# 256B: Algebraic Geometry

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*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

# Theme 1
# Building Cohomology

*Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.*

—Felix Klein, [Kle16]

## 1.1 January 17

Let's just get started.

### 1.1.1 Course Notes

Here are some notes about the course.

- The professor is Paul Vojta, whose email is `vojta@math.berkeley.edu`.
- The course webpage is `https://math.berkeley.edu/~vojta/256b.html`.
- The textbook is [Har77].
- We will assume algebraic geometry on the level of Math 256A, which is a prerequisite for this course.
- This course focuses on (Zariski) cohomology of schemes, so we will spend most of our time going through [Har77, Chapter III]. We will also discuss smoothness, which lives in [Har77, Chapter III] as well. Along our way, we will want to discuss some topics in [Har77, Chapter II] in more detail, such as on divisors.
- Grading will be based on homework. Homework will be weekly or biweekly, due on Wednesdays (in general).

### 1.1.2 Abelian Categories

We'll assume some basic category theory (monomorphisms, epimorphisms, equalizers, coequalizers, etc.). Abelian categories are somewhat complex, so we provide their definition. Roughly speaking, our end goal is to do cohomology, which arises from homological algebra, and homological algebra lives in abelian categories.

**Definition 1.1** (preadditive)**.** A *preadditive category* is a category $\mathcal{C}$ where the morphism set $\operatorname{Hom}_{\mathcal{C}}(A,B)$ forms an abelian group for any $A,B\in\mathcal{C}$, and composition distributes over addition. Explicitly, the composition map

$$\circ\colon \operatorname{Hom}_{\mathcal{C}}(B,C)\times\operatorname{Hom}_{\mathcal{C}}(A,B)\to\operatorname{Hom}_{\mathcal{C}}(A,C)$$

is bilinear.

It follows directly from having the preadditive structure that finite products and finite coproducts are canonically isomorphic. However, these (bi)products need not exist.

**Definition 1.2** (additive)**.** An *additive category* is a preadditive category admitting all finite products/coproducts.

**Definition 1.3** (abelian)**.** An *abelian category* is an additive category $\mathcal{C}$ in which the following hold.

- Every morphism admits a kernel and a cokernel; here, a (co)kernel is a (co)equalizer with the zero map.
- Every monomorphism is the kernel of some morphism.
- Every epimorphism is the cokernel of some morphism.

Let's give some examples.

**Example 1.4.** The following are abelian categories; we omit the checks.

- The category $\mathrm{Ab}$ of abelian groups is abelian.
- For a ring $A$, the category $\operatorname{Mod}(A)$ of $A$-modules is abelian. In particular, for a field $k$, the category $\operatorname{Vec}(k)$ of $k$-vector spaces is abelian.

**Example 1.5.** Here are more abelian categories, related to sheaves. All of their "abelian" hypotheses are done by passing to stalks or a similar local argument.

- For a topological space $X$, the category $\operatorname{Ab}(X)$ of sheaves of abelian groups on $X$ is abelian.
- Similarly, for a ringed space $(X,\mathcal{O}_X)$, the category $\operatorname{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules is abelian.
- For a scheme $X$, the category $\operatorname{QCoh}(X)$ of quasicoherent sheaves on $X$ is abelian.
- Similarly, for a scheme $X$, the category $\operatorname{Coh}(X)$ of coherent sheaves on $X$ is also abelian. Notably, we do not have infinite products here, but that's okay.

**Example 1.6.** For any abelian category $\mathcal{A}$, its opposite category $\mathcal{A}^{\mathrm{op}}$ is also abelian. One can see this by going through the conditions, all of which dualize.

### 1.1.3 Exact Functors

We will want to discuss exact functors in order to homological algebra in our abelian categories. Let's have at it.

**Definition 1.7** (additive)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if and only if the map

$$F\colon \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{D}}(FA, FB)$$

(of $F$ acting on morphisms $A \to B$) is a group homomorphism, for any $A, B \in \mathcal{C}$. Flipping arrows and using Example 1.6 produces the same definition for contravariant functors.

**Example 1.8.** Fix a topological space $X$. Then the functor $\Gamma(X, -)\colon \operatorname{Ab}(X) \to \operatorname{Ab}$ of global sections $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ is additive.

**Remark 1.9.** Being additive implies that the functor preserves biproducts. Roughly speaking, this holds because being a biproduct can be written as a set of equations for the object (and its inclusion/projection morphisms) to satisfy.

To define (left) exact for a functor, we need to define what it means to be exact.

**Definition 1.10** (exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. Then a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* $B$ if and only if $\ker g = \ker(\operatorname{coker} f)$ (up to some identification). Here, $\ker(\operatorname{coker} f)$ is intended to basically be the image.

**Definition 1.11** (left exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) additive functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left-exact* if and only if a left exact sequence

$$0 \to A' \to A \to A''$$

produces a left exact sequence

$$0 \to FA' \to FA \to FA''.$$

Reversing the arrows produces the dual notion of right exactness.

**Remark 1.12.** Being left exact equivalently means that $F$ preserves kernels, so by Remark 1.9 and a little category theory, $F$ actually preserves all finite limits.

**Example 1.13.** The functor of global sections from Example 1.8 is left exact by [Har77, Exercise II.1.8].

To get us set up, let's approximately describe what we are trying to do. Basically, fix an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of abelian groups on a topological space $X$. Then there is a sequence of "cohomology" functors $\{H^i(X, -)\}_{i \in \mathbb{N}}$ with $H^0(X, -) = \Gamma(X, -)$ and a "long" exact sequence as follows

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{F}') \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{F}'') \arrow[dll] & \\ & H^1(X, \mathcal{F}') \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & H^1(X, \mathcal{F}'') \arrow[r] & \cdots \end{tikzcd}$$

where the maps $H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}')$ take some work to describe.

**Remark 1.14.** These functors will have a number of magical properties, which will amount to the main theorems of this course. Let's give an example. Fix a projective scheme $X$ over a field $k$, where $i\colon X \to \mathbb{P}_k^n$ is the promised closed embedding; let $\mathcal{I}$ be the corresponding ideal sheaf of this closed embedding. Then we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}_k^n} \to i_*\mathcal{O}_X \to 0,$$

which one can do cohomology to. In fact, one can take the tensor product of this exact sequence with the twisting sheaves $\mathcal{O}_{\mathbb{P}_k^n}(m)$; for example, we will prove that $H^1(\mathbb{P}_k^n, \mathcal{I}(m)) = 0$ for sufficiently large $m$, which eventually implies that the map

$$\Gamma\left(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m)\right) \to \Gamma(X, \mathcal{O}_X(m))$$

is surjective for sufficiently large $m$. In other words, global sections of $\mathcal{O}_X(m)$ are all restrictions of global sections of $\mathcal{O}_{\mathbb{P}_k^n}(m)$!

## 1.2 January 19

We'll do some homological algebra today.

### 1.2.1 Homological Algebra on Complexes

Homological algebra is something that comes out of understanding complexes, which we will now define.

**Definition 1.15** (complex)**.** Fix an abelian category $\mathcal{A}$. A *complex* $(A^\bullet, d^\bullet)$ is a collection $\{A^i\}_{i\in\mathbb{Z}}$ together with some morphisms $d^i\colon A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$. We may abbreviate the differential $d^\bullet$ from the notation.

**Remark 1.16.** The above definition is usually a "cocomplex." We will have no need for the dual notion of a complex in this course.

**Remark 1.17.** By convention, if we state that we have a complex but only define $A^i$ for a subset of $\mathbb{Z}$, then the full bona fide complex simply sets the undefined terms to zero.

Now that we have a complex, we should define a morphism.

**Definition 1.18** (complex morphism)**.** Fix an abelian category $\mathcal{A}$. Given complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$, a morphism of complexes $\varphi^\bullet\colon A^\bullet \to B^\bullet$ is a collection of morphisms $\varphi^i\colon A^i \to B^i$ making the following diagram commute for each $i$.

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "\varphi^i"'] & A^{i+1} \arrow[d, "\varphi^{i+1}"] \\ B^i \arrow[r, "d^{i+1}"] & B^{i+1} \end{tikzcd}$$

Unsurprisingly, our definition of morphism provides us with a category of complexes, and in fact the category of complexes is an abelian category, where the point is that biproducts, kernels, and cokernels can all be computed pointwise at each term of the complex.

We are now ready to define cohomology.

**Definition 1.19** (cohomology)**.** Fix a complex $(A^\bullet, d^\bullet)$ valued in an abelian category $\mathcal{A}$. Then we define the $i$th *cohomology* as

$$h^i(A^\bullet) := \frac{\ker d^i}{\operatorname{im} d^{i-1}}.$$

Here, $\operatorname{im} d^{i-1}$ has an induced map to $\ker d^i$ because $d^i \circ d^{i-1} = 0$.

**Remark 1.20.** Quickly, recall that the image $\operatorname{im} d^{i-1}$ is in fact $\ker(\operatorname{coker} d^{i-1})$.

**Remark 1.21.** In fact, cohomology is functorial: a morphism $f^\bullet\colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of complexes induces a morphism $h^i(f^\bullet)\colon h^i(A^\bullet) \to h^i(B^\bullet)$ on the $i$th cohomology, and one can check that this makes $h^i$ into a functor. To be explicit, this morphism is induced by the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} d_A^{i-1} \arrow[r] \arrow[d, "f^i"] & \ker d_A^i \arrow[r] \arrow[d, "f^i"] & h^i(A^\bullet) \arrow[r] \arrow[d, dashed, "f^i"] & 0 \\
0 \arrow[r] & \operatorname{im} d_B^{i-1} \arrow[r] & \ker d_B^i \arrow[r] & h^i(B^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Namely, the morphisms on the left are well-defined because $f^\bullet$ is in fact a morphism.

The main result on these cohomology groups is the following.

**Proposition 1.22.** Fix an abelian category $\mathcal{A}$. Given a short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of complexes in $\mathcal{A}$, there are natural maps $\delta^i\colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ producing a long exact sequence as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & h^i(A^\bullet) \arrow[r] & h^i(B^\bullet) \arrow[r] & h^i(C^\bullet) \arrow[dll, "\delta^i"'] \\
& h^{i+1}(A^\bullet) \arrow[r] & h^{i+1}(B^\bullet) \arrow[r] & h^{i+1}(C^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

*Proof.* To produce the long exact sequence, use the Snake lemma. The proof is somewhat technical, so I will refer directly to [Elb22, Theorem 4.82], though the proof there is for the dual notion of homology instead of cohomology. (Note that we can replace $\mathcal{A}$ with $\mathcal{A}^{\mathrm{op}}$ to recover the result.) The naturality of the $\delta^\bullet$ can be checked directly from its construction. ■

We would like to measure a morphism of complexes based on what it does to cohomology: namely, two morphisms of complexes may induce the same map on cohomology despite being technically distinct. One way this might happen is by being "chain" homotopic.

**Definition 1.23** (chain homotopy)**.** Fix morphisms $f^\bullet, g^\bullet\colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of the chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. A *chain homotopy* is a sequence of maps $k^i\colon A^i \to B^{i-1}$ such that

$$f^i - g^i = k^{i+1} \circ d_A^i + d_B^{i-1} \circ k^i.$$

In this case, we say that $f^\bullet$ and $g^\bullet$ are chain homotopic.

**Remark 1.24.** One can check directly that being chain homotopic is an equivalence relation on chain morphisms.

And here is our result.

**Proposition 1.25.** Fix morphisms $f^\bullet, g^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. If $f^\bullet \sim g^\bullet$, then $h^i(f^\bullet) = h^i(g^\bullet)$ for all $i$.

*Proof.* By some embedding theorem, we may as well work in $\operatorname{Mod}(R)$ for some ring $R$. Now, fix some $\alpha \in \ker d_A^i$, and we want to show that

$$\left[f^i(\alpha) - g^i(\alpha)\right] = 0$$

in $h^i(B^\bullet)$. But now let $k^j \colon A^j \to B^{j-1}$ for $j \in \mathbb{Z}$ provide our chain homotopy, so we see

$$f^i(\alpha) - g^i(\alpha) = k^{i+1}\big(\underbrace{d_A^i(\alpha)}_{0}\big) + d_B^{i-1}\left(k^i(\alpha)\right)$$

vanishes in $h^i(B^\bullet)$, as desired. ■

### 1.2.2 Injective Resolutions

We would now like to use our homological algebra to say something concrete about functors, which requires building injective resolutions. Injective resolutions are built out of injectives, so here is that definition.

**Definition 1.26** (injective)**.** Fix an object $I$ in an abelian category $\mathcal{A}$. Then $I$ is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is right exact.

**Remark 1.27.** The functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is already left-exact (and contravariant), so it is equivalent to ask for this functor to be fully exact. Unwinding the definition, we may equivalently ask for short exact sequences

$$0 \to A' \to A \to A'' \to 0$$

to produce short exact sequences

$$0 \to \operatorname{Hom}_{\mathcal{A}}(A'', I) \to \operatorname{Hom}_{\mathcal{A}}(A, I) \to \operatorname{Hom}_{\mathcal{A}}(A', I) \to 0,$$

but this is already left-exact, so we are really only concerned about surjectivity on the right. So we may equivalently ask for injections $A' \hookrightarrow A$ to produce surjections $\operatorname{Hom}_{\mathcal{A}}(A', I) \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(A, I)$; i.e., any map $A' \to I$ can be extended to a full map $A \to I$.

We also have the following dual notion.

**Definition 1.28** (projective)**.** Fix an object $P$ in an abelian category $\mathcal{A}$. Then $P$ is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is right exact.

**Remark 1.29.** Exactly the dual arguments to Remark 1.27 say that being projective is equivalent to $\operatorname{Hom}_{\mathcal{A}}(P, -)$ being fully exact, or equivalently that any map $P \to A''$ can be pulled back to a map $P \to A$ whenever we have a surjection $A \twoheadrightarrow A''$.

And we now define our resolutions.

**Definition 1.30** (resolution)**.** Fix an object $A$ in an abelian category $\mathcal{A}$. A *coresolution* is an exact sequence

$$0 \to A \overset{\varepsilon}{\to} E^0 \to E^1 \to \cdots$$

in $\mathcal{A}$; we may write this as $0 \to A \to E^\bullet$. A *resolution* is an exact sequence

$$\cdots \to E_1 \to E_0 \overset{\varepsilon}{\to} A \to 0$$

in $\mathcal{A}$; again, we may write this as $E^\bullet \to A \to 0$. For any property $\mathcal{P}$ of objects in $\mathcal{A}$, we say that the resolution is $\mathcal{P}$ if and only if the $E$s are all $\mathcal{P}$.

Of interest to us right now are injective and projective resolutions, but we will find use for other kinds of resolutions.

We want to be able to build injective resolutions. The following provides the required adjective.

**Definition 1.31** (enough injectives)**.** An abelian category $\mathcal{A}$ has *enough injective* if and only if any object $A \in \mathcal{A}$ has a monomorphism to an injective object.

And here is the relevant result.

**Proposition 1.32.** Fix an abelian category $\mathcal{A}$ with enough injectives. Then every object $A \in \mathcal{A}$ has an injective resolution.

*Proof.* By induction, it is enough to show that, for any map $f\colon A \to E$, there exists a map $g\colon E \to I$ where $I$ is injective and the sequence $A \to E \to I$ is exact. Indeed, this will be enough because we can start with the sequence $0 \to A$, then extend to $0 \to A \to E^0$, then extend to $0 \to A \to E^0 \to E^1$, and so on.

Now, to show the claim of the previous paragraph, we note that we may find an injective object $I$ and a monomorphism $\overline{g}\colon \operatorname{coker} f \to I$ because $\mathcal{A}$ has enough injectives. Then we note that the composite

$$A \to E \to \operatorname{coker} f \hookrightarrow I$$

produces the exact sequence $A \to E \to I$, as desired. ∎

## 1.3 January 22

Today we will derive functors.

### 1.3.1 More on Injective Resolutions

A nice property of injective resolutions is that they are, in some sense, functorial in their object.

**Proposition 1.33.** Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$. Given injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$, one can find maps $g^i\colon E^i \to F^i$ for each $i$ inducing a chain morphism of the injective resolutions.

*Proof.* This is an exercise is induction and using the injective. ∎

In fact, this morphism is unique.

**Proposition 1.34.** Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$, and fix injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$. Then any two morphisms $f^\bullet$ and $g^\bullet$ of the injective resolutions, which agree on $A \to B$, are chain homotopic.

*Proof.* Set $h^\bullet := f^\bullet - g^\bullet$. Upon subtracting out $g$ suitably, we see that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\delta"] \arrow[d, "0"] & I^0 \arrow[r, "d_A^0"] \arrow[d, "h^0"] & I^1 \arrow[r, "d_A^1"] \arrow[d, "h^1"] & I^2 \arrow[r, "d_A^2"] \arrow[d, "h^2"] & \cdots \\
0 \arrow[r] & B \arrow[r, "\varepsilon"] & J^0 \arrow[r, "d_B^0"] & J^1 \arrow[r, "d_B^1"] & J^2 \arrow[r, "d_B^2"] & \cdots
\end{tikzcd}$$

commutes, and we want to show that the morphism $h^\bullet$ of the injective resolutions is chain homotopic to the zero map.

Now, we see $h^0 \circ \delta = 0$, so we may as well factor $h^0$ through $\operatorname{coker} \delta \subseteq I^1$. But $J^0$ is an injective object, so the map $\overline{h}^0 \colon \operatorname{coker} \delta \to J^i$ extends to a map $k^1 \colon I^1 \to J^0$. For completeness, we also define $k^0 \colon I^0 \to J^{-1}$ be the zero map. Anyway, we now compute

$$d_B^{-1} \circ k^0 + k^1 \circ d_A^0 = h^0$$

by construction.

Further, we see

$$\left(h^1 - d_B^0 \circ k^1\right) \circ d_A^0 = h^1 \circ d_A^0 - d_B^0 \circ h^0 = 0$$

by the commutativity of our diagram. As such, we have a map $\left(h^1 - d_B^0 \circ k^1\right) \colon \operatorname{coker} d_A^0 \to J^1$ which can be extended to a map $k^2 \circ I^2 \to J^1$ by the injectivity of $J^1$. In particular, we see that $h^1 - d_B^0 \circ k^1 = k^2 \circ d_A^1$ by construction. Explicitly, let $\pi^1 \colon I^1 \to \operatorname{coker} d^0$ and $i^1 \colon \operatorname{coker} d^0 \to I^2$ be the obvious maps, and we compute

$$d_B^0 \circ k^1 + k^2 \circ d_A^1 = h^1 - \overline{h}^1 \circ \pi^1 + k^2 \circ d^1 = h^1 - k^2 \circ i^2 \circ pi^1 + k^2 \circ d_A^1 = h^1.$$

We now iterate the construction of $k^{i+1}$ from $k^i$ provided in this paragraph inductively to complete the proof. ■

**Remark 1.35.** The proofs of the previous two proposition nowhere require that the resolutions on $A$ be injective. We will have no need to work in this generality though.

### 1.3.2 Right-Derived Functors

At long last, we can derive functors.

**Definition 1.36** (right-derived functor)**.** Fix a left-exact functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories. For each $i \in \mathbb{N}$, we define the *right derived functors*

$$R^i F(A, I^\bullet) := h^i(FI^\bullet),$$

where $0 \to A \to I^\bullet$ is an injective resolution of the object $A$. This construction is functorial: given a morphism $\varphi \colon A \to B$ in $\mathcal{A}$ equipped with injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, we define the morphism

$$R^i F(\varphi, f^\bullet) \colon h^i(FI^\bullet) \to h^i(FJ^\bullet)$$

as $h^i(F(f^\bullet))$ for any extension $f^\bullet \colon I^\bullet \to J^\bullet$ of $\varphi$.

We would like to remove the dependencies on the injective resolutions. This requires a couple checks. To begin, we get rid of the dependency of $R^i F(\varphi)$ on $f^\bullet$.

**Lemma 1.37.** Fix objects $A$ and $B$ in an abelian category $\mathcal{A}$, and equip them with injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$. For any two morphisms $f^\bullet, g^\bullet \colon I^\bullet \to J^\bullet$ extending a given morphism $\varphi \colon A \to B$, we have

$$R^i F(\varphi, f^\bullet) = R^i F(\varphi, g^\bullet).$$

*Proof.* We know that $f^\bullet$ and $g^\bullet$ are chain homotopic by Proposition 1.34. This chain homotopy is preserved by an additive functor, so $Ff^\bullet$ and $Fg^\bullet$ are still chain homotopic, so Proposition 1.25 implies the conclusion upon taking cohomology. ■

**Notation 1.38.** Fix everything as in Definition 1.36. We will write $R^iF(\varphi)$ for $R^iF(\varphi, f^\bullet)$ because it is independent of the choice of $f^\bullet$ by Lemma 1.37 (and an $f^\bullet$ always exists by Proposition 1.33). For now, $R^iF(\varphi)$ still should depend on the choice of injective resolutions, but we will suppress it from the notation anyway.

**Remark 1.39.** Perhaps we should check functoriality of our construction.

- For an object $A$ equipped with an injective resolution $0 \to A \to I^\bullet$, we can extend $\mathrm{id}_A\colon A \to A$ by $\mathrm{id}_{I^\bullet}\colon I^\bullet \to I^\bullet$. Passing through $F$ and taking cohomology reveals $R^iF(\mathrm{id}_A) = \mathrm{id}_{R^iF(A,I^\bullet)}$.
- Fix morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$ extending to maps of injective resolutions $f^\bullet\colon I^\bullet \to J^\bullet$ and $g^\bullet\colon J^\bullet \to K^\bullet$, respectively. Then one want to extend $(\psi \circ \varphi)\colon A \to C$ to a morphism $I^\bullet \to K^\bullet$ is via $g^\bullet \circ f^\bullet$, and doing so establishes that

$$\begin{tikzcd} R^iF(A, I^\bullet) \arrow[r, "R^iF(\varphi)"] \arrow[dr, "R^iF(\psi\circ\varphi)"'] & R^iF(B, J^\bullet) \arrow[d, "R^iF(\psi)"] \\ & R^i(C, K^\bullet) \end{tikzcd}$$

commutes, from which we can read off functoriality.

**Remark 1.40.** We can purchase that $R^iF$ does not depend on the choice of injective resolution from Remark 1.39: running the functoriality check on $0 \to A \to I^\bullet$ mapping to $0 \to A \to J^\bullet$ and then back to $0 \to A \to I^\bullet$ reveals that the maps $R^iF(A, I^\bullet) \to R^iF(A, J^\bullet$ and $R^iF(A, J^\bullet) \to R^iF(A, I^\bullet)$ are mutually inverse, so we get the needed isomorphism.

**Remark 1.41.** Note $R^iF$ is additive because all steps in the construction (passing through $F$ and then taking cohomology) are additive.

We can even compute our $0$th right-derived functor without tears.

**Example 1.42.** Fix an abelian category $\mathcal{A}$ with enough injectives. Then $F \simeq R^0F$. Indeed, on objects, fix an injective resolution $0 \to A \to I^\bullet$ for a given object $A \in \mathcal{A}$, and we see that

$$R^0F(A) = h^0(F(I^\bullet)) = \ker(FI^0 \to FI^1) = FA,$$

where the last equality follows from left-exactness of $F$. On morphisms $\varphi\colon A \to B$, we fix injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, and then we produce a morphism of left exact sequences as follows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[d, "f^1"] \\ 0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \end{tikzcd}$$

Passing through $F$ retains left exactness (and commutativity), allowing us to conclude $R^0F(\varphi) = F\varphi$.

## 1.4 January 24

Today we continue deriving functors.

### 1.4.1 The Long Exact Sequence

Here is the main result on cohomology.

**Theorem 1.43.** Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, there are natural morphisms $\delta^i\colon R^iF(A'') \to R^{i+1}F(A')$ for $i \geq 0$ (i.e., the $\delta^i$ are natural in the short exact sequence) such that there is a long exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & R^0F(A') \arrow[r] & R^0F(A) \arrow[r] & R^0F(A'') \arrow[dll, "\delta^0"] \\ & R^1F(A') \arrow[r] & R^1F(A) \arrow[r] & R^1F(A'') \arrow[r] & \cdots \end{tikzcd}$$

*Proof.* We use Proposition 1.22. The main obstacle is that we need to produce a short exact sequence of injective resolutions for $A'$, $A$, and $A''$. We begin by fixing injective resolutions $0 \to A' \to I^\bullet$ and $0 \to A'' \to J^\bullet$, which we would like to glue together into an injective resolution for $A$ as well. In particular, we would like a sequence of morphisms to go into the middle of the following diagram.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & A \arrow[r, dashed] \arrow[d] & I^0 \oplus J^0 \arrow[r, dashed] \arrow[d] & I^1 \oplus J^1 \arrow[r, dashed] \arrow[d] & \cdots \\ 0 \arrow[r] & A'' \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & 0 \end{tikzcd}$$

Here, the downward morphisms, except the ones on the far left, are all given by having a split short exact sequence. (Note $I^i \oplus J^i$ is injective for each $i$ because the sum of injective objects must be injective; this can be seen directly from the definition of injective objects.)

Working inductively, the main point is as follows: suppose we have a diagram as follows, where we would like to induce the vertical morphism $f$ making the diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "f'"'] & K \arrow[r] \arrow[d, dashed, "f"] & K'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I \arrow[r] & I \oplus J \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

Here, $I$ and $J$ are injective, and $f'$ and $f''$ is injective; the Snake lemma will imply that $f$ is injective too. Well, by summing, all one needs is maps $g'\colon K \to I$ and $g''\colon K \to J$ making the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "f'"'] & K \arrow[r] \arrow[dl, dashed, "g'"] \arrow[dr, dashed, "g''"'] & K'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & I \arrow[r] & I \oplus J \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

For this, we see that $g''$ is given by composition, and $g'$ is given because $K' \subseteq K$ and $I$ is injective object.

We now explain how the previous step proves the result. We immediately produce the needed map $A \to I^0 \oplus J^0$. Now to go from having the map $I^i \oplus J^i \to I^{i+1} \oplus J^{i+1}$ to having the map $I^{i+1} \oplus J^{i+1}$, we use the above paragraph on the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \frac{I^{i+1}}{I^i} \arrow[r] \arrow[d, hook] & \frac{I^{i+1}}{I^i} \oplus \frac{J^{i+1}}{J^i} \arrow[r] \arrow[d, dashed] & \frac{J^{i+1}}{J^i} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I^{i+2} \arrow[r] & I^{i+2} \oplus J^{i+2} \arrow[r] & J^{i+2} \arrow[r] & 0
\end{tikzcd}$$

This completes the construction of the needed short exact sequence of injective resolutions, from which the result follows upon using Proposition 1.22 on the short exact sequence of complexes

$$0 \to FI^\bullet \to FI^\bullet \oplus FJ^\bullet \to FJ^\bullet \to 0.$$

(This is still short exact because additive functors preserve split short exact sequences.) Note that we have not checked that the $\delta^\bullet$s are natural in the short exact sequence; this follows from the naturality of Proposition 1.22. ■

### 1.4.2 Acyclic Objects

We note the following computation.

**Proposition 1.44.** Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. If $I \in \mathcal{A}$ is injective, then $R^iF(I) = 0$ for all $i \geq 1$.

*Proof.* There is an injective resolution

$$0 \to I \to I \to 0 \to 0 \to \cdots$$

of $I$. Upon taking $F$, we see that $R^0F(I) = I/0$ and $R^1F(I) = 0/I$ and $R^iF(I) = 0/0$ for $i \geq 2$. This proves the result. ■

We now get the following definition.

**Definition 1.45** (acyclic)**.** Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. We say an object $A \in \mathcal{A}$ is *acyclic for $F$* if and only if $R^iF(A) = 0$ for all $i \geq 1$.

**Example 1.46.** If $A \in \mathcal{A}$ is injective, then Proposition 1.44 implies that $A$ is acyclic for any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$.

Here is the point of defining acyclic objects.

**Proposition 1.47.** Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with enough injectives. For any acyclic resolution $0 \to A \to I^\bullet$, there are canonical isomorphisms

$$R^iF(A) \cong h^i(FJ^\bullet).$$

*Proof.* Induct on $i$ using the long exact sequences. For example, there is nothing to say for $i = 0$. To get up to $i = 1$, use the exact sequence

$$0 \to A \xrightarrow{\varepsilon} J^0 \to \operatorname{coker} \varepsilon \to 0$$

to produce the needed long exact sequence

$$0 \to FA \to FJ^0 \to F \operatorname{coker} \varepsilon \to R^1F(A) \to 0,$$

and $h^1(FJ^\bullet)$ becomes the needed quotient. This process continues upwards. ■

### 1.4.3 A Little $\delta$-Functors

Here is our definition.

**Definition 1.48** ($\delta$-functor)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A $\delta$-*functor* consists of the data of some additive functors $T^i\colon \mathcal{A} \to \mathcal{B}$ for each $i \in \mathbb{N}$ and some morphisms $\delta^i\colon T^iA'' \to T^{i+1}A$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ such that there is a long exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & T^0A' \arrow[r] & T^0A \arrow[r] & T^0A'' \arrow[dll, "\delta^0"] \\ & T^1A' \arrow[r] & T^1A \arrow[r] & T^1A'' \arrow[r] & \cdots \end{tikzcd}$$

**Example 1.49.** If $\mathcal{A}$ has enough injective, the derived functors provide examples of $\delta$-functors by Theorem 1.43.

The following definition will be very helpful.

**Definition 1.50** (initial)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. A $\delta$-functor $(T^\bullet, \delta_T^\bullet)$ is *initial* if and only if any other $\delta$-functor $(U^\bullet, \delta_U^\bullet)$ together with a map $\varphi\colon T^0 \Rightarrow U^0$ has a unique sequence of natural transformations $\eta^\bullet\colon T^\bullet \Rightarrow U^\bullet$ extending $\varphi$ and commute with the formation of the long exact sequences. Explicitly, a short exact sequence $0 \to A' \to A \to A'' \to 0$ induces the following morphism of long exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & T^0A' \arrow[r] \arrow[d, "f^0"] & T^0A \arrow[r] \arrow[d, "f^0"] & T^0A'' \arrow[r, "\delta_T^0"] \arrow[d, "f^0"] & T^1A' \arrow[r] \arrow[d, "f^1"] & T^1A \arrow[r] \arrow[d, "f^1"] & \cdots \\ 0 \arrow[r] & U^0A' \arrow[r] & U^0A \arrow[r] & U^0A'' \arrow[r, "\delta_U^0"] & U^1A' \arrow[r] & U^1A \arrow[r] & \cdots \end{tikzcd}$$

Note that initial $\delta$-functors are unique up to unique isomorphism when they exist.

## 1.5 January 26

Today we will finish our discussion of right-derived functors.

### 1.5.1 Initial $\delta$-Functors

We will want to make some use of our discussion of $\delta$-functors.

**Definition 1.51** (effaceable)**.** Fix an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories. Then $F$ is *effaceable* if and only if each $A \in \mathcal{A}$ has a monomorphism $u\colon A \to M$ such that $Fu = 0$.

We have the following result which will help us check that right-derived functors are initial.

**Theorem 1.52.** Fix $\delta$-functor $(T^\bullet, \delta^\bullet)\colon \mathcal{A} \to \mathcal{B}$. If $T^\bullet$ is *effaceable* for all $i > 0$, then $(T^\bullet, \delta^\bullet)$ is initial.

*Proof.* Omitted. The proof is somewhat long and technical. We refer to [Wei94, Theorem 2.4.7] for most of the needed details. ■

**Corollary 1.53.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ has enough injectives. If $F\colon \mathcal{A} \to \mathcal{B}$ is left exact, the right-derived functors $(R^\bullet F, \delta^\bullet)$ is effaceable and thus initial.

*Proof.* By Theorem 1.52, it remains to show being effaceable. Well, for any object $A \in \mathcal{A}$, we can find a map $u\colon A \to I$ where $I$ is injective, so the map $R^i u\colon R^iU \to R^iI$ is the zero map for $i > 1$ because $R^iI = 0$ by Proposition 1.44. ■

**Corollary 1.54.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ has enough injectives. If $(T^\bullet, \delta^\bullet)$ is an initial $\delta$-functor, then $T^0$ is left exact, and $T^\bullet \simeq R^iT^0$ for all $i \geq 0$.

*Proof.* For any short exact sequence
$$0 \to A' \to A \to A'' \to 0,$$
being a $\delta$-fucntor implies the left exact sequence
$$0 \to T^0A' \to T^0A \to T^0A''.$$
Thus, $T^0$ is left exact. Now, the usual category theory arguments show that initial $\delta$-functors (when they exist) are unique up to unique isomorphism, so Corollary 1.53 completes the proof. ■

### 1.5.2 Having Enough Injectives

Let's show that some abelian categories have enough injectives. We begin with $\mathrm{Ab}$.

**Definition 1.55** (divisible)**.** An abelian group $A$ is *divisible* if and only if the multiplication-by-$n$ map $n\colon A \to A$ is surjective for all nonzero integers $n$.

**Example 1.56.** The groups $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{R}$, and $0$ are divisible.

Here is the point of this definition.

**Proposition 1.57.** An abelian group $A$ is injective in $\mathrm{Ab}$ if and only if $A$ is divisible.

*Proof.* We show our implications separately.

- Suppose $A$ is injective, and fix some $a \in A$ and nonzero integer $n \in \mathbb{Z}$ so that we want to find $a' \in A$ with $a = na'$. Well, we have the morphism $n\mathbb{Z} \to A$ given by $n \mapsto a$, but $n\mathbb{Z} \subseteq \mathbb{Z}$ means that the injectivity of $A$ forces $n\mathbb{Z} \to A$ to extend to $\mathbb{Z} \to A$, as follows.
$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[dr, "n\mapsto a"'] & \mathbb{Z} \arrow[d, dashed] \\ & & A \end{tikzcd}$$
Now, the image of $1$ along $\mathbb{Z} \to A$ can be called $a'$ and has $na' = a$ by construction.
- Suppose $A$ is divisible. We will use Zorn's lemma. Well, for our setup, suppose that we have an inclusion $M' \subseteq M$ and a map $\varphi\colon M' \to A$ which we would like to extend up to $M$.

  Let $\Phi$ be the collection of extensions of $\varphi\colon M' \to A$ to some subgroup $N \subseteq M$ containing $M'$, and order $\Phi$ by extension: we have $(N_1, \varphi_1) \preceq (N_2, \varphi_2)$ if and only if $N_1 \subseteq N_2$ and $\varphi_2|_{N_1} = \varphi_1$. Now, $\Phi$ is nonempty (it has $(M', \varphi)$), and its ascending chains are upper-bounded (the union of an extension of group homomorphisms will continue to be a group homomorphism), so Zorn's lemma provides $\Phi$ with a maximal element $(M'', \varphi'')$.

We claim that $M'' = M$, which will complete the proof. Well, we will show a contrapositive: suppose $(N, \psi) \in \Phi$ has $N \neq M$; then we claim that $(N, \psi)$ is not maximal. Well, given any $x \in M \setminus N$, we will extend $\psi$ to $N + \mathbb{Z}x$. Set $H := \{n \in \mathbb{Z} : nx \in N\}$. We have two cases.

- Suppose $H = 0$. Then $N + \mathbb{Z}x = N \oplus \mathbb{Z}x$, so we can extend $\psi$ by just setting $\psi(x) := 0$.
- Suppose $H = n\mathbb{Z}$ for some positive integer $n > 0$. Divisibility promises us some $a \in A$ such that $\psi(nx) = na$, so we would like to extend $\psi$ by $\psi(x) = a$. Namely, we would like to define $\widetilde{\psi} \colon (N + \mathbb{Z}x) \to A$ by
$$\widetilde{\psi}(m + kx) := \psi(m) + ka.$$
Of course, this will be a group homomorphism extending $\widetilde{\psi}$ provided that it is well-defined. Well, suppose $m + kx = m' + k'x$, and we want to show that $\psi(m) + ka = \psi(m') + k'a$, or equivalently, $\psi(m - m') = (k' - k)a$. We now note that $(k' - k)x = m - m' \in N$, so $k' - k = n\ell$ for some integer $\ell$ by construction of $n$, so we computed
$$(k' - k)a = n\ell a = \psi(n\ell x) = \psi((k' - k)x) = \psi(m - m'),$$
as needed. ■

**Theorem 1.58.** Fix a ring $R$. The category $\operatorname{Mod}(R)$ has functorial injectives.

*Proof.* We proceed in steps.

1. As an intermediate step, set $J := \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. Then we note that
$$\operatorname{Hom}_R(-, J) \simeq \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$$
by the hom–tensor adjunction. Additionally, if $M \neq 0$, we see that $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is nonzero by being an injective object, so the left-hand side is also nonzero. Lastly, the right-hand functor is exact, so the left-hand functor is exact, so we see that $J$ is injective.
We now set $A^\vee := \operatorname{Hom}_R(A, J)$.
2. So we will want to show that the map
$$\operatorname{ev}_\bullet \colon A \to A^{\vee\vee}$$
given by $\operatorname{ev}_a \colon \varphi \mapsto \varphi(a)$ is injective. Well, let $K := \ker \operatorname{ev}_\bullet$, and we draw the following commutative diagram.
$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "\operatorname{ev}_\bullet"'] & M \arrow[d, "\operatorname{ev}_\bullet"] \\ K^{\vee\vee} \arrow[r] & M^{\vee\vee} \end{tikzcd}$$
Because $(-)^\vee$ is an exact functor, we see that the bottom row must be injective. But the diagonal composite is zero, so actually $\operatorname{ev}_\bullet \colon K \to K^{\vee\vee}$ must be fully the zero map. Thus, $K = 0$ by the check in the previous step.
3. We actually construct the needed injection. Note we have a surjection
$$\bigoplus_{x \in A^\vee} R \twoheadrightarrow A^\vee,$$
so we have an injection
$$A \hookrightarrow A^{\vee\vee} \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}\left(\bigoplus_{x \in A^\vee} R, J\right) = J^{A^\vee}.$$

The right-hand side can be seen to be injective, so we are essentially done; notably, our construction is functorial in $A$. Explicitly, given a map $A \to B$, we induce a map $B^\vee \to A^\vee$, and taking fibers of this map induces a map $J^{A^\vee} \to J^{B^\vee}$. (Any coordinate in $A^\vee$ not in the image of $B^\vee$ can just get sent to $0$.) ■

## 1.6 January 29

Today we continue to show that categories have enough injectives.

### 1.6.1 Exactness in Abelian Categories

Let's say a few more things about abelian categories.

**Example 1.59.** Fix an abelian category $\mathcal{A}$. Then $\mathcal{A}$ has an empty biproduct $0$, which is both initial and final by its definition. We will not bother to write out the identification of biproducts in additive categories.

**Remark 1.60.** Fix an abelian category $\mathcal{A}$. Any morphism $\varphi\colon A \to B$ can be factored as $\nu \circ \eta$ where $\eta\colon A \to X$ is epic and $\nu\colon X \to B$ is monic. To see that this factorization exists, we can set $\eta = \operatorname{coker}(\ker \varphi)$ and $\nu = \ker(\operatorname{coker} \varphi)$. Additionally, the factorization $\nu \circ \eta$ is unique in the following sense: if $\eta'\colon A \to X'$ and $\nu'\colon X' \to B$ is another such factorization, there is a unique isomorphism $\psi\colon X \to X'$ making the following diagram commute.

$$\begin{tikzcd} & X \arrow[dd, "\psi"] \arrow[dr, "\nu"] & \\ A \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & & B \\ & X' \arrow[ur, "\nu'"'] & \end{tikzcd}$$

**Remark 1.61.** The previous remark implies that being an isomorphism is equivalent to being both monic and epic. Namely, one just factors the given morphism $\varphi\colon A \to B$ in the two ways $\operatorname{id}_B \circ \varphi = \varphi \circ \operatorname{id}_A$ to conclude that $\varphi$ has an inverse.

The prior two remarks allow us to make sense of exactness in a meaningful way.

**Definition 1.62** (exact)**.** Fix morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, and factor these as $\varphi = \nu \circ \eta$ and $\psi = \mu \circ \varepsilon$ where $\nu$ and $\mu$ are epic and $\eta$ and $\varepsilon$ are monic. Here is the diagram.

$$\begin{tikzcd} & X \arrow[dr, "\nu"] & & Y \arrow[dr, "\varepsilon"] & \\ A \arrow[ur, "\eta"] \arrow[rr, "\varphi"'] & & B \arrow[ur, "\mu"] \arrow[rr, "\psi"'] & & C \end{tikzcd}$$

Then the sequence

$$A \to B \to C$$

is *exact* if and only if $\nu = \ker \varepsilon$; this is equivalent to asking for $\varepsilon = \operatorname{coker} \nu$.

The equivalence of these two notions follows by the uniqueness of the factorization. Note that this is approximately the correct notion because we really want to say that $\varphi$ surjects onto the kernel of $\psi$. But then we note $\nu$ basically acts as the image of $\varphi$, and $\varepsilon$ basically acts as the kernel of $\psi$.

### 1.6.2 Sheaves Have Enough Injectives

We now move up to sheaves.

**Theorem 1.63.** Fix a ringed space $(X, \mathcal{O}_X)$. Then the category $\operatorname{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules has functorial injectives.

*Proof.* Fix a sheaf $\mathcal{F} \in \operatorname{Mod}(\mathcal{O}_X)$. For each $x \in X$, recall that $\operatorname{Mod}(\mathcal{O}_{X,x})$ has functorial injectives by Theorem 1.58, so we let $I_x(\mathcal{F}_x)$ be an injective module into which $\mathcal{F}_x$ injects. Letting $j_x : \{x\} \to X$ denote the inclusion map, we then define

$$\mathcal{I} \coloneqq \prod_{x \in X} (j_x)_* I_x(\mathcal{F}_x).$$

Note that this is an $\mathcal{O}_X$-module because it is the product of $\mathcal{O}_X$-modules. Note that there is a naturally defined map $i : \mathcal{F} \to (j_x)_* I_x(\mathcal{F}_x)$ defined by the composite

$$\mathcal{F}(U) \to \mathcal{F}_x \to I_x(\mathcal{F}_x)$$

for each $x \in U$ (and we get the zero map for $x \notin U$). This map $i$ is injective on stalks: we can see that $\mathcal{F}_x$ will embed into the coordinate $(j_x)_* I_x(\mathcal{F}_x)$. Additionally, this construction of $i$ is functorial.

As such, it just remains to show that $\mathcal{I}$ is injective. Suppose that $\mathcal{G} \in \operatorname{Mod}(\mathcal{O}_X)$, and we compute

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \simeq \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* I_x(\mathcal{F}_x)) \simeq \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x(\mathcal{F}_x)).$$

Now, each $I_x(\mathcal{F}_x)$ is an injective object, so the functors $\operatorname{Hom}_{\mathcal{O}_{X,x}}(-, I_x(\mathcal{F}_x))$ is an exact functor for each $x \in X$, so the total functor above is exact, as needed. ■

**Corollary 1.64.** Fix a topological space $X$. Then the category $\operatorname{Ab}(X)$ of category of sheaves of abelian groups on $X$ has functorial injectives.

*Proof.* Set $\mathcal{O}_X$ to be the constant sheaf $\underline{\mathbb{Z}}$ on $X$. Then $\mathcal{O}_X$ is a sheaf of rings, and $\mathcal{O}_X$-modules are exactly sheaves of abelian groups, so the result follows from Theorem 1.63. ■

### 1.6.3 Sheaf Cohomology

We can finally define sheaf cohomology.

**Definition 1.65** (sheaf cohomology)**.** Fix a topological space $X$. Because $\operatorname{Ab}(X)$ has enough injectives (by Corollary 1.64) and $\Gamma(X, -) : \operatorname{Ab}(X) \to \operatorname{Ab}$ is left exact, we define the *sheaf cohomology functors* as

$$H^\bullet(X, -) \coloneqq R^\bullet \Gamma(X, -).$$

**Remark 1.66.** It is rather hard to compute $H^\bullet(X, -)$ directly from the definition. For example, it will be helpful to build a large class of acyclic objects and then use Proposition 1.47.

To realize the above remark, we have the following definition.

**Definition 1.67** (flasque)**.** Fix a sheaf $\mathcal{F}$ on a topological space $X$. Then $\mathcal{F}$ is *flasque* if and only if its restriction maps are surjective.

## 1.7 January 31

Here we go.

### 1.7.1 Flasque Resolutions

We have "already" seen many examples of flasque sheaves, as explained in the following lemma.

**Lemma 1.68.** Fix a ringed space $(X, \mathcal{O}_X)$. Then any injective $\mathcal{O}_X$-module is flasque.

*Proof.* Fix an open subset $U \subseteq X$, and let $j\colon U \hookrightarrow X$ be the inclusion so that we may set $\mathcal{O}_U := j_!(\mathcal{O}_X|_U)$. Notably, we are realizing $\mathcal{O}_X$ as an $\mathcal{O}_U$-module.

Let's quickly review $j_!$. Explicitly, for a sheaf $\mathcal{F}$ on $U$, we defined $j_!\mathcal{F}$ as "extension by zero": it is the sheafification of the presheaf

$$\mathcal{O}_U^{\text{pre}}(W) :=\mapsto \begin{cases} \mathcal{F}(W) & \text{if } W \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

Notably, the stalks of $j_!\mathcal{F}$ are $\mathcal{F}_x$ if $x \in U$ and $0$ otherwise, which we can see by working with the above sheaf. Notably, there is a canonical map $\mathcal{F} \to (j_!\mathcal{F})|_U$, which we can see is an isomorphism by checking on stalks.

We now proceed with the proof. For each open $V \subseteq U$, there is an injection $\mathcal{O}_V \hookrightarrow \mathcal{O}_U$ (we can see that this is an injection by checking on stalks). As such, for our injective sheaf $\mathcal{I}$, we get the following commutative diagram.

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I}) \arrow[d] \\ \mathcal{I}(U) \arrow[r] & \mathcal{I}(V) \end{tikzcd}$$

We claim that we can place vertical isomorphisms, which will complete the proof because the top row is surjective because $\mathcal{I}$ is an injective $\mathcal{O}_X$-module.

Well, for the vertical morphisms, we write

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) &= \operatorname{Hom}_{\mathcal{O}_X}\left(\mathcal{O}_U^{\text{pre}}, \mathcal{I}\right) \\ &\overset{*}{=} \operatorname{Hom}_{\mathcal{O}_X|_U}\left(\mathcal{O}_U^{\text{pre}}|_U, \mathcal{I}|_U\right) \\ &= \operatorname{Hom}_{\mathcal{O}_X|_U}\left(\mathcal{O}_X|_U, \mathcal{I}|_U\right) \\ &= \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U), \mathcal{I}(U)) \\ &= \mathcal{I}(U). \end{aligned}$$

Here, $\overset{*}{=}$ is just given by restricting an $\mathcal{O}_X$-morphism; it is injective because the map $\mathcal{O}_U \to \mathcal{I}$ can be determined by how it behaves on stalks, which are only seen on $U$, and it is surjective because one can extend a map $\mathcal{O}_U^{\text{pre}}|_U \to \mathcal{I}|_U$ to a full map $\mathcal{O}_U^{\text{pre}} \to \mathcal{I}$ by having the map $\mathcal{O}_U^{\text{pre}}(W) \to \mathcal{I}(W)$ just be zero (which of course is the only option!). ■

Anyway, let's put our flasque sheaves to good use.

**Lemma 1.69.** Fix a topological space $X$. Any flasque sheaf $\mathcal{F} \in \operatorname{Ab}(X)$ is acyclic for $H^\bullet(X, -)$.

*Proof.* This is a matter of dimension-shifting. We claim that $H^i(X, \mathcal{F}) = 0$ for all $i \geq 1$ and flasque sheaves $\mathcal{F}$. We proceed by induction on $i$, so we may assume the result for indices less than $i$. Now, find an injective sheaf with an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$. Letting $\mathcal{G}$ be the quotient, we produce the exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

The two middle terms are flasque (see Lemma 1.68), so the right term is flasque. Now, [Har77, Exercise 1.16] tells us that $\mathcal{G}$ is flasque, and the sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0$$

is exact. Now, the long exact sequence produces the exact sequence

$$H^{i-1}(X, \mathcal{I}) \to H^{i-1}(X, \mathcal{G}) \to H^i(X, \mathcal{F}) \to \underbrace{H^i(X, \mathcal{I})}_{0}.$$

The previous exact sequence shows that the map $H^{i-1}(X, \mathcal{I}) \to H^{i-1}(X, \mathcal{G})$ is surjective map for $i = 1$, and it continues to be surjective for other $i$ by the induction (namely, both terms will be zero). Thus, the map $H^{i-1}(X, \mathcal{G}) \to H^i(X, \mathcal{F})$ is the zero map, so we conclude that $H^i(X, \mathcal{F}) = 0$. ■

And here is the promised sanity check.

**Proposition 1.70.** Fix a ringed space $(X, \mathcal{O}_X)$. Then $H^\bullet(X, -) = R^\bullet\Gamma(X, -)$, where now $\Gamma(X, -)$ is a functor $\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Ab}$.

*Proof.* An injective resolution in $\mathrm{Mod}(\mathcal{O}_X)$ is a flasque resolution by Lemma 1.68 and hence an acyclic resolution in $\mathrm{Ab}(X)$ by Lemma 1.69. So Proposition 1.47 completes the proof. ■

**Remark 1.71.** A priori, the objects $H^\bullet(X, -)$ were just abelian groups, but Proposition 1.70 assures us that we can usually give this more structure. In particular, if $X$ is an $A$-scheme for a ring $A$, then actually $\Gamma(X, -)$ is a functor $\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(A)$, and the right-derived functors for this $\Gamma(X, -)$ agree with $H^\bullet(X, -)$ upon passing through the forgetful functor because the forgetful functor is exact (namely sending cohomology to cohomology).

### 1.7.2 Directed Colimits

We would like for our cohomology to vanish at high dimensions when $X$ is a finite-dimensional scheme. The following lemma will be useful.

**Lemma 1.72.** Fix a Noetherian topological space $X$ and a directed system $\{\mathcal{F}_\alpha\}_{\alpha \in \Lambda}$ of flasque sheaves. Then the directed limit $\varinjlim \mathcal{F}_\alpha$ is flasque.

*Proof.* Quickly, because colimits commute with colimits, we see that

$$\left(\varinjlim \mathcal{F}_\alpha\right)(U) = \varinjlim \mathcal{F}_\alpha(U).$$

(In particular, this is a sheaf, and then it satisfies the needed universal property by construction; the above equality requires $X$ to be Noetherian.) Now, fix open subsets $V \subseteq U$. Then $\varinjlim \mathcal{F}_\alpha(U) \to \varinjlim \mathcal{F}_\alpha(V)$ is surjective because it is surjective on the components (and colimits commute with colimits), so the above description of our sections completes the proof. ■

## 1.8 February 2

Let's just get this over with.

### 1.8.1 More on Directed Colimits

We continue our discussion towards Grothendieck vanishing. We can now see that directed colimits commutes with cohomology.

**Proposition 1.73.** Fix a Noetherian topological space $X$. Given a directed system $\{\mathcal{F}_\alpha\}_{\alpha\in\Lambda}$ of sheaves in $\operatorname{Ab}(X)$, there is a natural isomorphism

$$\varinjlim H^\bullet(X,\mathcal{F}_\alpha) = H^\bullet\left(X, \varinjlim \mathcal{F}_\alpha\right)$$

compatible in the long exact sequence.

*Proof.* For convenience, let $\mathcal{C}$ be the category of directed systems in $\operatorname{Ab}(X)$ indexed by $\Lambda$. We would like to exhibit an isomorphism

$$\varinjlim H^\bullet(X,-) \simeq H^\bullet\left(X, \varinjlim -\right)$$

of $\delta$-functors $\mathcal{C} \to \operatorname{Ab}$.

Quickly, we note that we can take a sheaf $\mathcal{F}$ and map it to its "sheaf of discontinuous sections" given by

$$U \mapsto \prod_{x\in U} \mathcal{F}_x.$$

This construction is functorial and can be repeated, so we get functorial flasque resolutions in $\mathcal{F}$.

In particular, let $\mathcal{G}_\alpha^\bullet$ be the produced flasque resolution of $\mathcal{F}_\alpha$. Thus, using Lemma 1.69 with Proposition 1.47 to compute our cohomology, we see

$$\varinjlim H^i(X,\mathcal{F}_\alpha) \simeq \varinjlim h^i(\Gamma(X,\mathcal{G}_\alpha^\bullet)).$$

Now, taking directed colimits is exact, so this is

$$h^i\left(\varinjlim \Gamma(X,\mathcal{G}_\alpha^\bullet)\right).$$

Taking global sections commutes with directed colimits (here we use that $X$ is Noetherian with [Har77, Exercise 1.11]), so this is

$$h^i\left(\Gamma\left(X, \varinjlim \mathcal{G}_\alpha^\bullet\right)\right).$$

Now, taking these directed colimits commutes with taking stalks, so it will be exact on sheaves, so we have the resolution

$$0 \to \varinjlim \mathcal{F}_\alpha \to \varinjlim \mathcal{G}_\alpha^\bullet,$$

so our last cohomology is the desired $H^i\left(X, \varinjlim \mathcal{F}_\alpha\right)$. Everything has been done on the level of resolutions, so we have produced a bona fide isomorphism of $\delta$-functors. ■

**Example 1.74.** Cohomology also commutes with infinite direct sums because these are directed colimits (of the finite sums!).

### 1.8.2 Cohomology on Closed Subsets

Next up, one reduction we will want to make is to go down closed subschemes, so we have the following.

**Lemma 1.75.** Fix a closed subset $j\colon Y \to X$ of a topological space. Given a sheaf $\mathcal{F} \in \mathrm{Ab}(Y)$, there is a natural isomorphism

$$H^\bullet(Y, \mathcal{F}) = H^\bullet(X, j_*\mathcal{F})$$

compatible in the long exact sequence.

*Proof.* We are asking for an isomorphism of $\delta$-functors $\mathrm{Ab}(Y) \to \mathrm{Ab}$. The point is that, by computing stalks, $j_*$ is an exact functor, and by computing sections, $j_*$ sends flasque sheaves to flasque sheaves. So we use the usual combination of Lemma 1.69 with Proposition 1.47 so that a flasque resolution $\mathcal{G}^\bullet$ of $\mathcal{F}$ produces the sequence of natural isomorphisms

$$H^i(Y, \mathcal{F}) \simeq h^i(\Gamma(Y, \mathcal{G}^\bullet)) = h^i(\Gamma(X, j_*\mathcal{G}^\bullet)) \simeq H^i(X, j_*\mathcal{F}).$$

Everything was done on the level of resolutions, so this is an isomorphism of $\delta$-functors. ■

**Remark 1.76.** If $Y \subseteq X$ is not closed, $j_*$ need not be exact.

With our closed subsets, we will want the notion of restricting sheaves.

**Definition 1.77.** Fix a topological space $X$ and closed subset $i\colon Z \to X$ and open subset $j\colon U \to X$ where $U = X \setminus Z$. For a sheaf $\mathcal{F}$ on $X$, we set $\mathcal{F}_Z := i_*(\mathcal{F}|_Z)$ and $\mathcal{F}_U := j_!(\mathcal{F}|_U)$.

**Remark 1.78.** Fix everything as above. Computing stalks, we see that there is an exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0$$

of sheaves on $X$, provided $\mathcal{F} \in \mathrm{Ab}(X)$. We will not bother to give the construction of the maps; they can be given on the level of presheaves, where the left map is essentially an inclusion, and the right map is essentially a restriction.

## 1.9 February 5

Here we go.

### 1.9.1 Grothendieck Vanishing

Last class we began the proof of the following result, which I have moved to today because it will be our focus for today.

**Theorem 1.79 (Grothendieck vanishing).** Fix a Noetherian topological space $X$ of dimension $n$. Given $\mathcal{F} \in \mathrm{Ab}(X)$, we have $H^i(X, \mathcal{F}) = 0$ for $i > n$.

*Proof.* We proceed by induction on the collection of pairs $(n, m) \in \{(-1, 0)\} \cup \mathbb{N} \times \mathbb{Z}^+$, where $n = \dim X$ and $m$ is the number of irreducible components of $X$. For our induction, we order $\{(-1, 0)\} \cup \mathbb{N} \times \mathbb{Z}^+$ lexicographically; here $\dim \varnothing = -1$. In other words, we will induct on the dimension, and within that induction, we will induct on the number of irreducible components.

Anyway, we proceed in steps.

1. We begin by reducing to $X$ being irreducible; fix $X$ of dimension $n$, and assume all lower results (lower dimension, fewer irreducible components if of dimension $n$). We may assume that $X$ is nonempty, so choose an irreducible component $Z \subseteq X$, and set $U := X \setminus Z$. Notably, for this paragraph (making the reduction), we are assuming the statement for $Z$, so any sheaf $\mathcal{F} \in \mathrm{Ab}(X)$ has the exact sequence

$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0$$

by Remark 1.78. By the long exact sequence, it will be enough to show that $H^i(X, \mathcal{F}_Z) = H^i(X, \mathcal{F}_U) = 0$ for all $i > n$. For one, note that $H^i(X, \mathcal{F}_Z) = 0$ for all $i > n$ because $H^i(X, \mathcal{F}_Z) = H^i(Z, \mathcal{F}|_Z)$ by Lemma 1.75, and we have assumed the conclusion for $Z$.

So it remains to discuss $H^\bullet(X, \mathcal{F}_U)$. We begin by claiming that there is a sheaf $\mathcal{G}$ on $\overline{U}$ such that $\mathcal{F}_U = j_*\mathcal{G}$ where $j \colon \overline{U} \hookrightarrow X$ is the inclusion. Indeed, Remark 1.78 provides an exact sequence

$$0 \to (\mathcal{F}_U)_{X \setminus \overline{U}} \to \mathcal{F}_U \to (\mathcal{F}_U)_{\overline{U}} \to 0,$$

but $(\mathcal{F}_U)_{X\setminus\overline{U}} = 0$ by computing stalks: any nonzero stalk must have $p \in X \setminus \overline{U}$ and $(\mathcal{F}|_U)_p \neq 0$ and hence $p \in U$ also, but no such $p$ suffices. Thus, we see

$$\mathcal{F}_U \cong (\mathcal{F}_U)_{\overline{U}} \cong j_*(\mathcal{F}_U|_{\overline{U}}),$$

so $\mathcal{G} := \mathcal{F}_U|_{\overline{U}}$ will do the trick.

We are now ready to show that $H^i(X, \mathcal{F}_U) = 0$ for $i > n$. Well, by the previous paragraph, we see

$$H^i(X, \mathcal{F}_U) = H^i(X, j_*(\mathcal{F}_U|_{\overline{U}})) = H^i(\overline{U}, \mathcal{F}_U|_{\overline{U}})$$

by Lemma 1.75. But now, by inductive hypothesis, this vanishes for $i < n$ because $\overline{U}$ has one fewer irreducible component than $X$ and no higher dimension.

2. We handle some base cases. When $\dim X = -1$, we have $X = \varnothing$, where there is nothing to do. We will also handle $\dim X = 0$ while we're here. The previous step allows us to assume that $X$ is irreducible.

Quickly, we remark that the only closed subsets of $X$ are $\{\varnothing, X\}$. Indeed, of course these sets are closed. Conversely, if $Z \subseteq X$ is a minimally closed subset, then minimality forces $Z$ to be irreducible, but then $\varnothing \subseteq Z \subseteq X$ requires $Z \in \{\varnothing, X\}$ because $\dim X = 0$.

Thus, the previous paragraph implies that $X$ has the indiscrete topology. In particular, all sheaves are flasque because evaluating a sheaf on $\varnothing$ makes a single point, so $H^i(X, -)$ vanishes for $i > 0$, as needed.

3. Fix a sheaf $\mathcal{F}$ of abelian groups on $X$; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case where $\mathcal{F}$ is finitely generated (as a sheaf—namely, there are finitely many sections such that the restrictions of those sections generate $\mathcal{F}(U)$ for all open $U \subseteq X$). Well, define the set

$$B := \bigcup_{\text{open } U \subseteq X} \mathcal{F}(U),$$

and let $A$ denote the collection of finite subsets. Notably, by restriction, $A$ becomes a set which is directed by the collection of open sets on $X$. Anyway, for $\alpha \in A$, let $\mathcal{F}_\alpha$ denote the sheaf generated by the sections in $\alpha$, and we conclude by noting that $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$, so

$$H^i(X, \mathcal{F}) = \varinjlim H^i(X, \mathcal{F}_\alpha)$$

by Proposition 1.73, which vanishes for $i > \dim X$ by assumption of this step.

4. Fix a finitely generated sheaf $\mathcal{F}$ of abelian groups on $X$; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case where $\mathcal{F}$ is generated by a single section (and its restrictions). Indeed, assuming we have the case of generated by a single section, we may proceed by induction: if $\mathcal{F}$ is generated

by $n$ sections $\alpha$ (so that $\mathcal{F} = \mathcal{F}_\alpha$), let $\alpha' \subsetneq \alpha$ be a proper subset of sections, and let $\mathcal{F}_{\alpha'}$ be the sheaf generated by $\alpha'$. Then we have the exact sequence

$$0 \to \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G} \to 0,$$

and we note that $\mathcal{G}$ is generated by fewer than $n$ elements; explicitly, $\mathcal{G}$ is generated by the images of the sections in $\alpha \setminus \alpha'$. To be explicit, one can see that $\mathcal{F}_{\alpha\setminus\alpha'} \twoheadrightarrow \mathcal{G}$ by checking on stalks. Thus, $H^i(X, \mathcal{F}_{\alpha'}) = H^i(X, \mathcal{G}) = 0$ for $i > \dim X$ by assumption, so the long exact sequence enforces $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.

5. Fix a sheaf $\mathcal{F}$ of abelian groups on $X$ generated by a single section $s \in \mathcal{F}(U)$ where $U \subseteq X$ is open; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case of subsheaves of $\underline{\mathbb{Z}}_U$.

   Well, we may assume that $U$ is nonempty (or else $\mathcal{F} = 0$, and there is nothing to be done). Now, there is a map $\underline{\mathbb{Z}}_U \to \mathcal{F}$ given by sending $1 \mapsto s$ on $U$ (working on the presheaf) and then appropriately restricting elsewhere. This map is surjective by hypothesis on $\mathcal{F}$ (indeed, it is surjective on the level of the presheaves), so we let $\mathcal{K}$ denote the kernel, providing the short exact sequence

$$0 \to \mathcal{K} \to \underline{\mathbb{Z}}_U \to \mathcal{F} \to 0.$$

   Now, $H^i(X, \mathcal{K}) = H^i(X, \underline{\mathbb{Z}}_U) = 0$ for $i > \dim X$ by assumption of this section, so $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$ by the long exact sequence.

6. In this reduction step, we will use that $X$ is irreducible. Fix a subsheaf $\mathcal{F}$ of $\underline{\mathbb{Z}}_U$ for open $U \subseteq X$; we need $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$. We reduce to the case $\mathcal{F} = \underline{\mathbb{Z}}_U$. If $\mathcal{F} = 0$, there is nothing to do.

   Otherwise, we may let $d$ be the smallest positive integer such that $d \in \mathcal{F}_x$ as $x \in U$ varies (notably, some $\mathcal{F}_x$ is nonzero, so a $d$ exists). Now,

$$V := \{x \in U : d \in \mathcal{F}_x\}$$

   is nonempty and open ($d \in \mathcal{F}_x$ means that $d \in \mathcal{F}(U')$ for some open $U' \subseteq U$), and $\mathcal{F}_x = d\mathbb{Z}$ for each $x \in V$, so $\mathcal{F}|_V = d\underline{\mathbb{Z}}$, so we have an equality $\mathcal{F}_V = d\underline{\mathbb{Z}}_V$. So we have an exact sequence

$$0 \to d\underline{\mathbb{Z}}_V \to \mathcal{F} \to \mathcal{F}/d\underline{\mathbb{Z}}_V \to 0$$

   of sheaves on $X$. By assumption, we have the result for $d\underline{\mathbb{Z}}_V$. Now, $\mathcal{F}/d\underline{\mathbb{Z}}_V$ is supported on $\overline{U \setminus V}$ by construction of $V$, and $\overline{U \setminus V}$ will have smaller dimension than $X$ because $X$ is irreducible, so we get the result for $\mathcal{F}/d\underline{\mathbb{Z}}_V$ by the inductive hypothesis on $X$. So the long exact sequence purchases the result for $\mathcal{F}$.

7. We complete the induction. We may assume that $X$ is irreducible of dimension $n$, and we may assume that $\mathcal{F} = \underline{\mathbb{Z}}_U$. Well, we have an exact sequence

$$0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{X\setminus U}$$

   by Remark 1.78. Because $X$ is irreducible, $X \setminus U$ has smaller dimension, $H^i(X, \underline{\mathbb{Z}}_{X\setminus U}) = 0$ for $i > \dim X - 1$ (also using Lemma 1.75). Additionally, $\underline{\mathbb{Z}}$ is flasque because $X$ is irreducible—all open subsets of $X$ are connected, so $\underline{\mathbb{Z}}(U) = \mathbb{Z}$ always—and hence acyclic by Lemma 1.69, so we conclude $H^i(X, \underline{\mathbb{Z}}_U) = 0$ for $i > \dim X$ by the long exact sequence. ■

While we're here, let's do an example computation to show Theorem 1.79 is sharp.

**Exercise 1.80.** Fix a field $k$, and set $X := \mathbb{A}^1_k$. Given distinct points $P, Q \in X$, set $U := X \setminus \{P, Q\}$, and we see

$$H^1(X, \underline{\mathbb{Z}}_U) \neq 0.$$

*Proof.* Let $j \colon U \hookrightarrow X$ denote the inclusion so that $\underline{\mathbb{Z}}_U = j_!(\underline{\mathbb{Z}})$. Note that $U$ is irreducible, so any open subsets are connected, so we may as well have

$$\underline{\mathbb{Z}}_U(V) = \begin{cases} \mathbb{Z} & \text{if } V \subseteq U \text{ and } V \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Anyway, note that we have the exact sequence

$$0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{\{P,Q\}} \to 0$$

by Remark 1.78, so we get a long exact sequence

$$H^0(X, \underline{\mathbb{Z}}_U) \to H^0(X, \underline{\mathbb{Z}}) \to H^0(X, \underline{\mathbb{Z}}_{\{P,Q\}}) \to H^1(X, \underline{\mathbb{Z}}_U) \to H^1(X, \underline{\mathbb{Z}}).$$

Because $X$ is irreducible, we see that $\underline{\mathbb{Z}}$ is flasque (all open subsets are connected), so the rightmost term vanishes by Lemma 1.69. Also, above we noted that $H^0(X, \underline{\mathbb{Z}}_U) = 0$, and computing global sections on the other sheaves implies that we have

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^1(X, \underline{\mathbb{Z}}_U) \to 0,$$

so the rightmost map cannot be surjective. ■

# Theme 2
# Cohomology on Schemes

## 2.1 February 7

Today we will compute cohomology on affine schemes.

### 2.1.1 Cohomology on Affine Schemes

To build our cohomology on $\operatorname{Spec} A$, we pick up the following checks.

**Proposition 2.1.** Fix an injective $A$-module $I$, where $A$ is Noetherian. Then $\widetilde{I}$ is a flasque sheaf on $\operatorname{Spec} A$.

*Proof.* This proof is somewhat annoying, so we omit and refer to [Har77, Proposition III.3.4]. The main idea is to do Noetherian induction on $\operatorname{Supp} \widetilde{I}$. ■

**Proposition 2.2.** Fix a Noetherian ring $A$. Then quasicoherent sheaves on $X := \operatorname{Spec} A$ are acyclic.

*Proof.* Fix an $A$-module $M$, and we want to show that $\widetilde{M}$ is acyclic. Well, fix an injective resolution $0 \to M \to I^\bullet$, which by Proposition 2.1 produces an acyclic resolution

$$0 \to \widetilde{M} \to \widetilde{I}^\bullet.$$

Lemma 1.69 followed by Proposition 1.47 allows us to compute cohomology using this resolution, but this is just

$$0 \to M \to I^\bullet$$

upon taking global sections, which is exact, so the cohomology of $\widetilde{M}$ must vanish. ■

Thus, we see quasicoherent sheaves on affine schemes are well-behaved. This turns out to characterize affine schemes. Before doing anything, we will want the following lemma.

**Lemma 2.3.** Fix a quasicompact scheme $X$. Then any closed subscheme $V \subseteq X$ has a closed point.

*Proof.* Fix a finite affine open cover $\{U_i\}_{i=1}^n$ of $X$, doable because $X$ is quasicompact, and we may suppose that no affine open subset is covered by the union of the other ones (for then we could remove this open subset from the finite collection). Then we see that

$$U_1 \setminus (U_2 \cup U_3 \cup \cdots \cup U_n)$$

is a closed nonempty subset of $U_1$, so it has a closed point $p \in U_1$ corresponding to a maximal ideal of $\Gamma(U_1, \mathcal{O}_{U_1})$ which contains the ideal cut out by the complement of $(U_2 \cup \cdots \cup U_n)$. We claim that $p$ is still closed in $X$.

Well,

$$X \setminus \{p\} = \bigcup_{i=1}^n (U_i \setminus \{p\}) = (U_1 \setminus \{p\}) \cup \bigcup_{i=2}^n U_i$$

is open, so we are okay. ■

**Theorem 2.4** (Serre)**.** Fix a Noetherian scheme $X$. Then the following are equivalent.

(i) $X$ is affine.

(ii) $H^i(X, \mathcal{F}) = 0$ for all quasicoherent sheaves $\mathcal{F}$ on $X$ and indices $i > 0$.

(iii) $H^1(X, \mathcal{I}) = 0$ for all quasicoherent sheaves $\mathcal{I}$ of ideals on $X$.

*Proof.* Note (i) implies (ii) is Proposition 2.2, and (ii) implies (iii) with no content. So the main content of the argument is (iii) implies (i). We proceed in steps.

1. To set ourselves up, we recall [Har77, Exercise II.2.17], which is on the homework, which asserts that $X$ is affine if and only if there is a finite set $\{f_1, \ldots, f_r\}$ of global sections generating $A := \Gamma(X, \mathcal{O}_X)$ such that the open subschemes
$$X_{f_i} := \{x \in X : (f_i)_x \notin \mathfrak{m}_x\}$$
are affine.

2. We claim that all closed points $p \in X$ have some $f \in A$ such that $X_f$ is affine and $p \in X_f$. Well, let $U \subseteq X$ be some affine open neighborhood of $p \in X$, and let $Y := X \setminus U$. Note that we have a short exact sequence
$$0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(p) \to 0$$
of sheaves on $X$, where $\mathcal{I}_\bullet$ is the ideal sheaf of a closed subscheme, and $k(p)$ refers to the skyscraper sheaf at $p$. Notably exactness of this sequence can be checked on stalks; more explicitly, the left map is just an isomorphism on $X \setminus \{p\}$, and on $\{p\}$ this is $0 \to \mathfrak{m}_p \to \mathcal{O}_{X,p} \to k(p) \to 0$, which is exact because $p$ is closed (!). Now, by (iii), we have an exact sequence
$$H^0(X, \mathcal{I}_Y) \to H^0(X, k(p)) \to \underbrace{H^1(X, \mathcal{I}_{Y \cup \{p\}})}_{0},$$
so we can find some $f \in \Gamma(X, \mathcal{I}_Y)$ such that $f \notin \mathfrak{m}_p$ by this surjectivity, so $p \notin X_f$ by construction. But now $f \in \mathcal{I}_Y$ means that $Y \cap X_f \neq \varnothing$, so $X_f \subseteq U$, and in fact, we see $U_f \subseteq X_f \subseteq U_f$, so $X_f$ is affine (because $U$ is affine means $U_f$ is affine), so we have completed the proof of the claim.

3. We exhibit a finite set $\{f_1, \ldots, f_r\} \subseteq A$ of global sections such that $X_{f_\bullet}$ are affine and cover $X$. Well, let $X_{\text{cl}}$ denote the set of closed points of $X$, and Lemma 2.3 shows that each $p \in X_{\text{cl}}$ has some $f_p \in A$ such that $p \in X_{f_p}$ and $X_{f_p}$ is affine.
We want $\{X_{f_p}\}_{p \in X_{\text{cl}}}$ to cover $X$, so we consider
$$Z := X \Bigg\backslash \bigcup_{p \in X_{\text{cl}}} X_{f_p},$$

which is a closed subset of $X$, so we give $Z$ the reduced subscheme structure. Well, suppose for the sake of contradiction that $Z$ is nonempty. Because $X$ is quasicompact, so is the closed subset $Z$, so $Z$ has a closed point $p \in Z$. But then $p$ is still closed in $X$: we know $Z \setminus \{p\}$ is open in $X$, so there is an open $U \subseteq X$ such that $Z \setminus \{p\} = Z \cap U$, so $X \setminus \{p\} = (X \setminus Z) \cup U$. This is a contradiction because $p \in X_{f_p}$ and so cannot be in $Z$.

We are now done: $X$ is quasicompact, so we can reduce $\{X_{f_p}\}_{p \in X_{\text{cl}}}$ to a finite subcover, which completes this step and hence the proof.

4. We complete the proof. In particular, to plug into the first step, fix $\{f_1, \ldots, f_r\}$ as in the previous step, and we must show that $(f_1, \ldots, f_r) = A$.

   We want to show that the map $\alpha \colon \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X$ given by $(a_1, \ldots, a_r) \mapsto (a_1 f_1 + \cdots + a_r f_r)$ is surjective on global sections. Certainly $\alpha$ is surjective on stalks: any $x \in X$ can be placed in some $X_{f_r}$, and then $f_r \colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is already an isomorphism. But now $\mathcal{K} := \ker \alpha$ produces the exact sequence

$$\Gamma\left(X, \mathcal{O}_X^{\oplus r}\right) \stackrel{\alpha}{\to} \Gamma(X, \mathcal{O}_X) \to H^1(X, \mathcal{K}).$$

   To complete this step, we would like to know that $H^1(X, \mathcal{K}) = 0$.

   Namely, we will claim that $H^1\left(X, \mathcal{K} \cap \mathcal{O}_X^{\oplus i}\right) = 0$ for each $i \in \{0, \ldots, r\}$ by induction. There is nothing to say for $i = 0$. Then if $H^1(X, \mathcal{F} \cap \mathcal{O}_X^{\oplus(i-1)}) = 0$, we have an exact sequence

$$0 \to \mathcal{K} \cap \mathcal{O}_X^{\oplus(i-1)} \to \mathcal{K} \cap \mathcal{O}_X^{\oplus i} \to \mathcal{Q}_i \to 0,$$

   where $\mathcal{Q}_i$ is the needed sheaf. One can see that $\mathcal{Q}_i$ is a subsheaf of $\mathcal{O}_X$ (namely, it is an ideal sheaf) by applying the Snake lemma to the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \cap \mathcal{O}_X^{\oplus(i-1)} \arrow[r] \arrow[d, hook] & \mathcal{K} \cap \mathcal{O}_X^{\oplus i} \arrow[r] \arrow[d, hook] & \mathcal{Q}_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus(i-1)} \arrow[r] & \mathcal{O}_X^{\oplus i} \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

   Thus, $H^1(X, \mathcal{Q}_i) = 0$ by our hypothesis (iii), so the long exact sequence implies that $H^1\left(X, \mathcal{K} \cap \mathcal{O}_X^{\oplus i}\right) = 0$, where we are now using the inductive hypothesis. ■

## 2.2 February 9

We spent most of the class completing the proof of Theorem 2.4. I have edited into those notes for continuity reasons.

**Remark 2.5.** Note the identity map $A \to \Gamma(X, \mathcal{O}_X)$ induces a map $\varphi \colon X \to \operatorname{Spec} A$ via the adjunction. So after we produce the sections $(f_1, \ldots, f_r)$ with $X_{f_i}$ affine, it might appear that we are done because we might be able to glue the $X_{f_i} \cong \operatorname{Spec} A_{f_i}$ into making $\varphi$ an isomorphism. But this does not work: indeed, $\varphi$ need not even be surjective!

### 2.2.1 More Cohomology on Affine Schemes

Let's see an application of some of the work we've done.

**Corollary 2.6.** Fix a Noetherian scheme $X$. Any quasicoherent sheaf $\mathcal{F}$ on $X$ can be embedded into a flasque quasicoherent sheaf.

*Proof.* The point is to reduce to the affine case and then use Proposition 2.1. Let $\{U_i\}_{i=1}^n$ be a finite affine open cover of $X$, where $U_i = \operatorname{Spec} A_i$. Because $\mathcal{F}$ is quasicoherent, we can find an $A_i$-module $M_i$ such that $\mathcal{F}|_{U_i} \cong M_i$, and then we may embed $M_i$ into an injective $A_i$-module $I_i$. So we have injections $\mathcal{F}|_{U_i} \to \widetilde{I}_i$ on $U_i$, so we can glue these into a map

$$\mathcal{F} \to \bigoplus_{i=1}^n (j_i)_* \widetilde{I}_i,$$

where $j_i\colon U_i \to X$ is the inclusion. Notably, $(j_i)_*$ sends quasicoherent sheaves to quasicoherent sheaves (note $X$ is Noetherian), so $\bigoplus_{i=1}^n (j_i)_* \widetilde{I}_i$ is still quasicoherent, and the above map is injective because we can check injectivity on stalks and so on the affine open cover $\{U_i\}_{i=1}^n$. Lastly, $(j_i)_*$ sends flasque sheaves to flasque sheaves, so our sum is still flasque. ■

## 2.3 February 12

Today we will discuss Čech cohomology.

### 2.3.1 Čech Cohomology to Groups

For today, $X$ will be a topological space, $\mathcal{F}$ will be a sheaf of abelian groups on $X$, and $\mathcal{U} := \{U_i\}_{i\in I}$ is an open cover of $X$, and we will fix a well-ordering on $I$. For indices $i_0, \ldots, i_p \in I$, we define the notation

$$U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

We can now define our complex.

**Definition 2.7** (Čech complex)**.** Fix notation as above. For each $p \geq 0$, define

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

and the map $d\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}},$$

where the hat means an omission of the index.

**Remark 2.8.** One can check directly that $d^{p+1} \circ d^p = 0$, which we will not write out. Thus, $(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}), d^\bullet)$ is in fact a complex of abelian groups.

We now define a convention our indices. Given a class $\alpha \in C^p(\mathcal{U}, \mathcal{F})$, for arbitrary indices $i_0, \ldots, i_p \in I$ (perhaps not in order), we define

$$\alpha_{i_0,\ldots,i_p} := \begin{cases} 0 & \text{if there is a repeated index,} \\ (-1)^{\operatorname{sgn}\sigma} \alpha_{\sigma(i_0),\ldots,\sigma(i_p)} & \text{where } \sigma(i_0) < \cdots < \sigma(i_p). \end{cases}$$

Note that even if $\sigma(i_0) < \cdots < \sigma(i_p)$ fails to hold, multiplicativity of the sign means that we still have the equation

$$\alpha_{i_0,\ldots,i_p} = (-1)^{\operatorname{sgn}\sigma} \alpha_{\sigma(i_0),\ldots,\sigma(i_p)},$$

so our notation makes sense.

**Remark 2.9.** Our differential also still makes sense with these indices. By multiplicativity of the sign, it will suffice to prove the result by induction on the length of $\sigma$. Namely, supposing that

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1}(-1)^j\alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}},$$

we will show that

$$(d^p\alpha)_{\sigma(i_0),\ldots,\sigma(i_{p+1})} := \sum_{j=0}^{p+1}(-1)^j\alpha_{\sigma(i_0),\ldots,\widehat{\sigma(i_j)},\ldots,\sigma(i_{p+1})},$$

where $\sigma$ is a transposition $(\ell,\ell+1)$. Namely, the left-hand side is multiplied by $-1$, so we need the right-hand side to also be multiplied by $-1$. For the terms $j\notin\{\ell,\ell+1\}$, then we get our sign on each term. Lastly, $j\in\{\ell,\ell+1\}$ swap in the summation, so their signs also suitably swap with each other.

**Remark 2.10.** Even if there are repeated indices, we still achieve

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1}(-1)^j\alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}.$$

The left-hand side is zero, and the right-hand side is zero at almost every term, except perhaps when $j$ is an index repeated exactly twice, but in this case, the two times $j$ is an index repeated exactly twice will have cancelling signs, so the entire thing still vanishes.

Anyway, we can now define our cohomology.

**Definition 2.11** (Čech cohomology)**.** Fix notation as above. We define $\check{H}^p(\mathcal{U},\mathcal{F}) := h^p(C^\bullet(\mathcal{U},\mathcal{F}))$.

Let's do so some sample computations.

**Example 2.12.** If $\mathcal{U}=\{X\}$, then we see that

$$C^p(\mathcal{U},\mathcal{F}) = \begin{cases}\Gamma(X,\mathcal{F}) & \text{if } p=0,\\ 0 & \text{else,}\end{cases}$$

so $\check{H}^p(\mathcal{U},\mathcal{F})$ is the same.

**Remark 2.13.** Čech cohomology frequently does not actually produce a long exact sequence, so perhaps it is not technically a cohomology theory. Indeed, using Example 2.12, it is not the case that a short exact sequence $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ of sheaves of abelian groups on $X$ will produce a short exact sequence

$$0\to\Gamma(X,\mathcal{F}')\to\Gamma(X,\mathcal{F})\to\Gamma(X,\mathcal{F}'')\to 0.$$

**Example 2.14.** We always have $\check{H}^0(\mathcal{U},\mathcal{F})=\Gamma(X,\mathcal{F})$. Indeed, $\Gamma(X,\mathcal{F})$ is by definition the kernel of the map

$$\prod_i \mathcal{F}(U_i) \xrightarrow{d^1} \prod_{i<j}\mathcal{F}(U_i\cap U_j),$$

where $(d^1\alpha)_{ij}=\alpha_i-\alpha_j$. But this is exactly $\Gamma(X,\mathcal{F})$ by the sheaf conditions: we can uniquely glue sections on the $U_\bullet$ which agree on the intersections.

**Exercise 2.15.** Fix a field $k$ and $X := \mathbb{P}^1_k = \operatorname{Proj} k[x,y]$, and let $U := D_+(x)$ and $V := D_+(y)$ make up the standard affine open cover $\mathcal{U} := \{U, V\}$ of $X$. We compute the Čech cohomology of the sheaf $\mathcal{F} = \mathcal{O}_X(1)$.

*Proof.* We begin by computing our complex.

- We compute $C^0(\mathcal{U}, \mathcal{O}_X(1)) = \Gamma(U, \mathcal{O}_X(1)) \times \Gamma(V, \mathcal{O}_X(1)) = xk[y/x] \oplus yk[x/y]$.
- We compute $C^1(\mathcal{U}, \mathcal{O}_X(1)) = xk[y/x, x/y]$.
- For $p \geq 2$, we have $C^2(\mathcal{U}, \mathcal{O}_X(1)) = 0$ because our cover has only two elements anyway.

The only nontrivial differential is the map $C^0(\mathcal{U}, \mathcal{O}_X(1)) \to C^1(\mathcal{U}, \mathcal{O}_X(1))$, which we see "restricts" $x$ and $y$ to their images in $xk[y/x, x/y] = \Gamma(D_+(xy), \mathcal{O}_X(1))$.

In total, we may compute

$$\check{H}^0(\mathcal{U}, \mathcal{O}_X(1)) = xk[y/x] \cap yk[x/y] = kx \oplus ky,$$

which is correctly the global sections. Continuing,

$$\check{H}^1(\mathcal{U}, \mathcal{O}_X(1)) = \frac{\ker d^1}{\operatorname{im} d^0} = \operatorname{coker} d^0 = 0$$

because $d^0$ is surjective: any element of $xk[y/x, x/y]$ can be separated into polynomials in $x$ and polynomials in $y$, so it can be realized from $C^0(\mathcal{U}, \mathcal{O}_X(1))$. Lastly, we note $\check{H}^p(\mathcal{U}, \mathcal{O}_X(1)) = 0$ for $p \geq 2$ because $C^p(\mathcal{U}, \mathcal{O}_X(1)) = 0$ there. ■

## 2.4 February 14

Today we compare Čech and derived cohomology.

### 2.4.1 Čech Cohomology to Sheaves

For today, $X$ will be a topological space, $\mathcal{F}$ will be a sheaf of abelian groups on $X$, and $\mathcal{U} := \{U_i\}_{i \in I}$ is an open cover of $X$, and we will fix a well-ordering on $I$. We fix notation as previous.

So we get a complex on sheaves as follows, upgrading our previous complex.

**Definition 2.16** (Čech complex)**.** Fix notation as above. For each $p \geq 0$, define

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} j_* \mathcal{F}|_{U_{i_0, \ldots, i_p}},$$

where $j$ is the needed inclusion, and the map $d \colon \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p \alpha)_{i_0, \ldots, i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}},$$

where the hat means an omission of the index.

One checks as usual that we do indeed have a complex, which again we will not write out.

**Example 2.17.** Fix everything as above. Then we can compute

$$\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F}).$$

We now begin doing our comparison.

**Lemma 2.18.** Fix notation as above. Then there is a natural transformation $\varepsilon\colon \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ given by $\varepsilon_V(s) := (s|_{U_i \cap V})_{U_i \in \mathcal{U}}$. In fact,

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$$

is an exact sequence of sheaves.

*Proof.* We won't bother to check that $\varepsilon$ is in fact a morphism of sheaves. For the exactness, note the sequence

$$0 \to \Gamma(V, \mathcal{F}) \to \Gamma\left(V, \mathcal{C}^0(\mathcal{U}, \mathcal{F})\right) \to \Gamma\left(V, \mathcal{C}^1(\mathcal{U}, \mathcal{F})\right)$$

is exact for all $V$ by the sheaf condition on $\mathcal{F}$. For exactness elsewhere, we need exactness of

$$\mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F}).$$

This can be checked on stalks, which is done by hand. We won't write out the details. ■

**Proposition 2.19.** Fix everything as above. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.

*Proof.* Fix some $p \geq 0$ for the time being. For $\mathcal{F}$ is flasque, the restrictions $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ will also be flasque, so the pushforward $j_*(\mathcal{F}|_{U_{i_0,\ldots,i_p}})$ will also continue to be flasque, so the product $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ will be flasque. Thus, Lemma 1.69 and Proposition 1.47 allow us to compute $H^p(X, \mathcal{F})$ via this resolution. To complete the proof, we note

$$H^p(X, \mathcal{F}) = h^p(\Gamma(X, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))) = h^\bullet(C^\bullet(\mathcal{U}, \mathcal{F})) = \check{H}^p(\mathcal{U}, \mathcal{F}).$$

The left-hand side vanishes by Lemma 1.69, so the right-hand side also vanishes. ■

### 2.4.2 The Čech Comparison Theorem

So we have some acyclic objects agreeing. We are now ready to construct the needed natural map.

**Lemma 2.20.** Fix everything as above. Then there is natural map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$.

*Proof.* Fix an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Because the $\mathcal{I}^\bullet$ are injective, an inductive argument produces a morphism of the complexes $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{I}^\bullet$. Then this morphism of complexes induces a morphism on cohomology, as desired. Choosing the injectives functorially in $\mathcal{F}$ promises that this map is natural.

Alternatively, one can check that this map does not depend on the choice of $\mathcal{I}$ by doing some homotopy computation using the argument of Proposition 1.34; naturality follows by choosing the injective resolutions to have maps between them a priori. Being explicit, we can produce a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{C}(\mathcal{U}, \mathcal{F})^\bullet \arrow[r] \arrow[d] & \mathcal{I}^\bullet \arrow[d] \\ 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{C}(\mathcal{U}, \mathcal{G})^\bullet \arrow[r] & \mathcal{J}^\bullet \end{tikzcd}$$

where the rightmost square commutes up to some homotopy. Taking cohomology produces the needed commuting square for our map to be natural. ■

We now check when this map is an isomorphism.

**Theorem 2.21.** Fix a Noetherian separated scheme $X$, and let $\mathcal{U}$ be an affine open cover of $X$, and let $\mathcal{F}$ be a quasicoherent sheaf on $X$. Then for all $p \geq 0$, the natural map $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 2.20 are isomorphisms.

*Proof.* We induct on $p$. For $p = 0$, one uses Example 2.14; we won't bother to check that the map is the natural one. Additionally, we remark that if $\mathcal{F}$ is flasque, we get the result for $p > 0$ by Proposition 2.19.

Now, fix some quasicoherent sheaf $\mathcal{F}$ for which we want to show that $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ is an isomorphism. Then Corollary 2.6 allows us to embed $\mathcal{F}$ into a flasque quasicoherent sheaf $\mathcal{F}$; letting $\mathcal{Q}$ be the quotient, we get the exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0.$$

Note that $X$ being separated implies that $U_{i_0,\ldots,i_p}$ is affine, so Proposition 2.19 implies $H^1(U_{i_0,\ldots,i_p}, \mathcal{F}) = 0$, so

$$0 \to \mathcal{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{R}(U_{i_0,\ldots,i_p}) \to 0$$

is exact. Thus,

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

is an exact sequence of the Čech complexes, so we get a long exact sequence of Čech cohomology. Notably, we even have a morphism of the exact sequences of the above sequence with injective resolutions of $\mathcal{F}$ and $\mathcal{G}$ and $\mathcal{Q}$. Being explicit, there is going to be a morphism of short exact sequences as follows.

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U}, \mathcal{Q}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{I}^\bullet) \arrow[r] & \Gamma(X, \mathcal{J}^\bullet) \arrow[r] & \Gamma(X, \mathcal{K}^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Now, if $p \geq 1$, then $\check{H}^p(\mathcal{U}, \mathcal{G}) = H^p(\mathcal{U}, \mathcal{G}) = 0$ by being flasque (see also Proposition 2.19) so we get the commutative diagram as follows.

$$\begin{tikzcd}
\check{H}^{p-1}(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d, "\varepsilon"'] & \check{H}^{p-1}(\mathcal{U}, \mathcal{Q}) \arrow[r] \arrow[d, "\varepsilon"'] & \check{H}^p(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\varepsilon"'] & 0 \\
H^{p-1}(X, \mathcal{G}) \arrow[r] & H^{p-1}(X, \mathcal{Q}) \arrow[r] & H^p(X, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

The two maps on the left are isomorphisms by the inductive hypothesis, so the map $\varepsilon$ (on cokernels!) must be an isomorphism as well by a Five lemma. ■

## 2.5 February 16

Here we go.

### 2.5.1 Upgrading Čech Comparison

Here is a quick remark.

**Remark 2.22.** Fix a scheme $X$ over $\operatorname{Spec} A$. Then the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of $A$-modules, so the cohomology $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$ are $A$-modules as well. Analogously, $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of quasicoherent $A$-modules, so the induced map Lemma 2.20 can be checked to be a map of $A$-modules by making everything into a morphism of $A$-modules. So Theorem 2.21 explains that $H^\bullet(X, \mathcal{F})$ is an $A$-module when $X$ is a Noetherian separated $A$-scheme.

We will want to upgrade Theorem 2.21 somewhat; notably, Theorem 2.21 has some strong hypotheses on $X$ and $\mathcal{F}$, which we will work to remove. We will succeed in removing them for $H^1$.

Our method will be based on allowing the open cover $\mathcal{U}$ to get finer. So we should define what is meant by a refinement.

**Definition 2.23** (refinement)**.** A *refinement* of an open cover $\mathcal{U}$ on $X$ is an open cover $\mathcal{V}$ such that there is a mp $\lambda\colon \mathcal{V} \to \mathcal{U}$ such that $V \subseteq \lambda(V)$ for each $V \in \mathcal{V}$. In practice, we may index $\mathcal{U}$ and $\mathcal{V}$ and view $\lambda$ as a function on indices.

Refinements allow us to improve Čech cohomology. To make this precise, we need to get morphisms on Čech cohomology.

**Lemma 2.24.** Fix a sheaf $\mathcal{F}$ of abelian groups on $X$. Given a refinement $\lambda\colon \mathcal{V} \to \mathcal{U}$ where $\mathcal{U} := \{U_i\}_{i\in I}$ and $\mathcal{V} := \{V_j\}_{j\in J}$, we get a natural map of complexes $\lambda^\bullet\colon C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{V},\mathcal{F})$ and hence a (very) natural map $\lambda^p\colon \check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{V},\mathcal{F})$.

*Proof.* Define $\lambda^p\colon C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{V},\mathcal{F})$ by

$$(\lambda^p\alpha)_{j_0,\ldots,j_p} := \alpha_{\lambda(j_0),\ldots,\alpha(j_p)}.$$

One can check that $\lambda^p$ upgrades to a morphism of complexes by checking that

$$\begin{tikzcd} C^p(\mathcal{U},\mathcal{F}) \arrow[r, "d^p_{\mathcal{U}}"] \arrow[d, "\lambda^p"'] & C^{p+1}(\mathcal{U},\mathcal{F}) \arrow[d, "\lambda^{p+1}"] \\ C^p(\mathcal{V},\mathcal{F}) \arrow[r, "d^p_{\mathcal{V}}"] & C^{p+1}(\mathcal{V},\mathcal{F}) \end{tikzcd}$$

commutes, so we upgrade to a morphism on cohomology. While we're here, we do a flurry of naturality checks.

- Note that $\lambda^p$ is also natural in $\mathcal{F}$ because the diagram

  $$\begin{tikzcd} C^p(\mathcal{U},\mathcal{F}) \arrow[r, "C^p\varphi"] \arrow[d, "\lambda^p"'] & C^p(\mathcal{U},\mathcal{G}) \arrow[d, "\lambda^p"] \\ C^p(\mathcal{V},\mathcal{F}) \arrow[r, "C^p\varphi"] & C^p(\mathcal{U},\mathcal{G}) \end{tikzcd}$$

  commutes for any sheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$.

- Note that $\lambda$ is functorial in the refinement. Indeed, given a refinement $\mu\colon \mathcal{W} \to \mathcal{V}$ where $\mathcal{W} := \{W_k\}_{k\in K}$, then we see that

  $$\begin{tikzcd} C^\bullet(\mathcal{U},\mathcal{F}) \arrow[rr, "(\mu\circ\lambda)^\bullet"] \arrow[dr, "\lambda^\bullet"'] & & C^\bullet(\mathcal{W},\mathcal{F}) \\ & C^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur, "\mu^\bullet"'] & \end{tikzcd}$$

  commutes by an explicit computation.

- Note that the morphism $\lambda^\bullet\colon \check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{V},\mathcal{F})$ is independent of the choice of refinement $\lambda$, which can be seen by taking a common refinement of two choices $\lambda,\lambda'\colon \mathcal{U} \to \mathcal{V}$; we won't write out the relevant diagram for this check. ■

**Remark 2.25.** Any two refinements of $\mathcal{U}$ have a common refinement by taking intersections, so we have a directed system, so we can construct a directed colimit

$$\varinjlim \check{H}^p(\mathcal{U}, \mathcal{F})$$

for each $p$.

The following naturality check for Lemma 2.24 will be especially important.

**Lemma 2.26.** Fix a sheaf $\mathcal{F}$ of abelian groups on $X$. Given a refinement $\lambda\colon \mathcal{V} \to \mathcal{U}$ where $\mathcal{U} := \{U_i\}_{i\in I}$ and $\mathcal{V} := \{V_j\}_{j\in J}$, the following diagram commutes, where the unlabeled arrows are from Lemma 2.20.

$$\begin{tikzcd}
\check{H}^\bullet(\mathcal{U},\mathcal{F}) \arrow[rr] \arrow[dr, "\lambda^\bullet"'] & & H^p(X,\mathcal{F}) \\
& \check{H}^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur] &
\end{tikzcd}$$

*Proof.* Fix an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Then one can build a commutative diagram of resolutions

$$\begin{tikzcd}
\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \arrow[rr] \arrow[dr] & & \mathcal{I}^\bullet \\
& \mathcal{C}^\bullet(\mathcal{V},\mathcal{F}) \arrow[ur] &
\end{tikzcd}$$

where the top arrow is induced by a choice of arrow in the bottom right. (Notably, $\lambda^\bullet$ is induced in basically the same way as Lemma 2.24.) So taking global sections and then cohomology produces the needed commutative diagram. ■

The point of Lemma 2.26 is that we can combine it with Remark 2.25 to produce a map

$$\varinjlim \check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F}).$$

And here is our result.

**Proposition 2.27.** Fix a sheaf $\mathcal{F}$ of abelian groups on $X$. Then the natural map

$$\varinjlim \check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

described above is an isomorphism for $p \in \{0, 1\}$.

*Proof.* For $p = 0$, then Example 2.14 tells us that everything involved is $\Gamma(X, \mathcal{F})$. So the main content will be with $p = 1$.

So we take $p = 1$. We would like to dimension-shift, but we will run into complications. Embed $\mathcal{F}$ into a flasque sheaf $\mathcal{I}$ and let $\mathcal{Q}$ be the quotient so that

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$$

is an exact sequence. Now, to access Čech cohomology, note any open cover $\mathcal{U}$ has an injection $C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{I})$, so we let $D^\bullet(\mathcal{U})$ be the quotient complex so that we have an exact sequence

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{I}) \to D^\bullet(\mathcal{U}) \to 0$$

which we have checked in Lemma 2.24 is natural in the refinement $\mathcal{U}$. (Naturality in $D^\bullet$ is induced.) Thus, for a refinement $\lambda\colon \mathcal{V}\to\mathcal{U}$, we get a commutative diagram as follows with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{U},\mathcal{I}) \arrow[r] \arrow[d] & h^0(D(\mathcal{U})) \arrow[r] \arrow[d] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^1(\mathcal{U},\mathcal{I}) \arrow[d] \\ 0 \arrow[r] & \check{H}^0(\mathcal{V},\mathcal{F}) \arrow[r] & \check{H}^0(\mathcal{V},\mathcal{I}) \arrow[r] & h^0(D(\mathcal{V})) \arrow[r] & \check{H}^1(\mathcal{V},\mathcal{F}) \arrow[r] & \check{H}^1(\mathcal{V},\mathcal{I}) \end{tikzcd} \tag{2.1}$$

Note $\check{H}^1(\mathcal{U},\mathcal{I})=\check{H}^1(\mathcal{V},\mathcal{I})=0$ by Proposition 2.19, so the ends are zero. Also, the left two maps are isomorphisms by the $p=0$ case (they are both $\Gamma(X,\mathcal{F})$). Now, the universal property of the cokernel produces the morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & h^0(D^\bullet(\mathcal{U})) \arrow[r] \arrow[d] & \check{H}^0(\mathcal{U},\mathcal{Q}) \arrow[r, equal] \arrow[d, equal] & \Gamma(X,\mathcal{Q}) \arrow[d, equal] \\ 0 \arrow[r] & h^0(D^\bullet(\mathcal{V})) \arrow[r] & \check{H}^0(\mathcal{V},\mathcal{Q}) \arrow[r, equal] & \Gamma(X,\mathcal{Q}) \end{tikzcd}$$

basically because $\Gamma(X,-)$ is already known to be left exact. Thus, we see that the left vertical map above is injective, so the Five lemma in (2.1) tells us that $\check{H}^1(\mathcal{U},\mathcal{F})\to\check{H}^1(\mathcal{V},\mathcal{F})$ is injective.

We now take colimits over everything (which is an exact operation because the colimits are filtered) to draw the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(X,\mathcal{I}) \arrow[r] \arrow[d] & \varinjlim h^0(D^\bullet(\mathcal{U})) \arrow[r] \arrow[d, dashed] & \varinjlim \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \varinjlim \check{H}^1(\mathcal{U},\mathcal{I}) \arrow[d] \\ 0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] & \Gamma(X,\mathcal{I}) \arrow[r] & \Gamma(X,\mathcal{R}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{I}) \end{tikzcd}$$

As before, the ends vanish, and the middle arrow is induced. More precisely, the Horseshoe lemma produces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{C}^\bullet(\mathcal{U},\mathcal{I}) \arrow[r] \arrow[d] & \mathcal{C}^\bullet(\mathcal{U},\mathcal{Q}) \arrow[d] & \\ 0 \arrow[r] & \mathcal{I}^\bullet \arrow[r] & \mathcal{I}^\bullet\oplus\mathcal{J}^\bullet \arrow[r] & \mathcal{J}^\bullet \arrow[r] & 0 \end{tikzcd} \tag{2.2}$$

where $0\to\mathcal{F}\to\mathcal{I}^\bullet$ and $0\to\mathcal{I}\to\mathcal{J}^\bullet$ are both injective resolutions, which then produces the morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U},\mathcal{I}) \arrow[r] \arrow[d] & D^\bullet(\mathcal{U}) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \Gamma(X,\mathcal{I}^\bullet) \arrow[r] & \Gamma(X,\mathcal{I}^\bullet\oplus J^\bullet) \arrow[r] & \Gamma(X,J^\bullet) \arrow[r] & 0 \end{tikzcd}$$

by taking global sections. Taking cohomology (and then taking colimits) is what produces the needed vertical map making everything commute.

We now continue staring at (2.2). Now, we know that the first and second vertical arrows are isomorphisms, so the Five lemma dictates that it is enough to show that the third (middle) vertical arrow is an isomorphism. It is already injective, so we just need surjectivity. Morally, this is because sections of the quotient sheaf will (on an open cover) come from sections of the quotient presheaf, and $h^0(D^\bullet(\mathcal{U}))$ is exactly these sections on the open cover $\mathcal{U}$.

To be explicit, we work on stalks. Let $q\colon\mathcal{I}\to\mathcal{Q}$ denote the quotient map and some $s\in\Gamma(X,\mathcal{Q})$ which we would like to hit. Well, looking on stalks, each $x\in X$ has some open neighborhood $U_x$ with a section $t_x\in\Gamma(U_x,\mathcal{I})$ such that $q_x((t_x)_x)=s|_{U_x}$. Setting $\mathcal{U}\coloneqq\{U_x\}_{x\in X}$, we have that $((t_x)_x)\in C^0(\mathcal{U},\mathcal{I})$, so we can take this along $q$ to get $(s|_{U_x})_{x\in X}\in C^0(\mathcal{U},\mathcal{Q})$, but in fact this lives in $D^\bullet(\mathcal{U})$ because it came from $C^\bullet(\mathcal{U},\mathcal{I})$, so we see that $s$ was in fact in the image. ■

## 2.6 February 21

We spent most class completing a proof from the previous class, and I have edited directly into the notes of the previous class for continuity.

### 2.6.1 Cohomology on Projective Space

We are now moving towards the proof of Serre duality, for which we will want to have computed the cohomology of some line bundles on projective space. Throughout, we will take $A$ to be a Noetherian ring (for example, a field), set $S := A[x_0, \ldots, x_r]$ to be the graded ring, and we set $X := \mathbb{P}^r_A = \operatorname{Proj} S$. Our goal is to compute the cohomology of the sheaves $\mathcal{O}_X(n) := \widetilde{S(n)}$. We also recall the following construction: for an $\mathcal{O}_X$-module $\mathcal{F}$, we can define

$$\Gamma_\bullet(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)),$$

which is a $\mathbb{Z}$-graded $S$-module. Let's go ahead and state our desired theorem.

**Theorem 2.28.** Fix a Noetherian ring $A$, and set $S := A[x_0, \ldots, x_r]$ (for $r \geq 1$) and $X := \mathbb{P}^r_A$.

(a) The natural map
$$S \to \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$$
is an isomorphism of $\mathbb{Z}$-graded $S$-modules.

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for all $0 < i < r$ and $n \in \mathbb{Z}$.

(c) $H^r(X, \mathcal{O}_X(-r-1)) = A$.

(d) For each $n \in \mathbb{Z}$, the natural map
$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-r-1-n)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$
is a perfect pairing of free $A$-modules of finite rank.

**Remark 2.29.** The standard affine open covering on $\mathbb{P}^n_A$ tells us that $H^i(X, \mathcal{F}) = 0$ for any $i > r$ and any quasicoherent sheaf $\mathcal{F}$. Notably, we are using Čech cohomology via the comparison theorem Theorem 2.21, which applies because $X$ is in fact Noetherian and separated.

## 2.7 February 23

Here we go.

### 2.7.1 More on Cohomology on Projective Space

Today we prove Theorem 2.28.

*Proof of Theorem 2.28.* As suggested by the remark, our proof of Theorem 2.28 will use Čech cohomology. It will be helpful to glue everything together into

$$\mathcal{F} := \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n),$$

which is a $\mathbb{Z}$-graded quasicoherent sheaf of $S$-modules. Taking cohomology, which commutes with infinite sums because taking infinite sums is exact, we see that

$$H^i(X,\mathcal{F})=\bigoplus_{n\in\mathbb{Z}}H^i(X,\mathcal{O}_X(n))$$

for each $i$. Notably, one has an $A$-module structure everywhere by Remark 2.22.

Now, for our open cover $\mathcal{U}$ (for Čech cohomology), we take $U_j:=D_+(x_j)$ for $0\le j\le r$; note then that $U_{i_0,\ldots,i_p}:=U_{i_0}\cap\cdots\cap U_{i_p}=D_+(x_{i_0}\cdots x_{i_p})$. In particular, $\mathbb{P}^r_A$ is Noetherian and separated (note $A$ is Noetherian), so

$$\check{H}^\bullet(\mathcal{U},\mathcal{O}_X(n))\cong H^\bullet(X,\mathcal{O}_X(n))$$

by Theorem 2.21. Notably, we find that

$$\mathcal{F}(U_{i_0\cdots i_p})=S_{x_{i_0}\cdots x_{i_p}}$$

by tracking through the localizations on $\mathcal{O}_X(n)$, meaning that our Čech complex looks like

$$0\to\prod_{i_0}S_{x_{i_0}}\overset{d^0}{\to}\prod_{i_0,i_1}S_{x_{i_0}x_{i_1}}\overset{d^1}{\to}\cdots\overset{d^{r-1}}{\to}S_{x_0\cdots x_r}\to 0$$

of graded $S$-modules.

We now proceed with our arguments.

(a) We compute

$$\check{H}^0(\mathcal{U},\mathcal{F})=\ker d^0=\bigcap_i S_{x_i}.$$

Here, the intersections are being taken in $S_{x_0\cdots x_r}$, legal because the relevant localization maps are injective. Now, this last $S$-module is just $S$ because (for example) $S=S_{x_0}\cap S_{x_1}$ by a direct computation.[1]

(c) We begin by claiming that $\check{H}^r(\mathcal{U},\mathcal{F})$ is the free graded $A$-module with basis given by elements of the form $x_0^{\ell_0}\cdots x_r^{\ell_r}$ where $\ell_i$ are negative integers. Notably, in degree $-r-1$, we are searching for solutions to $\ell_0+\cdots+\ell_r=-r-1$, which is only $\ell_0=\cdots=\ell_r=-1$, so we will have rank $1$, which is (c).

So it remains to show the claim. Well, looking at our Čech complex,

$$\check{H}^r(\mathcal{U},\mathcal{F})=\operatorname{coker}\left(\prod_j S_{x_0\cdots\widehat{x}_j\cdots x_r}\overset{d^{r-1}}{\to}S_{x_0\cdots x_r}\right).$$

Now, $S_{x_0\cdots x_r}$ is a free $A$-module with basis given by terms of the form $x_0^{\ell_0}\cdots x_r^{\ell_r}$ where $\ell_0,\ldots,\ell_r\in\mathbb{Z}$, so we want the cokernel to kill all undesired terms. Well, tracking through $d^{r-1}$, we find that it is just inclusion (up to sign), so the image is the free $A$-module with basis given by terms of the form $x_0^{\ell_0}\cdots x_r^{\ell_r}$ with at least one nonnegative exponent, so the claim follows.

(d) Let's begin by describing the pairing. Given some $s\in H^0(X,\mathcal{O}_X(n))$ (which is just a global section), we produce a map $s\colon\mathcal{O}_X(-r-1-n)\to\mathcal{O}_X(-r-1)$. Moving up to cohomology, we get an $A$-linear map $H^r(X,\mathcal{O}_X(-r-1-n))\to H^r(X,\mathcal{O}_X(-r-1))$, which upon letting $s$ vary produces the required bilinear pairing

$$H^0(X,\mathcal{O}_X(n))\to\operatorname{Hom}_A(H^r(X,\mathcal{O}_X(-r-1-n)),H^r(X,\mathcal{O}_X(-r-1))).$$

It remains to check that this pairing is perfect and that these are free $A$-modules of finite rank; note (a) and the computation in (c) above actually shows that these are free $A$-modules of finite rank. If $n<0$, then both terms in our pairing will vanish, so there will be nothing left to check. For example, $H^0(X,\mathcal{O}_X(n))=0$ by (a), and $H^r(X,\mathcal{O}_X(-r-1-n))=0$ by the claim in (c): we know $H^r(X,\mathcal{O}_X(-r-$

[1] Here we have used that $r\ge 1$.

$1-n)) = 0$ is a free $A$-module with basis given by $x_0^{\ell_0}\cdots x_r^{\ell_r}$ where the $\ell_\bullet$s are negative integers summing to $-r-1-n > -r-1$, but there is no such basis element.

So we may take $n \geq 0$. Tracking through the description of our pairing on Čech cohomology, we see that the basis element $x_0^{m_0}\cdots x_r^{m_r} \in H^0(X, \mathcal{O}_X(n))$ (of total degree $n$) will send the basis element $x_0^{\ell_0}\cdots x_r^{\ell_r} \in H^r(X, \mathcal{O}_X(-r-1-n))$ to the element $x_0^{m_0+\ell_0}\cdots x_r^{m_r+\ell_r} \in H^r(X, \mathcal{O}_X(-r-1))$ (which means $0$ if any exponent is nonnegative). Let's explain this. To begin, we need to show that we actually have a well-defined map. On Čech cohomology, we are attempting to describe a map

$$\left(\ker d^0\right)_n \otimes_A \frac{C^r(\mathcal{U},\mathcal{F})_{-r-1-n}}{\left(\operatorname{im} d^{r-1}\right)_{-r-1-n}} \to \frac{C^r(\mathcal{U},\mathcal{F})_{-r-1}}{\left(\operatorname{im} d^{r-1}\right)_{-r-1}}.$$

We can now see that $\ker d^0 = S$, so it does have basis elements in the form $x_0^{m_0}\cdots x_r^{m_r}$ of total degree $n$, and tensoring by this element will indeed send basis elements $x_0^{\ell_0}\cdots x_r^{\ell_r} \in H^r(X, \mathcal{O}_X(-r-1-n))$ as described. (Notably, we do go to $0$ if any exponent is nonnegative because this is the image of $d^{r-1}$. Perhaps one might also want to note that if we input some element $x_0^{\ell_0}\cdots x_r^{\ell_r}$ with a nonnegative exponent, then the corresponding product will have a nonnegative exponent in the same spot.) Formally, perhaps one should go through the following commutative diagram, as follows.

$$\begin{tikzcd} C^\bullet(\mathcal{U}, \mathcal{O}_X(-r-1-n)) \arrow[r, "(-\otimes s)"] \arrow[d, equal] & C^\bullet(\mathcal{U}, \mathcal{O}_X(-r-1)) \arrow[d, equal] \\ C^\bullet(\mathcal{U},\mathcal{F})_{-r-1-n} \arrow[r, dashed] & C^\bullet(\mathcal{U},\mathcal{F})_{-r-1} \end{tikzcd}$$

We are now able to show that the relevant pairing is perfect. We use the bases listed above to actually claim that our pairing makes these bases dual bases: the basis element $x_0^{m_0}\cdots x_r^{m_r} \in H^0(X, \mathcal{O}_X(n))$ has dual basis element $x_0^{-m_0-1}\cdots x_r^{-m_r-1} \in H^r(X, \mathcal{O}_X(-r-1-n))$, which we can check directly. To see this, certainly we have

$$x_0^{m_0}\cdots x_r^{m_r}\cdot x_0^{-m_0-1}\cdots x_r^{-m_r-1} = x_0^{-1}\cdots x_r^{-1},$$

which is the basis element of $H^r(X, \mathcal{O}_X(-r-1))$. Then for any other basis element $x_0^{\ell_0}\cdots x_r^{\ell_r} \in H^r(X, \mathcal{O}_X(-r-1-n))$, the only way for the pairing to send this to a nonzero element is for $m_i+\ell_i < 0$ for each $\ell_i$, meaning that $\ell_i \leq -m_i-1$ for each $i$, but then $\sum_i m_i = n$ and $\sum_i \ell_i = -r-1-n$ forces equality everywhere.

(b) We will induct on $r$. For $r=1$, there is nothing to show because there is no $i$ to check. We now show two separate claims.

- We claim that each $i > 0$ has each element of $H^i(X,\mathcal{F})$ annihilated by some power of $x_r$. It is enough to show that the localization $H^i(X,\mathcal{F})_{x_r}$ vanishes. Now, by the inductive step, we see that the cohomology of the restricted Čech complex $C^\bullet(\mathcal{U}\cap U_r, \mathcal{F}|_{U_r})$ vanishes for indices $i > 0$: indeed, by Theorem 2.21, we may check this on sheaf cohomology, for which the result follows from Theorem 2.4 because $U_r$ is affine and $\mathcal{F}$ is quasicoherent. Thus, by localizing, we see that $C^\bullet(\mathcal{U}\cap U_r, \mathcal{F}|_{U_r})_{x_r}$ continues to have vanishing cohomology for indices $i > 0$, which reduces to the needed claim. (Note the Čech complex does reduce to $\mathcal{U}\cap U_r$ because we are localizing at $x_r$.)
- For $0 < i < r$, we actually claim that $x_r\colon H^i(X,\mathcal{F}) \to H^i(X,\mathcal{F})$ is injective. Here is where we will use the induction. The point is that each $n \in \mathbb{Z}$ has an exact sequence

$$0 \to S(n-1) \stackrel{x_r}{\to} S(n) \to \frac{S(n)}{(x_r)} \to 0$$

of graded $S$-modules. So we let $H \subseteq \mathbb{P}_A^r$ denote the hyperplane cut out by $x_r = 0$ so that taking $\widetilde{\cdot}$ everywhere glues us into the short exact sequence

$$0 \to \mathcal{F}(-1) \stackrel{x_r}{\to} \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $\mathcal{F}_H := \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_H(n)$. (Formally, we first get an exact sequence of the form $0 \to \mathcal{O}_X(n-1) \to \mathcal{O}_X(n) \to \mathcal{O}_H(n) \to 0$ and then summing over all $n \in \mathbb{Z}$.) Thus, the long exact sequence produces

$$H^{i-1}(X,\mathcal{F}) \to H^{i-1}(X,\mathcal{F}_H) \to H^i(X,\mathcal{F}(-1)) \xrightarrow{x_r} H^i(X,\mathcal{F})$$

for $1 \le i < r$, but $H^{i-1}(X,\mathcal{F}_H) = H^{i-1}(H,\mathcal{F}_H) = 0$ by the inductive hypothesis for $1 < i < r$. (We have used Lemma 1.75.) Thus, the rightmost map is injective in these cases, as claimed. But even when $i = 1$, the leftmost map is the map $S \to S/(x_r)$ by (a), which is surjective, so the rightmost map continues to be injective.

We now see that (b) follows from the above claims because multiplication by $x_r^k$ is injective but also the zero map for $k$ sufficiently large. ■

**Remark 2.30.** The choice of isomorphism in (c) is notably not canonical. In particular, it depends on our choice of basis element for the cokernel.

## 2.8 February 26

We began class completing the proof of Theorem 2.28, for which I have edited the notes of last class for continuity reasons. We then spent most of the remaining class on a single result which we spent more time on in the following class, so I have moved the remaining discussion there.

## 2.9 February 28

We began class by completing the proof of a result.

### 2.9.1 Cohomology on Projective Schemes

We recall the following definition.

**Definition 2.31** (very ample)**.** Fix a morphism $f\colon X \to Y$ of schemes. A line bundle $\mathcal{L}$ on $X$ is *$f$-very ample* if and only if there is locally closed embedding $i\colon X \to \mathbb{P}^r_Y$ for some $r \ge 0$ such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}^r_Y}(1)$.

**Remark 2.32.** Under proper hypotheses, we will be able to upgrade the locally closed embedding to a closed embedding. Explicitly, fix a scheme $X$ over a Noetherian scheme $Y$. Then the following are equivalent.

(a) $X$ is projective over $Y$.

(b) $X$ is proper and has a $Y$-very ample sheaf.

For (a) implies (b), note projective implies proper, and projective grants a very ample sheaf by definition. For (b) implies (a), note having a very ample sheaf grants a locally closed embedding $i\colon X \to \mathbb{P}^r_Y$, and $X$ being proper forces the image of $i$ to be closed, so $i$ upgrades to a closed embedding, making $X$ projective over $Y$.

We now use Theorem 2.28 for fun and profit.

**Theorem 2.33.** Fix a Noetherian ring $A$ and projective $A$-scheme $X$. Further, fix a very ample line bundle $\mathcal{O}_X(1)$ on $X$ and coherent sheaf $\mathcal{F}$ on $X$.

(a) $H^i(X,\mathcal{F})$ is a finitely generated $A$-modules for all $i>0$.

(b) There exists an integer $n_0$ such that $H^i(X,\mathcal{F}(n))=0$ for $n>n_0$ and $i>0$.

*Proof.* We proceed in steps.

1. We reduce to $X=\mathbb{P}^r_A$ for some $r>0$. Indeed, fix some closed embedding $j\colon X\to\mathbb{P}^r_A$ for some $r>0$ such that $\mathcal{O}_X(1)=j^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Then $H^i(X,\mathcal{F}(n))=H^i(\mathbb{P}^r_A,j_*(\mathcal{F}(n)))$ by Lemma 1.75; now, everything around is Noetherian and separated, so $j_*\mathcal{F}$ is coherent on $\mathbb{P}^r_A$, so (a) will indeed follow from knowing the result for $\mathbb{P}^r_A$. Further,
$$j_*(\mathcal{F}(n))=j_*(\mathcal{F}\otimes\mathcal{O}_X(n))=j_*\mathcal{F}\otimes j_*j^*\mathcal{O}_{\mathbb{P}^r_A}(n)=(j_*\mathcal{F})(n),$$
where $\stackrel{*}{=}$ is by the projection formula [Har77, Exercise II.5.1]. Thus, (b) will also follow from the result for $\mathbb{P}^r_A$.

2. We show the result for $\mathcal{F}=\mathcal{O}_X(q)$ for $q\in\mathbb{Z}$. Indeed, we get this result directly from Theorem 2.28: (a) follows from the stated computations, and (b) follows because $H^r(X,\mathcal{O}_X(q))=0$ for $q$ sufficiently large.

3. We also note that one can glue the previous step together to achieve $\mathcal{F}$ isomorphic to a finite direct sum of sheaves of the form $\mathcal{O}_X(q)$ for $q\in\mathbb{Z}$.

4. We now tackle the general case by descending induction on the index $i$. Notably, Čech cohomology via Theorem 2.21 tells us that $H^i(X,\mathcal{F})=0$ for $i>r$ (see Remark 2.29), making (a) and (b) have no content. (Technically, for (b), $n_0$ is not supposed to depend on $i$, but this will in practice be no issue because the value of $n_0$ doesn't even matter in these cases.)

   For our induction, we take $i\le r$, assuming the result for $i+1$. The main point is that there is a sheaf $\mathcal{E}$ isomorphic to $\bigoplus_{i=1}^N\mathcal{O}_X(q_i)$ with a surjection $\pi\colon\mathcal{E}\to\mathcal{F}$ by [Har77, Corollary II.5.18]. So we set $\mathcal{K}:=\ker\pi$ to produce a short exact sequence
$$0\to\mathcal{K}\to\mathcal{E}\to\mathcal{F}\to 0 \tag{2.3}$$
   of coherent sheaves. Thus, for (a), we note that the long exact sequence produces the exact sequence
$$H^i(X,\mathcal{E})\to H^i(X,\mathcal{F})\to H^{i+1}(X,\mathcal{K}),$$
   so the middle term must be a finitely generated $A$-module because the end terms are by the previous step and the inductive hypothesis. (Explicitly, we are using some Noetherian submodule argument.) For (b), we twist (2.3) by $n\in\mathbb{Z}$ and then take the long exact sequence to get the exact sequence
$$H^i(X,\mathcal{E}(n))\to H^i(X,\mathcal{F}(n))\to H^{i+1}(X,\mathcal{K}(n)).$$
   Notably, for $n$ large enough the two terms on the end will vanish, so the middle term will vanish as well.

   Technically, the value of $n_0$ in (b) is not supposed to depend on $i$. Well, we have shown that each $i\le r$ has $m_i\in\mathbb{Z}$ such that $H^i(X,\mathcal{F}(n))=0$ for $i>m_i$, and we can analogously take $m_i=0$ for $i>r$, so we can just take $n_0$ to be the maximum of all these values (or perhaps their sum) to complete the proof of (b). ■

**Corollary 2.34.** Fix a Noetherian ring $A$ and projective $A$-scheme $X$. For any coherent sheaf $\mathcal{F}$ on $X$, the $A$-module $\Gamma(X,\mathcal{F})$ is finitely generated.

*Proof.* Take $i=0$ in (a) of Theorem 2.33. ■

**Corollary 2.35.** Fix a Noetherian ring $A$ and a closed subscheme $X \subseteq \mathbb{P}^r_A$ for some $r \geq 0$. Then the restriction map

$$\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \to \Gamma(X, \mathcal{O}_X(n))$$

is surjective for $n$ sufficiently large.

*Proof.* Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^r_A}$ be the ideal sheaf cutting out the closed subscheme $Y$. Notably, everything in sight is Noetherian, so $\mathcal{I}$ is coherent, so $H^1(\mathbb{P}^r_A, \mathcal{I}(n)) = 0$ for $n$ sufficiently large by Theorem 2.33. Now, we take the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^r_A} \to i_*\mathcal{O}_X \to 0,$$

twist by some sufficiently large $n \in \mathbb{Z}$ and take the long exact sequence so that

$$\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \to \Gamma(\mathbb{P}^r_A, i_*\mathcal{O}_X(n)) \to H^1(\mathbb{P}^r_A, \mathcal{I}(n)).$$

The rightmost term vanishes by construction of $n$, and the middle term is $\Gamma(X, \mathcal{O}_X(n))$ (say, by Lemma 1.75), so the result follows. ■

**Remark 2.36.** Because global sections of $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n))$ are homogeneous polynomials of degree $n$ in $k[x_0, \ldots, x_r]$, we see that we can write out any global section in $\mathcal{O}_X(n)$ in this way.

## Theme 3

# Line Bundles and Divisors

## 3.1 March 1

Today we discuss line bundles.

### 3.1.1 Ample Line Bundles

The following result helps motivate the notion of ample.

**Proposition 3.1** (Serre)**.** Fix a projective scheme $X$ over a Noetherian ring $A$, and let $\mathcal{O}_X(1)$ be some very ample line bundle. For any coherent quasicoherent sheaf $\mathcal{F}$ on $X$, there is an integer $n_0(\mathcal{F})$ such that $\mathcal{F}(n)$ is generated by finitely many global sections for $n \geq n_0(\mathcal{F})$.

*Proof.* We proceed in steps.

1. We reduce to the case of $X = \mathbb{P}^r_A$. Because $X$ is projective, and $\mathcal{O}_X(1)$ is very ample, we are promised some closed embedding $i\colon X \to \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Because $i$ is finite (and $A$ Noetherian), we see that $i_*\mathcal{F}$ continues to be coherent. Notably, $H^0(X, \mathcal{F}) = H^0(\mathbb{P}^r_A, i_*\mathcal{F})$.

   For all affine open subschemes $U = \operatorname{Spec} B$ of $\mathbb{P}^r_A$, the fact that $i$ is closed allows us to say $V := i^{-1}(U)$ is an affine open subscheme $\operatorname{Spec} B/i$ of $X$ (where $I \subseteq B$ is some ideal). Now, $\mathcal{F}$ is coherent, so $\mathcal{F}|_V = \widetilde{M}$ for some finitely generated $(B/I)$-module $M$. Further, $(i_*\mathcal{F})|_U = \widetilde{M}$, where $M$ is now viewed as a $B$-module via $B \twoheadrightarrow B/I$, so if we have global sections $s_0, \ldots, s_n \in H^0(\mathbb{P}^r_A, i_*\mathcal{F})$ globally generating $i_*\mathcal{F}$, they will restrict to some $m_1, \ldots, m_n \in M$ which generate $M$ as a $B$-module and hence generate as a $(B/I)$-module. Thus, $\mathcal{F}|_V$ is generated by these global sections for all affine open subschemes $V$, which is good enough.

2. We complete the proof. Cover $X = \mathbb{P}^r_A$ with the standard affine open subschemes $D_+(x_i)$ where $0 \leq i \leq r$; say $D_+(x_i) = B_i := A[x_0/x_i, \ldots, x_n/x_i]$. Notably, $\mathcal{F}|_{D_+(x_i)} \cong \widetilde{M_i}$ for some finitely generated $B_i$-module $M_i$, so we may let $\{s_{ij}\}_{j=1}^{m_i}$ be a set of generators. Notably, for each $s_{ij}$, we can imagine cancelling out any possible pole by multiplying by $x_i^{n_{ij}}$ for some integer $n_{ij}$, making $x_i^{n_{ij}}s_{ij}$ a global section of $\mathcal{F}(n_{ij})$. (See [Har77, Lemma II.5.14].) Let $n$ be the maximum of all the $n_{ij}$s so that $x^n s_{ij}$ is a global section of $\mathcal{F}(n)$.

   Now, $\mathcal{F}(n)|_{D_+(x_i)}$ is isomorphic to $\widetilde{M_i'}$ for some $B_i$-module $M_i$, and $x_i^n\colon \mathcal{F} \to \mathcal{F}(n)$ will produce an isomorphism $M_i \to M_i'$ by how twisting works, so the global sections $x_i^n s_{ij}$ are able to generate each $\widetilde{M_i'}$. So our global sections are able to generate $\mathcal{F}(n)$, as required. ■

So let's codify the conclusion of the above result.

**Definition 3.2** (globally generated)**.** Fix a scheme $X$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *globally generated* if and only if $\Gamma(X, \mathcal{F})$ generates $\mathcal{F}$.

**Remark 3.3.** Perhaps one is worried about finite generation. Well, if $\mathcal{F}$ is a coherent sheaf on a Noetherian scheme $X$, then being globally generated means that $\mathcal{F}$ is actually generated by a finite set of global sections. Morally, letting $S$ be a generating subset of global sections, we cover $X$ by finitely many open affine Noetherian subschemes, and then over each subscheme, only finitely many global sections of $S$ need to be used because $\mathcal{F}$ is coherent. So we only pick out finitely many global sections from $S$ that are needed to generate it.

This leads to the following definition.

**Definition 3.4** (ample)**.** A line bundle $\mathcal{L}$ on a Noetherian scheme $X$ is *ample* if and only if any coherent sheaf $\mathcal{F}$ on $X$ has some integer $n_0(\mathcal{F}) \in \mathbb{Z}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for any $n \geq n_0(\mathcal{F})$.

**Remark 3.5.** A very ample line bundle is always relative to the (locally closed) embedding of $X$ into projective space.

**Example 3.6.** Proposition 3.1 explains that very ample line bundles on projective schemes are ample.

**Example 3.7.** Fix an affine Noetherian scheme $X = \operatorname{Spec} A$. Then every coherent sheaf on $X$ is globally generated, so actually any line bundle $\mathcal{L}$ is ample: for any coherent sheaf $\mathcal{F}$ on $X$, we see $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ continues to be coherent for any $n \geq 0$ (say), so it is globally generated.

Let's run some basic checks on ample line bundles.

**Lemma 3.8.** Fix a line bundle $\mathcal{L}$ on a Noetherian scheme $X$. Then the following are equivalent.

(i) $\mathcal{L}$ is ample.

(ii) $\mathcal{L}^{\otimes m}$ is ample for all $m > 0$.

(iii) $\mathcal{L}^{\otimes m}$ is ample for some $m > 0$.

*Proof.* Note (ii) implies (iii) has little content. Also, for (i) implies (ii), we proceed via the definitions directly: for any coherent sheaf $\mathcal{F}$, we know that there is a nonnegative integer $n_0(\mathcal{F})$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for $n \geq n_0(\mathcal{F})$, but then $\mathcal{F} \otimes (\mathcal{L}^{\otimes m})^{\otimes n}$ for $n \geq n_0(\mathcal{F})$ also. Thus, $\mathcal{L}^{\otimes m}$ is ample.

So it remains to show that (iii) implies (i), which requires a trick. Fix a coherent sheaf $\mathcal{F}$, and we want to show that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for sufficiently large $n$. Well, $\mathcal{L}^{\otimes m}$ is ample, so for each $i \in \{0, \ldots, m-1\}$, there is some $n_i(\mathcal{F}) \geq 0$ such that

$$\left(\mathcal{F} \otimes \mathcal{L}^{\otimes i}\right) \otimes \left(\mathcal{L}^{\otimes m}\right)^{\otimes n} = \mathcal{F} \otimes \mathcal{L}^{\otimes (i+mn)}$$

is globally generated for $n \geq n_i(\mathcal{F})$. (Namely, we recall $\mathcal{F} \otimes \mathcal{L}^{\otimes i}$ is coherent!) Thus, for $n \geq m(\max\{n_i\}+1)$, we see that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated by finding the $i \in \{0, \ldots, m-1\}$ with $n \equiv i \pmod m$ and applying the previous sentence. ■

**Lemma 3.9.** Fix an ample line bundle $\mathcal{L}$ on a Noetherian scheme $X$. For any open subscheme $U \subseteq X$, the line bundle $\mathcal{L}|_U$ remains ample.

*Proof.* This is not as easy as it might look. Let $\mathcal{F}$ be a coherent sheaf on $U$. The difficulty is that we must use [Har77, Exercise II.5.15] in order to build a coherent sheaf $\mathcal{F}'$ on $X$ such that $\mathcal{F}'|_U = \mathcal{F}$; morally, one reduces to the affine case by some technical argument, and then some sort of extension by zero can work. Now, there is some $n_0(\mathcal{F}')$ such that $\mathcal{F}' \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \geq n_0(\mathcal{F}')$, meaning that

$$\left(\mathcal{F}' \otimes \mathcal{L}^{\otimes n}\right)|_U = \mathcal{F} \otimes (\mathcal{L}|_U)^{\otimes n}$$

continues to be globally generated for $n \geq n_0(\mathcal{F}')$, as desired. (The restriction is globally generated basically by viewing the global generation condition as asking for a surjective map $\mathcal{O}_X^{\oplus m} \twoheadrightarrow \mathcal{F}' \otimes \mathcal{L}^{\otimes n}$.) ■

We now use ample line bundles to build very ample line bundles. Because we are allowed to "take roots" of ample line bundles, we will have to "take powers" below.

**Proposition 3.10.** Fix a Noetherian scheme $X$ of finite type over a Noetherian ring $A$. Given a line bundle $\mathcal{L}$ on $X$, then $\mathcal{L}$ is ample if and only if and only if $\mathcal{L}^{\otimes m}$ is very ample for some positive integer $m > 0$.

*Proof.* The forward direction is essentially Proposition 3.1. Indeed, if $\mathcal{L}^{\otimes m}$ is very ample, find some locally closed embedding $i\colon X \to \mathbb{P}^r_A$ such that $\mathcal{L}^{\otimes m} \cong i^*\mathcal{O})\mathbb{P}^r_A(1)$. Because $i$ is locally closed, we can write $i$ as $i_2 \circ i_1$ where $i_1\colon X \to \overline{X}$ is open, and $i_2\colon \overline{X} \to \mathbb{P}^r_A$ is closed. Then $i_2^*\mathcal{O}_{\mathbb{P}^r_A}(1)$ is very ample on $\overline{X}$, hence ample by Proposition 3.1, so its restriction to $X$ is ample by Lemma 3.9, so $\mathcal{L}$ is ample by Lemma 3.8.

We will show the other direction next class. ■

## 3.2 March 4

Today we finish discussing line bundles.

### 3.2.1 More on Ample Line Bundles

We now use ample line bundles to build very ample line bundles. Because we are allowed to "take roots" of ample line bundles, we will have to "take powers" below.

**Proposition 3.11.** Fix a Noetherian scheme $X$ of finite type over a Noetherian ring $A$. Given a line bundle $\mathcal{L}$ on $X$, then $\mathcal{L}$ is ample if and only if and only if $\mathcal{L}^{\otimes m}$ is very ample for some positive integer $m > 0$.

*Proof.* The forward direction is essentially Proposition 3.1. Indeed, if $\mathcal{L}^{\otimes m}$ is very ample, find some locally closed embedding $i\colon X \to \mathbb{P}^r_A$ such that $\mathcal{L}^{\otimes m} \cong i^*\mathcal{O})\mathbb{P}^r_A(1)$. Because $i$ is locally closed, we can write $i$ as $i_2 \circ i_1$ where $i_1\colon X \to \overline{X}$ is open, and $i_2\colon \overline{X} \to \mathbb{P}^r_A$ is closed. Then $i_2^*\mathcal{O}_{\mathbb{P}^r_A}(1)$ is very ample on $\overline{X}$, hence ample by Proposition 3.1, so its restriction to $X$ is ample by Lemma 3.9, so $\mathcal{L}$ is ample by Lemma 3.8.

It remains to show the forward direction. The point is that $\mathcal{L}$ and some generating sections will determine a morphism to projective space, which we eventually want to be a closed embedding. In particular, we need to separate points and tangent vectors to be a closed embedding. Anyway, we proceed in steps.

1. As such, as a starting step, we claim that any $p \in X$ has $n > 0$ and a section $s \in \Gamma\left(X, \mathcal{L}^{\otimes n}\right)$ such that $X_s$ contains $p$ and is affine. (Recall $X_s$ consists of $q \in X$ such that $s_q \notin \mathfrak{m}_q\mathcal{L}_q^{\otimes n}$; in particular, $X_s$ is open.) To begin, let $U$ be an affine open neighborhood of $p$ such that $\mathcal{L}|_U = \mathcal{O}_U$, and let $Y := X \setminus U$ be the complement, which we give the reduced scheme structure. While we're here, we also let $\mathcal{I}_Y$ be the coherent ideal sheaf corresponding to $Y$ (note $\mathcal{I}_Y$ is coherent because $X$ is Noetherian).

   Thus, ampleness of $\mathcal{L}$ tells us that $\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n$ sufficiently large. In particular, we may find a global section $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes n})$ such that $s_p \notin \mathfrak{m}_p(\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n})_p$. Now, $\mathcal{I}_Y \subseteq \mathcal{O}_X$, so $\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is injective, so we may view $s$ as a global section of $\mathcal{L}^{\otimes n}$.

   Continuing, $p \notin Y$ means $(i_*\mathcal{O}_Y)_p = 0$ (here, $i\colon Y \to X$ is the embedding), so the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

being exact at the stalk at $p$ forces the inclusion $\mathcal{I}_{Y,p} \to \mathcal{O}_{X,p}$ to be an isomorphism. As such, we see $s_p \notin \mathfrak{m}_p \mathcal{L}_p^{\otimes n}$. However, the failure of this to be an isomorphism for $q \in Y$ means that $s_q \in \mathfrak{m}_q \mathcal{L}_q^{\otimes n}$ for $q \in Y$. So we are able to conclude that $p \in X_s$ and $X_s \subseteq U$; in particular, $X_s$ is now affine because it is a distinguished open subscheme of the affine scheme $U$, where $s|_U$ is being viewed as a global section of $\mathcal{O}_U$ because $\mathcal{L}^{\otimes n}|_U \cong \mathcal{O}_U$.

2. We upgrade the open cover produced by the previous step. Now, because $X$ is quasicompact, we can find finitely many sections $\{s_1, \ldots, s_k\}$ such that the $X_{s_i}$ are affine and cover $X$. Now, by construction, $s_i \in \Gamma(X, \mathcal{L}^{\otimes n_i})$ for some $n_i > 0$, so we set $n := \operatorname{lcm}(n_1, \ldots, n_k)$ and replace $s_i$ with a power so that $s_i \in \Gamma(X, \mathcal{L}^{\otimes n})$. We also note that $\mathcal{L}^{\otimes n}$ is ample here, so we may as well replace $\mathcal{L}$ with $\mathcal{L}^{\otimes n}$ so that actually $n = 1$, and the $s_\bullet$ are global sections of $\mathcal{L}$.

   Continue with the open cover $X_i := X_{s_i}$ where $1 \le i \le k$ as in the previous step. Because $X$ is of finite type over $A$, we may write $B_i = A[b_{i1}, \ldots, b_{ik_i}]$. As we stated earlier, [Har77, Lemma II.5.14] tells us that each $i$ and $j$ have some $n_{ij} > 0$ such that $s_i^{n_{ij}} b_{ij}$ is the restriction of a global section $c_{ij} \in \Gamma(X, \mathcal{L}^{\otimes n_{ij}})$; again, by taking $\operatorname{lcm}$s and replacing $\mathcal{L}$ with a power of itself, we may assume that $n_{ij} = 1$ for all $i$ and $j$. As such, we see that the $s_i$ and $c_{ij}$ are global sections which generate $\mathcal{L}$.

3. At this point, we have a list of global sections generating $\mathcal{L}$, so we produce a morphism $\varphi\colon X \to \mathbb{P}_A^N$ to projective space. In particular, we can write $\mathbb{P}_A^N = \operatorname{Proj} k[x_i, y_{ij}]$ so that $\varphi^* x_i = s_i$ and $\varphi^* y_{ij} = c_{ij}$ for each $i$ and $j$; for example, $X_{s_i}$ is the pre-image of $D_+(x_i)$. Notably, the induced map $X_{s_{i_0}} \to D_+(x_{i_0})$ is a closed embedding: by construction, on rings, this map looks like $A[x_i/x_{i_0}, y_{ij}/x_{i_0}] \to B_{i_0}$, which is surjective by construction of the map and the $c_{ij}$s. As such, $X$ is a closed subscheme of an open subscheme of $\mathbb{P}_A^N$ via the map given by global sections of $\mathcal{L}$, so we conclude that $\mathcal{L}$ is very ample. ■

Let's see some extra miscellaneous facts about ample and very ample line bundles.

**Lemma 3.12.** Fix a morphism $\pi\colon X \to Y$ of Noetherian schemes.

(a) If $\mathcal{L}$ is $\pi$-very ample, then $\mathcal{L}^{\otimes n}$ is $\pi$-very ample for any $n \ge 0$.

(b) If $\mathcal{L}$ and $\mathcal{M}$ are both very ample, then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

*Proof.* For (a), use the $n$-uple embedding, which is notably closed. More explicitly, if $\mathcal{L}$ is very ample, then it comes from some locally closed embedding $i\colon X \to \mathbb{P}_Y^N$, so we can post-compose with the closed embedding $\mathbb{P}_Y^N \to \mathbb{P}_Y^{nN}$, which effectively takes powers of $\mathcal{L}$ as needed. For (b), proceed as above, but now we note that there is a closed embedding $\mathbb{P}_A^N \times \mathbb{P}_A^M \to \mathbb{P}_A^{NM+M+N-1}$ which we want to post-compose by to get $\mathcal{L} \otimes \mathcal{M}$. ■

And here are some facts about ampleness.

**Lemma 3.13.** Fix a Noetherian scheme $X$.

(a) If $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes \mathcal{M}$ is also ample.

(b) If $\mathcal{L}$ is ample, and $i\colon X' \to X$ is a locally closed embedding, then $i^*\mathcal{L}$ continues to be ample.

(c) For $r > 0$ and $n \le 0$, then $\mathcal{O}_{\mathbb{P}_A^r}(n)$ is not ample on $\mathbb{P}_A^r$.

*Proof.* One proceeds directly from the definitions for (a), and one pulls back according to the definitions for (b). Lastly, for (c), we simply have to note that $\mathcal{O}_{\mathbb{P}_A^r}(n)$ never has global sections for $n < 0$, so no tensor power of it can succeed to be very ample, so it can never succeed to be ample. ■

## 3.3 March 6

Here we go.

### 3.3.1 Ample Line Bundles via Cohomology

Let's return to cohomology but now with some ample flavor.

**Proposition 3.14.** Fix a proper $X$ over a Noetherian ring $A$. Given a line bundle $\mathcal{L}$ on $X$, the following are equivalent.

(i) $\mathcal{L}$ is ample.

(ii) For any coherent sheaf $\mathcal{F}$ on $X$, there is an integer $n_0(\mathcal{F})$ such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $i > 0$ and $n > n_0(\mathcal{F})$.

*Proof.* We have two implications to show. Quickly, we show (i) implies (ii) so that $\mathcal{L}$ is ample. Because $X$ has an ample sheaf $\mathcal{L}$, there is a very ample power $\mathcal{L}^{\otimes m}$ by Proposition 3.11. Now using Theorem 2.33, for each $k$ between $0$ and $n-1$, we can find some $n_k$ such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes k}(n)) = 0$ for $n > n_k$, where $\mathcal{O}_X(1) := \mathcal{L}^{\otimes m}$. Taking $n_0(\mathcal{F}) := \max\{(m+1)(n_k+1) : 0 \le k \le n-1\}$ will now work.

We show (ii) implies (i). This is harder. We proceed in steps.

1. Fix a coherent sheaf $\mathcal{F}$ on $X$ a closed point $p \in X$. We claim that there is an integer $n_0$ (depending on $\mathcal{F}$ and $p$) such that any $n \ge n_0$ has an open neighborhood $U \subseteq X$ such that $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ generates $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})|_U$.

   To begin, the fact that $p$ is closed grants us an exact sequence
$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to k(p) \to 0,$$
   so
$$\mathcal{I}_p \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes k(p) \to 0$$
   is exact. Computing the image of $\mathcal{I}_p \otimes \mathcal{F}$ in $\mathcal{F}$, we see that
$$0 \to \mathcal{I}_p\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes k(p) \to 0$$
   is exact. The point is that (ii) tells us that some $n_0$ has $H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $n \ge n_0$, meaning
$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes k(p))$$
   is surjective for $n \ge n_0$. Thus, Nakayama's lemma tells us that $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ generates $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_p$.

   As such, there is a map $\pi \colon \mathcal{O}_X^{\oplus N} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ which is surjective at $p$. It remains to spread out from $p$. Well, choose $U$ to be the complement of the support of $\operatorname{coker} \pi$, meaning that $(\operatorname{coker} \pi)|_U$ vanishes, so $\pi$ continues to be surjective on $U$, and we are done.

2. We upgrade the previous step to make $U$ independent of $n$. We still make allow $U$ to depend on $\mathcal{F}$ and $p$, which we fix. Notably, the first claim does grant $m > 0$ and open $V \subseteq X$ such that $\mathcal{L}^{\otimes m}$ is globally generated over $V$.

   Now, using the previous step on $\mathcal{F} \otimes \mathcal{L}^{\otimes r}$ for each $0 \le r < m$, we get some $n_0$ very large and open subsets $U_r$ for each $r$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes(n_0+r)}$ is globally generated over $U_r$ for each $r$. We now set
$$U := V \cap \bigcap_{r=0}^{m-1} U_r,$$
   which is now independent of $n$: for any $n \ge n_0$, write $n = n_0 + km + r$ with $k \ge 0$ and $0 \le r < m$, and we see that
$$\mathcal{F} \otimes \mathcal{L}^n = \mathcal{F} \otimes \mathcal{L}^{\otimes(n_0+r)} \otimes \mathcal{L}^{\otimes mk}$$
   is the tensor product of sheaves globally generated over $U$, which continues to be globally generated.

3. We now finish the proof. Because $X$ is Noetherian, it is quasicompact, so any nonempty closed subset has a closed point by Lemma 2.3. So we claim that the open neighborhoods $U_p$ of the previous step cover $X$: if not, then the complement is closed, which has a closed point $p$, but then $U_p$ means that it can have no closed point.

   Thus, quasicompactness of $X$ grants finitely many closed points $\{p_1, \ldots, p_r\}$ such that the $U_{p_i}$ cover $X$, and each $p_i$ has been granted some $n_i$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated over $U_i$ for each $n \geq n_i$. Choosing $n := \max\{n_i : 1 \leq i \leq r\}$ completes the argument because we have an open cover. ■

**Remark 3.15.** A close examination of the proof of (ii) implies (i) shows that we only use (ii) in the first step, and we only use it to show $H^1$ vanishes.

### 3.3.2 The Euler Characteristic

While we're here, we talk about the Euler characteristic quickly.

**Definition 3.16** (Euler characteristic)**.** Fix a $k$-scheme $X$. Then given a coherent sheaf $\mathcal{F}$ on $X$, the *Euler characteristic* $\chi(\mathcal{F})$ is

$$\chi(\mathcal{F}) := \sum_{i \geq 0} \dim_k H^i(X, \mathcal{F}).$$

Note that these dimensions are finite by Theorem 2.33.

**Remark 3.17.** If one has an exact sequence

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

of coherent sheaves on $X$, then $\sum_{i=0}^n (-1)^i \chi(\mathcal{F}_i) = 0$. Indeed, if $n = 2$, then this is direct from the long exact sequence; for the general case, one inducts on $n$. (For example, if $n = 0$ or $n = 1$, there is nothing to do, and for higher $n$, one can divide up the long exact sequence into short ones.)

## 3.4 March 8

Bump, bump, bump.

### 3.4.1 The Hilbert Polynomial

There is a notion of Hilbert polynomial arising from the Euler characteristic.

**Proposition 3.18.** Fix a projective scheme $X$ over a Noetherian ring $A$, and let $\mathcal{O}_X(1)$ be a very ample line bundle. Given a coherent sheaf $\mathcal{F}$ on $X$, there is a polynomial $P_{\mathcal{F}} \in \mathbb{Q}[x]$ such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n$.

*Proof.* Morally, one inducts on the support of $\mathcal{F}$. Using the projectivity of $X$, we are granted a closed embedding $i \colon X \to \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Additionally, note that we may assume that $X = \mathbb{P}^r_A$ by replacing $\mathcal{F}$ with $i_*\mathcal{F}$ (which is still coherent because $i$ is proper and everything in sight is Noetherian).

We will use Noetherian induction on $\operatorname{supp} \mathcal{F}$. For our base case, if $\operatorname{supp} \mathcal{F}$ is empty, then $\mathcal{F}$ vanishes, so $\chi(\mathcal{F}(n)) = 0$ always, so $P = 0$ will work. Otherwise, we will take $\operatorname{supp} \mathcal{F}$ to be nonempty for our induction. Without loss of generality that $\operatorname{supp} \mathcal{F} \not\subseteq H_r$ (certainly the support cannot be contained in all the hyperplanes because they have empty intersection), where $H_r$ is cut out by $x_r = 0$. We now define $\mathcal{K}$ and $\mathcal{Q}$ to build a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}(-1) \xrightarrow{x_r} \mathcal{F} \to \mathcal{Q} \to 0.$$

Twisting, we get the short exact sequence

$$0 \to \mathcal{K}(n) \to \mathcal{F}(-n) \xrightarrow{x_r} \mathcal{F}(n) \to \mathcal{Q}(n) \to 0$$

for all integers $n$. Thus, Remark 3.17 tells us that

$$\chi(\mathcal{F}(n)) - \chi(\mathcal{F}(n-1)) = \chi(\mathcal{Q}(n)) - \chi(\mathcal{K}(n))$$

for all integers $n$. We would now like to use the inductive hypothesis on $\mathcal{Q}$ and $\mathcal{K}$, which we see is legal because their supports are contained in $\operatorname{supp}\mathcal{F}$ (by construction) but also in $H_r$ (because outside $H_r$ the map $x_r\colon \mathcal{F}(-1) \to \mathcal{F}$ is an isomorphism), and $H_r \cap \operatorname{supp}\mathcal{F} \subsetneq \operatorname{supp}\mathcal{F}$.[1]

Thus, we get polynomials $P_{\mathcal{K}}$ and $P_{\mathcal{Q}}$ so that $\chi(\mathcal{K}(n)) = P_{\mathcal{K}}(n)$ and $\chi(\mathcal{Q}(n)) = P_{\mathcal{Q}}(n)$ for all integers $n$. But now we see that $\chi(\mathcal{F}(n)) - \chi(\mathcal{F}(n-1))$ numerically agrees with a polynomial, so a quick computation with finite differences tells us that it too is a polynomial. ■

**Remark 3.19.** Hartshorne has a hint for this result [Har77, Exercise III.5.2], which is somewhat misleading. Namely, it suggested cutting by a generic hyperplane, but if $A$ is finite, there may be no hyperplane which actually cuts down the dimension. One can fix this (as above) by using Noetherian induction; alternatively, one can use hypersurfaces instead of hyperplanes to get enough ways to cut down our dimension. A harder approach would be to base-change $A$ to $\overline{K(A)}$ in the finite case (and then $\overline{K(A)}$ is certainly infinite). But to show this move is legal, one has to know that this base-change does not adjust cohomology, which we will establish later.

**Remark 3.20.** A careful reading of the above proof shows that $\deg P_{\mathcal{F}} \le \dim\operatorname{supp}\mathcal{F}$.

The above proposition permits the following definition.

**Definition 3.21** (Hilbert polynomial)**.** Fix a projective $k$-scheme $X$ with very ample line bundle $\mathcal{O}_X(1)$. Given a coherent sheaf $\mathcal{F}$, we let $P_{\mathcal{F}}(x)$ denote the *Hilbert polynomial* defined so that $P(n) = \chi(\mathcal{F}(n))$ for all $n \in \mathbb{Z}$.

Quickly, we establish that this Hilbert polynomial agrees with what is found in commutative algebra.

**Corollary 3.22.** Fix a field $k$ and set $X := \mathbb{P}^r_A$ with $r > 0$. Given a coherent sheaf $\mathcal{F}$, set

$$M := \Gamma_\bullet(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}(n)).$$

Then the Hilbert polynomial $P_{\mathcal{F}}$ is the Hilbert polynomial of the module $M$ defined so that $P_M(n) = \dim_k M_n$ for $n$ sufficiently large.

*Proof.* Set $S := k[x_0, \ldots, x_r]$ for brevity, and we see that $M$ is a graded $S$-module. Now, $\dim_k M_n = \dim_k H^0(X, \mathcal{F}(n))$ by definition, so we are asking for

$$\chi(\mathcal{F}(n)) \stackrel{?}{=} \dim_k H^0(X, \mathcal{F}(n))$$

[1] We are using "Noetherian induction," which means that it is enough for these supports to be proper subsets of $\operatorname{supp}\mathcal{F}$. The point is that there is no infinite descending chain

for $n$ large enough. Recalling what $\chi(\mathcal{F}(n))$ is, we are asking for

$$\sum_{i=0}^{\infty} \dim_k H^i(X, \mathcal{F}(n)) \stackrel{?}{=} \dim_k H^0(X, \mathcal{F}(n)).$$

But now Proposition 3.14 tells us that there is $n_0(\mathcal{F})$ so that $H^i(X, \mathcal{F}(n)) = 0$ for $i > 0$ and $n > n_0(\mathcal{F})$, so the higher terms of the sum vanish. ■

### 3.4.2 Introducing Divisors

We now escape our discussion of cohomology to discuss Weil divisors. Weil divisors only real make sense with sufficient regularity hypotheses.

**Definition 3.23** (regular in codimension one)**.** Fix a scheme $X$. Then $X$ is *regular in codimension one* if and only if $\mathcal{O}_{X,\eta}$ is regular for all generic points $\eta$ of codimension $1$.

**Remark 3.24.** A normal Noetherian scheme is regular in codimension $1$ because normal Noetherian local domains are regular (in fact, discrete valuation rings).

We will want to bring down our schemes a little more.

**Definition 3.25** (smoothish)**.** A scheme $X$ is *smoothish* if it is Noetherian, separated, and regular in codimension $1$.

Please note that "smoothish" is not a word used in the usual literature, but I would prefer to not have to write all the hypotheses all the time.

Weil divisors are built from schemes of codimension $1$, which we now give a name to.

**Definition 3.26** (Weil divisor)**.** Fix a smoothish scheme $X$. A *prime divisor* on $X$ is a closed integral subscheme of codimension $1$; we denote the set of prime divisors by $X^{(1)}$. A *Weil divisor* is an element of $\mathbb{Z}[X^{(1)}]$. We let $\operatorname{Div} X$ denote the set of Weil divisors.

**Definition 3.27** (effective)**.** A Weil divisor $D$ on a smoothish scheme $X$ is *effective* if and only if

$$D = \sum_{Y \in X^{(1)}} n_Y Y$$

has $n_Y \geq 0$ for all prime divisors $Y$.

**Notation 3.28.** Fix a prime divisor $Y$ of a smoothish scheme $X$. Letting $\eta$ be the generic point of $Y$, we see that $\mathcal{O}_{X,\eta}$ is a discrete valuation ring inside $K(X)$, so we let $\nu_Y : K(X)^\times \to \mathbb{Z}$ be the corresponding valuation.

**Example 3.29.** Fix a (smoothish) curve $X$ over $\mathbb{C}$. Notably, $X$ is smooth (it's a regular curve), so $X(\mathbb{C})$ is a Riemann surface. The prime divisors on $X$ are just the closed points, and $K(X)$ are the meromorphic functions on $X$. Given a point $p \in X$, one can now realize $\nu_p(f)$ as the order of vanishing of $f$ at $p$.

## 3.5 March 11

Today we continue with divisors.

### 3.5.1 Principal Divisors

Now that we've defined valuations, we can define zeroes and poles.

**Definition 3.30** (zero, pole). Fix a smoothish scheme $X$ and some $Y \in X^{(1)}$. We say that $f \in K(X)^\times$ has a *zero* at $Y$ if and only if $\nu_Y(f) > 0$, and $f$ has a *pole* at $Y$ if and only if $\nu_Y(f) < 0$.

We now imagine fixing the function $f$ and letting $Y$ vary.

**Lemma 3.31.** Fix a smoothish scheme $X$. Given $f \in K(X)^\times$, the set

$$\left\{Y \in X^{(1)} : \nu_Y(f) \neq 0\right\}$$

is finite.

*Proof.* Note that $K(X) = \mathcal{O}_{X,\xi}$ where $\xi$ is the generic point of $X$, so we are granted an open subset $U \subseteq X_f \cap X_{1/f}$ of $X$ such that $f, 1/f \in \mathcal{O}_X(U)$; we may even assume that $U$ is affine. Notably, if $Y \in X^{(1)}$ achieves $Y \cap U \neq \varnothing$, then $\mathcal{O}_X(U) \subseteq \mathcal{O}_{X,Y}$ because the generic point of $Y$ will live in $U$, meaning $\nu_Y(f) \geq 0$; a similar argument shows $1/f \in \mathcal{O}_{X,Y}$ thus yielding $\nu_Y(1/f) \geq 0$, so we are actually receiving $\nu_Y(f) = 0$.

Thus, $\nu_Y(f) \neq 0$ actually requires $Y \subseteq X \setminus U$. Because $X \setminus U$ has strictly smaller dimension than $X$ and $Y$ has codimension $1$, we see that $Y$ must actually be one of the irreducible components of $X \setminus U$, of which there are finitely many because $X$ is Noetherian. ■

The above lemma makes the following definition make sense.

**Definition 3.32** (principal divisor). Fix a smoothish scheme $X$ and some $f \in K(X)^\times$. Then we define the *principal divisor*

$$\operatorname{div}(f) := \sum_{Y \in X^{(1)}} \nu_Y(f) Y.$$

We say that a divisor $D$ is *principal* if and only if it takes the form $\operatorname{div}(f)$.

**Example 3.33.** On the smoothish scheme $X = \operatorname{Spec}\mathbb{Z}$, we see $f = 6$ shows that $(2) + (3)$ is a principal divisor.

One can generalize the above construction to use line bundles.

**Definition 3.34** (rational section). Fix a smoothish scheme $X$. Given a line bundle $\mathcal{L}$ on $X$, a *rational section* is an element $s$ of the stalk $\mathcal{L}_\xi$ where $\xi$ is the generic point of $X$.

**Remark 3.35.** Fix a smoothish scheme $X$ and a rational section $s$ of a line bundle $\mathcal{L}$ on $X$. Given a prime divisor $Y$ of $X$, let $U, U' \subseteq X$ be nonempty open subsets such that $U \cap Y \neq \varnothing$ and $U' \cap Y \neq \varnothing$ with isomorphisms $\varphi \colon \mathcal{L}|_U \to \mathcal{O}_U$ and $\varphi' \colon \mathcal{L}|_{U'} \to \mathcal{O}_{U'}$. Then there is some global section $g \in \mathcal{O}_{U \cap U'}^\times$ such that $\varphi' = g\varphi$. However, we see $\nu_Y(g) = 0$ by the argument of Lemma 3.31 (note $U \cap U' \cap Y \neq \varnothing$), so

$$\nu_Y(\varphi(s)) = \nu_Y(\varphi'(s)).$$

The point is that we can make sense of the term $\nu_Y(s)$ because this is independent of the choice of trivializing isomorphism $\nu_Y$.

**Remark 3.36.** Fix a smoothish scheme $X$ and a rational section $s$ of a line bundle $\mathcal{L}$ on $X$. Then the set

$$\left\{Y \in X^{(1)} : \nu_Y(f) \neq 0\right\}$$

remains finite: one can give $X$ a finite open cover $\{U_\alpha\}_{\alpha\in\kappa}$ trivializing $\mathcal{L}$ so that $\nu_Y(s)$ becomes $\nu_Y(s|_{U_\alpha})$, where we know our set is finite (for $U_\alpha$) by Lemma 3.31. Combining our sets for all $U_\alpha$ for $\alpha \in \kappa$ keeps our set finite.

**Definition 3.37.** Fix a smoothish scheme $X$ and a rational section $s$ of a line bundle $\mathcal{L}$ on $X$. Then we define

$$\operatorname{div}_{\mathcal{L}}(s) := \sum_{Y\in X^{(1)}} \nu_Y(f)Y.$$

**Example 3.38.** We note that $\operatorname{div} = \operatorname{div}_{\mathcal{O}_X}$.

**Remark 3.39.** Given rational sections $s$ and $s'$ of the line bundles $\mathcal{L}$ and $\mathcal{L}'$, we see

$$\operatorname{div}_{\mathcal{L}\otimes\mathcal{L}'}(s\otimes s') = \operatorname{div}_{\mathcal{L}}(s) + \operatorname{div}_{\mathcal{L}'}(s').$$

The point is that we can find an open cover $\mathcal{U}$ of $X$ which trivializes $\mathcal{L}$ and $\mathcal{L}'$ simultaneously, where $s\otimes s'$ just becomes the product of two sections, whereupon it is enough to note that $\nu_Y(ss') = \nu_Y(s)+\nu_Y(s')$ for each prime divisor $Y$ and then sum over all $Y$.

**Example 3.40.** By taking $\mathcal{L} = \mathcal{L}' = \mathcal{O}_X$ in the prior remark, we see that

$$\operatorname{div}(ss') = \operatorname{div} s + \operatorname{div} s',$$

so $\operatorname{div}$ is a homomorphism $K(X)^\times \to \operatorname{Div} X$. In particular, the set of principal divisors is a subgroup of $\operatorname{Div} X$.

### 3.5.2 The Class Group

Now that we have a subgroup, we can consider the quotient.

**Definition 3.41** (linearly equivalent)**.** Fix a smoothish scheme $X$. Then two Weil divisors $D$ and $D'$ are *linearly equivalent*, denoted $D \sim D'$ if and only if $D - D'$ is a principal divisor. In particular, we define the *class group* $\operatorname{Cl} X$ as the quotient of $\operatorname{Div}(X)$ by the principal divisors, so $D \sim D'$ if and only if $[D] = [D']$ in $\operatorname{Cl} X$.

**Example 3.42.** A Noetherian domain $A$ is a unique factorization domain if and only if $A$ is normal (and hence smoothish by an argument) with $\operatorname{Cl}(\operatorname{Spec} A) = 0$. Indeed, under normality hypotheses, commutative algebra shows that to check that $A$ is a unique factorization domain has it enough to check that the codimension $1$ primes of $A$ to be principal, which is exactly what $\operatorname{Cl}(\operatorname{Spec} A) = 0$ promises.

For example,

$$\operatorname{Cl} \mathbb{A}_k^n = \operatorname{Cl}(\operatorname{Spec} k[x_1, \ldots, x_n]) = 0$$

for any field $k$.

We also recall the following fact from commutative algebra.

**Proposition 3.43.** Fix a normal Noetherian domain $A$. Then

$$\bigcap_{\mathfrak{p}} A_{\mathfrak{p}} = A,$$

where the intersection is over all minimal nonzero primes.

*Proof.* Omitted. ■

Let's see some applications of Proposition 3.43.

**Corollary 3.44.** If $A$ is a normal Noetherian domain with $X := \operatorname{Spec} A$, then we see that $f \in K(X)^\times$ has $\operatorname{div}(f)$ effective if and only if $f \in A$.

*Proof.* Having $\nu_Y(f) \geq 0$ just means that $f \in A_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime cutting out $Y$, so the result now follows from Proposition 3.43. ■

We would like to compute $\operatorname{Cl} X$ in some non-affine cases. We will do this by reducing to the affine case, for which we want the following exact sequence.

**Proposition 3.45.** Fix a smoothish scheme $X$ and a nonempty open subscheme $U \subseteq X$. Setting $Z := X \setminus U$, there is an exact sequence

$$0 \to \bigoplus_{\substack{Y \in X^{(1)} \\ Y \subseteq Z}} \mathbb{Z}[Y] \to \operatorname{Div} X \to \operatorname{Div} U \to 0.$$

Here, the right-hand map is given by $Y \mapsto (Y \cap U)$.

*Proof.* Exactness on the left is just the definition of the map on the right: some prime divisor $Y$ vanishes on $\operatorname{Div} U$ if and only if $Y \cap U$ is empty, which is equivalent to $Y \subseteq Z$. Lastly, surjectivity is not so bad: for prime divisor $Y_0$ of $U$, we note that $\overline{Y}_0$ is a prime divisor of $X$ satisfying $\overline{Y}_0 \cap U = Y_0$ (indeed, these subsets share their unique generic point!). ■

**Proposition 3.46.** Fix a smoothish scheme $X$ and a nonempty open subscheme $U \subseteq X$, and set $Z := X \setminus U$.

(a) The induced map $\operatorname{Cl} X \to \operatorname{Cl} U$ is surjective.

(b) If $\operatorname{codim}_X Z \geq 2$, then the induced map of (a) is an isomorphism.

(c) If $Z$ is a prime divisor of $X$, then the sequence

$$\mathbb{Z}[Z] \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0$$

is exact.

*Proof.* Here we go.

(a) We already know that the map $\operatorname{Div} X \to \operatorname{Div} U$ is surjective, so it remains to check that the principal divisors of $X$ go to principal divisors of $U$. Well, for $f \in K(X)$, we note that $\nu_Y(f) = \nu_{Y \cap U}(f|_U)$ whenever $Y \cap U$ is nonempty because $Y$ and $Y \cap U$ share a generic point; for other $Y \subseteq Z$, the component simply vanishes as $Y \mapsto 0$ in $\operatorname{Div} U$ anyway. So $\operatorname{div}(f)$ does go to $\operatorname{div}(f|_U)$.

(b) The point is that every prime divisor $Y$ of $X$ fails to live in $Z$. As such, Proposition 3.45 produces an isomorphism $\operatorname{Div} X \to \operatorname{Div} U$, and (a) tells us that the principal divisors of $U$ exactly are the principal divisors of $X$.

(c) We already have surjectivity on the left for the map $\operatorname{Cl} X \to \operatorname{Cl} U$, so it remains to show that the kernel of this map is generated by $Z$. Well, any prime divisor of $X$ vanishing in $U$ will arise from $Z$ because $Z$ is a prime divisor, so we are okay. ■

## 3.6 March 13

Here we go.

### 3.6.1 Degree of Divisors

We begin by defining degree in the most geometric situation.

**Definition 3.47** (degree)**.** Fix a field $k$ and positive integer $n > 0$. Then let $Y$ be a prime divisor of $X := \mathbb{P}_k^n$, which we remark is smoothish. Because $Y$ is integral, we may write $Y = V((f))$ for some irreducible homogeneous polynomial $f$ (unique up to multiplication by an element of $k^\times$), so we define the *degree* as

$$\deg Y := \deg f.$$

**Remark 3.48.** To see that $Y$ takes the form $V((f))$, note that $Y \hookrightarrow X$ is an integral closed subscheme of codimension 1, so we can affine-locally realize it via a quotient of $k[x_{0/i}, \ldots, \widehat{x_{i/i}}, \ldots, x_{n/i}]$ of codimension 1, and here dimension theory tells us that we will be cut out by a single irreducible polynomial. Then we can glue together our irreducible polynomials to complete.

Let's understand this notion of degree.

**Proposition 3.49.** Fix a field $k$ and positive integer $n > 0$ so that $X := \mathbb{P}_k^n$ is smoothish. Let $H$ be the hyperplane cut out by $x_0$.

(a) For any divisor $D$ on $X$, we have $D \sim dH$ where $d = \deg D$.

(b) All principal divisors on $f$ have degree $0$.

(c) $\deg$ induces an isomorphism $\operatorname{Cl} X \to \mathbb{Z}$.

*Proof.* Here we go.

(a) By linearity, we may assume that $D$ is a prime divisor $Y$ of the form $V((f))$ for some irreducible homogeneous polynomial of degree $d$. Then $f/x_0^d \in K(X)^\times$ (it is a rational section defined on the distinguished open subscheme cut out by $x_0 \neq 0$), so the principal divisor associated to $f/x_0^d$ is $Y - dH$, so we are done.

(b) We omit this argument.

(c) By (a), we have a surjective homomorphism $\deg\colon \operatorname{Div} X \to \mathbb{Z}$. Note that this is well-defined up to equivalence: if $D \sim D'$, then we see $(\deg D)H \sim (\deg D')H$ by (a), so to show that $\deg D = \deg D'$, it suffices to show that $H$ is not equivalent to $0$. But $H$ has degree $1$ (it's cut out by $x_0$), so we get the claim by (b).

Thus, we have a well-defined surjection $\deg \operatorname{Cl} X \to \mathbb{Z}$. Setting $U := X \setminus H$ (which we note is isomorphic to $\mathbb{A}_k^n$), we see that Proposition 3.46 provides an exact sequence

$$\mathbb{Z}[H] \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0,$$

but the last term vanishes because $\mathbb{A}_k^n$ is the spectrum of a unique factorization domain. So the map $\mathbb{Z}[H] \to \operatorname{Cl} X$ is surjective and has trivial kernel because any kernel would give kernel in $\mathbb{Z}[H] \to \operatorname{Cl} X \to \mathbb{Z}$ (where the last map is $\deg$), but the composite $\mathbb{Z}[H] \to \mathbb{Z}$ is the identity. ■

**Remark 3.50.** One can show that $X \times \mathbb{A}^1$ satisfies $\operatorname{Cl}(X \times \mathbb{A}^1) \cong \operatorname{Cl} X$.

**Remark 3.51.** It is not generally true that $\operatorname{Cl}(X \times Y) \cong \operatorname{Cl} X \times \operatorname{Cl} Y$. For example, take $X \subseteq \mathbb{P}_k^2$ to be cut out by $y^2 z = x^3 - xz^2$, and let $\Delta \subseteq X \times_k X$ be the diagonal. Then $\Delta$ is a prime divisor but not linearly equivalent to any divisor coming from $\operatorname{Cl} X \times \operatorname{Cl} X$. Namely, we are claiming that $\Delta$ fails to equivalent to one of the form $\sum_i P_i \times X + \sum_j X \times Q_j$ for closed points $\{P_i\}$ and $\{Q_j\}$ of $X$. Otherwise, we get $f \in K(X \times X)$ such that

$$\Delta = \sum_i P_i \times X + \sum_j X \times Q_j + \operatorname{div} f.$$

Choosing $R_1, R_2 \in X$ distinct from the $P_i$ and $Q_j$, then $f|_{R_1 \times X}$ produces a rational function on $X$ with $\operatorname{div} f = R_1 - \sum_j Q_j$, and similarly one gets $f_2$ with $\operatorname{div} f_2 = R_2 - \sum_j Q_j$, but then $R_1 \sim R_2$, which contradicts $X$ not being isomorphic to $\mathbb{P}_k^1$.

### 3.6.2 Rational Maps

We will also want to discuss rational maps in some detail before continuing.

**Definition 3.52** (variety)**.** Fix an algebraically closed field $k$. Then a *variety* over $k$ is an integral separated $k$-scheme of finite type.

We want to define rational maps, which we do as follows.

**Definition 3.53** (dominant)**.** A morphism $f\colon X \to Y$ of integral schemes is *dominant* if and only if $f(X)$ is Zariski dense in $Y$.

**Remark 3.54.** Suppose that $X$ and $Y$ are integral schemes with generic points $\xi$ and $\eta$, respectively. Then we claim that $f\colon X \to Y$ is dominant if and only if $f(\xi) = \eta$. Certainly if $f(\xi) = \eta$ than the image of $f$ is Zariski dense. Conversely, if $\overline{\xi \text{ specializes}}$ to some $x \in X$, then we note that $f(\xi)$ will specialize to some $f(x)$ by continuity of $f$, so $\overline{f(X)} \subseteq \overline{\{f(\xi)\}}$, so for this to be all of $Y$ we must have $f(\xi) = \eta$.

**Definition 3.55** (rational)**.** Fix integral separated schemes $X$ and $Y$. Then a *rational map* is an equivalence class of maps $(U, \varphi)$ where $\varphi\colon U \to Y$ is a bona fide morphism, where we declare $(U, \varphi) \sim (V, \psi)$ if and only if $\varphi|_{U \cap V} = \psi|_{U \cap V}$. The rational map $\varphi\colon X \to Y$ is dominant if its representatives are.

**Remark 3.56.** Let's discuss how to check that this is an equivalence relation. Reflexivity and symmetry have little content, but transitivity requires us to remark that any two rational maps will agree on a closed subset of their domain (this is where being separated is used), so agreeing on a closed subset means that they will actually agree.

**Remark 3.57.** One can compose rational maps exactly as expected, with the caveat that we need to work in smaller and smaller Zariski open subsets.

**Remark 3.58.** Any rational map $\varphi\colon X \to Y$ has a unique largest open subset $U$ where it is defined; indeed, one can simply take the union of all the $U$s appearing in the equivalence class. The point here is that one can specify the entire rational map by specifying how it behaves on any open subscheme.

With maps in one direction, we should discuss having maps in both directions.

**Definition 3.59** (birational)**.** Fix integral separated schemes $X$ and $Y$. A *birational* map is a rational map $\varphi\colon X \to Y$ with a birational inverse map.

**Remark 3.60.** One can see that birational maps are necessarily dominant because the image needs to surject onto some Zariski open subset.

## 3.7 March 15

Today we'll continue our discussion of birational maps and then move on to divisors on curves.

### 3.7.1 More on Birational Maps

Last class we discussed birational maps. It will be helpful to have the following stronger notion.

**Definition 3.61** (birational)**.** A morphism $f\colon X \to Y$ of integral separated schemes is *birational* if and only if it is a birational map; i.e., it has a rational map as an inverse.

**Example 3.62.** Fix an algebraically closed field $k$. Let $X \subseteq \mathbb{A}^2_k \times \mathbb{P}^1_k$ be the closed $k$-subvariety

$$\left\{((x,y),[t:u]) \in \mathbb{A}^2_k \times \mathbb{P}^1_k : xu = yt\right\}.$$

(Formally, this is cut out by some equation.) Let $\varphi$ be the projection onto $\mathbb{A}^2_k$. Notably, if $(x,y) \neq (0,0)$, then the fiber is the single point $((x,y),[x:y])$, but if $(x,y) = (0,0)$, then the fiber is a full $\mathbb{P}^1_k$. Thus, $\varphi$ is a birational morphism: certainly it is a morphism, and its inverse can be represented by the rational map is given by $(x,y) \mapsto ((x,y),[x:y])$, defined on $\mathbb{A}^2_k \setminus \{(0,0)\}$.

**Definition 3.63** (graph)**.** Given a morphism $f\colon X \to Y$ of $S$-schemes, the *graph* $\Gamma_f$ of $f$ is the morphism $(\mathrm{id}_X, f)\colon X \to X \times_S Y$.

**Example 3.64.** If $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine $R$-schemes, then the graph is the map $\operatorname{Spec} A \to \operatorname{Spec}(A \otimes_R B)$ given by the map $A \otimes_R B \to A$ defined by $a \otimes b \mapsto a \cdot f^\sharp(b)$.

**Remark 3.65.** If $f$ is separated, then $(\mathrm{id}_X, f)\colon X \to X \times Y$ is a closed embedding. Indeed, the square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_S Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f,\mathrm{id}_Y)} \\ Y & \xrightarrow{\Delta_f} & Y \times_S Y \end{array}$$

is a pullback square, so $\Gamma_f$ is a closed embedding because $\Delta_f$ is.

**Example 3.66.** Continuing from Example 3.62, define $U := \mathbb{A}^2_k \setminus \{(0,0)\}$ and then $f\colon U \to \mathbb{P}^1_k$ given by $\varphi(x,y) := [x : y]$. (Notably, $f$ is a rational map $\mathbb{A}^2_k \to \mathbb{A}^1_k$.) Then we set $\psi := (\mathrm{id}_U, f)$ so that $\psi$ is the graph of $f$; notably, everything in sight is separated, so this graph is a closed embedding onto the closed subscheme $X \cap (U \times \mathbb{P}^1_k)$ of $U \times \mathbb{P}^1_k$. Notably, $X \subseteq \mathbb{A}^2_k \times \mathbb{P}^1_k$ is then a closed subset agreeing with this graph in a Zariski dense subset, so $X$ is the closure of the graph.

**Remark 3.67.** Continuing, the example, we can view $f$ as an element of $K(\mathbb{A}^2_k \times \mathbb{P}^1_k)$, whereupon we may compute $f(x,y) = y/x$ has $\operatorname{div}(f) = (y) - (x)$. More generally, for a scheme $X$, rational maps $X \to \mathbb{A}^1$ correspond to rational functions on $X$ by passing to an open subset and using the adjunction

$$\operatorname{Mor}(X, \mathbb{A}^1) \cong \operatorname{Hom}(\mathbb{Z}[x], \Gamma(X, \mathcal{O}_X)) \cong \Gamma(X, \mathcal{O}_X).$$

We now note that birational maps do indeed (locally) behave like isomorphisms.

**Proposition 3.68.** Fix a field $k$ and a birational map $\varphi\colon X \to Y$ of $k$-varieties. Then there are nonempty open subschemes $U \subseteq X$ and $V \subseteq Y$ such that $\varphi$ induces an isomorphism $\varphi\colon U \cong V$. Conversely, any isomorphism on nonempty open subschemes induces a birational map.

*Proof.* The second claim just uses the given isomorphism on nonempty open subschemes to define the birational maps. For the first claim, if $\varphi$ has rational inverse given by $\psi$, then take some open subscheme $U \subseteq X$ where $\varphi$ is defined and some open subscheme $V \subseteq Y$ where $\psi$ is defined. Then $U \cap \varphi^{-1}(V)$ and $V \cap \psi^{-1}(U)$ should do the trick because we know that composing $\varphi$ and $\psi$ (when defined) will yield the identity. ■

We now take a moment to note that rational maps are determined purely by the functions on the variety.

**Proposition 3.69.** Fix a field $k$. A dominant rational maps $f\colon X \to Y$ of $k$-varieties functorially defines a $k$-algebra homomorphism $f^\sharp\colon K(Y) \to K(X)$ given by taking stalks at the generic points.

*Proof.* All that remains to be checked is functoriality, which is true because $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$ as maps of sheaves, and taking stalks is functorial. ■

**Remark 3.70.** In fact, we note that a rational function $g \in K(Y)$ thought of as a rational map $g\colon Y \to \mathbb{A}^1$ will go to the induced rational map $(g \circ f)\colon X \to \mathbb{A}^1$. This is a matter of tracking through the adjunction. Indeed, assuming that everything is defined over $U \subseteq X$ and $V \subseteq Y$, we merely need to note that the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Mor}(V, \operatorname{Spec} k[x]) \arrow[r] \arrow[d, "(-\circ f)"'] & \Gamma(V, \mathcal{O}_V) \arrow[d, "f^\sharp"'] & g \arrow[r, mapsto] \arrow[d, mapsto] & g^\sharp(x) \arrow[d, mapsto] \\
\operatorname{Mor}(U, \operatorname{Spec} k[x]) \arrow[r] & \Gamma(U, \mathcal{O}_U) & (g \circ f) \arrow[r, mapsto] & (g \circ f)^\sharp(x)
\end{tikzcd}$$

**Theorem 3.71.** Let $k$ be an algebraically closed field. The functor $X \mapsto K(X)$ from the category of $k$-varieties (equipped with dominant rational maps) to the category of $k$-algebras (equipped with $k$-algebra homomorphisms) is fully faithful.

*Proof.* See [Har77, Theorem I.4.4]. ■

### 3.7.2 Proper Curves

Here is our definition.

**Definition 3.72** (curve)**.** Fix a field $k$. A *curve* over $k$ is a one-dimensional $k$-variety $X$. Then $X$ is a complete if and only if proper and nonsingular if and only if regular.

Having properness (and being dimension $1$) allows us to use the valuative criterion to produce the following result.

**Proposition 3.73.** Ant rational map $f\colon C \to X$ from a regular $k$-curve $C$ to a proper $k$-variety $X$ extends uniquely to a full morphism $C \to X$.

*Proof.* This is exactly the valuative criterion for properness for curves. ■

## 3.8 March 18

Today we continue discussing curves.

### 3.8.1 Properties of Curves

We quickly note that projective and proper are the same for curves.

**Proposition 3.74.** Fix a regular curve $X$ over an algebraically closed field $k$. Then the following are equivalent.

(i) $X$ is projective.

(ii) $X$ is proper.

(iii) $X \cong t(C_{K(x)})$ where $t\colon \mathrm{Var}_k \to \mathrm{Sch}_k$ is the equivalence of [Har77, Proposition 2.6].

*Proof.* We won't show the equivalences of (i) and (iii). The implication (i) to (ii) is simply because projective implies proper. The implication (ii) to (i) is by Chow's lemma.

**Lemma 3.75** (Chow)**.** Fix a scheme $X$ proper over a Noetherian scheme $S$. Then there exists a birational morphism $g\colon X \to X'$ where $X'$ is projective over $S$.

We won't prove Lemma 3.75 because it is pretty hard. To see how Lemma 3.75 shows the implication, note that $g$ extends to a full morphism $g\colon X \to X'$ which is an isomorphism on an open subscheme $U$ of $X$. Checking our isomorphism on stalks reveals that $X'$ continues to be regular, and then we finish by noting that the inverse rational map for $g$ also extends to a full map $X' \to X$, so we are done. ■

This allows us to understand morphisms of curves.

**Proposition 3.76.** Fix a morphism $f\colon X \to Y$ of curves over an algebraically closed field $k$, and suppose $X$ is proper and regular. Then exactly one of the following hold.

(i) $f(X)$ is a single point.

(ii) $f$ is surjective. In this case, $f^\sharp\colon K(Y) \to K(X)$ is a finite field extension, and $Y$ is proper, and $f$ is finite.

*Proof.* Note $f(X)$ must be a closed subset of $Y$ because $X$ is proper, and because $Y$ is a curve, all closed subsets are either finite sets of points or all of $Y$. However, $X$ is connected, so $f(X)$ is connected, so the image of $f(X)$ is either a point or all of $Y$ (and not both).

It remains to do the other checks of (ii) in the case that $f$ is surjective. Because $f$ is surjective, we see that $Y$ is proper (in particular, $Y$ is universally closed because the image of any morphism from $Y$ will continue to have closed image because we can realize this as an image from $X$). Now, $f$ is dominant, so $f^\sharp\colon K(Y) \to K(X)$ is injective, so we have an actual field extension; because these two field extensions have transcendence degree $1$ by Noether normalization, we see that this is an algebraic extension. Because these rings are finitely generated over $k$, we see that the extension $K(Y) \subseteq K(X)$ is also finite generated, so our degree is in fact finite.

It remains to check that $f$ is finite. This is somewhat involved. Let $V = \operatorname{Spec} B$ be some nonempty affine open subscheme of $Y$, and let $A$ be the integral closure of $B$ in $K(X)$. Note $A$ is finite over $B$ because our field extension is finite. As such, we would like to know that $U := \operatorname{Spec} A$ is $f^{-1}(V)$. Well, $K(U) = K(X)$, so $U$ is birational to $X$. Further $U$ is normal and hence regular by construction, so properness of $X$ means that the birational map $U \to X$ becomes a full morphism $i\colon U \to X$. Notably, the diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\ V \arrow[r, phantom, "\subseteq"] & Y \end{tikzcd}$$

commutes by construction of the map $i$, so $i(U) \subseteq X$ lands inside $f^{-1}(V)$. We would like for $i$ to be an open embedding, which we do as follows: note $U$ will embed as an open subset into some unique proper regular curve $\overline{U}$,[2] and universality then provides a rational map $X \to \overline{U}$ (which extends to a full morphism $g\colon X \to \overline{U}$) making the diagram

$$\begin{tikzcd} U \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[drr, "i"] & \overline{U} & \\ V \arrow[ddrr] & & X \arrow[ul, "g"'] \arrow[d, "f"] \\ & & Y \end{tikzcd}$$

commute. Tracking around this diagram reveals that $g^{-1}(U) \subseteq f^{-1}(V)$.

We quickly claim that $g^{-1}(U) = f^{-1}(V)$. Well, suppose for the sake of contradiction that we have strict containment. Because generic points map to generic points, we at least are a nonempty open subset, so at worst we are missing some finite set of closed points $\{p_1, \ldots, p_r\}$. Well, set $W := f^{-1}(V) \setminus \{p_2, \ldots, p_r\}$ to be $g^{-1}(U)$ with $p_1$ appended. Then $g$ sends $W \setminus \{p_1\}$ to $U$, and $f$ maps $W$ to $V$, so because $U \to V$ is proper (in fact finite), we know that $g$ must extend to a map $W \to U$ by the valuative criterion for properness (as usual). But $g'(p_1) \in U$ while $g(p) \notin U$, which contradicts the fact that the map $\overline{U} \to k$ is separated (again using a valuative criterion for points on $k$).

We now complete the proof that $f$ is finite. Note $g|_{g^{-1}(U)} \circ i$ sends $U$ back to $U$, and $\varphi \circ g$ sends $f^{-1}(V)$ to $f^{-1}(V)$, and these maps are over $X$ and $Y$ respectively, so we have mutually inverse isomorphisms. Namely, we have provided our isomorphism of $U$ with $f^{-1}(V)$ over $Y$, so $f^{-1}(V)$ is finite over $V$ because $U$ is. ■

This allows us to define the degree of a morphism.

**Definition 3.77** (degree)**.** Fix a finite morphism $f\colon X \to Y$ over a field $k$. Then the *degree* of $f$, denoted $\deg f$, is the degree of the field extension $f^\sharp\colon K(Y) \to K(X)$.

**Proposition 3.78.** Fix a regular curve $U$ over an algebraically closed field $k$. Then there is an open embedding $U$ into a proper regular curve $X$.

[2] More precisely, place $U$ into some projective space, and let $\overline{U}$ be the closure there.

*Proof.* Let $V \subseteq U$ be some nonempty affine open subscheme. Then we can place $V$ into some projective space and takes its closure, which we call $\overline{V}$. Let $\pi\colon X \to \overline{V}$ be the normalization map so that $\pi$ is finite (note that this is a birational map). Now, $X$ is normal and proper (in particular, it follows that $X$ is regular), so one can argue as in the previous point to see that the rational map

$$U \to V \subseteq \overline{V} \to X$$

extends uniquely to an open embedding. Indeed, this is the main content of the proof of Proposition 3.76. ■

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# List of Definitions