185: Introduction to Complex Analysis

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Contents

Theme 1: Introducing Complex Numbers

1.1 January 19

It is reportedly close enough to start.

1.1.1 Logistics

We are online for the first two weeks, as with the rest of Berkeley. We will be using `bCourses` a lot, so check it frequently. There is also a website. There is a homework due on Friday, but do not worry about it.

Here are some syllabus things.

- Our main text is *Complex Variables and Applications, 8th Edition* because it is the version that Professor Morrow used. There is a free copy online.
- Homeworks are readings (for each course day) and weekly problem sets. Late homeworks are never accepted.
- Lowest two homework scores are dropped.
- There are two midterms and a final. The final is culmulative, as usual. The final can replace one midterm if the score is higher.
- Regrade requests can be made in GradeScope within one week of being graded.
- The class is curved but usually only curved at the end. The average on exams is expected to be 80%–83%.

1.1.2 Complex Numbers

Welcome to complex analysis. What does that mean?

Idea 1.1. In complex analysis, we study functions $f : \mathbb{C} \to \mathbb{C}$, usually analytic to some extent.

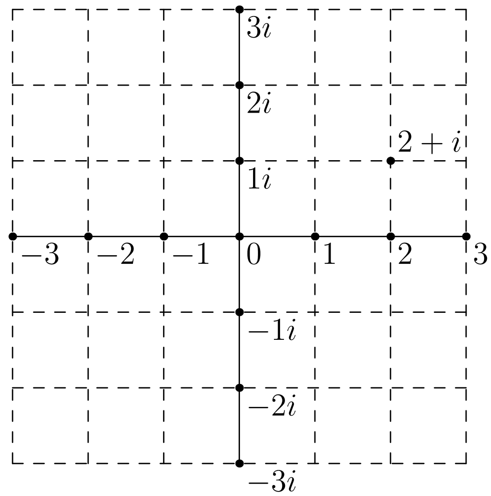
There are two pieces here: we should study $\mathbb{C}$ in themselves and then we will study the functions.

Complex numbers

Definition 1.2 (Complex numbers)**.** The set of complex numbers $\mathbb{C}$ is $\{a + bi : a, b \in \mathbb{R}\}$, where $i^2 = -1$.

Hopefully $\mathbb{R}$ is familiar from real analysis. As an aside, we see $\mathbb{R} \subseteq \mathbb{C}$ because $a = a + 0i \in \mathbb{C}$ for each $a \in \mathbb{R}$.

The complex numbers have an inherent geometry as a two-dimensional plane.



The point is that $\mathbb{C}$ looks like the real plane $\mathbb{R}^2$. More precisely, $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-vector space, where our basis is $\{1, i\}$.

We would like to understand $\mathbb{C}$ geometrically, "as a space." The first step here is to create a notion of size.

Norm on $\mathbb{C}$

Definition 1.3 (Norm on $\mathbb{C}$)**.** We define the **norm map** $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by $|z| := \sqrt{z\overline{z}}$. In other words,

$$|a + bi| := \sqrt{a^2 + b^2}.$$

Note that this agrees with the absolute value on $\mathbb{R}$: for $a \in \mathbb{R}$, we have $\sqrt{a^2} = |a|$.

Norm functions, as in the real case, give us a notion of distance.

Metric on $\mathbb{C}$

Definition 1.4 (Metric on $\mathbb{C}$)**.** We define the *metric on* $\mathbb{C}$ to be $d_{\mathbb{C}}(z_1, z_2) := |z_1 - z_2|$.

One can check that this is in fact a metric, but we will not do so here.

Remark 1.5. The distance in $\mathbb{C}$ is defined to match the distance in $\mathbb{R}^2$ under the basis $\{1, i\}$.

Again as we discussed in real analysis, having a metric gives us a metric topology by open balls. Lastly it is this topology that our geometry will follow from: we have turned $\mathbb{C}$ into a topological space.

1.1.3 Complex Functions

There are lots of functions on $\mathbb{C}$, and lots of them are terrible. So we would like to focus on functions with some structure. We'll start with *continuous functions*, which are more or less the functions that respect topology.

Then from continuous functions, we will be able to define *holomorphic functions*, which are complex differentiable. This intended to be similar to being real differentiable, but complex differentiable turns out to be a very strong condition. Nevertheless, everyone's favorite functions are holomorphic.

Example 1.6. Polynomials, $\exp$, $\sin$, and $\cos$ are all holomorphic.

To make concrete that complex differentiability is stronger than real differentiability, the Cauchy–Riemann equations which provides a partial differential equation to test complex differentiability.

From here we define *analytic* functions, which essentially are defined as taking the form

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Analytic functions are super nice in that we have an ability to physically write them down, so the following theorem is amazing.

Theorem 1.7. Holomorphic functions on $\mathbb{C}$ are analytic.

To prove this, we will need the following result, which is what Professor Morrow calls the most fundamental result in complex analysis, the *Cauchy integral formula*.

In short, the Cauchy integral formula lets us talk about the value of holomorphic functions (and derivatives) at a point in terms of integrals around the point. This will essentially let us build the power series for a holomorphic function by hand. But as described, we will need a notion of complex (path) integration to even be able to talk about the Cauchy integral formula.

The Cauchy integral formula has lots of applications; for example, *Liouville's theorem* on holomorphic functions and the *Fundamental theorem of algebra*.

Remark 1.8. It is very hard to speill Liouville.

Additionally, we remark that our study of holomorphic functions, via the Cauchy integral formula, will boil down to a study of complex path integrals. So we will finish out our story with the *Residue theorem*, which provides a very convenient way to compute such integrals.

Then as a fun addendum, we talk about automorphisms of the complex numbers.

Automorphisms of $\mathbb{C}$

Definition 1.9 (Automorphisms of $\mathbb{C}$)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is an *automorphism of* $\mathbb{C}$ if it is bijective and both f and f^{-1} are holomorphic.

What is amazing is that all of these functions have a concrete description in terms of *Möbius transformations*.

1.1.4 Why Care?

Whenever taking a class, it is appropriate to ask why one should care. Here are some reasons to care.

- Algebraic geometry in its study of complex analytic spaces uses complex analysis.
- Analytic number theory (e.g., the Prime number theorem) makes heavy use of complex analysis.
- Combinatorics via generating functions can use complex analysis.
- Physics uses complex analysis.

The first two Professor Morrow is more familiar with, the last two less so.

1.2 January 21

We're reviewing set theory today.

1.2.1 Set Theory Notation

We have the following definitions.

- $\varnothing$ means the empty set.
- $a \in X$ means that a is an element of the set X.
- $A \subseteq B$ means that A is a subset of B.
- $A \subsetneq B$ means that A is a proper subset of B.

- $A \cup B$ consists of the elements which are in at least one of A or B.
- $A \cap B$ consists of the elements which are in both A and B.
- $A \setminus B$ consits of the elements of A which are not in B.
- Two sets A and B are *disjoint* if and only if $A \cap B = \varnothing$.
- Given a set X, we define $\mathcal{P}(X)$ to be the set of all subsets of X.
- $|X| = \#X$ is the cardinality of X, or (roughly speaking) the number of elements of X.

As an example of unwinding notation, we have the following.

Proposition 1.10 (De Morgan's Laws)**.** Fix $\mathcal{S} \subseteq \mathcal{P}(X)$ a collection of subsets of a set X. Then

$$X \Bigg\backslash \bigcap_{S \in \mathcal{S}} S = \bigcup_{S \in \mathcal{S}} (X \setminus S) \qquad \text{and} \qquad X \setminus \bigcup_{S \in \mathcal{S}} S = \bigcap_{S \in \mathcal{S}} (X \setminus S).$$

Proof. We take these one at a time.

- Note $a \in X \setminus \bigcap \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcap \mathcal{S}$. However, $a \notin \bigcap \mathcal{S}$ is merely saying that a is not in all of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for one of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for some $S \in \mathcal{S}$, which is equivalent to $a \in \bigcup_{S \in \mathcal{S}} (X \setminus S)$.
- Note $a \in X \setminus \bigcup \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcup \mathcal{S}$. However, $a \notin \bigcup \mathcal{S}$ is merely saying that a is not in any of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for each of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for each $S \in \mathcal{S}$, which is equivalent to $a \in \bigcap_{S \in \mathcal{S}} (X \setminus S)$. ■

1.2.2 Some Conventions

In this class, we take the following names of standard sets.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Importantly, $0 \in \mathbb{N}$.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}$ and $q \neq\}$ is the set of rationals.
- $\mathbb{R}$ is the set of real numbers. We will not specify a construction here; see any real analysis class.
- $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq\}$ is the nonzero real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ is the positive real numbers.
- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ is the nonnegative real numbers.
- $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$ is the nonpositive real numbers.
- $\mathbb{C}$ is the complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$ is the set of nonzero complex numbes.

1.2.3 Relations

Let's review some set theory definitions.

Cartesian product

Definition 1.11 (Cartesian product)**.** Given two sets A and B, we define the *Cartesian product* $A \times B$ to be the set of orderd pairs (a, b) such that $a \in A$ and $b \in B$.

Binary relation

Definition 1.12 (Binary relation)**.** A *binary relation* on A is any subset $R \subseteq A^2 := A \times A$. We may sometimes notate $(x, y) \in R$ by xRy, read as "x is related to y."

Example 1.13. Equality is a binary relation on any set A; namely, it is the subset $\{(a, a) : a \in A\}$.

The best relations are equivalence relations.

Equivalence relation

Definition 1.14 (Equivalence relation)**.** An *equivalence relation* on A is a binary relation R satisfying the following three conditions.

- Reflexive: each $x \in A$ has $(x, x) \in R$.
- Symmetric: each $x, y \in A$ has $(x, y) \in R$ implies $(y, x) \in R$.
- Transitive: each $x, y, z \in A$ has $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Equivalence relations are nice because they allow us to partition the set into "equivalence classes."

Equivalence class

Definition 1.15 (Equivalence class)**.** Fix A a set and $R \subseteq A^2$ an equivalence relation. Then, for given $x \in A$, we define

$$[x]_R := \{y \in A : (x, y) \in R\}$$

to be the *equivalence class* of x.

The hope is that equivalence classes partition the set. What is a parition?

Parition

Definition 1.16 (Parition)**.** A *partition* of a set A is a collection of nonempty subsets $\mathcal{S} \subseteq \mathcal{P}(A)$ of A such that any two distinct $S_1, S_2 \in \mathcal{S}$ are disjoint while $A = \bigcup_{S \in \mathcal{S}} S$.

And now let's manifest our hope.

Lemma 1.17. Equivalence relations are in one-to-one correspondence with partitions of A.

Proof. Given an equivalence relation R, we define the collection

$$\mathcal{S}(R) = \{[x]_R : x \in A\}.$$

We claim that $R \mapsto \mathcal{S}(R)$ is our needed bijection. We have the following checks.

- Well-defined: observe that $\mathcal{S}(R)$ does partition A: if we have $[x]_R, [y]_R \in \mathcal{S}$, then $[x]_R \cap [y]_R \neq \varnothing$ implies there is some z with $(x, z) \in R$ and $(z, y) \in R$, so $x \in [y]_R$ and then $[x]_R \subseteq [y]_R$ follows. So by symmetry, $[y]_R \subseteq [x]_R$ as well, so we finish the disjointness check.

  Further, we see that
  $$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} [x]_R \subseteq A$$
  because $x \in [x]_R$, so indeed the equivalence classes cover A.

- Injective: suppose R_1 and R_2 have $\mathcal{S}(R_1) = \mathcal{S}(R_2)$. We show that $R_1 \subseteq R_2$, and $R_2 \subseteq R_1$ will follow by symmetry, finishing.

  We notice that, for any $\mathcal{S}$ paritioning A, being a partition, will have exactly one subset which contains x. But for $\mathcal{S}(R)$ for an equivalence relation R, we see $x \in [x]_R \in \mathcal{S}(R)$, so this equivalence class must be the one.

  So because $[x]_{R_1}$ and $[x]_{R_2}$ are the only subsets of $\mathcal{S}(R_1)$ and $\mathcal{S}(R_2)$ containing x (respectively), we must have $[x]_{R_1} = [x]_{R_2}$. Thus, $(x, y) \in R_1$ implies $y \in [x]_{R_1} = [x]_{R_2}$ implies $(x, y) \in R_2$.

- Surjective: suppose that $\mathcal{S}$ is a partition of A. As noted above, each $x \in A$ is a member of exactly one set $S \in \mathcal{S}$, which we call $[x]$. Then we define $R \subseteq A^2$ by $(x, y) \in R$ if and only if $y \in [x]$. One can check that this is an equivalence relation, which we will not do here in detail.[1]

  The point is that
  $$[x]_R = \{y : (x, y) \in R\} = \{y : y \in [x]\} = [x],$$
  so $\mathcal{S}(R) = \mathcal{S}$. So our mapping is surjective. ■

We continue our discussion.

Quotient set

Definition 1.18 (Quotient set)**.** Given an equivalence relation $R \subseteq A^2$, we define the *quotient set* A/R is the set of equivalence classes of R. In other words,

$$A/R = \{[x]_R : x \in A\}.$$

Intuitively, the quotient set is the set where we have gone ahead and identified the elements which are "similar"or "related."

We would like a more concrete way to talk about equivalence classes, for which we have the following.

Representatives

Definition 1.19 (Representatives)**.** Given an equivalence relation $R \subseteq A^2$, we say that $C \subseteq A$ is a *set of representatives of R-equivalence classes of A* if and only if C consists of exactly one element from each equivalence class in A/R.

1.2.4 Functions

To finish off, we discuss functions.

Functions

Definition 1.20 (Functions)**.** A *function* $f : X \to Y$ is a relation $f \subseteq X \times Y$ satisfying the following.

- For each $x \in X$, there is some $y \in Y$ such that $(x, y) \in f$. Intuitively, each $x \in X$ goes somewhere.
- For each $x \in X$ and given some $y_1, y_2 \in Y$ such that $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$. Intuitively, each $x \in X$ goes to at most one place.

We will write $f(x) = y$ as notational sugar for $(x, y) \in f$. Note this equality is legal because the value y with $(x, y) \in f$ is uniquely given.

We would like to create new functions from old. Here are two ways to do this.

Restriction

Definition 1.21 (Restriction)**.** Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define

$$f|_A = \{(x, y) \in f : x \in A\} \subseteq A \times Y$$

to be a function $f|_A : A \to Y$.

[1] Note $x \in [x]$ by definition of $[x]$. If $y \in [x]$, then note $y \in [y]$ as well, so $[x] = [y]$ is forced by uniqueness, so $x \in [y]$. If $y \in [x]$ and $z \in [y]$, then again by uniqueness $[x] = [y] = [z]$, so $z \in [x]$ follows.

We will not check that $f|_A$ is actually a function; it is, roughly speaking inherited from f.

Definition 1.22. Given two functions $f : X \to Y$ and $g : Y \to Z$, we define the *composition* of f and g to be some functino $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) := g(f(x)).$$

Again, we will not check that this makes a function; it is.

Functions can also help create new sets.

Image

Definition 1.23 (Image)**.** Given a function $f : X \to Y$, we define the *image* of f to be

$$\operatorname{im} f = f(X) := \{y \in Y : \text{there is } x \in X \text{ such that } f(x)y\}.$$

Namely, $\operatorname{im} f$ consists of all elements hit by someone in X hit by f.

Fiber, pre-image

Definition 1.24 (Fiber, pre-image)**.** Given a functino $f : X \to Y$ and some $y \in Y$, we define the *fiber* of f over y to be

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

In general, we define the *pre-image* of a subset $A \subseteq X$ to be

$$f^{-1}(A) := \{x \in A : f(x) \in A\} = \bigcup_{a \in A} \{x \in A : f(x) = a\} = \bigcup_{a \in A} f^{-1}(a).$$

Some functions have nicer properties than others.

Inj-, sur-, bijective

Definition 1.25 (Inj-, sur-, bijective)**.** Fix a function $f : X \to Y$. We have the following.

- Then f is *injective* or *one-to-one* if and only if, given $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- Then f is *surjective* or *onto* if and only if $\operatorname{im} f = Y$. In other words, for each $y \in Y$, there exists $x \in X$ with $f(x) = y$.
- Then f is *bijective* if and onlt if it is both injective and surjective.

Here is an example.

Identity

Definition 1.26 (Identity)**.** For a given set X, the function $\operatorname{id}_X : X \to X$ defined by $\operatorname{id}_X(x) := x$ is called the *identity function*.

For completeness, here are the checks that id_X is bijective.

- Injective: given $x_1, x_2 \in X$, we see $\operatorname{id}_X(x_1) = \operatorname{id}_X(x_2)$ implies $x_1 = \operatorname{id}_X(x_1) = \operatorname{id}_X(x_2) = x_2$.
- Surjective: given $x \in X$, we see that $x \in \operatorname{im} \operatorname{id}_X$ because $x = \operatorname{id}_X$.

We leave with some lemmas, to be proven once in one's life.

Lemma 1.27. Fix a finite sets X and Y such that $\#X = \#Y$. Then a function $f : X \to Y$ is bijective if and only if it is injective or surjective.

Proof. Certainly if f is bijective, then it is both injective and surjective, so there is nothing to say.

The reverse direction is harder. We proceed by induction on $\#X = \#Y$. If $\#X = \#Y = 0$, then $X = Y = \varnothing$, and all functions $f : \varnothing \to \varnothing$ are vacuously bijective: for injective, note that any $x_1, x_2 \in \varnothing$ have $x_1 = x_2$; for surjective, note that any $x \in \varnothing$ has $f(x) = x$.

Otherwise $\#X = \#Y > 0$. We have two cases.

- Take f injective; we show f is surjective. In this case, $\#X > 0$, so choose some $a \in X$. Note that $x \in X$ with $x \neq a$ will have $f(x) \neq f(a)$ by injectivity, so we may define the restriction

  $$f|_{X \setminus \{a\}} : X \setminus \{a\} \to Y \setminus \{f(a)\}.$$

  Observe that $f|_{X\setminus\{a\}}$ is injective because f is: if $x_1, x_2 \in X \setminus \{a\}$ have

  $$f(x_1) = f|_{X\setminus\{a\}}(x_1) = f|_{X\setminus\{a\}}(x_2) = f(x_2),$$

  then $x_1 = x_2$ follows.

  Now, $\#(X \setminus \{a\}) = \#(Y \setminus \{f(a)\}) = \#X - 1$, so by induction $f|_{X\setminus\{a\}}$ will be bijective because it is injective. In particular, f by way of $f|_{X\setminus\{a\}}$ fully hits $Y \setminus \{f(a)\}$ in its image, so because $f(a) \in \operatorname{im} f$ as well, we conclude $\operatorname{im} f = Y$. So f is surjective.

- Take f surjective; we show f is injective. Define a functiono $g : Y \to X$ as follows: for each $y \in Y$, the surjectivity of f promises some $x \in X$ such that $f(x) = y$, so choose any such x and define $g(y) := x$.[2] Observe that $f(g(y)) = y$ by construction.

  Now, we notice that g is injective: if $y_1, y_2 \in Y$ have $g(y_1) = g(y_2)$, then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$. So the previous case tells us that g is in fact bijective.

  So now choose any $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. The surjectivity of f promises some $y_1, y_2 \in Y$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$, so we see that

  $$x_1 = g(y_1) = g(f(g(y_1))) = g(f(x_1)) = g(f(x_2)) = g(f(g(y_2))) = g(y_2) = x_2,$$

  proving our injectivity. ■

Lemma 1.28. Fix $f : X \to Y$ a bijective function. Then there is a unique function $g : Y \to X$ such that $f \circ g = \operatorname{id}_Y$ and $g \circ f = \operatorname{id}_X$.

Proof. We show existence and uniqueness separately.

- We show existence. Note that, because $f : X \to Y$ is surjective, each $y \in Y$ has some $x \in X$ such that $f(x) = y$. In fact, this $x \in X$ is uniquely defined because $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so we may define $g(y)$ as the value x for which $f(x) = y$.

  By construction, $f(g(y)) = y$, so $f \circ g = \operatorname{id}_Y$. Additionally, we note that, given any $x \in X$, the value x_0 for which $f(x) = f(x_0)$ is $x = x_0$ by the injectivity, so $g(f(x)) = x$. Thus, $g \circ f = \operatorname{id}_X$, as claimed.

- We show uniqueness. Suppose that we have two functions $g_1, g_2 : Y \to X$ which satisfy

  $$f \circ g_1 = f \circ g_2 = \operatorname{id}_Y \qquad \text{and} \qquad g_1 \circ f = g_2 \circ f = \operatorname{id}_X .$$

  Then we see that

  $$g_1 = g_1 \circ \operatorname{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \operatorname{id}_X \circ g_2 = g_2,$$

  where we have used the fact that function composition associates. This finishes. ■

1.3 January 24

Good morning everyone.

[2] Technically we are using the Axiom of Choice here. One can remove this with an induction because all sets are finite, but I won't bother.

1.3.1 Algebraic Structure

Today we are reviewing the complex numbers (reportedly, "some basics"). Or at least it is hopefully mostly review. Here is our main character this semester.

Complex numbers

Definition 1.29 (Complex numbers)**.** The set $\mathbb{C}$ of *complex numbers* is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Here i is some symbol such that $i^2 = -1$ formally.

In particular, two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

The complex numbers also have some algebraic structure.

$+$ and $\times$ in $\mathbb{C}$

Definition 1.30 ($+$ and $\times$ in $\mathbb{C}$)**.** Given complex numbers $a_1 + b_1 i, a_2 + b_2 i \in \mathbb{C}$, we define

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i,$$

and

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i,$$

defined essentially by direct expansion, upon recalling $i^2 = -1$.

Here is the corresponding algebraic structure.

Proposition 1.31. The set $\mathbb{C}$ with the above operations is a two-dimensional $\mathbb{R}$-vector space with basis $\{1, i\}$.

Proof. The elements $\{1, i\}$ span $\mathbb{C}$ because all complex numbers in $\mathbb{C}$ can be written as $a + bi = a \cdot 1 + b \cdot i$ by definition.

To see that these elements are linearly independent, suppose $a + bi = 0$. If $b = 0$, then $a = 0$ follows, and we are done. Otherwise, take $b \neq 0$, but then we see $(-a/b) = i$, so

$$(-a/b)^2 = -1 < 0,$$

which does not make sense for real numbers. This finishes. ■

Proposition 1.32. The set $\mathbb{C}$ with the above operations is a field.

Proof. We have the following checks.

- The element $0 + 0i$ is our additive identity. Indeed, one can check that $(0 + 0i) + (a + bi) = (a + bi) + (0 + 0i) = a + bi$.
- The element $1 + 0i$ is our multiplicative identity. Indeed, one can check that $(1 + 0i)(a + bi) = (a + bi)(1 + 0i) = a + bi$.
- Commutativity of addition and multiplication follow from by expansion.
- The distributive laws can again be checked by expansion.
- The additive inverse of $a + bi$ is $(-a) + (-b)i$.
- The multiplicative inverse of $a + bi$ can be found by wishing really hard and writing

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

  Then one can check this works. ■

Sometimes we would like to extract our coefficients from our basis.

Re and Im

Definition 1.33 (Re and Im). Given $z := a + bi \in \mathbb{C}$, we define the operations

$$\operatorname{Re} z := a \qquad \text{and} \qquad \operatorname{Im} z := b.$$

Importantly, $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$.

Because we are merely doing basis extraction, it makes sense that these operations will preserve some (additive) structure.

Proposition 1.34. Fix $z = a + bi$ and $w = c + di$. Then the following.

(a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$.

(b) $\operatorname{Im}(z + w) = \operatorname{Im} z + \operatorname{Im} w$.

Proof. We proceed by direct expansion. Observe

$$\operatorname{Re}(z + w) = \operatorname{Re}((a + c) + (b + d)i) = a + c = \operatorname{Re} z + \operatorname{Re} w,$$

and

$$\operatorname{Im}(z + w) = \operatorname{Im}((a + c) + (b + d)i) = b + d = \operatorname{Im} z + \operatorname{Im} w.$$

This finishes. ■

It also turns out that the complex numbers have a very special transformation.

Conjugate

Definition 1.35 (Conjugate). Given $z := a + bi \in \mathbb{C}$, we define the *complex conjugate* to be $\overline{z} := a - bi \in \mathbb{C}$.

We promised conjugation would be special, so here are some special things.

Proposition 1.36. Fix $z = a + bi \in \mathbb{C}$. Then the following.

(a) $z + \overline{z} = 2 \operatorname{Re} z$.

(b) $z - \overline{z} = 2i \operatorname{Im} z$.

(c) $\overline{\overline{z}} = z$.

Proof. We take these one at a time.

(a) Write $a + bi + \overline{a + bi} = a + bi + a - bi = 2a$.

(b) Write $a + bi - \overline{a + bi} = a + bi - (a - bi) = 2bi$.

(c) Write $\overline{\overline{a + bi}} = \overline{a - bi} = a + bi$. ■

In fact, more is true.

Proposition 1.37. Fix $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$. Then the following.

(a) $\overline{z + w} = \overline{z} + \overline{w}$.

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$.

Proof. We take these one at a time.

- Write
$$\overline{z+w} = (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}.$$
- Write
$$\begin{aligned}\overline{z}\cdot\overline{w} &= (a-bi)(c-di)\\ &= (ac-bd)-(ad+bc)i\\ &= \overline{(ac-bd)+(ad+bc)i} \qquad = \overline{zw}.\end{aligned}$$

This finishes. ■

1.3.2 Defining Distance

Complex conjugation actually gives rise to a notion of size.

Norm on $\mathbb{C}$

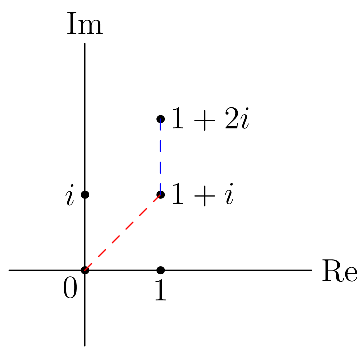
Definition 1.38 (Norm on $\mathbb{C}$). Given $z := a+bi$, we define the *norm function on* $\mathbb{C}$ by
$$|z| := \sqrt{a^2+b^2}.$$

Size actually gives distance.

Distance on $\mathbb{C}$

Definition 1.39 (Distance on $\mathbb{C}$). Given complex numbers $z = a+bi$ and $w = c+di$, we define the *distance* between z and w to be
$$|z-w| = \sqrt{(a-c)^2+(b-d)^2}.$$

Here are some examples.



One can ask what is the distance between $0+0i$ and $1+1i$, and we can compute directly that this is $\sqrt{1+1} = \sqrt{2}$. Similarly, the distance between $1+2i$ and $1+i$ is $|(1+2i)-(1+i)| = |i| = 1$. It should agree with our geometric intuition.

We mentioned complex conjugation is involved here, so we have the following lemma.

Lemma 1.40. Fix $z, w \in \mathbb{C}$. The following are true.

(a) $|z|^2 = z\overline{z}$.

(b) $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.

(c) $|z| = |\overline{z}| = |-z|$.

(d) $|z| = 0$ if and only if $z = 0$.

(e) $|zw| = |z|\cdot|w|$.

Proof. We take these one at a time. Set $z = a + bi$.

(a) We have
$$|z|^2 = a^2 + b^2 = (a + bi)(a - bi) = z\overline{z}.$$
Here we have used subtraction of two squares, which one can see when writing $a^2 + b^2 = a^2 - (ib)^2$.

(b) We have $a^2 \le a^2 + b^2$ and $b^2 \le a^2 + b^2$ by the Trivial inequality, so
$$|\operatorname{Re} z| = |a| \le \sqrt{a^2 + b^2} = |z|,$$
and similarly,
$$|\operatorname{Im} z| = |b| \le \sqrt{a^2 + b^2} = |z|.$$

(c) Note
$$|\overline{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|,$$
and
$$|-z| = |-a - bi| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(d) From (b), we know that $|\operatorname{Re} z|, |\operatorname{Im} z| \le |z|$, but $|z| = 0$ then forces $\operatorname{Re} z = \operatorname{Im} z = 0$, so $z = 0$.

(e) From (a), we can write $|zw|^2 = zw \cdot \overline{zw}$, which will expand out into
$$z \cdot w \cdot \overline{z} \cdot \overline{w}.$$
We can collect this into $z\overline{z} \cdot w\overline{w} = |z|^2|w|^2$. Thus, by (a) again, $|zw|^2 = |z|^2|w|^2$.But because all norms must be nonnegative real numbers, we may take square roots to conclude $|zw| = |z| \cdot |w|$. ■

Remark 1.41. Norms are actually more general constructions. For example, the requirement $|zw| = |z| \cdot |w|$ makes $|\cdot|$ into a "multiplicative" norm.

To finish off, we actually show that our distance function is good: we show the triangle inequality.

Lemma 1.42 (Triangle inequality)**.** For every $x, y, z \in \mathbb{C}$, we claim
$$|z - x| \le |z - y| + |y - z|.$$

This claim should be familiar from real analysis. Intuitively, it means that travelling between z and x cannot be made into a shorter trip by taking a detour to some other point y first.

Proof. Let $a := z - y$ and $b := y - z$ so that $a + b = z - x$. Thus, we are showing that
$$|a + b| \stackrel{?}{\le} |a| + |b|,$$
which is nicer because it only has two letters. For this, because everything is a nonnegative real numbers, it suffices to show the square of this requirement; i.e., we show
$$(|a| + |b|)^2 - |a + b|^2 \stackrel{?}{\ge} 0.$$
Fully expanding, it suffices to show
$$|a|^2 + |b|^2 + 2|a| \cdot |b| - |a + b|^2 \ge 0.$$
Expanding out $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$, we are showing
$$a\overline{a} + b\overline{b} + 2|a| \cdot |b| - (a + b)(\overline{a} + \overline{b}) \ge 0.$$

This is nice because the expansion of the rightmost term will induce some cancellation: it expands into $a\overline{a} + a\overline{b} + \overline{a}b + b\overline{b}$, so we are left with showing

$$2|a| \cdot |b| - a\overline{b} - b\overline{a} \overset{?}{\geq} 0.$$

Note that $\overline{a}b = \overline{a\overline{b}}$, so we can collect the final term as $2\,\mathrm{Re}(a\overline{b})$. Similarly, we can write $|a| \cdot |b| = |a| \cdot |\overline{b}| = |a\overline{b}|$, so we are showing

$$2|a\overline{b}| - 2\,\mathrm{Re}(a\overline{b}) \geq 0,$$

which is true because the real part does exceed the norm. This finishes. ■