261A: Lie Groups

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Fall 2024

Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Topological Background

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 28

Today we review differential topology. Here are some logistical notes.

- There will be weekly homeworks, of about 5 problems.
- There will be a final take-home exam.
- This course has a bCourses page.
- We will mostly follow Kirillov's book [Kir08].

1.1.1 Group Objects

The goal of this class is to study symmetries of geometric objects. As such, we are interested in studying (infinite) groups with some extra geometric structure, such as a real manifold or a complex manifold or a scheme structure. Speaking generally, we will have some category $\mathcal{C}$ of geometric objects, equipped with finite products (such as a final object), which allows us to have group objects in $\mathcal{C}$.

Definition 1.1 (group object)**.** Fix a category $\mathcal{C}$ with finite products, such as a final object $*$. A *group object* is the data (G, m, e, i) where $G \in \mathcal{C}$ is an object and $m\colon G \times G \to G$ and $e\colon * \to G$ and $i\colon G \to G$ are morphisms. We require this data to satisfy some associativity, identity, and inverse coherence laws.

For concreteness, we go ahead and write out the coherence diagrams, but they are not so interesting.

- Associative: the following diagram commutes.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}_G \times m} & G \times G \\ {\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array}$$

- Identity: the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}_G \times e"] \arrow[dr, equal] & G \times G \arrow[d, "m"] & G \arrow[l, "e \times \mathrm{id}_G"'] \arrow[dl, equal] \\ & G & \end{tikzcd}$$

- Inverses: the following diagram commutes.

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & & \\ G \arrow[ur, "i \times \mathrm{id}_G"] \arrow[r] \arrow[dr, "\mathrm{id}_G \times i"'] & * \arrow[r, "e"] & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Example 1.2. In the case where $\mathcal{C} = \mathrm{Set}$, we recover the notion of a group, where G is the set, m is the multiplication law, e is the identity, and i is the inverse.

Example 1.3. Group objects in the category of manifolds will be Lie groups.

1.1.2 Review of Topology

This course requires some topology as a prerequisite, but let's review these notions for concreteness. We refer to [Elb22] for most of these notions.

Definition 1.4 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ of a set X and collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of open subsets of X, which we require to satisfy the following axioms.

- $\varnothing, X \in \mathcal{T}$.
- Finite intersection: for $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Arbitrary unions: for a subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We will suppress the notation $\mathcal{T}$ from our topological space as much as possible.

Example 1.5. The set $\mathbb{R}$ equipped with its usual (metric) topology is a topological space.

Example 1.6. Given a topological space X and a subset $Z \subseteq X$, we can make Z into a topological space with open subsets given by $U \cap Z$ whenever $U \subseteq Z$ is open.

Definition 1.7 (closed)**.** A subset Z of a topological space X is *closed* if and only if $X \backslash Z$ is open.

One way to describe topologies is via a base.

Definition 1.8 (base)**.** Given a topological space X, a *base* $\mathcal{B} \subseteq \mathcal{P}(X)$ for the topology such that any open subset $U \subseteq X$ is the union of a subcollection of $\mathcal{B}$. Equivalently, for any open subset $U \subseteq X$ and $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example 1.9. The collection of open intervals $(a,b) \subseteq \mathbb{R}$ generates the usual topology. In fact, one can even restrict ourselves to open intervals (a,b) where $a,b \in \mathbb{Q}$, so $\mathbb{R}$ has a countable base.

Our morphisms are continuous maps.

Definition 1.10 (continuous)**.** A function $f\colon X \to Y$ between topological spaces is *continuous* if and only if $f^{-1}(V) \subseteq X$ is open for each open subset $V \subseteq Y$.

Thus, we can define Top as the category of topological spaces equipped with continuous maps as its morphisms. Thinking categorically allows us to make the following definition.

Definition 1.11 (homeomorphism)**.** A *homeomorphism* is an isomorphism in Top. Namely, a function $f\colon X \to Y$ between topological spaces which is continuous and has a continuous inverse.

Remark 1.12. There are continuous bijections which are not homeomorphisms! For example, one can map $[0,2\pi) \to S^1$ by sending $x \mapsto e^{ix}$, which is a continuous bijection, but the inverse is discontinuous at $1 \in S^1$.

Earlier, we wanted to have finite products in our category. Here is how we take products of pairs.

Definition 1.13 (product topology)**.** Given topological spaces X and Y, we define the topological space $X \times Y$ as having $X \times Y$ as its set and open subsets given by arbitrary unions of sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open.

Remark 1.14. Alternatively, we can say that the topology $X \times Y$ has a base given by the "rectangles" $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open. In fact, if $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, respectively, then we can check that the open subsets

$$\{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a base for $X \times Y$.

Remark 1.15. The final object in Top is the singleton space.

Now, group objects in Top are called topological groups, which are interesting in their own right. For example, locally compact topological groups have a good Fourier analysis theory.

Example 1.16. The group $\mathbb{R}$ under addition is a topological group. In fact, $\mathbb{Q}$ under addition is also a topological group, though admittedly a more unpleasant one.

Example 1.17. The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a topological group.

1.1.3 Review of Differential Topology

However, in this course, we will be more interested in manifolds, so let's define these notions. We refer to [Elb24] for (a little) more detail, and we refer to [Lee13] for (much) more detail. To begin, we note arbitrary topological spaces are pretty rough to handle; here are some niceness requirements. The following is a smallness assumption.

Definition 1.18 (separable)**.** A topological space X is *separable* if and only if it has a countable base.

The following says that points can be separated.

Definition 1.19 (Hausdorff)**.** A topological space X is *Hausdorff* if and only if any pair of distinct points $p, q \in X$ have disjoint open neighborhoods.

The following is another smallness assumption, which we will use frequently but not always.

Definition 1.20 (compact)**.** A topological space X is *compact* if and only if any open cover $\mathcal{U}$ (i.e., each $U \in \mathcal{U}$ is open, and $X = \bigcup_{U \in \mathcal{U}} U$) has a finite subcollection which is still an open cover.

We are now ready for our definition.

Definition 1.21 (topological manifold)**.** A *topological manifold of dimension* n is a topological space X satisfying the following.

- X is Hausdorff.
- X is separable.
- Locally Euclidean: X has an open cover $\{U_\alpha\}_{\alpha \in \kappa}$ such that there are open subsets $V_\alpha \subseteq \mathbb{R}^n$ and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$.

Remark 1.22. By passing to open balls, one can require that all the V_α are open balls. By doing a little more yoga with such open balls (noting $B(0,1) \cong \mathbb{R}^n$), one can require that $V_\alpha = \mathbb{R}^n$ always.

Remark 1.23. It turns out that open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ can only be homeomorphic if and only if $n = m$. This implies that the dimension of a connected component of X is well-defined without saying what n is in advance. However, we should say what n is in advance in order to get rid of pathologies like $\mathbb{R} \sqcup \mathbb{R}^2$.

To continue, we must be careful about our choice of U_αs and φ_αs.

Definition 1.24 (chart, atlas, transition function)**.** Fix a topological manifold X of dimension n.

- A *chart* is a pair (U, φ) of an open subset $U \subseteq X$ and homeomorphism φ of U onto an open subset of $\mathbb{R}^n$.
- An *atlas* is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$ such that $\{U_\alpha\}_{\alpha \in \kappa}$ is an open cover of X.
- The *transition function* between two charts (U, φ) and (V, ψ) is the composite homeomorphism

$$\varphi(U \cap V) \xleftarrow{\varphi} (U \cap V) \xrightarrow{\psi} \psi(U \cap V).$$

  Note that there is also an inverse transition map going in the opposite direction.

Let's see some examples.

Example 1.25. The space $\mathbb{R}^n$ is a topological manifold of dimension n. It has an atlas with the single chart $\mathrm{id} \colon \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.26. The singleton $\{*\}$ is a topological manifold of dimension 0. In fact, $\{*\} = \mathbb{R}^0$.

Example 1.27. The hypersurface $S^n \subseteq \mathbb{R}^{n+1}$ cut out by the equation

$$x_0^2 + \cdots + x_n^2 = 1$$

is a topological manifold of dimension n. It has charts given by stereographic projection out of some choice of north and south poles. Alternatively, it has charts given by the projection maps $\mathrm{pr}_i \colon S^n \to \mathbb{R}^n$ given by deleting the ith coordinate, defined on the open subsets

$$U_i^{\pm} := \{(x_0, \ldots, x_n) \in \mathbb{R}^n : \pm x_i > 0\}$$

for choice of index i and sign in $\{\pm\}$.

Calculus on our manifolds will come from our transition maps.

Definition 1.28. An atlas $\mathcal{A}$ on a topological manifold X is C^k, real analytic, or complex analytic (if $\dim X$ is even) if and only if the transition maps have the corresponding condition.

1.2 August 30

Today we finish our review of smooth manifolds. Once again, we refer to [Elb24] for a few more details and [Lee13] for many more details.

Notation 1.29. We will use the word *regular* to refer to one of the regularity conditions C^k, smooth, real analytic, or complex analytic. We may abbreviate complex analytic to “complex” when no confusion is possible. We use the field $\mathbb{F}$ to denote the “ground field,” which is $\mathbb{C}$ when considering the complex analytic case and $\mathbb{R}$ otherwise.

1.2.1 Smooth Manifolds

We now define a regular manifold.

Definition 1.30 (regular manifold)**.** A *regular manifold* of dimension n is a pair $(M, \mathcal{A})$ of a topological manifold M and a maximal regular atlas $\mathcal{A}$; a chart is called regular if and only if it is in $\mathcal{A}$. We will eventually suppress the $\mathcal{A}$ from our notation as much as possible.

The reason for using a maximal atlas is to ensure that it is more or less unique.

Remark 1.31. Here is perhaps a more “canonical” way to deal with atlas confusion. One can say that two regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if and only if the transition maps between them are also regular; this is the same as saying that $\mathcal{A}_1 \cup \mathcal{A}_2$ is regular. Compatibility forms an equivalence relation, and each equivalence class $[\mathcal{A}]$ has a unique maximal element, which one can explicitly define as

$$\mathcal{A}_{\max} := \{(U, \varphi) : \mathcal{A} \text{ and } (U, \varphi) \text{ are compatible}\}.$$

This explains why it is okay to just work with maximal atlases.

Example 1.32. One can give the topological manifold $\mathbb{R}^2$ many non-equivalent complex structures. For example, one has the usual choice of $\mathbb{R}^2 \cong \mathbb{C}$, but one can also make $\mathbb{R}^2$ homeomorphic to $B(0,1) \subseteq \mathbb{C}$.

Example 1.33. There are "exotic" smooth structures on S^7.

Example 1.34. Given regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, one can form the product manifold $X \times Y$. It should have maximal atlas compatible with the atlas

$$\{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B}\}.$$

1.2.2 Regular Functions

With any class of objects, we should have morphisms.

Definition 1.35. A function $f : X \to Y$ of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is regular if and only if any $p \in X$ has a choice of charts $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $f(U) \subseteq V$ and the composite

$$\varphi(U) \xleftarrow{\varphi} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is a regular function between open subsets of Euclidean space.

Remark 1.36. One can replace the single choice of charts above with any choice of charts satisfying $p \in U$ and $f(U) \subseteq V$.

Remark 1.37. Here is another way to state this: for any open $V \subseteq Y$ and smooth function $h : V \to \mathbb{F}$, the composite

$$f^{-1}(U) \xrightarrow{f} V \xrightarrow{h} \mathbb{F}$$

succeeds in being smooth (in any local coordinates).

Definition 1.38 (diffeomorphism)**.** A *diffeomorphism* of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is a regular map $f : X \to Y$ with regular inverse.

Remark 1.39. Alternatively, one can say that the charts in $\mathcal{A}$ and the charts in $\mathcal{B}$ are in natural bijection via f. Checking that these notions align is not too hard.

The above definition of regular map is a little rough to handle, so let's break it down into pieces.

Definition 1.40 (local coordinates)**.** Fix a regular manifold $(X, \mathcal{A})$ of dimension n. Then a system of *local coordinates* around some point $p \in X$ is a choice of regular chart $(U, \varphi) \in \mathcal{A}$ for which $\varphi(p) = 0$. From here, our local coordinates $(x_1, \ldots, x_n)$ are the composite of φ with a coordinate projection to the ground field. (In the complex analytic case, we want the ground field to be $\mathbb{C}$; otherwise, the ground field is $\mathbb{R}$.)

Now, we are able to see that a function $f : X \to \mathbb{R}$ is regular if and only if it becomes regular in local coordinates. One can even define regularity with respect to a subset of X.

Regularity allows us to produce lots of manifolds, as follows.

Theorem 1.41. Given regular maps $f_1, \ldots, f_m : X \to \mathbb{F}$, the subset

$$\{p \in \mathbb{F}^n : f_1(p) = \cdots = f_m(p) = 0 \text{ and } \{df_1(p), \ldots, df_n(p)\} \text{ are linearly independent}\}$$

is a manifold of dimension $n - m$.

Sketch. This is more or less by the implicit function theorem; for the $\mathbb{F} = \mathbb{R}$ cases, one can essentially follow [Lee13, Corollary 5.14]. ■

Example 1.42. The function $\mathbb{R}^{n+1} \to \mathbb{R}$ given by $(x_0, \ldots, x_n) \mapsto x_0^2 + \cdots + x_n^2$ is real analytic and sufficiently regular at the value 1, which establishes that S^n defined in Example 1.27 succeeds at being a real analytic manifold.

Functions to $\mathbb{F}$ have a special place in our hearts, so we take the following notation.

Notation 1.43. Give a regular manifold X and any open subset $U \subseteq X$, we let $\mathcal{O}_X(U)$ denote the set of regular functions $U \to \mathbb{F}$

Remark 1.44. One can check that the data $\mathcal{O}_X$ assembles into a sheaf. Namely, an inclusion $U \subseteq V$ produces restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Remark 1.45. Once we have all of our regular functions out of X, we note that some Yoneda-like philosophy explains that the sheaf of X determines its full regular structure. Here is an explicit statement: given a manifold X and two maximal regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ determining sheaves of regular functions $\mathcal{O}_1$ and $\mathcal{O}_2$, having $\mathcal{O}_1 = \mathcal{O}_2$ forces $\mathcal{A}_1 = \mathcal{A}_2$. Indeed, it is enough to show the inclusion $\mathcal{A}_1 \subseteq \mathcal{A}_2$, so suppose (U, φ) is a regular chart in $\mathcal{A}_1$. Then the corresponding local coordinates $(x_1, \ldots, x_n)$ all succeed at being regular for $\mathcal{A}_1$, so they are smooth functions in $\mathcal{O}_1$, so they live in $\mathcal{O}_2$ also, so (U, φ) will succeed at being a regular local diffeomorphism for $\mathcal{A}_2$ and hence be a regular chart.

Sheaf-theoretic notions tell us that we should be interested in germs.

Definition 1.46 (germ)**.** Fix a point p on a regular manifold X. A *germ* of a regular function $f \in \mathcal{O}_X(U)$ (where $p \in U$) is the equivalence class of functions $g \in \mathcal{O}_X(V)$ (for a possibly different open subset V containing p) such that $f|_{U\cap V} = g|_{U\cap V}$. The collection of equivalence classes is denoted $\mathcal{O}_{X,p}$ and is called the stalk at p.

1.3 September 4

Today we hope to finish our review of differential topology.

Convention 1.47. For the remainder of class, our manifolds will be smooth, real analytic, or complex analytic.

1.3.1 Tangent Spaces

Now that we are thinking locally about our functions via germs, we can think locally about our tangent spaces.

Definition 1.48 (derivation)**.** Fix a point p on a regular manifold X. A *derivation* at p is an $\mathbb{F}$-linear map $D\colon \mathcal{O}_{X,p} \to \mathbb{F}$ satisfying the Leibniz rule

$$D(fg) = g(p)D(f) + f(p)D(g).$$

Definition 1.49 (tangent space)**.** Fix a point p on a regular manifold X. Then the *tangent space* T_pX is the $\mathbb{F}$-vector space of all derivations on $\mathcal{O}_{X,p}$.

As with everything in this subject, one desires a local description of the tangent space.

Lemma 1.50. Fix an n-dimensional regular manifold X and a point $p \in X$. Equip p with a chart (U, φ) giving local coordinates $(x_1, \ldots, x_n)$. Then the maps $D_i \colon \mathcal{O}_{X,p} \to \mathbb{F}$ given by

$$D_i \colon [(V, f)] \mapsto \left.\frac{\partial f|_{U \cap V}}{\partial x_i}\right|_p$$

provide a basis for T_pX.

Proof. Checking that this is a derivation follows from the Leibniz rule on the chart. Linear independence of the $D_\bullet$s can also be checked locally by plugging in the germs $[(U, x_i)]$ into any linear dependence.

It remains to check that our derivations span. Well, fix any other derivation D which we want to be in the span of the $D_\bullet$s. By replacing D with $D - \sum_i D(x_i) D_i$, we may assume that $D(x_i) = 0$ for all i. We now want to show that $D = 0$. This amounts to some multivariable calculus. Fix a germ $[(V, f)]$, and shrink U and V enough so that f is defined on U; we want to show $D(f) = 0$. The fundamental theorem of calculus implies

$$f(x_1, \ldots, x_n) = f(0) + \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt.$$

However, one can expand out the derivative on the right by the chain rule to see that

$$f(x_1, \ldots, x_n) = f(0) + \sum_{i=1}^n x_i h_i(x_1, \ldots, x_n)$$

for some regular functions $h_1, \ldots, h_n \colon X \to \mathbb{F}$. Applying D, we see that

$$D(f) = \sum_{i=1}^n \underbrace{D(x_i)}_{0} h_i(p) + \underbrace{x_i(p)}_{0} D(h_i) = 0,$$

as required. ■

Tangent spaces have a notion of functoriality.

Definition 1.51. Fix a regular map $F \colon X \to Y$ of regular manifolds. Given $p \in X$, the *differential map* is the linear map $dF_p \colon T_pX \to T_{F(p)}Y$ defined by

$$dF_p(v)(g) := v(g \circ F)$$

for any $v \in T_pX$ and germ $g \in \mathcal{O}_{X,p}$. We may also denote $dF_p(v)$ by $F_* v$.

One has to check that dF_p is linear (which does not have much to check) and satisfies the Leibniz rule (which is a matter of expansion); we will omit these checks.

Remark 1.52. One also has a chain rule: for regular maps $F \colon X \to Y$ and $G \colon Y \to Z$, one has $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

1.3.2 Immersions and Submersions

This map at the tangent space is important enough to give us other definitions.

Definition 1.53 (submersion, immersion, embedding)**.** Fix a regular function $F\colon X \to Y$.

- The map F is a *submersion* if and only if dF_p is surjective for all $p \in X$.
- The map F is an *immersion* if and only if dF_p is injective for all $p \in X$.
- The map F is an *embedding* if and only if F is an immersion and a homeomorphism onto its image.

Remark 1.54. One can check that submersions $F\colon X \to Y$ have local sections $Y \to X$. Explicitly, for $Q \in Y$, the fiber $F^{-1}(\{Q\}) \subseteq X$ is a manifold, and if $Q \in \operatorname{im} F$, the fiber has codimension $\dim Y$.

Remark 1.55. If $F\colon X \to Y$ is an embedding, then the image $F(X) \subseteq Y$ inherits a unique manifold structure so that the inclusion $F(X) \subseteq Y$ is smooth.

Example 1.56. The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) := x$ is a submersion.

Example 1.57 (lemniscate)**.** The function $F\colon S^1 \to \mathbb{R}^2$ given by

$$F(\theta) := \left(\frac{\cos\theta}{1+\sin^2\theta}, \frac{\cos\theta\sin\theta}{1+\sin^2\theta}\right)$$

can be checked to be an immersion (namely, $F'(\theta) \neq 0$ always), but it fails to be injective because $F(\pi/4) = F(3\pi/4) = (0, 0)$, so it is not an embedding.

Example 1.58. The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$ is a smooth homeomorphism onto its image, but it is not an immersion.

Example 1.59. For any open subset $U \subseteq X$ of a manifold, the inclusion map $U \to X$ is an embedding. (In fact, it is also a submersion.)

We will want to distinguish between embeddings, notably to get rid of open embeddings.

Definition 1.60 (closed)**.** An embedding $F\colon X \to Y$ of regular manifolds is *closed* if and only if $F(X) \subseteq Y$ is closed.

Example 1.61. Fix a submersion $F\colon X \to Y$. A point $Q \in Y$ gives rise to a fiber $Z := F^{-1}(\{Q\})$, which Remark 1.54 explains is a closed submanifold of X of codimension $\dim Y$. One can check that T_pZ is exactly the kernel of $dF_p\colon T_pX \to T_pY$; see [Lee13, Proposition 5.37].

1.3.3 Lie Groups

We now may stop doing topology.

Definition 1.62 (Lie group)**.** A regular *Lie group* is a group object in the category of regular manifolds. For brevity, we may call (real) smooth Lie groups simply "Lie groups" or "real Lie groups," and we may call complex analytic Lie groups simply "complex Lie groups."

As with any object, we have a notion of morphisms.

Definition 1.63 (homomorphism)**.** A homomorphism of regular Lie groups is a regular map of the underlying manifolds and a homomorphism of the underlying groups; an isomorphism of regular Lie groups is a homomorphism with an inverse which is also a homomorphism.

Remark 1.64. If X is already a regular manifold, and we are equipped with continuous multiplication and inverse maps, to check that X becomes a regular Lie group, it is enough to check that merely the multiplication map is regular. See [Lee13, Exercise 7-3].

Remark 1.65. Hilbert's 5th problem asks when C^0 Lie groups can give rise to real Lie groups, and there is a lot of work in this direction. As such, we will content ourselves to focus on real Lie groups instead of any weaker regularity.

Remark 1.66. Any complex Lie group is also a real Lie group.

Here is a basic check which allows one to translate checks to the identity.

Lemma 1.67. Fix a regular Lie group G. For any $g \in G$, the maps $L_g \colon G \to G$ and $R_g \colon G \to G$ defined by $L_g(x) := gx$ and $R_g(x) := xg$ are regular diffeomorphisms.

Proof. Regularity follows from regularity of multiplication. Our inverses of L_g and R_g are given by $L_{g^{-1}}$ and $R_{g^{-1}}$, which verifies that we have defined regular diffeomorphisms. ■

1.4 September 6

Last time we defined a Lie group. Today and for the rest of the course, we will study them.

1.4.1 Examples of Lie Groups

Here are some examples of Lie groups and isomorphisms.

Example 1.68. For our field $\mathbb{F}$, the $\mathbb{F}$-vector space $\mathbb{F}^n$ is a Lie group over $\mathbb{F}$.

Example 1.69. Any finite (or countably infinite) group given the discrete topology becomes a real and complex Lie group.

Example 1.70. The groups $\mathbb{R}^\times$ and $\mathbb{R}^+$ (under multiplication) are real Lie groups. In fact, one has an isomorphism $\{\pm 1\} \times \mathbb{R}^+ \to \mathbb{R}^\times$ of real Lie groups given by $(\varepsilon, r) \mapsto \varepsilon r$.

Example 1.71. The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ (under multiplication) is a real Lie group.

Example 1.72. The group $\mathbb{C}^\times$ is a real Lie group. In fact, one has an isomorphism $S^1 \times \mathbb{R}^+ \to \mathbb{C}^\times$ of real Lie groups given again by $(\varepsilon, r) \mapsto \varepsilon r$.

Example 1.73. Over our field $\mathbb{F}$, the set $\mathrm{GL}_n(\mathbb{F})$ of invertible $n \times n$ matrices is a Lie group. Indeed, it is an open subset of $\mathbb{F}^{n^2}$ and thus a manifold, and one can check that the inverse and multiplication maps are rational and hence smooth.

Example 1.74. Consider the collection of matrices

$$\mathrm{SU}_2 := \left\{A \in \mathrm{GL}_2(\mathbb{C}) : \det A = 1 \text{ and } AA^\dagger = 1_2\right\},$$

where $A^\dagger$ is the conjugate transpose. Then SU_2 is an embedded submanifold of $\mathrm{GL}_2(\mathbb{C})$ (cut out by the given equations) and also a subgroup. By writing out $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, one can write out our equations on the coefficients as

$$\begin{cases} ad - bc = 1, \\ a\overline{a} + b\overline{b} = 1, \\ a\overline{c} + b\overline{d} = 0, \\ c\overline{c} + d\overline{d} = 1. \end{cases}$$

In particular, we see that the vector $(a, b) \in \mathbb{C}^2$ is orthogonal to the vector $(\overline{c}, \overline{d})$, so we can solve for this line as providing some $\lambda \in \mathbb{C}$ such that $(\overline{c}, \overline{d}) = \lambda(-b, a)$. But then the determinant condition requires $\lambda = 1$ from $|a|^2 + |b|^2 = 1$. By expanding out $a = w + ix$ and $b = y + iz$, one finds that SU_2 is diffeomorphic to S^3.

The classical groups provide many examples of Lie groups over our field $\mathbb{F}$.

- One has $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_n(\mathbb{F})$, which are subsets of matrices cut out by the conditions $\det A \neq 0$ and $\det A = 1$, respectively.
- Orthogonal: fix a non-degenerate symmetric 2-form Ω on $\mathbb{F}^n$. One can always adjust our basis of $\mathbb{F}^n$ so that Ω is diagonal, and by adjusting our basis by squares, we may assume that Ω has only $+1$ or -1s on the diagonal. If $\mathbb{F} = \mathbb{C}$, we can in fact assume that $\Omega = 1_n$, and then we find that our group is

  $$\mathrm{O}_n(\mathbb{C}) := \{A : A^\intercal A = 1_n\}.$$

  Otherwise, if $\mathbb{F} = \mathbb{R}$, then our adjustment (and rearrangement) of the basis allows us to assume that Ω takes the form $\Omega_{k,n-k} := \operatorname{diag}(+1, \ldots, +1, -1, \ldots, -1)$ with k copies of $+1$ and $n - k$ copies of -1, and we define

  $$\mathrm{O}_{k,n-k}(\mathbb{R}) := \{A : A^\intercal \Omega A = \Omega\}$$

- Special orthogonal: one can add the condition that $\det A = 1$ to all the above orthogonal groups, which makes the special orthogonal groups.
- Symplectic: for $\mathbb{F}^{2n}$, one can fix a non-degenerate symplectic 2-form Ω. It turns out that, up to basis, we find that $\Omega = \left[\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right]$, and we define

  $$\mathrm{Sp}_{2n}(\mathbb{F}) := \{A : A^\intercal \Omega A = 1_{2n}\}.$$

- Unitary: using the non-degenerate Hermitian forms, we can similarly define

  $$\mathrm{U}_n(\mathbb{C}) := \left\{A : A^\dagger A = 1_n\right\}$$

  is a real Lie group. (Conjugation is not complex analytic, so this is not a complex Lie group!)

1.4.2 Connected Components

We will want to focus on connected Lie groups in this class, so we spend a moment describing why one might hope that this is a reasonable reduction. The main point is that it is basically infeasible to classify finite groups, and allowing for disconnected Lie groups forces us to include all these groups in our study by Example 1.69.

Quickly, recall our notions of connectivity; we refer to [Elb22, Appendix A.1] for details.

Definition 1.75 (connected)**.** A topological space X is *disconnected* if and only if there exists disjoint nonempty open subsets $U, V \subseteq X$ covering X. If there exists no such pair of open subsets, then X is *connected*; in other words, the only subsets of X which are both open and closed are $\varnothing$ and X.

Definition 1.76 (connected component)**.** Given a topological space X and a point $p \in X$, the *connected component* of $p \in X$ is the union of all connected subspaces of X containing p.

Remark 1.77. One can check that the connected component is in fact connected and is thus the maximal connected subspace.

Definition 1.78 (path-connected)**.** A topological space X is *path-connected* if and only if any two points $p, q \in X$ have some (continuous) path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Definition 1.79. Given a topological space X and a point $p \in X$, the *path-connected component* of $p \in X$ is the collection of all $q \in X$ for which there is a path connecting p and q.

Remark 1.80. One can check that having a path connecting two points of X is an equivalence relation on the points of X. Then the path-connected components are the equivalence classes for this equivalence relation. From this, one can check that the path-connected components are the maximal path-connected subsets of a topological space.

One has the following lemmas.

Lemma 1.81. Fix a topological space X.

(a) If X is path-connected, then X is connected.

(b) If X is a connected topological n-manifold, then X is path-connected.

Proof. Part (a) is [Elb22, Lemma A.16]. Part (b) is [Elb24, Proposition 1.39]. ■

Lemma 1.82. Fix a continuous surjection $f\colon X \to Y$ of topological spaces. If X is connected, then Y is connected.

Proof. This is [Elb22, Lemma A.8]. ■

Anyway, we are now equipped to return to our discussion of Lie groups.

Lemma 1.83. Fix a Lie group G, and let $G^\circ \subseteq G$ be the connected component of the identity $e \in G$. For any $g \in G$, we see that gG° is the connected component of g.

Proof. Certainly gG° is a connected subset containing g by Lemma 1.82 (note multiplication is continuous), so it is contained in the connected component of g. On the other hand, any connected subset U around g must have $g^{-1}U$ be a connected subset around e, so $g^{-1}U \subseteq G^\circ$, so $U \subseteq gG^\circ$. In particular, the connected component of g is also contained in gG°. ■

Proposition 1.84. Fix a Lie group G, and let $G^\circ \subseteq G$ be the connected component of the identity $e \in G$.

(a) Then G° is a normal subgroup of G.

(b) The quotient $\pi_0(G) := G/G^\circ$ given the quotient topology from the surjection $G \twoheadrightarrow \pi_0(G)$ is a discrete countable group.

Proof. We show the parts independently.

(a) We check this in parts.

- Of course G° is a subgroup: it contains the identity, and the images of the maps $i\colon G^\circ \to G$ and $m\colon G^\circ \times G^\circ \to G$ must land in connected subsets of G containing the identity by Lemma 1.82, so we see that G° is contained
- We now must check that G° is normal. Fix some $g \in G$, and we want to show that $gG^\circ g^{-1} \subseteq G^\circ$. Well, define the map $G^\circ \to G$ by $a \mapsto gag^{-1}$, which we note is continuous because multiplication and inversion are continuous. Lemma 1.82 tells us that the image must be connected, and we see $e \mapsto e$, so the image must actually land in G°.

(b) One knows that $\pi_0(G)$ is a group because G° is normal, and it is discrete because connected components are both closed and open in G, so the corresponding points are closed and open in $\pi_0(G)$. (We are implicitly using Lemma 1.83.) This is countable because a separated topological space must have countably many connected components. ■

Remark 1.85. One can restate the above result as providing a short exact sequence

$$1 \to G^\circ \to G \to \pi_0(G) \to 1$$

of Lie groups. In this way, we can decompose our study of G into connected Lie groups and discrete countable groups. In this course, we will ignore studying discrete countable groups because they are too hard.

1.5 September 9

Today we talk more about subgroups and coverings.

1.5.1 Closed Lie Subgroups

Arbitrary subgroups of Lie groups may not inherit a manifold structure, so we add an adjective to acknowledge this.

Definition 1.86 (closed Lie subgroup)**.** Fix a Lie group G. A *closed Lie subgroup* is a subgroup $H \subseteq G$ which is also an embedded submanifold.

Remark 1.87. On the homework, we will show that closed Lie subgroups are in fact closed subsets of G. It is a difficult theorem (which we will not prove nor use in this class) that being a closed subset and a subgroup implies that it is an embedded submanifold.

Here are some checks on subgroups.

Lemma 1.88. Fix a connected topological group G. Given an open neighborhood U of G of the identity $e \in G$, the group G is generated by U is all of G.

Proof. Let H be the subgroup generated by U. For each $h \in H$, we see that $hU \subseteq H$, which is an open neighborhood (see Lemma 1.67), so $H \subseteq G$ is open. However, we also see that

$$G = \bigsqcup_{[g]} gH,$$

where $[g]$ varies over representatives of cosets. Thus, $G \backslash H$ is again the union of open subsets, so H is also closed, so $G = H$ because G is connected. ■

Lemma 1.89. Fix a homomorphism $f \colon G_1 \to G_2$ of connected Lie groups. If $df_e \colon T_eG_1 \to T_eG_2$ is surjective, then f is surjective.

Proof. By translating around (by Lemma 1.67), we see that f is a submersion. (Explicitly, for each $g \in G_1$, we see that $R_{f(g)} \circ f$ must continue to be a submersion at the identity, but this equals $f \circ R_g$, so f is a submersion at g too.) Because submersions are open [Lee13, Proposition 4.28], we see that f being a submersion means that its image is an open subgroup of G_2, which is all of G_2 by Lemma 1.88. ■

Here is a check to be a closed Lie subgroup.

Lemma 1.90. Fix a regular Lie group G of dimension n. A subgroup $H \subseteq G$ is a closed Lie subgroup of dimension k if and only if there is a single regular chart (U, φ) with $e \in U$ such that

$$U \cap H = \{g \in U : \varphi_{k+1}(g) = \cdots = \varphi_n(g) = 0\}$$

for some.

Proof. We have constructed a slice chart for the identity $e \in H$. We will translate this slice chart around to produce a slice chart for arbitrary $h \in H$, which will complete the proof by [Lee13, Theorem 5.8]. In particular, for any $h \in H$, we know that left translation $L_{h^{-1}} \colon G \to G$ is a diffeomorphism, so the composite

$$hU \xrightarrow{L_{h^{-1}}} U \xrightarrow{\varphi} \varphi(U)$$

continues to be a chart of G with $h \in hU$. Furthermore, we see that $g \in hU$ lives in H if and only if $L_{h^{-1}}g \in U \cap H$, which by hypothesis is equivalent to

$$\varphi_{k+1}\left(L_{h^{-1}}g\right) = \cdots = \varphi_n\left(L_{h^{-1}}g\right) = 0.$$

Thus, we have constructed the desired slice chart. ■

We may want some more flexibility with our subgroups.

Example 1.91. Fix an irrational number $\alpha \in \mathbb{R}$. Then there is a Lie group homomorphism $f \colon \mathbb{R} \to (\mathbb{R}/\mathbb{Z})^2$ defined by $f(t) := (t, \alpha t)$. One can check that $\operatorname{im} f \subseteq (\mathbb{R}/\mathbb{Z})^2$ is a dense subgroup, but it is not closed!

So we have the following definition.

Definition 1.92 (Lie subgroup)**.** Fix a Lie group G. A *Lie subgroup* is a subgroup $H \subseteq G$ which is an immersed submanifold.

1.5.2 Quotient Groups

Along with subgroups, we want to be able to take quotients.

Definition 1.93 (fiber bundle)**.** A *fiber bundle* with fiber F on a smooth manifold X is a surjective continuous map $\pi\colon Y \to X$ such that there is an open cover $\mathcal{U}$ of X and (local) homeomorphisms making the following diagram commute for all $U \in \mathcal{U}$.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & F \times U \arrow[dl, "\mathrm{pr}_2"] \\ & U & \end{tikzcd}$$

Fiber bundles are the correct way to discuss quotients.

Theorem 1.94. Fix a closed Lie subgroup H of a Lie group G.

(a) Then G/H is a manifold of dimension $\dim G - \dim H$ equipped with a quotient map $q\colon G \twoheadrightarrow G/H$.

(b) In fact, q is a fiber bundle with fiber H.

(c) If H is normal in G, then G/H is actually a Lie group (with the usual group structure).

(d) We have $T_e(G/H) \cong T_eG/T_eH$.

Proof. We construct the manifold structure on G/H as follows: for each $g \in G$, we produce a coset $\overline{g} \in G/H$, which we note as $q^{-1}(\overline{g}) = gH$. Now, $gH \subseteq G$ is an embedded submanifold because H is (we are using Lemma 1.67), so one can locally find a submanifold $M \subseteq G$ around g intersecting gH transversally, meaning that

$$T_gG = T_gM \oplus T_g(gH).$$

By shrinking M, we can ensure that the above map continues to be an isomorphism in a neighborhood of g, so the multiplication map $M \times H \to UH$ is a diffeomorphism. Now, MH is an open neighborhood of $g \in G$, and M projects down to an open subset of G/H, so $M \cong q(\overline{M})$ provides our chart.

Now, (a) and (d) follows by inspection of the construction. We see that (b) follows because we built our projection map $G \twoheadrightarrow G/H$ so that it locally looks like $U \times H \twoheadrightarrow \overline{U}$, so we get our fiber bundle. Lastly, (d) follows by the equality $T_gG = T_gM \oplus T_g(gH)$. ■

Remark 1.95. Writing the above out in detail would take several pages; see [Lee13, Theorem 21.10].

Access to quotients permits an isomorphism theorem, which we will prove later when we have talked a bit about Lie algebras.

Theorem 1.96 (Isomorphism)**.** Fix a Lie group homomorphism $f\colon G_1 \to G_2$.

(a) The kernel $\ker f$ is a normal closed Lie subgroup of G_1.

(b) The quotient $G_1/\ker f$ is a Lie subgroup of G_2.

(c) The image $\operatorname{im} f$ is a Lie subgroup of G_2. If $\operatorname{im} f$ is further closed, then $G_1/\ker f \to \operatorname{im} f$ is an isomorphism of Lie subgroups.

1.5.3 Actions

Groups are known by their actions, so let's think about how our actions behave.

Definition 1.97 (action)**.** Fix a Lie group G and regular manifold X. Then a *regular action* of G on X is a regular map $\alpha\colon G \times X \to X$ satisfying the usual constraints, as follows.

(a) Identity: $\alpha(e,x) := x$.

(b) Composition: $\alpha(g,\alpha(h,x)) = \alpha(gh,x)$.

This allows us to define the usual subsets.

Definition 1.98 (orbit, stabilizer)**.** Fix a regular action of a Lie group G on a regular manifold X.

(a) The *orbit* of $x \in X$ is the subspace $Gx := \{gx : g \in G\}$.

(b) The *stabilizer* of $x \in X$ is the subgroup $G_x := \{g \in G : gx = x\}$.

Here are some examples.

Example 1.99. The group $\mathrm{GL}_n(\mathbb{F})$ acts on the vector space $\mathbb{F}^n$.

Example 1.100. The group $\mathrm{SO}_3(\mathbb{R})$ preserves distances in its action on $\mathbb{R}^3$, so its action descends to an action on S^2.

Representations are special kinds of actions.

Definition 1.101 (representation)**.** Fix a Lie group G over $\mathbb{F}$. Then a *representation* of G is the (regular) linear action of G on a finite-dimensional vector space V over $\mathbb{F}$; namely, the map $v \mapsto g \cdot v$ for each $g \in G$ must be a linear map $V \to V$. A *homomorphism* of representations V and W is a linear map $A\colon V \to W$ such that $A(gv) = g(Av)$. These objects and morphisms make the category $\mathrm{Rep}_{\mathbb{F}}(G)$.

Remark 1.102. Equivalently, we may ask for the induced map $G \to \mathrm{GL}(V)$, given by sending $g \in G$ to the map $v \mapsto gv$, to be a Lie group homomorphism.

Remark 1.103. The category $\mathrm{Rep}_{\mathbb{F}}(G)$ comes with many nice operations.

(a) Duals: given a representation $\pi\colon G \to \mathrm{GL}(V)$, we can induce a G-action on $V^* := \mathrm{Hom}(V, \mathbb{F})$ by

$$((\pi^* g)(v^*))(v) := v^* \left(g^{-1} v\right).$$

(Here, $\pi^* g$ should be a map $V^* \to V^*$, so it takes a linear functional $v^* \in V^*$ as input and produces the linear functional $(\pi^* g)(v^*)$ as output.)

(b) Tensor products: given representations $\pi\colon G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, we can induce a G-action on $V \otimes W$ by

$$((\pi \otimes \pi')(g \otimes g'))(v \otimes v') := \pi(g)v \otimes \pi'(g')v'.$$

(c) Hom sets: given representations $\pi\colon\ G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, we can induce a G-action on $\mathrm{Hom}(V, W)$ by

$$(g\varphi)(v) := \pi'(g)\left(\varphi(\pi(g)^{-1}v)\right).$$

(d) Quotients: given representations $\pi\colon G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, where $V \subseteq V'$ is a G-representation, then we can induce a G-action on V'/V by

$$\pi'(g)(v' + V) := gv' + V.$$

One can check that these operations make $\mathrm{Rep}_{\mathbb{F}}(G)$ into a symmetric monoidal abelian category. Checking that these actually form actions is a matter of writing out the definitions, so we will omit it. (Notably, all of these actions are algebraic combinations of previous actions, so all regularity is inherited.)

Returning to group actions on manifolds, we remark that Theorem 1.96 can be seen as a version of the Orbit–stabilizer theorem.

Theorem 1.104 (Orbit–stabilizer)**.** Fix a regular action of a Lie group G on a regular manifold X. Further, fix $x \in X$.

(a) The orbit Gx is an immersed submanifold of X.

(b) The stabilizer G_x is a closed Lie subgroup of G.

(c) The quotient map $f\colon G/G_x \to X$ given by $g \mapsto gx$ is an injective immersion.

(d) If Gx is an embedded submanifold, then the map f of (c) is a diffeomorphism.

1.6 September 11

Today we talk more about homogeneous spaces.

1.6.1 Homogeneous Spaces

Let's see some applications of Theorem 1.104.

Example 1.105. Suppose a regular Lie group G acts smoothly and transitively on a regular manifold X. For each $x \in X$, we see that $G/G_x \to X$ is a bijective immersion. In particular, Sard's theorem implies that $\dim G/G_x = \dim X$, so we conclude that this map is in fact a bijective local diffeomorphism, which of course is just a diffeomorphism. Thus, Theorem 1.94 tells us that the map $G \to X$ given by $g \mapsto gx$ is a fiber bundle with fiber G_x.

The above situation is so nice that it earns a name.

Definition 1.106 (homogeneous space)**.** Fix a regular Lie group G acting smoothly and transitively on a regular manifold X. If the action of G is transitive, we say that X is a *homogeneous space* of G.

Here are many examples.

Example 1.107. Continuing from Example 1.100, we recall that $\mathrm{SO}_3(\mathbb{R})$ acts on S^2. In fact, one can check that the stabilizer of any $x \in S^3$ is isomorphic to S^1, so Example 1.105 tells us that $\mathrm{SO}_3(\mathbb{R}) \to S^2$ is a fiber bundle with fiber S^1. In general, we find that $\mathrm{SO}_n(\mathbb{R}) \to S^n$ is a fiber bundle with fiber $\mathrm{SO}_{n-1}(\mathbb{R})$.

Example 1.108. The group SU_2 acts on $\mathbb{CP}^1$ by matrix multiplication. We see that the stabilizer of some line in $\mathbb{CP}^1$ consists of the matrices in SU_2 with a nonzero eigenvector on the line. For example, using the computation of Example 1.74, we see that trying to stabilize $[1 : 0]$ gives rise to the matrices $\left[\begin{smallmatrix} a & -b \\ \overline{b} & \overline{a} \end{smallmatrix}\right]$ where we require $b = 0$. Thus, we see that our stabilizer is isomorphic to U_1. In particular, our orbits are compact immersed submanifolds of $\mathbb{CP}^1$ of dimension $\dim \mathrm{SU}_2 - \dim \mathrm{U}_1 = \dim \mathbb{CP}^1$, so the action must be transitive in order for orbits to be closed and the correct dimension.

Example 1.109. One can check that SU_n acts on $S^{2n-1} \subseteq \mathbb{C}^n$ with stabilizer isomorphic to SU_{n-1}.

Example 1.110 (flag varieties)**.** Let $\mathcal{F}_n$ be the set of "flags" of $\mathbb{F}^n$, which is an ascending chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{F}^n.$$

Then we see that $\mathrm{GL}_n(\mathbb{F})$ acts on $\mathcal{F}_n$ by matrix multiplication. On the homework, we check that this action is transitive with stabilizer (of the standard flag $\{\mathrm{span}_{\mathbb{F}}(e_1, \ldots, e_i)\}_{i=0}^n$) given by the matrix subgroup $B_n(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ of upper-triangular matrices. Thus, we see that we can realize $\mathcal{F}_n$ as the manifold quotient $\mathrm{GL}_n(\mathbb{F})/B_n(\mathbb{F})$, providing a manifold structure.

Example 1.111 (Grassmannians)**.** Let $\mathrm{Gr}_k(\mathbb{F}^n)$ be the set of vector subspaces $V \subseteq \mathbb{F}^n$ of dimension k. Then we see that $\mathrm{GL}_n(\mathbb{F})$ acts transitively on $\mathrm{Gr}_k(\mathbb{F}^n)$ with stabilizer of $\mathrm{span}(e_1, \ldots, e_k)$ given by matrices of the form

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where $A \in \mathbb{F}^{k \times k}$ and $B \in \mathbb{F}^{k \times (n-k)}$ and $D \in \mathbb{F}^{(n-k)\times(n-k)}$. Thus, we can realize $\mathrm{Gr}_k(\mathbb{F}^n)$ as the manifold quotient of $\mathrm{GL}_n(\mathbb{F})$, providing a manifold structure.

Example 1.112. There are many regular actions of G on itself.

- Regular left: define our action $R_\ell \colon G \times G \to G$ by $(g, x) \mapsto gx$.
- Regular right: define our action $R_r \colon G \times G \to G$ by $(g, x) \mapsto xg^{-1}$.
- Adjoint: define our action $\mathrm{Ad} \colon G \times G \to G$ by $(g, x) \mapsto gxg^{-1}$. (This action is rarely transitive!)

Example 1.113 (adjoint)**.** Fix a regular Lie group G. Note that $\operatorname{Ad}_g(1) = 1$, so we may take the differential to provide a map $(d\operatorname{Ad}_g)_e \colon T_eG \to T_eG$, so we get an adjoint representation

$$\begin{aligned} G \times T_eG &\to \quad T_eG \\ (g\ ,\ \ v) \quad &\mapsto (d\operatorname{Ad}_g)_e(v) \end{aligned}$$

which we will frequently abuse notation to abbreviate the above as providing some $\operatorname{Ad}_g \in \operatorname{GL}(T_eG)$. Expanding everything in sight into coordinates reveals that this action is smooth; in fact, one can check (again in coordinates) that the map $G \to \operatorname{GL}(T_eG)$ given by $g \mapsto (d\operatorname{Ad}_g)_e$ is smooth. Taking the differential of this last map produces a map $T_eG \to \operatorname{End}(T_eG)$, which is sometimes called the adjoint representation of T_eG.

1.6.2 Covering Spaces

It will help to recall some theory around covering spaces. See [Elb23] for (some) more detail about this theory or [Hat01] for (much) more detail.

Definition 1.114 (covering space)**.** A *covering space* is a fibration $p \colon Y \to X$ with discrete fiber S. The *degree* of p equals $\#S$.

In more words, we are asking for each $x \in X$ to have an open neighborhood U such that the restriction $p^{-1}(U) \to U$ is homeomorphic (over U) to $\bigsqcup_{s\in S} p^{-1}(U) \to U$ for some discrete set S.

Remark 1.115. If X is a regular manifold and $\deg p \leqslant |\mathbb{N}|$, then Y is also a regular manifold. Indeed, being a manifold is checked locally, so one can find neighborhoods as in the previous remark to witness the manifold structure.

We are interested in paths in topological spaces, but there are too many. To make this set smaller, we consider it up to homotopy.

Definition 1.116 (homotopy)**.** Fix a topological space X. Two paths $\gamma_0, \gamma_1 \colon [0,1] \to X$ are *homotopic* relative to their endpoints if and only if there is a continuous map $H_\bullet \colon [0,1]^2 \to X$ such that $H_0 = \gamma_0$ and $H_1 = \gamma_1$ and $H_s(0) = \gamma_0(0) = \gamma_1(0)$ and $H_s(1) = \gamma_0(1) = \gamma_1(1)$ for all s. The map H is called a *homotopy*.

Definition 1.117 (simply connected)**.** A topological space X is *simply connected* if and two paths with the same endpoints are homotopic relative to those endpoints.

Example 1.118. One can check that S^1 fails to be simply connected because the path going around the circle is not homotopic to the constant path.

It is important to know that one can lift paths.

Theorem 1.119. Fix a covering space $p \colon Y \to X$. Fix some point $x \in X$ and a path $\gamma \colon [0,1] \to X$ with $\gamma(0) = x$. Then each $\widetilde{x} \in p^{-1}(\{x\})$ has a unique path $\widetilde{\gamma} \colon [0,1] \to Y$ such that $\widetilde{\gamma}(0) = \widetilde{x}$ and making the following diagram commute.

$$\begin{tikzcd} {[0,1]} \arrow[r, "\widetilde{\gamma}"] \arrow[dr, "\gamma"'] & \widetilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

Remark 1.120. One can further check that having two homotopic paths $\gamma_1 \sim \gamma_2$ downstairs produce homotopic paths $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$.

Remark 1.121. More generally, fix a simply connected topological space Z. Then given a map $f\colon Z \to X$ and a choice of $\widetilde{x} \in p^{-1}(\{x\})$ and $z \in f^{-1}(\{x\})$, there will be a unique lift $\widetilde{f}\colon Z \to Y$ such that $\widetilde{f}(z) = \widetilde{x}$. In short, given any $z' \in Z$, find a path connecting z and z', send this path into X and then lift it up to Y. Because Z is simply connected (and the above theorem), the choice of path from z to z' does not really matter.

Anyway, we now define our collection of paths.

Definition 1.122 (fundamental group)**.** Fix a point x of a topological space X. Then the set of paths both of whose endpoints are x forms a monoid with operation given by composition (i.e., concatenation). If we take the quotient of this monoid by homotopy classes of paths, then we get a group of path homotopy classes, which we call $\pi_1(X, x)$. This is the *fundamental group*.

Remark 1.123. For any two $x, y \in X$ in the same path-connected component, the path $\alpha\colon [0,1] \to X$ connecting x to y produces an isomorphism $\pi_1(X, x) \cong \pi_1(X, y)$ by $\gamma \mapsto \alpha \cdot \gamma \cdot \alpha^{-1}$, where $\cdot$ denotes path composition.

Remark 1.124. The above remark allows us to verify that X is simply connected if and only $\pi_1(X, x)$ is trivial for all x. In fact, we only have to check this for one x in each path-connected component.

1.7 September 13

Today we continue our discussion of coverings.

1.7.1 The Universal Cover

There is more or less one covering space which produces all the other ones.

Definition 1.125 (universal cover)**.** Fix a path-connected topological space X. Then a covering space $p\colon Y \to X$ is the *universal cover* if and only if Y is connected and simply connected.

We now discuss an action of $\pi_1(X, b)$ on covering spaces in order to better understand this universal cover. Fix a covering space $p\colon Y \to X$ and a basepoint $x \in X$. Then we note that $\pi_1(X, x)$ acts on the fiber $p^{-1}(\{x\})$ as follows: for any $[\gamma] \in \pi_1(X, x)$ and $\widetilde{x} \in p^{-1}(\{x\})$, we define $\widetilde{\gamma}\colon [0,1] \to Y$ by lifting the path $\gamma\colon [0,1] \to X$ up to Y so that $\widetilde{\gamma}(0) = \widetilde{x}$; then

$$[\gamma] \cdot \widetilde{x} := \widetilde{\gamma}(1).$$

One can check that this action is well-defined (namely, it does not depend on the representative γ and does provide a group action). Here are some notes.

- If Y is path-connected, then the action is transitive: and $\widetilde{x}, \widetilde{x}' \in p^{-1}(\{x\})$ admit a path $\widetilde{\gamma}\colon [0,1] \to Y$ with $\widetilde{\gamma}(0) = \widetilde{x}$ and $\widetilde{\gamma}(1) = \widetilde{x}'$, so $\gamma := p \circ \widetilde{\gamma}$ has

$$[\gamma] \cdot \widetilde{x} := \widetilde{x}'$$

  by construction of γ.

- If Y is simply connected, then this action is also free. Indeed, choose two paths $\gamma_1, \gamma_2 \colon [0,1] \to X$ representing classes in $\pi_1(X,x)$. Now, suppose that $[\gamma_1] \cdot \widetilde{x} = [\gamma_2] \cdot \widetilde{x}$ for each $\widetilde{x} \in p^{-1}(\{x\})$, and we will show that $[\gamma_1] = [\gamma_2]$. Well, choosing lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, the hypothesis implies that they have the same endpoints. Thus, because Y is simply connected, we know $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$. We now see that $\gamma_1 \sim \gamma_2$ by composing the homotopy witnessing $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$ with p.

The conclusion is that $p^{-1}(\{x\})$ is in bijection with $\pi_1(X,x)$ when $p \colon Y \to X$ is the universal cover. Here are some examples.

Example 1.126. One has a covering space $p \colon S^n \to \mathbb{RP}^n$ given by

$$(x_0, \ldots, x_n) \mapsto [x_0 : \ldots : x_n].$$

For $n \geqslant 2$, we know S^n is simply connected, so it will be the universal cover, and we are able to conclude that $\pi_1(\mathbb{RP}^n)$ is isomorphic to a fiber of p, which has two elements, so $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.127. One has a covering space $p \colon \mathbb{R} \to S^1$ given by $p(t) := e^{2\pi i t}$. We can see that $\mathbb{R}$ is simply connected (it's convex), so this is a universal covering. This at least tells us that $\pi_1\left(S^1\right)$ is countable, and one can track through the group law through the above bijections to see that actually $\pi_1\left(S^1\right) \cong \mathbb{Z}$.

Example 1.128. One can show that $\pi_1(\mathbb{C}\backslash\{z_1, \ldots, z_n\})$ is the free group on n generators, basically corresponding to how one goes around each point.

Now, in the context of our Lie groups, we get the following result.

Theorem 1.129. Fix a regular Lie group G, and let $p \colon \widetilde{G} \to G$ be the universal cover.

(a) Then $\widetilde{G}$ has the structure of a regular Lie group.

(b) The projection p is a homomorphism of Lie groups.

(c) The kernel $\ker p \subseteq \widetilde{G}$ is discrete, central, and isomorphic to $\pi_1(G,e)$. In particular, $\pi_1(G,e)$ is commutative.

Proof. Here we go.

(a) Remark 1.115 tells us that $\widetilde{G}$ is a regular manifold, so it really only remains to exhibit the group structure. We will content ourselves with merely describing the group structure. Fix any $\widetilde{e} \in p^{-1}(\{e\})$, which will be our identity.

Now, $\widetilde{G}$ is simply connected, so $\widetilde{G} \times \widetilde{G}$ is also simply connected. Thus, Remark 1.115 explains that the composite

$$\widetilde{G} \times \widetilde{G} \to G \times G \xrightarrow{m} G$$

will lift to a unique map to the universal cover as a map $\widetilde{m}$ making the following diagram commute.

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, "\widetilde{m}"] \arrow[d, "(p,p)"'] & \widetilde{G} \arrow[d, "p"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} (\widetilde{e}, \widetilde{e}) \arrow[r, mapsto] \arrow[d, mapsto] & \widetilde{e} \arrow[d, mapsto] \\ (e,e) \arrow[r, mapsto] & e \end{tikzcd}$$

One can construct the inverse map similarly by lifting the map $\widetilde{G} \xrightarrow{p} G \xrightarrow{i} G$ to a map to $\widetilde{G}$ sending $\widetilde{e} \mapsto \widetilde{e}$. Uniqueness of lifting will guarantee that we satisfy the group law.

(b) We see that p is a homomorphism by construction of $\widetilde{m}$ above.

(c) This is on the homework. ■

Example 1.130. Recall that we have the fiber bundle $\mathrm{SO}_n(\mathbb{R}) \to S^{n-1}$ with fiber $\mathrm{SO}_{n-1}(\mathbb{R})$. Thus, the long exact sequence in homotopy groups produces

$$\pi_2\left(S^{n-1}\right) \to \pi_1(\mathrm{SO}_{n-1}(\mathbb{R})) \to \pi_1(\mathrm{SO}_n(\mathbb{R})) \to \pi_1\left(S^{n-1}\right) \to \pi_0\left(\mathrm{SO}_{n-1}(\mathbb{R})\right).$$

Now, for $n \geqslant 4$, one has that $\pi_2\left(S^{n-2}\right) = \pi_1\left(S^{n-1}\right) = 1$, so we have $\pi_1\left(\mathrm{SO}_{n-1}(\mathbb{R})\right) \cong \pi_1\left(\mathrm{SO}_n(\mathbb{R})\right)$. One can check that $\mathrm{SO}_3(\mathbb{R}) \cong \mathbb{RP}^3$, so we see that

$$\pi_1\left(\mathrm{SO}_n(\mathbb{R})\right) \cong \mathbb{Z}/2\mathbb{Z}$$

for $n \geqslant 4$. The universal (double) cover of $\mathrm{SO}_n(\mathbb{R})$ is called Spin_n, and Theorem 1.129 explains that we have a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n \to \mathrm{SO}_n(\mathbb{R}) \to 1.$$

Example 1.131. More concretely, one can show that $\mathrm{SU}_2(\mathbb{C})$ has an action on $\mathbb{R}^3$ preserving distances and orientation, so we get a homomorphism $\mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$. One can check that this map is surjective with kernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Remark 1.132. In general, Theorem 1.129 explains that we have a short exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$$

for any regular Lie group G, so it does not cost us too much to pass from G to $\widetilde{G}$, allowing us to assume that the Lie groups we study are simply connected. (Note that even though $\pi_1(G)$ is discrete, the short exact sequence does not split: $\widetilde{G}$ succeeds at being connected.)

1.7.2 Vector Fields

Fix a regular manifold X of dimension n. We may be interested in thinking about all our tangent spaces at once.

Definition 1.133 (tangent bundle)**.** Fix a regular manifold X. Then we define the *tangent bundle* as

$$TX := \{(x, v) : v \in T_xX\}.$$

Note that there is a natural projection map $TX \to X$ by $(x, v) \mapsto x$.

Remark 1.134. Locally on a chart (U, φ) of X, we see that φ provides coordinates $(x_1, \ldots, x_n)$ on U, so one has a bijection

$$U \times \mathbb{R}^n \to TU$$

by sending $(x, \partial/\partial x_i) \mapsto \left(\varphi^{-1}(x), d\varphi_x^{-1}(\partial/\partial x_i)\right)$ (In the future, we may conflate $d\varphi_x^{-1}(\partial/\partial x_i)$ with $\partial/\partial x_i$). This provides a chart for TU, and one can check that these charts are smoothly compatible by an explicit computation using the smooth compatibility of charts on X. The point is that $TX \to X$ is a vector bundle of rank n.

Vector bundles are interesting because of their sections.

Definition 1.135 (vector field)**.** Fix a regular manifold X. Then a *vector field* on X is a smooth section $\sigma\colon X \to TX$ of the natural projection map $TX \to X$.

Remark 1.136. Locally on a chart (U, φ) with coordinates $(x_1, \ldots, x_n)$, we see that we can think about a vector field σ locally as

$$\sigma(x) := \sum_{i=1}^{n} \sigma_i(x) \frac{\partial}{\partial x_i}\bigg|_x,$$

where the smoothness of σ enforces the $\sigma_\bullet$s to be smooth. Changing coordinates to (U', φ') with a coordinate expansion $\sigma(x) = \sum_i \sigma_i'(x) \frac{\partial}{\partial x_i'}$, one can change bases using the Jacobian of $\varphi' \circ \varphi^{-1}$ to find that

$$\sigma_i'(x) = \sum_{j=1}^{n} \frac{\partial x_i'}{\partial x_j'} \sigma_j(x).$$

Anyway, the point is that we can define a vector field locally on these coordinates and then going back and checking that we have actually defined something that will glue smoothly up to X.

The reason we care so much about tangent spaces in this class is because they give rise to our Lie algebras, whose representations are somehow our main focus.

Definition 1.137 (Lie algebra)**.** Fix a Lie group G. Then the *Lie algebra* of G is the vector space

$$\mathfrak{g} := T_eG.$$

We may also notate $\mathfrak{g}$ by $\operatorname{Lie}(G)$.

It is somewhat difficult to find structure in this tangent space immediately, so we note that T_eG is isomorphic with another vector space.

Definition 1.138 (invariant vector field)**.** Fix a Lie group G. Then a vector field $\xi\colon G \to TG$ is *left-invariant* if and only if

$$\xi(gx) = dL_g(\xi(x))$$

for any $x, g \in G$. One can define *right-invariant* analogously.

Remark 1.139. We claim that the vector space of left-invariant vector fields is isomorphic to $\mathfrak{g}$. Here are our maps.

- Given a left-invariant vector field ξ, one can produce the tangent vector $\xi(e) \in \mathfrak{g}$.
- Given some $\xi(e) \in \mathfrak{g}$, we define

$$\xi(g) := dL_g(\xi(e)) \in T_gG.$$

  It is not difficult to check that $\xi\colon G \to TG$ is at least a section of the natural projection $TG \to G$. We omit the check that ξ is smooth because it is somewhat involved.

Remark 1.140. As an aside, we note that the produced left-invariant vector fields parallelizes G after providing a basis of $\mathfrak{g}$; in particular, one has a canonical isomorphism $TG \cong G \times \mathfrak{g}$. One can actually show that TG is a Lie group with Lie group structure given by functoriality of the tangent bundle applied to the group operations of G, and one finds that $TG \cong G \rtimes \mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint action.

Next class we will go back and argue that our classical groups are actually Lie groups and compute their Lie algebras.

1.8 September 16

Today we will talk about Lie algebras of classical groups.

1.8.1 The Exponential Map: The Classical Case

Let's work through our examples by hand. Recall that our classical groups are our subgroups of $\mathrm{GL}_n(\mathbb{F})$ cut out by equations involving $\det$ and preserving a bilinear/sesquilinear form (symmetric, symplectic, or Hermitian).

Example 1.141. We show that $\mathrm{GL}_n(\mathbb{F})$ is a Lie group over $\mathbb{F}$ and compute its Lie algebra.

Proof. Note $\mathrm{GL}_n(\mathbb{F})$ is an open submanifold of $\mathrm{M}_n(\mathbb{F}) \cong \mathbb{F}^{n\times n}$. Matrix multiplication and inversion are rational functions of the coordinates and hence smooth, so $\mathrm{GL}_n(\mathbb{F})$ succeeds at being a Lie group. Lastly, we see that being open implies that our tangent space is

$$T_e \operatorname{GL}_n(\mathbb{F}) \cong T_e \operatorname{M}_n(\mathbb{F}) \cong \mathbb{F}^n,$$

as required. ■

We will postpone the remaining computations until we discuss the exponential. For these computations, we want the exponential map.

Definition 1.142 (exponential)**.** For $X \in \mathfrak{gl}_n(\mathbb{F})$, we define the exponential map $\exp\colon \mathfrak{gl}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{F})$ by

$$\exp(X) := \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Note that $\exp$ is an isomorphism at the identity, so the Inverse function theorem provides a smooth "local" inverse $\log(1_n + X)$ defined in an open neighborhood of 1_n. In fact, one can formally compute that

$$\log(1_n + X) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}.$$

We run a few small checks.

Remark 1.143. Note $\exp(0) = 1$. In fact, one can check that $d\exp_0(A) = A$ for any $A \in \mathfrak{gl}_n(\mathbb{F})$ by taking the derivative term by term.

Remark 1.144. We also see that $\exp\left(AXA^{-1}\right) = A\exp(X)A^{-1}$ and $\exp(X^\intercal) = \exp(X)^\intercal$ and $\exp(X^\dagger) = \exp(X)^\dagger$ by a direct expansion.

What's important about $\exp$ is the following multiplicative property.

Lemma 1.145. Fix $X, Y \in \mathfrak{gl}_n(\mathbb{F})$ which commute. Then

$$\exp(X + Y) = \exp(X)\exp(Y).$$

Proof. We check this in formal power series. Because everything in sight converges, this is safe. The main point is to just expand everything. Indeed,

$$\begin{aligned}
\exp(X+Y) &= \sum_{k=0}^{\infty} \frac{(X+Y)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{a+b=k} \binom{k}{a} X^a Y^b\right) \\
&= \sum_{a,b=0}^{\infty} \frac{1}{(a+b)!} \cdot \frac{(a+b)!}{a!b!} X^a Y^b \\
&= \sum_{a,b=0}^{\infty} \frac{X^a}{a!} \cdot \frac{Y^b}{b!} \\
&= \exp(X)\exp(Y),
\end{aligned}$$

as required. ■

Remark 1.146. For fixed X, the previous point implies that the map $\mathbb{F} \to \mathrm{GL}_n(\mathbb{F})$ given by $t \mapsto \exp(tX)$ is a Lie group homomorphism. (Smoothness is automatic by smoothness of $\exp$.) The image of this map is called the "1-parameter subgroup" generated by X.

Remark 1.147. Taking inverses shows that $\log(XY) = \log X + \log Y$ for X and Y close enough to the identity.

Here is another check which is a little more interesting.

Lemma 1.148. Fix $X \in \mathfrak{gl}_n(\mathbb{F})$. Then

$$\det\exp(X) = \exp(\operatorname{tr} X).$$

Proof. The computations do not change if we extend the base field, so we may work over $\mathbb{C}$ everywhere. Thus, we may assume that X is upper-triangular by conjugating (see Remark 1.144) say with diagonal entries $\{d_1, \ldots, d_n\}$. Now, for any $k \geqslant 0$, any X^k continues to be upper-triangular with diagonal entries $\{d_1^k, \ldots, d_n^k\}$. Thus, we see that $\exp(X)$ is upper-triangular with diagonal entries $\{\exp(d_1), \ldots, \exp(d_n)\}$, so

$$\begin{aligned}
\det\exp(X) &= \exp(d_1)\cdots\exp(d_n) && (1.1)\\
&= \exp(d_1 + \cdots + d_n) && (1.2)\\
&= \exp(\operatorname{tr} X), && (1.3)
\end{aligned}$$

as required. ■

1.8.2 The Classical Groups

For our classical groups, we will show the following result.

Theorem 1.149. For each classical group $G \subseteq \mathrm{GL}_n(\mathbb{F})$, there will exist a vector subspace $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$ (which can be identified with T_eG via the embedding $G \subseteq \mathrm{GL}_n(\mathbb{F})$) and open neighborhoods of the identity $U \subseteq \mathrm{GL}_n(\mathbb{F})$ and $\mathfrak{u} \subseteq \mathfrak{g}$ such that $\exp\colon (U \cap G) \to (\mathfrak{u} \cap \mathfrak{g})$ is a local isomorphism.

Before engaging with the examples, we note the following corollary.

Corollary 1.150. For each classical group G, we see that G is a Lie group with $T_eG = \mathfrak{g}$ and $\dim G = \dim \mathfrak{g}$.

Proof. It suffices to provide a slice chart of the identity for $G \subseteq \mathrm{GL}_n(\mathbb{F})$; we then get slice charts everywhere by translation. Well, ■

Let's now proceed with our examples. We begin with some general remarks.

Lemma 1.151. Let $G \subseteq \mathrm{GL}_n(\mathbb{F})$ be a closed Lie subgroup, and let $SG := \{g \in G : \det g = 1\}$. We show SG is a Lie subgroup and compute $T_1SG \subseteq T_1G$ as

$$T_1SG = \{g \in T_1G : \operatorname{tr} g = 0\}.$$

Proof. Let G act on $\mathbb{F}$ by $\mu\colon G \times \mathbb{F} \to \mathbb{F}$ by $\mu(g, c) := (\det g)c$. Note that μ is a polynomial and hence regular, so this is a regular action upon checking that $\mu(1, c) = c$ and $\mu(g, \mu(h, c)) = \mu(gh, c)$, which hold because $\det$ is a homomorphism.

Now, the stabilizer of $1 \in \mathbb{F}$ consists of the $g \in G$ such that $(\det g) \cdot 1 = 1$, which is equivalent to $\det g = 1$ and hence equivalent to $g \in SG$. Thus, $SG \subseteq G$ is a closed Lie subgroup with

$$T_1SG(\mathbb{F}) = \{v \in T_1G : (d\det)_1(v) = 0\},$$

where $\det\colon G \to \mathbb{F}$ is the determinant map. To compute $(d\det)_1(v)$, we identify $T_1G \subseteq T_1\,\mathrm{GL}_n(\mathbb{F}) = T_1M_n(\mathbb{F}) \cong M_n(\mathbb{F})$; then for any $X \in M_n(\mathbb{F})$, we note that the path $\gamma\colon \mathbb{R} \to M_n(\mathbb{R})$ defined by $\gamma(t) := 1 + tX$ has $\gamma(0) = 1$ and $\gamma'(0) = X$, so

$$(d\det)_1(X) = (d\det)_1(\gamma'(0)) = (\det \circ \gamma)'(0) = \frac{d}{dt}\det(1 + tX)\bigg|_{t=0}.$$

Thus, we are interested in the linear terms of the polynomial $\det(1+tX)$. Now, writing X out in coordinates as $X = [X_{ij}]_{1\leqslant i,j\leqslant n}$ and setting $A_{ij} = 1_{i=j} + tX_{ij}$, we note

$$\det(1 + tX) = \det A = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)A_{1\sigma(1)} \cdots A_{n\sigma(n)}.$$

Now, the only way a summand can produce linear terms is if there is at most one non-diagonal entry A_{ij}, which of course forces all entries to be diagonal. Thus,

$$\frac{d}{dt}\det(1 + tX)\bigg|_{t=0} = \frac{d}{dt}(1 + tX_{11}) \cdots (1 + tX_{nn})\bigg|_{t=0} \overset{*}{=} (X_{11} + \cdots + X_{nn}) = \operatorname{tr} X,$$

where $\overset{*}{=}$ holds by an expansion of the terms looking for linear terms. Thus,

$$T_1SG = \{X \in T_1G : \operatorname{tr} X = 0\}.$$

■

Lemma 1.152. Let $J \in M_n(\mathbb{F})$ be some matrix, and let $(-)^*$ denote either of the involutions $(-)^\intercal$ or $(-)^\dagger$. Then one has the subgroup

$$O_J(\mathbb{F}) := \{g \in \mathrm{GL}_n(\mathbb{F}) : g^*Jg = J\}.$$

We claim that $O_J(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ is a closed Lie subgroup (though if $(-)^* = (-)^\dagger$ and $\mathbb{F} = \mathbb{C}$, then $O_J(\mathbb{F})$ is a group over $\mathbb{R}$) and compute that

$$T_1O_J(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^*J + JX = 0\}.$$

Proof. Indeed, let $\mathrm{GL}_n(\mathbb{F})$ act on $M_n(\mathbb{F})$ by $\mu(g, A) := g^*Ag$. This (right!) action is polynomial and hence regular (with the previous parenthetical in mind), and we can check that it is an action because $\mu(1, A) = A$ and $\mu(g, \mu(h, A)) = g^*h^*Ahg = \mu(hg, A)$.

Now, the stabilizer of $J \in M_n(\mathbb{F})$ is precisely $O_J(\mathbb{F})$ by definition, so $O_J(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ is in fact a closed Lie subgroup. We also go ahead and compute $T_1O_J(\mathbb{F})$. Letting $f\colon \mathrm{GL}_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined by $f(g) := g^*Jg$, we see that

$$T_1O_J(\mathbb{F}) = \ker df_1,$$

so we want to compute df_1. As usual, we identify $T_1G \subseteq T_1\,\mathrm{GL}_n(\mathbb{F}) = T_1M_n(\mathbb{F}) \cong M_n(\mathbb{F})$; then for any $X \in M_n(\mathbb{F})$, we note that the path $\gamma\colon \mathbb{R} \to M_n(\mathbb{R})$ defined by $\gamma(t) := 1 + tX$ has $\gamma(0) = 1$ and $\gamma'(0) = X$, so

$$df_1(X) = df_1(\gamma'(0)) = (f \circ \gamma)'(0) = \frac{d}{dt} f(1 + tX)\bigg|_{t=0}.$$

Thus, we go ahead and compute

$$f(1 + tX) = (1 + tX)^*J(1 + tX) = J + t(X^*J + JX) + t^2X^*JX,$$

so

$$df_1(X) = \frac{d}{dt} f(1 + tX)\bigg|_{t=0} = X^*J + JX.$$

Thus,

$$T_1O_J(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^*J + JX = 0\},$$

as required. ■

We now execute our computations in sequence.

(a) Using the preceding remarks, we see that

$$T_1\,\mathrm{U}_{p,q}(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : X^*B_{p,q} + B_{p,q}X = 0\},$$

where $B_{p,q} = \left[\begin{smallmatrix} 1_p & \\ & -1_q \end{smallmatrix}\right]$ is a diagonal matrix. We now continue as in (c). Set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and then we compute

$$\begin{aligned} X^*B_{p,q} + B_{p,q}X &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^* & -C^* \\ B^* & -D^* \end{bmatrix} + \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &= \begin{bmatrix} A^* + A & B - C^* \\ B^* - C & -D^* - D \end{bmatrix}. \end{aligned}$$

In particular, this will vanish if and only if A and D are skew-Hermitian and $B = C^*$, so

$$\begin{aligned} T_1\,\mathrm{U}_{p,q}(\mathbb{C}) &= \left\{\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^*\right\}, \\ T_1\,\mathrm{U}_n(\mathbb{C}) &= \{A \in M_n(\mathbb{C}) : A = -A^*\}. \end{aligned}$$

Now, the space of $p \times p$ skew-Hermitian matrices A (namely, satisfying $A = -A^*$) is forced to have imaginary diagonal, and then the remaining entries are uniquely determined by their values strictly above the diagonal. Thus, the real dimension of this space is $p + p(p-1) = p^2$. We conclude that

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{U}_{p,q}(\mathbb{C}) &= p^2 + 2pq + q^2 = n^2, \\ \dim_{\mathbb{R}} \mathrm{U}_n(\mathbb{C}) &= n^2. \end{aligned}$$

From here, we address SU by recalling that

$$T_1\,\mathrm{SU}_{p,q}(\mathbb{R}) = \{X \in T_1\,\mathrm{U}_{p,q}(\mathbb{C}) : \operatorname{tr} X\}.$$

In particular,

$$T_1\,\mathrm{SU}_{p,q}(\mathbb{C}) = \left\{\begin{bmatrix} A & B \\ B^* & D\end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^*, \operatorname{tr} A + \operatorname{tr} D = 0\right\},$$
$$T_1\,\mathrm{SU}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = -A^*, \operatorname{tr} A = 0\}.$$

Now, tr continues is real and actually surjects onto $\mathbb{R}$ for $n \geqslant 1$, even for our family of matrices above (for example, for any real number r, the matrix $\operatorname{diag}(r, 0, \ldots, 0)$ has trace r and lives in the above families). Thus, the kernel has dimension one smaller than the total space, giving

$$\dim_{\mathbb{R}} \mathrm{SU}_{p,q}(\mathbb{C}) = n^2 - 1,$$
$$\dim_{\mathbb{R}} \mathrm{SU}_n(\mathbb{C}) = n^2 - 1.$$

Example 1.153. We will show that

$$\mathrm{SL}_n(\mathbb{F}) := \{A \in \mathrm{GL}_n(\mathbb{F}) : \det A = 1\}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \mathrm{SL}_n(\mathbb{F}) = n^2 - 1$.

Proof. We use Lemma 1.151. We see that

$$T_1\,\mathrm{SL}_n(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : \operatorname{tr} X = 0\}.$$

(Note $T_1\,\mathrm{GL}_n(\mathbb{F}) = T_1 M_n(\mathbb{F}) \cong M_n(\mathbb{F})$.) As such, we note that $\operatorname{tr}\colon M_n(\mathbb{F}) \to \mathbb{F}$ is surjective (for $n \geqslant 1$), so $\dim_{\mathbb{F}} T_1\,\mathrm{SL}_n(\mathbb{F}) = \dim_{\mathbb{F}} \ker \operatorname{tr} = \dim_{\mathbb{F}} M_n(\mathbb{F}) - 1 = n^2 - 1$. ■

We now begin our computations for bilinear forms.

Example 1.154. Let $B := 1_n$ be the standard bilinear form. We will show that

$$\mathrm{O}_n(\mathbb{F}) := \{A \in \mathrm{GL}_n(\mathbb{F}) : ABA^{\intercal}B\}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \mathrm{O}_n(\mathbb{F}) = \frac{1}{2}n(n-1)$.

Proof. We use Lemma 1.152. We see that

$$T_1\,\mathrm{O}_n(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^{\intercal} + X = 0\},$$

which is the space of alternating matrices. Thus, we see that the diagonal of $X \in T_1\,\mathrm{O}_n(\mathbb{F})$ vanishes, and the remaining entries are determined by the values strictly above the diagonal, of which there are $\frac{1}{2}n(n-1)$. Thus, $\dim \mathrm{O}_n(\mathbb{F}) = \frac{1}{2}n(n-1)$. ■

Example 1.155. We will show that

$$\mathrm{SO}_n(\mathbb{F}) := \{A \in \mathrm{O}_n(\mathbb{F}) : \det A = 1\}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \mathrm{SO}_n(\mathbb{F}) = \frac{1}{2}n(n-1)$.

Proof. Using Lemma 1.151, we see that

$$T_1\,\mathrm{SO}_n(\mathbb{F}) = \{X \in \mathrm{O}_n(\mathbb{F}) : \operatorname{tr} X = 0\}.$$

However, alternating matrices already have vanishing traces, so $T_1\,\mathrm{SO}_n(\mathbb{F})$ is simply the full space of alternating matrices, giving $\dim_{\mathbb{F}} \mathrm{SO}_n(\mathbb{F}) = \frac{1}{2}n(n-1)$. ■

Over $\mathbb{R}$, there are more bilinear forms.

Example 1.156. Let $B_{p,q} := 1_p \oplus 1_q$ where $n = p + q$. We will show that

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : AB_{p,q}A^\intercal B_{p,q}\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \mathrm{O}_{p,q}(\mathbb{R}) = \frac{1}{2}n(n-1)$.

Proof. By Lemma 1.152, we see that

$$T_1 \mathrm{O}_{p,q}(\mathbb{R}) = \{X \in M_n(\mathbb{R}) : X^\intercal B_{p,q} + B_{p,q}X = 0\},$$

where $B_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$ is a diagonal matrix. To compute the dimension of this space of matrices, we set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and then we compute

$$\begin{aligned} X^\intercal B_{p,q} + B_{p,q}X &= \begin{bmatrix} A^\intercal & C^\intercal \\ B^\intercal & D^\intercal \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^\intercal & -C^\intercal \\ B^\intercal & -D^\intercal \end{bmatrix} + \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &= \begin{bmatrix} A^\intercal + A & B - C^\intercal \\ B^\intercal - C & -D^\intercal - D \end{bmatrix}. \end{aligned}$$

In particular, this will vanish if and only if A and D are both alternating, and $B = C^\intercal$, yielding

$$T_1 \mathrm{O}_{p,q}(\mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ B^\intercal & D \end{bmatrix} : A \in M_p(\mathbb{R}) \text{ and } D \in M_q(\mathbb{R}) \text{ are alternating} \right\}.$$

Thus, the dimension of our space is

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{O}_{p,q}(\mathbb{R}) &= \underbrace{\frac{1}{2}p(p-1)}_{A} + \underbrace{pq}_{B=C^\intercal} + \underbrace{\frac{1}{2}q(q-1)}_{D} \\ &= \frac{1}{2}\left(p^2 + 2pq + q^2 - p - q\right) \\ &= \frac{1}{2}(p+q)(p+q-1) \\ &= \frac{1}{2}n(n-1), \end{aligned}$$

where the dimension computations for (the spaces of) A and D are as in. ■

Example 1.157. We will show that

$$\mathrm{SO}_{p,q}(\mathbb{R}) := \{A \in \mathrm{O}_{p,q}(\mathbb{F}) : \det A = 1\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \mathrm{SO}_{p,q}(\mathbb{R}) = \frac{1}{2}n(n-1)$.

Proof. We use Lemma 1.151, we note that

$$T_1 \mathrm{SO}_{p,q}(\mathbb{R}) : \{X \in T_1 \mathrm{O}_{p,q}(\mathbb{R}) : \operatorname{tr} X = 0\},$$

but our description of $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has A and D alternating, so $\operatorname{tr} X = \operatorname{tr} A + \operatorname{tr} D = 0$. Thus, we see $T_1 \mathrm{SO}_{p,q}(\mathbb{R}) = T_1 \mathrm{O}_{p,q}(\mathbb{R})$, so the above description of tangent space and dimension go through. ■

Example 1.158. Let $\Omega_{2n} := \left[\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right]$ be the standard symplectic form. We will show that

$$\operatorname{Sp}_{2n}(\mathbb{F}) := \{A \in \operatorname{GL}_n(\mathbb{F}) : A\Omega A^\intercal = A\}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \operatorname{Sp}_{2n}(\mathbb{F}) = 2n^2 + n$.

Proof. By Lemma 1.152, we see that

$$T_1 \operatorname{Sp}_{2n}(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^\intercal\Omega + \Omega X = 0\},$$

where $\Omega = \begin{bmatrix} & -1_n \\ 1_n & \end{bmatrix}$ is alternating. As usual, we set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and we compute

$$\begin{aligned} X^\intercal\Omega + \Omega X &= \begin{bmatrix} A^\intercal & C^\intercal \\ B^\intercal & D^\intercal \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} + \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} C^\intercal & -A^\intercal \\ D^\intercal & -B^\intercal \end{bmatrix} + \begin{bmatrix} -C & -D \\ A & B \end{bmatrix} \\ &= \begin{bmatrix} C^\intercal - C & -D - A^\intercal \\ A + D^\intercal & B - B^\intercal \end{bmatrix}. \end{aligned}$$

Thus, we see that

$$T_1 \operatorname{Sp}_{2n}(\mathbb{F}) = \left\{ \begin{bmatrix} A & B \\ C & -A^\intercal \end{bmatrix} : A, B, C \in M_n(\mathbb{F}), B = B^\intercal, C = C^\intercal \right\},$$

and our dimension is

$$\dim_{\mathbb{F}} \operatorname{Sp}_{2n}(\mathbb{F}) = \underbrace{n^2}_{A} + \underbrace{\frac{1}{2}n(n+1)}_{B} + \underbrace{\frac{1}{2}n(n+1)}_{C} = 2n^2 + n,$$

where we compute the dimension of space of symmetric matrices exactly analogously to the case of alternating matrices, except now the diagonal is permitted to be nonzero. ■

Lastly, we handle Hermitian forms.

Example 1.159. Let $B_{p,q} := 1_p \oplus 1_q$ where $n = p + q$. We will show that

$$\operatorname{U}_{p,q}(\mathbb{C}) := \left\{A \in \operatorname{GL}_n(\mathbb{F}) : AB_{p,q}A^\dagger B_{p,q}\right\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \operatorname{U}_{p,q}(\mathbb{C}) = n^2$.

Proof. Using Lemma 1.152, we see that

$$T_1 \operatorname{U}_{p,q}(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : X^* B_{p,q} + B_{p,q}X = 0\},$$

where $B_{p,q} = \left[\begin{smallmatrix} 1_p & \\ & -1_q \end{smallmatrix}\right]$ is a diagonal matrix. We now continue as in (c). Set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and then we compute

$$\begin{aligned} X^* B_{p,q} + B_{p,q}X &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^* & -C^* \\ B^* & -D^* \end{bmatrix} + \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &= \begin{bmatrix} A^* + A & B - C^* \\ B^* - C & -D^* - D \end{bmatrix}. \end{aligned}$$

In particular, this will vanish if and only if A and D are skew-Hermitian and $B = C^*$, so

$$\begin{aligned} T_1 \operatorname{U}_{p,q}(\mathbb{C}) &= \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^* \right\}, \\ T_1 \operatorname{U}_n(\mathbb{C}) &= \{A \in M_n(\mathbb{C}) : A = -A^*\}. \end{aligned}$$

Now, the space of $p \times p$ skew-Hermitian matrices A (namely, satisfying $A = -A^*$) is forced to have imaginary diagonal, and then the remaining entries are uniquely determined by their values strictly above the diagonal. Thus, the real dimension of this space is $p + p(p-1) = p^2$. We conclude that

$$\dim_{\mathbb{R}} \operatorname{U}_{p,q}(\mathbb{C}) = p^2 + 2pq + q^2 = n^2,$$

as required. ■

Example 1.160. We will show that

$$\operatorname{SU}_{p,q}(\mathbb{C}) := \{A \in \operatorname{SU}_{p,q}(\mathbb{C}) : \det A = 1_n\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \operatorname{SU}_{p,q}(\mathbb{C}) = n^2 - 1$.

Proof. By Lemma 1.151, we see

$$T_1 \operatorname{SU}_{p,q}(\mathbb{R}) = \{X \in T_1 \operatorname{U}_{p,q}(\mathbb{C}) : \operatorname{tr} X\}.$$

In particular,

$$\begin{aligned} T_1 \operatorname{SU}_{p,q}(\mathbb{C}) &= \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^*, \operatorname{tr} A + \operatorname{tr} D = 0 \right\}, \\ T_1 \operatorname{SU}_n(\mathbb{C}) &= \{A \in M_n(\mathbb{C}) : A = -A^*, \operatorname{tr} A = 0\}. \end{aligned}$$

Now, tr continues is real and actually surjects onto $\mathbb{R}$ for $n \geqslant 1$, even for our family of matrices above (for example, for any real number r, the matrix $\operatorname{diag}(r, 0, \ldots, 0)$ has trace r and lives in the above families). Thus, the kernel has dimension one smaller than the total space, giving

$$\dim_{\mathbb{R}} \operatorname{SU}_{p,q}(\mathbb{C}) = n^2 - 1,$$

as required. ■

THEME 2

PASSING TO LIE ALGEBRAS

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

—Emil Artin

2.1 September 18

Today we compute our Lie algebras.

2.1.1 The Exponential Map: The General Case

Given a Lie group G with Lie algebra $\mathfrak{g}$, we would like to define an exponential map $\exp\colon \mathfrak{g} \to G$. Recall that $\exp$ gave rise to our homomorphisms $\gamma\colon \mathbb{R} \to G$ with $\gamma(0) = e$ and $\gamma'(0)$ is specified. This will be our starting point.

Proposition 2.1. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, there exists a unique Lie group homomorphism $\gamma_X : \mathbb{R} \to G$ such that $\gamma_X'(0) = X$.

Proof. We use the theory of integral curves; see [Lee13, Chapter 9]. In particular, we see that we must satisfy $\gamma(s+t) = \gamma(t)\gamma(s)$ for all $s, t \in \mathbb{R}$, which yields

$$\gamma'(t) = \gamma(t)\gamma'(0),$$

where this multiplication really means $dL_{\gamma(t)}(\gamma'(0))$.

Thus, we see that we want to extend $X \in T_eG$ to a left-invariant vector field, and then we let $\gamma\colon \mathbb{R} \to G$ be the integral curve of this vector field satisfying $\gamma(0) = e$. (A priori, γ can only be defined in a neighborhood of the identity, but we can translate around in the group G to get a global solution. See [Lee13, Lemma 9.15] and in particular its corollary [Lee13, Theorem 9.18].) Then

$$\gamma'(t) = X(\gamma(t)) = dL_{\gamma(t)}(X(0)) = dL_{\gamma(t)}(X)$$

for each $t \in \mathbb{R}$.

Thus far we have shown that there is at most one Lie group homomorphism $\gamma_X\colon \mathbb{R} \to G$ satisfying $\gamma_X'(0) = X$; namely, it will be the above integral curve! It remains to check that the above integral curve actually satisfies $\gamma(t+s) = \gamma(t)\gamma(s)$. Well, for $s \in \mathbb{R}$, we define $\gamma_1(t) = \gamma(t+s)$ and $\gamma_2(t) = \gamma(s)\gamma(t)$. Then we see that γ_1 and γ_2 are both integral curves satisfying the ordinary differential equation

$$\widetilde{\gamma}'(t) = dL_{\widetilde{\gamma}(t)}(\widetilde{\gamma}'(0))$$

with initial condition $\widetilde{\gamma}(0) = \gamma(s)$, so the must be equal, completing the proof. ■

Remark 2.2. Here is one way to conclude without using [Lee13, Theorem 9.18]. The last paragraph of the proof provides a path $\gamma\colon (-\varepsilon, \varepsilon) \to G$ for some $\varepsilon > 0$ satisfying the homomorphism property. But then any $N > 0$ allows us to define $\widetilde{\gamma}\colon (-N, N) \to G$ given by

$$\widetilde{\gamma}(t) := \gamma(t/N)^N.$$

However, we can check that $\widetilde{\gamma}$ satisfies $\widetilde{\gamma}'(t) = dL_{\widetilde{\gamma}(t)}(X)$ with initial condition $\widetilde{\gamma}(0) = e$, so $\widetilde{\gamma}$ extends γ. Thus, we can extend γ to $\bigcup_{N>0}(-N\varepsilon, N\varepsilon) = \mathbb{R}$.

We now define $\exp$ motivated by the classical case.

Definition 2.3 (exponential)**.** Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, define γ_X via Proposition 2.1. Then we define $\exp_G\colon \mathfrak{g} \to G$ by

$$\exp_G(X) := \gamma_X(1).$$

We will omit the subscript from $\exp_G$ as much as possible.

Example 2.4. If $G \subseteq \mathrm{GL}_n(\mathbb{F})$ is classical, we can take $\gamma_X(t) = \exp(tX)$ where $\exp$ is defined as for GL_n. Thus, $\exp(X)$ matches with the above definition.

Example 2.5. Consider the Lie group $\mathbb{R}^n$. Then for each $X \in T_0\mathbb{R}^n$, we identify $T_0\mathbb{R}^n \cong \mathbb{R}^n$ to observe that we can take $\gamma_X(t) := tX$. Thus, $\exp(X) = X$.

Example 2.6. For any G, we can take $\gamma_0(t) := 0$, so $\exp(0) = 1$.

Example 2.7. We can directly compute that

$$(d\exp)_0(X) = \frac{d}{dt}\exp(tX)\bigg|_0 \overset{*}{=} \frac{d}{dt}\gamma_X(t)\bigg|_{t=0} = X.$$

The equality $\exp(tX) \overset{*}{=} \gamma_X(t)$ is explained as follows: we can check that $\gamma_{rX}(t) = \gamma_X(rt)$ for any $r, t \in \mathbb{R}$ by computing the derivative at 0, so $\exp(tX) = \gamma_{tX}(1) = \gamma_X(t)$ follows.

Here are some quick checks.

Proposition 2.8. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then $\exp\colon \mathfrak{g} \to G$ is regular and a local diffeomorphism.

Proof. Note that $\exp$ solves the differential equation given by Example 2.7, for which the theory of integral curves promises that this solution must be regular. Example 2.7 also tells us that $\exp$ is an isomorphism at the identity and hence a local diffeomorphism. ■

We would like to know something like $\exp(A+B) = \exp(A)\exp(B)$ when A and B commute, but one needs to be a little careful in how to state this. Here are some manifestations.

Proposition 2.9. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then

$$\exp((s+t)X) = \exp(sX)\exp(tX)$$

for any $s, t \in \mathbb{R}$ and $X \in \mathfrak{g}$.

Proof. This is a matter of following the definitions around. Let $\gamma_X : \mathbb{R} \to G$ be the one-parameter family for X. Then we see that $\gamma_{rX}(t) = \gamma_X(rt)$ for any $r \in \mathbb{R}$ as explained in Example 2.7, so

$$\exp((s+t)X) = \gamma_{(s+t)X}(1) = \gamma_X(s+t) = \gamma_X(s)\gamma_X(t) = \exp(sX)\exp(tX),$$

as desired. ■

Proposition 2.10. Fix a homomorphism $\varphi \colon G \to H$ of Lie groups. Then

$$\varphi(\exp_G(X)) = \exp_H(d\varphi_0(X))$$

for any $X \in T_eG$.

Proof. This follows from the definition. In particular, we claim that

$$\gamma_{d\varphi_0(X)}(t) \stackrel{?}{=} \varphi(\gamma_X(t)).$$

To see this, note that $t \mapsto \varphi(\gamma_X(t))$ is a Lie group homomorphism $\mathbb{R} \to H$, and we can compute the derivative at 0 to be $d\varphi_0(\gamma_X'(0)) = d\varphi_0(X)$, as required. Plugging in $t = 1$ to the above equation completes the proof. ■

Corollary 2.11. Fix homomorphisms $\varphi_1, \varphi_2 \colon G \to H$ of Lie groups. Suppose G is connected. If $d\varphi_1 = d\varphi_2$, then $\varphi_1 = \varphi_2$.

Proof. Using Proposition 2.10, we see that

$$\varphi(\exp(X)) = \exp(d\varphi_0(X))$$

produces the same answer for $\varphi \in \{\varphi_1, \varphi_2\}$. However, $\exp$ is a local diffeomorphism by Proposition 2.8, so we have determined the values of φ_1 and φ_2 on the image of $\exp$, which must contain an open neighborhood of the identity of G. Thus, because G is connected, we see that G is generated by this open neighborhood, so in fact we have fully determined the values of φ_1 and φ_2. ■

Proposition 2.12. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$, and let $\operatorname{Ad}_\bullet \colon G \to \operatorname{GL}(\mathfrak{g})$ be the adjoint representation of Example 1.113. For any $g \in G$ and $X \in \mathfrak{g}$, we have

$$g\exp(X)g^{-1} = \exp(\operatorname{Ad}_g X).$$

Proof. By Proposition 2.10, we see that

$$g\exp(X)g^{-1} = \operatorname{Ad}_g(\exp(X)) = \exp((d\operatorname{Ad}_g)_e X) = \exp(\operatorname{Ad}_g X),$$

where the last equality holds by definition of the adjoint representation. (Yes, the notation is somewhat confusing.) ■

While we are here, we note that there is a logarithm map.

Definition 2.13 (logarithm)**.** Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Because $\exp$ is a local diffeomorphism, there are open neighborhoods $U \subseteq G$ and $\mathfrak{u} \subseteq \mathfrak{g}$ of the identities so that $\log\colon U \to \mathfrak{u}$ is an inverse for $\exp$.

2.1.2 The Commutator

Define the form $\mu\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\mu(X, Y) := \log(\exp(X)\exp(Y)).$$

(Technically, μ is a priori only defined on an open neighborhood of the identity of $\mathfrak{g} \times \mathfrak{g}$.) Expanding out everything into coordinates, we see that μ has a Taylor series expansion as

$$\mu(X, Y) = c + \alpha_1(X) + \alpha_2(Y) + Q_1(X) + Q_2(Y) + \lambda(X, Y) + \cdots,$$

where c is constant, α_1 and α_2 are linear, Q_1 and Q_2 are quadratic, λ is bilinear, and $+\cdots$ denotes cubic and higher-order terms. However, we see that $\mu(X, 0) = 0$ and $\mu(0, Y) = 0$ for any $X, Y \in \mathfrak{g}$, so $c = Q_1 = Q_2 = 0$ and $\alpha_1(X) = X$ and $\alpha_2(Y) = Y$. Further, we claim that λ is skew-symmetric: it is enough to show that $\lambda(X, X) = 0$, for which we note that

$$2X = \log(\exp(2X)) = \log(\exp(X)\exp(X)) = \mu(X, X) = X + X + \lambda(X, X) + \cdots,$$

so $\lambda(X, X) = 0$ is forced.

This λ allows us to define the Lie bracket on $\mathfrak{g}$ in a purely group-theoretic way.

Definition 2.14 (Lie bracket)**.** Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then we define the *commutator* as the skew-symmetric form $\frac{1}{2}\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $[-,-]$. In particular, we see that

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right), \tag{2.1}$$

where $+\cdots$ denotes higher-order terms (as usual).

Remark 2.15. A priori, the commutator may only be defined on an open neighborhood of the identity of $\mathfrak{g} \times \mathfrak{g}$, so (2.1) only holds (a priori) for sufficiently small X and Y. However, bilinearity allows us to scale our definition of $[-,-]$ from this open neighborhood everywhere.

Example 2.16. We compute the commutator map for GL_n. We see that

$$\exp(X)\exp(Y) = 1 + X + Y + XY + \frac{1}{2}\left(X^2 + Y^2\right) + \cdots, \tag{2.2}$$

$$\exp\left(X + Y + \frac{1}{2}[X + Y] + \cdots\right) = 1 + X + Y + \frac{1}{2}\left(X^2 + Y^2\right) + \frac{1}{2}XY + \frac{1}{2}YX + \frac{1}{2}[X, Y] + \cdots, \tag{2.3}$$

giving $[X, Y] = XY - YX$ subtracting.

To compute the commutator for the classical groups, we need to check some functoriality.

Proposition 2.17. Fix a homomorphism $\varphi\colon G \to H$ of Lie groups. For any $X, Y \in T_eG$, we have

$$d\varphi_0([X, Y]) = [d\varphi_0(X), d\varphi_0(Y)].$$

Proof. We unravel the definitions. Everything in sight is linear, so we may assume that X and Y are sufficiently small, so $d\varphi_0(X)$ and $d\varphi_0(Y)$ are sufficiently small. We now compute

$$\begin{aligned}\exp\left(d\varphi_0(X)+d\varphi_0(Y)+\frac{1}{2}[d\varphi_0(X),d\varphi_0(Y)]\right) &= \exp\left(d\varphi_0(X)\right)\exp\left(d\varphi_0(Y)\right)\\ &\overset{*}{=} \varphi(\exp(X))\varphi(\exp(Y))\\ &= \varphi(\exp(X)\exp(Y))\\ &= \varphi\left(\exp\left(X+Y+\frac{1}{2}[X,Y]+\cdots\right)\right)\\ &\overset{*}{=} \exp\left(d\varphi_0(X)+d\varphi_0(Y)+\frac{1}{2}d\varphi_0([X,Y])+\cdots\right),\end{aligned}$$

where we have used Proposition 2.10 at the equalities $\overset{*}{=}$. Because $\exp$ is a diffeomorphism for X and Y sufficiently small, the desired equality follows. ■

Example 2.18. The embedding $\mathrm{SL}_n(\mathbb{F})\to\mathrm{GL}_n(\mathbb{F})$ implies by Proposition 2.17 that the Lie bracket on the Lie algebra $\mathfrak{sl}_n$ can be computed by restricting the commutator Lie bracket on $\mathfrak{gl}_n$ (given by Example 2.16). In particular, we see that $\mathfrak{sl}_n$ is closed under taking commutators, which is not totally obvious a priori! A similar operation permits computation of the Lie bracket of a Lie group G whenever given an embedding $G\subseteq\mathrm{GL}_n$ (such as for the classical groups).

Corollary 2.19. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathrm{Ad}_\bullet\colon G\to\mathrm{GL}(\mathfrak{g})$ denote the adjoint representation. For

$$\mathrm{Ad}_g([X,Y])=[\mathrm{Ad}_g(X),\mathrm{Ad}_g(Y)].$$

Proof. We simply apply Proposition 2.17 to $\mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g})$, which yields

$$(d\mathrm{Ad}_g)_e([X,Y])=[(d\mathrm{Ad}_g)_eX,(d\mathrm{Ad}_g)_eT],$$

which is the original equation after enough abuse of notation. ■

Proposition 2.20. Fix a Lie group G. For sufficiently small $X,Y\in T_eG$, we have

$$\exp(X)\exp(Y)\exp(X)^{-1}\exp(Y)^{-1}=\exp([X,Y]+\cdots).$$

Proof. This is a direct computation. We compute

$$\begin{aligned}\exp(X)\exp(Y)\exp(-X)\exp(-Y) &= \exp\left(X+Y+\frac{1}{2}[X,Y]+\cdots\right)\exp\left(-X-Y+\frac{1}{2}[X,Y]+\cdots\right)\\ &= \exp\left([X,Y]+\cdots\right),\end{aligned}$$

where we get some omitted cancellation of lower-order terms in the last equality (and there is a lot of higher-order terms). ■

Corollary 2.21. If G is abelian, then $[X,Y]=0$ for any X and Y.

Proof. It suffices to assume that X and Y are sufficiently small because the conclusion is linear. Now, Proposition 2.20 implies that

$$\exp([X,Y]+\cdots)=0,$$

so because $\exp$ is a diffeomorphism for small enough X and Y, so $[X,Y]=0$ follows. ■

2.2 September 20

Today we continue discussing the Lie bracket.

2.2.1 The Adjoint Action

Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Here is the standard example of a "Lie algebra representation."

Notation 2.22. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Note that the map $(d\mathrm{Ad}_g)_1 \colon G \to \mathrm{GL}(\mathfrak{g})$ is smooth, so we can consider the differential of this map, which we label $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Here are some checks on this map.

Proposition 2.23. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$.

(a) For $X, Y \in \mathfrak{g}$, we have $\mathrm{ad}_X(Y) = [X, Y]$.

(b) For $X \in \mathfrak{g}$, we have $\mathrm{Ad}_{\exp(X)} = \exp(\mathrm{ad}_X)$ as operators $\mathfrak{g} \to \mathfrak{g}$.

Proof. Here we go.

(a) By definition of our differential action, we have

$$(d\mathrm{Ad}_g)_1(Y) = \frac{d}{dt} g \exp(tY) g^{-1}\bigg|_{t=0}$$

for any $g \in G$ and $Y \in \mathfrak{g}$. We would like to compute a derivative of this map with respect to g (at the identity). As such, we plug in $g = \exp(sX)$ to compute

$$\begin{aligned}
\mathrm{ad}_X(Y) &= \frac{d}{ds}(d\mathrm{Ad}_{\exp(sX)})_1(Y)\bigg|_{s=0} \\
&= \frac{d}{ds}\frac{d}{dt}\exp(sX)\exp(tY)\exp(-sX)\bigg|_{t=0}\bigg|_{s=0} \\
&\overset{*}{=} \frac{d}{ds}\frac{d}{dt}\exp(tY + st[X,Y] + \cdots)\bigg|_{t=0}\bigg|_{s=0},
\end{aligned}$$

where in $\overset{*}{=}$ we have used the definition of our bracket. Upon expanding out $\exp$ as a series, we see that the lower-order terms are $1 + tY + st[X, Y] + \cdots$ (everything higher is at least quadratic) for small enough s and t, so the derivative evaluates to $[X, Y]$.

(b) This follows immediately from Proposition 2.10 upon setting $\varphi = (d\mathrm{Ad}_\bullet)_1$. ■

Here is an example computation of what all this adjoint business looks like for GL_n, more directly than appealing to the bracket.

Lemma 2.24. Identify $T\,\mathrm{GL}_n(\mathbb{F})$ with $\mathrm{GL}_n(\mathbb{F}) \times \mathfrak{gl}_n(\mathbb{F})$ via left-invariant vector fields. For $X \in \mathfrak{gl}_n(\mathbb{F})$, we have

$$\begin{cases} dL_g(X) = gX, \\ dR_g(X) = Xg^{-1}, \\ d\,\mathrm{Ad}_g(X) = gXg^{-1}. \end{cases}$$

Proof. Set $G := \mathrm{GL}_n(\mathbb{F})$ and $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{F})$. Note that the adjoint is the composite of L_g and R_g, so the last equality follows from the first two. For the first equality, we are computing the differential of the maps $L_g, R_g \colon G \to G$ at some $h \in G$. Well, L_g and R_g actually extend to perfectly fine linear maps $M_n(\mathbb{F}) \to M_n(\mathbb{F})$, and the differential of any linear map is simply itself upon identifying the tangent spaces of $M_n(\mathbb{F})$ with itself, so we conclude that $dL_g(X) = gX$ and $dR_g(X) = Xg^{-1}$, as required. ■

Lemma 2.25. Fix a homomorphism $\varphi \colon G \to H$ of Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. For any $g \in G$ and $X \in \mathfrak{g}$, we have

$$(d\mathrm{Ad}_{\varphi(g)})_e(d\varphi_1(X)) = d\varphi_1((d\mathrm{Ad}_g)_e(X)).$$

Proof. Simply take the differential (at 1) of the equation $\mathrm{Ad}_{\varphi(g)} \circ \varphi = \varphi \circ \mathrm{Ad}_g$, which is true because φ is a homomorphism. ■

Example 2.26. Given any embedding $G \subseteq \mathrm{GL}_n(\mathbb{F})$ with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$, we can use Lemma 2.25 to compute the adjoint action on $\mathfrak{g}$ by conjugation (via Lemma 2.24)!

Proposition 2.27. Let $(d\mathrm{Ad}_\bullet)_1 \colon \mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}(\mathfrak{gl}_n(\mathbb{F}))$ denote the adjoint representation. Then

$$\mathrm{ad}_X(Y) = XY - YX.$$

Proof. To parse the symbols, we note that $(d(d\mathrm{Ad}_\bullet)_1)_1 \colon \mathfrak{gl}_n(\mathbb{F}) \to \mathrm{End}(\mathfrak{gl}_n(\mathbb{F}))$, so the statement at least makes sense. Now, given $X \in \mathfrak{gl}_n(\mathbb{F})$, define $\gamma \colon \mathbb{F} \to M_n(\mathbb{F})$ by $\gamma(t) := 1 + tX$. Then $\gamma'(0) = X$. As such,

$$(d(d\mathrm{Ad}_\bullet)_1)_1(X) = (d(d\mathrm{Ad}_\bullet)_1)_1(\gamma'(0)) = ((d\mathrm{Ad}_\bullet)_1 \circ \gamma)'(0).$$

In particular, plugging in some $Y \in \mathfrak{gl}_n(\mathbb{F})$, we may use Lemma 2.24 to compute that

$$\begin{aligned}
\frac{d}{dt}((d\mathrm{Ad}_\bullet)_1 \circ \gamma)(t)(Y)\bigg|_{t=0} &= \frac{d}{dt}(d\mathrm{Ad}_{1+tX})_1(Y)\bigg|_{t=0} \\
&= \frac{d}{dt}(1+tX)Y(1+tX)^{-1}\bigg|_{t=0} \\
&= \frac{d}{dt}(1+tX)Y\left(1 - tX + t^2X^2 + \cdots\right)\bigg|_{t=0} \\
&= XY - YX,
\end{aligned}$$

where the series expansion takes t small enough for the series to converge. (For example, one can take t small enough so that all eigenvalues of tX are less than 1.) ■

Example 2.28. Given any embedding $G \subseteq \mathrm{GL}_n(\mathbb{F})$ with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$, we note that the action of G on $\mathfrak{g}$ actually extends to an action of G on $\mathfrak{gl}_n(\mathbb{F})$ (still by conjugation) which happens to stabilize $\mathfrak{g}$. Then the action $G \to \mathrm{GL}(\mathfrak{gl}_n(\mathbb{F}))$ is a restriction of the adjoint action $\mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}(\mathfrak{gl}_n(\mathbb{F}))$ given by conjugation still, whose differential action $\mathfrak{gl}_n(\mathbb{F}) \to \mathfrak{gl}(\mathfrak{gl}_n(\mathbb{F}))$ we computed above to be given by $\mathrm{ad}_X \colon Y \mapsto XY - YX$. This restricts back to the subspace $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$ (via the inclusion $G \subseteq \mathrm{GL}_n(\mathbb{F})$), where we know that the action must happen to stabilize $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$. The point is that we have computed our adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by the commutator. (Alternatively, one can redo the computation of the above proof.)

2.2.2 Lie Algebras

Here is a standard consequence of this theory.

Proposition 2.29 (Jacobi identity)**.** Fix a Lie group G with Lie algebra $\mathfrak{g}$. Then we have the Jacobi identity

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Proof. Doing some rearranging with Proposition 2.23 (and the skew-symmetry), we see that this is equivalent to plugging Z into the identity

$$\operatorname{ad}_{[X,Y]}\stackrel{?}{=}\operatorname{ad}_X\circ\operatorname{ad}_Y-\operatorname{ad}_Y\circ\operatorname{ad}_X.$$

To verify this, we note that the right-hand side is $[\operatorname{ad}_X,\operatorname{ad}_Y]$, where the commutator is taken in $\mathfrak{gl}(\mathfrak{g})$. Thus, we are trying to show that the adjoint preserves a commutator, which we do as follows: recall that $\operatorname{Ad}_\bullet\colon G\to\operatorname{GL}(\mathfrak{g})$ is a morphism of Lie groups, meaning that the differential map ad preserves the commutator by Proposition 2.17. ■

The Jacobi identity is important enough to earn the following definition.

Definition 2.30 (Lie algebra)**.** Fix a field k. Then a *Lie algebra* is a k-vector space $\mathfrak{g}$ equipped with a bilinear form $[-,-]\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying the following.

(a) Skew-symmetric: $[X,X]=0$ for all $X\in\mathfrak{g}$.

(b) Jacobi identity: for any $X,Y,Z\in\mathfrak{g}$, we have

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

A morphism of Lie algebras is a k-linear morphism preserving the forms.

Definition 2.31 (commutative)**.** A Lie algebra $\mathfrak{g}$ is *commutative* if and only if $[X,Y]=0$ for all $X,Y\in\mathfrak{g}$.

Example 2.32. For any k-algebra A, we produce a Lie bracket on A given by

$$[X,Y]:=XY-YX.$$

This map is of course linear in both X and Y (because multiplication is k-linear in a k-algebra), and $[X,X]=X^2-X^2=0$. Lastly, to see the Jacobi identity, we expand:

$$\begin{aligned}[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]&=[X,YZ-ZY]+[Y,ZX-XZ]+[Z,XY-YX]\\&=X(YZ-ZY)-(YZ-ZY)X\\&\quad+Y(ZX-XZ)-(ZX-XZ)Y\\&\quad+Z(XY-YX)-(XY-YX)Z\\&=0.\end{aligned}$$

For example, one can take A to be $\operatorname{End}_k(V)$ for some k-vector space V; this produces the Lie algebra $\mathfrak{gl}(V)$.

Example 2.33. Given a regular Lie group G, the tangent space at the identity $\mathfrak{g}$ is a Lie algebra according to the above definition.

The above example upgrades into a functor.

Proposition 2.34. Fix a regular Lie group G. For any morphism of Lie groups $\varphi\colon G_1 \to G_2$, the differential $d\varphi_e\colon T_eG_1 \to T_eG_2$ is a (functorial) morphism of Lie algebras. In fact, if G_1 is connected, the induced map

$$\mathrm{Hom}_{\mathrm{LieGrp}}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Lie}(k)}(T_eG_1, T_eG_2)$$

is injective. In other words, the functor $G \to T_eG$ from connected Lie groups to Lie algebras is faithful.

Proof. The differential being a homomorphism of Lie algebras follows from Proposition 2.17. Functoriality follows from the corresponding functoriality for differentials of more general smooth maps. The injectivity follows from Corollary 2.11. ■

Remark 2.35. It turns out that the functor above is also full, though we are not in a position to show this yet.

2.2.3 Subalgebras

Lie algebras are interesting enough to study on their own right, but we now note that we have sufficient motivation from Proposition 2.34.

Definition 2.36 (subalgebra, ideal)**.** Fix a Lie algebra $\mathfrak{g}$.

- A *Lie subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace closed under the Lie bracket of $\mathfrak{g}$; note that $\mathfrak{h}$ continues to be a Lie algebra.
- A *Lie ideal* is a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the stronger property that

$$[X, Y] \in \mathfrak{h}$$

for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

Definition 2.37 (representation)**.** A *representation* of a Lie algebra $\mathfrak{g}$ over a field k is a morphism $\mathfrak{g} \to \mathfrak{gl}(V)$ for some (finite-dimensional) vector space V over k. The representation is *faithful* if and only if the morphism $\mathfrak{g} \to \mathfrak{gl}(V)$ is injective.

Here is how these things relate back to Lie groups.

Proposition 2.38. Fix a regular Lie subgroup H of a regular Lie group G. Let their Lie algebras be $\mathfrak{h}$ and $\mathfrak{g}$, respectively.

(a) Then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.

(b) If H is normal in G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

(c) If G and H are connected, and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then H is normal in G.

Proof. Here we go.

(a) Certainly $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace, so we want to check that $[X, Y] \in \mathfrak{h}$ for $X, Y \in \mathfrak{h}$, where the target bracket is taken in $\mathfrak{g}$. Consider the embedding $\varphi\colon H \to G$ so that $\mathfrak{h} = \operatorname{im} d\varphi_0$. Thus, we use Proposition 2.17 to see that

$$d\varphi_0([X, Y]) = [d\varphi_0(X), d\varphi_0(Y)].$$

Thus, for any $X, Y \in \operatorname{im} d\varphi_0$, we see that $[X, Y] \in \operatorname{im} d\varphi_0$, as required.

(b) For any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, we want to check that $[X, Y] \in \mathfrak{h}$. By Proposition 2.23, we are asking to check that $\operatorname{ad}_X(Y) \in \mathfrak{h}$. Well, for any $g \in G$, we see that $gHg^{-1} \subseteq H$, so the adjoint $\operatorname{Ad}_g : G \to G$ restricts to $\operatorname{Ad}_g : H \to H$. In particular, by taking the differential, we see that the adjoint $(d\operatorname{Ad}_\bullet)_1 : G \to \operatorname{GL}(\mathfrak{g})$ restricts to $(d\operatorname{Ad}_\bullet)_1 : G \to \operatorname{GL}(\mathfrak{h})$. (Namely, $(d\operatorname{Ad}_g)_1(Y) \in \mathfrak{h}$ for any $Y \in \mathfrak{h}$.) Taking the differential of this, we see that we get our map $\operatorname{ad}_\bullet : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$, meaning that $\operatorname{ad}_X(Y) \in \mathfrak{h}$ for any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

(c) Recall from Proposition 2.23 that

$$\operatorname{Ad}_{\exp(X)}(Y) = \exp(\operatorname{ad}_X Y).$$

Thus, for any $X \in \mathfrak{g}$, we see that $\operatorname{Ad}_{\exp(X)}$ is an operator $\mathfrak{h} \to \mathfrak{h}$. Thus, for $g \in G$ sufficiently close to the identity, we see that $\operatorname{Ad}_g(Y) \in \mathfrak{h}$ for $Y \in \mathfrak{h}$. Taking the exponential, Proposition 2.12 tells us that $ghg^{-1} \in H$ for $g \in G$ and $h \in H$ both sufficiently close to the identity.

Concretely, we get an open neighborhood U of the identity of G such that $ghg^{-1} \in H$ for any $g \in U$ and $h \in H \cap U$. Now, the subset of G normalizing $U \cap H$ is a subgroup of G containing U, so we see that it must be all of G because G is connected. Then the subset of H normalized by G is again a subgroup of H containing $U \cap H$, so we see that it must be all of H because H is connected. Thus, H is normal in G. ∎

Here is some motivation for our definition of ideal.

Lemma 2.39. Fix a morphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras.

(a) The kernel $\ker \varphi \subseteq \mathfrak{g}$ is a Lie ideal.

(b) The image $\operatorname{im} \varphi \subseteq \mathfrak{h}$ is a Lie subalgebra.

Proof. Here we go.

(a) For any $X \in \ker \varphi$ and $Y \in \mathfrak{g}$, we need to check that $[X, Y] \in \ker \varphi$. Well,

$$\varphi([X, Y]) = [\varphi(X), Y] = [0, Y] = 0$$

by the bilinearity of $[-, -]$.

(b) For any $X, Y \in \operatorname{im} \varphi$, we must check that $[X, Y] \in \operatorname{im} \varphi$. Well, find $X_0, Y_0 \in \mathfrak{g}$ such that $X = \varphi(X_0)$ and $Y = \varphi(Y_0)$, and then we see that

$$[X, Y] = [\varphi(X_0), \varphi(Y_0)] = \varphi([X_0, Y_0])$$

is in the image of φ, as required. ∎

Here are some more ways to build Lie ideals.

Remark 2.40. Fix a collection $\{\mathfrak{g}_\alpha\}_{\alpha \in \kappa}$ of Lie ideals of $\mathfrak{g}$. Then we claim that the intersection $\bigcap_{\alpha \in \kappa} \mathfrak{g}_\alpha$ is still a Lie ideal of $\mathfrak{g}$. Indeed, for any $X \in \bigcap_{\alpha \in \kappa} \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}$, we see that $X \in \mathfrak{g}_\alpha$ and hence $[X, Y] \in \mathfrak{g}_\alpha$ for all $\alpha \in \kappa$; thus, $[X, Y] \in \bigcap_{\alpha \in \kappa} \mathfrak{g}_\alpha$.

Remark 2.41. For two Lie ideals I and J of a Lie algebra $\mathfrak{g}$, we claim that

$$[I, J] := \operatorname{span}\{[X, Y] : X \in I, Y \in J\}$$

is also a Lie ideal of $\mathfrak{g}$. Indeed, this is certainly a subspace (because it is a span). To check that $[\mathfrak{g}, [I, J]] \subseteq [I, J]$, we note that it is enough to check this for a spanning subset of I, so we pick up $Z \in \mathfrak{g}$ and $[X, Y] \in [I, J]$ and compute

$$[Z, [X, Y]] = -[X, [Y, Z]] - [[X, Z], Y] \in [I, J]$$

by the Jacobi identity, so we are done.

Lemma 2.42. Fix a Lie ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$. Then the quotient space $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra with bracket given by

$$[X+\mathfrak{h}, Y+\mathfrak{h}]_{\mathfrak{g}/\mathfrak{h}} := [X,Y]_{\mathfrak{g}} + \mathfrak{h}.$$

Proof. The main issue is checking that the bracket is well-defined. Well, if $X, Y \in \mathfrak{g}$ and $X', Y' \in \mathfrak{h}$, we must check that

$$[X+X', Y+Y'] + \mathfrak{h} \stackrel{?}{=} [X,Y] + \mathfrak{h},$$

where the bracket is taken in $\mathfrak{g}$. This is a matter of expanding with the bilinearity: note

$$\begin{aligned}[X+X', Y+Y'] &= [X+X', Y] + [X+X', Y'] \\ &= [X,Y] + [X',Y] + [X,Y'] + [X',Y'],\end{aligned}$$

and now we see that the last three terms live in $\mathfrak{h}$ because $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal.

Now, note that we have a canonical surjective linear map $\pi\colon \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ which satisfies

$$\pi([X,Y]) = [\pi(X), \pi(Y)].$$

Thus, the bilinearity, skew-symmetry, and Jacobi identity for $\mathfrak{g}/\mathfrak{h}$ are immediately inherited from the corresponding checks on $\mathfrak{g}$. Rigorously, perhaps one should note that (for example) the Jacobi identity corresponds to showing that some linear functional on $(\mathfrak{g}/\mathfrak{h})^3$ vanishes; however, this linear functional can be checked to vanish on the level of $\mathfrak{g}^3$. ■

Proposition 2.43. Fix a morphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ of Lie algebras. Then the induced quotient map

$$\overline{\varphi}\colon \mathfrak{g}/\ker\varphi \to \operatorname{im}\varphi$$

is an isomorphism.

Proof. Linear algebra implies that $\overline{\varphi}$ is already an isomorphism of vector spaces. Thus, it merely remains to check that $\overline{\varphi}$ is a morphism of Lie algebras. Well, for $X, Y \in \mathfrak{g} =$, we see

$$\begin{aligned}\overline{\varphi}([X+\ker\varphi, Y+\ker\varphi]) &= \overline{\varphi}([X,Y]+\ker\varphi) \\ &= \varphi([X,Y]) + \ker\varphi \\ &= [\varphi(X), \varphi(Y)] + \ker\varphi \\ &= [\overline{\varphi}(X), \overline{\varphi}(Y)],\end{aligned}$$

as required. ■

2.2.4 Lie Algebra of a Vector Field

One can in general provide a Lie algebra of a vector field. Fix a regular vector field $\xi\colon X \to TX$ on a regular manifold X. For any regular function f on an open subset $U \subseteq X$, we may define

$$(\xi f)(x) := \xi_x(f_x),$$

where we recall that $\xi_x \in T_xX$ is some derivation which outputs a number when fed a germ f_x. The point is that ξf is itself a regular function $X \to \mathbb{F}$! We are now able to define a bracket.

Proposition 2.44. Fix a regular manifold X. Given vector fields $\xi, \eta\colon X \to TX$, we define the Lie bracket

$$[\xi, \eta] := \xi\eta - \eta\xi.$$

Then $[-,-]$ is a Lie bracket.

Proof. At each $x \in X$, we have certainly defined a map taking regular functions f on X and outputting an element of $\mathbb{F}$ given by

$$[\xi, \eta]_x(f) := \xi_x(\eta f)_x - \eta_x(\xi f)_x.$$

This is certainly linear in f because ξ and η are. Further, the value of $[\xi, \eta]_x(f)$ only depends on the germ f_x because having $f_x = g_x$ for functions f and g implies $(f-g)_x = 0_x$, and then $\eta(f-g)$ and $\xi(f-g)$ both vanish in a neighborhood of x, so $[\xi, \eta]_x(f-g) = 0$.

It remains to check the product rule. Well, for regular functions f and g and some $y \in X$, we compute

$$(\eta f g)(y) = \eta_y(f_y g_y) = f(y)\eta_y(g_y) + g(y)\eta_y(g_y) = (f \cdot \eta g + g \cdot \eta f)(y),$$

and a similar computation works for ξ. Thus,

$$\begin{aligned}
\xi(\eta f g)(x) &= \xi(f\eta g + g\eta f)(x) \\
&= \xi(f\eta g)(x) + \xi(g\eta f)(x) \\
&= f(x)\xi(\eta g)(x) + (\xi f)(x)(\eta g)(x) + g(x)\xi(\eta f)(x) + (\xi g)(x)(\eta f)(x),
\end{aligned}$$

and a similar computation holds for $\eta(\xi f g)(x)$. Thus, we see that

$$\begin{aligned}
[\xi, \eta]_x(fg) = {} & f(x)\xi(\eta g)(x) + (\xi f)(x)(\eta g)(x) + g(x)\xi(\eta f)(x) + (\xi g)(x)(\eta f)(x) \\
& - (f(x)\eta(\xi g)(x) + (\eta f)(x)(\xi g)(x) + g(x)\eta(\xi f)(x) + (\eta g)(x)(\xi f)(x)) \\
= {} & f(x)[\xi, \eta]_x g + g(x)[\xi, \eta]_x f
\end{aligned}$$

after sufficient cancellation and rearranging. ■

Example 2.45. Fix regular functions f and g on some open subset of $U \subseteq \mathbb{R}^m$, and let x_i and x_j be two coordinates. Then we compute

$$\left[f\frac{\partial}{\partial x_i}, g\frac{\partial}{\partial x_j}\right] = f\frac{\partial g}{\partial x_i}\frac{\partial}{\partial x_j} - g\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}.$$

Proof. Fixing some $p \in U$ and regular germ h, we see

$$\begin{aligned}
\left[f\frac{\partial}{\partial x_i}, g\frac{\partial}{\partial x_j}\right]_p(h) &= f(p)\frac{\partial}{\partial x_i} g\frac{\partial h}{\partial x_j}\bigg|_p - g(p)\frac{\partial}{\partial x_j} f\frac{\partial h}{\partial x_i}\bigg|_p \\
&= f(p)\frac{\partial g}{\partial x_i}\bigg|_p \frac{\partial h}{\partial x_j}\bigg|_p + f(p)g(p)\frac{\partial h}{\partial x_i \partial x_j}\bigg|_p - g(p)\frac{\partial f}{\partial x_j}\bigg|_p \frac{\partial h}{\partial x_i}\bigg|_p - f(p)g(p)\frac{\partial h}{\partial x_i \partial x_j}\bigg|_p \\
&= f(p)\frac{\partial g}{\partial x_i}\bigg|_p \frac{\partial h}{\partial x_j}\bigg|_p - g(p)\frac{\partial f}{\partial x_j}\bigg|_p \frac{\partial h}{\partial x_i}\bigg|_p,
\end{aligned}$$

as required. ■

Remark 2.46. In local coordinates in some chart (U, φ) with $\varphi = (x_1, \ldots, x_m)$ of our regular manifold M, one can write vector fields as

$$\xi = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \eta = \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i},$$

where a_i and b_i are regular functions. Then one can expand the bilinearity to see that

$$[\xi, \eta] = \sum_{i,j=1}^{m} \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Indeed, after applying bilinearity, the main point is to compute $\left[f \frac{\partial}{\partial x}, g \frac{\partial}{\partial y} \right]$ for regular functions f and g and coordinates x and y, which we did in the previous example.

Remark 2.47. For example, if ξ and η are tangent to a regular submanifold $N \subseteq M$ of dimension k, then $[\xi, \eta]$ continues to be tangent. One can check this using a local slice chart, where the condition that ξ is tangent to Y is equivalent to having $a_i = 0$ for $i > k$. Combining this with the computation of the previous remark completes the argument.

2.3 September 23

Today we continue talking about vector fields.

2.3.1 Vector Fields on Lie Groups

Let's return to Lie groups.

Lemma 2.48. Fix a regular Lie group G. A vector field ξ on G is left-invariant if and only if

$$\xi(f \circ L_g) = \xi f \circ L_g$$

for any germ f defined in a neighborhood of g.

Proof. We show the two implications separately.

- If ξ is left-invariant, then $\xi_{gh} = (dL_g)_h(\xi_h)$ for any $g, h \in G$. Thus, for any $h \in G$, we see that

$$\begin{aligned} (\xi f \circ L_g)(h) &= \xi_{gh} f \\ &= ((dL_g)_h \xi_h) f \\ &= \xi_h (f \circ L_g), \end{aligned}$$

  as required.

- Suppose $\xi(f \circ L_g) = \xi f \circ L_g$ for any f. Then plugging in the identity tells us that

$$\xi_g f = (\xi f \circ L_g)(e) = \xi_e (f \circ L_g) = ((dL_g)_e(\xi_e))(f).$$

  Thus, $\xi_g = (dL_g)_e \xi_e$, as required. ∎

Lemma 2.49. Fix a left-invariant vector field ξ on a regular Lie group G. Then for a germ f at a point $g \in G$, one has

$$\xi_g f = \frac{d}{dt} f(g\exp(t\xi_e))\bigg|_{t=0}.$$

Proof. This is more or less the chain rule. For our $g \in G$, Lemma 2.48 tells us that

$$\xi_g f = \xi_e(f \circ L_g).$$

Now, the path $\gamma \colon \mathbb{F} \to G$ given by $\gamma(t) := \exp(t\xi_e)$ has $\gamma'(0) = \xi_e$, so

$$\xi_e(f \circ L_g) = d(f \circ L_g \circ \gamma)'(0) = \frac{d}{dt} f(g\exp(t\xi_e))\bigg|_{t=0},$$

as required. ■

Proposition 2.50. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then the collection of left-invariant vector fields $\operatorname{Vect}^L(G)$ is a Lie subalgebra of $\operatorname{Vect}(G)$ which is isomorphic to $\mathfrak{g}$.

Proof. By Remark 1.139, one certainly has an isomorphism $\operatorname{Vect}^L(G) \to \mathfrak{g}$ given by $\xi \mapsto \xi_e$, with inverse given by $X \mapsto \xi_X$, where ξ_X is the vector field $\xi_X(g) := dL_g(X)$. Now, by Lemma 2.48, ξ is left-invariant if and only if

$$\xi(f \circ L_g) = \xi f \circ L_g$$

for any germ f defined in a neighborhood of g. Thus, we see that $\operatorname{Vect}^L(G)$ is preserved by the commutator of $\operatorname{Vect}(G)$.

It remains to check that our isomorphism with $\mathfrak{g}$ is a morphism of Lie algebras. Fix $X, Y \in \mathfrak{g}$, and we would like to show that $[\xi_X, \xi_Y] = \xi_{[X,Y]}$. It is enough to check this equality after mapping back down to $\mathfrak{g}$, so we want to check that $[\xi_X, \xi_Y]_e = [X, Y]$. This is a direct computation: by Lemma 2.49, any germ f at e has

$$\begin{aligned}
[\xi_X, \xi_Y]_e f &= \frac{d}{dt}\left(\xi_Y f(\exp(tX)) - \xi_X f(\exp(tY))\right)\bigg|_{t=0} \\
&= \frac{\partial^2}{\partial s \partial t}\frac{d}{ds}\left(f(\exp(tX)\exp(sY)) - f(\exp(tY)\exp(sX))\right)\bigg|_{(s,t)=(0,0)} \\
&= \frac{\partial^2}{\partial s \partial t}\left(f\exp\left(tX + sY + \frac{1}{2}st[X,Y] + \cdots\right) - f\exp\left(tX + sY - \frac{1}{2}st[X,Y] + \cdots\right)\right)\bigg|_{(s,t)=(0,0)}.
\end{aligned}$$

Now, one can imagine taking a Taylor series expansion of $f \circ \exp \colon \mathfrak{g} \to \mathbb{R}$ in terms of Z, in which we see that the above derivative will only depend on the st term of the relevant expansion. More precisely, write $(f \circ \exp)(Z) = f(e) + \lambda(Z) + Q(Z) + C(Z)$, where λ is linear, Q is quadratic, and C has vanishing first- and second-order derivatives. Then, after cancellation within λ, we see that

$$\begin{aligned}
[\xi_X, \xi_Y]_e f = \frac{\partial^2}{\partial s \partial t} st\lambda(st[X,Y])\bigg|_{(s,t)=(0,0)} \\
&+ \frac{\partial^2}{\partial s \partial t} Q\left(tX + sY + \frac{1}{2}st[X,Y] + \cdots\right)\bigg|_{(s,t)=(0,0)} \\
&+ \frac{\partial^2}{\partial s \partial t} Q\left(tX + sY - \frac{1}{2}st[X,Y] + \cdots\right) + \cdots\bigg|_{(s,t)=(0,0)},
\end{aligned}$$

where $+\cdots$ denotes higher-order terms which will not affect the current derivative (for example, containing C). Now, the linear terms inside Q will produce cancelling terms after expansion, so the only term we are left to care about is

$$[\xi_X, \xi_Y]_e f = \lambda([X,Y]) = \frac{d}{dt}(f \circ \exp)(t[X,Y])\bigg|_{t=0} = (\xi_{[X,Y]})_e f,$$

as required. ■

2.3.2 Group Actions via Lie Algebras

In general, if G acts on a regular manifold M via the action $a\colon G \times X \to X$, one can define an action of $\mathfrak{g}$ on $\operatorname{Vect}(X)$ by analogy with Lemma 2.49.

Definition 2.51. Fix a left action $a\colon G \times M \to M$ of a regular Lie group G on a regular manifold M. Then we define $a_*\colon \mathfrak{g} \to \operatorname{Vect}(M)$ by

$$(a_*X)_p f := \frac{d}{dt} f(a(\exp(-tX), p))\bigg|_{t=0}$$

for any $p \in M$ and germ f at p.

Remark 2.52. Let's explain the sign in the above definition: the action of G on M induces a natural action of G on the regular functions $\mathcal{O}(M)$ by $(g \cdot f)(p) := f\left(g^{-1} \cdot p\right)$. It is this action of G on $\mathcal{O}(M)$ which motivates the above definition.

Let's run our checks on this definition.

Lemma 2.53. Let $a\colon G \times M \to M$ be an action of a regular Lie group G on a regular manifold M. Then that a_* is a homomorphism of Lie algebras.

Proof. We run our many checks in sequence. Throughout, $p, q \in M$ and $X, Y \in \mathfrak{g}$ and $s, t \in \mathbb{F}$ and f and g are regular functions on an open neighborhood of p.

1. For any regular function f defined in an open neighborhood of a point $p \in M$, we claim that

$$da_{(e,p)}(-X, 0)(f_p) \stackrel{?}{=} (a_*X)_p(f).$$

This is a matter of computation. Define $\gamma\colon \mathbb{F} \to G \times M$ by $\gamma(t) := (\exp(-tX), p)$. Then we see that $\gamma'(0) = (-X, 0)$ by definition of $\exp$. Thus, using the chain rule, we see that

$$\begin{aligned} da_{(e,p)}(-X,0)(f_p) &= da_{(e,p)}(-X,0)(f_p) \\ &= d(f \circ a)_{(e,p)}(-X, 0) \\ &= d(f \circ a)_{(e,p)}(\gamma'(0)) \\ &= (f \circ a \circ \gamma)'(0) \\ &= \frac{d}{dt}(f \circ a \circ \gamma)(t)\bigg|_{t=0} \\ &= \frac{d}{dt} f(a(\exp(-tX), p))\bigg|_{t=0} \\ &= (a_*X)_p(f), \end{aligned}$$

as required.

2. We check that $(a_*X)_p$ is a derivation T_pM. This follows essentially immediately from the previous step. We enumerate the checks for clarity.

   - Note that $(a_*X)_p(f)$ only depends on the germ f_p because it equals $da_{(e,p)}(-X,0)(f)$, and
   $$da_{(e,p)}(-X,0) \in T_pM$$
   only depends on the germ f_p. Thus, we may redefine $(a_*X)_p$ as taking germs as input.[1]
   - Now, we see that $(a_*X)_p$ is a function taking input as germs at p and outputting elements of $\mathbb{F}$; in particular, it equals the differential $da_{(e,p)}(-X,0)$, so $(a_*X)_p$ immediately becomes a linear map and satisfies the product rule, making it a derivation.

3. We check that a_*X is a vector field. Thus, far we know that we have $(a_*X)_p \in T_pM$ for each $p \in M$, so we have a section $a_*X\colon M \to TM$. It remains to check that a_*X is smooth. Well, the first step tells us that $(a_*X)_p = da_{(e,p)}(-X,0)$, so we see that a_*X equals the composite
$$\begin{array}{ccccccccc} M & \to & TM & \to & TG \times TM & \simeq & T(G\times M) & \overset{da}{\to} & TM \\ p & \mapsto & (p,0) & \mapsto & ((e,-X),(p,0)) & \mapsto & ((e,p),(-X,0)) & \mapsto & da_{(e,p)}(-X,0) \end{array}$$
of smooth maps, so $a_*X\colon M \to TM$ is smooth.

4. Thus far, we know that we have a well-defined map $a_*\colon \mathfrak{g} \to \operatorname{Vect}(X)$. It remains to check that this is a homomorphism of Lie algebras. We begin by checking that it is $\mathbb{F}$-linear. Well, for $X, Y \in \mathfrak{g}$ and $c, d \in \mathbb{F}$, we are asking to check that $a_*(cX + dY) = ca_*X + da_*Y$. For this, we check the equality of derivations at some point $p \in M$, for which the first step verifies
$$\begin{aligned} a_*(cX+dY)_p &= da_{(e,p)}(-cX-dY,0) \\ &= c\cdot da_{(e,p)}(-X,0) + d\cdot da_{(e,p)}(-Y,0) \\ &= (ca_*X + da_*Y)_p, \end{aligned}$$
as required.

5. We check that a_* is a homomorphism of Lie algebras. We already know that $a_*\colon \mathfrak{g} \to \operatorname{Vect}(M)$ is linear (and we know that everything in sight is a Lie algebra from class), so it only remains to check that a_* preserves the Lie bracket. Explicitly, we would like to show that $a_*[X,Y] = [a_*X, a_*Y]$ for given $X, Y \in \mathfrak{g}$. For this, we choose a germ f_p represented by a regular function f defined in an open neighborhood of p.

   To run our computations, we employ a trick motivated by one in the Etingof book. Namely, define $F\colon \mathfrak{g} \to \mathbb{F}$ by $F(Z) := f(a(\exp(Z),p))$. Now, we compute
$$\begin{aligned} (a_*X)_p(a_*Yf) &= \frac{d}{dt}(a_*Yf)(a(\exp(-tX),p))\bigg|_{t=0} \\ &= \frac{d}{dt}\frac{d}{ds} f\big(a(\exp(-sY), a(\exp(-tX),p))\big)\bigg|_{s=0}\bigg|_{t=0} \\ &= \frac{d}{dt}\frac{d}{ds} f\big(a(\exp(-sY)\exp(-tX),p)\big)\bigg|_{s=0}\bigg|_{t=0} \\ &= \frac{d}{dt}\frac{d}{ds} f\left(a\left(\exp\left(-sY - tX + \frac{1}{2}st[Y,X] + \cdots\right),p\right)\right)\bigg|_{s=0}\bigg|_{t=0} \\ &= \frac{\partial^2}{\partial s \partial t} F\left(-sY - tX + \frac{1}{2}st[Y,X] + \cdots\right)\bigg|_{(s,t)=(0,0)} \\ &= \frac{\partial^2}{\partial s \partial t} F\left(-tX - sY - \frac{1}{2}st[X,Y] + \cdots\right)\bigg|_{(s,t)=(0,0)}. \end{aligned}$$

[1] One can also check this directly: regular local functions f and g with $f_p = g_p$ has $f_p - g_p$ vanish in a neighborhood of p, permitting us to compute $(a_*X)_p(f) = (a_*X)_p(g)$.

By reversing the roles of X and Y in the above argument, we see that

$$(a_*Y)_p(a_*Xf) = \frac{\partial^2}{\partial s \partial t} F\left(-tX - sY + \frac{1}{2}st[X,Y] + \cdots\right)\Big|_{(s,t)=(0,0)}.$$

Thus, we see that we want to compute some particular derivatives of F. Now, f is regular, so F is a regular function $\mathfrak{g} \to \mathbb{F}$, so it will be approximately equal its Taylor expansion in a neighborhood of 0 as

$$F(Z) = f(0) + \lambda(Z) + Q(Z) + \cdots,$$

where λ is a linear functional, Q is a quadratic form, and $+\cdots$ refers to higher-order terms (with vanishing first- and second-derivatives). Plugging everything in and expanding, we see that

$$\begin{aligned}(a_*X)_p(a_*Yf) - (a_*Y)_p(a_*Xf) = \frac{\partial^2}{\partial s \partial t} &- st\lambda([X,Y])\Big|_{(s,t)=(0,0)} \\ &+ \frac{\partial^2}{\partial s \partial t} Q\left(-tX - sY + \frac{1}{2}st[X,Y] + \cdots\right)\Big|_{(s,t)=(0,0)} \\ &- \frac{\partial^2}{\partial s \partial t} Q\left(-tX - sY - \frac{1}{2}st[X,Y] + \cdots\right) + \cdots\Big|_{(s,t)=(0,0)},\end{aligned}$$

where $+\cdots$ continues to denote higher-order terms, but now we see that we are only going to care about $-\lambda([X,Y])$ when computing $\frac{\partial}{\partial s \partial t}\big|_{(s,t)=(0,0)}$. (Notably, the last two terms cancel out as a derivative of $Q(-tX - sY + \cdots) - Q(-tX - sY + \cdots)$.) But now we see that

$$(a_*X)_p(a_*Yf) - (a_*Y)_p(a_*Xf) = \lambda(-[X,Y]) = \frac{d}{dt}F(-t[X,Y])\Big|_{t=0} = a_*[X,Y]_pf,$$

as required. ■

We can now prove the Orbit–stabilizer theorem (Theorem 2.54) in the following more precise form.

Theorem 2.54 (Orbit–stabilizer)**.** Fix a left action $a\colon G \times M \to M$ of a regular Lie group G on a regular manifold M. Fix some $p \in M$.

(a) For all $p \in M$, the stabilizer G_p is a closed Lie subgroup with Lie algebra

$$\operatorname{Lie} G_p = \{X \in \mathfrak{g} : (a_*X)_p = 0\}.$$

(b) The induced map $G/G_p \to M$ given by $g \mapsto g \cdot p$ is an injective immersion. In particular, the orbit Go is an immersed submanifold.

(c) If the induced map $G/G_p \to M$ is an embedding, then G/G_p is diffeomorphic to Gp.

Proof. We begin with the proof of (a), which we do in steps.

1. Set

$$\mathfrak{g}_p := \{X \in \mathfrak{g} : (a_*X)_p = 0\}$$

for brevity. We claim that $\mathfrak{g}_p \subseteq \mathfrak{g}$ is a Lie subalgebra. Certainly $X \mapsto (a_*X)_p$ is a linear map $\mathfrak{g} \to \operatorname{Vect}(M) \to T_pM$, so $\mathfrak{g}_p$ is a linear subspace.

It remains to check that $\mathfrak{g}_p$ is preserved by the bracket. Fix $X, Y \in \mathfrak{g}_p$, and we want to check $[X,Y] \in \mathfrak{g}_p$. Well, because a_* is a homomorphism of Lie algebras, we see

$$a_*[X,Y]_pf = \underbrace{(a_*X)_p}_{0}(a_*Yf) - \underbrace{(a_*Y)_p}_{0}(a_*Xf) = 0$$

for any germ f at p. Thus, $a_*[X,Y] = 0$, so $[X,Y] \in \mathfrak{g}_p$.

2. For $X \in \mathfrak{g}_p$, we check that $\exp(X) \in G_p$. Indeed, we claim the two curves $\gamma_1(t) := \exp(-tX) \cdot p$ and $\gamma_2(t) := p$ are both integral curves for $a_* X$ with the same initial condition at 0. This completes the check because it implies that $\exp(X) \cdot p = \gamma_1(-1) = \gamma_2(-1) = p$ by uniqueness of integral curves.

   To prove the claim, we note that γ_2 is constant, so there is nothing to check there. For γ_1, we must check that
   $$\gamma_2'(t) \stackrel{?}{=} (a_* X)_{\gamma_2(t)}$$
   in $T_{\gamma_2(t)} M$. To check this, we pass through an arbitrary germ f to see that
   $$\gamma_2'(t) f = (f \circ \gamma_2)'(t) = \frac{d}{ds} f(\exp(-sX - tX) \cdot p)\bigg|_{s=0},$$
   and
   $$(a_* X)_{\gamma_2(t)} f = \frac{d}{ds} f(\exp(-tX - sX) \cdot p)\bigg|_{s=0},$$
   as required.

3. We attempt to control $\mathfrak{g}/\mathfrak{g}_p$. Choose a complement $\mathfrak{u}$ of $\mathfrak{g}_p \subseteq \mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{u}$. (We do not require that $\mathfrak{u}$ is a Lie subalgebra, despite the font.) Then the map $f \colon \mathfrak{u} \to T_p M$ given by $Z \mapsto (a_* Z)_p$ has kernel $\mathfrak{g}_p \cap \mathfrak{u} = 0$ and hence is injective. Thus, the Implicit function theorem tells us that the map $F \colon \mathfrak{u} \to M$ given by $v \mapsto \exp(-V) \cdot p$ must be an injective immersion for small v because $df_p(V) = dF_p(V)$.

   Instead of using the Implicit function theorem, we can argue using local diffeomorphisms as follows: fix a basis $\{e_1, \ldots, e_k\}$ of $\mathfrak{u}$, and extend the linearly independent set $\{dF_p(e_1), \ldots, df_p(e_k)\} \subseteq T_p M$ to a basis $\{dF_p(e_1), \ldots, dF_p(e_k)\} \sqcup \{e'_{k+1}, \ldots, e'_m\}$. Then define a local map $\widetilde{F} \colon \mathfrak{u} \times \mathbb{F}^{m-k} \to M$ by
   $$\widetilde{F}(a_1 e_1 + \cdots + a_m e_m) = F(a_1 e_1 + \cdots + a_k e_k) + a_{k+1} e'_{k+1} + \cdots + a_m e'_m,$$
   where the addition on the right-hand side is defined in a local chart of M around p. (Technically, $\widetilde{F}$ is only defined in a neighborhood of $0 \in \mathfrak{u}$.) Then $\widetilde{F}$ is a local diffeomorphism at 0 by construction, so F is an injective immersion in this same neighborhood of 0.

4. We construct a slice chart for $G_p \subseteq G$ at the identity, which will complete the proof (a) by Lemma 1.90. Note that the map $\exp^{\oplus} \colon \mathfrak{g}_p \oplus \mathfrak{u} \to G$ given by $(V, X) \mapsto \exp(V)\exp(X)$ is a local diffeomorphism at 0 (because the differential is simply the identity by checking what happens on each piece $\mathfrak{g}_p$ and $\mathfrak{u}$ separately). Thus, for $g \in G$ sufficiently close to e, we can write g uniquely as in the image of e and thus as $g = \exp(V)\exp(X)$ where $V \in \mathfrak{u}$ and $X \in \mathfrak{g}_p$. Now, we see that $g \in G_p$ if and only if $\exp(V) \in G_p$, which for small enough V is equivalent to $V \in \mathfrak{g}_p$ by the previous step.

   In total, we have constructed a very small open neighborhood $U \subseteq \mathfrak{g}_p \oplus \mathfrak{u}$ of the identity such that $e|_U$ is a diffeomorphism onto its image $\exp^{\oplus}(U) \subseteq G$ and
   $$G_p \cap \exp^{\oplus}(U) = \{(V, X) \in \mathfrak{g}_p \oplus \mathfrak{u} : V = 0\},$$
   which is a slice chart.

We now proceed with (b). Let $\overline{\varphi}$ denote the induced map $G/G_p \to M$ given by $\overline{\varphi}(g) := g \cdot p$, which we want to see is an injective immersion. Injectivity follows by definition of G_p: if $\varphi(g_1) = \varphi(g_2)$, then $g_1 \cdot p = g_2 \cdot p$, so $g_1^{-1} g_2 \in G_p$, so $g_1 G_p = g_2 G_p$. Being an immersion more or less follows from the proof. By translation, it suffices to show that $d\overline{\varphi}_e$ is injective.[2] Well, the Lie algebra of G/G_p is the quotient $\mathfrak{g}/\mathfrak{g}_p$ by Theorem 1.94, which is isomorphic to $\mathfrak{u}$ by construction of $\mathfrak{u}$. But we know that the action map is injective on $\mathfrak{u}$ by the third step above, so we are done.

Lastly, we note that (c) follows immediately from (b) because embeddings are diffeomorphic onto their images by the uniqueness of the smooth structure of embedded submanifolds. ■

[2] Once $d\overline{\varphi}_e$ is injective, we note that $\overline{\varphi} \circ L_g = L_g \circ \overline{\varphi}$ (where the first L_g is a map $G \to G$ and the second is a map $M \to M$, but both are diffeomorphisms), so $d\overline{\varphi}_g \circ d(L_g)_e = d(L_g)_p \circ d\overline{\varphi}_e$ verifies that $d\overline{\varphi}_g$ is injective.

Remark 2.55. We also remark that Theorem 1.96 follows quickly from the above result. Indeed, let G act on H via the homomorphism $\varphi\colon G \to H$: $g \cdot h := \varphi(g)h$. Then the stabilizer of any $h \in H$ is given by $\ker\varphi$, proving $\ker\varphi$ is in fact a closed Lie subgroup. Now, passing to $\overline{\varphi}$ as in the above proof shows that $G/\ker\varphi \to \operatorname{im}\varphi$ is an injective immersion.

2.4 September 25

We began class by finishing the proof of Theorem 2.54 and giving an example.

2.4.1 The Orbit–Stabilizer Theorem for Fun and Profit

Let's see an example of Theorem 2.54.

Example 2.56. Fix a finite-dimensional representation V of a regular Lie group G given by $\rho\colon G \to \mathrm{GL}(V)$. For $v \in V$, its stabilizer G_v has Lie algebra given by

$$\mathfrak{g}_v = \{X \in \mathfrak{g} : (\rho_* X)_v = 0\}.$$

Example 2.57. Fix a finite-dimensional algebra A over a field $\mathbb{F}$. Then we claim that $\mathrm{Aut}_k(A)$ is a closed Lie subgroup of $\mathrm{GL}(A)$, and we claim that

$$\mathrm{Lie}(\mathrm{Aut}\,A) = \mathrm{Der}(A) \subseteq \mathrm{End}(A).$$

Proof. Note that $\varphi \in \mathrm{GL}(A)$ is an automorphism if and only if φ also preserves the multiplication map $\mu\colon A \otimes A \to A$ of A. Now, $\mathrm{GL}(A)$ has a natural action $\rho\colon\ \mathrm{GL}(A) \to \mathrm{GL}(\mathrm{Hom}(A \otimes A, A))$ by

$$(\rho(g)\varphi)(x \otimes y) := g\varphi\left(g^{-1}x \otimes g^{-1}y\right).$$

Precisely speaking, this is the composite of the actions of G on the various pieces by Remark 1.103, so this is in fact a representation of G. Now, $g \in \mathrm{GL}(A)$ preserves the multiplication map μ if and only if

$$g(\mu(a \otimes b)) = \mu(g(a) \otimes g(b))$$

for all $a, b \in A$, which is equivalent to

$$(\rho(g)\mu)(a \otimes b) = g\mu\left(g^{-1}a \otimes g^{-1}b\right) = \mu(a \otimes b)$$

for all $a, b \in A$. Thus, $\mathrm{Aut}(A) \subseteq \mathrm{GL}(A)$ is the stabilizer of $\mu \in \mathrm{Hom}(A \otimes A, A)$ and hence a closed Lie subgroup by Theorem 2.54.

It remains to compute the Lie algebra, which Theorem 2.54 tells us is

$$\mathfrak{gl}(A)_\mu = \{X \in \mathfrak{gl}(A) : (\rho_* X)_\mu = 0\}.$$

Thus, we want to compute $(\rho_* X)_\mu$. Note that $\mathrm{Hom}(A \otimes A, A)$ is some finite-dimensional $\mathbb{F}$-vector space, so for any germ f defined around μ, we may use the chain rule to compute

$$\begin{aligned}(\rho_* X)_\mu f &= \frac{d}{dt} f(\rho(\exp(-tX), \mu))\bigg|_{t=0} \\ &= df_\mu\left(\frac{d}{dt}\rho(\exp(-tX), \mu)\bigg|_{t=0}\right).\end{aligned}$$

Thus, we see that $X \in \mathfrak{gl}(A)_\mu$ if and only if $\frac{d}{dt}\rho(\exp(-tX),\mu)\big|_{t=0} = 0$. Now, linear operators pass through derivatives, and evaluation is a linear operator on $\operatorname{Hom}(A\otimes A, A)$, so it suffices to check when

$$\frac{d}{dt}\rho(\exp(-tX),\mu)(a\otimes b)\Big|_{t=0}$$

vanishes, for arbitrary $a, b \in A$. Thus, we compute

$$\begin{aligned}\rho(\exp(-tX),\mu)(a\otimes b) &= \exp(-tX)\mu(\exp(tX)a, \exp(tX)b)\\ &= \sum_{n=0}^\infty \frac{(-1)^n t^n}{n!}X^n\left(\sum_{k=0}^\infty \frac{t^k}{k!}X^k a \cdot \sum_{\ell=0}\frac{t^\ell}{\ell!}X^\ell b\right)\\ &= \sum_{n,k,\ell=0}^\infty \frac{(-1)^n t^{n+k+\ell}}{n!k!\ell!}X^n\left(X^k a\cdot X^\ell b\right),\end{aligned}$$

where we have rearranged the sums with impunity because everything in sight converges absolutely. Furthermore, we can differentiate term-by-term to see that

$$\begin{aligned}\frac{d}{dt}\rho(\exp(-tX),\mu)(a\otimes b)\bigg|_{t=0} &= \frac{(-1)^1 t^{1+0+0}}{1!0!0!}X^1\left(X^0 a\cdot X^0 b\right)\\ &\quad + \frac{(-1)^0 t^{0+1+0}}{0!1!0!}X^0\left(X^1 a\cdot X^0 b\right)\\ &\quad + \frac{(-1)^0 t^{0+0+1}}{0!0!1!}X^0\left(X^0 a\cdot X^1 b\right)\\ &= -X(a\cdot b) + Xa\cdot b + a\cdot Xb.\end{aligned}$$

Thus,

$$\operatorname{Lie}(\operatorname{Aut} A) = \{X \in \mathfrak{gl}(A) : X(a\cdot b) = Xa\cdot b + a\cdot Xb\},$$

which of course is the set of derivations. ■

Remark 2.58. A close examination of the above proof finds that we only need μ to be an element of $\operatorname{Hom}(A\otimes A, A)$ for the argument to go through. Notably, we may replace (A,μ) above with a Lie algebra $(\mathfrak{g}, [-,-])$ to find that $\operatorname{Aut}_{\mathrm{LieAlg}}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$ is a Lie subgroup with Lie algebra given by the derivations

$$\{\varphi \in \mathfrak{gl}(\mathfrak{g}) : \varphi([X,Y]) = [\varphi(X),Y] + [X,\varphi(Y)] \text{ for all } X,Y \in \mathfrak{g}\}.$$

Remark 2.59. The adjoint map $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ actually lands in $\operatorname{Der}(\mathfrak{g})$: checking this is tantamount to checking that $X, Y, Z \in \mathfrak{g}$ has

$$\operatorname{ad}_X[Y,Z] \stackrel{?}{=} [\operatorname{ad}_X Y, Z] + [Y, \operatorname{ad}_X Z],$$

which one can check is equivalent to the Jacobi identity of Proposition 2.29.

Remark 2.60. Similarly, the adjoint action $\operatorname{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ actually lands in $\operatorname{Aut}_{\mathrm{LieAlg}}(\mathfrak{g})$. Indeed, for $g \in G$ and $X, Y \in \mathfrak{g}$, this amounts to checking that

$$\operatorname{Ad}_g[X,Y] \stackrel{?}{=} [\operatorname{Ad}_g X, \operatorname{Ad}_g Y],$$

which is Corollary 2.19.

Here is another application.

Definition 2.61 (center)**.** Fix a group G. Then the *center* of G is the subset

$$Z(G) := \{z \in G : zg = gz \text{ for all } g \in G\}.$$

Similarly, fix a Lie algebra $\mathfrak{g}$, then the *center* of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Remark 2.62. We will not bother to check that $Z(G)$ is a subgroup because this is a standard result of group theory. However, in order to do something, let's check that $\mathfrak{z}(\mathfrak{g})$ is a Lie ideal of $\mathfrak{g}$. Note that $\mathfrak{z}(\mathfrak{g})$ is the kernel of the collection of linear maps $X \mapsto [X, Y]$ as $Y \in \mathfrak{g}$ varies, so $\mathfrak{z}(\mathfrak{g})$ is an intersection of Lie ideals (by Lemma 2.39) and hence a Lie ideal by Remark 2.40.

Proposition 2.63. Fix a connected regular Lie group G. Then $Z(G)$ is a closed Lie subgroup with Lie algebra

$$\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Proof. We would like to see that $Z(G)$ is the kernel of the adjoint map $\operatorname{Ad}\colon G \to \operatorname{Aut} G$, but it is difficult to make sense of this argument because $\operatorname{Aut} G$ is not a manifold.

Instead, we note that $g \in G$ if and only if g commutes with an open neighborhood U of the identity: indeed, commuting with U implies commuting with the subgroup generated by U, but G is connected, so commuting with U is equivalent to commuting with G. Now, we can take U to be some neighborhood of the identity in the image of the local diffeomorphism $\exp\colon \mathfrak{g} \to G$, so $g \in Z(G)$ if and only if

$$g \exp(X) g^{-1} = \exp(X)$$

for all $X \in \mathfrak{g}$ in an open neighborhood of 0. Now, $\operatorname{Ad}_g \exp(X) = \exp(\operatorname{Ad}_g X)$ by Proposition 2.12, so the above equality is equivalent to having $\operatorname{Ad}_g X = X$ for X in a neighborhood of 0.

Thus, Remark 2.55 tells us that $Z(G)$ is the kernel of the representation $\operatorname{Ad}_\bullet\colon G \to \operatorname{GL}(\mathfrak{g})$. We conclude that its Lie algebra is the kernel of the differential of $\operatorname{Ad}_\bullet$, which of course is $\operatorname{ad}_\bullet\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Thus,

$$\operatorname{Lie} Z(G) = \{X \in \mathfrak{g} : \operatorname{ad}_X = 0\},$$

but Proposition 2.23 explains that $\operatorname{ad}_X = [X, -]$, so we see that this is simply $\mathfrak{z}(\mathfrak{g})$, as required. ■

Definition 2.64 (adjoint)**.** Fix a connected regular Lie group G. Then the *adjoint group* of G is $G^{\mathrm{ad}} := G/Z(G)$.

Example 2.65. For $G = \operatorname{GL}_n(\mathbb{F})$, one can check that $Z(G)$ is the subgroup $\{cI : c \in \mathbb{F}\}$. The adjoint group is then $\operatorname{PGL}_n(\mathbb{F})$.

2.4.2 The Baker–Campbell–Hausdorff Formula

For completeness, we mention the Baker–Campbell–Hausdorff formula. We will not need this result, so we will not prove it, and the discussion in this subsection will be quite terse. Fix a Lie group G with Lie algebra $\mathfrak{g}$. We would like to understand the group law on G purely in terms of $\mathfrak{g}$. As in our discussion of the commutator, we note that

$$\mu(X, Y) = \log(\exp(X)\exp(Y)),$$

defined in an open neighborhood of $\mathfrak{g}$ can be expanded out as

$$\mu(X,Y) = X + Y + \frac{1}{2}[X,Y] + \mu_3(X,Y) + \mu_4(X,Y) + \cdots = \sum_{n=1}^{\infty} \mu_n(X,Y),$$

where $\mu_n(X,Y)$ consists of the order-n terms in this Taylor expansion. Here is the main result. For example, as above, $\mu_1(X,Y) = X+Y$ and $\mu_2(X,Y) = \frac{1}{2}[X,Y]$.

Theorem 2.66 (Baker–Campbell–Hausdorff)**.** The polynomials μ_n above are independent of G.

One proves this basically by solving differential equations for the μ_n inductively in n.

Example 2.67. One could compute that

$$\mu_3(X,Y) = \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]).$$

2.4.3 The Fundamental Theorems of Lie Theory

To wrap up our transition to Lie algebras, we state the fundamental theorems of Lie theory, which we will mostly not prove.

Theorem 2.68. For a connected regular Lie group G with Lie algebra $\mathfrak{g}$, there is a bijection between Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$. This bijection sends $H \subseteq G$ to $\mathfrak{h} := \operatorname{Lie} H$.

Theorem 2.69. Fix a simply connected regular Lie group G with Lie algebra $\mathfrak{g}$. Then for any regular Lie group H with Lie algebra $\mathfrak{h}$, the map

$$\operatorname{Hom}_{\mathrm{LieGrp}}(G,H) \to \operatorname{Hom}_{\mathrm{LieAlg}}(\mathfrak{g},\mathfrak{h}),$$

given by taking the differential at the identity, is a bijection.

Theorem 2.70. Any finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of some simply connected regular Lie group.

Here is the consequence.

Corollary 2.71. The (full subcategory) of simply connected regular Lie groups is equivalent to the category of finite-dimensional Lie algebras, given by the Lie algebra functor.

Proof. Theorem 2.69 shows that this functor is fully faithful, and Theorem 2.70 shows that this functor is essentially surjective. This completes the proof. ■

We will begin with the proof of Theorem 2.68 next class. This requires the theory of distributions.

2.5 September 27

Today we continue our discussion of the fundamental theorems of Lie theory.

2.5.1 Distributions and Foliations

Here is the main definition.

Definition 2.72 (distribution)**.** Fix a regular manifold M. Then a *k-dimensional distribution* $\mathcal{D}$ on X is a k-dimensional (local) subbundle $\mathcal{D} \subseteq TX$.

Remark 2.73. Locally at a point $p \in M$, we can think about $\mathcal{D}_p$ as being spanned by k linearly independent differentials which spread out over a neighborhood.

Definition 2.74 (integrable)**.** A distribution $\mathcal{D}$ of dimension k on a regular manifold M is *integrable* if and only if each $p \in M$ has a regular chart (U, φ) with local coordinates $\varphi = (x_1, \ldots, x_n)$ such that

$$D|_U = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right\}.$$

Here is a more coordinate-free check for being integrable.

Definition 2.75 (foliation)**.** A distribution $\mathcal{D}$ of dimension k on a regular manifold M is a *foliation* if and only if $p \in M$ has an "integral" immersed submanifold $S_p \subseteq M$, meaning that $T_qS_p = D_q$ for all $q \in S_p$.

Foliations give rise to partitions of the manifold, called leaves.

Definition 2.76 (leaf)**.** Fix a foliation $\mathcal{D}$ of rank k on a smooth manifold M. Given $p \in M$, a *leaf* of $\mathcal{D}$ is the collection of points $q \in M$ such that there is a path γ connecting p and q with $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for all t.

Note that the leaves of $\mathcal{D}$ are connected and partition M.

Example 2.77. Orbits of the action of $\mathbb{R}$ on $\mathbb{R}^2/\mathbb{Z}^2$ by $r\colon (x, y) \mapsto (x + r, y)$ are leaves.

Example 2.78. For a fiber bundle with connected fibers, the leaves are fibers.

Example 2.79. For any connected closed Lie subgroup H of a regular Lie group G, the quotient $G \twoheadrightarrow G/H$ is a fiber bundle and hence produces leaves given by H.

Example 2.80. If $\mathcal{D}$ is a vector field (i.e., has dimension 1), then the leaves are integral submanifolds, which are the integral curves.

Anyway, here is our main theorem.

Theorem 2.81 (Frobenius)**.** A distribution $\mathcal{D}$ on a smooth manifold M is integrable if and only if $\mathcal{D}$ is closed under the Lie bracket.

Proof. The forward direction is not so bad: note $\mathcal{D}$ being integrable means that each $p \in M$ has an open neighborhood where $\mathcal{D}$ is just given by tangent spaces, and vector fields living in tangent spaces will be preserved by the Lie bracket. For the converse, see [Lee13, Theorem 19.12]. It proceeds by induction. ■

2.5.2 Sketches of the Fundamental Theorems

We begin with Theorem 2.68.

Proof of Theorem 2.68. The main point is to produce the reverse map producing Lie subgroups from Lie subalgebras. As such, fix some Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. For each $X \in \mathfrak{g}$, let ξ_X be the corresponding left-invariant vector field. We then let

$$\mathcal{D}^{\mathfrak{h}} := \operatorname{span}\{\xi_Y : Y \in \mathfrak{h}\}.$$

Here are some checks on $\mathcal{D}^{\mathfrak{h}}$.

- Quickly, we claim that $\mathcal{D}^{\mathfrak{h}}$ is integrable. By Theorem 2.81, it is enough to check that $\mathcal{D}^{\mathfrak{h}}$ is closed under the bracket. This is a matter of computation: for two vector fields $\sum_i f_i\xi_{Y_i}$ and $\sum_j g_j\xi_{Y_j}$ contained in $\mathcal{D}^{\mathfrak{h}}$, we find

$$\begin{aligned}\left[\sum_i f_i\xi_{Y_i}, \sum_j g_j\xi_{Y_j}\right] &= \sum_{i,j}\left(f_i(\xi_{Y_i}g_j)\xi_{X_i} - g_j(\xi_{Y_j}f_i)\xi_{h_i}\right) \\ &= \cdots .\end{aligned}$$

- In fact, we note that $\mathcal{D}^{\mathfrak{h}}$ is left-invariant.

We now let S_g be the integral submanifold corresponding to $g \in G$, and we note that we can take $H := S_1$ to complete the proof. ■

We now proceed with Theorem 2.69.

Proof of Theorem 2.69. Injectivity follows from Corollary 2.11, so we merely need to get the surjectivity. The point is to pass to the graph in order to produce morphisms when we already know how to produce objects (via Theorem 2.68).

Fix some homomorphism $\psi\colon \mathfrak{g} \to \mathfrak{h}$ of Lie algebras. Well, define $\theta\colon \mathfrak{g} \to \mathfrak{g}\oplus\mathfrak{k}$ by $\theta(X) := (X, \psi(X))$; note that this is still a Lie algebra homomorphism because it is the sum of Lie algebra homomorphisms. Now, $\operatorname{im}\theta$ is a Lie subalgebra of $\operatorname{Lie}(G \times H) = \mathfrak{g} \times \mathfrak{h}$ by Lemma 2.39, so Theorem 2.68 tells us that we can find some subgroup connected $\Gamma \subseteq G \times H$ with

$$\operatorname{Lie}\Gamma = \operatorname{im}\theta.$$

Let $\operatorname{pr}_1\colon \Gamma \to G$ and $\operatorname{pr}_2\colon \Gamma \to H$ be the projections. Note that $d(\operatorname{pr}_1)_e \circ \theta = \operatorname{id}_{\mathfrak{g}}$ by definition of θ, and $\theta \circ d(\operatorname{pr}_1)_e = \operatorname{id}_{\operatorname{Lie}\Gamma}$ by construction of Γ. Thus, we see that $d(\operatorname{pr}_1)_e$ is a bijection and hence a local isomorphism; in particular, $\operatorname{pr}_1\colon G \to \Gamma$ must be a covering space map, so we conclude that pr_1 is actually an isomorphism. We thus recover a map

$$G \overset{\operatorname{pr}_1}{\leftarrow} \Gamma \overset{\operatorname{pr}_2}{\to} H$$

which is ψ on the level of Lie algebras, as required. ■

We will largely omit the proof of Theorem 2.70. It follows from strong structure theory of Lie algebras. For example, one wants the following result.

Theorem 2.82 (Ado)**.** Any finite-dimensional Lie algebra $\mathfrak{g}$ has a faithful representation. In other words, there exists a finite-dimensional vector space V and an injective Lie algebra homomorphism $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$.

We will not show this, but we remark on a special case.

Remark 2.83. Suppose that $\mathfrak{g}$ is a Lie algebra with $\mathfrak{z}(\mathfrak{g}) = 0$. Then the adjoint representation $\operatorname{ad}_\bullet\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $X \mapsto [X, -]$ is a faithful representation.

With this in hand, we can prove Theorem 2.70.

Proof of Theorem 2.70. By passing to the universal cover of the connected component, it suffices to produce some regular Lie group G with $\operatorname{Lie} G = \mathfrak{g}$. Well, embed $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ for some finite-dimensional vector space V, and then we are done by Theorem 2.68 after noticing $\mathfrak{gl}(V) = \operatorname{Lie}\operatorname{GL}(V)$. ■

2.5.3 Complexifications

In the sequel, we will want to focus on Lie algebras of $\mathbb{C}$ instead of $\mathbb{R}$. For this, we make the following definition.

Definition 2.84 (complexification)**.** Fix a real Lie algebra $\mathfrak{g}$. Then we define the *complexification* as

$$\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}.$$

Then $\mathfrak{g}_\mathbb{C}$ is a Lie algebra over $\mathbb{C}$.

Example 2.85. One sees that $\mathfrak{gl}_n(\mathbb{R})_\mathbb{C}$ is simply $\mathfrak{gl}_n(\mathbb{C})$. However, $\mathfrak{gl}_n(\mathbb{C})$ is also $\mathfrak{u}_n(\mathbb{C})_\mathbb{C}$ after some care.

Example 2.86. One sees that $\mathfrak{so}_{k,\ell}(\mathbb{R})_\mathbb{C}$ is just $\mathfrak{so}_\mathbb{C}(\mathbb{C})$.

Definition 2.87 (complexification)**.** Fix a simply connected real Lie group H over $\mathbb{R}$. Then we let G be the unique simply connected complex Lie group G such that

$$\operatorname{Lie} G = \operatorname{Lie} H \otimes_\mathbb{R} \mathbb{C}.$$

Note that G certainly exists by Theorem 2.70.

Example 2.88. Note that $\mathrm{SL}_2(\mathbb{R})$ has a two-sheeted cover, which on the level of Lie algebras is given by $\mathfrak{sl}_2(\mathbb{R}) \otimes_\mathbb{R} \mathbb{C}$.

With care, one is able to go in the reverse direction.

Definition 2.89 (real form)**.** Fix a connected complex Lie group G. A real Lie subgroup $H \subseteq G$ is a *real form* of G such that the natural map

$$\operatorname{Lie} H \otimes_\mathbb{R} \mathbb{C} \to \operatorname{Lie} G$$

is an isomorphism of Lie algebras over $\mathbb{C}$.

Remark 2.90. It is not technically obvious that real forms always exist. One thing one may try for simply connected G is to take the fixed points of the map $G \to G$ induced by the complex conjugation morphism $\mathfrak{g} \to \mathfrak{g}$.

Let's see a few examples.

Example 2.91. Of course, for any real Lie algebra $\mathfrak{g}$, we see that $\mathfrak{g}$ is a real form of $\mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. For example, $\mathfrak{sl}_n(\mathbb{R})$ is a real form of $\mathfrak{sl}_n(\mathbb{C})$.

Example 2.92. Note that $\mathfrak{su}_n$ is a real form of $\mathfrak{sl}_n(\mathbb{C})$. Indeed, define the map $\mathfrak{su}_n \otimes_\mathbb{R} \mathbb{C} \to \mathfrak{sl}_n(\mathbb{C})$ by $X \otimes z \mapsto zX$. Certainly this map is well-defined because $\mathfrak{su}_n$ consists of traceless matrices already, it is linear by construction, and it preserves the Lie bracket because the Lie bracket is the matrix commutator everywhere. To check that we have an isomorphism, we note that our dimensions are equal by Example 1.160. Thus, for example, it is enough to note that our mapping is injective: any element of $\mathfrak{su}_n \otimes_\mathbb{R} \mathbb{C}$ can be written as $X \otimes 1 + Y \otimes i$, but if this goes to 0 in $\mathfrak{sl}_n(\mathbb{C})$, then $X + iY = 0$, and we conclude that $X = Y = 0$ by taking real and imaginary parts on the coordinates.

Theme 3
Building Representations

3.1 September 30

Today we will talk about representations.

3.1.1 Representations

Fix a ground field k which is an extension of $\mathbb{F}$. To review, recall that a representation of a regular Lie group G is a morphism $\rho_V : G \to \mathrm{GL}(V)$ of Lie groups; given the data of only the k-vector space V, we will assume that the representation is called ρ_V. A morphism $\varphi\colon V \to W$ of representations is one respecting the G-actions: we require φ to be linear and satisfying

$$\rho_W(g) \circ \varphi = \varphi \circ \rho_V(g)$$

for all $g \in G$. The category here is called $\mathrm{Rep}_k(G)$.

Similarly, for a Lie algebra $\mathfrak{g}$, a representation is a morphism $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras. A morphism $\varphi\colon V \to W$ of representations is one respecting the G-action again: again, we need

$$\rho_W(g) \circ \varphi = \varphi \circ \rho_V(g).$$

The category here is called $\mathrm{Rep}_k(\mathfrak{g})$.

Remark 3.1. As a quick aside, we note that a bijective morphism $\varphi\colon V \to W$ will be an isomorphism. Indeed, the inverse map $\psi\colon W \to V$ is an isomorphism of vector spaces by linear algebra, and we see that it is invariant under our action as follows: for any $w \in W$ and operator g in G or $\mathfrak{g}$, write $w = \varphi(v)$ for some unique $v \in V$ so that

$$\psi(gw) = \psi(g\varphi(v)) = \psi(\varphi(gv)) = gv = g\psi(w).$$

Note that if $\mathfrak{g} = \operatorname{Lie} G$, then we have a functor taking $\rho\colon G \to \mathrm{GL}(V)$ to $d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Let's explain this.

Lemma 3.2. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$.

(a) One has a functor $F\colon \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$ sending a representation $\rho\colon G \to \mathrm{GL}(V)$ to $d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

(b) The functor F is faithful.

Proof. For (a), we explain that $F\colon \operatorname{Rep}(G) \to \operatorname{Rep}(\mathfrak{g})$ is a functor. Here is the data.

- On objects, we send $\rho\colon G \to \operatorname{GL}(V)$ to the map $d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$, which we know is a morphism of Lie algebras because ρ is a group homomorphism.
- Further, we send morphisms $\varphi\colon V \to W$ of G-representations (namely, satisfying $\varphi\circ\rho_V(g) = \rho_W(g)\circ\varphi$ for all $g \in G$) to the morphism $d\varphi_0\colon V \to W$, which of course can be identified with the original map because φ is linear. For this to make sense, we should check that $\varphi\colon V \to W$ preserves the $\mathfrak{g}$-action if it preserves the G-action. Well, for $X \in \mathfrak{g}$ and $v \in V$, we must check that

$$\varphi(d(\rho_V)_e(X)v) \stackrel{?}{=} d(\rho_W)_e(X)\varphi(v).$$

  Well, define $\gamma\colon \mathbb{F} \to G$ by $\gamma(t) := \exp(tX)$. Then we note that linear maps (such as evaluation at v) pass through derivatives by their definition as a limit, so

$$\begin{aligned}
\varphi(d(\rho_V)_e(X)v) &= \varphi(d(\rho_V)_e(\gamma'(0))v) \\
&= \varphi\left((\rho_V \circ \gamma)'(0)v\right) \\
&= \varphi\left(\frac{d}{dt}\rho_V \circ \gamma(t)\bigg|_{t=0} v\right) \\
&= \frac{d}{dt}\varphi(\rho_V(\gamma(t))(v))\bigg|_{t=0} \\
&= \frac{d}{dt}\rho_W(\gamma(t))(\varphi(v))\bigg|_{t=0} \\
&= (\rho_W \circ \gamma)'(0)(\varphi(v)) \\
&= d\rho_e(X)(\varphi(v)),
\end{aligned}$$

  as required.

Here are the coherence checks.

- Identity: note that the identity map $\operatorname{id}_V\colon V \to V$ on G-representations (which is the identity linear map) gets sent to the identity linear map $V \to V$ on $\mathfrak{g}$-representations.
- Associativity: for morphisms $\varphi\colon V \to V'$ and $\varphi'\colon V' \to V''$ of G-representations, we note that we get the exact same maps out as $\mathfrak{g}$-representations, so $\psi \circ \varphi$ as a G-representation gets sent to $F(\psi\circ\varphi) = \psi \circ \varphi = F\psi \circ F\varphi$.

The previous point has given us our functor, so we now need to check that it is faithful for (b). Well, a G-invariant map $\varphi\colon V \to W$ goes to the same map $\varphi\colon V \to W$ as a $\mathfrak{g}$-representation by definition of F. Thus, given two maps $\varphi_1, \varphi_2\colon V \to W$ of G-representations, we see that $F\varphi_1 = F\varphi_2$ implies that

$$\varphi_1 = F\varphi_1 = F\varphi_2 = \varphi_2,$$

as required. ■

Remark 3.3. It will be helpful to remember in the sequel that

$$d\rho_e(X)v = \frac{d}{dt}\rho(\exp(tX))v\bigg|_{t=0},$$

which was proved in the argument above. Note that this derivative makes sense because it takes place in some Euclidean space.

3.1.2 Operations on Representations

We present some operations on representations of G and $\mathfrak{g}$. Note that these should always be related by the ambient functor $\operatorname{Rep}(G) \to \operatorname{Rep}(\mathfrak{g})$ which is an equivalence when G is simply connected by Proposition 3.28. As basic examples, here are some trivial representations.

Lemma 3.4. Fix a Lie group G and Lie algebra $\mathfrak{g}$ and a vector space V.

(a) We can make V into a "trivial" G-representation by $\rho_V(g) = \mathrm{id}_V$ for all $g \in G$.

(b) We can make V into a "trivial" $\mathfrak{g}$-representation by $\rho_V(X) := 0$ for all $X \in \mathfrak{g}$.

(c) Suppose $\mathfrak{g} = \operatorname{Lie} G$. Making V into a trivial G-representation, we see that $F(V)$ is the trivial $\mathfrak{g}$-representation.

Proof. Here we go.

(a) We have indeed defined a homomorphism $G \to \mathrm{GL}(V)$ because this is the trivial homomorphism. It is also regular because constant maps are regular.

(b) We have indeed defined a map $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$, and it is linear map of vector spaces. It remains to check that we have defined a map of Lie algebras, for which we note that

$$[\rho_V(X), \rho_V(Y)] = \rho_V(X) \circ \rho_V(Y) - \rho_V(Y) \circ \rho_V(X) = 0 = \rho_V([X, Y]).$$

(c) Fix the trivial representation as $\rho \colon G \to \mathrm{GL}(V)$. Then the induced map $d\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V)$ is given by

$$d\rho_1(X)v = \frac{d}{dt}\rho(\exp(-tX))v\bigg|_{t=0},$$

but of course $\rho(\exp(-tX))v = v$ for all $t \in \mathbb{R}$, so this derivative vanishes. Thus, $d\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V)$ is the zero map, as required. ∎

Example 3.5. We always have the trivial representation on the zero-dimensional vector space.

As something else easy to do, we note that there are complex conjugate representations.

Lemma 3.6. Fix a Lie group G and Lie algebra $\mathfrak{g}$.

(a) Given a representation $V \in \operatorname{Rep}_{\mathbb{C}}(G)$, we can make the complex conjugate vector space $\overline{V}$ into a representation of $\overline{G}$ by

$$\rho_{\overline{V}}(g)(\overline{v}) := \overline{\rho_V(g)\overline{v}}.$$

(b) Given a representation $V \in \operatorname{Rep}_{\mathbb{C}}(\mathfrak{g})$, we can make the complex conjugate vector space $\overline{V}$ into a representation of $\overline{\mathfrak{g}}$ by

$$\rho_{\overline{V}}(X)(v) := \overline{\rho_V(X)v}.$$

(c) Suppose $\mathfrak{g} = \operatorname{Lie}(G)$. Given a representation $V \in \operatorname{Rep}_{\mathbb{C}}(G)$, then $F\overline{V} = \overline{FV}$ as representations in $\operatorname{Rep}_{\mathbb{C}}(\mathfrak{g})$.

Proof. Here we go.

(a) For each $g \in G$, we note that $\rho_{\overline{V}}(g) \colon \overline{V} \to \overline{V}$ is $\mathbb{C}$-linear: for any $a, a' \in \mathbb{C}$ and $\overline{v}, \overline{v}' \in \overline{V}$, we see

$$\begin{aligned} \rho_{\overline{V}}(g)(a\overline{v} + a'\overline{v}') &= \rho_{\overline{V}}(g)(\overline{\overline{a}v + \overline{a}'v'}) \\ &= \overline{\rho_V(g)(\overline{a}v + \overline{a}'v')} \\ &= \overline{\overline{a}\rho_V(g)v + \overline{a}'\rho_V(g)v'} \\ &= a\rho_{\overline{V}}(g)(\overline{v}) + a'\rho_{\overline{V}}(g)\overline{v}'. \end{aligned}$$

To show that we have defined a group homomorphism, we see that

$$\rho_{\overline{V}}(gh)\overline{v} = \overline{\rho_V(gh)v} = \overline{\rho_V(g)\rho_V(h)v} = \rho_{\overline{V}}(g)\rho_{\overline{V}}(h)\overline{v}.$$

Lastly, we note that the map $\rho_{\overline{V}} \colon G \to \mathrm{GL}(\overline{V})$ is a regular map by expanding it on a basis: upon picking a $\mathbb{C}$-basis of V (which is also a $\mathbb{C}$-basis of $\overline{V}$), we see that the matrix $\rho_{\overline{V}}(g)$ is simply the complex conjugate of the matrix of $\rho(g)$, which will continue to be a regular map after keeping track of all of our conjugations.

(b) The same check as in (a) explains that $\rho_{\overline{V}}(X)$ is at least a $\mathbb{C}$-linear map for all $X \in \mathfrak{g}$. This map is also of course linear in X given by the linearity of ρ_V. Lastly, this is a homomorphism of Lie algebras by taking the conjugate of the identity

$$\rho_V([X, Y]) = \rho_V(X)\rho_V(Y) - \rho_V(Y) - \rho_V(X).$$

(c) Simply take the conjugate everywhere in sight. ■

To begin doing something with content, we handle direct sums.

Lemma 3.7. Fix a Lie group G and Lie algebra $\mathfrak{g}$.

(a) Given representations $V, W \in \mathrm{Rep}_k(G)$, we can make $V \otimes W$ into a representation of G via the coordinate-wise action

$$\rho_{V \oplus W}(g)(v \oplus w) := \rho_V(g)v \oplus \rho_W(g)w.$$

(b) Given representations $V, W \in \mathrm{Rep}_k(\mathfrak{g})$, we can make $V \oplus W$ into a representation of G via the coordinate-wise action

$$\rho_{V \oplus W}(X)(v \oplus w) = \rho_V(X)v \oplus \rho_W(X)w.$$

(c) Suppose $\mathfrak{g} = \mathrm{Lie}(G)$. Given representations $V, W \in \mathrm{Rep}_k(G)$, then $F(V \otimes W)$ is the direct sum representation in $\mathrm{Rep}_k(\mathfrak{g})$.

Proof. Here we go.

(a) By taking the direct sum of the homomorphisms $\rho_V \colon G \to \mathrm{GL}(V)$ and $\rho_W \colon G \to \mathrm{GL}(W)$, we obtain a regular homomorphism $G \to \mathrm{GL}(V) \oplus \mathrm{GL}(W)$. To finish, we note that $\mathrm{GL}(V) \oplus \mathrm{GL}(W)$ embeds into $\mathrm{GL}(V \oplus W)$ by sending (φ, ψ) to the linear map $V \oplus W \to V \oplus W$ acting by (φ, ψ) on the coordinates. To see that this last map is a regular homomorphism, we note that fixing an ordered basis of both V and W allows us to identify these GL groups with invertible matrices, in which case our map is given by

$$(A, B) \mapsto \begin{bmatrix} A & \\ & B \end{bmatrix}.$$

In particular, this map is regular in coordinates and hence regular; one can check that it is a homomorphism directly because $(A, B) \cdot (A', B')$ goes to the block-diagonal matrix $\operatorname{diag}(AA', BB')$. In total, we have obtained a composite of regular homomorphisms $G \to \mathrm{GL}(V) \oplus \mathrm{GL}(W) \to \mathrm{GL}(V \oplus W)$.

(b) We will simply proceed directly. We define a map $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ by

$$\rho(X) := \begin{bmatrix} \rho_V(X) & \\ & \rho_W(X) \end{bmatrix},$$

where we are thinking about endomorphisms of $V \oplus W$ in the above block-diagonal format. Linearity of ρ_V and ρ_W gives linearity of ρ. To check the bracket, we compute

$$\begin{aligned}
[\rho(X), \rho(Y)] &= \begin{bmatrix} \rho_V(X) & \\ & \rho_W(X) \end{bmatrix}\begin{bmatrix} \rho_V(Y) & \\ & \rho_W(Y) \end{bmatrix} - \begin{bmatrix} \rho_V(Y) & \\ & \rho_W(Y) \end{bmatrix}\begin{bmatrix} \rho_V(X) & \\ & \rho_W(X) \end{bmatrix} \\
&= \begin{bmatrix} \rho_V(X) \circ \rho_V(Y) - \rho_V(Y) \circ \rho_V(X) & \\ & \rho_W(X) \circ \rho_W(Y) - \rho_W(Y) \circ \rho_W(X) \end{bmatrix} \\
&= \begin{bmatrix} [\rho_V(X), \rho_V(Y)] & \\ & [\rho_W(X), \rho_W(Y)] \end{bmatrix} \\
&= \begin{bmatrix} \rho_V([X,Y]) & \\ & \rho_W([X,Y]) \end{bmatrix} \\
&= \rho([X,Y]).
\end{aligned}$$

(c) This is a direct computation. Given the representations $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ with direct sum $\rho_{V\oplus W}$, we need to compute the direct sum of the representations $d\rho_V$ and $d\rho_W$. Well, for any $X \in \mathfrak{g}$ and $(v, w) \in V \oplus W$, we note that evaluation at (v, w) is a linear map and hence passes through derivative computations (in Euclidean space!), so

$$\begin{aligned}
d\rho_{V\oplus W}(X)(v, w) &= \frac{d}{dt}\rho_{V\oplus W}(\exp(tX))\bigg|_{t=0}(v, w) \\
&= \frac{d}{dt}\begin{bmatrix} \rho_V(\exp(tX)) & \\ & \rho_W(\exp(tX)) \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}\Bigg|_{t=0} \\
&= \frac{d}{dt}\begin{bmatrix} \rho_V(\exp(tX))v & \\ & \rho_W(\exp(tX))w \end{bmatrix}\Bigg|_{t=0}.
\end{aligned}$$

Now, because we are in a Euclidean space, we can compute the derivative on each coordinate separately, which we see to be $\operatorname{diag}(d\rho_V(X), d\rho_W(X))$, as needed. ■

Next we handle the tensor product.

Lemma 3.8. Fix a Lie group G and Lie algebra $\mathfrak{g}$.

(a) Given representations $V, W \in \mathrm{Rep}_k(G)$, we can make $V \otimes W$ into a representation of G via the coordinate-wise action

$$\rho_{V\otimes W}(g)(v \otimes w) := \rho_V(g)v \otimes \rho_W(g)w.$$

(b) Given representations $V, W \in \mathrm{Rep}_k(\mathfrak{g})$, we can make $V \otimes W$ into a representation of G via the product rule action

$$\rho_{V\otimes W}(X)(v \otimes w) = \rho_V(X)v \otimes w + v \otimes \rho_W(X)w.$$

(c) Suppose $\mathfrak{g} = \mathrm{Lie}(G)$. Given representations $V, W \in \mathrm{Rep}_k(G)$, then $F(V \otimes W)$ is the tensor product representation in $\mathrm{Rep}_k(\mathfrak{g})$.

Proof. Here we go.

(a) For each $g \in G$, we need to provide a bilinear map $\rho(g)\colon (V \times W) \to (V \otimes W)$, for which we take

$$\rho(g)(v, w) := \rho_V(g)v \otimes \rho_W(g)w.$$

Linearity of $\rho_V(g)$ and $\rho_W(g)$ (and properties of the tensor product) verify that we have in fact defined a bilinear map, so we have in fact defined a map $G \to \mathrm{End}(V \otimes W)$. Here are our checks to make this map a representation.

- Group action: for the identity check, we note that

$$\rho(e)(v \otimes w) = (v \otimes w)$$

for any pure tensor $v \otimes w \in V \otimes W$. Thus, because maps out of $V \otimes W$ are determined by their action on pure tensors, we see that $\rho(e) = \mathrm{id}$. Similarly, for $g, h \in G$, we see that

$$\rho(gh)(v \otimes w) = (\rho_V(g)\rho_V(h)v \otimes \rho_W(g)\rho_W(h)w) = \rho(g)\rho(h)(v \otimes w),$$

so $\rho(gh)$ and $\rho(g) \circ \rho(h)$ are equal on pure tensors and hence equal as maps $V \otimes W \to V \otimes W$.

- Regular: we expand everything on a basis. Fix a basis $\{e_1, \ldots, e_m\}$ of V and $\{f_1, \ldots, f_m\}$ on W so that $\{e_i \otimes f_j\}_{i,j}$ is a basis of $V \otimes W$; let $\mathrm{pr}_\bullet$ be the appropriate projection whenever it appears. The previous step verifies that we have a group homomorphism $\rho\colon G \to \mathrm{GL}(V \otimes W)$, which we must now show to be regular. Notably, the matrix coefficients $\rho(g)_{i_1 j_1, i_2 j_2}$ of $\rho(g)$ are now computable as

$$\mathrm{pr}_{i_2 j_2} \rho(g)(e_{i_1} \otimes f_{j_1}) = \mathrm{pr}_{i_2 j_2}(\rho_V(g)e_{i_1} \otimes \rho_W(g)f_{j_1}) = \rho_V(g)_{i_1 i_2}\rho_W(g)_{j_1 j_2},$$

which is a product of regular functions and hence regular. Thus, ρ is regular on coordinates and hence regular.

(b) For each X, we need to provide a bilinear map $\rho(X)\colon (V \otimes W) \to (V \otimes W)$, for which we take

$$\rho(X)(v, w) := \rho_V(X)v \otimes w + v \otimes \rho_W(X)w.$$

Linearity of $\rho_V(X)$ and $\rho_W(X)$ (and properties of the tensor product) verify that we have in fact defined a bilinear map, so we have in fact defined a map $\mathfrak{g} \to \mathfrak{gl}(V \otimes W)$. Here are our checks to make this a representation.

- Linear: for $a, b \in \mathbb{F}$ and $X, Y \in \mathfrak{g}$, we should check that $\rho(aX + bY) = a\rho(X) + b\rho(Y)$. Because pure tensors span $V \otimes W$, it is enough to check this equality on pure tensors, for which we compute

$$\begin{aligned}
\rho(aX + bY)(v \otimes w) &= \rho_V(aX + bY)v \otimes w + v \otimes \rho_W(aX + bY)w \\
&= a(\rho_V(X)v \otimes w + v \otimes \rho_W(X)w) + b(\rho_V(Y)v \otimes w + v \otimes \rho_W(Y)w) \\
&= (a\rho(X) + b\rho(Y))(v \otimes w).
\end{aligned}$$

- Lie bracket: for $X, Y \in \mathfrak{g}$, we need to check $[\rho(X), \rho(Y)] = \rho([X, Y])$. It is enough to check this on pure tensors, for which we compute

$$\begin{aligned}
[\rho(X), \rho(Y)](v \otimes w) &= (\rho(X)\rho(Y) - \rho(Y)\rho(X))(v \otimes w) \\
&= \rho(X)\rho(Y)(v \otimes w) - \rho(Y)\rho(X)(v \otimes w) \\
&= \rho_V(X)\rho_V(Y)v \otimes w - \rho_V(Y)v \otimes \rho_W(X)w \\
&\qquad - \rho_V(X)v \otimes \rho_W(Y) + v \otimes \rho_W(X)\rho_W(Y)w \\
&\qquad - \rho_V(Y)\rho_V(X)v \otimes w + \rho_V(X)v \otimes \rho_W(Y)w \\
&\qquad + \rho_V(Y)v \otimes \rho_W(X)w - v \otimes \rho_W(Y)\rho_W(X)w \\
&= (\rho_V(X)\rho_V(Y) - \rho_V(Y)\rho_V(X))v \otimes w \\
&\qquad + v \otimes (\rho_W(X)\rho_W(Y) - \rho_W(Y)\rho_W(X))w \\
&= \rho([X, Y])(v \otimes w).
\end{aligned}$$

(c) This is a direct computation. Given the representations $\rho_V\colon G \to \mathrm{GL}(V)$ and $\rho_W\colon G \to \mathrm{GL}(W)$, we would like to compute $(d\rho_{V \otimes W})_e(X) \in \mathfrak{gl}(V \otimes W)$ for some $X \in \mathfrak{g}$. Well, it is enough to compute this on pure tensors $v \otimes w$, for which we note that evaluation is a linear map and hence can be moved inside a derivative in the computation

$$\begin{aligned}(d\rho_{V\otimes W})_e(X)(v\otimes w) &= \frac{d}{dt}\rho_{V\otimes W}(\exp(tX))\bigg|_{t=0}(v\otimes w) \\ &= \frac{d}{dt}\rho_{V\otimes W}(\exp(tX))(v\otimes w)\bigg|_{t=0} \\ &= \frac{d}{dt}\rho_V(\exp(tX))v\otimes\rho_W(\exp(tX))w\bigg|_{t=0} \\ &= \frac{d}{dt}(1+td\rho_V(X)+\cdots)v\otimes\rho_W(1+td\rho_W(X)+\cdots)w\bigg|_{t=0} \\ &= d\rho_V(X)v\otimes w+v\otimes d\rho_W(X)w,\end{aligned}$$

as required. Notably, we expanded our the Taylor series in order to the computation of the derivative at $t = 0$, but one can also indirectly apply some product rule after working more explicitly with coordinates. ■

Remark 3.9. By induction, we see that we can also define a tensor representation

$$V_1\otimes\cdots\otimes V_k$$

for any finite number of representations $V_1,\ldots,V_k$. One can compute the actions by simply extending the above ones to more terms inductively.

Example 3.10. We explain how to twist by a character.

- Fix a regular Lie group G. Given a representation $\rho\colon G\to\mathrm{GL}(V)$ and a character $\chi\colon G\to\mathrm{GL}_1(\mathbb{F})$, we see that we have a representation $\chi\otimes\rho$ on $\mathbb{F}\otimes V$. However, $\mathbb{F}\otimes V$ can be identified with V by the map $c\otimes v\mapsto cv$ (on pure tensors), so we have really defined a representation $\chi\rho\colon G\to\mathrm{GL}(V)$ given by
$$(\chi\rho)(g) := \chi(g)\rho(g).$$
- Fix a Lie algebra $\mathfrak{g}$. Given a representation $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ and a character $\chi\colon\mathfrak{g}\to\mathfrak{gl}(\mathbb{F})$, we again see that we have a representation $\chi\otimes\rho$ on $\mathbb{F}\otimes V$. Identifying $\mathbb{F}\otimes V$ with V as before, we see that we have defined a representation $\chi\rho\colon G\to\mathrm{GL}(V)$ by
$$(\chi\rho)(X) = \chi(X)+\rho(X).$$

We also have Hom sets.

Lemma 3.11. Fix a Lie group G and Lie algebra $\mathfrak{g}$.

(a) Given representations $V,W\in\mathrm{Rep}_k(G)$, we can make $\mathrm{Hom}(V,W)$ into a representation of G via
$$\rho_{\mathrm{Hom}(V,W)}(g)\varphi := \rho_W(g)\cdot\varphi\circ\rho_V(g)^{-1}.$$

(b) Given representations $V,W\in\mathrm{Rep}_k(\mathfrak{g})$, we can make $\mathrm{Hom}(V,W)$ into a representation of G via
$$\rho_{\mathrm{Hom}(V,W)}(X)\varphi := \rho_W(X)\circ\varphi-\varphi\circ\rho_V(X).$$

(c) Suppose $\mathfrak{g} = \mathrm{Lie}(G)$. Given representations $V,W\in\mathrm{Rep}_k(G)$, then $F(\mathrm{Hom}(V,W))$ is the corresponding in $\mathrm{Rep}_k(\mathfrak{g})$.

Proof. Here we go.

(a) Given finite-dimensional representations $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$, we explain how to build a representation $\rho\colon G \to \mathrm{Hom}(V, W)$. Indeed, for $g \in G$ and $\varphi \in \mathrm{Hom}(V, W)$, define

$$(\rho(g)\varphi)(v) := \rho_W(g)\varphi\left(\rho_V(g)^{-1}v\right).$$

In other words, $\rho(g)\varphi = \rho_W(g) \circ \varphi \circ \rho(g)^{-1}$. Here are our checks.

- Group action: for the identity check, we note

$$\rho(e)\varphi = \rho_W(e) \circ \varphi \circ \rho_V(e)^{-1} = \mathrm{id}_W \circ \varphi \circ \mathrm{id}_V^{-1},$$

as required. For the associativity check, we choose $g, h \in G$ and note

$$\rho(g)\rho(h)\varphi = \rho_W(g) \circ \rho_W(h) \circ \varphi \circ \rho(h)^{-1} \circ \rho(g)^{-1} = \rho(gh)\varphi.$$

- Regular: it is enough to show that we have given a regular map $G \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ by considering component-wise formulations of matrix entries. Well, our map is simply the composite

$$\begin{array}{ccccc} G \times \mathrm{Hom}(V, W) & \to & \mathrm{GL}(W) \times \mathrm{Hom}(V, W) \times \mathrm{GL}(V) & \mapsto & \mathrm{Hom}(V, W) \\ (g, \varphi) & \mapsto & \left(\rho_W(g), \varphi, \rho_V(g)^{-1}\right) & \mapsto & \rho_W(g) \circ \varphi \circ \rho_V(g)^{-1} \end{array}$$

which is regular as the composite of (products of) regular maps. For example, the last map is regular because it is simply matrix multiplication, which is polynomial on coordinates and hence regular.

(b) For any two Lie algebra representations V and W of $\mathfrak{g}$, we note that $\mathrm{Hom}(V, W)$ also has a Lie algebra representation structure given by

$$\rho_{\mathrm{Hom}(V,W)}(X)\varphi := \rho_W(X) \circ \varphi - \varphi \circ \rho_V(X).$$

Anyway, we now run our checks. Certainly $\rho_{\mathrm{Hom}(V,W)}(X)$ is a linear map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ (namely, our construction is linear in φ) because composition distributes over addition. Additionally, our construction is linear in X because $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W)$ and $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ should be linear. Lastly, we must check preservation of the bracket of our map $\mathfrak{g} \to \mathfrak{gl}(\mathrm{Hom}(V, W))$. Well, given $\varphi, \psi \in \mathrm{Hom}(V, W)$ and $X, Y \in \mathfrak{g}$, we compute

$$\begin{aligned} [\rho_{\mathrm{Hom}(V,W)}(X), \rho_{\mathrm{Hom}(V,W)}(Y)](\varphi) &= \rho_{\mathrm{Hom}(V,W)}(X) \circ \rho_{\mathrm{Hom}(V,W)}(Y)(\varphi) \\ &\quad - \rho_{\mathrm{Hom}(V,W)}(Y) \circ \rho_{\mathrm{Hom}(V,W)}(X)(\varphi) \\ &= \rho_{\mathrm{Hom}(V,W)}(X)(Y \circ \varphi - \varphi \circ Y) \\ &\quad - \rho_{\mathrm{Hom}(V,W)}(Y)(X \circ \varphi - \varphi \circ X) \\ &= X \circ Y \circ \varphi - Y \circ \varphi \circ X - X \circ \varphi \circ Y + \varphi \circ Y \circ X \\ &\quad - Y \circ X \circ \varphi + X \circ \varphi \circ Y + Y \circ \varphi \circ X - \varphi \circ X \circ Y \\ &= (X \circ Y - Y \circ X) \circ \varphi - \varphi \circ (X \circ Y - Y \circ X) \\ &= \rho_W([X, Y]) \circ \varphi - \varphi \circ \rho_V([X, Y]) \\ &= \rho_{\mathrm{Hom}(V,W)}([X, Y])(\varphi), \end{aligned}$$

where we have frequently but not always omitted our ρ_Vs and ρ_Ws.

(c) This is a direct computation. If $\varphi\colon V \to W$ were already a morphism of G-representations, then the action of (b) is simply $d\rho_{\mathrm{Hom}(V,W)}(X)(\varphi)$: indeed, the action should be

$$\begin{aligned} d\rho_{\mathrm{Hom}(V,W)}(X)(\varphi) &= \frac{d}{dt}\rho_{\mathrm{Hom}(V,W)}(\exp(tX))\varphi\bigg|_{t=0} \\ &= \frac{d}{dt}\rho_W(\exp(tX)) \circ \varphi \circ \rho_V(\exp(-tX))\bigg|_{t=0} \\ &= \frac{d}{dt}(1 + d\rho_W(tX) + \cdots) \circ \varphi \circ (1 - d\rho_W(tX) + \cdots)\bigg|_{t=0} \\ &= d\rho_W(X) \circ \varphi - \varphi \circ d\rho_V(X). \end{aligned}$$

As usual, $+\cdots$ denotes higher-order terms which cannot affect our derivative. Notably, we are using the fact that the linear term of a Taylor expansion (into some Euclidean space) is given by the derivative. ■

Example 3.12. Taking $W = k$ to be the trivial representation, we obtain duals as a special case of Lemma 3.11.

Here is also a good notion of subobjects.

Definition 3.13 (suprepresentation)**.** Fix a regular Lie group or Lie algebra. A *subrepresentation* of a representation is a subspace preserved by the G-action.

Remark 3.14. Let's make this notion more precise.

- For a regular Lie group G, we see that a subspace $U \subseteq V$ preserved by the G-action on a representation $\rho_V : G \to \mathrm{GL}(V)$ means that we can restrict the linear action map $G \times U \to V$ to an action $G \times U \to U$. Thus, we do indeed have a regular map $\rho_U : G \to \mathrm{GL}(U)$ by computing coordinates of matrices component-wise, and the natural inclusion map $U \hookrightarrow V$ is a morphism in $\mathrm{Rep}_k(G)$.
- For a Lie algebra $\mathfrak{g}$, we see that a subspace $U \subseteq V$ preserved by the $\mathfrak{g}$-action on a representation $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ means that we can restrict this linear map to $\rho_U : \mathfrak{g} \to \mathfrak{gl}(U)$. Notably, the Lie bracket of $\mathfrak{gl}(U)$ is more or less the restriction of the Lie bracket on $\mathfrak{gl}(V)$, so ρ_U continues to be a Lie algebra representation, and we see that the natural inclusion $U \hookrightarrow V$ is a morphism in $\mathrm{Rep}_k(\mathfrak{g})$.

Example 3.15. Let $\varphi \colon V \to W$ be a morphism in $\mathrm{Rep}_k(G)$. Then $\ker\varphi$ is a subrepresentation of V. Indeed, $\ker\varphi \subseteq V$ is certainly a linear subspace, and for the G-invariance, we note that any $v \ker\varphi$ has

$$\varphi(\rho_V(g)v) = \rho_W(g)(\varphi(v)) = 0$$

for any $g \in G$, so $\rho_V : G \to \mathrm{GL}(V)$ restricts to a subrepresentation $\rho_{\ker\varphi} \colon G \to \mathrm{GL}(\ker\varphi)$. (More precisely, we have restricted our regular action $G \times \ker\varphi \to V$ to a regular action $G \times \ker\varphi \to \ker\varphi$.)

Example 3.16. Let $\varphi \colon V \to W$ be a morphism in $\mathrm{Rep}_k(\mathfrak{g})$. Again, we see that $\ker\varphi \subseteq V$ is a subrepresentation for essentially the same reason: certainly $\ker\varphi \subseteq V$ is a linear subspace, and $v \in \ker\varphi$ has $\varphi(Xv) = X(\varphi(v)) = 0$ for any $X \in \mathfrak{g}$, so $\ker\varphi$ is closed under the G-action.

Example 3.17. Let $\varphi \colon V \to W$ be a morphism in $\mathrm{Rep}_k(G)$. Then $\operatorname{im}\varphi$ is a subrepresentation of W. Again, it is certainly a linear subspace, and it is preserved by the G-action because any $g \in G$ and $\varphi(v) \in \operatorname{im}\varphi$ has

$$\rho_W(g)(\varphi(v)) = \varphi(\rho_V(g)v) \in \operatorname{im}\varphi.$$

Example 3.18. Let $\varphi \colon V \to W$ be a morphism in $\mathrm{Rep}_k(\mathfrak{g})$. Then $\operatorname{im}\varphi$ is a subrepresentation of W. As usual, we have a linear subspace, and it is fixed by the G-action because $X \in \mathfrak{g}$ and $\varphi(v) \in \operatorname{im}\varphi$ has $X \cdot \varphi(v) = \varphi(X \cdot v) \in \operatorname{im}\varphi$.

Invariants provide an important example of subrepresentations.

Definition 3.19 (invariants)**.** Fix a regular Lie group G or Lie algebra $\mathfrak{g}$.

- We denote the G-invariants of a representation $V \in \operatorname{Rep}_k(G)$ by
$$V^G := \{v \in V : \rho_V(g)v = v \text{ for all } g \in G\}.$$
- We denote the $\mathfrak{g}$-invariants of a representation $V \in \operatorname{Rep}_k(\mathfrak{g})$ by
$$V^{\mathfrak{g}} := \{v \in V : \rho_V(g)v = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Remark 3.20. We won't bother to check that invariants provide subrepresentations right now. It follows from the more general Lemma 3.24.

Example 3.21. Note that $\operatorname{Hom}(V, W)^G = \operatorname{Hom}_G(V, W)$ for G-representations V and W. Indeed, a linear map $\varphi \in \operatorname{Hom}(V, W)$ is fixed by the G-action if and only if
$$g^{-1} \cdot \varphi(g \cdot v) = \left(g^{-1}\varphi\right)(v) = \varphi(v)$$
for all $g \in G$, which of course rearranges into φ being G-equivariant.

Example 3.22. Note again note that $\operatorname{Hom}(V, W)^{\mathfrak{g}} = \operatorname{Hom}_{\mathfrak{g}}(V, W)$ for this Lie algebra representation structure. Namely, we can see that $X \cdot \varphi(v) = \varphi(X \cdot v)$ for any X and v if and only if $X\varphi = 0$.

Example 3.23. Let V be a vector space, and fix a nonnegative integer $k \geqslant 0$. Then S_k acts on the tensor power $V^{\otimes k}$ by permuting the coordinates. Explicitly, permuting the coordinates provides a bilinear map $V^k \to V^{\otimes k}$, so it extends to a linear map $V^{\otimes k} \to V^{\otimes k}$ defined by
$$\sigma\colon (v_1 \otimes \cdots \otimes v_k) \mapsto (v_{\sigma 1} \otimes \cdots \otimes v_{\sigma k})$$
for any pure tensor. We won't bother to check that this is actually a group action, though it is not a lengthy check. The fixed points of this S_k-action is the symmetric power $\operatorname{Sym}^k(V)$.

Here is a more general notion of invariants.

Lemma 3.24. Fix a regular Lie group G or Lie algebra $\mathfrak{g}$, and let $X^*(G)$ and $X^*(\mathfrak{g})$ denote the set of regular homomorphisms $G \to \mathbb{F}^\times$ and $\mathfrak{g} \to \mathfrak{gl}(\mathbb{F})$, respectively. For the statements, select $R \in \{G, \mathfrak{g}\}$.

(a) Fix some $\chi \in X^*(R)$. For any representation V, the subspace
$$V^\chi := \{v \in V : \rho_V(r)v = \chi(r)v \text{ for all } r \in R\}$$
is a subrepresentation of V.

(b) For distinct characters $\chi_1, \ldots, \chi_k \in X^*(R)$ and any representation V, the subspaces $V^{\chi_1}, \ldots, V^{\chi_k}$ are linearly disjoint.

Proof. Here we go.

(a) Note that V^χ is the kernel of the family of linear maps $V \to V$ defined by $\{v \mapsto \rho_V(r)v - \chi(r)v\}_{r \in R}$, so V^χ is the intersection of linear subspaces and hence a linear subspace. To see that V^χ is preserved by the G-action, we note that any $v \in V^\chi$ and $r \in R$ will have $\rho_V(r)v \in V^\chi$: for any $s \in R$, we see
$$\rho_V(s)\rho_V(r) = \chi(r)\rho_V(s)v = \chi(s)\chi(r)v = \chi(s)\rho_V(r)v.$$

(b) Suppose for the sake of contradiction that there exists a nontrivial relation $v_1 + \cdots + v_k = 0$ where $v_i \in V^{\chi_i}$ for $i \in \{1, \ldots, k\}$. By possibly making k smaller, we may assume that all the $v_\bullet$s are nonzero, and in fact, we may assume that there does not exist such a relation with fewer than k characters $\chi_1, \ldots, \chi_k \in X^*(R)$. Now, if $k = 1$, then we are simply asserting that $v_1 = 0$, so there is nothing to say. Otherwise, we may assume that $k > 1$. Then there is $r \in R$ such that $\chi_k(r) \neq \chi_1(r)$, and we see that multiplying our relation by $\rho_V(r)$ produces the equation

$$\chi_1(r)v_1 + \cdots + \chi_k(r)v_k = 0.$$

But now we can subtract this relation from $\chi_k(r)v_k + \cdots + \chi_k(r)v_k = 0$, which produces a strictly smaller relation with at least one term $(\chi_1(r) - \chi_k(r))v$, which is a contradiction to the minimality of our relation. ■

Remark 3.25. If G is a finite group acting on a vector space V, and χ is a character of G, then we can define an operator $\pi_\chi \colon V \to V$ by

$$\pi_\chi(v) := \frac{1}{|G|} \sum_{g \in G} \chi^{-1}(g)\rho(g)v.$$

We have the following checks on π_χ.

- Note π_χ is a linear map (as the sum of linear maps).
- By rearranging the sum, we see that $\rho_V(h)\pi_\chi(v) = \chi(h)\pi_\chi(v)$ for any $h \in G$, so $\operatorname{im} \pi_\chi \subseteq V^\chi$.
- On the other hand, if $v \in V^\chi$ already, then $\pi_\chi(v)$ is just a sum with $|G|$ copies of v, so π_χ fixes V^χ pointwise.

In conclusion, we see that $\operatorname{im} \pi_\chi = V^\chi$ by Example 3.17. This is an alternate way to see that V^χ is a subrepresentation.

Example 3.26. Suppose V is a representation of a regular Lie group G or Lie algebra $\mathfrak{g}$. Given some nonnegative integer k, we recall that S_k acts on V. Thus, for a character χ of S_k, we note that the map $\pi_\chi \colon V^{\otimes k} \to V^{\otimes k}$ is a projection. In fact, π_χ respects the ambient action on V.

- In the case of a Lie group G, we see that

$$\rho_{V^{\otimes k}}(g)\pi_\chi(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \chi(\sigma)(\rho_V(g)v_{\sigma 1} \otimes \cdots \otimes \rho_V(g)v_{\sigma k}) = \pi_\chi \rho_{V^{\otimes k}}(g)(v_1 \otimes \cdots \otimes v_k),$$

so the equality $\rho_{V^{\otimes k}}(g) \circ \pi_\chi = \pi_\chi \circ \rho_{V^{\otimes k}}(g)$ follows by linearity.

- In the case of Lie algebra $\mathfrak{g}$, we see that

$$\begin{aligned}
\rho_{V^{\otimes k}}(X)\pi_\chi(v_1 \otimes \cdots \otimes v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \chi(\sigma)\rho_{V^{\otimes k}}(X)(v_{\sigma 1} \otimes \cdots \otimes v_{\sigma k}) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \chi(\sigma)(Xv_{\sigma 1} \otimes \otimes \cdots \otimes v_{\sigma k} + \cdots + v_{\sigma 1} \otimes \cdots \otimes Xv_{\sigma k}) \\
&= \pi_\chi \pi_{V^{\otimes k}}(g)(v_1 \otimes \cdots \otimes v_k).
\end{aligned}$$

Thus, $\left(V^{\otimes k}\right)^\chi \subseteq V^{\otimes k}$ continues to be a subrepresentation in all cases by Example 3.17. When $\chi = 1$, this is the symmetric power representation $\operatorname{Sym}^k(V)$. When $\chi = \operatorname{sgn}$, this is the alternating representation $\operatorname{Alt}^k(V)$.

3.1.3 Lie's Theorems for Representation Theory

We now discuss how to pass the representation theory for G to the representation theory of $\mathfrak{g}$. We want the following lemma.

Lemma 3.27. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. For a representation $V \in \operatorname{Rep}_k(G)$, give V the natural $\mathfrak{g}$-action via $d\rho$. Further, fix some character $\chi\colon G \to \mathrm{GL}_1(\mathbb{F})$ inducing a character $d\chi\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{F})$.

(a) We always have $V^\chi \subseteq V^{d\chi}$.

(b) If G is connected, then $V^\chi = V^{d\chi}$.

Proof. Quickly, we reduce to the case where $\chi = 1$ and thus $d\chi = 0$ (because χ is constant). By Example 3.10, we may consider the representation $\chi^{-1}\rho$. On one hand, we see that $v \in V^\chi$ if and only if $\left(\chi^{-1}\rho\right)(g)v = v$ for all $g \in G$; on the other hand, we see similarly that $v \in V^{d\chi}$ if and only if $d\left(\chi^{-1}\rho\right)(X)v = v$ for all $X \in \mathfrak{g}$. Thus, for our arguments, we will take $\chi = 1$ so that we may consider V^G and $V^{\mathfrak{g}}$.

(a) For $v \in V^G$, we must show that $v \in V^{\mathfrak{g}}$. Well, fix any $X \in \mathfrak{g}$, and we would like to show that $d\rho_e(X)(v) = v$. For this, define the path $\gamma\colon \mathbb{F} \to \mathfrak{g}$ given by $\gamma(t) := \exp(tX)$ so that $\gamma(0) = e$ and $\gamma'(0) = X$. Then

$$d\rho_e(X) = d\rho_e(\gamma'(0)) = (\rho \circ \gamma)'(0) \in \mathfrak{gl}(V)$$

by the chain rule, where this last derivative makes technical sense because we are outputting to a Euclidean space. To compute $(\rho \circ \gamma)'(0)(v)$, we note that applying an endomorphism in $\mathfrak{gl}(V)$ to a vector $v \in V$ is a linear map, and linear maps pass through the definition of the derivative, so we find that

$$\begin{aligned}
(\rho \circ \gamma)'(0)(v) &= \frac{d}{dt}(\rho \circ \gamma)(t)\bigg|_{t=0}(v) \\
&= \frac{d}{dt}(\rho \circ \gamma)(t)(v)\bigg|_{t=0} \\
&= \frac{d}{dt}\rho(\exp(tX))(v)\bigg|_{t=0} \\
&\overset{*}{=} \frac{d}{dt}v\bigg|_{t=0} \\
&= 0,
\end{aligned}$$

where $\overset{*}{=}$ holds because $v \in V^G$.

(b) We already showed one inclusion in (a), so now we just have to show that any $v \in V^{\mathfrak{g}}$ is fully fixed by G. Well, let $H \subseteq G$ be the subgroup of G stabilizing v, which we know to be a closed Lie subgroup. In fact, by our more precise isomorphism theorem, we know that its Lie algebra $\mathfrak{h}$ can be described by

$$\mathfrak{h} = \{X \in \mathfrak{g} : (\rho_* X)_v = 0\}.$$

However, we can compute

$$(\rho_* X)_v f = \frac{d}{dt} f(\exp(-tX)v)\bigg|_{t=0} = df_p((\rho \circ \exp)'(0)v) = df_p(d\rho_e(X)v)$$

for any germ f, but this derivative is of course 0 because $d\rho_e(X)v = 0$ for all $X \in \mathfrak{g}$ by assumption. Thus, $\mathfrak{h} = \mathfrak{g}$, so the exponential map $\exp\colon \mathfrak{h} \to G$ will be a local diffeomorphism. In particular, H contains in an open neighborhood of the identity, so H must equal G because G is connected. Thus, $v \in V^G$. ■

And here is our main result.

Proposition 3.28. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Recall the functor $F\colon \operatorname{Rep}(G) \to \operatorname{Rep}(\mathfrak{g})$ sending a representation $\rho\colon G \to \operatorname{GL}(V)$ to $d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

(a) If G is connected, then F is fully faithful.

(b) If G is connected and simply connected, then F is essentially surjective and hence an equivalence.

Proof. Here we go.

(a) Suppose that G is connected, and we want to show that F is fully faithful. In Lemma 3.2, we showed that F is faithful, so we now must show that F is full. Well, for G-representations V and W, we must show that any linear map $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(V, W)$ is in fact G-invariant. Well, we simply note that

$$\operatorname{Hom}_{\mathfrak{g}}(V, W) = \operatorname{Hom}(V, W)^{\mathfrak{g}},$$

which equals $\operatorname{Hom}(V, W)^G = \operatorname{Hom}_G(V, W)$ by Lemma 3.27, so we are done.

(b) Suppose that G is connected and simply connected. We want to show that F is essentially surjective in order to finish the proof that F is an equivalence of categories. Well, fix a representation of $\mathfrak{g}$ given by some Lie algebra homomorphism $\overline{\rho}\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Then Theorem 2.69 tells us that the differential provides a bijection

$$F\colon \operatorname{Hom}_{\mathrm{LieGrp}}(G, \operatorname{GL}(V)) \cong \operatorname{Hom}_{\mathrm{LieAlg}}(\mathfrak{g}, \mathfrak{gl}(V))$$

because G is simply connected. In particular, there is a Lie algebra homomorphism $\rho\colon G \to \operatorname{GL}(V)$ such that $\overline{\rho} = F\rho$, as required. ■

Remark 3.29. Given any connected Lie group G with universal cover $\widetilde{G}$, one can attempt to recover the representation theory of G from $\widetilde{G}$ via the short exact sequence in Remark 1.132.

Remark 3.30. For any Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, one sees that $\operatorname{Rep}_{\mathbb{C}}(\mathfrak{g}) = \operatorname{Rep}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$. (Another perspective is that we can reduce the complex representation theory of a complex Lie algebra to a real form.) To see this, note that any morphism $\mathfrak{g} \to \mathfrak{gl}(V)$ where V is a complex vector space canonically upgrades to a map $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{gl}(V)$ by taking the tensor product with the canonical inclusion $\mathbb{C} \to \mathfrak{gl}(V)$ given by $c \mapsto c\,\mathrm{id}_V$. In the reverse direction, any representation $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{gl}(V)$ can simply forget about the $\mathbb{C}$ factor to define a representation $\mathfrak{g} \to \mathfrak{gl}(V)$. This remark does not have the space or motivation to check that we have actually defined an equivalence.

3.1.4 Decomposing Representations

With direct sums, we have notions of irreducibility.

Definition 3.31 (indecomposable)**.** Fix a representation V. Then V is *indecomposable* if and only if any direct sum decomposition $V = V_1 \oplus V_2$ must have $V_1 = 0$ or $V_2 = 0$.

Definition 3.32 (irreducible)**.** Fix a representation V. Then V is *irreducible* if and only if V is nonzero, and any subrepresentation $U \subseteq V$ has $U = 0$ or $U = V$.

Example 3.33. The standard representation V of $\operatorname{GL}(V)$ is irreducible. Indeed, any nonzero subrepresentation $U \subseteq V$ has a nonzero vector $v \in U$. But then the orbit of v under $\operatorname{GL}(V)$ is $V\setminus\{0\}$, so U must contain $V\setminus\{0\}$, so $U = V$.

It will also turn out that $\operatorname{Sym}^k(V)$ and $\operatorname{Alt}^k(V)$ are irreducible representations of $\operatorname{GL}(V)$, but this is not so obvious. We will be able to show this with more ease later in the course.

These notions are related but not the same.

Remark 3.34. Any irreducible representation V is indecomposable. Indeed, writing $V = V_1 \oplus V_2$ has $V_1 \subseteq V$, so $V_1 = 0$ or $V_1 = V$.

Example 3.35. Consider the representation $\rho\colon \mathbb{C} \to \mathrm{GL}_2(\mathbb{C})$ given by

$$\rho(x) := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Then $\operatorname{span}\{e_1\}$ is a nontrivial proper subrepresentation of ρ because $\rho(e_1) = e_1$; thus, ρ fails to be irreducible.

However, ρ is indecomposable! Indeed, our vector space is two-dimensional, so a nontrivial decomposition of ρ into $\rho_1 \oplus \rho_2$ must have underlying vector spaces V_1 and V_2 with dimensions $\dim V_1 = \dim V_2 = 1$. But the action of $\mathbb{C}$ on $\mathbb{C}$ must be linear, so V_1 and V_2 must be eigenspaces. As such, we can see from the definition of ρ that all eigenvalues are 1, so ρ_1 and ρ_2 would have to be the trivial representation, meaning that ρ would have to be the sum of trivial representations and hence trivial, which is false because $\rho(1) \neq \mathrm{id}$.

Do note that there is something that we can always do for our decomposition, but it is not always as satisfying as a direct sum.

Remark 3.36 (Jordan–Holder)**.** For any representation V, one can always find a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1} \subseteq V_k = V$$

where each quotient V_i/V_{i-1} is irreducible. Indeed, we can proceed by induction on $\dim V$. As a base case, $\dim V = 1$ has nothing to do because V is irreducible for dimension reasons.

If V is already irreducible, then our filtration is $0 \subseteq V$. Otherwise, V is not irreducible, so we can find a nontrivial proper subrepresentation $V' \subseteq V$; choosing a minimal such representation (by dimension) must have V' irreducible. Then we can apply the inductive hypothesis to V/V' (which has smaller dimension than V) to build the required filtration.

In particular, filtrations means that we would have to build representations by short exact sequences, which may be difficult to handle especially when iterated.

We would like to decompose representations into irreducible parts because dealing with filtrations is difficult.

Definition 3.37 (completely reducible)**.** A representation V is *completely reducible* if and only if it is the direct sum of irreducible representations.

Remark 3.38. Technically, we have not required that the decomposition into irreducibles is unique. This is the content of Corollary 3.48.

3.2 October 2

Today we will continue talking about representations.

3.2.1 Schur's Lemma

The following result is our first interesting result about representations.

Proposition 3.39 (Schur's lemma)**.** Fix representations V and W (over a field k) over a Lie group G or Lie algebra $\mathfrak{g}$, denoted R.

(a) If V is irreducible, then any nonzero morphism $\varphi\colon V \to W$ is injective.

(b) If W is irreducible, then any nonzero morphism $\varphi\colon V \to W$ is surjective.

(c) If V and W are both irreducible, then any nonzero morphism $\varphi\colon V \to W$ is an isomorphism.

(d) If V and W are both irreducible, then the endomorphism algebra $\operatorname{End}_R(V)$ is a finite-dimensional division algebra over k. In particular, if k is algebraically closed, then the map $k \cong \operatorname{End}_R(V)$ defined by $\lambda \mapsto \lambda \operatorname{id}_V$ is a ring isomorphism.

Proof. Here we go.

(a) Note that $\ker\varphi \subseteq V$ is a subrepresentation by Examples 3.15 and 3.16. Thus, either $\ker\varphi = 0$ in which case φ is injective, or $\ker\varphi = V$ in which case $\varphi = 0$.

(b) Note that $\operatorname{im}\varphi \subseteq V$ is a subrepresentation by Examples 3.17 and 3.18. Thus, either $\operatorname{im}\varphi = 0$ in which case $\varphi = 0$, or $\operatorname{im}\varphi = W$ in which case φ is surjective.

(c) This follows by combining the previous two parts with Remark 3.1.

(d) Note that $\operatorname{End}_R(V)$ is certainly an algebra (possibly non-commutative). Part (c) explains that all nonzero elements have inverses, so this algebra becomes a division algebra. It remains to check the claim when k is algebraically closed. In fact, we show that any morphism $\varphi\colon V \to V$ must be a scalar, which will complete the proof because it shows that the natural map

$$k \to \operatorname{End}_R(V)$$

given by $c \mapsto c \operatorname{id}_V$ is an isomorphism.[1] Note that φ will have an eigenvector v with eigenvalue λ. Then $\varphi - \lambda \operatorname{id}_V$ is a morphism with a nontrivial kernel, so it must be the zero map because it is not an isomorphism! Thus, we conclude that $\varphi = \lambda \operatorname{id}_V$ is a scalar. ■

This result (and in particular (d)) is important enough to warrant its own subsection. To explain why, here are some interesting corollaries.

Corollary 3.40. Fix an algebraically closed field k.

(a) For any injective irreducible representation $\rho\colon G \to \operatorname{GL}(V)$ of a regular Lie group G, the center of G is

$$Z(G) = \{g \in G : \rho(g) = \lambda \mathrm{id}_V \text{ for some } \lambda \in \mathbb{C}\}.$$

(b) For any injective representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ over a field k. Then the center of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} : \rho(X) = \lambda \mathrm{id}_V \text{ for some } \lambda \in k\}.$$

Proof. The point is that living in the center implies commuting with the ambient action, which Proposition 3.39 explains implies the element must be a scalar. The injectivity of the representations implies that this characterizes the center.

[1] Certainly this is a ring map, and it is injective because V is nonzero, so we are really interested in showing that this map is surjective.

(a) In one direction, if $g \in G$ has $\rho(g) = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$, then any $h \in G$ has

$$\begin{aligned}\rho\left(hgh^{-1}\right) &= \rho(h)\rho(g)\rho(h)^{-1} \\ &= \rho(h) \circ \lambda \operatorname{id}_V \circ \rho(h)^{-1} \\ &= \lambda\rho(h)\rho(h)^{-1} \\ &= \lambda \operatorname{id}_V \\ &= \rho(g),\end{aligned}$$

so injectivity of ρ implies that $hgh^{-1} = g$; thus, $g \in Z(G)$.

Conversely, suppose $g \in Z(G)$. Then $\rho(g) \colon V \to V$ is an operator on an irreducible representation of G. In fact, $\rho(g)$ commutes with the action of G: for any $h \in G$, we see that

$$\rho(g) \circ \rho(h) = \rho(gh) = \rho(hg) = \rho(h) \circ \rho(g)$$

because $g \in Z(G)$. Thus, Proposition 3.39 implies that $\rho(g) = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$.

(b) In one direction, if $X \in \mathfrak{g}$ has $\rho(X) = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$, then any $Y \in \mathfrak{g}$ has

$$\begin{aligned}\rho([X,Y]) &= \rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X) \\ &= \lambda\rho(Y) - \lambda\rho(Y) \\ &= 0,\end{aligned}$$

so the injectivity of ρ implies that $[X,Y] = 0$; thus, $X \in \mathfrak{z}(\mathfrak{g})$.

Conversely, suppose $X \in \mathfrak{z}(\mathfrak{g})$. Then $\rho(X) \colon V \to V$ is an operator on an irreducible representation of $\mathfrak{g}$ which commutes with the $\mathfrak{g}$ action: any $Y \in \mathfrak{g}$ has

$$\rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X) = \rho([X,Y]) = \rho(0) = 0.$$

Thus, Proposition 3.39 implies that $\rho(X) = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$. ■

Example 3.41. By Corollary 3.40, we see that $Z(\mathrm{GL}_n(\mathbb{F}))$ consists of scalar matrices. One can do similar computations for all the classical groups.

Example 3.42. Note that $Z(\mathfrak{sl}_n(\mathbb{F})) = 0$ for $n \geqslant 2$. (If $n = 1$, then $\mathfrak{sl}_1(\mathbb{F}) = 0$ already.) Indeed, the main point is that the standard representation $\mathfrak{sl}_n(\mathbb{F}) \subseteq \mathrm{gl}_n(\mathbb{F})$ is irreducible. Well, for any nonzero subrepresentation $V \subseteq \mathbb{F}^n$, say $v \in V \setminus \{0\}$, and we may assume that $v = e_1$ upon changing basis. Now, for any $w \in \mathbb{F}^n$, we see that there is a traceless matrix $X \in \mathfrak{sl}_n(\mathbb{F})$ such that $Xv = w$, thus proving that $w \in V$, so $W = \mathbb{R}^n$. Applying this irreducible representation to Corollary 3.40, we conclude that

$$\mathfrak{sl}_n(\mathbb{F}) = \{\lambda 1_n \in \mathfrak{sl}_n(\mathbb{F}) : \lambda \in \mathbb{F}\} = 0$$

because $\operatorname{tr} \lambda 1_n = 0$ requires $\lambda = 0$.

Corollary 3.43. Fix an abelian Lie group G or Lie algebra $\mathfrak{g}$, denoted R. Then all irreducible complex representations are one-dimensional.

Proof. Let V be an irreducible complex representation of R with structure morphism ρ. Then for any $g \in R$, we see that $\rho(g) \colon V \to V$ is an operator commuting with the action of G: for any $h \in R$, we see that

$$\rho(g) \circ \rho(h) = \rho(h) \circ \rho(g)$$

because R is abelian. (Each case with R requires a slightly different argument, but the conclusion is the same: both equal $\rho(gh) = \rho(hg)$ when $R = G$, and the difference equals $\rho([g,h]) = 0$ when $R = \mathfrak{g}$.)

Thus, Proposition 3.39 implies that $\rho(g)$ is a scalar operator $\lambda_g \operatorname{id}_V$ for each $g \in G$. In particular, for any nonzero vector $v \in V$, we see that $\rho(g)$ acts a scalar on $\operatorname{span}\{v\}$ and hence preserves this subspace. Thus, $\operatorname{span}\{v\}$ is a nonzero subrepresentation of V, forcing $V = \operatorname{span}\{v\}$ by irreducibility. ■

Let's compute the representations of some abelian groups/algebras.

Example 3.44. The complex representations of the abelian Lie algebra $\mathbb{F} = \mathfrak{gl}(\mathbb{F})$ are just arbitrary $\mathbb{C}$-vector spaces V with a chosen endomorphism by $\rho\colon \mathfrak{gl}(\mathbb{F}) \to \mathfrak{gl}(V)$. In particular, the irreducible Lie algebra representations $\rho\colon \mathfrak{gl}(\mathbb{F}) \to \mathfrak{gl}(V)$ have $V = \mathbb{C}$ by Corollary 3.43, and then we see we are just asking for a linear map $\mathbb{F} \to \mathbb{C}$.

Now, applying the equivalence of Proposition 3.28, we construct the representation $\rho_\lambda\colon \mathbb{F} \to \mathrm{GL}(\mathbb{C})$ by $\rho_\lambda(t)(v) := \exp(\lambda t v)$ (for any $\lambda \in \mathbb{C}$), and we note that $d(\rho_\lambda)_e\colon \mathbb{F} \to \mathfrak{gl}(\mathbb{C})$ is the multiplication-by-λ irreducible representation of the previous paragraph. In particular, the equivalence of categories establishes these as our irreducible representations of $\mathbb{F}$.

Example 3.45. Consider the real Lie group $G := \mathbb{R}^\times$. Note that $\mathbb{R}^\times \cong \{\pm 1\} \times \mathbb{R}^+$ by the multiplication map, and $\mathbb{R}^+ \cong \mathbb{R}$ by taking the exponential. Now, this Lie group is abelian, so all irreducible representations are one-dimensional, so we can classify irreducible representations $\rho\colon G \to \mathrm{GL}(\mathbb{C})$ as

$$\mathrm{Hom}_{\mathrm{LieGrp}}(\{\pm 1\} \times \mathbb{R}, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{LieGrp}}(\{\pm 1\}, \mathbb{C}^\times) \times \mathrm{Hom}_{\mathrm{LieGrp}}(\mathbb{R}, \mathbb{C}^\times)$$

by tracking the universal property of the product (for both manifolds and groups). Now, $\mathrm{Hom}(\{\pm 1\}, \mathbb{C})$ is just looking for elements of $\mathbb{C}^\times$ of order dividing 2, which we know are only $\{\pm 1\}$. Continuing, we note $\mathrm{Hom}_{\mathrm{LieGrp}}(\mathbb{R}, \mathbb{C}^\times)$ was classified as the maps $t \mapsto \exp(\lambda t)$ for some $\lambda \in \mathbb{C}$ in Example 3.44 (because such representations must be irreducible by virtue of being one-dimensional). As such, we see that $\mathrm{Hom}_{\mathrm{LieGrp}}(\mathbb{R}^\times, \mathbb{C}^\times)$ consists of the maps $t \mapsto \operatorname{sgn}(t)^\varepsilon\, |t|^\lambda$ for some $\varepsilon \in \{0,1\}$ and $\lambda \in \mathbb{C}$.

Example 3.46. Consider the real Lie group S^1 equipped with the projection $\pi\colon \mathbb{R} \to S^1$ given by $\pi(t) := e^{2\pi i t}$. Then π is a smooth surjection with kernel $\ker \pi = \mathbb{Z}$. Thus, a representation $\rho\colon S^1 \to \mathrm{GL}(\mathbb{C})$ (as usual, all irreducible representations are 1-dimensional by Corollary 3.43) induces a representation $\widetilde{\rho}\colon \mathbb{R} \to \mathrm{GL}(\mathbb{C})$ as $\widetilde{\rho} := \rho \circ \pi$. Now, Example 3.44 tells us that $\widetilde{\rho}(t) = \exp(t\lambda) \in \mathbb{C}^\times$ for some $\lambda \in \mathbb{C}$. However, $\widetilde{\rho}$ must have $\ker \pi = \mathbb{Z}$ in its kernel, so $\exp(\lambda) = 1$, so $\lambda = 2\pi i n$ for some $n \in \mathbb{Z}$. Going back through π, we thus see that $\rho(z) = z^n$ for some $n \in \mathbb{Z}$, and we can check that these are all in fact polynomial (and hence smooth) representations $S^1 \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$.

One can upgrade Proposition 3.39 for arbitrary representations.

Corollary 3.47. Fix a Lie group G or Lie algebra $\mathfrak{g}$, denoted R. Fix complex completely reducible representations

$$V = \bigoplus_{i=1}^n V_i^{\oplus m_i} \qquad \text{and } W \cong \bigoplus_{i=1}^n V_i^{\oplus n_i},$$

where the set $\{V_i\}_{i=1}^n$ consists of pairwise non-isomorphic complex irreducible representations. Then

$$\mathrm{Hom}_R(V, W) = \bigoplus_{i=1}^n \mathbb{C}^{n_i \times m_i}.$$

Proof. Finite sums move outside Hom by the universal properties involved, so

$$\begin{aligned}\operatorname{Hom}_R(V,W) &= \operatorname{Hom}_R\left(\bigoplus_{i=1}^n V_i^{\oplus m_i}, \bigoplus_{i=1}^n V_i^{\oplus n_i}\right) \\ &= \bigoplus_{i,j=1}^n \operatorname{Hom}_R\left(V_i^{\oplus n_i}, V_j^{\oplus m_i}\right) \\ &= \bigoplus_{i,j=1}^n \operatorname{Hom}_R(V_i,V_j)^{n_i\times m_i}.\end{aligned}$$

Now, Proposition 3.39 tells us that $\operatorname{Hom}_R(V_i,V_j) = \mathbb{C}1_{i=j}$, so the result follows. ■

Corollary 3.48. Fix a Lie group G or Lie algebra $\mathfrak{g}$. If V is a completely reducible complex representation, then V has a unique decomposition into irreducibles up to isomorphism and permutation of the factors.

Proof. Because V is finite-dimensional, any two decompositions of V into irreducibles will end up using only finitely many irreducible components, which we can list out as $\{V_1,\ldots,V_n\}$. Then we are given two decompositions

$$\bigoplus_{i=1}^n V_i^{\oplus m_i} \cong V \cong \bigoplus_{i=1}^n V_i^{\oplus n_i}$$

for nonnegative integers $m_\bullet$s and $n_\bullet$s. We want to check that $m_i = n_i$ for each i. Well, for each i, Corollary 3.47 implies that

$$\dim \operatorname{Hom}_G(V_i,V) = \dim \operatorname{Hom}_G\left(V_i, \bigoplus_{i=1}^n V_i^{\oplus m_i}\right) = m_i$$

and similarly $\dim\operatorname{Hom}_G(V_i,V) = n_i$, so $m_i = n_i$ follows. ■

3.2.2 The Unitarization Trick

We would like tools to show that all representations are completely reducible. One place to start is with unitary representations.

Definition 3.49. Fix a Lie group G or Lie algebra $\mathfrak{g}$, denoted R. Then a representation V of R is *unitary* if and only if it has a positive-definite Hermitian inner product $\langle\cdot,\cdot\rangle$ commuting with the R-action. More precisely, we have the following.

- If R is a Lie group, then we want $\langle gv, gw\rangle = \langle v,w\rangle$ for all $g \in G$ and $v,w\in V$.
- If R is a Lie algebra, then we want $\langle Xv,w\rangle + \langle v,Xw\rangle = 0$ for all $X\in\mathfrak{g}$ and $v,w\in V$.

Here's a quick coherence check for the definition.

Lemma 3.50. Fix a Lie group G with Lie algebra $\mathfrak{g}$, and let $\rho\colon G\to\operatorname{GL}(V)$ be a complex representation inducing a representation $d\rho_e\colon\mathfrak{g}\to\mathfrak{gl}(V)$. Suppose that V has an $\mathbb{R}$-bilinear product $\langle -,-\rangle\colon V\times V\to\mathbb{C}$ (possibly Hermitian).

(a) For any $X\in\mathfrak{g}$, we have

$$\frac{d}{dt}\langle \exp(tX)v, \exp(tX)w\rangle = \langle Xv,w\rangle + \langle v,Xw\rangle.$$

(b) If ρ is unitary, then $d\rho_e$ is unitary.

Proof. Here we go.

(a) This is essentially the product rule. In V, we compute

$$\begin{aligned}\frac{d}{dt}\langle\exp(tX)v,\exp(tX)w\rangle &= \lim_{t\to 0}\frac{\langle\exp(tX)v,\exp(tX)w\rangle-\langle v,w\rangle}{t}\\ &= \lim_{t\to 0}\frac{\langle\exp(tX)v,\exp(tX)w\rangle-\langle\exp(tX)v,w\rangle}{t}+\lim_{t\to 0}\frac{\langle v,w\rangle-\langle\exp(tX)v,w\rangle}{t}\\ &= \lim_{t\to 0}\left\langle\exp(tX)v,\frac{\exp(tX)w-w}{t}\right\rangle+\lim_{t\to 0}\left\langle\frac{\exp(tX)v-v}{t},w\right\rangle.\end{aligned}$$

Now, linearity of the bilinear product implies its continuity, so we can bring the limit inside the bilinear products. Doing so and using Remark 3.3 shows that these limits evaluate to $\langle Xv,w\rangle+\langle v,Xw\rangle$.

(b) If ρ is unitary, then $\langle\exp(tX)v,\exp(tX)w\rangle=\langle v,w\rangle$ always, so the derivative in (a) always vanishes, so $\langle Xv,w\rangle+\langle v,Xw\rangle=0$ always. ■

We take a moment to acknowledge that, as usual, inner products have applications to duality.

Lemma 3.51. Fix a complex vector space V, and recall that we can define a complex vector space $\overline{V}$ as having the conjugate action. If V has Hermitian inner product $\langle-,-\rangle$, then the map $\overline{V}\to\overline{V^\vee}$ given by $v\mapsto\langle-,v\rangle$ is an isomorphism of vector spaces. If V is a unitary representation over a real Lie group or algebra, then this map is also an isomorphism of representations.

Proof. Well, for any $v\in V$, we see that $\langle-,v\rangle$ is a linear operator $V\to\mathbb{C}$ because $\langle-,-\rangle$ is Hermitian. In fact, this gives an $\mathbb{R}$-linear map $i_\bullet\colon\overline{V}\to V^\vee$ defined by $i_v:=\langle-,v\rangle_\bullet$, and it has trivial kernel because v nonzero implies that $\langle v,v\rangle>0$. Thus, our linear map $V\to V^\vee$ is a vector space isomorphism in light of the fact that $\dim V=\dim V^\vee$. Lastly, we note that this upgrades to an isomorphism of $\mathbb{C}$-vector spaces because

$$\langle-,a\overline{v}\rangle=\overline{a}\langle-,v\rangle.$$

Now, if V is a unitary representation, we need to check that this isomorphism is invariant.

- If V is a representation of a group G, then we note that

$$i_{gv}(w)=\langle gv,w\rangle=\langle v,g^{-1}w\rangle=(gi_v)(w).$$

- If V is a representation of a Lie algebra $\mathfrak{g}$, then we note that

$$i_{Xv}(w)=\langle Xv,w\rangle=-\langle v,Xw\rangle=(Xi_v)(w).$$

The above checks complete the proof. ■

As another coherence check, we note that the choice of invariant product is more or less unique.

Proposition 3.52. Fix a complex irreducible representation V of a Lie group G. Then there is at most one invariant Hermitian form on V, up to a positive scalar.

Proof. Fixing some invariant Hermitian forms $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ of a representation V, then we claim that we can find a function $\varphi\colon V\to V$ such that

$$\langle v,w\rangle_1=\langle\varphi(v),w\rangle_2$$

for all $v,w\in V$. Indeed, φ is simply the composite of the representation isomorphisms $\overline{V}\cong V^\vee\cong\overline{V}$ provided by Lemma 3.51.

In particular, taking conjugates, we see that φ is an automorphism of an irreducible representation, so Proposition 3.39 implies that $\varphi = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$. In particular, we conclude that

$$\langle -, - \rangle_1 = \lambda \langle -, - \rangle_2.$$

It remains to check that λ is a positive real number. Well, choose a nonzero vector $v \in V$, and then we see that $\lambda = \langle v, v \rangle_1 / \langle v_v \rangle_2 > 0$. ■

Anyway, here is the reason for defining the notion of unitary.

Proposition 3.53. Let V be a unitary representation of a Lie group G or Lie algebra $\mathfrak{g}$, denoted R. If $W \subseteq V$ is a subrepresentation, then so

$$W^\perp := \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

In fact, $V = W \oplus W^\perp$ as representations.

Proof. We run our checks in sequence.

- We claim that $W^\perp \subseteq V$ is a subrepresentation. Well, for each $v \in V$, we note that $w \mapsto \langle w, v \rangle$ is a linear map $V \to \mathbb{C}$ because $\langle -, - \rangle$ is Hermitian, so $W^\perp = \bigcap_{w \in W} \ker \langle -, w \rangle$ is a linear subspace. To see that this is a subrepresentation, we pick up some $v \in W^\perp$, and we want to show that $gv \in W^\perp$ for any $g \in R$. To continue, we do casework on R.
  - If $R = G$, then note that
  $$\langle gv, w \rangle = \left\langle v, g^{-1} w \right\rangle = 0$$
  for all $w \in W$ because $g^{-1} w \in W$ as well.
  - If $R = \mathfrak{g}$, then set $X = g$ and note that
  $$\langle Xv, w \rangle = -\langle v, Xw \rangle = 0$$
  for all $w \in W$ because $Xw \in W$ as well.
- We claim that the summation map $W \oplus W^\perp \to V$ is an isomorphism. Because W and $W^\perp$ are both subspaces of V, we certainly have a linear summation map $W \oplus W^\perp \to V$, so it is merely a matter of checking that we have an isomorphism.
  - Trivial kernel: suppose that $(w, v) \in W \oplus W^\perp$ has $w + v = 0$. Then $w = -v$ lives in $W \cap W^\perp$. In particular, $\langle w, w \rangle = 0$, which implies $w = 0$ (and hence $v = 0$) because $\langle -, - \rangle$ is Hermitian and non-degenerate.
  - Surjective: by a dimension count, it is now enough to check that $\dim V \leqslant \dim W + \dim W^\perp$.[2] Well, the presence of an inner product allows us to begin with an orthonormal basis $\{e_1, \ldots, e_k\}$ of W and then extend it to an orthonormal basis $\{e_{k+1}, \ldots, e_n\}$ of V. However, the condition of being orthonormal implies that $\{e_{k+1}, \ldots, e_n\} \subseteq W^\perp$, so this orthonormal subset provides a lower bound
  $$\dim W^\perp \geqslant n - k = \dim V - \dim W,$$
  as required. ■

Corollary 3.54. Let V be a unitary representation of a Lie group G or Lie algebra $\mathfrak{g}$. Then V is completely reducible.

[2] Explicitly, the image of $W + W^\perp \subseteq V$ has dimension $\dim W + \dim W'$ because the summation map already has trivial kernel by the previous point.

Proof. We induct on $\dim V$ by using Proposition 3.53. If $\dim V = 0$, then $V = 0$, so V is the empty sum of irreducible representations. Otherwise, for our inductive step, take $\dim V > 0$. If V is irreducible already, then there is nothing to do. Otherwise, let $W \subseteq V$ be a proper nontrivial subrepresentation of V, and then Proposition 3.53 implies that $V \cong W \oplus W^\perp$. Now, $0 < \dim W, \dim W^\perp < \dim V$, so W and $W^\perp$ are unitary representations with strictly smaller dimension than V, so W and $W^\perp$ are completely irreducible, so V is also completely irreducible (by taking the sum of the decompositions for W and $W^\perp$). ■

Example 3.55. Let $\mathfrak{g}$ be an abelian complex Lie algebra. Then the adjoint representation $\operatorname{ad}_\bullet \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is unitary no matter what Hermitian inner product $\langle -, - \rangle$ we give $\mathfrak{g}$: indeed, we see that

$$\langle \operatorname{ad}_X Y, Z\rangle + \langle Y, \operatorname{ad}_X Z\rangle = \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0 + 0 = 0$$

by Proposition 2.23. Thus, the adjoint representation is completely reducible by Corollary 3.54.

3.2.3 Compact Lie Groups

The main application of Corollary 3.54 is to compact groups. To explain this, we need some notion of an integration theory. Fix a regular Lie group G of dimension n with Lie algebra $\mathfrak{g}$. Remark 1.140 provides a parallelization of $TG \cong G \times \mathfrak{g}$ by right-invariant vector fields. Choosing a right-invariant global frame $\{\xi_1, \ldots, \xi_n\}$ of TG, we define

$$\omega \coloneqq \xi_1 \wedge \cdots \wedge \xi_n$$

to be a right-invariant top-degree differential form in $\Omega^n G = \wedge^n TG$. Then differential topology explains how to integrate regular compactly supported functions $f \colon G \to \mathbb{F}$ against ω. In particular, tracking through all the definitions, one finds that

$$\int_G (R_g f)\,\omega = \int_G f\,\omega$$

for any $g \in G$ and $f \colon G \to \mathbb{F}$. Indeed, integration is linear, so we may assume that f is supported in a single chart (U, φ) of G. Letting the coordinates be $\varphi = (x_1, \ldots, x_n)$, we see that $\omega = r(x)\,dx_1 \wedge \cdots \wedge dx_n$ for some regular function $r \colon U \to \mathbb{F}$. Then

$$\int_G f\,\omega = \int_U f(x) r(x)\,dx_1 \wedge \cdots \wedge dx_n.$$

However, the G-invariance of ω implies that we can translate everything by g to get the same value of the integral, which is the desired conclusion.

The point of having an integration theory is that we are able to take "averages." The following is our main application.

Proposition 3.56. Fix a complex representation V of a compact Lie group G. Then there is an invariant Hermitian inner product $\langle -, - \rangle$ on V. Thus, Corollary 3.54 implies that representations of a compact Lie group are completely reducible.

Proof. Begin with any Hermitian inner product $\langle -, - \rangle_0$ on V. Then we define $\langle -, - \rangle \colon V \times V \to \mathbb{C}$ by

$$\langle v, w\rangle \coloneqq \int_G \langle gv, gw\rangle_0\,\omega,$$

where ω is a right-invariant top differential form on G, scaled so that $\int_G \omega = 1$.[3] Note that G being compact implies that the integral certainly converges; notably, the function $g \mapsto \langle gv, gw\rangle$ is smooth because $\langle -, - \rangle$ is bilinear, and the representation is regular.

We now claim that $\langle -, - \rangle$ is the required invariant Hermitian inner product.

[3] Because G is compact, we can cover it in finitely many charts to conclude that $\int_G \omega$ is finite, and then we can scale ω by this integral to conclude that we can choose ω so that $\int_G \omega = 1$.

- Conjugate-symmetric: for any $v, w \in V$, we note that

$$\langle w, v\rangle = \int_G \langle gw, gv\rangle_0\, \omega = \overline{\int_G \langle gv, gw\rangle_0\, \omega} = \overline{\langle v, w\rangle}.$$

- Linear: for any $v, v', w \in V$ and $a, a' \in \mathbb{C}$, we note that

$$\begin{aligned}\langle av + a'v', w\rangle &= \int_G \langle g(av + a'v'), gw\rangle_0\, \omega \\ &= a\int_G \langle gv, gw\rangle_0\, \omega + a'\int_G \langle gv', gw\rangle_0\, \omega \\ &= a\langle v, w\rangle + a'\langle v', w\rangle.\end{aligned}$$

- Non-degenerate: for any $v \in V$, we note that the function $G \to \mathbb{C}$ given by $g \mapsto \langle gv, gv\rangle_0$ is a function which is always positive because $\langle -, -\rangle_0$ is Hermitian. Because G is compact, this function must have a minimum value $m_v > 0$, so we conclude that

$$\langle v, v\rangle = \int_G \langle gv, gv\rangle\, \omega \geqslant m_v > 0.$$

- Invariant: for any $v \in V$ and $h \in G$, we note that

$$\langle hv, hv\rangle = \int_G \langle ghv, ghv\rangle\, \omega \stackrel{*}{=} \int_G \langle gv, gv\rangle\, \omega = \langle v, v\rangle,$$

  where $\stackrel{*}{=}$ holds because ω is right-invariant.

The above checks complete the proof. ■

Here is an interesting example.

Example 3.57. The group SU_n is a compact real Lie group, so all its representations are completely reducible by Proposition 3.56. In fact, it is simply connected, so $\mathrm{Rep}_{\mathbb{C}}(\mathrm{SU}_n) = \mathrm{Rep}_{\mathbb{C}}(\mathfrak{su}_n)$ by Proposition 3.28. However, $\mathfrak{su}_n$ is also a real form of $\mathfrak{sl}_n(\mathbb{C})$ by Example 2.92, so we can use Remark 3.30 to note that

$$\mathrm{Rep}_{\mathbb{C}}(\mathfrak{su}_n) = \mathrm{Rep}_{\mathbb{C}}(\mathfrak{su}_n \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Rep}_{\mathbb{C}}(\mathfrak{sl}_n(\mathbb{C})) = \mathrm{Rep}_{\mathbb{C}}(\mathfrak{sl}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Rep}_{\mathbb{C}}(\mathrm{SL}_n(\mathbb{R})).$$

Thus, the complex representations of $\mathrm{SL}_n(\mathbb{R})$ are also completely reducible!

3.3 October 7

Today we finish classifying the representations of $\mathfrak{sl}_2$.

3.3.1

Recall that the Casimir operator $C := ef + fe + \frac{1}{2}h^2$ will commute with the action of sl_2 in any representation. We are using C to show that any representation is sum of the V_ns. By an induction, we may assume that we are in the situation where we have an extension

$$0 \to V_n \to W \to V_n^{\oplus(m-1)} \to 0,$$

and we need to check that $W \cong V_n^{\oplus m}$. For this, we note that we can split W as

$$W = \bigoplus_{\lambda \in \mathbb{C}} W(\lambda),$$

where $W(\lambda)$ is the generalized eigenspace for λ. However, one checks that there is a highest weight n of W, implying we can write

$$W = \bigoplus_{i=0}^{n} W(n-2i).$$

Notably, the commutator relations imply that $eW(\lambda) = W(\lambda+2)$, so $W(n) \subseteq \ker e|_W$.

We complete our classification by showing that h acts diagonally on $\ker e$. Well, for any u, we note that $f^m u = 0$ for some m large enough (by looking at weights), but the commutator relations find that

$$e^m f^m u = e^{m-1} f^{m-1} m(h-m+1)u = \cdots = m!h(h-1)\cdots(h-(m+1))u.$$

Thus, the polynomial $h(h-1)\cdots(h-(m+1))$ acts by 0 on $\ker e$, so the minimal polynomial of h has no repeated roots, so h acts diagonally.

To apply the fact that h acts diagonally on $W(n) \subseteq \ker e$, we note that $W(n-2i)$ is simply $W(n)$ shifted over by f some number of times, so we complete. Here are some corollaries.

Corollary 3.58. The element h acts diagonally on any finite-dimensional complex representation of $\mathfrak{sl}_2$.

Corollary 3.59. Let $N\colon V \to V$ be a nilpotent operator on a finite-dimensional vector space V. Then there is a unique (up to isomorphism) way to make V into a representation of $\mathfrak{sl}_2$ so that $N = e|_V$.

Proof. Put N into Jordan normal form. Then the Jordan blocks communicate how e should operate. The commutator relations imply how the rest of $\mathfrak{sl}_2$ should operate. ■

We would also like to understand tensor products.

Definition 3.60. Fix a finite-dimensional complex representation V of $\mathfrak{sl}_2$. Then the *character* of V is

$$\chi_V(z) := \sum_{m\in\mathbb{Z}} \dim V(m) z^m.$$

Remark 3.61. This relates to the usual character by noting that $\chi_V(e^t) = \operatorname{tr}_V(\exp(th))$.

One can check that $\chi_{V\oplus W} = \chi_V + \chi_W$ and $\chi_{V\otimes W} = \chi_V \chi_W$ by a direct computation with the eigenspaces. One can also compute that

$$\chi_n(z) := \chi_{V_n}(z) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}.$$

Thus, we see that these characters are linearly independent (say, over $\mathbb{Q}$), so we can recover any representation by writing it as a sum of the characters χ_{V_n}. (One can algorithmically remove highest-order terms in order to get our decomposition.)

This character theory allows us to prove the following result.

Theorem 3.62. One has

$$V_m \otimes V_n = \bigoplus_{0\leqslant i\leqslant \min\{m,n\}} = V_{|m-n|+2i}.$$

Proof. Decompose the characters. ■

Example 3.63. One can compute directly that

$$\chi_2\chi_3 = \chi_5 + \chi_3 + \chi_1.$$

Example 3.64. One finds that $\chi_n^2 = \chi_0 + \cdots + \chi_{2n}$.

3.3.2 The Universal Enveloping Algebra

We have shown that the category $\operatorname{Rep}_{\mathbb{C}} \mathfrak{g}$ is abelian for any Lie algebra $\mathfrak{g}$, so we may expect to be able to realize this category as a category of modules over some (possibly non-commutative) ring. This is the role of the universal enveloping algebra.

To start, we would have the universal algebra

$$T\mathfrak{g} := \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k},$$

which is some graded ring, where the multiplication $\mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{\otimes \ell} \to \mathfrak{g}^{\otimes(k+\ell)}$ is given by concatenation. We now take a quotient to remember the Lie bracket.

Definition 3.65 (universal enveloping algebra)**.** We define the *universal enveloping algebra* $U\mathfrak{g}$ as the quotient of $T\mathfrak{g}$ by the two-sided ideal $I\mathfrak{g}$ generated by the elements

$$x \otimes y - y \otimes x - [x, y].$$

Here are some properties of this definition.

Proposition 3.66. Fix a Lie algebra $\mathfrak{g}$.

(a) For any associative algebra A, we have

$$\operatorname{Hom}_{\mathrm{LieAlg}}(\mathfrak{g}, A) \simeq \operatorname{Hom}_{\mathrm{Alg}}(U\mathfrak{g}, A).$$

(b) The category $\operatorname{Rep}_k \mathfrak{g}$ is equivalent to the category $\operatorname{Mod}(U\mathfrak{g})$.

In order to actually compute $U\mathfrak{g}$, one can fix a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ and then note that $T\mathfrak{g}$ will be the free polynomial ring in these variables, so we can take a quotient to recover $U\mathfrak{g}$.

Example 3.67. If $\mathfrak{g}$ is abelian, then we see that $U\mathfrak{g} = k[x_1, \ldots, x_n]$.

Example 3.68. We see that

$$U(\mathfrak{sl}_2) = \frac{\mathbb{C}\langle e, f, h\rangle}{(ef - fe - h, he - eh - 2e, hf - fh + 2f)}.$$

As we had with our Lie algebra, we note that $U\mathfrak{g}$ can act on itself in some interesting ways. For example, it has the usual left multiplication structures, but there is also an adjoint action.

Lemma 3.69. The action of $\mathfrak{g}$ on $T\mathfrak{g}$ by derivations descends to $U\mathfrak{g}$.

Proof. For any $Z \in \mathfrak{g}$, we want to check that $\operatorname{ad}_Z(I) \subseteq I$. Well, it is enough to check this on generators of I, for which we compute

$$\operatorname{ad}_Z(X \otimes Y - Y \otimes X - [X,Y]) = ([Z,X] \otimes Y - Y \otimes [Z,X]) + (X \otimes [Z,Y] - [Z,Y] \otimes X) - [Z,[X,Y]],$$

and we are done after applying the Jacobi identity and rearranging. ■

3.4 October 9

Today we continue discussing the universal enveloping algebra.

3.4.1 More on the Universal Enveloping Algebra

Last time we showed that $\mathfrak{g}$ acts on $U\mathfrak{g}$ by derivations. In fact, for any $Z \in \mathfrak{g}$, one can check that

$$\operatorname{ad}_Z(a) = aZ - Za$$

for all $a \in U\mathfrak{g}$.

Remark 3.70. One can also have $\mathfrak{g}$ act on the symmetric algebra $S\mathfrak{g}$, which is the quotient of $T\mathfrak{g}$ by the two-sided ideal generated by the elements $x \otimes y - y \otimes x$. The point is that we have produced some interesting infinite-dimensional representations.

Note that there is a natural filtration on $T\mathfrak{g}$ which can sometimes "temper" our infinite-dimensional representations. Here is the definition.

Definition 3.71 (filtered)**.** An algebra A is *filtered* if and only if there exists a filtration

$$0 = \mathcal{F}_0 A \subseteq \mathcal{F}_1 A \subseteq \cdots \subseteq$$

such that $A = \bigcup_{i \geqslant 0} \mathcal{F}_i A$, and the multiplication in A sends $\mathcal{F}_i A \otimes \mathcal{F}_j A \to \mathcal{F}_{i+j} A$.

A filtration permits us to pay close attention to the quotients.

Definition 3.72 (associated graded algebra)**.** Fix a filtered algebra A with filtration $\{\mathcal{F}_i A\}_{i \geqslant 0}$. Then the *associated graded algebra* is

$$\operatorname{gr} A := \bigoplus_{i \geqslant 1} \frac{\mathcal{F}_i A}{\mathcal{F}_{i+1} A}.$$

Remark 3.73. One can check that $\operatorname{gr} A$ is a graded algebra. Conversely, given a graded algebra A with grading $A = \bigoplus_{i=1}^{\infty} A_i$, one can produce a filtered algebra with the filtration

$$\mathcal{F}_i A := \bigoplus_{j \leqslant i} A_j.$$

Remark 3.74. Fix a filtered algebra A. If $\operatorname{gr} A$ has no zero divisors, then one can check that A also has no zero divisors.

Now, the point is that $T\mathfrak{g} = \bigoplus_{i \geqslant 0} \mathfrak{g}^{\otimes i}$ is graded, but we cannot expect $U\mathfrak{g}$ to be graded because the relation $x \otimes y - y \otimes x = [x,y]$ is a relation between the second and first degree! But we can still pass to the filtration: let $\mathcal{F}_i U\mathfrak{g}$ simply be the image of the natural grading $\mathcal{F}_i T\mathfrak{g}$ under the surjection $T\mathfrak{g} \twoheadrightarrow U\mathfrak{g}$.

Remark 3.75. In contrast, one can check that $S\mathfrak{g}$ is not just filtered but graded!

The definition of the filtration quickly implies that $\mathcal{F}_i U\mathfrak{g}\cdot\mathcal{F}_j U\mathfrak{g} \subseteq \mathcal{F}_{i+j} U\mathfrak{g}$ and thus $[\mathcal{F}_i U\mathfrak{g}, \mathcal{F}_j U\mathfrak{g}] \subseteq \mathcal{F}_{i+j-1} U\mathfrak{g}$. One can also see that $\operatorname{gr} U\mathfrak{g}$ is then commutative, and there is a canonical map $S\mathfrak{g} \to \operatorname{gr} U\mathfrak{g}$.

Theorem 3.76 (Poincaré–Birkoff–Witt)**.** Fix a Lie algebra $\mathfrak{g}$. Then the canonical map $S\mathfrak{g} \to \operatorname{gr} U\mathfrak{g}$ is an isomorphism.

Remark 3.77. Intuitively, one can say that ordered monomials are linearly independent in $U\mathfrak{g}$.

This is an important theorem, but we will not prove it because it is somewhat tedious.

The remarks above tell us that the map is surjective and well-defined, so the main content is in the injectivity of the map. Here are some corollaries.

Corollary 3.78. The map $\mathfrak{g} \to U\mathfrak{g}$ is injective.

Remark 3.79. In fact, defining $U\mathfrak{g}$ for an arbitrary bilinear map $[-,-]\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$. It turns out that the induced map $\mathfrak{g} \to U\mathfrak{g}$ will succeed at being injective if and only if $[-,-]$ is skew-symmetric and satisfies a Jacobi identity. For example, we need

$$x \otimes x - x \otimes x = [x, x]$$

to vanish.

Corollary 3.80. The algebra $U\mathfrak{g}$ has no zero divisors.

One may a map directly between $U\mathfrak{g}$ and $S\mathfrak{g}$ instead of passing to gr.

Proposition 3.81. The map $\operatorname{sym}\colon S\mathfrak{g} \to U\mathfrak{g}$ defined by

$$\operatorname{sym}(x_1 \cdots x_p) := \frac{1}{p!} \sum_{\omega \in S_p} X_{\sigma(1)} \cdots x_{\sigma(p)}$$

is an isomorphism of $\mathfrak{g}$-modules (via the adjoint action).

Remark 3.82. As a consequence, we see that the invariant subspace of $U\mathfrak{g}$ by the adjoint action is the same as for $S\mathfrak{g}$, which of course is $Z(U\mathfrak{g})$. For example, if G is connected with $\mathfrak{g} = \operatorname{Lie} G$, then one sees that these are also the G-invariants.

Example 3.83. One can check that the Casimir element $C = ef + fe + \frac{1}{2}h^2$ lives in the center of $Z(\mathfrak{sl}_2)$. In fact, one can show that $Z(\mathfrak{sl}_2)$ is generated by C.

3.4.2 Ideals and Commutants

One can check that sums of ideals are ideals. Also, one sees that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the maximal abelian quotient: if $I \subseteq \mathfrak{g}$ is an ideal with $\mathfrak{g}/I$ abelian, then we must have $[\mathfrak{g},\mathfrak{g}] \subseteq I$.

Example 3.84. One can check that $[\mathfrak{gl}_n, \mathfrak{gl}_n] \subseteq \mathfrak{sl}_n$ because the trace of $XY - YX$ is zero for any $X, Y \in \mathfrak{gl}_n$. In fact, this is an equality, which one can check by hand.

These commutants provide a derived series: we define $\{D^i\mathfrak{g}\}_{i\geqslant 0}$ inductively by $D^0\mathfrak{g} := \mathfrak{g}$ and

$$D^{i+1}\mathfrak{g} := \left[D^i\mathfrak{g}, D^i\mathfrak{g}\right]$$

for all $i \geqslant 0$. This derived series plays the role of derived series in group theory. For example, one can use this to define solvability.

Proposition 3.85. Fix a Lie algebra $\mathfrak{g}$. Then the following are equivalent.

(a) $D^n\mathfrak{g} = 0$ for n sufficiently large.

(b) There exists a sequence of subalgebras

$$\mathfrak{g} = \mathfrak{a}^0 \supseteq \mathfrak{a}^1 \supseteq \cdots \supseteq \mathfrak{a}^k = 0$$

such that $\mathfrak{a}^{i+1}$ is an ideal in $\mathfrak{a}^i$ with abelian quotient.

(c) For every n sufficiently large and sequence of elements $\{x_1, \ldots, x_{2^n}\} \subseteq \mathfrak{g}$, the n-fold commutator

$$[\cdots[[x_1, x_2], [x_3, x_4]], \cdots]$$

vanishes.

Proof. The equivalence of (a) and (c) has no content. Note that (a) implies (b) because one may take $\mathfrak{a}^i = D^i\mathfrak{g}$. One achieves (b) implies (a) by showing that $\mathfrak{a}^i \supseteq D^i\mathfrak{g}$ inductively. ■

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List of Definitions