202A: Introduction to Topology and Analysis

Nir Elber

Fall 2022

Contents

Theme 1
Metric Spaces

My personal view on spaces is that every space I ever work with is either metrizable or is the Zariski topology.

—Evan Chen, [Che22]

1.1 August 24

Good morning everyone. This is my first class of the semester.

1.1.1 Administrative Notes

Here are some housekeeping remarks.

- The webpage for this class is `math.berkeley.edu/ rieffel/202AannF22.html`.
- The midterm date is negotiable. We will have a vote on Friday. The possible dates are Friday 14 October, Monday 17 October, or Wednesday 19 October.
- There will be no vote on the final exam. It is on 15 December at 7PM.
- Homework will be due Fridays by midnight, approximately every week.
- There is no particular text for this course, and any given text covers more than we have time for. That said, we will (very) loosely follow [Lan12], but it is helpful to have a number of different expositions around.
- Please wear a mask during lectures and office hours.

Here is a summary of the course.

- We will spend the next couple of lectures talking about metric spaces.
- We will then spend the first half of the course on general topology. The second half of the course will be on measure and integration.
- Throughout we will see a little on functional analysis.

1.1.2 Metric Spaces

Hopefully we remember something about metric spaces. Here's the definition.

Definition 1.1 (Metric)**.** A *metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following rules for any $x, y, z \in X$.

(a) Zero: $d(x, x) = 0$.

(b) Zero: $d(x, y) = 0$ implies $x = y$.

(c) Symmetry: $d(x, y) = d(y, x)$.

(d) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

We call (X, d) a *metric space*.

Remark 1.2. It is occasionally helpful to think about a "reversed" triangle inequality: note $d(x, z) \leq d(x, y) + d(y, z)$ implies $d(x, z) - d(x, y) \leq d(y, z)$. Similarly, $d(x, y) - d(x, z) \leq d(y, z)$, so it follows

$$|d(y, x) - d(x, z)| \leq d(y, z).$$

We will want some "almost" metrics as well. Here are their names.

Definition 1.3 (Semi-metric)**.** A *semi-metric* d on a set X satisfies (a), (c), and (d) of Definition 1.1. We call (X, d) a *semi-metric space*.

Definition 1.4 (Extended metric)**.** An *extended metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}^{\infty}$ satisfying (a)–(d) of Definition 1.1. We call (X, d) an *extended metric space*.

Intuitively, we might want extended metrics if we have points that we never want to be able to get to from other ones.

We can turn spaces with a semi-metric into a space with a metric.

Lemma 1.5. Fix a semi-metric space (X, d), and define the relation $\sim$ on X by $x \sim y$ if and only if $d(x, y) = 0$. Then $\sim$ is an equivalence relation.

Proof. We run these checks by hand. Fix any $x, y, z \in X$.

- Reflexive: $d(x, x) = 0$ means that $x \sim x$.
- Symmetry: if $x \sim y$, then $d(x, y) = 0$, so $d(y, x) = 0$, so $y \sim x$.
- Transitive: if $x \sim y$ and $y \sim z$, then

$$0 \leq d(x, z) \leq d(x, y) + d(y, z) = 0,$$

so $d(x, z) = 0$, so $x \sim z$. ■

As such, given a semi-metric space (X, d), we may look at the set of equivalence classes under $\sim$, which we will denote $X/{\sim}$.[1]

[1] The notation of $/{\sim}$ is intended to make us think of quotients.

Proposition 1.6. Fix a semi-metric space (X,d) and define $\sim$ as in Lemma 1.5. Then d naturally descends to a metric $\widetilde{d}$ on $X/\sim$.

Proof. Let $[x]$ denote the equivalence class of $x \in X$ under $\sim$. We claim that the function

$$\widetilde{d}([x],[y]) := d(x,y)$$

is a well-defined metric. We have the following checks; fix any $x,y,z \in X$.

- Well-defined: if $x \sim x'$ and $y \sim y'$, then note that

$$d(x,y) \le d(x,x') + d(x',y) = d(x',y) \le d(x',y') + d(y',y) = d(x',y').$$

  By symmetry, we also have $d(x',y') \le d(x,y)$, so equality follows. So d does descent properly to the quotient $X/\sim$.
- Zero: note that $\widetilde{d}([x],[y]) = 0$ if and only if $d(x,y) = 0$ if and only if $x \sim y$ if and only if $[x] = [y]$.
- Symmetry: note that

$$\widetilde{d}([x],[y]) = d(x,y) = d(y,x) = \widetilde{d}([y],[x]).$$

- Triangle inequality: note that

$$\widetilde{d}([x],[z]) = d(x,z) \le d(x,y) + d(y,z) = \widetilde{d}([x],[y]) + \widetilde{d}([y],[z]),$$

  which finishes. ■

Here are some examples of metric spaces.

Example 1.7. Given a connected graph $G = (V,E)$ with a weighting function $w\colon E \to \mathbb{R}_{\ge 0}$, we can build a metric as follows: define the "shortest-path" function $d\colon V \times V \to \mathbb{R}_{\ge 0}$ sending two vertices $v,w \in V$ to the length of the shortest path. If the graph G is not connected, we merely have an extended metric.

Example 1.8 (Euclidean metric)**.** The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$

$$d\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

is a metric.

Observe that it is not completely obvious that Example 1.8 satisfies the triangle inequality, but this will follow from the theory of the next subsections.

1.1.3 Norms on Vector Spaces

Norms provide convenient ways to build metrics.

Definition 1.9 (Norm)**.** Fix a vector space V over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|\colon V \to \mathbb{R}_{\ge 0}$ is a function satisfying the following, for any $r \in \mathbb{R}$ and $v,w \in V$.

(a) Zero: $\|v\| = 0$ if and only if $v = 0$.

(b) Scaling: $\|rv\| = |r| \cdot \|v\|$.

(c) Triangle inequality: $\|v+w\| \le \|v\| + \|w\|$.

Remark 1.10. We can probably work with a more general normed field instead of "merely" $\mathbb{R}$ or $\mathbb{C}$.

And here is our result.

Proposition 1.11. Given a metric space V with a norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$, then the function

$$d(v, w) := \|v - w\|$$

defines a metric on V.

Proof. We run the checks directly. Let $x, y, z \in V$ be points.

- Zero: note that $d(x, y) = 0$ if and only if $\|x - y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$.
- Symmetry: note that

$$d(x, y) = \|x - y\| = |-1| \cdot \|y - x\| = 1 \cdot \|y - x\| = d(y, x).$$

- Triangle inequality: note that

$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z),$$

which finishes the check. ■

Here are the usual examples.

Example 1.12. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then the following are norms on V.

- $\|(x_1, \ldots, x_n)\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.
- $\|(x_1, \ldots, x_n)\|_1 := \sum_{i=1}^n |x_i|$.

Here are some more esotetric examples.

Example 1.13. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then

$$\|(x_1, \ldots, x_n)\|_\infty := \sup\{|x_1|, \ldots, |x_n|\}$$

provides a norm on V.

Example 1.14. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then, given $p \geq 1$,

$$\|(x_1, \ldots, x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

provides a norm on V.

Remark 1.15. Taking the limit as $p \to \infty$ of $\|f\|_p$ gives $\|f\|_\infty$. This justifies the notation.

Remark 1.16. Despite having lots of examples, all of these norms are equivalent in a topological sense.

These normed vector spaces actually allow us to define a metric on any subset.

Proposition 1.17. Given a metric space (X,d) and a subset $Y \subseteq X$, the restriction of d to $Y \times Y$ is a metric.

Proof. All the requirements for d on $Y \times Y$ are satisfied for any points in X, so we are done by doing no work. ■

Example 1.18. Any subset $X \subseteq \mathbb{R}^n$ has an induced metric by restricting the (say) Euclidean metric.

1.1.4 A Hint of L^p Spaces

Here is a more complicated example of a metric.

Example 1.19. Define $V := C([0,1])$ to be the $\mathbb{R}$-vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0,1]$. The following are norms.

- $\|f\|_\infty := \sup\{|f(x)| : x \in [0,1]\}$.
- $\|f\|_1 := \int_0^1 |f(t)|\,dt$.
- $\|f\|_2 := \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$.
- More generally, given $p \geq 1$
$$\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

These integrals are finite because $[0,1]$ is compact, forcing f to achieve a finite maximum on $[0,1]$.

Remark 1.20. We can tell the same story for $C(X)$, for any measurable compact space X.

Remark 1.21. Note the analogy of Example 1.19 with Example 1.14. To see this more rigorously, set X to be the finite set $\{1,\ldots,n\}$ so that $C(X) = \mathbb{R}^n$.

We should probably justify the claims of this subsection, so here is our result.

Proposition 1.22. Define $V := C([0,1])$ to be the vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0,1]$. Then, given $p \geq 1$, the function $\|\cdot\|_p : C \to \mathbb{R}_{\geq 0}$ by

$$\|f\| := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}$$

is a norm.

Proof. We run the checks directly.

- Zero: if $f = 0$, then of course $\int_0^1 |f(t)|^p\,dt = 0$.
- Zero: suppose that $f \in C([0,1])$ has $f(t_0) \neq 0$ for any $t_0 \in [0,1]$; set $y := f(t_0)$. Then $f^{-1}((y/2, 3y/2))$ is a nonempty open subset of X and hence contains a nonempty open interval (a,b) with $a < b$. As such,
$$\int_X |f(t)|^p\,dt \geq \int_a^b |f(t)|^p\,dt \geq \int_a^b |y/2|^p\,dt > 0,$$

so we are done.

- Scaling: given $f \in C([0,1])$ and a scalar r, we have

$$\|rf\| = \left(\int_0^1 |rf(t)|^p\,dt\right)^{1/p} = \left(|r|^p \int_0^1 |f(t)|^p\,dt\right)^{1/p} = |r| \cdot \|f\|.$$

- Triangle inequality: we borrow from [Tao09]. Given $f, g \in C([0,1])$, for psychological reasons we will assume that f and g are nonzero (else this is clear); then $\|f\|, \|g\| \neq 0$, so we may scale everything so that $\|f\| + \|g\| = 1$. In fact, we may again use scaling to find $a, b \in V$ such that

$$f = (1-\theta)a \qquad \text{and} \qquad g = \theta b$$

where $\theta \in (0,1)$ and $\|a\| = \|b\| = 1$. Now, the triangle inequality translates into showing

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p\,dt = \|(1-\theta)a + \theta b\|_p^p \stackrel{?}{\leq} \left(\|(1-\theta)a\|_p + \|\theta b\|_p\right)^p = 1.$$

Well, because $p \geq 1$, the function $t \mapsto t^p$ is convex, so we get to write

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p\,dt \leq (1-\theta)\int_0^1 |a(t)|^p\,dt + \theta \int_0^1 |b(t)|^p\,dt,$$

which is what we wanted.

The above checks complete the proof; note that the proof of the triangle inequality was nontrivial. ■

Remark 1.23. Now, to show Remark 1.21, replace all $\int_0^1$ with $\sum_{i=1}^n$ and adjust all the language accordingly. The point is that "integrating over $[0,1]$" is analogous to "integrating over $\{1,\ldots,n\}$." A more thorough understanding of measure theory will allow us to rigorize this.

Next class we will talk about completeness.

1.2 August 26

Today we're talking about completeness of metric spaces.

1.2.1 Isometries

In mathematics, we are interested in objects not in isolation but as they relate to each other. Namely, we are interested also in the maps between our objects.

The philosophy here comes from category theory, where one is really most interested in the "morphisms" between "objects" instead of the objects themselves. For concreteness, here is a definition of a category.

Definition 1.24 (Category)**.** A *category* $\mathcal{C}$ consists of a class of objects $\operatorname{Ob}\mathcal{C}$ and class of morphisms $\operatorname{Mor}\mathcal{C}$ such that any two objects $A, B \in \operatorname{Ob}\mathcal{C}$ have a morphism class $\operatorname{Mor}(A, B)$. This data satisfy the following properties.

- Composition: given objects $A, B, C \in \operatorname{Ob}\mathcal{C}$, there is a binary composition operation
$$\circ\colon \operatorname{Mor}(B, C) \times \operatorname{Mor}(A, B) \to \operatorname{Mor}(A, C).$$
Explicitly, given $f \in \operatorname{Mor}(A, B)$ and $g \in \operatorname{Mor}(B, C)$, there is a composition $(g \circ f) \in \operatorname{Mor}(A, C)$.
- Given $A \in \operatorname{Ob}\mathcal{C}$, there is an identity morphism $\operatorname{id}_A \in \operatorname{Mor}(A, A)$.
- Identity: any $f \in \operatorname{Mor}(A, B)$ has $f \circ \operatorname{id}_A = f = \operatorname{id}_B \circ f$.
- Associativity: any $f \in \operatorname{Mor}(A, B)$ and $g \in \operatorname{Mor}(B, C)$ and $h \in \operatorname{Mor}(C, D)$ has $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 1.25. There is a category of groups, where the morphisms are group homomorphisms. The identity function gives the identity morphism, and composition of functions gives the required composition.

For completeness, we check that composition is well-defined: given homomorphisms $f\colon A \to B$ and $g\colon B \to C$, we need $(g \circ f)\colon A \to C$ to be a group homomorphism. Well,
$$(g \circ f)(a \cdot a') = g(f(a \cdot a')) = g(f(a) \cdot f(a')) = g(f(a)) \cdot g(f(a')) = (g \circ f)(a) \cdot (g \circ f)(a').$$

In our discussion of metric spaces, there are many possible kinds of morphisms for us to consider. Here is the strongest type.

Definition 1.26 (Isometry)**.** Given metric spaces (X, d_X) and (Y, d_Y), an *isometry* is a function $f\colon X \to Y$ preserving the metric as
$$d_Y(f(x), f(x')) = d_X(x, x').$$

Example 1.27. The $90°$ rotation $r\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $r(x, y) \mapsto (y, -x)$ is an isometry, where $\mathbb{R}^2$ is given the Euclidean metric. Indeed, any $(x, y), (x', y') \in \mathbb{R}^2$ have
$$\begin{aligned} d\big(r(x, y), r(x', y')\big) &= d\big((y, -x), (y', -x')\big) \\ &= \sqrt{(y - y')^2 + (-x - -x')^2} \\ &= \sqrt{(x - x')^2 + (y - y')^2} \\ &= d\big((x, y), (x', y')\big). \end{aligned}$$

Notation 1.28. Fix two metric spaces (X, d_X) and (Y, d_Y). Given a function $f\colon X \to Y$ with extra structure respecting some aspect of the metric, we might write $f\colon (X, d_X) \to (Y, d_Y)$ to emphasize this.

To show that isometries are valid morphisms, we need to check that the identity function $\operatorname{id}_X\colon X \to X$ is an isometry (which of course it is) and that the composition of two isometries is an isometry. We check this last one in a quick lemma.

Lemma 1.29. Given two isometries $f\colon (X, d_X) \to (Y, d_Y)$ and $g\colon (Y, d_Y) \to (Z, d_Z)$, the composition $g \circ f$ is an isometry.

Proof. Well, any two points $x, x' \in X$ have
$$d_Z(g(f(x)), g(f(x'))) = d_Y(f(x), f(x')) = d_X(x, x'),$$

which is what we wanted. ■

One can restrict further to surjective isometries, where the main point is that (again) the composition of two surjective functions remains surjective. (Note that the identity is of course surjective.) The following is the reason why a surjective isometry is a good notion.

Lemma 1.30. A surjective isometry $f\colon (X,d_X)\to(Y,d_Y)$ is bijective, and its inverse function is also an isometry.

Proof. To see that f is bijective, we only need to know that f is injective. Well, given $x,x'\in X$, note that $f(x)=f(x')$ if and only if $d_Y(f(x),f(x'))=0$ if and only if $d(x,x')=0$ if and only if $x=x'$.[2]

Thus, f is indeed bijective; let $g\colon Y\to X$ be its inverse. We now need to show that g is an isometry. Well, given $y,y'\in Y$, we may find $x,x'\in X$ such that $f(x)=y$ and $f(x')=y'$. Then

$$d_X(g(y),g(y'))=d_X((g\circ f)(x),(g\circ f)(x'))=d_X(x,x')\stackrel{*}{=}d_Y(f(x),f(x'))=d_Y(y,y'),$$

where in $\stackrel{*}{=}$ we have used the fact that f is an isometry. ■

Remark 1.31. The above result is somewhat subtle in its importance: the inverse function of a bijection is only an inverse in the category of sets. The above result is saying that this inverse morphism in the category of sets is lifting to an inverse morphism in the category of metric spaces with isometries as morphisms. In general, it is not always true that bijective morphisms are invertible, as we shall soon see.

1.2.2 Lipschitz Continuity

Isometries are somewhat restrictive, so we might weaken this as follows.

Definition 1.32 (Lipschitz continuous)**.** Given metric spaces (X,d_X) and (Y,d_Y), a function $f\colon X\to Y$ is a *Lipschitz continuous* if and only if there is a constant $c\in\mathbb{R}$ such that

$$d_Y(f(x),f(x'))\le c\cdot d_X(x,x').$$

Remark 1.33. Equivalently, we are asking for the ratio

$$\frac{d_Y(f(x),f(x'))}{d_X(x,x')}$$

to be uniformly bounded above for all $x\neq x'$. Notably, the inequality is trivially satisfied whenever $x=x'$, or equivalently whenever $d(x,x')=0$.

Example 1.34. Any isometry $f\colon (X,d_X)\to(Y,d_Y)$ is Lipschitz continuous: indeed, set $c:=1$ so that, for any $x,x'\in X$,

$$d_Y(f(x),f(x'))=d_X(x,x')\le 1\cdot d_X(x,x').$$

[2] In fact, this argument shows that all isometries are injective. We will shortly see that all actually Lipschitz continuous functions are injective.

Example 1.35. Provide $\mathbb{R}$ and $\mathbb{R}^2$ their usual Euclidean metrics. Then the projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ by $\pi\colon (x,y) \mapsto x$ is Lipschitz continuous: indeed, set $c := 1$ so that, for any $(x,y),(x',y') \in \mathbb{R}^2$, we have

$$d_{\mathbb{R}^2}\big((x,y),(x',y')\big) = \sqrt{(x-x')^2+(y-y')^2} \geq \sqrt{(x-x')^2} = d_{\mathbb{R}}(x,x') = d_{\mathbb{R}}\big(\pi((x,y)),\pi((x',y'))\big).$$

Again, one can see that the identity function $\operatorname{id}_X\colon (X,d_X) \to (X,d_X)$ is Lipschitz continuous (with $c := 1$), and here is our composition check.

Lemma 1.36. If $f\colon (X,d_X) \to (Y,d_Y)$ and $g\colon (Y,d_Y) \to (Z,d_Z)$ are Lipschitz continuous, then the composition $(g \circ f)\colon (X,d_X) \to (Z,d_Z)$ is also Lipschitz continuous.

Proof. We are given constants c and d such that any $x,x' \in X$ and $y,y' \in Y$ have

$$d_Y(f(x),f(x')) \leq c \cdot d_X(x,x') \qquad \text{and} \qquad d_Z(g(y),g(y')) \leq d \cdot d_Y(y,y').$$

As such, we use the constant cd to witness our Lipschitz continuity: any $x,x' \in X$ have

$$d_Z(g(f(x)),g(f(x'))) \leq d \cdot d_Y(f(x),f(x')) \leq cd \cdot d_X(x,x'),$$

which is what we wanted. ■

It will be shortly worth our time to talk about the constant c appearing in Definition 1.32.

Lemma 1.37. Fix a Lipschitz continuous function $f\colon (X,d_X) \to (Y,d_Y)$. Then there exists a constant c_f (possibly $-\infty$) such that any real number $c \geq c_f$ is equivalent to the following property: any $x,x' \in X$ have

$$d_Y(f(x),f(x')) \leq c \cdot d_X(x,x').$$

Proof. Let S denote the set of all constants c such that any $x,x' \in X$ have

$$d_Y(f(x),f(x')) \leq c \cdot d_X(x,x').$$

Equivalently, using Remark 1.33, S is the set of upper-bounds for

$$R := \left\{ \frac{d_Y(f(x),f(x'))}{d_X(x,x')} : x,x' \in X, x \neq x' \right\}.$$

Now, S is nonempty because f is Lipschitz continuity, so we set $c_f := \sup R$ to be the least upper bound for R—observe that $c_f = -\infty$ is permissible when X has one point. It is now pretty clear that $S = [c_f, \infty)$. ■

Note that c_f the property stated in the lemma automatically implies that c_f is the least possible constant and is unique. Being least is immediate (by the backwards direction), and being unique follows from being least. So because we have some uniqueness, we get a definition.

Definition 1.38 (Lipschitz constant)**.** Given a Lipschitz continuous function $f\colon (X,d_X) \to (Y,d_Y)$, the *Lipschitz constant c_f for f* is the least real number c such that

$$d_Y(f(x),f(x')) \leq c \cdot d_X(x,x').$$

We could, as before, look at surjective Lipschitz continuous functions, but these need not be bijective anymore as shown by Example 1.35. What's worse is that, as warned possible in Remark 1.31, bijective Lipschitz continuous functions need not even have a Lipschitz continuous inverse.

Exercise 1.39. We exhibit a function between metric spaces which is bijective and Lipschitz continuous, but its inverse function is not Lipschitz continuous.

Proof. Set $X := (0,1)$ and $Y := (1,\infty)$, both metric spaces with the Euclidean (subspace) metric, and set $f\colon (0,\infty) \to (0,\infty)$ by $f\colon x \mapsto 1/x$. Notably, $x \in X$ implies $f(x) \in Y$, and $y \in Y$ implies $f(y) \in X$.

- Note $f|_Y$ is bijective with inverse $f|_X$ because $f(f(x)) = f(1/x) = x$ for all $x \in (0,\infty)$.
- Note $f|_Y$ is Lipschitz continuous: set $c := 1$ and note that any $y, y' \in Y$ have
$$|f(y) - f(y')| = \left|\frac{1}{y} - \frac{1}{y'}\right| = \left|\frac{y-y'}{yy'}\right| \le |y - y'|.$$
- But $f|_X$ is not Lipschitz continuous: suppose for contradiction that f_X is Lipschitz continuous, and use Lemma 1.37 to recover the needed constant c_0. Then set $c := \max\{c_0, 4\}$, which must also work as a constant, and set $x := 1/c$ and $x' := 1/(3c)$ so that
$$|f(x) - f(x')| = |c - 3c| = 2c > c \cdot |x - x'|.$$
This is a contradiction, so we are done. ■

Remark 1.40 (Nir)**.** In some sense, the problem here is that the definition of Lipschitz continuity allows $d_Y(f(x), f(x'))$ to be "too small," which permits the inverse function to have distances which blow up.

In light of Exercise 1.39, we introduce a new definition.

Definition 1.41 (Lipschitz isomorphism)**.** Give metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *Lipschitz isomorphism* if and only if f is Lipschitz continuous and has an inverse function which is also Lipschitz continuous.

Remark 1.42. A good reason to care about this notion of continuity (and isomorphism) is that all normed $\mathbb{R}$-vector spaces of some finite dimension n are Lipschitz isomorphic.

1.2.3 Fun with Continuity

Here is yet a weaker notion of morphism.

Definition 1.43 (Uniformly continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *uniformly continuous* if and only if every $\varepsilon > 0$ has some $\delta > 0$ such that
$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$
for all $x, x' \in X$.

Example 1.44. Any Lipschitz continuous function $f\colon (X, d_X) \to (Y, d_Y)$ is also uniformly continuous: indeed, for any $\varepsilon > 0$, set $\delta := \max\{c_f, 1\}\varepsilon > 0$ (where c_f is the Lipschitz constant) so that
$$d_X(x, x') < \varepsilon \implies d_Y(f(x), f(x')) \le c_f \cdot d(x, x') < \delta.$$

Example 1.45. Give $[0,1]$ the Euclidean (subspace) metric, and set $f\colon [0,1]\to[0,1]$ by $f(x) := \sqrt{x}$.

- Note f is uniformly continuous because it is continuous on a compact set.
- However, f is not Lipschitz continuous: for any constant $c>0$, set $x = 1/(c+1)^2$ and $x'=0$ so that
$$\left|\frac{f(x)-f(x')}{x-x'}\right| = \left|\frac{1/(c+1)}{1/(c+1)^2}\right| = |c+1| > c,$$
so Remark 1.33 tells us that we are not Lipschitz continuous.

By rearranging quantifiers, we get another useful (but weaker) notion.

Definition 1.46 (Continuous)**.** Given metric spaces (X,d_X) and (Y,d_Y), a function $f\colon X\to Y$ is *continuous at $x\in X$* if and only if all $\varepsilon>0$ have some $\delta_x>0$ such that
$$d_X(x,x')<\delta_x \implies d_Y(f(x),f(x'))<\varepsilon.$$
Then f is *continuous* if and only if it is continuous at all $x\in X$.

Example 1.47. All uniformly continuous functions $f\colon (X,d_X)\to(Y,d_Y)$ are continuous. Indeed, at any $x_0\in X$ with $\varepsilon>0$, uniform continuity promises $\delta>0$ so that
$$|x-x'|<\delta \implies |f(x)-f(x')|<\varepsilon$$
for all $x,x'\in X$. Setting x' to x_0 recovers continuity.

Example 1.48. Give $\mathbb{R}$ the usual Euclidean metric, and set $f\colon\mathbb{R}\to\mathbb{R}$ by $f(x):=x^2$.

- Note $f(x)$ is continuous because it is a polynomial.
- However, $f(x)$ is not uniformly continuous: take $\varepsilon=1$. Now, for any $\delta>0$, set $x=1/\delta$ and $x'=1/\delta+\delta/2$ so that $|x-x'|<\delta$, but
$$|f(x)-f(x')| = \left(\frac{1}{\delta}+\frac{\delta}{2}\right)^2 - \frac{1}{\delta^2} = 1+\frac{\delta^2}{4} > \varepsilon.$$

As usual, the identity function is uniformly continuous and continuous (it's an isometry), and these continuities are preserved by composition. We will have a different way to see that continuous functions remain continuous under composition later, so for now we will focus on uniform continuity.

Lemma 1.49. Fix uniformly continuous morphisms $f\colon (X,d_X)\to(Y,d_Y)$ and $g\colon(Y,d_Y)\to(Z,d_Z)$. Then the function $(g\circ f)$ is uniformly continuous.

Proof. For any $\varepsilon>0$, the uniform continuity of g promises $\delta_g>0$ such that
$$d_Y(y,y')<\delta_g \implies d_Z(g(y),g(y'))<\varepsilon$$
for any $y,y'\in Y$. Continuing, the uniform continuity of f promises $\delta_f>0$ such that
$$d_X(x,x')<\delta_X \implies d_Y(f(x),f(x'))<\delta_Y \implies d_Z(g(f(x)),g(f(x')))<\varepsilon$$
for any $x,x'\in X$, which is what we wanted. ■

Remark 1.50. In some sense, isometries and Lipschitz continuous functions have their definition fundamentally interrelated with the metric. In contrast, the weaker notion of continuity will readily generalize to general topological spaces. Uniform continuity also generalizes to "uniformities," which is a different notion.

1.2.4 Convergence and Completeness

To discuss completeness, we need to talk about convergence.

Definition 1.51 (Converge)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n > N \implies d(x_n, x) < \varepsilon.$$

We might write this as "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$." In this event, we may say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges*, and its limit is x.

Remark 1.52 (Nir)**.** As a sanity check, the limit of a sequence is unique: if $x_n \to x$ and $x_n \to x'$ as $n \to \infty$, then any $\varepsilon > 0$ can find some large n so that $d(x_n, x), d(x_n, x') < \varepsilon/2$. As such,

$$d(x, x') < d(x_n, x) + d(x_n, x') = \varepsilon$$

for any $\varepsilon > 0$, so $d(x, x') = 0$ and thus $x = x'$ is forced.

We have no reason yet to be convinced that any of our morphisms described previously are good notions, so let's start with continuity.

Lemma 1.53. Fix a continuous function between metric spaces $f: (X, d_X) \to (Y, d_Y)$. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ converges to $f(x) \in Y$.

Proof. For any $\varepsilon > 0$,the continuity of f implies that we can find $\delta_x > 0$ so that

$$d_X(x_n, x) < \delta_x \implies d_Y(f(x_n), f(x)) < \varepsilon$$

for any x_n. But the fact that $x_n \to x$ as $n \to \infty$ means that there is $N > 0$ so that

$$n > N \implies d_X(x_n, x) < \delta_x \implies d_Y(f(x_n), f(x)) < \varepsilon,$$

so indeed, $f(x_n) \to f(x)$ as $n \to \infty$. ■

In fact, the converse also holds.

Lemma 1.54. Fix metric spaces (X, d_X) and (Y, d_Y), and fix a point $x \in X$. Then suppose a function $f: X \to Y$ satisfies that any convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x$ as $n \to \infty$ has $f(x_n) \to f(x)$ as $n \to \infty$. Then f is continuous at x.

Proof. We proceed by contraposition. If f is not continuous at x, then any $n \in \mathbb{N}$ can find x_n such that $d_X(x, x_n) < 1/n$ even though $d_Y(f(x_n), f(x)) \geq 1$. In particular, $x_n \to x$ as $n \to \infty$ (for any ε, choose $N = 1/\varepsilon$), but the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ does not converge to $f(x)$ because no n has $d_Y(f(x), f(x_n)) < 1$. ■

We would like a notion of convergence which only uses data internal to the sequence, and this leads to the following definition.

Definition 1.55 (Cauchy)**.** Fix a metric space (X,d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy sequence* if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n,m > N \implies d(x_n, x_m) < \varepsilon.$$

It would be rude if continuity was always the best kind of morphism, so this time around preserving Cauchy-ness requires something stronger.

Lemma 1.56. Fix a uniformly continuous function between metric spaces $f\colon (X,d_X) \to (Y,d_Y)$. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is Cauchy, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ is also Cauchy.

Proof. For any $\varepsilon > 0$, the uniform continuity of f promises $\delta > 0$ so that

$$d_X(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \varepsilon$$

for any x_n, x_m. However, the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy promises N so that

$$n,m > N \implies d_X(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \varepsilon,$$

which is what we wanted. ∎

Example 1.57. Continuous functions do not need to preserve Cauchy sequences: $f\colon (0,\infty) \to (0,\infty)$ by $f(x) := 1/x$ is continuous, and the sequence $\{1/n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ is Cauchy (it converges to 0 in $\mathbb{R}$) even though $\{f(1/n)\}_{n\in\mathbb{N}} = \{n\}_{n\in\mathbb{N}}$ certainly does not converge.

Anyway, it is quick to check that convergent sequences are Cauchy.

Lemma 1.58. Fix a metric space (X,d). Then all convergent sequences are Cauchy.

Proof. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$. Then, for any $\varepsilon > 0$, find N so that

$$d(x_n, x) < \varepsilon/2$$

for all $n > N$. Then any $n, m > N$ has

$$d(x_n, x_m) \le d(x_n, x) + d(x_m, x) < \varepsilon,$$

so the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. ∎

We in general hope that our Cauchy sequences converge. As such, we have the following definition.

Definition 1.59 (Complete)**.** A metric space (X,d) is *complete* if and only if every Cauchy sequence in X converges to a point in X.

We are sad when a metric space is not complete, so we hope to have a way to make it complete. The most natural way to do this is by using the notion of density.

Definition 1.60 (Dense)**.** Fix a metric space (X,d). Then $S \subseteq X$ is *dense* if and only if, given any $x \in X$ and $\varepsilon > 0$, we may find $x' \in S$ with $d(x,x') < \varepsilon$.

And here is our completion.

Definition 1.61 (Completion)**.** A *completion* of the metric space (X,d) is a metric space $(\overline{X}, \overline{d})$ equipped with an isometry $\iota\colon X \to \overline{X}$ such that $(\overline{X}, \overline{d})$ is complete and $\operatorname{im}\iota$ is dense in $\overline{X}$.

One can show that any metric space has a completion and that they are all isometric and therefore in some sense the same. We'll do these separately.

1.2.5 Existence of Completions

Let's start with existence.

Theorem 1.62. Any metric space (X, d) has a completion.

Proof. Let $\widetilde{X}$ denote the set of all Cauchy sequences in X. We hope to make $\widetilde{X}$ into our completion, but this requires a little care. To begin, we have the following lemma.

Lemma 1.63. Given a metric space (X, d) with two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, then the sequence

$$\{d(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$$

converges.

Proof. Because $\mathbb{R}$ is a complete metric space, it suffices to show that the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is Cauchy. Well, for any $\varepsilon > 0$, find a sufficiently large N so that

$$n, m > N \implies d(x_n, x_m), d(y_n, y_m) < \varepsilon/2.$$

Then any $n, m > N$ has

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \varepsilon + d(y_m, y_n),$$

and $d(x_m, y_m) < d(x_n, y_n) + \varepsilon$ as well by symmetry. It follows that any $n, m > N$ has

$$\left|d(x_n, y_n) - d(x_m, y_m)\right| < \varepsilon,$$

verifying that our sequence is Cauchy. ■

Remark 1.64. Here is a quick motivational remark for the definition of our metric below: if (X, d) is a metric space with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then we claim $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. Indeed, for any $\varepsilon > 0$, we can find N large enough so that $d(x_n, x), d(y_n, y) < \varepsilon/2$ for any $n > N$. As such,

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n) < d(x, y) + \varepsilon.$$

By symmetry, we get $d(x, y) \le d(x_n, y_n) + \varepsilon$ as well, finishing.

Thus, we define $\widetilde{d}\colon \widetilde{X} \times \widetilde{X} \to \mathbb{R}_{\ge 0}$ by

$$\widetilde{d}\left(\{x_n\}, \{y_n\}\right) := \lim_{n\to\infty} d(x_n, y_n).$$

We claim that $\widetilde{d}$ is a semi-metric on $\widetilde{X}$. We have the following checks; fix Cauchy sequences $\{x_n\}, \{y_n\}, \{z_n\}$.

- Zero: note
$$\widetilde{d}(\{x_n\}, \{x_n\}) = \lim_{n\to\infty} d(x_n, x_n) = 0.$$

- Symmetry: note
$$\widetilde{d}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \widetilde{d}(\{y_n\}, \{x_n\}).$$

- Triangle inequality: note

$$\begin{aligned}\widetilde{d}(\{x_n\},\{y_n\})+\widetilde{d}(\{y_n\},\{z_n\}) &= \lim_{n\to\infty} d(x_n,y_n)+\lim_{n\to\infty} d(y_n,z_n)\\ &= \lim_{n\to\infty}(d(x_n,y_n)+d(y_n,z_n))\\ &\ge \lim_{n\to\infty} d(x_n,z_n)\\ &= \widetilde{d}(x_n,z_n),\end{aligned}$$

  where we have implicitly used a number of limit laws.

So because $\widetilde{d}$ is a semi-metric, Proposition 1.6 tells us that $\widetilde{d}$ will descend naturally to a metric $\overline{d}$ on $\overline{X} := \widetilde{X}/\sim$, where $\{x_n\}\sim\{y_n\}$ if and only if $\widetilde{d}(\{x_n\},\{y_n\})=0$. We will let $[\{x_n\}]$ denote the equivalence class of the Cauchy sequence $\{x_n\}\in\widetilde{X}$ in $\overline{X}$.

We now show that $(\overline{X},\overline{d})$ can be made into a completion for X.

- Given $x\in X$, note that the constant sequence $\{x\}$ is Cauchy (for any $\varepsilon>0$, set $N=0$), so we define $\iota\colon X\to\overline{X}$ by
$$\iota(x) := [\{x\}].$$
  To see that ι is an isometry, note any $x,x'\in X$ have
$$\overline{d}(\iota(x),\iota(x'))=\widetilde{d}(\{x\},\{y\})=\lim_{n\to\infty} d(x,y)=d(x,y).$$
- We show that $\operatorname{im}\iota$ is dense in $\overline{X}$. Indeed, fix some $[\{x_n\}]\in\overline{X}$ and $\varepsilon>0$. Then there is some N so that $n,m>N$ has
$$d(x_n,x_m)<\varepsilon/2.$$
  Fixing a particular n_0 with $n_0>N$, we set $x:=x_{n_0}$ so that
$$\overline{d}([\{x_n\}],\iota(x))=\widetilde{d}(\{x_n\},x_{n_0})=\lim_{n\to\infty} d(x_n,x_{n_0}).$$
  Now, for $n>N$, we have $d(x_n,x_{n_0})<\varepsilon/2$, so we conclude that this limit must be less than ε.
- We show that $(\overline{X},\overline{d})$ is a complete metric space. Fix a Cauchy sequence $\{\overline{x}_k\}$ in $\overline{X}$. To find the Cauchy sequence we are supposed to converge to, we use our density result: for each $k\in\mathbb{N}$, we can find $y_k\in X$ such that $\overline{d}(\overline{x}_k,\iota(y_k))<1/k$.

  We claim that $\{y_k\}$ is Cauchy. Indeed, for any $\varepsilon>0$, we can find N such that $k,\ell>N_0$ has
$$\overline{d}(\overline{x}_k,\overline{x}_\ell)<\varepsilon/3.$$
  Then, setting $N:=\max\{3/\varepsilon,N_0\}$, we note that $k,\ell>N$ has
$$d(y_k,y_\ell)=\overline{d}(\iota(y_k),\iota(d_\ell))\le\overline{d}(\overline{x}_k,\iota(y_k))+\overline{d}(\overline{x}_\ell,\iota(y_\ell))+\overline{d}(\overline{x}_k,\overline{x}_\ell)<\varepsilon.$$
  Lastly, we claim that $\overline{x}_k\to[\{y_n\}]$ in $\overline{X}$. Indeed, for any $\varepsilon>0$, find some sufficiently large N so that
$$k,\ell>N \implies d(y_k,y_\ell)<\varepsilon/2.$$
  Then $k>\max\{N,2/\varepsilon\}$ has
$$\overline{d}(\overline{x}_k,[\{y_n\}])\le\overline{d}(\overline{x}_k,\iota(y_k))+\overline{d}([\{y_n\}],\iota(y_k))<\frac{\varepsilon}{2}+\lim_{n\to\infty} d(y_n,y_k).$$
  Because $k>N$, we have $d(y_n,y_k)<\varepsilon/2$ for any $n>N$, so the entire right-hand side must be upper-bounded by ε. This finishes.

The above checks complete the proof. ■

Remark 1.65 (Nir)**.** One might complain that we used the completeness of $\mathbb{R}$ in this proof because one common way to construct the real numbers is as the completion of $\mathbb{Q}$ under the Euclidean metric. To remedy this, one ought to define the equivalence relation on Cauchy sequences more directly, saying that two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of real numbers are equivalent under $\sim$ if and only if

$$\lim_{n\to\infty} d_{\mathbb{R}}(x_n, y_n) = 0.$$

1.2.6 Uniqueness of Completions

We now show that any two completions of a metric space (X,d) are isometric, which is our uniqueness result. Here is the main intermediate result.

Lemma 1.66. Fix a metric space (X,d) and a completion $(\overline{X},\overline{d})$ with its isometry $\iota\colon (X,d)\to(\overline{X},\overline{d})$. Then, for any complete metric space (Y,d') and isometry $\varphi\colon (X,d)\to(Y,d')$, there is a unique isometry $\psi\colon (\overline{X},\overline{d})\to(Y,d')$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \overline{X} \arrow[d, dashed, "\psi"] \\ & Y \end{tikzcd}$$

Proof. We start by showing the uniqueness of ψ. Well, for any $\overline{x}\in\overline{X}$, note that any $n\in\mathbb{N}$ allows us to find $x_n\in X$ with

$$\overline{d}(\overline{x},\iota(x_n)) < 1/n$$

because $\operatorname{im}\iota$ is dense in $\overline{X}$. Now, we notice that $\iota(x_n)\to\overline{x}$ as $n\to\infty$ because any $\varepsilon>0$ can set $N=1/\varepsilon$. As such, we see that Lemma 1.53 applied to any possible $\psi\colon\overline{X}\to Y$ forces

$$\psi(\overline{x}) = \psi\Big(\lim_{n\to\infty}\iota(x_n)\Big) = \lim_{n\to\infty}\psi(\iota(x_n)) = \lim_{n\to\infty}\varphi(x_n).$$

Note that, a priori, we do not know if the sequence $\{\varphi(x_n)\}_{n\in\mathbb{N}}$ converges, but this argument tells us that it must; the limit is unique by Remark 1.52, so $\psi(\overline{x})$ is unique as well.

We now show that ψ exists. As before, any $\overline{x}\in\overline{X}$ can find a sequence $\{x_n\}\subseteq X$ such that $\iota(x_n)\to\overline{x}$ as $n\to\infty$. Thus, we note that $\{\varphi(x_n)\}$ is Cauchy by Lemma 1.56, so the completeness of Y gives it a limit; we set

$$\psi(\overline{x}) \coloneqq \lim_{n\to\infty}\varphi(x_n).$$

We have the following checks on ψ.

- Well-defined: if we have two sequences $\{x_n\}$ and $\{x'_n\}$ such that $\iota(x_n)\to x$ and $\iota(x'_n)\to x$ as $n\to\infty$, we need to show that

  $$\lim_{n\to\infty}\varphi(x_n) = \lim_{n\to\infty}\varphi(x'_n).$$

  For brevity, set y and y' to be the limits of $\{\varphi(x_n)\}$ and $\{\varphi(x'_n)\}$, respectively. Then, for any $\varepsilon>0$, we note that there is a sufficiently large N such that

  $$n>N \implies d_Y(y,\varphi(x_n)), d_Y(y',\varphi(x'_n)) < \varepsilon/4.$$

  Further, we can make N even larger so that

  $$n>N \implies \overline{d}(\overline{x},\iota(x_n)), \overline{d}(\overline{x},\iota(x'_n)) < \varepsilon/4.$$

As such, any $n > N$ has

$$\begin{aligned} d_Y(y,y') &\le d_Y(y,\varphi(x_n)) + d_Y(\varphi(x_n),\varphi(x_n')) + d_Y(y',\varphi(x_n')) \\ &< \varepsilon/4 + d_X(x_n,x_n') + \varepsilon/4 \\ &= \varepsilon/2 + \overline{d}(\iota(x_n),\iota(x_n')) \\ &\le \varepsilon/2 + \overline{d}(\overline{x},\iota(x_n)) + \overline{d}(\overline{x},\iota(x_n')) \\ &< \varepsilon. \end{aligned}$$

It follows $d_Y(y,y') = 0$, so $y = y'$.

- Isometry: given $\overline{x},\overline{x}' \in \overline{X}$, find sequences $\{x_n\}$ and $\{x_n'\}$ in X so that $\iota(x_n) \to \overline{x}$ and $\iota(x_n') \to \overline{x}'$ as $n \to \infty$. Thus,

$$\begin{aligned} d_Y(\psi(\overline{x}),\psi(\overline{x}')) &= d_Y\Big(\lim_{n\to\infty}\varphi(x_n), \lim_{n\to\infty}\varphi(x_n')\Big) \\ &\overset{*}{=} \lim_{n\to\infty} d_Y(\varphi(x_n),\varphi(x_n')) \\ &= \lim_{n\to\infty} d(x_n,x_n') \\ &= \lim_{n\to\infty} \overline{d}(\iota(x_n),\iota(x_n')) \\ &= \overline{d}\Big(\lim_{n\to\infty}\iota(x_n), \lim_{n\to\infty}\iota(x_n')\Big) \\ &\overset{*}{=} \overline{d}(\overline{x},\overline{x}'), \end{aligned}$$

where we have used Remark 1.64 at the $\overset{*}{=}$.

- For any $x \in X$, we see that the (constant) Cauchy sequence $\{\iota(x)\}$ converges to $\iota(x)$, so

$$\psi(\iota(x)) = \lim_{n\to\infty} \varphi(x) = \varphi(x).$$

It follows $\psi \circ \iota = \varphi$.

Thus, we have finished establishing the existence of an isometry $\psi\colon \overline{X} \to Y$ such that $\varphi = \psi \circ \iota$. ■

Remark 1.67. One can also replace all isometries with uniformly continuous functions in the statement.

And here is our uniqueness result.

Theorem 1.68. Fix a metric space (X,d) and two completions $\iota\colon (X,d) \to (\overline{X},\overline{d})$ and $\iota'\colon (X,d) \to (\overline{X}',\overline{d}')$. Then there is a surjective isometry $\psi\colon (\overline{X},\overline{d}) \to (\overline{X}',\overline{d}')$.

Proof. Applying Lemma 1.66 twice, we get isometries $\psi\colon (\overline{X},\overline{d}) \to (\overline{X}',\overline{d}')$ and $\psi'\colon (\overline{X}',\overline{d}') \to (\overline{X},\overline{d})$ making the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \overline{X} \arrow[d, "\psi"] \\ & \overline{X}' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\iota'"] \arrow[dr, "\iota"'] & \overline{X}' \arrow[d, "\psi'"] \\ & \overline{X} \end{tikzcd}$$

In particular, we see that $\psi' \circ \psi$ makes the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\iota"'] & \overline{X} \arrow[d, "\psi'\circ\psi"] \\ & \overline{X} \end{tikzcd}$$

However, using Lemma 1.66 again, this isometry $\psi' \circ \psi$ is unique to make the diagram commute, and we could of course put the isometry $\operatorname{id}_{\overline{X}}$ here if we wanted to. Thus,

$$\psi' \circ \psi = \operatorname{id}_{\overline{X}}.$$

By symmetry, $\psi \circ \psi' = \operatorname{id}_{\overline{X'}}$, so we do see that ψ and ψ' are inverse isometries. This finishes the proof. ■

1.3 August 29

Good morning everyone.

1.3.1 Some Examples

Let's give some more examples of metric spaces. Let's start with spaces of continuous functions.

Definition 1.69. Given a (normed) topological field k, such as $\mathbb{R}$ or $\mathbb{C}$, we denote the k-vector space of k-valued continuous function from a topological space X as $C(X)$. By convention, we will take $k = \mathbb{C}$ unless otherwise specified.

And here are our two examples. The first is of a complete metric space.

Exercise 1.70. Give $V := C([0,1])$ the uniform norm

$$\|f\|_\infty := \sup\{|f(t)| : t \in [0,1]\}.$$

Then V is complete.

Proof. This is merely the statement that a sequence of continuous functions which are uniformly Cauchy will converge uniformly to a continuous function. We will prove this for completeness. Fix a sequence of continuous function $\{f_n\}_{n\in\mathbb{N}}$ which are Cauchy with respect to $\|\cdot\|_\infty$. In other words, for each $\varepsilon > 0$, there exists N_ε so that

$$n, m > N_\varepsilon \implies \|f_n - f_m\|_\infty < \varepsilon,$$

which means that $|f_n(t) - f_m(t)| < \varepsilon$ for all $t \in [0,1]$.

In particular, for any fixed $t \in [0,1]$, the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ (using the same N_ε), so we use the completeness of $\mathbb{R}$ to let this sequence converge to $f(t) \in \mathbb{R}$. We have the following checks.

- To see that $f_n \to f$ as $n \to \infty$ (under our metric), select any $\varepsilon > 0$, and then find N so that

  $$n, m > N \implies \|f_n - f_m\|_\infty < \varepsilon/3.$$

  Further, for any $t \in [0,1]$, we see that we can find a large enough $n_t > N$ so that $|f(t) - f_{n_t}(t)| < \varepsilon/3$. But then $n > N$ has

  $$|f_n(t) - f(t)| \le |f_n(t) - f_{n_t}(t)| + |f_{n_t}(t) - f(t)| < 2\varepsilon/3,$$

  so $\|f - f_n\|_\infty \le 2\varepsilon/3 < \varepsilon$.

- To see that f is continuous, fix $t \in [0,1]$ so that we want to show f is continuous at t. Well, for any $\varepsilon > 0$, find N large enough so that

  $$n, m > N \implies \|f_n - f_m\|_\infty < \varepsilon/4.$$

  Now, select $n_t > N$ large enough so that $|f(t) - f_{n_t}(t)| < \varepsilon/4$, and the continuity of f_{n_t} promises us $\delta > 0$ so that

  $$|t - t'| < \delta \implies |f_{n_t}(t) - f_{n_t}(t')| < \varepsilon/4.$$

In particular, for any t' with $|t-t'|<\delta$, find $n_{t'}>N$ large enough so that $|f(t')-f_{n_{t'}}(t')|<\varepsilon/4$, and then we see

$$|f(t)-f(t')| \le |f(t)-f_{n_t}(t)|+|f_{n_t}(t)-f_{n_t}(t')|+|f_{n_t}(t')-f_{n_{t'}}(t')|+|f_{n_{t'}}(t')-f(t')|<\varepsilon,$$

which is what we wanted. ■

The second example is the same space, but it is no longer complete.

Example 1.71. Fix $p \ge 1$ finite. Give $V := C([0,1])$ the L^p norm as

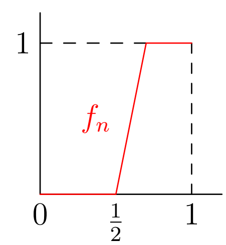
$$\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

Then V is not complete.

Proof. For each $n \ge 2$, define f_n as the piecewise continuous function

$$f_n(t) := \begin{cases} 0 & 0 \le t \le \frac12, \\ n(t-\frac12) & \frac12 \le t \le \frac12 + \frac1n, \\ 1 & \frac12 + \frac1n \le t \le 1. \end{cases}$$

Here is the image.



The point is that f_n is trying to converge to a discontinuous function. To help us with the proof here, we pick up the following lemma.

Lemma 1.72. Fix $V := C([0,1])$ and some finite $p \ge 1$. If we have a convergent sequence $f_n \to f$ as $n \to \infty$ in the $\|\cdot\|_p$ metric, and $f_n(t) = g(t)$ for all sufficiently large n and $t \in U$ for some open $U \subseteq C([0,1])$, then $f|_U(t) = g(t)$.

Proof. Suppose for the sake of contradiction that we have $t_0 \in U$ with $f(t_0) \ne g(t_0)$; we show that $\{f_n\}$ does not converge to f. Set $\varepsilon := |f(t_0)-g(t_0)|$, which is nonzero. The continuity of $f-g$ now promises that there is $\delta > 0$ for which

$$|t-t_0|<\delta \implies |(f-g)(t_0)-(f-g)(t)|<\varepsilon/2,$$

so in particular $|(f-g)(t)| \ge \varepsilon/2$. It follows that, for sufficiently large n, we have

$$\|f-f_n\|_p^p = \int_0^1 |f(t)-f_n(t)|^p\,dt \ge \int_U |(f-g)(t)|\,dt \ge \int_{U\cap(t_0-\delta,t_0+\delta)} \frac{\varepsilon}{2}\,dt.$$

Because $U \cap (t_0-\delta, t_0+\delta)$ is open, it has nonzero measure, so this entire right-hand quantity is nonzero, thus violating that $f_n \to f$ as $n \to \infty$. ■

Now suppose for the sake of contradiction that $f_n \to f$ as $n \to \infty$ for some $f \in V$. Then, using $U = (0, 1/2)$, we conclude that $f(t) = 0$ for all $t \in (0, 1/2)$. Similarly, for any n, we set $U_n = (1/2 + 1/n, 1)$, so $f_m|_{U_n}$ returns 1 always for sufficiently large m; this then implies $f(t) = 1$ for any $t \in U_n$ for any n, so $f(t) = 1$ for any $t \in (1/2, 1)$.

However, the sequences $a_n := \frac{1}{2} - \frac{1}{n}$ and $b_n := \frac{1}{2} + \frac{1}{n}$ (for $n \geq 3$) have $a_n \to \frac{1}{2}$ and $b_n \to \frac{1}{2}$ both as $n \to \infty$ while the continuity of f would require

$$0 = \lim_{n\to\infty} f(a_n) = f(1/2) = \lim_{n\to\infty} f(b_n) = 1,$$

which is a contradiction. ■

Remark 1.73. In an attempt to make this metric space complete, we can try to specify which functions we want to look at, which motivates the theory of measure and integration.

Remark 1.74. The $\|\cdot\|_2$ norm on $C(X)$ for some (say) subset $X \subseteq \mathbb{R}$ with finite measure as coming from an inner product

$$\langle f, g \rangle := \int_X f(t)\overline{g(t)}\, dt.$$

When $\|\cdot\|_2$ is complete, we would then get a Hilbert space, which are very nice normed vector spaces, and we'll see more of them in Math 202B.

Remark 1.75 (Nir)**.** In contrast to the finite case, we see that the $\|\cdot\|_\infty$ norm induces a different (metric) topology on $C([0,1])$ than the $\|\cdot\|_p$ norms with p finite because the former is complete while the latter are not. In fact, all the norms $\|\cdot\|_p$ induce different topologies on $C([0,1])$.

Part I

Topology

THEME 2

BUILDING TOPOLOGIES

Sets are not doors.

—Munkres

2.1 August 29

We continue lecture by shifting to topology.

2.1.1 Metric Topology

We close our discussion of metric spaces with a taste of topology. Recall the following definition.

Definition 1.46 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous at* $x \in X$ if and only if all $\varepsilon > 0$ have some $\delta_x > 0$ such that

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$

Then f is *continuous* if and only if it is continuous at all $x \in X$.

We are going to want to extend this definition to more general topological spaces. To step in that direction, we will want to talk about open sets, so we start with open balls.

Definition 2.1 (Ball)**.** Fix a metric space (X, d). Then the *open ball of radius* r *centered at* $x_0 \in X$ is

$$B(x_0, r) \coloneqq \{x \in X : d(x, x_0) < r\}.$$

The *closed ball* is $\overline{B(x_0, r)} \coloneqq \{x \in X : d(x, x_0) \le r\}$.

We can now restate continuity as follows.

Definition 2.2 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous* at $x \in X$ if and only if, given any nonempty open ball $B(f(x_0), \varepsilon)$, there exists a nonempty open ball $B(x_0, \delta)$ such that

$$f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon).$$

Namely, we've really only restated our inequalities.

To continue our generalization, we define the pre-image.

Definition 2.3 (Pre-image)**.** Fix a function $f\colon X \to Y$. Then we define the *pre-image* $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

Note that our pre-image notation matches with the notation of an inverse function. In general, no confusion will arise by confusing these two.

As such, let's restate continuity again: observe that $A \subseteq X$ and $B \subseteq Y$ has $f(A) \subseteq B$ if and only if all $a \in A$ have $f(a) \in B$ if and only if all $a \in A$ have $a \in f^{-1}(B)$ if and only if $A \subseteq f^{-1}(B)$.

Definition 2.4 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous* at $x \in X$ if and only if, given any nonempty open ball $B(f(x), \varepsilon)$, there exists a nonempty open ball $B(x, \delta)$ such that

$$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)).$$

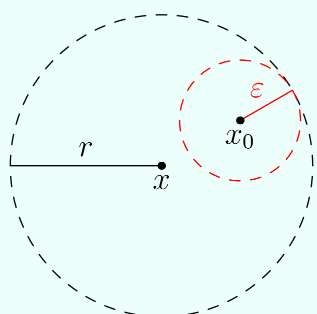
We defined open balls and promised open sets, so now let's define our open sets.

Definition 2.5 (Open set)**.** Fix a metric space (X, d). Then a subset $U \subseteq X$ is *open* if and only if, for each $x \in U$, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. In other words, each point in U has an open ball around it.

Example 2.6. Open balls are open sets. Indeed, given an open ball $B(x, r)$, note that any $x_0 \in B(x, r)$ has $d(x_0, x) < r$, so we take $\varepsilon := r - d(x_0, x)$. To see this works, observe $x' \in B(x_0, \varepsilon)$ will have

$$d(x', x) \le d(x', x_0) + d(x_0, x) < \varepsilon + (r - \varepsilon) = r,$$

so $B(x_0, \varepsilon) \subseteq B(x, r)$ follows. Here is the image for what just happened.



And here is our definition of corresponding definition of continuity.

Lemma 2.7. Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is continuous at $x \in X$ if and only if, given any open set $U \subseteq Y$ with $f(x) \in U$, there is an open ball $B(x, \delta)$, such that

$$B(x, \delta) \subseteq f^{-1}(U).$$

Proof. Taking f to be continuous, note that we can find $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$ because U is open. Thus, continuity promises $\delta > 0$ such that

$$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)) \subseteq f^{-1}(U).$$

Conversely, if f satisfies the conclusion of the statement, we can take $U = B(f(x), \varepsilon)$ for any $\varepsilon > 0$ by Example 2.6, and the conclusion promises $\delta > 0$ such that

$$B(x, \delta) \subseteq f^{-1}(U) = f^{-1}(B(f(x), \varepsilon)),$$

which is what we wanted. ■

It is cleaner to talk about the entire function being continuous instead of at a point.

Lemma 2.8. Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is continuous if and only if, given any open set $U \subseteq Y$ with $f(x) \in U$, the pre-image $f^{-1}(U)$ is open.

Proof. This is a matter of rearranging our quantifiers correctly. Lemma 2.7 tells us that, for all $x \in X$, all open $U \subseteq Y$ with $f(x) \in U$ has some $\delta > 0$ such that $B(x, \delta) \subseteq U$. Equivalently, for all open $U \subseteq Y$, any $x \in X$ with $x \in f^{-1}(U)$ has some $\delta > 0$ such that $B(x, \delta) \subseteq U$. But by definition of being open, we're just saying that all open $U \subseteq Y$ has $f^{-1}(U)$ also open. ■

So we have the following definition.

Definition 2.9 (Continuous)**.** A function $f : X \to Y$ between metric spaces is *continuous* if and only if, for any open set $U \subseteq Y$, the pre-image $f^{-1}(U)$ is open.

The philosophy here is to try to understand open sets instead of trying to understand the metrics. This is the idea of topology.

2.1.2 Open Sets

Thus, we are motivated to understand open sets. Here are some basic properties.

Proposition 2.10. Fix a metric space (X, d), and let $\mathcal{T}$ be the collection of open sets.

(a) We have $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union

$$\bigcup_{U \in \mathcal{U}} U$$

is open.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \in \mathcal{T}$, we have

$$\bigcap_{i=1}^{n} U_i$$

is open.

Proof. We go in sequence.

(a) To show $X \in \mathcal{T}$, note that any $x \in X$ has $B(x, 1) \subseteq X$ by definition. To show $\varnothing \in \mathcal{T}$, note that any $x \in \varnothing$ has $B(x, 1) \subseteq \varnothing$ because there is no $x \in \varnothing$ at all.

(b) For any $x \in \bigcup_{U \in \mathcal{U}} U$, we have $x \in V$ for some particular $V \in \mathcal{U}$. Then the openness of V tells us we can find $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subseteq V \subseteq \bigcup_{U \in \mathcal{U}} U,$$

which finishes.

(c) Fix x in the common intersection. Then, for any i, we have $x \in U_i$, so we have some $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subseteq U$, and so we set

$$\varepsilon := \min_{1 \le i \le n} \varepsilon_i.$$

In particular, $\varepsilon > 0$ because n is finite, and we have

$$B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq U_i$$

for each i, so $B(x, \varepsilon)$ is a subset of our intersection. ■

Remark 2.11. The arbitrary intersection of open sets need not be open: working in $\mathbb{R}$ with the usual metric,

$$\bigcap_{i=1}^{\infty} B(0, 1/n) = \{0\},$$

which is not open. (Namely, no $\varepsilon > 0$ has $B(x, \varepsilon) \subseteq \{0\}$.)

Motivated by Proposition 2.10, we have the following definition.

Definition 2.12 (Topology)**.** Fix a set X. Then a *topology* $\mathcal{T}$ on X is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following.

(a) We have $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union $\bigcup_{U \in \mathcal{U}} U$ lives in $\mathcal{T}$.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $\bigcap_{i=1}^{n} U_i$ lives in $\mathcal{T}$.

We will say that the ordered pair $(X, \mathcal{T})$ is a *topological space*. We say that the sets in $\mathcal{T}$ are *open*.

Example 2.13. By Proposition 2.10, metric spaces with their open sets form a topological space.

Here are some more basic examples.

Definition 2.14 (Discrete topology)**.** Given a set X, the *discrete topology* is the topology $\mathcal{P}(X)$.

Definition 2.15 (Indiscrete topology)**.** Given a set X, the *indiscrete topology* is the topology $\{\varnothing, X\}$.

It is fairly routine to check that the above collections form topologies. In fact, they are closed under both arbitrary union and arbitrary intersection.

Remark 2.16. The discrete topology can be defined by the metric $d \colon X \times X \to \mathbb{R}_{\ge 0}$ by

$$d(x, x') := \begin{cases} 1 & x \ne x', \\ 0 & x = x'. \end{cases}$$

Indeed, for any $x \in X$, we see $B(x, 1/2) = \{x\}$, so any subset $U \subseteq X$ is the open set

$$U = \bigcup_{x \in U} \{x\} = \bigcup_{x \in U} B(x, 1/2).$$

Remark 2.17. If $\#X \geq 2$, the indiscrete topology cannot be given a metric. Indeed, find distinct points $a, b \in X$ and set $r := d(a,b)$, so $a \neq b$ implies $r > 0$. Now, $a \in B(a,r)$, but $b \notin B(a,r)$, so $B(a,r)$ is an open set distinct from both $\varnothing$ and X.

Remark 2.18. One can give topologies a partial order by inclusion. Then the discrete topology is the maximal one (definitionally, any topology is a subset of $\mathcal{P}(X)$), and the indiscrete topology is the minimal one (definitionally, any topology contains $\varnothing$ and X).

And so here is our general definition of continuity.

Definition 2.19 (Continuous)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Then a function $f \colon X \to Y$ is *continuous* if and only if, for any $U_Y \in \mathcal{T}_Y$, we have $f^{-1}(U_Y) \in \mathcal{T}_X$.

2.2 August 31

It is once again the morning.

2.2.1 Intersections of Topologies

We will want to have lots of topologies to work with. Here is a basic way to build them.

Proposition 2.20. Let X be a set, and pick up some collection of topologies $\{\mathcal{T}_\alpha\}_{\alpha \in \lambda}$. Then the intersection

$$\mathcal{T} := \bigcap_{\alpha \in \lambda} \mathcal{T}_\alpha$$

is also a topology on X.

Proof. This is mostly a matter of writing out the axioms.

(a) Note that $\varnothing, X \in \mathcal{T}_\alpha$ for each α, so $\varnothing, X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, we have $\mathcal{U} \subseteq \mathcal{T}_\alpha$ for each α, so $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}_\alpha$ for each α, so

$$\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$$

as well.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, we have $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}_\alpha$ for each α, so $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha$ for each α, so

$$\bigcap_{i=1}^n U_i \in \mathcal{T}$$

follows. ■

Corollary 2.21. Fix a set X. Given a collection $\mathcal{S} \subseteq \mathcal{P}(X)$, there is a smallest topology $\mathcal{T}$ containing $\mathcal{S}$.

Proof. Certainly there is some topology containing $\mathcal{S}$, namely the discrete topology $\mathcal{P}(X)$. Thus, we can set our topology to be

$$\mathcal{T}(\mathcal{S}) := \bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S} \\ \mathcal{T}\text{ a topology}}} \mathcal{T},$$

which is a topology (by Proposition 2.20) which contains $\mathcal{S}$ (because each topology in the intersection contains $\mathcal{S}$), and of course any topology $\mathcal{T}$ containing $\mathcal{S}$ will have $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}$. ■

To codify this idea, we have the following idea.

Definition 2.22 (Generated topology)**.** Fix a set X. We say that a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ *generates* its smallest topology $\mathcal{T}$. We will write $\mathcal{T}(\mathcal{S})$ for this topology.

Remark 2.23 (Nir)**.** The topology $\mathcal{T}(\mathcal{S})$ is unique. Indeed, suppose two topologies $\mathcal{T}$ and $\mathcal{T}'$ are minimal topologies containing $\mathcal{S}$. Then $\mathcal{T}\cap\mathcal{T}'$ is also a topology containing $\mathcal{S}$ by Proposition 2.20, but $\mathcal{T}\cap\mathcal{T}' \subseteq \mathcal{T},\mathcal{T}'$ forces $\mathcal{T} = \mathcal{T}\cap\mathcal{T}' = \mathcal{T}'$.

Remark 2.24 (Nir)**.** Given collections $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S}')$. Indeed, we have

$$\mathcal{T}(\mathcal{S}) = \bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S} \\ \mathcal{T}\text{ a topology}}} \mathcal{T} \subseteq \bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S}' \\ \mathcal{T}_0\text{ a topology}}} \mathcal{T} = \mathcal{T}(\mathcal{S}').$$

Remark 2.25 (Nir)**.** If $\mathcal{T}$ is already a topology on X, then $\mathcal{T}(\mathcal{T}) = \mathcal{T}$. Indeed, of course $\mathcal{T} \subseteq \mathcal{T}(\mathcal{T})$, but then also

$$\mathcal{T}(\mathcal{T}) = \bigcap_{\substack{\mathcal{T}'\supseteq\mathcal{T} \\ \mathcal{T}'\text{ a topology}}} \mathcal{T}' \subseteq \mathcal{T}$$

because $\mathcal{T}$ is a topology containing $\mathcal{T}$.

2.2.2 Sub-bases

On the other side of things, we pick up the following definition.

Definition 2.26 (Sub-base)**.** Let $(X,\mathcal{T})$ be a topological space. A collection $\mathcal{S} \subseteq \mathcal{T}$ is a *sub-base* for $\mathcal{T}$ if and only if the following hold.

(a) $\mathcal{S}$ covers X, in that $X = \bigcup_{U\in\mathcal{S}} U$.

(b) $\mathcal{T}$ is generated by $\mathcal{S}$.

The point is that collections $\mathcal{S}$ are easy to find, so we have therefore found many topologies.

It will be useful to give a more concrete description of the topology generated by a collection $\mathcal{S}$. We start by taking finite intersections.

Lemma 2.27. Fix a set X and a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with $X = \bigcup_{U\in\mathcal{S}} U$. Then set

$$\mathcal{I}^{\mathcal{S}} := \left\{ \bigcap_{i=1}^n U_i : \{U_i\}_{i=1}^n \subseteq \mathcal{S} \right\}.$$

Then $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$ and $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection. Further, the topology generated by $\mathcal{I}^{\mathcal{S}}$ is also the topology generated by $\mathcal{S}$.

Proof. We show the claims in sequence

- That $\{U\} \subseteq \mathcal{S}$ for any $U \in \mathcal{S}$ implies that $U \in \mathcal{I}^{\mathcal{S}}$ for any $U \in \mathcal{S}$, so $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$ follows.
- To show $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection, pick up some finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{I}^{\mathcal{S}}$. Then, for each i, we can find some finite collection $\mathcal{U}_i \subseteq \mathcal{S}$ such that

  $$U_i = \bigcap_{V \in \mathcal{U}_i} V.$$

  Setting $\mathcal{U} := \bigcup_{i=1}^n \mathcal{U}_i$, we see that $\mathcal{U}$ is finite and that

  $$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n \bigcap_{V \in \mathcal{U}_i} V = \bigcap_{V \in \mathcal{U}} V$$

  must live in $\mathcal{I}^{\mathcal{S}}$.
- Because $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$, Remark 2.24 tells us $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right)$. In the other direction, note that any finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{S}$ also lives in $\mathcal{T}(\mathcal{S})$, so

  $$\bigcap_{i=1}^n U_i \in \mathcal{T}(\mathcal{S}).$$

  It follows $\mathcal{I}^{\mathcal{S}} \subseteq \mathcal{T}(\mathcal{S})$, so $\mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right) \subseteq \mathcal{T}(\mathcal{T}(\mathcal{S})) = \mathcal{T}(\mathcal{S})$ by Remark 2.25. ■

After taking finite intersections, we take arbitrary unions.

Lemma 2.28. Fix a set X and a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ closed under finite intersection with $\bigcup_{U \in \mathcal{I}} U = X$. Then the collection of (arbitrary) unions of elements in $\mathcal{I}$, denoted

$$\mathcal{T} := \left\{ \bigcup_{U \in \mathcal{U}} U : \mathcal{U} \subseteq \mathcal{I} \right\},$$

is $\mathcal{T}(\mathcal{I})$.

Proof. If $\mathcal{T}'$ is a topology containing $\mathcal{I}$, then note any collection $\mathcal{U} \subseteq \mathcal{I}$ lives in $\mathcal{T}'$, so the arbitrary union

$$\bigcup_{U \in \mathcal{U}} U$$

lives in $\mathcal{T}'$. It follows that $\mathcal{T} \subseteq \mathcal{T}'$, so

$$\mathcal{T} \subseteq \bigcap_{\substack{\mathcal{T}' \supseteq \mathcal{T} \\ \mathcal{T}' \text{ a topology}}} \mathcal{T}' = \mathcal{T}(\mathcal{I}).$$

Thus, it remains to show that $\mathcal{T}$ is in fact a topology, which will imply from $\mathcal{I} \subseteq \mathcal{T}$ that $\mathcal{T}(\mathcal{I}) \subseteq \mathcal{T}(\mathcal{T}) = \mathcal{T}$ by Remark 2.24. Here are our checks.

- Setting $\mathcal{U} = \varnothing \subseteq \mathcal{I}$, we see that $\bigcup_{U \in \mathcal{U}} U = \varnothing$, so $\varnothing \in \mathcal{T}$. Also, by hypothesis, we have

  $$X = \bigcup_{U \in \mathcal{I}} U \in \mathcal{T}.$$

- Arbitrary union: let $\mathcal{U} \subseteq \mathcal{T}$ be a subcollection. For any $U \in \mathcal{U}$, we can find a collection $\mathcal{V}_U \subseteq \mathcal{I}$ such that
$$U = \bigcup_{V \in \mathcal{V}_U} V.$$
Now, we set $\mathcal{V}$ to be the union of all the collections of $\mathcal{V}_U$ for each $U \in \mathcal{U}$, which is still contained in $\mathcal{I}$, so that
$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} \bigcup_{V \in \mathcal{V}_U} V = \bigcup_{V \in \mathcal{V}} V \in \mathcal{T}.$$
- Finite intersection: by induction, it suffices to pick up two sets $U, V \in \mathcal{T}$ and show $U \cap V \in \mathcal{T}$. Well, we can find collections $\mathcal{U}, \mathcal{V} \subseteq \mathcal{I}$ such that
$$U = \bigcup_{U' \in \mathcal{U}} U' \qquad \text{and} \qquad V = \bigcup_{V' \in \mathcal{V}} V',$$
from which it follows (by distribution) that
$$U \cap V = \left(\bigcup_{U' \in \mathcal{U}} U'\right) \cap \left(\bigcup_{V' \in \mathcal{V}} V'\right) = \bigcup_{U' \in \mathcal{U}} \left(U' \cap \bigcup_{V' \in \mathcal{V}} V'\right) = \bigcup_{\substack{U' \in \mathcal{U} \\ V' \in \mathcal{V}}} (U' \cap V').$$
Now, $\mathcal{I}$ is closed under finite intersection, so $U' \cap V' \in \mathcal{I}$, so we have witnessed $U \cap V$ as an arbitrary union of elements of $\mathcal{I}$, so $U \cap V \in \mathcal{T}$ follows. ■

Corollary 2.29. Fix a set X and a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with $X = \bigcup_{U \in \mathcal{S}} U$. Letting $\mathcal{I}^{\mathcal{S}}$ be the collection of finite intersections of $\mathcal{S}$ and then $\mathcal{T}$ be the collection of arbitrary unions of $\mathcal{I}^{\mathcal{S}}$, we have that $\mathcal{T} = \mathcal{T}(\mathcal{S})$.

Proof. By Lemma 2.27, we have $\mathcal{T}(\mathcal{S}) = \mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right)$. Plugging $\mathcal{I}^{\mathcal{S}}$ into Lemma 2.28 (which applies because $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection and covers X because $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$), we see that $\mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right) = \mathcal{T}$, finishing. ■

We quickly point out that the point of discussing sub-bases is that we will be allowed to check continuity on only a sub-base.

Lemma 2.30. Fix a topological space $(X, \mathcal{T}_X)$ and a set Y. Given a function $f \colon X \to Y$, the collection
$$\mathcal{T}(f) \coloneqq \left\{U \subseteq Y : f^{-1}(U) \in \mathcal{T}_X\right\}$$
forms a topology on Y.

Proof. Here are our checks.

- Note $f^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$, so $\varnothing \in \mathcal{T}(f)$. Also, $f^{-1}(Y) = X \in \mathcal{T}_X$, so $Y \in \mathcal{T}(f)$.
- Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}(f)$, we see that
$$f^{-1}\left(\bigcup_{U \in \mathcal{U}} U\right) = \bigcup_{U \in \mathcal{U}} f^{-1}(U)$$
is a union of elements of $\mathcal{T}_X$ and therefore in $\mathcal{T}_X$. Thus, $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}(f)$.
- Finite intersection: this is identical to the previous check. Given a finite collection $\{U_1, \ldots, U_n\} \in \mathcal{T}(f)$, we see that
$$f^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n f^{-1}(U_i)$$
is a finite intersection of elements of $\mathcal{T}_X$ and therefore in $\mathcal{T}_X$. Thus, $\bigcap_{i=1}^n U_i \in \mathcal{T}(f)$. ■

Proposition 2.31. Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and let $\mathcal{S}$ be a sub-base for $\mathcal{T}_Y$. Then a function $f: X \to Y$ is continuous if and only if

$$f^{-1}(U) \in \mathcal{T}_X$$

for all $U \in \mathcal{S}$.

Proof. Certainly if f is continuous then the pre-image of any open set $U \in \mathcal{S} \subseteq \mathcal{T}_Y$ must be open. On the other hand, let $\mathcal{T}(f) \subseteq \mathcal{P}(Y)$ be the collection of subsets U for which $f^{-1}(U) \in \mathcal{T}_X$. This is a topology by Lemma 2.30, and it contains $\mathcal{S}$ by hypothesis, so it follows

$$\mathcal{T}_Y = \mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(f).$$

Thus, $f^{-1}(U) \in \mathcal{T}_X$ for any $U \in \mathcal{T}_Y$, so f is continuous. ■

2.2.3 Bases

Having defined a sub-base, we should be rightly upset that we have not defined a base.

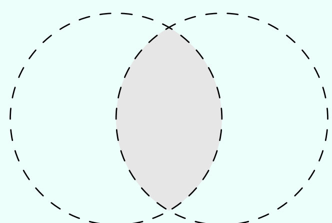
Definition 2.32 (Base)**.** Fix a set X. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *base* (for a topology on X) if and only if the collection of arbitrary unions of $\mathcal{B}$ form a topology on X.

This definition is a little hard to access because we still don't have a good notion of what a topology is.

Example 2.33. Fix a set X. Given any collection $\mathcal{S} \subseteq \mathcal{P}(X)$, the collection of finite intersections $\mathcal{I}^{\mathcal{S}}$ is a base by Lemma 2.28.

However, in general we do not require a base to be closed under finite intersection.

Example 2.34. Fix a metric space (X, d). Then the collection of open balls $\mathcal{B}$ forms a topology by Example 2.13. Notably, the intersection of two open balls need not be an open ball, as follows.



Even though bases are not closed under finite intersection, we do have the following.

Proposition 2.35. Fix a set X and a collection $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a base if and only if

(a) $X = \bigcup_{B \in \mathcal{B}} B$, and

(b) any $B_1, B_2 \in \mathcal{B}$ has some collection $\mathcal{U} \subseteq \mathcal{B}$ such that

$$B_1 \cap B_2 = \bigcup_{B \in \mathcal{U}} B.$$

Proof. In one direction, suppose that $\mathcal{B}$ is a base generating the topology $\mathcal{T}$.

(a) Because $X \in \mathcal{T}$, we see that X is the union of some subcollection $\mathcal{U} \subseteq \mathcal{B}$, so it follows

$$X = \bigcup_{U \in \mathcal{U}} U \subseteq \bigcup_{B \in \mathcal{B}} B \subseteq X.$$

(b) Given $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$, we see that $B_1 \cap B_2 \in \mathcal{T}$, so because $\mathcal{T}$ is made of arbitrary unions of $\mathcal{B}$, there is a collection $\mathcal{U} \subseteq \mathcal{B}$ such that

$$B_1 \cap B_2 = \bigcup_{B \in \mathcal{U}} B.$$

We now go in the other direction. Suppose $\mathcal{B}$ satisfies (a) and (b), and define

$$\mathcal{T} := \left\{ \bigcup_{U \in \mathcal{U}} U : \mathcal{U} \subseteq \mathcal{B} \right\}.$$

We now check that $\mathcal{T}$ is a topology.

- Using $\mathcal{U} = \varnothing \subseteq \mathcal{B}$, so we see that $\bigcup_{U \in \mathcal{U}} U = \varnothing$ is in $\mathcal{T}$. Also, by (a), we have

$$X = \bigcup_{B \in \mathcal{B}} B \in \mathcal{T}.$$

- Arbitrary union: this is the same as the check in Lemma 2.28. Given a collection $\mathcal{U} \subseteq \mathcal{T}$, each $U \in \mathcal{U}$ has some collection $\mathcal{V}_U \subseteq \mathcal{B}$ such that $\bigcup_{V \in \mathcal{V}_U} V = U$. Letting $\mathcal{V} \subseteq \mathcal{B}$ be the union of all the $\mathcal{V}_U$, we see

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} \bigcup_{V \in \mathcal{V}_U} V = \bigcup_{V \in \mathcal{V}} V$$

  lives in $\mathcal{T}$.

- Finite intersection: by induction, it suffices to pick up $U_1, U_2 \in \mathcal{T}$ and show $U_1 \cap U_2 \in \mathcal{T}$. Well, find $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$ such that

$$U_1 = \bigcup_{B_1 \in \mathcal{B}_1} B_1 \qquad \text{and} \qquad U_2 = \bigcup_{B_2 \in \mathcal{B}_2} B_2,$$

  which implies

$$U_1 \cap U_2 = \bigcup_{\substack{B_1 \in \mathcal{B}_1 \\ B_2 \in \mathcal{B}_2}} (B_1 \cap B_2).$$

  Now, (b) implies that $B_1 \cap B_2$ for any $B_1, B_2 \in \mathcal{B}$ is a union of elements in $\mathcal{B}$, so $B_1 \cap B_2 \in \mathcal{T}$. Thus, $U_1 \cap U_2$ is the arbitrary union of elements in $\mathcal{T}$, so $U_1 \cap U_2 \in \mathcal{T}$ by the previous check. ■

Remark 2.36 (Nir)**.** Careful readers might realize that we could rearrange the given exposition to show that, given a sub-base $\mathcal{S}$, the collection of finite intersections $\mathcal{I}^{\mathcal{S}}$ is a base instead of going through Lemma 2.28.

Remark 2.37. Of course, any base is also a sub-base. Notably, sub-bases only require that $X = \bigcup_{U \in \mathcal{S}} U$, which must be satisfied for bases.

Example 2.38. Set $X = \mathbb{R}$ with the usual topology $\mathcal{T}$. Then the collection $\mathcal{B}$ of open intervals (a, b) form a base for the usual topology (these are our open balls). In contrast, the collection

$$\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$

forms a sub-base for the usual topology. Namely, certainly $\mathcal{S} \subseteq \mathcal{T}$, and $\mathcal{B} \subseteq \mathcal{T}(\mathcal{S})$ because of the finite intersection $(-\infty, b) \cap (a, \infty) = (a, b)$ for any $a, b \in \mathbb{R}$. Namely, $\mathcal{T} = \mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}(\mathcal{T}(\mathcal{S})) = \mathcal{T}(\mathcal{S})$ follows.

2.2.4 Induced Topologies

We start with the following motivating example.

Example 2.39. Fix a set X, and give it the discrete topology. Then, for any topological space $(Y, \mathcal{T}_Y)$, any function $f\colon X \to Y$ is continuous because the pre-image of any open subset $U_Y \subseteq Y$ is open in X.

In general, we might have some smallish collection of functions which we want to force to be continuous, so we might ask what topology is forced by their continuity.

Definition 2.40 (Induced topology)**.** Fix a set X and a collection of topologies $\{(Y_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$ with some functions $f_\alpha\colon X \to Y_\alpha$ for each $\alpha \in \lambda$. Then

$$\bigcup_{\alpha\in\lambda} \{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha\}$$

is a sub-base for an *induced topology*.

The one thing to check is that X belongs to the arbitrary unions of our collection, which is clear because $X = f_\alpha^{-1}(Y_\alpha)$.

Definition 2.41 (Relative topology)**.** Fix $(Y, \mathcal{T})$ a topological space. Then the *relative topology* for a subset $X \subseteq Y$ is the topology induced by the natural embedding $\iota\colon X \hookrightarrow Y$.

We have the following more concrete description.

Lemma 2.42. Fix $(Y, \mathcal{T}_Y)$ a topological space. Then the relative topology for a subset $X \subseteq Y$ consists of the subsets

$$\{X \cap U : U \in \mathcal{T}_Y\}.$$

Proof. Let $\iota\colon X \hookrightarrow Y$ be the natural embedding. Then we are given the sub-base

$$\mathcal{S} := \left\{\iota^{-1}(U) : U \in \mathcal{T}_Y\right\}.$$

Now, $\iota^{-1}(U) = X \cap U$, and then we can check directly that this collection $\mathcal{S}$ gives a topology and finish by Remark 2.25. Here are the checks, which should be completely routine by now.

- Note $\varnothing \in \mathcal{T}_Y$ implies $\varnothing = X \cap \varnothing \in \mathcal{S}$. Also, $Y \in \mathcal{T}_Y$ implies $X = X \cap Y \in \mathcal{S}$.
- Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{S}$, for each $U \in \mathcal{U}$ find $U_V \in \mathcal{T}_Y$ such that $U = X \cap U_V$. Then

$$\bigcup_{U\in\mathcal{U}} = U = \bigcup_{U\in\mathcal{U}} X \cap U_V = X \cap \underbrace{\bigcup_{U\in\mathcal{U}} U_V}_{\in\mathcal{T}_Y}$$

  lives in $\mathcal{S}$.
- Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{S}$, find $V_i \in \mathcal{T}_Y$ such that $U_i = X \cap V_i$. Then

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (X \cap V_i) = X \cap \underbrace{\bigcap_{i=1}^n V_i}_{\in\mathcal{T}_Y}$$

  lives in $\mathcal{S}$. ■

2.3 September 2

There are no questions about anything.

2.3.1 Closed Sets

We begin, as always, with a definition.

Definition 2.43 (Closed)**.** Fix a topological space $(X, \mathcal{T})$. A subset $V \subseteq X$ is *closed* if and only if $(X \setminus V) \in \mathcal{T}$.

Here are some basic properties.

Lemma 2.44. Fix a topological space $(X, \mathcal{T})$.

(a) The set $\varnothing$ and X are both closed.

(b) Arbitrary intersection: given a collection of closed sets $\mathcal{V}$, the intersection $\bigcap_{V \in \mathcal{V}} V$ is closed.

(c) Finite union: given a finite collection of closed sets $\{V_1, \ldots, V_n\}$, the union $\bigcup_{i=1}^n V_i$ is closed.

Proof. We proceed in sequence.

(a) Note that $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ are both open so $\varnothing$ and X are closed.

(b) Arbitrary intersection: observe that

$$X \Bigg\backslash \bigcap_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} (X \setminus V)$$

is an arbitrary union of open sets and therefore open. Thus, $\bigcap_{V \in \mathcal{V}} V$ is closed.

(c) Finite union: observe that

$$X \Bigg\backslash \bigcup_{i=1}^n V_i = \bigcap_{i=1}^n (X \setminus V_i)$$

is the finite intersection of open sets and therefore open. Thus, $\bigcup_{i=1}^n V_i$ is closed. ■

Remark 2.45. Observe that both X and $\varnothing$ are both open and closed. This is allowed.

Example 2.46. Fix a metric space (X, d). Then any closed ball $\overline{B(x_0, r)}$ is closed: we need to show

$$U \coloneqq X \setminus \overline{B(x_0, r)} = \{x \in X : d(x, x_0) > r\}$$

is open. Well, for any $y \in U$, we see $d(y, x_0) > r$, so set $\varepsilon_y \coloneqq d(y, x_0) - r$, so $y' \in B(y, \varepsilon_y)$ has $d(x_0, y') \ge d(x_0, y) - d(y, y') > r$. Thus, any $y \in U$ has $B(y, \varepsilon_y) \subseteq U$, finishing.

Remark 2.47. In $\mathbb{R}^2$ with the Euclidean metric,

$$\bigcup_{\varepsilon < 1}^{\infty} \overline{B(0, \varepsilon)} = \left\{x \in \mathbb{R}^2 : d(0, x) < \varepsilon \text{ for some } \varepsilon < 1\right\} = B(0, 1)$$

is not closed. Indeed, we need to show $U \coloneqq X \setminus B(0, 1) = \left\{x \in \mathbb{R}^2 : d(0, x) \ge 1\right\}$ is not open. Well, note $(1, 0) \in U$, but any $\varepsilon > 0$ has $(1 - \varepsilon/2, 0) \in B((1, 0), \varepsilon)$ despite $(1 - \varepsilon/2, 0) \notin U$. Thus, U is not open.

Remark 2.48. One can define a topology by defining its closed sets to satisfy the axioms of Lemma 2.44. Then one defines the open sets as the complements of open sets.

Remark 2.49. Aligned with Remark 2.48, one can show that a function $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is continuous if and only if $f^{-1}(V)$ is closed for all closed subsets $V\subseteq Y$.

- If f is continuous, then note any closed subset $V\subseteq Y$ has $Y\setminus V$ open, so $f^{-1}(Y\setminus V)=X\setminus f^{-1}(V)$ is open, so $f^{-1}(V)$ is closed.
- If f preserves closed sets, then any open subset $U\subseteq Y$ has $Y\setminus U$ closed, so $f^{-1}(Y\setminus U)=X\setminus f^{-1}(U)$ is closed, so $f^{-1}(U)$ is open.

In the case of metric spaces, we also have the following characterization of metric spaces.

Lemma 2.50. Fix a metric space (X,d) and $V\subseteq X$. The following are equivalent.

(a) V is closed.

(b) Any sequence $\{x_n\}_{n\in\mathbb{N}}$ in V which converges to a point $x\in X$ actually converges to $x\in V$.

Proof. In one direction, suppose V is closed, and suppose $x_n\to x$ as $n\to\infty$ with $x\notin V$. Then we show that some $n\in\mathbb{N}$ has $x_n\notin V$. Well, $x\in X\setminus V$, and $X\setminus V$ is open, so there is some $\varepsilon>0$ with

$$B(x,\varepsilon)\subseteq X\setminus V.$$

However, $x_n\to x$ as $n\to\infty$ promises some large n such that $d(x,x_n)<\varepsilon$, implying that $x_n\in X\setminus V$ and so $x_n\notin V$.

In the other direction, suppose V is not closed. Then $X\setminus V$ is not open, so we can find $x\in X\setminus V$ for which there is no $\varepsilon>0$ with $B(x,\varepsilon)\subseteq X\setminus V$. As such, $x\notin V$ but $B(x,1/n)\cap V\neq\varnothing$ for all $n\in\mathbb{N}$, so just pick up some

$$x_n\in B(x,1/n)\cap V$$

for each $n\in\mathbb{N}$. As such, $d(x,x_n)<1/n$ for all $n\in\mathbb{N}$, so $x_n\to x$ as $n\to\infty$ (take $N=1/\varepsilon$), and $x_n\in V$ for all $n\in\mathbb{N}$, but the limit x does not live in V. ■

Remark 2.51. The reason we are not generalizing the above lemma to arbitrary topological spaces is because we haven't generalized convergence yet.

Corollary 2.52. Fix a complete metric space (X,d). Then a closed subset $V\subseteq X$ given the restricted metric is also complete.

Proof. Suppose a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in V is Cauchy. Embedding back in X, this sequence is still Cauchy in X, so it has a limit $x\in X$. But Lemma 2.50 then promises $x\in V$, so $\{x_n\}_{n\in\mathbb{N}}$ does in fact have a limit x in V. ■

2.3.2 Closures

Given a general set, we can define the closure as follows.

Definition 2.53 (Closure). Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we define the *closure* as

$$\overline{S} := \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V.$$

Lemma 2.54. Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, the closure $\overline{S}$ is the unique smallest closed set containing S.

Proof. Note that

$$\overline{S} := \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V$$

is closed as the arbitrary intersection of closed sets, by Lemma 2.44. To see that $\overline{S}$ is a minimal such closed set, note that any closed V containing S must have $\overline{S} \subseteq V$ by definition of $\overline{S}$.

Lastly, to see that $\overline{S}$ is unique, note that if we have two minimal closed sets $\overline{S}_1$ and $\overline{S}_2$ containing S, then note $\overline{S}_1 \cap \overline{S}_2$ are both closed sets containing S by Lemma 2.44, so minimality forces $\overline{S}_1 = \overline{S}_1 \cap \overline{S}_2 = \overline{S}_2$. ■

Example 2.55. If $S \subseteq X$ is closed, then we see

$$S \subseteq \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V \subseteq S$$

because S is a closed set containing S. Thus, $S = \overline{S}$.

Here is a more concrete way to work with the closure.

Lemma 2.56. Fix a topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$. Then $x \in \overline{A}$ if and only if every open subset $U \subseteq X$ containing x has $U \cap A \neq \varnothing$.

Proof. In one direction, if there exists an open subset $U \subseteq X$ containing x such that $U \cap A \neq \varnothing$, then $A \subseteq X \setminus U$. By definition of the closure, it follows $\overline{A} \subseteq X \setminus U$, so $x \notin X \setminus U$ ensures $x \notin \overline{A}$.

In the other direction, suppose $x \notin \overline{A}$. Then $X \setminus \overline{A}$ is an open subset containing x (note $\overline{A}$ is closed by Lemma 2.54), and

$$A \cap (X \setminus \overline{A}) \subseteq \overline{A} \cap (X \setminus \overline{A}) = \varnothing,$$

so we have found an open set containing x disjoint from A. ■

With the notation, we note that we can move our notion of density from metric spaces to general topology.

Lemma 2.57. Fix a metric space (X, d). Then $S \subseteq X$ is dense if and only if $\overline{S} = X$.

Proof. In one direction, suppose that S is not dense in X, and we show $\overline{S} \subsetneq X$. Well, we are granted $x \in X$ and $\varepsilon > 0$ such that $S \cap B(x, \varepsilon) = \varnothing$, so $S \subseteq X \setminus B(x, \varepsilon)$. However, $X \setminus B(x, \varepsilon)$ is closed, so

$$\overline{S} \subseteq X \setminus B(x, \varepsilon) \subsetneq X,$$

as needed.

In the other direction, suppose $\overline{S} \subsetneq X$, and we show that S is not dense in X. Well, find $x \in X \setminus \overline{S}$. Because $X \setminus \overline{S}$ is open, we may find $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq X \setminus \overline{S}$, implying that

$$B(x, \varepsilon) \cap S \subseteq B(x, \varepsilon) \cap \overline{S} = \varnothing,$$

making S not dense in X. ■

Thus, we can generalize our definition as follows.

Definition 2.58 (Dense)**.** Fix a topological space $(X, \mathcal{T})$. Given subsets $A \subseteq B$, we say A is *dense* in B if and only if $B \subseteq \overline{A}$.

Remark 2.59. We are not requiring that B be closed for the definition of density. For example, $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{Q}$.

2.3.3 The Product Topology

Let's see more examples of induced topologies. We start with the easiest example of the product topology.

Definition 2.60 (Product topology)**.** Fix topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. The *product topology* on $X_1 \times X_2$ is the topology induced by the canonical projection mappings

$$\pi_1 \colon X_1 \times X_2 \to X_1 \qquad \text{and} \qquad \pi_2 \colon X_1 \times X_2 \to X_2.$$

We now give the following more concrete description of the product topology.

Lemma 2.61. Fix topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. The product topology $\mathcal{T}$ on $X := X_1 \times X_2$ has a base given by

$$\mathcal{B} := \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}.$$

Proof. The product topology is the minimal topology making $\pi_1 \colon X_1 \times X_2 \to X_1$ and $\pi_2 \colon X_1 \times X_2 \to X_2$ continuous. Namely, the product topology has a sub-base given by the sets

$$\pi_1^{-1}(U_1) = U_1 \times X_2 \qquad \text{and} \qquad \pi_2^{-1}(U_2) = X_1 \times U_2$$

for any $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$. Using Example 2.33, we let $\mathcal{I}$ denote the finite intersections of these open sets and note $\mathcal{I}$ is a base for our topology.

Now, we finish by claiming $\mathcal{B} = \mathcal{I}$. On one hand, any $U_1 \times U_2 \in \mathcal{B}$ with $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$ can be written as the finite intersection

$$U_1 \times U_2 = (U_1 \times X_2) \cap (X_1 \times U_2) = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \in \mathcal{I}.$$

On the other hand, pick finitely many sets of the form $\pi_1^{-1}(U_1)$ and $\pi_2^{-1}(U_2)$; dividing them into their classes, we can write our finite collection of sets as in $\{U_1^{(i)} \times X_2\}_{i=1}^m$ or $\{X_1 \times U_2^{(j)}\}_{j=1}^n$. Their intersection is

$$\left(\bigcap_{i=1}^m U_1^{(i)} \times X_2\right) \cap \left(\bigcap_{j=1}^n X_1 \times U_2^{(j)}\right) = \underbrace{\left(\bigcap_{i=1}^m U_1^{(i)}\right)}_{U_1 :=} \cap \underbrace{\left(\bigcap_{j=1}^n U_2^{(j)}\right)}_{U_2 :=}.$$

Now, $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ are finite intersection of open sets and therefore open, so our finite intersection takes the form $U_1 \times U_2$ and thus lives in $\mathcal{B}$. ■

Remark 2.62. Later in life we will discuss measurable sets, which are not quite topologies but will have similar ideas in spirit. For example, they will also care deeply about "rectangles."

We can define this more generally.

Definition 2.63 (Product topology)**.** Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. The *product topology* on $X := \prod_{\alpha\in\lambda} X_\alpha$ is induced by the canonical projection maps

$$\pi_\alpha \colon X \to X_\alpha.$$

Here is our more concrete description.

Lemma 2.64. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. The product topology on $X := \prod_{\alpha\in\lambda}$ has a base

$$\mathcal{B} := \left\{ \prod_{\alpha\in\lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha, U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \right\}.$$

Proof. We are immediately given the sub-base of $\mathcal{S} := \{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha\}$. Using Example 2.33, we let $\mathcal{I}$ denote the finite intersections of $\mathcal{S}$ so that $\mathcal{I}$ is a base for our product topology.

As before, we finish by claiming $\mathcal{I} = \mathcal{B}$. To stay organized, we proceed in steps.

- We show $\mathcal{B} \subseteq \mathcal{I}$. Namely, for any $\prod_{\alpha\in\lambda} U_\alpha$ in $\mathcal{B}$, we set $\lambda' := \{\alpha : U_\alpha \neq X_\alpha\}$, which we know must be finite. Then
$$\prod_{\alpha\in\lambda} U_\alpha = \bigcap_{\alpha\in\lambda} \pi^{-1}(U_\alpha) = \bigcap_{\alpha\in\lambda'} \pi_\alpha^{-1}(U_\alpha)$$
because $\pi^{-1}(X_\alpha) = X$. The right-hand side is indeed a finite intersection of elements of $\mathcal{S}$ and therefore in $\mathcal{I}$.

- We show $\mathcal{S} \subseteq \mathcal{B}$. For a given β and $U_\beta \in \mathcal{T}_\beta$, set $U_\alpha := X_\alpha$ for each $\alpha \neq \beta$. Then we see that
$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha\in\lambda} U_\alpha$$
is in $\mathcal{B}$ because $U_\alpha = X_\alpha$ for all but a single $\alpha \in \lambda$.

- We show $\mathcal{B}$ is closed under finite intersection. By induction, it suffices to pick up $U, U' \in \mathcal{B}$ and show that $U \cap U' \in \mathcal{B}$. Indeed, write
$$U = \prod_{\alpha\in\lambda} U_\alpha \qquad \text{and} \qquad U' = \prod_{\alpha\in\lambda} U'_\alpha,$$
where $\lambda_0 = \{\alpha : U_\alpha \neq X_\alpha\}$ and $\lambda'_0 = \{\alpha : U'_\alpha \neq X_\alpha\}$ are both finite. Then
$$U \cap U' = \prod_{\alpha\in\lambda} (U_\alpha \cap U'_\alpha),$$
and we have $U_\alpha \cap U'_\alpha = X_\alpha$ whenever $\alpha \notin (\lambda_0 \cup \lambda'_0)$, which is only finitely many exceptions because both λ_0 and λ_0 are finite.

- We show $\mathcal{I} \subseteq \mathcal{B}$. Indeed, $\mathcal{I}$ is made of the finite intersections of $\mathcal{S}$, and we see that $\mathcal{B}$ does indeed contain the finite intersections of $\mathcal{S}$ because $\mathcal{B}$ contains the finite intersections of itself, and $\mathcal{S} \subseteq \mathcal{B}$. ■

Remark 2.65. If λ is finite, then the arguments of Lemma 2.61 generalize to give the cleaner base

$$\left\{\prod_{\alpha\in\lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha\right\}.$$

This also follows directly from Lemma 2.64, where we note that the "finitely many exceptions" actually permits all $\alpha \in \lambda$ to be an exception because λ is finite.

Example 2.66. Give $\{0,1\}$ the discrete topology. Then the space $X := \{0,1\}^{\mathbb{N}}$ given the product topology does not have

$$U := \prod_{n\in\mathbb{N}} \{0\}$$

open in X even though $\{0\} \subseteq \{0,1\}$ is always open. To see this, we note U has only a single element. On the other hand, for U to be open, Lemma 2.64 tells us U must contain a basis element B of the form

$$B := \prod_{n\in\mathbb{N}} U_n$$

where $U_n = \{0,1\}$ for all but finitely many n. However, B is infinite as the infinite product of sets containing more than 1 element, so $B \not\subseteq U$.

We quickly remark that the product topology satisfies the following universal property.

Lemma 2.67. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$, and give the product $X := \prod_{\alpha\in\lambda} X_\alpha$ the projections $\pi_\alpha \colon X \to X_\alpha$ and the product topology $\mathcal{T}$. Given a topological space $(Y, \mathcal{T}_Y)$ and continuous maps $f_\alpha \colon Y \to X_\alpha$, there is a unique continuous map $f \colon Y \to X$ such that $f_\alpha = \pi_\alpha \circ f$ for each $\alpha \in \lambda$.

Proof. We show uniqueness and existence separately.

- Uniqueness: suppose both f and f' satisfy that $f_\alpha = \pi_\alpha \circ f = \pi_\alpha \circ f'$ for each $\alpha \in \lambda$. Then, for some $y \in Y$, we see that $f(y) = (x_\alpha)_{\alpha\in\lambda}$ and $f'(y) = (x'_\alpha)_{\alpha\in\lambda}$ have

$$x_\beta = (\pi_\beta \circ f)(y) = f_\beta(y) = (\pi_\beta \circ f')(y) = x'_\beta$$

for each $\beta \in \lambda$. So we conclude that $f(y) = f'(y)$ on all inputs. Observe that we have not used continuity anywhere.

- Existence: define $f \colon Y \to X$ by

$$f(y) := (f_\alpha(y))_{\alpha\in\lambda}.$$

We now need to check that f is continuous. By Proposition 2.31, it suffices to check this on the subbase of Lemma 2.64. In particular, pick up some finite $\lambda' \subseteq \lambda$ and set $U_\alpha \in \mathcal{T}_\alpha$ for each $\alpha \in \lambda$ while $U_\alpha = X_\alpha$ for $\alpha \notin \lambda'$. Then our basis element is

$$U := \prod_{\alpha\in\lambda} U_\alpha.$$

In particular,

$$\begin{aligned} f^{-1}(U) &= \{y \in Y : f_\alpha(y) \in U_\alpha \text{ for all } \alpha \in \lambda\} \\ &= \bigcap_{\alpha\in\lambda} f_\alpha^{-1}(U_\alpha) \\ &= \left(\bigcap_{\alpha\in\lambda'} f_\alpha^{-1}(U_\alpha)\right) \cap \left(\bigcap_{\alpha\notin\lambda'} f_\alpha^{-1}(\underbrace{U_\alpha}_{X_\alpha})\right), \end{aligned}$$

which is open because the left term is a finite intersection of open sets and the right term is just Y. ■

Corollary 2.68. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Give the product $X := \prod_{\alpha\in\lambda} X_\alpha$ the projections $\pi_\alpha \colon X \to X_\alpha$ and the product topology $\mathcal{T}$. Given a topological space $(Y, \mathcal{T}_Y)$, a function $f \colon Y \to X$ is continuous if and only if the compositions $\pi_\alpha \circ f$ are continuous.

Proof. Certainly if f is continuous, then the continuity of π_α means that each $\pi_\alpha \circ f$ is continuous.

Conversely, set $f_\alpha := \pi_\alpha \circ f$ to be a continuous map $f_\alpha \colon Y \to X_\alpha$. Then Lemma 2.67 promises us a unique continuous map $\widetilde{f} \colon Y \to X$ such that

$$\pi_\alpha \circ \widetilde{f} = f_\alpha = \pi_\alpha \circ f.$$

However, the uniqueness proof of Lemma 2.67 showed that there is in fact one unique map of sets whose projections under π_α are f_α, so we conclude $f = \widetilde{f}$. Thus, f is continuous. ■

2.3.4 Comments on the Dual Space

Given a vector space V with a norm $\|\cdot\|$, we might be interested in the linear functionals on V, but because V is a metric space, we should actually be looking at the continuous linear functional. One can show (in Math 202B) that one has "plenty" of continuous linear functionals. Here is a lemma we will use a few times.

Lemma 2.69. Let $\|\cdot\|$ be a norm on an $\mathbb{R}$-vector space V. Then a linear functional $f \colon V \to \mathbb{R}$ is continuous if and only if there exists a real number $c > 0$ such that

$$|f(v)| \le c\,\|v\| \tag{2.1}$$

for all $v \in V$.

Proof. In one direction, suppose that we can find a real number $c > 0$ satisfying (2.1) for all $v \in V$. To show f is continuous, we use Lemma 1.54: suppose that we have a sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \to v$ as $n \to \infty$. Then, for any $\varepsilon > 0$, find N such that $n > N$ implies

$$\|v - v_n\| < \varepsilon/c$$

so that

$$|f(v) - f(v_n)| \le c\,\|v - v_n\| < \varepsilon.$$

Conversely, suppose that f is continuous. Note that we don't have to worry about $v = 0$ because this gives equality. Now, we can find $\delta > 0$ such that $\|v\| < \delta$ implies $|f(v)| < 1$. It follows that any nonzero $v \in V$ will have

$$\left\| \frac{\delta}{2\,\|v\|} v \right\| < \delta,$$

so we see

$$|f(v)| = \frac{2\,\|v\|}{\delta} \left| f\left(\frac{\delta}{2\,\|v\|} v \right) \right| \le \frac{2}{\delta} \cdot \|v\|,$$

so $c := 2/\delta$ will do the trick. ■

Here is an example.

Exercise 2.70. Give $V := C([0,1])$ a p-norm $\|\cdot\|_p$ for some $p \ge 1$ or $p = \infty$. Then $g \in C([0,1])$ defines a continuous linear functional

$$\varphi_g \colon f \mapsto \int_0^1 f(t)g(t)\,dt.$$

Proof. To show φ_g is linear, pick up any $r_1, r_2 \in \mathbb{R}$ and $f_1, f_2 \in V$; then

$$\varphi_g(r_1 f_1 + r_2 f_2) = \int_0^1 (r_1 f_1 + r_2 f_2)(t) g(t)\, dt = r_1 \int_0^1 f_1(t) g(t)\, dt + r_2 \int_0^1 f_2(t) g(t)\, dt = r_1 \varphi_g(f_1) + r_2 \varphi_g(f_2).$$

Checking continuity is a little more involved. Note $|g|$ is a continuous function on a compact set $[0,1]$ and therefore has a maximum M. We now use Lemma 2.69; we have two cases.

- Suppose $p = \infty$. Then, for any $f \in V$, we see

  $$|\varphi_g(f)| = \left| \int_0^1 f(t) g(t)\, dt \right| \le M \int_0^1 |f(t)|\, dt \le M \|f\|_\infty ,$$

  which finishes by Lemma 2.69.
- Suppose $p \ge 1$ is finite. To begin, we note

  $$|\varphi_g(f)| = \left| \int_0^1 f(t) g(t)\, dt \right| \le M \int_0^1 |f(t)|\, dt.$$

  Now, because the function $x \mapsto x^p$ is convex, we see that

  $$\left(\int_0^1 |f(t)|\, dt \right)^p \le \int_0^1 |f(t)|^p\, dt = \|f\|_p^p ,$$

  so $|\varphi_g(f)| \le M \|f\|_p$. Lemma 2.69 finishes. ■

Even though the linear functionals we found were continuous for all $\|\cdot\|_p$, it is possible to find linear functionals continuous for some of our norms but not others.

Exercise 2.71. Fix $V := C([0,1])$, and select some $t_0 \in [0,1]$. Then

$$\varphi \colon f \mapsto f(t_0)$$

defines a linear functional on V which is continuous for $\|\cdot\|_\infty$ but not for $\|\cdot\|_p$ for any finite $p \ge 1$.

Proof. To see continuity with $\|\cdot\|_\infty$, we note that any $f \in V$ has

$$|\varphi(f)| = |f(t_0)| \le \|f\|_\infty ,$$

so Lemma 2.69 finishes.

We now show that φ is not continuous for a fixed $\|\cdot\|_p$, where $p \ge 1$ is finite. Using Lemma 2.69, we just have to show that the ratio $|\varphi(v)| / \|v\|_p$ is unbounded for $v \in V$. For this, we define $f_c \colon [0,1] \to \mathbb{R}$ by

$$f(t) := \max\left\{0, c - c^{2p+1}(t - t_0)^2\right\}.$$

The idea here is that f has a sharp bump at t_0. Now, f is a continuous function on $[0,1]$ because it is the composition of continuous functions, so $f \in V$. We can compute

$$\|f\|_p = \left(\int_0^1 |f(t)|^p\, dt \right)^{1/p}.$$

Now, $f(t)$ will only be nonzero when $c - c^{2p+1}(t - t_0)^2 \ge 0$, which is equivalent to $t - t_0 \in (-c^{-p}, c^{-p})$, so we bound

$$\|f\|_p^p = \int_0^1 |f(t)|^p\, dt \le \int_{-c^{-p}}^{c^{-p}} \left(c - c^{2p+1} z^2\right) dz \le 2c^{1-p}.$$

Notably, as $c \to \infty$, we have that $\|f\|_p \le 2^{1/p} \cdot c^{1/p-1}$ is bounded, but $|\varphi(f)| = c$ grows unbounded. Thus, φ is discontinuous. ■

Remark 2.72. Now, we have exhibited many continuous functions

$$\varphi_g\colon C([0,1])\to\mathbb{R},$$

so we can ask for the topology on $C([0,1])$ induced by these. It turns out that this induced topology is much weaker than any individual norm topology; this topology is often called the weak topology determined by $C([0,1])$.

Remark 2.73. By the end of the class, we will have a reasonable notion of the dual space of $\|\cdot\|_1$ and $\|\cdot\|_2$. The dual space for $\|\cdot\|_\infty$ will come up in Math 202B.

Remark 2.74. Still working with $C([0,1])$ given a specific norm $\|\cdot\|_p$, one can show that any $g\in C([0,1])$ has some $r_g\in\mathbb{R}$ with

$$\varphi_g(B(0,1))\subseteq B(0,r_g).$$

It turns out to be helpful to be able to consider the product topology on the (very large) product

$$\prod_{g\in C([0,1])} B(0,r_g).$$

2.4 September 7

It's another day of sun.

2.4.1 Quotient Spaces

Here is a different way to induce a topology, the reverse of the induced topology.

Definition 2.75 (Final topology)**.** Fix a set Y and some topological spaces $\{(X_\alpha,\mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Given functions $f_\alpha\colon X_\alpha\to Y$, we define the *final topology* on Y to be the "strongest" (i.e., with the most open sets) making the f_α continuous.

Remark 2.76. Note that certainly some topology on Y exists making the f_α continuous because we can give Y the indiscrete topology, where $f_\alpha^{-1}(\varnothing)=\varnothing$ and $f_\alpha^{-1}(Y)=X_\alpha$ are open for each $\alpha\in\lambda$.

Here is a more concrete description.

Lemma 2.77. Fix a set Y and some topological spaces $\{(X_\alpha,\mathcal{T}_\alpha)\}_{\alpha\in\lambda}$, with functions $f_\alpha\colon X_\alpha\to Y$. Then the final topology is

$$\mathcal{T}\coloneqq\bigcap_{\alpha\in\lambda}\left\{S\subseteq Y: f_\alpha^{-1}(S)\in\mathcal{T}_\alpha\right\}.$$

Proof. Certainly each $\left\{S\subseteq Y: f_\alpha^{-1}(S)\in\mathcal{T}_\alpha\right\}$ is a topology by Lemma 2.30, as is their intersection by Proposition 2.20. Thus, $\mathcal{T}$ is a topology.

It remains to show that $\mathcal{T}$ is the strongest topology making each of the f_α continuous. Well, suppose $\mathcal{T}'$ is a topology making each of the f_α continuous. Then, for each $U\in\mathcal{T}'$, we have

$$f_\alpha^{-1}(U)\in\mathcal{T}_\alpha\text{ for each }\alpha\in\lambda,$$

so $U\in\mathcal{T}$ follows. Thus, $\mathcal{T}'\subseteq\mathcal{T}$. ■

We will be primarily interested in the case with just one function.

Remark 2.78. In the case of one function, which is Lemma 2.30, note that we might as well assume that $f\colon X \to Y$ is onto for otherwise we might as well just pass to the relative topology on $\operatorname{im} f$. To be explicit, we see $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open if and only if $f^{-1}(U \cap \operatorname{im} f)$ is open if and only if $U \cap \operatorname{im} f$ is open.

We are now ready to define the quotient space.

Lemma 2.79. Given sets $f\colon X \to Y$, there is an equivalence relation $\sim$ on X with $x \sim x'$ if and only if $f(x) = f(x')$.

Proof. We check the conditions one at a time. Find $x, x', x'' \in X$.

- Reflexive: note $f(x) = f(x)$, so $x \sim x$.
- Symmetric: if $x \sim x'$, then $f(x) = f(x')$, so $f(x') = f(x)$, so $x' \sim x$.
- Transitive: if $x \sim x'$ and $x' \sim x''$, then $f(x) = f(x') = f(x'')$, so $f(x) = f(x'')$, so $x \sim x''$. ■

With an equivalence relation, we may consider the set of equivalence classes $X/{\sim}$.

Remark 2.80. Conversely, given some partition $P \subseteq \mathcal{P}(X)$ of X, we can define $f\colon X \to P$ by $f\colon x \mapsto [x]$, where $[x] \in P$ is the element of P containing x. (Note $[x] \in P$ exists and is well-defined because P is a partition.) The point is that surjective functions give rise to equivalence relations, and equivalence relations give rise to surjective functions.

Anyway, here is our definition.

Definition 2.81 (Quotient topology)**.** Fix an equivalence relation $\sim$ on a set X with a topology $\mathcal{T}$. Then the *quotient topology* on $X/{\sim}$ is the final topology for the natural projection $X \twoheadrightarrow X/{\sim}$.

It turns out that we can talk about the quotient space by universal property as well.

Proposition 2.82. Fix an equivalence relation $\sim$ on a set X with a topology $\mathcal{T}$; let $\pi\colon X \twoheadrightarrow (X/{\sim})$ be the natural projection. Then, for any continuous map $f\colon X \to Z$ such that any $x \sim x'$ has $f(x) = f(x')$, there is a unique continuous map $\overline{f}\colon (X/{\sim}) \to Z$ such that

$$f = \overline{f} \circ \pi.$$

Proof. We show uniqueness and existence separately.

- Uniqueness: for any $[x] \in (X/{\sim})$, we see that we must have
$$\overline{f}([x]) = \overline{f}(\pi(x)) = f(x),$$
so $\overline{f}([x])$ is forced by our other data.
- Existence: for each $[x] \in (X/{\sim})$, define $\overline{f}([x]) := f(x)$. Note that this is well-defined: if $[x] = [x']$, then $x \sim x'$, so $f(x) = f(x')$ by hypothesis.
  It remains to show that $\overline{f}$ is continuous. Well, for an open set $U \subseteq Z$, we note that
$$\overline{f}^{-1}(U) = \{[x] : \overline{f}([x]) \in U\} = \{[x] : f(x) \in U\} = \pi\left(f^{-1}(U)\right).$$
  Now, $\pi^{-1}\left(\pi\left(f^{-1}(U)\right)\right) = f^{-1}(U)$ because $x \in \pi^{-1}\left(\pi\left(f^{-1}(U)\right)\right)$ if and only if $\pi(x) \in \pi\left(f^{-1}(U)\right)$, which is equivalent to there being $x' \in f^{-1}(U)$ with $\pi(x) = \pi(x')$, which is equivalent to there being x' with $x \sim x'$ while $f(x) = f(x') \in U$.
  Thus, $\pi^{-1}\left(\pi\left(f^{-1}(U)\right)\right)$ is open, so it follows $\pi\left(f^{-1}(U)\right) \subseteq (X/{\sim})$ is open. ■

2.4.2 Homeomorphism

Homeomorphisms are isomorphisms in our category Top. To be technical, here is our definition.

Definition 2.83 (Homeomorphism)**.** A function $f: X \to Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a *homeomorphism* if and only if f is continuous and has a continuous inverse. Formally, we require a continuous map $g: Y \to X$ such that

$$f \circ g = \mathrm{id}_Y \qquad \text{and} \qquad g \circ f = \mathrm{id}_X.$$

Warning 2.84. It is not enough for f to be continuous and bijective to be a homeomorphism. The hypothesis that the inverse function be continuous is necessary.

Remark 2.85. The definition above does not require that f be bijective, but this follows from f having an inverse.

Here are some examples.

Example 2.86. Fix a nonzero real number a and a real number b. Then the function $\varphi_{a,b}: \mathbb{R} \to \mathbb{R}$ by $\varphi_{a,b}(x) := ax + b$ is continuous: checking this on the subbase (which is enough by Proposition 2.31), we compute $\varphi_{a,b}^{-1}((c,d)) = ((c-b)/a, (d-b)/a)$. The inverse function is $\varphi_{1/a,-b/a}$—note $\varphi_{1/a,-b/a}(\varphi_{a,b}(x)) = \varphi_{a,b}(\varphi_{1/a,-b/a}(x)) = x$—which is continuous for the same reason, so this function $\varphi_{a,b}$ is a homeomorphism.

Lemma 2.87. Fix a homeomorphism $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$. Further, for any subset $S \subseteq X$, give S and $f(S)$ their respective relative topologies. Then the restriction $f|_S: S \to f(S)$ is a homeomorphism.

Proof. For clarity, let $g: Y \to X$ be the inverse function for f; note that $g(f(S)) = \{g(f(x)) : x \in S\} = S$, so $g|_{f(S)}: f(S) \to S$. Observe that we still have $g(f(x)) = x$ and $f(g(y))$ for each $x \in X$ and $y \in Y$, so $f|_S$ and $g|_S$ are inverse functions by restricting these equations.

It remains to see that f and g are continuous. We will show that f is continuous, and g will follow by symmetry. Well, for an open subset $U \cap f(S) \subseteq f(S)$ (where $U \subseteq X$ is open), we see

$$f|_S^{-1}(U \cap f(S)) = \{x \in S : f(x) \in U \cap f(S)\} = S \cap \{x \in X : f(x) \in U\} \cap \{x \in S : f(x) \in f(S)\} = S \cap f^{-1}(U),$$

which is indeed open in the relative topology of S. ■

Example 2.88. Fix real numbers $b > a$. Continuing from Example 2.86, $\varphi_{a,b}: \mathbb{R} \to \mathbb{R}$ restricts by Lemma 2.87 to a homeomorphism

$$\varphi_{b-a,a}|_{[0,1]}: [0,1] \to [a,b].$$

Namely, $x \in [0,1]$ if and only if $0 \le x \le 1$ if and only if $a \le (b-a)x + a \le b$ if and only if $\varphi_{b-a,a}(x) \in [a,b]$.

Example 2.89. Give $\mathbb{R}$ the Euclidean topology, and let $\mathbb{R}_d$ be the real numbers with the discrete topology. Then the identity function $\iota: \mathbb{R}_d \to \mathbb{R}$ is continuous because all functions from the discrete topology are continuous. However, ι is its own inverse, and the inverse function

$$\pi: \mathbb{R} \to \mathbb{R}_d$$

(which is also the identity on $\mathbb{R}$) is not continuous. For example, $\pi^{-1}(\{0\}) = \{0\}$ is not open in $\mathbb{R}$ (by Remark 2.11) even though $\mathbb{R} \setminus \{0\} \subseteq \mathbb{R}_d$ is open.

Here are some more exotic examples.

Exercise 2.90. Give $X := [0,1]$ the subspace topology, and define the equivalence relation $\sim$ as having equivalence classes $\{0,1\}$ and $\{r\}$ for each $r \in (0,1)$. Then the quotient topology $X/{\sim}$ is homeomorphic to $S^1 \subseteq \mathbb{C}$.

Proof. We note that $\sim$ is an equivalence relation because its equivalence classes are a partition. Now, we define the maps

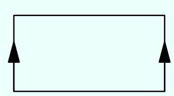
$$\begin{array}{ccc} (X/{\sim}) & \cong & S^1 \\ t & \mapsto & e^{2\pi i t} \\ \theta/2\pi & \leftarrow\!\shortmid & e^{i\theta} \end{array}$$

which we can see to be well-defined inverse. Note that $\mathbb{R} \to \mathbb{C}$ by $t \mapsto e^{it}$ is continuous by complex analysis (it's in fact holomorphic). Restricting, we get the continuous map $[0,1] \to S^1$, and then we can see that we can mod out by $0 \sim 1$ because they both go to the same place (using Proposition 2.82). One can check by hand that the inverse map is continuous, but we won't bother. ■

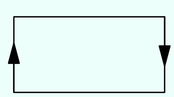
Remark 2.91 (Nir)**.** Here is a quick way to see that the inverse map is continuous: any continuous bijection $f\colon (X/{\sim}) \to S^1$ with $(X/{\sim})$ compact—which is true because X is compact—and S^1 Hausdorff will send closed subsets $V \subseteq (X/{\sim})$ (which are compact) to compact subsets of S^1 (which are closed). Thus, f is a closed map, so its inverse is continuous because f is bijective.

For the next few examples, we won't be very rigorous because we haven't provided good definitions of the relevant spaces.

Example 2.92. Give $X := [0,2] \times [0,1]$ the subspace topology, and define the equivalence relation $\sim$ as requiring $(0,r) \sim (2,r)$ only. Then X is homeomorphic to a circle by gluing its edges. One might draw X as follows.



Example 2.93. Continuing with the drawing style of Example 2.92, we have that



is the Möbius strip.

Remark 2.94. Note that these homeomorphisms do not care for the metric of our spaces. All that matters is the continuity.

Example 2.95. Let X be the unit sphere in $\mathbb{R}^3$ with the subspace topology, and define the equivalence relation on X by equivalence classes $\{v, -v\}$ for each $v \in X$. Then $X/{\sim}$ turns out to be $\mathbb{RP}^2$, which is hard to visualize.

2.4.3 Group Actions

A space might even have interesting homeomorphisms to itself.

Example 2.96. Fix a real number θ. The circle S^1 in $\mathbb{C}$ (given the subspace topology) has the rotation homeomorphism

$$r_\theta \colon e^{it} \mapsto e^{i(t+\theta)}.$$

Remark 2.97. In general, given a topological space $(X, \mathcal{T})$, we can make the group of homeomorphisms $\operatorname{Aut}(X)$ of homeomorphisms whose operation is composition.

This gives the following definition.

Definition 2.98 (Group action)**.** A *group action* by a group G on a topological space X is a group homomorphism

$$\varphi_\bullet \colon G \to \operatorname{Aut}(X).$$

Example 2.99. The group $\langle\sigma\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on a normed vector space $(V, \|\cdot\|)$ by sending σ^k to

$$\varphi_{\sigma^k} \cdot v := (-1)^k v.$$

Notably, φ_{σ^k} is continuous and its own inverse for any k, so it is a homeomorphism. In fact, we can see directly that $\varphi_{\sigma^k} \circ \varphi_{\sigma^\ell} = \varphi_{\sigma^{k+\ell}}$.

Notably, with a group action comes a partition.

Definition 2.100 (Orbit)**.** Let G act on a topological space X by $\varphi_\bullet \colon G \to \operatorname{Aut}(X)$. Then the G-*orbit* Gx of a point $x \in x$ is the set

$$Gx := \{\varphi_g(x) : g \in G\}.$$

We denote the set of all orbits $\mathcal{O}_x$ be X/G.

Remark 2.101. Note that the map $x \mapsto \mathcal{O}_x$ is a well-defined (surjective) map $X \to X/G$. In particular, we need to know that $x \in \mathcal{O}_{x'}$ implies that $\mathcal{O}_x = \mathcal{O}_{x'}$ so that there is exactly one orbit containing x. Well, $x \in \mathcal{O}_{x'}$ means we can find $g_0 \in G$ such that $x = \varphi_{g_0}(x')$, so

$$\mathcal{O}_x = \{\varphi_g(x) : g \in G\} = \{\varphi_g(\varphi_{g_0}(x')) : g \in G\} = \{\varphi_{gg_0}(x') : g \in G\} \subseteq \mathcal{O}_{x'}.$$

Conversely, we note that $x' = \varphi_{g_0^{-1}}(x)$, so $\mathcal{O}_{x'} \subseteq \mathcal{O}_x$ follows, giving equality.

Thus, the G-orbits partition X, so we can give the set X/G the quotient topology as the final topology of the natural projection $X \twoheadrightarrow X/G$.

THEME 3

BUILDING FUNCTIONS

I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial.

—Andre Weil, [Wei59]

3.1 September 9

The fun continues. The next problem set is going to be long but only in words, not in what we actually have to prove. We are being told not to be intimidated.

Remark 3.1. We are about to transition from making topologies to coming up with adjectives which will give "lots" of continuous maps to, say, the real numbers. A rigorization of this shall be provided shortly.

3.1.1 Normal Spaces

Last class we briefly mentioned the Hausdorff property.

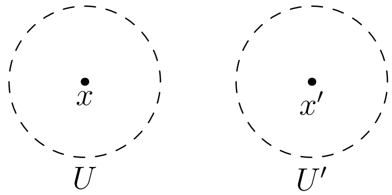
Definition 3.2 (Hausdroff)**.** Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is *Hausdorff* if and only if, for any two distinct points $x, x' \in X$, there are disjoint open sets U and U' such that $x \in U$ and $x' \in U'$.

Example 3.3. A metric space (X, d) is Hausdorff. Indeed, given distinct points $x, x' \in X$, we have $d(x, x') > 0$, so we set $r := \frac{1}{2}d(x, x')$. Then $x \in B(x, r)$ and $x' \in B(x', r)$ (which are open sets by Example 2.6), we see $B(x, r) \cap B(x', r) = \varnothing$. Indeed, if we had $y \in B(x, r) \cap B(x', r)$, then we must have

$$d(x, x') \le d(x, y) + d(x', y) < 2r = d(x, x'),$$

which is a contradiction.

Here is the image



Here is another adjective.

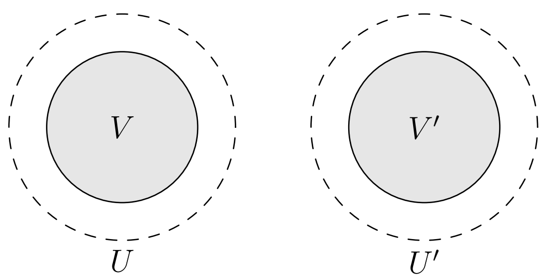
Definition 3.4 (Normal)**.** Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is *Hausdorff* if and only if, for any two disjoint closed sets $V, V' \subseteq X$, there are disjoint open sets U and U' such that $V \in U$ and $V' \in U'$.

Remark 3.5. Intuitively, Hausdorff is approximately the normal property with singleton sets. In particular, some authors require "Hausdorff" in the definition of a normal space. We will not do this.

Example 3.6. Any set X given the indiscrete topology is normal. The problem here is that the only closed sets $\{\varnothing, X\}$, so the only possible pair of disjoint closed sets are $V_1 := \varnothing$ and $V_2 := \varnothing$, for which the open sets $U_1 := \varnothing$ and $U_2 := \varnothing$ are disjoint and cover these.

Example 3.7. A set X with more than 2 elements is normal, as shown in the previous example, but it is not Hausdorff. Namely, finding distinct points $x_1, x_2 \in X$, the only open subset of X containing x_1 or x_2 is X, so there are no disjoint open subsets U_1 containing x_1 and U_2 containing x_2.

Here is the image.



It is not completely obvious that metric spaces are normal, but we will see that they are.

Here is the main result for today.

Theorem 3.8 (Urysohn's lemma)**.** Fix a topological space $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is normal, then for any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f \colon X \to [0,1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

So the point here is to realize Remark 3.1, where being normal is implying that we have "lots" of continuous functions.

Remark 3.9. Certainly if a topological space $(X, \mathcal{T})$ satisfies the conclusion of Theorem 3.8, then $(X, \mathcal{T})$ is normal. Indeed, for any disjoint closed subsets $V_0, V_1 \subseteq X$, pick up the promised continuous function f. Then

$$V_0 \subseteq f^{-1}((-1/2, 1/2)) \qquad \text{and} \qquad V_1 \subseteq f^{-1}((1/2, 3/2))$$

are disjoint open sets; namely, these are open because f is continuous, and they are disjoint because $f^{-1}((-1/2,1/2)) \cap f^{-1}((1/2,3/2)) = f^{-1}\big((-1/2,1/2) \cap (1/2,3/2)\big) = f^{-1}(\varnothing) = \varnothing$.

3.1.2 Urysohn's Lemma: Metric Spaces

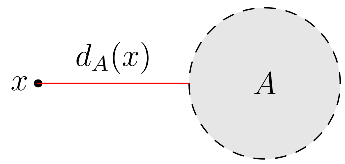
Let's see Theorem 3.8 for metric spaces, which will prove that metric spaces are normal by Remark 3.9. We pick up the following definition.

Definition 3.10. Fix a metric space (X, d). Then we define, for any $x \in X$ and nonempty subset $A \subseteq X$,

$$d_A(x) := \inf_{a \in A} d(x, a).$$

Remark 3.11. The infimum here exists because A is nonempty, so the set $\{d(x, a) : a \in A\}$ is nonempty (and bounded below by 0).

The image is that $d_A(x)$ is the distance from x to A.



We have the following continuity check.

Lemma 3.12. Fix a metric space (X, d). Then, for any nonempty subset $A \subseteq X$, the function $d_A \colon X \to \mathbb{R}$ is Lipschitz continuous.

Proof. Fix any $x, y \in X$. Then, for any given $a \in A$, we find that

$$d_A(x) \le d(x, a) \le d(x, y) + d(y, a).$$

Thus, $d_A(x) - d(x, y) \le d(y, a)$ for all $a \in A$, so we conclude that

$$d_A(x) - d(x, y) \le \inf_{a \in A} d(y, a) = d_A(y),$$

so $d_A(x) - d_A(y) \le d(x, y)$. By symmetry, we also have $d_A(y) - d_A(x) \le d(x, y)$, so it follows

$$\left|d_A(x) - d_A(y)\right| \le d(x, y),$$

which is what we need for our Lipschitz continuous. ■

As a sanity-check that this function behaves like it should, we pick up the following.

Lemma 3.13. Fix a metric space (X, d). Then, for any nonempty subset $A \subseteq X$, we have

$$d_A^{-1}(\{0\}) = \overline{A}.$$

Proof. Certainly $A \subseteq d_A^{-1}(\{0\})$ because $d_A(a) = 0$ for all $a \in A$. (In particular, $d_A(x) \ge 0$ everywhere, and $a \in A$ implies that $d_A(a) \le d(a, a) = 0$.) Because d_A is continuous by Lemma 3.12, we see $d_A^{-1}(\{0\})$ is closed, so containing A forces

$$\overline{A} \subseteq d_A^{-1}(\{0\}).$$

Conversely, suppose that $x \notin X \setminus \overline{A}$, and we show that $d_A(x) > 0$. Indeed, $X \setminus \overline{A}$ is open, so there is some open ball $B(x, \varepsilon)$ with $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq X \setminus \overline{A}$. It follows $B(x, \varepsilon) \cap \overline{A} = \varnothing$, so

$$d(a, x) \ge \varepsilon$$

for all $a \in A$. Thus, $d_A(x) \ge \varepsilon > 0$, so $d_A(x) \ne 0$. ■

Example 3.14. If $A \subseteq X$ is a dense subset, then $\overline{A} = X$, so $d_A : X \to \mathbb{R}$ is the constantly zero function.

Example 3.15. If $A \subseteq X$ is closed, then $\overline{A} = A$ by Example 2.55, so $d_A^{-1}(\{0\}) = A$. In other words, we have $x \in A$ if and only if $d_A(x) = 0$.

Let's now show Theorem 3.8 for metric spaces.

Proposition 3.16. Fix a metric space (X, d). For any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f : X \to [0,1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

Proof. The point is to use the Lipschitz continuous functions d_{V_0}, d_{V_1}. Then we define

$$f(x) := \frac{d_{V_0}(x)}{d_{V_0}(x) + d_{V_1}(x)}.$$

Note that defining $f : X \to \mathbb{R}$ does not have division-by-zero problems: because $d_{V_0}(x), d_{V_1}(x) \geq 0$, the only way to get zero in the denominator is by $d_{V_0}(x) = d_{V_1}(x) = 0$. However, this forces $x \in V_0 \cap V_1$ by Lemma 3.13 because V_0 and V_1 are closed, but in fact $V_0 \cap V_1 = \varnothing$.

We now run our checks on f.

- Because the quotient of two continuous functions is still continuous, we see that f is continuous.
- Using the fact that $d_A(x) \geq 0$ for any nonempty $A \subseteq X$ and $x \in X$, we find

  $$f(x) = \frac{d_{V_0}(x)}{d_{V_0}(x) + d_{V_1}(x)} \geq 0,$$

  and

  $$f(x) = 1 - \frac{d_{V_1}(x)}{d_{V_0}(x) + d_{V_1}(x)} \leq 1,$$

  so $\operatorname{im} f \subseteq [0,1]$.
- If $x \in V_0$, then $d_{V_0}(x) = 0$, so $f(x) = 0/(0 + d_{V_1}(x)) = 0$. If $x \in V_1$, then $d_{V_1}(x) = 0$, so $f(x) = d_{V_0}(x)/(d_{V_0}(x) + 0) = 1$. ■

And here is our check.

Corollary 3.17. Any metric space (X, d) is normal.

Proof. Plug Proposition 3.16 into Remark 3.9. ■

3.1.3 Urysohn's Lemma: The General Case

We will not prove the general case of Theorem 3.8 today, but we will make some progress. Here is a useful lemma.

Lemma 3.18. Fix a normal topological space $(X, \mathcal{T})$. Given a closed subset $V \subseteq X$ and an open subset $U_0 \subseteq X$ with $V \subseteq U_0$, there is an open set U such that

$$V \subseteq U \subseteq \overline{U} \subseteq U_0.$$

Proof. Because $V \subseteq U_0$, we define $V' := X \setminus U_0$, which is closed because U_0 is open. Further, $V' \subseteq X \setminus U_0 \subseteq X \setminus V$ forces $V \cap V' = \varnothing$. Thus, using the normality of $(X, \mathcal{T})$, we are promised disjoint open sets U and U' such that

$$V \subseteq U \qquad \text{and} \qquad V' \subseteq U'.$$

In particular, we see that

$$U \subseteq X \setminus U'$$

while $X \setminus U'$ is closed by definition. Thus, by definition of the closure, $\overline{U} \subseteq X \setminus U' \subseteq X \setminus V' = U_0$. This finishes the proof. ■

3.2 September 12

There are still no questions.

3.2.1 Urysohn's Lemma: The General Case

We continue the proof from last class.

> **Theorem 3.8** (Urysohn's lemma)**.** Fix a topological space $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is normal, then for any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f\colon X \to [0,1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

Proof. To begin, define $U_1 := X \setminus V_1$, which is open because V_1 is closed; notably $V_0 \subseteq U_1$. The idea here is that the points of U_1 will take value at most 1. Now, by Lemma 3.18, we find $U_{1/2}$ with

$$V_0 \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_1.$$

Intuitively, we are going to let f take values at most $1/2$ on $U_{1/2}$. Using Lemma 3.18 again, we can find $U_{1/2}$ with

$$V_0 \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2},$$

and now our function will take values at most $1/4$ on $U_{1/4}$. On the other side, we can use the containment $\overline{U_{1/2}} \subseteq U_1$ in Lemma 3.18 to find $U_{3/4}$ such that

$$\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U_1,$$

and here $U_{3/4}$ our function should take values less than $3/4$.

We can then continue the process for eights and then off to infinity. Let's describe what we have at the end of this inductive process. Set $\Delta := \{k/2^n : 0 < k \le 2^n\}$ to be the set of "dyadic" rationals in $(0,1]$; notably Δ is dense in $[0,1]$.[1] Then each $r \in \Delta$, we get an open set $U_r \subseteq X$. These have the following properties.

- Any $r, s \in \Delta$ with $r < s$ has $\overline{U_r} \subseteq U_s$.
- By construction $U_1 = X \setminus V_1$.
- Also, $V_0 \subseteq U_r$ for all $r \in \Delta$.

We now define

$$f(x) := \begin{cases} 1 & x \in V_1, \\ \inf\{r \in \Delta : x \in U_r\} & x \notin V_1, \end{cases}$$

where $x \notin V_1$ in the second case promises $x \in U_1$ so that the infimum in the second line makes sense. We now run the following checks on f.

[1] The fact we need is that $a, b \in [0,1]$ with $a < b$ have $r \in \Delta$ between them. Well, multiply $b - a$ by a suitably large power of 2 so that $2^n(b-a) > 1$, so there is an integer k in this interval between $2^n a$ and $2^n b$, so $a < k/2^n < b$.

- Note that $\operatorname{im} f(x) \subseteq \overline{\Delta} = [0,1]$.
- By the construction of these open sets, we have $f(x) = 1$ if $x \in V_1$.
- Further, $f(x) < r$ for all $r \in \Delta$ if $x \in V_0$, so $f(x) = 0$ for $x \in V_0$.
- It remains to check that f is continuous. For this, we use Proposition 2.31 to check the continuity on a subbase. Specifically, we use sets of the form $[0,a)$ and $(a,1]$ for $a \in (0,1)$. Indeed, note $[0,a) \cap (b,1] = (a,b)$, so intersections of these can give all open intervals strictly contained $[0,1]$; adding in the "open" intervals $[0,a)$ and $(a,0]$ make all the open intervals in $[0,1]$, which are a basis for our topology.

  We now proceed with our check; fix some $a \in (0,1)$.

  - Note that $x \in X$ has $f(x) < a$ if and only if there is some $r \in \Delta$ such that $f(x) < r < a$ (by density of Δ) if and only if there is some $r \in \Delta$ such that $x \in U_r$ and $r < a$ (by definition of the infimum). As such,
  $$f^{-1}([0,a)) = \bigcup_{r<a} U_r.$$
  - Note that $x \in X$ has $f(x) > a$ if and only if there is an $r, s \in \Delta$ with $f(x) > r > s > a$ (by density). It follows $x \notin U_r$, which contains $\overline{U_s}$, so $x \notin \overline{U_s}$ for some $s \in \Delta$ with $s > a$.
  On the other hand, $x \notin \overline{U_s}$ for some $s \in \Delta$ with $s > a$ implies that $x \notin U_r$ for any $r \in \Delta$ with $r > s > a$, so it follows $f(x) \geq s > a$.
  Thus, $f(x) > a$ if and only if $x \notin \overline{U_s}$ for $s \in \Delta$ with $s > a$, implying
  $$f^{-1}((a,1]) = \bigcup_{s>a} (X \setminus \overline{U_s}).$$

The above checks complete the proof. ∎

Remark 3.19. We could not have f output to $\mathbb{Q} \cap [0,1]$ because we used the completeness of $\mathbb{R}$ in the construction of f.

Remark 3.20. It is somewhat noticeable that we have not discussed sequences at all in this class yet, even though they were featured prominently in metric space topology. The reason we have been avoiding them is that we prefer to use open sets and not points to study general topological spaces.

3.2.2 Bounded Functions

We are going to want a little functional analysis before we continue.

Definition 3.21 (Bounded)**.** Fix a metric space (X,d) and a nonempty set A. A subset $A \subseteq X$ is *bounded* if and only if there is an open ball $B(x,r)$ containing A. More generally, a function $f\colon A \to X$ is *bounded* if and only if $\operatorname{im} f \subseteq X$ is bounded, and we let $B(A,X)$ denote the set of all bounded functions $f\colon A \to X$.

We will be particularly interested in the case where X is a normed vector space.

The point of defining bounded functions is that we can provide them with a metric.

Definition 3.22 (Uniform metric)**.** Fix a nonempty set X and a metric space (Y,d). Then the *uniform metric* is the function $d_u\colon B(X,Y)^2 \to \mathbb{R}_{\geq 0}$ defined by
$$d_u(f,g) \coloneqq \sup\{d(f(x),g(x)) : x \in X\}.$$

Lemma 3.23. Fix a set X and a metric space (Y, d). Then the uniform metric d_u on $B(X, Y)$ is a metric.

Proof. Here are our checks; fix $f, g, h \in B(X, Y)$.

- Well-defined: because f and g bounded, we can find open balls $B(a, r)$ and $B(b, s)$ containing $\operatorname{im} f$ and $\operatorname{im} g$ respectively. It follows that, for any $x \in X$, we have

$$d(f(x), g(x)) \le d(f(x), a) + d(a, b) + d(b, g(x)) \le r + d(a, b) + s,$$

so the set $\{d(f(x), g(x)) : x \in X\}$ has an upper bound and hence a supremum.

- Nonnegative: fixing a particular $x \in X$, note $d_u(f, g) \ge d(f(x), g(x)) \ge 0$.
- Zero: note $d_u(f, f)$ is $\sup\{d(f(x), f(x)) : x \in X\} = \sup\{0 : x \in X\} = 0$.
- Zero: note $d_u(f, g) = 0$ implies that $\sup\{d(f(x), g(x)) : x \in X\} = 0$, so $d(f(x), g(x)) \le 0$ for all $x \in X$, so $d(f(x), g(x)) = 0$ for all $x \in X$, so $f(x) = g(x)$ for all $x \in X$.
- Symmetric: note

$$d_u(f, g) = \sup\{d(f(x), g(x)) : x \in X\} = \sup\{d(g(x), f(x)) : x \in X\} = d_u(g, f).$$

- Triangle inequality: note that

$$d(f(x), h(x)) \le d(f(x), g(x)) + d(g(x), h(x)) = d_u(f, g) + d_u(g, h)$$

for all $x \in X$, so it follows $d_u(f, h) \le d_u(f, g) + d_u(g, h)$ by taking the supremum. ∎

Here is why we like this metric.

Proposition 3.24. Fix a set X and a complete metric space (Y, d). Then $B(X, Y)$ given the uniform metric is complete.

Proof. Fix a Cauchy sequence $\{f_n\}_{n \in \mathbb{N}}$ in $B(X, Y)$. Namely, for all $\varepsilon > 0$, there exists some N so that

$$n, m > N \implies d(f_n(x), f_m(x)) < \varepsilon$$

for all $x \in X$. In particular, fixing some particular $x \in X$, we see that $\{f_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Y, so the completeness of Y promises some limit $f(x)$.

It remains to check that the data of f assembles to a function $f \in B(X, Y)$. Well, any (fixed) $\varepsilon > 0$ promises an N so that $n, m > N$ forces $d(f_n(x), f_m(x)) < \varepsilon$ for all $x \in X$. Now, fixing some $x \in X$, any $\delta > 0$ has some N' large enough so that $m > N'$ has $d(f_m(x), f(x)) < \delta$, meaning that $n, m > \max\{N, N'\}$ gives

$$d(f_n(x), f(x)) \le d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \varepsilon + \delta$$

for all $\delta > 0$. Thus, fixing some $n > N$, we see $d(f_n(x), f(x)) \le \varepsilon$ for all $x \in X$.

To finish, we note $f_n \in B(X, Y)$ is bounded, so there is an open ball $B(a, r)$ containing $\operatorname{im} f_n$. Thus, for all $x \in X$,

$$d(a, f(x)) \le d(a, f_n(x)) + d(f_n(x), f(x)) < r + \varepsilon,$$

so $\operatorname{im} f \subseteq B(a, r + \varepsilon)$. ∎

We close with the following result.

Proposition 3.25. Fix a topological space $(X, \mathcal{T})$ and a metric space (Y, d). Let $B_c(X, Y) \subseteq B(X, Y)$ denote the metric subspace of bounded continuous functions $f\colon X \to Y$. Then $B_c(X, Y)$ is a closed subspace of $B(X, Y)$. In particular, if (Y, d) is complete, then $B_c(X, Y)$ is also complete.

Proof. Note that the second claim follows from the first claim by Corollary 2.52; thus, we focus on the first claim. For this, we use Lemma 2.50: fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ of bounded continuous functions such that $f_n \to f$ as $n \to \infty$ where $f\colon X \to Y$ is just some bounded function. We need to show that f is continuous.

Well, fix an open set $U \subseteq Y$ so that we need to show $f^{-1}(U) \subseteq X$ is open. For this, we pick up any element $x \in f^{-1}(U)$, and we find an open neighborhood $U_x \subseteq f^{-1}(U)$ containing x; this will finish because it shows

$$f^{-1}(U) \subseteq \bigcup_{x\in U} U_x \subseteq f^{-1}(U),$$

so $f^{-1}(U)$ is the arbitrary union of open sets.

We now proceed with the proof directly.

1. Because $f(x) \in U$, and U is open, there is some $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$.

2. Because $\{f_n\}_{n\in\mathbb{N}}$ converges to f, there is a sufficiently large N so that $n > N$ has $d(f_n(y), f(y)) < \varepsilon/2$ for all $y \in X$. Fix some $n > N$.

3. Now, for all $y \in f_n^{-1}(B(f(x), \varepsilon/2))$, we see

$$d(f(y), f(x)) \le d(f(y), f_n(y)) + d(f_n(y), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so $f(y) \in U$. As such, we see that $f_n^{-1}(B(f(x), \varepsilon/2))$ is open (because f_n is continuous), it contains x, and it is contained in $f^{-1}(U)$.

The above open neighborhood completes the proof of the first claim. ■

3.3 September 14

The march continues.

3.3.1 The Tietze Extension Theorem

Here is the main result for today.

Theorem 3.26 (Tietze extension)**.** Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to \mathbb{R}$, there exists a continuous function $\widetilde{f}\colon X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$. In fact, if $\operatorname{im} f \subseteq [a, b]$, then we may enforce $\operatorname{im} \widetilde{f} \subseteq [a, b]$ as well.

This property is quite special to $\mathbb{R}$ shared by a few other spaces.

Example 3.27. Take $X := \overline{B(0, 1)} \subseteq \mathbb{R}^2$ given the relative topology, and let $A = \partial X$ be the boundary, which is the unit circle. Then the identity function $\operatorname{id}_A\colon A \to A$ does not extend continuously to a function $\widetilde{\operatorname{id}_A}\colon X \to A$. To see this rigorously, take a course in algebraic topology.

Example 3.28. Of course, any set Y given the indiscrete topology will be such that a continuous function $f\colon A \to Y$ can be extended to continuously to a function $\widetilde{f}\colon X \to Y$ because all functions to Y are continuous for free.

Remark 3.29. The condition of $\operatorname{im} f \subseteq [a,b]$ might as well be replaced by $\operatorname{im} f \subseteq [0,1]$ by using the homeomorphism $\mathbb{R} \to \mathbb{R}$ by $x \mapsto (x-a)/(b-b)$ which will send $[a,b]$ to $[0,1]$.

Here is a lemma which will help the proof of Theorem 3.26.

Lemma 3.30. Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to [0,r]$ (where $r > 0$), there exists a continuous function $g\colon X \to [0, r/3]$ such that

$$0 \le f(a) - g(a) \le 2r/3$$

for each $a \in A$.

Proof. Set $B := \{x \in A : f(x) \le r/3\} = f^{-1}([0, r/3])$ and $C := \{x \in A : f(x) \ge 2r/3\} = f^{-1}([(2r/3, r])$. Both $B, C \subseteq A$ and C are closed because they are the pre-image of closed subsets under $f\colon A \to \mathbb{R}$. In fact, by the relative topology, we can write $B = B' \cap A$ where $B' \subseteq X$ is closed. However, B' and A are both closed in X, so $B \subseteq X$ is closed. Similar holds for C.

Thus, so Urysohn's lemma provides (Theorem 3.8) a continuous function $g\colon X \to [0,1]$ such that $g|_B = 0$ and $g|_C = 1$. As such, we define $g\colon X \to [0, r/3]$ by

$$g(x) := (r/3) \cdot g(x),$$

which is still continuous because the map $x \mapsto (r/3)x$ is a homeomorphism $[0,1] \to [0, r/3]$ by Example 2.88. We can now see that g satisfies the needed properties. Fix some $a \in A$.

- If $a \in B$, then $g(a) = 0$ while $f(a) \le r/3$, so $0 \le f(a) - g(a) \le r/3$.
- If $a \in C$, then $g(a) = r/3$ while $f(a) \in [2r/3, r]$, so $0 \le f(a) - g(a) \le 2r/3$.
- Lastly, $a \notin B$ and $a \notin C$ means that $r/3 < f(a) < 2r/3$ while $0 \le g(a) \le r/3$, so it follows $0 \le f(a) - g(a) \le 2r/3$ still.

The above checks finish. ■

We now show the following special case of Theorem 3.26.

Proposition 3.31. Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to [0,1]$, there exists a continuous function $\widetilde{f}\colon X \to [0,1]$ such that $\widetilde{f}|_A = f$.

Proof. For brevity, define $\sigma := 2/3$. Taking $r = 1$ in Lemma 3.30, we get a function $g_1\colon X \to [0, 1/3]$ with

$$0 \le f(a) - g_1(a) \le \sigma$$

for all $a \in A$, so define $\widetilde{f}_1 := g_1$. Next applying Lemma 3.30 to $(f - \widetilde{f}_1|_A)\colon A \to [0, \sigma]$ with $r = \sigma$, we get promised a function $g_2\colon X \to [0, \sigma/3]$ with

$$0 \le f(a) - \widetilde{f}_1(a) - g_2(a) \le \sigma^2$$

for any $a \in A$, so define $\widetilde{f}_2 := \widetilde{f}_1 + g_2$.

In general, suppose given a function $\widetilde{f}_n\colon X \to [0,1]$ with

$$0 \le f(a) - \widetilde{f}_n(a) \le \sigma^n$$

for $a \in A$, we can use Lemma 3.30 to $(f - \widetilde{f}_n|_A)\colon A \to [0, \sigma^n]$ to get a function $g_{n+1}\colon X \to [0, \sigma^n/3]$ with

$$0 \le f(a) - \widetilde{f}_n(a) - g_{n+1}(a) \le \sigma^{n+1}$$

for $a \in A$, allowing us to then set $\widetilde{f}_{n+1} := \widetilde{f}_n + g_{n+1}$.

Applying the above process inductively, we get a function

$$\widetilde{f}_n = \sum_{k=1}^{n} g_k$$

going to $[0,1]$ such that $\|g_k\|_\infty \le \sigma^{k-1}/3$ and $0 \le f(a) - \widetilde{f}_n(a) \le (2/3)^n$ for each $a \in A$ and $n \ge 1$. Notably, using the uniform metric d_u, we see that any $n \ge m$ has

$$d_u(\widetilde{f}_n, \widetilde{f}_m) = \sup_{x\in X}\left(\sum_{k=m+1}^{n} g_k(x)\right) \le \sum_{k=m+1}^{n} \frac{1}{3}\sigma^{k-1} \le \frac{\sigma^m}{3}\sum_{k=0}^{\infty}\sigma^k = \frac{\sigma^m}{3}\cdot\frac{1}{1-\sigma} = \left(\frac{2}{3}\right)^m,$$

which gets arbitrarily small. Thus, $\{\widetilde{f}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence: for any $\varepsilon > 0$, we can find N with $n > N$ having $(2/3)^n < \varepsilon$, meaning $n, m \ge N$ will have $d_u(\widetilde{f}_n, \widetilde{f}_m) < \varepsilon$. Now, because $[0,1] \subseteq \mathbb{R}$ is a closed subset of a complete metric space and hence complete by Corollary 2.52, the sequence $\{\widetilde{f}_n\}_{n\in\mathbb{N}}$ converges to a continuous function $\widetilde{f}\colon X \to [0,1]$ by Proposition 3.25.

It remains to check that $\widetilde{f}|_A = f$. Well, any $a \in A$ and $n \in \mathbb{N}$ have

$$|f(a) - \widetilde{f}(a)| \le |f(a) - \widetilde{f}_n(a)| + |\widetilde{f}_n(a) - \widetilde{f}(a)| \le \left(\frac{2}{3}\right)^n + |\widetilde{f}_n(a) - f(a)|.$$

Because $\widetilde{f}_n \to f$ as $n \to \infty$ under the metric d_u, we see that $|\widetilde{f}_n(a) - f(a)| \to 0$ as $n \to \infty$. Additionally, $(2/3)^n \to 0$ as $n \to \infty$, so the entire right-hand side goes to 0 as $n \to \infty$, meaning that $|f(a) - \widetilde{f}(a)| < \varepsilon$ for all $\varepsilon > 0$. Thus, $f(a) = \widetilde{f}(a)$ for each $a \in A$. ■

3.4 September 16

We continue the proof from last class.

3.4.1 The Tietze Extension Theorem: Proof

And here is the proof of the general case of Theorem 3.26.

> **Theorem 3.26** (Tietze extension)**.** Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to \mathbb{R}$, there exists a continuous function $\widetilde{f}\colon X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$. In fact, if $\operatorname{im} f \subseteq [a,b]$, then we may enforce $\operatorname{im}\widetilde{f} \subseteq [a,b]$ as well.

Proof. Fix a continuous function $f\colon A \to \mathbb{R}$. Note that there is a homeomorphism $\varphi\colon \mathbb{R} \cong (-1,1)$, so we name composite

$$A \xrightarrow{f} \mathbb{R} \overset{\varphi}{\cong} (-1,1) \subseteq [0,1]$$

g and then extend it to a function $\widetilde{g}_0\colon X \to [-1,1]$ by Proposition 3.31. We would like to go back to $(-1,1)$ and then back to $\mathbb{R}$, but it is possible for $-1, 1 \in \operatorname{im} g_0$.

Isolating the problem, we set $B := \widetilde{g}_0^{-1}(\{-1,1\})$ and note that $A \cap (B_0 \cup B_1) = \varnothing$ because $\widetilde{g}_0(A) = g(A) \subseteq (-1,1)$. Now, by normality of X, we get promised by Theorem 3.8 a continuous function $\delta\colon X \to \mathbb{R}$ such that $\delta|_B = 0$ and $\delta|_A = 1$. Thus, we define

$$\widetilde{g}(x) := \delta(x)\widetilde{g}_0(x).$$

Notably, $\widetilde{g}|_A = \delta|_A \cdot \widetilde{g}_0|_A = 1 \cdot g = g$. But now $|\widetilde{g}(x)| = 1$ would force $|\widetilde{g}_0(x)| = 1$, but this implies $\delta(x) = 0$ by construction and so $\widetilde{g}(x) = 0$; thus, $\pm 1 \notin \operatorname{im}\widetilde{g}$, so we can pull back $\widetilde{g}$ through $\varphi\colon \mathbb{R} \cong (-1,1)$ to $\mathbb{R}$. ■

3.4.2 Existence of Completions, Again

We quickly provide another proof of the existence of completions. We begin with the following example.

Example 3.32. Given any topological space $(X, \mathcal{T})$, the metric space $(B_c(X, \mathbb{R}), d_u)$ of bounded continuous functions is complete by Proposition 3.25 because $\mathbb{R}$ is complete.

More generally, we will want to remember the following definition.

Definition 3.33 (Banach space)**.** A normed vector space $(V, \|\cdot\|)$ is a *Banach space* if and only if it is complete.

As such, we pick up the following tool.

Lemma 3.34. Fix an isometry $f\colon (X,d) \to (Y, d_Y)$ of metric spaces such that (Y, d_Y) is complete. Then $\overline{f(X)}$ equipped with the induced metric from Y is a complete metric space, and it is actually a completion of (X,d) when equipped with the natural embedding $\iota\colon X \to \overline{f(X)}$ from f.

Proof. For brevity, define $\overline{X} := \overline{f(X)}$ and set $\overline{d}$ to be the metric on $\overline{X}$ induced by (Y, d_Y). In particular, $\overline{X} \subseteq Y$ is a closed subset, and so $(\overline{X}, \overline{d})$ is complete by Corollary 2.52. Now, note that $\iota\colon (X,d) \to (\overline{X}, \overline{d})$ is an isometry because, for any $x, x' \in X$,

$$d(x,x') = d_Y(f(x), f(x')) = d_Y(\iota(x), \iota(x')) = \overline{d}(\iota(x), \iota(x'))$$

using our various restriction maps.

Lastly, we have to show that $\operatorname{im}\iota \subseteq \overline{X}$ is dense. Well, by Lemma 2.57, it suffices to note

$$\overline{\operatorname{im}\iota} = \overline{f(X)} = \overline{X},$$

which is what we wanted. ■

We are now ready to prove Theorem 1.62.

Theorem 1.62. Any metric space (X,d) has a completion.

Proof. Let our metric space be (X,d). For each $x \in X$, define $f_x(y) := d(x,y)$. To embed f_x into $B_c(X,\mathbb{R})$, we would need f_x to be bounded, but it need not be. To fix this, we choose a base-point $x_0 \in X$, and define

$$h_x := f_x - f_{x_0}.$$

In particular, any $y \in X$ will have $|h_x(y)| = |d(x,y) - d(x_0,y)| \le d(x,x_0)$, so h_x is bounded, and it is continuous as the sum of two continuous functions. More explicitly, for any $\varepsilon > 0$, take $\delta = \varepsilon$ so that $d(x_1,x_2) < \delta$ implies

$$|h_x(x_1) - h_x(x_2)| = |d(x,x_1) - d(x,x_2)| \le d(x_1,x_2) < \delta = \varepsilon.$$

We now need to show that the map $h_\bullet\colon (X,d) \to (B_c(X,\mathbb{R}), d_u)$ is an isometry. Indeed,

$$d_u(h_{x_1}, h_{x_2}) = \sup_{x\in X}\{h_{x_1}(x) - h_{x_2}(x)\} = \sup_{x \in X}\{d(x_1,x) - d(x_2,x)\}.$$

This is certainly upper-bounded by $d(x_1,x_2)$ by the triangle inequality, and we do achieve $d(x_1,x_2)$ at $x = x_2$ because $d(x_1,x_2) - d(x_2,x_2) = d(x_1,x_2)$. So indeed, $d_u(h_{x_1}, h_{x_2}) = d(x_1,x_2)$.

Thus, we have provided an isometry $h_\bullet\colon (X,d) \to (B_c(X,\mathbb{R}), d_u)$ from (X,d) to the complete metric space $(B_c(X,\mathbb{R}), d_u)$ (see Example 3.32), so $\overline{h_\bullet(X)}$ is a completion for (X,d) by Lemma 3.34. ■

Remark 3.35. Despite the above construction, it is actually fairly non-obvious what functions really are in $\overline{h_\bullet(X)}$.

THEME 4

COMPACTNESS

That something so small could be so beautiful.

—Anthony Doerr, [Doe14]

4.1 September 16

We continue the lecture, into compactness.

4.1.1 Compactness

The following is perhaps the most important definition in point-set topology.

Definition 4.1 (Open cover). Fix a topological space $(X, \mathcal{T})$. An *open cover* of X is a collection $\mathcal{U} \subseteq \mathcal{T}$ of open sets such that

$$X = \bigcup_{U \in \mathcal{U}} U.$$

Definition 4.2 (Open subcover). Fix a topological space $(X, \mathcal{T})$. An *(open) subcover* $\mathcal{U}'$ of an open cover $\mathcal{U}$ is an open cover $\mathcal{U}'$ of X such that $\mathcal{U}' \subseteq \mathcal{U}$.

And here is the relevant definition.

Definition 4.3 (Compact). Fix a topological space $(X, \mathcal{T})$. We say that $(X, \mathcal{T})$ is *compact* if and only if every open cover of X has a finite subcover.

Example 4.4. The subset $[0, 1] \subseteq \mathbb{R}$ given the relative topology is compact.

In light of the previous example, it is helpful to extend our definition to subsets of a topological space.

Definition 4.5 (Compact). Fix a topological space $(X, \mathcal{T})$. A subset $A \subseteq X$ is *compact* if and only if A is compact when given the relative topology from X.

Lemma 4.6. Fix a topological space $(X,\mathcal{T})$. Then A is compact if and only if any $\mathcal{U} \subseteq \mathcal{T}$ covering A has a finite subcover covering A.

Proof. The point is to use Lemma 2.42. In one direction, suppose A is compact. Then a cover $\{U_\alpha\}_{\alpha\in\lambda} \subseteq \mathcal{T}$ of A provides the open cover by

$$V_\alpha := A \cap U_\alpha$$

of A. Indeed, $A \cap U_\alpha \subseteq A$ is open, and $\bigcup_{\alpha\in\lambda} V_\alpha = \bigcup_{\alpha\in\lambda}(A\cap U_\alpha) = A$. Thus, compactness provides a finite subset $\lambda' \subseteq \lambda$ such that $\{V_\alpha\}_{\alpha\in\lambda'}$ still covers A, so

$$A = \bigcup_{\alpha\in\lambda'} (A \cap U_\alpha) \subseteq \bigcup_{\alpha\in\lambda'} U_\alpha,$$

meaning that the finite subcover $\{U_\alpha\}_{\alpha\in\lambda'} \subseteq \{U_\alpha\}_{\alpha\in\lambda}$ still covers A.

In the other direction, suppose that each open cover of A from $\mathcal{T}$ has a finite subcover. Now, give A some open cover $\{V_\alpha\}_{\alpha\in\lambda}$ from the relative topology on A. Each open subset V_α can be written as $U_\alpha \cap A$ where $U_\alpha \subseteq X$ is open by Lemma 2.42, so we define

$$\mathcal{U} := \{U_\alpha\}_{\alpha\in\lambda}.$$

Notably, $\bigcup_{\alpha\in\lambda} U_\alpha$ contains $\bigcup_{\alpha\in\lambda} V_\alpha$, which is A, so $\mathcal{U}$ covers A and hence has a finite subset $\lambda' \subseteq \lambda$ such that $\{U_\alpha\}_{\alpha\in\lambda'}$ covers A. But then

$$A = \bigcup_{\alpha\in\lambda'} (A \cap U_\alpha) = \bigcup_{\alpha\in\lambda'} V_\alpha,$$

so $\{V_\alpha\}_{\alpha\in\lambda'}$ provides a finite subcover of $\{V_\alpha\}_{\alpha\in\lambda}$. ■

In light of the above proof, it will be helpful to extend our notion of an open cover.

Notation 4.7. Given a topological space $(X,\mathcal{T})$, we will say that some open sets $\mathcal{U} \subseteq \mathcal{T}$ form an open cover for a subset $A \subseteq U$ if and only if

$$A \subseteq \bigcup_{U\in\mathcal{U}} U.$$

Remark 4.8. We will freely use Lemma 4.6 as a "definition" of compactness without reference.

Example 4.9. Given compact subsets $A_1, A_2 \subseteq X$ of a topological space $(X,\mathcal{T})$, we see that $A_1 \cup A_2$ is also compact. Indeed, given an open cover $\mathcal{U}$ of $A_1 \cup A_2$, we see that $\mathcal{U}$ is an open cover for both A_1 and A_2, so we can find our finite subcovers $\mathcal{U}_1 \subseteq \mathcal{U}$ and $\mathcal{U}_2 \subseteq \mathcal{U}$ by the compactness of A_1 and A_2, respectively. Thus, $\mathcal{U}_1 \cup \mathcal{U}_2 \subseteq \mathcal{U}$ is a finite collection covering A_1 and A_2 and therefore covering $A_1 \cup A_2$.

Here is a quick fact about compactness.

Lemma 4.10. Fix a compact topological space $(X,\mathcal{T})$. Then any closed subset $A \subseteq X$ is compact.

Proof. By Lemma 4.6, pick up an open cover $\mathcal{U}$ of A, and we would like to find a finite subcover. Then we set

$$\mathcal{V} := \mathcal{U} \cup \{X \setminus A\}.$$

Notably, $X \setminus A$ is open in X because A is closed, so we see

$$\bigcup_{U\in\mathcal{V}} U = (X\setminus A) \cup \bigcup_{U\in\mathcal{U}} U \supseteq (X \setminus A) \cup A = X,$$

so $\mathcal{V}$ is an open cover for X. As such, we can find a finite subcover $\mathcal{V}'$ for X, and we set $\mathcal{U}' := \mathcal{V} \cap \mathcal{U}$.

We claim that $\mathcal{U}'$ is a finite subcover of $\mathcal{U}$; indeed, $\mathcal{U}' \subseteq \mathcal{V}$ is finite, and $\mathcal{U}' \subseteq \mathcal{U}$ is a subset. It remains to check that $\mathcal{U}'$ covers A. Well, for any $a \in A$, we can find some $U' \in \mathcal{V}'$ containing a because $\mathcal{V}'$ covers X. However, $a \notin X \setminus A$, so $U' \neq X \setminus A$, so actually $U' \in \mathcal{U}'$. Thus,

$$A \subseteq \bigcup_{U \in \mathcal{U}'} U,$$

which is what we wanted. ■

Example 4.11. Give $X = \mathbb{R}$ the indiscrete topology. Then X has only two open sets, so any nonempty subset $S \subseteq X$ can only be covered by $\{X\}$, which is its own finite subcover. For example, $\{0\}$ is compact in X, but it is not closed because $\mathbb{R} \setminus \{0\}$ is not open.

4.2 September 16

There are questions today.

4.2.1 Compact Hausdorff Spaces

Last class we saw in Example 4.11 that compact subsets of a topological space need not be compact. It turns out that compact subsets of Hausdorff spaces are in fact closed. Let's see this.

Lemma 4.12. Fix a Hausdorff topological space $(X, \mathcal{T})$, and let $A \subseteq X$ be compact. Then, for any $x \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $x \in V$.

Proof. For each $y \in (X \setminus A)$, the Hausdorff condition promises disjoint open sets V_y and U_y such that $y \in U_y$ and $x \in V_y$. We would like to take the union of all the U_y and the intersection of all the U_x, but the arbitrary intersection of open sets need not be open.

To fix this, we note that $\{U_y\}_{y \in A}$ are some open sets which cover A, so the compactness of A allows us some finite subset $Y \subseteq A$ such that $\{U_y\}_{y \in Y}$ covers A. As such, we set

$$U := \bigcup_{y \in Y} U_y \qquad \text{and} \qquad V := \bigcap_{y \in Y} V_y.$$

Here are our checks.

- Both U and V are open because these are a finite union and a finite intersection of open sets, respectively.
- By construction of Y, we see that $A \subseteq U$.
- Note $x \in V_y$ for all $y \in Y \subseteq A$, so $x \in V$ as well.
- Lastly, we see that U and V are disjoint: for each $z \in U$, we can find some $y \in Y$ such that $z \in U_y$, but then $z \notin V_y$ by construction, so $z \notin V$. ■

Corollary 4.13. Fix a Hausdorff topological space $(X, \mathcal{T})$, and let $A \subseteq X$ be compact. Then A is closed.

Proof. For each $x \notin A$, Lemma 4.12 grants us an open subset V_x containing x which is disjoint from A. It follows $V_x \subseteq X \setminus A$, so we may say

$$(X \setminus A) \subseteq \bigcup_{y \in X \setminus A} V_y \subseteq \bigcup_{y \in X \setminus A} (X \setminus A) = X \setminus A,$$

so $X \setminus A = \bigcup_{y \in X \setminus A} V_y$ shows that $X \setminus A$ is a union of open sets and therefore open. It follows that A is closed. ■

Corollary 4.14. Fix a compact Hausdorff topological space $(X, \mathcal{T})$. Then all closed subsets $A \subseteq X$ and $x \notin A$ have disjoint open subsets U and V with $A \subseteq U$ and $x \in V$.

Proof. Lemma 4.10 says that A is compact, so Lemma 4.12 finishes. ■

The above property is useful enough to deserve a definition.

Definition 4.15 (Regular)**.** A topological space $(X, \mathcal{T})$ is *regular* if and only if each closed subset $A \subseteq X$ and $x \notin A$ have disjoint open subsets U and V with $A \subseteq U$ and $x \in V$.

Example 4.16. Every compact Hausdorff space is regular by Corollary 4.14.

Example 4.17. Any normal, Hausdorff space is regular. For example, metric spaces are regular.

In fact, compact Hausdorff spaces are not just regular but also normal.

Proposition 4.18. Fix a compact Hausdorff space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is normal.

Proof. Fix disjoint closed subsets A and B. Then A and B are compact by Lemma 4.10.

Now, for any $y \in B$, we see $y \notin A$, so Lemma 4.12 grants us disjoint open subsets U_y and V_y such that U_y contains A and V_y contains y. As before, we see $\{V_y\}_{y \in B}$ forms an open cover of B, so the compactness of B promises a finite subset $Y \subseteq B$ such that $\{V_y\}_{y \in Y}$ still covers B. Thus, we set

$$U := \bigcap_{y \in Y} U_y \qquad \text{and} \qquad V := \bigcup_{y \in Y} V_y.$$

Here are our checks again.

- Note U is open as a finite intersection of open sets. Similarly, V is open as a union of open sets.
- By construction $A \subseteq U_y$ for each y, so $A \subseteq U$.
- By construction $\{V_y\}_{y \in Y}$ covers B, so $B \subseteq V$.
- Lastly, to see that U and V are disjoint, note that any $z \in V$ has $z \in V_y$ for some $y \in Y$, so $z \notin U_y$, so $z \notin U$. ■

4.2.2 Compact Images

We continue our fact-collection for compact spaces.

Lemma 4.19. Fix a continuous map $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$. If $(X,\mathcal{T}_X)$ is compact, then $\operatorname{im} f\subseteq Y$ is also compact.

Proof. For psychological reasons, we may assume that $\operatorname{im} f = Y$, though we will not do this.

Suppose we have an open cover $\{V_\alpha\}_{\alpha\in\lambda}\subseteq\mathcal{T}_Y$ for $\operatorname{im} f$. Then we set

$$\mathcal{U} \coloneqq \left\{f^{-1}(V_\alpha)\right\}_{\alpha\in\lambda}$$

In particular, the continuity of f promises that everyone is $\mathcal{U}$ is open. We claim $\mathcal{U}$ covers X: for any $x\in X$, we see $f(x)\in\operatorname{im} f$, so $f(x)\in V_\alpha$ for some $\alpha\in\lambda$, so $x\in f^{-1}(V_\alpha)\in\mathcal{U}$.

Thus, the compactness of X promises a finite subset $\lambda'\subseteq\lambda$ so that $\left\{f^{-1}(V_\alpha)\right\}_{\alpha\in\lambda'}$ is still an open cover for X. Thus, we can see that the finite collection of open subsets

$$\{V_\alpha\}_{\alpha\in\lambda'}\subseteq\{V_\alpha\}_{\alpha\in\lambda}$$

still covers $\operatorname{im} f$. Indeed, for any $y\in\operatorname{im} f$, find $x\in X$ with $f(x)=y$, so place $x\in f^{-1}(V_\alpha)$ for some $\alpha\in\lambda'$, so $y\in V_\alpha$. ■

Corollary 4.20. Fix a continuous function $f\colon\mathbb{R}\to\mathbb{R}$. Then, for any closed interval $[a,b]$, f achieves its maximum on $[a,b]$.

Proof. Note that $f([a,b])$ is compact, and $\mathbb{R}$ is Hausdorff, so $f([a,b])$ is also closed. Further, $f([a,b])$ is bounded because it is compact. Thus, $f([a,b])$ has all of its limit points and in particular contains its supremum. ■

We take a moment to use this machinery to build an easier test for homeomorphisms; namely, we manifest Remark 2.91.

Proposition 4.21. Fix a compact topological space $(X,\mathcal{T}_X)$ and a Hausdorff topological space $(Y,\mathcal{T}_Y)$. Then any continuous bijection $f\colon X\to Y$ is a homeomorphism.

Proof. The bijectivity of f promises some inverse function $g\colon Y\to X$, which we need to show is continuous. Well, for an open subset $U\subseteq X$, we need to show that $g^{-1}(U)$ is open. But because g is the inverse of f, we see

$$g^{-1}(U)=\{y\in Y: g(y)\in U\}=\{f(x)\in Y: g(f(x))\in U\}=f(U),$$

so we need to show that $f(U)$ is open. Taking compliments, we set $A\coloneqq X\setminus U$ so that A is closed, and we will show that $f(A)$ is closed; this will finish because the bijectivity of f forces

$$f(U)=f(X\setminus A)=f(X)\setminus f(A)=Y\setminus f(A)$$

to be open.

We are now ready to finish the proof. Because $(X,\mathcal{T}_X)$ is compact, A being closed implies that A is compact by Lemma 4.10. It follows by Lemma 4.19 that $f(A)$ is compact, so because $(Y,\mathcal{T}_Y)$ is Hausdorff, we see Lemma 4.12 forces $f(A)$ to be closed. This finishes. ■

4.2.3 Compactness via Closed Sets

It will be helpful to be able to discuss compact sets in terms of closed sets.

Lemma 4.22. A set X is covered by a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ if and only if

$$\bigcap_{S\in\mathcal{S}} (X \setminus S) = \varnothing.$$

Proof. Note

$$\bigcap_{S\in\mathcal{S}} (X \setminus S) = X \Big\backslash \bigcup_{S\in\mathcal{S}} X,$$

which is empty if and only if $\bigcup_{S\in\mathcal{S}} X = X$. ∎

Corollary 4.23. Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is compact if and only if any collection of closed subsets $\mathcal{V}$ with $\bigcap_{V\in\mathcal{V}} V = \varnothing$ has some finite subcollection $\mathcal{V}' \subseteq \mathcal{V}$ with $\bigcap_{V\in\mathcal{V}'} V = \varnothing$.

Proof. If X is compact, then note any collection of closed subsets $\mathcal{V}$ with $\bigcap_{V\in\mathcal{V}} V = \varnothing$ has

$$X = X \Big\backslash \bigcap_{V\in\mathcal{V}} V = \bigcup_{V\in\mathcal{V}} (X \setminus V),$$

so $\mathcal{U} = \{(X \setminus V) : V \in \mathcal{V}\}$ is an open cover. Thus, we can find a finite subset $\mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{U}' = \{(X \setminus V) : V \in \mathcal{V}'\}$ covers X, so it follows that $\bigcap_{V\in\mathcal{V}'} V = \varnothing$ by taking complements, as above.

Conversely, we show that X is compact. Well, pick up an open cover $\mathcal{U}$ of X. Then Lemma 4.22 says that $\mathcal{V} = \{(X \setminus V) : V \in \mathcal{V}\}$ has $\bigcap_{V\in\mathcal{V}} V = \varnothing$. By hypothesis on X, we get some finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $\bigcap_{U\in\mathcal{U}'} (X \setminus U) = \varnothing$, so Lemma 4.22 says $\mathcal{U}'$ covers X. ∎

It will be useful to have some language to describe this.

Definition 4.24 (Finite intersection property). Fix a set X. A collection $\mathcal{S} \subseteq \mathcal{P}(X)$ has the *finite intersection property* if and only if any nonempty finite subcollection $\mathcal{S}' \subseteq \mathcal{S}$ has

$$\bigcap_{S\in\mathcal{S}'} S \neq \varnothing.$$

In particular, we get the following.

Proposition 4.25. Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is compact if and only if any collection $\mathcal{V}$ of closed subsets with the finite intersection property has

$$\bigcap_{V\in\mathcal{V}} V \neq \varnothing.$$

Proof. Applying contraposition to the conclusion, we are saying that any collection $\mathcal{V}$ with $\bigcap_{V\in\mathcal{V}} V = \varnothing$ has some finite subcollection $\mathcal{V}' \subseteq \mathcal{V}$ with $\bigcap_{V\in\mathcal{V}'} V = \varnothing$. This is equivalent to $(X, \mathcal{T})$ being compact by Corollary 4.23. ∎

Remark 4.26. It is somewhat important to notice that the proof of Proposition 4.25 does not require the Axiom of Choice to prove. It is purely moving around definitions cleverly.

4.3 September 21

Today we begin talking about Tychonoff's theorem.

4.3.1 Comments on Choice

Here is our main result for today.

Theorem 4.27 (Tychonoff)**.** Fix a collection $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$ of compact topological spaces, and give the product space $X := \prod_{\alpha\in\lambda} X_\alpha$ the product topology. Then X is compact.

Notably, we are not requiring the spaces X_α to be Hausdorff.

Warning 4.28. The proof of Theorem 4.27 will be the hardest part of this course.

Remark 4.29. The reason for Warning 4.28 is that we need to at least know that X is nonempty to say anything about X at all, and an arbitrary product being nonempty is equivalent to the Axiom of Choice. In fact, Theorem 4.27 (notably not assuming that the X_α are Hausdorff!) actually implies the Axiom of Choice, as shown by John Kelly.

To prepare ourselves, we will point out a few of the main ingredients we will use. We will use the Axiom of Choice, which we will go ahead and state now.

Axiom 4.30 (Choice)**.** Given a collection of nonempty sets $\{S_\alpha\}_{\alpha\in\lambda}$, the product $\prod_{\alpha\in\lambda} S_\alpha$ is nonempty.

We will also use Zorn's lemma. To state Zorn's lemma, we begin by defining a partially ordered set and its chains.

Definition 4.31 (Poset)**.** A *partially ordered set* or *poset* is a set P equipped with a reflexive, antisymmetric, and transitive relation $\le \subseteq P \times P$.

Example 4.32. Given a set X, the power set $\mathcal{P}(X)$ is a partially ordered set under inclusion $\subseteq$. Here are the checks.

- Reflexive: for $A \in \mathcal{P}(X)$, we see $A \subseteq A$.
- Antisymmetric: for $A, B \in \mathcal{P}(X)$, we see $A \subseteq B$ and $B \subseteq A$ implies $A = B$.
- Transitive: for $A, B, C \in \mathcal{P}(X)$, we see $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.

Replacing all the $\subseteq$s with $\supseteq$s shows that $\mathcal{P}(X)$ is also a partially ordered set under containment $\supseteq$.

Posets have very natural subposets.

Definition 4.33 (Subposet)**.** Given a partially ordered $(P, \le)$, a *subposet* is a subset $S \subseteq P$ equipped with the restricted partial order $\le \cap (S \times S)$.

All the checks for $(S, \le \cap (S \times S))$ being a partially ordered set are inherited directly from P, so the proof amounts to just writing them down.

Example 4.34. Given a topological space $(X, \mathcal{T})$, we see that $\mathcal{T}$ is a subposet of $\mathcal{P}(X)$, where $\mathcal{P}(X)$ can be given the partial order $\subseteq$ or $\supseteq$ from Example 4.32.

And here are our chains.

Definition 4.35 (Chain)**.** Fix a partially ordered set $(P, \leq)$. Then a *chain* is a subset $C \subseteq P$ such that the subposet $(C, \leq)$ is totally ordered.

Zorn's lemma is interested in special kinds of partially ordered sets.

Definition 4.36 (Inductively ordered)**.** A partially ordered set $(P, \leq)$ is *inductively ordered* if and only if every chain $C \subseteq P$ has an upper bound in P. In other words, there is an element $p \in P$ such that $c \leq p$ for all $c \in C$.

And here is Zorn's lemma.

Axiom 4.37 (Zorn's lemma)**.** An inductively ordered partially ordered set $(P, \leq)$ has a maximal element.

Remark 4.38. It turns out that the Axiom of Choice (in the form of Zorn's lemma) is also equivalent to every vector space having a basis. (In one direction, given a vector space V, one can build a basis by taking a maximal linearly independent set of vectors in V.) One can get a feeling for the other direction because the $\mathbb{Q}$-vector space $\mathbb{R}$ doesn't have any "constructible" basis.

Remark 4.39. The fact that every (commutative) ring has a maximal ideal containing any given proper ideal is also equivalent to the Axiom of Choice (in the form of Zorn's lemma). Here are two examples.

- Given any set S, finding a maximal ideal of the ring $R := \mathbb{F}_2^S$ (whose operations are pointwise from $\mathbb{F}_2$) which contains the ideal $\mathbb{F}_2^{\oplus S}$ requires knowing that R is nonempty.
- The ring $R := C([0, \infty))$ of continuous $\mathbb{R}$-valued functions has the ideal
$$I := \left\{ f \in R : \lim_{x \to \infty} f(x) = 0 \right\}$$
doesn't have any constructible maximal ideals containing it.

For our next example, we define a filter.

Definition 4.40 (Filter)**.** Fix a set X. A *filter* $\mathcal{F}$ on X is a collection of nonempty subsets of X satisfying the following conditions.

(a) $\mathcal{F}$ is closed under finite intersection.

(b) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

Example 4.41. Given a topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$, the subposet $\mathcal{T}$ of $(\mathcal{P}(X), \subseteq)$ has a filter $\mathcal{F}$ of all those open subsets containing A.

Example 4.42. Given a set X, the collection of subsets containing a given point $p \in X$ is a filter and in fact a "maximal" filter.

The point is that Zorn's lemma automatically promises us maximal filters, or "ultrafilters."

Example 4.43. Fix $X := [0, \infty)$. Then the collection $\mathcal{F}$ of the subsets of $A \subseteq X$ which contain $[n, \infty)$ for some integer n is a filter. However, there is no obvious maximal filter.

4.4 September 23

We continue discussing Tychonoff's theorem.

4.4.1 Tychonoff's Theorem

Here is our statement.

Theorem 4.27 (Tychonoff)**.** Fix a collection $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$ of compact topological spaces, and give the product space $X := \prod_{\alpha\in\lambda} X_\alpha$ the product topology. Then X is compact.

Proof. We will use Proposition 4.25. For each α, let $\pi_\alpha\colon X \to X_\alpha$ denote the canonical projection. Let $\mathcal{V}$ be a collection of closed subsets of X satisfying the finite intersection property, and we will show that $\bigcap_{V\in\mathcal{V}} V$ is nonempty. We proceed in steps.

1. The beginning of this proof does not use topology. Let $\Omega_\mathcal{V}$ be the collection of families of subsets $\mathcal{F}$ of X which contain $\mathcal{V}$ and have the finite intersection property. We claim that $W_\mathcal{V}$ is inductively ordered under $\supseteq$.

   Well, let $\Omega \subseteq \Omega_\mathcal{V}$ be some chain, and we define the collection
   $$\mathcal{U} := \bigcup_{\mathcal{F}\in\Omega} \mathcal{F},$$
   which we claim is the required upper bound for Ω. Of course, each $\mathcal{F}$ contains $\mathcal{V}$, and $\mathcal{U} \supseteq \mathcal{F}$ for each $\mathcal{F}$, so $\mathcal{U}$ both contains $\mathcal{V}$ and is an upper bound for Ω. It remains to show $\mathcal{U} \in \Omega_\mathcal{V}$, for which we need to show that $\mathcal{U}$ has the finite intersection property.

   For this, find some finite subcollection of nonempty subsets $\{A_k\}_{k=1}^n \subseteq \mathcal{U}$ which we would like to show have nonempty intersection. Now, for each k, there is some $\mathcal{F}_k \in \Omega$ containing A_k, by construction of $\mathcal{U}$ as the union over Ω. Because the number of subsets is finite, and because Ω is totally ordered, we may find the largest of the $\mathcal{F}_k$, which we call $\mathcal{F}$.

   Now, $\mathcal{F} \in \Omega \subseteq \Omega_\mathcal{V}$ must have the finite intersection property, so $\{A_k\}_{k=1}^n \subseteq \mathcal{F}$ forces
   $$\bigcap_{k=1}^n A_k \neq \varnothing,$$
   which is what we wanted. This completes the proof.

2. From the previous step, Zorn's lemma promises a maximal family $\mathcal{M}$. We claim that $\mathcal{M}$ is closed under taking finite intersections. Indeed, define $\mathcal{M}'$ as the set of all finite intersections of $\mathcal{M}$, and we will show that $\mathcal{M}' = \mathcal{M}$.

   Well, certainly $\mathcal{M} \subseteq \mathcal{M}'$ because intersections of exactly one set $F \in \mathcal{M}$ will just recover $F \in \mathcal{M}'$. Thus, if we can show $\mathcal{M}' \in \Omega_\mathcal{V}$, the desired equality $\mathcal{M}' = \mathcal{M}$ will follow by maximality.

   Certainly, we of course have $\mathcal{M} \supseteq \mathcal{V}$, so $\mathcal{M}' \supseteq \mathcal{V}$ as well. So to show $\mathcal{M}' \in \Omega_\mathcal{V}$, it remains to show the finite intersection property. Well, let $\{A_k\}_{k=1}^n \subseteq \mathcal{M}'$ be some finite subcollection of nonempty subsets, and we show their intersection is nonempty. By definition of $\mathcal{M}'$, each k lets us write
   $$A_k = \bigcap_{\ell=1}^{n_k} B_{k,\ell}$$
   for some subsets $B_{k,\ell} \in \mathcal{M}$; because $A_k \in \mathcal{M}'$ is nonempty, we see that $B_{k,\ell} \in \mathcal{M}$ is nonempty, so the finite intersection property on $\mathcal{M}$ tells us that
   $$\bigcap_{k=1}^n A_k = \bigcap_{k=1}^n \bigcap_{\ell=1}^{n_k} B_{k,\ell}$$
   is nonempty, which is what we wanted.

3. We claim that if a subset $B \subseteq X$ has $B \cap A \neq \varnothing$ for each $A \in \mathcal{M}$, then in fact $B \in \mathcal{M}$. Indeed, define $\mathcal{M}'' := \mathcal{M} \cup \{B\}$, and we show $\mathcal{M}'' = \mathcal{M}$.

   Certainly $\mathcal{M} \subseteq \mathcal{M}''$, so it is enough by maximality of $\mathcal{M}$ to show $\mathcal{M}'' \in \Omega_{\mathcal{V}}$. Certainly $\mathcal{V} \subseteq \mathcal{M} \subseteq \mathcal{M}''$, so it remains to show that $\mathcal{M}''$ satisfies the finite intersection property.

   For this, pick up some finite subcollection of nonempty subsets $\{A_k\}_{k=1}^n \subseteq \mathcal{M}''$, and we show their intersection is nonempty. If none of these subsets are B, then in fact $\{A_k\}_{k=1}^n \subseteq \mathcal{M}$, so the finite intersection property for $\mathcal{M}$ forces

$$\bigcap_{k=1}^n A_k \neq \varnothing.$$

   Otherwise, say $B = A_1$ without loss of generality. Then we may assume $B \neq A_k$ for each $k > 1$, so $A_k \in \mathcal{M}$ for each $k > 1$, so we note

$$\bigcap_{k=1}^n A_k = B \cap \bigcap_{k=2}^n A_k.$$

   However, $\mathcal{M}$ is closed under finite intersection, so in fact $\bigcap_{k=2}^n A_k \in \mathcal{M}$, and by the finite intersection property, we have that $\bigcap_{k=2}^n A_k$ is nonempty. Thus, by hypothesis on B, we see

$$B \cap \bigcap_{k=2}^n A_k \neq \varnothing,$$

   which is what we wanted.

4. We now begin touching our product. For given $\alpha \in \lambda$ and $\mathcal{F} \in \Omega_{\mathcal{V}}$, we claim that

$$\pi_\alpha(\mathcal{F}) := \{\pi_\alpha(A) : A \in \mathcal{F}\}$$

   satisfies the finite intersection property. Fix a finite subcollection of nonempty subsets $\{\pi_\alpha(A_k)\}_{k=1}^n$ of $\pi_\alpha(\mathcal{F})$, and we will show its intersection is nonempty. Then we must have A_k being nonempty for each k, so the finite intersection property on $\mathcal{F}$ forces

$$\bigcap_{k=1}^n A_k \neq \varnothing.$$

   Finding some a in this intersection, we see $\pi_\alpha(a) \in \pi_\alpha(A_k)$ for each k, so $\pi_\alpha(a)$ belongs in

$$\bigcap_{k=1}^n \pi_\alpha(A_k),$$

   thus making this intersection nonempty.

5. And now the topology begins. For given α, note that

$$\overline{\mathcal{M}}_\alpha := \{\overline{\pi_\alpha(A)} : A \in \mathcal{M}\}$$

   has the finite intersection property by the previous step. Namely, any finite subcollection of nonempty subsets $\{\pi_\alpha A_k\}_{k=1}^n$ has a nonempty intersection, so writing

$$\varnothing \neq \bigcap_{k=1}^n \pi_\alpha(A_k) \subseteq \bigcap_{k=1}^n \overline{\pi_\alpha(A_k)}$$

   gives what we want. However, X_α is compact (!), so Proposition 4.25 tells us that

$$\bigcap_{A \in \overline{\mathcal{M}}_\alpha} A \neq \varnothing.$$

6. Directly invoking the Axiom of Choice, we may find some $x_\alpha \in \bigcap_{A \in \overline{\mathcal{M}_\alpha}} A$ for each α. Set $x := (x_\alpha)_{\alpha\in\lambda}$ to be the corresponding element of X.

   We claim that each nonempty $A \in \mathcal{M}$ has $x \in \overline{A}$. By Lemma 2.56, it suffices to show that every open subset U containing x has nonempty intersection with A. Because each open subset U containing x has a(n open) basis set $B \subseteq U$ containing x, it suffices to check $B \cap A \neq \varnothing$ for basis elements, and $B \cap A \subseteq U \cap A$ will give the result.

   There are three steps. Observe that we must invoke the definition of the product topology on X to talk topologically about X, so we do so here.

   (a) We begin by checking this on the sub-base. For each $\alpha \in \lambda$, fix some sub-base element $\pi_\alpha^{-1}(U_\alpha)$ (where $U_\alpha \subseteq X_\alpha$ is open) containing x, and we claim

   $$\pi_\alpha^{-1}(U_\alpha) \cap A \overset{?}{\neq} \varnothing.$$

   Well, $x \in \pi_\alpha^{-1}(U_\alpha)$ requires $x_\alpha \in U_\alpha$, but $A \in \mathcal{M}$ forces $x_\alpha \in \overline{\pi_\alpha(A)}$. Thus, $U_\alpha \cap \pi_\alpha(A) \neq \varnothing$ by Lemma 2.56, so there is some $a \in A$ with $\pi_\alpha(a) \in U_\alpha$, so $\pi_\alpha^{-1}(U_\alpha) \cap A$ is in fact nonempty.

   (b) We show that each basis set containing x lives in $\mathcal{M}$. Part (a) above added to item 3 directly shows that every sub-base open set containing x lives in $\mathcal{M}$. Thus, item 2 tells us that any finite intersection of sub-basic sets containing x live in $\mathcal{M}$ as well, but these are exactly the basic sets containing x. (Namely, any basic set is the intersection of sub-basic sets, and x living in the basic set forces x to still live in those sub-basic sets.)

   (c) It follows from the finite intersection property for $\mathcal{M}$ that any basic set B containing x has $B \in \mathcal{M}$ and therefore $A \cap B \neq \varnothing$ because A is nonempty.

   The above steps finish this part.

7. We finish the proof. Any $V \in \mathcal{V}$ is closed and has $V \in \mathcal{M}$. By the above point, we see $x \in \overline{V}$, so $x \in V$ by Example 2.55, so we have exhibited

   $$\bigcap_{V \in \mathcal{V}} V \neq \varnothing.$$

The above steps have showed that $\bigcap_{V\in\mathcal{V}} \mathcal{V} \neq \varnothing$ from $\mathcal{V}$ having the finite intersection property, so we conclude that X is compact by Proposition 4.25. ■

4.5 September 26

We begin class by finishing the proof of Tychonoff's theorem (Theorem 4.27). I have gone ahead and just edited Friday's lecture for continuity.

4.5.1 Remarks on Tychonoff's Theorem

Here are some remarks.

Remark 4.44. Intuitively, the maximal element $\mathcal{M}$ is constructed in order to become some filter focused around the single point x. Similar to maximal ideals corresponding to points, adding in all the "maximality" constraints for $\mathcal{M}$ hones in our focus to the single constructed point x.

Remark 4.45. Here is an application of Theorem 4.27. One can show that any normed vector space $(V, \|\cdot\|)$ has "lots" of continuous functionals by extending those found on a finite-dimensional subspace; let V' be the complete normed vector space of continuous linear functionals. (The norm of some $v' \in V'$ is its Lipschitz constant, using Lemma 2.69.) Fixing the unit ball B of V', one can give V' the weakest topology making all the linear functionals "from V" continuous (this is the weak-* topology), which one can show is both Hausdorff and compact (!). This is the Banach–Alaoglu theorem, and it follows from Theorem 4.27 by showing the space we want is a closed subspace of the compact space

$$\prod_{v\in V}[-\|v\|, \|v\|].$$

Remark 4.46 (β-compactification)**.** Let $A := C([0,\infty))$ be the space of bounded continuous function $[0,\infty) \to \mathbb{R}$, which we can see directly is an $\mathbb{R}$-algebra by taking r to the constant function r. Let A' be the set of continuous functions $A \to \mathbb{R}$. Notably, any $x \in \mathbb{R}$ gives a continuous ring homomorphism $A \to \mathbb{R}$ by $f \mapsto f(x)$, so we let Y be the set of all homomorphisms $A \to \mathbb{R}$. Again, A' is compact using the weak-* topology, and so Y as a closed subset of A' can be given a compact topology. Then one can show that A is homeomorphic to $C(Y)$.

4.5.2 Tychonoff's Theorem and Choice

We now show that Tychonoff's theorem implies the Axiom of Choice.

Theorem 4.47 (Kelley)**.** Tychonoff's theorem implies the Axiom of Choice.

Proof. Assume Theorem 4.27 is true. Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of nonempty sets. We want to show that

$$X := \prod_{\alpha\in\lambda} X_\alpha$$

is nonempty.

The trick is to enlarge the X_α to be able to give them a suitable topology. Choose some (set) ω which does not live in $\bigcup_\alpha X_\alpha$; for example, setting ω to be equal to this set will do (using the Axiom of Foundation). Then we set

$$Y_\alpha := X_\alpha \cup \{\omega\},$$

which we give the topology $\mathcal{T}_\alpha := \{Y_\alpha, \varnothing, X_\alpha, \{\omega\}\}$. We quickly check that this is a topology.

- Note $\varnothing$ and Y_α are open.
- Arbitrary union: let $\mathcal{U} \subseteq \mathcal{T}_\alpha$ be a collection. Note that $\mathcal{U}$ is necessarily finite, so it suffices by induction to show that $U \cup U' \in \mathcal{T}_\alpha$ for any $U, U' \in \mathcal{T}_\alpha$. We have the following cases.
  - If $U = \varnothing$ or $U' = \varnothing$, then we get $U \cup U' \in \{U, U'\} \subseteq \mathcal{T}_\alpha$.
  - If $U = Y_\alpha$ or $U' = Y_\alpha$, then $U \cup U' = Y_\alpha \in \mathcal{T}_\alpha$.
  - Note $U = U'$ gives $U \cup U' = U \in \mathcal{T}_\alpha$.
  - We have left to deal with $\{U, U'\} \subseteq \{X_\alpha, \{\omega\}\}$ where U and U' are distinct, which means we just have to check $\mathcal{T}_\alpha \cup \{\omega\} = Y_\alpha$ is open.
- Finite intersection: note that $U \in \mathcal{T}_\alpha$ implies $Y_\alpha \setminus U \in \mathcal{T}_\alpha$ because $Y_\alpha \setminus \{\omega\} = X_\alpha$ and $Y_\alpha \setminus \varnothing = Y_\alpha$, and the other checks follow. Thus, we note any finite collection $\mathcal{U} \subseteq \mathcal{T}_\alpha$ has

$$Y_\alpha \Bigg\backslash \bigcap_{U\in\mathcal{U}} U = \bigcup_{\alpha\in\lambda} (Y_\alpha \setminus U)$$

  is a union of open sets and hence open. It follows that our intersection also lives in $\mathcal{T}_\alpha$.

Additionally, because $\mathcal{T}_\alpha$ has only finitely many sets, the space $(Y_\alpha, \mathcal{T}_\alpha)$ is compact: any subcollection of $\mathcal{T}_\alpha$ is finite, so all open covers of Y_α are automatically finite. It follows that the product

$$Y \coloneqq \prod_{\alpha\in\lambda} Y_\alpha$$

is compact by applying Theorem 4.27 (!).

We will now extract out our element of X using compactness of Y via Proposition 4.25. Let $\pi_\alpha \colon Y \to Y_\alpha$ be the canonical projection. Note that $Y_\alpha \setminus X_\alpha = \{\omega\}$ is open in Y_α, so $X_\alpha \subseteq Y_\alpha$ is closed, so $V_\alpha \coloneqq \pi_\alpha^{-1}(X_\alpha)$ is a closed subset of Y by the continuity of π_α (using Remark 2.49).

We now claim that the closed sets $\{V_\alpha\}_{\alpha\in\lambda}$ satisfy the finite intersection property: given a finite subcollection $\{V_{\alpha_i}\}_{i=1}^n$, one may finitely (!) choose a point $x_{\alpha_i} \in X_{\alpha_i}$. So we define

$$y_\alpha \coloneqq \begin{cases} x_{\alpha_i} & \alpha \in \{\alpha_1, \ldots, \alpha_n\}, \\ \omega & \alpha \notin \{\alpha_1, \ldots, \alpha_n\}, \end{cases}$$

so the point $(y_\alpha)_{\alpha\in\lambda} \in Y$ has $\pi_{\alpha_i}(y) \in X_{\alpha_i}$ for each α_i, so $y_{\alpha_i} \in \pi_{\alpha_i}^{-1}(X_\alpha) = V_\alpha$, so

$$y \in \bigcap_{i=1}^n V_{\alpha_i}.$$

So we have verified the finite intersection property.

It follows from Proposition 4.25 that we can find

$$y \in \bigcap_{\alpha\in\lambda} V_\alpha.$$

However, this implies that each $\alpha \in \lambda$ has $y \in V_\alpha$ and so $\pi_\alpha(y) \in X_\alpha$. It follows that

$$y \in \prod_{\alpha\in\lambda} X_\alpha,$$

which finishes the proof. ■

Remark 4.48. The topology on Y_α need not be Hausdorff, so we needed Theorem 4.27 to allow non-Hausdorff spaces.

Theme 5
Compactness for Metric Spaces

Rarely is a picture a proof, but I hope a good picture will cement your understanding of why something is true. Seeing is believing.

—Charles C. Pugh, [Pug15]

5.1 September 28

Today we discuss compactness for metric spaces.

5.1.1 Totally Bounded Spaces

Here is a quick lemma.

Lemma 5.1. Fix a compact metric space (X,d). For any $\varepsilon > 0$, there are finitely many points $\{x_i\}_{i=1}^n$ such that

$$X = \bigcup_{i=1}^n B(x_i,\varepsilon).$$

Proof. Note that of course

$$X = \bigcup_{x\in X}\{x\} \subseteq \bigcup_{x\in X} B(x,\varepsilon) = X,$$

so $\{B(x,\varepsilon)\}_{x\in X}$ is an open cover for X (see Example 2.6). The result follows by extracting a finite subcover. ■

This is a pretty nice finiteness property for a metric space to have, so we give it a name.

Definition 5.2 (Totally bounded)**.** Fix a metric space (X,d). A subset $A \subseteq X$ is *totally bounded* if and only if any $\varepsilon > 0$ has a finite set $\{x_i\}_{i=1}^n \subseteq A$ for which

$$A \subseteq \bigcup_{i=1}^n B(x_i,\varepsilon).$$

If X is totally bounded, we say that (X,d) is totally bounded.

Example 5.3. Any compact metric space is totally bounded by Lemma 5.1.

It's going to turn out that totally bounded is pretty close to compactness. Here is a quick sanity check.

Lemma 5.4. A totally bounded metric space (X,d), and $A \subseteq X$, then A with is totally bounded.

Proof. For any $\varepsilon > 0$, we see that there is a finite set $S \subseteq X$ for which

$$X = \bigcup_{x\in S}^n B(x,\varepsilon)$$

because (X,d) is totally bounded. Now, let $T \subseteq S$ be the subset for which $B(x,\varepsilon) \cap A \neq \varnothing$ for each $x \in S$, and we then find some $y_x \in B(x,\varepsilon) \cap A$ for each $x \in T$. We now claim that

$$A \subseteq \bigcup_{x\in T}^m B(y_x,\varepsilon),$$

which will finish the proof. Indeed, if $a \in A$, then $a \in X$, so we can find some $x_0 \in S$ with $a \in B(x_0,\varepsilon/2)$. It follows that

$$d(a,y_{x_0}) \le d(a,x_0) + d(x_0,y_{x_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so we get

$$a \in B(y_{x_0},\varepsilon) \subseteq \bigcup_{x\in T} B(y_x,\varepsilon),$$

which is what we wanted. ■

Lemma 5.5. Fix a metric space (X,d) and a subset $A \subseteq X$ which is totally bounded. Then $\overline{A}$ is also totally bounded.

Proof. Fix any $\varepsilon > 0$. Because A is totally bounded, we may find $\{a_i\}_{i=1}^n \subseteq A$ for which

$$A \subseteq \bigcup_{i=1}^n B(a_i,\varepsilon/2).$$

We now claim that

$$\overline{A} \stackrel{?}{\subseteq} \bigcup_{i=1}^n B(a_i,\varepsilon),$$

which will finish the proof. Indeed, if $x \in \overline{A}$, then Lemma 2.56 tells us that $B(x,\varepsilon/2) \cap \overline{A}$ is nonempty, so place $a \in A \cap B(x,\varepsilon/2)$. By hypothesis on the a_i, there exists some a_i such that $a \in B(a_i,\varepsilon/2)$ as well, so

$$d(x,a_i) \le d(x,a) + d(a,a_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so

$$x \in B(a_i,\varepsilon) \subseteq \bigcup_{i=1}^n B(a_i,\varepsilon).$$

The claim follows. ■

5.1.2 Nets

It will be beneficial to us to be able to talk about nets for convergence instead of just sequences.

Definition 5.6 (Directed set)**.** A partially ordered set Λ is a *directed set* if and only if any $a, b \in \Lambda$ have some $c \in \Lambda$ for which $c \geq a, b$.

Example 5.7. Any totally ordered set is a directed set. In particular, any $a, b \in \Lambda$ will have $a \geq b$ or $b \geq a$, so we just set c to be the larger of the two.

Definition 5.8 (Net)**.** Fix a topological space $(X, \mathcal{T})$. Given a directed set Λ, a net is a Λ-indexed sequence $\{x_\alpha\}_{\alpha \in \Lambda}$ in X.

Definition 5.9 (Cluster point)**.** Fix a topological space $(X, \mathcal{T})$ and a net $\{x_\alpha\}_{\alpha\in\Lambda}$. Then $x \in X$ is a *cluster point* if and only if, for any open subset U containing x and $\alpha \in \Lambda$, there is some $\alpha' > \alpha$ for which $x_{\alpha'} \in U$.

Remark 5.10. Fix a metric space (X, d). Then a cluster point x of a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is in fact a limit point. Indeed, for any $\varepsilon > 0$, find some N_1 for which $m, n \geq N_1$ has $d(x_m, x_n) < \varepsilon/2$. Additionally, being a cluster point means there is N_2 with $d(x, x_{N_2}) < \varepsilon/2$. Thus, setting $N := \max\{N_1, N_2\}$, any $n > \max\{N_1, N_2\}$ has

$$d(x_n, x) \leq d(x_n, x_{N_2}) + d(x_{N_2}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Here is the application to metric spaces.

Proposition 5.11. Fix a compact topological space $(X, \mathcal{T})$. Then any net $\{x_\alpha\}_{\alpha\in\Lambda}$ has a cluster point.

Proof. Define

$$A_\alpha := \{x_\beta : \beta > \alpha\}.$$

Observe $\beta \geq \alpha$ implies $A_\beta \subseteq A_\alpha$, so $A_\beta \subseteq \overline{A_\alpha}$, so $\overline{A_\beta} \subseteq \overline{A_\alpha}$.

Additionally, we note that any finite subset of the A_α have a nonempty intersection. Indeed, for any finite $S \subseteq \Lambda$, inductively applying the fact that Λ is a directed set promises us some $\omega \in \Lambda$ with $\omega \geq \alpha$ for each $\alpha \in S$. It follows that $x_\omega \in A_\alpha$ for each $\alpha \in S$, so

$$\bigcap_{n\in S} A_n,$$

contains x_ω and hence is not empty.

Now, because $A_\alpha \subseteq \overline{A_\alpha}$, we see that the $\overline{A_\alpha}$ also have the finite intersection property: for any finite $S \subseteq \Lambda$, see

$$\varnothing \neq \bigcap_{\alpha\in S} A_\alpha \subseteq \bigcap_{\alpha\in S} \overline{A_\alpha},$$

But now the $\overline{A_\alpha}$ are closed, so the compactness of X (!) tells us that there is an element

$$x \in \bigcap_{\alpha\in\Lambda} \overline{A_\alpha}$$

by Proposition 4.25.

It remains to check that x is a cluster point. Indeed, for any open set U containing x, we see that $x \in \overline{A_\alpha}$ and so $U \cap A_\alpha \neq \varnothing$ for each α by Lemma 2.56. As such, for any $\alpha \in \Lambda$, we are being promised $U \cap A_\alpha \neq \varnothing$, so there is x_β with $\beta \geq \alpha$ with $x_\beta \in U$. This finishes. ∎

Corollary 5.12. Any compact metric space (X, d) is complete.

Proof. Fix a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of X. Because X is compact as a topological space, Proposition 5.11 promises us some cluster point $x \in X$. But then x is our limit point by Remark 5.10. ■

5.1.3 A "Metric" Completeness

Here is our capstone result: a converse for Lemma 5.1 combined with Corollary 5.12.

Theorem 5.13. Fix a metric space (X, d). If X is complete and totally bounded, then X is compact.

Proof. Suppose that X is compact and totally bounded. We show that X is not complete. Because X is not compact, we can find an open cover $\mathcal{U}$ of X with no finite subcover.

Notice that, for any fixed $\varepsilon > 0$, being totally bounded means we can find some finite $S \subseteq X$ for which X is covered by the $\{B(x,\varepsilon)\}_{x\in S}$. If it were the case that each $x \in S$ has $B(x,\varepsilon)$ covered by some finite cover $\{U_{x,i}\}_{i=1}^{n_x} \in \mathcal{U}$, then we could write

$$X \subseteq \bigcup_{x\in S} B(x,\varepsilon) \subseteq \bigcup_{x\in S}\left(\bigcup_{i=1}^{n_x} U_{x,i}\right),$$

giving our finite subcover of $\mathcal{U}$. However, this violates the fact that $\mathcal{U}$ has no finite subcover, so there must be some $x \in S$ not covered by any finite subset of $\mathcal{U}$.

We can run the above argument starting with $\varepsilon = 1/2$ and find our x_1. Then we replace X with $B(x_1, 1/2)$ where $B(x_1, 1/2)$ has no finite subcover by $\mathcal{U}$, so running the argument with $\varepsilon = 1/2^2$ on the totally bounded space $B(x_1, 1/2)$ grants us $x_2 \in B(x_1, 1/2)$ such that $B(x_2, 1/2^2)$ still has no finite subcover by $\mathcal{U}$. Going again, we run the argument with $\varepsilon = 1/2^3$ on the totally bounded space $B(x_2, 1/2^2)$, so we get a totally bounded ball $B(x_3, 1/2^3)$ with no finite subcover by $\mathcal{U}$.

We can continue this process inductively, which gives a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that each $n \in \mathbb{N}$ has $B(x_n, 1/2^n)$ with no finite cover by $\mathcal{U}$ and

$$d(x_n, x_{n+1}) \le 1/2^n.$$

A standard argument shows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.[1] To finish the proof, we claim that it has no limit point.

Indeed, suppose for the sake of contradiction that $x_n \to x$ as $n \to \infty$. Then we find some $U \in \mathcal{U}$ containing x, and by definition of a set being open, we can find some open ball $B(x,\varepsilon)$ contained in U. We now find some n large enough so that $1/2^n < \varepsilon/2$ and $d(x_n, x) < \varepsilon/2$ so that any $y \in B(x_n, 1/2^n)$ has

$$d(x,y) \le d(x,x_n) + d(x_n,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $y \in B(x,\varepsilon)$. It follows $B(x_n, 1/2^n) \subseteq B(x,\varepsilon) \subseteq U$, which is a contradiction to the construction of $B(x_n, 1/2^n)$. This completes the proof. ■

Corollary 5.14. Fix a complete metric space (X, d). Then a subset $A \subseteq X$ is compact if and only if A is closed and totally bounded.

Proof. In the forward direction, if A is compact, then A is totally bounded by Lemma 5.1, and A is closed by Corollary 4.13 because (X, d) is a metric space and thus Hausdorff. In the reverse direction, if A is closed and totally bounded, then A is complete by Corollary 2.52 and therefore compact by Theorem 5.13. ■

[1] For any $m \ge n$, note that $d(x_m, x_n) \le \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \le \sum_{k=n}^{m-1} 1/2^k < \sum_{k=n}^{\infty} 1/2^k = 1/2^{n-1}$. Namely, we see that $m, n \to \infty$ makes $d(x_m, x_n) \to 0$.

5.2 September 30

There are no questions.

5.2.1 Totally Bounded for Function Spaces

We continue our discussion of compactness in metric spaces. Fix a topological space $(X, \mathcal{T})$ and a metric space (M, d) so that we can give the space of bounded continuous functions $B_c(X, M)$ the uniform metric d_u by Proposition 3.25. We would like to understand the compact subset of $B_c(X, M)$, so Corollary 5.14 tells us that we are really interested in totally bounded subsets, and we'll take the closure afterward to get our compact sets.

Here are a few lemmas.

Lemma 5.15. Fix a topological space $(X, \mathcal{T})$ and a metric space (M, d) so that we can give the space of bounded continuous functions $B_c(X, M)$ the uniform metric d_u. Fixing a totally bounded subset $\mathcal{F} \subseteq B_c(X, M)$, the set

$$\{s(x) : s \in \mathcal{F}\}$$

totally bounded for any fixed $x \in X$.

Proof. For any $\varepsilon > 0$ has a finite set $\{f_1, \ldots, f_n\} \subseteq \mathcal{F}$ so that $\mathcal{F}$ is covered by the $B(s_i, \varepsilon)$. This is equivalent to saying that any $f \in \mathcal{F}$ has some f_i with

$$d(f(x), f_i(x)) \le \varepsilon/2$$

for all $x \in X$, so

$$\{f(x) : s \in \mathcal{F}\} \subseteq \bigcup_{i=1}^{n} B(f_i(x), \varepsilon),$$

so the claim follows. ■

Lemma 5.15 motivates the following definition.

Definition 5.16 (Pointwise totally bounded). Fix topological spaces $(X, \mathcal{T}_X)$ and a metric space (M, d), and let $\mathcal{F}$ be a family of continuous functions $f \colon X \to M$. Then $\mathcal{F}$ is *pointwise totally bounded* if and only if any $x \in \mathcal{F}$ makes the set

$$\{f(x) : f \in \mathcal{F}\}$$

totally bounded.

Example 5.17. By Lemma 5.18, any totally bounded subset of $B_c(X, M)$ is pointwise totally bounded.

Lemma 5.18. Fix a topological space $(X, \mathcal{T})$ and a metric space (M, d) so that we can give the space of bounded continuous functions $B_c(X, M)$ the uniform metric d_u. Fixing a totally bounded subset $\mathcal{F} \subseteq B_c(X, M)$ and a point $x \in X$, any $\varepsilon > 0$ has some open subset $U \subseteq X$ containing x such that

$$d(f(x), f(y)) < \varepsilon$$

for any $y \in X$ and $f \in \mathcal{F}$.

Proof. For any $f \in \mathcal{F}$, find some f_i with $d(f_i, f) < \varepsilon/3$, we see that any $y \in \mathcal{F}$ can write

$$d(f(x), f(y)) \le d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y)) \le 2\varepsilon/3 + d(f_i(x), f_i(y)).$$

Now, by the continuity of f_i, we see that there is an open subset U_i containing x such that $y \in U_j$ implies $d(f_i(x), f_i(y)) < \varepsilon/3$, so $d(f(x), f(y)) < \varepsilon$ follows.

We now let f vary, which allows the U_i to vary. Defining

$$U := \bigcap_{i=1}^n U_i,$$

we see U is an open subset of X containing x, and each $y \in U$ has $d(f(x), f(y)) < \varepsilon$ for any (!) $f \in \mathcal{F}$. ■

Lemma 5.18 motivates the following definition.

Definition 5.19 (Equicontinuous)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and a metric space (M, d), and let $\mathcal{F}$ be a family of continuous functions $f\colon X \to M$. We say that the family $\mathcal{F}$ is *equicontinuous* at some $x \in X$ if and only if any $\varepsilon > 0$ has some open subset $U \subseteq X$ such that $y \in U$ has

$$d(f(y), f(x)) < \varepsilon$$

for all $f \in \mathcal{F}$. The entire family $\mathcal{F}$ is *equicontinuous* if any only if it is equicontinuous at all $x \in X$.

Example 5.20. By Lemma 5.18, any totally bounded subset of $B_c(X, M)$ is equicontinuous.

5.2.2 Arzelá–Ascoli's Theorem

We might hope for a converse of our given lemmas. Here is the result.

Theorem 5.21 (Arzelá–Ascoli)**.** Fix a compact topological space $(X, \mathcal{T})$ and a metric space (M, d) so that we can give the space of bounded continuous functions $B_c(X, M)$ the uniform metric d_u. Then any equicontinuous and pointwise totally bounded family $\mathcal{F} \subseteq B_c(X, M)$ is totally bounded.

Proof. Fix some $\varepsilon > 0$ so that we want to cover $\mathcal{F}$ with finitely balls of radius $\varepsilon > 0$.

The point is to use compactness on the equicontinuous statement. Indeed, for any $x \in X$, we are promised an open subset $U_x \subseteq X$ such that any $y \in U_x$ and $f \in \mathcal{F}$ has $d(f(x), f(y)) < \varepsilon/4$. However, this means

$$X \subseteq \bigcup_{x \in X} U_x$$

gives us an open cover of X, so compactness tells us that there is some finite sequence of points $\{x_i\}_{i=1}^n$ such that the $U_i := U_{x_i}$ cover X.

Now, fixing any particular i, we use the pointwise totally bounded condition to note

$$\{f(x_i) : f \in \mathcal{F}\}$$

is totally bounded, so we get a finite subset $S_i \subseteq \mathcal{F}$ such that

$$\{f(x_i) : f \in \mathcal{F}\} \subseteq \bigcup_{f \in S_i} B(f(x_i), \varepsilon/4).$$

We now define S as the union of all the S_i, which is finite as the finite union of finite sets.

To finish the proof, we will need to do a little bookkeeping. Let Ψ denote the set of functions from $\{1, \ldots, n\}$ to S so that we can set

$$\mathcal{F}_\psi := \{f \in \mathcal{F} : f(x_i) \in B(\psi(i), \varepsilon/4) \text{ for each } 1 \le i \le n\}$$

for any $\psi \in \Psi$. By construction, the $\mathcal{F}_\psi$ cover $\mathcal{F}$: fix some $f \in \mathcal{F}$. Note that any $y \in U_i$ implies that $d(f(y), f(x_i)) \le \varepsilon/4$ by construction of U_i, and for any given $f(x_i)$, there is an element $s_i \in S_i$ such that

$d(f(x_i), s_i) < \varepsilon/4$ by construction of the $S_i \subseteq S$. Defining ψ by $\psi(i) := s_i$ for this particular s_i, we see that any $y \in U_i$ for any i has

$$d(f(y), f(x_i)) \le d(f(y), s_i) + d(s_i, f(x_i)) \le \varepsilon/2.$$

Letting i vary, we recall that the U_i cover X, so we have found $\psi \in \Psi$ with $f \in \mathcal{F}_\psi$, which is what we wanted.

We will finish upon showing that $\mathcal{F}_\psi$ has diameter less than ε. Well, for any $f, g \in \mathcal{F}_\psi$, we need to show that $d_u(f, g) < \varepsilon$. Well, fix any $x \in X$ and find some j with $x \in U_j$. Then we see

$$d(f(x), g(x)) \le d(f(x), f(x_j)) + d(f(x_j), g(x_j)) + d(g(x_j), g(x)) \le \varepsilon/2 + d(f(x_j), g(x_j)).$$

Now, by construction of ψ, we see

$$d(f(x_j), g(x_j)) \le d(f(x_j), \psi(j)) + d(\psi(j), g(x_j)) < \varepsilon/2,$$

so we see that $d(f(x), g(x)) < \varepsilon$ in total. It follows $\|f - g\|_\infty \le \varepsilon$, so, say, dividing all εs by two will give $\mathcal{F}_\psi$ all with radius less than ε. ■

5.3 October 3

It's spooky season. We begin class by finishing the proof of Theorem 5.21. I have edited the proof from yesterday for continuity reasons.

5.3.1 Locally Compact Spaces

Here is our definition.

Definition 5.22 (Locally compact)**.** A topological space $(X, \mathcal{T})$ is *locally compact* if and only if each point $x \in X$ has some open subset $U \in \mathcal{T}$ containing x such that $\overline{U}$ is compact.

Example 5.23. The set of real numbers $\mathbb{R}$ with the usual topology is locally compact. Indeed, any $x \in \mathbb{R}$ has the open neighborhood $(x-1, x+1)$ with closure $[x-1, x+1]$, and $[x-1, x+1]$ is compact.

Example 5.24. For the same reason, the space $[a, b)$ is also locally compact.

Remark 5.25. Even though compact Hausdorff spaces are normal (by Proposition 4.18), locally compact Hausdorff spaces do not have to be.

For today, we are going to look at only locally compact Hausdorff spaces.

Lemma 5.26. Fix a locally compact Hausdorff space $(X, \mathcal{T})$. Then any $x \in X$ and open subset $U \in \mathcal{T}$ containing x has some open subset $U_x \subseteq X$ containing x such that $\overline{U_x}$ is compact and $\overline{U_x} \subseteq U$.

Proof. We begin by finding our promised U' containing x with $\overline{U'}$ compact. Thus, it suffices to find some open subset V containing x such that $\overline{V}$ is compact and $\overline{V} \subseteq U \cap U'$, but now we see that

$$\overline{U \cap U'} \subseteq \overline{U'}$$

is a closed subset of the compact space $\overline{U'}$ and therefore compact by Lemma 4.10. In particular, we can replace U with $U \cap U'$ and assume that $\overline{U}$ is compact.

Now, let $\partial U := \overline{U} \setminus U$ be the boundary of U. Notably, ∂U is a closed subset of the compact space $\overline{U}$, so ∂U is compact by Lemma 4.10. Because $\{x\}$ is a closed subset in U (note $X \setminus \{x\}$ is open, so $U \setminus \{x\}$ is open in the relative topology), the fact that compact Hausdorff spaces are normal (Proposition 4.18) grants open subsets U_x and U_∂ of $\overline{U}$ with $x \in U_x$ and $\partial U \subseteq U_\partial$.

Now, $U_x \subseteq \overline{U} \setminus U_\partial \subseteq \overline{U} \setminus \partial U$, so we see $\overline{U_x} \subseteq \overline{U} \setminus U_\partial$ because $\overline{U} \setminus U_\partial$ is a closed subset of $\overline{U}$. Further, $\overline{U_x}$ is a closed subset of a compact space $\overline{U}$, so $\overline{U_x}$ is compact by Lemma 4.10, so we are done. ■

Remark 5.27. Lemma 5.26 basically says that open subspaces of locally compact Hausdorff spaces are locally compact.

We can extend the previous result past points to full compact sets.

Proposition 5.28. Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and some compact subset $C \subseteq X$. Then any open subset U containing C has some open subset U_C containing C such that $\overline{U_C}$ is compact and $\overline{U_C} \subseteq U$.

Proof. We use Lemma 5.26. For each $x \in C$, find some U_x by Lemma 5.26 with U_x containing x with $\overline{U_x}$ compact and $\overline{U_x} \subseteq U$. Then we see that

$$C \subseteq \bigcup_{x \in C} U_x,$$

so we have provided an open cover for C, so we can choose finitely many $\{x_i\}_{i=1}^n \subseteq C$ with $U_i := U_{x_i}$ so that

$$C \subseteq \bigcup_{i=1}^n U_i \subseteq U_C.$$

Now, we see that

$$\overline{\bigcup_{i=1}^n U_i} = \bigcup_{i=1}^n \overline{U_i}$$

is a compact subset of U because being compact is closed under finite unions (by inductively applying Example 4.9), so $\bigcup_{i=1}^n U_i$ is the required open subset. ■

5.3.2 Supports

A nice thing about locally compact Hausdorff spaces is that they let us talk about supports.

Definition 5.29 (Support)**.** Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and a normed vector space $(V, \|\cdot\|)$. Then the *support* of a continuous function $f : X \to V$ is

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

Notably, $\{x \in X : f(x) \neq 0\} = f^{-1}(V \setminus \{0\})$ is the pre-image of an open subset and is therefore open by the continuity of f. In particular, normed vector spaces are metric spaces and therefore Hausdorff, so $\{0\} \subseteq V$ is in fact a closed subset.

Here are some quick checks about the support.

Lemma 5.30. Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and a normed k-vector space $(V, \|\cdot\|)$. Then, given two continuous functions $f, g \in C(X, V)$ and $a, b \in k$, we have that

$$\operatorname{supp}(af + bg) \subseteq (\operatorname{supp} f \cup \operatorname{supp} g)$$

Proof. Because $\operatorname{supp} f \cup \operatorname{supp} g$ is the union of two closed sets, it's closed, so it suffices by definition of the closure to show that

$$\{x \in X : (af + bg)(x) \neq 0\} \stackrel{?}{\subseteq} (\operatorname{supp} f \cup \operatorname{supp} g).$$

Well, if $f(x) = 0$ and $g(x) = 0$, then we see $(af + bg)(x) = af(x) + bg(x) = 0$, so $x \notin \operatorname{supp}(af + bg)$. Applying contraposition, we see $x \in \operatorname{supp}(af + bg)$ implies $f(x) \neq 0$ or $g(x) \neq 0$, so $x \in \operatorname{supp} f$ or $x \in \operatorname{supp} g$. ■

Lemma 5.31. Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and a normed k-algebra $(R, \|\cdot\|)$. Then, given two continuous functions $f, g \in C(X, R)$ and $a, b \in k$, we have that

$$\operatorname{supp} fg \subseteq (\operatorname{supp} f \cap \operatorname{supp} g)$$

Proof. Again, because $\operatorname{supp} f \cap \operatorname{supp} g$ is the intersection of closed sets, it's closed, so it suffices to show that

$$\{x \in X : (fg)(x) \neq 0\} \stackrel{?}{\subseteq} (\operatorname{supp} f \cap \operatorname{supp} g).$$

Well, if $f(x) = 0$ or $g(x) = 0$, then $(fg)(x) = f(x)g(x) = 0$. Thus, by contraposition, if $(fg)(x) = 0$, then $f(x) \neq 0$ and $g(x) \neq 0$, so $x \in (\operatorname{supp} f \cap \operatorname{supp} g)$. ∎

We tend to like small things, so here are our small functions.

Definition 5.32 (Compact support)**.** Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and a normed vector space $(V, \|\cdot\|)$. A continuous function $f \colon X \to V$ has *compact support* if and only if its support is compact. We let $C_c(X, V)$ denote the continuous functions of compact support.

Here is a quick sanity check.

Lemma 5.33. Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and a normed $k$vector space $(V, \|\cdot\|)$. Then $C_c(X, V)$ is a k-subspace of $C(X, V)$. If V is a normed k-algebra, then $C_c(X, V)$ is a k-subalgebra.

Proof. We have the following checks.

- Zero: note that the zero function $z \colon X \to V$ by $z(x) = 0$ for all $x \in X$ has

  $$\{x \in X : z(x) \neq 0\} = \varnothing.$$

  The closure of the empty set is still empty (certainly $\overline{\varnothing} \subseteq \varnothing$ by definition of the closure), so we conclude that $\operatorname{supp} z = \varnothing$. Now, $\varnothing$ is compact because any open cover can take the empty subcover, which is certainly finite. Thus, $z \in C_c(X, C)$.

- Linear combination: given $f, g \in C_c(X, V)$ and $a, b \in k$, we see from Lemma 5.30 that $\operatorname{supp}(af + bg)$ is a closed subset of $\operatorname{supp} f \cup \operatorname{supp} g$. However, $\operatorname{supp} f \cup \operatorname{supp} g$ is the union of two compact sets and therefore compact by Example 4.9, so $\operatorname{supp}(af + bg)$ is a closed subset of a compact space and hence compact by Lemma 4.10.

- Multiplication: given $f, g \in C_c(X, V)$, we see from Lemma 5.31 that $\operatorname{supp}(fg)$ is a closed subset of

  $$\operatorname{supp} f \cap \operatorname{supp} g \subseteq \operatorname{supp} f.$$

  However, $\operatorname{supp} f$ is compact, so $\operatorname{supp} fg$ is a closed subset of a compact space and hence compact by Lemma 4.10.

The first two checks tell us that we have a subspace, and the last check uses the algebra structure to get a subalgebra. ∎

Of course, we would like to know that there are a nontrivial number of functions of compact support, so here we go.

Proposition 5.34. Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and a normed vector space $(V, \|\cdot\|)$. For any compact subset $C \subseteq X$ and open subset $U \subseteq X$ containing C, there is a continuous function $f \colon X \to \mathbb{R}$ of compact support such that $f|_C = 1$ and $f|_{X \setminus U} = 0$.

Proof. The point is to apply Theorem 3.8. By Proposition 5.28, we may find an open subset V containing C such that $\overline{V}$ is compact and $\overline{V} \subseteq U$. Then we see C and $\overline{V} \setminus V$ are disjoint closed subsets of $\overline{V}$—note C is closed because X is Hausdorff, using Corollary 4.13.

Thus, because $\overline{V}$ is a normal space (it's compact and Hausdorff, so Proposition 4.18 applies), we are promised a continuous function $f_{\overline{V}} \colon \overline{V} \to \mathbb{R}$ such that $f_{\overline{V}}|_C = 1$ and $f_{\overline{V}}|_{\overline{V} \setminus V} = 0$. We now extend $\overline{V}$ to all of X by

$$f(x) := \begin{cases} f_{\overline{V}}(x) & x \in \overline{V}, \\ 0 & x \notin \overline{V}. \end{cases}$$

Indeed, if $x \in C$, we see $x \in \overline{V}$, so $f(x) = 1$; similarly, if $x \notin U$, then $x \notin \overline{V}$ and so $f(x) = 0$. Lastly, to see that f is continuous, we pick up some open closed $W \subseteq V$; we have the following cases.

- If $0 \notin W$, then we see that $f(x) \in W$ forces $x \in \overline{V}$, so

$$f^{-1}(W) = f_{\overline{V}}^{-1}(W)$$

  is a closed subset of $\overline{V}$ by the continuity of $f_{\overline{V}}$. Closed subsets of closed subspaces are still closed, though, so we see that $f^{-1}(W)$ is closed in X.

- If $0 \in W$, then we do casework. If $x \in \overline{V}$, then actually $x \in f_{\overline{V}}^{-1}(W)$, which is closed in $\overline{V}$ and hence closed in X by continuity of $f_{\overline{V}}$. Otherwise, $x \notin \overline{V}$, but then we see that $x \in X \setminus V$ as well; conversely, if $x \in \overline{V} \setminus V$, then either $x \in \overline{V}$ and so $f_{\overline{V}}(x) = 0 \in W$, or $x \notin \overline{V}$ and so $f(x) = 0 \in W$.

  In total, we see

$$f^{-1}(\{0\}) = f_{\overline{V}}(W) \cup (X \setminus V)$$

  is the union of closed sets and thus closed.

Lastly, we see that $\operatorname{supp} f \subseteq \overline{V}$, so $\operatorname{supp} f$ is a closed subset of a compact set, so we conclude $\operatorname{supp} f$ is compact by Lemma 4.10, so f has compact support. ■

Remark 5.35. By Proposition 3.25, the space of bounded continuous functions $X \to \mathbb{R}$ is complete under $\|\cdot\|_\infty$. We note that $C_c(X, \mathbb{R})$ is a subalgebra, but it is not a closed subset. It turns out that its closure is $C_\infty(X, \mathbb{R})$, which is the space of functions which vanish at infinity: namely, for any $\varepsilon > 0$, there is a compact set $C \subseteq X$ such that $|f(x)| \leq \varepsilon$ for each $x \notin C$.

Part II

Measure Theory

THEME 6

TOWARDS MEASURES

One fish, two fish, red fish, blue fish.

—Dr. Suess, [Gei60]

6.1 October 5

We begin today by making some motivating remarks on C^*-algebras and the like. I hope it's not important because I didn't understand it very well.

6.1.1 Evaluation Maps

For this subsection, we will want to work with the fields $\mathbb{R}$ and $\mathbb{C}$ at the same time, so we pick up the following definition.

Definition 6.1. An *archimedean field* is either $\mathbb{R}$ or $\mathbb{C}$.

We now recall the following piece of notation, which we will state in the case we now care about.

Notation 6.2. Fix an archimedean field k and a compact Hausdorff space X. Then we let $C(X)$ denote the continuous functions $X \to k$.

Remark 6.3. Note that $C(X)$ is a k-subalgebra of k^X because the constantly one function is continuous and the sum and product of two continuous functions is still continuous.

It will turn out that $C(X)$ can tell us a lot about X. For example, homomorphisms we can use X to build homomorphisms $C(X) \to k$.

Example 6.4. Given any $x \in X$, the function $\operatorname{ev}_x \colon C(X) \to k$ by $f \mapsto f(x)$ is a homomorphism. To see that this is a homomorphism, note that $\operatorname{ev}_x(1) = 1$, and $\operatorname{ev}_x(f+g) = (f+g)(x) = f(x) + g(x)$, and $\operatorname{ev}_x(fg)(x) = (fg)(x) = f(x)g(x)$.

In fact, these are all the homomorphisms!

Theorem 6.5. Fix an archimedean field k and a compact Hausdorff space X. Then all homomorphisms $C(X) \to k$ take the form ev_x for some $x \in X$.

Proof. Fix some homomorphism $\varphi\colon C(X) \to k$, and suppose for the sake of contradiction that $\varphi := \operatorname{ev}_x$ for each $x \in X$. To relate our geometry and our algebra, we will use the fact that the "algebraic" set $k \setminus \{0\}$ is open.

Now, we can find $f_x \in C(X)$ with $\varphi(f_x) \neq f_x(x)$ for each $x \in X$. However, $(f_x - \varphi(f_x)1_X)\colon X \to k$ is a continuous function, so

$$U_x := \{y \in X : (f_x - \varphi(f_x)1_X)(y) \neq 0\}$$

is the preimage of the open subset $k \setminus \{0\}$ through the continuous function $(f_x - \varphi(f_x)1_X)$.

Further, $x \in U_x$ because $(\varphi(f_x)1_X)(x) = \varphi(f_x) \neq f_x(x)$, so the open sets $\{U_x\}_{x\in X}$ produce an open cover of X, so we can finitely many of these points in $\{x_1, \ldots, x_n\}$ so that the open sets $U_i := U_{x_i}$ cover X. Thus, the function

$$f := \sum_{i=1}^{n} (f_x - \varphi(f_x)1_X)^2$$

is nonzero everywhere and thus a unit in $C(X)$. On the other hand, $\varphi(f_x - \varphi(f_x)1_X) = \varphi(f_x) - \varphi(f_x)\varphi(1_X) = 0$, so summing gives $\varphi(f) = 0$, which is a contradiction because ring homomorphisms send units to units! ■

This motivates us to work in a little more generality: fix a local field k. Given a k-algebra A, set $A^* := \operatorname{Hom}_k(A, k)$, and given $a \in A$, define the homomorphism $\operatorname{ev}_a\colon A^* \to k$ by $f \mapsto f(a)$. Then, by convention, we will give A^* the weakest topology making the $a \mapsto \operatorname{ev}_a$ continuous.

Example 6.6. Using the k-algebra $A = C(X)$, the map $X \to A^*$ by $x \mapsto \operatorname{ev}_x$ is a homeomorphism. Namely, this is a bijection by Theorem 6.5, and it is a homeomorphism essentially by the definition of the topology on A^*.

The point of the above example is that the algebra $C(X)$ and its evaluation maps are able to fully recover the topological space X!

6.1.2 The Gelfand–Naimark Theorem

By adding in a little more data, we can read even more information off $C(X)$.

Remark 6.7. With $k = \mathbb{C}$, note that complex conjugation extends to a function $C(X) \to C(X)$ by $f \mapsto \overline{f}$. Then one can check that

$$\left\|\overline{f} \cdot f\right\|_\infty = \|f\|_\infty^2 .$$

In fact, the converse is true!

Theorem 6.8 (Gelfand–Naimark)**.** Suppose that A is a commutative Banach $\mathbb{R}$-algebra or $\mathbb{C}$-algebra equipped with an involution $a \mapsto a^*$ such that $\|aa^*\| = \|a\|^2$. Then there is an isomorphism

$$A \simeq C(A^*).$$

In particular, all of these Banach algebras come from a topological space!

Example 6.9. When X is locally compact, set $C_\infty(X)$ to be the set of continuous functions $X \to k$ which vanish at infinity. Even though $C_\infty(X)$ has no multiplicative unit, it is still the case that $C_\infty(A^*) \cong C_\infty(X)$, and in fact $A^* \cong X$. Not having a unit turns out to not be a problem because we can have a function be 1 over a large interval, which is topologically close enough to a unit.

Example 6.10. In contrast, the bounded continuous functions $A := C_b(X)$ have $A \cong C(A^*)$ still, even though A^* is compact. This is weird: the embedding $X \hookrightarrow A^*$ is going to have elements not live in the image, but the elements outside the image require the Axiom of Choice to see.

The above example is why we prefer to work with $C_\infty(X)$ when we talk about locally compact spaces X.

Before jumping into measure theory, we will want to pick up the following definition.

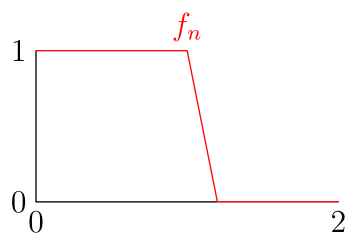
Definition 6.11. A *Hilbert space* is a complete inner product $\mathbb{R}$- or $\mathbb{C}$-vector space.

Example 6.12. Given a Hilbert space H, the set of linear operators $B(H)$ on H has a conjugation again, giving us an involution $T \mapsto \overline{T}$. One still has $\|T\overline{T}\| = \overline{T}^2$, so Theorem 6.8 applies, and we can think about these spaces as spaces of functions.

The above example will generalize to the study of C^* algebras, but we won't discuss this further.

6.1.3 Finitely Additive Measures

We begin with a motivating example. Consider the set of functions $f_n\colon [0,2] \to \mathbb{R}$, given by the following image.



More precisely, we can write

$$f_n(x) := \min\{1, \max\{0, 1 - n(x-1)\}\}.$$

These functions are all continuous by definition, but we can also give them a piecewise definition as

$$f_n(x) := \begin{cases} 1 & x \le 1, \\ 1 - n(x-1) & 1 \le x \le 1 + 1/n, \\ 0 & 1 + 1/n \le x \le 2. \end{cases}$$

In particular, we can see that $f_n \to 1_{[0,1]}$ as $n \to \infty$ with respect to the $\|\cdot\|_p$ norm for $p \in [1, \infty)$: the error is

$$\left\|1_{[0,1]} - f_n\right\|_p^p = \int_0^2 |1_{[0,1/2]}(t) - f_n(t)|^p\,dt = \int_1^{1+1/n} |f_n(t)|^p\,dt \le \int_1^{1+1/n} dt = 1/n,$$

which goes to 0 as $n \to \infty$. Namely, to complete the set of our continuous functions $C([0,1])$ equipped with $\|\cdot\|_p$ for $p \in [1, \infty)$, we need to add in these indicator functions. Nonetheless, we just integrated over $1_{[0,1]}$ just fine above, so we will want to build a class of functions which includes $1_{[0,1]}$ both for completeness reasons but also for integration reasons.

It turns out that not all sets should be able to be integrated over; this leads to the notion of measurable sets. So we will have some collection of subsets $\mathcal{R} \subseteq \mathcal{P}(\mathbb{R})$ and then some measuring function $\mu\colon \mathcal{R} \to [0, \infty]$ (we must allow infinity!). Let's discuss what we want to be true of μ.

- Additivity: if $E, F \in \mathcal{R}$ are disjoint, then $E \sqcup F$ should be in $\mathcal{R}$, and we had better have $\mu(E \sqcup F) = \mu(E) + \mu(F)$. Namely, the sum of the sizes of two disjoint sets had better just be size of the disjoint union.

- Splitting: if $E, F \in \mathcal{R}$ with $F \subseteq E$, then we want (from the above)
$$\mu(E) = \mu(E \cap F) + \mu(E \setminus F),$$
where the idea is that we can look at just the size of $E \cap F$ and $E \setminus F$ individually to more locally compute our sizes.

We can view the above rules as first dictating what sets should be measured at all. As such, we have the following definition.

Definition 6.13 (Ring)**.** Fix a set X. A ring is a nonempty collection $\mathcal{R} \subseteq \mathcal{P}(X)$ with the following properties.

- Union: if $E, F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$.
- Subtraction: if $E, F \in R$, then $E \setminus F \in \mathcal{R}$.

Example 6.14. Of course, the full collection $\mathcal{P}(X)$ is a ring. More generally, given a subset $S \subseteq X$, the collection of subsets of S is a ring: if E, F are subsets of S, we see $E \cup F$ and $E \setminus F$ are both subsets of F.

Example 6.15. Of course, $\{\varnothing\}$ is a ring.

Example 6.16. The set of all finite subsets of X is a ring. Indeed, if $E, F \subseteq X$ are finite, then both $E \cup F$ and $E \setminus F$ are finite as well.

Remark 6.17. Fix a ring $\mathcal{R}$ and some $E, F \in \mathcal{R}$. Note that $E \cap F = E \setminus (E \setminus F)$, so $E \cap F \in \mathcal{R}$ as well.

Remark 6.18. Given a ring $\mathcal{R}$, we note that $\varnothing \in \mathcal{R}$: we know there is some $E \in \mathcal{R}$, so it follows $\varnothing = E \setminus E \in \mathcal{R}$.

Adding in the desired properties for our μ, we can now define "small" measures.

Definition 6.19 (Finitely additive measure)**.** Fix a set X and ring $\mathcal{R} \subseteq \mathcal{P}(X)$. Then a *finitely additive measure* is a function $\mu \colon \mathcal{R} \to [0, \infty]$ such that any disjoint $E, F \in \mathcal{R}$ have
$$\mu(E \sqcup F) = \mu(E) + \mu(F)$$

Remark 6.20. Note that $\mu(\varnothing) = \mu(\varnothing \sqcup \varnothing) = 2\mu(\varnothing)$, so it follows $\mu(\varnothing) = 0$.

It turns out that being finitely additive is not good enough.

Example 6.21. We use the usual measure μ on $\mathbb{R}$. Fix a sequence of disjoint intervals $\{E_i\}_{i=1}^\infty$ in $[0, 1]$, and we see that we should have
$$\sum_{i=1}^\infty \mu(E_i) < \infty.$$
Defining $F_n := \bigsqcup_{i \le n} E_i$ and $F := \sqcup_{i=1}^\infty E_i$, we see that the characteristic functions 1_{F_n} is a Cauchy sequence converging to 1_F, but we don't immediately have access to 1_F because it's an infinite union!

So next class we will discuss how adding a countably additive condition will help us.

6.2 October 5

We continue our discussion into measure theory.

6.2.1 σ-Things

Motivated by Example 6.21, we see that we want to be able to measure countable unions. As such, we have the following definitions.

Definition 6.22 (σ-ring)**.** Fix a set X. Then a ring $\mathcal{R} \subseteq \mathcal{P}(X)$ is a σ*-ring* if and only if R is closed under countable unions.

Definition 6.23 (σ-algebra)**.** Fix a set X. Then a ring $\mathcal{R} \subseteq \mathcal{P}(X)$ is a σ*-algebra* if and only if R is a σ-ring and contains X.

Example 6.24. Given a set X, we see $\mathcal{P}(X)$ is a σ-ring because a countable union of subsets of X is still a subset of X. Further, $\mathcal{P}(X)$ is a σ-algebra because $X \in \mathcal{P}(X)$.

Example 6.25. Fix a set X. Then the collection $\mathcal{S} \subseteq \mathcal{P}(X)$ of countable subsets of X is a σ-ring; here are our checks.

- Countable union: suppose $\{E_i\}_{i=1}^\infty \subseteq \mathcal{S}$. Then
$$\bigcup_{i=1}^\infty E_i$$
is the countable union of countable subsets of X and therefore countable. It follows that $\bigcup_{i=1}^\infty E_i \in \mathcal{S}$.
- Subtraction: if $E, F \in \mathcal{S}$, then E and F are both countable, so $E \setminus F \subseteq E$ is still countable, so $E \setminus F \in \mathcal{S}$.

Notably, if X itself is not an uncountable set, then $X \notin \mathcal{S}$, so $\mathcal{S}$ is not a σ-algebra.

As usual, we may give the collection of all σ-rings (and σ-algebras) the subposet structure coming from inclusion on $\mathcal{P}(X)$. For example, $\mathcal{P}(X)$ is the largest collection in $\mathcal{P}(X)$ and is thus the largest σ-ring and also the largest σ-algebra.

Analogous to our discussion of topologies in Proposition 2.20, we pick up the following lemma to make our σ-rings smaller.

Lemma 6.26. Fix a set X, and fix a collection Σ of rings, σ-rings, or σ-algebra. Then
$$\mathcal{S} := \bigcap_{\mathcal{R}\in\Sigma} \mathcal{R}$$
is another ring, σ-ring, or σ-algebra, respectively.

Proof. We show the axioms get inherited individually.

(a) Suppose that each $\mathcal{R} \in \Sigma$ is closed under finite unions. Then for any $E, F \in \mathcal{S}$, we see $E, F \in \mathcal{R}$ for each $\mathcal{R} \in \Sigma$, so $E \cup F \in \mathcal{R}$ for each $\mathcal{R} \in \Sigma$, so $E \cup F \in \mathcal{S}$.

(b) Suppose that each $\mathcal{R} \in \Sigma$ is closed under subtraction. Then for any $E, F \in \mathcal{S}$, we see $E, F \in \mathcal{R}$ for each $\mathcal{R} \in \Sigma$, so $E \setminus F \in \mathcal{R}$ for each $\mathcal{R} \in \Sigma$, so $E \setminus F \in \mathcal{S}$.

(c) Suppose that each $\mathcal{R} \in \Sigma$ is closed under countable union. Then for any countable collection $\{E_i\}_{i=1}^\infty \in \mathcal{S}$, we see $\{E_i\}_{i=1}^\infty \in \mathcal{R}$ for each $\mathcal{R} \in \Sigma$, so $\bigcup_{i=1}^\infty E_i \setminus F \in \mathcal{R}$ for each $\mathcal{R} \in \Sigma$, so $\bigcup_{i=1}^\infty E_i \in \mathcal{S}$.

(d) Suppose that each $\mathcal{R} \in \Sigma$ contains X. Then $X \in \mathcal{S}$.

The above checks complete the proof. For example, if Σ contains σ-rings, then checks (a)–(c) show $\mathcal{S}$ is still a σ-ring. ■

Corollary 6.27. Fix a set X and a collection $\mathcal{C} \subseteq \mathcal{P}(X)$. Then there is a unique smallest ring, σ-ring, or σ-algebra containing $\mathcal{C}$.

Proof. Let Σ denote the collection of all rings, σ-rings, or σ-algebras containing $\mathcal{C}$. We want to show that Σ contains a unique minimum element. Well, we set

$$\mathcal{S} := \bigcap_{\mathcal{R} \in \Sigma} \mathcal{R}.$$

Notably, $\mathcal{S} \in \Sigma$ by Lemma 6.26, and $\mathcal{S}$ is its minimum somewhat directly: for any $\mathcal{R} \in \Sigma$, we have $\mathcal{S} \subseteq \mathcal{R}$ by construction of $\mathcal{S}$. ■

This gives us the following definition.

Definition 6.28 (σ-ring generated by)**.** Fix a set X. Then give a collection $\mathcal{C}$, we let $\mathcal{S}(\mathcal{C})$ denote the *σ-ring generated by* $\mathcal{C}$, as conjured by Corollary 6.27.

There are analogous definitions for ring and σ-algebra, but we won't state them explicitly.

Remark 6.29. As usual, we note that $\mathcal{C} \subseteq \mathcal{C}'$ implies $\mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}(C')$ because $\mathcal{S}(\mathcal{C}')$ is a σ-ring containing $\mathcal{C}$.

Remark 6.30. Also as usual, if $\mathcal{S}$ is already a σ-ring, then $\mathcal{S}(\mathcal{S}) = \mathcal{S}$. Of course, $\mathcal{S} \subseteq \mathcal{S}(\mathcal{S})$, but also $\mathcal{S}$ is a σ-ring containing $\mathcal{S}$, so $\mathcal{S}(\mathcal{S}) \subseteq \mathcal{S}$ follows.

Example 6.31. Fix a set X. We claim σ-ring generated by the collection $\mathcal{F}$ finite subsets of X is the σ-ring $\mathcal{S}$ of countable subsets of X. Certainly $\mathcal{S}(\mathfrak{F}) \subseteq \mathcal{S}$ because $\mathcal{S}$ is a σ-ring by Example 6.25. On the other hand, any countable subset $E \subseteq X$ has

$$E = \bigcup_{x \in E} \{x\}$$

while $\{x\} \in \mathcal{F} \subseteq \mathcal{S}(\mathcal{F})$ and therefore $E \in \mathcal{S}(\mathcal{F})$. Thus, $\mathcal{S} \subseteq \mathcal{S}(\mathcal{F})$.

6.2.2 Measures

We are now ready to define measures.

Definition 6.32 (Countably additive)**.** Fix a set X and a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(X)$. A function $\mu\colon \mathcal{C} \to [0,\infty]$ is *countably additive* if and only if any pairwise disjoint subcollection $\{E_i\}_{i=1}^\infty \subseteq \mathcal{C}$ with $\bigsqcup_{i=1}^\infty E_i \in \mathcal{C}$ has

$$\mu\left(\bigsqcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i).$$

Notably, we are allowed to have the right-hand side diverge to ∞ if the left-hand side is ∞.

Remark 6.33. In general, it is pretty difficult to actually show that a function is countably additive, but one can take advantage of the fact that

$$\bigsqcup_{i=1}^\infty E_i$$

might not actually be in $\mathcal{C}$.

And here is our definition.

Definition 6.34 (Measure)**.** Fix a set X and σ-ring $\mathcal{S}$. Then a *measure* on $\mathcal{S}$ is a function $\mu\colon \mathcal{S} \to [0,\infty]$ which is countably additive.

Remark 6.35. Note that the countable unions of sets in $\mathcal{S}$ to check the countably additive condition are always in $\mathcal{S}$ because $\mathcal{S}$ is a σ-ring. Namely, the trick suggested in Remark 6.33 doesn't help us.

Remark 6.36. In general, it is not a good idea to ask for unions larger than countable. Approximately speaking, we really want to have countable unions, but we need to be careful adding any other infinities. The main problem is that those infinite sums don't have easy notions of convergence. Even if we don't want to work with something like nets to allow larger convergences, then allowing arbitrary unions for $E \subseteq X$ gives

$$\mu(E) = \sum_{x\in X} \mu(\{x\}),$$

which intuitively should vanish if we make our points have measure 0.

Remark 6.37. Fix a set X and measure $\mu\colon \mathcal{S} \to [0,\infty]$. If $\mu(\varnothing) < \infty$, then note $\varnothing = \bigsqcup_{i=1}^\infty \varnothing$ implies that the sum $\sum_{i=1}^\infty \mu(\varnothing) = \mu(\varnothing)$ converges, so $\mu(\varnothing) = 0$ is forced. Otherwise, if $\mu(\varnothing) = \infty$, then any $E \in \mathcal{S}$ has $E = E \sqcup \varnothing$, so $\mu(E) = \mu(E) + \mu(\varnothing) = \infty$.

Let's see some examples.

Exercise 6.38. More generally, fix a set X using the σ-ring $\mathcal{S} := \mathcal{P}(X)$ of countable subsets of X. For a function $f\colon X \to [0,\infty)$, we define

$$\mu_f(E) := \sum_{x\in E} f(x)$$

for each countable subset $E \subseteq X$. Then μ_f is a measure.

Proof. Note that the order of the sum over $x \in X$ doesn't matter because if the sum converges, then it absolutely converges because all the terms in the sum are positive. Now, to see that we have a measure, pick up some countably many pairwise disjoint countable subsets $\{E_i\}_{i=1}^\infty$ of X. Then

$$\mu_f\left(\bigsqcup_{i=1}^\infty E_i\right) = \sum_{x\in\bigsqcup_{i=1}^\infty E_i} f(x) \stackrel{*}{=} \sum_{i=1}^\infty \sum_{x\in E_i} f(x) = \sum_{i=1}^\infty \mu_f(E_i),$$

where $\overset{*}{=}$ holds because each $x \in \bigsqcup_{i=1}^{\infty} E_i$ lives in exactly one of the E_i. ■

Example 6.39. Fix a set X with σ-ring $\mathcal{S} := \mathcal{P}(X)$. Then we set $\mu(E) := \#E$ for each $E \subseteq X$; namely, if $\mu(E) = \infty$ if and only if E is infinite. We claim that μ is a measure: if $\{E_i\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint subsets of X, then it's a property of cardinality that the cardinality of the (disjoint) union is the sum of the cardinalities.

6.2.3 Premeasures

We are going to want to build measures, but this is somewhat difficult. So we begin with something a little weaker. We begin by weakening our rings.

Definition 6.40 (Prering)**.** Fix a set X. A *prering* of a set X is a nonempty collection $\mathcal{P} \subseteq \mathcal{P}(X)$ satisfying the following.

- Intersection: if $E, F \in \mathcal{P}$, then $E \cap F = \mathcal{P}$.
- Decomposition: if $E, F \in \mathcal{P}$, then we can write

$$E \setminus F = \bigsqcup_{i=1}^{n} G_i$$

  for some finite disjoint union on the right-hand side with $G_i \in \mathcal{P}$ for each i.

Remark 6.41. Fix a prering $\mathcal{P}$. Note any $E \in \mathcal{P}$ has $E \setminus E = \varnothing$, so $\varnothing \in \mathcal{P}$ always because $\mathcal{P}$ is required to be nonempty.

And now here are our weaker measures.

Definition 6.42 (Premeasure)**.** Fix a set X and a prering $\mathcal{P} \subseteq \mathcal{P}(X)$. A *premeasure* on $\mathcal{P}$ is a countably additive function $\mu \colon \mathcal{P} \to [0, \infty]$.

It will turn out that premeasures on prering will give measures on the generated σ-ring. This is nicer because the countably additive condition might be easier to check on a prering, using ideas of Remark 6.33.

Here is our main example.

Exercise 6.43. Fix our set $X := \mathbb{R}$, and let $\mathcal{P}$ be the collection of half-open intervals $[a, b)$ where $a, b \in \mathbb{R}$. Then $\mathcal{P}$ is a prering.

Proof. We begin by checking that $\mathcal{P}$ is a prering.

- Intersection: suppose that $[a, b), [a', b') \in \mathcal{P}$; without loss of generality, take $a \leq a'$ so that $x \in [a, b) \cap [a', b')$ requires $a' \leq x$. Now, note

$$\begin{aligned} [a, b) \cap [a', b') &= \{x \in \mathbb{R} : a \leq x \text{ and } a' \leq x \text{ and } x < b \text{ and } x < b'\} \\ &= [\max\{a, a'\}, \min\{b, b'\}). \end{aligned}$$

- Decomposition: suppose that $[a, b), [a', b') \in \mathcal{P}$. Now, note

$$\begin{aligned} [a, b) \setminus [a', b') &= \{x \in \mathbb{R} : a \leq x \text{ and } x < b \text{ and } (a' > x \text{ or } x \geq b')\} \\ &= \{x \in \mathbb{R} : a \leq x \text{ and } x < b \text{ and } x < a'\} \cup \{x \in \mathbb{R} : a \leq x \text{ and } x < b \text{ and } b' \leq x\} \\ &= [a, \min\{b, a'\}) \cup [\max\{a, b'\}, b). \end{aligned}$$

The above checks complete the proof. ■

Continuing from Exercise 6.43, it will turn out that the function $\mu\colon \mathcal{P} \to \mathbb{R}$ given by

$$\mu([a, b)) := b - a$$

will give a premeasure, but we will not show this today. (We will say that one should use ideas of Exercise 6.43.) This is surprisingly annoying to prove.

Example 6.44. Give $\mathbb{Q} \cap [0, 1)$ an enumeration $\{q_k\}_{k\in\mathbb{N}}$. Then define the interval $F_k := [q_k, q_{k+1}) \cup [q_{k+1}, q_k)$ and "disjoint-ize" these intervals by taking

$$E_k := F_k \Big\backslash \bigcup_{\ell=1}^{k-1} F_\ell$$

and then decompose E_k into a finite disjoint union of G_ks so that the G_ks are now disjoint. Any proof that μ is a premeasure must account for pathologies like this.

Bibliography

[Wei59] Andre Weil. “Correspondence”. In: *Annals of Mathematics* 69.1 (1959), pp. 247–251.

[Gei60] Theodor Geisel. *One Fish, Two Fish, Red Fish, Blue Fish*. Beginner Books. Beginner Books, 1960.

[Tao09] Terrence Tao. *245B, notes 3: L^p spaces*. 2009. URL: https://terrytao.wordpress.com/2009/01/09/245b-notes-3-lp-spaces/.

[Lan12] Serge Lang. *Real and Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 2012. URL: https://books.google.com/books?id=KOMlBQAAQBAJ.

[Doe14] Anthony Doerr. *All the Light We Cannot See*. Scribner, 2014.

[Pug15] Charles C. Pugh. *Real Mathematical Analysis*. Undergraduate Texts in Mathematics. Springer International Publishing, 2015.

[Che22] Evan Chen. *An Infinitely Large Napkin*. 2022. URL: https://venhance.github.io/napkin/Napkin.pdf.

List of Definitions