18.755: Lie Groups and Lie Algebras II

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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Here we go.

1.1.1 Review of Lie Groups

We start with some quick review. Here are our groups.

Definition 1.1 (Lie group)**.** A *Lie group* is a group object G in the category of manifolds. One may specify a "real" or "complex" Lie group, which means that we are taking the category of real or complex manifolds. Explicitly, we are asking for G to be equipped with regular maps $m\colon G \times G \to G$, $i\colon G \to G$, and an identity. A *homomorphism of Lie groups* is a morphism of the group objects.

Example 1.2. One has the usual examples: $\mathbb{R}^n$, $\mathrm{U}(n)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{O}(p,q)$, and $\mathrm{SU}(n)$ are all real Lie groups.

Example 1.3. There are classical groups over $\mathbb{C}$, such as $\mathrm{SL}_n(\mathbb{C})$, which are all Lie groups.

Definition 1.4. If G is a Lie group, then its connected component G° is a normal Lie subgroup.

Remark 1.5. The quotient $\pi_0 G \coloneqq G/G^\circ$ is a discrete topological group.

Remark 1.6. Given a Lie group G, the universal cover $\widetilde{G} \to G$ can be checked to a Lie group via some universal properties, so we receive a homomorphism $\pi\colon \widetilde{G} \to G$. It turns out that the kernel is a central discrete subgroup $Z \subseteq \widetilde{G}$. It notably follows that $\pi_1(G)$ is abelian.

Remark 1.7. One can check that G° is generated by any open neighborhood of the identity. Indeed, the generated subgroup can be seen to be both open and closed.

Example 1.8. With $G = S^1$, we have the universal cover $\widetilde{G} = \mathbb{R}$, and the kernel is $\mathbb{Z} \subseteq \mathbb{R}$.

We also have subgroups.

Definition 1.9 (Lie subgroup)**.** A *Lie subgroup* is an immersed submanifold $H \subseteq G$ which is also a subgroup, meaning that $H \hookrightarrow G$ admits injective differentials. A *closed Lie subgroup* is an embedded submanifold $H \subseteq G$ which is also a subgroup.

Remark 1.10. It turns out that closed Lie subgroups are in fact closed subsets, which can be checked locally.

Example 1.11. The subgroup $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is a Lie subgroup, but it is not a closed Lie subgroup. The only closed Li subgroups are vector spaces.

Example 1.12. The subgroup $\mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ is a closed real Lie subgroup.

Remark 1.13. It turns out that a closed subgroup of G is in fact a closed Lie subgroup. We will prove this later in the semester.

Definition 1.14 (quotient)**.** Fix a closed Lie subgroup $H \subseteq G$. Then G/H is a manifold with transitive G-action. If H is normal, then G/H is further a Lie group.

Remark 1.15. In general, if G acts transitively on a manifold X, then for any $x \in X$, $\mathrm{Stab}_G(x) \subseteq G$ is a closed Lie subgroup, and the quotient is isomorphic to X.

Remark 1.16. If G acts on a space X which is not transitive, then for any $x \in X$, the subset $Gx \subseteq X$ is at least an immersed submanifold.

Example 1.17. The group $\mathbb{R}$ has an action on $\mathbb{R}^2/\mathbb{Z}^2$ by $t\colon x \mapsto tx$. The orbit of (say), $(1/2, \sqrt{2}/2)$ is an immersed but not closed submanifold.

Definition 1.18 (representation)**.** Fix a Lie group G. A *representation* of a Lie group is a homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

Example 1.19. Let G act on itself by conjugation. Then each $g \in G$ acts on $T_1G \to T_1G$, so we receive an adjoint representation $\mathrm{Ad}_\bullet\colon G \to \mathrm{GL}(T_1G)$.

As usual, one can define morphisms of representations, subrepresentations, direct sums, duals, tensor products, irreducible representations, and so on. We also have a Schur's lemma.

Lemma 1.20. Fix irreducible representations V and W of G.

(a) Then a G-equivariant map $\varphi\colon V \to W$ is either zero or an isomorphism.

(b) Any G-equivariant map $A \to A$ is a scalar.

Proof. Omitted. ∎

Definition 1.21 (unitary)**.** A *unitary representation* is one admitting a G-invariant positive-definite Hermitian form.

Remark 1.22. Any unitary representation admits a decomposition into irreducible representations by taking orthogonal complements.

Non-Example 1.23. Let $B \subseteq \mathrm{GL}_2(\mathbb{C})$ be the subgroup of upper-triangular matrices. Then the standard representation of B does not admit a decomposition into irreducibles, so it cannot be made unitary.

Example 1.24. If G is finite, then any representation V admits a unitary structure: given any unitary structure $\langle -, - \rangle_0$, one can define an invariant unitary structure

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle_0,$$

where dg is a choice of Haar measure.

Theorem 1.25 (Maschke)**.** Fix a finite group G. Then all representations admit decomposition into irreducible representations.

Proof. This follows from Example 1.24. ■

1.1.2 Review of Lie Algebras

We now linearize our story.

Remark 1.26. Note that G acts on itself by left translations ℓ_g, so the tangent bundle TG can be given a global frame by the induced isomorphisms $d\ell_g \colon T_1 G \to T_g G$.

Notation 1.27. For each $a \in T_1 G$, we define the vector field L_a by

$$L_a := ga \in T_a G.$$

Remark 1.28. One can check that all left-invariant vector fields take the form L_a.

Definition 1.29 (commutator)**.** Fix a Lie group G. For each $a, b \in T_1 G$, we may take the commutator $[L_a, L_b]$ to produce another left-invariant vector field, which we label $L_{[a,b]}$.

Remark 1.30. The formalism of the commutator tells us that $[-, -]$ is antisymmetric and satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

Definition 1.31 (Lie algebra)**.** Fix a vector space $\mathfrak{g}$ over a field k. Then a *Lie algebra* is such a vector space $\mathfrak{g}$ equipped with an antisymmetric pairing $[-, -] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

Example 1.32. For any Lie group G, we have seen that we may equip $\operatorname{Lie} G := T_1 G$ with the structure of a Lie group.

Example 1.33. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathfrak{g} = M_n(\mathbb{C})$, and one can check that $[X, Y] = XY - YX$.

We now define Lie subalgebras and morphisms of Lie algebras in the expected way.

Definition 1.34 (Lie ideal)**.** Fix a Lie algebra $\mathfrak{g}$. Then a *Lie ideal* $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace for which $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Example 1.35. For any closed Lie subgroup $H \subseteq G$, we see that $\operatorname{Lie} H \subseteq \operatorname{Lie} G$ is a Lie subalgebra. If H is normal, then $\operatorname{Lie} H$ is a Lie ideal.

As expected, there is some representation theory.

Definition 1.36. Fix a Lie algebra $\mathfrak{g}$ over a field k. Then a *representation* of $\mathfrak{g}$ is a morphism $\mathfrak{g} \to \mathfrak{gl}_n(k)$.

One can relate $\operatorname{Lie} G$ to G more directly via exponentiation.

Definition 1.37 (exponential)**.** Fix a Lie group G with Lie algebra $\mathfrak{g}$. We define a map $\exp\colon \mathfrak{g} \to G$ as follows. For each $a \in \mathfrak{g}$, one can check that the differential equation

$$\begin{cases} e'(t) = e(t) \cdot a, \\ e(0) = 1, \end{cases}$$

admits a unique solution; we then define $\exp(ta) := e(t)$. (This is independent of the choice of t.) It turns out that $t \mapsto \exp(ta)$ is a group homomorphism.

Example 1.38. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\exp\colon M_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is the usual matrix exponential.

Remark 1.39. It turns out that $\exp$ is a local diffeomorphism (though not necessarily injective), so there is a local inverse $\log\colon U \to \mathfrak{g}$, where U is some open neighborhood of the identity.

Remark 1.40. For small a and b, it turns out that

$$\log(\exp(a)\exp(b)) = a + b + \frac{1}{2}[a, b] + \cdots,$$

where $\cdots$ denotes cubic terms. For example, if G is commutative, then we see that the Lie bracket $[-, -]$ vanishes; conversely, if $[-, -]$ vanishes, then G can be checked to commute in an open neighborhood of the identity, so G commutes.

1.1.3 Fundamental Theorems

In a first course, one checks the following two fundamental theorems.

Theorem 1.41. Fix a Lie group G. Then there is a bijection between connected closed Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \operatorname{Lie} G$.

Theorem 1.42. Fix Lie groups G and K, with G simply connected. Then taking the differential

$$\operatorname{Hom}(G, K) \to \operatorname{Hom}(\operatorname{Lie} G, \operatorname{Lie} K)$$

is an isomorphism.

There is a third fundamental theorem, which we will prove later.

Theorem 1.43. For any finite-dimensional Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$), then there is a Lie group G with $\operatorname{Lie} G \cong \mathfrak{g}$.

The three theorems provide an equivalence between the category of simply connected Lie groups and the category of Lie algebras, thereby classifying the former.

Remark 1.44. It follows that one may classify connected Lie groups as quotients of simply connected Lie groups by discrete central subgroups.

1.1.4 Representations of Lie Algebras

Let's start with the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 1.45. Fix the usual basis $e := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $f := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, and $h := [e, f]$ of $\mathfrak{sl}_2(\mathbb{C})$.

(a) Then all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ can be parameterized as $\{V_n\}_{n \geq 0}$, where V_n is the representation of homogeneous polynomials in x and y of degree n.

(b) Every representation is a direct sum of irreducible representations.

(c) Clebsch–Gordon rule: for any n and m, we have

$$V_n \otimes V_m = \bigoplus_{i=0}^{\min\{m,n\}} V_{|m-n|+2i}.$$

It will be helpful to turn representation theory of Lie algebras into a module category.

Definition 1.46 (universal enveloping algebra)**.** Fix a Lie algebra $\mathfrak{g}$. Then we define $U\mathfrak{g}$ as the quotient of the tensor algebra by the relation

$$[x, y] = x \otimes y - y \otimes x.$$

Remark 1.47. It turns out that $\operatorname{Rep} \mathfrak{g}$ is the same category as $\operatorname{Mod} U\mathfrak{g}$.

Even though we have taken a quotient by an inhomogeneous relation, $U\mathfrak{g}$ still receives a natural filtration by degree.

Theorem 1.48 (Poincaré–Birkhoff–Witt)**.** Fix a Lie algebra $\mathfrak{g}$, and equip $U\mathfrak{g}$ with the natural filtration. For any basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$, the ordered monomials in the basis form a basis of $U\mathfrak{g}$.

To continue our story, we need some adjectives for Lie algebras.

Definition 1.49 (solvable)**.** A Lie algebra $\mathfrak{g}$ is *solvable* if and only if the derived series eventually vanishes. Here, the derived series is defined inductively by $D^0(\mathfrak{g}) := \mathfrak{g}$ and $D^{n+1}(\mathfrak{g}) := [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$ for each $n \geq 0$.

Definition 1.50 (nilpotent)**.** A Lie algebra $\mathfrak{g}$ is *nilpotent* if and only if the lower central series eventually vanishes. Here, the derived series is defined inductively by $L_0(\mathfrak{g}) \coloneqq \mathfrak{g}$ and $L_{n+1}(\mathfrak{g}) \coloneqq [L_n(\mathfrak{g}), \mathfrak{g}]$ for each $n \geq 0$.

Remark 1.51. One can see that nilpotent implies solvable.

The representation theory of solvable Lie algebras is quite easy.

Theorem 1.52 (Lie)**.** Fix a finite-dimensional solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero.

(a) Then every irreducible representation of $\mathfrak{g}$ is one-dimensional.

(b) Every representation admits a basis on which $\mathfrak{g}$ acts by upper-triangular matrices.

Theorem 1.53 (Engel)**.** Fix a finite-dimensional Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ is nilpotent for all $X \in \mathfrak{g}$.

Thus, we see that we will want to ignore solvable and nilpotent pieces.

Definition 1.54 (radical)**.** Fix a Lie algebra $\mathfrak{g}$. Then the *radical* $\operatorname{rad} \mathfrak{g}$ is the sum of all solvable ideals of $\mathfrak{g}$.

Remark 1.55. One can check that $\operatorname{rad} \mathfrak{g}$ is a solvable ideal, so it is automatically the largest solvable ideal.

Definition 1.56 (semisimple)**.** Fix a Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is *semisimple* if and only if $\operatorname{rad} \mathfrak{g} = 0$.

Remark 1.57. It turns out that

$$\mathfrak{g}_{\mathrm{ss}} \coloneqq \frac{\mathfrak{g}}{\operatorname{rad} \mathfrak{g}}$$

is always semisimple. It turns out that the induced exact sequence splits, so there is a decomposition $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \ltimes \operatorname{rad} \mathfrak{g}$, which is known as the Levi decomposition; we will prove this later.

Having defined semisimple, we should define "simple."

Definition 1.58 (simple)**.** A Lie algebra $\mathfrak{g}$ is *simple* if and only if its only ideals are 0 and $\mathfrak{g}$.

Remark 1.59. One can check that semisimple Lie algebras are precisely the sums of simple Lie algebras.

It turns out to be convenient to allow a little radical.

Definition 1.60 (reductive)**.** A Lie algebra $\mathfrak{g}$ is *reductive* if and only if its radical is its center.

Example 1.61. One can check that $\mathfrak{sl}_n(\mathbb{C})$ is simple, and $\mathfrak{gl}_n(\mathbb{C})$ is reductive.

To test for a Lie algebra being semisimple (and other adjectives), we introduce the Killing form.

Definition 1.62 (Killing form)**.** Fix a Lie algebra $\mathfrak{g}$. Then we define the *Killing form* by

$$K(x, y) \coloneqq \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y).$$

Remark 1.63. One can check that K is $\mathfrak{g}$-invariant.

Theorem 1.64 (Cartan criteria)**.** Fix a Lie algebra $\mathfrak{g}$.

(a) $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subseteq K$.

(b) $\mathfrak{g}$ is semisimple if and only if K is non-degenerate.

Proposition 1.65. A Lie algebra $\mathfrak{g}$ is reductive if and only if it admits a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ for which the bilinear form

$$B_V(X, Y) := \operatorname{tr}(\rho_X \circ \rho_Y)$$

is non-degenerate.

We may as well state one of the main theorems of our representation theory.

Theorem 1.66. Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.

1.1.5 Structure Theory of Lie Algebras

Here is another piece of structure theory.

Definition 1.67 (adjoint)**.** Fix a semisimple Lie algebra $\mathfrak{g}$. Then we define the *adjoint* Lie group G^{ad} by $G^{\mathrm{ad}} := \operatorname{Aut}(\mathfrak{g})^\circ \subseteq \operatorname{GL}(\mathfrak{g})$.

Remark 1.68. It turns out that $\operatorname{Lie} G^{\mathrm{ad}} = \mathfrak{g}$.

In our setting, one can generalize the Jordan decomposition.

Definition 1.69 (semisimple, nilpotent)**.** An element $X \in \mathfrak{g}$ is *semisimple* or *nilpotent* if and only if the operator $\operatorname{ad}_X X$ is.

Theorem 1.70. Fix a Lie algebra $\mathfrak{g}$. Then any $X \in \mathfrak{g}$ can be written uniquely as a sum of a semisimple and nilpotent element.

Remark 1.71. It turns out that semisimple elements always act semisimply on representations, and nilpotent elements always act nilpotently on representations.

The notion of semisimple elements is important to define Cartan subalgebras.

Definition 1.72 (Cartan)**.** Fix a semisimple Lie algebra $\mathfrak{g}$. Then a *Cartan subalgebra* is a maximal commutative subalgebra of

Proposition 1.73. Fix a semisimple Lie algebra $\mathfrak{g}$. All Cartan subalgebras are conjugate by G^{ad}.

Definition 1.74. Fix a semisimple Lie algebra $\mathfrak{g}$. Then the *rank* of $\mathfrak{g}$ is the dimension of the Cartan subalgebras.

A choice of Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ produces a root decomposition, which we write as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha.$$

Definition 1.75 (root system)**.** Fix a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Then the *root system* of $\mathfrak{g}$ consists of those nonzero eigenvalues $\alpha \in \mathfrak{h}^*$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We write $\mathfrak{g}_\alpha$ for this eigenspace, and we write $\Phi(G)$ for the root system.

Remark 1.76. One can check that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{[\alpha,\beta]}$. In fact, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal for the Killing form except when $\alpha = -\beta$, where it is a perfect pairing.

Remark 1.77. It turns out that $\dim \mathfrak{g}_\alpha = 1$ for each α. It follows that

$$\#\Phi(\mathfrak{g}) = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}.$$

Remark 1.78. There are the usual pictures of root systems of various types.

1.1.6 Root Systems

It is useful to write down what properties are satisfied by these root systems.

Definition 1.79 (root system)**.** Fix a Euclidean space E. Then a finite subset $\Phi \subseteq E$ is a *root system* if and only if

(a) Φ spans E,

(b) for each $\alpha, \beta \in \Phi$, the number

$$n_{\alpha\beta} := \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer,

(c) for each $\alpha, \beta \in \Phi$, the reflection

$$s_\alpha(\beta) := \beta - n_{\alpha\beta}\beta$$

is in Φ.

We say that Φ is *reduced* if and only if $\alpha \in \Phi$ implies that $2\alpha \notin \Phi$.

Definition 1.80 (reducible)**.** A root system Φ is *reducible* if and only if it can be written as a disjoint union of root systems coming from a decomposition of the Euclidean space into a product of Euclidean spaces.

The reflections are important enough to be placed into a group.

Definition 1.81 (Weyl group)**.** Fix a root system $\Phi \subseteq E$. Then the *Weyl group* W is the subgroup of $\mathrm{GL}(E)$ generated by the reflections.

Example 1.82. The Weyl group associated to the root system of $\mathfrak{sl}_{n+1}$ consists of the permutation matrices in $\mathrm{GL}_n(\mathbb{R})$. Indeed, each reflection corresponds to a transposition. This root system is said to be of type A_n, where n refers to the rank.

Remark 1.83. There is a determinant $\det$ on the Weyl group, which we define to be the determinant of the natural representation of W in $\mathrm{GL}(E)$.

Example 1.84. The root system associated to $\mathfrak{so}_{2n+1}$ is B_n. The root system associated to $\mathfrak{sp}_{2n}$ is C_n. Lastly, the root system associated to $\mathfrak{so}_{2n}$ is D_n.

Remark 1.85. There are also various exceptional reduced root systems, which we may say something about later.

We can even break down irreducible root systems into more controlled pieces.

Definition 1.86 (positive)**.** Fix a root system $\Phi \subseteq E$. For a choice of $t \in E$ for which $(t, \alpha) \neq 0$ for all $\alpha \in E$, we say that a root in Φ is *positive* if and only if $(t, \alpha) > 0$. Similarly, α is *negative* if and only if $(t, \alpha) < 0$. We let Φ^+ and Φ^- denote the sets of positive and negative roots, respectively.

Definition 1.87. Fix a root system $\Phi \subseteq E$. A positive root is *simple* if and only if it is not a sum of other positive roots (with positive integer coefficients). We let Π denote the set of simple roots.

Proposition 1.88. Fix a root system $\Phi \subseteq E$. Then Π is a basis, and every positive root $\alpha \in \Phi^+$ can be written as a unique sum of elements of Π with positive integer coefficients.

Each root system also admits a dual.

Definition 1.89 (dual root system)**.** Fix a root system $\Phi \subseteq E$. Then we define the *dual root system* $\Phi^\vee \subseteq E^\vee$ to be given by the points

$$\alpha^\vee = \frac{2(\alpha, -)}{(\alpha, \alpha)}$$

for each $\alpha \in \Phi$.

Remark 1.90. The reduced root system B_n is dual to C_n.

1.2 February 4

We continue our review today.

1.2.1 Weights

It will be helpful to have some lattices from our root systems.

Definition 1.91. Fix a root system $\Phi \subseteq E$.

- The *root lattice* Q is spanned by Φ.
- The *coroot lattice* $Q^\vee$ is spanned by the $\alpha^\vee$.
- The *weight* lattice $P \subseteq E$ is $(Q^\vee)^*$.
- The *coweight lattice* $P^\vee \subseteq E^*$ is Q^*.

In general, $Q \subseteq P$, but equality does not have to hold.

Example 1.92. For $\mathfrak{sl}_n$, the quotient P/Q is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

It is useful to have a basis for the weight lattice

Definition 1.93 (fundamental weight)**.** Fix a root system $\Phi \subseteq E$. A *fundamental weight* is an element of the dual basis (in the weight lattice P) of the $\alpha_\bullet^\vee \in Q^\vee$ where α is a simple root. Explicitly, if $\{\alpha_1, \ldots, \alpha_r\}$ are the simple roots, then the fundamental weights $\{\omega_1, \ldots, \omega_r\}$ satisfy

$$(\omega_i, \alpha_j^\vee) = 1_{i=j}.$$

Definition 1.94 (dominant)**.** Fix a root system $\Phi \subseteq E$. A weight λ is *dominant* if and only if $\langle \lambda, \alpha_i^\vee \rangle$ is a nonnegative integer for all i.

Remark 1.95. One can check that the dominant weights are exactly the $\mathbb{Z}_{\geq 0}$-span of the fundamental weights.

1.2.2 The Dynkin Diagram

Let's start trying to classify our Lie algebras.

Definition 1.96 (Cartan matrix)**.** Fix a root system $\Phi \subseteq E$, and order the simple roots as $\{\alpha_1, \ldots, \alpha_r\}$. The matrix A with entries

$$a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

is the *Cartan matrix*.

Remark 1.97. The Cartan matrix satisfies the following properties, all essentially by construction.

- We have $a_{ii} = 2$ for each i.
- For distinct i and j, we have $a_{ij} \leq 0$, and a_{ij} vanishes if and only if a_{ji} vanishes.
- Because the inner product $(-,-)$ on the $\alpha_\bullet$s is positive definite, the pairing $(v, w) := v^\intercal A w$ is positive definite.

Remark 1.98. For any i and j, a piece of the Cartan matrix

$$\begin{bmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{bmatrix}$$

must have positive determinant by the positive-definiteness. Thus, $4 - a_{ij}a_{ji} > 0$, so $a_{ij}a_{ji} \in [0, 1, 2, 3]$ follows.

Example 1.99. The Cartan matrix for $\mathfrak{sl}_4$ is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The data of a Cartan matrix can be encoded combinatorially into a Dynkin diagram.

Definition 1.100 (Dynkin diagram)**.** Fix a root system $\Phi \subseteq E$, and order the simple roots as $\{\alpha_1, \ldots, \alpha_r\}$. Let A be the Cartan matrix. Then we define the *Dynkin diagram* to have r vertices and the following connections.

- If $a_{ij} = 0$, then vertices i and j have no connection.
- If $a_{ij} = -1$ and $a_{ji} = -1$, we draw a single line between i and j.
- If $a_{ij} \leq -2$, then we draw an arrow from vertex i to j with $|a_{ij}|$ heads.

There are the usual pictures of all Dynkin diagrams.

Example 1.101. For example, A_{n-1} is a path with $n-1$ vertices.

$$\bullet \text{———} \bullet \text{———} \bullet \text{———} \cdots \text{———} \bullet$$

Remark 1.102. It turns out that the root system determines the Lie algebra, the Cartan matrix determines the root system, and the Dynkin diagram determines the Cartan matrix.

Remark 1.103 (Coxeter group)**.** One can read the Weyl group off of the Cartan matrix: namely, the reflection s_is_j has order

$$m_{ij} := \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0, \\ 3 & \text{if } a_{ij}a_{ji} = 1, \\ 4 & \text{if } a_{ij}a_{ji} = 2, \\ 6 & \text{if } a_{ij}a_{ji} = 3. \end{cases}$$

Indeed, this claim amounts to checking that s_is_j is a rotation with specified degree, which can be seen on the inner products. It turns out that the Weyl group is generated by the relations $s_i^2 = 1$ and $(s_is_j)^{m_{ij}} = 1$ for each pair of distinct i and j.

To prove Remark 1.102, a difficult step is to show that every Dynkin diagram does in fact give rise to a Lie algebra. This is the content of the following theorem.

Theorem 1.104 (Serre presentation)**.** Fix a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with root system Φ, and choose an ordered set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Select basis vectors e_i and f_i in each space $\mathfrak{g}_{\alpha_i}$ and $\mathfrak{g}_{-\alpha_i}$, and define $h_i := [e_i, f_i]$. Then we have the following relations.

(a) We have $[h_i, h_j] = 0$, $[h_i, e_j] = a_{ij}e_j$, and $[h_i, f_j] = -a_{ij}f_j$.

(b) We have $[e_i, f_j] = 1_{i=j}h_i$.

(c) Serre relations: $\operatorname{ad}_{e_i}^{1-a_{ij}} e_j = 0$ and $\operatorname{ad}_{f_i}^{1-a_{ij}} f_j = 0$.

In fact, the free Lie algebra defined by these generators and relations (for any root system Φ) produces a finite-dimensional Lie algebra.

1.2.3 Back to Representations of Lie Algebras

Throughout, we fix a finite-dimensional Lie algebra $\mathfrak{g}$, choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Then we choose a collection of positive roots for the induced root system, which lets us split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where $\mathfrak{n}_+$ is the sum of the positive root spaces, and $\mathfrak{n}_-$ is the sum of the negative root spaces.

Definition 1.105 (Verma module)**.** Fix a weight $\lambda \in \mathfrak{h}^*$. Then we define the *Verma module* M_λ to be generated as a $U\mathfrak{g}$-module by the vector v_λ, given the relations $ev_\lambda = 0$ for $e \in \mathfrak{n}_+$ and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$.

Remark 1.106. In other words, $M_\lambda = U\mathfrak{g} \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the representation of $\mathfrak{h} \oplus \mathfrak{n}_+$ on which $\mathfrak{h}$ acts by λ and $\mathfrak{n}_+$ acts by zero. For example, the PBW theorem gives us an isomorphism

$$U\mathfrak{n}_- \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+) \to U\mathfrak{g}$$

of vector spaces, so M_λ is a free module over $U\mathfrak{n}_-$ of rank 1.

Remark 1.107. For generic λ, it turns out that M_λ is irreducible.

Definition 1.108 (highest weight)**.** Fix a representation V of $\mathfrak{g}$. Then a weight λ is a *highest weight* if and only if V is a nonzero quotient of M_λ. In other words, there is a vector $v \in V$ on which $\mathfrak{h}$ acts by λ and $\mathfrak{n}_+$ acts by zero.

Our classification result is as follows.

Theorem 1.109. Fix everything as above.

(a) The Verma modules M_λ admit a unique irreducible quotient.

(b) The representation L_λ is finite-dimensional if and only if λ is a dominant weight.

(c) Every irreducible finite-dimensional representation of $\mathfrak{g}$ takes the form L_λ for a dominant weight λ.

Proof. The proof of (a) is not too hard, and it is also not hard to show that L_λ being finite-dimensional, then λ is dominant. The hard part is to show that λ being dominant implies that L_λ is finite-dimensional!

Lastly, by iteratively acting by $\mathfrak{n}_+$, one can show that every irreducible finite-dimensional representation V of $\mathfrak{g}$ admits some highest weight λ. Indeed, given a finite-dimensional irreducible representation V of $\mathfrak{g}$, the representation theory of $\mathfrak{sl}_2$ implies that the action of $\mathfrak{h}$ on V diagonalizes and admits integer eigenvalues. It follows that there is a surjection $M_\lambda \twoheadrightarrow V$, so there is a surjection $L_\lambda \twoheadrightarrow V$, which must then be an isomorphism. ■

Remark 1.110. Here is another way to select the highest weight: one can give the weight lattice a partial ordering, and then the highest one is the maximal element among the weights λ for which $V[\lambda] \neq 0$. In particular, the maximality implies that $\mathfrak{n}_+$ acts by zero on $V[\lambda]$.

There is also a construction of the quotient L_λ by taking relations.

Theorem 1.111. Fix everything as above, and choose a dominant weight λ. Then L_λ is the quotient of M_λ by the relations

$$f_i^{(\lambda,\alpha_i^\vee)+1} v_\lambda = 0.$$

Remark 1.112. These relations are forced by passing to the representation theory of $\mathfrak{sl}_2$.

1.2.4 Weyl Formulae

Our construction of L_λ is not so explicit; for example, what is the dimension of L_λ? To answer this question, we will need the Weyl dimension formula. We start with the Weyl character formula.

Definition 1.113 (character)**.** Fix a representation V of a semisimple Lie algebra $\mathfrak{g}$ admitting a weight decomposition

$$V = \bigoplus_{\mu \in P} V[\mu].$$

Then we define the *character*

$$\operatorname{ch} V := \sum_{\mu \in P} \dim V[\mu] e^\mu$$

living in the free power series vector space with basis given by the e^μs, with a relation $e^{\mu+\lambda}$.

Remark 1.114. If V is finite-dimensional, then it admits a weight decomposition (by the representation theory of $\mathfrak{sl}_2$), and the character is a finite sum.

Remark 1.115. Let's explain why this is a character. Suppose that V is a representation of a Lie group G with Lie algebra $\mathfrak{g}$. For $h \in \mathfrak{h}$, the exponential $\exp(h) \in G$ acts on V diagonally, with trace

$$\operatorname{tr}_V \exp(h) = \sum_{\mu \in P} \dim V[\mu] e^{\mu(h)},$$

which is exactly $\operatorname{ch} V(h)$. In fact, these formulae determine the entire character because semisimple elements can all be found in Cartan subalgebras and are dense in G.

We are now ready to state the Weyl character formula.

Theorem 1.116 (Weyl character formula)**.** Fix a semisimple Lie algebra $\mathfrak{g}$ with root system Φ, and let Φ^+ be a set of positive roots. Further, set $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Then for each dominant weight λ, we have

$$\operatorname{ch} L_\lambda = \frac{\sum\limits_{w\in W}\det(w)e^{w(\lambda+\rho)-\rho}}{\prod\limits_{\alpha\in\Phi_+}(1-e^{-\alpha})}.$$

Proof. Omitted. The idea is that the character of M_λ is easy to compute, and one can build a resolution of L_λ in terms of Verma modules. ■

Remark 1.117. One can write the right-hand side more symmetrically as

$$\frac{\sum\limits_{w\in W}\det(w)e^{w(\lambda+\rho)}}{\prod\limits_{\alpha\in\Phi_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right)}.$$

Remark 1.118. A priori, the right-hand side is a rational function, but one can use the anti-symmetry of the Weyl group action in the numerator to show that the denominator does in fact divide the numerator.

Example 1.119 (Weyl denominator)**.** If $\lambda = 0$, then L_λ is the trivial representation, so it follows that

$$\sum_{w\in W}\det(w)e^{w\rho} = \prod_{\alpha\in\Phi_+}\left(e^{\alpha/2}-e^{-\alpha/2}\right).$$

Example 1.120 (Vandermonde determinant)**.** Suppose that $\mathfrak{g} = \mathfrak{sl}_n$ so that $\alpha_{ij} = x_i - x_j$. Then we set $X_i := e^{x_i}$, so we find that

$$\sum_{w\in S_n}\left(\operatorname{sgn}(w)\prod_{i=1}^{n}X_{w(i)}^{i-1}\right) = \prod_{i<j}(X_i - X_j).$$

Note that the right-hand side is precisely the determinant of the matrix

$$\begin{bmatrix} 1 & X_1 & \cdots & X_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_n & \cdots & X_n^{n-1} \end{bmatrix}.$$

By Theorem 1.116, we could attempt to compute $\dim L_\lambda$ by plugging in $h = 0$ to the polynomial.

Theorem 1.121 (Weyl dimension formula)**.** Fix a semisimple Lie algebra $\mathfrak{g}$ with root system Φ, and let Φ^+ be a set of positive roots. Then for any dominant weight λ,

$$\dim L_\lambda = \prod_{\alpha\in\Phi_+}\frac{(\alpha^\vee, \lambda+\rho)}{(\alpha^\vee, \rho)}.$$

Proof. This is slightly complicated because the given denominator vanishes at $h = 0$, so we are left to compute the limit

$$\dim L_\lambda = \lim_{h \to 0} \frac{\sum_{w \in W} \det(w) e^{(w(\lambda+\rho),h)}}{\prod_{\alpha \in \Phi_+} \left(e^{(\alpha,h)/2} - e^{-(\alpha,h)/2}\right)}.$$

This limit looks difficult to compute, but we may merely compute it on the line $h := 2th_\rho$, where h_ρ is the dual element for $\rho \in h^*$. In particular, it turns out that the numerator will factor in this case! To see this, note $\alpha(h_\rho) = (\alpha, \rho)$, so

$$\sum_{w \in W} \det(w) e^{(w(\lambda+\rho),2th_\rho)} = \sum_{w \in W} \det(w) e^{2t(w(\lambda+\rho),\rho)},$$

which upon moving the w around and plugging into Example 1.119 gives

$$\prod_{\alpha \in \Phi_+} \left(e^{t(\alpha,\lambda+\rho)} - e^{-t(\alpha,\lambda+\rho)}\right).$$

The denominator is now

$$\prod_{\alpha \in \Phi_+} \left(e^{(\alpha,h)/2} - e^{-(\alpha,h)/2}\right) = \prod_{\alpha \in \Phi_+} \left(e^{t(\alpha,\rho)} - e^{-t(\alpha,\rho)}\right).$$

Sending $t \to 0$ (and cancelling out a factor of two) proves the result. ■

1.2.5 Representation Theory of $\mathfrak{sl}_n(\mathbb{C})$

The dominant weights ω_i are the "simplest," so the simplest interesting representations will be the L_{ω_i}s. For example, we may hope to find all other representations inside them.

Proposition 1.122. Fix a semisimple Lie algebra $\mathfrak{g}$ with root system Φ, and let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be an ordered set of simple roots. Further, let $\{\omega_1, \ldots, \omega_r\}$ be the set of dominant weights. Then the category finite-dimensional representations of $\mathfrak{g}$ is $\otimes$-generated by the representations $\{L_{\omega_i}\}_{i=1}^r$.

Proof. Because any finite-dimensional representation is a direct sum of the irreducible representations of the form L_λ, where λ is a dominant weight, it is enough to find these representations inside some tensor product of L_{ω_i}s. Well, λ is dominant, so we may write $\lambda = \sum_{i=1}^r k_i\omega_i$ for some nonnegative integers $\{k_1, \ldots, k_r\}$. We then consider the representation

$$V := \bigotimes_{i=1}^r L_{\omega_i}^{k_i}.$$

For any dominant weight μ, recall that μ is the highest weight of L_μ, meaning that the weights of L_μ are concentrated in $\mu - P^+$, where P^+ consists of the dominant weights. Additionally, the nature of the $\mathfrak{g}$-action on the tensor product implies that $W[\mu] \otimes W'[\mu'] = (W \otimes W')[\mu + \mu']$ for any W and W' and μ and μ'. Thus, we can see that the highest weight of V is $\lambda = \sum_{i=1}^r k_i\omega_i$, and it has multiplicity 1. It follows that there is a map $M_\lambda \to V$, which must descend to an embedding $L_\lambda \hookrightarrow V$, as desired. ■

Remark 1.123. The proof also shows that the other L_μs appearing in the representation V have $\mu < \lambda$. Indeed, one can simply calculate the weights appearing in V/L_λ and note that they are all strictly smaller than λ. The multiplicities of the various L_μ appearing in V can be interesting to calculate.

Remark 1.124. In fact, we can see that L_λ is the irreducible subrepresentation generated by the highest weight vector

$$v_\lambda = v_{\omega_1}^{\otimes k_1} \otimes \cdots \otimes v_{\omega_r}^{\otimes k_r}$$

in V. Indeed, the image of L_λ in V certainly contains this vector because this is the unique line with weight λ, so it follows that the generated subrepresentation must be exactly the irreducible representation L_λ.

Let's work out our general theory in an interesting example.

Example 1.125. Let V_{std} be the standard representation of $\mathfrak{sl}_n(\mathbb{C})$. There is a choice of ordered simple roots of $\mathfrak{sl}_n(\mathbb{C})$ for which

$$L_{\omega_i} \cong \wedge^i V_{\text{std}}.$$

Proof. We proceed in steps.

1. We set up notation. Here, $\mathfrak{h}$ is the set of diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$, which we embed into $\mathbb{C}^n$ as the subspace with sum zero. Let $\{e_1, \ldots, e_n\}$ be the natural basis of $\mathbb{C}^n$, so we find that

$$\Phi = \{e_i - e_j : i \neq j\}.$$

   Indeed, the elementary matrix E_{ij} has weight $e_i - e_j$. By choosing the vector $t \coloneqq (n, n-1, n-2, \ldots, 1, 0)$ in $\mathbb{C}^n$, we see that Φ^+ consists of those $e_i - e_j$ with $i < j$. Then the simple roots are given by $\alpha_i :- e_i - e_{i+1}$, which we can check is in fact a basis.

   For example, we see that this implies that the positive spaces in $\mathfrak{g}$ are given by

$$\mathfrak{n}_+ \coloneqq \bigoplus_{p<q} \mathbb{C}E_{pq}.$$

2. We calculate the fundamental weights. Now, the dual space $\mathfrak{h}^*$ is then the quotient of $\mathbb{C}^n$ by the diagonal subspace $\mathbb{C}(1, \ldots, 1)$. To compute our fundamental weights ω_i, we see that we are on the hunt for some vectors ω_i for which

$$(\omega_i, e_j - e_{j+1}) = 1_{i=j}.$$

   Well, $\omega_i = (1, \ldots, 1, 0, \ldots, 0)$ works, where there are i total 1s. Accordingly, our dominant weights are the $\mathbb{Z}_{\geq 0}$-linear combinations of the ω_i, so they are the decreasing sequences of nonnegative integers.

3. We claim that the representation $\wedge^i V_{\text{std}}$ is irreducible. Well, let $W \subseteq \wedge^i V_{\text{std}}$ be some nonzero subrepresentation, and we want to show that $W = \wedge^i V_{\text{std}}$. Give V_{std} the standard basis $\{v_1, \ldots, v_n\}$, and we note that this basis is an eigenbasis for the $\mathfrak{h}$-action on V_{std}. Thus,

$$\{v_{n_1} \wedge \cdots \wedge v_{n_i} : n_1 < \cdots < n_i\}$$

   is an eigenbasis for the $\mathfrak{h}$-action on $\wedge^i V_{\text{std}}$. Because $W \subseteq V$ is a $\mathfrak{g}$-subrepresentation, we conclude that it is an $\mathfrak{h}$-subrepresentation, so W must contain some eigenvector $v_{n_1} \wedge \cdots \wedge v_{n_i}$.

   Now, for any $p \neq q$, we see that $E_{pq} \in \mathfrak{sl}_n(\mathbb{C})$ acts by zero on all basis vectors $v_\bullet$ except it sends $v_q \mapsto v_p$. Thus, we see that we may apply such elementary matrices to any given eigenvector $v_{n_1} \wedge \cdots \wedge v_{n_i}$ to move it to $v_1 \wedge \cdots \wedge v_i$ (namely, apply E_{1n_1}, then E_{2n_2}, and so on), and one can apply the process in reverse to move any $v_1 \wedge \cdots \wedge v_i$ to any other eigenvector. We conclude that W being stable under $\mathfrak{g}$ forces $W = \wedge^i V$.

4. We are now ready to show that $L_{\omega_i} \cong \wedge^i V_{\text{std}}$. Because $\wedge^i V_{\text{std}}$ is already irreducible, it is enough to check that its highest weight is ω_i. Well, we claim that the highest weight vector is

$$w \coloneqq v_1 \wedge \cdots \wedge v_i.$$

Well, for some $p < q$, we can see that $E_{pq}w = 0$ because E_{pq} can only ever send a basis vector v_q to the "earlier" basis vector v_p, which either causes w to vanish outright (if $> i$) or introduces a multiplicity into w, still causing $E_{pq}w = 0$. Thus, we do indeed see that $\mathfrak{n}_+ w = 0$. As for the weight, one merely needs to calculate

$$\operatorname{diag}(x_1, \ldots, x_n)w = (x_1 + \cdots + x_i)w,$$

which is precisely the action by ω_i! ■

Remark 1.126. The moral is the story is that the category $\operatorname{Rep}\mathfrak{sl}_n(\mathbb{C})$ is $\otimes$-generated by the representations $\wedge^i V_{\text{std}}$. In fact, because $\wedge^i V_{\text{std}}$ is a quotient of $V_{\text{std}}^{\otimes i}$, it follows that the category is $\otimes$-generated by V_{std}.

Remark 1.127. There is a perfect pairing

$$\wedge^i V_{\text{std}} \otimes \wedge^{n-i} V_{\text{std}} \to \wedge^n V_{\text{std}} = \mathbb{C}$$

given simply by $w \otimes w' \mapsto w \wedge w'$; indeed, we can see this is a perfect pairing where a basis vector $v_{n_1} \wedge \cdots \wedge v_{n_i}$ has dual basis vector $v_{m_1} \wedge \cdots \wedge v_{m_{n-i}}$, where $\{n_1, \ldots, n_i\} \cup \{m_1, \ldots, m_{n-i}\} = \{1, \ldots, n\}$. Thus, we see that $L_{\omega_i}^* = L_{\omega_{n-i}}$ for each i.

Here is another interesting family of representations.

Example 1.128. Let V_{std} be the standard representation of $\mathfrak{sl}_n(\mathbb{C})$. There is a choice of ordered simple roots of $\mathfrak{sl}_n(\mathbb{C})$ for which

$$L_{m\omega_1} \cong \operatorname{Sym}^m V_{\text{std}}.$$

Proof. We continue from Example 1.125.

1. We claim that the representation $\operatorname{Sym}^m V_{\text{std}}$ is irreducible. As in Example 1.125, we see that

   $$\{v_{i_1} \cdots v_{i_m} : i_1 \leq \cdots \leq i_m\}$$

   is an eigenbasis for the action of $\mathfrak{h}$ on V_{std}. Thus, any nonzero subrepresentation W of V_{std} contains such an eigenvector. However, given one such eigenvector $v_{i_1} \cdots v_{i_m}$, one can iteratively apply E_{1i_1}, then E_{1i_2}, and so on, eventually proving that $v_1^m \in W$. Moving this basis vector around, we see that $v_i^m \in W$ for each i.

   We now claim that each basis vector $v_{i_1} \cdots v_{i_m}$ lives in W. Indeed, by hitting v_1^m with $E_{11} - E_{22} + \sum_{i=2}^n c_i E_{1i}$, we see that

   $$\left(v_1 + \sum_{i=2}^n c_i v_2\right)^m \in W.$$

   Each coefficient varies with a different multivariate polynomial in the $c_\bullet$s, so by choosing specializations carefully (and generically), we see that each monomial lives in W. Formally, one can inductively allow more and more of the c_is to be nonzero, one at a time.

2. We complete the proof. In light of the previous step, it is enough to show that v_1^m is the highest weight vector of $\operatorname{Sym}^m V_{\text{std}}$ and has weight $m\omega_1$. Certainly E_{pq} kills v_1^m for each $p < q$ because $q > 1$. As for the weight calculation, we note that

   $$\operatorname{diag}(x_1, \ldots, x_n) \cdot v_1^m = mx_1 \cdot v_1^m,$$

   as desired. ■

Remark 1.129. Another way to check that $\operatorname{Sym}^m V_{\text{std}}$ is irreducible would be to use the Weyl dimension formula (Theorem 1.121) to compute that $\dim L_{m\omega_1} = \dim \operatorname{Sym}^m V_{\text{std}}$. Then the second step provides a canonical map $L_{m\omega_1} \to V_{\text{std}}$, which must be injective and hence an isomorphism.

1.3 February 9

Today we discuss Schur–Weyl duality.

1.3.1 Representations of $\mathrm{GL}_n(\mathbb{C})$

Here is our result.

Proposition 1.130. Irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ are parameterized by pairs (m, λ), where $m \in \mathbb{Z}$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a decreasing sequence of nonnegative integers with $\lambda_n = 0$, and

$$m = \sum_i \lambda_i + nr$$

for some integer r. The highest weight of the corresponding irreducible representation $L_{m,\lambda}$ is $(\lambda_1 + r, \ldots, \lambda_n + r)$.

Proof. It is not quite the case that $\mathrm{GL}_n(\mathbb{C})$ is $\mathbb{G}_m \times \mathrm{SL}_n(\mathbb{C})$, but the natural covering map

$$\mathbb{G}_m(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

has finite kernel $\mu_n \subseteq \mathbb{C}^\times$. Thus, the irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ consist of the irreducible representations of $\mu_n \times \mathrm{SL}_n(\mathbb{C})$ for which μ_n acts by the identity. Notably, such an irreducible representation can be described as a tensor product of a representation of $\mathbb{G}_m(\mathbb{C})$ and a representation of $\mathrm{SL}_n(\mathbb{C})$, with some condition on the diagonal.

However, irreducible representations of $\mathbb{C}^\times$ are one-dimensional. Passing to the Lie algebra, we see that we are basically given the data of an operator h for which $e^{2\pi i h} = 1$, which means that the operator admits integer eigenvalues, so we see that the corresponding representations of $\mathbb{C}^\times$ are given by the characters $\chi_m(z) \coloneqq z^m$.

Thus, irreducible representations of $\mathbb{C}^\times \mathrm{SL}_n(\mathbb{C})$ can be described as tensor products

$$L_{m,\lambda} \coloneqq \chi_m \otimes L_\lambda,$$

where λ is some weight of $\mathfrak{sl}_n(\mathbb{C})$. Recall that such weights can be described as decreasing sequences of n integers with $\lambda_n = 0$. Notably, μ_n acts on $L_{m,\lambda}$ by sending the scalar z to $z^{-m+\sum_i \lambda_i}$, which needs to be divisible by n for our representation to descend to n. We can compute the highest weight of $L_{m,\lambda}$ as $(\lambda_1 + r, \ldots, \lambda_n + r)$. ■

Remark 1.131. Equivalently, we may say that representations of $\mathrm{GL}_n(\mathbb{C})$ are parameterized by decreasing sequences of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$, which is the highest weight.

Example 1.132. We can compute that there is now a fundamental weight $(1, \ldots, 1)$, which is the determinant representation.

Here is an important class of our representations.

Definition 1.133 (polynomial)**.** An irreducible representation L_λ of $\mathrm{GL}_n(\mathbb{C})$ is *polynomial* if and only if $\lambda_n \geq 0$.

Remark 1.134. We can think about the sequence λ as instead being a partition

$$\mu \coloneqq (\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n)$$

of λ_1. We may then write $L_\mu \coloneqq L_\lambda$.

Remark 1.135. Partitions are labeled by Young diagrams, which we notably have no use for.

Remark 1.136. One can check that L_λ is polynomial if and only if L_λ embeds into $V^{\otimes N}$ for some N, where V is the standard representation. In particular, we see that L_λ is contained in $V^{\otimes|\mu|}$, which we can see by the same argument as in Proposition 1.122. The point is that the determinant representation can still be found in $V^{\otimes n}$, but the inverse of the determinant cannot!

Remark 1.137. One can check that polynomial representations are precisely those which extend continuously to $M_n(\mathbb{C})$; equivalently, we may say that these representations are those for which the matrix coefficients are polynomial (not just rational). Again, the point is that the determinant representation has polynomial matrix coefficients, but the inverse of the determinant does not.

1.3.2 Schur–Weyl Duality

Having said something about the representations of $\mathrm{GL}_n(\mathbb{C})$, we are ready to relate them to the symmetric group. Set $V := \mathbb{C}^n$. Then we note that both $\mathrm{GL}_n(\mathbb{C})$ and S_N act on $V^{\otimes N}$, where notably S_N acts by permuting the coordinates.

Theorem 1.138 (Schur–Weyl duality)**.** Fix positive integers n and N, and set $V := \mathbb{C}^n$. Let A be the image of the natural map $U\mathfrak{gl}(V) \to \operatorname{End}_{\mathbb{C}} V^{\otimes N}$, and let B be the image of the natural map $\mathbb{C}[S_N] \to \operatorname{End}_{\mathbb{C}} V^{\otimes N}$.

(a) The algebras A and B are centralizers of each other.

(b) There is a decomposition
$$V^{\otimes N} = \bigotimes_{\lambda \vdash N} L_\lambda \otimes \pi_\lambda$$
of $V^{\otimes N}$ into irreducible representations of $A \times B$; the indexing consists of partitions of N with at most n parts.

(c) As λ varies over partitions of at most n parts, then the π_λs are pairwise non-isomorphic irreducible representations of S_N.

(d) If $n \geq N$, then the π_λs exhaust all irreducible representations of S_N.

Remark 1.139. Note that $n = N$ implies that all permutations of N have at most n parts, so both (c) and (d) apply.

Remark 1.140. The existence of the decomposition in (b) should not be a surprise: for any group G and completely reducible representation Y of G, the natural map
$$\bigoplus_{W \in \mathrm{IrRep}(G)} W \otimes \operatorname{Hom}_G(W, Y) \to Y$$
is an isomorphism; indeed, to prove this, one simply decomposes Y and checks that the statement is true on irreducibles, where it follows from Schur's lemma. For (b), we see that we can simply let π_λ be the space $\operatorname{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$.

Definition 1.141 (Schur algebra)**.** The algebra A in Theorem 1.138 is the *Schur algebra*, denoted $\mathcal{S}_{n,N}$. The algebra B is then justly called the *centralizer algebra*, denoted $C_{n,N}$.

We are going to need a few lemmas. Our first goal is to show that A is the centralizer of B.

Lemma 1.142. Let U be a complex vector space. Then $S^N U$ is spanned by the pure tensors

$$\{x \otimes \cdots \otimes x : x \in U\}.$$

Proof. Any $u \in S^N U$ has finitely many terms, so by taking the finite number of vectors contained therein, we may reduce to the finite-dimensional case. Now, the span of our pure tensors forms a nonzero subrepresentation of $S^N U$ (for the group $\mathrm{GL}(U)$), but this representation is already known to be irreducible by Example 1.128, so the span must cover everything. ■

Lemma 1.143. Let R be an algebra over $\mathbb{C}$, and let $S^N R$ be the Nth symmetric power, which we note is also an algebra. Now, define $\Delta\colon R \to S^N R$ by

$$\Delta(x) \coloneqq (x \otimes 1 \otimes \cdots \otimes 1) + (1 \otimes x \otimes \cdots \otimes 1) + \cdots + (1 \otimes 1 \otimes \cdots \otimes x).$$

Then R is generated as an algebra by elements of the form $\Delta(x)$.

Proof. By Newton's identities, there is a polynomial P_N (with rational coefficients) for which

$$z_1 \cdots z_N = P_N\left(\sum_{i=1}^N z_i^1, \ldots, \sum_{i=1}^N z_i^N\right).$$

(This is an identity that takes place in $\mathbb{Q}[z_1, \ldots, z_N]$.) It follows that

$$x \otimes \cdots \otimes x = P_N\left(\Delta(x), \ldots, \Delta(x^N)\right)$$

for any $x \in R$, so we may conclude by Lemma 1.142. ■

Lemma 1.144. In the context of Theorem 1.138, the algebra A is the centralizer of B.

Proof. Note that certainly A and B commute with each other with no effort because B only permutes the factors of V. On the other hand, note that the centralizer Z_B of B is

$$\begin{aligned} Z_B &= \operatorname{End}\left(V^{\otimes N}\right)^{S_N} \\ &\overset{*}{=} \left(\operatorname{End}_{\mathbb{C}}(V)^{\otimes N}\right)^{S_N} \\ &= S^N \operatorname{End}_{\mathbb{C}}(V). \end{aligned}$$

Here, $\overset{*}{=}$ holds because there is a natural map going upwards, and it can be seen to be an isomorphism on a basis. Thus, by Lemma 1.143, it is enough to show that the elements $\Delta(\varphi)$ are in A for each $\varphi \in \operatorname{End}_{\mathbb{C}}(V)$, which is true by definition of A. ■

The rest of Schur–Weyl duality follows from the following piece of algebra.

Lemma 1.145 (Double centralizer)**.** Fix a finite-dimensional vector space V over any field k, and choose two algebras $A, B \subseteq \operatorname{End}_k V$. Suppose that B is a sum of matrix algebras, and A is the centralizer of B.

(a) The algebra A is isomorphic to a sum of matrix algebras.

(b) The algebra B is the centralizer of A.

(c) There is a decomposition
$$V = \bigoplus_{i\in I}(X_i \otimes Y_i),$$
where the lists $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are exactly lists of irreducible representations of A and B, respectively.

Proof. Start by letting $\{Y_i\}_{i\in I}$ be the list of irreducible representations of B so that we have a decomposition
$$V = \bigoplus_{i\in I}(X_i \otimes Y_i),$$
where $X_i := \operatorname{Hom}_B(Y_i, V)$. Because B embeds in $\operatorname{End}_k V$, we see that Y_i does admit an embedding into V: note that B is a finite-dimensional semisimple algebra, so one can see from the structure theory that any faithful representation contains every irreducible representation. Now, A is the centralizer of B, which is $\operatorname{End}_B(V)$, but this is
$$\operatorname{End}_B(V) = \bigoplus_{i\in I}\operatorname{End}_k(W_i)$$
because the V_is cannot map to distinct other representations. Then (a) is immediate, and (b) and (c) follow by a computation reversing everything in sight to compute decompositions over A. Namely, one will find that $B = \bigoplus_{i\in I}\operatorname{End}_k Y_i$. ■

Proof of Theorem 1.138. Note that B is a quotient of $\mathbb{C}[S_N]$ and is therefore a sum of matrix algebras by the representation theory of finite groups. We now apply Lemma 1.145, which we may do by Lemma 1.144. The decomposition in (b) follows immediately because the irreducible representations of A are labeled by partitions λ of at most n parts, and we see that (c) also immediately follows.

It remains to show (d). It is enough to show that the projection $\mathbb{C}[S_N] \to B$ is injective so that irreducible representations of B are irreducible representations of S_N. Well, let $\{e_1, \ldots, e_n\}$ be a basis of V, and then we see that $\mathbb{C}[S_N]$ acts faithfully on $e_1 \otimes \cdots \otimes e_N$ because the S_N-action produces linearly independent vectors! Thus, the map $\mathbb{C}[S_N] \to \operatorname{End}_{\mathbb{C}} V^{\otimes N}$ is injective, and we are done. ■

Remark 1.146. Theorem 1.138 provides a parameterization of irreducible representations of S_N by partitions of N for every $n \geq N$. This parameterization does not depend on the choice of n, which one can see via the natural diagonal embedding $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$. This is not too interesting, so we will not write it out.

1.3.3 Schur Functors

We would now like to describe the representations L_λ of $\mathrm{GL}_n(\mathbb{C})$ more explicitly.

Definition 1.147 (Schur functor)**.** Fix a partition λ of a positive integer N. Then we define the *Schur functor* by
$$S^\lambda := \operatorname{Hom}_{S_N}\left(\pi_\lambda, (-)^{\otimes N}\right).$$

Example 1.148. If $\lambda = (N)$, then $L_\lambda = S^N V$, so π_λ is the trivial representation, so $S^\lambda V = S^N V$.

Example 1.149. If $\lambda = (1, \ldots, 1)$, then $L_\lambda = \wedge^N V$ (which is the determinant representation when $\dim V = N$), so π_λ is the sign representation, so $S^\lambda V = \wedge^N V$.

Remark 1.150. In general, Theorem 1.138 has given us a decomposition

$$V^{\otimes N} = \bigoplus_{\lambda \vdash N} S^\lambda V \otimes \pi_\lambda.$$

Example 1.151. We have a decomposition

$$V^{\otimes 2} = \left(S^2 V \otimes \mathbb{C}_{\mathrm{triv}}\right) \oplus \left(\wedge^2 V \otimes \mathbb{C}_{\mathrm{sgn}}\right).$$

Note that we have silently applied Examples 1.148 and 1.149.

Example 1.152. We have a decomposition

$$V^{\otimes 3} = \left(S^3 V \otimes \mathbb{C}_{\mathrm{triv}}\right) \oplus \left(\wedge^3 V \otimes \mathbb{C}_{\mathrm{sgn}}\right) \oplus \left(S^{(2,1)} V \otimes \pi_{(2,1)}\right).$$

By referencing a character table of S_3, we see that $\pi_{(2,1)}$ must be the standard representation of S_3, which is its action on the trace-zero hyperplane in $\mathbb{C}^3$. For example, if we restrict to the group $\mathrm{GL}_3(\mathbb{C})$, then we have a decomposition

$$V^{\otimes 3} = S^3 V \oplus \wedge^3 V \oplus S^{(2,1)} V^{\oplus 2}.$$

On the other hand, the left-hand side is $V^{\otimes 2} \otimes V = \left(S^2 V \otimes V\right) \oplus \left(\wedge^2 V \otimes V\right)$. The former factor has a copy of $S^3 V$, and the latter factor has a copy of $\wedge^3 V$; none of these inclusions are equalities, so it follows that each has one copy of $S^{(2,1)} V$. This allows us to describe $S^{(2,1)} V \subseteq S^2 V \otimes V$ as the subset whose total symmetrization is zero. (There is a similar description of $S^{(2,1)} V$ as embedded in $\wedge^2 V \otimes V$.)

While it is a little annoying to describe the $S^\lambda V$s explicitly (though Example 1.152 gives a method), there is a formula for their dimension.

Proposition 1.153. Fix $V \coloneqq \mathbb{C}^n$. Then

$$\dim S^\lambda V = \prod_{1 \le i < j \le n} \frac{(\lambda_i - \lambda_j) + (j - i)}{j - i}.$$

Proof. Recall that $\rho = (n-1, n-2, \ldots, 0)$ because the positive roots are $e_i - e_j$ where $i < j$. Thus, we see that $(\rho, e_i - e_j) = (j - i)$, so $(\lambda + \rho, e_i - e_j) = (\lambda_i - \lambda_j) + (j - i)$, and Theorem 1.121 gives the result. ■

We take a moment to give some simplifications. If λ has only k parts, then we could split the product into

$$\prod_{1 \le i < j \le k} \frac{(\lambda_i - \lambda_j) + (j - i)}{j - i} \cdot \prod_{1 \le i \le k < j \le n} \frac{\lambda_i + (j - i)}{j - i}.$$

The latter product now telescopes after λ_i terms, allowing us to rewrite this as

$$\prod_{1 \le i < j \le k} \frac{(\lambda_i - \lambda_j) + (j - i)}{j - i} \cdot \prod_{i=1}^{k} \frac{(n + 1 - i) \cdots (n + \lambda_i - i)}{(k + 1 - i) \cdots (k + \lambda_i - i}.$$

For example, it follows that the dimension is a polynomial P_λ with rational coefficients, whose roots are integers contained in $[1 - \lambda, \ldots, k - 1]$.

1.4 February 11

Today we discuss the fundamental theorem of invariant theory.

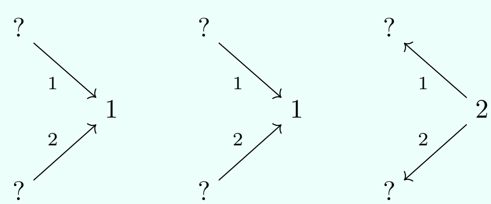
1.4.1 Finding Invariant Polynomials

Here is the basic problem: suppose that we have a finite-dimensional complex vector space V, and we are given various tensors $T_i \in V^{\otimes m_i} \otimes (V^*)^{\otimes n_i}$. Then we would like to find the invariant polynomial functions of $T_1, \ldots, T_k$, where "invariant" means that they are invariant under the ambient action of $\mathrm{GL}(V)$.

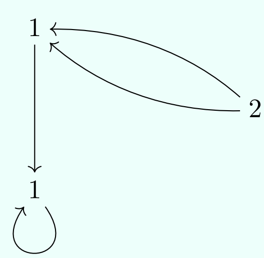
Suppose for concreteness that we are hunting for homogeneous polynomial functions $F(T_1, \ldots, T_k)$ which is homogeneous of degree d_i in the variable T_i. Here is one method of construction.

1. Set down some vertices i, which we equip with m_i inward arrows and n_i outward arrows, which we number.
2. We then draw d_i such vertices of color i, and we connect only those vertices which can preserve the numbering. This then gives a graph Γ, which exists if and only if the number of incoming and outgoing arrows is the same, meaning that
$$\sum_i d_i m_i = \sum_i d_i n_i.$$
(Certainly this condition is necessary by a degree argument. The converse is some graph theory.) If Γ does not exist, then notably there is no such polynomial because hitting $F(T_1, \ldots, T_k)$ with the scalar λ will not be invariant.
3. Now, once we have found the graph Γ, we construct the desired polynomial by contracting the tensors along the edges.

Example 1.154. Suppose we are given $T_1 \in V^{\otimes 2} \otimes V^*$ and $T_2 \in V^{*\otimes 2}$. This means that we start with the following.



One possible way to connect these arrows is as follows.



Accordingly, if we write $T_1 = (u_k^{ij})$ and $T_2 = (v_{pq})$, then our polynomial is
$$u_{i_1}^{i_1 j_1} u_{j_2}^{i_2 j_2} u_{j_1 j_2}.$$

Theorem 1.155. Fix everything as above. Then the constructed polynomials F_Γ span the algebra of invariant polynomials.

Proof. Let A be the full algebra of invariant polynomials, and let A_d be the subset which are homogeneous of degree $d = (d_1, \ldots, d_k)$. By taking a dual, a polynomial F can be viewed as an element of

$$\bigotimes_i \left(V^{*\otimes m_i} \otimes V^{\otimes n_i}\right) = V^{\otimes \sum d_i n_i} \otimes V^{*\otimes \sum_i d_i m_i}.$$

We are on the hunt for $\mathrm{GL}(V)$-invariants, so the space becomes

$$\mathrm{Hom}_{\mathrm{GL}(V)}\left(V^{\otimes \sum d_i m_i}, V^{\otimes \sum d_i n_i}\right).$$

Note that this vanishes for weight reasons if $\sum_i d_i m_i \neq \sum_i d_i n_i$. On the other hand, if the two sums equal some N, then Theorem 1.138 tells us that this space is

$$\mathrm{Hom}_{\mathrm{GL}(N)}\left(V^{\otimes N}, V^{\otimes N}\right) = \mathbb{C}[S_N]/I$$

for some ideal I. Accordingly, $A_{d_1,\ldots,d_k}$ is spanned by some polynomials which correspond to permutations $\sigma \in S_N$. These permutations σ amount to the data of a graph Γ because we are basically matching where certain edges go. Explicitly, we can number all the ingoing and outgoing edges "at the vertex" and then use the permutation σ to tell us which ingoing edge should match with which outgoing edge. ■

Remark 1.156. Given N, if $\dim V$ is large enough (namely, $\dim V \geq N$), then it turns out that the polynomials F_Γ are equal if and only if the graphs Γ are isomorphic. (Indeed, the ideal I vanishes when $\dim V \geq N$ by Theorem 1.138, so different permutations σ will give linearly independent polynomials. One then needs to pass to graphs.) Here, a graph isomorphism is required to preserve the bi-numbering of the edges (introduced in the proof) and the labels of the vertices.

Example 1.157. Suppose $T_1, \ldots, T_k$ are all in $V \otimes V^*$. Then all vertices look like $\to 1 \to$, so all the labels do not matter. By tracking the possible graphs, one can see that the algebra of invariant polynomials is generated by traces T_w of cyclic words w in the $T_\bullet$s. For example, the two-cycle $1 \to 2$ produces a factor of $\operatorname{tr}(T_1T_2)$, and the three-cycle $1 \to 2 \to 3$ produces a factor of $\operatorname{tr}(T_1T_2T_3)$. As a further specialization, note with just $k = 1$, we recover the fact that the algebra of functions on $V \otimes V^* \cong M_{\dim V}(\mathbb{C})$ which are invariant for the $\mathrm{GL}(V)$-action is generated by the trace.

Remark 1.158. Continuing the above example, it further turns out that these generators are "asymptotically algebraically independent," in the sense that any polynomial P which vanishes on all the T_ws for all possible dimensions of V must have $P = 0$. Indeed, P has only finitely many terms in the spanning set of Theorem 1.155, and these terms become linearly independent as $\dim V$ goes to infinity (namely, $\dim V$ larger than the total degree of $P(T_{w_1}, \ldots, T_{w_k})$ will suffice). Thus, $P = 0$ is forced!

Corollary 1.159. There is no polynomial identity P that holds for all matrices of all size. In other words, choose a polynomial $P \in \mathbb{C}\langle X_1, \ldots, X_r\rangle$ in the free non-commutative polynomial ring. If we have $P(A_1, \ldots, A_r) = 0$ for all choices of square matrices $A_1, \ldots, A_r$ of given dimension, then $P = 0$.

Proof. If $P(A_1, \ldots, A_r) = 0$, then note that

$$\operatorname{tr} P(A_1, \ldots, A_r)A_{r+1} = 0$$

for any other choice of matrix A_{r+1}. Remark 1.158 now implies that P must vanish! ■

Remark 1.160. If one fixes the size of matrices, then there are in fact many polynomial relations. For example, all 1×1 matrices satisfy the relation $AB - BA = 0$. As another example, for 2×2 matrices, $(AB - BA)^2$ is a scalar matrix, so $(AB - BA)^2 C - C(AB - BA)^2 = 0$. In general, for any n and commutative ring R, then any $X_1, \ldots, X_{2n} \in M_n(R)$ satisfies

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(2n)} = 0.$$

1.4.2 Schur Polynomials

We now introduce some Schur polynomials.

Definition 1.161 (Schur polynomial)**.** Let λ be a partition of n, and let s_λ be the trace of $\operatorname{diag}(x_1, \ldots, x_n) \in \mathrm{GL}_n(\mathbb{C})$ acting on the irreducible representation L_λ. Then s_λ is a *Schur polynomial*.

Remark 1.162. By Theorem 1.116, we find that

$$s_\lambda(x_1, \ldots, x_n) = \frac{\displaystyle\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)}^{\lambda_1 + n - 1} \cdots x_{\sigma(n)}^{\lambda_n}}{\displaystyle\prod_{i<j} (x_i - x_j)}.$$

(To make the formula look correct, one should multiply the numerator and denominator of Theorem 1.116 by e^ρ, which makes the denominator into $\prod_{\alpha \in \Phi_+} (e^{\alpha/2} - e^{-\alpha/2})$.) Note that the numerator may be written as the Vandermonde determinant $\det \left[x_i^{\lambda_i + n - j}\right]_{ij}$.

Remark 1.163. One can consider the Schur polynomials as polynomials in infinitely many variables by imagining that $s_\lambda(x_1, \ldots, x_n)$ is a polynomial $s_\lambda(x_1, \ldots, x_n, 0, \ldots, 0)$.

Example 1.164. The Schur polynomial s_λ for $\lambda = (N)$ corresponds to the representation $L_\lambda = S^N V$. Thus, by expanding $\operatorname{diag}(x_1, \ldots, x_n)$ on a basis $e_{i_1} \cdots e_{i_N}$ of V, we see that

$$s_\lambda(x_1, \ldots, x_n) = \sum_{i_1 + \cdots + i_N = n} x_1^{i_1} \cdots x_N^{i_N}.$$

This is a complete symmetric function. By rewriting a partition, we can also write this as

$$\sum_{1 \le j_1 \le \cdots \le j_N \le n} x_{j_1} \cdots x_{j_N}.$$

Example 1.165. The Schur polynomial s_λ for $\lambda = (1, \ldots, 1)$ corresponds to the representation $L_\lambda = \wedge^N V$. Computing the trace on a basis $e_{i_1} \wedge \cdots \wedge e_{i_N}$ of V recovers the elementary symmetric polynomial

$$s_\lambda(x_1, \ldots, x_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_N \le n} x_{j_1} \cdots x_{j_N}.$$

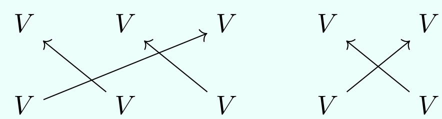
We now use these Schur polynomials to compute the characters of S_N.

Proposition 1.166. Fix a representation π_λ of S_N, where λ is a partition of N with at most n parts. Let $\sigma \in S_N$ be a permutation with m_i cycles of length i, and set $\chi_\lambda(\sigma) := \operatorname{tr}(\sigma;\pi_\lambda)$. Then $\chi_\lambda(\sigma)$ is the coefficient of the monomial $x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n}$ in the polynomial

$$\prod_{i<j}(x_i-x_j)\prod_k\left(x_1^i+\cdots+x_n^i\right)^{m_i}.$$

Proof. The idea is to compute the trace of the operator $g\otimes\sigma := g^{\otimes N}\circ\sigma$ on $V^{\otimes N}$, where $V := \mathbb{C}^n$. We also may as well suppose that $g = \operatorname{diag}(x_1,\ldots,x_n)$. Thus, σ permutes the Vs, and g then acts diagonally on a corresponding basis.

Example 1.167. The permutation $\sigma = (123)(45)$ will move around the Vs as in the following diagram.



We now compute our trace two ways.

- One way to compute the trace of the action of $g\otimes\sigma$ is to decompose σ into cycles via its action on $V^{\otimes N}$. As such, we may assume that σ is a single cycle of length m acting on $V^{\otimes m}$. Then σ permutes the standard basis of pure tensors $e_{i_1}\otimes\cdots\otimes e_{i_m}$ along cycles (while $g^{\otimes m}$ acts diagonally), so the only way for such a basis vector to produce a nontrivial contribution to the trace would be if σ fixes $e_{i_1}\otimes\cdots\otimes e_{i_m}$, which means $i_1=\cdots=i_m$. It follows that the trace of $g\otimes\sigma$ is $x_1^m+\cdots+x_n^m$.

  Now returning to the case of general σ, we multiply through the cycle contributions to find

  $$\operatorname{tr}\left(g\otimes\sigma;V^{\otimes N}\right)=\prod_i\left(x_1^i+\cdots+x_n^i\right)^{m_i}.$$

- Alternatively, we use Schur–Weyl duality. Theorem 1.138 tells us that we have a decomposition

  $$V^{\otimes N}=\bigoplus_{\lambda\vdash N}L_\lambda\otimes\pi_\lambda,$$

  where g only acts on L_λ, and σ only acts on π_λ. Thus,

  $$\operatorname{tr}\left(g\otimes\sigma;V^{\otimes N}\right)=\sum_{\lambda\vdash N}s_\lambda(x_1,\ldots,x_n)\chi_\lambda(\sigma).$$

Comparing the two formulae gives

$$\prod_i\left(x_1^i+\cdots+x_n^i\right)^{m_i}=\sum_{\lambda\vdash N}s_\lambda(x_1,\ldots,x_n)\chi_\lambda(\sigma).$$

We now multiply through by the Weyl denominator, which gives

$$\prod_{i<j}(x_i-x_j)\prod_i\left(x_1^i+\cdots+x_n^i\right)^{m_i}=\sum_{\lambda\vdash N}\left(\sum_{\tau\in S_n}\operatorname{sgn}(\tau)x_{\tau(1)}^{\lambda_1+n-1}\cdots x_{\tau(n)}^{\lambda_n}\chi_\lambda(\sigma)\right).$$

Taking coefficients completes the proof. In particular, we note that the monomial $x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n}$ is featured only once on the right-hand side: if one has $x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n}=x_{\tau(1)}^{\mu_1+n-1}\cdots x_{\tau(n)}^{\mu_n}$, then comparing the highest degrees forces $\lambda_1+n-1=\mu_1+n-1$ and $\tau(1)=1$, so one can remove x_1 from both sides and induct. ■

1.4.3 Howe Duality

Here is another application of Schur–Weyl duality.

Proposition 1.168 (Howe duality)**.** Fix complex vector spaces V and W. Then

$$S^n(V \otimes W) = \bigoplus_{\lambda \vdash n} S^\lambda V \otimes S^\lambda W.$$

Proof. Recall that $(V \otimes W)^{\otimes n} = V^{\otimes n} \otimes W^{\otimes n}$. But then Theorem 1.138 tells us that

$$(V \otimes W)^{\otimes n} = \left(\bigoplus_{\lambda \vdash n} S^\lambda V \otimes \pi_\lambda \right) \otimes \left(\bigoplus_{\mu \vdash n} S^\mu W \otimes \pi_\mu \right).$$

Now, we may take S_n-invariants on both sides to see that

$$S^n(V \otimes W) = \bigoplus_{\lambda, \mu \vdash n} S^\lambda V \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n}.$$

It remains to get rid of the factor of $(\pi_\lambda \otimes \pi_\mu)^{S_n}$. Well, χ_λ has integral values by Proposition 1.166, so it is self-dual, so $(\pi_\lambda \otimes \pi_\mu)^{S_n} = \operatorname{Hom}_{S_n}(\pi_\lambda, \pi_\mu)$, which we see vanishes if $\lambda \neq \mu$ by Schur's lemma (and is $\mathbb{C}$ otherwise). The result follows. ■

Corollary 1.169. Let A_V and A_W be the subalgebras of $\operatorname{End}_{\mathbb{C}}(S^n(V \otimes W))$ generated by $\operatorname{End}(V)$ and $\operatorname{End}(W)$. Then A_V and A_W are centralizers of each other.

Proof. By Proposition 1.168, we see that $S^n(V \otimes W)$ decomposes into a tensor product of irreducibles $S^\lambda V \otimes S^\lambda W$ for $A_V \times A_W$, so it is enough to check the result for these irreducibles. For example, any endomorphism of $S^\lambda V \otimes S^\lambda W$ which preserves A_W must then be a scalar on $S^\lambda W$ and therefore acts only on $S^\lambda V$, so it comes from A_V. ■

We can give a purely combinatorial application of Howe duality.

Proposition 1.170 (Cauchy identity)**.** One has

$$\sum_\lambda s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_m) z^{|\lambda|} = \prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - z x_i y_j}.$$

We will want the following lemma.

Lemma 1.171 (Molien)**.** Let $A \colon V \to V$ be an operator on a finite-dimensional vector space. Then

$$\sum_{r \geq 0} \operatorname{tr}(S^r A; S^r V) T^r = \frac{1}{\det(1 - zA)}.$$

Proof. It is enough to show this over $\mathbb{C}$ because we can reduce this to an equality of some polynomials, which can be checked on complex points. Then we are trying to check an equality of two continuous functions on the complex space $\operatorname{GL}(V)$, so it is enough to check it on the dense subset of diagonalizable matrices $A = \operatorname{diag}(x_1, \ldots, x_n)$. Then the trace of $S^r A$ is simply

$$\sum_{i_1 + \cdots + i_r = n} x_1^{i_1} \cdots x_n^{i_n},$$

so our left-hand side is

$$\sum_{i_1,\ldots,i_n\geq 0} x_1^{i_1}\cdots x_n^{i_n} z^{i_1+\cdots+i_n}.$$

However, we can rearrange this to

$$\left(\sum_{i_1\geq 0}(x_1z)^{i_1}\right)\cdots\left(\sum_{i_n\geq 0}(x_nz)^{i_n}\right)=\frac{1}{1-x_1z}\cdots\frac{1}{1-x_nz},$$

which is exactly $1/\det(1-zA)$. ■

Proof of Proposition 1.170. We will compute the $\sum_{N\geq 0}\operatorname{tr} S^N(x\otimes y)z^N$, where $x := \operatorname{diag}(x_1,\ldots,x_n)$ and $y := \operatorname{diag}(y_1,\ldots,y_m)$ are in $\operatorname{GL}(V)$ and $\operatorname{GL}(W)$, respectively.

- By Lemma 1.171, we see that

  $$\sum_{N\geq 0}\operatorname{tr} S^N(x\otimes y)z^N=\frac{1}{\det(1-z(x\otimes y))},$$

  which is

  $$\prod_{i=1}^n\prod_{j=1}^m\frac{1}{1-zx_iy_j}.$$

- On the other hand, by Proposition 1.168, we find that

  $$\operatorname{tr} S^N(x\otimes y)=\sum_{\lambda\vdash N}s_\lambda(x)s_\lambda(y)z^{|\lambda|}.$$

Setting these equal completes the proof. ■

1.5 February 17

Today we discuss miniscule weights.

1.5.1 Miniscule Weights

Here is our definition.

Definition 1.172 (miniscule). Fix a semisimple Lie algebra $\mathfrak{g}$. A dominant integral weight ω of $\mathfrak{g}$ is *miniscule* if and only if $(\omega,\beta)\leq 1$ for all coroots β.

Remark 1.173. It is enough to check the condition $(\omega,\beta)\leq 1$ for positive coroots.

Example 1.174. The weight 0 is always miniscule.

Example 1.175. For $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{C})$, all the fundamental weights ω_i are miniscule. Indeed, the fundamental weights are

$$\omega_i=(\underbrace{1,\ldots,1}_{i},0,\ldots,0).$$

But now $(\omega_i,e_j-e_k)\in\{0,+1\}$ for each $j<k$, depending on the relative placement of i between j and k. Indeed, it is zero if j and k are both below or both above i, and it is one if i is between them.

Remark 1.176. Any nonzero miniscule weight ω is fundamental; we may assume that $\mathfrak{g}$ is simple. Let $\{\alpha_1, \ldots, \alpha_r\}$ be a collection of simple roots, and let θ^* be the maximal coroot, which takes the form $\sum_i m_i\alpha_i^\vee$ where each m_i is positive. (One way to find this (co)weight is as the highest weight of the (co)adjoint representation. It then turns out that this θ^* is a coroot—the weights of the adjoint representation are all roots—for which $\theta^* - \alpha_i^\vee$ is a positive coroot always.) Then

$$(\omega, \theta^*) = \sum_i m_i(\omega, \alpha_i^\vee).$$

Thus, there is at most one index i with $(\omega, \alpha_i^\vee) = 1$, which means that $\omega = 0$ or ω equals one of the ω_is (because the fundamental weights form a dual basis).

It is helpful to characterize our miniscule weights.

Lemma 1.177. Fix a semisimple Lie algebra $\mathfrak{g}$, and choose a fundamental weight ω_i. Then ω_i is miniscule if and only if $m_i = 1$, where m_i comes from the expansion $\theta^* = \sum_j m_j\alpha_j^\vee$ of the maximal coroot.

Proof. Certainly if ω_i is miniscule, then $(\omega, \theta^*) \le 1$ implies that $m_i \le 1$ and so $m_i = 1$. Conversely, if $m_i = 1$, then for any positive coroot

$$\beta = \sum_j n_j\alpha_j^\vee,$$

we see that we must have $n_j \le m_j$ for each j, so $(\omega_i, \beta) \le 1$ follows. ■

Example 1.178. A direct calculation of the maximal coroot shows that the Lie algebras with exceptional types E_2, F_4, and E_8 admit no miniscule weights.

Lemma 1.179. Fix a semisimple Lie algebra $\mathfrak{g}$. Suppose that we have $\omega \in Q$ for which $(\omega, \beta) \le 1$ for all coroots β. Then $\omega = 0$.

Proof. Because ω is in the root lattice, we may write

$$\omega = \sum_i k_i\alpha_i$$

for some integers $k_\bullet$. For the sake of contradiction, assume that ω is nonzero; we may assume that ω is a minimal counterexample, in the sense that it minimizes $\sum_i |k_i|$. Because the inner product is positive-definite, we know that (ω, ω) is positive, so

$$\sum_i k_i(\omega, \alpha_i) > 0.$$

Thus, one of the summands is positive, so we may find an index j such that k_j and (ω, α_j) have the same sign; by possibly replacing ω with $-\omega$, we may assume that $k_j > 0$ and $(\omega, \alpha_j) > 0$. It follows that $(\omega, \alpha_j^\vee) > 0$, so $(\omega, \alpha_j^\vee) = 1$ by hypothesis.

Thus, $s_i\omega = \omega - \alpha_i$, so $s_i\omega$ is another vector in the root lattice, and we can see that $s_i\omega$ also satisfies the hypothesis. Indeed, the hypothesis $(\omega, \beta) \le 1$ and being in the root lattice is invariant under the Weyl group because the Weyl group preserves the set of coroots and the inner product. As such, we have arrived at our contradiction because the sum of the coordinates of $s_i\omega$ is strictly smaller than the sum of the coordinates of ω. ■

1.5.2 Miniscule Representations

Miniscule weights are interesting to us because their corresponding representations are easy to understand.

Proposition 1.180. Fix a semisimple Lie algebra $\mathfrak{g}$, and choose a dominant integral weight ω. Then the following are equivalent.

(a) The weight ω is miniscule.

(b) All weights of L_ω are contained in the orbit $W\omega$.

(c) If λ is a dominant integral weight such that $\omega - \lambda \in Q_+$, then $\lambda = \omega$.

Proof. We show our implications in sequence.

- We show (a) implies (c), which we will do by induction on the rank of $\mathfrak{g}$. (There is nothing to do if the rank is zero.) We proceed in steps.
  1. If $\omega = 0$, then $\lambda \in -Q_+$, so $(\lambda, \rho) \le 0$; but because λ is dominant, if it is nonzero, then $(\lambda, \rho) > 0$. We may now suppose that ω is nonzero.
  2. Then ω is a fundamental weight ω_i (by Remark 1.176). Now, write $\omega - \lambda = \sum_j n_j \alpha_j$, where the n_js are nonnegative. If n_j vanishes for some j with $j \ne i$, then we may pass to a smaller Lie algebra (by deleting the jth vertex of the Dynkin diagram), so we are done by the induction.
     Thus, we may assume that $n_j > 0$ for each $j \ne i$.
  3. Now, for any positive coroot β, we know that
     $$(\omega_i - \lambda, \beta) = (\omega_i, \beta) - (\lambda, \beta) \le (\omega_i, \beta) \le 1.$$
     Additionally, if $\alpha_i^\vee$ is not found in the expansion of β, then we must actually have $(\omega_i, \beta) \le 0$; for example, $(\omega_i - \lambda, \alpha_j^\vee) \le 0$ for all $j \ne i$.
  4. Now, observe that
     $$(\omega_i - \lambda, \omega_i - \lambda) = \sum_j n_j (\omega_i - \lambda, \alpha_i^\vee).$$
     If this is nonpositive, then $\omega_i - \lambda$ vanishes, and we are done. But each of the n_js are nonnegative for each $j \ne i$, and we just showed $(\omega_j - \lambda, \alpha_j^\vee) \le 0$ for $j \ne i$, so the only way to avoid nonpositivity is for $n_i > 0$ and $(\omega_i - \lambda_i, \alpha_i^\vee) > 0$.
     We are thus reduced to the case that $n_j > 0$ for all j, including $j = i$. We are now ready to conclude. Recall θ^* is a dominant coweight, so $(\omega_i - \lambda, \theta^*) \ge 1$ because we just showed that $(\omega_i - \lambda, \alpha_i^\vee) > 0$ for all i. (In a few more words, one can expand $\omega_i - \lambda = \sum_j n_j \alpha_j$ and $\theta^* = \sum_j p_j \omega_j^\vee$, where the n_js are positive and the p_js are nonnegative, with at least one nonzero.) On the other hand, $(\omega_i, \theta^*) \le 1$, so $(\lambda, \theta^*) \le 0$, so $\lambda = 0$, so $\omega_i \in Q_+$, which contradicts Lemma 1.179.
- We show that (c) implies (b). Let μ be a weight of L_ω. Then we can find $w \in W$ to push μ into the dominant chamber, meaning that $\lambda := w\mu$ is a dominant weight. But the construction of L_ω implies that $\omega - \lambda \in Q_+$, so $\omega = \lambda$ follows by (c).
- We show that (b) implies (a) by contraposition. Indeed, suppose that ω is not miniscule, which implies that there is a positive root α such that $(\omega, \alpha^\vee) > 1$. Unravelling $\alpha^\vee$, we see that $2(\omega, \alpha) > (\alpha, \alpha)$. Thus, $\omega - \alpha$ is a weight of L_ω, witnessed by the weight vector $f_\alpha v_\omega$, which we note is nonzero by the representation theory of $\mathfrak{sl}_{2,\alpha}$ (because $2(\omega, \alpha) > (\alpha, \alpha)$).
  We will be done as soon as we can check that $\omega - \alpha$ is not Weyl-conjugate to ω, but this is not hard: indeed,
  $$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega).$$

The above implications complete the proof. ∎

Corollary 1.181. Fix a miniscule weight ω of a semisimple Lie algebra $\mathfrak{g}$. Then

$$\operatorname{ch} L_\omega = \sum_{\gamma \in W\omega} e^\gamma.$$

Proof. This follows from part (b) of Proposition 1.180. Note that the multiplicities of these weight spaces must be 1 because the multiplicity of the highest weight space needs to be 1, and the Weyl group permutes the weight spaces $L[\gamma]$ as γ varies over $W\omega$ (by the definition of the Weyl group action). ■

Proposition 1.182. Fix a dominant integral weight ω of a semisimple Lie algebra $\mathfrak{g}$. Then ω is miniscule if and only if the restriction of L_ω to any simple $\mathfrak{sl}_{2,\alpha} \subseteq \mathfrak{g}$ is a direct sum of one-dimensional and two-dimensional representations for each α.

Proof. In the forward direction, we may suppose that ω is miniscule. Then any weight vector $v \in L_\omega$ which is a highest weight vector for some $\mathfrak{sl}_{2,\alpha}$ has $h_\alpha v = (\omega, \alpha^\vee)v$. But then $|(\omega, \alpha^\vee)| \le 1$, so the result follows.

In the backward direction, if ω is not miniscule, then we may find a positive root α such that $(\omega, \alpha^\vee) \ge 2$. Then the highest weight vector $v \in L_\omega$ admits $h_\alpha v = (\omega, \alpha^\vee)v$, so v generates a $\mathfrak{sl}_{2,\alpha}$-representation of dimension $(\omega, \alpha^\vee) + 1$. ■

1.5.3 Applications of Miniscule Representations

We will get significant mileage out of the following application of the Weyl character formula.

Corollary 1.183. Fix a miniscule weight ω of a semisimple Lie algebra $\mathfrak{g}$. Then for all dominant integral weights λ,

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma},$$

where $L_{\lambda+\gamma}$ vanishes (by convention) if and only if $\lambda + \gamma$ is not dominant.

Proof. By iteratively removing highest weight vectors, it is enough to check that the two representations have the same character. Well, by the Weyl character formula, we see that

$$\begin{aligned}
\operatorname{ch}(L_\omega \otimes L_\lambda) &= \operatorname{ch}(L_\omega)\operatorname{ch}(L_\lambda) \\
&= \sum_{\gamma \in W\omega} e^\gamma \cdot \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} \\
&= \frac{1}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} \sum_{\substack{w \in W \\ \gamma \in W\omega}} \operatorname{sgn}(w) e^{w(\lambda+\rho)+\gamma}.
\end{aligned}$$

We now replace γ with $w\gamma$, merely moving around some orbit, which gives

$$\operatorname{ch}(L_\omega \otimes L_\lambda) = \sum_{\gamma \in W\omega} \frac{1}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho+\gamma)}.$$

Each internal sum does look like $\operatorname{ch} L_{\lambda+\gamma}$, so we will be done as soon as we check that it vanishes if $\lambda+\gamma$ fails to be dominant. Well, suppose that $\lambda+\gamma$ is not dominant; then we can find $\alpha_i^\vee$ such that $(\lambda+\gamma, \alpha_i^\vee) < 0$. On the other hand, $(\gamma, \alpha_i^\vee) \ge -1$ because $\gamma \in W\omega$, and ω is miniscule, so $(\lambda+\gamma, \alpha_i^\vee) = -1$, so $(\lambda+\gamma+\rho, \alpha_i^\vee) = 0$. Thus, s_i fixes $\lambda+\gamma+\rho$, so the terms $\operatorname{sgn}(w)e^{w(\lambda+\rho+\gamma)}$ and $\operatorname{sgn}(s_i w)e^{w(\lambda+\rho+\gamma)}$ in our sum will cancel each other out. ■

Let's apply this to $\mathfrak{sl}_n(\mathbb{C})$.

Example 1.184. We work with $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Let V be the standard representation of $\mathrm{GL}_n(\mathbb{C})$, which is a miniscule representation by Example 1.175. The weights of V are the standard basis of the diagonal Cartan $\mathfrak{h} \subseteq \mathfrak{sl}_n(\mathbb{C})$. Then for any partition λ, we see that

$$V \otimes L_\lambda = \bigoplus_{i=1}^{n} L_{\lambda+e_i},$$

where $L_{\lambda+e_i}$ vanishes by convention if $\lambda + e_i$ fails to be a partition. In other words, $V \otimes L_\lambda$ consists of those L_μs, where the Young diagram for μ is obtained by adding a box (suitably) to the Young diagram for λ. For example,

$$V \otimes L_{(6,4,4,2,1)} = L_{(7,4,4,2,1)} \oplus L_{(6,5,4,2,1)} \oplus L_{(6,4,4,3,1)} \oplus L_{(6,4,4,2,2)} \oplus L_{(6,4,4,2,1,1)}.$$

Example 1.185. Continue in the setting of Example 1.184. Then $\wedge^i V$ is the miniscule representation attached to the fundamental weight ω_i (see Example 1.175). Thus, for any partition λ, we see that $\wedge^i V \otimes L_\lambda$ contains those permutations μ which are obtained by adding one to i different coordinates in the partition λ, in a way that ensures that μ is still a partition. Combinatorially, we are trying to add k different boxes to different rows to the Young diagram of λ in a way that ensures that we still have a Young diagram afterward.

We can now combine this discussion with Schur–Weyl duality.

Corollary 1.186. Fix some positive integer N.

(a) For any partition $\lambda \vdash N$, we have

$$\mathbb{C}[S_{N+1}] \otimes_{\mathbb{C}[S_N]} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu,$$

where μ varies over those Young diagrams obtained by suitably adding a box to the Young diagram of λ.

(b) For any partition $\mu \vdash N+1$, we have

$$\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \square} \pi_\lambda,$$

where λ varies over those Young diagrams obtained by suitably removing a box to the Young diagram of λ.

Proof. We quickly explain that (a) implies (b) by Frobenius reciprocity. (A similar argument also shows that (b) implies (a).) Indeed, by Frobenius reciprocity, some π_λ has multiplicity m in the restriction $\pi_\mu|_{S_N}$ if and only if π_μ has multiplicity m in the induction $\mathbb{C}[S_{N+1}] \otimes_{\mathbb{C}[S_N]} \pi_\lambda$. But (a) tells us exactly that this multiplicity vanishes except when μ is obtained by adding a single box to λ (in which case the multiplicity is one). Unwinding this produces the required decomposition of π_μ.

Thus, we will content ourselves with showing (a). Well, define $V := \mathbb{C}^n$, where n is large (e.g., larger than $N+1$). Frobenius reciprocity implies that

$$\mathrm{Hom}_{S_{N+1}}\left(\mathbb{C}[S_{N+1}] \otimes_{\mathbb{C}[S_N]} \pi_\lambda, V^{\otimes(N+1)}\right) = \mathrm{Hom}_{S_N}\left(\pi_\lambda, V \otimes V^{\otimes N}\right),$$

and this latter representation is simply $V \otimes S^\lambda V$ by definition of the Schur functor S^λ (and the fact that S_N

is acting trivially on V). On the other hand,

$$\operatorname{Hom}_{S_{N+1}}\left(\bigoplus_{\mu\in\lambda+\square}\pi_\mu, V^{\otimes(N+1)}\right)=\bigoplus_{\mu\in\lambda+\square}S^\mu V$$

by definition of the Schur functors. Now, Example 1.184 has shown that our two calculations have produced the same representation of $\mathrm{GL}(V)$, so Theorem 1.138 tells us that the corresponding representations of S_{N+1} must also be the same. ■

1.6 February 18

We continue providing some applications of miniscule weights.

1.6.1 The Conjugate Partition

Partitions come with a duality.

Definition 1.187 (conjugate)**.** Given a partition λ, we define the *conjugate partition* $\lambda^\dagger$ to have Young diagram which is the transpose of λ. Explicitly, the ith component of $\lambda^\dagger$ is the number of components of λ which are greater than or equal to i.

Example 1.188. We have that $(3,3,2,1)^\dagger=(4,3,2)$, which we can see as follows.

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \quad\Rightarrow\quad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \end{array}$$

Proposition 1.189. Fix a partition $\lambda\vdash N$. Then

$$\pi_\lambda\otimes\mathbb{C}_{\mathrm{sgn}}=\pi_{\lambda^\dagger}.$$

Proof. This is fairly tricky due to our construction of π_λ. We induct on N. With $N=1$, this has no content because all representations are trivial.

For the induction, we may suppose that $\nu\vdash(N+1)$, and remove a box of the Young diagram of ν to obtain some $\lambda\in\nu-\square$. Now, note that

$$(\mathbb{C}[S_{N+1}]\otimes_{\mathbb{C}[S_N]}\pi_\lambda)\otimes\mathbb{C}_{\mathrm{sgn}}=\mathbb{C}[S_{N+1}]\otimes_{\mathbb{C}[S_N]}(\pi_\lambda\otimes\mathbb{C}_{\mathrm{sgn}}),$$

which by the induction is $\mathbb{C}[S_{N+1}]\otimes_{\mathbb{C}[S_N]}\pi_\lambda^\dagger$. But Corollary 1.186 then explains that this representation is

$$\bigoplus_{\eta\in\lambda^\dagger+\square}\pi_\eta.$$

It follows that $\pi_\nu\otimes\mathbb{C}_{\mathrm{sgn}}=\pi_{\overline{\nu}}$ for some $\overline{\nu}\in\lambda^\dagger+\square$.

It remains to identify which partition in $\lambda^\dagger+\square$. For this, we recall from the homework that there is a Jucys–Murphy element $c:=\sum_{i<j}(ij)\in\mathbb{C}[S_{N+1}]$ which is central and acts on π_ν by the scalar

$$c(\nu):=\sum_{(i,j)\in\nu}(j-i).$$

Accordingly, it acts on $c(\overline{\nu})$ by $-c(\nu)$ (because $-\otimes\mathbb{C}_{\mathrm{sgn}}$ sends (ij) to $-(ij)$). Similarly, c acts on $\pi_{\nu^\dagger}$ by $c(\nu^\dagger)=-c(\nu)$, which we can see by a direct calculation (namely, flipping the Young diagram and thus flipping (i,j) to (j,i)). Thus, $c(\overline{\nu})=c(\nu^\dagger)$.

We are thus done because the contents $c(\mu)$ are all different for all $\mu\in\lambda^\dagger+\square$: indeed, each place to add a square to the Young diagram of $\lambda^\dagger$ adds a different amount to the content. To see this, note that the places to add a box go from the top-right to the bottom-left, and any such move downwards or leftwards strictly decreases the content in the box. ■

Corollary 1.190 (Skew Howe duality)**.** Fix finite-dimensional complex representations V and W. Then there is a decomposition into $\mathrm{GL}(V)\times\mathrm{GL}(W)$-representations

$$\lambda^N(V\otimes W)=\bigoplus_{\lambda\vdash N}S^\lambda V\otimes S^{\lambda^\dagger}W.$$

Proof. This is on the homework. The idea is to argue as in Proposition 1.168. ■

1.6.2 More on Miniscule Weights in Classical Types

We continue with a semisimple Lie algebra $\mathfrak{g}$ with root lattice Q and weight lattice P. We also fix a collection (α_i) of simple roots so that we have our Cartan matrix $A=((\alpha_i,\alpha_j^\vee))_{ij}$. Notably,

$$\alpha_i=\sum_j A_{ij}\omega_j,$$

so $P=AQ$, so $|P/Q|=\det A$.

Proposition 1.191. Fix a semisimple Lie algebra $\mathfrak{g}$. Then every coset in P/Q admits a unique miniscule weight.

Proof. Let's start with existence. Fix a coset $C\subseteq P/Q$, and choose some $\omega\in C\cap P_+$ (which exists by adding enough fundamental weights to any representative of C). We may even minimize ω so that $(\omega,\rho^\vee)$ is minimal, where we recall $\rho^\vee:=\frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha^\vee$.

We claim that ω is miniscule. Well, let λ be any dominant weight present in the weight decomposition of L_ω. We want to show that $\omega=\lambda$, which completes the proof by Proposition 1.180. Because λ is a weight in L_ω, it follows that $\lambda-\omega\in Q_+$, so $\lambda\in C\cap P_+$. Thus, $(\lambda,\rho^\vee)\geq(\omega,\rho^\vee)$ by minimality, so

$$(\lambda-\omega,\rho^\vee)\geq 0.$$

On the other hand, $\omega-\lambda\in Q_+$, so we may expand $\omega-\lambda=\sum_i m_i\alpha_i$ for nonnegative m_is, so $(\omega-\lambda,\rho^\vee)\leq 0$. Thus, $(\omega-\lambda,\rho^\vee)$ actually vanishes, and all the m_is are required to vanish, so $\omega=\lambda$.

We now turn to uniqueness. Suppose we have two distinct miniscule weights ω_1 and ω_2 in our coset C. Then $\omega_1-\omega_2\in Q$ is nonzero, so Lemma 1.179 grants us a coroot β with $(\omega_1-\omega_2,\beta)\geq 2$. Thus, $(\omega_1,\beta)=1$ and $(\omega_2,\beta)=-1$ because these weights are miniscule, which imply that β is positive and negative, respectively (by dominance). This is a contradiction, so we are done. ■

Remark 1.192. For example, it follows that the number of miniscule weights is $|P/Q|=\det A$.

1.6.3 Miniscule Representations

Let's write down some fundamental weights for our classical types. We will not write out all the calculations.

Example 1.193. We write down the fundamental weights for type C_n.

Proof. Here, $\mathfrak{g} = \mathfrak{sp}_{2n}$. The positive roots are given by

$$\{e_i \pm e_j : i < j\} \sqcup \{2e_i\}_i.$$

As such, our simple positive roots are given by $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = 2e_{2n}$. The coroots are the same, except $\alpha_n^\vee = e_n$. We can thus see that the fundamental weights are

$$\omega_i = (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0)$$

because $(\omega_i, \alpha_j^\vee) = 1_{i=j}$.

It follows that P/Q has two classes,[1] so there is a unique nonzero miniscule weight. We can see that L_{ω_1} is the standard representation of $\mathfrak{sp}_{2n}$, whose weights are given by $\{\pm e_i\}_i$ (indeed, choose the diagonal torus of $\mathfrak{sp}_{2n}$), which is exactly the orbit of the dominant weight e_1 under the Weyl group $S_n \ltimes \{\pm 1\}^n$. Thus, ω_1 is the unique nonzero miniscule weight by Proposition 1.180. ∎

Example 1.194. We write down the fundamental weights and miniscule representations for type B_n.

Proof. Here, $\mathfrak{g} = \mathfrak{so}_{2n+1}$. The positive roots are given by

$$\{e_i \pm e_j : i < j\} \sqcup \{e_i\}_i.$$

As such, our simple positive roots are given by $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{2n}$. The coroots are the same, except $\alpha_n^\vee = 2e_n$. We can thus see that the fundamental weights are

$$\omega_i = \begin{cases} (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0) & \text{if } i < n, \\ (1/2, \ldots, 1/2) & \text{if } i = n, \end{cases}$$

because $(\omega_i, \alpha_j^\vee) = 1_{i=j}$.

It follows that P/Q has two classes (the present situation is dual to type C_n), so there is a unique nonzero miniscule weight. This time, the miniscule weight is ω_n, which can be checked directly by computing the inner product with all the coroots (which have already been described). Thus, by Proposition 1.180 shows finds that L_{ω_n} has character with weights which are just signs applied to ω_n.

This is the "spin" representation S. We claim that S does not lift to a representation of $\mathrm{SO}_{2n+1}(\mathbb{C})$ but instead lifts to a representation of its universal cover $\mathrm{Spin}_{2n+1}(\mathbb{C})$. Indeed, weights of all representations of $\mathrm{SO}_{2n+1}(\mathbb{C})$ should be integral: using the bilinear form $\sum z_i z_{n+i} + z_{n+1}^2$, we see that any

$$g := \operatorname{diag}\left(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}, 1\right)$$

acts on a weight space $L[\lambda]$ by $a_1^{\lambda_1} \cdots a_n^{\lambda_n}$. Thus, if L comes from a representation of $\mathrm{SO}_{2n+1}(\mathbb{C})$, then the $\lambda_\bullet$s are required to be integers for the representation to be holomorphic. ∎

[1] The root lattice consists of the tuples in $\mathbb{Z}^n$ with even sum, but the weight lattice has everything in $\mathbb{Z}^n$.

Example 1.195. We write down the fundamental weights for type D_n.

Proof. Here, $\mathfrak{g} = \mathfrak{so}_{2n}$. The positive roots are given by

$$\{e_i \pm e_j : i < j\}.$$

As such, our simple positive roots are given by $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$. The coroots are the same. We can thus see that the fundamental weights are

$$\omega_i = \begin{cases} (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0) & \text{if } i \leq n-2, \\ (1/2, \ldots, 1/2, -1/2) & \text{if } i = n, \\ (1/2, \ldots, 1/2, 1/2) & \text{if } i = n, \end{cases}$$

because $(\omega_i, \alpha_j^\vee) = 1_{i=j}$.

It turns out that P/Q has up to four elements,[2] which can be computed from $\det A$. Thus, there are three nonzero miniscule weights, which are L_{ω_1} (which is the standard representation) and the two "spin representations" $S_+ := L_{\omega_{n-1}}$ and $S_- := L_{\omega_n}$. The fact that these are miniscule can be computed in the usual way by taking inner products with coroots.

Now, by taking a Weyl action, one can compute that the weights of $L_{\omega_{n-1}}$ (respectively, L_{ω_n}) are simple tuples of $\pm 1/2$ with an odd number (respectively, even number) of negative signs. Indeed, the Weyl group consists is the semidirect product of S_n acting on the kernel of $\det\colon \{\pm 1\}^n \to \{\pm 1\}$. As before, S_+ and S_- do not lift to $\mathrm{SO}_{2n}(\mathbb{C})$. ■

While we're here, let's say something about our exceptional types.

Example 1.196. For G_2, F_4, and E_8, one directly calculates that $\det A = 1$, so $P = Q$, so there are no nonzero miniscule weights.

Example 1.197. For E_6, one finds that $\det A = 3$, so there are two nonzero miniscule weights. On the Dynkin diagram, they are the weights ω_1 and ω_6 as follows.

$$\begin{array}{ccccccccc} \omega_1 & - & \bullet & - & \bullet & - & \bullet & - & \omega_6 \\ & & & & | & & & & \\ & & & & \bullet & & & & \end{array}$$

It turns out that L_{ω_1} has dimension 27, which is related to the lines on a cubic surface; it further turns out that L_{ω_6} is the dual of L_{ω_1}. There are many ways to construct L_{ω_1}.

Example 1.198. For E_7, one finds that $\det A = 2$, so there is only one nonzero miniscule weight. It turns out to be the weight ω_1 as follows.

$$\begin{array}{ccccccccccc} \omega_1 & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\ & & & & & & | & & & & \\ & & & & & & \bullet & & & & \end{array}$$

One has $\dim L_{\omega_1} = 56$.

1.6.4 Fundamental Representations

Let's work out our fundamental representations.

[2] It may actually be easier to calculate some candidate miniscule weights first, which we do in the following sentence, and then one can check that they represent all cosets.

Example 1.199. We work out the fundamental representations for type C_n.

Proof. Here, $\mathfrak{g} = \mathfrak{sp}_{2n}$. We already know that $L_{\omega_1} = V$, where V is the standard representation. We also know that $\wedge^2 V$ contains a copy of L_{ω_2}, but $\wedge^2 V$ is not irreducible: note that the bilinear form is in $\wedge^2 V^*$, so there is a copy of B^{-1} in $\wedge^2 V$. It turns out that the embedding

$$L_{\omega_2} \to \wedge^2 V \twoheadrightarrow \wedge_0^2 V,$$

for example by comparing with the dimension formula.

The general discussion is on the homework. Namely, for any $i \geq 2$, there is a contraction $\iota_B \colon \wedge^i V \to \wedge^{i-2} V$ given by contracting along B. Its kernel is a subrepresentation $\wedge_0^i V \subseteq \wedge^i B$, and finds that $\wedge_0^i B \cong L_{\omega_i}$. In fact, there is more structure: there is another operator $m_B \colon \wedge^i V \to \wedge^{i+2} V$ given by taking a wedge product with B^*. Suitably normalized, $f := \iota_B$ and $e := m_B$ produce an $\mathfrak{sl}_2$-triple on the exterior algebra $\wedge^\bullet V$, so $\wedge^\bullet V$ receives an action by $\mathfrak{sl}_2 \times \mathfrak{sp}_{2n}$. Thus, we may decompose

$$\wedge^\bullet V = \bigoplus_{i=1}^n \wedge_0^i V \otimes L_i,$$

where L_i is an irreducible representation of $\mathfrak{sl}_2$ with highest weight $m - i$. (Accordingly, we have an instance of Lemma 1.145.) ■

Remark 1.200. For abelian varieties A of dimension n, we see that $\mathfrak{sp}_{2n}$ admits a standard action on $V = \mathrm{H}^1(A; \mathbb{C})$. The decomposition $\wedge^\bullet V$ turns out to agree with the Hodge decomposition.

Remark 1.201. It follows from this discussion and Proposition 1.122 that all finite-dimensional representations of $\mathfrak{sp}_{2n}$ are found in tensor powers of the standard representation.

Example 1.202. We work out the fundamental representations for type B_n.

Proof. Let V be the standard representation of $\mathfrak{so}_{2n+1}$. On the homework, we will show that the representations $\wedge^i V$ are all irreducible for $i \leq n-1$, so $L_{\omega_i} = \wedge^i V$. ■

Example 1.203. We work out the fundamental representations for type D_n.

Proof. Let V be the standard representation of $\mathfrak{so}_{2n+1}$. Again, on the homework, one shows that $\wedge^i V$ all irreducible for $i \leq n-2$, so $L_{\omega_i} = \wedge^i V$. ■

Remark 1.204. In type D_n, it also turns out that $\wedge^{n-1} V$ is irreducible, but it is not fundamental because $(1, \ldots, 1, 0)$ is not a fundamental weight: it equals $\omega_{n-1} + \omega_n$. It follows that

$$\wedge^{n-1} V \subseteq S_+ \otimes S_-.$$

(Indeed, recall the construction of Proposition 1.122.)

Remark 1.205. Note that the spin representations are not found in tensor powers of the standard representation!

1.7 February 23

Here we go.

1.7.1 Spin Groups

Here is the easiest example of the spin representation.

Example 1.206. By the "belt trick," one has that $\pi_1(\mathrm{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$, and it turns out that its universal cover is $\mathrm{SL}_2(\mathbb{C})$. (Indeed, $\mathfrak{sl}_2 = \mathfrak{so}_3$.) In particular, the adjoint representation defines a covering $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{C})$, and it turns out that the kernel is $\{\pm 1\}$. (We will also show that $\mathrm{SL}_2(\mathbb{C})$ is simply connected in Example 1.209.) Thus, $\mathrm{Spin}_3(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$, and the spin representation is just the standard representation.

It remains to handle $n \geq 3$.

Proposition 1.207. For $n \geq 3$, we have $\pi_1(\mathrm{SO}_n(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$.

We will want the following lemma.

Lemma 1.208. Let $X_n \subseteq \mathbb{C}^n$ be the hypersurface cut out by the equation

$$z_1^2 + \cdots + z_n^2 = 1.$$

For any $k \in \{1, \ldots, n-2\}$, one has $\pi_k(X_n) = 0$.

Proof. One can retract $X_n(\mathbb{C})$ to $X_n(\mathbb{R})$, which is S^{n-1}, and then the result follows from calculations of lower-dimensional homotopy groups. To do the contraction, define $f_\bullet \colon X_n \times [0,1] \to X_n$ by

$$f_t(x+iy) = \frac{x+tiy}{\sqrt{x^2-t^2y^2}}.$$

Indeed, note that $x + iy \in X_n$ (where $x, y \in \mathbb{R}^n$) is equivalent to having $|x|^2 - |y|^2 = 1$ and $xy = 0$, and we can see that the output continues to have this property. But now f_1 is the identity and f_0 has real output, so we are done. ■

Example 1.209. Adjusting basis, one sees that

$$\mathrm{SL}_2(\mathbb{C}) = \{(a,b,c,d) : ad - bc = 1\}$$

is simply connected (by taking $n = 4$). In fact, we also see that $\pi_2(\mathrm{SL}_2(\mathbb{C})) = 0$.

Example 1.210. At $n = 3$, there is a short exact sequence

$$1 \to \mu_2 \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{C}) \to 1.$$

This is a homotopy fiber sequence because the latter action is free, so taking the long exact sequence in homotopy and using Example 1.209 shows that $\pi_1(\mathrm{SO}_3(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.

Proof of Proposition 1.207. We proceed by induction; Example 1.210 allows us to assume $n \geq 4$.

We now show the general case. Note that $\mathrm{SO}_n(\mathbb{C})$ acts transitively on X_n, and the stabilizer of the vector $(1, 0, \ldots, 0)$ is exactly $\mathrm{SO}_{n-1}(\mathbb{C})$ embedded diagonally. Thus, we have a homotopy fiber sequence

$$\mathrm{SO}_{n-1}(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C}) \to X_n.$$

Taking the long exact sequence in homotopy yields the exact sequence

$$\pi_2(X_n) \to \pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \to \pi_1(\mathrm{SO}_n(\mathbb{C})) \to \pi_1(X_n).$$

The end terms vanish for $n \geq 4$ by Lemma 1.208, so we are done by induction. ■

We may thus make the following definition.

Definition 1.211 (spin)**.** For a positive integer $n \geq 3$, let $\mathrm{Spin}_n(\mathbb{C})$ be the universal cover of $\mathrm{SO}_n(\mathbb{C})$ for each n.

Remark 1.212. By Proposition 1.207, we see that the canonical projection $\mathrm{Spin}_n(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C})$ is a double cover.

Example 1.213. The short exact sequence in Example 1.210 shows that $\mathrm{Spin}_3(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$.

Example 1.214. Note that $\mathfrak{so}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (e.g., by looking at Dynkin diagrams), so $\mathrm{Spin}_4(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ because there is a unique simply connected Lie group attached to a given finite-dimensional Lie algebra. One can calculate that the spin representations are given by $S_+ = \mathbb{C}^2 \boxtimes \mathbb{C}$ and $S_- = \mathbb{C} \boxtimes \mathbb{C}^2$. By tracking through the isomorphism $\mathfrak{so}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, one finds that the projection $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{C})$ has kernel $(-1, -1)$. Additionally, the standard representation of $\mathrm{SO}_4(\mathbb{C})$ is $\mathbb{C}^2 \boxtimes \mathbb{C}^2$.

Example 1.215. One has that $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, which one can see by looking at Dynkin diagrams. One can show as before that Sp_{2n} is simply connected, so $\mathrm{Spin}_5(\mathbb{C}) = \mathrm{Sp}_4(\mathbb{C})$, and the double cover $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C})$ has kernel $\{\pm 1\}$. (This is the kernel because the kernel needs to be central, and the nontrivial element needs to act by -1 on the spin representations in order to avoid descending to SO.) The spin representation of $\mathrm{SO}_5(\mathbb{C})$ turns out to be the standard representation of $\mathrm{Sp}_4(\mathbb{C})$.

Example 1.216. One has $\mathfrak{so}_6 \cong \mathfrak{sl}_4$, so $\mathrm{Spin}_6(\mathbb{C}) = \mathrm{SL}_4(\mathbb{C})$ because $\mathrm{SL}_4(\mathbb{C})$ is simply connected. The double cover $\mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C})$ has kernel $\{\pm 1\}$. The spin representations S_+ and S_- are the standard representation and its dual. (Note that these are the only possible irreducible representations of $\mathrm{SL}_4(\mathbb{C})$ with the correct dimension.) On the other hand, the standard representation of $\mathrm{SO}_6(\mathbb{C})$ is the second exterior power of the standard representation.

For $n \geq 7$, the group $\mathrm{Spin}_n(\mathbb{C})$ is harder to describe, and it is our next goal.

1.7.2 Clifford Algebras

Throughout, V is an inner product space over a field k which has characteristic not equal to 2.

Definition 1.217. Fix an inner product space V over a field k with $\operatorname{char} k \neq 2$. Then the *Clifford algebra* $\mathrm{Cl}\, V$ is the quotient of the universal tensor algebra by the ideal generated by the elements

$$v \otimes v - \frac{1}{2}(v, v).$$

Remark 1.218. For any $a, b \in V$, it follows that $2(ab+ba) = (a+b)^2 - a^2 - b^2 = (a+b, a+b) - (a,a) - (b,b) = 2(a,b)$, so $ab + ba = (a,b)$. Setting $a = b$ recovers the relation $2a^2 = (a,a)$.

Remark 1.219. The Clifford algebra admits a natural filtration $\{F_i\}$ induced by the standard filtration on the tensor algebra. In particular, $F_0 = k$, $F_1 = k + V$, $F_2 = (k+V) \cdot (k+V)$, and so on.

Remark 1.220. Because the relations we took a quotient by for $\operatorname{Cl} V$ live in even degree, there is a $\mathbb{Z}/2\mathbb{Z}$-grading on $\operatorname{Cl} V$, which we label $\operatorname{Cl}_+ V \oplus \operatorname{Cl}_- V$.

Remark 1.221. Recall that one can define the exterior algebra $\wedge V$ as the quotient of the tensor algebra TV by the ideal generated by the elements v^2. Thus, the natural map $TV \to \operatorname{Cl} V$ descends to a map $\wedge V \to \operatorname{gr} \operatorname{Cl} V$. This map is further a surjection: fixing an orthonormal basis $\{e_i\}$ of V, we see that $\operatorname{Cl} V$ is spanned by the vectors of the form

$$e_{i_1} \cdots e_{i_r}.$$

Using the relations $e_i e_j + e_j e_i = 1_{i=j}$, we see that we may even assume $i_1 < \cdots < i_r$. But then these vectors are found from $\wedge V$, so we are done.

Here is a basis, which is a variant of the Poincaré–Birkhoff–Witt theorem.

Theorem 1.222. Fix an inner product space V.

(a) If $\dim V = 2n$, then $\operatorname{Cl} V \cong M_{2^n}(\mathbb{C})$.

(b) If $\dim V = 2n+1$, then $\operatorname{Cl} V \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$.

Proof. Let's start with the case $\dim V = 2n$, and choose a basis $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ so that $(a_i, a_j) = 0$, $(b_i, b_j) = 0$ and $(a_i, b_j) = 1_{i=j}$. We now define operators on $M := \wedge(a_1, \ldots, a_n)$ as follows: for each i, we define $A_i v := a_i v$ and $B_i v := \frac{\partial}{\partial a_i} v$, where

$$\frac{\partial}{\partial a_i}(a_{i_1} \wedge \cdots \wedge a_{i_n}) := \begin{cases} 0 & \text{if } i \notin \{i_1, \ldots, i_n\}, \\ (-1)^{k-1}(a_{i_1} \wedge \cdots \wedge \widehat{a}_{i_k} \wedge \cdots \wedge a_{i_n}) & \text{if } i = i_k. \end{cases}$$

One can check that $A_i B_j + B_j A_i = 1_{i-j}$ and $A_i A_j + A_j A_i = 0$ and $B_j B_i + B_i B_j = 0$, so we get a natural map $\rho\colon \operatorname{Cl} V \to \operatorname{End} M$ given by $\rho(a_i) := A_i$ and $\rho(b_i) := B_i$.

Now, on the homework, we will show that increasing sequences $\{i_\bullet\}$ and $\{j_\bullet\}$ produce linearly independent elements

$$A_{i_1} \cdots A_{i_r} B_{j_1} \cdots B_{j_s}.$$

It follows that ρ is an embedding because the elements $a_{i_1} \cdots a_{i_r} b_{j_1} \cdots b_{j_s}$ are already known to be a spanning set, and we can see that they are sent to a linearly independent set. But then note that $\dim \operatorname{Cl} V = 2^{2n}$ while $\dim M = 2^{2n}$, so it actually follows that ρ is an isomorphism.

We now turn to the case where $\dim V = 2n+1$. Then we can pick a basis $\{a_1, \ldots, a_n, b_1, \ldots, b_n, z\}$, where the $a_\bullet$s and $b_\bullet$s have relations as above, and $(a_i, z) = (b_i, z) = 0$ for all i and $(z,z) = 2$. In particular, we see that $za_i = -a_i z$ and $zb_i = -b_i z$ and $z^2 = 1$. Now, set $M := \wedge(a_1, \ldots, a_n)$. There are two ways to extend the action of $\operatorname{Cl} V_0$ (where $V_0 := \operatorname{span}\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$) to $\operatorname{Cl} V$. Indeed, either one can take

$$z(a_{i_1} \cdots a_{i_r}) = \pm(-1)^r (a_1 \wedge \cdots \wedge a_{i_r}).$$

Notably, the choice of sign $\pm$ produces non-isomorphic modules $M_\pm$ for $\operatorname{Cl} V$ because there is a unique vector v with $b_i v = 0$ for all i, but then $zv = \pm v$ in $M_\pm$.

Thus, one receives a natural map

$$\rho \colon \operatorname{Cl} V \to \operatorname{End} M_+ \oplus \operatorname{End} M_-,$$

and one checks as before that it is an isomorphism. For example, one can check that M_+ and M_- are irreducible, which makes ρ a surjection, and then it becomes an isomorphism via some dimension calculations. ■

Remark 1.223. The above proof has actually shown that the natural map $\wedge V \to \operatorname{gr} \operatorname{Cl} V$ is an isomorphism.

Remark 1.224. Here is another approach: one can try to check that M is an irreducible module for $\operatorname{Cl} V$ instead of trying to show that the elements $A_{i_1} \cdots A_{i_r} B_{j_1} \cdots B_{j_s}$ are linearly independent. In particular, irreducibility implies that ρ is surjective by some density theorem.

1.7.3 Spin Representations

We are now ready to construct some spin representations.

Proposition 1.225. Fix an inner product space V over a field of characteristic not 2. View $\mathfrak{so}(V)$ as $\wedge^2 V$ via the identification $V \cong V^*$. Then the natural map $\xi \colon \wedge^2 V \to \operatorname{Cl} V$ defined by

$$\xi(a \wedge b) = \frac{1}{2}(ab - ba)$$

is a Lie algebra homomorphism. Here, $\operatorname{Cl} V$ has been given the natural Lie bracket from being an associative algebra.

Proof. This is a direct calculation. Note

$$\begin{aligned}
[\xi(a \wedge b), \xi(c \wedge d)] &= \left[ab - \frac{1}{2}(a, b), cd - \frac{1}{2}(c, d)\right] \\
&\overset{*}{=} [ab, cd] \\
&= abcd - cdab,
\end{aligned}$$

where $\overset{*}{=}$ has used the Clifford algebra relations. Now, one can successively switch vectors around with some error terms. For example, $abcd = abcd + acbd - acbd = (b, c)ad - acbd$. Continuing this process until we get to the permutation $cdab$, we see that this equals

$$(b, c)ad - (b, d)ac + (a, c)db - (a, d)cb,$$

which can be seen to be

$$(b, c)\xi(a \wedge b) - (b, d)\xi(a \wedge c) + (a, c)\xi(d \wedge b) - (a, d)\xi(c \wedge b),$$

which is $\xi([a \wedge b, c \wedge d])$ in $\mathfrak{so}(V)$. ■

Notably, one now sees that the map $U\mathfrak{so}(V) \to \operatorname{Cl}_+ V$ is surjective, so we can use the modules M constructed in Theorem 1.222 to find our spin representations.

- When $\dim V = 2n$, it turns out that the $\mathbb{Z}/2\mathbb{Z}$-grading on $\operatorname{Cl} V$ produces a $\mathbb{Z}/2\mathbb{Z}$-grading $M = M_0 \oplus M_1$ on M. Thus, we may view $M_0 \oplus M_1$ as a representation of $\mathfrak{so}_{2n}$, and these representations have dimension 2^{n-1}. By finding a suitable highest weight vector and comparing dimensions, one finds that these representations are S_+ and S_-, respectively.
- When $\dim V = 2n + 1$, it turns out that both M_+ and M_- are both the same spin representation S, which one again checks by find a suitable highest weight vector and comparing dimensions. Notably, it turns out that M_+ and M_- remain irreducible when restricted $\mathfrak{so}(V)$.

1.7.4 Dual Representations

We return to the setting where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$.

Remark 1.226. Fix an irreducible representation L_λ of $\mathfrak{g}$, where (as usual) λ is some dominant integral weight. Then $L_\lambda^* \cong L_{\overline{\lambda}}$ for some other dominant weight $\overline{\lambda}$. Notably, the highest weight of L_λ^* is just the negative of the lowest weight of L_λ. To compute this, recall that there is a maximal element $w_0 \in W$ of length $|\Phi_+|$, meaning that $w_0(\Phi_+) = \Phi_-$. Thus, w_0 maps the dominant chamber P_+ to $-P_+$. It follows that $w_0\lambda$ is exactly the lowest weight of L_λ, so $\overline{\lambda} = -w_0\lambda$. (Indeed, writing $w_0\mu = \lambda - \sum_i k_i\alpha_i$ for nonnegative k_i, then $\mu = w_0\lambda - \sum_i k_i w_0\alpha_i = w_0\lambda + \sum_i k_i\alpha_{\sigma(i)}$, where σ is some permutation of the simple roots induced by w_0. The minimality of $w_0\lambda$ follows.)

It is potentially interesting to calculate w_0 in some special cases. Note that w_0 maps the positive roots to the negative roots, so $-w_0$ permutes the simple roots, so it is a permutation of the vertices of the Dynkin diagram. Further, because Weyl group elements preserve the inner product, we see that $-w_0$ will further preserve the edges of the Dynkin diagram. Thus, $-w_0$ is a graph automorphism!

Example 1.227. For the types A_1, B_n, C_n, G_2, F_4, E_7, and E_8, the Dynkin diagram admits no automorphism, so $-w_0$ is the identity.

The remaining cases are slightly trickier. Let's remove a few more. Note that W acts trivially on the quotient P/Q: indeed, we see that

$$s_i\lambda = \lambda - (\lambda, \alpha_i^\vee)\alpha_i,$$

and the latter term is in Q. Thus, $-w_0$ acts by an inversion on P/Q, so we conclude that $-w_0 \neq 1$ whenever P/Q is not a product of $(\mathbb{Z}/2\mathbb{Z})$s.

Example 1.228. In type A_{n-1} for $n \geq 3$, we see that $P/Q \cong \mathbb{Z}/n\mathbb{Z}$, so $-w_0$ must act by flipping the entire Dynkin diagram. One can also see explicitly that this longest Weyl element is the full flip in S_n.

The remaining cases in D_n are harder. If n is odd, then $P/Q \cong (\mathbb{Z}/4\mathbb{Z})$, and so $-w_0$ acts by -1 already, so it is nontrivial. If n is even, then $P/Q \cong (\mathbb{Z}/2\mathbb{Z})^2$, but one can calculate that $-w_0$ acts by 1. This last calculation must be done explicitly. The Weyl group in this case is $S_n \ltimes \{\pm 1\}_0^n$, where $\{\pm 1\}_0^n$ is the kernel of the multiplication map $\{\pm 1\}^n \to \{\pm 1\}$. Then w_0 is the element sending positive roots to negative roots, which when n is even is simply $-\operatorname{id}$. (When n is odd, this element is not in the Weyl group!) Thus, $-w_0$ is trivial!

Example 1.229. For $\mathfrak{so}(4k+2)$, one can calculate $S_+^* = S_-$. But for $\mathfrak{so}(4k)$, one finds that $S_+^* = S_+$ and $S_-^* = S_-$.

Example 1.230. For type E_6, one finds that $P/Q \cong \mathbb{Z}/3\mathbb{Z}$, so $-w_0$ is nontrivial and flips the whole Dynkin diagram. In particular, it switches the two miniscule weights.

1.7.5 Maximal Root

We have already recalled a little work with the maximal root when we discussed miniscule representations, but we will now study the maximal root in more detail.

Definition 1.231 (maximal root)**.** Fix a simple Lie algebra $\mathfrak{g}$. Then the *maximal root* θ of $\mathfrak{g}$ is the highest weight of the adjoint representation of $\mathfrak{g}$.

Remark 1.232. Because the adjoint representation has weights equal to the root system, we see that θ is in fact a root. Its maximality follows because it is the highest weight among the roots.

Example 1.233. In type A_{n-1}, we can see that $\theta = (1, 0, \ldots, 0, -1)$, which is not fundamental.

Example 1.234. In type C_n, one sees that the adjoint representation $\mathfrak{g}$ is S^2V, which is $L_{2\omega_1}$. Thus, θ is not fundamental.

Proposition 1.235. For all simple Lie algebras $\mathfrak{g}$ which are not in types A or C, the maximal root θ is fundamental.

Proof. In the case $\mathfrak{g} = \mathfrak{so}_N$, we need to have $N \geq 7$ to avoid the exceptional isomorphisms. In this case, one finds that $\mathfrak{g} = \wedge^2 V$, which is L_{ω_2}. We will not say much about the exceptional types, but they can be computed explicitly. ■

Remark 1.236. There is a notion of extended Dynkin diagram which allows one to provide a more algebraic proof of this fact.

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We continue.

1.8.1 The Principal $\mathfrak{sl}_2$-triple

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra.

Definition 1.237. Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra. Then we define $e := \sum_i e_i$ and $h := 2\rho^\vee$. There are coefficients

$$f := \sum_i c_i f_i$$

such that (e, f, h) is an $\mathfrak{sl}_2$-triple. This is the *principal* $\mathfrak{sl}_2$*-triple*.

Remark 1.238. The coefficients are given by $c_i := (2\rho^\vee, \omega_i)$, which one can check because this is what is needed for $[e, f] = \sum_i c_i h_i$ to equal $h = \sum_i (2\rho^\vee, \omega_i) h_i$.

Example 1.239. For $\mathfrak{gl}_n$, one finds that

$$e = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and $h = \operatorname{diag}(n-1, n-3, \ldots, 3-n, 1-n)$. On the homework, we will calculate f.

It is somewhat important to decompose $\mathfrak{g}$ as a representation of this principal $\mathfrak{sl}_2$.

Definition 1.240 (height). Fix a semisimple finite-dimensional Lie algebra $\mathfrak{g}$. Then the *height* $|\alpha|$ of a root α is the sum of the coefficients when α is expressed a sum of simple roots. We will let r_m be the number of positive roots with height m. By convention, we let r_0 be the rank of $\mathfrak{g}$.

Remark 1.241. View $\mathfrak{g}$ as a representation of the principal $\mathfrak{sl}_2$-triple. One can check that $(2\rho^\vee, \alpha) = |\alpha|$, so $[h, e_\alpha] = 2\,|\alpha|\,e_\alpha$. Thus, the weights of $\mathfrak{g}$ are all even, and

$$r_m = \dim \mathfrak{g}[2m].$$

Thus, the representation theory of $\mathfrak{sl}_2$ assures us that $\{r_m\}$ is a decreasing sequence.

Tracking through the representation of $\mathfrak{sl}_2$, we are thus motivated to study the differences $r_m - r_{m+1}$.

Definition 1.242 (exponent). Fix a semisimple finite-dimensional Lie algebra $\mathfrak{g}$. Then an *exponent* of $\mathfrak{g}$ is a positive integer m for which $r_m > r_{m+1}$. Its *multiplicity* is $r_m - r_{m+1}$. We may enumerate the exponents (with multiplicity) by $m_1, m_2, \ldots$.

Example 1.243. Note that r_1 is the rank of $\mathfrak{g}$ because roots of height one are (by definition) simple.

Example 1.244. Suppose $\mathfrak{g}$ is simple. Note that r_2 is the number of roots of the form $\alpha_i + \alpha_j$, which happens exactly when i and j are connected. (Indeed, if there is no connection, then the roots are perpendicular.) But the Dynkin diagram is a tree, so $r_2 = r_1 - 1$, so 1 is an exponent with multiplicity 1.

Example 1.245. For n large enough, we see that $r_n = 0$, so the number of exponents (enumerated with multiplicity) is exactly r.

Example 1.246. The maximal root θ corresponds to the exponent $m_r = (\theta, \rho^\vee)$. By definition, $(\theta, \rho^\vee) + 1$ is the "Coxeter number."

Example 1.247. For $\mathfrak{g} = \mathfrak{sl}_n$, we see that roots of height ℓ are those of the form $e_i - e_{i+\ell}$. Thus, the exponents are $\{1, 2, \ldots, n-1\}$, and the Coxeter number is n.

Remark 1.248. By the representation theory of $\mathfrak{sl}_2$, we see that

$$\mathfrak{g} = \bigoplus_{i=1}^{r} L_{2m_i}.$$

For example, this implies that $\sum_{i=1}^{r} (2m_i + 1) = \dim \mathfrak{g}$, so $\sum_{i=1}^{r} m_i$ is the number of positive roots.

Example 1.249. For example, $\mathfrak{sl}_n = L_2 \oplus L_4 \oplus \cdots \oplus L_{2n-2}$. Another way to see this decomposition is to apply the Clebsch–Gordan rule to compute $L_{n-1} \otimes L_{n-1}^* - \mathbb{C}$, where "$-\mathbb{C}$" refers to the fact that we are removing the trivial representation.

Here are the exponents for the other Lie algebras.

Remark 1.250. It turns out that the exponents depends only on the undirected graph isomorphism of the Dynkin diagram, which one can see by Chevalley's theorem.

Example 1.251. Here are the exponents for the classical types.

- The exponents of $\mathfrak{sp}_{2n}$ are $\{1, 3, \ldots, 2n-1\}$.
- The exponents of $\mathfrak{so}_{2n+1}$ are $\{1, 3, \ldots, 2n-1\}$.
- The exponents of $\mathfrak{so}_{2n+2}$ are $\{1, 3, \ldots, 2n-1\} \sqcup \{n\}$, where the extra n at the end is added with multiplicity when n is odd.

Example 1.252. Here are the exponents for the exceptional types.

- The exponents of F_4 are $\{1, 5, 7, 11\}$.
- The exponents of E_6 are $\{1, 4, 5, 7, , 11\}$.
- The exponents of E_7 are $\{1, 5, 7, 9, 11, 13, 17\}$.
- The exponents of E_8 are $\{1, 7, 11, 13, 17, 19, \}$. These are the

Remark 1.253. Here is a curious property of these exponents: they are symmetric, in the sense that $m_i + m_{r-i}$ is constant. We will prove this later.

We close our discussion by saying a little more about the aforementioned Coxeter number.

Definition 1.254 (Coxeter number)**.** Fix a simple Lie algebra $\mathfrak{g}$. Then the *Coxeter number* $h_{\mathfrak{g}}$ is $(\theta, \rho^\vee)+1$, which is one more than the height of the maximal root θ. The *dual Coxeter number* $h_{\mathfrak{g}}^\vee$ is $(\theta^\vee, \rho) + 1$.

Remark 1.255. If $(-,-)$ is normalized so that $(\theta, \theta) = 2$, then we see that the dual Coxeter number is also $(\theta, \rho) + 1$. This normalization is natural because the quadratic Casimir element C for this normalization acts on the adjoint representation by

$$(\theta, \theta + 2\rho) = 2h_{\mathfrak{g}}^\vee.$$

Remark 1.256. If $\mathfrak{g}$ is simply laced (i.e., among the types A, D, or E), then $h_{\mathfrak{g}} = h_{\mathfrak{g}}^\vee$ because θ and $\theta^\vee$ are the same in these simply laced cases. However, they are not the same in general. Indeed, $h_{B_n}^\vee = 2n-1$, $h_{C_n}^\vee = n+1$, $h_{G_2}^\vee = 4$, and $h_{F_4}^\vee = 9$. Notably, this dual Coxeter number is also not the dual Coxeter number of the dual!

1.8.2 Real, Complex, and Quaternionic Representations

As an application of our principal $\mathfrak{sl}_2$, we say something about dividing our representations into various types.

Definition 1.257 (real, complex, quaternionic)**.** Fix a finite-dimensional irreducible representation of a group G. Then V has *complex type* if and only if $V \not\cong V^*$. Continuing, given an isomorphism $A\colon V \to V^*$, this induces a dual map $A^*\colon V^* \to V$, so there is a scalar λ such that $A^* = \lambda A$ after these identifications. If $A^* = A$, then V has *real type*, and if $A^* = -A$, then V has *quaternionic type*.

Remark 1.258. If we forget about the complex structure of V, merely viewing it in $\mathrm{Rep}_{\mathbb{R}}(V)$, then it turns out that $\mathrm{End}_{\mathbb{R}[G]}(V)$ is $\mathbb{C}$ for complex type, $M_2(\mathbb{R})$ for the real type, and $\mathbb{H}$ for quaternionic type. We will prove this on the homework.

Remark 1.259. On the homework, we will show that when G is finite, V has real type if and only if V admits a basis in which all elements of G act by matrices with real coefficients.

Example 1.260. Consider $G = \mathrm{SL}_2(\mathbb{C})$, and consider the irreducible representation $L_n = \mathrm{Sym}^n \mathbb{C}^2$. Then L_1 is of quaternionic type, so one can check that L_n has quaternionic type for n odd and real type for n even.

Example 1.261. For $G = \mathrm{SL}_n(\mathbb{C})$ for $n \geq 3$, one has that V and V^* are non-isomorphic, so V has complex type.

It is thus interesting to know which representations of a Lie group admit which types.

Proposition 1.262. Fix a Lie simply connected Lie group G with Lie algebra $\mathfrak{g}$, and choose a dominant weight λ.

(a) If $\lambda \neq -w_0\lambda$, then L_λ is of complex type.

(b) If $\lambda = -w_0\lambda$, then L_λ is of real type if $(\lambda, 2\rho^\vee)$ is even and is of quaternionic type if $(\lambda, 2\rho^\vee)$ is odd.

Proof. Part (a) follows because $L_\lambda^* = L_{-w_0\lambda}$ by Remark 1.226.

For (b), we are granted an isomorphism $A\colon L_\lambda \to L_\lambda^*$. We now restrict L_λ to the principal $\mathfrak{sl}_2$-triple (e, f, h) of $\mathfrak{g}$. Then the largest eigenvalue of h on L_λ comes from its maximal weight, so this eigenvalue is $m := (2\rho^\vee, \lambda)$; furthermore, we see that $L_\lambda[m]$ is one-dimensional. The other weight vectors in L_λ have weights

$$(2\rho^\vee, \lambda - \beta) = m - 2\sum_i (\omega_i, \beta),$$

which we note is strictly less than m. Thus, A must carry L_m to L_m^*, so we are done by Example 1.260! ■

Let's use Proposition 1.262 to determine the types of the spin representations.

Example 1.263. Consider $\mathfrak{g} = \mathfrak{so}_{2n}$ so that $\rho^\vee = \rho = (n-1, n-2, \ldots, 0)$. Then one sees that $S_+ = L_{\omega_n}$ and $S_- = L_{\omega_{n-1}}$, where $\omega_n = (1/2, \ldots, 1/2)$ and $\omega_{n-1} = (1/2, \ldots, 1/2, -1/2)$. We know that S_+ and S_- are self-dual only when n is even (by looking at the action of $-w_0$). For example, we can calculate

$$(\omega_n, 2\rho^\vee) = \frac{n(n-1)}{2},$$

so S_+ is of real type for $n \equiv 0 \pmod 4$ and of quaternionic type for $n \equiv 2 \pmod 4$.

Example 1.264. Consider $\mathfrak{g} = \mathfrak{so}_{2n+1}$ so that $\rho^\vee = (n, n-1, \ldots, 1)$ and $S = L_{\omega_n}$ has $\omega_n = (1/2, \ldots, 1/2)$. Then S is always self-dual because $-\omega_0$ is trivial, and

$$(\omega_n, 2\rho^\vee) = \frac{n(n+1)}{2}.$$

Thus, S is of quaternionic type when $n \equiv 1, 2 \pmod 4$ and is of real type when $n \equiv 0, 3 \pmod 4$.

The above two examples combine into the following.

Theorem 1.265 (Real Bott periodicity)**.** The type of the spin representations of $\mathfrak{so}_n$ is uniquely determined by $n \pmod 8$.

Proof. Combine Examples 1.263 and 1.264. ■

Remark 1.266. This is related to Bott periodicity in homotopy theory, but I do not know how.

1.8.3 Review of Topology

We end our class by saying something about integration.

Definition 1.267 (locally compact)**.** Fix a Hausdorff topological space X. Then X is *locally compact* if and only if any $p \in X$ admits an open neighborhood U for which the closure $\overline{U}$ is compact.

Example 1.268. The base of balls in $\mathbb{R}^n$ shows that $\mathbb{R}^n$ is locally compact. Because manifolds are locally homeomorphic to $\mathbb{R}^n$, we see that manifolds are locally compact.

Lemma 1.269. Fix a second countable locally compact space X. Then X admits a filtration $\{K_i\}$ of compact sets for which $X = \bigcup_i K_i$, and each $x \in X$ admits an open neighborhood contained in some K_i.

Proof. The idea is that X being second countable means that any open cover can be refined (and shrunken) into a countable one. Indeed, for an open cover $\mathcal{U}$, define a function from the base $\mathcal{B}$ to $\mathcal{U}$ by sending a basic open subset B to one in $\mathcal{U}$ which contains it, if one exists. The image of this map is the required subcover.

Thus, for each $x \in X$, we find an open neighborhood U_x such that $\overline{U}_x$ is compact. Then $\{U_x\}_{x\in X}$ can be refined to a countable subcover $\{U_i\}_{i=1}^n$, and defining

$$K_j := \bigcup_{i\le j} \overline{U_i}$$

will do the trick. ■

Definition 1.270 (locally finite)**.** An open cover $\mathcal{U}$ of a topological space X is *locally finite* if and only if each $x \in X$ admits an open neighborhood V for which

$$\{U \in \mathcal{U} : U \cap V \neq \varnothing\}$$

is finite for each $x \in U$.

Lemma 1.271. Fix a second countable locally compact topological space X. Then every base of X has a countable, locally finite subcover.

Proof. By Lemma 1.269, we may filter X as $X = \bigcup_i K_i$ by compact sets as described.

Let our base be $\mathcal{B}$, and we will build our subcover $\mathcal{U}$ by removing elements from $\mathcal{B}$, starting with $\mathcal{U} = \mathcal{B}$. We may immediately assume that it is countable because X is second countable, so we may enumerate $\mathcal{U}$ as $\{U_i\}_{i\in\mathbb{N}}$. We now apply the following inductive process for each n, starting with $n = 1$.

1. Because K_n is compact, we are granted a positive integer N for which $\{U_i\}_{i\le N}$ covers K_n, which we keep in $\mathcal{U}$.
2. We then remove from $\mathcal{U}$ all other open subsets which intersect with K_n, and go back to the previous step now with K_{n+1}. Note that because $\mathcal{B}$ was a base, any element $x \notin K_n$ will admit an open neighborhood avoiding K_n, so $\mathcal{U}$ continues to be a base on $X \setminus K_n$.

At the end of this process, we see that $\mathcal{U}$ is certainly a countable cover of X, and it is locally finite because any point $x \in X$ admits an open neighborhood U contained in one of the K_ns, and each K_n only has intersection with finitely many open subsets in $\mathcal{U}$. ■

Let's also say something about differential forms.

Definition 1.272 (differential form)**.** Fix a manifold M of dimension n. For a positive integer $k \leq n$, we define $\Omega^k(M)$ to be the space of *differential forms* of degree k. Explicitly, $\Omega^k(M) := \wedge^k T^*M$.

Remark 1.273. In local coordinates $(x_1, \ldots, x_n)$ of M, we see that a local frame of T^*M is given by $\{dx_1, \ldots, dx_n\}$. Thus, some $\omega \in \Omega^k(M)$ takes the form

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \ldots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

If we wanted to change coordinates to some $(y_1, \ldots, y_n)$, where we now think of x_i as a function of the ys, then

$$\omega = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_k}} f_{i_1 \ldots i_k} \det \left[\frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(y_{j_1}, \ldots, y_{j_k})} \right] dy_{j_1} \wedge \cdots \wedge dy_{j_k}.$$

Definition 1.274 (de Rham cohomology)**.** Fix a manifold M. There is a differential map $d\colon \mathcal{O}(M) \to \Omega(M)$ given by sending a function f to the covector df; this extends to a map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$. Then the *$k$th de Rham cohomology* of M is given by

$$\mathrm{H}^k_{\mathrm{dR}}(M) = \frac{\ker\left(d\colon \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\operatorname{im}\left(d\colon \Omega^{k-1}(M) \to \Omega^k(M)\right)}.$$

BIBLIOGRAPHY

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List of Definitions