258: Harmonic Analysis

Nir Elber

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

1.1 August 28

Why am I here?

1.1.1 Logistics

Here are the usual logistics notes.

- The professor is Ruixiang Zhang.
- There will be three assignments, which determine the grade. They will be rather hard.
- Office hours are on Wednesday, during 10:30AM–11:30AM, 2PM–3PM, and 3PM–4PM.

1.1.2 Convergence of Fourier Series

The point of the course is to study differentiable functions on a space which has an action by a group. Last class we proved the following result.

Theorem 1.1 (Riemann localization principle)**.** Fix a 1-periodic function $f \in L^1(\mathbb{R}/\mathbb{Z})$ which vanishes in a neighborhood of $x \in \mathbb{R}$. Then

$$\lim_{N\to\infty} S_N f(x) = 0.$$

Here,

$$S_N f(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x},$$

where

$$\hat{f}(k) := \int_0^1 f(x) e^{-2\pi i k x}\, dx.$$

Anyway, here is a quick sketch.

Sketch of Theorem 1.1. One can show that $\hat{f}(n) \to 0$ as $|n| \to \infty$ by approximating $f \in L^1(\mathbb{R}/\mathbb{Z})$ by simple integrable functions. Then one uses a geometric series style argument to get cancellation, writing

$$S_N f(x) = \int_0^1 \frac{\sin(N+1)\pi t}{\sin \pi t} \cdot f(x-t)\, dt$$

and then expressing the integral as a sum of Fourier coefficients of functions in $L^1(\mathbb{R}/\mathbb{Z})$. ■

We are now ready to show Dini's criterion.

Theorem 1.2 (Dini's criterion)**.** Fix a function $f \in L^1(\mathbb{R}/\mathbb{Z})$ and $x \in \mathbb{R}$. Then suppose that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty$$

for all $\delta > 0$. Then $S_N f(x) \to f(x)$ as $N \to \infty$.

Proof. We take $\delta < 1/2$. Using the Dirichlet kernel

$$D_N(x) := \sum_{|k| \le N} e^{2\pi i k x} = \frac{\sin(2N+1)\pi x}{\sin \pi x},$$

one has

$$\begin{aligned} S_N f(x) - f(x) &= \int_{-1/2}^{1/2} f(x-t) D_N(t)\, dt - f(x) \\ &= \int_{-1/2}^{1/2} (f(x-t) - f(x)) D_N(t)\, dt \\ &= \underbrace{\int_{|t|<\delta} (f(x-t) - f(x)) D_N(t)\, dt}_{I_1} + \underbrace{\int_{\delta \le |t| \le 1/2} (f(x-t) - f(x)) D_N(t)\, dt}_{I_2}. \end{aligned}$$

The argument of Theorem 1.1 establishes that $I_2 \to 0$ as $N \to \infty$, so it is safe, or one can directly see that we have essentially constructed a function which vanishes on an interval around x and took its Fourier transform. For I_1, we bound by absolute value, we see

$$|I_1| \le \int_{|t|<\delta} \left| \frac{f(x-t) - f(x)}{\sin \pi t} \right| dt \ll \int_{|t|<\delta} \left| \frac{f(x-t) - f(x)}{t} \right| dt,$$

which disappears as we take δ small. Namely, taking $\delta' \le \delta$, the hypothesis tells us that

$$\int_{|t|<\delta'} \left| \frac{f(x-t) - f(x)}{t} \right| dt < \infty,$$

so finiteness of the integral at $\delta = \delta'$ enforces it to go to 0 as $\delta' \to 0^+$. ■

It is not clear what the hypothesis in Theorem 1.2 is good for, but we will use it shortly; as an example application, Hölder continuous functions satisfy the condition. But notably, continuity is not good enough to give us convergence. Anyway, here is another criterion.

Theorem 1.3 (Jordan's criterion)**.** Fix a function $f \in L^1(\mathbb{R}/\mathbb{Z})$ and $x \in \mathbb{R}$. Further, suppose that f is of bounded variation in $(x-\delta, x+\delta)$ for some $\delta > 0$. Then

$$\lim_{N \to \infty} S_n f(x) = \frac{f(x_-) + f(x_+)}{2},$$

where $f(x_\pm)$ denotes the value of $f(a)$ as $a \to x^\pm$.

Proof. Being bounded variation here roughly means that it is the difference of two monotonic functions. Again, we take $\delta < 1/2$. Then Theorem 1.1, we may also assume that f vanishes outside $(x-\delta, x+\delta)$.

(Namely, the convergence is local to x, so we can subtract out $g(t) \coloneqq f(t)1_{|t-x|>\delta}(t)$.) Now,

$$\begin{aligned} S_N f(x) &= \int_{-1/2}^{1/2} f(x-t) D_N(t)\, dt \\ &= \int_0^{1/2} (f(x+t) + f(x-t)) D_N(t)\, dt. \end{aligned}$$

We now set $g(t) \coloneqq f(x+t) + f(x-t)$, essentially fixing x, so we want to show

$$\lim_{N\to\infty} \int_0^{1/2} g(t) D_N(t)\, dt = \frac{1}{2} g(0_+).$$

Subtracting f by $\frac{1}{2}g(0_+)$, we may assume that $g(0_+) = 0$. Also, f is the difference of two monotonic functions, and the above condition is linear, so we may as well assume that g is monotonic.

As before, take $\delta' < \delta$, and we split the integral into two parts, writing

$$\int_0^{1/2} g(t) D_N(t)\, dt = \underbrace{\int_0^{\delta'} g(t) D_N(t)\, dt}_{I_1 :=} + \underbrace{\int_{\delta'}^{\delta} g(t) D_N(t)\, dt}_{I_2 :=}.$$

Theorem 1.1 tells us that $I_2 \to 0$ as $N \to \infty$ because we are away from 0. Using a Mean value theorem argument, one finds

$$\int_0^{\delta'} g(t) D_N 9t)\, dt = g(\delta'_-) \int_v^{\delta'} D_N(t)\, dt$$

for some $v \in [0, \delta']$. To get convergence as $N \to \infty$, one needs to use cancellation within D_N. Well, we find

$$\int_v^{\delta'} D_N(t)\, dt = \int_v^{\delta'} \frac{\sin(2N+1)\pi t}{\sin \pi t}\, dt.$$

One would like to replace $\sin \pi t$ with t so that dt/t is the multiplicative Haar measure on $\mathbb{R}^\times$. Explicitly,

$$\left| \int_v^{\delta'} D_N(t)\, dt \right| = \left| \int_v^{\delta'} \sin(2N+1)\pi t \cdot \left(\frac{1}{\sin \pi t} - \frac{1}{\pi t} \right)\, dt \right| + \left| \int_v^{\delta'} \frac{\sin(2N+1)\pi t}{t}\, dt \right|.$$

We now see $\frac{1}{\sin \pi t} - \frac{1}{\pi t}$ is bounded by a constant in $[v, \delta']$, so the entire integral is also bounded by a constant; notably, this constant vanishes as $\delta' \to 0^+$. Applying a change of variables to the second term, we see that it is bounded by

$$\sup_{0<c_1<c_2<\delta'} \left| \int_{c_1}^{c_2} \frac{\sin \pi t}{t}\, dt \right|,$$

which also vanishes as $\delta' \to 0^+$, completing the proof. ■

BIBLIOGRAPHY

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.