18.906: Algebraic Topology II

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
∞-Categories

Language turns us all into jesters.

—Savannah Brown, [Bro24]

1.1 September 4

Here are some administrative notes.

- Office hours will be on Tuesday and Thursday immediately after class in 2-374.
- The syllabus will be posted to the course website later.
- The syllabus will contain some recommended textbooks, which are some free online texts that contain supersets of our class material.
- The grade will be 20% from a fifty-minute exam and 80% coming from problem sets. The exam will probably occur shortly before the drop deadline.

We hope to cover simplicial sets, ∞-categories, homotopy theory, Eilenberg–MacLane spaces, Postnikov towers, the Serre spectral sequence, and a little on vector bundles and characteristic classes. In particular, we see that the first part of the class is some purely formal nonsense, which we then use to set up the ∞-category of spaces, which is the natural setting for homotopy theory.

1.1.1 Category Theory

Here are some examples of categories to keep in mind for this course. We refer to Appendix A for the definitions.

Example 1.1. Any category $\mathcal{C}$ gives rise to a core groupoid $\operatorname{Core}\mathcal{C}$, which is the subcategory with the same objects but only taking the morphisms which are isomorphisms. One can check that this is in fact a subcategory.

In mathematics, one frequently encounters a category $\mathcal{C}$, and we are then interested in classifying the objects up to isomorphism.

Example 1.2. If $\mathcal{C} = \mathrm{Set}$, then isomorphisms are bijections, so sets "up to bijection" are simply given by their cardinalities.

Example 1.3. A commutative ring R gives rise to a category Mod_R of (left) R-modules. If R is a field, then this is a category of vector spaces, and objects up to isomorphism are given by their dimensions.

Example 1.4. One can consider the category Top of topological spaces, whose morphisms are continuous maps. (We will frequently restrict our category of topological spaces with some nicer subcategories, such as CW complexes or manifolds.) It is rather hard to classify objects up to isomorphism (here, isomorphisms are homeomorphisms), but there are some tools. For example, there are homology functors

$$\mathrm{H}_i(-; R)\colon \mathrm{Top} \to \mathrm{Mod}_R.$$

Because functors preserve isomorphisms, homeomorphic spaces must have isomorphic homology.

The definition of homology finds itself focused on continuous maps $|\Delta^n| \to X$, where $|\Delta^n|$ is the (topological) n-simplex.

Definition 1.5 (n-simplex)**.** The *(topological) n-simplex* $|\Delta^n|$ is the subspace

$$|\Delta^n| := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} t_i = 1 \right\}.$$

We will soon upgrade this topological n-simplex $|\Delta^n|$, which explains why we are writing $|\Delta^n|$ instead of Δ^n.

1.1.2 Homotopy Types, Intuitively

The homology functors $\mathrm{H}_i(-; R)\colon \mathrm{Top} \to \mathrm{Mod}_R$ factors through the homotopy category $\mathrm{Ho}(\mathrm{Top})$, which is obtained from Top by declaring homotopic maps to be equal.

We are going to be homotopy theorists for the most part, which means that we will be interested in understanding invariants of the category $\mathrm{Ho}(\mathrm{Top})$. One may complain that only studying spaces up to homotopy is an over-simplification. However, there are good reasons to only be interested in these "homotopy types" because such things also come up in other areas of mathematics.

Approximately speaking, a homotopy type is a collection of objects and morphisms between them. To ensure some level of homogeneity, one may require that any pair $f, g\colon A \to B$ of morphisms has a collection of "2-isomorphisms" $f \Rightarrow g$. Furthermore, there should be "3-isomorphisms" between these 2-isomorphisms, and this thinking continues inductively.

Example 1.6. Given two objects A and B with an isomorphism $f\colon A \to B$, one may think about these objects as being identified. Similarly, if we have a third isomorphism $g\colon B \to C$, then we can canonically identify all three objects. Here is diagram for this situation.

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad \simeq \qquad A$$

Example 1.7. Given two objects A and B, there may be two isomorphisms $f, g\colon A \to B$. One may want to identify these two objects via either isomorphism, but then we don't want to forget about the other isomorphism, so perhaps we are thinking about an object with an automorphism. Here is a diagram for this situation.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \qquad \simeq \qquad \begin{tikzcd} A \arrow[loop above, "g"] \end{tikzcd}$$

If one wanted to identify f and g, then there should be a "2-isomorphism" identifying f and g.

Example 1.8. We can think about a set as a homotopy type where all isomorphisms, 2-isomorphisms, and so on are all just the identity maps.

Example 1.9. We can think about a groupoid as a homotopy type where all 2-isomorphisms, 3-isomorphisms, and so on are all just the identity maps.

The above two examples will let us think about $\mathrm{Ho}(\mathrm{Top})$ as a category of ∞-groupoids. This is more or less why homotopy types are relevant to other areas of mathematics: one is frequently interested in not just isomorphisms between objects but also the uniqueness of those isomorphisms, and also the uniqueness of the isomorphisms identifying the isomorphisms, and so on.

Of course, we started with topological spaces, so let's explain how then make some ∞-groupoid.

Example 1.10. Given a topological space X, we can build a corresponding ∞-groupoid as follows.

- The points provide objects in our ∞-groupoids; these are functions $\left|\Delta^0\right| \to X$.
- The maps between points are given by paths; these are functions $\left|\Delta^1\right| \to X$.
- The maps between paths are given by homotopies of paths; these are functions $\left|\Delta^2\right| \to X$. Technically speaking, $\left|\Delta^2\right|$ gives two paths f and g whose composite should be homotopic to h. Thus, the structure of such a map tells us something about how composition should behave!

1.1.3 Simplices

After building up some intuition, we are now forced to do some combinatorics in order to get ourselves off of the ground.

Notation 1.11. For each integer $n \geq 0$, we define the category $[n]$ whose objects are the elements of $\{0, 1, \ldots, n\}$ and whose morphisms are given by

$$\mathrm{Hom}_{[n]}(i, j) := \begin{cases} \varnothing & \text{if } i < k, \\ * & \text{if } i \leq j, \end{cases}$$

where $*$ simply refers to some one-element set.

We remark that identities and the composition maps are then all uniquely defined (because everything is unique in the one-element set $*$); similarly, the coherence checks of identity and associativity have no content because everything is equal in $*$.

Remark 1.12. Combinatorially, $[n]$ is the poset category given by the totally ordered set

$$0 \le 1 \le 2 \le \cdots \le n.$$

Definition 1.13 (simplex)**.** The *simplex category* Δ has objects given by the categories $[n]$, and the morphisms are given by the collection of functors between any two such categories.

Remark 1.14. Combinatorially, we see that a functor $F\colon [n] \to [m]$ amounts to the data of an increasing map. Indeed, whenever $i \le j$ in $[n]$, which is equivalent to having a morphism $i \to j$, we see that there is a morphism $Fi \to Fj$, which is equivalent to the requirement $Fi \le Fj$.

Example 1.15. There are six morphisms $[1] \to [2]$, as follows.

- If $0 \mapsto 0$, then $1 \in [1]$ can go anywhere.
- If $0 \mapsto 1$, then 1 maps to 1 or 2 in $[2]$.
- If $0 \mapsto 2$, then 1 maps to 2.

Example 1.16. For each nonnegative integer n, there is a unique map $[n] \to [0]$ for each n. Indeed, everything must go to 0.

Remark 1.17. There is an important functor $F\colon \Delta \to \mathrm{Top}$ given by sending $[n] \mapsto |\Delta^n|$. Let's explain what this functor is on morphisms: given an increasing map $f\colon [n] \to [m]$, then we need to provide a continuous map $Ff\colon |\Delta^n| \to |\Delta^m|$. Well, we may identify $[n]$ and $[m]$ with bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, so f is now a function on bases, so it upgrades uniquely to a linear map $\mathbb{R}^n \to \mathbb{R}^m$ given by

$$Ff\left(\sum_{i=0}^n t_ie_i\right) \coloneqq \sum_{i=0}^n t_ie_{f(i)}.$$

Thus, we see that Ff does restrict to a map $|\Delta^n| \to |\Delta^m|$. Functoriality follows by the uniqueness of the construction of Ff: given two increasing maps $f\colon [n] \to [n']$ and $g\colon [n'] \to [n'']$, we see $Fg \circ Ff$ and $F(g \circ f)$ definitionally are both defined as $g \circ f$ on the basis of $\mathbb{R}^n$. (Of course, we should mention $\mathrm{id}_{[n]}\colon [n] \to [n]$ defines the identity on $\mathbb{R}^n$.)

1.1.4 Simplicial Sets

The following is the first important definition of this course.

Definition 1.18 (simplicial set)**.** A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. We let sSet denote the category of such functors. In other words, $\mathrm{sSet} = \mathrm{PSh}(\Delta)$.

Note that this "functor category" is in fact a category by Lemma A.13. Here are some examples of simplicial sets.

Example 1.19 ($\mathrm{Sing}(X)$)**.** There is a functor $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$ such that

$$\mathrm{Sing}(X)\colon [n] \mapsto \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X).$$

Proof. We have many checks to do, which we handle in sequence.

- We define $\mathrm{Sing}(X)$ on morphisms. Well, given an increasing map $f\colon [n] \to [m]$, the functor F of Remark 1.17 provides a continuous map $Ff\colon |\Delta^m| \to |\Delta^n|$, so there is a map
$$(- \circ Ff)\colon \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X) \to \mathrm{Mor}_{\mathrm{Top}}(|\Delta^m|, X).$$
- We check that $\mathrm{Sing}(X)$ is a functor. First, the identity morphism $\mathrm{id}_{[n]}\colon [n] \to [n]$ goes to the map
$$(- \circ F\mathrm{id}_{[n]})\colon \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X) \to \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X),$$
which is the identity because $F\mathrm{id}_{[n]} = \mathrm{id}_{|\Delta^n|}$. Second, given increasing maps $f\colon [n] \to [n']$ and $g\colon [n'] \to [n'']$, we need to check that
$$(- \circ F(g \circ f)) = (- \circ Ff) \circ (- \circ Fg),$$
which is true because $F(g \circ f) = Fg \circ Ff$.
- We define Sing on morphisms. Well, given a continuous map $f\colon X \to Y$, we use the map
$$(f \circ -)\colon \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X) \to \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, Y).$$
- We check that Sing is a functor. First, the identity $\mathrm{id}_X\colon X \to X$ goes to the map $(\mathrm{id}_X \circ -)$, which is just the identity composition. Second, given continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$, we note that
$$(g \circ -) \circ (f \circ -) = ((g \circ f) \circ -)$$
by the associativity of composition. ■

Remark 1.20. It turns out that not all simplicial sets arise from this construction. In particular, it turns out that the image of Sing has many nice properties.

Remark 1.21. It will turn out that the homotopy type of X is uniquely determined by $\mathrm{Sing}(X)$. This is remarkable because one expects Top to be a difficult category, even taken up to homotopy, but sSet just looks like some combinatorial data.

We are also interested in generalizing categories, so we pick up the following example.

Example 1.22 (nerve). Fix a category $\mathcal{C}$. Then there is a "nerve" functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ such that
$$N(\mathcal{C})\colon [n] \mapsto \mathrm{Fun}([n], \mathcal{C}).$$
The proof of this claim is exactly the same as in Example 1.19 (note that $\mathrm{Fun}([n],\mathcal{C}) = \mathrm{Mor}_{\mathrm{Cat}}([n],\mathcal{C})$), except now there is no need for the auxiliary functor $F\colon \Delta \to \mathrm{Top}$ because Δ is already a category. (Being brazen, one can copy the same proof but erasing all Fs, replacing Top with Cat throughout, and replacing $|\Delta^\bullet|$s with $[\bullet]$s throughout.)

Check: fully faithful.

Remark 1.23. As in Remark 1.20, nerves of categories have some nice properties which prevent them from producing all simplicial sets. It turns out that ∞-categories will be some kind of simultaneous generalization of Sings and nerves.

1.1.5 Simplicial Sets by Combinatorics

Even though we will avoid doing so as much as possible in the sequel, it can be worthwhile to have a purely combinatorial description of a simplicial set. Let's begin by classifying increasing maps. We will get some utility out of the following lemma, which allows us to think about increasing maps f in terms of the multi-set $\operatorname{im} f$.

Lemma 1.24. Let $f, g\colon [n] \to [m]$ be increasing maps. Suppose that

$$\#f^{-1}(\{k\}) = \#g^{-1}(\{k\})$$

for all $k \in [m]$. Then $f = g$.

Proof. We proceed by induction on n. If $n = 0$, then $[n]$ is a singleton, so there is a unique $k \in [m]$ for which $f^{-1}(\{k\})$ and $g^{-1}(\{k\})$ are nonempty, namely $f(0)$ and $g(0)$ respectively, so the result follows.

For the induction, we are given two increasing maps $f, g\colon [n+1] \to m$. There are two steps.

1. The main claim is that $f(n+1) = g(n+1)$. To show this, note that $\operatorname{im} f = \operatorname{im} g$ because these sets are just the $k \in [m]$ with nonempty fibers. Thus, because $n+1$ is the maximum of $[n+1]$, we see that $f(n+1)$ and $g(n+1)$ are maximal elements of $\operatorname{im} f$ and $\operatorname{im} g$, respectively, so $f(n+1) = g(n+1)$ follows.

2. We now complete the proof. Note that

$$\#f|_{[n]}^{-1}(\{k\}) = \begin{cases} \#f^{-1}(\{k\}) & \text{if } f(n+1) \neq k, \\ \#f^{-1}(\{k\}) - 1 & \text{if } f(n+1) = k, \end{cases}$$

and similar for g, so $f|_{[n]}$ and $g|_{[n]}$ have fibers of the same cardinality, so $f|_{[n]} = g|_{[n]}$ by the induction, so $f = g$ follows because they are already equal on $n+1$. ■

Let's now classify injective maps.

Definition 1.25 (face maps)**.** Given some $i \in [n]$, we define the *face map* $\delta^i\colon [n-1] \to [n]$ to be the embedding which omits i by sending the set $\{0, \ldots, i-1\}$ to itself and sending the set $\{i, \ldots, n\}$ to one more than each element.

Lemma 1.26. Every injective increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite

$$f = \delta^{i_1} \circ \cdots \circ \delta^{i_r},$$

where $i_1 > i_2 > \cdots > i_r$.

Proof. We proceed in steps.

1. Given a decreasing sequence $i_1 > i_2 > \cdots > i_r$, we claim that the map

$$(\delta^{i_1} \circ \cdots \circ \delta^{i_r})\colon [n] \to [n+r]$$

avoids the set $\{i_1, \ldots, i_r\}$. We proceed by induction on r; for $r = 0$, the statement is vacuous. For the induction, we are given a decreasing sequence $i_1 > i_2 > \cdots > i_r > i_{r+1}$. By the induction, the map

$$(\delta^{i_2} \circ \cdots \circ \delta^{i_{r+1}})\colon [n] \to [n+r]$$

already avoids the set $\{i_2, \ldots, i_{r+1}\}$. Then δ^{i_1} preserves the set $\{0, \ldots, i_1 - 1\}$ (and in particular preserves the omitted set $\{i_2, \ldots, i_{r+1}\}$) while going on to omit i_1, so the total composite $\delta^{i_1} \circ \cdots \circ \delta^{i_{r+1}}$ successfully omits $\{i_1, \ldots, i_{r+1}\}$.

2. We show that any injective f is a composite of $\delta^\bullet$s as given. Well, given an injective increasing map $f\colon [n] \to [m]$, set $I := [m] \setminus \operatorname{im} f$, and arrange the elements of I as $\{i_1, \ldots, i_r\}$ in decreasing order. Then $\delta^{i_1} \circ \cdots \circ \delta^{i_r}$ is another injective increasing map which omits I by the previous step, so it equals f by Lemma 1.24.

3. We show that two composites of decreasing $\delta^\bullet$s are equal if and only if the indices are equal. More precisely, suppose that
$$\delta^{i_1} \circ \cdots \delta^{i_r} = \delta^{i'_1} \circ \cdots \delta^{i'_{r'}}$$
as maps $[n] \to [m]$, and the sequence of indices are both strictly decreasing; denote this map by f for brevity. By the first step, the size of the fibers of f can be read off of the indices $i_\bullet$ or $i'_\bullet$ (an index is present exactly when not in $\operatorname{im} f$), so these sequences must be equal. ■

Remark 1.27. It follows that δ^i is the unique injection $[n] \to [n+1]$ omitting a given element of $[n+1]$.

Remark 1.28. The requirement that the indices are strictly decreasing is necessary for the uniqueness. Indeed, if $i \le j$, then $\delta^i \circ \delta^j$ avoids i and $j+1$, so it equals $\delta^{j+1} \circ \delta^i$.

Analogously, we have should handle surjective increasing maps.

Definition 1.29 (degeneracy maps)**.** Given some $j \in [n+1]$, we define the *degeneracy map* $\sigma^j\colon [n+1] \to [n]$ to be the surjection which hits j twice by sending the set $\{0, \ldots, j\}$ to itself and sending $\{j+1, \ldots, n+1\}$ to one less than each element.

Lemma 1.30. Every surjective increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite
$$f = \sigma^{j_1} \circ \cdots \circ \sigma^{j_r},$$
where $j_1 \ge j_2 \ge \cdots \ge j_r$.

Proof. The structure of this proof is similar to Lemma 1.26, but the technical core requires a couple modifications.

1. Given a decreasing sequence $j_1 \ge j_2 \ge \cdots \ge j_r$, we claim that the map
$$(\sigma^{j_1} \circ \cdots \circ \sigma^{j_r})\colon [n] \to [n-r]$$
has fiber over $k \in [n-r]$ of size equal to $1 + \#\{t : j_t = k\}$. We proceed by induction on r; for $r = 0$, the statement is vacuous. For the induction, we are given a decreasing sequence $j_1 \ge \cdots \ge j_{r+1}$. By induction, we know that the fiber of $\sigma := \sigma^{j_2} \circ \cdots \circ \sigma^{j_{r+1}}$ over k is $1 + \{2 \le t \le r+1 : j_t = k\}$.

   Now, $\sigma^{j_1} \circ \sigma$ has the same-size fibers over any $k < j_1$ as σ because σ^{j_1} preserves $\{0, \ldots, j_1\}$. For $k > j_1$, we note that the fibers of σ over each such k is 1 because $k > j_i$ for each i (by the induction), so $\sigma^{j_1} \circ \sigma$ also has fiber of size 1 over this k. Lastly, for $k = j_1$, we see that the fiber increases in size by 1 because σ^{j_1} sends $j_1 + 1$ (whose fiber has size 1 for σ) to j_1. This casework completes the proof.

2. We show that any surjective f is a composite of $\sigma^\bullet$s as given. Well, let J be the multi-subset of $[m]$ hit multiple times by f, counted with multiplicity, and we may arrange the elements of J as $\{j_1, \ldots, j_r\}$ in decreasing order. Then $\sigma^{j_1} \circ \cdots \circ \sigma^{j_r}$ and f have fibers of the same size by the previous step, so they are equal functions by Lemma 1.24.

3. We show that two composites of decreasing $\sigma^\bullet$s are equal if and only if the indices are equal. More precisely, suppose that
$$\sigma^{j_1} \circ \cdots \circ \sigma^{j_r} = \sigma^{j'_1} \circ \cdots \circ \sigma^{j'_{r'}}$$

as maps $[n] \to [m]$, and the sequences of indices are both decreasing; denote this map by f. Well, the fibers of f can be read off the indices $\{j_\bullet\}$ or $\{j'_\bullet\}$ by the first step, so these sequences must be equal. ■

Remark 1.31. As in Remark 1.27, we note that the requirement that the indices are decreasing is necessary for the uniqueness. Indeed, if $i < j$, then $\sigma^i \circ \sigma^j$ hits $j-1$ twice and i twice (counted with multiplicity), so $\sigma^i \circ \sigma^j = \sigma^{j-1} \circ \sigma^i$.

We are now ready to classify general maps.

Lemma 1.32. Every increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite

$$f = (\delta^{i_1} \circ \cdots \circ \delta^{i_r}) \circ (\sigma^{j_1} \circ \cdots \sigma^{j_s}),$$

where $i_1 > \cdots > i_r$ and $j_1 \geq \cdots \geq j_s$.

Proof. The main point is to show that any increasing map f admits a unique decomposition as $\delta \circ \sigma$ where $\delta\colon [k] \to [m]$ is injective and $\sigma\colon [n] \to [k]$ is surjective. The existence and uniqueness of the required decomposition now follows by the existence and uniqueness of the decomposition $f = \delta \circ \sigma$ with Lemmas 1.26 and 1.30. For example, to get the uniqueness, if

$$\delta^{i_1} \circ \cdots \circ \delta^{i_r} \circ \sigma^{j_1} \circ \cdots \sigma^{j_s} = \delta^{i'_1} \circ \cdots \circ \delta^{i'_{r'}} \circ \sigma^{j'_1} \circ \cdots \sigma^{j'_{s'}},$$

then the composites of the $\delta^\bullet$s and of the $\sigma^\bullet$s must each be equal (because those are injections and surjections, respectively), and then the equalities of the indices follows from using Lemmas 1.26 and 1.30, respectively.

It remains to show the main claim. We show existence and uniqueness separately.

- Existence: note $\operatorname{im} f \subseteq [m]$ is some totally ordered subset, so we let its cardinality be $k+1$. By suitably ordering the elements of $\operatorname{im} f$, we receive a totally ordered bijection $[k] \to \operatorname{im} f$. Then we see that f decomposes into

$$\underbrace{[n] \xrightarrow{f} \operatorname{im} f \leftarrow [k]}_{\sigma} = \underbrace{[k] \to \operatorname{im} f \subseteq [m]}_{\delta},$$

as required.

- Uniqueness: suppose we have two equal decompositions $f = \delta \circ \sigma = \delta' \circ \sigma'$ where $\sigma\colon [n] \to [k]$ and $\sigma'\colon [n] \to [k']$ and $\delta\colon [k] \to [m]$ and $\delta\colon [k'] \to [m]$. To begin, note that the injectivity of δ and δ' implies that $k+1$ and $k'+1$ are both the cardinality of $\operatorname{im} f$, so $k = k'$ follows. Now, because δ and δ' have the same image, and both are injective, it follows that all their fibers from $[m]$ have the same size (as either 0 or 1)! Thus, $\delta = \delta'$ follows from Lemma 1.24. The injectivity of δ now shows that $\delta \circ \sigma = \delta \circ \sigma'$ implies $\sigma = \sigma'$. ■

Remark 1.33. As in Remarks 1.27 and 1.31, we note that putting δs before σs is important for the uniqueness. Suppose we have some $\sigma^j \circ \delta^i$, and then we have the following cases.

- If $j > i$, then σ^j fixes $\{0, \ldots, i+1\}$, so $\sigma^j \circ \delta^i$ avoids i and hits j twice. This is the same as $\delta^i \circ \sigma^{j-1}$.
- If $j = i$ or $j = i-1$, then $\sigma^j \circ \delta^i$ fixes $\{0, \ldots, i-1\}$ throughout, and the elements at least i get $+1$ from δ^i and -1 from σ^j. Thus, $\sigma^j \circ \delta^i = \mathrm{id}$.
- If $j < i-1$, then $\sigma^j \circ \delta^i$ avoids $i-1$ (-1 from σ^j) and hits j twice. This is the same as $\delta^{i-1} \circ \sigma^j$.

Having access to generators of these maps and some relations between them allows us to provide a combinatorial definition of a simplicial set.

Definition 1.34. A *combinatorial simplicial set* is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with face maps $d_0, \ldots, d_n \colon X_n \to X_{n-1}$ and degeneracy maps $s_0, \ldots, s_n \colon X_n \to X_{n+1}$ (for each n) satisfying the following simplicial identities

$$\begin{cases} d_j d_i = d_i d_{j+1} & \text{if } i \le j, \\ s_j s_i = s_i s_{j-1} & \text{if } i < j, \end{cases} \qquad \text{and} \qquad \begin{cases} d_i s_j = s_{j-1} d_i & \text{if } i < j, \\ d_i s_j = \mathrm{id} & \text{if } i = j \text{ or } i = j+1, \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

A morphism $f \colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets is a function $f_n \colon X_n \to Y_n$ for each n commuting with the face and degeneracy maps; i.e., $f_{n-1} \circ d_n = d_n \circ f_n$ and $f_{n+1} \circ s_n = s_n \circ f_n$.

Remark 1.35. One can check that there is a category of combinatorial simplicial sets. In particular, the identity is given by $(\mathrm{id}_X)_n := \mathrm{id}_{X_n}$, and composition is defined by $(g \circ f)_n := g_n \circ f_n$ (which commutes with the face and degeneracy maps because g and f do).

Proposition 1.36. There is an isomorphism of categories from the category of simplicial sets to the category of combinatorial simplicial sets by sending $X \in \mathrm{sSet}$ to a combinatorial simplicial set given by

$$\begin{cases} X_n := X([n]), & \\ d_\bullet := X(\delta^\bullet) & \text{for each } n \in \mathbb{N}, \\ s_\bullet := X(\sigma^\bullet) & \text{for each } n \in \mathbb{N}. \end{cases}$$

Proof. We run our many checks in sequence.

- To check that $X \in \mathrm{sSet}$ is sent to a combinatorial simplicial set $\{X_n\}$, we just need to check that the $d_\bullet$s and $s_\bullet$s satisfy the simplicial identities. This follows from the functoriality of X and Remarks 1.27, 1.31 and 1.33.

- We define $X \mapsto \{X_n\}$ on morphisms. Well, a functor $f \colon X \Rightarrow Y$ of simplicial sets defines maps $f_{[n]} \colon X([n]) \to Y([n])$, which we claim assembles into a morphism $f \colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets by $f_n := f_{[n]}$. To check this, we need to check compatibility with the face and degeneracy maps. Well, $f_{n-1} \circ d_n = d_n \circ f_n$ and $f_{n+1} \circ s_n = s_n \circ f_n$ follow by naturality of f because these amount to requiring

$$f_{[n-1]} \circ X(\delta^n) = Y(\delta^n) \circ f_{[n]} \qquad \text{and} \qquad f_{[n+1]} \circ X(\sigma^n) = Y(\sigma^n) \circ f_{[n]}.$$

- We show that $X \mapsto \{X_n\}$ is functorial. To begin, note $\mathrm{id} \colon X \Rightarrow X$ goes to the identity maps $\mathrm{id}_n \colon X_n \to X_n$. Then given $f \colon X \Rightarrow Y$ and $g \colon Y \Rightarrow Z$, we see that the composite $(g \circ f) \colon \{X_n\} \to \{Z_n\}$ is given by $(g \circ f)_n = (g \circ f)_{[n]} = g_{[n]} \circ f_{[n]} = g_n \circ f_n$, as required.

- We define a map from combinatorial simplicial sets back to simplicial sets. Well, given a combinatorial simplicial set $\{X_n\}$, we begin defining our functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ by $X([n]) := X_n$. On morphisms $f \colon [n] \to [m]$, we need to define some map $Xf \colon X_m \to X_n$. For this, we note that Lemma 1.32 allows us to write f uniquely as a composite

$$f = (\delta^{i_1} \circ \cdots \circ \delta^{i_r}) \circ (\sigma^{j_1} \circ \cdots \circ \sigma^{j_s}),$$

where $i_\bullet$ is strictly decreasing and $j_\bullet$ is decreasing. Thus, we define

$$Xf := (s_{j_s} \circ \cdots \circ s_{j_1}) \circ (d_{i_r} \circ \cdots \circ d_{i_1}).$$

For example, $f = \mathrm{id}_{[n]}$ is equal to the empty composite everywhere, so $X\mathrm{id}_{[n]} = \mathrm{id}_{X_n}$.

To complete our functoriality check, because any morphisms can be written as a composite of $\delta^\bullet$s and $\sigma^\bullet$s, it is enough to check functoriality for such morphisms. Namely, we have to check that

$$\begin{cases} X(\delta_i\delta_j) = X(\delta_j)X(\delta_i), \\ X(\sigma_i\sigma_j) = X(\sigma_j)X(\sigma_i), \\ X(\delta_i\sigma_j) = X(\sigma_j)X(\delta_i), \\ X(\sigma_j\delta_i) = X(\delta_i)X(\sigma_j). \end{cases}$$

For the first, this is by definition when $i > j$ and follows from the simplicial identities otherwise; the second is similar. The third is also automatic, and the last follows from the simplicial identities again.

- We define our map on morphisms. Well, given a morphism $F\colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets, we already have our component morphisms $F_n\colon X_n \to Y_n$ which will become our morphisms $F_{[n]}\colon X([n]) \to Y([n])$. It remains to check the naturality of $F\colon X \Rightarrow Y$. Well, let $f\colon [n] \to [m]$ be an increasing map, we should check that $Xf \circ F_m = F_n \circ Yf$. Because f can be written as a composite of $\delta^\bullet$s and $\sigma^\bullet$s (by Lemma 1.32), it is enough to check this for $f \in \{\delta^\bullet, \sigma^\bullet\}$, which now follows because F started its life as a morphism of combinatorial simplicial sets.
- We show that $\{X_n\} \mapsto X$ is functorial. To begin, note $\mathrm{id}\colon \{X_n\} \to \{X_n\}$ goes to the identity maps $\mathrm{id}_{[n]}\colon X([n]) \to X([n])$. Then given $f\colon \{X_n\} \to \{Y_n\}$ and $g\colon \{Y_n\} \to \{Z_n\}$, we see that the composite $(g \circ f)\colon X \Rightarrow Z$ is given by $(g\circ f)_{[n]} = (g \circ f)_n = g_n \circ f_n = g_{[n]} \circ f_{[n]}$.
- We complete the check that we have defined inverse equivalences. For concreteness, let $A\colon \{X_n\} \mapsto X$ and $B\colon X \mapsto \{X_n\}$ be our functors.

  Let's check $BA = \mathrm{id}$. On an object $\{X_n\}$, we see that $BA\{X_n\}$ has $(BA\{X_n\})_n = A\{X_n\}([n]) = X_n$ and simplicial maps d_i and s_j given by $A(\{X_n\})(\delta^i)$ and $A(\{X_n\})(\sigma^j)$ which are d_i and s_j, respectively. On morphisms, we see $BAf = f$ because $(BAf)_n = Af_{[n]} = f_n$ for each n.

  Lastly, let's check $AB = \mathrm{id}$. On an object X, we analogously see that $ABX([n]) = BX_n = X([n])$; further, to check that $ABX(f) = X(f)$ for an increasing map f, we note that Lemma 1.32 reduces this check to $\delta^\bullet$ and $\sigma^\bullet$ by functoriality, which similarly follows by construction of A and B (which turns $\delta^\bullet$s and $\sigma^\bullet$s to $d_\bullet$s and $s_\bullet$s and vice versa). Lastly, on morphisms, we see $(ABf)_{[n]} = Bf_n = f_{[n]}$ for each n. ■

Remark 1.37. In light of Proposition 1.36, we will occasionally identify simplicial sets and combinatorial simplicial sets. In particular, the term "combinatorial simplicial set" will not appear again.

Remark 1.38. There is also a notion of "semi-simplicial set" where we remove all the data associated to the $s_\bullet$s. This notion is sufficient to work with homology, but because we are now homotopy theorists, we work with simplicial sets.

1.2 September 9

The first problem set will be posted in about a day.

1.2.1 Some Simplicial Sets

We are now allowed to remove the absolute value bars from our Δ^n.

Definition 1.39 (simplex)**.** For each $n \geq 0$, we define the n-*simplex* Δ^n as the simplicial set よ$([n])$.

Remark 1.40. As in Remark A.18, we see $\Delta^n([m])$ is $\operatorname{Mor}_{\mathrm{sSet}}([m], \Delta^n)$, which is $\text{よ}_{[n]}([m])$ or $\operatorname{Fun}([m],[n])$. Of course, this is just the collection of order-preserving maps $[m] \to [n]$. Then given an increasing map $f\colon [m] \to [m']$, we see that $\Delta^n(f)\colon \Delta^n([m']) \to \Delta^n([m])$ is

$$\text{よ}_{[n]}(f)\colon \operatorname{Mor}_{\mathrm{sSet}}([m'], \Delta^n) \to \operatorname{Mor}_{\mathrm{sSet}}([m], \Delta^n),$$

which of course is just $(- \circ f)$. All the prior identifications chain to show that we are still looking at $(- \circ f)$ on the level of order-preserving maps.

Example 1.41. We see $\left(\Delta^2\right)_0$ has three elements, and $\left(\Delta^2\right)_1$ has six elements. Here are three of the elements of $\left(\Delta^2\right)_1$.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

Remark 1.42. On the other hand, for a simplicial set X, Remark A.18 tells us that $\operatorname{Mor}_{\mathrm{sSet}}(\Delta^n, X)$ is in bijection with X_n, and this bijection takes $\varphi\colon \Delta^n \to X$ to $\varphi_{[n]}(\mathrm{id}_{[n]}) \in X_n$. We even know that the inverse map takes some $x \in X$ and outputs a natural transformation $\Delta^n \to X$ defined by sending $f\colon [m] \to [n]$ in $\Delta^n(m)$ to $Xf(x) \in X_m$.

Example 1.43. The maps $\delta^i\colon [n-1] \to [n]$ and $\sigma^i\colon [n+1] \to [n]$ induce maps $\text{よ}\delta^i\colon \Delta^{n-1} \to \Delta^n$ and $\text{よ}\sigma^i\colon \Delta^{n+1} \to \Delta^n$ given by $(- \circ \delta^i)$ and $(- \circ \sigma^i)$, respectively. We continue to label our face maps by $d_i\colon \Delta^{n-1} \to \Delta^n$ and degeneracy maps by $s_i\colon \Delta^{n+1} \to \Delta^n$.

Here is the sort of thing that this language allows us to prove.

Lemma 1.44. A map $f\colon X \to Y$ of simplicial sets is monic if and only if $f_n\colon X_n \to Y_n$ is injective for all $n \geq 0$.

Proof. In one direction, if f_n is injective for all $n \geq 0$, then one can see directly that f is monic: given maps $g_1, g_2\colon Y \to Z$, we see that $f \circ g_1 = f \circ g_2$ implies that $g_1 = g_2$, as we can see by passing to n-simplices for each n.

In the reverse direction, suppose that f is monic. Then f must induce injections

$$\operatorname{Mor}_{\mathrm{sSet}}(\Delta_n, X) \to \operatorname{Mor}_{\mathrm{sSet}}(\Delta_n, Y).$$

However, as described in Remark 1.42, this is (naturally) isomorphic to the map $f_n\colon X_n \to Y_n$, which is therefore also injective! ■

Here are two more important simplicial sets.

Definition 1.45 (boundary)**.** For each $n \geq 0$, we define the *boundary* $\partial\Delta^n \in \mathrm{sSet}$ to be the subfunctor of Δ^n with $\partial\Delta^n(m)$ given by the non-surjective maps $[m] \to [n]$.

Remark 1.46. There are canonical inclusions $\partial\Delta^n(m) \subseteq \Delta^n(m)$ for each m, which we claim upgrades into an embedding of simplicial sets. Well, for each $g\colon [m] \to [m']$ and $f \in \partial\Delta^n(m)$, we see that $(f \circ g)$ continues to not be surjective, so $(f \circ g) \in \partial\Delta^n(m')$, so the claim follows from Remark A.16.

Definition 1.47 (horn)**.** For each $i \in [n]$, we define the *ith horn* $\Lambda_i^n \in \mathrm{sSet}$ to be the subfunctor of Δ^n with Λ_n^i given by the maps $[m] \to [n]$ which avoid an element not equal to i. We say that Λ_i^n is an *inner horn* if and only if $0 < i < n$; otherwise, Λ_i^n is an *outer horn*.

Remark 1.48. As in Remark 1.46, there are canonical inclusions $\Lambda_n^i(m) \subseteq \partial\Delta^n(m)$ for each i and n and m with $0 \le i \le n$. We check that this upgrades to an inclusion of simplicial sets with Remark A.16: we have to check that any maps $g\colon [m] \to [m']$ and $f \in \partial\Delta^n(m)$ has $(f \circ g) \in \partial\Delta^n(m)$, which is true because $i \notin \operatorname{im} f$.

Example 1.49. Intuitively, Λ_i^n deletes the face opposite i. For example, here is Λ_0^2.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd}$$

One can similarly draw Λ_1^2 (which omits $0 \to 2$) and Λ_2^2 (which omits $0 \to 1$).

1.2.2 Dimension

We take a moment to introduce a complexity measure of simplicial sets, which will be helpful in the sequel when we are making inductive arguments.

Definition 1.50 (dimension)**.** Fix a simplicial set X. Then X has *dimension* d if and only if d is the smallest nonnegative integer for which all simplices in X_k are in the image of the degeneracy maps for $k > d$. (We may say that such a simplex is *degenerate*.) If there is no such nonnegative integer d, we say that X is infinite-dimensional. We may write the dimension as $\dim X$.

Remark 1.51. Given a morphism $f\colon X \to Y$, if $x \in X_n$ is degenerate, then so is $f(x) \in Y_n$. Indeed, we are provided with $x' \in X_{n-1}$ for which $x = f(x')$, so the claim follows from the commutativity of the following diagram.

$$\begin{tikzcd} X_{n-1} \arrow[r, "f"] \arrow[d, "s_i"'] & Y_{n-1} \arrow[d, "s_i"'] \\ X_n \arrow[r, "f"] & Y_n \end{tikzcd} \qquad \begin{tikzcd} x' \arrow[r, mapsto] \arrow[d, mapsto] & f(x') \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

Example 1.52. We claim that $\dim \Delta^n = n$. In fact, we will show that a map $[m] \to [n]$ in $\Delta^n(m)$ is non-degenerate if and only if it is injective, which completes the proof of the claim ($\mathrm{id}_{[n]}$ is injective, but nothing will be injective for $m > n$). In one direction, for any $m \ge 0$, the degeneracy map $s_i\colon X_m \to X_{m+1}$ sends an increasing map $[m] \to [n]$ to the composite $[m+1] \twoheadrightarrow [m] \to [n]$; in particular, the composite necessarily fails to be injective. Conversely, any non-injective map sends two consecutive inputs i and $i+1$ to the same output because it is still increasing. Thus, $[k] \to [n]$ factors through $\sigma_i\colon [k] \to [k-1]$ and is therefore degenerate!

Example 1.53. We claim that $\dim \Lambda_i^n = n - 1$. As in Example 1.52, it is enough to show that a map $[m] \to [n]$ in $\Lambda_i^n(m)$ is non-degenerate if and only if it is injective, which completes the proof of the claim (δ_i is injective, but nothing will be injective for $m \ge n$). The arguments are now exactly the same as in Example 1.52, with the caveat that the converse argument must ensure that we avoid an element away from i in the image at all times.

Intuitively, we expect that a simplicial set X can be (canonically) built out of its k-simplices where k varies over all nonnegative integers bounded by $\dim X$. Here, "built" means that we are gluing via colimits.

Lemma 1.54. Let $F\colon \mathcal{I} \to \mathrm{sSet}$ be a functor with colimit X. Then

$$\dim A \le \sup_{i\in\mathcal{I}} F(i).$$

Proof. Let the given supremum be d. We have to show that $d \le n$ implies that $\dim A \le n$ for any n. To show that $\dim A \le n$, we have to show that any element of A_k for $k > n$ is degenerate. Because colimits are computed pointwise, we know that

$$A_k = \operatorname*{colim}_{i\in I} F(i),$$

so the construction of colimits in Set implies that a is in the image of one of the canonical maps $F(i)_k \to A_k$. But all elements in $F(i)_k$ are degenerate, and the fact that the elements in $F(i)_k$ are degenerate implies that the elements of A_k by Remark 1.51. ■

Now, we will show that simplicial sets can be built out of their k-simplices for $k \le \dim X$ in steps. We will require the following technical lemma.

Lemma 1.55. Fix some map $\sigma\colon \Delta^n \to X$ of simplicial sets. Then σ factors uniquely into a composite

$$\Delta^n \twoheadrightarrow \Delta^m \hookrightarrow X,$$

where the map $\Delta^n \to \Delta^m$ is induced by a surjection $[n] \twoheadrightarrow [m]$ and the map $\Delta^m \hookrightarrow X$ maps to a non-degenerate m-simplex.

Proof. It is easier to show the uniqueness of the factorization: simply set m to be the smallest positive integer for which σ factors into

$$\Delta^n \overset{\pi}{\twoheadrightarrow} \Delta^m \overset{\iota}{\hookrightarrow} X.$$

We have now two checks.

- Surjective: suppose for the sake of contradiction, suppose that the map $[n] \to [m]$ induced by π fails to be surjective. Then we can factor the map $[n] \to [m]$ into some composite $[n] \to [m-1] \to [m]$, so π can be factored into a composite $\Delta^n \to \Delta^{m-1} \to \Delta^m$, which violates the minimality of m.
- Non-degenerate: suppose that the image of ι is not a non-degenerate m-simplex. Then x is in the image of some degeneracy map $X_{m-1} \to X_m$; say $x = s_i(x')$. Thus, the commutative diagram in Remark 1.51 shows that ι factors as some composite $\Delta^{m-1} \overset{s_i}{\to} \Delta^m \to X$, which again violates the minimality of m.

It remains to show the uniqueness of the given factorization. Well, suppose that we have another such factorization

$$\Delta^n \overset{\pi'}{\twoheadrightarrow} \Delta^{m'} \overset{\iota'}{\hookrightarrow} X$$

such that π' is induced by a surjection $[n] \twoheadrightarrow [m']$, and ι' maps $[m']$ to a non-degenerate m'-simplex.

In fact, the main claim will be that π factors through π'. Let's explain why this is enough. Say that $\pi = \pi''\pi'$, where $\pi''\colon \Delta^{m'} \to \Delta^m$ is some map. We will end up showing that π'' is the identity. Now, we choose a section α' of π' (so that $\pi'\alpha' = \mathrm{id}_{\Delta^{m'}}$), and $\iota\pi''\pi' = \iota'\pi'$ implies that $\iota\pi''\pi'\alpha = \iota'\pi'\alpha$ and so

$$\iota\pi'' = \iota'.$$

We now see that having π'' is the identity implies that $m = m'$ and $\pi = \pi'$ and $\iota = \iota'$, so we focus our efforts on showing π'' is the identity. Because π is induced by a surjective map $[n] \twoheadrightarrow [m]$, we conclude that π'' is induced by a surjective map $[m'] \twoheadrightarrow [m]$. It remains to show that $m \le m'$, which will then force π'' to be induced by the identity map. Well, if π'' fails to be injective, then π'' has degenerate image, so $\iota\pi''$ has degenerate image, so ι' will have degenerate image! This contradicts the construction of ι', so we are done.

We now turn our attention to showing the main claim, which is some hands-on combinatorics. To show that π factors through π', we will identify these maps with their induced maps $[n] \twoheadrightarrow [m]$ and $[n] \twoheadrightarrow [m']$. Then π factors through π' if and only if $\pi'(i) = \pi'(j)$ implies $\pi(i) = \pi(j)$. Assume this is not the case for the sake of contradiction. In other words, we have some i and j for which $\pi'(i) = \pi'(j)$ while $\pi(i) \neq \pi(j)$; note that i and j must be distinct. Now, find a section $\alpha\colon [m] \hookrightarrow [n]$ of π whose image includes both i and j, which is possible because $\pi(i) \neq \pi(j)$. Then $\iota\pi = \iota'\pi'$ implies that $\iota'\pi'\alpha = \iota$, but $\pi'\alpha$ is not injective (because $\pi'(i) = \pi'(j)$), so $\iota'\pi'\alpha$ does not map to a non-degenerate simplex, which contradicts the construction of ι. ■

Proposition 1.56. Fix a simplicial set X and a nonnegative integer d. Then the following are equivalent.

(a) $\dim X \leq d$.

(b) X is the colimit of some functor $F\colon \mathcal{I} \to \mathrm{sSet}$, where $\dim F(i) \leq d$ for all i.

(c) X is the colimit of some functor $F\colon \mathcal{I} \to \mathrm{sSet}$, and each i has some $k \leq d$ for which $F(i) = \Delta^k$.

(d) Let $\mathcal{I}$ be the category of pairs $([k], x)$, where $k \leq d$ and $x \in X_k$ is some map, where morphisms are increasing maps $[k] \to [k']$ for which the induced map $X_k \to X_{k'}$ sends x to x'. Then X is the colimit of the natural functor $F\colon \mathcal{I} \to \mathrm{sSet}$ given by $F(([k], x)) = \Delta^k$.

Proof. Note that (d) implies (c) with no effort, and (c) implies (b) with no effort as soon as we recall Example 1.52. Similarly, (b) implies (a) is exactly Lemma 1.54.

It remains to show that (a) implies (d), for which we will use Lemma 1.55. Quickly, we remark that the natural functor $F\colon \mathcal{I} \to \mathrm{sSet}$ simply sends the pair $([k], x)$ to the element $x \in X_k$. Note that there is a natural map

$$f\colon \operatorname*{colim}_{i \in \mathcal{I}} F(i) \to X$$

induced by the maps $F(([k], x)) \to X$ defined by $x\colon \Delta^k \to X_k$. (Indeed, we have induced a map from the colimit because the maps $F(i) \to X$ automatically commute with the internal maps $F(i) \to F(j)$ for each $i \to j$ in $\mathcal{I}$ by the definition of the category $\mathcal{I}$.) We would like to show that f is an isomorphism.

It is enough to show that f is injective and surjective; then the inverse can be constructed on the level of the simplices by hand. Quickly, we show that f is surjective. This uses the existence assertion in Lemma 1.55, which shows that $\operatorname{im} f$ contains every non-degenerate m-simplex of X for each $m \leq d$. We would like to show that these simplices generate X (via the degeneracy maps). Well, for any $x \in X_n$, either x is non-degenerate or in the image of a degeneracy map. Continuing inductively, we conclude that x equals $\sigma(x')$ where $x' \in X_m$ is non-degenerate, and σ is some composite of degeneracy maps. Note then that $m \leq d$ because $\dim X \leq d$, so $x' \in \operatorname{im} f$. Thus, $x \in \operatorname{im} f$ as well because f commutes with the degeneracy maps!

We will now show that f is injective, which will complete the proof. We will use the uniqueness assertion in lemma 1.55. Well, suppose that we have two elements y and y' in the colimit for which $f(y) = f(y')$; we would like to show that $y = y'$. We may assume that y and y' are both ℓ-simplices for the same ℓ because they both must map to the same graded piece of X.

We now apply some reductions to simplify the presentation of y and y'. Set $x \coloneqq f(y) = f(y')$ for brevity.

- Because colimits are computed pointwise, we see that y arises as an ℓ-simplex of some $F(([k], x)) = \Delta^k$. This means that we have an increasing map $[\ell] \to [k]$, and $f(y)$ is induced by the map $\Delta^\ell \to \Delta^k \to X$. By factoring $[\ell] \to [k]$ into a surjection and an injection via Lemma 1.32, we may as well assume that the map $[\ell] \to [k]$ is a surjection (simply by changing the pair $([k], x)$). One similarly finds y' in the image of some $F([k'], x')$, where the promised map $[\ell'] \to [k']$ is surjective.

- We reduce to the case where x is non-degenerate. Note $x\colon \Delta^k \to X$ factors as

$$\Delta^k \twoheadrightarrow \Delta^m \hookrightarrow X,$$

where $\Delta^k \twoheadrightarrow \Delta^m$ is induced by a surjection, and $\Delta^m \to X$ goes to a non-degenerate m-simplex. We can now replace k with m so that $F(([k],x))$ also has x non-degenerate. One can do something similar for y' to assume that x' is non-degenerate.

Now, the moral is that $f(y) = f(y')$ implies that the composites $\Delta^\ell \twoheadrightarrow \Delta^k \hookrightarrow X$ and $\Delta^\ell \twoheadrightarrow \Delta^{k'} \hookrightarrow X$ produces the same ℓ-simplex, but this factorization is unique by Lemma 1.55. This shows that $([k],x) = ([k'],x')$, and in fact we see that $y = y'$ because y and y' amount to the data of the map $\Delta^\ell \to \Delta^k$. ■

The category $\mathcal{I}$ may look like it admits a rather complicated definition, but it can be used effectively in practice. Indeed, it is essentially a combinatorial gadget which tells us how to glue simplices together.

Example 1.57. We write Λ^2_1 as a colimit of Δ^0s and Δ^1s. By Example 1.53, the non-degenerate m-simplices are given by injective maps $[m] \to [n]$ avoiding an element away from 1, so of course $m \in \{0,1\}$. In particular, our 0-simplices are 0, 1, and 2; our 1-simplices are $0 \le 1$ and $1 \le 2$. Keeping track of the maps between these simplices, we see that Λ^2_1 is the colimit of the following diagram, where we are identifying an m-simplex of Λ^2_1 with Δ^m.

$$\begin{tikzcd}
 & (0 \le 1) \arrow[dl, shift left, "\delta^1"] \arrow[dr, shift left, "\delta^0"] & & (1 \le 2) \arrow[dl, shift left, "\delta^2"] \arrow[dr, shift left, "\delta^1"] & \\
(0) \arrow[ur, shift left, "\sigma^0"] & & (1) \arrow[ul, shift left, "\sigma^1"] \arrow[ur, shift left, "\sigma^1"] & & (2) \arrow[ul, shift left, "\sigma^2"]
\end{tikzcd}$$

In this way, we see that a map $\Lambda^2_1 \to X$ amounts to the data of three objects $e_0, e_1, e_2 \in X$ and two morphisms $f_{01}, f_{12} \in X$ such that $d_1 f_{01} = e_0$, $d_0 f_{01} = d_1 f_{12} = e_1$, and $d_0 f_{12} = e_2$. In fact, one can throw away the data of e_0 and e_2 because it is determined by f_{01} and f_{12} by the face maps.

1.2.3 More on Nerves

It is worthwhile to explain nerves a little more. In this section, we will characterize which simplicial sets appear as nerves of categories.

Exercise 1.58. Fix a category $\mathcal{C}$. We work our $N(\mathcal{C})_i$ for $i \in \{0,1,2\}$.

Proof. Here we go.

- We see $N(\mathcal{C})_0$ consists of functors from the category $\{\bullet\}$, which are just objects of $\mathcal{C}$.
- Similarly, $N(\mathcal{C})_1$ consists of functors from the category $\{\bullet \to \bullet\}$ to $\mathcal{C}$, which are just morphisms of $\mathcal{C}$.
- Lastly, we note $N(\mathcal{C})_2$ consists of functors from the category $\{\bullet \to \bullet \to \bullet\}$ to $\mathcal{C}$, which amounts to the data of a diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] \\
 & \bullet
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
c \arrow[r, "f"] \arrow[dr, "(g \circ f)"'] & c' \arrow[d, "g"] \\
 & c''
\end{tikzcd}$$

so that the nerve is required to know something about composition!

We can also describe N on some easy morphisms. For example, there is a unique map $[n] \to [0]$ for any $n \ge 0$: it sends all objects of $[n]$ to 0 and all morphisms to id_0. For example, the corresponding map $N(\mathcal{C})_0 \to N(\mathcal{C})_1$ needs to send the object $c \in \mathcal{C}$, which corresponds to the constant functor $[0] \to \mathcal{C}$, to the identity map $\mathrm{id}_c \colon c \to c$, which indeed is the image of the morphism in $[1]$ when passed through the composite $[1] \to [0] \xrightarrow{c} \mathcal{C}$. ■

Remark 1.59. More generally, an element of $N(\mathcal{C})_m$ is a functor $F\colon [m] \to \mathcal{C}$. Because $[m]$ is a totally ordered set, this amounts to having objects $\{F(0), \ldots, F(m)\}$ and morphisms $F(i) \to F(i+1)$ for each $i \in [m-1]$; from here, we see that the morphism $i \to i+j$ equals the composite $i \to i+1 \to \cdots \to i+j$ and thus goes to the composite

$$F(i) \to F(i+1) \to \cdots \to F(i+j)$$

by functoriality. The construction of these maps provides functoriality automatically. (We should send the identity maps to identity maps, of course.)

Remark 1.60. Given an element $F \in N(\mathcal{C})_m$, it is possible to use naturality to extract out the objects $F(i)$ and morphisms $F(i) \to F(j)$ (where $i \le j$).

- There is a morphism $\varepsilon_i\colon [0] \to [m]$ defined by $0 \mapsto i$, which then induces a map $N(\mathcal{C})\varepsilon_i\colon N(\mathcal{C})_m \to N(\mathcal{C})_0$ by functoriality. This map sends F to $(F \circ \varepsilon_i)\colon [0] \to \mathcal{C}$, which is the functor which picks out the object $F(i)$.
- There is a morphism $\varepsilon_{ij}\colon [1] \to [m]$ defined by $0 \mapsto i$ and $1 \mapsto j$, which then induces a map $N(\mathcal{C})\varepsilon_{ij}\colon N(\mathcal{C})_m \to N(\mathcal{C})_1$ by functoriality. This map sends F to $(F \circ \varepsilon_{ij})\colon [1] \to \mathcal{C}$, which is the functor which picks out the morphism $F(i) \to F(j)$.

We will get some utility out of the following lemmas. Roughly speaking, the point is that a map to a nerve is determined by what it does on the level of morphisms.

Lemma 1.61. Fix a category $\mathcal{C}$. Suppose that we have a simplicial set X and two maps $\varphi, \psi\colon X \to N(\mathcal{C})$. If $\varphi_0 = \psi_0$ and $\varphi_1 = \psi_1$, then $\varphi = \psi$.

Proof. The main point is to use Remark 1.60. We need to show that $\varphi_m = \psi_m$ for all m. Note that there is nothing to do for $m \in \{0, 1\}$, so we may assume that $m \ge 2$. Thus, we begin by choosing some $x \in X_m$. We claim that the value of $\varphi_m(x)$ only depends on $\varphi_0(X\varepsilon_i(x))$s and $\varphi_1(X\varepsilon_{ij}(x))$s; an analogous claim holds for ψ by symmetry, so the proof will be complete.

Well, as in Remark 1.59, $\varphi_m(x)$ amounts to the data of some chain

$$c_0 \to \cdots \to c_m$$

of morphisms in $\mathcal{C}$. As explained in Remark 1.60, we see that $c_i = N(\mathcal{C})\varepsilon_i(\varphi_m(x))$ for each i, which is simply $\varphi_1(X\varepsilon_i(x))$ by naturality. Similarly, the map $c_i \to c_j$ for $i \le j$ is $N(\mathcal{C})\varepsilon_{ij}(\varphi_m(x))$, which is simply $\varphi_1(X\varepsilon_{ij}(x))$ by naturality again! ■

We even remark that we have the following existence result.

Lemma 1.62. Fix a category $\mathcal{C}$ along with $n+1$ objects $\{c_0, \ldots, c_n\}$ and morphisms $f_{i,i+1}\colon c_i \to c_{i+1}$ for each i. Suppose we have a simplicial subset $S \subseteq \Delta^n$ such that S_1 contains all maps $\varepsilon_i\colon [1] \to [n]$ of the form $x \mapsto (i+x)$ for each $0 \le i < n$. Then there is a map $\varphi\colon S \to N(\mathcal{C})$ for which $\varphi_0(x) = c_x$ for each x and $\varphi_1(\varepsilon_i) = f_{i,i+1}$ for each i.

Proof. Let's begin with existence so that we know what we are expecting. For the existence argument, we may as well assume that $S = \Delta^n$ because the problem only becomes harder with a larger simplicial set. With Remark 1.59 in mind, we define $f_{i,i+j}$ for any $j \ge 0$ as being the composite $f_{i+j-1,i+j} \circ \cdots \circ f_{i,i+1}$. Then we send any increasing map $g\colon [m] \to [n]$ in $\Delta^n(m)$ to the functor $[m] \to \mathcal{C}$ defined by

$$c_{g(0)} \to c_{g(1)} \to \cdots \to c_{g(m)},$$

where the intermediate morphisms are $f_{g(i),g(i+1)}\colon c_{g(i)} \to c_{g(i+1)}$. To show that this is natural, we choose a map $h\colon [m] \to [m']$ and observe that the diagram

$$\begin{tikzcd} \Delta^n(m') \arrow[r] \arrow[d, "(-\circ h)"'] & \mathrm{Fun}([m'],\mathcal{C}) \arrow[d, "(-\circ h)"] \\ \Delta^n(m) \arrow[r] & \mathrm{Fun}([m],\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto] \arrow[d, mapsto] & c_{g(0)} \to \cdots \to c_{g(m')} \arrow[d, mapsto] \\ (g\circ h) \arrow[r, mapsto] & c_{g(h(0))} \to \cdots \to c_{g(h(m))} \end{tikzcd}$$

commutes. ■

Non-Example 1.63. One does not expect any map $\Lambda^2_0 \to N(\mathcal{C})$ to always extend to Δ^2. Indeed, Λ^2_0 only has the maps $0 \to 1$ and $0 \to 2$, but there is no obvious way to then produce a map $1 \to 2$ in the nerve!

Lemma 1.64. The functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful.

Proof. We will show this directly.

- Faithful: given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ such that $NF = NG$, we need to check that $F = G$. Well, for any object $c \in \mathcal{C}$, we see that $c \in N(\mathcal{C})_0$, and by construction $(NF)_0(c) = Fc$ and $(NG)_0(c) = Gc$. (Explicitly, $(NF)_0$ is the composite functor $[0] \xrightarrow{c} \mathcal{C} \to \mathcal{D}$, which simply picks out the object Fc.) Similarly, for any morphism $f\colon c \to c'$, we note that $f \in N(\mathcal{C})_1$, whereupon we find that the construction of the nerve has $NF(f) = Ff$ and $NG(f) = Gf$. (Again, one explicitly has that $(NF)_1$ is the composite functor $[1] \xrightarrow{f} \mathcal{C} \to \mathcal{D}$.)

- Full: fix a map $\varphi\colon N\mathcal{C} \to N\mathcal{D}$ of simplicial sets, and we need to go back and define a functor $F\colon \mathcal{C} \to \mathcal{D}$. As in the previous point, we see that $\varphi_0\colon (N\mathcal{C})_0 \to (N\mathcal{D})_0$ is a map of objects, so we define $Fc \coloneqq \varphi_0(c)$ for all $c \in \mathcal{C}$. Similarly, we see that $\varphi_1\colon (N\mathcal{C})_1 \to (N\mathcal{D})_1$ is a map of morphisms, so we define $Ff \coloneqq \varphi_1(f)$ for all morphisms $f\colon c \to c'$.

  It remains to check that these data assemble into a functor F and that $NF = \varphi$. For example, as in Exercise 1.58, the identity map $\mathrm{id}_c\colon c \to c$ is the image of c along the canonical map $N(\mathcal{C})_0 \to N(\mathcal{C})_1$. Because φ is a natural transformation, we see $\varphi_1(\mathrm{id}_c)$ must then be the image of $\varphi_0(c)$ along the canonical map $N(\mathcal{D})_0 \to N(\mathcal{D})_1$, which of course is $\mathrm{id}_{\varphi_0(c)}$; we conclude $F\mathrm{id}_c = \mathrm{id}_{Fc}$.

  Next, we need to check associativity. Choose maps $f_{01}\colon c_0 \to c_1$ and $f_{12}\colon c_1 \to c_2$, and we need to check $F(f_{12} \circ f_{01}) = Ff_{12} \circ Ff_{01}$; set $f_{02}\colon c_0 \to c_2$ for brevity. Well, consider the functor $G\colon [2] \to \mathcal{C}$ given by $c_0 \to c_1 \to c_2$ (as in Remark 1.59); this will go to some element $\varphi_2(G)\colon [2] \to \mathcal{D}$ which we would like to describe. This is a diagram of the form $\bullet \to \bullet \to \bullet$, and as described in Remark 1.60, we can extract out the ith object as $N(\mathcal{D})\varepsilon_i(\varphi_2(G))$ and the morphism between the ith and jth object as $N(\mathcal{D})\varepsilon_{ij}(\varphi_2(G))$. By naturality, we see that $N(\mathcal{D})\varepsilon_i(\varphi_2(G)) = \varphi_1(c_i) = Fc_i$ for each i. Similarly, we see and that $N(\mathcal{D})\varepsilon_{ij}(\varphi_2(G)) = \varphi(f_{ij}) = Ff_{ij}$ for each $i \le j$. In total, $\varphi_2(G)$ is a diagram of the form

$$\begin{tikzcd} Fc_0 \arrow[r, "Ff_{01}"] \arrow[dr, "Ff_{02}"'] & Fc_1 \arrow[d, "Ff_{12}"] \\ & Fc_2 \end{tikzcd}$$

  thereby completing the argument because the triangle must commute.

  Lastly, we have to check that $NF = \varphi$. But these are equal in degrees 0 and 1 by construction, so we are done by Lemma 1.61. ■

These horns allow us to state a special property of nerves.

Proposition 1.65. Fix a category $\mathcal{C}$. Then any map $\Lambda^n_i \to N(\mathcal{C})$ from an inner horn Λ^n_i extends uniquely to a map $\Delta^n \to N(\mathcal{C})$, as in the following diagram.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed, "!"] & \end{tikzcd}$$

Proof. If $n \geq 3$, then any map $[1] \to [n]$ avoids at least 2 elements, so $(\Lambda^n_i)_0 = (\Delta^n)_0$ and $(\Lambda^n_i)_1 = (\Delta^n)_1$. Thus, uniqueness follows from Lemma 1.61, and existence follows from Lemma 1.62. We note that the same argument even works for $n = 2$ because we are only worried about the inner horn Λ^2_1, which still has the morphisms $0 \to 1$ and $1 \to 2$. Lastly, there are no inner horns for $n \in \{0, 1\}$, so the statement is vacuous, and we are done. ■

Proposition 1.66. Fix a simplicial set N. Suppose that every map $\Lambda^n_i \to N$ from an inner horn extends uniquely to a map $\Delta^n \to N$. Then there is a category $\mathcal{C}$ such that $N \cong N(\mathcal{C})$.

Proof. We proceed in steps.

1. We define the objects and morphisms of the category $\mathcal{C}$. Indeed, simply take the objects to be N_0 and the collection of all morphisms to be N_1. More specifically, for any $x, y \in \mathcal{C}$, we use Remark 1.60 to motivate the definition

   $$\operatorname{Mor}_{\mathcal{C}}(x, y) := \{f \in N_1 : N\varepsilon_0(f) = x \text{ and } N\varepsilon_1(f) = y\}.$$

   For example, the canonical map $\sigma_0 \colon [1] \to [0]$ induces a special morphism $s_0(x)$ for each $x \in N_0$ such that $N\varepsilon_0(s_0(x)) = N\varepsilon_1(s_0(x)) = x$. Accordingly, we define $\operatorname{id}_x := s_0(x)$.

2. We define composition in the category of $\mathcal{C}$. Well, suppose we have two morphisms $f_{01} \colon c_0 \to c_1$ and $f_{12} \colon c_1 \to c_2$ which we would like to compose.

   The point is that these data assemble into a map $\Lambda^2_1 \to N$ which maps $i \mapsto c_i$ in degree 0 and maps $(i \to j)$ to f_{ij} (for $(i, j) \in \{(0, 1), (1, 2)\}$) in degree 1; this in fact gives a map $\Lambda^1_2 \to N$ by Example 1.57. Thus, the hypothesis on N gives us a unique $f_{12} \odot f_{01} \in N$ such that $d_2(f_{12} \odot f_{01}) = f_{01}$ and $d_0(f_{12} \odot f_{01}) = f_{12}$, so we define

   $$f_{12} \circ f_{01} := d_1(f_{12} \odot f_{01}).$$

   By construction, we see that $(f_{12} \circ f_{01}) \colon c_0 \to c_2$.

3. We check that composition is unital. Well, start with some map $f \colon c_0 \to c_1$, and we need to show that $f \circ \operatorname{id}_{c_0} = f$ and $\operatorname{id}_{c_1} \circ f = f$. We will content ourselves with showing that $f \circ \operatorname{id}_{c_0} = f$ because the argument is symmetric for the other identity. For this, we claim that $s_0(f) = (f \odot \operatorname{id}_{c_0})$, which will complete the proof because then $d_1(s_0(f)) = f$ by the simplicial identities, thereby implying that $f \circ \operatorname{id}_{c_0} = f$. Now, to check that $s_0(f) = (f \odot \operatorname{id}_{c_0})$, we use the uniqueness of our lifting: we must check that $d_2(s_0(f)) = \operatorname{id}_{c_0}$ and $d_0(s_0(f)) = f$, which both follow from the simplicial identities.

4. We check that composition is associative. This will be a little technical because it requires us to work with inner horns of Δ^3. We are given three morphisms $f_{01} \colon c_0 \to c_1$ and $f_{12} \colon c_1 \to c_2$ and $f_{23} \colon c_2 \to c_3$, and we want to check that $f_{23} \circ (f_{12} \circ f_{01}) = (f_{23} \circ f_{12}) \circ f_{01}$. We go ahead and set $f_{02} := f_{12} \circ f_{01}$ and $f_{13} := f_{23} \circ f_{12}$.

   Next, we build some 2-simplices. By hypothesis on N, it is enough to define a map from an inner horn Λ^3_i. As in Example 1.57, it is enough to provide three suitable elements of N_2. Well, for any $i < j < k$ in $\{0, 1, 2, 3\}$, we define the 2-simplex $f_{ijk} := f_{jk} \odot f_{ij}$, which we note is always well-defined because $\{k - j, j - i\} \subseteq \{1, 2\}$.

We now build a 3-simplex to do our bidding. We can build a map $\Lambda^3_1 \to N$ via Proposition 1.56. To start, we send $i \mapsto c_i$ for each i and $i \le j$ to f_{ij} for each i and j. Lastly, on 2-simplices, we will glue f_{012}, f_{013}, and f_{123}, which cohere in the colimit because we only have to check that the edges $0 \le 1$ and $1 \le 2$ and $1 \le 3$ and $2 \le 3$ agree in Λ^3_1. Thus, Λ^3_1 now extends to a unique 3-simplex $e_1 \in N_3$.

Now, on one hand, we note that $d_1(e_1)$ is a 2-simplex with $d_2(d_1(e_1)) = f_{23}$ and $d_0(d_1(e_1)) = f_{02}$, so $d_1(e_1) = f_{23} \odot f_{02}$, so $d_1(d_1(e_1)) = f_{23} \circ f_{02}$. In the same way, we calculate that $d_2(e_1) = f_{13} \odot f_{01}$, so $d_1(d_2(e_1)) = f_{13} \circ f_{01}$.

However, $(d_1 \circ d_1)$ and $(d_1 \circ d_2)$ are the same map $N_3 \to N_1$, so the claim follows.

5. We define a natural transformation $\eta\colon N \to N(\mathcal{C})$. Thus, for each n, we must define a map $N_n \to N(\mathcal{C})_n$, so each $x \in N_n$ must produce some functor $\eta_n(x)\colon [n] \Rightarrow \mathcal{C}$.

   Well, on objects, we should send $i \in [n]$ to $(x \circ \varepsilon_i) \in N_0$, and on morphisms, we should send $i \to j$ to the map $(x\varepsilon_{ij}) \in N_1$; note that $(x\varepsilon_{ij})\colon (x\varepsilon_i) \to (x\varepsilon_j)$ by construction of $\mathcal{C}$. The construction of the identities in $\mathcal{C}$ shows that $(x\varepsilon_{ii}) = s_i(x\varepsilon_i) = \mathrm{id}_{x\varepsilon_i}$. Lastly, given two maps $\varepsilon_{ij}\colon i \to j$ and $\varepsilon_{jk}\colon j \to k$, we need to check that $x\varepsilon_{jk} \circ x\varepsilon_{ij} = x\varepsilon_{ik}$. Well, there is a map $\varepsilon_{ijk}\colon \Delta^3 \to \Delta^n$ given by $i \le j \le k$, which we notice has

   $$d_2(x\varepsilon_{ijk}) = x\varepsilon_{ij} \qquad \text{and} \qquad d_2(x\varepsilon_{ijk}) = x\varepsilon_{jk}$$

   by an explicit calculation, so we see that $x\varepsilon_{ijk} = x\varepsilon_{jk} \odot x\varepsilon_{ik}$ and thus

   $$x\varepsilon_{jk} \circ x\varepsilon_{ij} = d_1(x\varepsilon_{ijk}),$$

   which is $x\varepsilon_{ik}$, as required.

   We now must check that the map $\eta_n\colon N_n \to N(\mathcal{C})_n$ is natural. Well, for any increasing map $f\colon [m] \to [n]$ and $x \in N_n$, we need to check that $\eta_m(f(x)) = f(\eta_n(x))$ in $N(\mathcal{C})_m$. As explained in Remark 1.59, the data of a functor $[m] \to N(\mathcal{C})_m$ amounts to the data in degrees 0 and 1, so we should check the equalities against ε_is and ε_{ij}s. Unwinding all the abuse of notation, we see that $\eta_m(f(x))$ on the morphism ε_{ij} is $Nf(x) \circ \varepsilon_{ij}$, which is $x \circ f \circ \varepsilon_{ij}$ or $x\varepsilon_{f(i),f(j)}$. On the other hand, $f(\eta_n(x))$ on $i \le j$ is $\eta_n(x)$ on $f(i) \le f(j)$ (by the functoriality described in Remark 1.59), which is $x\varepsilon_{f(i),f(j)}$ again.

6. We show that η is an isomorphism of simplicial sets. Because we have already described a morphism of simplicial sets, we must merely check that η_n is a bijection for each n. We will do this by induction, where the cases $n \in \{0,1\}$ have no content. For $n \ge 2$, we may choose some i with $0 < i < n$, and then we see that the inclusion defines a natural map

   $$(-)_n = \mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, -) \to \mathrm{Mor}_{\mathrm{sSet}}\left(\Lambda^n_i, -\right),$$

   which is an isomorphism for both N and $N(\mathcal{C})$ by the uniqueness of the lifting. Now, Proposition 1.56 (combined with Example 1.53) shows that Λ^n_i is the colimit of some functor $F\colon \mathcal{I} \to \mathrm{sSet}$, where $F(i) = \Delta^{k_i}$ for some $k_i < n$ for each i. Thus, we see that there is a natural map

   $$(-)_n \to \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathrm{sSet}}(\Delta^{k_i}, -) = \lim_{i \in \mathcal{I}} (-)_{k_i},$$

   which is still an isomorphism for both N and $N(\mathcal{C})$. Thus, the map $\eta\colon N \to N(\mathcal{C})$ fits into a commuting diagram

   $$\begin{tikzcd} N_n \arrow[r] \arrow[d, "\eta_n"'] & \lim\limits_{i \in \mathcal{I}} N_{k_i} \arrow[d, "\lim \eta"] \\ N(\mathcal{C})_n \arrow[r] & \lim\limits_{i \in \mathcal{I}} N(\mathcal{C})_{k_i} \end{tikzcd}$$

   where we know that the horizontal arrows are isomorphisms as described above. But the right arrow is an isomorphism by the inductive hypothesis, so the left arrow is as well, so we are done. ■

Remark 1.67. In fact, a simplicial set is the nerve of a category if and only if it satisfies the conclusion of Proposition 1.65. Thus, we have a characterization of the image of the fully faithful nerve functor! Amusingly, this allows one to give an alternate definition of a category in terms of simplicial sets; this is not circular because one can define simplicial sets as combinatorial simplicial sets.

Remark 1.68. One can check that a category is a groupoid if and only if the outer horns also admit horn fillings. The point is that being a groupoid allows one to reverse all the arrows, so coherence of composition allows one to do the filling.

1.2.4 More on Sing

We now turn to Sing.

Proposition 1.69. The functor $\mathrm{Sing}\colon \mathrm{sSet} \to \mathrm{Top}$ admits a left adjoint $|\cdot| : \mathrm{Top} \to \mathrm{sSet}$. In fact, $|\Delta^n|$ is defined to be the topological n-simplex.

It is worthwhile to know how to construct adjoints.

Theorem 1.70. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories, where $\mathcal{D}$ admits colimits. For any functor $F\colon \mathcal{C} \to \mathcal{D}$, there is a unique functor $G\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{D}$ preserving colimits for which the composite

$$\mathcal{C} \overset{よ}{\to} \mathrm{PSh}(\mathcal{C}) \overset{G}{\to} \mathcal{D}.$$

In fact, G is a left adjoint.

Remark 1.71. This property characterizes Sing: indeed, for any topological space Y, we need to have $\mathrm{Sing}(Y)(n)$ to be

$$\mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, \mathrm{Sing}(Y)) = \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, Y).$$

We are now able to characterize the image of Sing.

Proposition 1.72. Fix a topological space Y. Then any map $\Lambda^n_i \to \mathrm{Sing}\, Y$ admits a lift to a map $\Delta^n \to \mathrm{Sing}\, Y$.

Proof. By the adjunction, it is enough to lift a map $|\Lambda^n_i| \to Y$ to a map $|\Delta^n| \to Y$. But this is not hard because there are projection maps $|\Delta^n| \to |\Lambda^n_i|$. ■

1.2.5 Kan Complexes

Proposition 1.72 motivates the following definition.

Definition 1.73 (Kan complex)**.** A *Kan complex* is a simplicial set X in which every $\Lambda^n_i \to X$ admits a lift to a map $\Delta^n \to X$.

Example 1.74. By Proposition 1.72, we see that $\mathrm{Sing}\, Y$ is always a Kan complex.

Example 1.75. By Remark 1.68, we see that $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$

At long last, we may define ∞-categories, which is intended to simultaneously generalize nerves and Kan complexes.

Definition 1.76 (∞-category, quasicategory)**.** An ∞-*category* or *quasicategory* is a simplicial set X where every inner horn $\Lambda_i^n \to X$ admits a lift to $\Delta^n \to X$. We may call X_0 the *objects*, call X_1 the *morphisms*, and call X_n the n-morphisms for $n \geq 1$. More concretely, for any $E \in \mathcal{C}_2$, we may say that $d_1 E$ exhibits a 2-*isomorphism* between $d_0 E$ and $d_2 E$.

Definition 1.77 (homotopic)**.** Two maps $f, g\colon X \to Y$ are *homotopic* if and only if there is a map $h\colon X \times \Delta^1 \to Y$ such that the composites with $d_0\colon X \times \Delta^0 \to X \times \Delta^1$ and $d_1\colon X \times \Delta^0 \to X \times \Delta^1$ are g and f, respectively.

Remark 1.78. It turns out that being homotopic is an equivalence relation; the symmetry check uses the fact that Y is a Kan complex.

Definition 1.79 (homotopy equivalent)**.** Two Kan complexes X and Y are homotopy equivalent if and only if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identities.

We will make use of the following hard(!) theorem.

Theorem 1.80 (Quillen)**.** If X is a CW complex, then $|\operatorname{Sing} X|$ is homotopy equivalent to X. Similarly, if X is a Kan complex, then $\operatorname{Sing}|X|$ is homotopy equivalent to X.

Corollary 1.81. The homotopy category of topological spaces is equivalent to the homotopy category of Kan complexes.

This theorem is a purely motivational statement: it allows us to pass from topological spaces to just Kan complexes.

1.3 September 11

We continue discussing our quasicategories.

1.3.1 More on Kan Complexes

Let's say a little more about Kan complexes.

Remark 1.82. Fix a Kan complex C and a morphism $f\colon c \to c'$ in C_1. Then one can construct an inverse for f as follows: the outer horn

$$\begin{tikzcd} & c' & \\ c \arrow[ur, "f"] \arrow[rr, equal] & & c \end{tikzcd}$$

to C must fill out to a map $\Delta^2 \to C$, which provides us with the data of some map $g\colon B \to A$. Then the composite $g \circ f$ can be seen to be homotopic to the identity via the map $\Delta^2 \to C$.

Here is a more general notion.

Definition 1.83 (Kan fibration)**.** A morphism $X \to Y$ of simplicial sets is a *Kan fibration* if and only if, for any n and $i \in [n]$, any two maps $\Lambda^n_i \to X$ and $\Delta^n \to Y$ admits a lifting map $\Delta^n \to X$ making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Example 1.84. A simplicial set X has a canonical map to the terminal object Δ^0, so the map $X \to \Delta^0$ is a Kan fibration if and only if X is a Kan complex.

For today, we will be interested in what sorts of maps $A \to B$ admit lifts against Kan fibrations. Of course, this includes the horn inclusions $\Lambda^n_i \to \Delta^n$, but there are more ways to generate such morphisms.

Definition 1.85 (saturated)**.** A nonempty class Σ of morphisms of simplicial sets is *saturated* if and only if it is closed under pushouts, retracts, coproducts, composition, and transfinite composition.

We will explain the terms in this definition shortly, but let's start by explaining why we should care.

Definition 1.86 (anodyne)**.** The smallest saturated class of maps containing all horn inclusions is the class of *anodyne* maps.

Proposition 1.87. Suppose that a map $f\colon A \to B$ is anodyne. Then any Kan fibration $g\colon X \to Y$ fitting into the solid square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

will admit a dashed map $B \to X$ making the diagram commute.

Remark 1.88. It turns out that a morphism $A \to B$ is anodyne if and only if it is monic and the induced map $|A| \to |B|$ is a homotopy equivalence. We will not prove this, so we will not use it.

Let's now explain what it means for a class $\mathcal{P}$ of morphisms to be saturated.

Definition 1.89. Let $\mathcal{P}$ be a class of morphisms in a category. If $f\colon A \to B$ is in $\mathcal{P}$, and we are given any map $A \to A'$ living in a pushout diagram as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

Then $\mathcal{P}$ is *closed under pushouts* if we always have $f' \in \mathcal{P}$ for all such diagrams.

Example 1.90. We claim that the class of anodyne maps is closed under pushouts. Indeed, given any map $A \to A'$ producing a pushout map $f' : A' \to B'$, we would like to know if we can always fill in for the dashed arrow.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "f'"] & X \arrow[d] \\ B \arrow[r] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Well, we are granted a map $B \to X$ making the diagram commute, so there is a map $B' \to X$ making the diagram commute because B' is a pushout.

Definition 1.91 (retract)**.** A map $f' : A' \to B'$ is a *retract* of a map $f : A \to B$ if and only if it fits into a diagram as follows.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] \arrow[rr, bend left, equal] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B' \arrow[r] \arrow[rr, bend right, equal] & B \arrow[r] & B' \end{tikzcd}$$

Definition 1.92 (coproduct)**.** Given a collection of maps $f_i : A_i \to B_i$ (as i varies over an index set I), then the coproduct map is defined as the induced map

$$\coprod_{i \in I} f_i : \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i.$$

Definition 1.93 (transfinite composition)**.** Suppose that there is a diagram

$$A_0 \to A_1 \to A_2 \to \cdots$$

of maps. Then the transfinite composition is the colimit of the diagram

$$A_0 \to \operatorname*{colim}_i A_i.$$

One can check that the class of anodyne maps has all the above closure properties.

Here is are a few more useful facts about anodyne maps.

Remark 1.94. All anodyne maps are monomorphisms.

Next, we will want to prove that anodyne maps are closed under products. For this, we pick up the following definition.

Definition 1.95 (pushout product)**.** Given two maps $f : A \to B$ and $g : C \to D$ in a category $\mathcal{C}$, the *pushout product* is the induced map

$$(B \times C) \coprod_{A \times C} (A \times D) \to (B \times D).$$

We may write this map as $f \boxtimes g$.

Example 1.96. The pushout product of a map $f : A \to B$ and the initial map $\varnothing \to C$ is the induced map

$$A \times C \to B \times C.$$

Remark 1.97. Pushout products are associative in a suitable way, which we will not prove in general.

Theorem 1.98. Fix a monomorphism $f\colon A \to B$ of simplicial sets and an anodyne map $g\colon C \to D$. Then the pushout product $f \boxtimes g$ is anodyne.

The rest of this class will be spent proving this result.

Remark 1.99 (Jeremy Hahn)**.** If you try to visualize this with geometric realizations, then you will recover some exercise that is in Hatcher somewhere.

Let's give the main claim for Theorem 1.98.

Lemma 1.100. The class of anodyne maps is the smallest saturated class containing all maps of the form $f \boxtimes i$, where $f\colon A \to B$ is monic, and $i\colon \Delta^0 \to \Delta^1$ is some map.

Proof of Theorem 1.98 from Lemma 1.100. Fix a monomorphism i, and we consider the class $\mathcal{P}$ of maps j such that $i \boxtimes j$ is anodyne. We want to check that $\mathcal{P}$ contains all anodyne maps. One can check that this class is saturated, so by Lemma 1.100, it remains to prove that $\mathcal{P}$ contains the maps $i \boxtimes (j \boxtimes k)$ where $j\colon A \to B$ is monic, and $k\colon \Delta^0 \to \Delta^1$ is some map. Well,

$$i \boxtimes (j \boxtimes k) = (i \boxtimes j) \boxtimes k,$$

and $i \boxtimes j$ continues to be monic, so this is anodyne by Lemma 1.100! ■

It remains to prove Lemma 1.100. Let $\mathcal{P}$ be the saturated class of maps in the statement, and we would like to show that it coincides with the saturated class of anodyne maps. This will be done via two inclusions, so it suffices to show that each class contains the other's generators.

Lemma 1.101. Let $\mathcal{P}$ be the smallest saturated class containing all maps of the form $f \boxtimes i$, where $f\colon A \to B$ is monic, and $i\colon \Delta^0 \to \Delta^1$ is some map. Then every anodyne map is in $\mathcal{P}$.

Proof. It is enough to show that the horn inclusions $\Lambda^n_i \to \Delta^n$ are in $\mathcal{P}$, which amounts to some explicit combinatorics. By symmetry, we may assume that $i < n$. We now claim that the diagram

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, "i"'] & (\Lambda^n_i \times \Delta^1) \coprod_{\Lambda^n_i \times \Delta^0} (\Delta^n \times \Delta^0) \arrow[r] \arrow[d, "i \boxtimes \varepsilon_0"'] & \Lambda^n_i \arrow[d] \\
\Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n
\end{tikzcd}$$

is a retract diagram, which will complete the proof. Because the vertical maps are all monic, we only have to construct the bottom maps, and then one needs to check that one can induce the top maps accordingly. Well, the first map $\Delta^n \to \Delta^n \times \Delta^1$ is just $\mathrm{id}_{\Delta^n} \times \varepsilon_1$. The second map requires some more work to define. In a picture, it is defined via the n-simplex

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \cdots \arrow[r] & i-1 \arrow[r] \arrow[d] & i \arrow[r] \arrow[d] & i \arrow[r] \arrow[d] & \cdots \arrow[r] & i \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & i-1 \arrow[r] & i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & n
\end{tikzcd}$$

of $\Delta^n \times \Delta^1$. ■

For the second inclusion, we will use the following result.

Proposition 1.102. Suppose $A \to B$ is a monomorphism of simplicial sets. Then there is a canonical sequence of morphisms

$$A \to A_0 \to A_1 \to \cdots$$

with $A = A_{-1}$ such that B is the colimit of this diagram, and there are pushout squares as follows.

$$\begin{tikzcd} \coprod_{I_i} \partial\Delta^i \arrow[r] \arrow[d] & A_{i-1} \arrow[d] \\ \coprod_{I_i} \Delta^i \arrow[r] & A_i \end{tikzcd}$$

Proof. Intuitively, we are attaching i-simplices to A_{i-1} to produce B. More precisely, one sets A_i to be the smallest simplicial subset of B containing A and for which the maps $A_i(j) \to B(j)$ are isomorphisms for $j \leq i$. ■

Lemma 1.103. Let $\mathcal{P}$ be the smallest saturated class containing all maps of the form $f \boxtimes i$, where $f\colon A \to B$ is monic, and $i\colon \Delta^0 \to \Delta^1$ is some map. Then every map in $\mathcal{P}$ is anodyne.

Proof. Given any monomorphism $f\colon A \to B$ and map $i\colon \Delta^0 \to \Delta^1$, we must check that $f \boxtimes i$ is anodyne. By Proposition 1.102, we reduce to the case where f is the inclusion $\partial\Delta^n \to \Delta^n$. This is rather complicated, so we will content ourselves with $n = 1$. Here, we are looking at the following map.

$$(\partial\Delta^1 \times \Delta^1) \coprod_{\partial\Delta^1 \times \Delta^0} (\Delta^1 \times \Delta^0) \to (\Delta^1 \times \Delta^1).$$

The right-hand side is a square, and the left-hand side is the following boundary.

$$\begin{tikzcd} \bullet & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

To show that this is anodyne, we note that we can fill in the lower-right triangle via a pushout against $\Lambda^2_1 \to \Delta^2$, and then we can fill in the upper-left triangle by pushing out against $\Lambda^2_0 \to \Delta^2$. Thus, our inclusion is anodyne. ■

This concludes the proof of Lemma 1.100 and thus the proof of Theorem 1.98.

1.3.2 Internal Mor

Let's explain an application.

Definition 1.104. Fix simplicial sets X and Y. Then we define the simplicial set $\underline{\mathrm{Mor}}(X, Y)$ as having n-simplices

$$\underline{\mathrm{Mor}}(X, Y)_n := \mathrm{Mor}_{\mathrm{sSet}}(X \times \Delta^n, Y).$$

Remark 1.105. This gives sSet "internal Mors," making it a Cartesian closed category. Namely, one can see that we have a natural isomorphism

$$\mathrm{Mor}_{\mathrm{sSet}}(A, \underline{\mathrm{Mor}}(B, C)) \simeq \mathrm{Mor}_{\mathrm{sSet}}(A \times B, C).$$

To motivate our definition, we note that requiring this would require

$$\underline{\mathrm{Mor}}(X, Y)_n = \mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, \underline{\mathrm{Mor}}(X, Y)) = \mathrm{Mor}_{\mathrm{sSet}}(\Delta^n \times X, Y),$$

which is the given definition up to commutativity of the product.

Proposition 1.106. Fix simplicial sets X and Y such that Y is a Kan complex. Then $\underline{\mathrm{Mor}}(X, Y)$ is a Kan complex.

Proof. Given any map $\Lambda^n_i \to \underline{\mathrm{Mor}}(X, Y)$, we need to exhibit a lifted map $\Delta^n \to \underline{\mathrm{Mor}}(X, Y)$. By Remark 1.105, we need to exhibit a lift in the following diagram.

$$\begin{tikzcd} \Lambda^n_i \times X \arrow[r] \arrow[d] & Y \\ \Delta^n \times X \arrow[ur] & \end{tikzcd}$$

However, the left-hand map is anodyne by Theorem 1.98, so we are done! ■

Remark 1.107. This is an incarnation of the fact that the category of Kan complexes is much nicer than Top: we have internal Mors, which is much harder to come by for topological spaces.

Let's keep working with Kan complexes.

Definition 1.108 (isomorphism)**.** Two objects in a Kan complex are *isomorphic* if and only if there is a map between them.

Definition 1.109 (equivalence)**.** A map $f: X \to Y$ of Kan complexes is a *homotopy equivalence* if and only if there is a map $g: Y \to X$ such that $f \circ g$ is isomorphic to id_Y in $\underline{\mathrm{Mor}}(Y, Y)$ and that $g \circ f$ is isomorphic to id_X in $\underline{\mathrm{Mor}}(X, X)$.

Definition 1.110. Fix a Kan complex X. Then $\pi_0(X)$ is the set of isomorphism classes of objects in X.

Appendix A
Category Theory

In this appendix, we review some category theory as fast as possible.

Warning A.1. We will mostly ignore size issues. If it makes the reader feel better, we are willing to assume the existence of a countable ascending chain of inaccessible cardinals throughout the class.

A.1 Basic Definitions

Let's recall some starting notions of category theory.

Definition A.2 (category)**.** A *category* $\mathcal{C}$ is a collection of objects, a collection of morphisms $\operatorname{Mor}_{\mathcal{C}}(A, B)$ for each pair of objects, a distinguished identity element id_A in $\operatorname{Mor}_{\mathcal{C}}(A, A)$, and a composition law

$$\circ\colon \operatorname{Mor}_{\mathcal{C}}(B, C) \times \operatorname{Mor}_{\mathcal{C}}(A, B) \to \operatorname{Mor}_{\mathcal{C}}(A, C).$$

We then require the composition law to be associative and unital with respect to the identity maps.

Remark A.3. We will use the notation Hom for Mor whenever the category $\mathcal{C}$ is additive, meaning that these collections of morphisms are abelian groups, and the composition law is $\mathbb{Z}$-bilinear.

The following two examples are most important for this class.

Example A.4. There is a category Set of all sets. The morphisms are given by functions of sets, and the identity maps are compositions are all as usual. The fact that function composition is associative and unital implies that Set succeeds at being a category.

Example A.5. We let Δ to be the category whose objects are the nonnegative integers n, written $[n] \in \Delta$, where $\operatorname{Mor}_{\Delta}([m], [n])$ consists of the increasing maps $\{0, \ldots, m\} \to \{0, \ldots, n\}$. Composition of functions and identities are defined as for Set, allowing one to check that Δ is a category in exactly the same way.

The following example provides a useful technical tool.

Example A.6. Given a category $\mathcal{C}$, there is an opposite category $\mathcal{C}^{\mathrm{op}}$ whose objects are the same, but the morphisms are given by

$$\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(c, c') := \mathrm{Mor}_{\mathcal{C}}(c', c).$$

We may write $f^{\mathrm{op}}\colon c' \to c$ for the morphism corresponding to $f\colon c \to c'$. Identities remain the same, but composition is defined by $f^{\mathrm{op}} \circ (f') = (f' \circ f)^{\mathrm{op}}$. The fact that $\mathcal{C}$ is a category makes $\mathcal{C}^{\mathrm{op}}$ into a category.

Definition A.7 (isomorphism)**.** A morphism $f\colon A \to B$ in a category $\mathcal{C}$ is an *isomorphism* if and only if there is a morphism $g\colon B \to A$ for which $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Definition A.8 (groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

It is worthwhile to have maps between categories as well.

Definition A.9 (functor)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the data of a map on the level of objects and maps

$$F\colon \mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathcal{D}}(Fc, Fc')$$

for any $c, c' \in \mathcal{C}$. Furthermore, we require $F\mathrm{id}_c = F\,\mathrm{id}_{Fc}$ and $F(f' \circ f) = Ff' \circ Ff$ for any $c \in \mathcal{C}$ and compose-able f and f'.

Here are some adjectives that a functor can have.

Definition A.10 (isomorphism)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *isomorphism* if and only if there is an inverse functor $G\colon \mathcal{D} \to \mathcal{C}$ for which FG and GF are both the identity functors.

Definition A.11 (full, faithful)**.** Fix a functor $F\colon \mathcal{C} \to \mathcal{D}$ and consider the maps

$$F\colon \mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathcal{D}}(Fc, Fc')$$

for any $c, c' \in \mathcal{C}$. We say that F is *full* if and only if these maps are always surjective, and we say that F is *faithful* if and only if these maps are always injective.

A.1.1 Natural Transformations

We are shortly going to get a lot of mileage out of the next example, so we spend some time to prove it in detail. We would like to define a category of functors between two given categories,[1] but this requires us to have a notion of morphism between functors.

Definition A.12 (natural transformation)**.** Given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta\colon F \Rightarrow G$ is the data of a morphism $\eta_A\colon FA \to GA$ for each object $A \in \mathcal{C}$. We further require that $Gf \circ \eta_A = \eta_B \circ Ff$ for any morphism $f\colon A \to B$. A *natural isomorphism* is a natural transformation η in which each morphism η_A is an isomorphism.

Diagrammatically, the equation $Gf \circ \eta_A = \eta_B \circ Ff$ amounts to the commutativity of the following square.

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"] & GB \end{tikzcd}$$

Anyway, here is our result.

[1] For those who are choosing to think about size issues, we remark that we will typically have one of $\mathcal{C}$ or $\mathcal{D}$ be locally small.

Lemma A.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then there is a functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ where the objects are functors $\mathcal{C} \to \mathcal{D}$ and the morphisms are natural transformations.

Proof. We have explained our objects and morphisms, but we still have to provide identities and composition laws and check that everything works.

- Identities: given a functor $F\colon \mathcal{C} \to \mathcal{D}$, there is an identity natural transformation $\mathrm{id}_F\colon F \to F$ given by $(\mathrm{id}_F)_A \coloneqq \mathrm{id}_{FA}$; checking that this is a natural transformation amounts to noting that $Ff \circ \mathrm{id}_{FA} = \mathrm{id}_{FB} \circ Ff$ for any morphism $f\colon A \to B$.

- Composition: given two natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, we define the composite natural transformation $(\beta \circ \alpha)\colon F \Rightarrow H$ by $(\beta \circ \alpha)_A \coloneqq \beta_A \circ \alpha_A$ for each $A \in \mathcal{A}$. Checking that this is a natural transformation amounts to checking the commutativity of the outer rectangle of

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"'] \arrow[rr, bend left, "(\beta\circ\alpha)_A"] \arrow[d, "Ff"'] & GA \arrow[r, "\beta_A"'] \arrow[d, "Gf"'] & HA \arrow[d, "Hf"] \\ FB \arrow[r, "\alpha_B"] \arrow[rr, bend right, "(\beta\circ\alpha)_B"'] & GB \arrow[r, "\beta_B"] & HB \end{tikzcd}$$

  which indeed commutes: the top and bottom triangles commute by definition of $\beta \circ \alpha$, and the two inner squares commute by naturality of α and β.

- Identities: given a natural transformation $\eta\colon F \Rightarrow G$, we need to check that $\mathrm{id}_G \circ \eta = \eta \circ \mathrm{id}_F = \eta$. Well, for any object A, we see that

$$(\mathrm{id}_G \circ \eta)_A = (\mathrm{id}_G \circ \eta)_A = \mathrm{id}_{G(A)} \circ \eta_A = \eta_A,$$

  and

$$(\eta \circ \mathrm{id}_F)_A = \eta_A \circ \mathrm{id}_{FA} = \eta_A.$$

- Associativity: given natural transformations α, β, and γ with appropriate domains and codomains, we must check that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$. Well, for any object A, we see that

$$((\alpha \circ \beta) \circ \gamma) = (\alpha_A \circ \beta_A) \circ \gamma_A = \alpha_A \circ (\beta_A \circ \gamma_A) = (\alpha \circ (\beta \circ \gamma))_A,$$

  as required. ∎

Example A.14. For any $c \in \mathcal{C}$, there is a functor $\mathrm{ev}_c\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ given by $\mathrm{ev}_c(F) \coloneqq Fc$ on objects. On morphisms, we send a natural transformation $\eta\colon F \Rightarrow F'$ to the morphism $\mathrm{ev}_c(\eta) \coloneqq \eta_c$. For example, $\mathrm{ev}_c(\mathrm{id}_F) = (\mathrm{id}_F)_c = \mathrm{id}_{Fc}$. Lastly, to check functoriality, we pick up two natural transformations $\eta\colon F \Rightarrow F'$ and $\eta'\colon F' \Rightarrow F''$, and we note that $\mathrm{ev}_c(\eta'\eta) = (\eta'\eta)_c = \eta'_c\eta_c = \mathrm{ev}_c(\eta')\,\mathrm{ev}_c(\eta)$.

A.2 The Yoneda Lemma

We are going to get some mileage out of using presheaf categories.

Definition A.15 (presheaf)**.** Fix a category $\mathcal{C}$. Then a *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Accordingly, the presheaf category $\mathrm{PSh}(\mathcal{C})$ is the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

Remark A.16. It will be worthwhile to have a way to build "subpresheaves." Given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, suppose that we have a collection of sets $\{Gc\}_{c\in\mathcal{C}}$ such that $Gc \subseteq Fc$ for each $c \in \mathcal{C}$. Furthermore, suppose that each map $f\colon c \to c'$ has $Ff(Gc) \subseteq Gc'$. Then we can define $Gf \coloneqq Ff|_{Gc}$, and it follows that G is a functor because F is a functor, and the inclusions $Gc \subseteq Fc$ now assemble into a natural transformation.

Here is our main result.

Theorem A.17 (Yoneda lemma)**.** Fix a category $\mathcal{C}$. Then there is a functor $\text{よ}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ which is defined on objects by

$$\text{よ}_c \coloneqq \mathrm{Mor}_{\mathcal{C}}(-,c).$$

Furthermore, よ is fully faithful.

Proof. This is purely formal. We proceed with our checks in sequence.

- For $c \in \mathcal{C}$, we show that $\text{よ}_c\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a functor. To review, on morphisms $g\colon d \to d'$, we define $\text{よ}(g)\colon \mathrm{Mor}_{\mathcal{C}}(d',c) \to \mathrm{Mor}_{\mathcal{C}}(d,c)$ by $\text{よ}_c(g) \coloneqq (-\circ g)$. For example, $\text{よ}_c(\mathrm{id}_d) = (-\circ \mathrm{id}_d)$, which is just the identity map. Lastly, for maps $g\colon d \to d'$ and $g'\colon d' \to d''$, we see that $\text{よ}_c(g'\circ g)$ and $\text{よ}_c(f)\circ \text{よ}_c(f')$ both equal

  $$(-\circ(g'\circ g)) = (-\circ g)\circ(-\circ g')$$

  by the associativity of composition.

- We construct よ on morphisms. Given $f\colon c \to c'$, we need a natural transformation $\text{よ}_f\colon \text{よ}_c \Rightarrow \text{よ}_{c'}$. Well, for any object $d \in \mathcal{D}$, we define the component map $(\text{よ}_f)_d\colon \mathrm{Mor}_{\mathcal{C}}(d,c) \to \mathrm{Mor}_{\mathcal{C}}(d,c')$ by $(f\circ -)$. To see that よ_f assembles into a natural transformation, we see that any morphism $g\colon d \to d'$ makes the following diagram commute.

$$\begin{tikzcd} \text{よ}_c(d') \arrow[r, "\text{よ}_c(g)"] \arrow[d, "(\text{よ}_f)_{d'}"'] & \text{よ}_c(d) \arrow[d, "(\text{よ}_f)_d"] \\ \text{よ}_{c'}(d') \arrow[r, "\text{よ}_{c'}(g)"] & \text{よ}_c(d') \end{tikzcd} \qquad \begin{tikzcd} h \arrow[r, mapsto] \arrow[d, mapsto] & h\circ g \arrow[d, mapsto] \\ f\circ h \arrow[r, mapsto] & f\circ h\circ g \end{tikzcd}$$

- We show that よ is a functor. For example, よ_{id_c} is the identity natural transformation because $(\text{よ}_{\mathrm{id}_c})_d$ is just $(\mathrm{id}_c\circ -)$ for each $d \in \mathcal{C}$, which is in fact the identity. Continuing, for any $f\colon c \to c'$ and $f'\colon c' \to c''$, we see that any object $d \in \mathcal{C}$ makes both $(\text{よ}_{f'\circ f})_d$ and $(\text{よ}_{f'}\circ \text{よ}_f)_d$ equal to

  $$((f'\circ f)-) = (f'\circ -)\circ(f\circ -)$$

  by associativity.

- Fix any functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. We define an injective map $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(\text{よ}_c, F) \to Fc$. To define our map, we send a natural transformation $\eta\colon \text{よ}_c \Rightarrow F$ to $\eta_c(\mathrm{id}_c) \in Fc$. Let's check that this is injective: if η and η' have $\eta_c(\mathrm{id}_c) = \eta'_c(\mathrm{id}_c)$, then we need to show that $\eta_d(h) = \eta'_d(h)$ for any $d \in \mathcal{C}$ and $h\colon d \to c$. Well, the commutativity of the diagram

$$\begin{tikzcd} \text{よ}_c(c) \arrow[r, "\eta_c"] \arrow[d, "\text{よ}_c(h)"'] & Fc \arrow[d, "Fh"] \\ \text{よ}_c(d) \arrow[r, "\eta_d"] & Fd \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_c \arrow[r, mapsto] \arrow[d, mapsto] & \eta_c(\mathrm{id}_c) \arrow[d, mapsto] \\ h \arrow[r, mapsto, dashed] & Fh(\eta_c(\mathrm{id}_c)) \end{tikzcd}$$

  reveals that $\eta_d(h) = Fh(\eta_c(\mathrm{id}_c))$. A similar argument holds for η', so we conclude that $\eta_d(h) = \eta'_d(h)$.

- We show that the map of the previous point is in fact surjective. The end of the argument informs our construction: for any $x \in Fc$, we define $\eta\colon よ_c \Rightarrow F$ by $\eta_d(h) := Fh(x)$ for any $d \in \mathcal{C}$ and $h\colon d \to c$. This η of course satisfies $\eta_c(\mathrm{id}_c) = x$, so it only remains to check that η is actually a natural transformation. Well, for any morphism $g\colon d \to d'$, we see that the following diagram commutes.

$$\begin{tikzcd} よ_c(d') \arrow[r, "\eta_d"] \arrow[d, "よ_c(g)"'] & Fd' \arrow[d, "Fg"] \\ よ_c(d) \arrow[r, "\eta_{d'}"] & Fd \end{tikzcd} \qquad \begin{tikzcd} h \arrow[r, mapsto] \arrow[d, mapsto] & Fh(x) \arrow[d, mapsto] \\ h \circ g \arrow[r, mapsto] & Fg(Fh(x)) \end{tikzcd}$$

- We complete the proof. We need to show that よ gives bijections $よ\colon \operatorname{Mor}_{\mathcal{C}}(c, c') \to \operatorname{Mor}_{\mathrm{PSh}}(よ_c, よ_{c'})$. Well, the two previous points shows that the target is in bijection with $よ_{c'}(c)$ via $\eta \mapsto (よ_{c'})_c(\mathrm{id}_c)$. But the total composite
$$\operatorname{Mor}_{\mathcal{C}}(c, c') \to \operatorname{Mor}_{\mathrm{PSh}}(よ_c, よ_{c'}) \to よ_{c'}(c)$$
sends a map $f\colon c \to c'$ to the natural transformation $(f \circ -)$ and then back to the map f. We conclude that よ is the inverse bijection for the map of the previous two points. ■

Remark A.18. It is worth noting that the previous point provides us with a bijection
$$\operatorname{Mor}_{\mathrm{PSh}}(よ_c, F) \to Fc$$
by $\eta \mapsto \eta_c(\mathrm{id}_c)$. In fact, we also exhibited an inverse map by sending $x \in Fc$ to the natural transformation η defined by $\eta_d(h) := Fh(x)$ for any $h\colon d \to c$.

Motivated by algebraic geometry, one has the following definition.

Definition A.19 (representable)**.** A presheaf $\mathcal{F}$ on a category $\mathcal{C}$ is *representable* if and only if there is an object $c \in \mathcal{C}$ for which $\mathcal{F}$ is isomorphic to $よ(c)$.

A.3 Limits and Colimits

Some objects in categories can be characterized by their special properties. Limits and colimits provide a convenient language for this.

Definition A.20 (limit)**.** Fix a functor $F\colon \mathcal{I} \to \mathcal{C}$. An object $c \in \mathcal{C}$ is a *limit* of F if and only if c is "universal" with respect to having morphisms $Fi \to c$ for each $i \in \mathcal{I}$ making the diagrams

$$\begin{tikzcd} c \arrow[r] \arrow[dr] & Fi \arrow[d] \\ & Fi' \end{tikzcd}$$

commute for any $i \to i'$. In other words, for any other object c' equipped with such morphisms, there is a unique map $c' \to c$ commuting with these morphisms. A *colimit* of F is the same notion, but the maps from F go into c instead of out of c.

Example A.21 (product)**.** A product is a limit of a functor $F\colon \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ has no non-identity morphisms. The colimit is called the coproduct.

Example A.22 (equalizer)**.** An equalizer is a limit of a functor $F\colon \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ has exactly two objects and one morphism between them. The colimit is called the co-equalizer.

Lemma A.23. Fix a functor $F\colon \mathcal{I} \to \mathcal{C}$. Then any two limits of F are isomorphic.

Proof. Let c and c' be limits of F. Then there are unique maps $c \to c'$ and $c' \to c$ commuting with the morphisms from $F(\mathcal{I})$ by definition of a limit. But then there is also a unique morphism $c \to c$ commuting with these morphisms, which must be the identity, so the composite

$$c \to c' \to c$$

must also be the identity. Similarly, we see that $c' \to c \to c'$ is the identity, thereby completing the proof. ■

The above lemma allows us to give some notation for our limits: we may write the limit of F as $\lim_{\mathcal{I}} F$ (when it exists!) and the colimit as $\operatorname{colim}_{\mathcal{I}} F$.

Remark A.24. Lemma A.23 also holds for colimits, which we can see by passing to opposite categories.

Example A.25. The category Set admits all limits and colimits of functors $F\colon \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ has at most κ many objects and morphisms for some cardinal κ.

Proof. We explicitly construct the limit and colimit of such a functor F.

- We handle the limit. Define the set

$$L \coloneqq \left\{ (x_i) \in \prod_{i \in \mathcal{I}} Fi : Ff(x_i) = x_j \text{ for all } f\colon i \to j \right\}.$$

  We claim that L is the limit. Because L is a subset of the product, there are projection maps $\mathrm{pr}_i\colon L \to Fi$ for each $i \in \mathcal{I}$. Furthermore, for any $f\colon i \to j$ and $(x_i) \in L$, we see that $\mathrm{pr}_j(x_i) = f(\mathrm{pr}_i(x_i))$ by construction of L.

  It remains to check that L is universal. Well, if L' is any object equipped with such maps $\varphi_i\colon L' \to Fi$ for each i, then we must construct a unique map $\varphi\colon L' \to L$ commuting with everything. For the uniqueness, we see that this commuting requires $\mathrm{pr}_i \varphi(x') = \varphi_i(x')$ for all $i \in \mathcal{I}$, so we are forced to define

$$\varphi(x') \coloneqq (\varphi_i(x'))$$

  for all $i \in \mathcal{I}$. It remains to check that this map works. We already know that it commutes with all the given maps, so we only have to check that φ is well-defined (i.e., outputs to L). Namely, for any $f\colon i \to j$, we need to check that $Ff(\varphi_i(x')) = \varphi_j(x')$, which is true by hypothesis on the $\varphi_\bullet$s.

- We handle the colimit, which is similar. We will only give the construction. Define the set

$$C \coloneqq \left(\bigsqcup_{i \in \mathcal{I}} F_i \right) /\sim,$$

  where $\sim$ is the equivalence relation generated by having $x_i \in Fi$ and $x_j \in Fj$ similar if and only if there is a map $f\colon i \to j$ for which $Ff(x_i) = x_j$. Because L is a quotient of a disjoint union, there are inclusion maps $\iota_i\colon F_i \to L$ for each $i \in \mathcal{I}$. Furthermore, for any $f\colon i \to j$ and $\iota_i(x_i) \in L$, we see that $\iota_j(Ff(x_i)) = \iota_i(x_i)$ by definition of the equivalence relation.

  It remains to check that C is universal, which we omit. ■

Example A.26. Fix categories $\mathcal{C}$ and $\mathcal{D}$, and suppose that $\mathcal{D}$ admits all colimits from categories of cardinality at most κ. Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits all limits and colimits of functors $F\colon \mathcal{I} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where the category $\mathcal{I}$ of cardinality at most κ.

Proof. We explicitly construct the colimit "pointwise." Namely, we need to define some functor A to be the colimit of F, so we ought to construct $Ac \in \mathcal{D}$ for each $c \in \mathcal{C}$. With this in mind, we set

$$Ac \coloneqq \operatorname*{colim}_{i \in \mathcal{I}} F(i)(c),$$

which exists as an element of $\mathcal{D}$ because $\mathcal{D}$ admits a colimit of the functor

$$\mathcal{I} \xrightarrow{F} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\mathrm{ev}_c} \mathcal{D}.$$

We will let $(\eta_i)_c\colon Fi(c) \to Ac$ be the induced morphism. Now, for each morphism $f\colon c \to c'$, we need to define a morphism $Af\colon Ac \to Ac'$. Well, we claim that the maps $F(i)(c) \to F(i)(c')$ assemble into a morphism

$$\operatorname*{colim}_{i \in \mathcal{I}} F(i)(c) \to \operatorname*{colim}_{i \in \mathcal{I}} F(i)(c').$$

Well, to map out of $\mathrm{colim}_{i \in \mathcal{I}} F(i)(c)$, we need to check that the maps $F(j)(c) \to F(j)(c') \to \mathrm{colim}\, F(i)(c')$ commute with the internal maps of $\mathcal{I}$, which amounts to the commutativity of the following diagram.

$$\begin{tikzcd}
Fj(c) \arrow[r, "Fj(f)"] \arrow[d, "Ff(c)"'] & Fj(c') \arrow[r] \arrow[d, "Ff(c')"] & \operatorname*{colim}_{i \in \mathcal{I}} Fi(c') \\
Fj'(c) \arrow[r, "Fj'(f)"] & Fj'(c') \arrow[ur] &
\end{tikzcd}$$

Here, the square commutes because the internal map $j \to j'$ goes to a natural transformation $Fj \Rightarrow Fj'$. The point is that Af is the unique map making the diagram

$$\begin{tikzcd}
Fj(c) \arrow[r, "Fj(f)"] \arrow[d] & Fj(c') \arrow[d] \\
Ac \arrow[r, "Af"] & Ac'
\end{tikzcd}$$

commute for every $j \in \mathcal{I}$; note that this is equivalent to saying that $\eta_j\colon Fj \Rightarrow A$ is a natural transformation for each $j \in \mathcal{J}$. For example, if $f = \mathrm{id}_c$, then certainly having $Af = \mathrm{id}_{Ac}$ will make the diagram commute. Similarly, given two maps $f\colon c \to c'$ and $f'\colon c' \to c''$, the commutativity of the diagram

$$\begin{tikzcd}
Fj(c) \arrow[r, "Fj(f)"'] \arrow[rr, bend left, "Fh(f'f)"] \arrow[d] & Fj(c') \arrow[r, "Fj(f')"'] \arrow[d] & Fj(c'') \arrow[d] \\
Ac \arrow[r, "Af"] \arrow[rr, bend right, "A(f'f)"'] & Ac' \arrow[r, "Af'"] & Ac''
\end{tikzcd}$$

for all $j \in \mathcal{I}$ implies that $A(f'f) = Af' \circ Af$ by the uniqueness of a map making the outer rectangle commute.

We thus see that we have constructed a functor $A\colon \mathcal{C} \to \mathcal{D}$. It remains to actually show that A is the colimit. Well, suppose we have an object $B\colon \mathcal{C} \to \mathcal{D}$ equipped with natural transformations $\varphi_i\colon Fi \Rightarrow B$ for each $i \in \mathcal{I}$ commuting with the induced maps from $\mathcal{I}$. We would like to induce a unique map $\varphi\colon A \Rightarrow B$ such that $\varphi\eta_i = \varphi_i$ for all $i \in \mathcal{I}$.

Let's show uniqueness by showing why φ is forced. Having $\varphi\eta_i = \varphi_i$ implies that $\varphi_c \circ (\eta_i)_c = (\varphi_i)_c$ for each $c \in \mathcal{C}$. On the other hand, these maps $(\eta_i)_c$ cause the diagram

$$\begin{tikzcd}
Fi(c) \arrow[r, "(\eta_i)_c"] \arrow[d] & Bc \\
Fj(c) \arrow[ur, "(\eta_j)(c)"'] &
\end{tikzcd}$$

to commute for each map $i \to j$ (by applying ev_c to the commutativity relation for the η_is). Thus, there is a unique map $\varphi_c \colon Ac \to Bc$ satisfying $\varphi_c \circ (\eta_i)_c = (\varphi_i)_c$. This proves uniqueness.

It remains to show that the data of these maps $\varphi_c \colon Ac \to Bc$ actually assemble into a natural transformation $\varphi \colon A \Rightarrow B$; this would directly imply that $\varphi \circ \eta_i = \varphi_i$ for each $i \in \mathcal{I}$ because we already know this to be true at each $c \in \mathcal{C}$. For our naturality check, we choose some map $f \colon c \to c'$, and we draw the following diagram.

$$\begin{tikzcd} Fi(c) \arrow[r, "(\eta_i)_c"'] \arrow[d, "Fi(f)"'] \arrow[rr, bend left, "(\varphi_i)_c"] & Ac \arrow[r, "\varphi_c"'] \arrow[d, "Af"] & Bc \arrow[d, "Bf"] \\ Fi(c') \arrow[r, "(\eta_i)_{c'}"] \arrow[rr, bend right, "(\varphi_i)_{c'}"'] & Ac' \arrow[r, "\varphi_{c'}"] & Bc' \end{tikzcd}$$

The left square and outer rectangle commute by naturality. Now, by the same colimit argument executed in the previous paragraph, there is at most one map $Ac \to Bc'$ factoring through the maps $Fi(c) \to Bc \to Bc'$ for all $i \in \mathcal{I}$; however, both $Bf \circ \varphi_c$ and $\varphi_{c'} \circ Af$ satisfy this property, so we conclude! ■

Remark A.27. There is an analogous statement if $\mathcal{D}$ admits all limits, whose proof is the same. (Indeed, one can recover the limit version by passing from $\mathcal{D}$ to $\mathcal{D}^{\mathrm{op}}$.)

Here is an application of the Yoneda lemma.

Example A.28. Fix a category $\mathcal{C}$ and presheaf $A \in \mathrm{PSh}(\mathcal{C})$. Construct a category $\mathcal{I}$ of pairs (c, η), where $c \in \mathcal{C}$, and $\eta \colon \text{よ}(c) \Rightarrow A$. Then the canonical map

$$\operatorname*{colim}_{(c,\eta)\in\mathcal{I}} \text{よ}(c) \to A$$

is an isomorphism.

Proof. In other words, we would like to show that the given maps $\eta \colon \text{よ}(c) \to A$ for each pair (c, η) makes into A a colimit of the functor $F \colon \mathcal{I} \to \mathrm{PSh}(\mathcal{C})$. We run many checks.

1. We check that F is actually a functor. On objects, we are sending $F((c,\eta)) \coloneqq \text{よ}(c)$. On morphisms, we send a morphism $f \colon (c,\eta) \to (c', \eta')$ to the morphism $\text{よ}(f) \colon \text{よ}(c) \to \text{よ}(c')$. Because $\text{よ} \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is already functorial, we see that F becomes functorial.

2. We note that each pair $(c,\eta) \in \mathcal{I}$ has $\eta \colon \text{よ}(c) \Rightarrow A$. For A to be a candidate colimit of F, we need to check that any map $f \colon (c,\eta) \to (c',\eta')$ makes the diagram

$$\begin{tikzcd} \text{よ}(c) \arrow[r, "\eta"] \arrow[d, "\text{よ}(f)"'] & A \\ \text{よ}(c') \arrow[ur, "\eta'"'] & \end{tikzcd}$$

commute, which of course is true by definition of the morphisms in $\mathcal{I}$.

3. We run the uniqueness part of the universality check for A. Indeed, suppose that we have some B with maps $\varphi_{(c,\eta)} \colon \text{よ}(c) \Rightarrow B$ for each $(c,\eta) \in \mathcal{I}$ such that $\text{よ}(f)\varphi_{(c,\eta)} = \varphi_{(c',\eta')}$ for each $f \colon (c,\eta) \to (c',\eta')$ in $\mathcal{I}$. Then we show that there is at most one map $\varphi \colon A \to B$ making the diagram

$$\begin{tikzcd} \text{よ}(c) \arrow[r, "\eta"] \arrow[dr, "\varphi_{(c,\eta)}"'] & A \arrow[d, dashed, "\varphi"] \\ & B \end{tikzcd}$$

commute for each pair $(c,\eta)\in\mathcal{I}$. Well, fix some object $c\in\mathcal{C}$ and element $a\in Ac$, and we need to show that $\varphi(a)$ has at most one value. Well, $a\in Ac$ has equivalent data to some natural transformation $\eta\colon \text{よ}(c)\to A$ satisfying $\eta_c(\mathrm{id}_c)\in Ac$ by Remark A.18, so we have a pair $(c,\eta)\in\mathcal{I}$. Similarly, the natural transformation $\varphi_{(c,\eta)}\colon \text{よ}(c)\to B$ has equivalent data to the element $(\varphi_{(c,\eta)})_c(\mathrm{id}_c)$. But now the commutativity of the above diagram

$$\begin{tikzcd} \text{よ}(c) \arrow[r, "\eta"] \arrow[dr, "\varphi_{(c,\eta)}"'] & A \arrow[d, dashed, "\varphi"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_c \arrow[r, mapsto] \arrow[dr, mapsto] & a \arrow[d, mapsto] \\ & \eta_c(\mathrm{id}_c) \end{tikzcd}$$

requires that $\varphi(a)=\eta_c(\mathrm{id}_c)$. Thus, there is at most one map φ.

4. We show that the recipe for $\varphi\colon A\to B$ defined in the previous step is actually a morphism of presheaves. Thus far, we have defined maps $\varphi_c\colon Ac\to Bc$ for each $c\in\mathcal{C}$, and we remark that these maps are well-defined by the uniqueness properties of Remark A.18. It now remains to show naturality. Well, fix a morphism $f\colon c\to c'$ and some $a'\in Ac'$, and we want to show that

$$Bf(\varphi_{c'}(a'))\stackrel{?}{=}\varphi_c(Af(a')).$$

Well, $a\in Ac$ corresponds (via Remark A.18) to some natural transformation $\eta\colon \text{よ}(c')\Rightarrow A$ such that $\eta_{c'}(\mathrm{id}_{c'})=a'$, so we have a pair $(c',\eta')\in\mathcal{I}$; namely, for any $h\colon d\to c'$, we have $\eta'_d(h)=Ah(a')$. Similarly, $Af(a')\in Ac'$ corresponds to some natural transformation $\eta\colon \text{よ}(c)\Rightarrow A$ given by $\eta_d(h)=Ah(Af(a))$ for any $h\colon d\to c$. By definition of よ, we also have a morphism of pairs $f\colon (c,\eta)\to(c',\eta')$. We are now able to chase around the following diagram.

$$\begin{tikzcd} Ac' \arrow[r, "Af"] & Ac & \\ \text{よ}(c')(c') \arrow[u, "\eta'_{c'}"] \arrow[d, "(\varphi_{(c',\eta')})_{c'}"'] \arrow[r, "\text{よ}(c')(f)"] & \text{よ}(c')(c) \arrow[u, "\eta'_c"'] \arrow[d, "(\varphi_{(c',\eta')})_c"'] & \text{よ}(c)(c) \arrow[l, "\text{よ}(f)(c)"'] \arrow[ul, "\eta_c"'] \arrow[dl, "(\varphi_{(c,\eta)})_c"] \\ Bc' \arrow[r, "Bf"] & Bc & \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, mapsto] & Af(a) & \\ \mathrm{id}_{c'} \arrow[u, mapsto] \arrow[d, mapsto] & & \mathrm{id}_c \arrow[ul, mapsto] \arrow[dl, mapsto] \\ \varphi_{c'}(a') \arrow[r, dashed, mapsto, "Bf"] & \varphi_c(Af(a)) & \end{tikzcd}$$

The triangles commute because we have morphisms of pairs. The squares are naturality squares. The $\varphi_{c'}(a')$ and $\varphi_c(Af(a))$ appear at the bottom by construction of $\varphi_{c'}$ and φ_c by definition of those maps. We are now done by commutativity! ■

A.4 Adjoints

Here is the main definition.

Definition A.29 (adjoint)**.** Fix two functors $F\colon\mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{D}\to\mathcal{C}$. Then F and G are *adjoint functors* if and only if there are bijections

$$\mathrm{Mor}_{\mathcal{D}}(Fc,d)=\mathrm{Mor}_{\mathcal{C}}(c,Gd)$$

for all $c\in\mathcal{C}$ and $d\in\mathcal{D}$ which are functorial in c and d. In this situation, we say that F is the *left adjoint* and that G is the *right adjoint*.

Here is the main result on adjoints.

Proposition A.30. Fix adjoint functors $F\colon\mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{D}\to\mathcal{C}$. Then F preserves colimits.

Proof. ■

Remark A.31. Similarly, G preserves limits; one can see this by passing to the opposite category everywhere.

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List of Definitions