250B: Commutative Algebra
For the Morbidly Curious

Nir Elber

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Contents

THEME 1

INTRODUCTION TO DIMENSION

In this sense the algebraic geometers have never left paradise: There is no snake (that is, Peano curve) in the garden.

—David Eisenbud

1.1 April 5

Welcome back.

1.1.1 Regular Rings

Today we are mostly talking about regular local rings of dimension 1. Concretely, we have a ring R with a unique maximal ideal $\mathfrak{m}$ (by being local), and because R is regular, we have

$$\mathfrak{m} = (\pi)$$

for some $\pi \in R$. As we showed last time, all regular local rings are domains, but in fact we now claim that regular local rings of dimension 1 are principal ideal domains. In fact, we have stronger.

Proposition 1.1. Fix a Noetherian, regular, local ring R of dimension 1, with maximal ideal (π). Then all nonzero ideals of R are of the form $\left(\pi^k\right)$ for some natural number k.

Proof. Set $K(R)$ to be the quotient field of R. We show that any nonzero element of the quotient field $r \in K(R)^\times$ can be written as $u\pi^n$ for some $u \in R \setminus (\pi)$ and $n \in \mathbb{Z}$. This will be enough to finish the proof because we more or less have a very good unique prime factorization.

We start by taking $r \in K(R)$. By the Krull intersection theorem, we see that

$$\bigcap_{n \geq 1} (\pi)^n = (0),$$

so because $r \neq 0$, we deduce that r must live in $(\pi^n) \setminus \left(pi^{n+1}\right)$ for some n. Thus, setting $r = u\pi^n$, we see that $u \in R \setminus (\pi)$, so u is a unit, as needed.

For general elements $\frac{r}{s} \in K(R)$, we simply take the quotient of the representations of r and s to finish. This finishes the proof. ■

These rings are actually called discrete valuation rings. Let's explain this terminology.

Definition 1.2. A group Γ endowed with a total order $\geq$ is a *totally ordered group* if and only if the set

$$\{\gamma : \gamma \geq 0\}$$

is closed under the operation of Γ, and $\gamma_1 \geq \gamma_2$ is implied by $\gamma_1\gamma_2^{-1} \geq 0$.

Example 1.3. The rings $\mathbb{Z}$ and $\mathbb{R}$ are ordered groups.

It happens that the discrete, countable totally ordered groups are all $\mathbb{Z}$.

Definition 1.4 (Valuation)**.** Fix a domain R and totally ordered group Γ. A *valuation* ν is a group homomorphism $\nu : K(R)^\times \to \Gamma$ satisfying the following.

- We have $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.
- We have $R = \{x \in K(R) : \nu(x) \geq 0\}$.

Definition 1.5 (Discrete valuation ring)**.** A *valuation ring* is an integral domain R equipped with a valuation $\nu : K(R)^\times \to \mathbb{Z}$.

And here are our examples.

Example 1.6. Fix a regular, local ring R of dimension 1. Then we described in the proof of Proposition 1.1 a way to write elements $x \in K(R)^\times$ in the form $u\pi^n$ where $n \in \mathbb{Z}$. Defining $\nu(x) := n$ equips R with a discrete valuation.

Example 1.7. The ring $\mathbb{Z}_p$ is a discrete valuation ring. Indeed, $K(\mathbb{Z}_p) = \mathbb{Q}_p$, and in the same way above we can write any $x \in \mathbb{Z}_p$ as up^n for a unit u and integer n. Setting $\nu(x) := n$ provides our valuation. In fact, the function

$$d(a,b) := p^{-\nu(a-b)}$$

turns $\mathbb{Q}_p$ into a metric space, in fact complete with respect to this metric d.

Remark 1.8 (Nir)**.** It also turns out that discrete valuation rings are regular, local rings of dimension 1 as well.

1.1.2 Normal Domains

Just for fun, let's provide a criterion to have a normal domain. To start, recall that unique factorization domains are normal already. As such, we recall the following statement.

Proposition 1.9. A Noetherian domain R is a unique factorization domain if and only if every prime $\mathfrak{p}$ minimal over some principal ideal (a) is itself principal.

The idea is to weaken this condition to give us normality. In particular, recall that a prime $\mathfrak{p}$ is associated to the ideal I if and only if $\mathfrak{p} \in \operatorname{Ass} R/I$ if and only if there exists $x \in R$ such that

$$\mathfrak{p} = \{r : rx \in I\}.$$

Notably, we this would imply that $x \notin I$ and hence $[x]_I \neq [0]_I$.

Anyway, here is our statement.

Theorem 1.10. Fix a Noetherian domain R. Then R is normal if and only if either of the following conditions hold.

(a) All primes $\mathfrak{p}$ associated to a principal ideal $(a) \subseteq R$ have $\mathfrak{p}R_\mathfrak{p} \subseteq R_\mathfrak{p}$ is a principal ideal.

(b) Every localization of R at a codimension-1 prime $\mathfrak{p}$ is a regular local ring.

Proof. The condition (a) is equivalent to $R_\mathfrak{p}$ being a discrete valuation ring: note that we already know that $R_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Now, because codimension-1 primes are the ones minimal over principal rings, we see that (a) and (b) are equivalent.

We now show the backwards direction. We pick up the following lemma.

Lemma 1.11. Fix a Noetherian domain R. Then $x \in K(R)$ has $x \in R$ if and only if $x \in R_\mathfrak{p}$ for all primes $\mathfrak{p}$ associated to some principal ideal. In other words,

$$R = \bigcap_{\mathfrak{p}} R_\mathfrak{p},$$

where $\mathfrak{p}$ varies over primes associated to a principal ideal.

Proof. Of course, $x \in R$ implies that $x \in R_\mathfrak{p}$ for each $\mathfrak{p}$.

In the reverse direction, suppose $x \notin R$ has $x = \frac{a}{b}$. Then we are given $a \notin (b)$. Now, we showed a while ago that, in an R-module M, we have $m = 0$ if and only if $\frac{m}{1} = \frac{0}{1}$ in $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$. As such, working with $M := R/(x)$, we see that $[a]_{(x)} \neq [0]_{(x)}$, so there exists a prime $\mathfrak{p}$ associated to $R/(b)$ (i.e., associated to the ideal (b)) with $a \notin (b)_\mathfrak{p}$, so $x \notin R_\mathfrak{p}$. ■

Thus, the hypothesis tells us that each $R_\mathfrak{p}$ is a discrete valuation ring and hence a principal ideal domain and hence a unique factorization domain and hence normal.[1] Thus, because the intersection of normal domains is normal, we deduce that R is normal.

We now show the forwards direction. Suppose that R is normal, and let $\mathfrak{p}$ be some prime associated to a principal ideal (a). We would like to show that $\mathfrak{p}R_\mathfrak{p}$ is principal; because $\mathfrak{p}R_\mathfrak{p}$ is still associated to $(a)R_\mathfrak{p}$, we see that we may replace R and $\mathfrak{p}$ with $R_\mathfrak{p}$ and $\mathfrak{p}R_\mathfrak{p}$ so that R is local with maximal ideal $\mathfrak{p}$.

To continue, we pick up the following definition.

Definition 1.12. A *fractional ideal* is an R-submodule of $K(R)$.

As such, we set

$$\mathfrak{p}^{-1} := \{x \in K(R) : x\mathfrak{p} \subseteq R\}.$$

Notably, $\mathfrak{p}\mathfrak{p}^{-1}$ will be contained in R, and it contains $\mathfrak{p}$. So it is either $\mathfrak{p}$ or R.

Suppose for the sake of contradiction that $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}$. Well, any $x \in \mathfrak{p}^{-1}$ is integral by the Cayley–Hamilton theorem, so $x \in R$, so we have shown $\mathfrak{p}^{-1} \subseteq R$. But this does not make sense: $\mathfrak{p}$ is associated to (a) by some element $[b]_{(a)}$, but then $b\mathfrak{p} \subseteq (a)$, so $a^{-1}b\mathfrak{p} \subseteq R$, so $a^{-1}b \in R \setminus \mathfrak{p}$.

But now $\mathfrak{p}^{-1}\mathfrak{p} = R$ shows that there exists $\frac{a}{b}$ such that $\frac{x}{y}\mathfrak{p} = R$ for some unit x, so $\frac{1}{y}\mathfrak{p} = R$, so $\mathfrak{p} = (y)$. This finishes the proof. ■

Remark 1.13. It is possible for $\mathfrak{p}$ in the proof to not be principal but still have $\mathfrak{p}R_\mathfrak{o}$ be principal.

As a corollary of the proof, we get the following results.

[1] One can show this somewhat more directly by building a monic polynomial with some $u\pi^m$ as a root and then arguing about the maximal ideal, but we won't bother.

Corollary 1.14. Fix a Noetherian domain R. If R is normal, then

$$R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}},$$

where the intersection is over all primes $\mathfrak{p}$ of codimension 1.

Corollary 1.15. Fix X an affine algebraic variety such that $A(X)$ is a normal domain. If we have a subvariety $Y \subseteq X$ is such that $A(Y)$ is of codimension at least 2, then $A(X - Y) = A(X)$.

Proof. Suppose that $\mathfrak{q}$ is the prime ideal corresponding to the variety Y. Then $A(X - Y) = A(X)_{\mathfrak{q}}$, so taking the intersection finishes. ■

1.1.3 Invertible Modules

For the following discussion, we will take R to be a Noetherian domain, for intuition. We have the following definition.

Definition 1.16 (Invertible module)**.** An R-module M is *invertible* if and only if all prime ideals $\mathfrak{p} \subseteq R$ has $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}$.

It turns out that these are all fractional ideals in the case where R is a Noetherian domain. Before that, here are some examples.

Example 1.17. A principal ideal $(f) \subseteq R$ is invertible.

Example 1.18. If M and N are invertible R-modules, then any prime $\mathfrak{p}$ will have

$$(M \otimes_R N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}} \cong R_{\mathfrak{p}},$$

so $M \otimes_R N$ is also invertible.

Example 1.19. If M is an invertible, finitely generated R-module, then $M^* := \operatorname{Hom}_R(M, R)$ is also invertible. In particular, because R is Noetherian, M is finitely presented, so

$$R_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong \operatorname{Hom}_R(M, R)_{\mathfrak{p}}.$$

To start our discussion, here is a lemma.

Lemma 1.20. Fix a Noetherian domain R. An R-module M is invertible if and only if the map

$$\mu : M^* \otimes_R M \to R$$

by $\varphi \otimes m \mapsto \varphi(m)$ is an isomorphism.

Proof. It suffices to work with the case that $\mu_{\mathfrak{p}}$ is an isomorphism for all primes $\mathfrak{p}$. By running through the isomorphisms in the examples, we see that we are asking for

$$\mu_{\mathfrak{p}} : (M_{\mathfrak{p}})^* \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \to R_{\mathfrak{p}}$$

is an isomorphism for all primes $\mathfrak{p}$.

In particular, we are allowed to assume that R is local with maximal ideal $\mathfrak{p}$. In one direction, suppose that μ is an isomorphism. By surjectivity, we are promised some

$$\mu\left(\sum_{i=1}^{n}\varphi_i\otimes a_i\right)=1.$$

In particular, there exists i such that $\varphi_i(a_i)\notin\mathfrak{p}$, but $R\setminus\mathfrak{p}$ are all units, so we can force $\varphi(a)=1$ for some φ and a. Now, living in a local ring thus forces by φ to show that

$$M\cong R\oplus\ker\varphi,$$

but $\ker\varphi$ is trivial because any kernel would have to show up in the kernel of μ, which is trivial by hypothesis.

We don't discuss the other direction. ■

Remark 1.21. We can see that M will be generated by the elements a_i in the summation

$$\mu\left(\sum_{i=1}^{n}\varphi_i\otimes a_i\right)=1.$$

Thus, M should be finitely generated.

This discussion gives us the following definition.

Definition 1.22 (Picard group)**.** Fix a Noetherian domain R. Then $\operatorname{Pic}R$ is the group of isomorphism classes of invertible R-modules.

Remark 1.23. The Picard group loosely corresponds to line bundles.

To be explicit, the group operation of $\operatorname{Pic}R$ is by

$$[X]\cdot[Y]:=[X\otimes_R Y],$$

our identity element is $[R]$, and the inverses are $[X]^{-1}:=[X^*]$.

1.1.4 The Class Group

To close out class, we discuss the connection to fractional ideals.

Lemma 1.24. Fix a Noetherian domain R. Then M is invertible if and only if M is isomorphic to some nonzero fractional ideal.

Proof. The idea is to embed M into $K(R)$ to extract our fractional ideal. Well, the embedding $R\to K(R)$ gives us an embedding

$$M\to K(R)\otimes_R M.$$

But now, $K(R)\otimes_R M\cong K(R)$ because $K(R)\otimes_R M$ is an invertible module over $K(R)$, which must be isomorphic to $K(R)$ because $K(R)$ only has the localization at the prime (0) (which does nothing).

As such, we have placed M as an R-submodule of $K(R)$ and hence is isomorphic to a nonzero fractional ideal. ■

As such, we can give an alternate characterization of the Picard group.

Lemma 1.25. Fix a Noetherian domain R. If I and J are fractional ideals, then

$$IJ \cong I \otimes_R J \qquad \text{and} \qquad I^{-1}J \cong \operatorname{Hom}(I.J).$$

Proof. The isomorphism $I \otimes_R J \cong IJ$ is by $a \otimes b \mapsto ab$. That $I^{-1}J \cong \operatorname{Hom}_R(I, J)$ follows from carefully considering the localizations. ■

Thus, modding out by principal ideals from the fractional ideals gives us the Picard group back again.

List of Definitions