261A: Lie Groups

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

INTRODUCTION

1.1 August 28

Today we review differential topology. Here are some logistical notes.

- There will be weekly homeworks, of about 5 problems.

- There will be a final take-home exam.

- This course has a bCourses page.

- We will mostly follow Kirillov's book [Kir08].

1.1.1 Group Objects

The goal of this class is to study symmetries of geometric objects. As such, we are interested in studying (infinite) groups with some extra geometric structure, such as a real manifold or a complex manifold or a scheme structure. Speaking generally, we will have some category $\mathcal{C}$ of geometric objects, equipped with finite products (such as a final object), which allows us to have group objects in $\mathcal{C}$.

> **Definition 1.1** (group object). Fix a category $\mathcal{C}$ with finite products, such as a final object $*$. A *group object* is the data (G, m, e, i) where $G \in \mathcal{C}$ is an object and $m\colon G \times G \to G$ and $e\colon * \to G$ and $i\colon G \to G$ are morphisms. We require this data to satisfy some associativity, identity, and inverse coherence laws.

For concreteness, we go ahead and write out the coherence diagrams, but they are not so interesting.

- Associative: the following diagram commutes.

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mathrm{id}_G \times m} & G \times G \\
{\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow{\quad m \quad} & G
\end{array}
$$

- Identity: the following diagram commutes.

$$
\begin{array}{ccccc}
G & \xrightarrow{\mathrm{id}_G \times e} & G \times G & \xleftarrow{e \times \mathrm{id}_G} & G \\
 & \searrow & \downarrow{\scriptstyle m} & \swarrow & \\
 & & G & &
\end{array}
$$

- Inverses: the following diagram commutes.

$$
\begin{array}{ccc}
 & G \times G & \\
{\scriptstyle i \times \mathrm{id}_G} \nearrow & & \searrow {\scriptstyle m} \\
G \xrightarrow{\quad} * \xrightarrow{\ e\ } & & G \\
{\scriptstyle \mathrm{id}_G \times i} \searrow & & \nearrow {\scriptstyle m} \\
 & G \times G &
\end{array}
$$
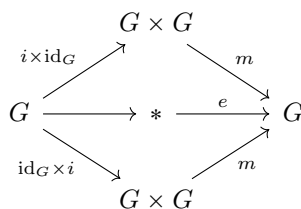
Example 1.2. In the case where $\mathcal{C} = \mathrm{Set}$, we recover the notion of a group, where G is the set, m is the multiplication law, e is the identity, and i is the inverse.

Example 1.3. Group objects in the category of manifolds will be Lie groups.

1.1.2 Review of Topology

This course requires some topology as a prerequisite, but let's review these notions for concreteness. We refer to [Elb22] for most of these notions.

Definition 1.4 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ of a set X and collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of open subsets of X, which we require to satisfy the following axioms.

- $\varnothing, X \in \mathcal{T}$.

- Finite intersection: for $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.

- Arbitrary unions: for a subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We will suppress the notation $\mathcal{T}$ from our topological space as much as possible.

Example 1.5. The set $\mathbb{R}$ equipped with its usual (metric) topology is a topological space.

Example 1.6. Given a topological space X and a subset $Z \subseteq X$, we can make Z into a topological space with open subsets given by $U \cap Z$ whenever $U \subseteq Z$ is open.

Definition 1.7 (closed)**.** A subset Z of a topological space X is *closed* if and only if $X \backslash Z$ is open.

One way to describe topologies is via a base.

Definition 1.8 (base)**.** Given a topological space X, a *base* $\mathcal{B} \subseteq \mathcal{P}(X)$ for the topology such that any open subset $U \subseteq X$ is the union of a subcollection of $\mathcal{B}$. Equivalently, for any open subset $U \subseteq X$ and $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example 1.9. The collection of open intervals $(a, b) \subseteq \mathbb{R}$ generates the usual topology. In fact, one can even restrict ourselves to open intervals (a, b) where $a, b \in \mathbb{Q}$, so $\mathbb{R}$ has a countable base.

Our morphisms are continuous maps.

> **Definition 1.10** (continuous)**.** A function $f \colon X \to Y$ between topological spaces is *continuous* if and only if $f^{-1}(V) \subseteq X$ is open for each open subset $V \subseteq Y$.

Thus, we can define Top as the category of topological spaces equipped with continuous maps as its morphisms. Thinking categorically allows us to make the following definition.

> **Definition 1.11** (homeomorphism)**.** A *homeomorphism* is an isomorphism in Top. Namely, a function $f \colon X \to Y$ between topological spaces which is continuous and has a continuous inverse.

> **Remark 1.12.** There are continuous bijections which are not homeomorphisms! For example, one can map $[0, 2\pi) \to S^1$ by sending $x \mapsto e^{ix}$, which is a continuous bijection, but the inverse is discontinuous at $1 \in S^1$.

Earlier, we wanted to have finite products in our category. Here is how we take products of pairs.

> **Definition 1.13** (product topology)**.** Given topological spaces X and Y, we define the topological space $X \times Y$ as having $X \times Y$ as its set and open subsets given by arbitrary unions of sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open.

> **Remark 1.14.** Alternatively, we can say that the topology $X \times Y$ has a base given by the "rectangles" $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open. In fact, if $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, respectively, then we can check that the open subsets
> $$\{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$
> is a base for $X \times Y$.

> **Remark 1.15.** The final object in Top is the singleton space.

Now, group objects in Top are called topological groups, which are interesting in their own right. For example, locally compact topological groups have a good Fourier analysis theory.

> **Example 1.16.** The group $\mathbb{R}$ under addition is a topological group. In fact, $\mathbb{Q}$ under addition is also a topological group, though admittedly a more unpleasant one.

> **Example 1.17.** The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a topological group.

1.1.3 Review of Differential Topology

However, in this course, we will be more interested in manifolds, so let's define these notions. We refer to [Elb24] for (a little) more detail, and we refer to [Lee13] for (much) more detail. To begin, we note arbitrary topological spaces are pretty rough to handle; here are some niceness requirements. The following is a smallness assumption.

> **Definition 1.18** (separable)**.** A topological space X is *separable* if and only if it has a countable base.

The following says that points can be separated.

> **Definition 1.19** (Hausdorff)**.** A topological space X is *Hausdorff* if and only if any pair of distinct points $p, q \in X$ have disjoint open neighborhoods.

The following is another smallness assumption, which we will use frequently but not always.

> **Definition 1.20** (compact)**.** A topological space X is *compact* if and only if any open cover $\mathcal{U}$ (i.e., each $U \in \mathcal{U}$ is open, and $X = \bigcup_{U \in \mathcal{U}} U$) has a finite subcollection which is still an open cover.

We are now ready for our definition.

> **Definition 1.21** (topological manifold)**.** A *topological manifold of dimension n* is a topological space X satisfying the following.
>
> - X is Hausdorff.
>
> - X is separable.
>
> - Locally Euclidean: X has an open cover $\{U_\alpha\}_{\alpha \in \kappa}$ such that there are open subsets $V_\alpha \subseteq \mathbb{R}^n$ and homeomorphisms $\varphi_\alpha \colon U_\alpha \to V_\alpha$.

> **Remark 1.22.** By passing to open balls, one can require that all the V_α are open balls. By doing a little more yoga with such open balls (noting $B(0,1) \cong \mathbb{R}^n$), one can require that $V_\alpha = \mathbb{R}^n$ always.

> **Remark 1.23.** It turns out that open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ can only be homeomorphic if and only if $n = m$. This implies that the dimension of a connected component of X is well-defined without saying what n is in advance. However, we should say what n is in advance in order to get rid of pathologies like $\mathbb{R} \sqcup \mathbb{R}^2$.

To continue, we must be careful about our choice of U_αs and φ_αs.

> **Definition 1.24** (chart, atlas, transition function)**.** Fix a topological manifold X of dimension n.
>
> - A *chart* is a pair (U, φ) of an open subset $U \subseteq X$ and homeomorphism φ of U onto an open subset of $\mathbb{R}^n$.
> - An *atlas* is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$ such that $\{U_\alpha\}_{\alpha \in \kappa}$ is an open cover of X.
> - The *transition function* between two charts (U, φ) and (V, ψ) is the composite homeomorphism
>
> $$\varphi(U \cap V) \xleftarrow{\varphi} (U \cap V) \xrightarrow{\psi} \psi(U \cap V).$$
>
> Note that there is also an inverse transition map going in the opposite direction.

Let's see some examples.

> **Example 1.25.** The space $\mathbb{R}^n$ is a topological manifold of dimension n. It has an atlas with the single chart $\mathrm{id} \colon \mathbb{R}^n \to \mathbb{R}^n$.

> **Example 1.26.** The singleton $\{*\}$ is a topological manifold of dimension 0. In fact, $\{*\} = \mathbb{R}^0$.

> **Example 1.27.** The hypersurface $S^n \subseteq \mathbb{R}^{n+1}$ cut out by the equation
>
> $$x_0^2 + \cdots + x_n^2 = 1$$
>
> is a topological manifold of dimension n. It has charts given by stereographic projection out of some choice of north and south poles. Alternatively, it has charts given by the projection maps $\mathrm{pr}_i \colon S^n \to \mathbb{R}^n$ given by deleting the ith coordinate, defined on the open subsets
>
> $$U_i^\pm := \{(x_0, \ldots, x_n) \in \mathbb{R}^n : \pm x_i > 0\}$$
>
> for choice of index i and sign in $\{\pm\}$.

Calculus on our manifolds will come from our transition maps.

> **Definition 1.28.** An atlas $\mathcal{A}$ on a topological manifold X is C^k, real analytic, or complex analytic (if $\dim X$ is even) if and only if the transition maps have the corresponding condition.

1.2 August 30

Today we finish our review of smooth manifolds. Once again, we refer to [Elb24] for a few more details and [Lee13] for many more details.

> **Notation 1.29.** We will use the word *regular* to refer to one of the regularity conditions C^k, smooth, real analytic, or complex analytic. We may abbreviate complex analytic to "complex" when no confusion is possible. We use the field $\mathbb{F}$ to denote the "ground field," which is $\mathbb{C}$ when considering the complex analytic case and $\mathbb{R}$ otherwise.

1.2.1 Smooth Manifolds

We now define a regular manifold.

> **Definition 1.30** (regular manifold)**.** A *regular manifold* of dimension n is a pair $(M, \mathcal{A})$ of a topological manifold M and a maximal regular atlas $\mathcal{A}$; a chart is called regular if and only if it is in $\mathcal{A}$. We will eventually suppress the $\mathcal{A}$ from our notation as much as possible.

The reason for using a maximal atlas is to ensure that it is more or less unique.

> **Remark 1.31.** Here is perhaps a more "canonical" way to deal with atlas confusion. One can say that two regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if and only if the transition maps between them are also regular; this is the same as saying that $\mathcal{A}_1 \cup \mathcal{A}_2$ is regular. Compatibility forms an equivalence relation, and each equivalence class $[\mathcal{A}]$ has a unique maximal element, which one can explicitly define as
>
> $$\mathcal{A}_{\max} := \{(U, \varphi) : \mathcal{A} \text{ and } (U, \varphi) \text{ are compatible}\}.$$
>
> This explains why it is okay to just work with maximal atlases.

> **Example 1.32.** One can give the topological manifold $\mathbb{R}^2$ many non-equivalent complex structures. For example, one has the usual choice of $\mathbb{R}^2 \cong \mathbb{C}$, but one can also make $\mathbb{R}^2$ homeomorphic to $B(0,1) \subseteq \mathbb{C}$.

Example 1.33. There are "exotic" smooth structures on S^7.

Example 1.34. Given regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, one can form the product manifold $X \times Y$. It should have maximal atlas compatible with the atlas

$$\{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B}\}.$$

1.2.2 Regular Functions

With any class of objects, we should have morphisms.

Definition 1.35. A function $f \colon X \to Y$ of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is regular if and only if any $p \in X$ has a choice of charts $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $f(U) \subseteq V$ and the composite

$$\varphi(U) \xleftarrow{\varphi} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is a regular function between open subsets of Euclidean space.

Remark 1.36. One can replace the single choice of charts above with any choice of charts satisfying $p \in U$ and $f(U) \subseteq V$.

Remark 1.37. Here is another way to state this: for any open $V \subseteq Y$ and smooth function $h \colon V \to \mathbb{F}$, the composite

$$f^{-1}(U) \xrightarrow{f} V \xrightarrow{h} \mathbb{F}$$

succeeds in being smooth (in any local coordinates).

Definition 1.38 (diffeomorphism)**.** A *diffeomorphism* of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is a regular map $f \colon X \to Y$ with regular inverse.

Remark 1.39. Alternatively, one can say that the charts in $\mathcal{A}$ and the charts in $\mathcal{B}$ are in natural bijection via f. Checking that these notions align is not too hard.

The above definition of regular map is a little rough to handle, so let's break it down into pieces.

Definition 1.40 (local coordinates)**.** Fix a regular manifold $(X, \mathcal{A})$ of dimension n. Then a system of *local coordinates* around some point $p \in X$ is a choice of regular chart $(U, \varphi) \in \mathcal{A}$ for which $\varphi(p) = 0$. From here, our local coordinates $(x_1, \ldots, x_n)$ are the composite of φ with a coordinate projection to the ground field. (In the complex analytic case, we want the ground field to be $\mathbb{C}$; otherwise, the ground field is $\mathbb{R}$.)

Now, we are able to see that a function $f \colon X \to \mathbb{R}$ is regular if and only if it becomes regular in local coordinates. One can even define regularity with respect to a subset of X.

Regularity allows us to produce lots of manifolds, as follows.

Theorem 1.41. Given regular maps $f_1, \ldots, f_m \colon X \to \mathbb{F}$, the subset

$$\{p \in \mathbb{F}^n : f_1(p) = \cdots = f_m(p) = 0 \text{ and } \{df_1(p), \ldots, df_n(p)\} \text{ are linearly independent}\}$$

is a manifold of dimension $n - m$.

Sketch. This is more or less by the implicit function theorem; for the $\mathbb{F} = \mathbb{R}$ cases, one can essentially follow [Lee13, Corollary 5.14]. ∎

> **Example 1.42.** The function $\mathbb{R}^{n+1} \to \mathbb{R}$ given by $(x_0, \ldots, x_n) \mapsto x_0^2 + \cdots + x_n^2$ is real analytic and sufficiently regular at the value 1, which establishes that S^n defined in Example 1.27 succeeds at being a real analytic manifold.

Functions to $\mathbb{F}$ have a special place in our hearts, so we take the following notation.

> **Notation 1.43.** Give a regular manifold X and any open subset $U \subseteq X$, we let $\mathcal{O}_X(U)$ denote the set of regular functions $U \to \mathbb{F}$

> **Remark 1.44.** One can check that the data $\mathcal{O}_X$ assembles into a sheaf. Namely, an inclusion $U \subseteq V$ produces restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

> **Remark 1.45.** Once we have all of our regular functions out of X, we note that some Yoneda-like philosophy explains that the sheaf of X determines its full regular structure. Here is an explicit statement: given a manifold X and two maximal regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ determining sheaves of regular functions $\mathcal{O}_1$ and $\mathcal{O}_2$, having $\mathcal{O}_1 = \mathcal{O}_2$ forces $\mathcal{A}_1 = \mathcal{A}_2$. Indeed, it is enough to show the inclusion $\mathcal{A}_1 \subseteq \mathcal{A}_2$, so suppose (U, φ) is a regular chart in $\mathcal{A}_1$. Then the corresponding local coordinates $(x_1, \ldots, x_n)$ all succeed at being regular for $\mathcal{A}_1$, so they are smooth functions in $\mathcal{O}_1$, so they live in $\mathcal{O}_2$ also, so (U, φ) will succeed at being a regular local diffeomorphism for $\mathcal{A}_2$ and hence be a regular chart.

Sheaf-theoretic notions tell us that we should be interested in germs.

> **Definition 1.46** (germ). Fix a point p on a regular manifold X. A *germ* of a regular function $f \in \mathcal{O}_X(U)$ (where $p \in U$) is the equivalence class of functions $g \in \mathcal{O}_X(V)$ (for a possibly different open subset V containing p) such that $f|_{U \cap V} = g|_{U \cap V}$. The collection of equivalence classes is denoted $\mathcal{O}_{X,p}$ and is called the stalk at p.

1.3 September 4

Today we hope to finish our review of differential topology.

> **Convention 1.47.** For the remainder of class, our manifolds will be smooth, real analytic, or complex analytic.

1.3.1 Tangent Spaces

Now that we are thinking locally about our functions via germs, we can think locally about our tangent spaces.

> **Definition 1.48** (derivation). Fix a point p on a regular manifold X. A *derivation* at p is an $\mathbb{F}$-linear map $D \colon \mathcal{O}_{X,p} \to \mathbb{F}$ satisfying the Leibniz rule
> $$D(fg) = g(p)D(f) + f(p)D(g).$$

> **Definition 1.49** (tangent space). Fix a point p on a regular manifold X. Then the *tangent space* T_pX is the $\mathbb{F}$-vector space of all derivations on $\mathcal{O}_{X,p}$.

As with everything in this subject, one desires a local description of the tangent space.

> **Lemma 1.50.** Fix an n-dimensional regular manifold X and a point $p \in X$. Equip p with a chart (U, φ) giving local coordinates $(x_1, \ldots, x_n)$. Then the maps $D_i \colon \mathcal{O}_{X,p} \to \mathbb{F}$ given by
>
> $$D_i \colon [(V, f)] \mapsto \left. \frac{\partial f|_{U \cap V}}{\partial x_i} \right|_p$$
>
> provide a basis for T_pX.

Proof. Checking that this is a derivation follows from the Leibniz rule on the chart. Linear independence of the $D_\bullet$s can also be checked locally by plugging in the germs $[(U, x_i)]$ into any linear dependence.

It remains to check that our derivations span. Well, fix any other derivation D which we want to be in the span of the $D_\bullet$s. By replacing D with $D - \sum_i D(x_i)D_i$, we may assume that $D(x_i) = 0$ for all i. We now want to show that $D = 0$. This amounts to some multivariable calculus. Fix a germ $[(V, f)]$, and shrink U and V enough so that f is defined on U; we want to show $D(f) = 0$. The fundamental theorem of calculus implies

$$f(x_1, \ldots, x_n) = f(0) + \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n) \, dt.$$

However, one can expand out the derivative on the right by the chain rule to see that

$$f(x_1, \ldots, x_n) = f(0) + \sum_{i=1}^n x_i h_i(x_1, \ldots, x_n)$$

for some regular functions $h_1, \ldots, h_n \colon X \to \mathbb{F}$. Applying D, we see that

$$D(f) = \sum_{i=1}^n \underbrace{D(x_i)}_{0} h_i(p) + \underbrace{x_i(p)}_{0} D(h_i) = 0,$$

as required. $\blacksquare$

Tangent spaces have a notion of functoriality.

> **Definition 1.51.** Fix a regular map $F \colon X \to Y$ of regular manifolds. Given $p \in X$, the *differential map* is the linear map $dF_p \colon T_pX \to T_{F(p)}Y$ defined by
>
> $$dF_p(v)(g) := v(g \circ F)$$
>
> for any $v \in T_pX$ and germ $g \in \mathcal{O}_{X,p}$. We may also denote $dF_p(v)$ by F_*v.

One has to check that dF_p is linear (which does not have much to check) and satisfies the Leibniz rule (which is a matter of expansion); we will omit these checks.

> **Remark 1.52.** One also has a chain rule: for regular maps $F \colon X \to Y$ and $G \colon Y \to Z$, one has $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

1.3.2 Immersions and Submersions

This map at the tangent space is important enough to give us other definitions.

> **Definition 1.53** (submersion, immersion, embedding)**.** Fix a regular function $F\colon X \to Y$.
>
> - The map F is a *submersion* if and only if dF_p is surjective for all $p \in X$.
>
> - The map F is an *immersion* if and only if dF_p is injective for all $p \in X$.
>
> - The map F is an *embedding* if and only if F is an immersion and a homeomorphism onto its image.

> **Remark 1.54.** One can check that submersions $F\colon X \to Y$ have local sections $Y \to X$. Explicitly, for $Q \in Y$, the fiber $F^{-1}(\{Q\}) \subseteq X$ is a manifold, and if $Q \in \operatorname{im} F$, the fiber has codimension $\dim Y$.

> **Remark 1.55.** If $F\colon X \to Y$ is an embedding, then the image $F(X) \subseteq Y$ inherits a unique manifold structure so that the inclusion $F(X) \subseteq Y$ is smooth.

> **Example 1.56.** The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) := x$ is a submersion.

> **Example 1.57** (lemniscate)**.** The function $F\colon S^1 \to \mathbb{R}^2$ given by
> $$F(\theta) := \left(\frac{\cos \theta}{1 + \sin^2 \theta}, \frac{\cos \theta \sin \theta}{1 + \sin^2 \theta} \right)$$
> can be checked to be an immersion (namely, $F'(\theta) \neq 0$ always), but it fails to be injective because $F(\pi/4) = F(3\pi/4) = (0, 0)$, so it is not an embedding.

> **Example 1.58.** The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$ is a smooth homeomorphism onto its image, but it is not an immersion.

> **Example 1.59.** For any open subset $U \subseteq X$ of a manifold, the inclusion map $U \to X$ is an embedding. (In fact, it is also a submersion.)

We will want to distinguish between embeddings, notably to get rid of open embeddings.

> **Definition 1.60** (closed)**.** An embedding $F\colon X \to Y$ of regular manifolds is *closed* if and only if $F(X) \subseteq Y$ is closed.

> **Example 1.61.** Fix a submersion $F\colon X \to Y$. A point $Q \in Y$ gives rise to a fiber $Z := F^{-1}(\{Q\})$, which Remark 1.54 explains is a closed submanifold of X of codimension $\dim Y$. One can check that $T_p Z$ is exactly the kernel of $dF_p\colon T_p X \to T_p Y$; see [Lee13, Proposition 5.37].

1.3.3 Lie Groups

We now may stop doing topology.

> **Definition 1.62** (Lie group)**.** A regular *Lie group* is a group object in the category of regular manifolds. For brevity, we may call (real) smooth Lie groups simply "Lie groups" or "real Lie groups," and we may call complex analytic Lie groups simply "complex Lie groups."

As with any object, we have a notion of morphisms.

> **Definition 1.63** (homomorphism)**.** A homomorphism of regular Lie groups is a regular map of the underlying manifolds and a homomorphism of the underlying groups; an isomorphism of regular Lie groups is a homomorphism with an inverse which is also a homomorphism.

> **Remark 1.64.** If X is already a regular manifold, and we are equipped with continuous multiplication and inverse maps, to check that X becomes a regular Lie group, it is enough to check that merely the multiplication map is regular. See [Lee13, Exercise 7-3].

> **Remark 1.65.** Hilbert's 5th problem asks when C^0 Lie groups can give rise to real Lie groups, and there is a lot of work in this direction. As such, we will content ourselves to focus on real Lie groups instead of any weaker regularity.

> **Remark 1.66.** Any complex Lie group is also a real Lie group.

Here is a basic check which allows one to translate checks to the identity.

> **Lemma 1.67.** Fix a regular Lie group G. For any $g \in G$, the maps $L_g \colon G \to G$ and $R_g \colon G \to G$ defined by $L_g(x) := gx$ and $R_g(x) := xg$ are regular diffeomorphisms.

Proof. Regularity follows from regularity of multiplication. Our inverses of L_g and R_g are given by $L_{g^{-1}}$ and $R_{g^{-1}}$, which verifies that we have defined regular diffeomorphisms. ■

1.4   September 6

Last time we defined a Lie group. Today and for the rest of the course, we will study them.

1.4.1   Examples of Lie Groups

Here are some examples of Lie groups and isomorphisms.

> **Example 1.68.** For our field $\mathbb{F}$, the $\mathbb{F}$-vector space $\mathbb{F}^n$ is a Lie group over $\mathbb{F}$.

> **Example 1.69.** Any finite (or countably infinite) group given the discrete topology becomes a real and complex Lie group.

> **Example 1.70.** The groups $\mathbb{R}^\times$ and $\mathbb{R}^+$ (under multiplication) are real Lie groups. In fact, one has an isomorphism $\{\pm 1\} \times \mathbb{R}^+ \to \mathbb{R}^\times$ of real Lie groups given by $(\varepsilon, r) \mapsto \varepsilon r$.

> **Example 1.71.** The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ (under multiplication) is a real Lie group.

Example 1.72. The group $\mathbb{C}^\times$ is a real Lie group. In fact, one has an isomorphism $S^1 \times \mathbb{R}^+ \to \mathbb{C}^\times$ of real Lie groups given again by $(\varepsilon, r) \mapsto \varepsilon r$.

Example 1.73. Over our field $\mathbb{F}$, the set $\mathrm{GL}_n(\mathbb{F})$ of invertible $n \times n$ matrices is a Lie group. Indeed, it is an open subset of $\mathbb{F}^{n^2}$ and thus a manifold, and one can check that the inverse and multiplication maps are rational and hence smooth.

Example 1.74. Consider the collection of matrices

$$\mathrm{SU}_2 := \left\{ A \in \mathrm{GL}_2(\mathbb{C}) : \det A = 1 \text{ and } AA^\dagger = 1_2 \right\},$$

where $A^\dagger$ is the conjugate transpose. Then SU_2 is an embedded submanifold of $\mathrm{GL}_2(\mathbb{C})$ (cut out by the given equations) and also a subgroup. By writing out $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, one can write out our equations on the coefficients as

$$\begin{cases} ad - bc = 1, \\ a\bar{a} + b\bar{b} = 1, \\ a\bar{c} + b\bar{d} = 0, \\ c\bar{c} + d\bar{d} = 1. \end{cases}$$

In particular, we see that the vector $(a, b) \in \mathbb{C}^2$ is orthogonal to the vector $(\bar{c}, \bar{d})$, so we can solve for this line as providing some $\lambda \in \mathbb{C}$ such that $(\bar{c}, \bar{d}) = \lambda(-b, a)$. But then the determinant condition requires $\lambda = 1$ from $|a|^2 + |b|^2 = 1$. By expanding out $a = w + ix$ and $b = y + iz$, one finds that SU_2 is diffeomorphic to S^3.

The classical groups provide many examples of Lie groups over our field $\mathbb{F}$.

- One has $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_n(\mathbb{F})$, which are subsets of matrices cut out by the conditions $\det A \neq 0$ and $\det A = 1$, respectively.

- Orthogonal: fix a non-degenerate symmetric 2-form Ω on $\mathbb{F}^n$. One can always adjust our basis of $\mathbb{F}^n$ so that Ω is diagonal, and by adjusting our basis by squares, we may assume that Ω has only $+1$ or -1s on the diagonal. If $\mathbb{F} = \mathbb{C}$, we can in fact assume that $\Omega = 1_n$, and then we find that our group is

$$\mathrm{O}_n(\mathbb{C}) := \{A : A^\mathsf{T} A = 1_n\}.$$

  Otherwise, if $\mathbb{F} = \mathbb{R}$, then our adjustment (and rearrangement) of the basis allows us to assume that Ω takes the form $\Omega_{k,n-k} := \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1)$ with k copies of $+1$ and $n - k$ copies of -1, and we define

$$\mathrm{O}_{k,n-k}(\mathbb{R}) := \{A : A^\mathsf{T} \Omega A = \Omega\}$$

- Special orthogonal: one can add the condition that $\det A = 1$ to all the above orthogonal groups, which makes the special orthogonal groups.

- Symplectic: for $\mathbb{F}^{2n}$, one can fix a non-degenerate symplectic 2-form Ω. It turns out that, up to basis, we find that $\Omega = \left[\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right]$, and we define

$$\mathrm{Sp}_{2n}(\mathbb{F}) := \{A : A^\mathsf{T} \Omega A = 1_{2n}\}.$$

- Unitary: using the non-degenerate Hermitian forms, we can similarly define

$$\mathrm{U}_n(\mathbb{C}) := \left\{ A : A^\dagger A = 1_n \right\}$$

  is a real Lie group. (Conjugation is not complex analytic, so this is not a complex Lie group!)

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1.4.2   Connected Components

We will want to focus on connected Lie groups in this class, so we spend a moment describing why one might hope that this is a reasonable reduction. The main point is that it is basically infeasible to classify finite groups, and allowing for disconnected Lie groups forces us to include all these groups in our study by Example 1.69.

   Quickly, recall our notions of connectivity; we refer to [Elb22, Appendix A.1] for details.

> **Definition 1.75** (connected). A topological space X is *disconnected* if and only if there exists disjoint nonempty open subsets $U, V \subseteq X$ covering X. If there exists no such pair of open subsets, then X is *connected*; in other words, the only subsets of X which are both open and closed are $\varnothing$ and X.

> **Definition 1.76** (connected component). Given a topological space X and a point $p \in X$, the *connected component* of $p \in X$ is the union of all connected subspaces of X containing p.

> **Remark 1.77.** One can check that the connected component is in fact connected and is thus the maximal connected subspace.

> **Definition 1.78** (path-connected). A topological space X is *path-connected* if and only if any two points $p, q \in X$ have some (continuous) path $\gamma \colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

> **Definition 1.79.** Given a topological space X and a point $p \in X$, the *path-connected component* of $p \in X$ is the collection of all $q \in X$ for which there is a path connecting p and q.

> **Remark 1.80.** One can check that having a path connecting two points of X is an equivalence relation on the points of X. Then the path-connected components are the equivalence classes for this equivalence relation. From this, one can check that the path-connected components are the maximal path-connected subsets of a topological space.

One has the following lemmas.

> **Lemma 1.81.** Fix a topological space X.
>
>   (a) If X is path-connected, then X is connected.
>
>   (b) If X is a connected topological n-manifold, then X is path-connected.

Proof. Part (a) is [Elb22, Lemma A.16]. Part (b) is [Elb24, Proposition 1.39]. ∎

> **Lemma 1.82.** Fix a continuous surjection $f \colon X \to Y$ of topological spaces. If X is connected, then Y is connected.

Proof. This is [Elb22, Lemma A.8]. ∎

Anyway, we are now equipped to return to our discussion of Lie groups.

> **Lemma 1.83.** Fix a Lie group G, and let $G^\circ \subseteq G$ be the connected component of the identity $e \in G$. For any $g \in G$, we see that gG° is the connected component of g.

Proof. Certainly gG° is a connected subset containing g by Lemma 1.82 (note multiplication is continuous), so it is contained in the connected component of g. On the other hand, any connected subset U around g must have $g^{-1}U$ be a connected subset around e, so $g^{-1}U \subseteq G^\circ$, so $U \subseteq gG^\circ$. In particular, the connected component of g is also contained in gG°. ∎

> **Proposition 1.84.** Fix a Lie group G, and let $G^\circ \subseteq G$ be the connected component of the identity $e \in G$.
>
> (a) Then G° is a normal subgroup of G.
>
> (b) The quotient $\pi_0(G) := G/G^\circ$ given the quotient topology from the surjection $G \twoheadrightarrow \pi_0(G)$ is a discrete countable group.

Proof. We show the parts independently.

(a) We check this in parts.

- Of course G° is a subgroup: it contains the identity, and the images of the maps $i\colon G^\circ \to G$ and $m\colon G^\circ \times G^\circ \to G$ must land in connected subsets of G containing the identity by Lemma 1.82, so we see that G° is contained

- We now must check that G° is normal. Fix some $g \in G$, and we want to show that $gG^\circ g^{-1} \subseteq G^\circ$. Well, define the map $G^\circ \to G$ by $a \mapsto gag^{-1}$, which we note is continuous because multiplication and inversion are continuous. Lemma 1.82 tells us that the image must be connected, and we see $e \mapsto e$, so the image must actually land in G°.

(b) One knows that $\pi_0(G)$ is a group because G° is normal, and it is discrete because connected components are both closed and open in G, so the corresponding points are closed and open in $\pi_0(G)$. (We are implicitly using Lemma 1.83.) This is countable because a separated topological space must have countably many connected components. ∎

> **Remark 1.85.** One can restate the above result as providing a short exact sequence
>
> $$1 \to G^\circ \to G \to \pi_0(G) \to 1$$
>
> of Lie groups. In this way, we can decompose our study of G into connected Lie groups and discrete countable groups. In this course, we will ignore studying discrete countable groups because they are too hard.

1.5 September 9

Today we talk more about subgroups and coverings.

1.5.1 Closed Lie Subgroups

Arbitrary subgroups of Lie groups may not inherit a manifold structure, so we add an adjective to acknowledge this.

> **Definition 1.86** (closed Lie subgroup)**.** Fix a Lie group G. A *closed Lie subgroup* is a subgroup $H \subseteq G$ which is also an embedded submanifold.

> **Remark 1.87.** On the homework, we will show that closed Lie subgroups are in fact closed subsets of G. It is a difficult theorem (which we will not prove nor use in this class) that being a closed subset and a subgroup implies that it is an embedded submanifold.

Here are some checks on subgroups.

> **Lemma 1.88.** Fix a connected topological group G. Given an open neighborhood U of G of the identity $e \in G$, the group G is generated by U is all of G.

Proof. Let H be the subgroup generated by U. For each $h \in H$, we see that $hU \subseteq H$, which is an open neighborhood (see Lemma 1.67), so $H \subseteq G$ is open. However, we also see that

$$G = \bigsqcup_{[g]} gH,$$

where $[g]$ varies over representatives of cosets. Thus, $G\backslash H$ is again the union of open subsets, so H is also closed, so $G = H$ because G is connected. ∎

> **Lemma 1.89.** Fix a homomorphism $f\colon G_1 \to G_2$ of connected Lie groups. If $df_e\colon T_eG_1 \to T_eG_2$ is surjective, then f is surjective.

Proof. By translating around (by Lemma 1.67), we see that f is a submersion. (Explicitly, for each $g \in G_1$, we see that $R_{f(g)} \circ f$ must continue to be a submersion at the identity, but this equals $f \circ R_g$, so f is a submersion at g too.) Because submersions are open [Lee13, Proposition 4.28], we see that f being a submersion means that its image is an open subgroup of G_2, which is all of G_2 by Lemma 1.88. ∎

We may want some more flexibility with our subgroups.

> **Example 1.90.** Fix an irrational number $\alpha \in \mathbb{R}$. Then there is a Lie group homomorphism $f\colon \mathbb{R} \to (\mathbb{R}/\mathbb{Z})^2$ defined by $f(t) := (t, \alpha t)$. One can check that $\operatorname{im} f \subseteq (\mathbb{R}/\mathbb{Z})^2$ is a dense subgroup, but it is not closed!

So we have the following definition.

> **Definition 1.91** (Lie subgroup). Fix a Lie group G. A *Lie subgroup* is a subgroup $H \subseteq G$ which is an immersed submanifold.

1.5.2 Quotient Groups

Along with subgroups, we want to be able to take quotients.

> **Definition 1.92** (fiber bundle). A *fiber bundle* with fiber F on a smooth manifold X is a surjective continuous map $\pi\colon Y \to X$ such that there is an open cover $\mathcal{U}$ of X and (local) homeomorphisms making the following diagram commute for all $U \in \mathcal{U}$.
>
> $$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\sim\quad} & F \times U \\ & {\scriptstyle \pi}\searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & U & \end{array}$$

Fiber bundles are the correct way to discuss quotients.

Theorem 1.93. Fix a closed Lie subgroup H of a Lie group G.

(a) Then G/H is a manifold of dimension $\dim G - \dim H$ equipped with a quotient map $q\colon G \twoheadrightarrow G/H$.

(b) In fact, q is a fiber bundle with fiber H.

(c) If H is normal in G, then G/H is actually a Lie group (with the usual group structure).

(d) We have $T_e(G/H) \cong T_eG/T_eH$.

Proof. We construct the manifold structure on G/H as follows: for each $g \in G$, we produce a coset $\overline{g} \in G/H$, which we note as $q^{-1}(\overline{g}) = gH$. Now, $gH \subseteq G$ is an embedded submanifold because H is (we are using Lemma 1.67), so one can locally find a submanifold $M \subseteq G$ around g intersecting gH transversally, meaning that
$$T_gG = T_gM \oplus T_g(gH).$$
By shrinking M, we can ensure that the above map continues to be an isomorphism in a neighborhood of g, so the multiplication map $M \times H \to UH$ is a diffeomorphism. Now, MH is an open neighborhood of $g \in G$, and M projects down to an open subset of G/H, so $M \cong q(\overline{M})$ provides our chart.

Now, (a) and (d) follows by inspection of the construction. We see that (b) follows because we built our projection map $G \twoheadrightarrow G/H$ so that it locally looks like $U \times H \twoheadrightarrow \overline{U}$, so we get our fiber bundle. Lastly, (d) follows by the equality $T_gG = T_gM \oplus T_g(gH)$. ∎

Remark 1.94. Writing the above out in detail would take several pages; see [Lee13, Theorem 21.10].

Access to quotients permits an isomorphism theorem, which we will prove later when we have talked a bit about Lie algebras.

Theorem 1.95 (Isomorphism)**.** Fix a Lie group homomorphism $f\colon G_1 \to G_2$.

(a) The kernel $\ker f$ is a normal closed Lie subgroup of G_1.

(b) The quotient $G_1/\ker f$ is a Lie subgroup of G_2.

(c) The image $\operatorname{im} f$ is a Lie subgroup of G_2. If $\operatorname{im} f$ is further closed, then $G_1/\ker f \to \operatorname{im} f$ is an isomorphism of Lie subgroups.

1.5.3 Actions

Groups are known by their actions, so let's think about how our actions behave.

Definition 1.96 (action)**.** Fix a Lie group G and regular manifold X. Then a *regular action* of G on X is a regular map $\alpha\colon G \times X \to X$ satisfying the usual constraints, as follows.

(a) Identity: $\alpha(e, x) := x$.

(b) Composition: $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$.

This allows us to define the usual subsets.

Definition 1.97 (orbit, stabilizer)**.** Fix a regular action of a Lie group G on a regular manifold X.

(a) The *orbit* of $x \in X$ is the subspace $Gx := \{gx : g \in G\}$.

(b) The *stabilizer* of $x \in X$ is the subgroup $G_x := \{g \in G : gx = x\}$.

Here are some examples.

> **Example 1.98.** The group $\mathrm{GL}_n(\mathbb{F})$ acts on the vector space $\mathbb{F}^n$.

> **Example 1.99.** The group $\mathrm{SO}_3(\mathbb{R})$ preserves distances in its action on $\mathbb{R}^3$, so its action descends to an action on S^2.

Representations are special kinds of actions.

> **Definition 1.100** (representation). Fix a Lie group G over $\mathbb{F}$. Then a *representation* of G is the (regular) linear action of G on a finite-dimensional vector space V over $\mathbb{F}$; namely, the map $v \mapsto g \cdot v$ for each $g \in G$ must be a linear map $V \to V$. A *homomorphism* of representations V and W is a linear map $A\colon V \to W$ such that $A(gv) = g(Av)$. These objects and morphisms make the category $\mathrm{Rep}_{\mathbb{F}}(G)$.

> **Remark 1.101.** Equivalently, we may ask for the induced map $G \to \mathrm{GL}(V)$, given by sending $g \in G$ to the map $v \mapsto gv$, to be a Lie group homomorphism.

> **Remark 1.102.** The category $\mathrm{Rep}_{\mathbb{F}}(G)$ comes with many nice operations.
>
> (a) Duals: given a representation $\pi\colon G \to \mathrm{GL}(V)$, we can induce a G-action on $V^* := \mathrm{Hom}(V, \mathbb{F})$ by
> $$\left((\pi^*g)(v^*)\right)(v) := v^*\left(g^{-1}v\right).$$
> (Here, π^*g should be a map $V^* \to V^*$, so it takes a linear functional $v^* \in V^*$ as input and produces the linear functional $(\pi^*g)(v^*)$ as output.)
>
> (b) Tensor products: given representations $\pi\colon G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, we can induce a G-action on $V \otimes W$ by
> $$\left((\pi \otimes \pi')(g \otimes g')\right)(v \otimes v') := \pi(g)v \otimes \pi'(g')v'.$$
>
> (c) Quotients: given representations $\pi\colon G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, where $V \subseteq V'$ is a G-representation, then we can induce a G-action on V'/V by
> $$\pi'(g)(v' + V) := gv' + V.$$
>
> One can check that these operations make $\mathrm{Rep}_{\mathbb{F}}(G)$ into a symmetric monoidal abelian category.

Returning to group actions on manifolds, we remark that Theorem 1.95 can be seen as a version of the Orbit–stabilizer theorem.

> **Theorem 1.103** (Orbit–stabilizer). Fix a regular action of a Lie group G on a regular manifold X. Further, fix $x \in X$.
>
> (a) The orbit Gx is an immersed submanifold of X.
>
> (b) The stabilizer G_x is a closed Lie subgroup of G.
>
> (c) The quotient map $f\colon G/G_x \to X$ given by $g \mapsto gx$ is an injective immersion.
>
> (d) If Gx is an embedded submanifold, then the map f of (c) is a diffeomorphism.

1.6 September 11

Today we talk more about homogeneous spaces.

1.6.1 Homogeneous Spaces

Let's see some applications of Theorem 1.103.

Example 1.104. Suppose a regular Lie group G acts smoothly and transitively on a regular manifold X. For each $x \in X$, we see that $G/G_x \to X$ is a bijective immersion. In particular, Sard's theorem implies that $\dim G/G_x = \dim X$, so we conclude that this map is in fact a bijective local diffeomorphism, which of course is just a diffeomorphism. Thus, Theorem 1.93 tells us that the map $G \to X$ given by $g \mapsto gx$ is a fiber bundle with fiber G_x.

The above situation is so nice that it earns a name.

Definition 1.105 (homogeneous space). Fix a regular Lie group G acting smoothly and transitively on a regular manifold X. If the action of G is transitive, we say that X is a *homogeneous space* of G.

Here are many examples.

Example 1.106. Continuing from Example 1.99, we recall that $\mathrm{SO}_3(\mathbb{R})$ acts on S^2. In fact, one can check that the stabilizer of any $x \in S^3$ is isomorphic to S^1, so Example 1.104 tells us that $\mathrm{SO}_3(\mathbb{R}) \to S^2$ is a fiber bundle with fiber S^1. In general, we find that $\mathrm{SO}_n(\mathbb{R}) \to S^n$ is a fiber bundle with fiber $\mathrm{SO}_{n-1}(\mathbb{R})$.

Example 1.107. The group SU_2 acts on $\mathbb{CP}^1$ by matrix multiplication. We see that the stabilizer of some line in $\mathbb{CP}^1$ consists of the matrices in SU_2 with a nonzero eigenvector on the line. For example, using the computation of Example 1.74, we see that trying to stabilizer $[1 : 0]$ gives rise to the matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where we require $b = 0$. Thus, we see that our stabilizer is isomorphic to U_1. In particular, our orbits are compact immersed submanifolds of $\mathbb{CP}^1$ of dimension $\dim \mathrm{SU}_2 - \dim \mathrm{U}_1 = \dim \mathbb{CP}^1$, so the action must be transitive in order for orbits to be closed and the correct dimension.

Example 1.108. One can check that SU_n acts on $S^{2n-1} \subseteq \mathbb{C}^n$ with stabilizer isomorphic to SU_{n-1}.

Example 1.109 (flag varieties). Let $\mathcal{F}_n$ be the set of "flags" of $\mathbb{F}^n$, which is an ascending chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{F}^n.$$

Then we see that $\mathrm{GL}_n(\mathbb{F})$ acts on $\mathcal{F}_n$ by matrix multiplication. On the homework, we check that this action is transitive with stabilizer (of the standard flag $\{\mathrm{span}_{\mathbb{F}}(e_1, \ldots, e_i)\}_{i=0}^n$) given by the matrix subgroup $B_n(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ of upper-triangular matrices. Thus, we see that we can realize $\mathcal{F}_n$ as the manifold quotient $\mathrm{GL}_n(\mathbb{F})/B_n(\mathbb{F})$, providing a manifold structure.

Example 1.110 (Grassmannians). Let $\mathrm{Gr}_k(\mathbb{F}^n)$ be the set of vector subspaces $V \subseteq \mathbb{F}^n$ of dimension k. Then we see that $\mathrm{GL}_n(\mathbb{F})$ acts transitively on $\mathrm{Gr}_k(\mathbb{F}^n)$ with stabilizer of $\mathrm{span}(e_1, \ldots, e_k)$ given by matrices of the form

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where $A \in \mathbb{F}^{k \times k}$ and $B \in \mathbb{F}^{k \times (n-k)}$ and $D \in \mathbb{F}^{(n-k) \times (n-k)}$. Thus, we can realize $\mathrm{Gr}_k(\mathbb{F}^n)$ as the manifold quotient of $\mathrm{GL}_n(\mathbb{F})$, providing a manifold structure.

Example 1.111. There are many regular actions of G on itself.

- Regular left: define our action $R_\ell\colon G \times G \to G$ by $(g, x) \mapsto gx$.

- Regular right: define our action $R_r\colon G \times G \to G$ by $(g, x) \mapsto xg^{-1}$.

- Adjoint: define our action $\mathrm{Ad}\colon G \times G \to G$ by $(g, x) \mapsto gxg^{-1}$. (This action is not transitive in general!) Notably, $\mathrm{Ad}_g(1) = 1$, so this also provides an action of G on T_eG.

1.6.2 Covering Spaces

It will help to recall some theory around covering spaces. See [Elb23] for (some) more detail about this theory or [Hat01] for (much) more detail.

Definition 1.112 (covering space). A *covering space* is a fibration $p\colon Y \to X$ with discrete fiber S. The *degree* of p equals $\#S$.

In more words, we are asking for each $x \in X$ to have an open neighborhood U such that the restriction $p^{-1}(U) \to U$ is homeomorphic (over U) to $\bigsqcup_{s \in S} p^{-1}(U) \to U$ for some discrete set S.

Remark 1.113. If X is a regular manifold and $\deg p < \infty$, then Y is also a regular manifold. Indeed, being a manifold is checked locally, so one can find neighborhoods as in the previous remark to witness the manifold structure.

We are interested in paths in topological spaces, but there are too many. To make this set smaller, we consider it up to homotopy.

Definition 1.114 (homotopy). Fix a topological space X. Two paths $\gamma_0, \gamma_1\colon [0, 1] \to X$ are *homotopic* relative to their endpoints if and only if there is a continuous map $H_\bullet\colon [0, 1]^2 \to X$ such that $H_0 = \gamma_0$ and $H_1 = \gamma_1$ and $H_s(0) = \gamma_0(0) = \gamma_1(0)$ and $H_s(1) = \gamma_0(1) = \gamma_1(1)$ for all s. The map H is called a *homotopy*.

Definition 1.115 (simply connected). A topological space X is *simply connected* if and two paths with the same endpoints are homotopic relative to those endpoints.

Example 1.116. One can check that S^1 fails to be simply connected because the path going around the circle is not homotopic to the constant path.

It is important to know that one can lift paths.

Theorem 1.117. Fix a covering space $p\colon Y \to X$. Fix some point $x \in X$ and a path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$. Then each $\widetilde{x} \in p^{-1}(\{x\})$ has a unique path $\widetilde{\gamma}\colon [0, 1] \to Y$ such that $\widetilde{\gamma}(0) = \widetilde{x}$ and making the following diagram commute.

$$
\begin{array}{ccc}
[0,1] & \xrightarrow{\ \widetilde{\gamma}\ } & \widetilde{X} \\
& \searrow{\scriptstyle \gamma} & \downarrow{\scriptstyle p} \\
& & X
\end{array}
$$

Remark 1.118. One can further check that having two homotopic paths $\gamma_1 \sim \gamma_2$ downstairs produce homotopic paths $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$.

Remark 1.119. More generally, fix a simply connected topological space Z. Then given a map $f: Z \to X$ and a choice of $\widetilde{x} \in p^{-1}(\{x\})$ and $z \in f^{-1}(\{x\})$, there will be a unique lift $\widetilde{f}: Z \to Y$ such that $\widetilde{f}(z) = \widetilde{x}$. In short, given any $z' \in Z$, find a path connecting z and z', send this path into X and then lift it up to Y. Because Z is simply connected (and the above theorem), the choice of path from z to z' does not really matter.

Anyway, we now define our collection of paths.

Definition 1.120 (fundamental group). Fix a point x of a topological space X. Then the set of paths both of whose endpoints are x forms a monoid with operation given by composition (i.e., concatenation). If we take the quotient of this monoid by homotopy classes of paths, then we get a group of path homotopy classes, which we call $\pi_1(X, x)$. This is the *fundamental group*.

Remark 1.121. For any two $x, y \in X$ in the same path-connected component, the path $\alpha: [0, 1] \to X$ connecting x to y produces an isomorphism $\pi_1(X, x) \cong \pi_1(X, y)$ by $\gamma \mapsto \alpha \cdot \gamma \cdot \alpha^{-1}$, where $\cdot$ denotes path composition.

Remark 1.122. The above remark allows us to verify that X is simply connected if and only $\pi_1(X, x)$ is trivial for all x. In fact, we only have to check this for one x in each path-connected component.

There is more or less one covering space which produces all the other ones.

Definition 1.123 (universal cover). Fix a path-connected topological space X. Then a covering space $p: Y \to X$ is the *universal cover* if and only if Y is connected and simply connected.

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List of Definitions