185: Introduction to Complex Analysis

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Contents

Theme 1: Introducing Complex Numbers

1.1 January 19

It is reportedly close enough to start.

1.1.1 Logistics

We are online for the first two weeks, as with the rest of Berkeley. We will be using `bCourses` a lot, so check it frequently. There is also a website. There is a homework due on Friday, but do not worry about it.

Here are some syllabus things.

- Our main text is *Complex Variables and Applications, 8th Edition* because it is the version that Professor Morrow used. There is a free copy online.
- Homeworks are readings (for each course day) and weekly problem sets. Late homeworks are never accepted.
- Lowest two homework scores are dropped.
- There are two midterms and a final. The final is culmulative, as usual. The final can replace one midterm if the score is higher.
- Regrade requests can be made in GradeScope within one week of being graded.
- The class is curved but usually only curved at the end. The average on exams is expected to be 80%–83%.

1.1.2 Complex Numbers

Welcome to complex analysis. What does that mean?

Idea 1.1. In complex analysis, we study functions $f : \mathbb{C} \to \mathbb{C}$, usually analytic to some extent.

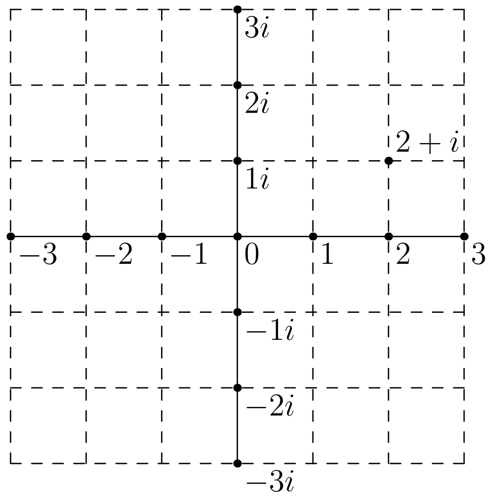
There are two pieces here: we should study $\mathbb{C}$ in themselves and then we will study the functions.

Complex numbers

Definition 1.2 (Complex numbers)**.** The set of complex numbers $\mathbb{C}$ is $\{a + bi : a, b \in \mathbb{R}\}$, where $i^2 = -1$.

Hopefully $\mathbb{R}$ is familiar from real analysis. As an aside, we see $\mathbb{R} \subseteq \mathbb{C}$ because $a = a + 0i \in \mathbb{C}$ for each $a \in \mathbb{R}$.

The complex numbers have an inherent geometry as a two-dimensional plane.



The point is that $\mathbb{C}$ looks like the real plane $\mathbb{R}^2$. More precisely, $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-vector space, where our basis is $\{1, i\}$.

We would like to understand $\mathbb{C}$ geometrically, "as a space." The first step here is to create a notion of size.

Norm on $\mathbb{C}$

Definition 1.3 (Norm on $\mathbb{C}$)**.** We define the **norm map** $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by $|z| := \sqrt{z\bar{z}}$. In other words,

$$|a + bi| := \sqrt{a^2 + b^2}.$$

Note that this agrees with the absolute value on $\mathbb{R}$: for $a \in \mathbb{R}$, we have $\sqrt{a^2} = |a|$.

Norm functions, as in the real case, give us a notion of distance.

Metric on $\mathbb{C}$

Definition 1.4 (Metric on $\mathbb{C}$)**.** We define the *metric on* $\mathbb{C}$ to be $d_{\mathbb{C}}(z_1, z_2) := |z_1 - z_2|$.

One can check that this is in fact a metric, but we will not do so here.

Remark 1.5. The distance in $\mathbb{C}$ is defined to match the distance in $\mathbb{R}^2$ under the basis $\{1, i\}$.

Again as we discussed in real analysis, having a metric gives us a metric topology by open balls. Lastly it is this topology that our geometry will follow from: we have turned $\mathbb{C}$ into a topological space.

1.1.3 Complex Functions

There are lots of functions on $\mathbb{C}$, and lots of them are terrible. So we would like to focus on functions with some structure. We'll start with *continuous functions*, which are more or less the functions that respect topology.

Then from continuous functions, we will be able to define *holomorphic functions*, which are complex differentiable. This intended to be similar to being real differentiable, but complex differentiable turns out to be a very strong condition. Nevertheless, everyone's favorite functions are holomorphic.

Example 1.6. Polynomials, $\exp$, $\sin$, and $\cos$ are all holomorphic.

To make concrete that complex differentiability is stronger than real differentiability, the Cauchy–Riemann equations which provides a partial differential equation to test complex differentiability.

From here we define *analytic* functions, which essentially are defined as taking the form

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Analytic functions are super nice in that we have an ability to physically write them down, so the following theorem is amazing.

Theorem 1.7. Holomorphic functions on $\mathbb{C}$ are analytic.

To prove this, we will need the following result, which is what Professor Morrow calls the most fundamental result in complex analysis, the *Cauchy integral formula*.

In short, the Cauchy integral formula lets us talk about the value of holomorphic functions (and derivatives) at a point in terms of integrals around the point. This will essentially let us build the power series for a holomorphic function by hand. But as described, we will need a notion of complex (path) integration to even be able to talk about the Cauchy integral formula.

The Cauchy integral formula has lots of applications; for example, *Liouville's theorem* on holomorphic functions and the *Fundamental theorem of algebra*.

Remark 1.8. It is very hard to speill Liouville.

Additionally, we remark that our study of holomorphic functions, via the Cauchy integral formula, will boil down to a study of complex path integrals. So we will finish out our story with the *Residue theorem*, which provides a very convenient way to compute such integrals.

Then as a fun addendum, we talk about automorphisms of the complex numbers.

Automorphisms of $\mathbb{C}$

Definition 1.9 (Automorphisms of $\mathbb{C}$)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is an *automorphism of* $\mathbb{C}$ if it is bijective and both f and f^{-1} are holomorphic.

What is amazing is that all of these functions have a concrete description in terms of *Möbius transformations*.

1.1.4 Why Care?

Whenever taking a class, it is appropriate to ask why one should care. Here are some reasons to care.

- Algebraic geometry in its study of complex analytic spaces uses complex analysis.
- Analytic number theory (e.g., the Prime number theorem) makes heavy use of complex analysis.
- Combinatorics via generating functions can use complex analysis.
- Physics uses complex analysis.

The first two Professor Morrow is more familiar with, the last two less so.

1.2 January 21

We're reviewing set theory today.

1.2.1 Set Theory Notation

We have the following definitions.

- $\varnothing$ means the empty set.
- $a \in X$ means that a is an element of the set X.
- $A \subseteq B$ means that A is a subset of B.
- $A \subsetneq B$ means that A is a proper subset of B.

- $A \cup B$ consists of the elements which are in at least one of A or B.
- $A \cap B$ consists of the elements which are in both A and B.
- $A \setminus B$ consits of the elements of A which are not in B.
- Two sets A and B are *disjoint* if and only if $A \cap B = \varnothing$.
- Given a set X, we define $\mathcal{P}(X)$ to be the set of all subsets of X.
- $|X| = \#X$ is the cardinality of X, or (roughly speaking) the number of elements of X.

As an example of unwinding notation, we have the following.

Proposition 1.10 (De Morgan's Laws)**.** Fix $\mathcal{S} \subseteq \mathcal{P}(X)$ a collection of subsets of a set X. Then

$$X \Bigg\backslash \bigcap_{S \in \mathcal{S}} S = \bigcup_{S \in \mathcal{S}} (X \setminus S) \qquad \text{and} \qquad X \setminus \bigcup_{S \in \mathcal{S}} S = \bigcap_{S \in \mathcal{S}} (X \setminus S).$$

Proof. We take these one at a time.

- Note $a \in X \setminus \bigcap \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcap \mathcal{S}$. However, $a \notin \bigcap \mathcal{S}$ is merely saying that a is not in all of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for one of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for some $S \in \mathcal{S}$, which is equivalent to $a \in \bigcup_{S \in \mathcal{S}} (X \setminus S)$.
- Note $a \in X \setminus \bigcup \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcup \mathcal{S}$. However, $a \notin \bigcup \mathcal{S}$ is merely saying that a is not in any of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for each of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for each $S \in \mathcal{S}$, which is equivalent to $a \in \bigcap_{S \in \mathcal{S}} (X \setminus S)$. ■

1.2.2 Some Conventions

In this class, we take the following names of standard sets.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Importantly, $0 \in \mathbb{N}$.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}$ and $q \neq\}$ is the set of rationals.
- $\mathbb{R}$ is the set of real numbers. We will not specify a construction here; see any real analysis class.
- $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq\}$ is the nonzero real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ is the positive real numbers.
- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ is the nonnegative real numbers.
- $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$ is the nonpositive real numbers.
- $\mathbb{C}$ is the complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$ is the set of nonzero complex numbes.

1.2.3 Relations

Let's review some set theory definitions.

Cartesian product

Definition 1.11 (Cartesian product)**.** Given two sets A and B, we define the *Cartesian product* $A \times B$ to be the set of orderd pairs (a, b) such that $a \in A$ and $b \in B$.

Binary relation

Definition 1.12 (Binary relation)**.** A *binary relation* on A is any subset $R \subseteq A^2 := A \times A$. We may sometimes notate $(x, y) \in R$ by xRy, read as "x is related to y."

Example 1.13. Equality is a binary relation on any set A; namely, it is the subset $\{(a, a) : a \in A\}$.

The best relations are equivalence relations.

Equivalence relation

Definition 1.14 (Equivalence relation)**.** An *equivalence relation* on A is a binary relation R satisfying the following three conditions.

- Reflexive: each $x \in A$ has $(x, x) \in R$.
- Symmetric: each $x, y \in A$ has $(x, y) \in R$ implies $(y, x) \in R$.
- Transitive: each $x, y, z \in A$ has $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Equivalence relations are nice because they allow us to partition the set into "equivalence classes."

Equivalence class

Definition 1.15 (Equivalence class)**.** Fix A a set and $R \subseteq A^2$ an equivalence relation. Then, for given $x \in A$, we define

$$[x]_R := \{y \in A : (x, y) \in R\}$$

to be the *equivalence class* of x.

The hope is that equivalence classes partition the set. What is a parition?

Parition

Definition 1.16 (Parition)**.** A *partition* of a set A is a collection of nonempty subsets $\mathcal{S} \subseteq \mathcal{P}(A)$ of A such that any two distinct $S_1, S_2 \in \mathcal{S}$ are disjoint while $A = \bigcup_{S \in \mathcal{S}} S$.

And now let's manifest our hope.

Lemma 1.17. Equivalence relations are in one-to-one correspondence with partitions of A.

Proof. Given an equivalence relation R, we define the collection

$$\mathcal{S}(R) = \{[x]_R : x \in A\}.$$

We claim that $R \mapsto \mathcal{S}(R)$ is our needed bijection. We have the following checks.

- Well-defined: observe that $\mathcal{S}(R)$ does partition A: if we have $[x]_R, [y]_R \in \mathcal{S}$, then $[x]_R \cap [y]_R \neq \varnothing$ implies there is some z with $(x, z) \in R$ and $(z, y) \in R$, so $x \in [y]_R$ and then $[x]_R \subseteq [y]_R$ follows. So by symmetry, $[y]_R \subseteq [x]_R$ as well, so we finish the disjointness check.

  Further, we see that

  $$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} [x]_R \subseteq A$$

  because $x \in [x]_R$, so indeed the equivalence classes cover A.

- Injective: suppose R_1 and R_2 have $\mathcal{S}(R_1) = \mathcal{S}(R_2)$. We show that $R_1 \subseteq R_2$, and $R_2 \subseteq R_1$ will follow by symmetry, finishing.

  We notice that, for any $\mathcal{S}$ paritioning A, being a partition, will have exactly one subset which contains x. But for $\mathcal{S}(R)$ for an equivalence relation R, we see $x \in [x]_R \in \mathcal{S}(R)$, so this equivalence class must be the one.

  So because $[x]_{R_1}$ and $[x]_{R_2}$ are the only subsets of $\mathcal{S}(R_1)$ and $\mathcal{S}(R_2)$ containing x (respectively), we must have $[x]_{R_1} = [x]_{R_2}$. Thus, $(x, y) \in R_1$ implies $y \in [x]_{R_1} = [x]_{R_2}$ implies $(x, y) \in R_2$.

- Surjective: suppose that $\mathcal{S}$ is a partition of A. As noted above, each $x \in A$ is a member of exactly one set $S \in \mathcal{S}$, which we call $[x]$. Then we define $R \subseteq A^2$ by $(x, y) \in R$ if and only if $y \in [x]$. One can check that this is an equivalence relation, which we will not do here in detail.[1]

  The point is that
  $$[x]_R = \{y : (x, y) \in R\} = \{y : y \in [x]\} = [x],$$
  so $\mathcal{S}(R) = \mathcal{S}$. So our mapping is surjective. ■

We continue our discussion.

Quotient set

Definition 1.18 (Quotient set)**.** Given an equivalence relation $R \subseteq A^2$, we define the *quotient set* A/R is the set of equivalence classes of R. In other words,

$$A/R = \{[x]_R : x \in A\}.$$

Intuitively, the quotient set is the set where we have gone ahead and identified the elements which are "similar"or "related."

We would like a more concrete way to talk about equivalence classes, for which we have the following.

Representatives

Definition 1.19 (Representatives)**.** Given an equivalence relation $R \subseteq A^2$, we say that $C \subseteq A$ is a *set of representatives of R-equivalence classes of A* if and only if C consists of exactly one element from each equivalence class in A/R.

1.2.4 Functions

To finish off, we discuss functions.

Functions

Definition 1.20 (Functions)**.** A *function* $f : X \to Y$ is a relation $f \subseteq X \times Y$ satisfying the following.

- For each $x \in X$, there is some $y \in Y$ such that $(x, y) \in f$. Intuitively, each $x \in X$ goes somewhere.
- For each $x \in X$ and given some $y_1, y_2 \in Y$ such that $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$. Intuitively, each $x \in X$ goes to at most one place.

We will write $f(x) = y$ as notational sugar for $(x, y) \in f$. Note this equality is legal because the value y with $(x, y) \in f$ is uniquely given.

We would like to create new functions from old. Here are two ways to do this.

Restriction

Definition 1.21 (Restriction)**.** Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define

$$f|_A = \{(x, y) \in f : x \in A\} \subseteq A \times Y$$

to be a function $f|_A : A \to Y$.

[1] Note $x \in [x]$ by definition of $[x]$. If $y \in [x]$, then note $y \in [y]$ as well, so $[x] = [y]$ is forced by uniqueness, so $x \in [y]$. If $y \in [x]$ and $z \in [y]$, then again by uniqueness $[x] = [y] = [z]$, so $z \in [x]$ follows.

We will not check that $f|_A$ is actually a function; it is, roughly speaking inherited from f.

Definition 1.22. Given two functions $f : X \to Y$ and $g : Y \to Z$, we define the *composition* of f and g to be some functino $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) := g(f(x)).$$

Again, we will not check that this makes a function; it is.

Functions can also help create new sets.

Image

Definition 1.23 (Image). Given a function $f : X \to Y$, we define the *image* of f to be

$$\operatorname{im} f = f(X) := \{y \in Y : \text{there is } x \in X \text{ such that } f(x)y\}.$$

Namely, $\operatorname{im} f$ consists of all elements hit by someone in X hit by f.

Fiber, pre-image

Definition 1.24 (Fiber, pre-image). Given a functino $f : X \to Y$ and some $y \in Y$, we define the *fiber* of f over y to be

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

In general, we define the *pre-image* of a subset $A \subseteq X$ to be

$$f^{-1}(A) := \{x \in A : f(x) \in A\} = \bigcup_{a \in A} \{x \in A : f(x) = a\} = \bigcup_{a \in A} f^{-1}(a).$$

Some functions have nicer properties than others.

Inj-, sur-, bijective

Definition 1.25 (Inj-, sur-, bijective). Fix a function $f : X \to Y$. We have the following.

- Then f is *injective* or *one-to-one* if and only if, given $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- Then f is *surjective* or *onto* if and only if $\operatorname{im} f = Y$. In other words, for each $y \in Y$, there exists $x \in X$ with $f(x) = y$.
- Then f is *bijective* if and onlt if it is both injective and surjective.

Here is an example.

Identity

Definition 1.26 (Identity). For a given set X, the function $\operatorname{id}_X : X \to X$ defined by $\operatorname{id}_X(x) := x$ is called the *identity function*.

For completeness, here are the checks that id_X is bijective.

- Injective: given $x_1, x_2 \in X$, we see $\operatorname{id}_X(x_1) = \operatorname{id}_X(x_2)$ implies $x_1 = \operatorname{id}_X(x_1) = \operatorname{id}_X(x_2) = x_2$.
- Surjective: given $x \in X$, we see that $x \in \operatorname{im} \operatorname{id}_X$ because $x = \operatorname{id}_X$.

We leave with some lemmas, to be proven once in one's life.

Lemma 1.27. Fix a finite sets X and Y such that $\#X = \#Y$. Then a function $f : X \to Y$ is bijective if and only if it is injective or surjective.

Proof. Certainly if f is bijective, then it is both injective and surjective, so there is nothing to say.

The reverse direction is harder. We proceed by induction on $\#X = \#Y$. If $\#X = \#Y = 0$, then $X = Y = \varnothing$, and all functions $f : \varnothing \to \varnothing$ are vacuously bijective: for injective, note that any $x_1, x_2 \in \varnothing$ have $x_1 = x_2$; for surjective, note that any $x \in \varnothing$ has $f(x) = x$.

Otherwise $\#X = \#Y > 0$. We have two cases.

- Take f injective; we show f is surjective. In this case, $\#X > 0$, so choose some $a \in X$. Note that $x \in X$ with $x \neq a$ will have $f(x) \neq f(a)$ by injectivity, so we may define the restriction

  $$f|_{X\setminus\{a\}} : X \setminus \{a\} \to Y \setminus \{f(a)\}.$$

  Observe that $f|_{X\setminus\{a\}}$ is injective because f is: if $x_1, x_2 \in X \setminus \{a\}$ have

  $$f(x_1) = f|_{X\setminus\{a\}}(x_1) = f|_{X\setminus\{a\}}(x_2) = f(x_2),$$

  then $x_1 = x_2$ follows.

  Now, $\#(X \setminus \{a\}) = \#(Y \setminus \{f(a)\}) = \#X - 1$, so by induction $f|_{X\setminus\{a\}}$ will be bijective because it is injective. In particular, f by way of $f|_{X\setminus\{a\}}$ fully hits $Y \setminus \{f(a)\}$ in its image, so because $f(a) \in \operatorname{im} f$ as well, we conclude $\operatorname{im} f = Y$. So f is surjective.

- Take f surjective; we show f is injective. Define a functiono $g : Y \to X$ as follows: for each $y \in Y$, the surjectivity of f promises some $x \in X$ such that $f(x) = y$, so choose any such x and define $g(y) := x$.[2] Observe that $f(g(y)) = y$ by construction.

  Now, we notice that g is injective: if $y_1, y_2 \in Y$ have $g(y_1) = g(y_2)$, then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$. So the previous case tells us that g is in fact bijective.

  So now choose any $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. The surjectivity of f promises some $y_1, y_2 \in Y$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$, so we see that

  $$x_1 = g(y_1) = g(f(g(y_1))) = g(f(x_1)) = g(f(x_2)) = g(f(g(y_2))) = g(y_2) = x_2,$$

  proving our injectivity. ■

Lemma 1.28. Fix $f : X \to Y$ a bijective function. Then there is a unique function $g : Y \to X$ such that $f \circ g = \operatorname{id}_Y$ and $g \circ f = \operatorname{id}_X$.

Proof. We show existence and uniqueness separately.

- We show existence. Note that, because $f : X \to Y$ is surjective, each $y \in Y$ has some $x \in X$ such that $f(x) = y$. In fact, this $x \in X$ is uniquely defined because $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so we may define $g(y)$ as the value x for which $f(x) = y$.

  By construction, $f(g(y)) = y$, so $f \circ g = \operatorname{id}_Y$. Additionally, we note that, given any $x \in X$, the value x_0 for which $f(x) = f(x_0)$ is $x = x_0$ by the injectivity, so $g(f(x)) = x$. Thus, $g \circ f = \operatorname{id}_X$, as claimed.

- We show uniqueness. Suppose that we have two functions $g_1, g_2 : Y \to X$ which satisfy

  $$f \circ g_1 = f \circ g_2 = \operatorname{id}_Y \qquad \text{and} \qquad g_1 \circ f = g_2 \circ f = \operatorname{id}_X .$$

  Then we see that

  $$g_1 = g_1 \circ \operatorname{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \operatorname{id}_X \circ g_2 = g_2,$$

  where we have used the fact that function composition associates. This finishes. ■

1.3 January 24

Good morning everyone.

[2] Technically we are using the Axiom of Choice here. One can remove this with an induction because all sets are finite, but I won't bother.

1.3.1 Algebraic Structure

Today we are reviewing the complex numbers (reportedly, "some basics"). Or at least it is hopefully mostly review. Here is our main character this semester.

Complex numbers

Definition 1.29 (Complex numbers)**.** The set $\mathbb{C}$ of *complex numbers* is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Here i is some symbol such that $i^2 = -1$ formally.

In particular, two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

The complex numbers also have some algebraic structure.

$+$ and $\times$ in $\mathbb{C}$

Definition 1.30 ($+$ and $\times$ in $\mathbb{C}$)**.** Given complex numbers $a_1 + b_1 i, a_2 + b_2 i \in \mathbb{C}$, we define

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i,$$

and

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i,$$

defined essentially by direct expansion, upon recalling $i^2 = -1$.

Here is the corresponding algebraic structure.

Proposition 1.31. The set $\mathbb{C}$ with the above operations is a two-dimensional $\mathbb{R}$-vector space with basis $\{1, i\}$.

Proof. The elements $\{1, i\}$ span $\mathbb{C}$ because all complex numbers in $\mathbb{C}$ can be written as $a + bi = a \cdot 1 + b \cdot i$ by definition.

To see that these elements are linearly independent, suppose $a + bi = 0$. If $b = 0$, then $a = 0$ follows, and we are done. Otherwise, take $b \neq 0$, but then we see $(-a/b) = i$, so

$$(-a/b)^2 = -1 < 0,$$

which does not make sense for real numbers. This finishes. ■

Proposition 1.32. The set $\mathbb{C}$ with the above operations is a field.

Proof. We have the following checks.

- The element $0 + 0i$ is our additive identity. Indeed, one can check that $(0 + 0i) + (a + bi) = (a + bi) + (0 + 0i) = a + bi$.
- The element $1 + 0i$ is our multiplicative identity. Indeed, one can check that $(1 + 0i)(a + bi) = (a + bi)(1 + 0i) = a + bi$.
- Commutativity of addition and multiplication follow from by expansion.
- The distributive laws can again be checked by expansion.
- The additive inverse of $a + bi$ is $(-a) + (-b)i$.
- The multiplicative inverse of $a + bi$ can be found by wishing really hard and writing

$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

Then one can check this works. ■

Sometimes we would like to extract our coefficients from our basis.

Re and Im

Definition 1.33 (Re and Im)**.** Given $z := a + bi \in \mathbb{C}$, we define the operations

$$\operatorname{Re} z := a \qquad \text{and} \qquad \operatorname{Im} z := b.$$

Importantly, $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$.

Because we are merely doing basis extraction, it makes sense that these operations will preserve some (additive) structure.

Proposition 1.34. Fix $z = a + bi$ and $w = c + di$. Then the following.

(a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$.

(b) $\operatorname{Im}(z + w) = \operatorname{Im} z + \operatorname{Im} w$.

Proof. We proceed by direct expansion. Observe

$$\operatorname{Re}(z + w) = \operatorname{Re}((a + c) + (b + d)i) = a + c = \operatorname{Re} z + \operatorname{Re} w,$$

and

$$\operatorname{Im}(z + w) = \operatorname{Im}((a + c) + (b + d)i) = b + d = \operatorname{Im} z + \operatorname{Im} w.$$

This finishes. ■

It also turns out that the complex numbers have a very special transformation.

Conjugate

Definition 1.35 (Conjugate)**.** Given $z := a + bi \in \mathbb{C}$, we define the *complex conjugate* to be $\overline{z} := a - bi \in \mathbb{C}$.

We promised conjugation would be special, so here are some special things.

Proposition 1.36. Fix $z = a + bi \in \mathbb{C}$. Then the following.

(a) $z + \overline{z} = 2 \operatorname{Re} z$.

(b) $z - \overline{z} = 2i \operatorname{Im} z$.

(c) $\overline{\overline{z}} = z$.

Proof. We take these one at a time.

(a) Write $a + bi + \overline{a + bi} = a + bi + a - bi = 2a$.

(b) Write $a + bi - \overline{a + bi} = a + bi - (a - bi) = 2bi$.

(c) Write $\overline{\overline{a + bi}} = \overline{a - bi} = a + bi$. ■

In fact, more is true.

Proposition 1.37. Fix $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$. Then the following.

(a) $\overline{z + w} = \overline{z} + \overline{w}$.

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$.

Proof. We take these one at a time.

- Write
$$\overline{z+w} = (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}.$$

- Write
$$\begin{aligned}\overline{z}\cdot\overline{w} &= (a-bi)(c-di)\\ &= (ac-bd)-(ad+bc)i\\ &= \overline{(ac-bd)+(ad+bc)i}\\ &= \overline{zw}.\end{aligned}$$

This finishes. ■

1.3.2 Defining Distance

Complex conjugation actually gives rise to a notion of size.

Norm on $\mathbb{C}$

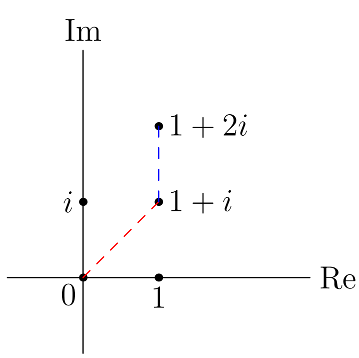
Definition 1.38 (Norm on $\mathbb{C}$)**.** Given $z := a+bi$, we define the *norm function on* $\mathbb{C}$ by
$$|z| := \sqrt{a^2+b^2}.$$

Size actually gives distance.

Distance on $\mathbb{C}$

Definition 1.39 (Distance on $\mathbb{C}$)**.** Given complex numbers $z = a+bi$ and $w = c+di$, we define the *distance* between z and w to be
$$|z-w| = \sqrt{(a-c)^2+(b-d)^2}.$$

Here are some examples.



One can ask what is the distance between $0+0i$ and $1+1i$, and we can compute directly that this is $\sqrt{1+1}=\sqrt{2}$. Similarly, the distance between $1+2i$ and $1+i$ is $|(1+2i)-(1+i)| = |i| = 1$. It should agree with our geometric intuition.

We mentioned complex conjugation is involved here, so we have the following lemma.

Lemma 1.40. Fix $z, w \in \mathbb{C}$. The following are true.

(a) $|z|^2 = z\overline{z}$.

(b) $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.

(c) $|z| = |\overline{z}| = |-z|$.

(d) $|z| = 0$ if and only if $z = 0$.

(e) $|zw| = |z| \cdot |w|$.

Proof. We take these one at a time. Set $z = a + bi$.

(a) We have
$$|z|^2 = a^2 + b^2 = (a+bi)(a-bi) = z\overline{z}.$$
Here we have used subtraction of two squares, which one can see when writing $a^2 + b^2 = a^2 - (ib)^2$.

(b) We have $a^2 \le a^2 + b^2$ and $b^2 \le a^2 + b^2$ by the Trivial inequality, so
$$|\operatorname{Re} z| = |a| \le \sqrt{a^2 + b^2} = |z|,$$
and similarly,
$$|\operatorname{Im} z| = |b| \le \sqrt{a^2 + b^2} = |z|.$$

(c) Note
$$|\overline{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|,$$
and
$$|-z| = |-a - bi| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(d) From (b), we know that $|\operatorname{Re} z|, |\operatorname{Im} z| \le |z|$, but $|z| = 0$ then forces $\operatorname{Re} z = \operatorname{Im} z = 0$, so $z = 0$.

(e) From (a), we can write $|zw|^2 = zw \cdot \overline{zw}$, which will expand out into
$$z \cdot w \cdot \overline{z} \cdot \overline{w}.$$
We can collect this into $z\overline{z} \cdot w\overline{w} = |z|^2|w|^2$. Thus, by (a) again, $|zw|^2 = |z|^2|w|^2$.But because all norms must be nonnegative real numbers, we may take square roots to conclude $|zw| = |z| \cdot |w|$. ■

Remark 1.41. Norms are actually more general constructions. For example, the requirement $|zw| = |z| \cdot |w|$ makes $|\cdot|$ into a "multiplicative" norm.

To finish off, we actually show that our distance function is good: we show the triangle inequality.

Lemma 1.42 (Triangle inequality)**.** For every $x, y, z \in \mathbb{C}$, we claim
$$|z - x| \le |z - y| + |y - z|.$$

This claim should be familiar from real analysis. Intuitively, it means that travelling between z and x cannot be made into a shorter trip by taking a detour to some other point y first.

Proof. Let $a := z - y$ and $b := y - z$ so that $a + b = z - x$. Thus, we are showing that
$$|a + b| \overset{?}{\le} |a| + |b|,$$

which is nicer because it only has two letters. For this, because everything is a nonnegative real numbers, it suffices to show the square of this requirement; i.e., we show

$$(|a|+|b|)^2 - |a+b|^2 \stackrel{?}{\geq} 0.$$

Fully expanding, it suffices to show

$$|a|^2 + |b|^2 + 2|a|\cdot|b| - |a+b|^2 \stackrel{?}{\geq} 0.$$

Expanding out $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$, we are showing

$$a\overline{a} + b\overline{b} + 2|a|\cdot|b| - (a+b)(\overline{a}+\overline{b}) \stackrel{?}{\geq} 0.$$

This is nice because the expansion of the rightmost term will induce some cancellation: it expands into $a\overline{a} + a\overline{b} + \overline{a}b + b\overline{b}$, so we are left with showing

$$2|a|\cdot|b| - (a\overline{b} + b\overline{a}) \stackrel{?}{\geq} 0.$$

Note that $\overline{a}b = \overline{a\overline{b}}$, so we can collect the final term as $2\operatorname{Re}(a\overline{b})$. Similarly, we can write $|a|\cdot|b| = |a|\cdot|\overline{b}| = |a\overline{b}|$, so we are showing

$$2|a\overline{b}| - 2\operatorname{Re}(a\overline{b}) \geq 0,$$

which is true because the real part does exceed the norm. This finishes. ■

1.4 January 26

In-person class should start on Monday. Homework #2 will be released on Friday.

1.4.1 Geometry on $\mathbb{C}$

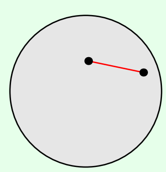
So let's try to build a topology on $\mathbb{C}$ today. We pick up the following definition.

Convex

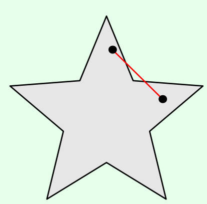
Definition 1.43 (Convex). A subset $X \subseteq \mathbb{C}$ is *convex* if and only if, for every $z, w \in X$ and $t \in [0,1]$, we have that $w + t(z-w) \in X$.

Intuitively, "convex" means that X contains the line segment of any two points in X.

Example 1.44. The circle is convex: any line with endpoints in the circle lives in the circle.



Non-Example 1.45. The star-shape is not convex: the given line goes outside the star.



To define our open sets, we will define balls first.

Open ball

Definition 1.46 (Open ball). Given some $z_0 \in \mathbb{C}$, then *open ball* centered at z_0 with radius $r > 0$ is

$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Observe $z_0 \in B(z_0, r)$.

Open balls let us define all sorts of properties.

Isolated

Definition 1.47 (Isolated). Fix $X \subseteq \mathbb{C}$. A point $z \in X$ is isolated in X if and only if there exists $r > 0$ such that

$$B(z, r) \cap X = \{z\}.$$

Discrete

Definition 1.48 (Discrete). A subset $X \subseteq \mathbb{C}$ is *discrete* if and only if every point is isolated.

Example 1.49. Any finite subset of $X \subseteq \mathbb{C}$ is discrete. Namely, any point $z \in X$ can take

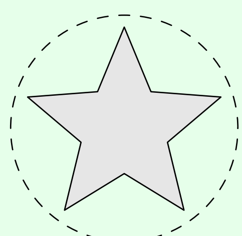
$$r = \frac{1}{2} \min_{w \in X \setminus \{z\}} |z - x|.$$

Example 1.50. The subset $\mathbb{Z} \subseteq \mathbb{C}$ is isolated. Namely, take $r = \frac{1}{2}$ for any given point.

Bounded

Definition 1.51 (Bounded). A subset $X \subseteq \mathbb{C}$ is *bounded* if and only if there is an M such that $X \subseteq B(0, M)$.

Example 1.52. The star from earlier fits into a large circle and is therefore bounded.



And here is our fundamental definition for our topology.

Open

Definition 1.53 (Open). A subset $X \subseteq \mathbb{C}$ is *open* if and only if, for each $z \in X$, there exists $r > 0$ such that $B(z, r) \subseteq X$.

Remark 1.54 (Nir). We should probably show that open balls are open; let $B(z, r)$ be an open ball. Well, for any $w \in B(z, r)$, set $r_w := r - |z - w|$, which is positive because $w \in B(z, r)$ requires $|z - w| < r$. Now, $w' \in B(w, r_w)$ implies that $|w - w'| < r - |z - w|$, so by the triangle inequality,

$$|z - w'| \leq |z - w| + |w - w'| < r,$$

so $w' \in B(z, r)$ follows. So indeed, each $w \in B(z, r)$ has $B(w, r_w) \subseteq B(z, r)$.

Open lets us define closed.

Closed

Definition 1.55 (Closed). A subset $X \subseteq \mathbb{C}$ is *closed* if and only if $\mathbb{C} \setminus X$ is open.

Warning 1.56. Sets are not doors: a set can be both open and closed.

1.4.2 Unions and Intesections

Here are some basic properties of our topology.

Lemma 1.57. The subsets $\varnothing$ and $\mathbb{C}$ are open and closed in $\mathbb{C}$.

Proof. It suffices to show that $\varnothing$ and $\mathbb{C}$ are both open, by definition of closed. That $\varnothing$ is open holds vacuously because one cannot find any $z \in \varnothing$ anyways. That $\mathbb{C}$ is open holds because open balls are subsets of $\mathbb{C}$, so any $z \in \mathbb{C}$ can take $r = 1$ so that

$$B(z, r) \subseteq \mathbb{C}.$$

So we are done. ■

Lemma 1.58. Fixing some $z \in \mathbb{C}$, the set $\{z\}$ is closed.

Proof. We show that $U := \mathbb{C} \setminus \{z\}$ is open. Well, fix any $w \in U$, and because $w \neq z$, we note $|z - w| > 0$, so we set $r := \frac{1}{2}|z - w|$. It follows that

$$z \notin B(w, r)$$

because $|z - w| > r$. But this is equivalent to $B(w, r) \subseteq \mathbb{C} \setminus \{x\} = U$, so we are done. ■

We would like to make new open and closed subsets from old ones. Here is one way to do so.

Lemma 1.59. The following are true.

(a) Arbitrary union: if $\mathcal{U}$ is any collection of open subsets of $\mathbb{C}$, then the union $\bigcup_{U \in \mathcal{U}} U$ is also open.

(b) Arbitrary intersection: if $\mathcal{V}$ is any collection of closed subsets of $\mathbb{C}$, then intersection $\bigcap_{V \in \mathcal{V}} V$ is also closed.

Proof. We take these one at a time.

(a) Fix $z \in \bigcup_{U \in \mathcal{U}} U$. We need to show there is some $r > 0$ such that

$$B(z, r) \stackrel{?}{\subseteq} \bigcup_{U \in \mathcal{U}} U.$$

Well, we know there must be some $U_z \in \mathcal{U}$ such that $z \in U_z$ by definition of the union. But now U_z is open, and therefore we are promised an $r > 0$ such that

$$B(z, r) \subseteq U_z \subseteq \bigcup_{U \in \mathcal{U}} U,$$

so we are done.

(b) Fix $\mathcal{V}$ a collection of closed subsets of $\mathbb{C}$. We want to show that

$$\mathbb{C} \Bigg\backslash \bigcap_{V \in \mathcal{V}} V$$

is open, which by de Morgan's law is equivalent to

$$\bigcup_{V \in \mathcal{V}} (\mathbb{C} \setminus V)$$

being open. However, each $V \in \mathcal{V}$ is closed, so $\mathbb{C} \setminus V$ will be open, so we are done by (a). ■

Lemma 1.60. The following are true.

(a) Finite intersection: if $\{U_k\}_{k=1}^n$ is a finite collection of open subsets of $\mathbb{C}$, then the intersection $\bigcap_{k=1}^n U_k$ is also open.

(b) Finite union: if $\{V_k\}_{k=1}^n$ is a finite collection of closed subsets of $\mathbb{C}$, then $\bigcup_{k=1}^n V_k$ is also clsed.

Proof. We take these one at a time.

(a) Fix $z \in \bigcap_{k=1}^n U_k$ so that we need to find $r > 0$ such that

$$B(z, r) \overset{\subseteq}{\bigcup}{}_{k=1}^{n} U_k.$$

Well, $z \in U_k$ for each k, and each U_k is open, so there is an $r_k > 0$ such that $B(z, r_k) \subseteq U_k$. Thus, we set $r := \min_k\{r_k\}$; because there are only finitely many r_k, we are assured that $r > 0$. Now, we observe that

$$B(z, r) \subseteq B(z, r_k) \subseteq U_k.$$

(Explicitly, $|w - z| < r$ implies $|w - z| < r_k$ because $r \le r_k$.) Thus, it follows that

$$B(z, r) \subseteq \bigcap_{k=1}^n U_k,$$

as desired.

(b) We use de Morgan's laws. We want to show that

$$\mathbb{C} \Bigg\backslash \bigcup_{k=1}^n V_k$$

is open, which by de Morgan's laws is the same thing as showing that

$$\bigcap_{k=1}^n (\mathbb{C} \setminus V_k)$$

is open. However, each $\mathbb{C} \setminus V_k$ is open by hypothesis on the V_k, so the full intersection is open by (a). This finishes. ■

Remark 1.61. The finiteness is in fact necessary. For example,

$$\bigcap_{n \in \mathbb{N}} B(0, 1/n) = \{0\}.$$

Then one can check that each open ball is open while singletons in $\mathbb{C}$ are not.

1.4.3 Interior, Closure

Let's see more definitions.

Interior

Definition 1.62 (Interior). Given a subset $X \subseteq \mathbb{C}$, we define the *interior* X° of X to be the union of all open sets contained in X (which will be open by Lemma 1.59).

Remark 1.63. In fact, X° is the largest open subset of X, for any open subset $U_0 \subseteq \mathbb{C}$ contained in X will have

$$U_0 \subseteq \bigcup_{\text{open } U \subseteq X} U = X^\circ.$$

It follows X is open if and only if $X = X^\circ$: if $X = X^\circ$, then X is open because X° is open; if X is open, then X is the largest open subset of $\mathbb{C}$ contained in X, so $X = X^\circ$.

Closure

Definition 1.64 (Closure). Given a subset $X \subseteq \mathbb{C}$, we define the *closure* $\overline{X}$ of X to be the intersection of all closed sets containing X (which will be closed by Lemma 1.59).

Remark 1.65. In fact, X° is the smallest closed set containing X, for any closed subset $V_0 \subseteq \mathbb{C}$ containing X will have

$$V_0 \supseteq \bigcap_{\text{open } V \supseteq X} V = \overline{X}.$$

It follows X is closed if and only if $X = \overline{X}$: if $X = \overline{X}$, then X is open because $\overline{X}$ is closed; if X is closed, then X is the smallest closed subset of $\mathbb{C}$ containing X, so $X = \overline{X}$.

By the above definitions, it is not too hard to see that $X^\circ \subseteq X \subseteq \overline{X}$.

The interior and closure also let us define the boundary.

Frontier, boundary

Definition 1.66 (Frontier, boundary). Given a subset $X \subseteq \mathbb{C}$, we define the *frontier* or *boundary* ∂X of X to be $\overline{X} \setminus X^\circ$.

1.4.4 Connectivity

Disconnected

Definition 1.67 (Disconnected). A subset $X \subseteq \mathbb{C}$ is *disconnected* if and only if there exists nonempty disjoint open substets U_1 and U_2 such that $X \subseteq U_1 \cup U_2$ and $X \cap U_1, X_\cap U_2 \neq \varnothing$. (In other words, the subspace of $X \subseteq \mathbb{C}$ is (topologically) disconnected.) In this case, we say that U_1 and U_2 *disconnect* X.

Lastly, we say X is *connected* if and only if it is not disconnected.

Example 1.68. The set $\varnothing$ is connected because it is impossible for $U \cap \varnothing \neq \varnothing$ for any open set U of $\mathbb{C}$.

Example 1.69. Any singleton $\{z\}$ is connected. In fact, one cannot decompose $\{x\}$ into two disjoint sets at all, much less into disjoint sets of the form $U \cap \{x\}$ with U open.

Example 1.70. Any open ball $B(z, r)$ is connected. This is surprisingly annoying to check.

Example 1.71. The set $\{1,2\}$ is disconnected by $U_1 = B(1,1/2)$ and $U_2 = B(2,1/2)$.

Connectivity plays nicely with the rest of our definitions as well.

Lemma 1.72. A given subset $X \subseteq \mathbb{C}$ is connected if and only if the only subsets of X which are both open and closed (in the subspace topology) are $\varnothing$ and X.

Proof. We take the directions independently. For the forwards direction, take X connected, and suppose that $U \subseteq X$ is open and closed. In ths subspace topology, we get that $X \setminus U$ will also be open, and then the subsets U and $X \setminus U$ are both open, disjoint and have

$$X = U \cup (X \setminus U).$$

Thus, we require $U = \varnothing$ or $X \setminus U = \varnothing$, so $U \in \{\varnothing, X\}$.

We leave the reverse direction as an exercise. Suppose that X is disconnected, and we show that there is a nonempty proper closed and open subset of X. Well, because X is disconnected, we have disjoint open sets U_1 and U_2 of $\mathbb{C}$ such that $X \cap U_1, X \cap U_2 \neq \varnothing$ and $X \subseteq U_1 \cup U_2$. It follows that

$$X = (U_1 \cap X) \cup (U_2 \cap X). \tag{$*$}$$

However, now consider the open subset $U := U_1 \cap X$ of X. We note that $(U_1 \cap X) \cap (U_2 \cap X) = \varnothing$, so by $(*)$ we see that $U_1 \cap X = X \setminus (U_2 \cap X)$, so $U_1 \cap X$ is closed as well.

To finish, we note that $U \neq \varnothing$ is nonempty, and its complement is $X \setminus U = U_2 \cap X$ is also nonempty, so $U \neq X$ is proper. Thus, U is a proper nonempty closed and open subset of X. This finishes. ■

Remark 1.73 (Nir)**.** It is actually important that the open substes in the above lemma are in the subspace topology and are not required to be $\mathbb{C}$-open. For example, $X = \{1,2\}$ is disconnected, but it has no nonempty $\mathbb{C}$-open subsets to witness this.

Lemma 1.74. Fix $\mathcal{S}$ a collection of connected subsets of $\mathbb{C}$. If $\bigcap_{S \in \mathcal{S}} S$ is nonempty, then $\bigcup_{S \in \mathcal{S}} S$ will be connected.

Proof. Suppose $\bigcup_{S \in \mathcal{S}} S$ is contained in the disjoint open subsets U_1 and U_2 of $\mathbb{C}$; we claim $U_1 \cap \left(\bigcup_{S \in \mathcal{S}} S\right) = \varnothing$ or $U_2 \cap \left(\bigcup_{S \in \mathcal{S}} S\right) = \varnothing$, which will finish.

Pick up some

$$z \in \bigcap_{S \in \mathcal{S}} S,$$

which exists because the intersection is nonempty. Without loss of generality, we may assume that $z \in U_1$.

Now, $z \in S$ for each $S \in \mathcal{S}$, so we see $U_1 \cap S \neq \varnothing$, so because $(U_1 \cap S) \cup (U_2 \cap S) = S$, we see that $U_2 \cap S = \varnothing$ by hypothesis on S's connectivity. Thus, unioning over the $U_2 \cap S = \varnothing$,

$$U_2 \cap \left(\bigcup_{S \in \mathcal{S}} S\right) = \varnothing,$$

which finishes the proof. ■

Remark 1.75. The condition with nonempty intersection is necessary: $\{0\}$ and $\{1\}$ are connected, but $\{0\} \cup \{1\}$ is not.

1.5 January 28

Hopefully we'll be in-person on Monday. Homework 2 will be released later today, due next Friday.

1.5.1 Sequences

Today we're talking about sequences, building towards a theory of sequences and series. Next week we will begin studying holomorphic functions and actually doing complex analysis.

Anyways, here is a series of definitions.

Sequence

Definition 1.76 (Sequence)**.** A *sequence* of complex numbers is a function $f : \mathbb{N} \to \mathbb{C}$. Often we will notate this by $\{z_n\}_{n\in\mathbb{N}}$ where $z_n := f(n)$.

By convention, all of our sequences will be sequences of complex numbers unless otherwise stated.

Subsequence

Definition 1.77 (Subsequence)**.** A sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a *subsequence* of a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ if and only if there is some strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $w_n = z_{g(n)}$.

Bounded

Definition 1.78 (Bounded)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is *bounded* if and only if there exists a positive real number $M > 0$ such that

$$\{z_n\}_{n\in\mathbb{N}} \subseteq B(0, M).$$

In other words, $|z_n| < M$ for each $n \in \mathbb{N}$.

We are in particular interested in convergence in analysis.

Converges

Definition 1.79 (Converges)**.** A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *converges* to some $z \in \mathbb{C}$ if and only if, for each $\varepsilon > 0$, there exists some N such that $n > N$ implies

$$|z - z_n| < \varepsilon.$$

We will notate this by $z_n \to z$ or $\lim_{n\to\infty} z_n = z$.

Note that the definition of the limit above says that

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} |z_n - z| = 0.$$

Intuitively, the distance between the z_n and the z has to "narrow in" on z.

We would like some real-analytic tools for our complex analysis. Here is a convergence lemma.

Lemma 1.80. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a sequence. Then, letting $z_n := x_n + y_n i$, we have that $z_n \to z$ where $z = x + yi$ if and only if $x_n \to x$ and $y_n \to y$.

Proof. This is essentially by definition of the metric on $\mathbb{C}$. We take the directions one at a time.

- Suppose that $z_n \to z$ in $\mathbb{C}$. Then we claim that $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z_n$ in $\mathbb{R}$. Indeed, for any $\varepsilon > 0$, there is N such that

  $$n > N \implies |z - z_n| < \varepsilon.$$

  But now we see that $|\operatorname{Re} z_n - \operatorname{Re} z|, |\operatorname{Im} z_n - \operatorname{Im} z| \le \sqrt{(\operatorname{Re} z_n - \operatorname{Re} z)^2 + (\operatorname{Im} z_n - \operatorname{Im} z)^2}$, so it follows

  $$n > N \implies |\operatorname{Re} z_n - \operatorname{Re} z|, |\operatorname{Im} z_n - \operatorname{Im} z| < \varepsilon,$$

  finishing.

- Suppose that $\operatorname{Re} z_n \to x$ and $\operatorname{Im} z_n \to y$. We claim that $z_n \to x + yi$. Indeed, for any $\varepsilon > 0$, there exists N_x such that
$$n > N_x \implies |\operatorname{Re} z_n - x| < \varepsilon/2$$
and N_y such that
$$n > N_y \implies |\operatorname{Im} z_n - y| < \varepsilon/2.$$
It follows that
$$n > \max\{N_x, N_y\} \implies |z_n - (x+yi)| = \sqrt{|\operatorname{Re} z_n - x|^2 + |\operatorname{Im} z_n - y|^2} \le \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2} < \varepsilon.$$
This finishes. ■

Essentially, this means that checking convergence of complex numbers is the same as checking real and imaginary parts individually, so we can turn convergence questions into ones from real analysis.

We also have the following basic properties about convergence.

Proposition 1.81. Fix $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ a convergent sequence. The following are true.

(a) $\{z_n\}_{n\in\mathbb{N}}$ is bounded.

(b) The limit of $\{z_n\}_{n\in\mathbb{N}}$ is unique.

(c) Every subsequence of $\{z_n\}_{n\in\mathbb{N}}$ converges to z.

Proof. We take the claims one at a time. Let $z \in \mathbb{C}$ be so that $z_n \to z$.

(a) Fix $\varepsilon = 1$ so that there exists N so that $n > N$ implies $|z_n - z| < 1$. Now set
$$M := \max(\{|z_n| + 1 : n \le N\} \cup \{|z| + 1\}).$$
We claim that $|z_n| < M$ for each $n \in \mathbb{N}$. We have two cases.
  - If $n \le N$, then $|z_n| < |z_n| + 1 \le M$.
  - Otherwise $n > N$ so that
$$|z_n| \le |z_n - z| + |z| < |z| + 1 \le M,$$
  so we are done.

(b) Suppose that $z_n \to z'$ for some $z' \in \mathbb{C}$, and we show $z = z'$. Indeed, if $z = z'$, then we are done, so suppose that $z \ne z'$ so that $|z - z'| \ne 0$. Then we set $\varepsilon := \frac{1}{2}|z - z'| > 0$, and we are promised some N, N' such that
$$n > N \implies |z - z_n| < \frac{\varepsilon}{2} \quad \text{and} \quad n > N' \implies |z' - z_n| < \frac{\varepsilon}{2}.$$
In particular, we see that, for $n > \max\{N, N'\}$, we have
$$|z - z'| \le |z - z_n| + |z_n - z'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \frac{1}{2}|z - z'|.$$
But because $0 \le |z - z'|$, we see that this forces $|z - z'| = 0$, so $z = z'$ follows. (Technically we have hit contradiction, but we do not need to use this.)

(c) Note that subsequences can be characterized by choosing a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ so that we want to show $z_{f(n)} \to z$. Indeed, for any $\varepsilon > 0$, we are promised some N so that
$$n > N \implies |z - z_n| < \varepsilon.$$
Now, for each $n \in \mathbb{N}$, we have[3] $f(n) \ge n$, so we see that
$$n > N \implies f(n) > N \implies |z - z_{f(n)}| < \varepsilon,$$
which finishes. ■

Sequences themselves have an arithmetic.

[3] We can show this by induction on n, for $f(0) \ge 0$ and $f(n+1) > f(n)$ forces $f(n+1) \ge f(n) + 1$.

Proposition 1.82. Fix $\{z_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ sequences such that $z_n \to z$ and $w_n \to w$. Then the following hold.

(a) $z_n + w_n \to z + w$.

(b) $z_n w_n \to zw$.

(c) If $w \neq 0$ and $w_n \neq 0$ for each $n \in \mathbb{N}$, then $\frac{1}{w_n} \to \frac{1}{w}$.

Proof. We take these one at a time, essentially borrowing the proof from metric spaces.

(a) Fix some $\varepsilon > 0$. We can find some N_z such that

$$n > N_z \implies |z - z_n| < \varepsilon/2$$

and some N_w such that

$$n > N_w \implies |w - w_n| < \varepsilon/2.$$

Now, taking $N := \max\{N_z, N_w\}$ so that the triangle inequality gives

$$n > N \implies |(z+w) - (z_n + w_n)| \leq |z - z_n| + |w - w_n| < \varepsilon,$$

which finishes.

(b) We have to use the fact that the sequences are bounded here. Because $w_n \to w$, the sequence is bounded, so there is an M so that $|w_n| < M$ for each $n \in \mathbb{N}$. Now, the key inequality is that

$$|z_n w_n - zw| \leq |z_n w_n - z w_n| + |z w_n - zw| \leq M|z_n - z| + |z| \cdot |w_n - w|. \qquad (*)$$

So with this in mind, fix any $\varepsilon > 0$, and we see that we are promised N_z such that

$$n > N_z \implies |z_n - z| < \frac{\varepsilon}{2M}$$

and some N_w such that

$$n > N_w \implies |w_n - w| < \frac{\varepsilon}{2|z|}$$

so that $(*)$ implies

$$n > \max\{N_x, N_w\} \implies |z_n w_n - zw| < \varepsilon,$$

finishing.

(c) We begin with some motivating arithmetic. Observe that

$$\left|\frac{1}{w} - \frac{1}{w_n}\right| = \frac{|w_n - w|}{|w w_n|}.$$

We can upper-bound the numerator without tears, so we see the main difficulty is lower-bounding the denominator. Well, because $w \neq 0$, we can set $\varepsilon = |w|/2$ so that there exists N_0 such that

$$n > N_0 \implies |w_n - w| < |w|/2.$$

In particular, it follows that $|w_n| \geq |w| - |w - w_n| = |w|/2$ for $n > N_0$.
With this in mind, fix any $\varepsilon > 0$. Then we are promised some N_1 such that

$$n > N_1 \implies |w_n - w| < |w|^2\varepsilon/2$$

so that we see

$$n > \max\{N_0, N_1\} \implies \left|\frac{1}{w} - \frac{1}{w_n}\right| = \frac{|w_n - w|}{|w| \cdot |w_n|} \leq \frac{|w|^2\varepsilon/2}{|w| \cdot |w|/2} = \varepsilon,$$

finishing. ■

1.5.2 Limit Points

Here is our main character.

Limit point

Definition 1.83 (Limit point)**.** Fix $X \subseteq \mathbb{C}$ and some $z \in \mathbb{C}$. Then we say that z is a *limit point* if and only if there exists some sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to z$.

Accumulation point

Definition 1.84 (Accumulation point)**.** Fix $X \subseteq \mathbb{C}$ and some $z \in \mathbb{C}$. Then we say that z is an *accumulation point* if and only if there exists some sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq X \setminus \{z\}$ such that $z_n \to z$.

Essentially accumulation points do not allow isolated points while limit points do.

The above essentially gives a more directly topological definition of "closed set." It also gives us a more directly topological definition of the closure.

Lemma 1.85. Fix $X \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. The following are equivalent.

(a) We have that $z \in \overline{X}$.

(b) For all $\varepsilon > 0$, we have $B(z,\varepsilon) \cap X \neq \varnothing$.

(c) There exists $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to z$.

Proof. We show our directions one at a time.

- We show (a) implies (b). Suppose $z \in \overline{X}$, and for the sake of contradiction suppose we have $\varepsilon > 0$ such that $B(z,\varepsilon) \cap X = \varnothing$. In particular, $z \notin X$.

  Now, $z \in \overline{X}$ implies that z is contained in every closed set containing X by definition of $\overline{X}$. But because $B(z,\varepsilon)$ is open and is disjoint from X, we see
  $$z \in \overline{X} \subseteq \mathbb{C} \setminus B(z,\varepsilon),$$
  which is a contradiction.

- We show (b) implies (c). For each $n \in \mathbb{N}$, we know that $B(z,1/n) \cap X \neq \varnothing$, so we find some $z_n \in B(z,1/n)$. Now, for any $\varepsilon > 0$, choose $N := 1/\varepsilon$ so that
  $$n > N \implies |z_n - z| < \frac{1}{n} < \frac{1}{N} = \varepsilon,$$
  so indeed $z_n \to z$.

- We show (b) implies (a). We proceed by contraposition. Suppose that $z \notin \overline{X}$. It follows that $z \in \mathbb{C} \setminus \overline{X}$, which is open, so there exists an $r > 0$ such that
  $$B(z,r) \subseteq \mathbb{C} \setminus \overline{X} \subseteq \mathbb{C} \setminus X.$$
  It follows that $B(z,r) \cap X = \varnothing$.

- We show (c) implies (b). Suppose $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ has $z_n \to z$ for some $z \in \mathbb{C}$. For any $\varepsilon > 0$, there exists N such that
  $$n > N \implies |z_n - z| < \varepsilon,$$
  so in particular, choosing any $n := \lceil N \rceil + 1$ has $z_n \in B(z,\varepsilon) \cap X$, so $B(z,\varepsilon) \cap X \neq \varnothing$. ■

The above discussion can give us a more directly topological definition of "closed."

Lemma 1.86. A subset $X \subseteq \mathbb{C}$ is closed in $\mathbb{C}$ if and only if X contains all of its limit points.

Proof. By the previous lemma, we see that $z \in \overline{X}$ if and only if z is a limit point of X, so $\overline{X}$ is the set of limit points of X. Now, X is closed if and only if $X = \overline{X}$, so X is closed if and only if all limit points of X are in fact points of X. (Note that all points of X are automatically limit points essentially because $X \subseteq \overline{X}$ for free.) ■

While we're here, we can pick up a nice topological result.

Lemma 1.87. Fix $X \subseteq \mathbb{C}$ a connected subset. Then $\overline{X}$ is also connected.

Proof. This argument is purely topological. We proceed by contraposition: suppose $\overline{X}$ is disconnected by $U_1, U_2 \subseteq \mathbb{C}$. We claim that U_1, U_2 disconnect X. Well, we already know that $A \subseteq \overline{A} \subseteq U_1 \cup U_2$, and we already know that U_1 and U_2 are disjoint.

We claim that, for $U \subseteq \mathbb{C}$ an open subsets, if $U \cap \overline{X} \neq \varnothing$, then $U \cap X \neq \varnothing$ as well. Indeed, we proceed by contraposition: if $U \cap X = \varnothing$, then $X \subseteq \mathbb{C} \setminus U$, but $\mathbb{C} \setminus U$ is closed, so

$$\overline{X} \subseteq \mathbb{C} \setminus U,$$

so $\overline{X} \cap U = \varnothing$.

Thus, it follows from $U_1 \cap \overline{X}, U_2 \cap \overline{X} \neq \varnothing$ that $U_1 \cap X, U_2 \cap X \neq \varnothing$. This finishes the proof that U_1 and U_2 disconnect X. Indeed, ■

1.5.3 Cauchy Sequences

Just like in real analysis, we will be interested in Cauchy sequences.

Cauchy sequence

Definition 1.88 (Cauchy sequence). A sequence $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is a *Cauchy sequence* if and only if, for each $\varepsilon > 0$, there exists an N such that

$$n, m > N \implies |z_n - z_m| < \varepsilon.$$

We have the following results on Cauchy sequences.

Proposition 1.89. Fix $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ a sequence. If $\{z_n\}_{n \in \mathbb{N}}$ converges, it is Cauchy.

Proof. This proof uses no special properties of $\mathbb{C}$. If $z_n \to z$, then for a given $\varepsilon > 0$, there exists N such that

$$n > N \implies |z_n - z| < \varepsilon/2.$$

It follows that

$$n, m > N \implies |z_n - z_m| < |z_n - z| + |z_m - z| < \varepsilon,$$

finishing. ■

Proposition 1.90. Every Cauchy sequence in $\mathbb{C}$ converges.

Proof. If $\{z_n\}_{n \in \mathbb{N}}$ is Cauchy, then we claim $\{\operatorname{Re} z_n\}_{n \in \mathbb{N}}$ and $\{\operatorname{Im} z_n\}_{n \in \mathbb{N}}$ are Cauchy sequnces. Indeed, for any $\varepsilon > 0$, there exists N so that

$$n, m > N \implies |z_n - z_m| < \varepsilon,$$

but then $|\operatorname{Re} z_n - \operatorname{Re} z_m| < |z_n - z_m|$ and $|\operatorname{Im} z_n - \operatorname{Im} z_m| < |z_n - z_m|$, so the same N witnesses that $\{\operatorname{Re} z_n\}_{n \in \mathbb{N}}$ and $\{\operatorname{Im} z_n\}_{n \in \mathbb{N}}$ are Cauchy in $\mathbb{R}$.

Now, Cauchy sequences in $\mathbb{R}$ converge, so there are reals $x, y \in \mathbb{R}$ such that $\operatorname{Re} z_n \to x$ and $\operatorname{Im} z_n \to w$. It follows that $z_n \to x + yi$, finishing. ■

1.5.4 A Little More Topology

We close with one more topological definition.

Sequentially compact

Definition 1.91 (Sequentially compact). A subset $X \subseteq \mathbb{C}$ is *sequentially compact* if and only if every $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ has a convergent subsequence which converges in X.

Remark 1.92. This happens to be equivalent to X is compact because $\mathbb{C} \cong \mathbb{R}^2$ satisfies the fact that all compact sets are closed and bounded.

Example 1.93. Every finite set is compact.

And here is a last definition.

Tends to infinity

Definition 1.94 (Tends to infinity). A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ *tends to infinity* (notated $z_n \to \infty$) if and only if each $M > 0$ has some $N \in \mathbb{N}$ such that

$$n > N \implies |z_n| > M.$$

Essentially the points of $\{z_n\}_{n\in\mathbb{N}}$ wander infinitely away.

1.6 January 31

So we are lecturing in-person today. Good morning everyone.

Quote 1.95. If I don't fall off the stage, I will consider it a major accomplishment.

Homework 2 is due Friday, the 4th of February. Office hours will occur at the usual times, but they will now occur in-person at Evans 749.

1.6.1 Series

Today we're mostly talking about series, and on Friday we'll talk about continuous functions.

Series

Definition 1.96 (Series). An *infinite series* over $\mathbb{C}$ is an infinite sum

$$S := \sum_{n=1}^{\infty} z_n$$

where $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a sequence of complex numbers.

With respect to series, we really want to know when various series converge so that the series is well-defined.

Converge, diverge

Definition 1.97 (Converge, diverge). Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers, we define the mth partial sum to be

$$S_m := \sum_{n=0}^{m} z_m.$$

Then we say that the infinite series *converges* if and only if the sequence $\{S_m\}$ of partial sums converges. Otherwise, we say that S is *divergent*.

As usual, we start with some basic examples.

Exercise 1.98. Fix some $z \in \mathbb{C}$ with $|z| < 1$, we define $z_n := z^n$. Then we have

$$S = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Proof. Fix some partial sum

$$S_N := \sum_{k=0}^{N} z^k = 1 + z + z^2 + \cdots + z^N.$$

Multiplying by z, we see that

$$zS_n = z + z^2 + \cdots + z^N + z^{N+1}.$$

It follows that

$$S_N - zS_N = 1 - z^{N+1}.$$

Because $|z| < 1$, we have $z \neq 1$, so we may write

$$S_N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}.$$

However, we may note that as $N \to \infty$, the bad term z^{N+1} will have size

$$\left|z^{N+1}\right| = |z|^{N+1},$$

which goes to 0 (because $|z| < 1$).[4] It follows that

$$\lim_{N\to\infty} S_N = \frac{1}{1-z},$$

which is what we wanted. ■

Anyways, here are some basic lemmas.

Lemma 1.99 (Divergence test)**.** Suppose that $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers such that $\sum z_n$ converges. Then $z_n \to 0$ as $n \to \infty$.

Proof. Let S_n be the nth partial sum so that we are given $S_n \to L$ for some $L \in \mathbb{C}$. But now we see that

$$z_{n+1} = \left(\sum_{k=0}^{N+1} z_k\right) - \left(\sum_{k=0}^{N} z_k\right) = S_{n+1} - S_n.$$

Using limit laws, we see that

$$\lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = L - L = 0.$$

Shifting the indices back gives $z_n \to 0$ as $N \to \infty$. ■

Here is an important example of a divergent series.

[4] This is surprisingly annoying to rigorize with a ε-δ proof, so we won't do so here. The interested can try to use induction to manually bound $|z|^n$ by $\frac{c}{n}$ for some c.

Exercise 1.100. We claim that

$$S=\sum_{k=1}^{\infty}\frac{1}{k}$$

does not converge.

Proof. We will show that the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is not Cauchy, which will show that the series diverges. Well, observe that

$$S_{2^{n+1}}-S_{2^n}=\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k}$$

after cancelling out all of our terms. However, each term in the sum is at least $\frac{1}{2^{n+1}}$, so we may bound

$$S_{2^{n+1}}-S_{2^n}\geq\frac{1}{2^{n+1}}\left(2^{n+1}-2^n\right)=\frac{1}{2}.$$

We now show that the partial sums are not Cauchy. Fix ε. Supposing for the sake of contradiction that the sequence is Cauchy, there exists N so that $n,m>N$ has

$$|S_n-S_m|<\frac{1}{2}.$$

However, we can find some power of 2 named 2^r which exceeds N, in which case we find $2^{r+1},2^r>N$ and

$$|S_{2^{r+1}}-S_{2^r}|\geq\frac{1}{2},$$

which is our contradiction. ■

Remark 1.101. Because a sequence will converge if and only if its real and imaginary parts do, we can turn a convergene test into a real-number test by taking the real and imaginary parts of the sum.

1.6.2 The Comparison Test

Recall the comparison test in $\mathbb{R}$.

Theorem 1.102 (Comparison test). Fix $\{x_n\}_{n\in\mathbb{N}},\{y_n\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ sequences of real numbers. Further, suppose that we there exists a positive constatnt $c>0$ such that $0\leq x_n\leq cy_n$. Then the following hold.

- If $\sum y_n$ converges, then $\sum x_n$ converges as well.
- If $\sum x_n$ diverges, then $\sum y_n$ diverges as well.

Proof. We appeal to real analysis. The interested can see Theorem 2.1.21 in Eterović. The main point is to use the Monotone sequence theorem. ■

Here is an example.

Exercise 1.103. Fix $s>1$ an integer. Then the series

$$S=\sum_{k=1}^{\infty}\frac{1}{k^s}$$

converges.

Proof. Because s is an integer, we have $s \geq 2$. Namely, $\frac{1}{k^s} \leq \frac{1}{k^2}$, so by the comparison test it suffices to just show the convergence of

$$S' := \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

For this, we apply some trickery. In particular, for $k > 1$, we can bound

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

In particular,

$$S' = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} < 1 + \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right).$$

Thus, by the comparison test, it suffices to show the convergence of

$$T := \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right).$$

But the nth partial sum will telescope, giving

$$T_n := \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 - \frac{1}{n},$$

so $T_n \to 1$ as $n \to \infty$, and $T = 1$. It follows that S' is upper-bounded by $1 + T \leq 2$. ■

1.6.3 Absolute Convergence

The following kind of converence is nontrivially stronger, but that makes it better.

Absolute convergence

Definition 1.104 (Absolute convergence)**.** Fix a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of complex numbers. Then the sum $S := \sum z_n$ converges *absolutely* if and only if the series

$$\sum_{n=0}^{\infty} |z_n|$$

also converges. In other words, the partial sums of the above series converges.

We have the following quick lemma to justify naming this "convergence."

Lemma 1.105. If a series converges absolutely, then the series also converges.

Proof. The idea is to use the triangle inequality. Fix our series

$$S := \sum_{n=0}^{\infty} z_n$$

for which

$$T := \sum_{n=0}^{\infty} |z_n|$$

converges. Let S_n be the nth partial sum of S and T_n the n the partial sum of T.

Our goal is to show that $\{S_n\}_{n\in\mathbb{N}}$ is Cauchy. Observe $\{T_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers because $|z| \geq 0$ always. To start off our arithmetic, we note that, for $n, m \in \mathbb{N}$ with $n > mn$, we have

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} z_k \right|,$$

which by the triangle inequality can be bounded by

$$|S_n - S_m| \leq \sum_{k=m+1}^{n} |z_k| = T_m - T_n.$$

But now we can use the fact that $\{T_n\}_{n\in\mathbb{N}}$ must be Cauchy to finish: for any $\varepsilon > 0$, there exists some N such that $n > m > N$ implies $T_m - T_n < \varepsilon$. But then this same N promises $n > m > N$ implies

$$|S_n - S_m| < T_m - T_n < \varepsilon,$$

which is what we wanted. ■

Here is a surprise tool that will help us later.

Lemma 1.106. Fix a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of nonzero complex numbers. Further suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ tends to infinity (i.e., $|a_n| \to \infty$ as $n \to \infty$), then for any positive real number $r \in \mathbb{R}^+$, the series

$$\sum_{k=0}^{\infty} \left(\frac{r}{|a_k|} \right)^k$$

converges.

Proof. We need the a_n to be nonzero in order to allow division, so the real puzzle is to determine how to use the fact $|a_n| \to \infty$. Well, there exists some N such that $n > N$ has

$$|a_n| > 2r.$$

But then $\frac{r}{|a_n|} < \frac{1}{2}$ for each $n > N$, so we can use the comparison test as follows: observe that

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

will converge, and there will exist some $c > 1$ so that

$$\frac{r}{|a_k|} < \frac{c}{2^k}$$

for $0 \leq k \leq N$; and then for $n > N$, we get the above inequality anyways as discussed earlier (observe we took $c > 1$). ■

Quote 1.107. I can't break math on the first day of class. I can do it later on.

Lemma 1.108. Suppose that we have two series $S := \sum_{k\in\mathbb{N}} z_k$ and $T := \sum_{k\in\mathbb{N}} w_k$ are both absolutely convergent. Then the sum

$$P := \sum_{k=0}^{\infty} \left(\sum_{i+j=k} z_i w_j \right)$$

is absolutely convergent as well. In fact, P will converge to ST.

Proof. We sketch the result, and the remaining details are in Eterović. As usual, consider the partial sums

$$A_n := \sum_{k=0}^{n} |z_k| \qquad \text{and} \qquad B_n = \sum_{k=0}^{n} |w_k|,$$

both of which will converge as $n \to \infty$. Brazenly multiplying these together, we see that

$$A_n B_n = \sum_{i,j=0}^{n} |z_i w_j| = \sum_{k=0}^{n} \sum_{\substack{i+j=k \\ 0 \le i,j \le n}} |z_i w_j| + \sum_{k>n} \sum_{\substack{i+j=k \\ 0 \le i,j \le n}} |z_i w_j|.$$

In the first sum, observe that any time $i + j = k$, we will automatically have $i, j \le k \le n$. It follows that

$$A_n B_n = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_i w_j \right) + \underbrace{\sum_{\substack{i+j>n \\ 0 \le i,j \le n}} |z_i w_j|}_{R_n}.$$

The key claim is that $R_n \to 0$. The main idea is that $i + j > n$ implies that $i \ge n/2$ or $j \ge n/2$, so we can write

$$|R_n| \le \sum_{i=0}^{n} \sum_{j=n/2}^{n} |z_i w_j| + \sum_{i=n/2}^{n} \sum_{j=0}^{n} |z_i w_j| = \left(\sum_{i=0}^{n} |z_i| \right) \left(\sum_{j=n/2}^{n} |w_j| \right) + \left(\sum_{i=n/2}^{n} |z_i| \right) \left(\sum_{j=0}^{n} |w_j| \right).$$

Now, fix any $\varepsilon > 0$, and we show there exists X so that $n > X$ has $|R_n| < \varepsilon$. Note $A := \sum |z_k|$ and $B := \sum |w_k|$ both converge and hence have Cauchy partial sums. Because the partial sums are increasing, we may bound

$$|R_n| \le A \left(\sum_{j=n/2}^{n} |w_j| \right) + B \left(\sum_{i=n/2}^{n} |z_i| \right)$$

So there exists N such that $n > m > N$ has

$$\sum_{i=m+1}^{n} |z_i| < \frac{\varepsilon}{2B}$$

Similarly there exists M so that $n > m > M$ has

$$\sum_{j=m+1}^{n} |w_j| < \frac{\varepsilon}{2A},$$

from which it follows that $n > n/2 > \max\{N, M\}$ will have

$$|R_n| \le A \cdot \frac{\varepsilon}{2A} + B \cdot \frac{\varepsilon}{2B} = \varepsilon,$$

which finishes.

Now, because $R_n \to 0$, we see

$$\lim_{n \to \infty} \sum_{k=0}^{n} \left(\sum_{i+j=k} |z_i| \cdot |w_j| \right) = \lim_{n \to \infty} A_n B_n - \lim_{n \to \infty} R_n,$$

which does indeed converge, so indeed the series

$$\sum_{k=0}^{\infty} \left(\sum_{i+j=k} |z_i| \cdot |w_j| \right)$$

will converge. By the comparison test (using the triangle inequality), it follows that

$$P = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} z_i w_j \right)$$

will also absolutely converge.

To show that P converges to ST, we observe that the difference of the nth partial sum is

$$P_n - S_n T_n = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_i w_j \right) - \sum_{i,j=0}^{n} z_i w_j = \sum_{k=0}^{n} \left(\sum_{i+j=k} z_i w_j \right) - \sum_{k=0}^{n} \left(\sum_{i+j=k}^{n} z_i w_j \right) + \sum_{\substack{0 \le i,j \le n \\ i+j<n}} z_i w_j,$$

so

$$P_n - S_n T_n = \sum_{\substack{0 \le i,j \le n \\ i+j<n}} z_i w_j.$$

But by the triangle inequality, we see $|P_n - S_n T_n| \le R_n$, so $P_n - S_n T_n \to 0$ as $n \to \infty$. It follows P_n and $S_n T_n$ have the same limit as $n \to \infty$ (which exists because S_n and T_n have a limit). So indeed, $P = ST$. ■