256B: Algebraic Geometry

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Introduction

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

1.1 January 17

Let's just get started.

1.1.1 Course Notes

Here are some notes about the course.

- The professor is Paul Vojta, whose email is vojta@math.berkeley.edu.
- The course webpage is https://math.berkeley.edu/ṽojta/256b.html.
- The textbook is [Har77].
- We will assume algebraic geometry on the level of Math 256A, which is a prerequisite for this course.
- This course focuses on (Zariski) cohomology of schemes, so we will spend most of our time going through [Har77, Chapter III]. We will also discuss smoothness, which lives in [Har77, Chapter III] as well. Along our way, we will want to discuss some topics in [Har77, Chapter II] in more detail, such as on divisors.
- Grading will be based on homework. Homework will be weekly or biweekly, due on Wednesdays (in general).

1.1.2 Abelian Categories

We'll assume some basic category theory (monomorphisms, epimorphisms, equalizers, coequalizers, etc.). Abelian categories are somewhat complex, so we provide their definition. Roughly speaking, our end goal is to do cohomology, which arises from homological algebra, and homological algebra lives in abelian categories.

Definition 1.1 (preadditive)**.** A *preadditive category* is a category $\mathcal{C}$ where the morphism set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ forms an abelian group for any $A, B \in \mathcal{C}$, and composition distributes over addition. Explicitly, the composition map

$$\circ\colon \operatorname{Hom}_{\mathcal{C}}(B, C) \times \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{C}}(A, C)$$

is bilinear.

It follows directly from having the preadditive structure that finite products and finite coproducts are canonically isomorphic. However, these (bi)products need not exist.

Definition 1.2 (additive)**.** An *additive category* is a preadditive category admitting all finite products/coproducts.

Definition 1.3 (abelian)**.** An *abelian category* is an additive category $\mathcal{C}$ in which the following hold.

- Every morphism admits a kernel and a cokernel; here, a (co)kernel is a (co)equalizer with the zero map.
- Every monomorphism is the kernel of some morphism.
- Every epimorphism is the cokernel of some morphism.

Let's give some examples.

Example 1.4. The following are abelian categories; we omit the checks.

- The category Ab of abelian groups is abelian.
- For a ring A, the category $\operatorname{Mod}(A)$ of A-modules is abelian. In particular, for a field k, the category $\operatorname{Vec}(k)$ of k-vector spaces is abelian.

Example 1.5. Here are more abelian categories, related to sheaves. All of their "abelian" hypotheses are done by passing to stalks or a similar local argument.

- For a topological space X, the category $\operatorname{Ab}(X)$ of sheaves of abelian groups on X is abelian.
- Similarly, for a ringed space $(X, \mathcal{O}_X)$, the category $\operatorname{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules is abelian.
- For a scheme X, the category $\operatorname{QCoh}(X)$ of quasicoherent sheaves on X is abelian.
- Similarly, for a scheme X, the category $\operatorname{Coh}(X)$ of coherent sheaves on X is also abelian. Notably, we do not have infinite products here, but that's okay.

Example 1.6. For any abelian category $\mathcal{A}$, its opposite category $\mathcal{A}^{\mathrm{op}}$ is also abelian. One can see this by going through the conditions, all of which dualize.

1.1.3 Exact Functors

We will want to discuss exact functors in order to homological algebra in our abelian categories. Let's have at it.

Definition 1.7 (additive)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if and only if the map

$$F\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{D}}(FA,FB)$$

(of F acting on morphisms $A \to B$) is a group homomorphism, for any $A, B \in \mathcal{C}$. Flipping arrows and using Example 1.6 produces the same definition for contravariant functors.

Example 1.8. Fix a topological space X. Then the functor $\Gamma(X,-)\colon \operatorname{Ab}(X) \to \operatorname{Ab}$ of global sections $\mathcal{F} \mapsto \Gamma(X,\mathcal{F})$ is additive.

Remark 1.9. Being additive implies that the functor preserves biproducts. Roughly speaking, this holds because being a biproduct can be written as a set of equations for the object (and its inclusion/projection morphisms) to satisfy.

To define (left) exact for a functor, we need to define what it means to be exact.

Definition 1.10 (exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. Then a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* B if and only if $\ker g = \ker(\operatorname{coker} f)$ (up to some identification). Here, $\ker(\operatorname{coker} f)$ is intended to basically be the image.

Definition 1.11 (left exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) additive functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left-exact* if and only if a left exact sequence

$$0 \to A' \to A \to A''$$

produces a left exact sequence

$$0 \to FA' \to FA \to FA''.$$

Reversing the arrows produces the dual notion of right exactness.

Remark 1.12. Being left exact equivalently means that F preserves kernels, so by Remark 1.9 and a little category theory, F actually preserves all finite limits.

Example 1.13. The functor of global sections from Example 1.8 is left exact by [Har77, Exercise II.1.8].

To get us set up, let's approximately describe what we are trying to do. Basically, fix an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of abelian groups on a topological space X. Then there is a sequence of "cohomology" functors $\{H^i(X,-)\}_{i\in\mathbb{N}}$ with $H^0(X,-) = \Gamma(X,-)$ and a "long" exact sequence as follows

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{F}') \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{F}'') \arrow[dll] & \\ & H^1(X,\mathcal{F}') \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}'') \arrow[r] & \cdots \end{tikzcd}$$

where the maps $H^i(X,\mathcal{F}'') \to H^{i+1}(X,\mathcal{F}')$ take some work to describe.

Remark 1.14. These functors will have a number of magical properties, which will amount to the main theorems of this course. Let's give an example. Fix a projective scheme X over a field k, where $i\colon X \to \mathbb{P}_k^n$ is the promised closed embedding; let $\mathcal{I}$ be the corresponding ideal sheaf of this closed embedding. Then we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}_k^n} \to i_*\mathcal{O}_X \to 0,$$

which one can do cohomology to. In fact, one can take the tensor product of this exact sequence with the twisting sheaves $\mathcal{O}_{\mathbb{P}_k^n}(m)$; for example, we will prove that $H^1(\mathbb{P}_k^n, \mathcal{I}(m)) = 0$ for sufficiently large m, which eventually implies that the map

$$\Gamma\left(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m)\right) \to \Gamma(X, \mathcal{O}_X(m))$$

is surjective for sufficiently large m. In other words, global sections of $\mathcal{O}_X(m)$ are all restrictions of global sections of $\mathcal{O}_{\mathbb{P}_k^n}(m)$!

1.2 January 19

We'll do some homological algebra today.

1.2.1 Homological Algebra on Complexes

Homological algebra is something that comes out of understanding complexes, which we will now define.

Definition 1.15 (complex)**.** Fix an abelian category $\mathcal{A}$. A *complex* $(A^\bullet, d^\bullet)$ is a collection $\{A^i\}_{i\in\mathbb{Z}}$ together with some morphisms $d^i\colon A^i \to A^{i+1}$ such that $d^{i+1}\circ d^i = 0$. We may abbreviate the differential $d^\bullet$ from the notation.

Remark 1.16. The above definition is usually a "cocomplex." We will have no need for the dual notion of a complex in this course.

Remark 1.17. By convention, if we state that we have a complex but only define A^i for a subset of $\mathbb{Z}$, then the full bona fide complex simply sets the undefined terms to zero.

Now that we have a complex, we should define a morphism.

Definition 1.18 (complex morphism)**.** Fix an abelian category $\mathcal{A}$. Given complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$, a morphism of complexes $\varphi^\bullet\colon A^\bullet \to B^\bullet$ is a collection of morphisms $\varphi^i\colon A^i \to B^i$ making the following diagram commute for each i.

$$\begin{tikzcd}
A^i \arrow[r, "d^i"] \arrow[d, "\varphi^i"'] & A^{i+1} \arrow[d, "\varphi^{i+1}"] \\
B^i \arrow[r, "d^{i+1}"] & B^{i+1}
\end{tikzcd}$$

Unsurprisingly, our definition of morphism provides us with a category of complexes, and in fact the category of complexes is an abelian category, where the point is that biproducts, kernels, and cokernels can all be computed pointwise at each term of the complex.

We are now ready to define cohomology.

Definition 1.19 (cohomology)**.** Fix a complex $(A^\bullet, d^\bullet)$ valued in an abelian category $\mathcal{A}$. Then we define the ith *cohomology* as

$$h^i(A^\bullet) := \frac{\ker d^i}{\operatorname{im} d^{i-1}}.$$

Here, $\operatorname{im} d^{i-1}$ has an induced map to $\ker d^i$ because $d^i \circ d^{i-1} = 0$.

Remark 1.20. Quickly, recall that the image $\operatorname{im} d^{i-1}$ is in fact $\ker(\operatorname{coker} d^{i-1})$.

Remark 1.21. In fact, cohomology is functorial: a morphism $f^\bullet\colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of complexes induces a morphism $h^i(f^\bullet)\colon h^i(A^\bullet) \to h^i(B^\bullet)$ on the ith cohomology, and one can check that this makes h^i into a functor. To be explicit, this morphism is induced by the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} d_A^{i-1} \arrow[r] \arrow[d, "f^i"] & \ker d_A^i \arrow[r] \arrow[d, "f^i"] & h^i(A^\bullet) \arrow[r] \arrow[d, dashed, "f^i"] & 0 \\
0 \arrow[r] & \operatorname{im} d_B^{i-1} \arrow[r] & \ker d_B^i \arrow[r] & h^i(B^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Namely, the morphisms on the left are well-defined because $f^\bullet$ is in fact a morphism.

The main result on these cohomology groups is the following.

Proposition 1.22. Fix an abelian category $\mathcal{A}$. Given a short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of complexes in $\mathcal{A}$, there are natural maps $\delta^i\colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ producing a long exact sequence as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & h^i(A^\bullet) \arrow[r] & h^i(B^\bullet) \arrow[r] & h^i(C^\bullet) \arrow[dll, "\delta^i"] \\
h^{i+1}(A^\bullet) \arrow[r] & h^{i+1}(B^\bullet) \arrow[r] & h^{i+1}(C^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. To produce the long exact sequence, use the Snake lemma. The proof is somewhat technical, so I will refer directly to [Elb22, Theorem 4.82], though the proof there is for the dual notion of homology instead of cohomology. (Note that we can replace $\mathcal{A}$ with $\mathcal{A}^{\text{op}}$ to recover the result.) The naturality of the $\delta^\bullet$ can be checked directly from its construction. ■

We would like to measure a morphism of complexes based on what it does to cohomology: namely, two morphisms of complexes may induce the same map on cohomology despite being technically distinct. One way this might happen is by being "chain" homotopic.

Definition 1.23 (chain homotopy)**.** Fix morphisms $f^\bullet, g^\bullet\colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of the chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. A *chain homotopy* is a sequence of maps $k^i\colon A^i \to B^{i-1}$ such that

$$f^i - g^i = k^{i+1} \circ d_A^i + d_B^{i-1} \circ k^i.$$

In this case, we say that $f^\bullet$ and $g^\bullet$ are chain homotopic.

Remark 1.24. One can check directly that being chain homotopic is an equivalence relation on chain morphisms.

And here is our result.

Proposition 1.25. Fix morphisms $f^\bullet, g^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. If $f^\bullet \sim g^\bullet$, then $h^i(f^\bullet) = h^i(g^\bullet)$ for all i.

Proof. By some embedding theorem, we may as well work in $\operatorname{Mod}(R)$ for some ring R. Now, fix some $\alpha \in \ker d_A^i$, and we want to show that

$$\left[f^i(\alpha) - g^i(\alpha)\right] = 0$$

in $h^i(B^\bullet)$. But now let $k^j \colon A^j \to B^{j-1}$ for $j \in \mathbb{Z}$ provide our chain homotopy, so we see

$$f^i(\alpha) - g^i(\alpha) = k^{i+1}\big(\underbrace{d_A^i(\alpha)}_{0}\big) + d_B^{i-1}\left(k^i(\alpha)\right)$$

vanishes in $h^i(B^\bullet)$, as desired. ■

1.2.2 Injective Resolutions

We would now like to use our homological algebra to say something concrete about functors, which requires building injective resolutions. Injective resolutions are built out of injectives, so here is that definition.

Definition 1.26 (injective)**.** Fix an object I in an abelian category $\mathcal{A}$. Then I is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is right exact.

Remark 1.27. The functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is already left-exact (and contravariant), so it is equivalent to ask for this functor to be fully exact. Unwinding the definition, we may equivalently ask for short exact sequences

$$0 \to A' \to A \to A'' \to 0$$

to produce short exact sequences

$$0 \to \operatorname{Hom}_{\mathcal{A}}(A'', I) \to \operatorname{Hom}_{\mathcal{A}}(A, I) \to \operatorname{Hom}_{\mathcal{A}}(A', I) \to 0,$$

but this is already left-exact, so we are really only concerned about surjectivity on the right. So we may equivalently ask for injections $A' \hookrightarrow A$ to produce surjections $\operatorname{Hom}_{\mathcal{A}}(A', I) \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(A, I)$; i.e., any map $A' \to I$ can be extended to a full map $A \to I$.

We also have the following dual notion.

Definition 1.28 (projective)**.** Fix an object P in an abelian category $\mathcal{A}$. Then P is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is right exact.

Remark 1.29. Exactly the dual arguments to Remark 1.27 say that being projective is equivalent to $\operatorname{Hom}_{\mathcal{A}}(P, -)$ being fully exact, or equivalently that any map $P \to A''$ can be pulled back to a map $P \to A$ whenever we have a surjection $A \twoheadrightarrow A''$.

And we now define our resolutions.

Definition 1.30 (resolution)**.** Fix an object A in an abelian category $\mathcal{A}$. A *coresolution* is an exact sequence

$$0 \to A \overset{\varepsilon}{\to} E^0 \to E^1 \to \cdots$$

in $\mathcal{A}$; we may write this as $0 \to A \to E^\bullet$. A *resolution* is an exact sequence

$$\cdots \to E_1 \to E_0 \overset{\varepsilon}{\to} A \to 0$$

in $\mathcal{A}$; again, we may write this as $E^\bullet \to A \to 0$. For any property $\mathcal{P}$ of objects in $\mathcal{A}$, we say that the resolution is $\mathcal{P}$ if and only if the Es are all $\mathcal{P}$.

Of interest to us right now are injective and projective resolutions, but we will find use for other kinds of resolutions.

We want to be able to build injective resolutions. The following provides the required adjective.

Definition 1.31 (enough injectives)**.** An abelian category $\mathcal{A}$ has *enough injective* if and only if any object $A \in \mathcal{A}$ has a monomorphism to an injective object.

And here is the relevant result.

Proposition 1.32. Fix an abelian category $\mathcal{A}$ with enough injectives. Then every object $A \in \mathcal{A}$ has an injective resolution.

Proof. By induction, it is enough to show that, for any map $f\colon A \to E$, there exists a map $g\colon E \to I$ where I is injective and the sequence $A \to E \to I$ is exact. Indeed, this will be enough because we can start with the sequence $0 \to A$, then extend to $0 \to A \to E^0$, then extend to $0 \to A \to E^0 \to E^1$, and so on.

Now, to show the claim of the previous paragraph, we note that we may find an injective object I and a monomorphism $\overline{g}\colon \operatorname{coker} f \to I$ because $\mathcal{A}$ has enough injectives. Then we note that the composite

$$A \to E \to \operatorname{coker} f \hookrightarrow I$$

produces the exact sequence $A \to E \to I$, as desired. ■

1.3 January 22

Today we will derive functors.

1.3.1 More on Injective Resolutions

A nice property of injective resolutions is that they are, in some sense, functorial in their object.

Proposition 1.33. Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$. Given injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$, one can find maps $g^i\colon E^i \to F^i$ for each i inducing a chain morphism of the injective resolutions.

Proof. This is an exercise is induction and using the injective. ■

In fact, this morphism is unique.

Proposition 1.34. Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$, and fix injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$. Then any two morphisms $f^\bullet$ and $g^\bullet$ of the injective resolutions, which agree on $A \to B$, are chain homotopic.

Proof. Set $h^\bullet := f^\bullet - g^\bullet$. Upon subtracting out g suitably, we see that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{\delta} & I^0 & \xrightarrow{d_A^0} & I^1 & \xrightarrow{d_A^1} & I^2 & \xrightarrow{d_A^2} & \cdots \\ & & {\scriptstyle 0}\downarrow & & {\scriptstyle h^0}\downarrow & & {\scriptstyle h^1}\downarrow & & {\scriptstyle h^2}\downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow{\varepsilon} & J^0 & \xrightarrow{d_B^0} & J^1 & \xrightarrow{d_B^1} & J^2 & \xrightarrow{d_B^2} & \cdots \end{array}$$

commutes, and we want to show that the morphism $h^\bullet$ of the injective resolutions is chain homotopic to the zero map.

Now, we see $h^0 \circ \delta = 0$, so we may as well factor h^0 through $\operatorname{coker} \delta \subseteq I^1$. But J^0 is an injective object, so the map $\overline{h}^0 \colon \operatorname{coker} \delta \to J^i$ extends to a map $k^1 \colon I^1 \to J^0$. For completeness, we also define $k^0 \colon I^0 \to J^{-1}$ be the zero map. Anyway, we now compute

$$d_B^{-1} \circ k^0 + k^1 \circ d_A^0 = h^0$$

by construction.

Further, we see

$$\left(h^1 - d_B^0 \circ k^1\right) \circ d_A^0 = h^1 \circ d_A^0 - d_B^0 \circ h^0 = 0$$

by the commutativity of our diagram. As such, we have a map $\left(h^1 - d_B^0 \circ k^1\right) \colon \operatorname{coker} d_A^0 \to J^1$ which can be extended to a map $k^2 \circ I^2 \to J^1$ by the injectivity of J^1. In particular, we see that $h^1 - d_B^0 \circ k^1 = k^2 \circ d_A^1$ by construction. Explicitly, let $\pi^1 \colon I^1 \to \operatorname{coker} d^0$ and $i^1 \colon \operatorname{coker} d^0 \to I^2$ be the obvious maps, and we compute

$$d_B^0 \circ k^1 + k^2 \circ d_A^1 = h^1 - \overline{h}^1 \circ \pi^1 + k^2 \circ d^1 = h^1 - k^2 \circ i^2 \circ pi^1 + k^2 \circ d_A^1 = h^1.$$

We now iterate the construction of k^{i+1} from k^i provided in this paragraph inductively to complete the proof. ■

Remark 1.35. The proofs of the previous two proposition nowhere require that the resolutions on A be injective. We will have no need to work in this generality though.

1.3.2 Right-Derived Functors

At long last, we can derive functors.

Definition 1.36 (right-derived functor)**.** Fix a left-exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories. For each $i \in \mathbb{N}$, we define the *right derived functors*

$$R^i F(A, I^\bullet) := h^i(FI^\bullet),$$

where $0 \to A \to I^\bullet$ is an injective resolution of the object A. This construction is functorial: given a morphism $\varphi \colon A \to B$ in $\mathcal{A}$ equipped with injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, we define the morphism

$$R^i F(\varphi, f^\bullet) \colon h^i(FI^\bullet) \to h^i(FJ^\bullet)$$

as $h^i(F(f^\bullet))$ for any extension $f^\bullet \colon I^\bullet \to J^\bullet$ of φ.

We would like to remove the dependencies on the injective resolutions. This requires a couple checks. To begin, we get rid of the dependency of $R^i F(\varphi)$ on $f^\bullet$.

Lemma 1.37. Fix objects A and B in an abelian category $\mathcal{A}$, and equip them with injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$. For any two morphisms $f^\bullet, g^\bullet \colon I^\bullet \to J^\bullet$ extending a given morphism $\varphi \colon A \to B$, we have

$$R^i F(\varphi, f^\bullet) = R^i F(\varphi, g^\bullet).$$

Proof. We know that $f^\bullet$ and $g^\bullet$ are chain homotopic by Proposition 1.34. This chain homotopy is preserved by an additive functor, so $Ff^\bullet$ and $Fg^\bullet$ are still chain homotopic, so Proposition 1.25 implies the conclusion upon taking cohomology. ■

Notation 1.38. Fix everything as in Definition 1.36. We will write $R^iF(\varphi)$ for $R^iF(\varphi, f^\bullet)$ because it is independent of the choice of $f^\bullet$ by Lemma 1.37 (and an $f^\bullet$ always exists by Proposition 1.33). For now, $R^iF(\varphi)$ still should depend on the choice of injective resolutions, but we will suppress it from the notation anyway.

Remark 1.39. Perhaps we should check functoriality of our construction.

- For an object A equipped with an injective resolution $0 \to A \to I^\bullet$, we can extend $\mathrm{id}_A\colon A \to A$ by $\mathrm{id}_{I^\bullet}\colon I^\bullet \to I^\bullet$. Passing through F and taking cohomology reveals $R^iF(\mathrm{id}_A) = \mathrm{id}_{R^iF(A,I^\bullet)}$.
- Fix morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$ extending to maps of injective resolutions $f^\bullet\colon I^\bullet \to J^\bullet$ and $g^\bullet\colon J^\bullet \to K^\bullet$, respectively. Then one want to extend $(\psi \circ \varphi)\colon A \to C$ to a morphism $I^\bullet \to K^\bullet$ is via $g^\bullet \circ f^\bullet$, and doing so establishes that

$$\begin{tikzcd} R^iF(A, I^\bullet) \arrow[r, "R^iF(\varphi)"] \arrow[dr, "R^iF(\psi\circ\varphi)"'] & R^iF(B, J^\bullet) \arrow[d, "R^iF(\psi)"] \\ & R^i(C, K^\bullet) \end{tikzcd}$$

  commutes, from which we can read off functoriality.

Remark 1.40. We can purchase that R^iF does not depend on the choice of injective resolution from Remark 1.39: running the functoriality check on $0 \to A \to I^\bullet$ mapping to $0 \to A \to J^\bullet$ and then back to $0 \to A \to I^\bullet$ reveals that the maps $R^iF(A, I^\bullet) \to R^iF(A, J^\bullet$ and $R^iF(A, J^\bullet) \to R^iF(A, I^\bullet)$ are mutually inverse, so we get the needed isomorphism.

Remark 1.41. Note R^iF is additive because all steps in the construction (passing through F and then taking cohomology) are additive.

We can even compute our 0th right-derived functor without tears.

Example 1.42. Fix an abelian category $\mathcal{A}$ with enough injectives. Then $F \simeq R^0F$. Indeed, on objects, fix an injective resolution $0 \to A \to I^\bullet$ for a given object $A \in \mathcal{A}$, and we see that

$$R^0F(A) = h^0(F(I^\bullet)) = \ker(FI^0 \to FI^1) = FA,$$

where the last equality follows from left-exactness of F. On morphisms $\varphi\colon A \to B$, we fix injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, and then we produce a morphism of left exact sequences as follows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[d, "f^1"] \\ 0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \end{tikzcd}$$

Passing through F retains left exactness (and commutativity), allowing us to conclude $R^0F(\varphi) = F\varphi$.

1.4 January 24

Today we continue deriving functors.

1.4.1 The Long Exact Sequence

Here is the main result on cohomology.

Theorem 1.43. Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, there are natural morphisms $\delta^i\colon R^iF(A'') \to R^{i+1}F(A')$ for $i \geq 0$ (i.e., the δ^i are natural in the short exact sequence) such that there is a long exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & R^0F(A') \arrow[r] & R^0F(A) \arrow[r] & R^0F(A'') \arrow[dll, "\delta^0"'] \\ & R^1F(A') \arrow[r] & R^1F(A) \arrow[r] & R^1F(A'') \arrow[r] & \cdots \end{tikzcd}$$

Proof. We use Proposition 1.22. The main obstacle is that we need to produce a short exact sequence of injective resolutions for A', A, and A''. We begin by fixing injective resolutions $0 \to A' \to I^\bullet$ and $0 \to A'' \to J^\bullet$, which we would like to glue together into an injective resolution for A as well. In particular, we would like a sequence of morphisms to go into the middle of the following diagram.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & A \arrow[r, dashed] \arrow[d] & I^0 \oplus J^0 \arrow[r, dashed] \arrow[d] & I^1 \oplus J^1 \arrow[r, dashed] \arrow[d] & \cdots \\ 0 \arrow[r] & A'' \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & 0 \end{tikzcd}$$

Here, the downward morphisms, except the ones on the far left, are all given by having a split short exact sequence. (Note $I^i \oplus J^i$ is injective for each i because the sum of injective objects must be injective; this can be seen directly from the definition of injective objects.)

Working inductively, the main point is as follows: suppose we have a diagram as follows, where we would like to induce the vertical morphism f making the diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "f'"'] & K \arrow[r] \arrow[d, dashed, "f"] & K'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I \arrow[r] & I \oplus J \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

Here, I and J are injective, and f' and f'' is injective; the Snake lemma will imply that f is injective too. Well, by summing, all one needs is maps $g'\colon K \to I$ and $g''\colon K \to J$ making the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "f'"'] & K \arrow[r] \arrow[dl, dashed, "g'"'] \arrow[dr, dashed, "g''"] & K'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I \arrow[r] & I \oplus J \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

For this, we see that g'' is given by composition, and g' is given because $K' \subseteq K$ and I is injective object.

We now explain how the previous step proves the result. We immediately produce the needed map $A \to I^0 \oplus J^0$. Now to go from having the map $I^i \oplus J^i \to I^{i+1} \oplus J^{i+1}$ to having the map $I^{i+1} \oplus J^{i+1}$, we use the above paragraph on the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \frac{I^{i+1}}{I^i} \arrow[r] \arrow[d, hook] & \frac{I^{i+1}}{I^i} \oplus \frac{J^{i+1}}{J^i} \arrow[r] \arrow[d, dashed] & \frac{J^{i+1}}{J^i} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & I^{i+2} \arrow[r] & I^{i+2} \oplus J^{i+2} \arrow[r] & J^{i+2} \arrow[r] & 0 \end{tikzcd}$$

This completes the construction of the needed short exact sequence of injective resolutions, from which the result follows upon using Proposition 1.22 on the short exact sequence of complexes

$$0 \to FI^\bullet \to FI^\bullet \oplus FJ^\bullet \to FJ^\bullet \to 0.$$

(This is still short exact because additive functors preserve split short exact sequences.) Note that we have not checked that the $\delta^\bullet$s are natural in the short exact sequence; this follows from the naturality of Proposition 1.22. ■

1.4.2 Acyclic Objects

We note the following computation.

Proposition 1.44. Fix a left exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. If $I \in \mathcal{A}$ is injective, then $R^iF(I) = 0$ for all $i \geq 1$.

Proof. There is an injective resolution

$$0 \to I \to I \to 0 \to 0 \to \cdots$$

of I. Upon taking F, we see that $R^0F(I) = I/0$ and $R^1F(I) = 0/I$ and $R^iF(I) = 0/0$ for $i \geq 2$. This proves the result. ■

We now get the following definition.

Definition 1.45 (acyclic)**.** Fix a left exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. We say an object $A \in \mathcal{A}$ is *acyclic for* F if and only if $R^iF(A) = 0$ for all $i \geq 1$.

Example 1.46. If $A \in \mathcal{A}$ is injective, then Proposition 1.44 implies that A is acyclic for any left exact functor $F \colon \mathcal{A} \to \mathcal{B}$.

Here is the point of defining acyclic objects.

Proposition 1.47. Fix a left exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. For any acyclic resolution $0 \to A \to I^\bullet$, there are canonical isomorphisms

$$R^iF(A) \cong h^i(FJ^\bullet).$$

Proof. Induct on i using the long exact sequences. For example, there is nothing to say for $i = 0$. To get up to $i = 1$, use the exact sequence

$$0 \to A \xrightarrow{\varepsilon} J^0 \to \operatorname{coker} \varepsilon \to 0$$

to produce the needed long exact sequence

$$0 \to FA \to FJ^0 \to F \operatorname{coker} \varepsilon \to R^1F(A) \to 0,$$

and $h^1(FJ^\bullet)$ becomes the needed quotient. This process continues upwards. ■

1.4.3 A Little δ-Functors

Here is our definition.

Definition 1.48 (δ-functor)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A *δ-functor* consists of the data of some additive functors $T^i\colon \mathcal{A} \to \mathcal{B}$ for each $i \in \mathbb{N}$ and some morphisms $\delta^i\colon T^iA'' \to T^{i+1}A$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ such that there is a long exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & T^0A' \arrow[r] & T^0A \arrow[r] & T^0A'' \arrow[dll, "\delta^0"] & \\ & T^1A' \arrow[r] & T^1A \arrow[r] & T^1A'' \arrow[r] & \cdots \end{tikzcd}$$

Example 1.49. If $\mathcal{A}$ has enough injective, the derived functors provide examples of δ-functors by Theorem 1.43.

The following definition will be very helpful.

Definition 1.50 (initial)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A δ-functor $(T^\bullet, \delta_T^\bullet)$ is *initial* if and only if any other δ-functor $(U^\bullet, \delta_U^\bullet)$ together with a map $\varphi\colon T^0 \Rightarrow U^0$ has a unique sequence of natural transformations $\eta^\bullet\colon T^\bullet \Rightarrow U^\bullet$ extending φ and commute with the formation of the long exact sequences. Explicitly, a short exact sequence $0 \to A' \to A \to A'' \to 0$ induces the following morphism of long exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & T^0A' \arrow[r] \arrow[d, "f^0"'] & T^0A \arrow[r] \arrow[d, "f^0"'] & T^0A'' \arrow[r, "\delta_T^0"] \arrow[d, "f^0"'] & T^1A' \arrow[r] \arrow[d, "f^1"'] & T^1A \arrow[r] \arrow[d, "f^1"'] & \cdots \\ 0 \arrow[r] & U^0A' \arrow[r] & U^0A \arrow[r] & U^0A'' \arrow[r, "\delta_U^0"] & U^1A' \arrow[r] & U^1A \arrow[r] & \cdots \end{tikzcd}$$

Note that initial δ-functors are unique up to unique isomorphism when they exist.

1.5 January 26

Today we will finish our discussion of right-derived functors.

1.5.1 Initial δ-Functors

We will want to make some use of our discussion of δ-functors.

Definition 1.51 (effaceable)**.** Fix an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories. Then F is *effaceable* if and only if each $A \in \mathcal{A}$ has a monomorphism $u\colon A \to M$ such that $Fu = 0$.

We have the following result which will help us check that right-derived functors are initial.

Theorem 1.52. Fix δ-functor $(T^\bullet, \delta^\bullet)\colon \mathcal{A} \to \mathcal{B}$. If $T^\bullet$ is *effaceable* for all $i > 0$, then $(T^\bullet, \delta^\bullet)$ is initial.

Proof. Omitted. The proof is somewhat long and technical. We refer to [Wei94, Theorem 2.4.7] for most of the needed details. ∎

Corollary 1.53. Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ has enough injectives. If $F\colon \mathcal{A} \to \mathcal{B}$ is left exact, the right-derived functors $(R^\bullet F, \delta^\bullet)$ is effaceable and thus initial.

Proof. By Theorem 1.52, it remains to show being effaceable. Well, for any object $A \in \mathcal{A}$, we can find a map $u\colon A \to I$ where I is injective, so the map $R^i u\colon R^i U \to R^i I$ is the zero map for $i > 1$ because $R^i I = 0$ by Proposition 1.44. ■

Corollary 1.54. Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ has enough injectives. If $(T^\bullet, \delta^\bullet)$ is an initial δ-functor, then T^0 is left exact, and $T^\bullet \simeq R^i T^0$ for all $i \geq 0$.

Proof. For any short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

being a δ-fucntor implies the left exact sequence

$$0 \to T^0 A' \to T^0 A \to T^0 A''.$$

Thus, T^0 is left exact. Now, the usual category theory arguments show that initial δ-functors (when they exist) are unique up to unique isomorphism, so Corollary 1.53 completes the proof. ■

1.5.2 Having Enough Injectives

Let's show that some abelian categories have enough injectives. We begin with Ab.

Definition 1.55 (divisible)**.** An abelian group A is *divisible* if and only if the multiplication-by-n map $n\colon A \to A$ is surjective for all nonzero integers n.

Example 1.56. The groups $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{R}$, and 0 are divisible.

Here is the point of this definition.

Proposition 1.57. An abelian group A is injective in Ab if and only if A is divisible.

Proof. We show our implications separately.

- Suppose A is injective, and fix some $a \in A$ and nonzero integer $n \in \mathbb{Z}$ so that we want to find $a' \in A$ with $a = na'$. Well, we have the morphism $n\mathbb{Z} \to A$ given by $n \mapsto a$, but $n\mathbb{Z} \subseteq \mathbb{Z}$ means that the injectivity of A forces $n\mathbb{Z} \to A$ to extend to $\mathbb{Z} \to A$, as follows.

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[dr, "n\mapsto a"'] & \mathbb{Z} \arrow[d, dashed] \\ & & A \end{tikzcd}$$

  Now, the image of 1 along $\mathbb{Z} \to A$ can be called a' and has $na' = a$ by construction.

- Suppose A is divisible. We will use Zorn's lemma. Well, for our setup, suppose that we have an inclusion $M' \subseteq M$ and a map $\varphi\colon M' \to A$ which we would like to extend up to M.

  Let Φ be the collection of extensions of $\varphi\colon M' \to A$ to some subgroup $N \subseteq M$ containing M', and order Φ by extension: we have $(N_1, \varphi_1) \preceq (N_2, \varphi_2)$ if and only if $N_1 \subseteq N_2$ and $\varphi_2|_{N_1} = \varphi_1$. Now, Φ is nonempty (it has (M', φ)), and its ascending chains are upper-bounded (the union of an extension of group homomorphisms will continue to be a group homomorphism), so Zorn's lemma provides Φ with a maximal element (M'', φ'').

We claim that $M'' = M$, which will complete the proof. Well, we will show a contrapositive: suppose $(N, \psi) \in \Phi$ has $N \neq M$; then we claim that (N, ψ) is not maximal. Well, given any $x \in M \setminus N$, we will extend ψ to $N + \mathbb{Z}x$. Set $H := \{n \in \mathbb{Z} : nx \in N\}$. We have two cases.

- Suppose $H = 0$. Then $N + \mathbb{Z}x = N \oplus \mathbb{Z}x$, so we can extend ψ by just setting $\psi(x) := 0$.
- Suppose $H = n\mathbb{Z}$ for some positive integer $n > 0$. Divisibility promises us some $a \in A$ such that $\psi(nx) = na$, so we would like to extend ψ by $\psi(x) = a$. Namely, we would like to define $\widetilde{\psi}\colon (N + \mathbb{Z}x) \to A$ by
$$\widetilde{\psi}(m + kx) := \psi(m) + ka.$$
Of course, this will be a group homomorphism extending $\widetilde{\psi}$ provided that it is well-defined. Well, suppose $m + kx = m' + k'x$, and we want to show that $\psi(m) + ka = \psi(m') + k'a$, or equivalently, $\psi(m - m') = (k' - k)a$. We now note that $(k' - k)x = m - m' \in N$, so $k' - k = n\ell$ for some integer ℓ by construction of n, so we computed
$$(k' - k)a = n\ell a = \psi(n\ell x) = \psi((k' - k)x) = \psi(m - m'),$$
as needed. ■

Theorem 1.58. Fix a ring R. The category $\operatorname{Mod}(R)$ has functorial injectives.

Proof. We proceed in steps. Given an R-module A, define $A^\vee := \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ to be the dual, which we note is another R-module.

1. So we will want to show that the map
$$\mathrm{ev}_\bullet \colon A \to A^{\vee\vee}$$
given by $\mathrm{ev}_a \colon \varphi \mapsto \varphi(a)$ is injective. This will be enough because the above map is functorial in A. We now show that this map is injective, so suppose $a \in A$ is nonzero, and we want to show that a is not in the kernel of $\mathrm{ev}_\bullet$. Well, define $\varphi \colon \mathbb{Z}a \to \mathbb{Q}/\mathbb{Z}$ by
$$\varphi(a) := \begin{cases} 1/2 & \text{if } a \text{ has infinite order,} \\ 1/m & \text{if } a \text{ has order } m. \end{cases}$$
Note φ does define a group homomorphism $\mathbb{Z}a \to \mathbb{Q}/\mathbb{Z}$, so the injectivity of $\mathbb{Q}/\mathbb{Z}$ tells us that φ extends to a homomorphism $\widetilde{\varphi}\colon A \to \mathbb{Q}/\mathbb{Z}$, and we see that $\mathrm{ev}_a(\widetilde{\varphi}) = \varphi(a) \neq 0$ by construction.
2. We actually construct the needed injection. Note we have a surjection
$$\bigoplus_{x \in A^\vee} R \twoheadrightarrow A^\vee,$$
so we have an injection
$$A \hookrightarrow A^{\vee\vee} \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}\Bigg(\bigoplus_{x \in A^\vee} R, \mathbb{Q}/\mathbb{Z}\Bigg).$$
The right-hand side can be fixed to be injective, so we are essentially done; notably, our construction is functorial in A. Explicitly, given a map $A \to B$, we induce a map $B^\vee \to A^\vee$, and taking fibers of this map induces a map $(\mathbb{Q}/\mathbb{Z})^{A^\vee} \to (\mathbb{Q}/\mathbb{Z})^{B^\vee}$. (Any coordinate in $A^\vee$ not in the image of $B^\vee$ can just get sent to 0.) ■

1.6 January 29

Today we continue to show that categories have enough injectives.

1.6.1 Exactness in Abelian Categories

Let's say a few more things about abelian categories.

Example 1.59. Fix an abelian category $\mathcal{A}$. Then $\mathcal{A}$ has an empty biproduct 0, which is both initial and final by its definition. We will not bother to write out the identification of biproducts in additive categories.

Remark 1.60. Fix an abelian category $\mathcal{A}$. Any morphism $\varphi\colon A \to B$ can be factored as $\nu \circ \eta$ where $\eta\colon A \to X$ is epic and $\nu\colon X \to B$ is monic. To see that this factorization exists, we can set $\eta = \operatorname{coker}(\ker \varphi)$ and $\nu = \ker(\operatorname{coker} \varphi)$. Additionally, the factorization $\nu \circ \eta$ is unique in the following sense: if $\eta'\colon A \to X'$ and $\nu'\colon X' \to B$ is another such factorization, there is a unique isomorphism $\psi\colon X \to X'$ making the following diagram commute.

$$\begin{tikzcd} & X \arrow[dr, "\nu"] \arrow[dd, "\psi"] & \\ A \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & & B \\ & X' \arrow[ur, "\nu'"'] & \end{tikzcd}$$

Remark 1.61. The previous remark implies that being an isomorphism is equivalent to being both monic and epic. Namely, one just factors the given morphism $\varphi\colon A \to B$ in the two ways $\mathrm{id}_B \circ \varphi = \varphi \circ \mathrm{id}_A$ to conclude that φ has an inverse.

The prior two remarks allow us to make sense of exactness in a meaningful way.

Definition 1.62 (exact)**.** Fix morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, and factor these as $\varphi = \nu \circ \eta$ and $\psi = \mu \circ \varepsilon$ where ν and μ are epic and η and ε are monic. Here is the diagram.

$$\begin{tikzcd} & X \arrow[dr, "\nu"] & & Y \arrow[dr, "\varepsilon"] & \\ A \arrow[ur, "\eta"] \arrow[rr, "\varphi"'] & & B \arrow[ur, "\mu"] \arrow[rr, "\psi"'] & & C \end{tikzcd}$$

Then the sequence

$$A \to B \to C$$

is *exact* if and only if $\nu = \ker \varepsilon$; this is equivalent to asking for $\varepsilon = \operatorname{coker} \nu$.

The equivalence of these two notions follows by the uniqueness of the factorization. Note that this is approximately the correct notion because we really want to say that φ surjects onto the kernel of ψ. But then we note ν basically acts as the image of φ, and ε basically acts as the kernel of ψ.

1.6.2 Sheaves Have Enough Injectives

We now move up to sheaves.

Theorem 1.63. Fix a ringed space $(X, \mathcal{O}_X)$. Then the category $\operatorname{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules has functorial injectives.

Proof. Fix a sheaf $\mathcal{F} \in \operatorname{Mod}(\mathcal{O}_X)$. For each $x \in X$, recall that $\operatorname{Mod}(\mathcal{O}_{X,x})$ has functorial injectives by Theorem 1.58, so we let $I_x(\mathcal{F}_x)$ be an injective module into which $\mathcal{F}_x$ injects. Letting $j_x\colon \{x\} \to X$ denote the inclusion map, we then define

$$\mathcal{I} := \prod_{x \in X} (j_x)_* I_x(\mathcal{F}_x).$$

Note that this is an $\mathcal{O}_X$-module because it is the product of $\mathcal{O}_X$-modules. Note that there is a naturally defined map $i\colon \mathcal{F} \to (j_x)_* I_x(\mathcal{F}_x)$ defined by the composite

$$\mathcal{F}(U) \to \mathcal{F}_x \to I_x(\mathcal{F}_x)$$

for each $x \in U$ (and we get the zero map for $x \notin U$). This map i is injective on stalks: we can see that $\mathcal{F}_x$ will embed into the coordinate $(j_x)_* I_x(\mathcal{F}_x)$. Additionally, this construction of i is functorial.

As such, it just remains to show that $\mathcal{I}$ is injective. Suppose that $\mathcal{G} \in \operatorname{Mod}(\mathcal{O}_X)$, and we compute

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \simeq \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* I_x(\mathcal{F}_x)) \simeq \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x(\mathcal{F}_x)).$$

Now, each $I_x(\mathcal{F}_x)$ is an injective object, so the functors $\operatorname{Hom}_{\mathcal{O}_{X,x}}(-, I_x(\mathcal{F}_x))$ is an exact functor for each $x \in X$, so the total functor above is exact, as needed. ■

Remark 1.64. If one tracks through all the constructions, we see that the local rings of the injective sheaf $\mathcal{I} \in \operatorname{Mod}(\mathcal{O}_X)$ are all divisible abelian groups, so $\mathcal{I}$ is in fact also an injective sheaf in $\operatorname{Ab}(X)$. Thus, for example, the constructed injective resolutions in $\operatorname{Mod}(\mathcal{O}_X)$ are also injective resolutions in the larger category $\operatorname{Ab}(X)$.

Corollary 1.65. Fix a topological space X. Then the category $\operatorname{Ab}(X)$ of category of sheaves of abelian groups on X has functorial injectives.

Proof. Set $\mathcal{O}_X$ to be the constant sheaf $\underline{\mathbb{Z}}$ on X. Then $\mathcal{O}_X$ is a sheaf of rings, and $\mathcal{O}_X$-modules are exactly sheaves of abelian groups, so the result follows from Theorem 1.63. ■

1.6.3 Sheaf Cohomology

We can finally define sheaf cohomology.

Definition 1.66 (sheaf cohomology)**.** Fix a topological space X. Because $\operatorname{Ab}(X)$ has enough injectives (by Corollary 1.65) and $\Gamma(X, -)\colon \operatorname{Ab}(X) \to \operatorname{Ab}$ is left exact, we define the *sheaf cohomology functors* as

$$H^\bullet(X, -) := R^\bullet\Gamma(X, -).$$

Remark 1.67. It is rather hard to compute $H^\bullet(X, -)$ directly from the definition. For example, it will be helpful to build a large class of acyclic objects and then use Proposition 1.47.

To realize the above remark, we have the following definition.

Definition 1.68 (flasque)**.** Fix a sheaf $\mathcal{F}$ on a topological space X. Then $\mathcal{F}$ is *flasque* if and only if its restriction maps are surjective.

We have "already" seen many examples of flasque sheaves, as explained in the following lemma.

Lemma 1.69. Fix a ringed space $(X, \mathcal{O}_X)$. Then any injective $\mathcal{O}_X$-module is flasque.

We will prove this lemma next class.

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List of Definitions