

191: Analytic Number Theory

For the Very Patient

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Spring 2023

CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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THEME 1

ARITHMETIC PROGRESSIONS

1.1 January 18

Here we go.

1.1.1 House-Keeping

We're teaching analytic number theory. Here are some notes.

- We will be referencing [Dav80] mostly, but we will do some things that Davenport does not do. For example, we will discuss the circle method, for which we refer to [Dav05].
- We will assume complex analysis, at the level of Math 185. We will use some Fourier analysis, but we will discuss the relevant parts as we need them. Of course, because this is number theory, we will assume some algebra, such as characters on abelian groups.
- There is a website [here](#), which includes a list of topics. Notably, there is a website for a previous version of the course.
- Grading is still up in there, as is the syllabus. Tentatively, grading will be as follows: by around the middle of the semester, there will be a list of recommended papers to read. Then we will write a 2–6-page report and present it to Professor Zhang. We will not have problem sets.
- Tentatively, office hours will be 90 minutes before lecture on Monday and Wednesday, in Evans 813.
- We should all write an email to Professor Zhang to introduce ourselves; for example, say what you're looking forward to in the course.

1.1.2 Facts about Dirichlet Series

In this first part of the course, we will be moving towards the following result.

Theorem 1.1 (Dirichlet). Fix nonzero integers $a, q \in \mathbb{Z}$ such that $\gcd(a, q) = 1$. Then there exist infinitely many primes p such that $p \equiv a \pmod{q}$.

The statement of Theorem 1.1 is purely elementary, but the standard proof uses complex analysis. The functions we will do analysis on are generalizations of the Riemann ζ function, defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges absolutely for $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$. Indeed, we can show this.

Proposition 1.2. Let $f: \mathbb{N} \rightarrow \mathbb{C}$ denote a sequence of complex numbers such that $|f(n)| = O(n^\sigma)$ for some $\sigma \in \mathbb{R}$. Then the series

$$D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

converges absolutely for $s \in \mathbb{C}$ such that $\operatorname{Re} s > \sigma + 1$. Thus, $D(s)$ defines a holomorphic function in this region.

Proof. We are given $|f(n)| \leq Cn^\sigma$ for some $C > 0$. Thus, showing the absolute convergence is direct: note

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re}(s) - \sigma}},$$

which converges because $\operatorname{Re}(s) - \sigma > 1$.

We can now convert absolute convergence to uniform convergence of the partial sums $\{D_n\}_{n \in \mathbb{N}}$ of D , from which Lemma A.15 will finish. Fix some compact subset $D \subseteq U$, and we want to show $D_n \rightarrow D$ uniformly on D . Because D is compact, there exists $s_0 \in D$ with minimal $\operatorname{Re} s_0$; define $\sigma_0 := \operatorname{Re} s_0$. Now, the series

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma_0}}$$

converges by our absolute convergence.

As such, for any $\varepsilon > 0$, select N such that $n_0 > N$ implies

$$\sum_{n > n_0} \frac{|f(n)|}{n^{\sigma_0}} < \varepsilon.$$

Thus, for any $s \in \mathbb{C}$ and $n_0 > N$, we see

$$|D(s) - D_{n_0}(s)| = \left| \sum_{n > n_0} \frac{f(n)}{n^s} \right| \leq \sum_{n > n_0} \frac{|f(n)|}{n^{\operatorname{Re} s}} \leq \sum_{n > n_0} \frac{|f(n)|}{n^{\sigma_0}} < \varepsilon,$$

which is what we wanted. ■

It follows from Proposition 1.2 that $\zeta(s)$ defines a holomorphic function on $\operatorname{Re} s > 1$.

1.1.3 The Euler Product

The following factorization is due to Euler.

Definition 1.3 (multiplicative). Let $f: \mathbb{N} \rightarrow \mathbb{C}$ be a function. Then f is *multiplicative* if and only if $f(nm) = f(n)f(m)$ for any $n, m \in \mathbb{N}$ such that $\gcd(n, m) = 1$.

Proposition 1.4. Let $f: \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function such that $|f(n)| = O(n^\sigma)$. For any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \sigma + 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right).$$

In fact, the product converges absolutely and uniformly on compacts.

Proof. Fix $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma + 1$. Roughly speaking, this follows from unique prime factorization in \mathbb{Z} . For and N and M to be fixed later, define

$$P_{N,M} := \prod_{p < N} \left(\sum_{k=0}^{M-1} \frac{f(p^k)}{p^{ks}} \right),$$

and define $P_{N,\infty}$ analogously. Define $A_{N,M}$ to be the set of integers n such that the prime factorization of n includes primes less than N each to a power less than M , and define $A_{N,\infty}$ analogously. Note $A_{N,M}$ is a finite set, so the distributive law implies

$$P_{N,M} = \sum_{n \in A_{N,M}} \frac{f(n)}{n^s}.$$

To begin, we fix N and claim

$$P_{N,\infty} \stackrel{?}{=} \sum_{n \in A_{N,\infty}} \frac{f(n)}{n^s}.$$

Note $P_{N,\infty} = \lim_{M \rightarrow \infty} P_{N,M}$, so we fix some $M > 0$ and compute

$$\left| P_{N,M} - \sum_{n \in A_{N,\infty}} \frac{f(n)}{n^s} \right| = \left| \sum_{n \in A_{N,\infty} \setminus A_{N,M}} \frac{f(n)}{n^s} \right| \leq \sum_{n \notin A_{N,M}} \left| \frac{f(n)}{n^s} \right|.$$

Now, the smallest n such that $n \notin A_{N,M}$ is at least 2^M , so we see

$$\left| P_{N,M} - \sum_{n \in A_{N,\infty}} \frac{f(n)}{n^s} \right| \leq \sum_{n \geq 2^M} \left| \frac{f(n)}{n^s} \right|,$$

which now vanishes as $M \rightarrow \infty$ because $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely by Proposition 1.2. This completes the proof of the claim.

We now send $N \rightarrow \infty$ to finish the proof. For any $N > 0$, we use the claim to note

$$\left| P_{N,\infty} - \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right| = \left| \sum_{n \notin A_{N,\infty}} \frac{f(n)}{n^s} \right| \leq \sum_{n \notin A_{N,\infty}} \left| \frac{f(n)}{n^s} \right|.$$

Now, we note that the smallest $n \notin A_{N,\infty}$ is at least N because any $n < N$ has a prime factor less than N , so

$$\left| P_{N,\infty} - \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right| \leq \sum_{n \geq N} \left| \frac{f(n)}{n^s} \right|,$$

and now we see that the right-hand side goes to 0 as $N \rightarrow \infty$ because $\sum_{n=1}^{\infty} f(n)/n^s$ converges absolutely by Proposition 1.2. The equality follows.

It remains to show that the product converges absolutely and uniformly on compacts. We use Proposition A.25. Indeed, fix some compact $D \subseteq \{s : \operatorname{Re} s > \sigma + 1\}$. Now, we want to upper-bound

$$a_p(s) = -1 + \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} = \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}}$$

on D . Well, let σ_0 denote the (achieved!) minimum of the continuous function $\operatorname{Re}: D \rightarrow \mathbb{R}$, and note that $\sigma_0 > \sigma + 1$. Now, $f(n) = O(n^\sigma)$ promises some constant C such that $|f(n)| \leq Cn^\sigma$ for all n . Thus, we see

$$|a_p(s)| \leq \sum_{k=1}^{\infty} \left| \frac{f(p^k)}{p^{ks}} \right| \leq C \sum_{k=1}^{\infty} \frac{p^{k\sigma}}{p^{k\sigma_0}} = C \cdot \frac{p^{\sigma-\sigma_0}}{1-p^{\sigma-\sigma_0}} < Cp^{\sigma-\sigma_0}.$$

Notably, the geometric series converges because $p^{\sigma-\sigma_0} < p^{-1} < 1$. However, this finishes our check of absolute convergence by Proposition A.25 because

$$\sum_{p \text{ prime}} C p^{\sigma-\sigma_0} < C \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0-\sigma}}$$

converges because $\sigma_0 - \sigma > 1$. ■

Corollary 1.5. We have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Proof. By Proposition 1.4, we see

$$\zeta(s) = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \right) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

which is what we wanted. ■

We can now use Corollary 1.5 to give a proof of the infinitude of primes.

Theorem 1.6. There are infinitely many primes. In fact,

$$\sum_{p \text{ prime}} \frac{1}{p} = +\infty.$$

Proof. Throughout the proof, s will be a real number greater than 1. The key estimate is to note

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \int_1^{\infty} x^{-s} dx = -\frac{1}{1-s},$$

which goes to $+\infty$ as $s \rightarrow 1^+$. In particular, $\log \zeta(s) \rightarrow +\infty$ as $s \rightarrow 1^+$.

The last ingredient we need is to bound the Euler product of Corollary 1.5. In particular, we see

$$\log \zeta(s) = \log \left(\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \right) = \sum_{p \text{ prime}} -\log(1-p^{-s}).$$

(Formally, one should cap the number of factors and then send the number of factors to infinity.) Using the Taylor expansion of $-\log(1-x)$, we now see

$$\log \zeta(s) = \sum_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{1}{k p^{ks}} \right) = \left(\sum_{p \text{ prime}} \frac{1}{p^s} \right) + \sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{1}{k p^{ks}} \right).$$

We would like to focus on $\sum_p 1/p^s$, so we quickly show that the other sum converges. All terms are positive, so it suffices to show that it is bounded above, for which we see

$$\sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{1}{k p^{ks}} \right) \leq \sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{1}{p^k} \right) = \sum_{p \text{ prime}} \frac{1/p^2}{1-1/p} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1,$$

where we have telescoped in the last equality. Letting the value of this sum be $S(s)$, we see

$$\log \zeta(s) - S(s) = \sum_{p \text{ prime}} \frac{1}{p^s} < \sum_{p \text{ prime}} \frac{1}{p}.$$

Now, as $s \rightarrow 1^+$, we see $\log \zeta(s) - S(s) \rightarrow +\infty$, so the theorem follows. ■

The proof of Theorem 1.1 more or less imitates the argument of Theorem 1.6. Roughly speaking, we will show that

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{q}}} \frac{1}{p} = +\infty,$$

from which our infinitude follows. Finding a way to extract out the equivalence class $a \pmod{q}$ will use a little character theory.

1.1.4 Characters

Throughout, our groups will be finite and abelian, and actually we will be most interested in the abelian groups $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^\times$ for integers n . Formally, here is our definition.

Definition 1.7. Fix a positive integer n . Then we define $(\mathbb{Z}/n\mathbb{Z})^\times$ as the units in $\mathbb{Z}/n\mathbb{Z}$, which is $\{a \pmod{n} : \gcd(a, n) = 1\}$.

Remark 1.8. It is a fact that $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic for any prime p . This is nontrivial to prove, but we will not show it here.

Notably, given a prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$, there is an isomorphism of rings

$$\mathbb{Z}/n\mathbb{Z} \cong \prod_{p|n} (\mathbb{Z}/p^{\nu_p(n)})$$

and hence also an isomorphism of multiplicative groups, upon taking units.

Having said all that, the theory is most cleanly build working with general finite abelian groups.

Definition 1.9 (dual group). Let G be a group. Then the *dual group* is $\widehat{G} := \text{Hom}(G, \mathbb{C}^\times)$, where the operation is pointwise. Its elements are called *characters*.

Notation 1.10 (principal character). There is a "trivial" character $1: G \rightarrow \mathbb{C}^\times$ sending $g \mapsto 1$, which is the identity. We might call 1 the *principal character*; we might also denote 1 by χ_0 .

Notation 1.11 (conjugate character). If $\chi: G \rightarrow \mathbb{C}^\times$ is a character, then note that $\overline{\chi}: G \rightarrow \mathbb{C}^\times$ defined by $\overline{\chi}(g) := \overline{\chi(g)}$ is also a character. Indeed, conjugation is a field homomorphism.

Remark 1.12. If G is a finite group, we note that any $\chi \in \widehat{G}$ and $g \in G$ has

$$\chi(g)^{\#G} = \chi(g^{\#G}) = 1,$$

so $\chi(g)$ is a $(\#G)$ th root of unity. In particular, $|\chi(g)| = 1$, so $\overline{\chi(g)} = \chi(g)^{-1} = \chi(g^{-1})$.

It will be helpful to have the following notation.

Notation 1.13. We might write $e: \mathbb{C} \rightarrow \mathbb{C}$ for the function $e(z) := \exp(2\pi iz)$.

We now begin computing \widehat{G} for finite abelian groups.

Lemma 1.14. Suppose G and H are groups. Then $\widehat{G} \times \widehat{H} \cong \widehat{G \times H}$ by sending (χ_G, χ_H) to $(g, h) \mapsto \chi_G(g)\chi_H(h)$.

Proof. We have the following checks. Let e_G and e_H be the identities of G and H , respectively.

- **Well-defined:** given $(\chi_G, \chi_H) \in \widehat{G} \times \widehat{H}$, define $\varphi(\chi_G, \chi_H): G \times H \rightarrow \mathbb{C}^\times$ by $\varphi(\chi_G, \chi_H): (g, h) \mapsto \chi_G(g)\chi_H(h)$. Note $\varphi(\chi_G, \chi_H)$ is a homomorphism: we have

$$\begin{aligned} \varphi(\chi_G, \chi_H)((g, h) \cdot (g', h')) &= \varphi(\chi_G, \chi_H)(gg', hh') \\ &= \chi_G(gg')\chi_H(hh') \\ &= \chi_G(g)\chi_H(h)\chi_G(g')\chi_H(h') \\ &= \varphi(\chi_G, \chi_H)(g, h) \cdot \varphi(\chi_G, \chi_H)(g', h'). \end{aligned}$$

- **Homomorphism:** to show φ is a homomorphism, we have

$$\begin{aligned} \varphi((\chi_G, \chi_H) \cdot (\chi'_G, \chi'_H))(g, h) &= \chi_G(g)\chi'_G(g)\chi_H(h)\chi'_H(h) = \varphi(\chi_G, \chi_H)(g, h) \cdot \varphi(\chi'_G, \chi'_H)(g, h), \\ \text{so } \varphi((\chi_G, \chi_H) \cdot (\chi'_G, \chi'_H)) &= \varphi(\chi_G, \chi_H) \cdot \varphi(\chi'_G, \chi'_H). \end{aligned}$$

- **Injective:** if $\varphi(\chi_G, \chi_H) = 1$, then

$$\chi_G(g)\chi_H(h) = \varphi(\chi_G, \chi_H)(g, h) = 1$$

for all $g \in G$ and $h \in H$. Setting $g = e_G$ shows that $\chi_H = 1$, and similarly setting $h = e_H$ shows that $\chi_G = 1$. Thus, $(\chi_G, \chi_H) = (1, 1)$.

- **Surjective:** given a character $\chi: (G \times H) \rightarrow \mathbb{C}^\times$, define $\chi_G(g) := \chi(g, e_H)$ and $\chi_H(h) := \chi(e_G, h)$. Note χ_G is a character because

$$\chi_G(gg') = \chi(gg', e_H) = \chi(g, e_H)\chi(g', e_H) = \chi_G(g)\chi_G(g').$$

Switching the roles of G and H shows that χ_H is also a character. Lastly, we note $\varphi(\chi_G, \chi_H) = \chi$ because

$$\varphi(\chi_G, \chi_H)(g, h) = \chi_G(g)\chi_H(h) = \chi(g, e_H)\chi(e_G, h) = \chi(g, h).$$

This completes the proof. ■

Lemma 1.15. Suppose $G = \mathbb{Z}/n\mathbb{Z}$ for a positive integer n . Then $\chi_\bullet: \mathbb{Z}/n\mathbb{Z} \cong \widehat{G}$ by sending $[k]$ to the character $\chi_k: [\ell] \mapsto e(k\ell/n)$.

Proof. To begin, note $\chi_k: \mathbb{Z} \rightarrow \mathbb{C}^\times$ defines a homomorphism because

$$\chi_k(\ell + \ell') = e\left(\frac{k(\ell + \ell')}{n}\right) = e\left(\frac{k\ell}{n}\right)e\left(\frac{k\ell'}{n}\right) = \chi_k(\ell)\chi_k(\ell').$$

Further, note $\chi_k(n\ell) = e(k\ell) = 1$ for any $n\ell \in \mathbb{Z}$, so $n\mathbb{Z} \subseteq \ker \chi_k$. It follows that χ_k produces a homomorphism $\chi_k: G \rightarrow \mathbb{C}^\times$.

We now note that $\chi_\bullet: \mathbb{Z} \rightarrow \widehat{G}$ defines a homomorphism: for any $[\ell] \in G$, we see

$$\chi_{k+k'}([\ell]) = e\left(\frac{(k+k')\ell}{n}\right) = e\left(\frac{k\ell}{n}\right)e\left(\frac{k'\ell}{n}\right) = \chi_k([\ell])\chi_{k'}([\ell]).$$

Additionally, $\chi_{nk}([\ell]) = e(k\ell) = 1$, so $\chi_{nk} = 1$, so $nk \in \ker \chi_\bullet$. It follows that χ_\bullet produces a homomorphism $\chi_\bullet: \mathbb{Z}/n\mathbb{Z} \rightarrow \widehat{G}$.

It remains to show that χ_\bullet is a bijection. We have two checks.

- Injective: suppose $\chi_k = 1$ for $k \in \mathbb{Z}$. We must show $k \in n\mathbb{Z}$. Well, we must then have

$$1 = \chi_k([1]) = e(k/n),$$

which forces $n \mid k$.

- Surjective: given some character $\chi: G \rightarrow \mathbb{C}^\times$, we note $\chi([1])^n = \chi([0]) = 1$, so $\chi([1])$ is an n th root of unity. Thus, there exists k such that $\chi([1]) = e(k/n) = \chi_k([1])$. Thus, for any $\ell \in \{0, 1, \dots, n-1\}$, we see

$$\chi([\ell]) = \chi(\underbrace{[1] + \dots + [1]}_\ell) = \underbrace{\chi([1]) \cdot \dots \cdot \chi([1])}_\ell = \underbrace{\chi_k([1]) \cdot \dots \cdot \chi_k([1])}_\ell = \chi_k([\ell]),$$

so $\chi = \chi_k$ follows. ■

Proposition 1.16. Let G be a finite abelian group. Then $G \cong \widehat{\widehat{G}}$.

Proof. By the Fundamental theorem of finitely generated abelian groups, we may write

$$G \cong \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z}$$

for some positive integers n_i . Thus, using Lemma 1.14 and Lemma 1.15, we compute

$$\widehat{\widehat{G}} \cong \left(\widehat{\prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z}} \right) = \prod_{i=1}^n \widehat{\mathbb{Z}/n_i\mathbb{Z}} \cong \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z} \cong G,$$

which is what we wanted. ■

Proposition 1.16 might look like we now understand dual groups perfectly, but the isomorphism given there is non-canonical because the isomorphism of Lemma 1.15 is non-canonical. In other words, given some $g \in G$, there is in general no good way to produce character $\chi \in \widehat{\widehat{G}}$.

However, there is a natural map $G \rightarrow \widehat{\widehat{G}}$ which is an isomorphism.

Proposition 1.17. Fix a finite abelian group G . Define the map $\text{ev}_\bullet: G \rightarrow \widehat{\widehat{G}}$ by sending $g \in G$ to the map $\text{ev}_g \in \widehat{\widehat{G}}$ defined by $\text{ev}_g: \chi \mapsto \chi(g)$. Then ev_\bullet is an isomorphism.

Proof. We begin by checking that ev_\bullet is a well-defined homomorphism. For each $g \in G$, we see $\text{ev}_g: \widehat{G} \rightarrow \mathbb{C}^\times$ is a homomorphism because

$$\text{ev}_g(\chi\chi') = \chi(g)\chi'(g) = \text{ev}_g(\chi) \text{ev}_g(\chi').$$

Further, ev_\bullet is a homomorphism because

$$\text{ev}_{gg'}(\chi) = \chi(g)\chi(g') = \text{ev}_g(\chi) \text{ev}_{g'}(\chi).$$

It remains to show that ev_\bullet is an isomorphism. We claim that ev_\bullet is injective, which will be enough because $|G| = |\widehat{\widehat{G}}|$ by Proposition 1.16.

For this, we appeal to the following lemma.

Lemma 1.18. Fix a finite abelian group G with identity e . If $g \neq e$, then there exists $\chi \in \widehat{G}$ such that $\chi(g) \neq 1$.

Proof. Using the Fundamental theorem of finitely generated abelian groups, we may write

$$G \cong \prod_{i=1}^n \mathbb{Z}/n_i\mathbb{Z}$$

for positive integers $n_i \geq 2$. Moving our problem from G to the right-hand side, we are given some $(g_i)_{i=1}^n$ such that $[g_i] \neq [0]$ for at least one i , and we want a character χ such that $\chi((g_i)_{i=1}^n) \neq 1$. Without loss of generality, suppose that $g_1 \neq 0$ and define χ by

$$\chi((k_i)_{i=1}^n) := e(k_1/n_1).$$

Certainly $\chi((g_i)_{i=1}^n) = e(g_1/n_1) \neq 1$, so it remains to show that χ is a character. This technically follows from Lemma 1.14, but we can see it directly by computing

$$\chi((k_i)_{i=1}^n + (k'_i)_{i=1}^n) = e(k_1/n_1)e(k'_1/n_1) = \chi((k_i)_{i=1}^n) \chi((k'_i)_{i=1}^n).$$

This completes the proof. ■

The proof now follows quickly from Lemma 1.18. By contraposition, we see that any $g \in G$ such that $\chi(g) = 1$ for all $\chi \in \widehat{G}$ and must have $g = e$. But this is exactly the statement that $\text{ev}_\bullet: G \rightarrow \widehat{\widehat{G}}$ is injective. ■

1.1.5 Finite Fourier Analysis

We now proceed to essentially do Fourier analysis for finite abelian groups. Here is the idea.



Idea 1.19. We can write general functions $G \rightarrow \mathbb{C}$ as linear combinations of characters.

Remark 1.20. When G is not abelian, one must work with function $G \rightarrow \mathbb{C}$ which are “locally constant” on conjugacy classes of G .

Here is our Fourier transform.

Notation 1.21. Let G be a finite abelian group. Given a function $f: G \rightarrow \mathbb{C}$, we define $\widehat{f}: \widehat{G} \rightarrow \mathbb{C}$ by

$$\widehat{f}(\chi) := \sum_{g \in G} f(g) \overline{\chi(g)}.$$

Recall $\overline{\chi(g)} = \chi(g^{-1})$ by Remark 1.12.

To manifest Idea 1.19 properly, we need the following orthogonality relations.

Proposition 1.22. Let G be a finite abelian group.

- For any fixed $\chi \in \widehat{G}$, we have

$$\sum_{g \in G} \chi(g) = \begin{cases} 0 & \text{if } \chi \neq 1, \\ \#G & \text{if } \chi = 1. \end{cases}$$

- For any $g \in G$, we have

$$\sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} 0 & \text{if } g \neq e, \\ \#G & \text{if } g = e. \end{cases}$$

Proof. We show these directly.

(a) If $\chi = 1$, then the sum is $\sum_{g \in G} 1 = \#G$.

Otherwise, $\chi \neq 1$, so there exists $g_0 \in G$ such that $\chi(g_0) \neq 1$. It follows

$$\chi(g_0) \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(g_0 g) \stackrel{*}{=} \sum_{g \in G} \chi(g),$$

so we must have $\sum_{g \in G} \chi(g) = 0$. Note that we have re-indexed the sum at $\stackrel{*}{=}$.

(b) If $g = e$, then the sum is $\sum_{\chi \in \hat{G}} \chi(g) = \#(\hat{G})$, which is $\#G$ by Proposition 1.16.

Otherwise, $g \neq e$, so by Lemma 1.18, there exists χ_0 such that $\chi_0(g) \neq 1$. Employing the same trick, it follows

$$\chi_0 \sum_{\chi \in \hat{G}} \chi(g) = \sum_{\chi \in \hat{G}} (\chi_0 \chi)(g) \stackrel{*}{=} \sum_{\chi \in \hat{G}} \chi(g),$$

so we must have $\sum_{\chi \in \hat{G}} \chi(g) = 0$. Again, we re-indexed at $\stackrel{*}{=}$. ■

Now here is our result.

Theorem 1.23 (Fourier inversion). Let G be a finite abelian group. For any $f: G \rightarrow \mathbb{C}$, we have

$$f(g) = \frac{1}{\#G} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g)$$

for any $g \in G$.

Proof. This is direct computation with Proposition 1.22. Indeed, for any $g_0 \in G$, we see

$$\sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g_0) = \sum_{\chi \in \hat{G}} \sum_{g \in G} f(g) \chi(g^{-1}) \chi(g_0) = \sum_{g \in G} \left(f(g) \sum_{\chi \in \hat{G}} \chi(g^{-1} g_0) \right).$$

Now using Proposition 1.22, given $g \in G$, we see that the inner sum will vanish whenever $g \neq g_0$ and returns $\#G$ when $g = g_0$. In total, it follows

$$\frac{1}{\#G} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g_0) = f(g_0),$$

which is exactly what we wanted. ■

Here is our chief application.

Corollary 1.24. Let G be a finite abelian group. Fixing some $g_0 \in G$, we have

$$1_{g_0}(g) = \frac{1}{\#G} \sum_{\chi \in \hat{G}} \overline{\chi(g_0)} \chi(g)$$

for any $g \in G$.

Proof. Note

$$\hat{1}_{g_0}(\chi) = \sum_{g \in G} 1_{g_0}(g) \overline{\chi(g)} = \overline{\chi(g_0)}$$

because all terms except $g = g_0$ vanish. The result now follows from Theorem 1.23. ■

1.1.6 Dirichlet Characters

We want to extend our characters on $(\mathbb{Z}/q\mathbb{Z})^\times$ to work on all \mathbb{Z} , but this requires some trickery because, for example, 0 is not in general represented in $(\mathbb{Z}/q\mathbb{Z})^\times$. Here is our definition.

Definition 1.25 (Dirichlet character). Let q be a nonzero integer. A *Dirichlet character* $(\bmod q)$ is a function $\chi: \mathbb{Z} \rightarrow \mathbb{C}$ such that there exists a character $\tilde{\chi}: (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ for which

$$\chi(a) = \begin{cases} 0 & \text{if } \gcd(a, q) > 1, \\ \tilde{\chi}([a]) & \text{if } \gcd(a, q) = 1. \end{cases}$$

We might write this situation as $\chi \pmod{q}$. The Dirichlet character corresponding to 1 is denoted χ_0 and still called the *principal character*.

Remark 1.26. Note χ is periodic with period q .

We can finally define our generalization of ζ .

Definition 1.27 (Dirichlet L -function). Fix a Dirichlet character $\chi \pmod{q}$. Then we define the *Dirichlet L -function* as

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

By Proposition 1.2, we have absolute convergence for $\operatorname{Re} s > 1$, and $L(s, \chi)$ defines a holomorphic function there.

Remark 1.28. Continuing in the context of the definition, we note Proposition 1.4 gives

$$L(s, \chi) = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{\chi(p)^k}{p^{ks}} \right) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}}$$

for $\operatorname{Re} s > 1$.

In fact, the summation for $L(s, \chi)$ defines a holomorphic function for $\operatorname{Re} s > 0$, but seeing this requires a little care.

1.2 January 20

A syllabus was posted. There are some extra references posted.

1.2.1 Abel Summation

We are going to need the following technical result. Roughly speaking, it allows us to estimate infinite sums with a discrete part and a continuous part by summing the discrete part and integrating the continuous part. Oftentimes, a sum is difficult because of the way it mixes discrete and continuous portions, so it is useful to be able to separate them.

Theorem 1.29 (Abel summation). Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers, and define the partial sums be given by

$$A(t) := \sum_{1 \leq n \leq t} a_n.$$

For any real numbers $x, y \in \mathbb{R}$ with $x < y$ and continuously differentiable function $f: (0, x] \rightarrow \mathbb{C}$, we have

$$\sum_{0 < n \leq x} a_n f(n) = A(x)f(x) - \int_0^x A(t)f'(t) dt.$$

Proof. The idea is to write $a_n = A(n) - A(n-1)$, so we write

$$\begin{aligned} \sum_{n \leq x} a_n f(n) &= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x} A(n-1)f(n) \\ &= \sum_{0 < n \leq x} A(n)f(n) - \sum_{-1 < n \leq x-1} A(n)f(n+1) \\ &= A(\lfloor x \rfloor)f(\lfloor x \rfloor) - A(-1)f(0) - \sum_{0 < n \leq x-1} A(n)(f(n+1) - f(n)). \end{aligned}$$

Note $A(-1) = 0$. We now introduce an integral by noting $A(n)(f(n+1) - f(n)) = \int_n^{n+1} A(t)f'(t) dt$, which upon summing over n yields

$$\sum_{0 < n \leq x} a_n f(n) = A(\lfloor x \rfloor)f(\lfloor x \rfloor) - \int_0^{\lfloor x \rfloor} A(t)f'(t) dt.$$

To finish, we see

$$A(\lfloor x \rfloor)f(\lfloor x \rfloor) = A(x)f(x) + A(\lfloor x \rfloor)(f(\lfloor x \rfloor) - f(x)) = A(x)f(x) - \int_{\lfloor x \rfloor}^x A(t)f'(t) dt,$$

which when combined with the previous equality finishes. ■

Remark 1.30. One can use the theory of Riemann–Stieltjes integration to turn Theorem 1.29 into just an application of integration by parts, but we will not need this.

Here is a quick application of Theorem 1.29.

Proposition 1.31. The limit

$$\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right)$$

converges to a finite value.

Proof. Set $a_n = 1$ for each n and $f(t) := 1/t$ so that $A(t) = \lfloor t \rfloor$. Then Theorem 1.29 tells us

$$\begin{aligned} \sum_{0 < k \leq n} \frac{1}{k} &= \frac{\lfloor n \rfloor}{n} + \int_0^n \frac{\lfloor t \rfloor}{t^2} dt \\ &= 1 + \int_1^n \frac{\lfloor t \rfloor}{t^2} dt \\ &= 1 + \int_1^n \frac{1}{t} dt - \int_1^n \frac{\{t\}}{t^2} dt \\ &= \log n + 1 - \int_1^n \frac{\{t\}}{t^2} dt. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) = 1 - \int_1^{\infty} \frac{\{t\}}{t^2} dt,$$

and this integral converges because it is bounded above by $\int_1^{\infty} 1/t^2 dt = 1$. ■

Definition 1.32 (Euler–Mascheroni constant). The *Euler–Mascheroni constant* γ is the limit

$$\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right).$$

1.2.2 Continuing $L(s, \chi)$

As an example application of Theorem 1.29, we may give $L(s, \chi)$ an analytic continuation to $\{s : \operatorname{Re} s > 0\}$ when χ is not the principal character.

Proposition 1.33. Let $\chi \pmod{q}$ be a non-principal Dirichlet character. Then the function $L(s, \chi)$ admits an analytic continuation to $\{s : \operatorname{Re} s > 0\}$.

Proof. For given s with $\operatorname{Re} s > 1$, set $a_n := \chi(n)$ and $f(x) := 1/x^s$. Then the partial sums $A(t) := \sum_{1 \leq n \leq t} a_n$ have

$$\sum_{n=1}^{kq} \chi(n) = \sum_{a=0}^{k-1} \sum_{r=1}^q \chi(aq + r) = k \sum_{r=1}^q \chi(r) = k \sum_{\substack{1 \leq r \leq q \\ \gcd(r, q)=1}} \chi(r) = k \cdot 0$$

for any $k \geq 0$, where in the last equality we have used Proposition 1.22. Thus, for any $t \geq 0$, find $k \in \mathbb{Z}$ such that $kq \leq t < k(q+1)$, and we see

$$|A(t)| = \left| \sum_{1 \leq n \leq t} \chi(n) \right| = \left| \sum_{1 \leq n \leq kq} \chi(n) + \sum_{kq < n \leq t} \chi(n) \right| \leq \sum_{kq < n \leq t} |\chi(n)| \leq t - kq \leq q.$$

Now, finally using Theorem 1.29, we see

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \left(\lim_{x \rightarrow \infty} A(x)x^{-s} \right) - \lim_{x \rightarrow \infty} \int_0^x (A(t) \cdot -st^{-s-1}) dt.$$

Because $\operatorname{Re} s > 1$, we see $|A(x)x^{-s}| \leq qx^{-\operatorname{Re} s}$ goes to 0 as $x \rightarrow \infty$. Thus, we are left with

$$L(s, \chi) = s \int_0^{\infty} \frac{A(t)}{t^{s+1}} dt = s \underbrace{\int_1^{\infty} \frac{A(t)}{t^{s+1}} dt}_{I(s)}.$$

We claim that the right-hand side provides our analytic continuation to $\{s : \operatorname{Re} s > 0\}$. Indeed, it suffices to show that $I(s)$ is analytic on $\{s : \operatorname{Re} s > 0\}$. This is technical.

Roughly speaking, we want to write

$$\left| \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt \right| \leq q \int_1^{\infty} \frac{1}{t^{\operatorname{Re} s + 1}} dt = q \cdot \left. \frac{t^{\operatorname{Re} s}}{-\operatorname{Re} s} \right|_1^{\infty} = \frac{q}{\operatorname{Re} s}$$

for any $\operatorname{Re} s > 0$, meaning that the integral converges, so we ought to have a holomorphic function. To make this computation rigorous, we will show that $I(s)$ is holomorphic on $\{s : \operatorname{Re} s > \sigma\}$ for any $\sigma > 0$, which will

be enough by taking the union over all σ . Indeed, for some fixed σ , we define $g: [1, \infty)$ by $g(t) := q/t^{\sigma+1}$ for $t > 2$ and 0 elsewhere so that

$$\left| \frac{A(t)}{t^{s+1}} \right| = \left| \frac{A(t)}{t^{s+1}} \right| \leq g(t)$$

for $\operatorname{Re} s > \sigma$, and

$$\int_{\mathbb{R}} g(t) dt = q \int_1^{\infty} t^{-\sigma-1} dt < \infty$$

because $\sigma > 0$. Thus, Proposition A.18 implies that $I(s)$ is holomorphic on $\{s : \operatorname{Re} s > \sigma\}$, finishing the proof. ■

Remark 1.34. Using the notions and notations of the above proof, we see that

$$|L(s, \chi)| = \left| s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt \right| \leq \frac{q|s|}{\operatorname{Re} s}$$

for $\operatorname{Re} s > 0$. This upper-bound is occasionally helpful.

One might wonder what happens to the principal character χ_0 . It turns out its behavior is tied to ζ .

Lemma 1.35. Let $\chi_0 \pmod{q}$ be the principal Dirichlet character. Then for $\operatorname{Re} s > 1$, we have

$$L(s, \chi) = \left(\prod_{p|q} (1 - p^{-s}) \right) \zeta(s).$$

Proof. By Remark 1.28, we see

$$L(s, \chi) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}} = \prod_{p \nmid q} \frac{1}{1 - p^{-s}},$$

so

$$L(s, \chi) \prod_{p|q} \frac{1}{1 - p^{-s}} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \zeta(s)$$

by Corollary 1.5, which finishes. ■

Thus, we are interested in continuing ζ . With a little more effort than Proposition 1.33, we may provide $\zeta(s)$ a meromorphic continuation to $\{s : \operatorname{Re} s > 0\}$. The main difficulty here is that we have a pole to deal with.

Proposition 1.36. The function $\zeta(s)$ has a meromorphic continuation to $\{s : \operatorname{Re} s > 0\}$. It is holomorphic everywhere except at $s = 1$, where it has a simple pole of residue 1.

Proof. For given s with $\operatorname{Re} s > 1$, set $a_n := 1$ and $f(x) := 1/x^s$. Then the partial sums $A(t) := \sum_{1 \leq n \leq t} a_n$ have $A(t) = [t]$, so Theorem 1.29 grants

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\lim_{x \rightarrow \infty} [x] \cdot x^{-s} \right) - \lim_{x \rightarrow \infty} \int_0^x ([t] \cdot -st^{-s-1}) dt.$$

Because $\operatorname{Re} s > 1$, we see $|[x] \cdot x^{-s}| \leq x^{1-\operatorname{Re} s}$ goes to 0 as $x \rightarrow \infty$. Thus, we are left with

$$\zeta(s) = s \int_0^{\infty} \frac{[t]}{t^{s+1}} dt = s \int_1^{\infty} \frac{[t]}{t^{s+1}} dt.$$

To extract out a main term, we write $[t] = t + \{t\}$, giving

$$\zeta(s) = s \int_1^\infty t^{-s} dt + s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt = \frac{s}{s-1} + s \underbrace{\int_1^\infty \frac{\{t\}}{t^{s+1}} dt}_{I(s)}.$$

We claim that the above expression defines our meromorphic continuation. Notably, the function $s/(s-1) = 1 + 1/(s-1)$ is holomorphic everywhere except at $s = 1$, where it has a simple pole of residue 1.

Thus, it remains to show that $s \cdot I(s)$ is a holomorphic function for $\operatorname{Re} s > 0$, where it suffices to show that $I(s)$ is a holomorphic function for $\operatorname{Re} s > 0$. This is mildly technical. At a high level, we would like to just note that

$$\left| \int_1^\infty \frac{\{t\}}{t^{s+1}} dt \right| \leq \int_1^\infty \frac{1}{t^{\operatorname{Re} s + 1}} dt = \frac{t^{-\operatorname{Re} s}}{-\operatorname{Re} s} \Big|_1^\infty = \frac{1}{\operatorname{Re} s},$$

so the integral converges and ought to define a holomorphic function. To make this computation rigorous, we will show that $I(s)$ is holomorphic on $\{s : \operatorname{Re} s > \sigma\}$ for any $\sigma > 0$, which will be enough by taking the union over all σ . Indeed, for some fixed σ , we set $g(t) := 1/t^{\sigma+1}$ for $t > 1$ and 0 elsewhere so that

$$\left| \frac{\{t\}}{t^{s+1}} \right| \leq g(t)$$

for $\operatorname{Re} s > \sigma$, and

$$\int_{\mathbb{R}} g(t) dt = \int_1^\infty t^{-\sigma-1} dt < \infty$$

because $\sigma > 0$. Thus, Proposition A.18 implies that $I(s)$ is holomorphic on $\{s : \operatorname{Re} s > \sigma\}$, finishing the proof. ■

Remark 1.37. Using the notions and notation of the above proof, we see that

$$|\zeta(s)| \leq \frac{|s|}{|s-1|} + \left| s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt \right| \leq \frac{|s|}{|s-1|} + \frac{|s|}{\operatorname{Re} s}.$$

For example, if $\operatorname{Re} s > 1$, then we get $|\zeta(s)| \leq 1 + \frac{|s|}{\operatorname{Re} s} < |s| + 1$.

Remark 1.38. Doing repeated integration by parts, one can extend the continuations above further to the left, but we will not do this. Instead, we will use a functional equation to continue to all \mathbb{C} in one fell swoop.

Corollary 1.39. Let $\chi_0 \pmod{q}$ denote the principal Dirichlet character. Then $L(s, \chi)$ has a meromorphic continuation to $\{s : \operatorname{Re} s > 0\}$. It is holomorphic everywhere except for a simple pole at $s = 1$.

Proof. Note that the function $\prod_{p|q} (1 - p^{-s})$ is entire and has its only zero at $s = 0$. Combining Lemma 1.35 and Proposition 1.36 completes the proof. ■

1.2.3 Reducing to $L(1, \chi)$

We now attack Theorem 1.1 directly. As in Theorem 1.6, we will want to understand $\log L(s, \chi)$.

Lemma 1.40. Let $\chi \pmod{q}$ be a Dirichlet character. For any s with $\operatorname{Re} s > 1$, we have

$$\log L(s, \chi) = \sum_{p \text{ prime}} \frac{\chi(p)}{p^s} + E(s, \chi),$$

where $|E(s, \chi)| \leq 1$.

Proof. Fix s with $\operatorname{Re} s > 1$. Applying \log to the Euler product of Remark 1.28, we see

$$\log L(s, \chi) = \sum_{p \text{ prime}} -\log(1 - \chi(p)p^{-s}) = \sum_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{k p^{ks}} \right).$$

The $k = 1$ term of the right-hand sum is the main term present in the statement, so we need to bound the terms with $k > 1$. Thus, for $\operatorname{Re} s > 1$, we compute

$$\left| \sum_{p \text{ prime}} \left(\sum_{k=2}^{\infty} \frac{\chi(p)^k}{k p^{ks}} \right) \right| \leq \sum_{n=2}^{\infty} \left(\sum_{k=2}^{\infty} \frac{1}{n^k} \right) = \sum_{n=2}^{\infty} \frac{1/n^2}{1 - 1/n} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1,$$

where we have telescoped in the last equality. This completes the proof. \blacksquare

As an aside, we note that Lemma 1.40 provides us with a relatively large zero-free region for $L(s, \chi)$.

Corollary 1.41. Let $\chi \pmod{q}$ be a Dirichlet character. For any s with $\operatorname{Re} s > 1$, we have $L(s, \chi) \neq 0$.

Proof. By Lemma 1.40, we see

$$|\log L(s, \chi)| \leq \sum_{p \text{ prime}} \left| \frac{\chi(p)}{p^s} \right| + 1 \leq \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}} + 1,$$

which converges because $\operatorname{Re} s > 1$. Thus, $\log L(s, \chi)$ takes on a finite value for all s with $\operatorname{Re} s > 0$, which implies $L(s, \chi) \neq 0$. \blacksquare

Remark 1.42. Alternatively, we can recall from Proposition 1.4 that the Euler product for $L(s, \chi)$ converges absolutely for $\operatorname{Re} s > 1$, and in particular $L(s, \chi) = 0$ would require one of the Euler factors

$$\frac{1}{1 - \chi(p)p^{-s}}$$

to vanish by Remark A.24. However, none of these Euler factors vanish.

We now see that we can use Lemma 1.40 and Corollary 1.24 to extract a particular congruence class.

Lemma 1.43. Let q be an integer. For brevity, set $G := (\mathbb{Z}/q\mathbb{Z})^\times$, and fix some $a \in G$. For any s with $\operatorname{Re} s > 1$, we have

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{q}}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \in \hat{G}} \overline{\chi(a)} \log L(s, \chi) + E(s),$$

where $|E(s)| \leq 1$.

Proof. Corollary 1.24 tells us

$$1_{[a]}(p) = \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi}(a) \chi(p),$$

so

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{q}}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \left(\overline{\chi}(a) \sum_{p \text{ prime}} \frac{\chi(p)}{p^s} \right).$$

However, using the notation of Lemma 1.40, we see

$$\frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \left(\overline{\chi}(a) \sum_{p \text{ prime}} \frac{\chi(p)}{p^s} \right) = \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi}(a) \log L(s, \chi) + \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi}(a) E(s, \chi).$$

Because $\#\widehat{G} = \#G = \varphi(q)$ by Proposition 1.16, we conclude that the right-hand error term has magnitude bounded by 1, which completes the proof. ■

We can now reduce Theorem 1.1 to analyzing $L(1, \chi)$.

Proposition 1.44. Let q be an integer. Suppose that $L(1, \chi) \neq 0$ for each non-principal Dirichlet character $\chi \pmod{q}$. Then, for all $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, we have

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{q}}} \frac{1}{p} = +\infty.$$

In particular, there are infinitely many primes $p \equiv a \pmod{q}$.

Proof. Note that $L(1, \chi)$ is at least a complex number for non-principal characters $\chi \pmod{q}$ by Proposition 1.33.

Let χ_0 denote the principal character. By Corollary 1.39, we see $L(s, \chi_0) \rightarrow +\infty$ as $s \rightarrow 1^+$: indeed, we know $L(s, \chi_0)$ must go to something in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ because $L(s, \chi_0) \geq 1$ when $s > 1$ is real. But $L(s, \chi_0)$ cannot go to a finite value because then $L(s, \chi_0)$ would only have a removable singularity at $s = 1$.

Thus, we also have $\log L(s, \chi_0) \rightarrow +\infty$ as $s \rightarrow 1^+$. However, $\log L(s, \chi) \rightarrow \log L(1, \chi)$ as $s \rightarrow 1^+$ for non-principal characters χ , and by hypothesis, this is a finite limit. It follows that

$$\lim_{s \rightarrow 1^+} \frac{1}{\varphi(q)} \sum_{\chi \in \widehat{G}} \overline{\chi}(a) \log L(s, \chi) = +\infty,$$

so the result follows from Lemma 1.43. ■

So we want to understand $L(1, \chi)$ when χ is a non-principal character. By paying closer attention to the above proof, we can control most of our characters χ .

Lemma 1.45. Let q be an integer, and set $G := (\mathbb{Z}/q\mathbb{Z})^\times$ for brevity. For each Dirichlet character $\chi \pmod{q}$, let $v(\chi)$ denote the order of vanishing of $L(s, \chi)$ at $s = 1$. Then

$$\sum_{\chi \in \widehat{G}} v(\chi) \leq 0.$$

In other words, at most one non-principal character χ has $L(1, \chi) = 0$, in which case $L(s, \chi)$ has a simple zero at $s = 1$.

Proof. The idea here is that Lemma 1.43 has a certainly nonnegative sum on the left-hand side, so not too many of the $L(s, \chi)$ s on the right-hand side may be 0, for otherwise the right-hand side would go to $-\infty$.

We make a few quick remarks on $v(\chi)$. Note Corollary 1.39 implies $v(\chi_0) = -1$, where χ_0 is the principal character. Additionally, $v(\chi) \geq 0$ for all non-principal characters χ by Proposition 1.33, and $v(\chi)$ is finite because $L(s, \chi)$ is not constantly zero by Corollary 1.41.

Thus, for each character χ , we may write $L(s, \chi) = (s-1)^{v(\chi)} L_0(s, \chi)$ for some function $L_0(s, \chi)$ holomorphic on $\{s : \operatorname{Re} s > 0\}$ with $L_0(1, \chi) \neq 0$. Setting up our application of Lemma 1.43, we see

$$\sum_{\chi \in \widehat{G}} \log L(s, \chi) = \left(\sum_{\chi \in \widehat{G}} v(\chi) \right) \log(s-1) + \left(\sum_{\chi \in \widehat{G}} \log L_0(s, \chi) \right)$$

for $\operatorname{Re} s > 1$. However, we now plug into Lemma 1.43 with $a := 1$ so that $\overline{\chi(a)} = 1$ for all χ , giving

$$\sum_{\substack{p \text{ prime} \\ p \equiv 1 \pmod{q}}} \frac{1}{p^s} = \frac{1}{\varphi(q)} \left(\sum_{\chi \in \widehat{G}} v(\chi) \right) \log(s-1) + \frac{1}{\varphi(q)} \left(\sum_{\chi \in \widehat{G}} \log L_0(s, \chi) \right) + E(s)$$

for $\operatorname{Re} s > 0$. As $s \rightarrow 1^+$, the left-hand side remains nonnegative. On the right-hand side, the middle and right terms both remain finite, so the left term must also remain finite. However, $\log(s-1) \rightarrow -\infty$ as $s \rightarrow 1^+$, so we must have $\sum_{\chi} v(\chi) \leq 0$ to ensure this term is nonnegative.

We now show the last sentence. Indeed, we have

$$\sum_{\chi \in \widehat{G} \setminus \{\chi_0\}} v(\chi) \leq -v(\chi_0) = 1,$$

so at most one $\chi \in \widehat{G} \setminus \{\chi_0\}$ may have $v(\chi) > 0$, in which case χ has $v(\chi) = 1$. ■

For example, the above lemma lets us control “complex” characters.

Lemma 1.46. Let q be an integer. If $\chi \pmod{q}$ is a non-principal Dirichlet character with $\chi \neq \bar{\chi}$, then $L(1, \chi) \neq 0$.

Proof. If $L(1, \chi) = 0$, then we see

$$L(1, \bar{\chi}) = \lim_{s \rightarrow 1^+} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^s} = \overline{\lim_{s \rightarrow 1^+} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}} = \overline{L(s, \chi)} = 0.$$

But this grants two distinct characters χ and $\bar{\chi}$ with $L(1, \chi) = L(1, \bar{\chi}) = 0$, violating Lemma 1.45. ■

Thus, it remains to deal with the “real” non-principal characters χ with $\chi = \bar{\chi}$. This is genuinely difficult, so we will wait until next class for them.

1.3 January 23

Today we finish the proof of Theorem 1.1.

1.3.1 The Dirichlet Convolution

As motivation, we might be interested in the product of two Dirichlet series. Formally, we might write

$$\left(\sum_{k=1}^{\infty} \frac{a_k}{k^s} \right) \left(\sum_{\ell=1}^{\infty} \frac{b_{\ell}}{\ell^s} \right) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{a_k b_{\ell}}{(k\ell)^s} = \sum_{n=1}^{\infty} \left(\sum_{k\ell=n} a_k b_{\ell} \right) \frac{1}{n^s}.$$

Of course, we will want to formalize this intuitive argument to give the corresponding series the correct analytic properties, but we have at least arrived at the correct definition.

Definition 1.47 (Dirichlet convolution). Fix functions $f, g: \mathbb{N} \rightarrow \mathbb{C}$. Then the *Dirichlet convolution* $(f * g): \mathbb{N} \rightarrow \mathbb{C}$ is defined by

$$(f * g)(n) := \sum_{k\ell=n} f(k)g(\ell) = \sum_{d|n} f(d)g(n/d).$$

And we may now take products of Dirichlet series.

Proposition 1.48. Fix functions $f, g: \mathbb{N} \rightarrow \mathbb{C}$ such that $|f(n)|, |g(n)| = O(n^\sigma)$ for some $\sigma \in \mathbb{R}$. Then define the series

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \quad D(s) := \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}.$$

Then D converges absolutely for $\operatorname{Re} s > \sigma + 1$, where it defines a holomorphic function given by $D(s) = F(s)G(s)$.

Proof. Fix s with $\operatorname{Re} s > \sigma + 1$. We show that $D(s)$ converges absolutely and yields $D(s) = F(s)G(s)$, from which it follows that $D(s)$ is holomorphic over the region by using Proposition 1.2 on F and G . Let $F_n(s)$, $G_n(s)$, and $D_n(s)$ denote the n th partial sums. Then we see

$$F_N(s)G_N(s) = \left(\sum_{k=1}^N \frac{f(k)}{k^s} \right) \left(\sum_{\ell=1}^N \frac{g(\ell)}{\ell^s} \right) = \sum_{n=1}^N \underbrace{\left(\sum_{k\ell=n} f(k)g(\ell) \right)}_{D_N(s)} \frac{1}{n^s} + \underbrace{\sum_{\substack{1 \leq k, \ell \leq N \\ k\ell > N}} \frac{f(k)g(\ell)}{(k\ell)^s}}_{R_N(s) := \dots}.$$

Thus, the key claim is that $R_N(s) \rightarrow 0$ as $N \rightarrow \infty$. The main point is that $k\ell > N$ requires $k > \sqrt{N}$ or $\ell > \sqrt{N}$, so

$$|R_N(s)| \leq \sum_{\substack{1 \leq k, \ell \leq N \\ k\ell > N}} \frac{|f(k)| \cdot |g(\ell)|}{(k\ell)^{\operatorname{Re} s}} \leq \left(\sum_{k > \sqrt{N}} \frac{|f(k)|}{k^{\operatorname{Re} s}} \right) \left(\sum_{\ell \geq 1} \frac{|g(\ell)|}{\ell^{\operatorname{Re} s}} \right) + \left(\sum_{k \geq 1} \frac{|f(k)|}{k^{\operatorname{Re} s}} \right) \left(\sum_{\ell > \sqrt{N}} \frac{|g(\ell)|}{\ell^{\operatorname{Re} s}} \right).$$

The absolute convergence of F and G at s now causes the right-hand side to be

$$\left(\sum_{k=1}^{\infty} \frac{|f(k)|}{k^{\operatorname{Re} s}} \right) \cdot 0 + 0 \cdot \left(\sum_{\ell=1}^{\infty} \frac{|g(\ell)|}{\ell^{\operatorname{Re} s}} \right) = 0$$

as $N \rightarrow \infty$, so we conclude $R_N(s) \rightarrow 0$ as $N \rightarrow \infty$. Thus, we conclude

$$F(s)G(s) = \lim_{N \rightarrow \infty} (F_N(s)G_N(s)) = \lim_{N \rightarrow \infty} D_N(s) + \lim_{N \rightarrow \infty} R_N(s) = D(s).$$

Lastly, we need to show that $D(s)$ actually converges absolutely. Well, we note that we can replace f with $|f|$ and g with $|g|$ and s with $\operatorname{Re} s$ everywhere in the above bounding to show that

$$\sum_{n=1}^{\infty} \left| \frac{(f * g)(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{(|f| * |g|)(n)}{n^{\operatorname{Re} s}} = \left(\sum_{k=1}^{\infty} \frac{|f(k)|}{k^{\operatorname{Re} s}} \right) \left(\sum_{\ell=1}^{\infty} \frac{|g(\ell)|}{\ell^{\operatorname{Re} s}} \right),$$

and the right-hand side converges because $F(s)$ and $G(s)$ converge absolutely. Thus, $D(s)$ converges absolutely. ■

Example 1.49. Let $d(n)$ denote the number of divisors of n . Then we see

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{(1 * 1)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$

Here, $1: \mathbb{N} \rightarrow \mathbb{C}$ is the function which constantly returns 1.

We might be interested in an Euler product factorization for a product of two Dirichlet series (as in Proposition 1.4), but this notably requires the relevant functions to be multiplicative. Thus, we now show that the Dirichlet convolution sends multiplicative functions to multiplicative functions.

Lemma 1.50. Let $f, g: \mathbb{N} \rightarrow \mathbb{C}$ be multiplicative functions. Then $(f * g): \mathbb{N} \rightarrow \mathbb{C}$ is still multiplicative.

Proof. Let n and m be coprime positive integers. We must show $(f * g)(nm) = (f * g)(n) \cdot (f * g)(m)$. The key point is that there is a bijection between divisors $d \mid nm$ and pairs of divisors $d_n \mid n$ and $d_m \mid m$ by sending (d_n, d_m) to d . We quickly show formally that this is a bijection.

- Well-defined: certainly $d_n \mid n$ and $d_m \mid m$ implies $d_n d_m \mid nm$.
- Injective: suppose $d_n d_m = d'_n d'_m$ for $d_n, d'_n \mid n$ and $d_m, d'_m \mid m$. We show $d_n = d'_n$, and $d_m = d'_m$ follows by symmetry. Well, for each $p \mid n$, we see $p \nmid m$ because $\gcd(n, m) = 1$, so $p \nmid d_m, d'_m$ as well, meaning

$$\nu_p(d_n) = \nu_p(d_n d_m) = \nu_p(d'_n d'_m) = \nu_p(d'_n)$$

for all $p \mid n$. However, $p \mid d_n, d'_n$ implies $p \mid n$, so we see that the prime factorizations of d_n and d'_n are the same, so $d_n = d'_n$.

- Surjective: for each $d \mid nm$, define $d_n := \gcd(d, n)$ and $d_m := \gcd(d, m)$. Certainly $d_n \mid n$ and $d_m \mid m$, so it remains to show $d = d_n d_m$. Well, for each $p \mid n$, we see $\nu_p(d_n) = \nu_p(d)$ because $d \mid n$; and similarly, each $p \mid m$ has $\nu_p(d_m) = \nu_p(d)$. Because each prime $p \mid nm$ divides exactly one of n or m , we see that

$$\nu_p(d_n d_m) = \nu_p(d_n) + \nu_p(d_m) = \nu_p(d)$$

by doing casework on $p \mid n$ or $p \mid m$.

We have written down all of this so that we may compute

$$\begin{aligned} (f * g)(nm) &= \sum_{d \mid nm} f(d)g(nm/d) \\ &= \sum_{d_n \mid n} \sum_{d_m \mid m} f(d_n d_m)g\left(\frac{n}{d_n} \cdot \frac{m}{d_m}\right) \\ &\stackrel{*}{=} \left(\sum_{d_n \mid n} f(d_n)g(n/d_n) \right) \left(\sum_{d_m \mid m} f(d_m)g(m/d_m) \right) \\ &= (f * g)(n) \cdot (f * g)(m). \end{aligned}$$

Here, we have used the multiplicativity at $\stackrel{*}{=}$, noting that $d_n \mid n$ and $d_m \mid m$ implies $\gcd(d_n, d_m) = 1$ because $\gcd(n, m) = 1$. ■

1.3.2 The Mellin Transform

In this subsection, we pick up a few facts about the Mellin transform. Roughly speaking, we are doing Fourier analysis on the group \mathbb{R}^+ whose operation is multiplication. As such, the Haar measure is dx/x : for any Borel set $S \subseteq \mathbb{R}^+$ and $a \in \mathbb{R}^+$, we see

$$\int_{aS} \frac{dx}{x} = \int_S \frac{d(ax)}{ax} = \int_S \frac{a}{a} \cdot \frac{dx}{x} = \int_S \frac{dx}{x},$$

so dx/x is in fact a translation-invariant measure on \mathbb{R}^+ . Anyway, here is our definition of the Mellin transform.

Definition 1.51 (decaying). A function $\varphi: (0, \infty) \rightarrow \mathbb{C}$ is *decaying at a rate of* (α, β) for real numbers $\alpha < \beta$ if and only if the functions $x^\alpha \varphi(x)$ and $x^\beta \varphi(x)$ are bounded.

Example 1.52. If $\varphi: (0, \infty) \rightarrow \mathbb{C}$ has compact support, then φ decays at a rate of (α, β) for all $\alpha < \beta$. Indeed, for any γ , the function $x^\gamma \varphi(x)$ is a continuous function supported on a compact set and is thus bounded.

Definition 1.53 (Mellin transform). Let $\varphi: (0, \infty) \rightarrow \mathbb{C}$ be a continuous function decaying at a rate of (α, β) . Then the *Mellin transform* is the function $\mathcal{M}\varphi$ given by

$$(\mathcal{M}\varphi)(s) := \int_0^\infty \varphi(x) x^s \frac{dx}{x}$$

for $\alpha < \operatorname{Re} s < \beta$.

Remark 1.54. We quickly check that the integral $\mathcal{M}\varphi$ (absolutely) converges for $\alpha < \operatorname{Re} s < \beta$. For each $\gamma \in \{\alpha, \beta\}$, find a constant $C_\gamma \in \mathbb{R}$ such that $|x^\gamma \varphi(x)| \leq C_\gamma$ for all $x \in (0, \infty)$. For our absolute convergence, we set $\sigma := \operatorname{Re} s \in (\alpha, \beta)$ and compute

$$\int_0^\infty |\varphi(x) x^s| \frac{dx}{x} \leq \int_0^1 C_\alpha x^{-\alpha+\sigma-1} dx + \int_1^\infty C_\beta x^{-\beta+\sigma-1} dx,$$

so both of the right-hand integrals converge because $-\alpha + \sigma - 1 > -1$ and $-\beta + \sigma - 1 < -1$. Notably, this shows that $(\mathcal{M}\varphi)$ is uniformly bounded by

$$\int_0^1 C_\alpha x^{-\alpha+\alpha_0-1} dx + \int_1^\infty C_\beta x^{-\beta+\beta_0-1} dx$$

whenever $\sigma \in [\alpha_0, \beta_0]$.

Remark 1.55. Fixing some $\sigma \in (\alpha, \beta)$, let $\psi(u) := e^{-\sigma u} \varphi(e^{-u})$. Provided that ψ is Schwarz, changing variables by $x = e^{-u}$ gives

$$(\mathcal{M}\varphi)(\sigma + 2\pi it) = \int_0^\infty \varphi(x) x^{\sigma+2\pi it} \frac{dx}{x} = \int_{\mathbb{R}} \varphi(e^{-u}) e^{-\sigma u - 2\pi i t u} du = (\mathcal{F}\psi)(t).$$

Here is a basic result on the Mellin transform.

Lemma 1.56. Fix a differentiable function $\varphi: (0, \infty) \rightarrow \mathbb{C}$ such that φ decays at a rate of (α, β) . Defining $\psi(x) := x\varphi'(x)$, for any $\alpha < \operatorname{Re} s < \beta$, the integral defining $(\mathcal{M}\psi)(s)$ converges, and

$$(\mathcal{M}\psi)(s) = -s(\mathcal{M}\varphi)(s).$$

Proof. This is by integration by parts. Indeed, we compute

$$\begin{aligned} (\mathcal{M}\psi)(s) &= \int_0^\infty x\varphi'(x)x^s \frac{dx}{x} \\ &= x^s\varphi(x) \Big|_0^\infty - s \int_0^\infty \varphi(x)x^s \frac{dx}{x} \\ &= -s(\mathcal{M}\varphi)(s), \end{aligned}$$

which is what we wanted. Note, $x^s\varphi(x) \rightarrow 0$ as $x \rightarrow 0^+$ and $x \rightarrow \infty$ because φ decays at a rate of (α, β) and $\operatorname{Re} s \in (\alpha, \beta)$. ■

We will need two key properties of the Mellin transform.

Proposition 1.57. Let $\varphi: (0, \infty) \rightarrow \mathbb{C}$ be a continuous function decaying at a rate of (α, β) .

- (a) The function $\mathcal{M}\varphi$ is holomorphic on $\{s : \alpha < \operatorname{Re} s < \beta\}$.
- (b) Suppose that φ is infinitely differentiable, and the n th derivatives decays at a rate of $(\alpha - n, \beta - n)$. Then for any integer $A \geq 0$ and $[\alpha_0, \beta_0] \subseteq (\alpha, \beta)$, the set

$$\{|s|^A (\mathcal{M}\varphi)(s) : \alpha_0 \leq \operatorname{Re} s \leq \beta_0\}$$

is bounded.

Proof. These are essentially bounding results.

- (a) We use Proposition A.18. Here, $f(s, t) := \varphi(x)x^{s-1}$. We will show that $\mathcal{M}\varphi$ is holomorphic on the vertical strip $U := \{s : -\alpha_0 < \operatorname{Re} s < \beta_0\}$ for any $\alpha < \alpha_0 < \beta_0 < \beta$, and the result will follow by taking the union over all α_0 and β_0 .

By hypothesis on φ , we can find a constant C such that $|x^\alpha\varphi(x)| \leq C$ and $|x^\beta\varphi(x)| \leq C$ for each x . As such, we define $g: (0, \infty) \rightarrow \mathbb{R}$ by

$$g(t) := \begin{cases} Cx^{-\alpha+\alpha_0-1} & \text{if } x \leq 1, \\ Cx^{-\beta+\beta_0-1} & \text{if } x > 1. \end{cases}$$

Note $\int_{\mathbb{R}} g(t) dt < \infty$ because $-\alpha + \alpha_0 - 1 > -1$ and $-\beta + \beta_0 - 1 < -1$. Thus, we see that $x \in (0, 1]$ gives

$$|\varphi(x)x^{s-1}| \leq Cx^{-\alpha+\operatorname{Re} s-1} \leq Cx^{-\alpha+\alpha_0+1},$$

and similar for $x \in (1, \infty)$ comparing with β_0 . The result now follows from Proposition A.18.

- (b) This follows from Lemma 1.56. Define $\varphi_0 := \varphi$ and $\varphi_{n+1}(x) := x\varphi'_n(x)$ for each n . By induction, φ_n decays at a rate of (α, β) for each n , and for each n , we see

$$|s^n (\mathcal{M}\varphi)(s)| = |(\mathcal{M}\varphi_n)(s)|$$

by Lemma 1.56. However, for each n , we see that $(\mathcal{M}\varphi_n)$ is uniformly bounded on $[\alpha_0, \beta_0]$ by Remark 1.54, which is what we wanted. ■

Remark 1.58. The condition that $\varphi^{(n)}$ decay at a rate of $(\alpha - n, \beta - n)$ is essentially requiring that φ behave like a polynomial somewhat. These sorts of conditions more or less vanish for sufficiently good functions; for example, if φ is infinitely differentiable and has compact support, then all the derivatives have compact support, so $\varphi^{(n)}$ always decays at a rate of (α, β) for all $\alpha < \beta$ by Example 1.52.

Theorem 1.59. Let $\varphi: (0, \infty) \rightarrow \mathbb{C}$ be a function such that $\psi(u) := e^{-\sigma u} \varphi(e^{-u})$ is Schwarz. For any $\sigma \in \mathbb{R}$ and $x \in (0, \infty)$, we have

$$\varphi(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (\mathcal{M}\varphi)(s) x^{-s} ds.$$

Proof. We translate everything to the Fourier setting with Remark 1.55, where Theorem C.10 finishes. Following this outline, we compute

$$\begin{aligned} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (\mathcal{M}\varphi)(s) x^{-s} ds &= \int_{\mathbb{R}} (\mathcal{M}\varphi)(\sigma + 2\pi it) x^{-\sigma-2\pi it} dt \\ &= x^{-\sigma} \int_{\mathbb{R}} (\mathcal{F}\psi)(t) e^{2\pi i(-\log x)t} dt \\ &= x^{-\sigma} \cdot \psi(-\log x) \\ &= \varphi(x), \end{aligned}$$

which is what we wanted. ■

1.3.3 Finishing Dirichlet's Theorem

We finish the proof of Theorem 1.1. By Proposition 1.44 and Lemma 1.46, we have left to show $L(1, \chi) \neq 0$ for real characters χ . We provide a slick proof of this result.

Lemma 1.60. Let $\chi \pmod{q}$ be a “real” non-principal Dirichlet character, meaning $\chi = \bar{\chi}$. We show

Proof. We combine two techniques called “positivity” and “smoothing.” The main point is that $L(1, \chi) = 0$ implies that the zero of $L(s, \chi)$ at $s = 1$ is able to cancel the pole of $\zeta(s)$ as $s = 1$, implying that the function $\zeta(s)L(s, \chi)$ is holomorphic on $\{s : \operatorname{Re} s > 0\}$ by combining Propositions 1.33 and 1.36.

Anyway, we divide the proof in three steps.

1. Let's begin with our positivity result. Because we are interested in $\zeta(s)L(s, \chi)$, we will want to study the coefficients of this Dirichlet series, which are given by $(1 * \chi)$ by Proposition 1.48. Note $(1 * \chi)$ is multiplicative by Lemma 1.50.

To set up our bounding, we claim that $(1 * \chi)(n) \geq 0$ for all $n \in \mathbb{N}$, and $(1 * \chi)(n^2) \geq 1$. Because $(1 * \chi)$ is multiplicative, we may write

$$(1 * \chi)(n) = (1 * \chi) \left(\prod_{p|n} p^{\nu_p(n)} \right) = \prod_{p|n} (1 * \chi) \left(p^{\nu_p(n)} \right).$$

Thus, it suffices to show $(1 * \chi)(p^k) \geq 0$ for each prime-power p^k , and $(1 * \chi)(p^k) \geq 1$ when k is even. Well, we can compute this directly as

$$(1 * \chi)(p^k) = \sum_{d|p^k} \chi(d) = \sum_{\nu=0}^k \chi(p^\nu) = \sum_{\nu=0}^k \chi(p)^\nu.$$

Now, $\chi(p) = \overline{\chi(p)}$ by hypothesis on χ , so because $|\chi(p)| = 1$ by Remark 1.12, we conclude $\chi(p) \in \{\pm 1\}$. Thus, on one hand, if $\chi(p) = 1$, then $(1 * \chi)(p^\nu) = \nu + 1 \geq 1$ always. On the other hand, if $\chi(p) = -1$, then $(1 * \chi)(p^\nu)$ is 1 when ν is even and 0 if ν is odd. The claim follows.

To finish, our positivity claim is that

$$\sum_{x < n \leq 2x} (1 * \chi)(n) \geq \sum_{x < n^2 \leq 2x} (1 * \chi)(n^2) \geq \sum_{\sqrt{x} < n \leq \sqrt{2x}} 1 = \left\lfloor \sqrt{2x} \right\rfloor - \left\lfloor \sqrt{x} \right\rfloor \geq (\sqrt{2} - 1)\sqrt{x} - 2.$$

Thus, for x large enough, we see

$$\sum_{x < n \leq 2x} (1 * \chi)(n) \geq \frac{1}{3} \sqrt{x}.$$

2. We now apply smoothing Let $\varphi: (0, \infty) \rightarrow (0, \infty)$ be an infinitely differentiable function with support contained in $[0.9, 2.1]$ such that $\varphi(x) = 1$ for $x \in [1, 2]$. Then one sees

$$\sum_{n=1}^{\infty} \varphi(n/x) (1 * \chi)(n) \geq \sum_{x < n \leq 2x} (1 * \chi)(n) \geq \frac{1}{3} \sqrt{x}.$$

Note that this sum is finite because only finitely many n have $n/x \leq 2.1$.

We now use the Mellin transform $\mathcal{M}\varphi$. Indeed, note that φ is decaying at a rate of (α, β) for any $\alpha < \beta$ by Remark 1.58. Further, for any $\sigma > 0$, the function $\psi(u) := e^{-\sigma u} \varphi(e^{-u})$ has compact support and is infinitely differentiable, so $x^k \psi^{(\ell)}(x)$ is continuous of compact support for all k and ℓ and hence bounded. Thus, ψ is Schwarz, so we can use Theorem 1.59 to compute

$$\sum_{n=1}^{\infty} \psi(n/x) (1 * \chi)(n) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{2-i\infty}^{2+i\infty} \left((\mathcal{M}\varphi)(s) x^s \cdot \frac{(1 * \chi)(n)}{n^s} \right) ds.$$

Thus, we see that we would like to exchange the integral and the sum so that we can sum over $(1 * \chi)$ to finally make $\zeta(s)L(s, \chi)$ appear. It suffices to show that this iterated “integral” absolutely converges, so for any $\sigma > 0$, we may compute

$$I_{\sigma}(x) := \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n=1}^{\infty} \left| (\mathcal{M}\varphi)(s) x^s \cdot \frac{(1 * \chi)(n)}{n^s} \right| ds = \int_{\sigma-i\infty}^{\sigma+i\infty} |(\mathcal{M}\varphi)(s) x^s \cdot \zeta(s)L(s, \chi)| ds$$

by Proposition 1.48. To bound this, we see $|x^s| \leq x^{\operatorname{Re} s} = x^{\sigma}$ and

$$|\zeta(s)L(s, \chi)| \leq q \cdot \frac{|s|}{\sigma} \cdot |s| \left(\frac{1}{|1-\sigma|} + \frac{1}{\sigma} \right) = C_0(q, \sigma) |s|^2$$

by Remarks 1.34 and 1.37, where $C_0(q, \sigma)$ is some constant. Thus,

$$I_{\sigma}(x) \leq C_0(q, \sigma) x^{\sigma} \int_{\sigma-i\infty}^{\sigma+i\infty} (|\mathcal{M}\varphi)(s)| (\sigma^2 + (\operatorname{Im} s)^2) ds.$$

However, by Proposition 1.57 (and Remark 1.58), there is C such that $|(\mathcal{M}\varphi)(s)| \leq C|s|^{-4} \leq C(\operatorname{Im} s)^{-4}$ on the vertical strip of interest, so we bound

$$\begin{aligned} \frac{I(x)}{C_0(q, \sigma) x^{\sigma}} &\leq C \left(\int_{\sigma-i\infty}^{\sigma-i} \frac{(\sigma^2 + (\operatorname{Im} s)^2)}{(\operatorname{Im} s)^4} ds \right) + C \left(\int_{\sigma+i}^{\sigma+i\infty} \frac{(\sigma^2 + (\operatorname{Im} s)^2)}{(\operatorname{Im} s)^4} ds \right) \\ &\quad + \left(\int_{\sigma-i}^{\sigma+i} (|\mathcal{M}\varphi)(s)| (\sigma^2 + (\operatorname{Im} s)^2) ds \right). \end{aligned}$$

The integrals on the top row are finite by direct computation (they are improper integrals avoiding 0 of decaying on the order of x^{-2} or faster), and the bottom integral is finite because it is a finite integral of a continuous function. We conclude that $I(x)$ converges, so we have absolute convergence.

In fact, the entire right-hand side of the above bound is merely some function of σ , so we have actually shown that

$$\int_{\sigma-i\infty}^{\sigma+i\infty} |(\mathcal{M}\varphi)(s) x^s \cdot \zeta(s)L(s, \chi)| ds \leq C(q, \sigma) x^{\sigma} \quad (1.1)$$

for some constant $C(q, \sigma)$.

3. Anyway, we now know we can write

$$\frac{1}{3}\sqrt{x} \leq \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \underbrace{(\mathcal{M}\varphi)(s)x^s\zeta(s)L(s,\chi)}_{D(s)} ds$$

by exchanging the sum and the integral and using Proposition 1.48. In order to use (1.1), we would like to push the vertical line left from $\operatorname{Re} s = 2$ to $\operatorname{Re} s = 1/3$ (for example).

We will be allowed to do this by Cauchy's theorem because the function $D(s) = (\mathcal{M}\varphi)(s)x^s\zeta(s)L(s,\chi)$ is holomorphic on $\{s : \operatorname{Re} s > 0\}$. Indeed, the only possible pole among these functions is the pole of order 1 at $s = 1$ for $\zeta(s)$, but $L(s,\chi)$ has a zero there by assumption and thus cancels this out!

We now apply Cauchy's theorem. For any $T > 0$, we see

$$\left| \int_{1/3-iT}^{1/3+iT} D(s) ds - \int_{2-iT}^{2+iT} D(s) ds \right| \leq \int_{1/3+iT}^{2+iT} |D(s)| ds + \int_{1/3-iT}^{2-iT} |D(s)| ds.$$

We would like to show that this right-hand side vanishes as $T \rightarrow \infty$. Because the length of each of these paths is finite, it suffices to show that $|D(s)|$ vanishes as $\operatorname{Im} s \rightarrow \infty$ on these paths. Well, utilizing our bounds from before, we see

$$|D(s)| \leq |(\mathcal{M}\varphi)(s)| \cdot x^2 \cdot C_0(q, \sigma) (4 + (\operatorname{Im} s)^2).$$

Because $(\mathcal{M}\varphi)(s)$ is rapidly decaying as $\operatorname{Im} s \rightarrow \infty$ (recall Proposition 1.57), we see that this indeed goes to 0 as $\operatorname{Im} s \rightarrow \infty$.

In total, we see

$$\frac{1}{3}\sqrt{x} \leq \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D(s) ds = \frac{1}{2\pi i} \int_{1/3-i\infty}^{1/3+i\infty} D(s) ds \leq C(q, 1/3)x^{1/3},$$

where we have used (1.1) at the end. However, for x large enough, this is impossible: $x^{1/2-1/3} \rightarrow \infty$ as $x \rightarrow \infty$. So we have hit our contradiction. ■

Remark 1.61. The product $\zeta(s)L(s,\chi)$ is the Dedekind ζ -function associated to a real quadratic field.

1.3.4 A Little on Quadratic Forms

To say something in the direction of Dirichlet's class number formula, we discuss quadratic forms. In particular, we will discuss the reduction theory, which shows that there are finitely many classes of binary quadratic forms of given discriminant.

Definition 1.62 (binary quadratic form). A *binary quadratic form* is a function $f: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ where $f(x, y) := ax^2 + bxy + cy^2$ where $a, b, c \in \mathbb{Z}$. If $\gcd(a, b, c) = 1$, then we call the quadratic form *primitive*.

It is a problem of classical interest to determine when a quadratic form achieves a particular integer.

It is another problem of classical interest to count the number of binary quadratic forms. However, some binary quadratic forms are "the same," in the sense that they are just a variable change away.

Example 1.63. The quadratic forms $x_1^2 + x_2^2$ and $y_1^2 + 2y_1y_2 + 2y_2^2$ are roughly the same by the change of variables given by

$$(y_1, y_2) = (x_1 - x_2, x_2).$$

To define this correctly, we define a group action on the set of quadratic forms.

Lemma 1.64. Let \mathcal{Q} be the set of binary quadratic forms. Then $\mathrm{SL}_2(\mathbb{Z})$ acts on the set of binary quadratic forms by

$$(\gamma \cdot f) := f \circ \gamma^{-1},$$

where $f \in \mathcal{Q}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. We have the following checks.

- Identity: note $(\mathrm{id} \cdot f) = f \circ \mathrm{id}^{-1} = f \circ \mathrm{id} = f$.
- Composition: note $((\gamma\gamma') \cdot f) = f \circ (\gamma\gamma')^{-1} = f \circ (\gamma')^{-1} \circ \gamma^{-1} = \gamma \cdot (\gamma' \cdot f)$. ■

Definition 1.65 (equivalent). Two binary quadratic forms $f_1, f_2: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ are *equivalent* if and only if f_1 and f_2 live in the same orbit under the $\mathrm{SL}_2(\mathbb{Z})$ -action. In other words, f_1 and f_2 are equivalent if and only if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$f_1 = f_2 \circ \gamma.$$

Note that this is in fact an equivalence relation because the orbits of a group action form a partition.

Remark 1.66. For a binary quadratic form $f(x, y) := ax^2 + bxy + cy^2$, note that

$$f(v) = ax^2 + bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}}_{M:=} \begin{bmatrix} x \\ y \end{bmatrix} = v^T M v$$

for any $v = (x, y) \in \mathbb{Z}^2$. In fact, this symmetric matrix M is unique to f : if $v^T M v = v^T M' v$ for all $v = (x, y) \in \mathbb{Z}^2$, then writing $M = (a_{ij})$ and $M' = (a'_{ij})$, we see

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = v^T M v = v^T M' v = a'_{11}x^2 + 2a'_{12}xy + a'_{22}y^2.$$

Plugging in $(x, y) \in \{(1, 0), (0, 1), (1, 1)\}$ shows $M = M'$.

Remark 1.67. Associate a binary quadratic form f the matrix M as in Remark 1.66. Thus, for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$(\gamma \cdot f)(v) = f(\gamma^{-1}v) = (\gamma^{-1}v)^T M \gamma^{-1}v = v^T (\gamma^{-T} M \gamma^{-1}) v,$$

so we can associate $\gamma \cdot f$ to the matrix $\gamma^{-T} M \gamma^{-1}$. (Notably, this is still a symmetric matrix!) This allows for relatively easy computation of $\gamma \cdot f$.

So we would like to count the number of quadratic forms, up to equivalence. However, we will soon see that there are still infinitely many of equivalence classes, so we will want some stronger invariant to distinguish between them.

Definition 1.68 (discriminant). The *discriminant* of the binary quadratic form $f(x, y) := ax^2 + bxy + cy^2$ is given by $\mathrm{disc} f := b^2 - 4ac$. The number of equivalence classes of quadratic forms of discriminant d is notated by $h(-d)$.

Remark 1.69. By definition, note that the discriminant of the binary quadratic form f is $4 \det M$, where M is the matrix associated to f as in Remark 1.66. Using Remark 1.67, we see that the discriminant of $\gamma \cdot f$ is thus

$$4 \det (\gamma^{-T}) \det(M) \det (\gamma^{-1}) = 4 \det M$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Remark 1.69 shows that the discriminant is invariant to equivalence class. Thus, for example, for each $d \in \mathbb{Z}$, we set

$$f_d(x, y) := dxy$$

so that $\text{disc } f = d^2$. Now letting d vary of \mathbb{Z} , we see that there are infinitely many equivalence classes of quadratic forms.

But once we bound our discriminant, there will be finitely many quadratic forms. Here is our goal.

Theorem 1.70. Let $d < 0$ be an integer. Then $h(d)$ is finite.

Remark 1.71. It is also true that $h(d)$ is finite when $d \geq 0$, but we will not show it here.

1.3.5 The Upper-Half Plane

To show Theorem 1.70, we will want to relate the action of $\text{SL}_2(\mathbb{Z})$ on quadratic forms with the action of $\text{SL}_2(\mathbb{R})$ on $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z := \frac{az + b}{cz + d}.$$

Here are some checks on this action.

Lemma 1.72. Let $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$ denote the upper-half plane.

(a) The group $\text{SL}_2(\mathbb{R})$ acts on \mathbb{H} by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z := \frac{az + b}{cz + d}.$$

(b) The orbit of $i \in \mathbb{H}$ under $\text{SL}_2(\mathbb{R})$ is all of \mathbb{H} .

(c) The stabilizer of $i \in \mathbb{H}$ is $\text{SO}_2(\mathbb{R})$, the group of rotations.

Proof. We show the parts one at a time.

(a) To begin, we show that the action is well-defined: given z with $z \in \mathbb{H}$, we need to show that $\gamma \cdot z \in \mathbb{H}$ for any $\gamma \in \text{SL}_2(\mathbb{R})$. Well, giving coefficients to γ , we compute

$$\gamma \cdot z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} = \frac{(ac|z|^2 + bd) + (adz + bc\bar{z})}{|cz + d|^2}.$$

To check $\gamma \cdot z \in \mathbb{H}$, we must check that the imaginary part here is positive. Well, we see

$$\text{Im}(\gamma \cdot z) = \frac{(ad - bc) \text{Im}(z)}{|cz + d|^2} = \frac{\text{Im}(z)}{|cz + d|^2},$$

where the last equality is because $\det \gamma = 1$.

We now run our checks to have a group action.

- Identity: we compute

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z = \frac{z + 0}{0 + 1} = z.$$

- Composition: we compute

$$\begin{aligned}
 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} z \right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{a'z + b'}{c'z + d'} \\
 &= \frac{a \cdot \frac{a'z + b'}{c'z + d'} + b}{c \cdot \frac{a'z + b'}{c'z + d'} + d} \\
 &= \frac{a(a'z + b') + b(c'z + d')}{c(a'z + b') + d(c'z + d')} \\
 &= \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + (cb' + dd')} \\
 &= \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix} z \\
 &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \right) z.
 \end{aligned}$$

(b) Giving coefficients to some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we use the computation in (a) to see

$$\gamma \cdot i = \begin{bmatrix} a & b \\ c & d \end{bmatrix} i = \frac{(ac|i|^2 + bd) + (adi + bc\bar{i})}{|ci + d|^2} = \frac{(ac + bd) + (ad - bc)i}{c^2 + d^2} = \frac{(ac + bd) + i}{c^2 + d^2}.$$

Thus, for any $a + bi \in \mathbb{H}$, we see

$$\begin{bmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{bmatrix} i = \frac{a/b + i}{1/b} = a + bi,$$

so the orbit of i is indeed all of \mathbb{H} .

(c) Using the computation of (b), we see that $\gamma \cdot i = i$ if and only if the usual coefficients of γ have $ac + bd = 0$ and $c^2 + d^2 = 1$. Thus, we see that any $\theta \in [0, 2\pi)$ will give

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} i = i$$

because $(\cos \theta)(\sin \theta) + (\cos \theta)(-\sin \theta) = 0$ and $(\cos \theta)^2 + (\sin \theta)^2 = 1$. It follows that $\mathrm{SO}_2(\mathbb{R})$ is certainly contained in the stabilizer of i .

Conversely, suppose γ stabilizes i and has the usual coefficients. Note that the pair (c, d) with $c^2 + d^2 = 1$ has a unique $\theta \in [0, 2\pi)$ such that $c = \sin \theta$ and $d = \cos \theta$. To solve for a and b , we divide our work in two cases.

- If $c \neq 0$, then we see $a = -bd/c$. Further, $ad - bc = 1$, so we see $-bd^2/c - bc = 1$, which gives

$$b = -\frac{1}{d^2/c + c} = -\frac{c}{c^2 + d^2} = -c = -\sin \theta.$$

Thus, we see $a = -bd/c = d = \cos \theta$. Plugging everything in, we see $\gamma \in \mathrm{SO}_2(\mathbb{R})$.

- If $c = 0$, then $d \neq 0$, so we see $b = -ac/d$. Thus, $ad - bc = 1$, so we see $ad + ac^2/d = 1$, which gives

$$a = \frac{1}{d + c/d} = \frac{d}{c^2 + d^2} = d = \cos \theta.$$

Thus, we see $b = -ac/d = -c = -\sin \theta$. Plugging everything in, we again see $\gamma \in \mathrm{SO}_2(\mathbb{R})$.

The above cases complete the proof. ■

Remark 1.73. Parts (b) and (c) of Lemma 1.72 roughly show

$$\frac{\mathrm{SL}_2(\mathbb{R})}{\mathrm{SO}_2(\mathbb{R})} \cong \mathbb{H}.$$

Next class we will discuss how to build a fundamental domain for the induced action of $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{R})$ on \mathbb{H} .

1.4 January 25

Today we continue discussing quadratic forms.

1.4.1 A Fundamental Domain

Recall from Remark 1.73 that

$$\frac{\mathrm{SL}_2(\mathbb{Z})}{\mathrm{SO}_2(\mathbb{R})} \cong \mathbb{H}.$$

Now, $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{R})$ has a natural action on \mathbb{H} ; this is a “discrete subgroup,” so one might say that the action is discrete. (Concretely, we can see that the orbit of any $z \in \mathbb{H}$ under the action of $\mathrm{SL}_2(\mathbb{Z})$ is a discrete set.) We will be interested in a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{H} . Here is an example.

Proposition 1.74. Define the subset

$$D := \{z \in \mathbb{H} : |z| > 1, -1/2 \leq \operatorname{Re} z < 1/2\} \cup \{z \in \mathbb{H} : |z| = 1, -1/2 \leq \operatorname{Re} z \leq 0\}.$$

Then D is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{H} . In other words, for each $z \in \mathbb{H}$, there exists a unique $z_0 \in \mathbb{H}$ such that there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $z = \gamma \cdot z_0$.

Proof. Omitted. Roughly speaking, one has to show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements

$$S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad T := \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then one can use T to push all elements of \mathbb{H} to $\{z \in \mathbb{H} : -1 \leq \operatorname{Re} z < 1\}$ and use S to push what’s left over to S . We refer to [Ser12] for details. ■

1.4.2 Gauss Reduced Forms

We now use Proposition 1.74 for fun and profit.

Theorem 1.70. Let $d < 0$ be an integer. Then $h(d)$ is finite.

Proof. Roughly speaking, a quadratic form $f(x, y) := ax^2 + bxy + cy^2$ where $a, c > 0$ without loss of generality, we can study $f(x, 1)$ to have a root

$$z_f := \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{d}}{2a}.$$

Now, in our case of interest, we have $d < 0$, so this describes an element of \mathbb{H} . (There is also a negative root, but we focus on z_f .) In fact, one can check that $z_{\gamma f} = \gamma z_f$, which is how we relate quadratic forms to \mathbb{H} .

In fact, by Proposition 1.74, we know there is some γ_f such that $z_{\gamma_f} \in D$. The point here is that the number of quadratic forms up to equivalence is bounded above by the number of points in D with imaginary part $\sqrt{|d|}$. For example, the condition $|z_f| \geq 1$ implies that

$$\frac{b^2 - d}{4a^2} = \frac{c}{a},$$

so $a \leq c$. Further, the condition $-1/2 \leq \operatorname{Re} z \leq 1/2$ implies $|b| \leq 2a$. Thus, we are counting the number of triples (a, b, c) with $a, c > 0$ such that $b^2 - 4ac = d$ and $|b| \leq a \leq c$, which we can see immediately is finite. Indeed, $b^2 \leq d$, so there are only finitely many possible b , but then for each b , we see $4ac = b^2 - d$, so there are only finitely many possible a and c . ■

Remark 1.75. A quadratic form satisfying the above conditions on a, b, c is called "Gauss reduced."

1.4.3 Dirichlet's Class Number Formula

We take a moment to record Dirichlet's class number formula for completeness, though we will not prove it.

Theorem 1.76 (class number formula). Let d be a "fundamental discriminant," meaning that $d \equiv 1 \pmod{4}$ and is squarefree or $d = 4q$ where $q \equiv 2, 3 \pmod{4}$ and is squarefree. Let $\chi_d = \left(\frac{d}{\bullet}\right)$ be the Kronecker symbol.

(a) If $d < 0$,

$$h(d) = \frac{w_d |d|^{1/2}}{2\pi} \cdot L(1, \chi_d),$$

where $w_d = 2$ if $d < -4$ and $w_d = 4$ if $d = -4$ and $w_d = 6$ if $d = -3$. (Namely, w_d is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$.)

(b) If $d > 0$, then

$$h(d) \log \varepsilon_d = |d|^{1/2} L(1, \chi_d),$$

where ε_d is a fundamental unit for $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. (Namely, $\varepsilon_d = (t_0 + u_0 \sqrt{d})/2$ yields the least positive solution to $t_0^2 - du_0^2 = 4$.)

The point behind the fundamental discriminant is that $\operatorname{disc} \mathcal{O}_{\mathbb{Q}(\sqrt{d})} = d$.

Remark 1.77. The interested should now be able to do the first part of the first problem set.

THEME 2

THE PRIME NUMBER THEOREM

2.1 January 25

We now shift gears and move towards the Prime number theorem. Today, we begin by discussing Riemann's original paper on the topic.

Remark 2.1. For the rest of this course, any sum or product over an unnamed p will be a sum over primes.

2.1.1 The Statement

So far we have established the following facts about ζ .

- By Corollary 1.5, for $\operatorname{Re} s > 1$, there is an Euler product factorization

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

- By Proposition 1.36, there is a meromorphic continuation of $\zeta(s)$ to $\operatorname{Re} s > 1$, where $\zeta(s)$ is analytic everywhere except for a pole of order 1 at $s = 1$.

Roughly speaking, we will show the Prime number theorem by being able to study $\zeta'(s)/\zeta(s) = \frac{d}{ds} \log \zeta(s)$. Let's establish some notation.

Definition 2.2. For $x \in \mathbb{R}$, we define the following functions.

$$\begin{aligned} \pi(x) &:= \sum_{p \leq x} 1, \\ \vartheta(x) &:= \sum_{p \leq x} \log p, \\ \Lambda(n) &:= \begin{cases} \log p & \text{if } n = p^\nu \text{ for } \nu \in \mathbb{Z}^+, \\ 0 & \text{else,} \end{cases} \\ \psi(x) &:= \sum_{n \leq x} \Lambda(n). \end{aligned}$$

Quickly, we note that the prime-powers we have included in $\Lambda(n)$ and $\psi(x)$ don't actually matter.

Lemma 2.3. For any $x \geq 2$, we have

$$\psi(x) = \vartheta(x) + O(\sqrt{x}(\log x)^2).$$

Proof. Note

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{k=1}^{\infty} \left(\sum_{p^k \leq x} \log p \right).$$

Now, note $k \geq \log_2 x$ implies that $p^k \geq 2^k \geq x$ for all primes p , so we only need to sum up to $\log_2 x$. As such, we upper-bound the $k > 1$ sum as

$$\left| \sum_{k=2}^{\log_2 x} \left(\sum_{p^k \leq x} \log p \right) \right| \leq |\log_2 x - 1| \cdot \left| \sum_{n \leq \sqrt{x}} \log(\sqrt{n}) \right| \leq \frac{\sqrt{x}(\log x)^2}{2 \log 2}.$$

Adding the $k = 1$ sum back in, we see that

$$\psi(x) = \sum_{p \leq x} \log p + O(\sqrt{x}(\log x)^2),$$

which is what we wanted. ■

Remark 2.4. Doing logarithmic differentiation, one finds

$$\frac{d}{ds}(-\log \zeta(s)) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

This explains why ψ is a “better” prime-counting function than π .

Now, here is our statement.

Theorem 2.5 (Prime number). We have $\pi(x) \sim x/\log x$ as $x \rightarrow \infty$.

Here is why we mentioned ϑ and ψ .

Lemma 2.6. Define $g(x) := \vartheta(x) - x$. Then

$$\pi(x) = \int_2^x \frac{1}{\log t} dt + \frac{2}{\log 2} + \frac{g(x)}{\log x} + \int_2^x \frac{g(t)}{t(\log t)^2} dt.$$

Proof. This is summation by parts. Let $a_n = \log n$ be 1 if n is prime and 0 otherwise so that the partial sum up to x of a_n is given by $\vartheta(x)$. Further, let $f(n) := 1/\log n$ if $n > 1$ and 0 at $n = 1$. Then Theorem 1.29 tells us

$$\begin{aligned} \pi(x) &= \sum_{n \leq x} a_n f(n) \\ &= \vartheta(x)f(x) - \int_0^x \vartheta(t)f'(t) dt \\ &= \frac{x}{\log x} + \int_2^x \frac{t}{t(\log t)^2} dt + \frac{g(x)}{\log x} + \int_2^x \frac{g(t)}{t(\log t)^2} dt \\ &= \int_2^x \frac{1}{\log t} dt + \frac{2}{\log 2} + \frac{g(x)}{\log x} + \int_2^x \frac{g(t)}{t(\log t)^2} dt, \end{aligned}$$

which is what we wanted. ■

Proposition 2.7. The following are equivalent.

- (a) $\pi(x) \sim x/\log x$ as $x \rightarrow \infty$.
- (b) $\vartheta(x) \sim x$ as $x \rightarrow \infty$.
- (c) $\psi(x) \sim x$ as $x \rightarrow \infty$.

Proof. By Lemma 2.3, we see

$$\frac{\vartheta(x)}{x} - \frac{\psi(x)}{x} = O\left(x^{-1/2}(\log x)^2\right) = o(1),$$

so

$$\lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} = \lim_{x \rightarrow \infty} \frac{\psi(x)}{x},$$

provided that either limit exists. The equivalence of (b) and (c) follows.

We now show that (a) and (b) are equivalent to finish.

- Showing (a) implies (b) is by summation by parts. Let a_n denote the prime indicator at n so that the partial sum up to x is $\pi(x)$. Further, define $f(x) := \log x$ for $x \geq 1$ and 0 otherwise. Thus, Theorem 1.29 implies

$$\begin{aligned} \vartheta(x) &= \sum_{n \leq x} a_n f(n) \\ &= \pi(x)f(x) - \int_0^x \pi(t)f'(t) dt \\ &= \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt, \end{aligned}$$

so

$$\frac{\vartheta(x)}{x} = \frac{\pi(x)}{x/\log x} - \frac{1}{x} \int_2^x \frac{\pi(t)}{t} dt.$$

Given $\pi(x) \sim x/\log x$, the main term here has $\pi(x)/(x/\log x) \rightarrow 1$ as $x \rightarrow \infty$, so we have left to show that the right term vanishes. Well, $|\pi(x)/(x/\log x)| \rightarrow 1$ as $x \rightarrow \infty$ implies that this function has a maximum on $[2, \infty)$, so we let M denote the maximum. Thus,

$$\left| \int_2^x \frac{\pi(t)}{t} dt \right| \leq M \int_2^x \frac{\log t}{t} dt.$$

To finish, we use L'Hôpital's rule to note

$$\lim_{x \rightarrow \infty} \frac{M \int_2^x \log t/t dt}{x} = \lim_{x \rightarrow \infty} \frac{M \log x}{x} = 0.$$

- Showing (b) implies (a) follows from Lemma 2.6. Indeed, note

$$\frac{\pi(x)}{x/\log x} = \frac{\int_2^x 1/\log t dt}{x/\log x} + \frac{g(x)}{x} + \frac{\log x}{x} \int_2^x \frac{g(t)}{t(\log t)^2} dt, \quad (2.1)$$

where $g(x) := \vartheta(x) - x$. Using L'Hôpital's rule, the main term has

$$\lim_{x \rightarrow \infty} \frac{\int_2^x 1/\log t dt}{x/\log x} = \lim_{x \rightarrow \infty} \frac{1/\log x}{(\log x - 1)/(\log x)^2} = \lim_{x \rightarrow \infty} \frac{1}{1 - 1/\log x} = 1.$$

It remains to show that everything else on the right-hand side of (2.1) vanishes. We are given $g(x)/x \rightarrow 0$ as $x \rightarrow \infty$, so we have nothing to worry about there. For the last term, $g(x)/x \rightarrow 0$ implies that we can choose N so that $x > N$ enforces $|g(x)| \leq 2x$, meaning

$$\left| \int_2^x \frac{g(t)}{t(\log t)^2} dt \right| \leq \left| \int_2^N \frac{g(t)}{t(\log t)^2} dt \right| + \int_N^x \frac{1}{(\log t)^2} dt.$$

The left term is constant so vanishes as $x \rightarrow \infty$ when multiplied through by $\log x/x$. The right term will also vanish similarly: by L'Hôpital's rule, we see

$$\lim_{x \rightarrow \infty} \frac{\int_N^x \frac{1}{(\log t)^2} dt}{x/\log x} = \lim_{x \rightarrow \infty} \frac{1/(\log x)^2}{(\log x - 1)/(\log x)^2} = \lim_{x \rightarrow \infty} \frac{1}{\log x - 1} = 0.$$

This completes the proof. ■

2.1.2 Poisson Summation

Starting with the easier parts of Riemann's paper, we will show the functional equation for $\zeta(s)$. For this, we use the Poisson summation formula.

Theorem 2.8 (Poisson summation). Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a Schwarz function. Then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n),$$

and both sums converge absolutely.

Proof. Consider the function

$$F(x) := \sum_{n \in \mathbb{Z}} f(x+n).$$

The point is to compute the Fourier series of $F: \mathbb{R} \rightarrow \mathbb{C}$. Thus, we divide the proof into steps.

1. Note that F is continuous. Indeed, we will essentially see that the series (absolutely) converges uniformly on compact sets: let F_N denote the N th partial sum, where $N \geq 1$. Thus, to show that F is continuous on some closed interval $[a, b]$, it suffices to show that $F_N \rightarrow F$ uniformly on $[a, b]$ because each F_N is continuous. This will be enough because each $x \in \mathbb{R}$ is contained in some closed interval $[x-1, x+1]$, so F is continuous at each $x \in \mathbb{R}$.

Before doing anything, note $x \in [a, b]$ implies $|x| \leq m$ where $m := \max\{|a|, |b|\}$, so we will take $N > 2m$ throughout. Now, the Schwartz condition on f lets us find a constant $C \in \mathbb{R}$ such that $|x^2 f(x)| \leq C$, so

$$|F(x) - F_N(x)| \leq \sum_{|n| > N} |f(x+n)| \leq 2C \sum_{|n| > N} \frac{1}{(x+n)^2}.$$

The sum now splits into

$$|F(x) - F_N(x)| \leq \sum_{n < -N} \frac{1}{(x+n)^2} + \sum_{n > N} \frac{1}{(x+n)^2} = \sum_{n > N} \left(\frac{1}{(n-x)^2} + \frac{1}{(n+x)^2} \right).$$

The summand is now decreasing in n , so we may upper-bound this by the integral test, writing

$$|F(x) - F_N(x)| \leq \int_{N-1}^{\infty} \left(\frac{1}{(t-x)^2} + \frac{1}{(t+x)^2} \right) dt = \frac{1}{2(N-1-x)} + \frac{1}{2(N-1+x)},$$

which does vanish as $N \rightarrow \infty$.

As an aside, note that the above bounding has also shown that the series $F(x)$ absolutely converges because we showed that $\sum_{|n| > N} |f(x+n)|$ converges for some N depending on x (though this dependency is irrelevant here).

2. Note F is 1-periodic because rearranging the sum gives

$$F(x+1) = \sum_{n \in \mathbb{Z}} f(x+n+1) = \sum_{n \in \mathbb{Z}} f(x+n) = F(x).$$

3. The next step is compute the Fourier coefficients of F , which for some $n \in \mathbb{Z}$ is

$$a_n(F) = \int_0^1 \left(\sum_{k \in \mathbb{Z}} f(x+k) e^{-2\pi i n x} \right) dx.$$

We would like to exchange the integral and the sum, so we check the absolute convergence as

$$\int_0^1 \left(\sum_{k \in \mathbb{Z}} |f(x+k) e^{-2\pi i n x}| \right) dx = \int_0^1 \left(\sum_{k \in \mathbb{Z}} |f(x+k)| \right) dx.$$

Now, we showed that the series $x \mapsto \sum_{k \in \mathbb{Z}} |f(x+k)|$ converges uniformly on compact closed intervals $[a, b]$, so it defines a continuous function on the closed interval $[0, 1]$, so this integral converges. As such, we may now apply Fubini's theorem to get

$$a_n(F) = \sum_{k \in \mathbb{Z}} \int_0^1 f(x+k) e^{-2\pi i n x} dx = \sum_{k \in \mathbb{Z}} \int_k^{k+1} f(x) e^{-2\pi i n x} dx = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = (\mathcal{F}f)(n).$$

4. We would like to build the Fourier series using Theorem C.20, but for this we must show that S_F converges absolutely and uniformly. Well, by Lemma C.6, we see that $n \neq 0$ have

$$(\mathcal{F}f)(n) = \frac{1}{2\pi i n} \cdot (\mathcal{F}f')(n) = \frac{1}{-4\pi^2 n^2} \cdot (\mathcal{F}f'')(n).$$

Now, $(\mathcal{F}f'')$ is bounded by Remark C.5, so find M such that $|(\mathcal{F}f'')(s)| \leq M$ for all s . Checking the absolute and uniform convergence, we see $N > 0$ lets us upper-bound

$$\sum_{|n| > N} |a_n(F) e^{2\pi i n x}| \leq \frac{M}{4\pi^2} \sum_{|n| > N} \frac{1}{n^2} = \frac{2M}{4\pi^2} \sum_{n > N} \frac{1}{n^2} = \frac{2M}{4\pi^2} \int_N^{\infty} \frac{1}{x^2} dx = \frac{2M}{4\pi^2 N},$$

which does vanish as $N \rightarrow \infty$.

5. The previous step gives our absolute and uniform convergence, so Theorem C.20 tells us

$$\sum_{n \in \mathbb{Z}} f(x+n) = F(x) = \sum_{n \in \mathbb{Z}} a_n(F) e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n) e^{-2\pi i n x}$$

for all $x \in \mathbb{R}$. Setting $x = 0$ completes the proof. ■

Example 2.9. Let f be a Schwarz function, and define $f_x(t) := f(tx)$ for any $x > 0$. Then $(\mathcal{F}f_x)(s) = \frac{1}{x} (\mathcal{F}f)\left(\frac{s}{x}\right)$, so Theorem 2.8 yields

$$\sum_{n \in \mathbb{Z}} f(nx) = \sum_{n \in \mathbb{Z}} f_x(n) = \sum_{n \in \mathbb{Z}} (\mathcal{F}f_x)(n) = \frac{1}{x} \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n/x)$$

for any $x > 0$.

2.2 January 27

We began class finishing the proof of Theorem 2.8. I have edited directly into that proof for continuity.

2.2.1 An Abstract Functional Equation

We now use Theorem 2.8 in order to show the functional equation for ζ , which provides us with its meromorphic continuation.

There is a usual functional equation, but we will take a moment to point out that there is nothing particularly special about the functional equation we are about to construct. Indeed, we can build a family of functional equations as follows.

Proposition 2.10. Call a Schwarz function $f: \mathbb{R} \rightarrow \mathbb{R}$ “slow” if and only if the function

$$S_f(x) := \sum_{n \in \mathbb{Z}} f(nx)$$

is defined on $(0, \infty)$ and decays at a rate of (α, β) for all $0 < \alpha < \beta$. If f is slow, then $I(f, s) := (\mathcal{M}S_f)(s)$ converges absolutely to a holomorphic function on $\{s : \operatorname{Re} s > 0\}$. In fact, if $\mathcal{F}f$ is also slow, then

$$I(\mathcal{F}f, 1 - s) = I(f, s)$$

for $0 < \operatorname{Re} s < 1$.

Proof. The second sentence follows from Proposition 1.57.

It remains to show the last sentence. By Example 2.9, we see

$$\sum_{n \in \mathbb{Z}} f(nx) = \frac{1}{x} \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n/x)$$

for any $x > 0$. It follows that

$$\begin{aligned} I(f, s) &= \int_0^\infty \left(\sum_{n \in \mathbb{Z}} f(nx) \right) x^s \frac{dx}{x} \\ &= \int_0^\infty \left(\sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n/x) \right) x^{s-1} \frac{dx}{x} \\ &= \int_0^\infty \left(\sum_{n \in \mathbb{Z}} (\mathcal{F}f)(nx) \right) x^{1-s} \frac{dx}{x} \\ &= I(\mathcal{F}f, 1 - s), \end{aligned}$$

which is what we wanted. ■

Corollary 2.11. Continue in the context of Proposition 2.10, but further assume that the (double) integrals $I(f, s)$ and $I(\mathcal{F}f, s)$ both absolutely converge for $\operatorname{Re} s > 0$. Then

$$\zeta(s)(\mathcal{M}f)(s) = (\mathcal{M}\mathcal{F}f)(1 - s)\zeta(1 - s).$$

Proof. Because the (double) integral $I(f, s)$ absolutely converges, we may use Fubini's theorem to write

$$\begin{aligned} I(f, s) &= \int_0^\infty \left(\sum_{n \in \mathbb{Z}} f(nx) \right) x^s \frac{dx}{x} \\ &= \sum_{n \in \mathbb{Z}} \left(\int_0^\infty f(nx) x^s \frac{dx}{x} \right) \\ &\stackrel{*}{=} 2 \sum_{n=1}^\infty \left(\frac{1}{n^s} \int_0^\infty f(x) x^s \frac{dx}{x} \right) \\ &= 2\zeta(s)(\mathcal{M}f)(s), \end{aligned}$$

Note at $\stackrel{*}{=}$ we have assumed that $f(0) = 0$, which holds because $S_f(x)$ converges absolutely. (Indeed, if $|f(0)| > 0$, then as $x \rightarrow 0^+$, we would have $S_f(x)$ diverge: we may say $|f(x)| > |f(0)|/2$ for $|x| < \delta$, but then the absolute sum is bounded below by $n|f(0)|/2$ at $x = \delta/n$.) To finish, we plug into the functional equation of Proposition 2.10. ■

Remark 2.12. We could spend time searching for a function f satisfying all of our various hypotheses, but we are about to show a more concrete functional equation, so there is little point.

2.2.2 Facts about Γ

In this subsection, we will collect a few facts about Γ which will be helpful shortly. We will loosely follow [Tao14, Section 1].

Definition 2.13. For $\operatorname{Re} s > 0$, we define

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

Remark 2.14. In some sense, Γ is a continuous version of a Gauss sum: it's an integral of an additive character multiplied by a multiplicative character, over a suitable Haar measure.

Remark 2.15. Define $f: [0, \infty) \rightarrow \mathbb{R}$ by $f(t) := e^{-t}$ so that $\Gamma = \mathcal{M}f$ by definition. Notably, for any $c > 0$, the function $t \mapsto t^c f(t)$ is bounded on $[0, \infty)$ because

$$\lim_{t \rightarrow \infty} t^c f(t) = \lim_{t \rightarrow \infty} \frac{t^c}{e^t} = 0.$$

(Explicitly, find N such that $|t^c f(t)| < 1$ for any $x > N$; for $x \leq N$, note $t \mapsto t^c f(t)$ has a maximum on the compact set $[0, N]$.) Thus, f decays at a rate of (α, β) for any $0 < \alpha < \beta$, so Proposition 1.57 implies that Γ converges absolutely and is a holomorphic function on $\{s : \operatorname{Re} s > 0\}$ by taking the union over all such (α, β) .

Remark 2.15 assures us that Γ is holomorphic on $\operatorname{Re} s > 0$, but we quickly note that we can provide Γ with a meromorphic continuation to the left, at the cost of some poles.

Lemma 2.16. For any $\operatorname{Re} s > 0$, we have $\Gamma(s+1) = s\Gamma(s)$.

Proof. This is integration by parts. Indeed, we compute

$$\begin{aligned}
 \Gamma(s+1) &= \int_0^\infty e^{-t} t^s dt \\
 &= -e^{-t} t^s \Big|_0^\infty + \int_0^\infty e^{-t} s t^{s-1} dt \\
 &= 0 + s \int_0^\infty e^{-t} t^{s-1} dt \\
 &= s \Gamma(s),
 \end{aligned}$$

which is what we wanted. ■

Example 2.17. For any positive integer n , applying Lemma 2.16 inductively yields

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)\Gamma(1).$$

Notably, $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$, so we see $\Gamma(n) = (n-1)!$ for any positive integer n .

Remark 2.18. We now describe how to (inductively) continue Γ using Lemma 2.16. Fix some $n \in \mathbb{N}$ and set $U_n := \{s : \operatorname{Re} s > -n, -s \notin \mathbb{Z}_{\geq 0}\}$. Then we define $\Gamma_n := U_n \rightarrow \mathbb{C}$ by

$$\Gamma_n(s) := \frac{\Gamma(s+n)}{s(s+1)(s+2)\cdots(s+n-1)}.$$

Because Γ is holomorphic on $\operatorname{Re} s > 0$, we see Γ_n is holomorphic on U_n . Further, Lemma 2.16 implies that $\Gamma_n(s) = \Gamma(s)$ for $\operatorname{Re} s > 0$, so we have indeed defined a continuation of Γ . Sending $n \rightarrow \infty$ provides our meromorphic continuation of Γ to all of \mathbb{C} .

Remark 2.18 is in some sense analogous to defining a continuation for $\zeta(s)$ to all of \mathbb{C} using the repeated integration by parts mentioned in Remark 1.38. However, just as with ζ , there is a “functional equation” for Γ which does not require the sort of inductive arguments of Remark 2.18. We begin by upgrading Lemma 2.16.

Lemma 2.19. For any $s_1, s_2 \in \mathbb{C}$ such that $\operatorname{Re} s_1, \operatorname{Re} s_2 > 0$, we have

$$\Gamma(s_1 + s_2) \int_0^1 u^{s_1-1} (1-u)^{s_2-1} du = \Gamma(s_1) \Gamma(s_2).$$

Proof. Remark 2.15 tells us that the integral defining Γ converges absolutely, so Fubini’s theorem lets us write

$$\Gamma(s_1) \Gamma(s_2) = \int_0^\infty \int_0^\infty e^{-t_1-t_2} t_1^{s_1-1} t_2^{s_2-1} dt_1 dt_2.$$

We would like to combine the t_1 and t_2 into a single t . Thus, we set $t_1 = ut$ and $t_2 = (1-u)t$ for $u \in [0, 1]$ and $t \in (0, \infty]$. More precisely, for $t_1, t_2 > 0$, we have $u = t_1/(t_2 + t_1)$ and $t = t_1 + t_2$, which makes our Jacobian

$$\det \begin{pmatrix} \partial t_1 / \partial u & \partial t_1 / \partial t \\ \partial t_2 / \partial u & \partial t_2 / \partial t \end{pmatrix} = \det \begin{pmatrix} t & u \\ -t & 1-u \end{pmatrix} = t.$$

Thus,

$$\Gamma(s_1) \Gamma(s_2) = \int_0^1 \int_0^\infty e^{-t} t^{s_1+s_2-1} u^{s_1-1} (1-u)^{s_2-1} dt du.$$

This still absolutely converges (indeed, we can just change coordinates back to $dt_1 dt_2$ to see this), so a last application of Fubini's theorem reveals

$$\Gamma(s_1)\Gamma(s_2) = \left(\int_0^1 u^{s_1-1} (1-u)^{s_2-1} du \right) \left(\int_0^1 e^{t} t^{s_1+s_2-1} dt \right) = \Gamma(s_1 + s_2) \int_0^1 u^{s_1-1} (1-u)^{s_2-1} du,$$

which is what we wanted. ■

Remark 2.20. Because $\Gamma(1) = 1$ and $\int_0^1 u^s du = \frac{1}{s+1}$ for $\operatorname{Re} s > -1$, we see Lemma 2.19 implies

$$\Gamma(s) = \Gamma(s)\Gamma(1) = \Gamma(s+1) \int_0^1 u^s du = \frac{1}{s+1} \Gamma(s+1),$$

thus recovering Lemma 2.16.

Proposition 2.21 (Functional equation for Γ). For any s with $0 < \operatorname{Re} s < 1$, we have

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

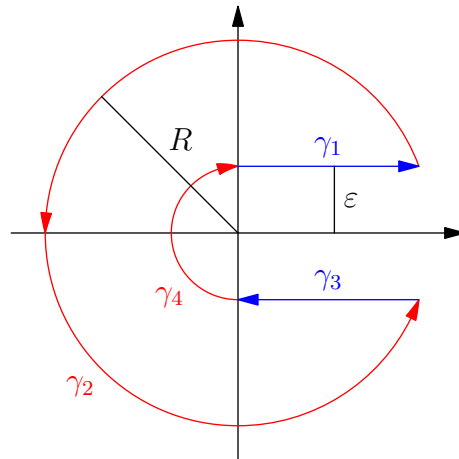
Proof. By Lemma 2.19, we see

$$\Gamma(s)\Gamma(1-s) = \Gamma(1) \int_0^1 u^{s-1} (1-u)^{-s} du = \int_0^1 (1-u) \left(\frac{u}{1-u} \right)^{s-1} du.$$

As such, we have reduced to compute some integral. This is done via contour integration. Thus, we set $t := \frac{u}{1-u} = \frac{1}{1-u} - 1$ so that $u = \frac{t}{t+1}$ and $dt = \frac{1}{(1-u)^2} du = (1+t)^2 du$, which gives

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{t^{s-1}}{1+t} dt.$$

We are now ready to use contour integration. Being careful, the function $t \mapsto t^{s-1}$ is given a meromorphic continuation to $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ by $t \mapsto \exp((s-1)\log t)$, where $\log t$ has a branch cut at $\mathbb{R}_{\geq 0}$; explicitly, $\operatorname{Im}(\log t) \in [0, 2\pi)$. Now, for fixed R, ε with $R > 1 > \varepsilon > 0$, draw the following contour γ , split into four pieces.



Notably, the function $f(z) := z^{s-1}/(1+z)$ is meromorphic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ with a single simple pole at $z = -1$, with residue $(-1)^{s-1} = e^{\pi i(s-1)}$. Thus, the Residue theorem yields

$$\frac{1}{2\pi i} \oint_\gamma f(z) dz = e^{\pi i(s-1)} = -e^{\pi i s}$$

because $z = -1$ lies within the interior of γ . We now compute the integral $\oint_{\gamma} f(z) dz$ on each of the γ_i independently.

- For $i \in \{1, 3\}$, we compute

$$\int_{\gamma_i} f(z) dz = \int_0^R \frac{\lambda_i(t \pm \varepsilon i)^{s-1}}{1 + (t \pm \varepsilon i)} (\pm dt),$$

where $\lambda_i = 1$ (and we use $+$) if $i = 1$, and $\lambda_i = e^{2\pi i(s-1)} = e^{2\pi i s}$ (and we use $-$) if $i = 3$. Now, for each $\varepsilon \in (0, 1)$, we see

$$\left| \frac{(t \pm \varepsilon i)^{s-1}}{1 + (t \pm \varepsilon i)} \right| \leq \frac{(t+1)^{\operatorname{Re} s - 1}}{1+t},$$

and the right-hand function has finite integral over $[0, R]$ because $\operatorname{Re} s > 0$. Thus, we may apply the Dominated convergence theorem to see that sending $\varepsilon \rightarrow 0^+$ tells us that

$$\lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_i} f(z) dz = \lim_{R \rightarrow \infty} \pm \lambda_i \int_0^R \frac{t^{s-1}}{1+t} dt = \pm \lambda_i \int_0^{\infty} \frac{t^{s-1}}{1+t} dt.$$

- On γ_2 , we bound

$$\left| \int_{\gamma_2} f(z) dz \right| \leq 2\pi R \cdot \max_{z \in \operatorname{im} \gamma_2} |f(z)|.$$

To compute this maximum, we use the fact that $|z| = R > 1$ to see

$$\left| \frac{z^{s-1}}{1+z} \right| \leq \frac{R^{\operatorname{Re} s - 1}}{R-1},$$

so

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \frac{2\pi R^{\operatorname{Re} s}}{R-1} = \frac{2\pi R^{\operatorname{Re} s - 1}}{1 - 1/R}.$$

Sending $R \rightarrow \infty$ has this integral go to 0/1 because $\operatorname{Re} s < 1$.

- On γ_4 , we bound

$$\left| \int_{\gamma_4} f(z) dz \right| \leq \pi \varepsilon \cdot \max_{z \in \operatorname{im} \gamma_4} |f(z)|.$$

To compute this maximum, we use the fact that $|z| = \varepsilon < 1$ to see

$$\left| \frac{z^{s-1}}{1+z} \right| \leq \frac{\varepsilon^{\operatorname{Re} s - 1}}{1 - \varepsilon},$$

so

$$\left| \int_{\gamma_4} f(z) dz \right| \leq \frac{\pi \varepsilon^{\operatorname{Re} s}}{1 - \varepsilon}.$$

Sending $\varepsilon \rightarrow 0^+$ has this integral go to 0/1 because $\operatorname{Re} s > 0$.

Combining the above integrals, we see

$$-2\pi i e^{\pi i s} = \oint_{\gamma} f(z) dz = (1 - e^{2\pi i s}) \int_0^{\infty} \frac{t^{s-1}}{1+t} dt$$

upon sending $\varepsilon \rightarrow 0^+$ and then $R \rightarrow \infty$. Rearranging,

$$\Gamma(s)\Gamma(1-s) = \int_0^{\infty} \frac{t^{s-1}}{1+t} dt = \frac{-2\pi i e^{\pi i s}}{1 - e^{2\pi i s}} = \frac{\pi}{(e^{-\pi i s} - e^{\pi i s})/(2i)} = \frac{\pi}{\sin(\pi s)},$$

which is what we wanted. ■

Example 2.22. Plugging in $s = 1/2$ into Proposition 2.21, we see $\Gamma(1/2)^2 = \pi$. However, the definition of Γ surely has $\Gamma(1/2) > 0$, so we must have $\Gamma(1/2) = \sqrt{\pi}$.

Corollary 2.23. The function Γ has a meromorphic continuation to all \mathbb{C} , and Γ has no zeroes. The only poles are simple poles occurring at all nonpositive integers, and the residue of the pole at $-n$ is $(-1)^n/n!$ for each positive integer n .

Proof. For completeness, we use the functional equation to produce an analytic continuation. Let Z be the zeroes of Γ in $\{s : \operatorname{Re} s > 0\}$, which we know is an isolated set because Γ is a nonconstant holomorphic function. Now, define $U := \mathbb{C} \setminus (\mathbb{Z}_{\leq 0} \cup \{s : 1 - s \in Z\})$, and define a function $U \rightarrow \mathbb{C}$ by

$$s \mapsto \begin{cases} \Gamma(s) & \text{if } \operatorname{Re} s > 0, \\ \pi/(\Gamma(1-s)\sin(\pi s)) & \text{if } \operatorname{Re} s < 1. \end{cases}$$

Note Proposition 2.21 tells us that this function is well-defined on the overlapping region $\{s : 0 < \operatorname{Re} s < 1\}$. Thus, gluing these meromorphic functions together, we define a single meromorphic function $\Gamma : U \rightarrow \mathbb{C}$.

It remains to show the other listed properties of Γ . Note that $\Gamma(s+1) = s\Gamma(s)$ holds on $\{s : \operatorname{Re} s > 0\}$ and hence everywhere by analytic continuation. Thus, for the residue computation, we fix some nonnegative integer n and write

$$\lim_{s \rightarrow 0} s\Gamma(s-n) = \lim_{s \rightarrow 0} \frac{s\Gamma(s-n+1)}{(s-n)} = \cdots = \lim_{s \rightarrow 0} \frac{s\Gamma(s+1)}{(s-n)(s-n+1)\cdots(s-1)(s)} = \frac{\Gamma(1)}{(-1)^n n!}.$$

Thus, Γ has a pole of residue $(-1)^n/n!$ at $-n$ for each nonnegative integer n .

Lastly, we show that Γ has no zeroes. Note Γ has no zeroes on the positive integers by Example 2.17, and Γ isn't even defined on the nonpositive integers, so Γ has no zeroes on \mathbb{Z} . Additionally, we note that we surely have an analytic continuation of Γ by $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ by Remark 2.18. Thus, we see each $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ has $1-s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, thus giving

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \neq 0,$$

which forces $\Gamma(s) \neq 0$. (Note this functional equation extends from $\{s : 0 < \operatorname{Re} s < 1\}$ to all $\mathbb{C} \setminus \mathbb{Z}$ by uniqueness of analytic continuation.) ■

Remark 2.24. Corollary 2.23 tells us that $1/\Gamma(s)$ is an entire function. Indeed, the poles of Γ becomes 0s, and Γ has no zeroes to become poles! Notably, Corollary 2.23 tells us that $1/\Gamma(s)$ is entire with only simple zeroes at the nonpositive integers n .

While we're here, we give another application of Lemma 2.19.

Proposition 2.25. For any $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \sqrt{\pi}2^{1-s}\Gamma(s).$$

Proof. By the uniqueness of analytic continuation, it suffices to show this for $0 < \operatorname{Re} s < 1$. Now Lemma 2.19

lets us write

$$\begin{aligned}
 \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma(s)} &= \int_0^1 u^{s-1}(1-u)^{s-1} du \\
 &= \int_{-1}^1 \left(\frac{1+t}{2}\right)^{s/2-1} \left(\frac{1-t}{2}\right)^{s/2-1} \frac{dt}{2} \\
 &= \frac{1}{2^{s-1}} \int_{-1}^1 (1-t^2)^{s/2-1} dt \\
 &= \frac{1}{2^s} \int_0^1 (1-t^2)^{s/2-1} dt \\
 &= \frac{1}{2^{s-1}} \int_0^1 u^{1/2-1}(1-u)^{s/2-1} dt \\
 &= \frac{1}{2^{s-1}} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}.
 \end{aligned}$$

By Corollary 2.23, $\Gamma(s/2) \neq 0$ for all s , so we may rearrange the above into

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \Gamma\left(\frac{1}{2}\right) 2^{1-s}\Gamma(s).$$

Plugging in $\Gamma(1/2) = \sqrt{\pi}$ from Example 2.22 completes the proof. ■

2.2.3 Bounds on Γ

While Γ is still fresh in our mind, we will prove a few bounds about it. We continue to roughly follow [Tao14]. From Theorem B.22, we know that a lower bound on Γ will produce an upper bound on $1/\Gamma$ and thus a factorization of $1/\Gamma$. However, it will be convenient to actually provide this factorization first and then use it to produce bounds.

Proposition 2.26. For any $s \in \mathbb{C}$, we have

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} E_1(s/n).$$

Here, γ is the Euler–Mascheroni constant, and $E_1(z) = (1-z)e^z$. In fact, this product converges absolutely and uniformly.

Proof. The infinite product converges absolutely and uniformly to an entire function by Lemma B.18. Indeed, we see that

$$\#\{k : -k < r\} = \lfloor r \rfloor + 1 < r + 1 \ll r^{1+\varepsilon}$$

for any $\varepsilon > 0$. (Formally, note $(r+1)r^{-1-\varepsilon} \rightarrow 0$ as $r \rightarrow \infty$, so this continuous function is bounded.) So indeed, we have all the correct convergence.

It follows from the uniqueness of analytic continuation that we can just check the identity for $s \in \mathbb{R}_{>0}$. Quickly, we remove the γ term by writing

$$\begin{aligned}
 se^{\gamma s} \prod_{n=1}^{\infty} E_1(s/n) &= \lim_{n \rightarrow \infty} se^{\sum_{k=1}^n s/k - s \log n} \prod_{k=1}^n (1 + s/k) e^{-s/k} \\
 &= \lim_{n \rightarrow \infty} sn^{-s} \prod_{k=1}^n (1 + s/k) \\
 &= \lim_{n \rightarrow \infty} \frac{s(s+1) \cdots (s+n)}{n^s n!}.
 \end{aligned}$$

Thus, it suffices to show that

$$\Gamma(s) \stackrel{?}{=} \lim_{n \rightarrow \infty} \frac{n^s n!}{s(s+1) \cdots (s+n)}.$$

Using the functional equation $\Gamma(s+1) = s\Gamma(s)$ from Lemma 2.16 inductively tells us that $\Gamma(n+1) = n!$ and $\Gamma(s+n+1) = (s+n) \cdots (s+1)s\Gamma(s)$, so we see

$$\lim_{n \rightarrow \infty} \frac{n^s n!}{s(s+1) \cdots (s+n)} = \lim_{n \rightarrow \infty} \left(n^s \cdot \frac{\Gamma(s)\Gamma(n+1)}{\Gamma(s+n+1)} \right).$$

Now using Lemma 2.19, this is

$$\begin{aligned} \lim_{n \rightarrow \infty} n^s \cdot \frac{\Gamma(s)\Gamma(n+1)}{\Gamma(s+n+1)} &= \lim_{n \rightarrow \infty} n^s \int_0^1 t^{s-1} (1-t)^n dt \\ &= \lim_{n \rightarrow \infty} \int_0^1 (nt)^{s-1} (1-t)^n n dt \\ &= \lim_{n \rightarrow \infty} \int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt. \end{aligned}$$

We would like to use the Dominated convergence to compute this limit of integrals as $\Gamma(s)$. As such, we for each n , define

$$f_n(t) := t^{s-1} \left(1 - \frac{t}{n}\right)^n \mathbf{1}_{\leq n}(t).$$

We would like to use the Dominated convergence theorem on the f_n . Well, for $t \geq n$, we see $f_n(t) = 0$, and for $t < n$, we see

$$\log f_n(t) = (s-1) \log t + n \log \left(1 - \frac{t}{n}\right) \stackrel{*}{\leq} (s-1) \log t + n \left(1 - \frac{t}{n}\right) = (s-1) \log t - t,$$

where $\stackrel{*}{\leq}$ holds by using the power series $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$. Thus, $f_n(t) \leq t^{s-1} e^{-t}$, which has a finite integral over $(0, \infty)$ bounded by $\Gamma(s)$. Thus, by the Dominated convergence theorem, we see

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(t) dt = \int_0^\infty \left(\lim_{n \rightarrow \infty} f_n(t) \right) dt = \int_0^\infty t^{s-1} e^{-t} dt = \Gamma(s),$$

where we have used the fact that $\left(1 - \frac{t}{n}\right)^n \rightarrow e^{-t}$ as $n \rightarrow \infty$. (This also can be seen by taking logs and bounding the error term.) This completes the proof. \blacksquare

Corollary 2.27. For $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have

$$\frac{\Gamma'(s)}{\Gamma(s)} = \lim_{n \rightarrow \infty} \left(\log n - \sum_{k=0}^n \frac{1}{s+k} \right).$$

Proof. Note that the product of Proposition 2.26 converges absolutely and uniformly. Thus, Corollary A.28 applies and tells us that

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \frac{(1/\Gamma)'(s)}{(1/\Gamma)(s)} = \underbrace{\frac{1}{s}}_s + \underbrace{\gamma}_{e^{\gamma s}} + \sum_{k=-1}^{-\infty} \underbrace{\frac{1}{k} \cdot \frac{E_1'(s/k)}{E_1(s/k)}}_{E_1(s/k)}$$

wherever $1/\Gamma(s) \neq 0$, which is exactly $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ by Remark 2.24. Now, we see

$$\frac{E_1'(z)}{E_1(z)} = \frac{1}{z-1} + 1,$$

so

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma + \sum_{k=-1}^{-\infty} \left(\frac{1}{-k} - \frac{1}{s-k} \right) = -\frac{1}{s} - \gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{s+k} \right),$$

where we must be very careful about signs. To finish, we use the definition of γ to write

$$\frac{\Gamma'(s)}{\Gamma(s)} = \lim_{n \rightarrow \infty} \left(-\frac{1}{s} + \log n - \sum_{k=1}^n \frac{1}{k} + \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{s+k} \right) \right) = \lim_{n \rightarrow \infty} \left(\log n - \sum_{k=0}^n \frac{1}{s+k} \right),$$

which is what we wanted. ■

Thus, to estimate Γ'/Γ , we want to know about the rate of growth of harmonic numbers. It turns out that Abel summation is not quite good enough for our purposes, so we will have to integrate by parts one more time.

Lemma 2.28 (Trapezoid rule). Fix a continuously twice-differentiable function $f: [m, n] \rightarrow \mathbb{C}$, where $m < n$ are integers. Then

$$\sum_{k=m}^n f(k) = \int_m^n f(t) dt + \frac{f(m) + f(n)}{2} + O\left(\int_m^n |f''(t)| dt\right).$$

Proof. This is integration by parts twice. Indeed, we see

$$\begin{aligned} \int_m^n f(t) dt &= \sum_{k=m}^{n-1} \int_k^{k+1} f(t) dt \\ &= \sum_{k=m}^{n-1} \left(\frac{1}{2} f(k+1) - \frac{1}{2} f(k) - \int_k^{k+1} \left(t - n - \frac{1}{2} \right) f'(t) dt \right) \\ &= \sum_{k=m}^{n-1} \left(\frac{f(k+1) + f(k)}{2} - \int_k^{k+1} \left(t - n - \frac{1}{2} \right) f'(t) dt \right) \\ &= \sum_{k=m}^{n-1} \left(\frac{f(k+1) + f(k)}{2} - \left(\frac{1}{2} \left(n+1 - n - \frac{1}{2} \right)^2 - \frac{1}{8} \right) f'(n+1) \right. \\ &\quad \left. + \left(\frac{1}{2} \left(n - n - \frac{1}{2} \right)^2 - \frac{1}{8} \right) f'(n) + \int_k^{k+1} \left(\frac{1}{2} \left(t - n - \frac{1}{2} \right)^2 - \frac{1}{8} \right) f''(t) dt \right) \\ &= \sum_{k=m}^{n-1} \left(\frac{f(k+1) + f(k)}{2} + \frac{1}{2} \int_k^{k+1} (\{t\}^2 - \{t\}) f''(t) dt \right) \\ &= \sum_{k=m}^n f(k) - \frac{f(m) + f(n)}{2} + \frac{1}{2} \int_m^n (\{t\}^2 - \{t\}) f''(t) dt. \end{aligned}$$

Rearranging the above equality finishes upon noting that the function $\frac{\{t\}^2 - \{t\}}{2}$ has a maximum (for example, it is bounded in magnitude by $\frac{1+1}{2} = 1$). ■

And here is our estimate.

Proposition 2.29. Fix $\varepsilon \in (0, \pi)$. For $s \in \{z \in \mathbb{C} : |\arg z| < \pi - \varepsilon\}$, we have

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O_\varepsilon(1/|s|^2).$$

Proof. We use Corollary 2.27. Using Lemma 2.28, we set $f(t) := 1/(s+t)$ so that

$$\begin{aligned} \sum_{k=0}^n \frac{1}{s+k} &= \int_0^n \frac{1}{s+t} dt + \frac{f(0)+f(n)}{2} + O\left(\int_0^n \frac{2}{|s+t|^3} dt\right) \\ &= \log(n+s) - \log s + \frac{1}{2s} + \frac{1}{2(n+s)} + O\left(\int_0^n \frac{2}{|s+t|^3} dt\right). \end{aligned}$$

We would like for the integral to be $O_\varepsilon(1/|s|^2)$ as $n \rightarrow \infty$. We have two cases.

- If $\operatorname{Re} s \geq 0$, then $|s+t| \geq \operatorname{Re}(s+t) \geq t$ and $|s+t| \geq |s|$ for each $t \geq 0$, so we can easily upper-bound

$$\begin{aligned} \int_0^n \frac{2}{|s+t|^3} dt &\leq \int_0^1 \frac{2}{|s+t|^3} dt + \int_1^\infty \frac{2}{|s+t|^3} dt \\ &\leq \int_0^1 \frac{2}{|s|^3} dt + \int_1^\infty \frac{2}{|t|^3} dt \\ &= \frac{2}{|s|^3} + \left(-\frac{1}{t^2}\right)_1^\infty \\ &= \frac{2}{|s|^3} - 1, \end{aligned}$$

which is in fact $O(1/|s|^2)$.

- If $\operatorname{Re} s < 0$, then we note $\operatorname{Im} s > 0$ because $\arg z \neq \pi$. Here the bounding is harder; take $n > 2|s|$ for convenience. For large values of t , we note $t \geq 2|s|$ will make $|s+t| \geq t - |s|$ fairly large, so

$$\int_{2|s|}^n \frac{2}{|s+t|^3} dt \leq \int_{2|s|}^\infty \frac{2}{(t-|s|)^3} dt = -\frac{1}{(t-|s|)^2} \Big|_{2|s|}^\infty = \frac{1}{|s|^2},$$

which is $O(1/|s|^2)$.

Now, the interval $t \in [0, 2|s|]$ is more difficult to handle. Because t is real, we note that $|s+t| = \sqrt{(t+\operatorname{Re} s)^2 + (\operatorname{Im} s)^2} \geq |\operatorname{Im} s|$, which is perhaps the best we can do because $s+t$ can have arbitrarily small real part in this interval. However, letting $\theta := \arg z$, we note that

$$\frac{\operatorname{Im} s}{|s|} = |\sin \theta| \geq \sin(\pi - \varepsilon)$$

by assumption on $\arg s$. Thus,

$$\int_0^{2|s|} \frac{2}{|s+t|^3} dt \leq \int_0^{2|s|} \frac{2}{|\operatorname{Im} s|^3} dt = 2|s| \cdot \frac{2}{|\operatorname{Im} s|^3} \leq 4|\sin(\pi - \varepsilon)|^3 \cdot \frac{1}{|s|^2},$$

which is still $O_\varepsilon(1/|s|^2)$, so we are safe. Totaling our integrals finishes.

In total, we see that

$$\sum_{k=0}^n \frac{1}{s+k} = \log(n+s) - \log s + \frac{1}{2s} + \frac{1}{2(n+s)} + O_\varepsilon(1/|s|^2).$$

Thus, by Corollary 2.27, we see

$$\begin{aligned} \frac{\Gamma'(s)}{\Gamma(s)} &= \lim_{n \rightarrow \infty} \left(\log n - \sum_{k=0}^n \frac{1}{s+k} \right) \\ &= \lim_{n \rightarrow \infty} \left(\log n - \log(n+s) + \log s - \frac{1}{2s} - \frac{1}{2(n+s)} + O_\varepsilon(1/|s|^2) \right) \\ &= \log s - \frac{1}{2s} + \lim_{n \rightarrow \infty} \left(\log \left(1 - \frac{s}{n+s} \right) - \frac{1}{2(n+s)} \right) + O_\varepsilon(1/|s|^2) \\ &= \log s - \frac{1}{2s} + O_\varepsilon(1/|s|^2), \end{aligned}$$

which is what we wanted. ■

Taking the integral of this allows us to recover a version of Stirling's approximation.

Proposition 2.30 (Stirling's approximation). Fix $\varepsilon \in (0, \pi/2)$. For $s \in \{z \in \mathbb{C} : |\arg z| < \pi - \varepsilon\}$, we have

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O_\varepsilon(1/|s|).$$

Proof. Set $\Omega_\varepsilon := \{z \in \mathbb{C} : |\arg z| < \pi - \varepsilon\}$; note that Ω_ε is convex because (roughly speaking) any complex nonzero complex number on the line segment connecting two nonzero complex numbers α, β will have argument between the two arguments of α and β . Anyway, we proceed in steps.

1. The function Γ is holomorphic on Ω_ε and does not vanish there by Remark 2.24, so Lemma A.16 grants us a logarithm. In fact, using Remark A.17 to get our explicit logarithm, we see

$$\log \Gamma(s) = \underbrace{\log \Gamma(1)}_0 + \int_1^s \frac{\Gamma'(z)}{\Gamma(z)} dz,$$

where the integral here is along the straight line from 1 to s (which does live in Ω_ε because Ω_ε is convex). Thus, we see we do in fact want to integrate the bound given by Proposition 2.29.

2. Being a little careful, we set

$$E_\varepsilon(s) := \frac{\Gamma'(s)}{\Gamma(s)} - \log s + \frac{1}{2s}.$$

Notably, E_ε is holomorphic on Ω_ε because the right-hand side here is holomorphic on Ω_ε (for suitably chosen \log), so E_ε is in particular integrable. Additionally, we note that $|E_\varepsilon(s)| \leq C_\varepsilon/|s|^2$ for some constant C_ε and $|s|$ large enough. Thus, fixing s and some large $N > |s|$, we compute for $|s|$ large enough that

$$\int_1^s E_\varepsilon(z) dz = \int_1^N E_\varepsilon(z) dz - \int_s^N E_\varepsilon(z) dz.$$

The left integral here converges absolutely as $N \rightarrow \infty$ because

$$\int_1^\infty |E_\varepsilon(z)| dz \leq C_\varepsilon \int_1^\infty \frac{1}{z^2} dz = C_\varepsilon.$$

We would like to show that the right integral is $O_\varepsilon(1/|s|)$. However, if s is close to the negative real axis, there are potentially large contributions of the integral when z is roughly 0, so we change our path.

Instead of using the straight line from s to N , we follow the arc of a circle with center at $z = 0$ and radius $|s|$ until we hit the positive real axis (moving clockwise if $\operatorname{Re} s > 0$ and counterclockwise otherwise); then we move along the positive real axis from $|s|$ to N . Letting γ denote this arc, we see

$$\begin{aligned} \left| \int_s^N E_\varepsilon(z) dz \right| &\leq \left| \int_\gamma E_\varepsilon(z) dz \right| + \left| \int_{|s|}^N E_\varepsilon(z) dz \right| \\ &\leq \ell(\gamma) \cdot \max_{z \in \operatorname{im} \gamma} \{|E_\varepsilon(z)|\} + \int_{|s|}^N |E_\varepsilon(z)| dz \\ &\leq \pi |s| \cdot \frac{C_\varepsilon}{|s|^2} + C_\varepsilon \int_{|s|}^N \frac{1}{z^2} dz \\ &= \frac{C_\varepsilon \pi}{|s|} + \frac{C_\varepsilon}{|s|} - \frac{C_\varepsilon}{N}. \end{aligned}$$

In total, we see

$$\int_1^s E_\varepsilon(z) dz = \int_1^N E_\varepsilon(z) dz + O\left(\frac{C_\varepsilon \pi}{|s|} + \frac{C_\varepsilon}{|s|} - \frac{C_\varepsilon}{N}\right).$$

Sending $N \rightarrow \infty$ shows that this is

$$\int_1^s E_\varepsilon(z) dz = \underbrace{\int_1^\infty E_\varepsilon(z) dz}_{C:=} + O_\varepsilon(1/|s|),$$

which is good enough for our purposes.

3. Integrating over E_ε , we see

$$\begin{aligned} \log \Gamma(s) &= \int_1^s \frac{\Gamma'(z)}{\Gamma(z)} dz \\ &= \int_1^s \left(\log z - \frac{1}{2z} \right) dz + \int_1^s E_\varepsilon(z) dz \\ &= s \log s - s - \frac{1}{2} \log s + C + O_\varepsilon(1/|s|) \\ &= \left(s - \frac{1}{2} \right) \log s - s + C + O_\varepsilon(1/|s|) \end{aligned}$$

for some constant C chosen above.

4. It remains to show $C = \frac{1}{2} \log 2\pi$. For this, we use Proposition 2.21. We restrict our attention to $\Omega'_\varepsilon := \{z \in \mathbb{C} : \varepsilon < |\arg z| < \pi - \varepsilon\}$. For $t > 0$, set $s := \frac{1}{2} + it$; notably, $s, 1-s \in \Omega_\varepsilon$ for $t > \frac{1}{2}$ because this gives $\arg s, \arg(1-s) \in (-\pi/2, \pi/2)$. Also, $|s| = |1-s| \geq t$. Thus, on one hand, we can use our bound above to show

$$\begin{aligned} \log \Gamma(s) \Gamma(1-s) &= \log \Gamma(s) + \log \Gamma(1-s) \\ &= it \log \left(\frac{1}{2} + it \right) - \left(\frac{1}{2} + it \right) + C - it \log \left(\frac{1}{2} - it \right) - \left(\frac{1}{2} - it \right) + C + O_\varepsilon(1/|t|) \\ &= 2C + it \log \left(\frac{1/(2t) + i}{1/(2t) - i} \right) - 1 + O_\varepsilon(1/|t|). \end{aligned}$$

We are going to need to understand the log term here more carefully. Choosing a suitable branch of \log (say, now away from the positive reals), we write $x := 1/t$ so that we are interested in the behavior of the holomorphic function $f(x) := \log \left(\frac{x/2+i}{x/2-i} \right)$ at $x = 0$. Notably, $f(0) = \log(-1) = \pi i$ (for some choice of branch of \log). Additionally, we see

$$f'(x) = \frac{1/2}{x/2+i} - \frac{1/2}{x/2-i}$$

yields $f'(0) = -i$. Thus, our power series for f is given by $f(x) = \pi i - it + \dots$, so

$$\lim_{t \rightarrow \infty} \left(\pi t + it \log \left(\frac{1/(2t) + it}{1/(2t) - i} \right) \right) = \lim_{x \rightarrow 0} \frac{if(x) + \pi}{x} = 1.$$

On the other hand, we see

$$\begin{aligned} \log \left(\frac{\pi}{\sin(\pi s)} \right) &= \log \pi - \log \sin \left(\frac{\pi}{2} + \pi it \right) \\ &= \log \pi - \log \left(\frac{e^{i\pi/2 - \pi t} - e^{-i\pi/2 + \pi t}}{2i} \right) \\ &= \log \pi - \log \left(\frac{e^{\pi t} + e^{-\pi t}}{2} \right) \\ &= -\pi t + \log 2\pi - \log(1 + e^{-2\pi t}), \end{aligned}$$

up to multiples of $2\pi i$, so

$$2C + it \log \left(\frac{1/(2t) + i}{1/(2t) - i} \right) - 1 + O_\varepsilon(1/|t|) = -\pi t + \log 2\pi - \log(1 + e^{-2\pi t}).$$

Quickly, we rearrange this to

$$2C + \left(\pi t + it \log \left(\frac{1/(2t) + i}{1/(2t) - i} \right) - 1 \right) + O_\varepsilon(1/|t|) = \log 2\pi - \log(1 + e^{-2\pi t}).$$

Thus, sending $t \rightarrow \infty$ makes almost all terms vanish, leaving us with $C = \frac{1}{2} \log 2\pi$ (up to a multiple of $2\pi i$). This completes the proof. ■

Corollary 2.31. Fix real numbers $a < b$. For any $\sigma \in [a, b]$, we have

$$|\Gamma(\sigma + it)| \sim_{a,b} \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma-1/2}$$

as $|t| \rightarrow \infty$.

Proof. For psychological reasons, we quickly reduce to the case where $t > 0$. By definition of Γ , we see that $\Gamma(s) \in \mathbb{R}$ if $s \in \mathbb{R}_{>0}$, so $\Gamma(s) = \overline{\Gamma(\overline{s})}$. However, these are both holomorphic functions, so the uniqueness of analytic continuation enforces

$$|\Gamma(\sigma + it)| = |\overline{\Gamma(\sigma + it)}| = |\Gamma(\overline{\sigma + it})| = |\Gamma(\sigma - it)|.$$

Thus, adjusting for the sign appropriately, we may assume $t > 0$ in the argument which follows.

Now, this bound is an application of Proposition 2.30. Set $\varepsilon := \pi/4$ and assume that $t > \max\{|a|, |b|\}$ throughout so that $\arg s \in (\pi/4, 3\pi/4)$. Thus, noting $|s| \geq t$, we get the estimate

$$\begin{aligned} \log \Gamma(\sigma + it) &= \left(\sigma + it - \frac{1}{2} \right) \log(\sigma + it) - (\sigma + it) + \frac{1}{2} \log 2\pi + O(1/t) \\ &= \left(\sigma - \frac{1}{2} \right) \log t + it \log \left(\frac{\sigma}{t} + i \right) - \sigma + \frac{1}{2} \log 2\pi \\ &\quad + \left(\sigma - \frac{1}{2} \right) \log \left(\frac{\sigma}{t} + i \right) + it(\log t - 1) + O(1/t). \end{aligned}$$

Because we want $|\Gamma(\sigma + it)| = \exp(\operatorname{Re} \log \Gamma(\sigma + it))$, we are primarily interested in the real part of the above expression. Notably, $\log \left(\frac{\sigma}{t} + i \right) \rightarrow \log i = \pi i/2$ as $t \rightarrow \infty$, so this term contributes no real part. Similarly, $it(\log t - 1)$ is purely imaginary and doesn't matter.

The hardest term left to understand is $it \log \left(\frac{\sigma}{t} + i \right)$. Well, set $x := 1/t$ and $f(x) := \log(\sigma x + i)$, and we want to understand the behavior of f around $x = 0$. Notably, for suitably chosen \log , we are holomorphic at $x = 0$ with $f(0) = \log i = i\pi/2$ and $f'(0) = \frac{\sigma}{\sigma \cdot 0 + i} = -\sigma i$. Thus,

$$\lim_{t \rightarrow \infty} \left(it \log \left(\frac{\sigma}{t} + i \right) - \sigma + \frac{\pi}{2} t \right) = \lim_{x \rightarrow 0} \frac{if(x) + \frac{\pi}{2} - \sigma x}{x} = 0,$$

so in total,

$$\lim_{t \rightarrow \infty} \log \left| \frac{\Gamma(\sigma + it)}{\sqrt{2\pi} e^{-\pi t/2} t^{\sigma-1/2}} \right| = \lim_{t \rightarrow \infty} \left(\left(\sigma - \frac{1}{2} \right) \log t - \log t^{\sigma-1/2} + it \log \left(\frac{\sigma}{t} + i \right) - \sigma + \frac{\pi}{2} t \right) = 0.$$

Taking exp of both sides completes the proof. ■

2.2.4 The Functional Equation

We now return to discussing the functional equation for ζ . Being concrete, we will want to fix a particular Schwarz function $f: \mathbb{R} \rightarrow \mathbb{R}$. Staring at Corollary 2.11, we see that it will be helpful to have control over both f and $\mathcal{F}f$, so we will take $f(x) := e^{-\pi x^2}$, even this of course does not satisfy all the hypotheses. The associated function S_f has a name.

Definition 2.32 (Γ). Define the function $\Theta: (0, \infty) \rightarrow \mathbb{R}$ by

$$\Theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}.$$

Remark 2.33. Note that series defining Θ converges absolutely and uniformly on any interval $[\varepsilon, 0)$ for $\varepsilon > 0$ by the Weierstrass M -test: indeed, we may upper-bound

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \leq 1 + 2 \sum_{n=1}^{\infty} e^{-\pi \varepsilon n^2} \leq 1 + 2 \sum_{n=1}^{\infty} e^{-\pi \varepsilon n} = 1 + 2 \cdot \frac{e^{-\pi \varepsilon}}{1 - e^{-\pi \varepsilon}} < \infty.$$

In particular, the uniform convergence confirms that Θ defines a continuous function $(0, \infty) \rightarrow \mathbb{R}$.

The functional equation for ζ will come from the following functional equation for Θ .

Proposition 2.34. For any $t > 0$, we have

$$\Theta(t) = \frac{1}{\sqrt{t}} \Theta\left(\frac{1}{t}\right).$$

Proof. For fixed $t > 0$, define $f(x) := e^{-\pi(\sqrt{t}x)^2}$, which is Schwarz by Lemma C.6 and Exercise C.7; in particular, $(\mathcal{F}f)(s) = \frac{1}{\sqrt{t}} f(s)$. Thus, Theorem 2.8 implies that $\Theta(t)$ converges absolutely, and

$$\Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n) = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{t}} e^{-\pi n^2 / t} = \frac{1}{\sqrt{t}} \Theta\left(\frac{1}{t}\right),$$

which is what we wanted. ■

Remark 2.35. For $z \in \mathbb{H}$, set $q := e^{2\pi iz}$ so that $|q| < 1$. Then

$$f(z) := \sum_{n \in \mathbb{Z}} q^{n^2/2} = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$$

converges absolutely and satisfies $f(it) = \Theta(t)$. One can check, for example using Proposition A.18, that f is holomorphic on \mathbb{H} . In fact, f is a modular form: note that $f(z+2) = f(z)$. Further, hitting Proposition 2.34 with the uniqueness of analytic continuation, we see that $f(-1/z) = (z/i)^{1/2} f(z)$ for $z \in \mathbb{H}$, for suitably defined square root. Thus, (with a growth condition we haven't mentioned) f is a modular form of weight $1/2$ and level

$$\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

We next describe how ζ relates Θ . This requires us to “complete” ζ , as follows.

Definition 2.36 (ξ). For $\operatorname{Re} s > 0$, we define

$$\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

Remark 2.37. Note ξ is meromorphic on $\{s : \operatorname{Re} s > 0\}$ with only a simple pole at $s = 1$ because $s \mapsto \pi^{-s/2} \Gamma(s/2)$ is holomorphic here (see Remark 2.15), and ζ is meromorphic with only a simple pole at $s = 1$ (see Proposition 1.36).

Remark 2.38. In some sense, we want to write

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Here, $\pi^{-s/2} \Gamma(s/2)$ is an “archimedean local factor” corresponding to the infinite place ∞ of \mathbb{Q} , and each of the $(1 - p^{-s})^{-1}$ are “nonarchimedean local factors.” Roughly speaking, the rigorization of this intuition is Tate’s thesis [Tat10].

Now here is how Θ enters the picture.

Lemma 2.39. For $\operatorname{Re} s > 1$, we have

$$\xi(s) = \int_0^\infty \left(\frac{\Theta(t) - 1}{2} \right) t^{s/2} \frac{dt}{t}.$$

Proof. This argument is similar to Corollary 2.11. Note

$$\frac{\Theta(t) - 1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 t}$$

for any $t > 0$, so we are looking at

$$\int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t} t^{(s-2)/2} \right) dt.$$

We would like to switch the sum and integral, so we check for absolute convergence. Well, to check absolute convergence, it’s enough to check after we exchange the integral and sum, so we compute

$$\begin{aligned} \sum_{n=1}^\infty \left(\int_0^\infty \left| e^{-\pi n^2 t} t^{s/2} \cdot \frac{1}{t} \right| dt \right) &= \sum_{n=1}^\infty \left(\int_0^\infty e^{-\pi n^2 t} t^{\operatorname{Re} s/2} \frac{dt}{t} \right) \\ &= \sum_{n=1}^\infty \left(\int_0^\infty e^{-t} \left(\frac{t}{\pi n^2} \right)^{\operatorname{Re} s/2} \frac{dt}{t} \right) \\ &= \pi^{-\operatorname{Re} s/2} \sum_{n=1}^\infty \left(\frac{1}{n^{\operatorname{Re} s}} \int_0^\infty e^{-\pi t} t^{\operatorname{Re} s/2} \frac{dt}{t} \right) \\ &= \pi^{-\operatorname{Re} s/2} \zeta(\operatorname{Re} s) \Gamma(\operatorname{Re} s/2). \end{aligned}$$

Now, $\operatorname{Re} s > 1$, so all terms are finite, so we have absolute convergence. Thus, our integral converges absolutely, so we can exchange the integral and sum. Repeating the above equalities but removing the absolute value signs (and hence removing $\operatorname{Re} s$ with just s everywhere) shows

$$\int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t} t^{(s-2)/2} \right) dt = \sum_{n=1}^\infty \left(\int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} \right) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \xi(s)$$

for $\operatorname{Re} s > 1$, which is what we wanted. ■

We are now ready to prove our functional equation.

Theorem 2.40 (Functional equation for ξ). The function ξ has a meromorphic continuation to all \mathbb{C} , with only simple poles at $s = 1$ and $s = 0$ of residue 1 and -1 , respectively. In fact, ξ satisfies the equation

$$\xi(s) = \xi(1-s)$$

for $s \in \mathbb{C} \setminus \{0, 1\}$.

Proof. We combine Proposition 2.34 with Lemma 2.39. We proceed in steps.

1. The integral in Lemma 2.39 is poorly behaved for $\operatorname{Re} s < 1$ because of the integral over $t \in (0, 1)$, so we define

$$I(s) := \int_1^\infty \left(\frac{\Theta(t) - 1}{2} \right) t^{s/2} \frac{dt}{t}.$$

We claim that $I(s)$ defines an entire function; more precisely, we will show that $I(s)$ defines holomorphic function on $\{s : \operatorname{Re} s > \sigma\}$ for any $\sigma \in \mathbb{R}$, and taking the union over all σ will finish.

We use Proposition A.18. For one, the function $t \mapsto \left(\frac{\Theta(t) - 1}{2} \right) t^{s/2}$ is continuous (recall Θ is continuous by Remark 2.33) and hence measurable. Lastly, we must upper-bound

$$\begin{aligned} \int_1^\infty \left| \left(\frac{\Theta(t) - 1}{2} \right) t^{s/2} \right| \frac{dt}{t} &= \frac{1}{2} \int_1^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} \\ &\leq \frac{1}{2} \int_1^\infty \left(\sum_{n=1}^\infty e^{-\pi n t} \right) t^{s/2} \frac{dt}{t} \\ &= \frac{1}{2} \int_1^\infty \frac{e^{-\pi t} t^{s/2}}{1 - e^{-\pi t}} \frac{dt}{t} \\ &\leq \frac{1}{2(1 - e^{-\pi})} \int_1^\infty e^{-\pi t} t^{s/2} \frac{dt}{t}. \end{aligned}$$

Thus, we take $g: (1, \infty) \rightarrow \mathbb{C}$ by $g(t) := e^{-\pi t} t^{s/2-1} / (2(1 - e^{-\pi}))$. Using Proposition A.18, it remains to show that $\int_1^\infty g(t) dt < \infty$. Well, $e^{-\pi t} t^{s/2+1} \rightarrow 0$ as $t \rightarrow \infty$, so this function achieves a maximum on $[1, \infty)$,¹ which we will call M . It follows

$$\int_1^\infty g(t) dt \leq \frac{1}{2(1 - e^{-\pi})} \int_1^\infty \frac{M}{t} \frac{dt}{t} < \infty.$$

2. Having controlled the $(1, \infty)$ part of the integral in Lemma 2.39, we turn to the $(0, 1)$ part. The idea here is to use Proposition 2.34 to transform the $(0, 1)$ part back into a well-behaved $(1, \infty)$ part. Indeed, for $\operatorname{Re} s > 1$, we may evaluate

$$\begin{aligned} \int_0^1 \left(\frac{\Theta(t) - 1}{2} \right) t^{s/2} \frac{dt}{t} &= \int_1^\infty \left(\frac{\Theta(1/t) - 1}{2} \right) t^{-s/2} \frac{dt}{t} \\ &= \int_1^\infty \left(\frac{\sqrt{t}\Theta(t) - 1}{2} \right) t^{-s/2} \frac{dt}{t} \\ &= \int_1^\infty \left(\frac{\sqrt{t}\Theta(t) - \sqrt{t}}{2} \right) t^{-s/2} \frac{dt}{t} - \frac{1}{2} \int_1^\infty t^{-s/2} \frac{dt}{t} + \frac{1}{2} \int_1^\infty t^{(1-s)/2} \frac{dt}{t} \\ &= \int_1^\infty \left(\frac{\Theta(t) - 1}{2} \right) t^{(1-s)/2} \frac{dt}{t} - \int_1^\infty t^{-s} \frac{dt}{t} + \int_1^\infty t^{1-s} \frac{dt}{t} \\ &= I(1-s) - \frac{1}{s} - \frac{1}{1-s}. \end{aligned}$$

¹ Find N such that $g(t) < g(1)$ for $t < N$. Then the maximum of g is its maximum on $[1, N]$.

3. Synthesizing the previous steps, Lemma 2.39 grants

$$\xi(s) = I(s) + I(1-s) - \frac{1}{s} - \frac{1}{1-s}$$

on $\operatorname{Re} s > 1$. However, $I(s)$ is fully entire, so the right-hand side is a meromorphic function on \mathbb{C} with simple poles at $s = 1$ (of residue $\operatorname{Res}_{s=1} -\frac{1}{1-s} = \operatorname{Res}_{s=1} \frac{1}{s-1} = 1$) and at $s = 0$ (of residue $\operatorname{Res}_{s=1} -\frac{1}{s} = -1$). Viewing the right-hand side as our continuation of ξ completes the analysis of ξ . Lastly, the above equation tells us that

$$\xi(s) = \xi(1-s)$$

for $s \in \mathbb{C} \setminus \{0, 1\}$, which completes the proof. ■

2.2.5 Corollaries of the Functional Equation

We quickly establish the following more asymmetric version of the functional equation.

Corollary 2.41 (Functional equation for ζ). The function ζ has a meromorphic continuation to \mathbb{C} with only a simple pole at $s = 1$ of residue 1. In fact, for $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s).$$

Proof. We begin by discussing the meromorphic continuation of ζ . Note Theorem 2.40 lets us continue ζ by writing

$$\zeta(s) = \frac{\xi(s)}{\pi^{-s/2} \Gamma(s/2)}$$

for any $s \in \mathbb{C} \setminus \{0, 1\}$. Notably, the denominator is never nonzero, and even though $\Gamma(s/2)$ has a simple pole at nonpositive even integers $-2n$ by Corollary 2.23 at these points $\xi(s)$ will have at worst simple pole by Theorem 2.40 as well, so we can just multiply the numerator and denominator by $(s - 2n)$ until the denominator is nonzero.

It remains to deal with $s \in \{0, 1\}$. At $s = 0$, we write

$$\zeta(s) = \frac{s\xi(s)}{\pi^{-s/2} \cdot s\Gamma(s/2)}$$

so that we have written $\zeta(s)$ as the quotient of holomorphic functions nonzero at $s = 0$. (Note $s \cdot \Gamma(s/2)$ has no pole and is nonzero at $s = 0$ by Corollary 2.23.) However, at $s = 1$, we already know that ζ has a simple pole of residue 1 by Proposition 1.36.

To finish the proof, we must produce the functional equation. By uniqueness of the functional equation, it suffices to focus on $0 < \operatorname{Re} s, 1$. Here, Theorem 2.40 grants

$$\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s) = \xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s). \quad (2.2)$$

Multiplying both sides by $\Gamma\left(\frac{1+s}{2}\right)$, we see Proposition 2.21 implies

$$\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\sin\left(\pi \cdot \frac{1+s}{2}\right)} = \frac{\pi}{\cos(\pi s/2)}.$$

On the other hand, Proposition 2.25 implies

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) = \sqrt{\pi} 2^{1-s} \Gamma(s).$$

In total, we may rearrange (2.2) into

$$\frac{\pi^{-(1-s)/2+1} \zeta(1-s)}{\cos(\pi s/2)} = \pi^{1/2-s/2} \cdot 2^{1-s} \zeta(s),$$

which rearranges into the desired equation. ■

Example 2.42. Note

$$\zeta(0) = \lim_{s \rightarrow 1} \zeta(1-s) = \lim_{s \rightarrow 1} \left(2(2\pi)^{-s} \cdot \frac{\cos(\pi s/2)}{s-1} \cdot \Gamma(s) \cdot (s-1)\zeta(s) \right).$$

Using L'Hôpital's rule, we see $\cos(\pi s/2)/(s-1) \rightarrow -\pi/2$ as $s \rightarrow 1$. By Proposition 1.36, we see $(s-1)\zeta(s) \rightarrow 1$ as $s \rightarrow 1$. Plugging everything else in, we see $\zeta(0) = 2 \cdot (2\pi)^{-1} \cdot (\pi/2) \cdot 1 \cdot 1 = -1/2$.

This functional equation grants us some basic knowledge about the zeroes of ζ .

Corollary 2.43. We have the following.

- (a) Conjugate symmetry: if $\zeta(s) = 0$, then $\zeta(\bar{s}) = 0$.
- (b) Trivial zeroes: the function ζ is zero at $-2n$ for each positive integer n .
- (c) Critical strip: if $\zeta(s) = 0$ and $-s/2 \notin \mathbb{N}$, then $0 \leq \operatorname{Re} s \leq 1$.
- (d) Horizontal symmetry: if $\zeta(s) = 0$ and $0 \leq \operatorname{Re} s \leq 1$, then $\zeta(1-s) = 0$.

Proof. Here we go.

- (a) More generally, we claim that $\overline{\zeta(\bar{s})} = \zeta(s)$ for $s \in \mathbb{C} \setminus \{1\}$, from which the claim will follow.

Because ζ is holomorphic in this region, we see that $s \mapsto \overline{\zeta(\bar{s})}$ is also holomorphic. (Formally, we can just check that $\zeta(x+yi) = u(x+yi) + iv(x+yi)$ satisfying the Cauchy–Riemann equations implies that $u(x-yi) - iv(x-yi)$ does as well.) So by the uniqueness of analytic continuation, it suffices to check the result for $s \in \mathbb{R}_{>1}$, which is clear because ζ is real on this line, so

$$\zeta(s) = \zeta(\bar{s}) = \overline{\zeta(\bar{s})}.$$

- (b) For any positive integer n , write

$$\zeta(s) = \frac{(s+2n)\xi(s)}{\pi^{-s/2} \cdot (s+2n)\Gamma(s/2)}.$$

As $s \rightarrow -2n$, the numerator vanishes because ξ is holomorphic at $s = -2n$ by Theorem 2.40. However, the denominator is finite and nonzero: $\pi^{-s/2}$ vanishes nowhere, and $\Gamma(s/2)$ has a simple pole at $s = -2n$ by Corollary 2.23 which is cancelled by the factor of $(s+2n)$. In total, we conclude

$$\zeta(-2n) = \lim_{s \rightarrow -2n} \zeta(s) = 0.$$

- (c) If $\operatorname{Re} s > 1$, then $\zeta(s) \neq 0$ by Corollary 1.41 already. Thus, it remains to discuss $\operatorname{Re} s < 0$. Well, for $\operatorname{Re} s > 1$, we see

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s),$$

and for $\operatorname{Re} s > 1$ this right-hand side will only vanish when $\cos(\pi s/2) = 0$, which is equivalent to $s \in 2\mathbb{Z}_{\geq 0} + 1$. (Namely, $\Gamma(s)$ never vanishes by Corollary 2.23, and $\zeta(s)$ does not vanish in this region as just discussed.) Unwinding, we see that $\zeta(s) = 0$ and $\operatorname{Re} s < 0$ implies that $1-s \in 2\mathbb{Z}_{\geq 0} + 1$, which is equivalent to $s \in -2\mathbb{Z}_{\leq 0}$. This is what we wanted.

- (d) Note that $\zeta(1)$ isn't defined, and $\zeta(0) \neq 0$ by Example 2.42, so we may safely ignore $s \in \{0, 1\}$. Otherwise, we stare at

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s),$$

which is valid on $\{s : 0 \leq \operatorname{Re} s \leq 1\} \setminus \{0, 1\}$. (In particular, Γ is holomorphic here by Corollary 2.23.) Thus, $\zeta(s) = 0$ implies $\zeta(1-s) = 0$. ■

2.3 January 30

Today we began by completing the proof of the functional equation. I have directly edited into last class's notes for continuity reasons.

2.3.1 Zeroes of ζ

We would like to understand the zeroes of $\zeta(s)$, for which we use Cauchy's formula. Roughly speaking, we will study integrals

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{\zeta'(s)}{\zeta(s)} ds,$$

where γ is a contour over a very tall vertical strip in \mathbb{C} .

We will want the following bounds.

Lemma 2.44 (Stirling). We have

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s + \frac{1}{2} \log 2\pi + O_{\delta}(1/|s|),$$

and

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O_{\delta}(1/|s|)$$

as $|s| \rightarrow \infty$ for $-\pi + \delta < \arg s < \pi - \delta$.

Proof. Omitted. ■

Remark 2.45. For $\sigma \in [a, b]$ for some fixed $a < b$, we have $\Gamma(\sigma + it) \sim e^{-\pi|t|/2} |t|^{\sigma-1/2}$ as $\rightarrow \infty$. Again, we omit this proof.

We also want access to the Hadamard factorization theorem.

Theorem 2.46. Fix an entire function $\varphi: \mathbb{C} \rightarrow \mathbb{C}$. Let $n(r)$ denote the number of zeroes z such that $|z| < r$ and α the order of an entire function φ , one has $n(R) = O_{\varepsilon}(R^{\alpha+\varepsilon})$ for any $\varepsilon > 0$.

Notably, for $\operatorname{Re} s > 1/2$, one has $|s(s-1)\zeta(s)| \ll |s|^3$. Further, one can check that Γ has order 1 as an entire function, so $s(1-s)\xi(s)$ has order at most 1. Thus, Hadamard's factorization theorem enforces

$$s(1-s)\xi(s) = e^{A+Bs} \prod_{\zeta(\rho)=0} \left(\left(1 - \frac{s}{\rho}\right) e^{s/\rho} \right).$$

Notably, this product will converge absolutely. For example, absolute convergence tells us

$$\sum_{\zeta(\rho)=0} \frac{1}{|\rho|^{1+\varepsilon}} < \infty$$

for any $\varepsilon > 0$. One also has the following result on the distribution of our ρ .

Theorem 2.47. Define

$$N(T) := \#\{\rho : 0 \leq \operatorname{Re} \rho \leq 1, \operatorname{Im} \rho \geq 0, \zeta(\rho) = 0\}.$$

Then

$$N(T) = \frac{T}{2\pi} \log \left(\frac{T}{2\pi} \right) - \frac{T}{2\pi} + O(\log T)$$

as $T \rightarrow \infty$.

We will first show the following lemma.

Lemma 2.48. We have

$$\sum_{\rho} \frac{1}{1 + |\operatorname{Im} \rho - T|^2} \ll \log(T + 3).$$

Proof. This is by smoothing. By taking logarithmic differentiation

$$\frac{\Xi'(s)}{\Xi(s)} = B + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

However, by using the above estimates, we see

$$\frac{\Xi'(s)}{\Xi(s)} = \frac{1}{s} - \frac{1}{1-s} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}$$

by definition of ξ . Now, the term ζ'/ζ is well-behaved for $\operatorname{Re} s$ large: we set $s := 2 + it$, and one can see that $|\zeta'(s)/\zeta(s)|$ is bounded by an absolute constant. Thus, we understand on the right-hand side here.

Continuing, we see

$$\operatorname{Re} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right) = \frac{2 - \operatorname{Im} s}{(2 - \operatorname{Im} s)^2 + (T - \operatorname{Im} s)^2} + \frac{\beta}{(\operatorname{Re} s + \operatorname{Im} s)^2} \gg \frac{1}{1 + |T - \operatorname{Im} s|^2}.$$

However, $\Gamma'(s)/\Gamma(s) \ll \log(T + 3)$ by Stirling, so the result follows. ■

2.4 February 1

Today we move towards a proof of the explicit formula.

Notation 2.49. A sum/product over ρ is over the zeroes of $\zeta(s)$.

2.4.1 Zeroes of ζ , Again

Let's provide a few applications of Lemma 2.48.

Corollary 2.50. We have

$$\#\{\rho : \zeta(\rho) = 0, \operatorname{Im} \rho \in [T, T + 1], \operatorname{Re} \rho \in (0, 1)\} = O(\log T).$$

Proof. This follows from Lemma 2.48 by separating out our zeroes into intervals. ■

We will be interested in contours γ_T which look like large vertical rectangles; namely, they are the boundary of the rectangle $[-\varepsilon, 1 + \varepsilon] \times [-T, T]$. Notably, the top and bottom of the rectangle's contours will cancel out by the functional equation, so we only need to pay attention to the vertical parts of this contour.

Lemma 2.51. For $t > 3$, we have

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{s - \rho} + O(\log t)$$

for $\operatorname{Re} s \in [-1, 2]$.

Proof. We consider

$$\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(2+it)}{\zeta(2+it)}.$$

Thus, we recall

$$\frac{\Xi'(s)}{\Xi(s)} = \frac{1}{s} - \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)},$$

so we can just bound everything. Notably, we can use the infinite product to bound Ξ'/Ξ and then compare everything. For example,

$$\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(2+it)}{\zeta(2+it)} = \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log t),$$

where the $O(\log t)$ includes the trivial zeroes of ζ . Now, we notice

$$\left| \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right| \ll \frac{1}{|\operatorname{Im} \rho - t|^2}$$

for $|\operatorname{Im} \rho - t| \geq 1$, so we can use Lemma 2.48 to absorb most terms into $O(\log t)$. In total, we see

$$\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(2+it)}{\zeta(2+it)} = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{s-\rho} + O(\log t),$$

and now the $2+it$ term can also be absorbed to $O(\log t)$. ■

Remark 2.52. One can give more accurate bounding than the above, but we will not need it.

We now return to the proof of Theorem 2.47.

Proof of Theorem 2.47. Use the argument principle on Ξ on the box $[-1, 2] \times [-T, T]$. In particular, by the functional equation, it suffices to just look at the right and top edges of this box. The hope is that we can use Lemma 2.51 and the ideas in its proof to do the bounding for us. In particular, we will be working with the equation

$$\frac{\Xi'(s)}{\Xi(s)} = \frac{1}{s} - \frac{1}{1-s} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}.$$

Now, the main term in the argument will come from $\Gamma'(s)/\Gamma(s)$, which one can see using Stirling's asymptotics. Most of these terms are not going to matter on our contour. It turns out that the only difficulty is integrating ζ'/ζ over the line $\{a+Ti : a \in [-1, 2]\}$. Well, using the above estimates, we recall

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\operatorname{Im} \rho - T| \leq 1} \frac{1}{s-\rho} + O(\log T),$$

where now the integral of the $1/(s-\rho)$ term is bounded by a constant, and the number of terms is $O(\log T)$ by Corollary 2.50, so everything is absorbed into the error term. ■

2.4.2 The Explicit Formula

Let's move towards the explicit formula. Here is our statement.

Theorem 2.53. When x is not a prime-power, we have

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1-x^{-2}).$$

Remark 2.54. Ignoring convergence issues, we may compute

$$\psi(x) = \sum_{n \leq x} \Gamma(n) = \sum_{n=1}^{\infty} 1_{[0,1]}(n/x) \Gamma(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) x^s \frac{ds}{s}.$$

Now, if we imagine that we could push this integral all the way to the left of \mathbb{C} , we will eventually vanish and only pick up on the poles of ζ'/ζ . As such, we expect to achieve a formula of the form

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho},$$

where the sum is over the roots ρ of ζ . Thus, we see that having more control over the zeroes of ζ will be able to get good bounds on $\psi(x) - x$. In particular, the Riemann hypothesis is equivalent to $\psi(x) = x + O(\sqrt{x})$. As another application, the discontinuity of ψ will imply that ζ must have infinitely many roots.

Here is a lemma.

Lemma 2.55. We have

$$\psi(x) - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} x^s ds = O(x(\log x)^2/T).$$

Proof. We first describe a heuristic. The main idea is to use contour integration, noting that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } y < 1, \\ 1 & \text{if } y > 1, \end{cases} \quad (2.3)$$

where $c > 1$ and $y \in (0, \infty) \setminus \{1\}$. The proof of this is essentially complex analysis where we “complete the contour” of this vertical line either off to $-\infty$ or off to $+\infty$ depending on $y < 1$ or $y > 1$.

Now, the point is that we can write

$$\psi(x) = \sum_{n \leq x} \Lambda(n) \approx \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} x^s \frac{ds}{s},$$

where we have ignored convergence issues to exchange the Dirichlet series for ζ'/ζ with this integral. The point now is that we can integrate ζ'/ζ appropriately to give the formula.

To make this more rigorous, we need to do only finite computations. Thus, we define

$$I(y, T) := \frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \frac{ds}{s}.$$

We now note that our extra variable c will be later set to $1 + 1/\log T$, so it is important to have this degree of freedom. Now, the proof of (2.3) grants

$$|I(y, T) - 1_{>1}(y)| \ll y^c \min\{1, 1/|T \log y|\},$$

where the implied constant is absolute. Integrating over this, we see

$$\left| \psi(x) - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} x^s ds \right| \leq \left(\sum_{n=1}^{\infty} \Lambda(n) (x/n)^c \min \left\{ 1, \frac{1}{|T \log(x/n)|} \right\} \right).$$

Now, setting $c = 1 + 1/\log T$, we get an upper-bound of $O(x(\log x)^2/T)$. Roughly speaking, we note that x away from particular n are small; however, when n is close to x , we can explicitly evaluate the logarithm as about $1/(n - x)$, and there the sum is roughly harmonic and thus grants a logarithmic growth. ■

2.5 February 3

Last time we were in the middle of the proof of the explicit formula. I have edited directly into yesterday's notes for continuity reasons.

2.5.1 The Explicit Formula, Continued

We now do a contour shift on the integral

$$\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} x^s ds.$$

Roughly speaking, we will expand our box to look like $[-U, c] \times [-T, T]$, sending $U \rightarrow \infty$ for fixed T . We will then send $T \rightarrow \infty$, always remembering to choose T avoiding zeroes of $\zeta(s)$. In particular, by Corollary 2.50, we can be at least $1/\log T$ away from any particular zeroes. We will finish the proof next lecture.

On this contour, the point is that $|x^s| \leq x^{\operatorname{Re} s}$, so for most of this contour, we don't have to care. For example, it will be enough to only care about $\operatorname{Re} s > -1$. By the functional equation, it's enough to just look at the integral from $c + iT$ to $-1 + iT$. To bound the size here, we change T so that $\operatorname{Im} s = T$ is at most $\gg 1/\log T$ away from zeroes. Now, to bound, we see

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\operatorname{Im} \rho - T| \leq 1} \underbrace{\frac{1}{s - \rho}}_{O(\log T)} + O(\log T)$$

for, say, $-1 \leq \sigma \leq 2$. Thus, the contribution of the integral over $-1 \leq \sigma \leq c$ is given by

$$O\left((\log T)^2 \int_{-1}^c \frac{x^{\sigma+iT}}{|\sigma+iT|} d\sigma\right) = O\left(\frac{x(\log T)^2}{T}\right),$$

so this also goes to 0 as $T \rightarrow \infty$.

Remark 2.56. It is helpful for computations to have the functional equation

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\pi s/2) \Gamma(s) \zeta(s),$$

where we have notably used the reflection formula for Γ .

As such, the value of ζ'/ζ on $\{s : \operatorname{Re} s \leq -1\}$ is bounded by

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \ll \log |s| + 1,$$

where s avoids circle of radius $1/2$ around the zeroes (including the trivial ones). (Note because the trivial zeroes only occur at the negative even integers, we can indeed choose U at odd integers to be okay here.)

We now report the bounds on the other parts of the contour, for completeness. Indeed, the entire contribution for $\operatorname{Re} s \leq -1$ is given by

$$O\left(\frac{x(\log T)^2}{T}\right),$$

where U is a very large odd positive integer. Thus, we use residue calculus to see

$$\psi(x) = x - \sum_{|\operatorname{Im} s| \leq T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-s}) + O\left(\frac{x \log(xT)^2}{T}\right),$$

where x is not a prime power. Note the contributions of $-\frac{1}{2} \log(1 - x^{-s})$ are coming from the trivial zeroes of ζ . This completes the proof.

2.5.2 A Zero-Free Region

We are going to construct a zero-free region slightly to the left of (and including) $\operatorname{Re} s = 1$. In some sense, the explicit formula tells us that the Prime number theorem is equivalent to requiring $\zeta(1 + it) \neq 0$ for $t \in \mathbb{R}$, where the point of the zero-free region is to control the nontrivial zeroes in the explicit formula.

We are going to use positivity to create our zero-free region. We begin with a slick but weak proof.

Proposition 2.57. Fix some $t_0 \in \mathbb{R}$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Defining $\sigma_z(n) := \sum_{d|n} d^z$, we have

$$\sum_{n=1}^{\infty} \frac{|\sigma_{it_0}(n)|^2}{n^s} = \frac{\zeta(s)^2 \zeta(s + it_0) \zeta(s - it_0)}{\zeta(2s)}$$

Proof. Direct expansion with Euler factors. ■

The point is that we can provide a meromorphic continuation of this function to $s \in \mathbb{C}$, whose power we can plug into the following result.

Lemma 2.58 (Landau). Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers, and define

$$D(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Further, let $\sigma_0 \in \mathbb{R}$ be the smallest real numbers such that D absolutely converges on $\operatorname{Re} s > \sigma_0$. Then D does not extend to an analytic function past σ_0 .

Proof. This is just complex analysis, so we omit it. ■

Thus, if we can find t_0 such that $\zeta(1 + it_0) = 0$, then we also have a zero at $\zeta(1 - it_0)$, so in fact the function

$$\frac{\zeta(s)^2 \zeta(s + it_0) \zeta(s - it_0)}{\zeta(2s)}$$

is analytic on $\operatorname{Re} s > 1/2$ and is zero at $1/2$. But this is an obvious contradiction because the series must absolutely converge by Lemma 2.58, but we cannot vanish at $s = 1/2$ by just staring at it. Thus, we could not actually have continued it any further.

Remark 2.59. Essentially the same proof can show that $L(s, \chi)\zeta(s)$ does not vanish at $s = 0$, provided we give $L(s, \chi)$ an analytic continuation. We will do this later.

2.6 February 6

Today we construct our zero-free region for ζ .

2.6.1 A General Lemma

The above zero-free region is technically enough to prove the Prime number theorem, but to get an error term, we will want to do better. As such, we pick up the following lemma.

Lemma 2.60. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers, and define

$$D(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Suppose D satisfies the following conditions.

- $D(s)$ converges absolutely on $\operatorname{Re} s > 1$.
- $D(s)$ has a pole of order $m > 0$ at $s = 1$.
- We can define

$$\Xi(s) := (s(1-s))^m D^s \left(\prod_{i=1}^{\ell} \Gamma_{\mathbb{R}}(s + \alpha_i) \right) D(s)$$

for some D and m . Here, $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$. Then $\Xi(s)$ is entire, order 1, and satisfies $\Xi(s) = \Xi(1-s)$.

Then D has at most m zeroes in $(1 - c(\ell, m)/\log M, 1) \subseteq \mathbb{R}$, where $M := (D+1) \prod_{i=1}^{\ell} (|\alpha_i| + 1)$ is the “analytic conductor,” and $c(\ell, m)$ is a constant computable from ℓ and m .

Remark 2.61. More complex L -functions might have “complex” Γ -factors $\Gamma_{\mathbb{C}}$. Roughly speaking, such factors arise from taking the Mellin transform of a Gaussian, so over \mathbb{R} we get $\Gamma_{\mathbb{R}}$, but over \mathbb{C} we will get something a little different. For details, see [Tat10].

It might look concerning that Lemma 2.60 only gives us an interval in the real numbers, but we can more carefully select our $D(s)$ to get a more comprehensive region.

Example 2.62. Given some $t_0 \in \mathbb{R}$, set

$$D(s) := \zeta(s)^3 \zeta(s + it_0)^2 \zeta(s - it_0)^2 \zeta(s + 2it_0) \zeta(s - it_0).$$

Some trigonometry shows that $D(s)$ has nonnegative coefficients. Then one can use Lemma 2.60 on $D(s)$: note $m = 3$, $\ell = 9$, so $M \ll \log(|t_0| + 1)$, meaning that we have at most 3 zeroes in the interval $(1 - c/\log(|t_0| + 1), 1)$ for some computable constant c . However, given $\beta \in (1 - c/\log(|t_0| + 1), 1)$, if $\zeta(\beta + it_0) = 0$, then $\zeta(\beta - it_0) = 0$ as well, so D actually gets four zeroes, which is contradiction. In particular, we get a zero-free region which looks like

$$\left\{ s = \sigma + it : \sigma > 1 - \frac{c}{\log(|t| + 1)} \right\}.$$

Remark 2.63. The magical $D(s)$ from Example 2.62 does come from a larger structure, but it is somewhat advanced to explain.

Remark 2.64. It might look upsetting that Example 2.62 does not achieve a full zero-free region of the form $\{s : \operatorname{Re} s > c\}$ for some $c > 0$, but the proof of such a region is not known.

Now that we care about Lemma 2.60, let’s prove it.

Proof of Lemma 2.60. Note $\Xi(0) \neq 0$ by the functional equation. Now, factoring, we see

$$\Xi(s) = e^{A+Bs} \prod_{\Xi(\rho)=0} \left(1 - \frac{s}{\rho} \right) e^{s/\rho}.$$

Taking the logarithmic derivative, we see

$$\frac{\Xi'(s)}{\Xi(s)} = B + \sum_{\Xi(\rho)=0} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

As an intermediate step, we claim

$$\operatorname{Re} B = - \sum_{\Xi(\rho)=0} \operatorname{Re}(1/\rho).$$

Note that summing over these zeroes in conjugate pairs will give us absolute convergence. Anyway, this follows from the functional equation: note $\Xi(s) = \Xi(1-s)$ grants

$$B + \sum_{\Xi(\rho)=0} \left(\frac{1}{1-s-\rho} + \frac{1}{\rho} \right) = -B - \sum_{\Xi(\rho)=0} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

The zeroes are symmetric by the functional equation, so the contribution given by $1/(1-s-\rho) = -1/(s-(1-\rho))$ and $1/(s-\rho)$ will cancel. This completes the proof of the claim.

As such, the product definition of Ξ allows us to expand

$$\sum_{\Xi(\rho)=0} \frac{1}{s-\rho} = \frac{m}{s} + \frac{m}{s-1} + \frac{G'(s)}{G(s)} + \frac{D'(s)}{D(s)},$$

where

$$G(s) := D^s \left(\prod_{j=1}^{\ell} \Gamma_{\mathbb{R}}(s + \alpha_j) \right).$$

Now, for $s > 1$, we know $D'(s)/D(s) < 0$ by hypothesis on $D(s)$. As such, we are now in good shape because we more or less understand everything on our right-hand side, so we can translate it into knowledge about Ξ . In particular, we can show

$$\sum_{\Xi(\rho)=0} \frac{1}{s-\rho} \leq \frac{m}{s-1} + \frac{m}{s} + C_1 \log M,$$

where C_1 is some constant depending on ℓ and m . In particular, if we send $s \rightarrow 1^+$ and have too many zeroes of Ξ close to 1, then the left-hand side should explode while the right-hand side grows slower.

As such, let

$$R_c := \{\rho \in (1 - c/\log M) : \Xi(\rho) = 0\},$$

where $c > 0$ is some constant we will fix later. We see

$$\sum_{\rho \in R_c} \frac{m}{s-\rho} \leq \frac{m}{s-1} + C_2 \log M$$

as $s \rightarrow 1^+$, for some perhaps different constant C_2 . As such, for some $\delta > 0$ we will fix later, we set $s := 1 + \delta/\log M$ so that

$$\sum_{\rho \in R_c} \frac{1}{c + \delta} \leq \frac{m}{\delta} + C_2$$

after cancelling out C_2 . But taking, say, $c < 1/(100mC_2)$ and $\delta < 1/2C_2$ will enforce $\#R_c < m + 1$, which is what we wanted. ■

2.6.2 The Prime Number Theorem, Finally

We are now ready to prove the Prime number theorem.

Theorem 2.5 (Prime number). We have $\pi(x) \sim x / \log x$ as $x \rightarrow \infty$.

Proof. We show $\psi(x) \sim x$, which is one of our equivalent formulations; we use the explicit formula. Namely, taking $T > 3$, we recall

$$\psi(x) = x - \sum_{|\operatorname{Im} \rho| \leq T} \frac{x^\rho}{\rho} + O(1) + O\left(\frac{x(\log T)^2}{T}\right).$$

However, we can upper-bound

$$\left| \sum_{|\operatorname{Im} \rho| \leq T} \frac{x^\rho}{\rho} \right| \leq \sum_{|\operatorname{Im} \rho| \leq T} \frac{x^\rho}{|\rho|} \leq x \sum_{|\operatorname{Im} \rho| \leq T} \frac{x^{-c/\log(|t|+1)}}{|\rho|} \ll x^{1-c/\log T} (\log T)^2.$$

Here, this last inequality follows from “dyadic decomposition.” Note that the number of terms is $\ll T \log T$; then decomposing \mathbb{R}^+ into $\bigcup_{k \in \mathbb{Z}} (2^k, 2^{k+1}]$, the number of roots here is $\ll 2^k k$, so we get that our sum is bounded by

$$\sum_{\substack{k \geq 0 \\ 2^k \leq T}} 2^{-k} \cdot 2^k k + O(1) \ll (\log T)^2.$$

Now, taking $T = e^{c\sqrt{\log x}}$ gets a bound $|\psi(x) - x| \ll x \exp(-c\sqrt{\log x})$, which is enough. ■

2.7 February 8

We began class by proving the Prime number theorem. I have edited directly into those notes for continuity.

2.7.1 Quadratic Residue Speedrun

Fix a prime p , for simplicity. Let $\chi \pmod{p}$ be a Dirichlet character. Our goal is to prove an analytic continuation and functional equation for $L(s, \chi)$.

Proposition 2.65. Fix a prime p . Then \mathbb{F}_p^\times is cyclic.

Proof. Standard. ■

Corollary 2.66. Fix a prime p and some $d \in \mathbb{Z}^+$.

- (a) If $\gcd(d, p-1) = 1$, then the function $\mathbb{F}_p^\times \rightarrow \mathbb{F}_p^\times$ defined by $x \mapsto x^d$ is an isomorphism.
- (b) If $d \mid p-1$, then each $a \in \mathbb{F}_p^\times$ makes $x^d \equiv a \pmod{p}$ with either 0 or d solutions.

Proof. Here we go.

- (a) Let g be a generator of \mathbb{F}_p^\times . Then g^d is also a generator and lives in the image of the map.
- (b) The map $x \mapsto x^d$ has kernel of size d . Thus, any element of the image has d solutions to the equation; otherwise there are no solutions. ■

This motivates the Legendre symbol.

Definition 2.67 (quadratic residue). Fix an odd prime p . And some $a \in \mathbb{Z}$ not divisible by p .

- If $x^2 \equiv a \pmod{p}$ has a solution, then a is a *quadratic residue*.
- If $x^2 \equiv a \pmod{p}$ does not have a solution, then a is a *nonquadratic residue*.

We will be silent about the case of $p \mid a$.

Definition 2.68. Fix an odd prime p and some $a \in \mathbb{Z}$. Then we define the *Legendre symbol* by

$$\left(\frac{a}{p}\right) := \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue,} \\ -1 & \text{if } a \text{ is a nonquadratic residue.} \end{cases}$$

Proposition 2.69 (Euler's criterion). Fix an odd prime p . For any $a \in \mathbb{Z}$, we have

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}.$$

Proof. If $a \equiv 0 \pmod{p}$, there is nothing to say. If $a \equiv x^2 \pmod{p}$ for some x , then $a^{(p-1)/2} \equiv x^{p-1} \equiv 1$. Lastly, if a is a nonquadratic residue, then we see

$$\left(a^{(p-1)/2} - 1\right) \left(a^{(p-1)/2} + 1\right) \equiv 0 \pmod{p},$$

but $x^{(p-1)/2} \equiv 1 \pmod{p}$ has already been granted $(p-1)/2$ roots as the quadratic residues, so we must instead have $a^{(p-1)/2} \equiv -1 \pmod{p}$. ■

Corollary 2.70. Fix an odd prime p . Then $\left(\frac{-1}{p}\right) = 1$ if $p \equiv 1 \pmod{4}$, and $\left(\frac{-1}{p}\right) = -1$ if $p \equiv -1 \pmod{4}$.

Proof. Use Proposition 2.69. ■

Remark 2.71. Note that $\left(\frac{\bullet}{p}\right)$ is the unique non-principal real character \pmod{p} . One can see this because the characters $\mathbb{F}_p^\times \rightarrow \{\pm 1\}$ must have order 2, so we are looking for elements of order 2 in $\widehat{\mathbb{F}_p^\times} \cong \mathbb{F}_p^\times$. But \mathbb{F}_p^\times has only one element of order 2, so we know we have found the only such real non-principal character.

Remark 2.72. More generally studying when $f(x) \equiv 0 \pmod{p}$ has solutions (and how it factors) has connections directly with the Langlands program and similar. We will not say more because this is (very) far outside the scope of the course.

2.7.2 Gauss Sums

We mentioned in Remark 2.14 that Γ is more or less a continuous version of a Gauss sum: it's some kind of multiplicative Fourier transform of an additive character. Well, here are the usual Gauss sums.

Definition 2.73 (Gauss sum). Fix a prime p and integer m . Then the Gauss sum is

$$\tau(\chi, m) := \sum_{n \in \mathbb{F}_p^\times} e\left(\frac{nm}{p}\right) \chi(n).$$

Namely, $\psi_m: n \mapsto e\left(\frac{nm}{p}\right)$ is our additive character, our measure is the counting measure, so we are indeed just looking at the multiplicative Fourier transform of an additive character.

Let's show a few basic facts.

Lemma 2.74. Fix a prime p and integer m . Then for a nontrivial character χ , we have

$$\tau(\chi, m) = \bar{\chi}(m) \tau(\chi, 1).$$

Proof. This is just a change of variables. ■

Lemma 2.75. Fix a prime p . Then $|\tau(\chi, 1)|^2 = p$.

Proof. This is essentially the Plancherel formula. Namely, we see

$$\begin{aligned} (p-1)|\tau(\chi, 1)|^2 &= \sum_{m=0}^{p-1} |\tau(\chi, m)|^2 \\ &= \sum_{k, \ell \in \mathbb{F}_p^\times} \left(\chi(k) \bar{\chi}(\ell) \sum_{m=0}^{p-1} e\left(\frac{m(k-\ell)}{p}\right) \right) \\ &= p \sum_{k \in \mathbb{F}_p^\times} |\chi(k)|^2 \\ &= p(p-1), \end{aligned}$$

where we have used Corollary 1.24. ■

2.8 February 10

We continue discussing applications of the Gauss sum.

2.8.1 The Pólya–Vinogradov Inequality

As an aside, we set up the Pólya–Vinogradov inequality.

Theorem 2.76 (Pólya–Vinogradov inequality). Fix a prime p and a nontrivial character $\chi \pmod{p}$. Then for any a, b , we have

$$\left| \sum_{a \leq n \leq b} \chi(n) \right| \ll \sqrt{p} \log p,$$

where the implicit constant does not depend on anything.

Proof. Roughly speaking, we are computing the inner product of χ and the indicator function of an interval. Using “Plancherel’s formula” to bound completes the proof. The trick here is to “complete the sum.” Because $\sum_{n=0}^{p-1} \chi(n) = 0$, we may assume that $a, b \leq p$. (If shifting yields $a \leq p \leq b \leq p + a$, then we can flip the entire sum to make it $b - p \leq p - a \leq p$.)

Now, the main point is to take the Fourier transform

$$\widehat{1_{[a,b]}}(m) = \sum_{a \leq n \leq b} e\left(-\frac{mn}{p}\right) \ll \frac{p}{m},$$

where we have expanded out the geometric series to get this bound; namely, we are noting $\frac{1}{1-e(-1/p)} \approx p$. As such, we use the Fourier inversion formula Corollary 1.24 to see

$$\left| \sum_{x \in \mathbb{F}_p} 1_{[a,b]}(x) \chi(x) \right| = \frac{1}{p} \left| \sum_{m, x \in \mathbb{F}_p} \widehat{1_{[a,b]}}(m) e\left(\frac{mx}{p}\right) \chi(x) \right|.$$

Now, by Lemma 2.75, this is bounded above by

$$\frac{1}{p} \sum_{m=1}^{p-1} \widehat{1_{[a,b]}} \sqrt{p} \ll \frac{1}{\sqrt{p}} \cdot \sqrt{p} \sum_{m=1}^{p-1} \frac{1}{m} \ll \sqrt{p} \log p.$$

(Notably, the $m = 0$ term provides no contribution.) This completes the proof. ■

Corollary 2.77. The least nonquadratic residue is $O(\sqrt{x} \log x)$.

Proof. This is direct from Theorem 2.76. ■

Remark 2.78. For “short” intervals, one can do better, which is the point of the Burgess bound.

2.8.2 A Functional Equation for $L(s, \chi)$

Fix a prime p . Given a Dirichlet character $\chi \pmod{p}$, we would like to provide a functional equation akin to Theorem 2.40. Roughly speaking, there will be three cases.

- χ could be the trivial character χ_0 .
- χ could be the (unique) nontrivial real quadratic character $\left(\frac{\cdot}{p}\right)$.
- χ could be a complex character satisfying $\chi \neq \bar{\chi}$.

We separate out these cases because it turns out that our knowledge of the second case is the worst.

Remark 2.79. Note that the case of $\chi = \chi_0$ is essentially just ζ with a few Euler factors, so its functional equation can be derived directly from Theorem 2.40.

In light of Remark 2.79, we will focus on the case of $\chi \neq \chi_0$. Now, for $\alpha \in \mathbb{R}$, an argument similar to Example 2.9 yields

$$\sum_{n \in \mathbb{Z}} e^{-\pi(n+\alpha)^2/x} = x^{1/2} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x + 2\pi i n \alpha}.$$

In particular, taking $\alpha = m/p$ for $m \in \mathbb{Z}$ grants

$$\sum_{n \in \mathbb{Z}} e^{-\pi(n+m/p)^2/x} = \left(\frac{x}{p}\right)^{1/2} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x/p + 2\pi i n m/p}.$$

To continue, we will work with $\chi(-1) = 1$.² Here, we set

$$\Theta_\chi(x) := \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi n^2 x/p}.$$

Roughly speaking, in the case where $\chi(-1) = -1$, this sum would completely vanish, so we would have to add a factor of n or similar to make this summation behave. We will not say more about this case.

Using Corollary 1.24, we see

$$\begin{aligned} \Theta_\chi(x) &= \sum_{n \in \mathbb{Z}} \frac{1}{\tau(\bar{\chi}, 1)} \left(\sum_{m=0}^{p-1} \bar{\chi}(m) \right) e^{mn/p} e^{-\pi n^2 x/p} \\ &= \frac{1}{\tau(\bar{\chi}, 1)} \sum_{m=0}^{p-1} \chi(m) \sum_{n \in \mathbb{Z}} e^{2\pi i mn/p - \pi n^2 x/p} \\ &\stackrel{*}{=} \frac{1}{\tau(\bar{\chi}, 1)} \cdot \left(\frac{x}{p} \right)^{1/2} \sum_{m=0}^{p-1} \chi(m) \sum_{n \in \mathbb{Z}} e^{-\pi(n+m/p)^2 p/x} \\ &= \frac{1}{\tau(\bar{\chi}, 1)} \cdot \left(\frac{x}{p} \right)^{1/2} \sum_{m=0}^{p-1} \chi(m) \sum_{n \in \mathbb{Z}} e^{-\pi(pn+m)^2/(px)}, \end{aligned}$$

where we have applied Poisson summation at $\stackrel{*}{=}$. Now, the summations loop over all residue classes in $pn + m$, so we see

$$\Theta_\chi(x) = \frac{1}{\tau(\bar{\chi}, 1)} \cdot \left(\frac{x}{p} \right)^{1/2} \sum_{t \in \mathbb{Z}} \chi(t) e^{-\pi t^2/(px)} = \frac{1}{\tau(\bar{\chi}, 1)} \cdot \left(\frac{x}{p} \right)^{1/2} \Theta_{\bar{\chi}}(1/x),$$

where we are also using the fact that χ is periodic (mod p).

Now, to find our functional equation, we write

$$\Xi_\chi(s) := p^{s/2} \pi^{-s/2} \Gamma(s/2) L(s, \chi).$$

Here, the factor of p roughly comes from some kind of conductor, and the $\pi^{-s/2} \Gamma(s/2)$ is our real Γ -factor. In particular, our functional equation will turn out to be the following result.

Theorem 2.80 (Functional equation for Ξ_χ). Fix a nontrivial Dirichlet character $\chi \pmod{p}$ such that $\chi(-1) = -1$. Then $\Xi_\chi(s)$ is entire and satisfies the functional equation

$$\Xi_\chi(s) = \varepsilon(\bar{\chi}) \Xi_{\bar{\chi}}(1-s),$$

where $\varepsilon_\chi := \sqrt{q}/\tau(\chi)$.

Proof. We follow the proof of Theorem 2.40. One can compute the integral

$$\Gamma(s/2) (p/\pi)^{s/2} n^{-s} = \int_0^\infty e^{-\pi n^2 x/p} x^{s/2} \frac{dx}{x}$$

for $\operatorname{Re} s > 1$. Summing, we see

$$\Xi_\chi(s) = \frac{1}{2} \int_0^\infty \Theta_\chi(x) x^{s/2} \frac{dx}{x}.$$

At this point, one can see directly that this right-hand side is entire for all $s \in \mathbb{C}$: indeed, $\Theta_\chi(x)$ rapidly decays at both 0 and ∞ , so its Mellin transform is safe for all $s \in \mathbb{C}$. Thus, we already see that Ξ_χ is entire. In particular, this equality now holds for all $s \in \mathbb{C}$.

² This is called the “unramified at ∞ case” because the place here at infinity is totally real.

Anyway, applying the usual variable change $x \mapsto 1/x$, we see

$$\begin{aligned}\Xi_\chi(s) &= \frac{1}{2} \int_0^\infty \Theta_\chi(1/x) x^{-s/2} \frac{dx}{x} \\ &= \frac{1}{2} \int_0^\infty \left(\frac{p^{1/2}}{\tau(\bar{\chi})} \Theta_{\bar{\chi}}(x) \right) x^{-s/2} \frac{dx}{x},\end{aligned}$$

where we have used the functional equation for Θ_χ at the last equality. Upon using the analytic continuation for $\Xi_{\bar{\chi}}$ provided by the previous paragraph, we get

$$\Xi_\chi(s) = \varepsilon_{\bar{\chi}} \Xi_{\bar{\chi}}(1-s).$$

This completes the proof. ■

Remark 2.81. The element ε_χ in the functional equation is called “the root number.” There is a wealth of research trying to understand their behavior.

Remark 2.82. We omit the case of $\chi(-1) = -1$.

2.9 February 13

Today we conclude discussing the functional equation for $L(s, \chi)$.

2.9.1 All Functional Equations

We will want to state our functional equation for primitive Dirichlet characters, so here is the definition of primitive characters.

Definition 2.83 (primitive). A Dirichlet character $\chi \pmod{q}$ is *primitive* if and only if $\chi: (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ has no smaller period than q .

Remark 2.84. In more general contexts, q is called the “conductor” of χ .

Here is our statement.

Theorem 2.85. Fix a primitive Dirichlet character $\chi \pmod{q}$ for $q > 1$. Then set $a_\chi := \frac{1}{2}(1 - \chi(-1)) \in \{0, 1\}$, and define

$$\Xi_\chi(s) := \left(\frac{q}{\pi} \right)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi).$$

Then Ξ_χ is entire and satisfies the functional equation

$$\Xi_\chi(s) = \varepsilon_{\bar{\chi}} \Xi_{\bar{\chi}}(1-s),$$

where $\varepsilon_\chi := i^a q^{1/2} / \tau(\chi)$, where $\tau(\bar{\chi})$ is a Gauss sum.

Proof. Omitted. ■

We should probably say a few words about these more general Gauss sums.

Lemma 2.86. Fix a primitive Dirichlet character $\chi \pmod{q}$. Then $\tau(\chi, m) = \overline{\chi}(m)\tau(\chi, 1)$.

Proof. If $\gcd(m, q) = 1$, then the same proof from earlier goes through, so we may assume that $\gcd(m, q) > 1$. Further, if $q \mid m$, then everything vanishes, so we may assume that $q \nmid m$.

In this case, we would like to show that $\tau(\chi, m) = 0$, or in other words,

$$\sum_{k=0}^{q-1} \chi(k) e\left(\frac{mk}{q}\right) \stackrel{?}{=} 0.$$

For this, we must use the fact that χ is primitive. Now, take $d := \gcd(m, q)$ and $m' := m/d$ and $q' := q/d$ so that $\gcd(m', q') = 1$. As such, we compute

$$\begin{aligned} \tau(\chi, m) &= \sum_{k=0}^{q-1} \chi(k) e\left(\frac{mk}{q}\right) \\ &= \sum_{k=0}^{q'-1} \sum_{r=0}^{d-1} \chi(kq' + r) e\left(\frac{m'(kq' + r)}{q'}\right) \\ &= \sum_{r=0}^{d-1} \left(e\left(\frac{m'r}{q'}\right) \sum_{k=0}^{q'-1} \chi(kq' + r) \right). \end{aligned}$$

Because χ is primitive, $\chi \pmod{q'}$ is not periodic, so we can find $r_1 \equiv r_2 \pmod{q'}$ such that $\chi(r_1)$ and $\chi(r_2)$ are nonzero and not equal. It follows from the above expression that

$$\chi(r_1)\tau(\chi, m) = \chi(r_2)\tau(\chi, m)$$

by taking and then re-indexing the sums over r in our last summation.

Alternatively, if $\chi(tq' + 1) = 1$ for all $t \in \mathbb{Z}$, then it actually follows that χ is periodic $\pmod{q'}$, which is a contradiction, so it is enough to check that

$$\tau(\chi, m) = \chi(tq' + 1)\tau(\chi, m)$$

for all t by again just re-indexing the relevant sum. ■

Lemma 2.87. Fix a primitive Dirichlet character $\chi \pmod{q}$. Then $|\tau(\chi)|^2 = \sqrt{q}$.

Proof. Again sum over the $\tau(\chi, m)$ as in Lemma 2.75. ■

The rest of the proof of Theorem 2.85 now follows exactly as we proceeded last class.

2.9.2 A Zero-Free Region for Complex Characters

We continue to let $\chi \pmod{q}$ be a primitive character.

Remark 2.88. When $\chi \pmod{q}$ is not primitive, then $L(s, \chi)$ is equal, up to a few Euler factors, to some $L(s, \chi')$, where χ' has smaller modulus. These finite Euler factors do not affect our zero-free region.

Example 2.89. Fix

$$D(s) := \zeta(s)^3 L(s, \chi)^2 L(s, \bar{\chi})^2 L(s, \chi^2) L(s, \bar{\chi}^2).$$

When χ is a complex character, one can check that this has nonnegative real coefficients, so Lemma 2.60 goes through and grants us a zero-free region using the same argument we used for ζ . Namely, we still get a zero-free region which looks like

$$\left\{ \sigma + it : \sigma > 1 - \frac{c}{\log |q|(|t| + 2)} \right\},$$

where c is some fixed constant. In particular, we get a smaller zero-free region for larger q .

2.9.3 The Quadratic Gauss Sum

We take a moment to evaluate some Gauss sums.

Lemma 2.90. Fix a prime p . Then

$$\tau := \sum_{k=1}^{p-1} \left(\frac{k}{p} \right) e \left(\frac{k}{p} \right) = \sum_{k=0}^{p-1} e \left(\frac{k^2}{p} \right).$$

Proof. Quickly, note that $\sum_{k=0}^{p-1} e(k/p) = 0$, so we can add this to our original sum to see

$$\tau = \sum_{k=0}^{p-1} e \left(\frac{k^2}{p} \right)$$

by counting the number of times quadratic and nonquadratic residues appear. Explicitly, $k^2 = 0$ happens once, k^2 hits quadratic residues twice, and k^2 hits nonquadratic residues zero times; this exactly matches $1 + \left(\frac{a}{p} \right)$. ■

The benefit of Lemma 2.90 is that we can use this as our “definition” of the Gauss sum, namely remove the hypothesis that p is prime.

Proposition 2.91. Fix some odd integer q . Then

$$\sum_{r=0}^{q-1} e \left(\frac{r^2}{q} \right) = \frac{1 - i^q}{1 - i} \cdot \sqrt{q}.$$

Proof. We will use Poisson summation; the point is to turn our Gauss sum into an infinite sum by adding some dampening factor. As such, define $f(t) := \Theta(t/\sqrt{\pi})$ so that Proposition 2.34 tells us that

$$f(t) = \left(\frac{\pi}{t} \right)^{1/2} f \left(\frac{\pi^2}{t} \right) \tag{2.4}$$

for $t > 0$. Everything here is holomorphic on $\operatorname{Re} s > 0$, so we can extend this identity to work for $\operatorname{Re} s > 0$ by the uniqueness of analytic continuation.

Notably, for $\varepsilon > 0$, we see

$$\begin{aligned} f \left(\varepsilon + \frac{2\pi i}{q} \right) &= 1 + 2 \sum_{n=1}^{\infty} e^{-n^2} e^{-2\pi i n^2 / q} \\ &= 1 + 2 \sum_{r=0}^{q-1} \left(e^{-2\pi i r^2 / q} \sum_{m=0}^{\infty} e^{-(r+mq)^2 \varepsilon} \right). \end{aligned}$$

Now, one can check that $\varepsilon \rightarrow 0^+$ enforces

$$\sum_{m=0}^{\infty} e^{-(r+mq)^2 \varepsilon} \sim \frac{\sqrt{\pi}}{2q\sqrt{\varepsilon}}$$

by doing Poisson summation and only focusing on the leading term. In particular, we see that

$$f\left(\varepsilon + \frac{2\pi i}{q}\right) \sim \frac{\sqrt{\pi}}{2q\sqrt{\varepsilon}} \sum_{r=1}^q e^{-2\pi i r^2/q}$$

as $\varepsilon \rightarrow 0^+$. As such, using the functional equation (2.4) for f , as well as the computation

$$\frac{\pi^2}{\varepsilon + 2\pi i/q} = -\frac{\pi i q}{2} + \frac{q^2}{4}\varepsilon + O(\varepsilon^2)$$

to take the asymptotics on the other side. This will complete the proof. \blacksquare

Remark 2.92. One can use Proposition 2.91 to prove the law of quadratic reciprocity, essentially by comparing Gauss sums \pmod{p} and \pmod{q} for distinct odd primes p and q . One can also use these techniques to prove the supplement $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.

2.10 February 15

We open class by remarking that the arguments of Gauss sums turn out to be equidistributed. As references, we mention “Sato–Tate Theorems for Finite Field Mellin Transforms” and “Gauss sums, Kloosterman Sums, and Monodromy Groups.” Let’s give a quick algebraic proof of the supplement.

Proposition 2.93. Fix an odd prime p . Then $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.

Proof. Let $\zeta \in \overline{\mathbb{F}_p}$ be an eighth root of unity. (We can also find ζ in \mathbb{F}_{p^4} because ζ is a root of the equation $\zeta^4 + 1 = 0$.) Because $\zeta^4 = -1$, we see $\zeta^2 = \zeta^{-2}$, so

$$(\zeta + \zeta^{-1})^2 = \zeta^2 + \zeta^{-2} + 2 = 2.$$

Now, we see $\left(\frac{2}{p}\right) = 1$ is equivalent to having $\zeta + \zeta^{-1} \in \mathbb{F}_p$, which because $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is topologically generated by the Frobenius, is equivalent to

$$\zeta^p + \zeta^{-p} = (\zeta + \zeta^{-1})^p \stackrel{?}{=} \zeta + \zeta^{-1}.$$

Now, this is equivalent to $p \equiv \pm 1 \pmod{8}$; indeed, if $p \equiv \pm 3 \pmod{8}$, then we see $\zeta^3 + \zeta^{-3} = -\zeta - \zeta^{-1}$ is the negative root. \blacksquare

2.10.1 Counting Zeroes of $L(s, \chi)$

We would like to generalize Theorem 2.47. For this, we have the following definition.

Notation 2.94. Fix a primitive character $\chi \pmod{q}$. Then we define

$$N_\chi(T) := \#\{|\rho| \leq T : \Xi_\chi(\rho) = 0\}.$$

Remark 2.95. Note that the functional Theorem 2.85 does tell us that all zeroes live in the critical strip, but we no longer have conjugate symmetry because χ might not be real.

Arguing as before, the set

$$\left\{ \frac{L'(2 + iT, \chi)}{L(2 + iT, \chi)} : T \in \mathbb{R}, q, \chi \pmod{q} \right\}$$

is uniformly bounded above, and one can show as in Lemma 2.48 that

$$\sum_{\Xi_\chi(\rho)=0} \frac{1}{1 + |\operatorname{Im} \rho - T|^2} \ll \log(|T| + 2) + \log q,$$

where the implied constant is absolute. Following the rest of the proof of Theorem 2.47 from the argument principle again gives us

$$\frac{1}{2} N_\chi(T) = \frac{T}{2\pi} \log \left(\frac{qT}{2\pi} \right) - \frac{T}{2\pi} + O(\log q(|T| + 2)),$$

where the implied constants are still absolute.

2.10.2 Solovay–Strassen Primality Testing

As an application of what we've done so far, we describe a primality test assuming the Generalized Riemann Hypothesis (GRH).

Question 2.96. Can one determine if an integer n is prime in $\operatorname{Poly}(\log n)$ time?

This is known unconditionally (via the AKS algorithm), and there are fast probabilistic algorithms, but we describe an algorithm which works assuming GRH. Here is the statement of GRH.

Conjecture 2.97 (Generalized Riemann Hypothesis). For any primitive character $\chi \pmod{q}$, the zeroes of $\Xi_\chi(s)$ all lie on the vertical line

$$\operatorname{Re} s = \frac{1}{2}.$$

And now here is our result.

Theorem 2.98 (Miller–Rabin). Assume GRH. Then we can test if an integer n is prime in $\operatorname{Poly}(\log n)$ time.

For this, we describe the Miller–Rabin primality test, which is one of the more efficient probabilistic primality tests.

Lemma 2.99 (Fermat's little theorem). Fix a prime p . If $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.

Proof. This follows from Lagrange's theorem: the order of $a \in \mathbb{F}_p^\times$ divides $|\mathbb{F}_p^\times| = p - 1$. ■

Remark 2.100. It turns out that there are infinitely many integers n such that

$$a^n \equiv a \pmod{n}.$$

For example, $n = 561$ works. Thus, one cannot really use Lemma 2.99 to test for primality.

Instead, we will want to use Proposition 2.69, which implies

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \pmod{p}$$

for $\gcd(a, p) = 1$. In order to compute Legendre symbols efficiently, we must introduce the Jacobi symbol.

Definition 2.101 (Jacobi). Fix an odd integer n . Then for integers a , we define the *Jacobi symbol*

$$\left(\frac{a}{n}\right) = \prod_p \left(\frac{a}{p}\right)^{\nu_p(n)}.$$

Remark 2.102. The Jacobi symbol, like the Legendre symbol, is multiplicative in the numerator, but it is also multiplicative in the denominator. One can use this to show “Jacobi reciprocity,” which asserts that odd $a, b \in \mathbb{Z}$ grant

$$\left(\frac{a}{b}\right) \left(\frac{b}{a}\right) = (-1)^{(a-1)(b-1)/4}.$$

One also gets the supplement $\left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8}$.

Remark 2.103. Remark 2.102 allows us to compute Jacobi symbols efficiently via reciprocity. Roughly speaking, we are basically just doing the Euclidean algorithm and keeping track of some signs.

We now do primality testing with the Euler criterion.

Lemma 2.104. Fix an odd integer n . If n is not prime, there exists some a such that

$$a^{(n-1)/2} \not\equiv \left(\frac{a}{n}\right) \pmod{n}.$$

Proof. Omitted. The idea is to split this into two cases: if n is squarefree, one can divide the result via the Chinese remainder theorem into various $(\text{mod } p)$ statements; if n is not squarefree, then one can look $(\text{mod } p^2)$ somewhere to get the result. ■

This suggests the following algorithm to test if n an odd integer n is prime.

1. Choose some random $a \in [1, n)$.
2. Compute the Jacobi symbol $\left(\frac{a}{n}\right)$. Via the Euclidean algorithm, one can compute this in $O((\log n)^2)$ time.
3. Compute $a^{(n-1)/2} \pmod{n}$ using exponentiation by repeated squaring. This will run in $O((\log n)^3)$ time.
4. If the above do not match, then n is not prime. If the above match, then return to step 1 and try and another a .

In fact, we will show (using GRH), that one only has to check $a \ll (\log n)^2$, so the entire algorithm runs in $O((\log n)^5)$.

2.11 February 17

The next two classes (Wednesday and Friday) will be recorded and posted online.

2.11.1 Deterministic Solovay–Strassen

Here is our main theorem, which tells us that the Solovay–Strassen primality test can be made deterministic.

Theorem 2.105. Suppose that q is an odd integer which is not prime. Assuming GRH, then there exists $a \ll (\log q)^2$ such that

$$\left(\frac{a}{q}\right) \not\equiv a^{(q-1)/2} \pmod{q}.$$

We will require the following result.

Proposition 2.106. Let q be an odd integer. Given a subgroup $A \subseteq (\mathbb{Z}/q\mathbb{Z})^\times$. Assuming GRH, there exists an absolute constant c such that

$$\{[k] \in (\mathbb{Z}/n\mathbb{Z})^\times : k \leq c(\log q)^2\} \subseteq A$$

implies $A = (\mathbb{Z}/q\mathbb{Z})^\times$.

One can quickly prove Theorem 2.105 from Proposition 2.106.

Proof of Theorem 2.105. Indeed, the set

$$A := \left\{ a \in (\mathbb{Z}/q\mathbb{Z})^\times : \left(\frac{a}{q}\right) \not\equiv a^{(n-1)/2} \pmod{q} \right\}$$

is a subgroup of $(\mathbb{Z}/q\mathbb{Z})^\times$. Because q is not prime, we know that $A \neq (\mathbb{Z}/q\mathbb{Z})^\times$, so it follows from Proposition 2.106 that A does not contain all $[k] \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $k \leq c(\log q)^2$. The conclusion follows. ■

We now attack Proposition 2.106.

Proof of Proposition 2.106. Suppose for the sake of contradiction that $A \neq (\mathbb{Z}/q\mathbb{Z})^\times$. Now, the quotient $(\mathbb{Z}/q\mathbb{Z})^\times / A$ is nontrivial and abelian, so it has some nonzero character. Pulling this character to $(\mathbb{Z}/q\mathbb{Z})^\times$, we have a Dirichlet character $\chi \pmod{q}$ such that

$$\chi(k) = 1$$

for each $0 \leq k \leq c(\log q)^2$ coprime to n .

As in the proof of the Prime number theorem, we want to consider the infinite sum

$$\sum_{n \leq x} \Lambda(n) \chi(n) = \frac{1}{2\pi i} \int_{\text{Re } s=c} \left(-\frac{L'(s, \chi)}{L(s, \chi)} \right) x^s \frac{ds}{s}$$

and shift the contour over to the left. To do this, we apply smoothing ψ to get a smooth function compactly supported on $[1/4, 3/4]$. Arguing as in Dirichlet's theorem, we see

$$\sum_{n \geq 0} \Lambda(n) \chi(n) \psi\left(\frac{n}{x}\right) = \frac{1}{2\pi i} \int_{\text{Re } s=2} \left(-\frac{L'(s, \chi)}{L(s, \chi)} \right) (\mathcal{M}\psi)(s) x^s ds.$$

Shifting the contour is somewhat delicate, but it can be done similarly as in our proof of the Prime number theorem. This gives

$$\sum_{n \geq 0} \Lambda(n) \chi(n) \psi\left(\frac{n}{x}\right) = - \sum_{L(\rho, \chi)=0} (\mathcal{M}\psi)(\rho_\chi) x^{\rho_\chi}$$

plus some smaller error terms. By GRH, we may assume that all the roots lie on $\text{Re } s = \frac{1}{2}$, so this is bounded (up to a constant) by

$$\sqrt{x} \log q.$$

Notably, the number of zeroes does not increase very much, especially in comparison to the rapid decay of $\mathcal{M}\psi$. As such, we see

$$x \ll \left| \sum_{n \geq 0} \Lambda(n) \chi(n) \psi\left(\frac{n}{x}\right) \right| \ll \sqrt{x} \log q,$$

so $x \ll (\log q)^2$. However, setting $x = c(\log q)^2$ for c large enough will break this bound, which is our contradiction. ■

2.11.2 Imprimitive Characters

Let's talk a little more about our characters.

Definition 2.107 (conductor). Fix a Dirichlet character $\chi \pmod{q}$. Then the *conductor* $f(\chi)$ is the minimal period of χ restricted $\{n \in \mathbb{Z} : \gcd(n, q) = 1\}$. If $f(\chi) \neq q$, then χ is said to be *imprimitive*.

Roughly speaking, one can take characters and reduce them to the primitive case by pretending they are Dirichlet characters modulo their conductor.

Definition 2.108 (induces). Fix a Dirichlet character $\chi \pmod{q}$ with conductor f . Then the Dirichlet character " $\chi \pmod{f}$ " is primitive (by construction) and is said to *induce* $\chi \pmod{q}$.

Example 2.109. The principal Dirichlet characters are induced by the constantly 1 character.

Remark 2.110. The point is that a Dirichlet character $\chi \pmod{q}$ induced by a primitive Dirichlet character $\chi' \pmod{f}$ has L -function given by

$$L(s, \chi') = L(s, \chi) \prod_{\substack{p \nmid f \\ p \mid q}} \frac{1}{1 - \chi'(p)p^{-s}}.$$

Note that these finitely many Euler factors do not add any zeroes or poles or similar.

Remark 2.111. Under our philosophy that the real characters are the hard ones. By the Chinese remainder theorem, it suffices to understand characters modulo prime powers p^ν . If p is odd, then $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$ is cyclic, so the only real characters are either principal or is $\left(\frac{\cdot}{p}\right)$ depending on if the generator $g \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times$ gets sent to -1 or 1 . If $p = 2$, then $(\mathbb{Z}/2^\nu\mathbb{Z}) \cong \langle -1 \rangle \times \langle 5 \rangle = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{\nu-2}\mathbb{Z}$ for $\nu \geq 2$, and we can decompose this as one might expect. Namely, there is one modulo 4 and two modulo 8.

2.12 February 22

This lecture was recorded.

2.12.1 Real Primitive Characters

We continue discussing primitive characters χ modulo prime powers p^ν . Recall that these are controlled for p odd because $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$ is still cyclic. In particular, there is exactly one real primitive Dirichlet character \pmod{p} and none for p^ν for $\nu > 1$.

For $p = 2$, one must be a little more careful.

- There is a unique primitive real character $(\text{mod } 2)$ which is the trivial one.
- There is a real primitive character $\chi_4 \pmod{4}$ given by $(\frac{-1}{\bullet})$.
- Further, there are two real primitive characters $(\text{mod } 8)$ given by $(\frac{2}{\bullet})$ and $(\frac{-2}{\bullet})$.
- There are no more real primitive characters $(\text{mod } 2^\nu)$.

Roughly speaking, these characters can give all primitive characters by the Chinese remainder theorem. We can be a little more explicit about this.

Remark 2.112. Using the Kronecker symbol, one can write all real primitive Dirichlet characters as $(\frac{d}{\bullet})$. The values d yielding primitive characters are the ones which are fundamental discriminants; namely,

$$d \equiv \begin{cases} N & \text{if } N \equiv 1 \pmod{4} \text{ and is squarefree,} \\ 4N & \text{if } N \equiv 2, 3 \pmod{4} \text{ and is squarefree.} \end{cases}$$

Note that we permit $N < 0$.

Remark 2.113. Fix a real primitive character $\chi = (\frac{d}{\bullet})$. Then $\zeta(s)L(s, \chi)$ is the Dedekind ζ -function for the quadratic field $\mathbb{Q}(\sqrt{d})$. Roughly speaking, this explains why $\zeta(s)L(s, \chi)$ should have positive coefficients. See [Dav80, Chapter 6] for details.

2.12.2 Zero-Free Regions for $L(s, \chi)$

We now establish some zero-free regions for $L(s, \chi)$. For complex primitive Dirichlet characters χ , one can use the product

$$\zeta(s)^3 L(s + it_0, \chi)^2 L(s - it_0, \bar{\chi})^2 L(s + 2it_0, \chi^2) L(s - 2it_0, \bar{\chi}^2)$$

combined with Lemma 2.60 to see that $L(s, \chi)$ has no zeroes in the region

$$\left\{ s \in \mathbb{C} : \operatorname{Re} s > 1 - \frac{c}{\log(q(|t| + 2))} \right\}.$$

Note that any imprimitive character χ can be reduced to a primitive one by adjusting finitely many Euler factors of $L(s, \chi)$, which do not change a vanishing region.

In the case of real primitive Dirichlet characters χ , one does not do as well. Notably, $\chi^2 = \bar{\chi}^2$ is the principal character, so $L(s + 2it_0, \chi^2)$ is lacking a pole at $s = 1$. Nonetheless, an argument will still work except at $t_0 = 0$, where we see that we have at most one zero in the real numbers in the region

$$\left\{ s \in \mathbb{C} : \operatorname{Re} s > 1 - \frac{c}{\log(q(|t| + 2))} \right\}.$$

However, at most one zero is still not good enough for our purposes. Well, to deal with a possibly real zero, we can apply Lemma 2.60 to

$$\zeta(s)L(s, \chi),$$

and we can produce a lower bound (using the proof of $L(1, \chi) \neq 0$) to produce the lower bound

$$|L(1, \chi)| > cq^{-1/2}.$$

In particular, from the summation, our Dirichlet series is 1 on squares, which is where the square root comes from.

Remark 2.114. In contrast, we can upper-bound $L(s, \chi)$ relatively easily as

$$|L'(\sigma, \chi)| \ll (\log q)^2 \quad \text{for} \quad 1 - \frac{1}{\log q} \leq \sigma \leq 1,$$

and

$$|L(\sigma, \chi)| \ll \log q, \quad \text{for} \quad 1 - \frac{1}{\log q} \leq \sigma \leq 1.$$

For details, see [Dav80, Chapter 14]. Roughly speaking, one can just use the Dirichlet series for $L(s, \chi)$. In particular, early terms rotate quickly and can be bounded as sines and cosines, and the later terms are small.

The idea above is that we can use the above derivative combined with our lower bound for $L(1, \chi)$ in order to get some very small interval in the real numbers where we are nonzero. This is indeed technically a zero-free region; in the next lecture, we will cover Siegel's theorem, which is ineffective but will do a little better.

In total, we get that any problematic real zero β of $L(s, \chi)$ must satisfy

$$\beta < 1 - \frac{c}{(\log q)^2 \sqrt{q}}.$$

Of course, this is much worse when compared to $c/\log q$, but it does give us a zero-free region to work with.

Remark 2.115 (Landau). Fix $\chi_d := \left(\frac{d}{\bullet}\right)$. Then one can use

$$\zeta(s) L(s, \chi_{d_1}) L(s, \chi_{d_2}) L(s, \chi_{d_1 d_2}),$$

for $d_1 \neq d_2$, which is the ζ function of a biquadratic field. Now, because each of these L -functions have at most one real zero in the desired region $\left(1 - \frac{c}{\log |d_1 d_2|}, 1\right]$, we note that a zero for $L(s, \chi_{d_1})$ will force the other L -functions to not have zeroes! An idea like this is able to produce a zero-free region, and it is the key input to Siegel's theorem.

APPENDIX A

COMPLEX ANALYSIS

Our reality isn't about what's real, it's about what we pay attention to.

—Hank Green, [Gre20]

In this chapter, we review some basic facts of complex analysis. We do not provide proofs of all statements.

A.1 Holomorphic Functions

Complex analysis is the study of holomorphic functions, so we quickly provide a definition.

Definition A.1 (holomorphic). Fix a complex function $f: \Omega \rightarrow \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is some subset. We say that f is *differentiable* at $z \in \Omega$ if and only if the limit

$$f'(z) := \lim_{w \rightarrow z} \frac{f(z) - f(w)}{z - w}$$

exists. If f is differentiable at all $z \in \Omega$, then we say f is *holomorphic* on Ω .

Here is the main test on holomorphic functions.

Theorem A.2 (Cauchy–Riemann equations). Fix a complex function $f: \Omega \rightarrow \mathbb{C}$, where Ω is a nonempty open subset. Writing $f(x + yi) := u(x, y) + iv(x, y)$, then f is differentiable at $z_0 = x_0 + iy_0$ implies that

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0), \\ v_x(x_0, y_0) = -u_y(x_0, y_0). \end{cases}$$

Proof. See [Elb22, Theorem 3.19]. Intuitively, we are saying that f is locally a scaled rotation, which is what multiplication by a complex numbers. ■

Remark A.3. Under suitably hypotheses, satisfying the Cauchy–Riemann equations implies that f is differentiable at the point z_0 . See [Elb22, Theorem 3.26].

A.2 Path Integrals

To do calculus on complex functions, we also want to know how to integrate them.

Definition A.4 (path integral). Fix a piecewise continuous function $f: \Omega \rightarrow \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is some subset. Given a piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$, we define

$$\int_{\gamma} f(z) dz := \int_0^1 \operatorname{Re}(f(\gamma(t))\gamma'(t)) dt + i \int_0^1 \operatorname{Im}(f(\gamma(t))\gamma'(t)) dt.$$

If γ is closed (i.e., $\gamma(0) = \gamma(1)$), then we might write $\oint_{\gamma} f(z) dz$.

As usual, limits commute with integrals under suitably uniformity hypotheses.

Lemma A.5. Fix an open subset $\Omega \subseteq \mathbb{C}$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ of piecewise continuous function $\Omega \rightarrow \mathbb{C}$. Given a piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$, if $f_n \rightarrow f$ uniformly for some $f: \Omega \rightarrow \mathbb{C}$, then

$$\lim_{n \rightarrow \infty} \oint_{\gamma} f_n(z) dz = \oint_{\gamma} f(z) dz.$$

Proof. See [Elb22, Lemma 4.62]. Roughly speaking, the point is that we can upper-bound

$$\left| \oint_{\gamma} f(z) dz - \oint_{\gamma} f_n(z) dz \right| \leq \oint_{\gamma} |f(z) - f_n(z)| dz \leq \sup_{z \in \operatorname{im} \gamma} |f(z) - f_n(z)| \cdot \ell(\gamma),$$

which goes to 0 as $n \rightarrow \infty$ by the uniformity. ■

Proposition A.6. Fix an open, connected subset $\Omega \subseteq \mathbb{C}$. Fix a piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$ and a function $f: \Omega \rightarrow \mathbb{C}$ continuous on $\operatorname{im} \gamma$. For any $z_0 \in \Omega$, we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - w} dz = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - w)^{n+1}} dz \right) (w - z_0)^n$$

for w in some open neighborhood of z_0 .

Proof. See [Elb22, Proposition 4.61]. Roughly speaking, we begin by translating so that $z_0 = 0$; then for $|w|$ small enough (for example, avoiding $\operatorname{im} \gamma$ and $|w| < \frac{1}{2}|z|$), we can write

$$\frac{f(z)}{z - w} = \frac{f(z)}{z} \cdot \frac{1}{1 - (w/z)} = \sum_{n=0}^{\infty} \frac{f(z)w^n}{z^{n+1}},$$

and this geometric series converges absolutely and uniformly because $|w| < \frac{1}{2}|z|$. Thus, the Weierstrass M -test grants uniform convergence, so we can integrate both sides over \oint_{γ} and finish by switching the infinite sum and integral by Lemma A.5. ■

A.3 The Cauchy Integral Formula

In this section, we provide various formulations of the Cauchy integral formula. The most basic formulation is as follows.

Theorem A.7 (Cauchy–Goursat). Fix a simply connected open subset $\Omega \subseteq \mathbb{C}$. If $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic on Ω , then for any piecewise C^1 closed path $\gamma: [0, 1] \rightarrow \Omega$, we have

$$\oint_{\gamma} f(z) dz = 0.$$

Proof. See [Elb22, Theorem 4.65, Theorem 4.70]. Intuitively, this result follows from Green’s theorem combined with Theorem A.2. ■

We can extend this result in a few ways. For one, we can evaluate the function using integrals.

Theorem A.8 (Cauchy integral formula). Fix an open connected subset $\Omega \subseteq \mathbb{C}$ containing some $\overline{B(z_0, r)}$. If $f: \Omega \rightarrow \mathbb{C}$ is holomorphic, then

$$f(w) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - w} dz$$

for $w \in B(z_0, r)$, where γ is the counterclockwise path around $\partial B(z_0, r)$.

Proof. See [Elb22, Theorem 4.63]. Roughly speaking, one can use Theorem A.7 in order to allow us to send $r \rightarrow 0^+$ without changing the value of the integral. Then we note

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - w} dz = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z) - f(w)}{z - w} dz + f(w) \cdot \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - w} dz.$$

The integral on the right is just 1, so we want to show that the other integral goes to 0. Well, $\frac{f(z) - f(w)}{z - w}$ is roughly $f'(w)$ as $r \rightarrow 0^+$, so we can upper-bound this integral by (say) $2|f'(w)| \cdot 2\pi r$ for small r , which goes to 0 as $r \rightarrow 0^+$. ■

Remark A.9. Roughly speaking, using Theorem A.7 again, the above proof more or less says that we can replace γ with any counterclockwise path around z_0 provided that Ω is simply connected.

Remark A.10. Combining Theorem A.8 with Proposition A.6 implies that holomorphic $f: \Omega \rightarrow \mathbb{C}$ are locally equal to a power series.

In fact, we can evaluate derivatives using integrals.

Corollary A.11. Fix an open connected subset $\Omega \subseteq \mathbb{C}$ containing some $\overline{B(z_0, r)}$. If $f: \Omega \rightarrow \mathbb{C}$ is holomorphic, then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - w)^{n+1}} dz$$

for $w \in \Omega$ and $n \geq 0$, where γ is the counterclockwise path around $\partial B(z_0, r)$.

Proof. See [Elb22, Corollary 4.71]. By Theorem A.8 and Proposition A.6, we may write

$$f(w) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - w)^{n+1}} dz \right) (w - z_0)^n$$

for w in some open neighborhood of z_0 . (As usual, the deformation process with Theorem A.7 allows us to shrink γ as necessary.) But now we can, with some pain, take derivatives of this power series by hand (see [Elb22, Proposition 3.44]) to achieve the result. ■

As an application, we discuss analytic continuation.

Theorem A.12. Fix an open connected subset $\Omega \subseteq \mathbb{C}$. Given holomorphic functions $f_1, f_2: \Omega \rightarrow \mathbb{C}$, if

$$\{z \in \mathbb{C} : f_1(z) = f_2(z)\}$$

contains an accumulation point, then $f_1 = f_2$ on Ω .

Proof. See [Elb22, Theorem 5.1]. By working with $f_1 - f_2$, it suffices to show that, if $f^{-1}(\{0\})$ has an accumulation point, then $f = 0$. We show this by contraposition: suppose $f \neq 0$, and we will show that each $z_0 \in f^{-1}(\{0\})$ has some $r > 0$ such that $B(z_0, r) \cap f^{-1}(\{0\}) = \{z_0\}$. By shifting, we may assume that $z_0 = 0$. By Remark A.10, we see that f is locally a power series around $z = 0$. Because f is nonzero, this local power series cannot identically vanish. However, this implies that there is some m such that the power series for $f(z)/z^m$ has a nonzero constant term. However, $f(z)/z^m$ is then a continuous function which is nonzero at 0 and is thus nonzero in an open neighborhood of 0. ■

A.4 Building Primitives

We would like to build a converse for Theorem A.7.

Proposition A.13. Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a continuous function $f: \Omega \rightarrow \mathbb{C}$. Given that

$$\oint_{\gamma} f(z) dz = 0$$

for any closed piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$. Then there exists a holomorphic function $F: \Omega \rightarrow \mathbb{C}$ such that $F' = f$.

Proof. See [Elb22, Theorem 4.44]. By translating, we may assume $0 \in \Omega$, and the point is to define

$$F(z) = f(0) + \int_{\gamma} f(s) ds,$$

where γ is any closed piecewise C^1 path from 0 to z . The hypothesis on f implies that F is well-defined. To show that $F'(w) = f(w)$ for any $w \in \Omega$, work in some small open neighborhood of w so that we may assume Ω is convex. In particular, let γ_z denote the straight line from w to z so that

$$\left| \frac{F(z) - F(w)}{z - w} - f(w) \right| = \left| \frac{1}{z - w} \int_{\gamma_z} f(s) ds - f(w) \right| \leq \sup_{s \in \text{im } \gamma_z} |f(s) - f(w)|.$$

However, f is continuous, so this supremum goes to 0 as $z \rightarrow w$. ■

And here is our converse.

Theorem A.14 (Morera). Fix an open, connected subset $\Omega \subseteq \mathbb{C}$ and a continuous function $f: \Omega \rightarrow \mathbb{C}$. Given that

$$\oint_{\gamma} f(z) dz = 0$$

for any closed piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$. Then f is holomorphic.

Proof. By Proposition A.13, there exists holomorphic $F: \Omega \rightarrow \mathbb{C}$ such that $F' = f$. However, F is locally given by a power series by Remark A.10, so we can differentiate-term-by-term to tell us that F is infinitely differentiable. In particular, f is holomorphic. ■

Here are a couple useful corollaries of Theorem A.14.

Lemma A.15. Fix some open, connected subset $\Omega \subseteq \mathbb{C}$ and some function $f: \Omega \rightarrow \mathbb{C}$. Given holomorphic functions $f_n: \Omega \rightarrow \mathbb{C}$ for each $n \in \mathbb{N}$, if $f_n \rightarrow f$ uniformly on all compact subsets $D \subseteq \Omega$, then f is holomorphic. In fact, for each $w \in \Omega$, we have $f'_n(w) \rightarrow f'(w)$ as $n \rightarrow \infty$.

Proof. To show f is holomorphic, the point is to use Morera's theorem. Quickly, note that to show f is differentiable at some particular $z \in \Omega$, we may find $r > 0$ such that $B(z, r) \subseteq \Omega$ and then replace Ω with $B(z, r)$; in particular, we may assume that Ω is simply connected. Each f_n is continuous, so we see f is continuous as well by the uniform convergence. Thus, fixing any closed piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$, we would like to show

$$\oint_{\gamma} f(z) dz \stackrel{?}{=} 0.$$

Note $\text{im } \gamma$ is compact, so $f_n \rightarrow f$ uniformly on $\text{im } \gamma$. Thus, fixing any $\varepsilon > 0$, we can find some N such that

$$|f(z) - f_n(z)| < \varepsilon$$

for all $n > N$. Fixing any $n > N$, we use Theorem A.7 to see

$$\left| \oint_{\gamma} f(z) dz \right| = \left| \oint_{\gamma} f(z) dz - \oint_{\gamma} f_n(z) dz \right| \leq \oint_{\gamma} |f(z) - f_n(z)| dz \leq \varepsilon \ell(\gamma),$$

where $\ell(\gamma)$ is the length of γ . (Note $\ell(\gamma)$ is finite because γ is piecewise C^1 .) Sending $\varepsilon \rightarrow 0^+$ finishes the application of Theorem A.14.

It remains to show the last sentence. The point is to use Corollary A.11. Well, for any fixed $w \in \Omega$, we again find some $r > 0$ such that $B(w, r) \subseteq \Omega$. Then for some $\varepsilon \in (0, r)$, we let γ be the counterclockwise path around w with radius ε . Then Corollary A.11 grants

$$\begin{aligned} |f'(w) - f'_n(w)| &= \left| \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-w)^2} dz - \frac{1}{2\pi i} \oint_{\gamma} \frac{f_n(z)}{(z-w)^2} dz \right| \\ &\leq \frac{1}{2\pi} \cdot \sup_{z \in \Omega} \{|f(z) - f_n(z)|\} \cdot \oint_{\gamma} \frac{1}{|z-w|^2} dz \\ &= \frac{1}{2\pi} \cdot \sup_{z \in \Omega} \{|f(z) - f_n(z)|\} \cdot \frac{2\pi\varepsilon}{\varepsilon^2}. \end{aligned}$$

Now, as $n \rightarrow \infty$, we see that $|f(z) - f_n(z)| \rightarrow 0$ uniformly, so the final expression goes to 0. This completes the proof. ■

Lemma A.16. Fix some simply connected open subset $\Omega \subseteq \mathbb{C}$. Given a holomorphic function $f: \Omega \rightarrow \mathbb{C}$ which vanishes nowhere, there exists a holomorphic function $g: \Omega \rightarrow \mathbb{C}$ such that $f = \exp \circ g$.

Proof. By shifting Ω , we may assume that $0 \in \Omega$. By scaling f (which is the same as shifting g), we may assume that $f(0) = 1$. Now, note that f is locally a power series by Remark A.10, so f' is holomorphic, so f'/f is holomorphic because f vanishes nowhere on Ω . Thus, by Theorem A.7, we see that

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 0$$

for any closed piecewise C^1 path $\gamma: [0, 1] \rightarrow \Omega$, where here we are using the fact that Ω is simply connected. Thus, we let g be a primitive for f'/f ; by shifting g , we may assume that $g(0) = 0$. Now let $h(z) := \exp(g(z))/f(z)$ so that we want to show $h(z) = 1$ for each z ; note that f is always nonzero, so h is in fact holomorphic. Well, we see that

$$h'(z) = \frac{f(z) \cdot \exp(g(z))g'(z) - \exp(g(z))f'(z)}{f(z)^2} = 0,$$

so h is constant. But $h(0) = 1$ by construction of f and g , so $h(z) = 1$ for each z . ■

Remark A.17. Unwinding the proof of Proposition A.13 and Lemma A.16, we see that we can actually explicitly define g by

$$g(z) := \log f(z_0) + \int_{\gamma} \frac{f'(z)}{f(z)} dz,$$

where $z_0 \in \Omega$ is any point, and γ is any path connecting z_0 to z .

A.5 Differentiation Under the Integral

While we're here, we pick up the following very useful technical result from [Mat01].

Proposition A.18 (Differentiation under the integral sign). Let (X, \mathcal{S}, μ) be a measurable space, and let $U \subseteq \mathbb{C}$ be open, and let $f: U \times X \rightarrow \mathbb{C}$ and $g: X \rightarrow \mathbb{C}$ be functions satisfying the following properties.

- The function g is integrable; namely, $\int_X g(t) dt < \infty$.
- For fixed x , the function $z \mapsto f(z, x)$ is holomorphic on U and has $|f(z, x)| \leq g(x)$ for all z .
- For fixed z , the function $x \mapsto f(z, x)$ is measurable.

Then the function $F: U \rightarrow \mathbb{C}$ given by $F(z) := \int_X f(z, x) dx$ is holomorphic on U and satisfies

$$F'(z) = \int_X \frac{\partial f}{\partial z}(z, x) dx.$$

Proof. We use Morera's theorem to show F is holomorphic and the Cauchy integral formula to compute the derivative. The intuition here is that we can control integrals of F easier than its derivatives, so we will try to turn everything into an integral. For clarity, we proceed in steps.

1. We show F is continuous on U . Well, fix some $w \in U$, and we show F is continuous at w ; for concreteness, again find $r > 0$ such that $B(w, r) \subseteq U$. Indeed, for some distinct $w' \in B(w, r)$, we let $\gamma: [0, 1] \rightarrow U$ denote the straight line from w to w' . Thus, the Fundamental theorem of calculus and Cauchy's integral formula grants

$$\begin{aligned} F(w') - F(w) &= \int_X (f(w', x) - f(w, x)) dx \\ &= \int_X \left(\int_{\gamma} \frac{\partial f}{\partial z}(z, x) dz \right) dx \\ &= \frac{1}{2\pi i} \int_X \left(\int_{\gamma} \int_{\gamma_z} \frac{f(z', x)}{(z - z')^2} dz' dz \right) dx, \end{aligned}$$

where γ_z denotes the counterclockwise circle around z of radius $r - \frac{1}{2}|w - w'|$, which is inside $B(w, r) \subseteq U$ because z is on the line connecting w to w' . Now, taking absolute values everywhere, we see

$$|F(w') - F(w)| \leq \frac{1}{2\pi} \int_X \left(\int_{\gamma} \int_{\gamma_z} \frac{g(t)}{|r - \frac{1}{2}|w - w'|^2} dz' dz \right) dx \leq \frac{1}{2\pi} \int_X g(t) dt \cdot \frac{\ell(\gamma) \cdot 2\pi |r - \frac{1}{2}|w - w'||}{|r - \frac{1}{2}|w - w'|^2},$$

where we have used the computation $\ell(\gamma_z) = 2\pi |r - \frac{1}{2}|w - w'||$ for each z . Now, as $w' \rightarrow w$, we see $\ell(\gamma) = |w - w'|$ goes to 0, so the entire right-hand side goes to 0. This completes the proof of continuity at w .

2. We show F is holomorphic on U . It suffices to show that F is differentiable at some fixed $w \in U$ and has the given derivative. As such, we find $r > 0$ such that $B(w, r) \subseteq U$ and replace f with its restriction

to $B(w, r) \times X$ and F with its restriction to $B(w, r)$. In particular, we have reduced to the case where U is open and convex.

Now, we already know F is continuous, so we may use Morera's theorem. Well, let $\gamma: [0, 1] \rightarrow U$ be some closed curve, and we want to show

$$\int_{\gamma} F(z) dz = \int_{\gamma} \int_X f(z, x) dx dz \stackrel{?}{=} 0.$$

We would like to exchange the two integrals, so we note we have absolute convergence because

$$\int_{\gamma} \int_X |f(z, x)| dx dz \leq \int_{\gamma} \int_X g(x) dx dz \leq \ell(\gamma) \int_X g(x) dx < \infty.$$

Thus, Fubini's theorem lets us write

$$\int_{\gamma} F(z) dz = \int_X \int_{\gamma} f(z, x) dz dx = \int_X 0 dx = 0,$$

where we have used Cauchy's theorem to evaluate $\int_{\gamma} f(z, x) dz = 0$; recall we reduced to the case where U is convex above!

3. It remains to compute the derivative of F . Because F is holomorphic, we may use the Cauchy integral formula: for any $w \in U$, find r such that $B(w, r) \subseteq U$, and let γ be the loop of radius $r/2$ around w . Then

$$F'(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(z)}{(z-w)^2} dz = \frac{1}{2\pi i} \int_{\gamma} \int_X \frac{f(z, x)}{(z-w)^2} dx dz.$$

As usual, we would like to exchange the two integrals, so we note that we have absolute convergence because

$$\int_{\gamma} \int_X \left| \frac{f(z, x)}{(z-w)^2} \right| dx dz \leq \int_{\gamma} \int_X \frac{g(x)}{(r/2)^2} dx dz \leq \frac{\ell(\gamma)}{(r/2)^2} \int_X g(x) dx < \infty.$$

Thus, Fubini's theorem lets us write

$$F'(w) = \int_X \left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(z, x)}{(z-w)^2} dz \right) dx = \int_X \frac{\partial f}{\partial z}(w, x) dx,$$

where we have again applied the Cauchy integral formula. ■

Remark A.19. Proposition A.18 might look like needless abstract nonsense with the measure space floating around, but the point here is that we will be able to flexibly apply this result to exchange derivatives with both usual integrals and infinite sums.

A.6 Infinite Products

Throughout analytic number theory, it is useful to take infinite products for one reason or another. In this section, we follow [SS03a, Section 5.3]. We begin by discussing products of elements.

Definition A.20 (absolutely converges). Given a sequence of complex numbers $\{a_k\}_{k \in \mathbb{N}}$, the infinite product

$$\prod_{k=1}^{\infty} (a_k + 1)$$

converges absolutely if and only if the product $\prod_{k=1}^{\infty} (|a_k| + 1)$ converges.

Lemma A.21. Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of complex numbers such that

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

Then the infinite product $\prod_{k=1}^{\infty} (1 + a_k)$ converges and vanishes if and only if some factor vanishes.

Proof. If any factor vanishes, then the entire product converges to 0, so there is nothing to say. Otherwise, assume that $a_k \neq -1$ for all k , and we must show that the infinite product converges to a nonzero value. We have two cases.

1. Suppose that $|a_n| < 1/2$ for all n . Then we can use the power series to define \log . The main claim is that the infinite sum

$$\sum_{k=1}^{\infty} \log(1 + a_k)$$

converges. In fact, it converges absolutely: we compute

$$\begin{aligned} \sum_{k=1}^{\infty} |\log(1 + a_k)| &= \sum_{k=1}^{\infty} \left| \sum_{\ell=1}^{\infty} (-1)^{\ell} \frac{a_k^{\ell}}{\ell} \right| \\ &\leq \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{\infty} \frac{|a_k|^{\ell}}{\ell} \right) \\ &\leq \sum_{k=1}^{\infty} -\log(1 - |a_k|). \end{aligned}$$

Now, \log is concave down on $(0, \infty)$, so $-\log(1 - x)$ is concave up on $[0, 1/2]$, so comparing with a line segment, we see

$$-\log(1 - |a_k|) \leq \frac{(1/2 - |a_k|)(-\log(1 - 0)) + |a_k|(-\log(1 - 1/2))}{1/2} < |a_k| \cdot \frac{-\log(1 - 1/e)}{1/2} = 2|a_k|$$

for each $|a_k| \in [0, 1/2]$. Thus,

$$\sum_{k=1}^{\infty} |\log(1 + a_k)| \leq \sum_{k=1}^{\infty} -\log(1 - |a_k|) \leq \sum_{k=1}^{\infty} 2|a_k| = 2 \sum_{k=1}^{\infty} |a_k|,$$

completing the proof of the claim.

To complete the proof in this case, we use the fact that \exp is continuous to write

$$\begin{aligned} \prod_{k=1}^{\infty} (1 + a_k) &= \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k) \\ &= \lim_{n \rightarrow \infty} \exp \left(\sum_{k=1}^n \log(1 + a_k) \right) \\ &= \exp \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n \log(1 + a_k) \right) \\ &= \exp \left(\sum_{k=1}^{\infty} \log(1 + a_k) \right), \end{aligned}$$

which we already know converges (absolutely). Additionally, we converge to a nonzero value because $\exp(z) \neq 0$ for all $z \in \mathbb{C}$. This is what we wanted.

2. In the general case, note that the convergence of the sum $\sum_{k=1}^{\infty} |a_k|$ enforces $|a_k| \rightarrow 0$ as $k \rightarrow \infty$. Thus, there exists some n such that $|a_k| < 1/2$ for $k > N$, so we see

$$\prod_{k=1}^{\infty} (1 + a_k) = \prod_{k=1}^n (1 + a_k) \cdot \prod_{k=n+1}^{\infty} (1 + a_k).$$

The left product is finite, and the right product converges by the previous step: note $\sum_{k=n+1}^{\infty} |a_k| < \infty$ because $\sum_{k=1}^{\infty} |a_k| < \infty$. Thus, the entire product converges, and it converges to a nonzero value because the left and right factors above are both nonzero. ■

Remark A.22. In fact, by replacing $\{a_k\}_{k \in \mathbb{N}}$ with $\{|a_k|\}_{k \in \mathbb{N}}$, the lemma tells us that

$$\prod_{k=1}^{\infty} (1 + |a_k|)$$

also converges. In particular, the product converges absolutely.

We even have a converse to the above result.

Lemma A.23. Fix a sequence of complex numbers $\{a_k\}_{k \in \mathbb{N}}$ such that the infinite product

$$\prod_{k=1}^{\infty} (1 + a_k)$$

converges absolutely. Then $\sum_{k=1}^{\infty} |a_k|$ converges.

Proof. We may replace $\{a_k\}_{k \in \mathbb{N}}$ with $\{|a_k|\}_{k \in \mathbb{N}}$ so that we can assume that the a_k are positive real numbers. Now, the convergence of the infinite product allows us to use the continuity of \log to write

$$\log \left(\prod_{k=1}^{\infty} (1 + a_k) \right) = \log \left(\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k) \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \log(1 + a_k) = \sum_{k=1}^{\infty} \log(1 + a_k).$$

In particular, this infinite sum converges, so $\log(1 + a_k) \rightarrow 0$ as $k \rightarrow \infty$, so we can find N large enough so that $\log(1 + a_k) < \log(3/2)$ for each $k > N$, meaning that $1 + a_k < 3/2$ and so $a_k < 1/2$. But here we note that \log is concave down, so

$$\log(1 + a_k) \geq \frac{(1/2 - a_k) \log(1 + 0) + a_k \log(1 + 1/2)}{1/2} = a_k \cdot 2 \log(3/2).$$

For psychological reasons, we note that $3 \log(3/2) = \log(27/8) > \log 3 > 1$, so we see $\log(1 + a_k) > \frac{2}{3} a_k$ here. In total, we see

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^N a_k + \sum_{k>N} a_k \leq \sum_{k=1}^N a_k + \frac{3}{2} \sum_{k>N} \log(1 + a_k) < \infty,$$

so the sum converges. This completes the proof. ■

Remark A.24. Combining Lemmas A.21 and A.23 shows that an infinite product which converges absolutely will only vanish if any of the factors vanish.

Now that we know how to take infinite products of elements, we can also take infinite products of functions. Here is our analogue of the Weierstrass M -test.

Proposition A.25. Fix an open subset $\Omega \subseteq \mathbb{C}$ and a sequence $\{f_k\}_{k \in \mathbb{N}}$ of holomorphic functions $\Omega \rightarrow \mathbb{C}$. Suppose that we have constants $\{c_k\}_{k \in \mathbb{N}} \subseteq \mathbb{C}$ such that

$$\sum_{k=1}^{\infty} |c_k| < \infty \quad \text{and} \quad |f_k(z) - 1| < |c_k| \text{ for all } z.$$

Then the infinite product $f(z) := \prod_{k=1}^{\infty} f_k(z)$ converges absolutely and uniformly on compacts to a holomorphic function $\Omega \rightarrow \mathbb{C}$.

Proof. Set $a_k(z) := f_k(z) - 1$ for each k . Then we see

$$\sum_{k=1}^{\infty} |a_k(z)| \leq \sum_{k=1}^{\infty} |c_k| < \infty,$$

so Remark A.22 implies that the infinite product $f(z)$ converges absolutely for all $z \in \mathbb{C}$.

Now, because each f_k is holomorphic, each partial product defining f is holomorphic, so it suffices by Lemma A.15 to show that the partial products converge uniformly to f . As in the previous proof, we have two cases.

1. Suppose that $|c_k| < 1/2$ for each k . The idea is to use \exp to turn our product into a sum. For technical reasons, we start by noting that $m > n$ gives

$$\left| \sum_{k=m}^n \log f_k(z) \right| \leq \sum_{k=m}^n |\log(1 + a_k(z))| \leq \sum_{k=m}^n -\log(1 - |a_k(z)|) \leq \sum_{k=m}^n -\log(1 - |c_k|) \leq 2 \sum_{k=m}^n |c_k|,$$

where we have bounded as in the previous proof. (Here, \log is defined using the power series.) In particular, this partial sum and even the full series has magnitude bounded by $C := 2 \sum_{k=1}^{\infty} |c_k|$.

For psychological reasons, we note that \exp has continuous and hence bounded derivative on the compact set $\overline{B(0, C)}$, so \exp is Lipschitz continuous; let L be the Lipschitz constant for \exp . The main computation is that any $z \in D$ have

$$\begin{aligned} \left| f(z) - \prod_{k=1}^n f_k(z) \right| &= \left| \exp \left(\sum_{k=1}^{\infty} \log(1 + a_k(z)) \right) - \exp \left(\sum_{k=1}^n \log(1 + a_k(z)) \right) \right| \\ &\leq L \left| \sum_{k=n+1}^{\infty} \log(1 + a_k(z)) \right| \\ &\leq 2L \sum_{k=n+1}^{\infty} |c_k| \end{aligned}$$

by the above bounding. However, as $n \rightarrow \infty$, this right-hand side goes to 0 because $\sum_{k=1}^{\infty} |c_k|$ converges. The uniform convergence follows.

2. We reduce to the above case. Because $\sum_{k=1}^{\infty} |c_k|$ converges, we see $|c_k| \rightarrow 0$ as $k \rightarrow \infty$, so there exists n such that $|c_k| < 1/2$ for $k > n$. Thus,

$$f(z) = \prod_{k=1}^n f_k(z) \cdot \prod_{k=n+1}^{\infty} f_k(z).$$

By the previous step, the convergence in the right (infinite) product is uniform, and the left term is a finite product, so the convergence in the original product is also uniform. ■

Remark A.26. It might look concerning that we have full uniform convergence instead of the usual more mild uniform convergence on compacts. However, the hypothesis requires that the f_k are bounded, so Ω will have to be pretty small anyway if the f_k are nonconstant.

Remark A.27. Note that the above proof shows that, for any $z \in \mathbb{C}$, the sequence $\{a_n(z)\}_{n \in \mathbb{N}}$ satisfies the hypotheses of Lemma A.21. Thus, $f(z) = 0$ if and only if $f_k(z) = 0$ for some k . As such, if $f(z) \neq 0$, we note that the above proof tells us that the equality

$$\log f(z) = \sum_{k=1}^{\infty} \log f_k(z)$$

makes sense, and the sum converges absolutely.

Because we are doing analysis, it will also be beneficial to be able to compute derivatives.

Corollary A.28. Fix an open subset $\Omega \subseteq \mathbb{C}$ and a sequence $\{f_k\}_{k \in \mathbb{N}}$ of holomorphic functions $\Omega \rightarrow \mathbb{C}$. Suppose that the infinite product $f(z) := \prod_{k=1}^{\infty} f_k(z)$ converges absolutely and uniformly on compacts. Then

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{\infty} \frac{f'_k(z)}{f_k(z)}$$

if $f(z) \neq 0$.

Proof. Let $p_n(z)$ denote the n th partial product. By Lemma A.15, we know that $p'_n(z) \rightarrow f'(z)$ as $n \rightarrow \infty$. Now, $f(z) \neq 0$ implies that $f_k(z) \neq 0$ for each k by Remark A.27, so $p_n(z) \neq 0$ for each n as well. Thus, we may add in the fact $p_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$ to compute

$$\frac{f'(z)}{f(z)} = \lim_{n \rightarrow \infty} \frac{p'_n(z)}{p_n(z)} \stackrel{*}{=} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{f'_k(z)}{f_k(z)} = \sum_{k=1}^{\infty} \frac{f'_k(z)}{f_k(z)}.$$

Note that $\stackrel{*}{=}$ holds because we can formally check that $\frac{(ab)'(z)}{(ab)(z)} = \frac{a'(z)}{a(z)} + \frac{b'(z)}{b(z)}$ for holomorphic functions a and b not vanishing at z . In particular, there is no need for a logarithm here. ■

APPENDIX B

ENTIRE FUNCTIONS

In this chapter, we provide enough of the theory of entire functions in order to state and prove the Hadamard factorization theorem. Throughout this section, entire functions will be nonzero. We follow [SS03a, Chapter 5].

B.1 Counting Zeroes

It will be useful to bound the number of zeroes of a nonzero entire function, so we establish some notation.

Notation B.1. Fix a nonzero entire function $f: \mathbb{C} \rightarrow \mathbb{C}$. Then we let $Z_f(r)$ denote the multiset of complex numbers $z \in B(0, r)$ such that $f(z) = 0$, counted with multiplicity. Additionally, we set $n_f(r) := \#Z_f(r)$.

Remark B.2. If $f: \mathbb{C} \rightarrow \mathbb{C}$ is a nonzero entire function, then $n_f(r)$ is indeed finite: indeed, if $n_f(r)$ is infinite, then we claim $f = 0$. To see this, we note f has infinitely many zeroes in the compact set $\overline{B(0, r)}$, which has two cases.

- If f has infinitely many distinct zeroes in $\overline{B(0, r)}$, then the zero-set of f must have a limit point because it is an infinite subset of a compact set. But this implies $f = 0$.
- If f has a zero w of order infinity, then the Taylor expansion of f around w vanishes identically. It follows that f vanishes in an open neighborhood of w , so f has infinitely many zeroes, reducing to the previous case.

We will shortly be able to roughly bound the number of zeroes of f by the growth rate of f , but for the purpose of this section, we will not place constraints on the growth rate of f . To begin this counting, we pick up the following lemma.

Lemma B.3. Fix a nonzero entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(0) \neq 0$. For any $R > 0$, we have

$$\int_0^R n_f(r) \frac{dr}{r} = \sum_{\substack{f(z)=0 \\ |z| < R}} \log \left| \frac{R}{z} \right|,$$

where the sum counts zeroes with multiplicity.

Proof. Note that the $f(0) \neq 0$ hypothesis is included to ensure that the sum is well-defined. For each $z \in Z_f(R)$, we will use the indicator $1_{>|z|}(r)$ to indicate $z \in B(0, r)$. In particular, we compute

$$\begin{aligned} \int_0^R n_f(r) \frac{dr}{r} &= \int_0^R \left(\sum_{z \in Z_f(R)} 1_{>|z|}(r) \right) \frac{dr}{r} \\ &\stackrel{*}{=} \sum_{z \in Z_f(R)} \left(\int_0^R 1_{>|z|}(r) \frac{dr}{r} \right) \\ &= \sum_{z \in Z_f(R)} \left(\int_{|z|}^R \frac{dr}{r} \right) \\ &= \sum_{z \in Z_f(R)} \log \left| \frac{R}{z} \right|, \end{aligned}$$

which is what we wanted. Notably, we are able to switch the sum and integral in $\stackrel{*}{=}$ because the sum is finite by Remark B.2. ■

As such, we are motivated to understand this summation of logarithms as a proxy to understand $n_f(r)$. This is the content of Jensen's theorem.

Theorem B.4 (Jensen). Fix a nonzero entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(0) \neq 0$. Further, suppose f does not vanish on $\partial B(0, R)$ for some $R > 0$. Then

$$\log |f(0)| = \sum_{\substack{f(z)=0 \\ |z|<R}} \log \left| \frac{z}{R} \right| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta.$$

Proof. This is essentially a consequence of the Cauchy integral formula. Again, note that $f(0) \neq 0$ is required for the sum to make sense. Anyway, we proceed in steps to build up to the general case.

1. Suppose that f does not vanish on $B(0, R)$ so that f does not vanish on $\overline{B(0, R)}$. Here, the sum is empty, so we would like to show

$$\log |f(0)| \stackrel{?}{=} \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta. \quad (\text{B.1})$$

The idea is to apply the Cauchy integral formula to a suitably defined logarithm of f . This logarithm will exist because f is nonzero in our region.

Quickly, we claim that f does not vanish on $B(0, R + \varepsilon)$ for some $\varepsilon > 0$. Indeed, suppose not. Then for each n , we can find $r_n e^{i\theta_n} \in B(0, R + 1/n)$ such that $f(z_n) = 0$. In particular, we see $R < r_n < R + 1/n$ for each n , and $e^{i\theta_n} \in S^1$.

Thus, $r_n \rightarrow R$ as $n \rightarrow \infty$, and having infinitely many elements $e^{i\theta_n}$ in the compact set S^1 forces the θ_n to have a subsequence $e^{i\theta_{n_k}}$ to converge to some $e^{i\theta}$. It follows that $r_{n_k} e^{i\theta_{n_k}} \rightarrow R e^{i\theta}$, so because $f^{-1}(\{0\})$ is closed, we see that $R e^{i\theta}$ lives in $f^{-1}(\{0\})$, so f vanishes on some point in $\partial B(0, R)$. This is a contradiction to the construction of R .

We now continue the proof with our $\varepsilon > 0$ such that f does not vanish $B(0, R + \varepsilon)$. Because f does not vanish, and $B(0, R + \varepsilon)$ is simply connected, we can use Lemma A.16 to define g so that g is holomorphic and satisfies $f(z) = \exp(g(z))$.

We now apply the Cauchy integral formula to $g(z)$. Let γ_R be the path $\theta \mapsto Re^{i\theta}$ so that the Cauchy integral formula grants

$$\begin{aligned} g(0) &= \frac{1}{2\pi i} \oint_{\gamma_R} \frac{g(z)}{z} dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(Re^{i\theta})}{Re^{i\theta}} \cdot iRe^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{i\theta}) d\theta. \end{aligned}$$

Now, we see $\log |f(z)| = \log |\exp(g(z))| = \log \exp(\operatorname{Re} g(z)) = \operatorname{Re} g(z)$, so taking real parts of the above equation yields (B.1), as desired.

2. Suppose $f(z) = z - w$ for some nonzero $w \in B(0, R)$. Notably, f has only the root w , so after some rearranging, we would like to show

$$0 \stackrel{?}{=} \frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - w/R| d\theta.$$

(Notably, $\log |f(0)| = \log |w|$.) We now would like to reduce to the previous case; set $\alpha := R/w$ so that $|\alpha| > 1$. Now, we send $\theta \mapsto -\theta$, so we see

$$\frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - 1/\alpha| d\theta = \int_0^{2\pi} \log |e^{-i\theta} - 1/\alpha| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}/\alpha| d\theta,$$

where we have factored out $\log |e^{-i\theta}| = \log 1 = 0$.

Thus, we set $g(z) := 1 - z/\alpha$ so that g is entire but does not vanish on $\overline{B(0, 1)}$ so that the previous case implies

$$0 = \log 1 = \log |g(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}/\alpha| d\theta,$$

which is what we wanted.

3. To set up the finish of the proof, suppose that the theorem is true for the nonzero entire functions $f_1, f_2: \mathbb{C} \rightarrow \mathbb{C}$. Then we claim that the theorem is true for $f_1 f_2$. Indeed, we see $(f_1 f_2)(0) \neq 0$ implies $f_1(0), f_2(0) \neq 0$. Further, for any $R > 0$, we see $f_1 f_2$ not vanishing on $\partial B(0, R)$ implies that $f_2 f_2$ does not vanish on $\partial B(0, R)$ also.

Thus, the theorem hypotheses hold for both f_1 and f_2 if they hold for $f_1 f_2$. Now, for $i \in \{1, 2\}$, let $Z_i(R)$ denote the multiset zeroes of f_i in $B(0, R)$ counted with multiplicity. Then the multiset $Z_1(R) \cup Z_2(R)$ is the multiset of zeroes of $f_1 f_2$ counted with multiplicity. Now, applying the theorem statement to both f_1 and f_2 , we compute

$$\begin{aligned} \log |(f_1 f_2)(0)| &= \log |f_1(0)| + \log |f_2(0)| \\ &= \sum_{z \in Z_1(R)} \log \left| \frac{z}{R} \right| + \frac{1}{2\pi} \int_0^{2\pi} \log |f_1(Re^{i\theta})| d\theta \\ &\quad + \sum_{z \in Z_2(R)} \log \left| \frac{z}{R} \right| + \frac{1}{2\pi} \int_0^{2\pi} \log |f_2(Re^{i\theta})| d\theta \\ &= \sum_{z \in Z_1(R) \cup Z_2(R)} \log \left| \frac{z}{R} \right| + \frac{1}{2\pi} \int_0^{2\pi} \log |(f_1 f_2)(Re^{i\theta})| d\theta, \end{aligned}$$

so the theorem statement also holds for $f_1 f_2$. This is what we wanted.

4. We now finish the proof in the general case. We define $g: \mathbb{C} \setminus Z_f(R) \rightarrow \mathbb{C}$ by

$$g(z) := \frac{f(z)}{\prod_{w \in Z_f(R)} (z - w)}.$$

Now, the right-hand side will only have removable singularities at each element of $Z_f(R)$, so in fact we may extend analytically g to all \mathbb{C} so that

$$f(z) = g(z) \prod_{w \in Z_f(R)} (z - w).$$

By the first step, the theorem statement holds for g , and by the second step, the theorem statement holds for each $z - w$. (Note $w \neq 0$ because $0 \notin Z_f(R)$ because $f(0) \neq 0$.) Thus, by the previous step, we may inductively take the product to show that the theorem statement holds for f , which is what we wanted. ■

Corollary B.5. Fix a nonzero entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(0) \neq 0$. For any $R > 0$ such that f does not vanish on $\partial B(0, R)$, we have

$$\int_0^R n_f(r) \frac{dr}{r} = -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta.$$

Proof. By Lemma B.3 and Theorem B.4, we see

$$\int_0^R n_f(r) \frac{dr}{r} = \sum_{\substack{f(z)=0 \\ |z| < R}} \log \left| \frac{R}{z} \right| = -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta,$$

which is what we wanted. ■

B.2 Functions of Bounded Order

Here is the central definition of this section.

Definition B.6 (order). A function $f: \mathbb{C} \rightarrow \mathbb{C}$ has *order bounded by ρ* for some $\rho > 0$ if and only if there are $A, B > 0$ such that

$$|f(z)| \leq Ae^{B|z|^\rho}.$$

Then we define the *order* $\rho(f)$ as the infimum of the real numbers $\rho \geq 0$ such that f has order bounded by ρ . Note that we are permitting $\rho(f) = +\infty$.

Here are some starting examples and remarks.

Example B.7. Any polynomial $f(z) \in \mathbb{C}[z]$ has order 0. Indeed, write $f(z) = \sum_{k=0}^n a_k z^k$ so that any $\rho > 0$ has

$$\lim_{|z| \rightarrow \infty} |f(z)|e^{-|z|^\rho} \leq \sum_{k=0}^n \lim_{r \rightarrow \infty} a_k r^k e^{-r^\rho} = \sum_{k=0}^n \lim_{r \rightarrow \infty} \frac{a_k r^{k/\rho}}{e^r} = 0,$$

where the last equality holds by, say, L'Hôpital's rule. Thus, $z \mapsto |f(z)|e^{-|z|^\rho}$ is a bounded function on \mathbb{C} (there exists R such that it is bounded by 1 for $|z| > R$, and the continuous function is certainly bounded on the compact set $\overline{B(0, R)}$), so we can find $A > 0$ such that $|f(z)| \leq Ae^{-|z|^\rho}$ for any $z \in \mathbb{C}$. It follows that f has order bounded by ρ for any $\rho > 0$, so $\rho(f) = 0$.

Example B.8. For any $A, B, \rho > 0$, the function $f(z) := Ae^{B|z|^\rho}$ has order ρ . We now claim that f has order bounded by ρ' if and only if $\rho' \geq \rho$, which will finish the proof. This has the following components.

- If $\rho' = \rho$, then the inequality $|f(z)| \leq Ae^{B|z|^\rho}$ tells us that f has order bounded by ρ' .
- For any $\rho' > \rho$, we claim that f has order bounded by ρ' . Indeed, we see that

$$\lim_{|z| \rightarrow \infty} |f(z)|e^{-|z|^{\rho'}} = A \exp \left(\lim_{|z| \rightarrow \infty} B|z|^\rho \cdot \lim_{|z| \rightarrow \infty} (1 - |z|^{\rho-\rho'}) \right) = 0$$

because $|z|^\rho \rightarrow \infty$ as $|z| \rightarrow \infty$. Thus, $|f(z)|e^{-|z|^{\rho'}}$ is bounded, so we can find $M > 0$ such that $|f(z)| \leq Me^{B|z|^{\rho'}}$ for any $z \in \mathbb{C}$.

- For any $0 < \rho' < \rho$, we claim that f does not have order bounded by ρ' . Indeed, suppose for the sake of contradiction that we have $A', B' > 0$ such that $|f(z)| \leq A'e^{B'|z|^{\rho'}}$. Then the function $e^{B|z|^\rho - B'|z|^{\rho'}}$ is bounded above by A'/A , but

$$\lim_{|z| \rightarrow \infty} e^{B|z|^\rho - B'|z|^{\rho'}} = \exp \left(\lim_{|z| \rightarrow \infty} B|z|^{\rho'} \cdot \lim_{|z| \rightarrow \infty} (1 - (B/B')|z|^{\rho'-\rho}) \right) = +\infty.$$

Remark B.9. For any function $f: \mathbb{C} \rightarrow \mathbb{C}$, if f has order bounded by ρ , then f has order bounded by ρ' for any $\rho' > \rho$. Indeed, we are granted $A, B > 0$ such that $|f(z)| \leq Ae^{B|z|^\rho}$, so it suffices to find constants $A', B' > 0$ such that $Ae^{B|z|^\rho} \leq A'e^{B'|z|^{\rho'}}$. But $Ae^{B|z|^\rho}$ has order bounded by ρ' by Example B.8, so we are done.

Remark B.10. If f has order bounded by α , and g has order bounded by β , then fg has order bounded by $\max\{\alpha, \beta\}$. Without loss of generality, take $\alpha \geq \beta$. Now, $|g(z)| \leq Ae^{B|z|^\beta}$ for some $A, B > 0$, so Example B.8 implies that there exist $A', B' > 0$ such that $|g(z)| \leq A'e^{B'|z|^\alpha}$ because $\alpha \geq \beta$. But f also has constants $A'', B'' > 0$ such that $|f(z)| \leq A''e^{B''|z|^\alpha}$, so

$$|(fg)(z)| \leq A'A''e^{(B'+B'')|z|^\alpha},$$

implying that fg has order bounded by α .

Remark B.11. If the entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ of order bounded by ρ has a zero at $z = 0$, then $g(z) := f(z)/z$ still has order bounded by ρ . Indeed, we do know that there are $A, B > 0$ such that $|f(z)| \leq Ae^{B|z|^\rho}$ for $z \in \mathbb{C}$. Now, $g(z)e^{-B|z|^\rho}$ is a continuous function and hence bounded by some constant $A' > 0$ on $\overline{B(0, 1)}$. However, $|z| > 1$, we see

$$|g(z)|e^{-B|z|^\rho} \leq |f(z)|e^{-B|z|^\rho} \leq A.$$

Thus, we see $|g(z)| \leq \max\{A, A'\}e^{B|z|^\rho}$, so g has order bounded by ρ .

It is a key fact, now, that we can count the number of zeroes of a function of bounded order.

Proposition B.12. Fix a nonzero entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ with order bounded by $\rho > 0$. Then there exists a constant $C, D > 0$ such that $n_f(R) \leq Cr^\rho$ for $R > D$.

Proof. We would like to use Corollary B.5. We handle two cases.

1. Suppose $f(0) \neq 0$ so that Corollary B.5 applies. In particular, the key observation is that $n_f(r)$ is an increasing function. As such, we compute

$$\begin{aligned} \int_R^{2R} n_f(r) \frac{dr}{r} &= -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(2Re^{i\theta})| d\theta \\ &\quad - \left(-\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta \right) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(2Re^{i\theta})| d\theta - \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta. \end{aligned}$$

However, because $|f(z)| \leq Ae^{B|z|^\rho}$ for some $A, B > 0$, we can upper-bound

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(2Re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \log |Ae^{BR^\rho}| d\theta = \log A + \frac{BR^\rho}{2\pi},$$

so it follows that

$$\int_R^{2R} n_f(r) \frac{dr}{r} \leq \left| \frac{1}{2\pi} \int_0^{2\pi} \log |f(2Re^{i\theta})| d\theta \right| + \left| \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta \right| \leq 2 \log A + \frac{B(1+2^\rho)}{2\pi} R^\rho.$$

On the other hand, because n_f is increasing, we can lower-bound

$$\int_R^{2R} n_f(r) \frac{dr}{r} \geq n_f(R) \int_R^{2R} \frac{dr}{r} = n_f(R) \log 2.$$

In total, we see

$$n_f(R)R^{-\rho} \leq \frac{2 \log A}{\log 2} \cdot R^{-\rho} + \frac{B(1+2^\rho)}{2\pi}.$$

As $R \rightarrow \infty$, the right-hand side approaches $B(1+2^\rho)/2\pi$, so we can find some D such that the right-hand side is less than or equal to $B(1+2^\rho)/\pi$ for $R > D$. Thus, $n_f(R) \leq \frac{B(1+2^\rho)}{\pi} R^\rho$ for $R > D$, which is what we wanted.

2. We now reduce to the case where $f(0) \neq 0$. Suppose f has a zero order of m at 0. If $m = 0$, then we are already done by the previous case. Otherwise, set $g(z) := f(z)/z^m$, which has only a removable singularity at $z = 0$ and thus extends to an entire function $g: \mathbb{C} \rightarrow \mathbb{C}$ such that $g(0) \neq 0$ while $f(z) = z^m g(z)$. Note that $n_f(R) = m + n_g(R)$ for any $R > 0$, and g still has order bounded by ρ by Remark B.11.

It follows from the previous step that there are $C, D > 0$ such that $R > D$ have

$$n_f(R)R^{-\rho} = mR^{-\rho} + n_g(R)R^{-\rho} = mR^{-\rho} + C.$$

Again, as $R \rightarrow \infty$, the right-hand side approaches C , so we can select $D' > 0$ such that the right-hand side is less than or equal to $2C$. Thus, $n_f(R) \leq 2CR^\rho$ for $R > \max\{D, D'\}$, which finishes. ■

Having a polynomial bound on the number of zeroes lets us bound sums with these zeroes.

Corollary B.13. Fix a sequence of nonzero complex numbers $\{z_k\}_{k \in \mathbb{N}}$ such that

$$n_f(r) := \#\{k : z_k < r\} \ll r^\rho$$

for some positive real number ρ . For any $\sigma > \rho$, the sum

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^\sigma}$$

converges. (Note that this sum may be finite.)

Proof. We use the dyadic decomposition. Because we are working with an infinite sum of positive numbers, it suffices to bound the sum from above. Now, from Proposition B.12, we see $n_f(R) \leq CR^\rho$ for some $R > D$; by replacing D with $2^{\lceil \log D \rceil} > D$ to no ill effect, we may assume that $D = 2^d$ for some positive integer d . Thus, we write

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{|z_k|^\sigma} &= \sum_{z \in Z_f(D) \setminus \{0\}} \frac{1}{|z|^\sigma} + \lim_{R \rightarrow \infty} \sum_{\substack{f(z)=0 \\ |z| \geq R}} \frac{1}{|z|^\sigma} \\
&\leq \sum_{z \in Z_f(D) \setminus \{0\}} \frac{1}{|z|^\sigma} + \sum_{k=d}^{\infty} \left(\sum_{\substack{f(z)=0 \\ 2^k \leq |z| < 2^{k+1}}} \frac{1}{|z|^\sigma} \right) \\
&\leq \sum_{z \in Z_f(D) \setminus \{0\}} \frac{1}{|z|^\sigma} + \sum_{k=d}^{\infty} \frac{n_f(2^{k+1})}{2^{k\sigma}} \\
&\leq \sum_{z \in Z_f(D) \setminus \{0\}} \frac{1}{|z|^\sigma} + C \sum_{k=d}^{\infty} \frac{2^{(k+1)\rho}}{2^{k\sigma}} \\
&\leq \sum_{z \in Z_f(D) \setminus \{0\}} \frac{1}{|z|^\sigma} + 2^\rho C \sum_{k=d}^{\infty} 2^{k(\rho-\sigma)} \\
&\leq \sum_{z \in Z_f(D) \setminus \{0\}} \frac{1}{|z|^\sigma} + 2^\rho C \cdot \frac{2^{d(\rho-\sigma)}}{1 - 2^{\rho-\sigma}},
\end{aligned}$$

which is indeed finite. ■

B.3 Elementary Factors

Hadamard's factorization theorem requires taking an infinite product of some special factors with prescribed zeroes. Here are those factors.

Definition B.14 (elementary factor). Given a nonnegative integer n , the elementary factor $E_n(z)$ is given by

$$E_n(z) := (1 - z)e^{z + z^2/2 + \cdots + z^n/n}.$$

In particular, $E_0(z) = 1 - z$.

The proof of Hadamard's factorization theorem will roughly amount to combining Lemma A.16 with various bounds on elementary factors. As such, we take the time to establish the needed bounds on our elementary factors.

Lemma B.15. Given a nonnegative integer n , we have $|1 - E_n(z)| \leq 2e|z|^{n+1}$ if $|z| \leq 1/2$.

Proof. Because $|z| \leq 1/2$, we may use the power series to define \log . Thus, we see

$$E_n(z) = \exp \left(\log(1 - z) + \sum_{k=1}^n \frac{z^k}{k} \right) = \exp \left(- \sum_{k=1}^{\infty} \frac{z^k}{k} + \sum_{k=1}^n \frac{z^k}{k} \right) = \exp \left(\sum_{k>n} \frac{z^k}{k} \right).$$

Now, for brevity, set $z' := \sum_{k>n} z^k/k$. The bound on z' we will need is

$$|z'| = \left| z^{n+1} \sum_{k=0}^{\infty} \frac{z^k}{k+n+1} \right| \leq |z|^{n+1} \sum_{k=0}^{\infty} \frac{|z|^k}{k+n+1} \leq |z|^{n+1} \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k = 2|z|^{n+1}.$$

On the other hand, we see

$$\begin{aligned}
 |1 - E_n(z)| &= |1 - \exp(z')| \\
 &= \left| - \sum_{k=1}^{\infty} \frac{(z')^k}{k!} \right| \\
 &\leq \sum_{k=1}^{\infty} \frac{|z'|^k}{k!} \\
 &= \exp(|z'|) - 1.
 \end{aligned}$$

However, $n \geq 0$, so $|z'| \in [0, 1]$. Thus, because \exp is concave up, we see that

$$\exp(|z'|) - 1 \leq (1 - |z'|)(\exp(0) - 1) + |z'|(\exp(1) - 1) < e|z'|.$$

In total, we conclude $|1 - E_n(z)| \leq e|z'| \leq 2e|z|^{n+1}$. ■

Lemma B.16. Given a nonnegative integer n , we have $|E_n(z)| \geq \exp(-2|z|^{n+1})$ if $|z| \leq 1/2$.

Proof. As in the previous proof, $|z| \leq 1/2$ allows us to define \log by power series so that

$$E_n(z) = \exp\left(\sum_{k>n} \frac{z^k}{k}\right).$$

Again, we set $z' := \sum_{k>n} \frac{z^k}{k}$ for brevity and compute

$$\begin{aligned}
 |E_n(z)| &= |\exp(z')| \\
 &= \exp(\operatorname{Re} z') \\
 &\geq \exp(-|z'|) \\
 &= \exp\left(-\sum_{k=n+1}^{\infty} \frac{|z|^k}{k}\right) \\
 &= \exp\left(-|z|^{n+1} \sum_{k=0}^{\infty} \frac{|z|^k}{k+n+1}\right) \\
 &\geq \exp\left(-|z|^{n+1} \sum_{k=0}^{\infty} (1/2)^k\right) \\
 &= \exp(-2|z|^{n+1}),
 \end{aligned}$$

which is what we wanted. ■

Lemma B.17. Given a nonnegative integer n , we have $|E_n(z)| \geq |1 - z| \exp(-2|z|^n)$ for $|z| \geq 1/2$.

Proof. Similar to the previous proof, we see

$$\left| \exp\left(\sum_{k=1}^n \frac{z^k}{k}\right) \right| = \exp\left(\operatorname{Re} \sum_{k=1}^n \frac{z^k}{k}\right) = \exp\left(\sum_{k=1}^n \frac{(\operatorname{Re} z)^k}{k}\right) \geq \exp\left(-\sum_{k=1}^n \frac{|z|^k}{k}\right).$$

However, we see

$$\sum_{k=1}^n \frac{|z|^k}{k} \leq |z|^n \sum_{k=1}^n \frac{|z|^{k-n}}{k} \leq |z|^n \sum_{k=1}^n \frac{(1/2)^{k-n}}{1} < |z|^n \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2|z|^n.$$

Combining,

$$|E_n(z)| = |1 - z| \left| \exp \left(\sum_{k=1}^n \frac{z^k}{k} \right) \right| \geq |1 - z| \exp \left(- \sum_{k=1}^n \frac{|z|^k}{k} \right) \geq |1 - z| \exp(-2|z|^n),$$

which is what we wanted. ■

B.4 Hadamard Factorization

Throughout this section, $f: \mathbb{C} \rightarrow \mathbb{C}$ will be a nonzero entire function of (finite) order ρ_0 ; set $n := \lfloor \rho_0 \rfloor$ for brevity. We will also let $\{z_k\}_{k \in \lambda}$ denote the nonzero zeroes of f (with multiplicity), where λ is either finite or \mathbb{N} , ordered by magnitude. Notably, Remark B.2 tells us that there are only finitely zeroes with fixed bounded magnitude, so such an ordering possible.

The key to the proof will be the following lower bound on a product of elementary factors.

Lemma B.18. Fix a nonnegative real number ρ_0 , and set $n := \rho_0$. Further, fix a sequence $\{z_k\}_{k \in \lambda}$ of countably many nonzero such that

$$n_f(r) := \#\{k : z_k < r\} \ll r^{\rho_0 + \varepsilon}$$

for any $\varepsilon > 0$. Then given a real number s such that $\rho_0 < s < n + 1$, there is some $c > 0$ such that

$$\left| \prod_{k \in \lambda} E_n(z/z_k) \right| \geq \exp(-c|z|^s) \quad \text{for} \quad |z - z_k| \geq |z_k|^{-n-1}$$

for $|z|$ sufficiently large, and the infinite product converges absolutely and uniformly on compacts to an entire function for all $z \in \mathbb{C}$.

Proof. We begin by showing that the infinite product converges absolutely and uniformly on compacts to an entire function; we use Proposition A.25. Indeed, for any compact subset $D \subseteq \mathbb{C}$, we know D is bounded, so find R such that $D \subseteq B(0, R)$. Then we are able to divide the infinite product into

$$\prod_{k \in \lambda} E_n(z/z_k) = \underbrace{\left(\prod_{|z_k| \leq 2R} E_n(z/z_k) \right)}_{P_1(z)} \underbrace{\left(\prod_{|z_k| > 2R} E_n(z/z_k) \right)}_{P_2(z)}$$

because the z_k have been ordered by magnitude. Now, the $P_1(z)$ factor is finite and will therefore not affect our convergence, so we focus on showing that P_2 converges absolutely and uniformly on D . Well, by Lemma B.15, we see that any z_k with $|z_k| > 2R$ gives $|z/z_k| < 1/2$, so

$$|1 - E_n(z/z_k)| \leq 2e|z/z_k|^{n+1} \leq \frac{2R^{n+1}}{|z_k|^{n+1}},$$

but

$$\sum_{k \in \lambda} \frac{2R^{n+1}}{|z_k|^{n+1}} = 2R^{n+1} \sum_{k \in \lambda} \frac{1}{|z_k|^{n+1}}$$

converges by hypothesis. Thus, Proposition A.25 kicks in to tell us that our infinite product converges absolutely and uniformly on D .

It remains to show the lower bound. Quickly, note that no term of the product will ever vanish by definition of the elementary factors and the fact that $|z - z_k| \geq |z_k|^{-n-1}$ for each z_k . This is somewhat technical.

As above, we divide the product into factors depending on $|z|$, though here we write

$$\prod_{k \in \lambda} E_n(z/z_k) = \underbrace{\left(\prod_{|z_k| \leq 2|z|} E_n(z/z_k) \right)}_{Q_1(z) :=} \underbrace{\left(\prod_{|z_k| > 2|z|} E_n(z/z_k) \right)}_{Q_2(z) :=}.$$

We now bound these terms independently.

- We control Q_2 using Lemma B.16. Indeed, we see

$$\begin{aligned} |Q_2(z)| &= \prod_{|z_k| > 2|z|} |E_n(z/z_k)| \\ &\geq \prod_{|z_k| > 2|z|} \exp(-2|z/z_k|^{n+1}) \\ &\stackrel{*}{=} \exp\left(\sum_{|z_k| > 2|z|} -2|z/z_k|^{n+1}\right) \\ &= \exp\left(-2|z|^{n+1} \sum_{|z_k| > 2|z|} \frac{1}{|z_k|^{n+1}}\right), \end{aligned}$$

and this last sum again converges by Corollary B.13; note $\stackrel{*}{=}$ is valid here by the continuity of \exp and the fact that our infinite product already converges.

To finish our bounding here, we see

$$\frac{1}{|z_k|^{n+1}} \leq \frac{1}{|z_k|^{n+1-s}} \cdot \frac{1}{|z_k|^s} \leq \frac{1}{(2|z|)^{n+1-s}} \cdot \frac{1}{|z_k|^s},$$

because $n+1-s > 0$, so

$$|Q_2(z)| \geq \exp\left(-2^{s-n}|z|^s \sum_{|z_k| > 2|z|} \frac{1}{|z_k|^s}\right) = \exp(-c_1|z|^s)$$

for $c_2 := 2^{s-n} \sum_{|z_k| > 2|z|} 1/|z_k|^s$. (The sum still converges by Corollary B.13.) Note $c_2 > 0$.

- We control Q_1 using Lemma B.17 and the fact that $|z - z_k| \geq |z_k|^{-n-1}$ for each z_k . Indeed, freely rearranging this finite product, we see

$$\begin{aligned} |Q_1(z)| &= \prod_{|z_k| \leq 2|z|} |E_n(z/z_k)| \\ &\geq \prod_{|z_k| \leq 2|z|} \left(\left| 1 - \frac{z}{z_k} \right| \exp(-2|z/z_k|^n) \right) \\ &= \prod_{|z_k| \leq 2|z|} \left| 1 - \frac{z}{z_k} \right| \cdot \exp\left(-2|z|^n \sum_{|z_k| \leq 2|z|} \frac{1}{|z_k|^n}\right). \end{aligned}$$

The left term here will be difficult to bound, but we can control the right term here as we did with Q_2 . Indeed, we see

$$\frac{1}{|z_k|^n} \leq \frac{1}{|z_k|^{n-s}} \cdot \frac{1}{|z_k|^s} \leq \frac{1}{(2|z|)^{n-s}} \cdot \frac{1}{|z_k|^s},$$

because $n-s < 0$, so

$$\exp\left(-2|z|^n \sum_{|z_k| \leq 2|z|} \frac{1}{|z_k|^n}\right) \geq \exp\left(-2^{s-n+1}|z|^s \sum_{|z_k| > 2|z|} \frac{1}{|z_k|^s}\right) = \exp(-c_2|z|^s)$$

for $c'_2 := 2^{s-n+1}|z|^s \sum_{|z_k| > 2|z|} 1/|z_k|^s$. Note $c'_1 > 0$.

We now focus on the left term. By hypothesis on z , we see

$$\begin{aligned} \prod_{|z_k| \leq 2|z|} \left| 1 - \frac{z}{z_k} \right| &= \prod_{|z_k| \leq 2|z|} \left| \frac{z - z_k}{z_k} \right| \\ &\geq \prod_{|z_k| \leq 2|z|} |z_k|^{-n-2} \\ &\geq ((2|z|)^{-n-2})^{n_f(2|z|)} \\ &= \exp(-(n+2) \log(2|z|) n_f(2|z|)). \end{aligned}$$

Now, f has order bounded by $s > \rho_0$ (formally, one must use Remark B.9 here), so hypothesis tells us that $n_f(2|z|) \leq C|z|^s$ for $|z|$ sufficiently large. In total, we set $c'_1 := -C(n+2) \log(2|z|)$ so that

$$\prod_{|z_k| \leq 2|z|} \left| 1 - \frac{z}{z_k} \right| \geq \exp(-c'_1 |z|^s).$$

Thus, for $c_1 := c'_1 + c''_1$, we see $|Q_1(z)| \geq \exp(-c_1 |z|^s)$ for $|z|$ sufficiently large.

In total, we see that

$$\left| \prod_{k \in \lambda} E_n(z/z_k) \right| = |Q_1(z)| \cdot |Q_2(z)| \geq \exp(-(c_1 + c_2) |z|^s)$$

for $|z|$ sufficiently large. This is what we wanted. ■

We will use Lemma B.18 in the following form.

Lemma B.19. Fix everything as above. Then there is an unbounded set of positive real numbers $R \subseteq \mathbb{R}$ and constant $c > 0$ such that

$$\left| \prod_{k \in \lambda} E_n(z/z_k) \right| \geq \exp(-c|z|^s) \quad \text{for} \quad |z| \in R.$$

Proof. The hypotheses on $\{z_k\}$ are satisfied by Proposition B.12 and Corollary B.13. Quickly, note that the infinite product makes sense by Lemma B.18. In order to use Lemma B.18, we let μ denote the Lebesgue measure and note

$$\mu \left(\underbrace{\bigcup_{k \in \lambda} (|z_k| - |z_k|^{-n-1}, |z_k| + |z_k|^{-n-1})}_{S :=} \right) \leq 2 \sum_{k \in \lambda} \frac{1}{|z_k|^{n+1}}$$

converges by Corollary B.13, so $\mu(S)$ is finite. Now, we note that $|z - z_k| < |z_k|^{-n-1}$ for some z_k implies $|z| \in (|z_k| - |z_k|^{-n-1}, |z_k| + |z_k|^{-n-1}) \subseteq S$, so we are roughly looking for real numbers r not in S .

Now, the “sufficiently large” condition on $|z|$ in Lemma B.18 amounts to requiring $|z| > D$ for some $D > 0$. Thus, we set

$$R := \{r > D : r \notin S\}.$$

If R were bounded above, then there would be some M such that any $r > M$ has either $r \leq D$ or $r \in S$, meaning that $(\max\{M, D\}, \infty) \subseteq S$, which is a contradiction because S has finite measure. Thus, R is not bounded above.

Lastly, note that any $z \in \mathbb{C}$ with $|z| \in R$ has $|z|$ sufficiently large and $|z| \notin S$ and thus $|z - z_k| \geq |z_k|^{-n-1}$ for each z_k , giving

$$\left| \prod_{k \in \lambda} E_n(z/z_k) \right| \geq \exp(-c|z|^s) \quad \text{for} \quad |z| = r_m$$

by Lemma B.18. This is what we wanted. ■

In particular, the above lemma will be plugged into the following proposition.

Proposition B.20. Fix an entire function $g: \mathbb{C} \rightarrow \mathbb{C}$ and positive real number $s > 0$. Suppose there is a constant $C > 0$ and an unbounded set of positive real numbers $R \subseteq \mathbb{R}$ such that $\operatorname{Re} g(z) \leq C|z|^s$ if $|z| \in R$. Then g is a polynomial of degree less than or equal to s .

Proof. This proof does not use the notation of the rest of the section. Because g is analytic (by Remark A.10), there is $\varepsilon > 0$ such that we may write

$$g(z) = \sum_{k=0}^{\infty} g_k z^k$$

for $z \in B(0, \varepsilon)$. By differentiating the power series by hand, we see that $g_n = g^{(n)}(0)/n!$ for each $n \geq 0$. However, by Cauchy's integral formula in the form Corollary A.11, we let γ_r denote the counterclockwise circle around $z = 0$ with radius $r \in R$ and see

$$\begin{aligned} g_n &= \frac{g^{(n)}(0)}{n!} \\ &= \frac{1}{2\pi i} \oint_{\gamma_r} \frac{g(z)}{z^{n+1}} dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(re^{i\theta})}{r^{n+1}e^{i\theta(n+1)}} \cdot rie^{i\theta} d\theta \\ &= r^{-n} \cdot \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

To access $\operatorname{Re} g$, we will want to take the conjugate of this. Well, the function $g(z)z^{n-1}$ is holomorphic for $n > 0$, so

$$\begin{aligned} 0 &= \oint_{\gamma_r} g(z)z^{n-1} dz \\ &= \oint_{\gamma_r} g(re^{i\theta}) \cdot r^{n-1}e^{i(n-1)\theta} \cdot rie^{i\theta} d\theta \\ &= r^n \oint_{\gamma_r} g(re^{i\theta}) e^{in\theta} d\theta. \end{aligned}$$

Thus, our conjugate integral is

$$\int_0^{2\pi} \overline{g(re^{i\theta})} e^{-in\theta} d\theta = \overline{\int_0^{2\pi} g(re^{i\theta}) e^{in\theta} d\theta} = 0$$

for $n > 0$. Summing, we see

$$r^n g_n = \frac{1}{2\pi} \int_0^{2\pi} \left(g(re^{i\theta}) + \overline{g(re^{i\theta})} \right) e^{-in\theta} d\theta = \frac{1}{\pi} \int_0^{2\pi} (\operatorname{Re} g)(re^{i\theta}) e^{-in\theta} d\theta$$

for $n > 0$. Now, we would like to use this integral to bound $|g_n|$, but we only have an upper bound on $\operatorname{Re} g$, so some trickery is required. In particular, we note that $\int_0^{2\pi} Cr^s e^{-in\theta} d\theta = 0$ for any $n > 0$, so we actually have

$$r^n g_n = \frac{1}{\pi} \int_0^{2\pi} ((\operatorname{Re} g)(re^{i\theta}) - Cr^s) e^{-in\theta} d\theta.$$

But now the integrand is always negative, so we can upper-bound the magnitude as

$$\begin{aligned} r^n |g_n| &\leq \frac{1}{\pi} \int_0^{2\pi} |(\operatorname{Re} g)(re^{i\theta}) - Cr^s| d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} Cr^s d\theta - \frac{1}{\pi} \int_0^{2\pi} (\operatorname{Re} g)(re^{i\theta}) d\theta \\ &= 2Cr^s - \frac{1}{\pi} \operatorname{Re} g_0. \end{aligned}$$

In particular, we see that $|g_n| \leq 2Cr^{s-n} - r^{-n} \cdot \frac{1}{\pi} \operatorname{Re} g_0$, so for any $n > s$, sending $r \rightarrow \infty$ (possible because R is unbounded) enforces $g_n = 0$. It follows that g is a polynomial of degree at most $\lfloor s \rfloor$ on $B(0, \varepsilon)$ and therefore a polynomial everywhere by analytic continuation. ■

Remark B.21. Intuitively, we expect entire functions to have growth rate which is exponential if they are not polynomial. And indeed, this sort of statement will follow from Hadamard's factorization theorem, so it is not surprising that this is a necessary ingredient to the proof.

We are finally able to state and prove Hadamard's factorization theorem.

Theorem B.22 (Hadamard's factorization). Fix a nonzero entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ of (finite) order ρ_0 ; set $n := \lfloor \rho_0 \rfloor$ for brevity. We also let $\{z_k\}_{k \in \lambda}$ denote the nonzero zeroes of f (with multiplicity), where λ is either finite or \mathbb{N} , ordered by magnitude. Further, let m denote the order of the zero of f at $z = 0$. Then

$$f(z) = e^{g(z)} z^m \prod_{k \in \lambda} E_n(z/z_k)$$

for some polynomial $g(z)$ of degree at most n .

Proof. Set

$$G(z) := \frac{f(z)}{z^m \prod_{k \in \lambda} E_n(z/z_k)}.$$

Note that G has a removable singularity at $z = 0$ because m is the order of the zero of f at $z = 0$. Further, the infinite product in the denominator converges absolutely and uniformly on compacts to an entire function by Lemma B.18. As such, by Proposition A.25, it will vanish exactly on the z_k , so G has only removable singularities at the z_k .

In total, we can continue G to an entire function due to these removable singularities, and G will have no zeroes because all zeroes of the numerator $f(z)$ are correctly cancelled out by the denominator. Thus, Lemma A.16 promises us an entire function $g: \mathbb{C} \rightarrow \mathbb{C}$ such that $G = \exp \circ g$. Thus, we see

$$f(z) = e^{g(z)} z^m \prod_{k \in \lambda} E_n(z/z_k),$$

so it remains to show that g is a polynomial of degree at most n . We would like to use Proposition B.20 to finish, so we want to bound g .

Choose some $s \in (\rho_0, n+1)$ so that f has order bounded by s by Remark B.9. By Remark B.11, we see that $f(z)/z^m$ still has order bounded by s , so we can find constants $A, B > 0$ such that

$$\left| \frac{f(z)}{z^m} \right| < A e^{B|z|^s}.$$

Now, by Lemma B.19, there is an unbounded set $R \subseteq \mathbb{R}$ of positive real numbers and constant $c > 0$ such that we can write

$$\left| \prod_{k \in \lambda} E_n(z/z_k) \right| \geq \exp(-c|z|^s)$$

if $|z| \in R$, so

$$|G(z)| = \left| \frac{f(z)}{z^m} \right| \cdot \left| \prod_{k \in \lambda} E_n(z/z_k) \right|^{-1} < A e^{(B+c)|z|^s}.$$

However, $|G(z)| = |\exp(g(z))| = \exp(\operatorname{Re} g(z))$, so we see that

$$\operatorname{Re} g(z) \leq \log A + (B+c)|z|^s$$

for $|z| \in R$. Replacing R with $R \cap (1, \infty)$ (which is an unbounded subset if R is unbounded), we may assume that $|z| > 1$, so actually

$$\operatorname{Re} g(z) \leq (\log A)|z|^s + (B+c)|z|^s,$$

so Proposition B.20 now kicks in and tells us that $g(z)$ is a polynomial of degree less than or equal to s . Because $\lfloor s \rfloor = n$, this finishes. ■

APPENDIX C

FOURIER ANALYSIS

*Ring around the rosie,
A pocket full of posies.
Ashes! Ashes!
We all fall down!*

—“Ring Around the Rosie”

C.1 The Fourier Transform

It will pay off for us later to have established a little Fourier analysis right now. Our exposition follows [SS03b, Chapter 5].

Definition C.1 (Schwarz). A function $f: \mathbb{R} \rightarrow \mathbb{C}$ is *Schwarz* if and only if f is infinitely differentiable and the n th derivative $f^{(n)}$ satisfies that the function $x^A \cdot f^{(n)}(x)$ is bounded for all nonnegative integers A .

Remark C.2. Because the linear combination of bounded sets remains bounded, we see that Schwarz functions form a \mathbb{C} -vector space. Also, by definition, if f is Schwarz, then any derivative is also Schwarz.

Remark C.3. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Schwarz, we note that $|x^A f(x)|$ is integrable over \mathbb{R} for any $A \geq 0$. Indeed, let k be an integer greater than $A + 2$, and we are granted a constant C such that $|x^k f(x)| \leq C$. Thus,

$$\int_{\mathbb{R}} |x^A f(x)| dx \leq \int_{[-1,1]} |x^A f(x)| dx + \int_{|x| \geq 1} \frac{C}{x^{A-k}} dx,$$

which is finite because $A - k < -2$.

Definition C.4 (Fourier transform). Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a Schwarz function. Then we define the *Fourier transform* to be the function $\mathcal{F}f: \mathbb{R} \rightarrow \mathbb{C}$ given by

$$(\mathcal{F}f)(s) := \int_{\mathbb{R}} f(x) e^{-2\pi i s x} dx.$$

Remark C.5. The integral converges because it absolutely converges: we note

$$\int_{\mathbb{R}} |f(x)e^{-2\pi isx}| dx = \int_{\mathbb{R}} |f(x)| dx$$

is finite by Remark C.3. In fact, this shows that $\mathcal{F}f$ is bounded.

We now run some quick checks on the Fourier transform.

Lemma C.6. Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a Schwartz function.

- (a) For some $\lambda > 0$, define $f_{\lambda}(x) := f(\lambda x)$. Then $f_{\lambda}: \mathbb{R} \rightarrow \mathbb{C}$ is Schwartz, and $(\mathcal{F}f_{\lambda})(s) = \frac{1}{\lambda}(\mathcal{F}f)\left(\frac{s}{\lambda}\right)$.
- (b) We have $(\mathcal{F}f')(s) = 2\pi is(\mathcal{F}f)(s)$.
- (c) The function $\mathcal{F}f$ is differentiable, and $(\mathcal{F}f)'(s)$ is the Fourier transform of the function $g(x) := -2\pi ix f(x)$.
- (d) The function $\mathcal{F}f$ is Schwarz.

Proof. We show these one at a time.

- (a) To show f_{λ} is Schwarz, we note $f_{\lambda}^{(n)}(x) = \lambda^n f^{(n)}(\lambda x)$ for all $n \geq 0$, so $x^n \cdot f_{\lambda}^{(n)}(x)$ is bounded because $(\lambda x)^n f^{(n)}(\lambda x)$ is. The last equality is a direct computation. We see

$$\begin{aligned} (\mathcal{F}f_{\lambda})(s) &= \int_{\mathbb{R}} f_{\lambda}(x)e^{-2\pi isx} dx \\ &= \int_{\mathbb{R}} f(\lambda x)e^{-2\pi i(s/\lambda)\lambda x} dx \\ &= \frac{1}{\lambda} \int_{\mathbb{R}} f(x)e^{-2\pi i(s/\lambda)x} dx \\ &= \frac{1}{\lambda}(\mathcal{F}f)\left(\frac{s}{\lambda}\right). \end{aligned}$$

- (b) Note f' is Schwarz by Remark C.2, so the statement at least makes sense. Now, by integration by parts, we see

$$\begin{aligned} (\mathcal{F}f')(s) &= \int_{\mathbb{R}} f'(x)e^{-2\pi isx} dx \\ &= f(x)e^{-2\pi isx} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)(2\pi is e^{-2\pi isx}) dx \\ &= 2\pi is(\mathcal{F}f)(s). \end{aligned}$$

To justify the last equality, we see that $f(x)e^{-2\pi isx} \rightarrow 0$ as $x \rightarrow \pm\infty$ because f is Schwarz: note $|f(x)e^{-2\pi isx}| = |f(x)|$, and $|xf(x)|$ is bounded, so there is a constant C such that $f(x) \leq C/|x|$ for all $x \neq 0$, meaning that $|f(x)| \rightarrow 0$ as $x \rightarrow \pm\infty$.

- (c) Note g is the product of infinitely differentiable functions and thus infinitely differentiable. Further, by induction, derivatives of g are the \mathbb{C} -linear of terms of the form $x^k f^{(\ell)}(x)$. Thus, for any integers $k, \ell \geq 0$, the function $|x|^k |g^{(k)}(x)|$ is a \mathbb{C} -linear combination of bounded functions because f is Schwarz, so it follows that $|x|^k |g^{(\ell)}(x)|$ is bounded, so g is Schwarz.

The rest of the proof is a direct computation. For $t, t' \in \mathbb{R}$, we see

$$\int_t^{t'} (\mathcal{F}g)(s) ds = \int_t^{t'} \left(\int_{\mathbb{R}} -2\pi ix f(x)e^{-2\pi isx} dx \right) ds.$$

We would like to exchange the two integrals. Well, by Fubini's theorem, we note that $\int_{\mathbb{R}} |xf(x)| dx < \infty$ is finite by Remark C.3, so

$$\int_t^{t'} \left(\int_{\mathbb{R}} |-2\pi i x f(x) e^{-2\pi i s x}| dx \right) ds \leq 2\pi |t' - t| \int_{\mathbb{R}} |xf(x)| dx < \infty.$$

Thus, we may write

$$\begin{aligned} \int_t^{t'} (\mathcal{F}g)(s) ds &= \int_{\mathbb{R}} \left(\int_t^{t'} -2\pi i x f(x) e^{-2\pi i s x} ds \right) dx \\ &= \int_{\mathbb{R}} f(x) \left(e^{-2\pi i t' x} - e^{-2\pi i t x} \right) dx \\ &= (\mathcal{F}f)(t') - (\mathcal{F}f)(t). \end{aligned}$$

Thus, by the Fundamental theorem of calculus, we see

$$(\mathcal{F}f)'(t) = \lim_{t' \rightarrow t} \frac{(\mathcal{F}f)(t') - (\mathcal{F}f)(t)}{t' - t} = \lim_{t' \rightarrow t} \left(\frac{1}{t' - t} \int_t^{t'} (\mathcal{F}g)(s) ds \right) = (\mathcal{F}g)(t).$$

- (d) By Remark C.5, the Fourier transform of a Schwarz function is bounded. Thus, it suffices to note that, for any nonnegative integers k and ℓ , the function $s \mapsto s^k (\mathcal{F}f)^{(\ell)}(s)$ is the Fourier transform of the function

$$x \mapsto \frac{1}{(2\pi i)^k} \left(\frac{d}{dx'} \right)^{(k)} \left((-2\pi i x')^{\ell} f(x') \right) \Big|_{x'=x}$$

by combining (b) and (c). This completes the proof. ■

As an application, we can compute the Fourier transform of the Gaussian.

Exercise C.7 (Gaussian). Define $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) := e^{-\pi x^2}$. Then g is Schwarz, and $(\mathcal{F}g)(x) = g(x)$.

Proof. We build a differential equation that $\mathcal{F}g$ solves, and then we solve that differential equation. Namely, by using Lemma C.6 repeatedly, we see

$$\begin{aligned} (\mathcal{F}g)'(s) &= \int_{\mathbb{R}} -2\pi i x g(x) e^{-2\pi i s x} dx \\ &= i \int_{\mathbb{R}} g'(x) e^{-2\pi i s x} dx \\ &= i(\mathcal{F}g')(s) \\ &= -2\pi x (\mathcal{F}g)(s), \end{aligned}$$

so $(\mathcal{F}g)$ solves the differential equation $y' + 2\pi y = 0$. To solve this differential equation, we define $f(x) := (\mathcal{F}g)(x)e^{\pi x^2}$ and use the differential equation to write

$$f'(x) = (\mathcal{F}g)(x) \cdot 2\pi x e^{\pi x^2} - 2\pi x (\mathcal{F}g)(x) \cdot e^{\pi x^2} = 0.$$

Thus, f is a constant function, so there exists $a \in \mathbb{C}$ such that $(\mathcal{F}g)(x) = a e^{-\pi x^2}$ for all $x \in \mathbb{R}$.

To finish, we need to introduce an initial condition. Well, we compute $(\mathcal{F}g)(0) = 1$ in the usual way,

writing

$$\begin{aligned}
 (\mathcal{F}g)(0)^2 &= \left(\int_{\mathbb{R}} e^{-\pi x^2} dx \right)^2 \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi(x^2+y^2)} dx dy \\
 &= \int_0^{2\pi} \int_0^\infty e^{-\pi r^2} r dr d\theta \\
 &= \int_0^{2\pi} \frac{1}{2\pi} d\theta \\
 &= 1.
 \end{aligned}$$

However, surely $(\mathcal{F}g)(0) \geq 0$, so we conclude $(\mathcal{F}g)(0) = 1$. It follows that $a = 1$, so $(\mathcal{F}g)(x) = e^{-\pi x^2}$ for all $x \in \mathbb{R}$. ■

C.2 Fourier Inversion

The goal of this subsection is to prove the Fourier inversion theorem; we continue to roughly follow [SS03b, Chapter 5]. Roughly speaking, this will follow from understanding the Gaussian. Here are the necessary tools.

Lemma C.8. Define the Gaussian $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) := e^{-\pi x^2}$. For any $\delta > 0$, we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{|t| \geq \delta} g(t/\varepsilon) dt = 0.$$

Proof. Changing variables, we see

$$\lim_{\varepsilon \rightarrow 0^+} \int_{|t| \geq \delta} g(t/\varepsilon) dt = \lim_{\varepsilon \rightarrow 0^+} \int_{|t| \geq \delta/\varepsilon} g(t) dt = \lim_{N \rightarrow \infty} \int_{|t| \geq N} g(t) dt,$$

where $N = \delta/\varepsilon$ in the last equality. However, g is Schwarz by Exercise C.7, so $\int_{\mathbb{R}} g(t) dt$ is finite by Remark C.3, so

$$\lim_{N \rightarrow \infty} \int_{|t| \leq N} g(t) dt = \int_{\mathbb{R}} g(t) dt.$$

Rearranging, we see

$$\lim_{N \rightarrow \infty} \int_{|t| \geq N} g(t) dt = 0,$$

which is what we wanted. ■

Lemma C.9. Define the Gaussian $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) := e^{-\pi x^2}$. For any bounded and continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$, we have

$$f(0) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}} f(t) g(t/\varepsilon) dt.$$

Proof. The point here is that, for any $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon} \int_{\mathbb{R}} g(t/\varepsilon) dt = \int_{\mathbb{R}} g(t) dt = (\mathcal{F}g)(0) = g(0) = 1 \tag{C.1}$$

by Exercise C.7. However, the functions $t \mapsto g(t/\varepsilon)$ concentrate at $t = 0$ as $\varepsilon \rightarrow 0^+$, so we expect that adding in an $f(t)$ to our integral will force the output to be $f(0)$.

As an aside, we go ahead and check that these integrals converge for each $\varepsilon > 0$. Indeed, they absolutely converge: because f is bounded, we may find $M_f \in \mathbb{R}$ such that $|f(x)| \leq M_f$ for each $x \in \mathbb{R}$, which gives

$$\int_{\mathbb{R}} |f(t)g(t/\varepsilon)| dt \leq M_f \int_{\mathbb{R}} g(t/\varepsilon) dt = \varepsilon M_f,$$

where we have used (C.1).

We now proceed with the proof, which is somewhat technical. For psychological reasons, we set $h(x) := f(x) - f(0)$ for all $x \in \mathbb{R}$. Note h is still bounded and continuous (it's a shift away from f). Further, for each $\varepsilon > 0$, we see

$$\frac{1}{\varepsilon} \int_{\mathbb{R}} h(t)g(t/\varepsilon) dt = \frac{1}{\varepsilon} \int_{\mathbb{R}} f(t)g(t/\varepsilon) dt - \frac{f(0)}{\varepsilon} \int_{\mathbb{R}} g(t/\varepsilon) dt = \frac{1}{\varepsilon} \int_{\mathbb{R}} f(t)g(t/\varepsilon) dt - f(0),$$

where we have used (C.1) in the last equality, so it suffices to show

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}} h(t)g(t/\varepsilon) dt \stackrel{?}{=} 0.$$

Well, fix any $\delta > 0$. Note h is continuous at 0 and has $h(0) = 0$, so we may find $\delta_0 > 0$ such that $|h(t)| < \delta$ for $|t| < \delta_0$. For the other values of t , we note h is bounded, so we may find $M_h \geq 0$ such that $|h(t)| < M_h$ for all t . Thus, we upper-bound

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}} h(t)g(t/\varepsilon) dt \right| &\leq \frac{1}{\varepsilon} \int_{|t| \leq \delta_0} |h(t)g(t/\varepsilon)| dt + \frac{1}{\varepsilon} \int_{|t| \geq \delta_0} |h(t)g(t/\varepsilon)| dt \\ &\leq \frac{\delta}{\varepsilon} \int_{|t| \leq \delta_0} g(t/\varepsilon) dt + \frac{M_h}{\varepsilon} \int_{|t| \geq \delta_0} g(t/\varepsilon) dt \\ &\leq \frac{\delta}{\varepsilon} \int_{\mathbb{R}} g(t/\varepsilon) dt + \frac{M_h}{\varepsilon} \int_{|t| \geq \delta_0} g(t/\varepsilon) dt \\ &= \delta + \frac{M_h}{\varepsilon} \int_{|t| \geq \delta_0} g(t/\varepsilon) dt. \end{aligned}$$

(As usual, we have used (C.1) in the last equality.) Thus, using Lemma C.8, sending $\varepsilon \rightarrow 0^+$ shows that

$$\lim_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}} h(t)g(t/\varepsilon) dt \right| \leq \delta$$

for any $\delta > 0$, so sending $\delta \rightarrow 0^+$ completes the proof. ■

And here is our main attraction.

Theorem C.10 (Fourier inversion). Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a Schwarz function. For any $x \in \mathbb{R}$, we have

$$f(x) = \int_{\mathbb{R}} (\mathcal{F}f)(s) e^{2\pi i x s} ds.$$

Proof. Expanding out the definition of $\mathcal{F}f$, we are computing

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t) e^{-2\pi i t s} dt \right) e^{2\pi i x s} ds.$$

We would like to exchange the two integrals, but we do not have absolute convergence. As such, we employ a trick: fix some $\varepsilon > 0$, and define the integral

$$f_{\varepsilon}(x) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) e^{2\pi i (x-t)s} e^{-\pi \varepsilon^2 s^2} dt ds.$$

Notably, we expect $f_\varepsilon(x) \rightarrow \int_{\mathbb{R}} (\mathcal{F}f)(s) e^{2\pi i x s} ds$ as $\varepsilon \rightarrow 0^+$. As such, we compute the behavior of $\varepsilon \rightarrow 0^+$ in two ways.

- We integrate over dt first. Namely, we would like to send $\varepsilon \rightarrow 0^+$, for which we use the Dominated convergence theorem. For each $\varepsilon > 0$, note that we have the bound

$$\left| \int_{\mathbb{R}} f(t) e^{2\pi i(x-t)s} e^{-\pi \varepsilon^2 s^2} dt \right| \leq e^{-\pi \varepsilon^2 s^2} \int_{\mathbb{R}} |f(t)| dt.$$

Now, $s \mapsto e^{-\pi \varepsilon^2 s^2}$ is Schwarz by Exercise C.7 (combined with (a) of Lemma C.6), so we may integrate the right-hand function over all $s \in \mathbb{R}$ by Remark C.3.

Thus, our integrand in $f_\varepsilon(x)$ is dominated by an integrable function, so the Dominated convergence theorem implies

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x) = \int_{\mathbb{R}} \left(\lim_{\varepsilon \rightarrow 0^+} e^{-\pi \varepsilon^2 s^2} \int_{\mathbb{R}} f(t) e^{2\pi i(x-t)s} dt \right) ds = \int_{\mathbb{R}} (\mathcal{F}f)(s) e^{2\pi i x s} ds.$$

- We integrate over ds first. As such, we begin by justifying our application of Fubini's theorem: checking for absolute convergence, we compute

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(t) e^{2\pi i(x-t)s} e^{-\pi \varepsilon^2 s^2}| dt ds = \left(\int_{\mathbb{R}} |f(t)| dt \right) \left(\int_{\mathbb{R}} e^{-\pi \varepsilon^2 s^2} ds \right).$$

Now, f is Schwarz by hypothesis, as is $s \mapsto e^{-\pi \varepsilon^2 s^2}$ by Exercise C.7, so both of these integrals are finite by Remark C.3.

Thus, we may switch the order of our integration. Setting up notation, we let $g(x) := e^{-\pi x^2}$ denote the Gaussian (so that $(\mathcal{F}g)(s) = g(s)$ for all $s \in \mathbb{R}$) and $g_\varepsilon(x) := g(\varepsilon x)$. Then we see

$$\begin{aligned} f_\varepsilon(x) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) e^{2\pi i(x-t)s} e^{-\pi \varepsilon^2 s^2} ds dt \\ &= \int_{\mathbb{R}} f(t) \left(\int_{\mathbb{R}} e^{-\pi(\varepsilon s)^2} e^{-2\pi i(t-x)s} ds \right) dt \\ &= \int_{\mathbb{R}} f(t) (\mathcal{F}g_\varepsilon)(t-x) dt \\ &\stackrel{*}{=} \frac{1}{\varepsilon} \int_{\mathbb{R}} f(t) g\left(\frac{t-x}{\varepsilon}\right) dt \\ &= \frac{1}{\varepsilon} \int_{\mathbb{R}} f(t+x) g(t/\varepsilon) dt, \end{aligned}$$

where we have used part (a) of Lemma C.6 at $\stackrel{*}{=}$. Sending $\varepsilon \rightarrow 0^+$, Lemma C.9 tells us that

$$f(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}} f(t+x) g(t/\varepsilon) dt = \lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x).$$

Combining the above two computations completes the proof. ■

C.3 Fourier Coefficients

In order to say that we've done some Fourier analysis, we will also say a few things about Fourier series. We follow [SS03b, Chapter 2].

The idea here is that the functions $e_n: x \mapsto e^{2\pi i n x}$ for $n \in \mathbb{Z}$ form an orthonormal set of continuous functions $\mathbb{R} \rightarrow \mathbb{C}$, where our (Hermitian) inner product is given by

$$\langle f, g \rangle := \frac{1}{2\pi i} \int_0^1 f(x) \overline{g(x)} dx.$$

Indeed, for any $n, m \in \mathbb{Z}$, we see

$$\langle e_n, e_m \rangle = \int_0^1 e^{2\pi i n x} \overline{e^{2\pi i m x}} dx = \int_0^1 e^{2\pi i (m-n)x} dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \quad (\text{C.2})$$

Now, the functions e_n are varied enough that we might hope that all sufficiently smooth 1-periodic functions $f: \mathbb{R} \rightarrow \mathbb{C}$ can be written in terms of our orthonormal functions as

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$$

for some coefficients $a_n \in \mathbb{C}$. Thus, we might hope we can extract out the n th coefficient by

$$\langle f, e_n \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

This motivates the following definition.

Definition C.11 (Fourier coefficient). Fix a continuous 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$. Then we define the n th Fourier coefficient as

$$a_n(f) := \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Remark C.12. Note that the integral defining $a_n(f)$ converges absolutely. Indeed, f is continuous on $[0, 1]$ and hence bounded because $[0, 1]$ is compact. Thus, we may find M such that $|f(x)| \leq M$ for $x \in [0, 1]$, which implies

$$|a_n(f)| \leq \int_0^1 |f(x) e^{-2\pi i n x}| dx \leq M \int_0^1 dx = M.$$

Of course, one can weaken the requirement that f be continuous, but we will have no need for these levels of generality.

Remark C.13. In fact, we note

$$a_n(f) = \int_t^{t+1} f(x) e^{2\pi i n x} dx$$

for any $t \in \mathbb{R}$. Because $x \mapsto f(x) e^{2\pi i n x}$ is 1-periodic, it suffices to show this for $t \in [0, 1)$. Then the integral over $[t, t+1) = [t, 1) \sqcup [1, 1+t)$ is equal to the integral over $[0, t) \sqcup [t, 1) = [0, 1)$, where we have used the 1-periodicity.

Here is some basic arithmetic with these coefficients.

Lemma C.14. Fix continuous 1-periodic functions $f, g: \mathbb{R} \rightarrow \mathbb{C}$.

- (a) For any $z, w \in \mathbb{C}$ and $n \in \mathbb{Z}$, we see $a_n(zf + wg) = za_n(f) + wa_n(g)$.
- (b) For any $n \in \mathbb{Z}$, we see $a_n(\bar{f}) = \overline{a_{-n}(f)}$.
- (c) Given $x_0 \in \mathbb{R}$, define $g(x) := f(x + x_0)$. Then $a_n(g) = e^{-2\pi i n x_0} a_n(f)$.

Proof. Here we go.

- (a) This follows from the fact that $\langle \cdot, \cdot \rangle$ is an inner product. Indeed,

$$a_n(zf + wg) = z \int_0^1 f(x) e^{-2\pi i n x} dx + w \int_0^1 g(x) e^{-2\pi i n x} dx = za_n(f) + wa_n(g).$$

(b) We compute

$$a_n(\bar{f}) = \int_0^1 \overline{f(x)} e^{-2\pi i n x} dx = \overline{\int_0^1 f(x) e^{2\pi i n x} dx} = \overline{a_{-n}(f)}.$$

(c) We compute

$$a_n(g) = \int_0^1 f(x + x_0) e^{2\pi i n x} dx = e^{-2\pi i n x_0} \int_0^1 f(x + x_0) e^{2\pi i n (x + x_0)} dx = e^{-2\pi i n x_0} a_n(f),$$

where the last inequality used Remark C.13. ■

Here is a slightly harder computation, still akin to Lemma C.6.

Lemma C.15. Fix a continuously differentiable 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$. For $n \neq 0$, we have

$$a_n(f') = -2\pi i n a_n(f).$$

Proof. This is by integration by parts. Indeed, we compute

$$\begin{aligned} a_n(f') &= \int_0^1 f'(x) e^{-2\pi i n x} dx \\ &= \left. \frac{f(x) e^{-2\pi i n x}}{-2\pi i n} \right|_0^1 - \frac{1}{-2\pi i n} \int_0^1 f(x) e^{-2\pi i n x} dx \\ &= 0 + \frac{1}{2\pi i n} \cdot a_n(f), \end{aligned}$$

which is what we wanted. ■

The following is our key result.

Lemma C.16. Fix a continuous 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(0) \neq 0$. Then $a_n(f) \neq 0$ for some $n \in \mathbb{Z}$.

Proof. Define the function $p: \mathbb{R} \rightarrow \mathbb{C}$ by $p(x) := e^{-2\pi i n x}$. Roughly speaking, the idea is that $a_n(f) = 0$ for all $n \in \mathbb{Z}$ implies that any “polynomial in p ” named $q \in \mathbb{C}[p, p^{-1}]$ written as

$$q := \sum_{n \in \mathbb{Z}} q_n p^n,$$

where all but finitely many of the q_n vanish, will have

$$\int_{-1/2}^{1/2} f(x) q(x) dx = \sum_{n \in \mathbb{Z}} \left(q_n \int_{-1/2}^{1/2} f(x) e^{2\pi i n x} dx \right) = \sum_{n \in \mathbb{Z}} q_n a_n(f) = 0$$

by Remark C.13. Indeed, we will be able to build a function $q \in \mathbb{C}[p, p^{-1}]$ which is “concentrated at 0” so that $f(0) \neq 0$ is incompatible with all these integrals vanishing.

We now proceed with the proof. Quickly, we replace $f(x)$ with $f(x)/f(0)$, which is still continuous, 1-periodic, and has $a_n(f/f(0)) = a_n(f)/f(0)$ for all $n \in \mathbb{Z}$, so $a_n(f/f(0)) \neq 0$ implies $a_n(f) \neq 0$. Thus, we may assume $f(0) = 1$, and we still want to show $a_n(f) \neq 0$ for some n .

We now set up some bounding, in steps.

1. Note f is continuous on the compact set $[-1/2, 1/2]$, so we may find some M_f such that $|f(x)| \leq M_f$ for all $x \in [-1/2, 1/2]$.

2. Because f is continuous, we may find $\delta_f > 0$ such that $|f(x) - 1| \leq 1/2$ for $|x| < \delta_f$. In particular, we see $f(x) \geq 1/2$ for $|x| < \delta_f$. By making δ_f smaller if necessary, we will enforce $\delta_f \leq 1/4$.
3. Now, define $q_1(x) := 2\varepsilon + p(x) + p(x)^{-1} = 2\varepsilon + \cos(2\pi x)$, for $\varepsilon := \frac{2}{3}(1 - \cos(2\pi\delta_f))$. Note $\cos(2\pi x)$ is decreasing in the region in $[\delta_f, 1/2]$, so in fact

$$\varepsilon \leq \frac{1}{3}(1 - \cos(2\pi\delta_f))$$

for $x \in [\delta_f, 1/2]$. Rearranging, we see

$$q_1(x) = 2\varepsilon + \cos(2\pi x) \leq 1 - \varepsilon$$

for $x \in [\delta_f, 1/2]$. In fact, because $q_1(x) \geq -1 + 2\varepsilon$, we see that $|q_1(x)| \leq 1 - \varepsilon$ for $x \in [\delta_f, 1/2]$. Lastly, because q_1 is even, these inequalities hold on $[-1/2, -\delta_f] \cup [\delta_f, 1/2]$.

4. Lastly, choose $\delta_q > 0$ such that $|q_1(x) - q_1(0)| \leq \varepsilon$ for $|x| < \delta_q$. In particular, $q_1(x) \geq 1 - \varepsilon$ for $|x| < \delta_q$. By making δ_q smaller if necessary, we may assume $\delta_q < \delta_f$, though this is actually implied.

To finish, we define $q_N := q_1^N$ for $N \in \mathbb{N}$. (Notably, $q_1 = q_1^1$.) The point is that $k \rightarrow \infty$ makes q_N blow up at 0 around points where f is bounded below by $1/2$, but q_N will vanish elsewhere. Indeed, using Remark C.13, we compute

$$\begin{aligned} \int_{-1/2}^{1/2} f(x)q_N(x) dx &= \int_{|x| \leq \delta_q} f(x)q_N(x) dx + \int_{\delta_q \leq |x| \leq \delta_f} f(x)q_N(x) dx + \int_{\delta_f \leq |x| \leq 1/2} f(x)q_N(x) dx \\ &\geq 2\delta_q \cdot \frac{1}{2} (1 + \varepsilon)^N + 2(\delta_f - \delta_q) \cdot \frac{1}{2} \cdot 0 - 2 \left(\frac{1}{2} - \delta_f \right) B (1 - \varepsilon)^N \\ &\geq \delta_q (1 + \varepsilon)^N - \delta_f B (1 - \varepsilon)^N. \end{aligned}$$

Thus, as $N \rightarrow \infty$, the integral goes to $+\infty$. In particular, we can (in theory) find an (explicit) N such that $\int_{-1/2}^{1/2} f(x)q_N(x) dx > 0$. Now, we may write

$$q_N = (2\varepsilon + p + p^{-1})^N = \sum_{n=-N}^N q_{N,n} p^n$$

for some coefficients $q_{N,n} \in \mathbb{R}$. Thus,

$$0 < \int_{-1/2}^{1/2} f(x)q_N(x) dx = \sum_{n=-N}^N \left(q_{N,n} \int_{-1/2}^{1/2} f(x) e^{-2\pi i n x} dx \right) = \sum_{n=-N}^N q_{N,n} a_n(f),$$

where we have used Remark C.13. Thus, there exists n with $|n| \leq N$ such that $a_n(f) \neq 0$. ■

Proposition C.17. Fix a continuous 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$ such that $a_n(f) = 0$ for all $n \in \mathbb{N}$. Then $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. This follows from Lemma C.16 and the following reductions.

- It suffices to show the result for real-valued functions f . Indeed, we may write $f(x) := u(x) + iv(x)$ for some real-valued, continuous, and 1-periodic functions $u, v: \mathbb{R} \rightarrow \mathbb{R}$. (Namely, $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, and each adjective is inherited from f .) However, for each $n \in \mathbb{N}$, we use Lemma C.14 to see

$$a_n(u) = a_n \left(\frac{f + \bar{f}}{2} \right) = \frac{1}{2} (a_n(f) + \overline{a_{-n}(f)}) = 0,$$

and

$$a_n(v) = a_n\left(\frac{f - \bar{f}}{2i}\right) = \frac{1}{2i} \left(a_n(f) - \overline{a_{-n}(f)} \right) = 0.$$

Thus, if we can prove the result for real-valued functions, we see $a_n(u) = a_n(v) = 0$ for all $n \in \mathbb{Z}$ forces $u = v = 0$, so $f = u + iv = 0$ also.

- It suffices to show that $f(0) = 0$, which is Lemma C.16. Indeed, for some fixed $x_0 \in \mathbb{R}$, we define $g(x) := f(x + x_0)$. Note g is still continuous and 1-periodic. Further, Lemma C.14 tells us that $a_n(g) = e^{2\pi i n x_0} a_n(f) = 0$ for each $n \in \mathbb{Z}$. Thus, Lemma C.16 implies $g(0) = 0$, so $f(x_0) = g(0) = 0$ follows. ■

The point is that we know the linear transformation sending a continuous 1-periodic function f to the tuple of its coefficients $\{a_n(f)\}_{n \in \mathbb{N}}$ is injective. We now expect that we can construct a partial inverse map by sending the tuple of coefficients to the corresponding Fourier series, which is what we show next.

C.4 Fourier Series

Now that we have our coefficients, we can define our Fourier series. We continue to follow [SS03b, Chapter 2].

Definition C.18 (Fourier series). Fix a continuous 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$. Then we define the N th partial sum of the *Fourier series* of f as

$$S_{f,N}(x) := \sum_{n=-N}^N a_n(f) e^{2\pi i n x}.$$

The *Fourier series* is defined as $S_f(x) := \lim_{N \rightarrow \infty} S_{f,N}(x)$, when this limit converges.

The main goal of this subsection is to provide smoothness conditions on f which will imply $f(x) = S(x)$ for all $x \in \mathbb{R}$.

We will begin by figuring out when this series will converge.

Lemma C.19. Fix a twice continuously differentiable 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$. Then the series $S_f(x)$ converges absolutely and uniformly.

Proof. This follows from Lemma C.15. Indeed, for $n \neq 0$, we see that

$$a_n(f) = \frac{1}{-2\pi i n} \cdot a_n(f') = \frac{a_n(f'')}{4\pi^2 n^2}.$$

Because f'' is continuous, Remark C.12 grants $M \in \mathbb{R}$ such that $|a_n(f'')| \leq M$, so it follows that $|a_n(f)| \leq M/(4\pi^2 n^2)$ for $n \neq 0$. Thus, we see the series S_f converges absolutely because

$$\sum_{n \in \mathbb{Z}} |a_n(f) e^{2\pi i n x}| \leq a_0(f) + \frac{2M}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

for any $x \in \mathbb{R}$. To get the uniform convergence, for any $N \in \mathbb{N}$, we compute

$$|S_f(x) - S_{f,N}(x)| = \left| \sum_{|n| > N} a_n(f) e^{2\pi i n x} \right| \leq \sum_{|n| > N} \frac{M}{4\pi^2 n^2} = \frac{2M}{4\pi^2} \sum_{n > N} \frac{1}{n^2} < \frac{2M}{4\pi^2} \int_N^{\infty} \frac{1}{t^2} dt = \frac{2M}{4\pi^2 N},$$

which does vanish as $N \rightarrow \infty$. ■

And in this situation, we can show that our Fourier series is well-behaved.

Theorem C.20. Fix a continuous 1-periodic function $f: \mathbb{R} \rightarrow \mathbb{C}$. If the series S_f converges absolutely and uniformly, then $S_f(x) = f(x)$ for all $x \in \mathbb{R}$.

Proof. The point is to show that $a_n(S_f) = a_n(f)$ for all $n \in \mathbb{Z}$ so that the result will follow from Proposition C.17.

Quickly, note that the uniform convergence provided by hypothesis implies that S_f is a continuous function because the partial sums $S_{f,N}$ are continuous. Further, S_f is 1-periodic: for any $x \in \mathbb{R}$, we see

$$S_f(x+1) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n(f) e^{2\pi i n(x+1)} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n(f) e^{2\pi i n x} = S_f(x).$$

Thus, we are allowed to compute the Fourier coefficients

$$a_n(S_f) = \int_0^1 \left(\sum_{m \in \mathbb{Z}} a_m(f) e^{2\pi i(m-n)x} \right) dx$$

for $n \in \mathbb{Z}$. We would like to exchange the sum and the integral, for which we use Fubini's theorem. Indeed, we see

$$\int_0^1 \left(\sum_{m \in \mathbb{Z}} |a_m(f) e^{2\pi i(m-n)x}| \right) dx = \left(\int_0^1 dx \right) \sum_{m \in \mathbb{Z}} |a_m(f)| = \sum_{m \in \mathbb{Z}} |a_m(f)|,$$

which converges because $S_f(0)$ converges absolutely by hypothesis. Thus, Fubini's theorem lets us write

$$a_n(S_f) = \sum_{m \in \mathbb{Z}} \left(\int_0^1 a_m(f) e^{2\pi i(m-n)x} dx \right) = a_n(f),$$

where we have used (C.2) in the last equality. To finish the proof, we note $a_n(S_f - f) = 0$ by Lemma C.14. As such, $S_f - f = 0$ by Proposition C.17, which finishes the proof. ■

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