunctor such that $\{F_i\}$ covers F. Here, covering means that each field K has $F(\operatorname{Spec} K) = \bigcup_i F_i(\operatorname{Spec} K)$.

The point is that each F_i is represented by some X_i, and we just want to glue these X_i together. This is the idea of the proof.

Remark 2.159. One can replace $\mathrm{Sch}^{\mathrm{op}}$ with Ring or $\mathrm{Sch}_S^{\mathrm{op}}$.

Remark 2.160. One can show that Lemma 2.158 implies that the fiber product exists. Namely, the fiber product forms a Zariski sheaf, which we can see from the part where we glued to make W in the key case. Then the F_i come from the purely affine case, which was comparatively easier. Lastly, the F_i cover F roughly speaking comes from the rest of the proof.

We will not need Lemma 2.158 for the time being.

2.7.2 Fibers

As a first application, we discuss fibers. Given a scheme morphism $\varphi\colon Y \to S$, we might be interested in the fibers here to pull-back. Namely, pulling back to a "subscheme" X of S, we can imagine the fibers of Y over

X as the fiber product, as in the following diagram.

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

For example, in the case of $Y = \operatorname{Spec} k[x,y,z]/\left(y^2 - x(x-1)(x-s)\right)$ and $S = \operatorname{Spec} k[s]$, we see that we have the obvious map $Y \to S$ sending $x, y \mapsto 0$.

Now, if we want to understand the fiber at a given point $s_0 \in S$ with $s_0 \in k$ for concreteness, the corresponding scheme is a $\operatorname{Spec} k$ over $\operatorname{Spec} S$ induced by the map $k[s] \to k$ by $s \mapsto s_0$. Then we can track our fibers in this affine case as given by

$$\begin{tikzcd} Y \otimes_{k[s]} \operatorname{Spec} k \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[s] \end{tikzcd}$$

where we can compute directly from commutative algebra that

$$Y \otimes_{k[s]} \operatorname{Spec} k \simeq \operatorname{Spec} \frac{k[x,y]}{(y^2 - x(x-1)(x-s_0))}$$

here.

As such, we are convinced that the following is a good definition of a fiber.

Definition 2.161 (Fiber)**.** Fix a scheme Y over a scheme S. Given a point $s_0 \in S$, the fiber product $Y \times_S \{s_0\}$ is the *fiber* of $Y \to S$ over s_0.

Remark 2.162. If $X \hookrightarrow S$ is a closed embedding, then $Y \times_S X \hookrightarrow Y$ is a closed embedding as well. In particular, if $s_0 \in S$ is a closed point, then our fiber is in fact a closed embedding. More generally, if S is irreducible with generic point η, we call $Y \times_S \{\eta\}$ the generic fiber.

Remark 2.163. Notably, we can check purely topologically that the fiber defined by the fiber product is the correct fiber at a point.

2.7.3 Base Extension

We again begin with a special case. Take $S = \operatorname{Spec} K$ to be our base and $X = \operatorname{Spec} K'$ where K'/K is a field extension; the embedding $K \hookrightarrow K'$ induces a map $X \to S$.

Now, if we have a scheme Y over S, we might want to pull Y back to a scheme over X, where we are applying some base-change operation. To do this, we unsurprisingly want the fiber product, as in the following diagram.

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

As an example, if we have $Y = \operatorname{Spec} K[x,y]/\left(y^2 - x^3 + x\right)$, we can compute that

$$X \times_S Y = \operatorname{Spec} \frac{K'[x,y]}{(y^2 - x^3 + x)},$$

which agrees with our intuition of what base-change should do.

Example 2.164. As another quick example, we can compute $\mathbb{P}^n_K \times_{\operatorname{Spec} K} \operatorname{Spec} K' = \mathbb{P}^n_{K'}$. For example, we could cleanly define projective n-space over a scheme S

$$\mathbb{P}^n_S = \mathbb{P}^n_{\operatorname{Spec}\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$$

This notation is a bit cumbersome, so we will abbreviate it.

Notation 2.165. Fix a morphism $S' \to S$. If X is a scheme over S, we might denote the base-change of X to a scheme over S' as $X_{S'} = X \times_S S'$.

Remark 2.166. Given a field extension K'/K and a scheme X over $\operatorname{Spec} K$, we can check that $X_{K'}(K') = X(K')$. This is purely formal: a morphism $\operatorname{Spec} K' \to X$ induces a unique morphism $\operatorname{Spec} K' \to X_{K'}$ making the diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, equal, bend right] & & \\
& X_{K'} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} K' \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

commute.

In applications, the base-change to the algebraic closure will be especially important because certain aspects become clear only once passing to an algebraic closure. This gives the following definition.

Definition 2.167 (Goemetrically irreducible, reduced, connected)**.** A scheme X over a field K is *geometrically irreducible/reduced/connected* if and only if $X_{\overline{K}}$ is irreducible/reduced/connected.

Example 2.168. Fix the scheme $X = \operatorname{Spec}\mathbb{Q}(\sqrt{2})$ over the scheme $\operatorname{Spec}\mathbb{Q}$. Even though X is irreducible, it is not geometrically irreducible because $X_{\overline{\mathbb{Q}}}$ becomes two copies of $\operatorname{Spec}\overline{\mathbb{Q}}$. Indeed,

$$X_{\overline{\mathbb{Q}}} = \operatorname{Spec}(\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) = \operatorname{Spec}(\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}) = \operatorname{Spec}\overline{\mathbb{Q}} \sqcup \operatorname{Spec}\overline{\mathbb{Q}}.$$

Namely, $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$ by decomposing some element $(a + b\sqrt{2}) \otimes \alpha$ as $1 \otimes a\alpha + \sqrt{2} \otimes b\alpha$.

Intuitively, being irreducible but not geometrically irreducible means that passing to the algebraic closure gives rise to Galois conjugate pieces. This allows us to separate geometric information from Galois information.

2.7.4 The Relative Frobenius

For concreteness, fix a scheme S over $\mathbb{F}_p$. Notably, a scheme X over S has a map $X \to S$, so because the sheaf $p\mathcal{O}_S$ vanishes, we have that $p\mathcal{O}_X$ will also vanish. The point is that the pth-power map $f \mapsto f^p$ is going to induce a scheme morphism $F_X : X \to X$.

To see morphism topologically, let's see an example. When $X = \operatorname{Spec} A$ is affine, we see that A is an $\mathbb{F}_p$-algebra, and the Frobenius mapping $\varphi \colon a \mapsto a^p$ we can see directly sends $\mathfrak{p}$ by φ^{-1} to itself. Thus, the Frobenius is indeed nothing at all topologically.

Remark 2.169. The above morphism is called the absolute Frobenius.

Now, we have another Frobenius morphism $F_S\colon S \to S$, and we can see that the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

commutes. However, the fiber product now promises us a morphism $X \to X \times_S S$, where the two Ss are different copies with a Frobenius morphism going between them.

Remark 2.170. The above morphism is called the relative Frobenius.

Example 2.171. With $S = \operatorname{Spec} \mathbb{F}_p[s]$ and $X = \operatorname{Spec} \mathbb{F}_p[s, x]$, the relative Frobenius keeps s fixed and sends $x \mapsto x^p$.

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