202A: Introduction to Topology and Analysis

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Contents

THEME 1
METRIC SPACES

My personal view on spaces is that every space I ever work with is either metrizable or is the Zariski topology.

—Evan Chen, [Che22]

1.1 August 24

Good morning everyone. This is my first class of the semester.

1.1.1 Administrative Notes

Here are some housekeeping remarks.

- The webpage for this class is `math.berkeley.edu/ rieffel/202AannF22.html`.
- The midterm date is negotiable. We will have a vote on Friday. The possible dates are Friday 14 October, Monday 17 October, or Wednesday 19 October.
- There will be no vote on the final exam. It is on 15 December at 7PM.
- Homework will be due Fridays by midnight, approximately every week.
- There is no particular text for this course, and any given text covers more than we have time for. That said, we will (very) loosely follow [Lan12], but it is helpful to have a number of different expositions around.
- Please wear a mask during lectures and office hours.

Here is a summary of the course.

- We will spend the next couple of lectures talking about metric spaces.
- We will then spend the first half of the course on general topology. The second half of the course will be on measure and integration.
- Throughout we will see a little on functional analysis.

1.1.2 Metric Spaces

Hopefully we remember something about metric spaces. Here's the definition.

Definition 1.1 (Metric)**.** A *metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following rules for any $x, y, z \in X$.

(a) Zero: $d(x, x) = 0$.

(b) Zero: $d(x, y) = 0$ implies $x = y$.

(c) Symmetry: $d(x, y) = d(y, x)$.

(d) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

We call (X, d) a *metric space*.

We will want some "almost" metrics as well. Here are their names.

Definition 1.2 (Semi-metric)**.** A *semi-metric* d on a set X satisfies (a), (c), and (d) of Definition 1.1. We call (X, d) a *semi-metric space*.

Definition 1.3 (Extended metric)**.** An *extended metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}^{\infty}_{\geq 0}$ satisfying (a)–(d) of Definition 1.1. We call (X, d) an *extended metric space*.

Intuitively, we might want extended metrics if we have points that we never want to be able to get to from other ones.

We can turn spaces with a semi-metric into a space with a metric.

Lemma 1.4. Fix a semi-metric space (X, d), and define the relation $\sim$ on X by $x \sim y$ if and only if $d(x, y) = 0$. Then $\sim$ is an equivalence relation.

Proof. We run these checks by hand. Fix any $x, y, z \in X$.

- Reflexive: $d(x, x) = 0$ means that $x \sim x$.
- Symmetry: if $x \sim y$, then $d(x, y) = 0$, so $d(y, x) = 0$, so $y \sim x$.
- Transitive: if $x \sim y$ and $y \sim z$, then

$$0 \leq d(x, z) \leq d(x, y) + d(y, z) = 0,$$

so $d(x, z) = 0$, so $x \sim z$. ■

As such, given a semi-metric space (X, d), we may look at the set of equivalence classes under $\sim$, which we will denote $X/{\sim}$.[1]

Proposition 1.5. Fix a semi-metric space (X, d) and define $\sim$ as in Lemma 1.4. Then d naturally descends to a metric $\widetilde{d}$ on $X/{\sim}$.

Proof. Let $[x]$ denote the equivalence class of $x \in X$ under $\sim$. We claim that the function

$$\widetilde{d}([x], [y]) := d(x, y)$$

is a well-defined metric. We have the following checks; fix any $x, y, z \in X$.

[1] The notation of $/{\sim}$ is intended to make us think of quotients.

- Well-defined: if $x \sim x'$ and $y \sim y'$, then note that
$$d(x,y) \le d(x,x') + d(x',y) = d(x',y) \le d(x',y') + d(y',y) = d(x',y').$$
By symmetry, we also have $d(x',y') \le d(x,y)$, so equality follows. So d does descent properly to the quotient $X/\sim$.
- Zero: note that $\widetilde{d}([x],[y]) = 0$ if and only if $d(x,y) = 0$ if and only if $x \sim y$ if and only if $[x] = [y]$.
- Symmetry: note that
$$\widetilde{d}([x],[y]) = d(x,y) = d(y,x) = \widetilde{d}([y],[x]).$$
- Triangle inequality: note that
$$\widetilde{d}([x],[z]) = d(x,z) \le d(x,y) + d(y,z) = \widetilde{d}([x],[y]) + \widetilde{d}([y],[z]),$$
which finishes. ■

Here are some examples of metric spaces.

Example 1.6. Given a connected graph $G = (V, E)$ with a weighting function $w\colon E \to \mathbb{R}_{\ge 0}$, we can build a metric as follows: define the "shortest-path" function $d\colon V \times V \to \mathbb{R}_{\ge 0}$ sending two vertices $v, w \in V$ to the length of the shortest path. If the graph G is not connected, we merely have an extended metric.

Example 1.7 (Euclidean metric)**.** The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$
$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) \coloneqq \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$
is a metric.

Observe that it is not completely obvious that Example 1.7 satisfies the triangle inequality, but this will follow from the theory of the next subsections.

1.1.3 Norms on Vector Spaces

Norms provide convenient ways to build metrics.

Definition 1.8 (Norm)**.** Fix a vector space V over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|\colon V \to \mathbb{R}_{\ge 0}$ is a function satisfying the following, for any $r \in \mathbb{R}$ and $v, w \in V$.

(a) Zero: $\|v\| = 0$ if and only if $v = 0$.

(b) Scaling: $\|rv\| = |r| \cdot \|v\|$.

(c) Triangle inequality: $\|v + w\| \le \|v\| + \|w\|$.

Remark 1.9. We can probably work with a more general normed field instead of "merely" $\mathbb{R}$ or $\mathbb{C}$.

And here is our result.

Proposition 1.10. Given a metric space V with a norm $\|\cdot\| : V \to \mathbb{R}_{\ge 0}$, then the function
$$d(v,w) \coloneqq \|v - w\|$$
defines a metric on V.

Proof. We run the checks directly. Let $x, y, z \in V$ be points.

- Zero: note that $d(x,y) = 0$ if and only if $\|x-y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$.
- Symmetry: note that
$$d(x,y) = \|x-y\| = |-1| \cdot \|y-x\| = 1 \cdot \|y-x\| = d(y,x).$$
- Triangle inequality: note that
$$d(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \le \|x-y\| + \|y-z\| = d(x,y) + d(y,z),$$
which finishes the check. ■

Here are the usual examples.

Example 1.11. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then the following are norms on V.

- $\|(x_1,\ldots,x_n)\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.
- $\|(x_1,\ldots,x_n)\|_1 := \sum_{i=1}^n |x_i|$.

Here are some more esotetric examples.

Example 1.12. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then
$$\|(x_1,\ldots,x_n)\|_\infty := \sup\{|x_1|,\ldots,|x_n|\}$$
provides a norm on V.

Example 1.13. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then, given $p \ge 1$,
$$\|(x_1,\ldots,x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$
provides a norm on V.

Remark 1.14. Taking the limit as $p \to \infty$ of $\|f\|_p$ gives $\|f\|_\infty$. This justifies the notation.

Remark 1.15. Despite having lots of examples, all of these norms are equivalent in a topological sense.

These normed vector spaces actually allow us to define a metric on any subset.

Proposition 1.16. Given a metric space (X,d) and a subset $Y \subseteq X$, the restriction of d to $Y \times Y$ is a metric.

Proof. All the requirements for d on $Y \times Y$ are satisfied for any points in X, so we are done by doing no work. ■

Example 1.17. Any subset $X \subseteq \mathbb{R}^n$ has an induced metric by restricting the (say) Euclidean metric.

1.1.4 A Hint of L^p Spaces

Here is a more complicated example of a metric.

Example 1.18. Define $V := C([0,1])$ to be the $\mathbb{R}$-vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0,1]$. The following are norms.

- $\|f\|_\infty := \sup\{|f(x)| : x \in [0,1]\}$.
- $\|f\|_1 := \int_0^1 |f(t)|\,dt$.
- $\|f\|_2 := \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$.
- More generally, given $p \ge 1$
$$\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

These integrals are finite because $[0,1]$ is compact, forcing f to achieve a finite maximum on $[0,1]$.

Remark 1.19. We can tell the same story for $C(X)$, for any measurable compact space X.

Remark 1.20. Note the analogy of Example 1.18 with Example 1.13. To see this more rigorously, set X to be the finite set $\{1,\ldots,n\}$ so that $C(X) = \mathbb{R}^n$.

We should probably justify the claims of this subsection, so here is our result.

Proposition 1.21. Define $V := C([0,1])$ to be the vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0,1]$. Then, given $p \ge 1$, the function $\|\cdot\|_p : C \to \mathbb{R}_{\ge 0}$ by
$$\|f\| := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}$$
is a norm.

Proof. We run the checks directly.

- Zero: if $f = 0$, then of course $\int_0^1 |f(t)|^p\,dt = 0$.
- Zero: suppose that $f \in C([0,1])$ has $f(t_0) \ne 0$ for any $t_0 \in [0,1]$; set $y := f(t_0)$. Then $f^{-1}((y/2, 3y/2))$ is a nonempty open subset of X and hence contains a nonempty open interval (a,b) with $a < b$. As such,
$$\int_X |f(t)|^p\,dt \ge \int_a^b |f(t)|^p\,dt \ge \int_a^b |y/2|^p\,dt > 0,$$
so we are done.
- Scaling: given $f \in C([0,1])$ and a scalar r, we have
$$\|rf\| = \left(\int_0^1 |rf(t)|^p\,dt\right)^{1/p} = \left(|r|^p \int_0^1 |f(t)|^p\,dt\right)^{1/p} = |r| \cdot \|f\|.$$
- Triangle inequality: we borrow from [Tao09]. Given $f, g \in C([0,1])$, for psychological reasons we will assume that f and g are nonzero (else this is clear); then $\|f\|, \|g\| \ne 0$, so we may scale everything so that $\|f\| + \|g\|$. In fact, we may again use scaling to find $a, b \in V$ such that
$$f = (1-\theta)a \qquad \text{and} \qquad g = \theta b$$

where $\theta \in (0,1)$ and $\|a\| = \|b\| = 1$. Now, the triangle inequality translates into showing

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p \, dt = \|(1-\theta)a + \theta b\|_p^p \overset{?}{\le} \left(\|(1-\theta)a\|_p + \|\theta b\|_p\right)^p = 1.$$

Well, because $p \ge 1$, the function $t \mapsto t^p$ is convex, so we get to write

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p \, dt \le (1-\theta)\int_0^1 |a(t)|^p \, dt + \theta \int_0^1 |b(t)|^p \, dt,$$

which is what we wanted.

The above checks complete the proof; note that the proof of the triangle inequality was nontrivial. ■

Remark 1.22. Now, to show Remark 1.20, replace all $\int_0^1$ with $\sum_{i=1}^n$ and adjust all the language accordingly. The point is that "integrating over $[0,1]$" is analogous to "integrating over $\{1, \ldots, n\}$." A more thorough understanding of measure theory will allow us to rigorize this.

Next class we will talk about completeness.

1.2 August 26

Today we're talking about completeness of metric spaces.

1.2.1 Many Kinds of Morphisms

In mathematics, we are interested in objects not in isolation but as they relate to each other. Namely, we are interested also in the maps between our objects.

The philosophy here comes from category theory, where one is really most interested in the "morphisms" between "objects" instead of the objects themselves. For concreteness, here is a definition of a category.

Definition 1.23 (Category)**.** A *category* $\mathcal{C}$ consists of a class of objects $\operatorname{Ob}\mathcal{C}$ and morphisms $\operatorname{Mor}\mathcal{C}$ such that any two objects $A, B \in \operatorname{Ob}\mathcal{C}$ have a morphism set $\operatorname{Mor}(A,B)$. This data satisfy the following properties.

- Given $f \in \operatorname{Mor}(A,B)$ and $g \in \operatorname{Mor}(B,C)$, there is a composition $(g \circ f) \in \operatorname{Mor}(A,C)$.
- Given $A \in \operatorname{Ob}\mathcal{C}$, there is an identity morphism $\operatorname{id}_A \in \operatorname{Mor}(A,A)$.
- Identity: any $f \in \operatorname{Mor}(A,B)$ has $f \circ \operatorname{id}_A = f = \operatorname{id}_B \circ f$.
- Associativity: any $f \in \operatorname{Mor}(A,B)$ and $g \in \operatorname{Mor}(B,C)$ and $h \in \operatorname{Mor}(C,D)$ has $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 1.24. There is a category of groups, where the morphisms are group homomorphisms.

Example 1.25. There is a category of metric spaces, where the morphisms will be continuous maps.

One can actually specify more carefully what kinds of morphisms we're paying attention to. Here are some kinds of morphisms.

Definition 1.26 (Isometry)**.** Given metric spaces (X, d_X) and (Y, d_Y), an *isometry* is a function $f\colon X \to Y$ preserving the metric as

$$d_Y(f(x), f(x')) = d_X(x, x').$$

One can check quickly that the composition of two isometries is an isometry.

Remark 1.27. Of course, one can then ask for surjective isometries as well, whose compositions remain surjective. It turns out that being surjective promises an inverse isometry.

Isometries are somewhat restrictive, so we might weaken this as follows.

Definition 1.28 (Lipschitz continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *Lipschitz continuous* if and only if there is a constant $c \in \mathbb{R}$ such that

$$d_Y(f(x), f(x')) \le c \cdot d_X(x, x').$$

The smallest constant c which works here is called the *Lipschitz constant* for f. If f has a Lipschitz continuous inverse, then f is a *Lipschitz isomorphism*.

Again, one checks that the composition of Lipschitz continuous functions and Lipschitz isomorphisms are preserved under composition.

Remark 1.29. A good reason to care about this notion of continuity is that all normed vector spaces of some finite dimension n are Lipschitz isomorphic.

Here is yet a weaker notion of morphism.

Definition 1.30 (Uniformly continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *uniformly continuous* if and only if every $\varepsilon > 0$ has some $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \delta$$

for all $x, x' \in X$.

By rearranging quantifiers, we get another useful (but weaker) notion.

Definition 1.31 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *continuous* if and only if, at any $x \in X$, all $\varepsilon > 0$ have some $\delta_x > 0$ such that

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$

As usual, composition is preserved and so on.

Remark 1.32. In some sense, isometries and Lipschitz continuous functions have their definition fundamentally interrelated with the metric. In contrast, the weaker notion of continuity will readily generalize to general topological spaces. Uniform continuity also generalizes to "uniformities," which is a different notion.

1.2.2 Completeness

To discuss completeness, we need to talk about convergence.

Definition 1.33 (Converge)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n > N \implies d(x_n, x) < \varepsilon.$$

We might write this as $x_n \to x$ as $n \to \infty$.

We would like a notion of convergence which only uses data internal to the sequence, and this leads to the following definition.

Definition 1.34 (Cauchy)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy sequence* if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n, m > N \implies d(x_n, x_m) < \varepsilon.$$

One can check that all convergent sequences are Cauchy, and we in general hope that our Cauchy sequences will converge. As such, we have the following definition.

Definition 1.35 (Complete)**.** A metric space (X, d) is *complete* if and only if every Cauchy sequence in X converges to a point in X.

We are sad when a metric space is not complete, so we hope to have a way to make it complete. The most natural way to do this is by using the notion of density.

Definition 1.36 (Density)**.** Fix a metric space (X, d). Then $S \subseteq X$ is *dense* if and only if, given any $x \in X$ and $\varepsilon > 0$, we may find $x' \in S$ with $d(x, x') < \varepsilon$.

And here is our completion.

Definition 1.37 (Completion)**.** A *completion* of the metric space (X, d) is a metric space $(\overline{X}, \overline{d})$ equipped with an isometry $\iota \colon X \to \overline{X}$ such that $(\overline{X}, \overline{d})$ is complete and $\operatorname{im} \iota$ is dense in $\overline{X}$.

One can show that any metric space has a completion and that they are all isometric and therefore in some sense the same. The uniqueness result will appear on the homework, so for now we will discuss existence.

Theorem 1.38. Any metric space (X, d) has a completion.

Sketch. Let $\widetilde{X}$ denote the set of all Cauchy sequences in X. We hope to make $\widetilde{X}$ into our completion, but this requires a little care. To begin, we have the following lemma.

Lemma 1.39. Given a metric space (X, d) with two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, then the sequence

$$\{d(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$$

converges.

Proof. Omitted. ■

Thus, we define $\widetilde{d} \colon \widetilde{X} \times \widetilde{X} \to \mathbb{R}_{\geq 0}$ be this well-defined function. One can show that $\widetilde{d}$ is a semi-norm, and so we use Proposition 1.5 to induce a metric $\overline{d}$ on $\overline{X} := \widetilde{X}/{\sim}$.

To finish the proof, one has to check that $\overline{X}$ is in fact a completion. It is somewhat annoying to check that $\overline{X}$ is complete (though it is not terribly tricky), and the required isometry $\iota\colon X \to \overline{X}$ is given by

$$\iota(x) := \{x\}_{n\in\mathbb{N}},$$

which is certainly Cauchy. ■

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List of Definitions