18.708: Topics in Algebra

Nir Elber

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

DE RHAM COHOMOLOGY IN MIXED CHARACTERISTIC

These talks were given by Alexander Petrov.

1.1 February 2

Here we go.

1.1.1 Algebraic de Rham Cohomology

Let's begin by describing what we mean by de Rham cohomology. We will consider a smooth variety X over an algebraically closed field F.

Definition 1.1 (smooth)**.** We say that a variety X over a field F is *smooth* if and only if $\Omega_{X/F}$ is a vector bundle of rank $\dim X$ on each connected component. Here, on an affine open subset $U \subseteq X$, recall that $\Omega_{X/F}(U)$ is spanned by symbols of the form $f\,dg$, where the symbol d is (as usual) F-linear and satisfies the Leibniz rule.

Definition 1.2 (algebraic de Rham cohomology)**.** Fix a smooth variety X over a field F. Then one can iterate the F-linear map $d\colon \mathcal{O}_X \to \Omega_{X/F}$ to a map $d\colon \Omega_{X/F}^i \to \Omega_{X/F}^{i+1}$ for each i, where $\Omega_{X/F}^i := \wedge^i \Omega_{X/F}$. We now define the *de Rham complex* to the complex

$$\Omega_{X/F}^\bullet \colon 0 \to \mathcal{O}_X \xrightarrow{d} \Omega_{X/F}^1 \xrightarrow{d} \cdots,$$

and we define the *de Rham cohomology* $\mathrm{H}_{\mathrm{dR}}^n(X/F)$ to be the nth hypercohomology of $\Omega_{X/F}^\bullet$. Here, hypercohomology means the total cohomology of some produced acyclic double complex which resolves the complex (e.g., a Čech resolution). Note that this hypercohomology is merely a vector space over F.

Example 1.3. The map $d\colon \Omega_{X/F}^1 \to \Omega_{X/F}^2$ is given by $d(f\,dg) = df \wedge dg$.

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Example 1.4. Suppose that X is affine. Then vector bundles are already acyclic, so the hypercohomology does nothing. Thus,

$$\mathrm{H}_{\mathrm{dR}}^n(X/F) = \mathrm{H}^n\left(X; 0 \to \mathcal{O}_X \xrightarrow{d} \Omega_{X/F}^1 \xrightarrow{d} \cdots\right).$$

As usual, this is $\ker\left(d|_{\Omega^n}\right) / \operatorname{im}\left(d|_{\Omega^{n-1}}\right)$.

Remark 1.5. If X is affine and $i > \dim X$, then $\Omega_{X/F}^i$ vanishes, so the algebraic de Rham cohomology also vanishes.

Remark 1.6. A different definition is required for non-smooth X. Roughly speaking, one should embed into a smooth variety and take cohomology there.

Here is one way to convince ourselves that this is a reasonable cohomology theory.

Theorem 1.7 (Grothendieck). Suppose that X is a smooth variety over $\mathbb{C}$. Then there is a canonical isomorphism

$$\mathrm{H}_{\mathrm{B}}^n(X(\mathbb{C}); \mathbb{C}) \to \mathrm{H}_{\mathrm{dR}}^n(X/\mathbb{C}).$$

Here, the left-hand side is Betti cohomology (also called singular cohomology).

Sketch. We argue in the case that X is affine. Then $X(\mathbb{C})$ already has a notion of $\Omega_{X/\mathbb{C}}^{i,\mathrm{an}}$ given by the holomorphic forms. Algebraic forms embed into holomorphic ones, which produces a morphism

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_X(X) & \longrightarrow & \Omega_{X/\mathbb{C}}^1(X) & \longrightarrow & \Omega_{X/\mathbb{C}}^2(X) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\mathrm{an}}(X) & \longrightarrow & \Omega_{X/\mathbb{C}}^{1,\mathrm{an}}(X) & \longrightarrow & \Omega_{X/\mathbb{C}}^{2,\mathrm{an}}(X) & \longrightarrow & \cdots
\end{array}
$$

of complexes. It then turns out that this is an isomorphism on cohomology, so we reduce to comparing analytic de Rham cohomology with singular cohomology.

This is now a problem of analysis. One can pass from holomorphic differentials to smooth differentials via a similar process, which produces another morphism

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_X^{\mathrm{an}}(X) & \longrightarrow & \Omega_{X/\mathbb{C}}^{1,\mathrm{an}}(X) & \longrightarrow & \Omega_{X/\mathbb{C}}^{2,\mathrm{an}}(X) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\infty(X(\mathbb{C}), \mathbb{C}) & \longrightarrow & \Omega_{C^\infty}^1(X(\mathbb{C})) & \longrightarrow & \Omega_{C^\infty}^2(X(\mathbb{C})) & \longrightarrow & \cdots
\end{array}
$$

of complexes, which is also an isomorphism on complexes. We are now reduced to the setting of de Rham's theorem for real manifolds. ∎

Example 1.8. Consider $X := \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} = \operatorname{Spec} k[t, 1/t]$.

- Our differential map $d\colon \mathbb{C}[t, 1/t] \to \mathbb{C}[t, 1/t]\, dt$ sends t^n to $nt^{n-1}\, dt$. Thus, $\mathrm{H}_{\mathrm{dR}}^0(X)$ is one-dimensional given by the constants, and $\mathrm{H}_{\mathrm{dR}}^1(X)$ is one-dimensional spanned by dt/t.

- The point above works also for holomorphic differentials. The interesting bit is in degree 1, where the point is that there is no global antiderivative for dx/x.

- On the other hand, $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle, so we expect its singular cohomology to be supported in degrees 0 and 1, where it should be one-dimensional.

> **Corollary 1.9** (Artin vanishing)**.** If X is an affine algebraic complex smooth variety, then $\mathrm{H}^n(X(\mathbb{C});\mathbb{C}) = 0$ for $n > \dim X$.

Proof. The algebraic de Rham cohomology complex vanishes above $\dim X$. ∎

> **Corollary 1.10.** Fix a smooth variety X over $\mathbb{C}$. Then $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C})$ is finite-dimensional.

Proof. Pass to singular cohomology. ∎

> **Remark 1.11.** This corollary still admits algebraic proofs in characteristic zero by working with holonomic $\mathcal{D}$-modules. Pavel Etingof claims that there is an algebraic proof using the fact that the direct image of a holonomic $\mathcal{D}$-module is a holonomic $\mathcal{D}$-module.

We would like to point out that our de Rham cohomology is algebraic but still interesting.

> **Remark 1.12.** Suppose that X is smooth over $\mathbb{Q}$. Base-changing by a field is exact, so
>
> $$\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{Q})_{\mathbb{C}} \cong \mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}).$$
>
> However, Theorem 1.7 grants an isomorphism to $\mathrm{H}^n(X(\mathbb{C});\mathbb{C}) \cong \mathrm{H}^n_{\mathrm{B}}(X(\mathbb{C});\mathbb{Z})_{\mathbb{C}}$. Notably, we then find a lattice and a rational structure over in some complex vector space, but the comparison between the two is quite interesting mathematically (and amounts to the study of periods).

> **Example 1.13.** In the case that $X = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\}$, the comparison between $\mathrm{H}^1_{\mathrm{dR}}(X/\mathbb{Q})_{\mathbb{C}}$ and $\mathrm{H}^1_{\mathrm{B}}(X(\mathbb{C});\mathbb{Z})$ is mediated by a constant $2\pi i$. Indeed, once unwinds the de Rham theorem, this amounts to the statement that a contour integral of dx/x going once around the origin is $2\pi i$.

1.1.2 Frobenius Structure

We now pass to positive characteristic. Let k be a perfect field of positive characteristic p, and we may still consider a smooth variety X.

> **Remark 1.14.** If k is perfect, then $\Omega^1_{X/k} = \Omega^1_{X/\mathbb{F}_p}$ by doing some thinking about inseparable extensions. The moral is that
>
> $$y^{1/p}\, dy = d\left((y^{1/p})^p\right),$$
>
> so the coefficients can be brought down when everything is a pth power.

This cohomology is rather strangely behaved.

> **Example 1.15.** Take $X := \mathbb{A}^1_k$. The de Rham cohomology still lives in degrees zero and one, so we would like to study the kernel and cokernel of the k-linear map $d\colon k[t] \to k[t]\, dt$ given by $t^n \mapsto nt^{n-1}$.
>
> - We see that $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \ker d$ is spanned by t^{pi} for each i.
>
> - We see that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_k/k) = \operatorname{im} d$ is infinite-dimensional because the differentials $t^{mp-1}\, dt$ fail to be in the image. In fact, these classes form a basis.

Let's try to view these infinite-dimensional groups as a feature instead of a bug. Indeed, it turns out that the de Rham complex has some extra structure. The de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots$$

is merely made of sheaves of k-vector spaces over X. In characteristic zero, this is all the structure present, but in characteristic p, we have more structure.

> **Notation 1.16.** Fix a variety X over a field k of characteristic p. For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we define
>
> $$\mathcal{F}^p := \{f^p : f \in \mathcal{O}_X\}$$
>
> to locally be given by the pth powers.

The moral is that $d(f^p) = 0$ always, so the de Rham complex is in fact $\mathcal{O}^p_X$-linear! Let's attempt to codify this.

> **Definition 1.17** (relative Frobenius)**.** Fix a scheme X over a field k of characteristic p. Then there is an *absolute Frobenius* $F_{\mathrm{abs}}\colon X \to X$ which is the identity on topological spaces and the pth power on sheaves. This is a morphism of schemes but not of k-schemes (in general). The *relative Frobenius* $F\colon X \to X^{(1)}$ is the morphism fitting into the following diagram.
>
>

> **Remark 1.18.** Note that $X^{(p)}$ is isomorphic to X as a scheme but not as a k-scheme! However, we now benefit because the relative Frobenius F is morphism of k-schemes.

> **Remark 1.19.** The relative Frobenius $F\colon X \to X^{(1)}$ is finite flat of degree $p^{\dim X}$

> **Example 1.20.** If $X = \operatorname{Spec} k[t_1, \ldots, t_n]$, then $X^{(1)} = \operatorname{Spec} k[t_1^p, \ldots, t_n^p]$. Thus, we see that the embedding
>
> $$k[t_1^p, \ldots, t_n^p] \subseteq k[t_1, \ldots, t_n]$$
>
> is indeed finite flat of degree p^n.

We now see that

$$0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/F} \to \cdots$$

is a complex of quasicoherent sheaves on $X^{(1)}$. In fact, because F is finite flat, these are all vector bundles: $F_*\mathcal{O}_X$ has rank $p^{\dim X}$ and $F_*\Omega^i_{X/k}$ has rank $p^{\dim X}\binom{\dim X}{i}$. Because $\mathcal{O}_{X^{(1)}} = (F_*\mathcal{O}_X)^p$, we see that this complex is in fact $\mathcal{O}_{X^{(1)}}$-linear.

> **Example 1.21.** Take $X = \operatorname{Spec} k[t]$. Then $X^{(1)} := \operatorname{Spec} k[t^p]$, and $d\colon k[t] \to k[t]\, dt$ is $k[t^p]$-linear! Thus, $\mathrm{H}^i_{\mathrm{dR}}(X/k)$ was required to be given by $k[t^p]$-modules, which explains why we received vector spaces of infinite dimension.

Note that passing through F_* is not going to adjust the underlying k-vector spaces, so

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \mathbb{H}^n_{\mathrm{Zar}}\left(X^{(1)}; 0 \to F_*\mathcal{O}_X \xrightarrow{d} F_*\Omega^1_{X/k} \xrightarrow{d} F_*\Omega^2_{X/k} \xrightarrow{d} \cdots\right).$$

To see why this has globalized the $\mathcal{O}_X^p$-linearity, we need the Cartier isomorphism.

Theorem 1.22 (Cartier isomorphism). Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism
$$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$
Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Remark 1.23. This is a reason why characteristic p may be more convenient than characteristic 0: one could still try to understand $\mathcal{H}^i(\Omega^\bullet_{X/k})$ when $\operatorname{char} k = 0$, but this has no easy answer.

Example 1.24. Consider $X = \mathbb{A}^1_k$. Then $\mathcal{H}^1$ is given by the module

$$\frac{k[t]\,dt}{d(k[t])},$$

which our formalism now remembers is a $k[t^p]$-module. And indeed, we can show that this is isomorphic to $k[t^p] \cdot t^{p-1}\,dt$. Setting $s := t^p$, we know that $\Omega^1_{X^{(1)}/k}$ is given by the module $k[s]\,ds$, so our isomorphism of modules is given by sending ds to $t^{p-1}\,dt$. One can even check that this isomorphism is canonical in the sense that it will not change under automorphisms of $\mathbb{A}^1$.

We will prove Theorem 1.22 later after a detour.

1.1.3 Crystalline Cohomology

We continue with our perfect field k of positive characteristic p. Our story so far has taken a variety X over a field k, and then we have produced some (total) complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$ in the derived category $D(\mathrm{Vec}_k)$. Crystalline cohomology will allow us to produce an answer in characteristic 0 instead of characteristic p. The idea is to "choose" a lift to characteristic p and then check that the answer is independent of the lift.

The correct formalism for this lifting is that of a "formal scheme."

Definition 1.25 (Witt ring). Fix a perfect field k of characteristic p. Then there is a ring $W(k)$ satisfying that

- $W(k)$ is p-torsion-free,

- $W(k)/p \cong k$, and

- $W(k)$ is the limit of the $W(k)/p^n$ as $n \to \infty$.

This ring $W(k)$ turns out to be unique up to unique isomorphism. We may write $W_n(k) := W(k)/p^n$.

Example 1.26. One can see that $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W(\overline{\mathbb{F}}_p)$ is its unramified closure.

Remark 1.27. There is a completely explicit construction of $W(k)$, but it is rather involved: given a p-torsion-free ring R, we identify $W(R) := R^{\mathbb{N}}$ but with ring structure chosen so that

$$(a_0, a_1, a_2, \ldots) \mapsto a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^n a_n$$

is a ring homomorphism $W(R) \to R$. It turns out that this ring structure is given by some polynomials (called "ghost coordinates"), so we are allowed to define $W(k)$. From a higher level, it turns out that $W(k)$ is the unique deformation of k, which exists because $\Omega^1_{k/\mathbb{F}_p} = 0$.

Definition 1.28 (formal scheme)**.** Fix a perfect field k of characteristic p. A p-*adic formal scheme* X is a collection of schemes X_n over $W_n(k)$ equipped with isomorphisms

$$X_{n+1} \times_{W_{n+1}(k)} W_n(k) \to X_n.$$

The structure sheaf $\widehat{\mathcal{O}}_X$ is the inverse limit of the $\mathcal{O}_{X_n}$s.

Example 1.29. Given a scheme Y over $W(k)$, we can produce a formal scheme $\widehat{Y}$ with $\widehat{Y}_n := Y \times_{W(k)} W_n(k)$ and the induced internal isomorphisms.

Remark 1.30. If X_1 is affine, then all the nilpotent thickenings are affine, so we may say that the full formal scheme is affine.

Remark 1.31. We can even define $\widehat{\Omega} X^1_{\widetilde{X}}$.

We can now describe crystalline cohomology.

Theorem 1.32. Fix a perfect field k of positive characteristic p. Then there is a functor sending smooth k-varieties X to a complex $\mathrm{R\Gamma}_{\mathrm{cris}}(X/W(k))$ in the derived category $D(\mathrm{Mod}_{W(k)})$ satisfying the following.

(a) There is a quasi-isomorphism $\mathrm{R\Gamma}_{\mathrm{cris}}(X/W(k)) \otimes^{\mathbb{L}}_{W(k)} k \cong \mathrm{R\Gamma}_{\mathrm{dR}}(X/k)$.

(b) If $\widetilde{X}$ is a smooth formal scheme over $W(k)$ (meaning that $\widetilde{X}_n$ is smooth over $W_n(k)$ for all n), then the scheme $X := \widetilde{X} \times_{W(k)} k$ has

$$\mathrm{R\Gamma}_{\mathrm{cris}}(X/W(k)) \cong \mathrm{R\Gamma}_{\mathrm{dR}}\left(X; \widehat{\mathcal{O}}_{\widetilde{X}} \xrightarrow{d} \widehat{\Omega}^1_{\widetilde{X}} \xrightarrow{d} \cdots\right).$$

This right-hand side can be thought of as $\mathrm{R\Gamma}_{\mathrm{dR}}(\widetilde{X}/W(k))$.

Remark 1.33. Here, (a) immediately tells us that the cohomology of $\mathrm{R\Gamma}_{\mathrm{cris}}(X/W(k))$ is not expected to be finitely generated.

Remark 1.34. There is something remarkable here, which is that choosing two different lifts of X to a smooth formal scheme produces the same cohomology!

Remark 1.35. It turns out that flatness is equivalent to smoothness in this context.

1.2   February 4

Today we continue our discussion of crystalline cohomology.

1.2.1   Example of Crystalline Cohomology

We continue working over a perfect field k of characteristic p. Let's run an example.

Remark 1.36. If our formal scheme $\widetilde{X}$ is smooth and affine, then $\mathrm{R}\Gamma_{\mathrm{dR}}(X/W(k))$ can be computed directly as the complex
$$0 \to \mathcal{O}(\widetilde{X}) \xrightarrow{d} \Omega^1(\widetilde{X}) \xrightarrow{d} \Omega^2(\widetilde{X}) \xrightarrow{d} \cdots .$$

Example 1.37. Take $X = \mathbb{A}_k^1$, which we can deform to a formal scheme $\widetilde{X} = \widehat{\mathbb{A}}_k^1$. Then
$$\mathcal{O}(\widetilde{X}) = \lim W_n[t],$$
which we may refer to as $W\langle t \rangle$. Here, $W\langle t \rangle$ contains those power series whose terms are more and more divisible by p. Similarly, we can compute that $\Omega^1(\widetilde{X}) = W\langle t \rangle\, dt$. For example, we now see that W has characteristic zero, so the kernel of the differential $d \colon W\langle t \rangle \to W\langle t \rangle\, dt$ is exactly the constants. On the other hand, Theorem 1.32 tells us that $\mathrm{H}^1_{\mathrm{dR}}(\widetilde{X}/W(k))$ is required to be quite interesting because the de Rham complex of $\mathbb{A}_k^1$ is interesting.

Remark 1.38. One can extend Theorem 1.32 slightly: even if some k-scheme X can only be deformed to a smooth scheme X_n over $W_n(k)$, then we still have
$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)) \otimes^{\mathbb{L}}_{W(k)} W_n(k) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X_n/W_n(k)).$$

The above example falls under the paradigm where the formal scheme comes from a genuine scheme. Here are some motivational remarks about this case.

Example 1.39. Let's consider the case where the formal scheme comes from a genuine scheme. Given a smooth scheme Y over $W(k)$ with formal completion $\widehat{Y}$, then $\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k))$ is the derived p-adic completion of $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$. Explicitly, one takes a complex representing $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ and takes a p-completion of each module. (This turns out to be a well-defined operation on the derived category.)

Example 1.40. We continue the previous example. Suppose further that Y is smooth and proper. Then it turns out that $\mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k))$ can be represented by finitely generated projective $W(k)$-modules, so the p-completion does nothing, so
$$\mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{Y}/W(k)) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(Y/W(k)).$$

1.2.2   The Mysterious Functor

Crystalline cohomology produces the following strange functoriality.

Corollary 1.41. Fix a smooth formal scheme $\widetilde{X}$ over $W(k)$. Given an endomorphism f of $\widetilde{X}$ such that f is the identity $(\mathrm{mod}\ p)$, then f^* is the identity on the cohomology groups $\mathrm{H}^n_{\mathrm{dR}}(\widetilde{X}/W(k))$.

Proof using Theorem 1.32. By Theorem 1.32, one may pass to crystalline cohomology of the reduction $X :=$ $X \times_k W(k)$, where the result has no content. ∎

Theorem 1.32 admits the following version in characteristic zero, where we heuristically replace the deformation $W(k) \to k$ with $\mathbb{C}[[t]] \to \mathbb{C}$.

> **Proposition 1.42.** Given a smooth formal scheme $\widetilde{X}$ over $\mathbb{C}[[t]]$ with reduction X over $\mathbb{C}$. Then
> $$\mathrm{R\Gamma}_{\mathrm{dR}}(\widetilde{X}/\mathbb{C}[[t]]) \cong \mathrm{R\Gamma}_{\mathrm{dR}}(X/\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[t]].$$

Sketch. The idea of the proof is to pass through the larger de Rham cohomology $\mathrm{R\Gamma}_{\mathrm{dR}}(\widetilde{X}/\mathbb{C})$; by definition, this is the inverse limit of the de Rham cohomology of the reductions $\widetilde{X}_n$. By the Poincaré lemma, it turns out that $\mathrm{R\Gamma}_{\mathrm{dR}}(\widetilde{X}/\mathbb{C})$ is quasi-isomorphic to $\mathrm{R\Gamma}_{\mathrm{dR}}(X/\mathbb{C})$. (Approximately speaking, this is saying that a tubular neighborhood of X is homotopic to X.) The result now follows by extending scalars. ∎

> **Remark 1.43.** This proof does not work in our mixed characteristic situation $W(k) \to k$ because there is no direct way to geometrically link $\widetilde{X}$ and X via a Poincaré lemma.

To prove Theorem 1.32, it will turn out that the key ideas can be used to instead prove Corollary 1.41.

Proof of Corollary 1.41 when $p > 2$. We attempt to give a version of Poincaré's lemma. The idea there is to use Cartan's formula, which arises by viewing differentials as living in flows of vector fields.

1. Our analog of this will be
   $$\log f := \sum_{n \geq 1} (-1)^{n+1} \frac{(f - \mathrm{id})^{\circ n}}{n},$$
   viewed as a W-linear endomorphism of $\mathcal{O}_{\widetilde{X}}$. To see that this makes sense, note that $f - \mathrm{id} = pA$ for some endomorphism A, so it is at least true that the terms of the series are divisible by larger and larger powers of p. Thus, the series descends to a compatible sequence of endomorphisms for each $\mathcal{O}_{\widetilde{X}_n}$ and hence an endomorphism of $\mathcal{O}_{\widetilde{X}}$.

2. It further turns out that $\log f$ is a derivation of $\mathcal{O}_{\widetilde{X}}$, meaning that
   $$(\log f)(ab) = (\log f)(a)b + a(\log f)(b).$$
   This is a formal consequence of the fact that f is a ring map, and $\log f$ takes products to sums. (Indeed, one can already find this in characteristic zero.)

3. To continue, we take $p > 2$ for simplicity. Indeed, then one can see $f = \exp(\log f)$, meaning that
   $$f \stackrel{?}{=} \sum_{n \geq 0} \frac{(\log f)^{\circ n}}{n!}.$$
   The right-hand sum makes sense because $\log f$ is divisible by p, which then implies that the sum converges. Thus, we may write $\log f = pD$ for some derivation D. The equality now follows from some formal calculation of the series.

4. We now pass to the de Rham complex. We would like to check that the endomorphism $f^* : \Omega_{\widetilde{X}}^\bullet \to \Omega_{\widetilde{X}}^\bullet$ is a quasi-isomorphism. Well, there is a "Lie derivative" L_D for which f^* acts on $\Omega_{\widetilde{X}}^i$ by
   $$\sum_{n \geq 0} \frac{p^n}{n!} L_D^{\circ n}.$$

Let's explain what this L_D is, which we may define affine-locally. For any $W(k)$-algebra A, a derivation $W \colon A \to A$ induces an endomorphism L_D on differentials as follows. Indeed, $\mathrm{id} + \varepsilon D$ is an automorphism of the thickening $A[\varepsilon]/\left(\varepsilon^2\right)$, so $\mathrm{id} + \varepsilon D$ acts on $\Omega^i_{A[\varepsilon]/(\varepsilon^2)}$. Certainly it must reduce to identity modulo ε, so we conclude that $(\mathrm{id} + \varepsilon D)^* = \mathrm{id} + \varepsilon L_D$ for some endomorphism L_D on differentials!

5. The previous step now tells us that $f^* = \mathrm{id} + L_D \circ G$ for some G. We will be done as soon as we can check that the endomorphism L_D of the de Rham complex is homotopic to zero. In other words, we are on the hunt for maps $\iota_D \colon \Omega^i_{\widetilde{X}} \to \Omega^{i-1}_{\widetilde{X}}$ fitting into the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\widetilde{X}} & \longrightarrow & \Omega^1_{\widetilde{X}} & \longrightarrow & \Omega^2_{\widetilde{X}} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle L_D} & {\scriptstyle \iota_D}\!\!\nearrow & \Big\downarrow{\scriptstyle L_D} & {\scriptstyle \iota_D}\!\!\nearrow & \Big\downarrow{\scriptstyle L_D} & & \\
0 & \longrightarrow & \mathcal{O}_{\widetilde{X}} & \longrightarrow & \Omega^1_{\widetilde{X}} & \longrightarrow & \Omega^2_{\widetilde{X}} & \longrightarrow & \cdots
\end{array}
$$

so that $L_D = d\iota_D + \iota_D D$. This is precisely Cartan's formula, which admits an algebraic proof. (Namely, one takes ι_D to be a contraction whose vector field is D.) To show this, one can show directly that $d\iota_D + \iota_D D$ is a derivation, and then one can check that there is an equality on functions and 1-forms. ∎

> **Remark 1.44.** The construction of L_D may appear ad-hoc. The intuition here is that our objects are not only functorial under endomorphisms but also under derivations.

It turns out that all the interesting algebra to prove Theorem 1.32 is already contained in the proof of Corollary 1.41, though this should not be obvious yet.

1.2.3 The Cartier Isomorphism

Recall that we wanted to prove Theorem 1.22.

> **Theorem 1.22** (Cartier isomorphism). Fix a smooth variety X over a perfect field k. Then there is a canonical isomorphism
> $$\mathcal{H}^i(F_*\Omega^\bullet_X) \cong \Omega^i_{X^{(1)}}.$$
> Here, the left-hand side is a coherent $\mathcal{O}_{X^{(1)}}$-module.

Proof. Let's start with $i = 0$. Note that there is a map

$$\mathcal{O}_{X^{(1)}} \to \ker\left(d \colon F_*\mathcal{O}_X \to F_*\Omega^1_X\right)$$

given by sending a function f to f^p. We would like to check that this is an isomorphism. Well, it is enough to check this on an open cover, but because X is smooth, we may pass étale-locally along some $f \colon U \to \mathbb{A}^n_k$ (where $U \subseteq X$ is some open subset) because the de Rham complex is immune to such deformations: indeed,

$$\left(F_*\Omega^0_X\right)\big|_{U^{(1)}} \cong f^{(1)*}\left(F_*\Omega^0_{\mathbb{A}^n_k}\right).$$

Thus, we may pass to $X = \mathbb{A}^n$.

We now work in general. By working locally, we may assume that X is affine and equal to some $\operatorname{Spec} A$. Then we would like an isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(X/k) \overset{?}{\cong} \Omega^i_{A^{(1)}/k}$$

of A-modules. Well, consider $\mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k)) := \mathrm{R}\Gamma_{\mathrm{cris}}(A/W(k)) \otimes_{W(k)} W_2(k)$, which we know to be isomorphic to $\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/W_2(k))$ for any deformation $\widetilde{X}$ of X to $W_2(k)$. Accordingly, we receive a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{dR}}(A/k) \to \mathrm{R}\Gamma_{\mathrm{cris}}(A/W_2(k)) \to \mathrm{R}\Gamma_{\mathrm{dR}}(A/k)$$

induced by the lifting $k \to W_2(k) \to k$, where the map $k \to W_2(k)$ is multiplication by p. Thus, we receive a long exact sequence

$$\cdots \to \mathrm{H}^i_{\mathrm{dR}}(A/k) \to \mathrm{H}^i_{\mathrm{dR}}(A/k) \xrightarrow{\beta_i} \mathrm{H}^{i+1}_{\mathrm{dR}}(A/k) \to \cdots .$$

It turns out that the graded map $\bigoplus_i \beta_i$ on the de Rham cohomology ring $\bigoplus_{i>0} \mathrm{H}^i_{\mathrm{dR}}(X/k)$ is a derivation; this is just some exercise in the homological algebra. Explicitly, one has to check that

$$\beta_{i+j}(x_i \wedge x_j) = \beta_i(x_i) \wedge x_j + (-1)^i x_i \wedge \beta_j(x_j)$$

for any x_i and x_j of degrees i and j, respectively. It further turns out that $\beta_{i+1} \circ \beta_i = 0$.

Now that we have found a derivation, we can make differentials appear. The universal property of differentials produce morphisms

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{A^{(1)}} & \xrightarrow{d} & \Omega^1_{A^{(1)}} & \xrightarrow{d} & \Omega^2_{A^{(1)}} & \xrightarrow{d} & \cdots \\
& & \downarrow{c_0} & & \downarrow{c_1} & & \downarrow{c_2} & & \\
0 & \longrightarrow & \mathrm{H}^0_{\mathrm{dR}}(A/k) & \xrightarrow{\beta} & \mathrm{H}^1_{\mathrm{dR}}(A/k) & \xrightarrow{\beta} & \mathrm{H}^2_{\mathrm{dR}}(A/k) & \xrightarrow{\beta} & \cdots
\end{array}
$$

where f_0 is induced by the degree-zero argument, and all the relevant morphisms intertwine β and d and produce maps of graded rings. For example, one finds that we need $c_1(df) = \beta_0(c_0(f))$ for functions f. One now checks that this in fact an isomorphism on the level of modules, which is checked étale-locally and then passed to affine spaces. ∎

> **Remark 1.45.** We can afford to be a little more explicit about our construction of the maps $c_\bullet$. Fix a lift $\widetilde{A}$ over $W_2(k)$ as well as a lift of the Frobenius $\widetilde{F}$. Given $\omega \in \Omega^i_{A^{(1)}/k}$, we pull it back along $\widetilde{F}$ to get a differential divisible by p in $\mathrm{H}^i_{\mathrm{dR}}(\widetilde{A}/W_2(k))$. Then one can check that the class $\widetilde{F}^*(\omega)/p$ is well-defined in $\mathrm{H}^i_{\mathrm{dR}}(A/k)$.

> **Remark 1.46.** Crystalline cohomology is not technically necessary because we could choose a lift $\widetilde{X}$ by hand using Remark 1.45. However, one needs to check that the constructed map is independent of the lift. This is not impossible (such lifts are well-understood in some cohomology group by deformation theory), but it is a little difficult.

1.3 February 9

Today we say more about the construction of crystalline cohomology.

1.3.1 $\mathcal{D}$-modules

We are going to use $\mathcal{D}$-modules to glue our de Rham complexes together.

> **Definition 1.47** ($\mathcal{D}$-module)**.** Fix a smooth scheme X over a ring R. Then we define the quasicoherent sheaf $\mathcal{D}_{X/R}$ of associative algebras on X defined explicitly as
>
> $$\mathcal{O}_X\{\partial_v : v \in T_X\},$$
>
> where the relations ∂_v satisfies the relations $\partial_{v_1+v_2} = \partial_{v_1} + \partial_{v_2}$, $\partial_{fv} = f\partial_v$, $[\partial_v, f] = L_v(f)$, and $[\partial_v, \partial_w] = \partial_{[v,w]}$. Here, T_X is the tangent bundle, which is the dual of $\Omega^1_{X/R}$ (equivalently, the sheaf of derivations $\mathcal{O}_X \to \mathcal{O}_X$). A $\mathcal{D}$-*module* is a quasicoherent sheaf for which the $\mathcal{O}_X$-action extends to the action of a module for $\mathcal{D}_{X/R}$.

Remark 1.48. Precisely, we have given a definition of an associative ring on affine opens, which then glue together on affines as a sheaf on a base.

Remark 1.49. There is a natural action of $\mathcal{D}_{X/R}$ on $\mathcal{O}_X$, where functions act by multiplication, and the derivations act by taking derivatives. If R contains $\mathbb{Q}$, then this natural map $\mathcal{D}_{X/R} \to \operatorname{End}_R(\mathcal{O}_X)$ is injective.

Example 1.50. In positive characteristic p, the action map $\mathcal{D}_{X/R} \to \operatorname{End}_R(\mathcal{O}_X)$ need not be injective. Indeed, with $X = \mathbb{A}^1_{\mathbb{F}_p}$ over $\mathbb{F}_p$, we see that the iterated derivation ∂_t^p kills all polynomials (because $p! = 0$).

Remark 1.51. The various relations make it so that a $\mathcal{D}$-module is exactly a vector bundle with a flat connection $\nabla \colon M \to M \otimes \Omega^1_{X/R}$.

Remark 1.52. The algebra $\mathcal{D}_{X/R}$ admits a filtration by the subgroups $\mathcal{D}_{X/R}^{\leq n}$ of differentials at most n. It turns out that $\operatorname{gr} \mathcal{D}_{X/R} = \operatorname{Sym}^\bullet T_X$, which is probably a variant of the PBW theorem.

Here is why $\mathcal{D}$-modules are relevant to us.

Lemma 1.53. Fix a smooth scheme X over a ring R. Then the de Rham complex $\Omega^\bullet_{X/R}$ is quasi-isomorphic to $\operatorname{RHom}_{\mathcal{D}_{X/R}}(\mathcal{O}_X, \mathcal{O}_X)$ in $D(\operatorname{Sh}(X))$.

Proof. We are going to find a locally projective resolution for $\mathcal{O}_X$ as a $\mathcal{D}_{X/R}$-module. To start, note that there is a projection
$$\mathcal{D}_{X/R} \to \mathcal{O}_X$$
given by sending $\delta \mapsto \delta(1)$. This is surjective because functions act by left multiplication. By definition, all the derivations ∂_v are in the kernel, and these are exactly what goes to zero, so we continue our sequence by writing
$$\mathcal{D}_{X/R} \otimes_{\mathcal{O}_X} T_{X/R} \to \mathcal{D}_{X/R} \to \mathcal{O}_X \to 0,$$
where the left map sends $\delta \otimes v \mapsto \delta \partial_v$. Then $\partial_v \otimes w - \partial_w \otimes v$ is in the kernel, and we can find that we should extend our complex by
$$\mathcal{D}_{X/R} \otimes \mathcal{O}_X \wedge^2 T_{X/R} \to \mathcal{D}_{X/R} \otimes_{\mathcal{O}_X} T_{X/R} \to \mathcal{D}_{X/R} \to \mathcal{O}_X \to 0,$$
where the left map sends $\delta \otimes (v_1 \wedge v_2)$ to $\delta \partial_{v_1} \otimes v_2 - \delta \partial_{v_2} \otimes \partial_{v_1} - \delta \otimes [v_1, v_2]$. This suggests that we are looking at a Koszul complex
$$\cdots \to \mathcal{D}_{X/R} \otimes \mathcal{O}_X \wedge^2 T_{X/R} \to \mathcal{D}_{X/R} \otimes_{\mathcal{O}_X} T_{X/R} \to \mathcal{D}_{X/R} \to \mathcal{O}_X \to 0,$$
and one can check that it is acyclic by passing to the associated graded everywhere.

We now compute RHom by hitting our complex with $\operatorname{Hom}_{\mathcal{D}_X}(-, \mathcal{O}_X)$. Then the point is that there is an isomorphism $\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes E, \mathcal{O}_X) = E^\vee$ in the obvious way, so taking $\operatorname{Hom}_{\mathcal{D}_X}(-, \mathcal{O}_X)$ produces the complex
$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/R} \xrightarrow{d} \Omega^2_{X/R} \xrightarrow{d} \cdots,$$
which is exactly the de Rham complex. ∎

Remark 1.54. One can replace the target $\mathcal{O}_X$ with any quasicoherent sheaf to obtain a similar result, which allows us to compute de Rham cohomology of an arbitrary quasicoherent sheaf.

1.3.2 The Category of Crystals

We are now ready to construct crystalline cohomology.

Theorem 1.55. Suppose $p > 2$. Fix any smooth variety X over a perfect field k of characteristic p. For each $n \geq 1$, then there is an abelian tensor category $\mathrm{Cris}(X/W_n(k))$ of crystals with unit object $\mathcal{O}_X^{\mathrm{cris}}$, with the following property: for any lift X_n of X to $W_n(k)$, there is an isomorphism

$$\mathrm{Cris}(X/W_n(k)) \cong \mathrm{Mod}(\mathcal{D}_{X_n/W_n})$$

sending $\mathcal{O}_X^{\mathrm{cris}}$ to $\mathcal{O}_X$.

Remark 1.56. This certainly implies Theorem 1.32 simply by defining

$$\mathrm{R\Gamma}_{\mathrm{cris}}(X/W_n(k)) := \mathrm{RHom}_{\mathrm{Cris}}(X/W_n(k))(\mathcal{O}_X^{\mathrm{cris}}, \mathcal{O}_X^{\mathrm{cris}}).$$

Namely, we may view the target as living in $\mathcal{D}_{X_n/W_n}$-modules for any lift X_n, which is then related to de Rham cohomology via Lemma 1.53. One can construct $\mathrm{R\Gamma}_{\mathrm{cris}}(X/W(k))$ by taking a suitable limit over n.

Remark 1.57. The category quasicoherent sheaves on X_n (as a tensor category) recovers X_n as a tensor category, so it is remarkable that the category of $\mathcal{D}$-modules does not!

Remark 1.58. The sheaf of algebras $\mathcal{D}_{X_n/W_n}$ does depend on the lift to X_n, so it is remarkable that we are able to construct $\mathrm{Cris}(X/W_n(k))$ at all!

We have been using a bit of deformation theory (for motivation) throughout, but let's state what we need more explicitly.

Theorem 1.59 (Deformation). Fix a commutative ring A and a nilpotent thickening $A \to A/I$ where $I^2 = 0$, and choose a smooth scheme Y over A/I.

(a) There is a (natural) class $\mathrm{ob}_{Y,A} \in \mathrm{H}^2(Y; T_{Y/(A/I)} \otimes_{A/I} I)$ which vanishes if and only if there is a smooth lift $\widetilde{Y}$ over A for which $\widetilde{Y}_{A/I} \cong Y$.

(b) Given a choice of lift $\widetilde{Y}$ with a choice of isomorphism $\rho \colon \widetilde{Y}_{A/I} \to Y$, then $\mathrm{H}^1(Y; T_{Y/(A/I)} \otimes I)$ parameterizes all lifts.

(c) Given a choice of lift $\widetilde{Y}$ with a choice of isomorphism $\rho \colon \widetilde{Y}_{A/I} \to Y$, then the group of automorphisms $f \colon \widetilde{Y} \to \widetilde{Y}$ which reduce to the identity modulo I is isomorphic to $\mathrm{H}^0(Y; T_{Y/(A/I)} \otimes_{A/I} I)$.

Example 1.60. By (a), we see that one can always lift smooth curves.

Remark 1.61. One can directly prove the commutativity of the group of automorphisms (roughly) as follows: f amounts to a deformation which looks like $\mathrm{id} + \varepsilon$ (e.g., on the sheaf of rings), which all commute with each other because $I^2 = 0$.

Let's explain how we use Theorem 1.59: if X is a smooth variety over a field k, then X is separated, so it admits an affine open cover $\{U_i\}_{i \in \Lambda}$ so that each intersection $U_i \cap U_j$ is still affine. By the first two parts of Theorem 1.59 (applied iteratively), we are granted a unique smooth lift $\widetilde{U}_\alpha$ over $W_n(k)$ (because higher cohomology of affine schemes will vanish). By the uniqueness, we are granted isomorphisms

$$f_{ij} \colon \mathcal{O}_{\widetilde{U}_i}|_{U_{ij}} \cong \mathcal{O}_{\widetilde{U}_j}|_{U_{ij}},$$

but the third part of Theorem 1.59 tells us that there are many choices for such an automorphism. Nonetheless, for any triple (i, i', i''), we see that the composite $f_{i''i}f_{i'i''}f_{ii'}$ is the identity $\pmod{p}$. But this does not mean that our sheaves glue! This explains why we cannot lift X.

However, we will be able to lift $\mathcal{D}$-modules to a category of $\mathcal{D}$-modules with W_n-coefficients without ever having to lift X to X_n. The key is the following, which asserts that such "automorphisms which are the identity $\pmod{p}$" are inner for $\mathcal{D}$-modules.

> **Lemma 1.62.** Suppose $p > 2$, and fix a smooth affine scheme Y_n over W_n. Then for any automorphism f of Y_n which is the identity $\pmod{p}$, there is a $D_f \in \mathcal{D}_{Y_n/W_n}(Y_n)^\times$ which is $1 \pmod{p}$, and
>
> $$f^*\delta = D_f \delta D_f^{-1}$$
>
> for any $\delta \in \mathcal{D}_{Y_n/W_n}(Y_n)$.

Proof. We may find Δ_f to satisfy
$$p\Delta_f = \log f^*,$$
where $\log f^*$ refers to the power series $\sum_{r \geq 1}(-1)^{r+1}(f - \mathrm{id})^{\circ r}/r$. (This exists because $p > 2$.) This logarithm is some endomorphism on $\mathcal{O}_{Y_n}$, and it is even a derivation of $\mathcal{O}_{Y_{n-1}}$, which one can check directly. We now define $D_f := \exp(p\partial_{\Delta_f})$, and one can check the identity directly from the identity $f^* = \exp(p\Delta_f)$. ∎

> **Remark 1.63.** The moral of Lemma 1.62 is that any $\mathcal{D}_{Y_n}$-module M admits a natural isomorphism $M \cong (f^*)_*M$, where the right-hand module means that $\mathcal{D}_{Y_n}$ acts on M through the automorphism f^*. Indeed, simply send m to $D_f m$.

> **Remark 1.64.** Lemma 1.62 even has a formulation in characteristic zero, which admits basically the same proof.

We are now ready to prove Theorem 1.55.

Proof of Theorem 1.55. Fix an affine open cover $\{U_i\}_{i \in \Lambda}$ on X with f_{ij}s as before. The category of $\mathcal{D}_{X/k}$-modules can be thought of as the category of big tuples $\{M_i\}_i$, where M_i is a module for $\mathcal{D}_{X/k}(U_i)$, and we are equipped with isomorphisms between $M_i|_{U_{ij}}$ and $M_j|_{U_{ij}}$ that satisfy some cocycle condition. In light of the discussion preceding Lemma 1.62, our problem is that we do not know how to fix the cocycle condition.

We now imitate this definition. Define our category $\mathrm{Cris}(X/W_n(k))$ of crystals as being the category of tuples $\{M_i\}_i$ of modules, where M_i is a module over $\mathcal{D}_{\widetilde{U}_i/W_n}(\widetilde{U}_i)$, equipped with isomorphisms

$$\alpha_{ij} \colon f_{ij}^* M_i|_{U_{ij}} \to M_j|_{U_{ij}}$$

such that the composites $\alpha_{i''i}\alpha_{i'i''}\alpha_{ii'}$ produces an automorphism $f_{i''i}^*f_{i'i''}^*f_{ii'}^* M_i \to M_i$, where this is an automorphism where we have identified the left module with M_i via Lemma 1.62.

It may appear that our construction depends rather poorly on the choice of cover. However, this is not the case: one can simply show that the uniqueness of our constructions means that our construction does not depend on refinements of open covers, so it does not actually depend on the choice of open cover. (It may look like this is so because the identifications depend on the choice of D_f in Lemma 1.62, but this construction was in fact canonical in the proof.) Professor Petrov seems to think that one will need to check some cocycle condition on intersections of four open subsets.

We now run our checks.

- For example, we note that the $\mathcal{O}_{\widetilde{U}_i}$s glue to an object $\mathcal{O}_X^{\mathrm{cris}}$ because all data is only every constructed $\pmod{p}$, and the cocycle condition is satisfied by the particular construction of D_f.

- Given a choice of lift X_n, we may actually lift everything to X_n, which shows that our crystals agree with $\mathcal{D}$-modules on X_n. ∎

Remark 1.65. One can relax the hypothesis that X is separated by further covering the U_{ij}s by affines.

Remark 1.66. These are called "crystals" because they are rigid and they grow. Namely, we constructed them locally, but they were rigid enough to be able to be glued to a global category.

Remark 1.67. Additionally, morphisms of varieties are finite type and separated, so it is not too hard to construct functoriality for our category of crystals. Explicitly, a morphism $f\colon X \to Y$ produces a morphism
$$f^*\colon \mathrm{R}\Gamma_{\mathrm{cris}}(Y/W(k)) \to \mathrm{R}\Gamma(X/W(k)).$$

1.4 February 11

Today, we give some applications of crystalline cohomology.

1.4.1 The Ax–Katz Theorem

Crystalline cohomology comes with a natural Frobenius.

Example 1.68. For any smooth scheme X over k, there is a Frobenius $F\colon X \to X^{(1)}$, which produces an endomorphism
$$F^*\colon \mathrm{R}\Gamma_{\mathrm{cris}}\left(X^{(1)}/W(k)\right) \to \mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k)).$$
For example, if X is defined over $\mathbb{F}_{q^r}$, then $X = X^{(r)}$, then we have an endomorphism $(F^*)^r$ of the complex $\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(k))$.

Remark 1.69. If X is smooth and proper over $\mathbb{F}_q$ where $q = p^r$, then $\mathrm{H}^n_{\mathrm{cris}}(X/W(\mathbb{F}_q))$ is finitely generated over $W(\mathbb{F}_q)$, and it turns out that
$$\#X(\mathbb{F}_q) = \sum_{i \geq 0}(-1)^i \,\mathrm{tr}\left((F^*)^r; \mathrm{H}^i_{\mathrm{cris}}(X/W(\mathbb{F}_q))\left[\tfrac{1}{p}\right]\right).$$
One has to check that crystalline cohomology is a Weil cohomology theory, which is true. It is not too difficult to check this formula directly when X is zero-dimensional, meaning that it is a finite disjoint union of points.

Theorem 1.70 (Ax–Katz)**.** Let X be a smooth proper geometrically connected variety over $\mathbb{F}_q$ for which $\mathrm{H}^i(X; \mathcal{O}_X) = 0$ for all $i > 0$. Then
$$\#X(\mathbb{F}_q) \equiv 1 \pmod{q}.$$

Proof. The idea is to use Remark 1.69. Note that $\mathrm{H}^0_{\mathrm{cris}}(X/W(\mathbb{F}_q))$ is the one-dimensional space with trivial Frobenius action, so it is enough to show that
$$\mathrm{tr}\left((F^*)^r; \mathrm{H}^i_{\mathrm{cris}}(X/W(\mathbb{F}_q))\left[\tfrac{1}{p}\right]\right) \equiv 0 \pmod{q}$$
for all $i \geq 1$. With an inversion of p, one will not be able to do better than $\pmod{p}$ information, so we see that we have to use the integral structure.

For a given module M over $W(\mathbb{F}_q)$, we let M_{tf} denotes the quotient by torsion so that M_{tf} is torsion-free. It is now enough to show that the map

$$F^*\colon \mathrm{H}^i_{\mathrm{cris}}(X^{(1)}/W(\mathbb{F}_q))_{\mathrm{tf}} \to \mathrm{H}^i_{\mathrm{cris}}(X/W(\mathbb{F}_q))_{\mathrm{tf}}$$

is divisible by p. Indeed, the trace after $(-)[1/p]$ is the same as the trace on the torsion-free part (because then inverting p does nothing), so we can just iterate this divisibility r times to prove the claim. We now reduce $\pmod p$, for which we recall that

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/W(\mathbb{F}_q)) \otimes^{\mathbb{L}}_{W(\mathbb{F}_q)} \mathbb{F}_q \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathbb{F}_q).$$

Now, the same argument as in Universal coefficient theorem yields an exact sequence

$$0 \to \mathrm{H}^n_{\mathrm{cris}}(X/W(\mathbb{F}_q))/p \to \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{F}_q) \to \mathrm{H}^{n+1}_{\mathrm{cris}}(X/W(\mathbb{F}_q))[p] \to 0.$$

By functoriality of the short exact sequence, if we could show that Frobenius was zero on de Rham cohomology (i.e., the map $F^*\colon \mathrm{H}^n_{\mathrm{dR}}(X^{(1)}/\mathbb{F}_q) \to \mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{F}_q)$ vanishes), then it would vanish on $\mathrm{H}^n_{\mathrm{cris}}(X/W(\mathbb{F}_q))/p$ and thus vanish on $\mathrm{H}^n_{\mathrm{cris}}(X/W(\mathbb{F}_q))_{\mathrm{tf}}/p$, which was the required claim.

We are thus reduced to computing some Frobenius action on the de Rham complex. Let's recall how this Frobenius action is defined: by pulling back along F^*, there are natural maps

$$
\begin{array}{ccccccc}
F^{-1}\mathcal{O}_{X^{(1)}} & \longrightarrow & F^{-1}\Omega^1_{X^{(1)}} & \longrightarrow & F^{-1}\Omega^2_{X^{(1)}} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\
\mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^2_X & \longrightarrow & \cdots
\end{array}
$$

which then descend to cohomology. But note that $F^*(f\,dg) = f^p\,d(g^p) = 0$, so F^* vanishes on all the higher terms. Thus, the above morphism F^* of complexes factors through the complex $\mathcal{O}_X \to 0 \to 0 \to \cdots$. Thus, after taking hypercohomology, we see that F^* factors through $\mathrm{H}^n(X;\mathcal{O}_X)$, which vanishes by assumption! ∎

Remark 1.71. We can remove the properness assumption, and one still finds that F^* finds on de Rham cohomology.

Remark 1.72 (Chevalley–Warning)**.** If $X \subseteq \mathbb{P}^N_{\mathbb{F}_q}$ is a hypersurface of degree at most N, then the hypothesis is satisfied. It is a classical result of Chevalley–Warning that $\#X(\mathbb{F}_q) \equiv 1 \pmod p$. A cohomological argument could proceed by using the Atiyah–Bott formula

$$\#X(\mathbb{F}_q) = \sum_{i \geq 0}(-1)^i \operatorname{tr}\left((F^*)^r; \mathrm{H}^n(X;\mathcal{O}_X)\right) \pmod p$$

and then argue as above. To access $\pmod q$ information, we need a different cohomology theory.

Remark 1.73. If one wants to prove that the Frobenius acting on crystalline cohomology was instead divisible by p^2, then there starts to be contributions of the cohomology of Ω_X, via some discussion of Hodge–Tate weights.

1.4.2   A Weak Hodge Decomposition

Here is another application.

Notation 1.74. Given a formal scheme $\widetilde{X}$ over $W(k)$, we define

$$\widetilde{X}^{(1)} := \widetilde{X} \times_{W(k)} W(k),$$

where the internal map $W(k) \to W(k)$ is the natural Frobenius on $W(k)$.

Proposition 1.75. Fix a smooth scheme X over a perfect field k of characteristic p. Suppose that we can lift X to a formal scheme $\widetilde{X}$ over $W(k)$ as well as a lift $\widetilde{F} \colon \widetilde{X} \to \widetilde{X}^{(1)}$ of F. Then the de Rham complex

$$\mathcal{O}_{\widetilde{X}} \xrightarrow{d} \Omega^1_{\widetilde{X}} \xrightarrow{d} \Omega^2_{\widetilde{X}} \xrightarrow{d} \cdots$$

is quasi-isomorphic to

$$\mathcal{O}_{\widetilde{X}^{(1)}} \xrightarrow{pd} \Omega^1_{\widetilde{X}^{(1)}} \xrightarrow{pd} \Omega^2_{\widetilde{X}^{(1)}} \xrightarrow{pd} \cdots.$$

Remark 1.76. If X is affine, then such lifts $\widetilde{X}$ and $\widetilde{F}$ always exist.

As an application, we get a version of the Hodge decomposition.

Corollary 1.77. Fix a smooth scheme X over a perfect field k of characteristic p. Suppose that we can lift X to a formal scheme $\widetilde{X}$ over $W(k)$ as well as a lift $\widetilde{F} \colon \widetilde{X} \to \widetilde{X}^{(1)}$ of F. Then the de Rham complex

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots$$

is quasi-isomorphic to

$$\mathcal{O}_{X^{(1)}} \xrightarrow{0} \Omega^1_{X^{(1)}} \to \Omega^2_{X^{(2)}} \to \cdots.$$

Thus,

$$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \bigoplus_{i+j=n} \mathrm{H}^i\left(X^{(1)}; \Omega^j_{X^{(1)}}\right).$$

Proof. To show the quasi-isomorphism, merely reduce Proposition 1.75 modulo p, and then we know that crystalline cohomology becomes de Rham cohomology. To prove the last claim, note that the hyperco-homology of the first complex is the de Rham cohomology, and hypercohomology of the second complex produces the right-hand side. Indeed, to take hypercohomology, one needs to resolve the second complex, but the "columns" in our resolution do not need to communicate with each other, so

$$\mathbb{H}^n\left(\mathcal{O}_{X^{(1)}} \xrightarrow{0} \Omega^1_{X^{(1)}} \to \Omega^2_{X^{(2)}} \to \cdots\right) = \bigoplus_{i+j=n} \mathrm{H}^i\left(X^{(1)}; \Omega^j_{X^{(1)}}\right)$$

follows after we keep track of our degrees. ∎

Proof of Proposition 1.75. We are interested in filling in the following diagram.

$$
\begin{array}{ccccccc}
\mathcal{O}_{\widetilde{X}} & \longrightarrow & \Omega^1_{\widetilde{X}} & \longrightarrow & \Omega^2_{\widetilde{X}} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle \widetilde{F}^*} & & \uparrow & & \uparrow & & \\
\mathcal{O}_{\widetilde{X}^{(1)}} & \longrightarrow & \Omega^1_{\widetilde{X}^{(1)}} & \longrightarrow & \Omega^2_{\widetilde{X}^{(1)}} & \longrightarrow & \cdots
\end{array}
$$

Staring at the diagram, we see that the map $\Omega^i_{\widetilde{X}^{(1)}} \to \Omega^i_{\widetilde{X}}$ had better be $p^{-i}F^*$. The proof of Theorem 1.70 checked that $p^{-1}F^*$ makes sense for $i > 0$, and by discussing Frobenius on higher wedges of differentials finds that even $p^{-i}F^*$ will always make sense.

We have thus provided our morphism of complexes, which we want to be a quasi-isomorphism. Because the terms are p-adically complete, it turns out that it is enough to check that it is a quasi-isomorphism $(\mathrm{mod}\ p)$.[1] It turns out that one can proceed as in the proof of Theorem 1.22. ∎

This decomposition remains true with weaker assumptions, which will be our next goal.

> **Theorem 1.78** (Berthelot–Ogus)**.** Fix a smooth scheme X over a perfect field k which can be lifted to a formal scheme $\widetilde{X}$ over $W(k)$. Suppose further that $\dim X < p$. Then
>
> $$\mathcal{O}_{\widetilde{X}} \xrightarrow{d} \Omega^1_{\widetilde{X}} \xrightarrow{d} \Omega^2_{\widetilde{X}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim X}_{\widetilde{X}}$$
>
> is quasi-isomorphic to
>
> $$\mathcal{O}_{\widetilde{X}^{(1)}} \xrightarrow{pd} \Omega^1_{\widetilde{X}^{(1)}} \xrightarrow{pd} \Omega^2_{\widetilde{X}^{(1)}} \xrightarrow{pd} \cdots \xrightarrow{pd} \Omega^{\dim X}_{\widetilde{X}^{(1)}}.$$

> **Remark 1.79.** The same argument as in Corollary 1.77 is able to show that
>
> $$\mathrm{H}^n_{\mathrm{dR}}(X/k) = \bigoplus_{i+j=n} \mathrm{H}^i(X^{(1)}; \Omega^j_{X^{(1)}}).$$
>
> It turns out that this decomposition remains true if X can merely be lifted to $W_2(k)$, and it even depends on this choice of lift.

More precisely, our goal will be to prove the above remark.

1.4.3 $\mathcal{D}$-modules by Crystals

In order to access our Hodge decomposition, we are going to realize $\mathcal{D}_X$-modules as quasicoherent sheaves on some stack X^{dR}; if X is affine, then X^{dR} will be able to be realized as the quotient of some formal scheme by a formal group scheme. Notably, quasicoherent sheaves can then just be thought of as certain equivariant sheaves on a formal scheme.

To understand where X^{dR} may come from, we return to characteristic 0.

> **Lemma 1.80.** Fix a smooth scheme F over a field F of characteristic zero. For a $\mathcal{D}_X$-module M and a point $x \in X(F)$, there is a natural isomorphism
>
> $$M|_{\widehat{\mathcal{O}}_{X,x}} \cong M_x \widehat{\otimes}_F \widehat{\mathcal{O}}_{X,x}.$$
>
> Here, the left-hand side is $\lim M \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/\mathfrak{m}_x^\bullet$, and $M_x = M \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/\mathfrak{m}_x$.

> **Remark 1.81.** The moral is that there is a canonical trivialization "M_x" of M in a formal neighborhood of a point.

Proof. We will produce an isomorphism $M_x \to \left(M|_{\widehat{\mathcal{O}}_{X,x}}\right)^{\nabla=0}$, where the target means that we are looking for sections with flat connection. Accordingly, choose $s \in M_x$, which can be lifted to some $\widetilde{s} \in M(U)$. Now, $\widetilde{s}$ has no reason to be flat, so we have to fix it. Accordingly, we use Taylor's formula to expand out U in local coordinates $(t_1, \ldots, t_d)$ and then define

$$\sum_{i_1,\ldots,i_d \geq 0} (-1)^{i_1+\cdots+i_d} \frac{t_1^{i_1} \cdots t_d^{i_d}}{i_1! \cdots i_d!} \nabla^{i_1}_{\partial_{t_1}} \circ \cdots \circ \nabla^{i_d}_{\partial_{t_d}}(\widetilde{s}),$$

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[1] This is a variant of Nakayama's lemma. For example, a sample statement is that a morphism $f\colon M \to N$ of torsion-free p-adically complete $\mathbb{Z}_p$-modules (meaning that $M \cong \lim M/p^\bullet$) can be checked to be an isomorphism $(\mathrm{mod}\ p)$. After deriving, it turns out that we don't have to check that it is torsion-free.

and one can hit this with ∇ to check that it is a flat section. Further, note that $i_1 = \cdots = i_d = 0$ reproduces s, so this section continues to lift s. We have thus defined our map, and one can check that it produces the required isomorphism by Nakayama's lemma. ∎

Example 1.82. Let's write this out for $d = 1$. Then in the first degree, we have replaced $\widetilde{s}$ with $t - \nabla_{\partial_t}\widetilde{s}$; hitting this with ∇_{∂_t}, we receive $-t\nabla^2_{\partial_t}(\widetilde{s})$. This is still nonzero, and we then further correct by $\frac{1}{2}t^2\nabla^2_{\partial_t}(\widetilde{s})$ and continue.

We may want to extend these canonical trivializations into a full trivialization of M, but then we have to explain how to relate the various points.

Notation 1.83. Fix a smooth scheme X over a field F of characteristic zero. Then we define $(X \times_F X)^\wedge_\Delta$ to be the union of the closed subschemes of $X \times X$ cut out by the powers of the quasicoherent ideal sheaf $\mathcal{I}_\Delta$ (of the diagonal). We can define other powers similarly.

Remark 1.84. Intuitively, $(X \times_F X)^\wedge_\Delta$ is the union of all nilpotent thickenings of Δ.

Theorem 1.85. Fix a smooth scheme X over a field F of characteristic zero. Define the formal scheme $(X \times_F X)^\wedge_\Delta$ to be the formal completion at the diagonal, which is the union of the nilpotent thickenings of the diagonal Δ. Then the data of a $\mathcal{D}_X$-module M is equivalent to the data of a quasicoherent sheaf M on X and an isomorphism $\alpha\colon \mathrm{pr}_1^* M \to \mathrm{pr}_2^* M$ on $(X \times_F X)^\wedge_\Delta$ making the diagram of sheaves on $(X \times X \times X)^\wedge_\Delta$

$$\mathrm{pr}_1^* M \xrightarrow{\mathrm{pr}_{12}^* \alpha} \mathrm{pr}_2^* M$$
$$\mathrm{pr}_{13}^* \alpha \searrow \quad \downarrow \mathrm{pr}_{23}^* \alpha$$
$$\mathrm{pr}_3^* M$$

commute.

Sketch. We describe how to turn a $\mathcal{D}_X$-module M into the desired data. Namely, we need to provide the data of α. For example, to define α along the fiber $\mathrm{pr}_1^{-1}(x) \cong \mathrm{Spf}\,\widehat{\mathcal{O}}_{X,x}$ needs to provide an isomorphism

$$\mathrm{pr}_1^* M|_{\mathrm{pr}_1^{-1} x} \to \mathrm{pr}_2^* M|_{\mathrm{pr}_1^{-1} x}.$$

The left-hand side is just $M_x \otimes_F \widehat{\mathcal{O}}_{X,x}$ because we start at the point x (which gives M_x) which is then expanded along the formal neighborhood. On the other hand, the right-hand side is $M|_{\widehat{\mathcal{O}}_{X,x}}$ because the pullback is able to remember some horizontal information.[2] It turns out that the maps of Lemma 1.80 now suitably glue over all points. ∎

Remark 1.86. The moral of α is that it lets us identify infinitesimally nearby fibers. Once one has identified "nearby" fibers, then one can expect to be able to take derivatives, which gives the $\mathcal{D}_X$-module structure.

Remark 1.87. The data (M, α) is sometimes called a "crystal."

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[2] This discussion is a little confusing. It becomes easier to think about in the case of $X = \mathbb{A}^1$, where we can think of Δ as the genuine diagonal of a square.

Example 1.88. Consider $X = \mathbb{A}^1_F$ (or any other algebraic group). Then we find that $\mathcal{D}_{\mathbb{A}^1}$-modules can be seen to be the same as quasicoherent sheaves on $\mathbb{A}^1$ with $\widehat{\mathbb{G}}_a$-equivariant structure. Well, we are asking for a suitable identification along the two projections

$$\left(\mathbb{A}^1 \times \mathbb{A}^1\right)^{\wedge}_{\Delta} \rightrightarrows \mathbb{A}^1.$$

However, the left-hand formal scheme is $\mathbb{A}^1 \times \widehat{\mathbb{G}}_a$ (given by sending (x, y) to $(x, x - y)$), and one can check that the cocycle condition corresponds to an equivariant structure. It follows that we are looking at quasicoherent sheaves on $\mathbb{A}^1/\widehat{\mathbb{G}}_a$.

Bibliography

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LIST OF DEFINITIONS