256A: Algebraic Geometry

Nir Elber

Spring 2022

Contents

Theme 1
Introduction

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 24

A feeling of impending doom overtakes your soul.

1.1.1 Administrative Notes

Here are housekeeping notes.

- Here are some housekeeping notes. There is a syllabus on `bCourses`.
- We hope to cover Chapter II of [Har77], mostly, supplemented with examples from curves.
- There are lots of books.
  - We use [Har77] because it is short.
  - There is also [Vak17], which has more words.
  - The book [Liu06] has notes on curves.
  - There are more books in the syllabus. Professor Tang has some opinions on these.
- Some proofs will be skipped in lecture. Not all of these will appear on homework.
- Some examples will say lots of words, some of which we won't have good definitions for until later. Do not be afraid of words.

Here are assignment notes.

- Homework is 70% of the class.
- Homework is due on noon on Fridays. There will be 6–8 problems, which means it is pretty heavy. The lowest homework score will be dropped.

- Office hours exist. Professor Tang also answers emails.
- The term paper covers the last 30% of the grade. They are intended to be extra but interesting topics we don't cover in this class.

1.1.2 Motivation

We're going to talk about schemes. Why should we care about schemes? The point is that schemes are "correct."

Example 1.1. In algebraic topology, there is a cup product map in homology, which is intended to algebraically measure intersections. However, intersections are hard to quantify when we aren't dealing with, say manifolds.

Here is an example of algebraic geometry helping us with this rigorization.

Theorem 1.2 (Bézout)**.** Let C_1 and C_2 be curves in $\mathbb{P}^2(k)$, for some algebraically closed k, where C_1 and C_2 are defined by homogeneous polynomials f_1 and f_2. Then the "intersection number" between the curves C_1 and C_2 is $(\deg f_1)(\deg f_2)$.

This is a nice result, for example because it automatically accounts for multiplicities, which would be difficult to deal with directly using (say) geometric techniques. Schemes will help us with this.

Example 1.3. Moduli spaces are intended to be geometric objects which represent a family of geometric objects of interest. For example, we might be interested in the moduli space of some class of elliptic curves.

It turns out that the correct way to define these objects is using schemes as a functor; we will see this in this class.

Remark 1.4. One might be interested in when a functor is a scheme. We will not cover this question in this class in full, but it is an interesting question, and we will talk about this in special cases.

1.1.3 Elliptic Curves

For the last piece of motivation, let's talk about elliptic curves, over a field k.

Definition 1.5 (Elliptic curve)**.** An *elliptic curve* over k is a smooth projective curve of genus 1, with a marked k-rational point.

Remember that we said that we not to be afraid of words. However, we should have some notion of what these words mean: being a curve means that we are one-dimensional, being smooth is intuitive, and having genus 1 roughly means that base-changing to a complex manifold has one hole. Lastly, the k-rational point requires defining a scheme as a functor.

Here's another (more concrete) definition of an elliptic curve.

Definition 1.6 (Elliptic curve)**.** An *elliptic curve* over k is an affine variety in $\mathbb{A}^2(k)$ cut out be a polynomial of the form

$$y^2 + a_1xy + a_3y^2 = x^3 + a_2x^2 + a_4x + a_6$$

with nonzero discriminant plus a point $\mathcal{O}$ at infinity.

Remark 1.7. Why are these equivalent? Well, the Riemann–Roch theorem approximately lets us take a smooth projective curve of genus 1 and then write it as an equation; the marked point goes to $\mathcal{O}$. In the reverse direction, one merely needs to embed our affine curve into projective space and verify its smoothness and genus.

Instead of working with affine varieties, we can also give a concrete description of an elliptic curve using projective varieties.

Definition 1.8 (Elliptic curve)**.** An *elliptic curve* over k is a projective variety in $\mathbb{P}^2(k)$ cut out be a polynomial of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with nonzero discriminant.

We get the equivalence of the previous two definitions via the embedding $\mathbb{A}_2(k) \hookrightarrow \mathbb{P}^2(k)$ by $(x, y) \mapsto [x : y : 1]$; the point at infinity $\mathcal{O}$ is $[0 : 1 : 0]$.

1.1.4 Crackpot Varieties

In order to motivate schemes, we should probably mention varieties, so we will spend some time in class discussing affine and projective varieties. For convenience, we work over an algebraically closed field k.

Definition 1.9 (Affine variety)**.** Given a field k, we define *affine n-space over k*, denoted $\mathbb{A}^n(k)$. An *affine variety* is a subset $Y \subseteq \mathbb{A}^n(k)$ of the form

$$Y = V(S) := \{p \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } f \in S\},$$

where $S \subseteq k[x_1, \ldots, x_n]$.

Remark 1.10. The set $S \subseteq k[x_1, \ldots, x_n]$ in the above definition need not be finite or countable. In certain cases, we can enforce this condition; for example, if $n = 1$, then $k[x]$ is a principal ideal domain, so we may force $\#S = 1$.

Note that we have defined vanishing sets $V(S)$ from subsets $S \subseteq k[x_1, \ldots, x_n]$. We can also go from vanishing sets to subsets.

Definition 1.11. Fix a field k and subset $Y \subseteq \mathbb{A}^n(k)$. Then we define the ideal

$$I(Y) := \{f \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } p \in Y\}.$$

Remark 1.12. One should check that this is an ideal, but we won't bother.

So we've defined some geometry. But we're in an algebraic geometry class; where's the algebra?

Theorem 1.13 (Hilbert's Nullstellensatz)**.** Fix an algebraically closed field k and ideal $J \subseteq k[x_1, \ldots, x_n]$. Then

$$I(V(J)) = \operatorname{rad} I,$$

where $\operatorname{rad} I$ is the radical of I.

Remark 1.14. The Nullstellensatz has no particularly easy proof.

The point of this result is that it ends up giving us a contravariant equivalence of posets of radical ideals and affine varieties.

Why do we care? In some sense, we prefer to work with ideals because it "remembers" more information than merely the points on the variety. To see this, note that elements $f \in k[x_1, \ldots, x_n]$ we are viewing as giving functions on $\mathbb{A}^n(k)$. However, when we work on a variety $Y \subseteq \mathbb{A}^n(k)$, then sometimes two functions will end up being identical on Y. So the correct ring of functions on Y is

$$k[x_1, \ldots, x_n]/I(Y),$$

so indeed keeping track of the (algebraic) ideal $V(Y)$ gets us some extra (geometric) information.

We will use this discussion as a jumping-off point to discuss affine schemes and then schemes. Affine schemes will have the following data.

- A commutative ring A, which we should think of as the ring of functions on a variety.
- A topological space $\operatorname{Spec} A$, which has more information than merely points on the variety.
- A structure sheaf of functions on $\operatorname{Spec} A$.

Remark 1.15. Our topological space $\operatorname{Spec} A$ will contain more points than just the points on the variety. In some sense, these extra points make the topology more apparent.

Remark 1.16. Going forward, one might hope to remove requirements that the field k is algebraically closed (e.g., to work with a general ring) or talk about ideals which are not radical. This is the point of scheme theory.

1.2 August 26

Let's finish up talking varieties, and then we'll move on to affine schemes.

1.2.1 Projective Varieties

We're going to briefly talk about projective varieties. Let's start with projective space.

Definition 1.17 (Projective space)**.** Given a field k, we define *projective n-space over k*, denoted $\mathbb{P}^n(k)$ as

$$\frac{k^{n+1} \setminus \{(0, \ldots, 0\}}{\sim},$$

where $\sim$ assigns two points being equivalent if and only if they span the same 1-dimensional subspace of k^{n+1}. We will denote the equivalence class of a point $(a_0, \ldots, a_n)$ by $[a_0 : \ldots : a_n]$.

To work with varieties, we don't quite cut out by general polynomials but rather by homogeneous polynomials.

Definition 1.18 (Projective variety)**.** Given a field k and a set of some homogeneous polynomials $T \subseteq k[x_1, \ldots, x_n]$, we define the *projective variety* cut out by T as

$$V(T) := \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in T\}.$$

Example 1.19. The elliptic curve corresponding to the affine algebraic variety in $\mathbb{A}^2(k)$ cut out by $y^2 - x^3 - 1$ becomes the projective variety in $\mathbb{P}^2(k)$ cut out by

$$Y^2Z - X^3 - Z^3 = 0.$$

Remark 1.20. One can give projective varieties some Zariski topology as well, which we will define later in the class.

What to remember about projective varieties is that we can cover $\mathbb{P}^2(k)$ (say) by affine spaces as

$$\begin{aligned}\mathbb{P}^2(k) &= \{[X:Y:Z] : X,Y,Z \in k \text{ not all } 0\} \\ &= \{[X:Y:Z] : X,Y,Z \in k \text{ and } X \neq 0\} \\ &\quad \cup \{[X:Y:Z] : X,Y,Z \in k \text{ and } Y \neq 0\} \\ &= \{[1:y:z] : y,z \in k\} \\ &\quad \cup \{[x:1:z] : x,z \in k\} \\ &\simeq \mathbb{A}^2(k) \cup \mathbb{A}^2(k).\end{aligned}$$

The point is that we can decompose $\mathbb{P}^2(k)$ into an affine cover.

Example 1.21. Continuing from Example 1.19, we can decompose $Z\left(Y^2Z - X^3 - Z^3\right)$ into having an affine open cover by

$$\underbrace{\left\{(x,y) : y^2 - x^3 - 1 = 0\right\}}_{z\neq 0} \cup \underbrace{\left\{(x,z) : z - x^3 - z^3 = 0\right\}}_{y \neq 0} \cup \underbrace{\left\{(y,z) : y^2 z - 1 - z^3 = 0\right\}}_{x\neq 0}.$$

Notably, we get almost everything from just one of the affine chunks, and we get the point at infinity by taking one of the other chunks.

Remark 1.22. It is a general fact that we only need two affine chunks to cover our projective curve.

1.2.2 The Spectrum

The definition of a(n affine) scheme requires a topological space and its ring of functions. We will postpone talking about the ring of functions until we discuss sheaves, so for now we will focus on the space.

Definition 1.23 (Spectrum)**.** Given a ring A, we define the *spectrum*

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}.$$

Example 1.24. With an algebraically closed field k, the spectrum of $k[x]$ consists of our prime ideals; using that $k[x]$ is a principal ideal domain, these all look like $(\pi(x))$ for either $\pi = 0$ or π an irreducible polynomial, but because $k[x]$ is algebraically closed, we find that

$$\operatorname{Spec} A = \{(0)\} \cup \{(x-a) : a \in k\}.$$

Notably, that each maximal ideal $\mathfrak{m} = (x-a)$ has a corresponding modulo map

$$k[x] \twoheadrightarrow \frac{k[x]}{\mathfrak{m}} \simeq k,$$

which really means "evaluation at a." The ideal (0) doesn't do anything interesting in its "evaluation" map.

This notion of having modulo being an evaluation will continue to be important.

Example 1.25. Fix a field k. Then $\operatorname{Spec} k = \{(0)\}$. Namely, non-isomorphic rings can have homeomorphic spectra.

Example 1.26. Similar to $k[x]$, we can classify $\operatorname{Spec}\mathbb{Z}$: all ideals are principal, so our primes look like (p) where $p = 0$ or is a rational prime.

Example 1.27. With k not algebraically closed, we can still classify $\operatorname{Spec} k[x]$ as

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(\pi) : \pi \text{ is irreducible}, \deg \pi > 0\}. \tag{1.2.1}$$

As always, one can spend the time to check this; the main point is that prime (for elements) is equivalent to irreducible in a principal ideal domain.

Remark 1.28. One can make the correspondence of (1.2.1) into a bijection by forcing π to be monic. Namely, $(\pi) = (\pi')$ if and only if they differ by a constant in $k^\times$ because $k[x]^\times = k^\times$.

Example 1.29. Concretely, we can imagine the prime point $(\pi) \in \operatorname{Spec}\mathbb{Q}[x]$ as a Galois equivalence class of algebraic integers. (We are working with algebraic integers to ensure our polynomials are monic.)

Here is a harder example, which we won't really spend the time to elaborate on.

Example 1.30. Let k be algebraically closed. Then $\operatorname{Spec} k[x,y]$ consists of $\{(0)\}$ and $\{(x-a, y-b) : a, b \in k\}$ as usual, but we also get

$$\{(\pi) : \pi \in k[x,y] \text{ irreducible polynomials}, \deg \pi > 0\}.$$

The main point is that $k[x,y]$ should have Krull dimension 2; now, (0) has dimension 2 (it's the full plane), $(x-a, y-b)$ has dimension 0 (they're points), and (π) have dimension 1 (they're curves). Proving that these are all the prime ideals requires some effort, and it is a special feature of our setting.

Remark 1.31. It is remarkable that the number of equations we need to cut out a variety of dimension d is $2 - d$. This is not always true.

The point is that we see working with our prime ideals allows us to realize some part of Remark 1.16 by working with spectra.

Definition 1.32 (Affine space)**.** Given a ring R, we define *affine n-space over R* as

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n].$$

Example 1.33. We classify $\operatorname{Spec} k[x]/\left(x^2\right)$. Notably, all prime ideals here must correspond to prime ideals of $k[x]$ containing $\left(x^2\right)$, which is only (x). So $\operatorname{Spec} k[x]/\left(x^2\right)$ has a single point.

Remark 1.34. In some sense, $\operatorname{Spec} k[x]/\left(x^2\right)$ will be able to let us talk about differential information algebraically: x here in some sense is a very small object x such that $x^2 = 0$. So we can study a "function" $f \in k[x]$ locally at a point p by studying $f(p+x)$.

1.2.3 The Zariski Topology

Thus far we've defined our space. Where's our topology.

Definition 1.35 (Zariski topology)**.** Fix a ring A. Then we define

$$V(T) := \{\mathfrak{p} \in \operatorname{Spec} A : T \subseteq \mathfrak{p}\}$$

These make the closed sets of the Zariski topology of $\operatorname{Spec} A$; one can show directly that they make a topology.

Remark 1.36. Note that $V(T) = V((T))$, where (T) is the ideal generated by $T \subseteq A$.

Example 1.37. We work with $\mathbb{A}^n_k$. Then, given $f \in k[x_1, \ldots, x_n]$, we want to be convinced that $V(\{f\})$ matches up with the affine k-points $(a_1, \ldots, a_n)$ which vanish on f. Well, $(a_1, \ldots, a_n)$ corresponds to the ideal $(x_1 - a_1, \ldots, x_n - a_n)$, and

$$\{f\} \subseteq (x_1 - a_1, \ldots, x_n - a_n)$$

is equivalent to f vanishing in the evaluation map

$$\frac{k[x_1, \ldots, x_n]}{(x_1 - a_1, \ldots, x_n - a_n)} \to k.$$

While we're here, let's also generalize Definition 1.11.

Definition 1.38. Fix a ring A. Then, given a subset $Y \subseteq \operatorname{Spec} A$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

And here is our nice version of Theorem 1.13.

Proposition 1.39. Fix a ring A and an ideal $\mathfrak{a} \subseteq A$ and a subset $Y \subseteq \operatorname{Spec} A$.

(a) The ideal $I(Y)$ is radical.

(b) We have that

$$I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a} \qquad \text{and} \qquad V(I(Y)) = \overline{Y},$$

where $\overline{Y}$ means the closure in the Zariski topology (i.e., the "Zariski closure").

(c) There is a one-to-one inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. Omitted. ■

Here are some more quick remarks.

Remark 1.40. A ring homomorphism $\varphi\colon A \to B$ will induce a continuous map

$$\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A.$$

We will state this more concretely later.

Example 1.41. Fix a ring A and ideal $\mathfrak{a} \subseteq A$. Then the surjection $A \twoheadrightarrow A/\mathfrak{a}$ induces a natural homeomorphism

$$\operatorname{Spec} A/\mathfrak{a} \cong V(\mathfrak{a}) = \{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{a}\}$$

because primes of $A/\mathfrak{a}$ are approximately primes of A containing $\mathfrak{a}$. This gives a description of the closed sets of $\operatorname{Spec} A$ as coming from other spectra.

Example 1.42. A ring homomorphism $A \to k$, where k is a field, induces a map from the single closed point of $\operatorname{Spec} k$ to $\operatorname{Spec} A$. We call the images of these maps the "k-points."

Example 1.43. Given a ring A and $f \in A$, the sets

$$D(f) \coloneqq (\operatorname{Spec} A) \setminus V(f) = \{\mathfrak{p} : f \notin \mathfrak{p}\}$$

form a base of the open sets in $\operatorname{Spec} A$. To see this, we can see directly that

$$(\operatorname{Spec} A) \setminus V(T) = \bigcup_{f \in T} D(f).$$

To turn $D(f)$ into a spectrum, we have $D(f) \cong \operatorname{Spec} A_f$, where A_f refers to the localization of A at $f^{\mathbb{N}}$. Notably, the prime ideals of $\operatorname{Spec} A_f$ are the primes of A which are disjoint from $\{f\}$. Notably, the homeomorphism

$$\operatorname{Spec} A_f \cong D(f)$$

is given by the localization map $A \to A_f$.

Remark 1.44. Not every open set is of the form $D(f)$ or even $D(S)$ where S is a multiplicative set. For example,

$$\mathbb{A}_k^2 \setminus \{(0,0)\} \subseteq \mathbb{A}_k^2$$

is an open set.

Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977.

[Liu06] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford graduate texts in mathematics. Oxford University Press, 2006. URL: https://books.google.com/books?id=pEGLDAEACAAJ.

[Vak17] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. 2017. URL: http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.

List of Definitions