

# 18.755: Lie Groups and Lie Algebras II

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*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

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# THEME 1

# INTRODUCTION

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## 1.1 February 2

Here we go.

### 1.1.1 Review of Lie Groups

We start with some quick review. Here are our groups.

**Definition 1.1** (Lie group). A *Lie group* is a group object  $G$  in the category of manifolds. One may specify a “real” or “complex” Lie group, which means that we are taking the category of real or complex manifolds. Explicitly, we are asking for  $G$  to be equipped with regular maps  $m: G \times G \rightarrow G$ ,  $i: G \rightarrow G$ , and an identity. A *homomorphism of Lie groups* is a morphism of the group objects.

**Example 1.2.** One has the usual examples:  $\mathbb{R}^n$ ,  $U(n)$ ,  $Sp_{2n}(\mathbb{R})$ ,  $O(p, q)$ , and  $SU(n)$  are all real Lie groups.

**Example 1.3.** There are classical groups over  $\mathbb{C}$ , such as  $SL_n(\mathbb{C})$ , which are all Lie groups.

**Definition 1.4.** If  $G$  is a Lie group, then its connected component  $G^\circ$  is a normal Lie subgroup.

**Remark 1.5.** The quotient  $\pi_0 G := G/G^\circ$  is a discrete topological group.

**Remark 1.6.** Given a Lie group  $G$ , the universal cover  $\tilde{G} \rightarrow G$  can be checked to a Lie group via some universal properties, so we receive a homomorphism  $\pi: \tilde{G} \rightarrow G$ . It turns out that the kernel is a central discrete subgroup  $Z \subseteq \tilde{G}$ . It notably follows that  $\pi_1(G)$  is abelian.

**Remark 1.7.** One can check that  $G^\circ$  is generated by any open neighborhood of the identity. Indeed, the generated subgroup can be seen to be both open and closed.

**Example 1.8.** With  $G = S^1$ , we have the universal cover  $\tilde{G} = \mathbb{R}$ , and the kernel is  $\mathbb{Z} \subseteq \mathbb{R}$ .

We also have subgroups.

**Definition 1.9** (Lie subgroup). A *Lie subgroup* is an immersed submanifold  $H \subseteq G$  which is also a subgroup, meaning that  $H \hookrightarrow G$  admits injective differentials. A *closed Lie subgroup* is an embedded submanifold  $H \subseteq G$  which is also a subgroup.

**Remark 1.10.** It turns out that closed Lie subgroups are in fact closed subsets, which can be checked locally.

**Example 1.11.** The subgroup  $\mathbb{Q}^n \subseteq \mathbb{R}^n$  is a Lie subgroup, but it is not a closed Lie subgroup. The only closed Lie subgroups are vector spaces.

**Example 1.12.** The subgroup  $O_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$  is a closed real Lie subgroup.

**Remark 1.13.** It turns out that a closed subgroup of  $G$  is in fact a closed Lie subgroup. We will prove this later in the semester.

**Definition 1.14** (quotient). Fix a closed Lie subgroup  $H \subseteq G$ . Then  $G/H$  is a manifold with transitive  $G$ -action. If  $H$  is normal, then  $G/H$  is further a Lie group.

**Remark 1.15.** In general, if  $G$  acts transitively on a manifold  $X$ , then for any  $x \in X$ ,  $Stab_G(x) \subseteq G$  is a closed Lie subgroup, and the quotient is isomorphic to  $X$ .

**Remark 1.16.** If  $G$  acts on a space  $X$  which is not transitive, then for any  $x \in X$ , the subset  $Gx \subseteq X$  is at least an immersed submanifold.

**Example 1.17.** The group  $\mathbb{R}$  has an action on  $\mathbb{R}^2/\mathbb{Z}^2$  by  $t: x \mapsto tx$ . The orbit of (say),  $(1/2, \sqrt{2}/2)$  is an immersed but not closed submanifold.

**Definition 1.18** (representation). Fix a Lie group  $G$ . A *representation* of a Lie group is a homomorphism  $G \rightarrow GL_n(\mathbb{C})$ .

**Example 1.19.** Let  $G$  act on itself by conjugation. Then each  $g \in G$  acts on  $T_1G \rightarrow T_1G$ , so we receive an adjoint representation  $Ad_\bullet: G \rightarrow GL(T_1G)$ .

As usual, one can define morphisms of representations, subrepresentations, direct sums, duals, tensor products, irreducible representations, and so on. We also have a Schur's lemma.

**Lemma 1.20.** Fix irreducible representations  $V$  and  $W$  of  $G$ .

- (a) Then a  $G$ -equivariant map  $\varphi: V \rightarrow W$  is either zero or an isomorphism.
- (b) Any  $G$ -equivariant map  $A \rightarrow A$  is a scalar.

*Proof.* Omitted. ■

**Definition 1.21 (unitary).** A *unitary representation* is one admitting a  $G$ -invariant positive-definite Hermitian form.

**Remark 1.22.** Any unitary representation admits a decomposition into irreducible representations by taking orthogonal complements.

**Non-Example 1.23.** Let  $B \subseteq \mathrm{GL}_2(\mathbb{C})$  be the subgroup of upper-triangular matrices. Then the standard representation of  $B$  does not admit a decomposition into irreducibles, so it cannot be made unitary.

**Example 1.24.** If  $G$  is finite, then any representation  $V$  admits a unitary structure: given any unitary structure  $\langle -, - \rangle_0$ , one can define an invariant unitary structure

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle_0,$$

where  $dg$  is a choice of Haar measure.

**Theorem 1.25 (Maschke).** Fix a finite group  $G$ . Then all representations admit decomposition into irreducible representations.

*Proof.* This follows from Example 1.24. ■

### 1.1.2 Review of Lie Algebras

We now linearize our story.

**Remark 1.26.** Note that  $G$  acts on itself by left translations  $\ell_g$ , so the tangent bundle  $TG$  can be given a global frame by the induced isomorphisms  $d\ell_g: T_1 G \rightarrow T_g G$ .

**Notation 1.27.** For each  $a \in T_1 G$ , we define the vector field  $L_a$  by

$$L_a := ga \in T_a G.$$

**Remark 1.28.** One can check that all left-invariant vector fields take the form  $L_a$ .

**Definition 1.29 (commutator).** Fix a Lie group  $G$ . For each  $a, b \in T_1 G$ , we may take the commutator  $[L_a, L_b]$  to produce another left-invariant vector field, which we label  $L_{[a,b]}$ .

**Remark 1.30.** The formalism of the commutator tells us that  $[-, -]$  is antisymmetric and satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

**Definition 1.31 (Lie algebra).** Fix a vector space  $\mathfrak{g}$  over a field  $k$ . Then a *Lie algebra* is such a vector space  $\mathfrak{g}$  equipped with an antisymmetric pairing  $[-, -]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

**Example 1.32.** For any Lie group  $G$ , we have seen that we may equip  $\text{Lie } G := T_1 G$  with the structure of a Lie group.

**Example 1.33.** If  $G = \text{GL}_n(\mathbb{C})$ , then  $\mathfrak{g} = M_n(\mathbb{C})$ , and one can check that  $[X, Y] = XY - YX$ .

We now define Lie subalgebras and morphisms of Lie algebras in the expected way.

**Definition 1.34** (Lie ideal). Fix a Lie algebra  $\mathfrak{g}$ . Then a *Lie ideal*  $\mathfrak{h} \subseteq \mathfrak{g}$  is a subspace for which  $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ .

**Example 1.35.** For any closed Lie subgroup  $H \subseteq G$ , we see that  $\text{Lie } H \subseteq \text{Lie } G$  is a Lie subalgebra. If  $H$  is normal, then  $\text{Lie } H$  is a Lie ideal.

As expected, there is some representation theory.

**Definition 1.36.** Fix a Lie algebra  $\mathfrak{g}$  over a field  $k$ . Then a *representation* of  $\mathfrak{g}$  is a morphism  $\mathfrak{g} \rightarrow \mathfrak{gl}_n(k)$ .

One can relate  $\text{Lie } G$  to  $G$  more directly via exponentiation.

**Definition 1.37** (exponential). Fix a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . We define a map  $\exp: \mathfrak{g} \rightarrow G$  as follows. For each  $a \in \mathfrak{g}$ , one can check that the differential equation

$$\begin{cases} e'(t) = e(t) \cdot a, \\ e(0) = 1, \end{cases}$$

admits a unique solution; we then define  $\exp(ta) := e(t)$ . (This is independent of the choice of  $t$ .) It turns out that  $t \mapsto \exp(ta)$  is a group homomorphism.

**Example 1.38.** If  $G = \text{GL}_n(\mathbb{C})$ , then  $\exp: M_n(\mathbb{C}) \rightarrow \text{GL}_n(\mathbb{C})$  is the usual matrix exponential.

**Remark 1.39.** It turns out that  $\exp$  is a local diffeomorphism (though not necessarily injective), so there is a local inverse  $\log: U \rightarrow \mathfrak{g}$ , where  $U$  is some open neighborhood of the identity.

**Remark 1.40.** For small  $a$  and  $b$ , it turns out that

$$\log(\exp(a) \exp(b)) = a + b + \frac{1}{2}[a, b] + \dots,$$

where  $\dots$  denotes cubic terms. For example, if  $G$  is commutative, then we see that the Lie bracket  $[-, -]$  vanishes; conversely, if  $[-, -]$  vanishes, then  $G$  can be checked to commute in an open neighborhood of the identity, so  $G$  commutes.

### 1.1.3 Fundamental Theorems

In a first course, one checks the following two fundamental theorems.

**Theorem 1.41.** Fix a Lie group  $G$ . Then there is a bijection between connected closed Lie subgroups  $H \subseteq G$  and Lie subalgebras  $\mathfrak{h} \subseteq \text{Lie } G$ .

**Theorem 1.42.** Fix Lie groups  $G$  and  $K$ , with  $G$  simply connected. Then taking the differential

$$\text{Hom}(G, K) \rightarrow \text{Hom}(\text{Lie } G, \text{Lie } K)$$

is an isomorphism.

There is a third fundamental theorem, which we will prove later.

**Theorem 1.43.** For any finite-dimensional Lie algebra  $\mathfrak{g}$  (over  $\mathbb{R}$  or  $\mathbb{C}$ ), then there is a Lie group  $G$  with  $\text{Lie } G \cong \mathfrak{g}$ .

The three theorems provide an equivalence between the category of simply connected Lie groups and the category of Lie algebras, thereby classifying the former.

**Remark 1.44.** It follows that one may classify connected Lie groups as quotients of simply connected Lie groups by discrete central subgroups.

### 1.1.4 Representations of Lie Algebras

Let's start with the representation theory of  $\mathfrak{sl}_2(\mathbb{C})$ .

**Theorem 1.45.** Fix the usual basis  $e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,  $f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ , and  $h := [e, f]$  of  $\mathfrak{sl}_2(\mathbb{C})$ .

- (a) Then all irreducible representations of  $\mathfrak{sl}_2(\mathbb{C})$  can be parameterized as  $\{V_n\}_{n \geq 0}$ , where  $V_n$  is the representation of homogeneous polynomials in  $x$  and  $y$  of degree  $n$ .
- (b) Every representation is a direct sum of irreducible representations.
- (c) Clebsch–Gordon rule: for any  $n$  and  $m$ , we have

$$V_n \otimes V_m = \bigoplus_{i=0}^{\min\{m,n\}} V_{|m-n|+2i}.$$

It will be helpful to turn representation theory of Lie algebras into a module category.

**Definition 1.46 (universal enveloping algebra).** Fix a Lie algebra  $\mathfrak{g}$ . Then we define  $U\mathfrak{g}$  as the quotient of the tensor algebra by the relation

$$[x, y] = x \otimes y - y \otimes x.$$

**Remark 1.47.** It turns out that  $\text{Rep } \mathfrak{g}$  is the same category as  $\text{Mod } U\mathfrak{g}$ .

Even though we have taken a quotient by an inhomogeneous relation,  $U\mathfrak{g}$  still receives a natural filtration by degree.

**Theorem 1.48 (Poincaré–Birkhoff–Witt).** Fix a Lie algebra  $\mathfrak{g}$ , and equip  $U\mathfrak{g}$  with the natural filtration. For any basis  $\{x_1, \dots, x_n\}$  of  $\mathfrak{g}$ , the ordered monomials in the basis form a basis of  $U\mathfrak{g}$ .

To continue our story, we need some adjectives for Lie algebras.

**Definition 1.49 (solvable).** A Lie algebra  $\mathfrak{g}$  is *solvable* if and only if the derived series eventually vanishes. Here, the derived series is defined inductively by  $D^0(\mathfrak{g}) := \mathfrak{g}$  and  $D^{n+1}(\mathfrak{g}) := [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$  for each  $n \geq 0$ .

**Definition 1.50 (nilpotent).** A Lie algebra  $\mathfrak{g}$  is *nilpotent* if and only if the lower central series eventually vanishes. Here, the derived series is defined inductively by  $L_0(\mathfrak{g}) := \mathfrak{g}$  and  $L_{n+1}(\mathfrak{g}) := [L_n(\mathfrak{g}), \mathfrak{g}]$  for each  $n \geq 0$ .

**Remark 1.51.** One can see that nilpotent implies solvable.

The representation theory of solvable Lie algebras is quite easy.

**Theorem 1.52 (Lie).** Fix a finite-dimensional solvable Lie algebra  $\mathfrak{g}$  over an algebraically closed field of characteristic zero.

- (a) Then every irreducible representation of  $\mathfrak{g}$  is one-dimensional.
- (b) Every representation admits a basis on which  $\mathfrak{g}$  acts by upper-triangular matrices.

**Theorem 1.53 (Engel).** Fix a finite-dimensional Lie algebra  $\mathfrak{g}$ . Then  $\mathfrak{g}$  is nilpotent if and only if  $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$  is nilpotent for all  $X \in \mathfrak{g}$ .

Thus, we see that we will want to ignore solvable and nilpotent pieces.

**Definition 1.54 (radical).** Fix a Lie algebra  $\mathfrak{g}$ . Then the *radical*  $\text{rad } \mathfrak{g}$  is the sum of all solvable ideals of  $\mathfrak{g}$ .

**Remark 1.55.** One can check that  $\text{rad } \mathfrak{g}$  is a solvable ideal, so it is automatically the largest solvable ideal.

**Definition 1.56 (semisimple).** Fix a Lie algebra  $\mathfrak{g}$ . Then  $\mathfrak{g}$  is *semisimple* if and only if  $\text{rad } \mathfrak{g} = 0$ .

**Remark 1.57.** It turns out that

$$\mathfrak{g}_{\text{ss}} := \frac{\mathfrak{g}}{\text{rad } \mathfrak{g}}$$

is always semisimple. It turns out that the induced exact sequence splits, so there is a decomposition  $\mathfrak{g} = \mathfrak{g}_{\text{ss}} \times \text{rad } \mathfrak{g}$ , which is known as the Levi decomposition; we will prove this later.

Having defined semisimple, we should define "simple."

**Definition 1.58 (simple).** A Lie algebra  $\mathfrak{g}$  is *simple* if and only if its only ideals are 0 and  $\mathfrak{g}$ .

**Remark 1.59.** One can check that semisimple Lie algebras are precisely the sums of simple Lie algebras.

It turns out to be convenient to allow a little radical.

**Definition 1.60 (reductive).** A Lie algebra  $\mathfrak{g}$  is *reductive* if and only if its radical is its center.

**Example 1.61.** One can check that  $\mathfrak{sl}_n(\mathbb{C})$  is simple, and  $\mathfrak{gl}_n(\mathbb{C})$  is reductive.

To test for a Lie algebra being semisimple (and other adjectives), we introduce the Killing form.

**Definition 1.62 (Killing form).** Fix a Lie algebra  $\mathfrak{g}$ . Then we define the *Killing form* by

$$K(x, y) := \text{tr}(\text{ad}_x \circ \text{ad}_y).$$

**Remark 1.63.** One can check that  $K$  is  $\mathfrak{g}$ -invariant.

**Theorem 1.64 (Cartan criteria).** Fix a Lie algebra  $\mathfrak{g}$ .

- (a)  $\mathfrak{g}$  is solvable if and only if  $[\mathfrak{g}, \mathfrak{g}] \subseteq K$ .
- (b)  $\mathfrak{g}$  is semisimple if and only if  $K$  is non-degenerate.

**Proposition 1.65.** A Lie algebra  $\mathfrak{g}$  is reductive if and only if it admits a representation  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  for which the bilinear form

$$B_V(X, Y) := \text{tr}(\rho_X \circ \rho_Y)$$

is non-degenerate.

We may as well state one of the main theorems of our representation theory.

**Theorem 1.66.** Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.

### 1.1.5 Structure Theory of Lie Algebras

Here is another piece of structure theory.

**Definition 1.67 (adjoint).** Fix a semisimple Lie algebra  $\mathfrak{g}$ . Then we define the *adjoint* Lie group  $G^{\text{ad}}$  by  $G^{\text{ad}} := \text{Aut}(\mathfrak{g})^\circ \subseteq \text{GL}(\mathfrak{g})$ .

**Remark 1.68.** It turns out that  $\text{Lie } G^{\text{ad}} = \mathfrak{g}$ .

In our setting, one can generalize the Jordan decomposition.

**Definition 1.69 (semisimple, nilpotent).** An element  $X \in \mathfrak{g}$  is *semisimple* or *nilpotent* if and only if the operator  $\text{ad}_X X$  is.

**Theorem 1.70.** Fix a Lie algebra  $\mathfrak{g}$ . Then any  $X \in \mathfrak{g}$  can be written uniquely as a sum of a semisimple and nilpotent element.

**Remark 1.71.** It turns out that semisimple elements always act semisimply on representations, and nilpotent elements always act nilpotently on representations.

The notion of semisimple elements is important to define Cartan subalgebras.

**Definition 1.72 (Cartan).** Fix a semisimple Lie algebra  $\mathfrak{g}$ . Then a *Cartan subalgebra* is a maximal commutative subalgebra of

**Proposition 1.73.** Fix a semisimple Lie algebra  $\mathfrak{g}$ . All Cartan subalgebras are conjugate by  $G^{\text{ad}}$ .

**Definition 1.74.** Fix a semisimple Lie algebra  $\mathfrak{g}$ . Then the *rank* of  $\mathfrak{g}$  is the dimension of the Cartan subalgebras.

A choice of Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  produces a root decomposition, which we write as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha.$$

**Definition 1.75 (root system).** Fix a semisimple Lie algebra  $\mathfrak{g}$  and a Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . Then the *root system* of  $\mathfrak{g}$  consists of those nonzero eigenvalues  $\alpha \in \mathfrak{h}^*$  for the adjoint action of  $\mathfrak{h}$  on  $\mathfrak{g}$ . We write  $\mathfrak{g}_\alpha$  for this eigenspace, and we write  $\Phi(\mathfrak{g})$  for the root system.

**Remark 1.76.** One can check that  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{[\alpha, \beta]}$ . In fact,  $\mathfrak{g}_\alpha$  and  $\mathfrak{g}_\beta$  are orthogonal for the Killing form except when  $\alpha = -\beta$ , where it is a perfect pairing.

**Remark 1.77.** It turns out that  $\dim \mathfrak{g}_\alpha = 1$  for each  $\alpha$ . It follows that

$$\#\Phi(\mathfrak{g}) = \dim \mathfrak{g} - \text{rank } \mathfrak{g}.$$

**Remark 1.78.** There are the usual pictures of root systems of various types.

### 1.1.6 Root Systems

It is useful to write down what properties are satisfied by these root systems.

**Definition 1.79 (root system).** Fix a Euclidean space  $E$ . Then a finite subset  $\Phi \subseteq E$  is a *root system* if and only if

- (a)  $\Phi$  spans  $E$ ,
- (b) for each  $\alpha, \beta \in \Phi$ , the number

$$n_{\alpha\beta} := \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer,

- (c) for each  $\alpha, \beta \in \Phi$ , the reflection

$$s_\alpha(\beta) := \beta - n_{\alpha\beta}\beta$$

is in  $\Phi$ .

We say that  $\Phi$  is *reduced* if and only if  $\alpha \in \Phi$  implies that  $2\alpha \notin \Phi$ .

**Definition 1.80 (reducible).** A root system  $\Phi$  is *reducible* if and only if it can be written as a disjoint union of root systems coming from a decomposition of the Euclidean space into a product of Euclidean spaces.

The reflections are important enough to be placed into a group.

**Definition 1.81 (Weyl group).** Fix a root system  $\Phi \subseteq E$ . Then the *Weyl group*  $W$  is the subgroup of  $\text{GL}(E)$  generated by the reflections.

**Example 1.82.** The Weyl group associated to the root system of  $\mathfrak{sl}_{n+1}$  consists of the permutation matrices in  $\mathrm{GL}_n(\mathbb{R})$ . Indeed, each reflection corresponds to a transposition. This root system is said to be of type  $A_n$ , where  $n$  refers to the rank.

**Remark 1.83.** There is a determinant  $\det$  on the Weyl group, which we define to be the determinant of the natural representation of  $W$  in  $\mathrm{GL}(E)$ .

**Example 1.84.** The root system associated to  $\mathfrak{so}_{2n+1}$  is  $B_n$ . The root system associated to  $\mathfrak{sp}_{2n}$  is  $C_n$ . Lastly, the root system associated to  $\mathfrak{so}_{2n}$  is  $D_n$ .

**Remark 1.85.** There are also various exceptional reduced root systems, which we may say something about later.

We can even break down irreducible root systems into more controlled pieces.

**Definition 1.86 (positive).** Fix a root system  $\Phi \subseteq E$ . For a choice of  $t \in E$  for which  $(t, \alpha) \neq 0$  for all  $\alpha \in E$ , we say that a root in  $\Phi$  is *positive* if and only if  $(t, \alpha) > 0$ . Similarly,  $\alpha$  is *negative* if and only if  $(t, \alpha) < 0$ . We let  $\Phi^+$  and  $\Phi^-$  denote the sets of positive and negative roots, respectively.

**Definition 1.87.** Fix a root system  $\Phi \subseteq E$ . A positive root is *simple* if and only if it is not a sum of other positive roots (with positive integer coefficients). We let  $\Pi$  denote the set of simple roots.

**Proposition 1.88.** Fix a root system  $\Phi \subseteq E$ . Then  $\Pi$  is a basis, and every positive root  $\alpha \in \Phi^+$  can be written as a unique sum of elements of  $\Pi$  with positive integer coefficients.

Each root system also admits a dual.

**Definition 1.89 (dual root system).** Fix a root system  $\Phi \subseteq E$ . Then we define the *dual root system*  $\Phi^\vee \subseteq E^\vee$  to be given by the points

$$\alpha^\vee = \frac{2(\alpha, -)}{(\alpha, \alpha)}$$

for each  $\alpha \in \Phi$ .

**Remark 1.90.** The reduced root system  $B_n$  is dual to  $C_n$ .

## 1.2 February 4

We continue our review today.

### 1.2.1 Weights

It will be helpful to have some lattices from our root systems.

**Definition 1.91.** Fix a root system  $\Phi \subseteq E$ .

- The *root lattice*  $Q$  is spanned by  $\Phi$ .
- The *coroot lattice*  $Q^\vee$  is spanned by the  $\alpha^\vee$ .
- The *weight lattice*  $P \subseteq E$  is  $(Q^\vee)^*$ .
- The *coweight lattice*  $P^\vee \subseteq E^*$  is  $Q^*$ .

In general,  $Q \subseteq P$ , but equality does not have to hold.

**Example 1.92.** For  $\mathfrak{sl}_n$ , the quotient  $P/Q$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ .

It is useful to have a basis for the weight lattice

**Definition 1.93 (fundamental weight).** Fix a root system  $\Phi \subseteq E$ . A *fundamental weight* is an element of the dual basis (in the weight lattice  $P$ ) of the  $\alpha_i^\vee \in Q^\vee$  where  $\alpha$  is a simple root. Explicitly, if  $\{\alpha_1, \dots, \alpha_r\}$  are the simple roots, then the fundamental weights  $\{\omega_1, \dots, \omega_r\}$  satisfy

$$(\omega_i, \alpha_j^\vee) = 1_{i=j}.$$

**Definition 1.94 (dominant).** Fix a root system  $\Phi \subseteq E$ . A weight  $\lambda$  is *dominant* if and only if  $\langle \lambda, \alpha_i^\vee \rangle$  is a nonnegative integer for all  $i$ .

**Remark 1.95.** One can check that the dominant weights are exactly the  $\mathbb{Z}_{\geq 0}$ -span of the fundamental weights.

## 1.2.2 The Dynkin Diagram

Let's start trying to classify our Lie algebras.

**Definition 1.96 (Cartan matrix).** Fix a root system  $\Phi \subseteq E$ , and order the simple roots as  $\{\alpha_1, \dots, \alpha_r\}$ . The matrix  $A$  with entries

$$a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

is the *Cartan matrix*.

**Remark 1.97.** The Cartan matrix satisfies the following properties, all essentially by construction.

- We have  $a_{ii} = 2$  for each  $i$ .
- For distinct  $i$  and  $j$ , we have  $a_{ij} \leq 0$ , and  $a_{ij}$  vanishes if and only if  $a_{ji}$  vanishes.
- Because the inner product  $(-, -)$  on the  $\alpha_i$ 's is positive definite, the pairing  $(v, w) := v^\top Aw$  is positive definite.

**Remark 1.98.** For any  $i$  and  $j$ , a piece of the Cartan matrix

$$\begin{bmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{bmatrix}$$

must have positive determinant by the positive-definiteness. Thus,  $4 - a_{ij}a_{ji} > 0$ , so  $a_{ij}a_{ji} \in [0, 1, 2, 3]$  follows.

**Example 1.99.** The Cartan matrix for  $\mathfrak{sl}_4$  is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The data of a Cartan matrix can be encoded combinatorially into a Dynkin diagram.

**Definition 1.100 (Dynkin diagram).** Fix a root system  $\Phi \subseteq E$ , and order the simple roots as  $\{\alpha_1, \dots, \alpha_r\}$ . Let  $A$  be the Cartan matrix. Then we define the *Dynkin diagram* to have  $r$  vertices and the following connections.

- If  $a_{ij} = 0$ , then vertices  $i$  and  $j$  have no connection.
- If  $a_{ij} = -1$  and  $a_{ji} = -1$ , we draw a single line between  $i$  and  $j$ .
- If  $a_{ij} \leq -2$ , then we draw an arrow from vertex  $i$  to  $j$  with  $|a_{ij}|$  heads.

There are the usual pictures of all Dynkin diagrams.

**Example 1.101.** For example,  $A_{n-1}$  is a path with  $n - 1$  vertices.



**Remark 1.102.** It turns out that the root system determines the Lie algebra, the Cartan matrix determines the root system, and the Dynkin diagram determines the Cartan matrix.

**Remark 1.103 (Coxeter group).** One can read the Weyl group off of the Cartan matrix: namely, the reflection  $s_i s_j$  has order

$$m_{ij} := \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0, \\ 3 & \text{if } a_{ij}a_{ji} = 1, \\ 4 & \text{if } a_{ij}a_{ji} = 2, \\ 6 & \text{if } a_{ij}a_{ji} = 3. \end{cases}$$

Indeed, this claim amounts to checking that  $s_i s_j$  is a rotation with specified degree, which can be seen on the inner products. It turns out that the Weyl group is generated by the relations  $s_i^2 = 1$  and  $(s_i s_j)^{m_{ij}} = 1$  for each pair of distinct  $i$  and  $j$ .

To prove Remark 1.102, a difficult step is to show that every Dynkin diagram does in fact give rise to a Lie algebra. This is the content of the following theorem.

**Theorem 1.104 (Serre presentation).** Fix a finite-dimensional simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  with root system  $\Phi$ , and choose an ordered set of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ . Select basis vectors  $e_i$  and  $f_i$  in each space  $\mathfrak{g}_{\alpha_i}$  and  $\mathfrak{g}_{-\alpha_i}$ , and define  $h_i := [e_i, f_i]$ . Then we have the following relations.

- (a) We have  $[h_i, h_j] = 0$ ,  $[h_i, e_j] = a_{ij}e_j$ , and  $[h_i, f_j] = -a_{ij}f_j$ .
- (b) We have  $[e_i, f_j] = \delta_{i,j}h_i$ .
- (c) Serre relations:  $\text{ad}_{e_i}^{1-a_{ij}} e_j = 0$  and  $\text{ad}_{f_i}^{1-a_{ij}} f_j = 0$ .

In fact, the free Lie algebra defined by these generators and relations (for any root system  $\Phi$ ) produces a finite-dimensional Lie algebra.

### 1.2.3 Back to Representations of Lie Algebras

Throughout, we fix a finite-dimensional Lie algebra  $\mathfrak{g}$ , choose a Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ . Then we choose a collection of positive roots for the induced root system, which lets us split  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ , where  $\mathfrak{n}_+$  is the sum of the positive root spaces, and  $\mathfrak{n}_-$  is the sum of the negative root spaces.

**Definition 1.105 (Verma module).** Fix a weight  $\lambda \in \mathfrak{h}^*$ . Then we define the *Verma module*  $M_\lambda$  to be generated as a  $U\mathfrak{g}$ -module by the vector  $v_\lambda$ , given the relations  $ev_\lambda = 0$  for  $e \in \mathfrak{n}_+$  and  $hv_\lambda = \lambda(h)v_\lambda$  for  $h \in \mathfrak{h}$ .

**Remark 1.106.** In other words,  $M_\lambda = U\mathfrak{g} \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$ , where  $\mathbb{C}_\lambda$  is the representation of  $\mathfrak{h} \oplus \mathfrak{n}_+$  on which  $\mathfrak{h}$  acts by  $\lambda$  and  $\mathfrak{n}_+$  acts by zero. For example, the PBW theorem gives us an isomorphism

$$U\mathfrak{n}_- \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+) \rightarrow U\mathfrak{g}$$

of vector spaces, so  $M_\lambda$  is a free module over  $U\mathfrak{n}_-$  of rank 1.

**Remark 1.107.** For generic  $\lambda$ , it turns out that  $M_\lambda$  is irreducible.

**Definition 1.108 (highest weight).** Fix a representation  $V$  of  $\mathfrak{g}$ . Then a weight  $\lambda$  is a *highest weight* if and only if  $V$  is a nonzero quotient of  $M_\lambda$ . In other words, there is a vector  $v \in V$  on which  $\mathfrak{h}$  acts by  $\lambda$  and  $\mathfrak{n}_+$  acts by zero.

Our classification result is as follows.

**Theorem 1.109.** Fix everything as above.

- (a) The Verma modules  $M_\lambda$  admit a unique irreducible quotient.
- (b) The representation  $L_\lambda$  is finite-dimensional if and only if  $\lambda$  is a dominant weight.
- (c) Every irreducible finite-dimensional representation of  $\mathfrak{g}$  takes the form  $L_\lambda$  for a dominant weight  $\lambda$ .

*Proof.* The proof of (a) is not too hard, and it is also not hard to show that  $L_\lambda$  being finite-dimensional, then  $\lambda$  is dominant. The hard part is to show that  $\lambda$  being dominant implies that  $L_\lambda$  is finite-dimensional!

Lastly, by iteratively acting by  $\mathfrak{n}_+$ , one can show that every irreducible finite-dimensional representation  $V$  of  $\mathfrak{g}$  admits some highest weight  $\lambda$ . Indeed, given a finite-dimensional irreducible representation  $V$  of  $\mathfrak{g}$ ,

the representation theory of  $\mathfrak{sl}_2$  implies that the action of  $\mathfrak{h}$  on  $V$  diagonalizes and admits integer eigenvalues. It follows that there is a surjection  $M_\lambda \twoheadrightarrow V$ , so there is a surjection  $L_\lambda \twoheadrightarrow V$ , which must then be an isomorphism. ■

**Remark 1.110.** Here is another way to select the highest weight: one can give the weight lattice a partial ordering, and then the highest one is the maximal element among the weights  $\lambda$  for which  $V[\lambda] \neq 0$ . In particular, the maximality implies that  $\mathfrak{n}_+$  acts by zero on  $V[\lambda]$ .

There is also a construction of the quotient  $L_\lambda$  by taking relations.

**Theorem 1.111.** Fix everything as above, and choose a dominant weight  $\lambda$ . Then  $L_\lambda$  is the quotient of  $M_\lambda$  by the relations

$$f_i^{(\lambda, \alpha_i^\vee) + 1} v_\lambda = 0.$$

**Remark 1.112.** These relations are forced by passing to the representation theory of  $\mathfrak{sl}_2$ .

## 1.2.4 Weyl Formulae

Our construction of  $L_\lambda$  is not so explicit; for example, what is the dimension of  $L_\lambda$ ? To answer this question, we will need the Weyl dimension formula. We start with the Weyl character formula.

**Definition 1.113 (character).** Fix a representation  $V$  of a semisimple Lie algebra  $\mathfrak{g}$  admitting a weight decomposition

$$V = \bigoplus_{\mu \in P} V[\mu].$$

Then we define the *character*

$$\text{ch } V := \sum_{\mu \in P} \dim V[\mu] e^\mu$$

living in the free power series vector space with basis given by the  $e^\mu$ 's, with a relation  $e^{\mu+\lambda}$ .

**Remark 1.114.** If  $V$  is finite-dimensional, then it admits a weight decomposition (by the representation theory of  $\mathfrak{sl}_2$ ), and the character is a finite sum.

**Remark 1.115.** Let's explain why this is a character. Suppose that  $V$  is a representation of a Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . For  $h \in \mathfrak{h}$ , the exponential  $\exp(h) \in G$  acts on  $V$  diagonally, with trace

$$\text{tr}_V \exp(h) = \sum_{\mu \in P} \dim V[\mu] e^{\mu(h)},$$

which is exactly  $\text{ch } V(h)$ . In fact, these formulae determine the entire character because semisimple elements can all be found in Cartan subalgebras and are dense in  $G$ .

We are now ready to state the Weyl character formula.

**Theorem 1.116 (Weyl character formula).** Fix a semisimple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$ , and let  $\Phi^+$  be a set of positive roots. Further, set  $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ . Then for each dominant weight  $\lambda$ , we have

$$\text{ch } L_\lambda = \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}.$$

*Proof.* Omitted. The idea is that the character of  $M_\lambda$  is easy to compute, and one can build a resolution of  $L_\lambda$  in terms of Verma modules. ■

**Remark 1.117.** One can write the right-hand side more symmetrically as

$$\frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho)}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}.$$

**Remark 1.118.** A priori, the right-hand side is a rational function, but one can use the anti-symmetry of the Weyl group action in the numerator to show that the denominator does in fact divide the numerator.

**Example 1.119 (Weyl denominator).** If  $\lambda = 0$ , then  $L_\lambda$  is the trivial representation, so it follows that

$$\sum_{w \in W} \det(w) e^{w\rho} = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

**Example 1.120 (Vandermonde determinant).** Suppose that  $\mathfrak{g} = \mathfrak{sl}_n$  so that  $\alpha_{ij} = x_i - x_j$ . Then we set  $X_i := e^{x_i}$ , so we find that

$$\sum_{w \in S_n} \left( \text{sgn}(w) \prod_{i=1}^n X_{w(i)}^{i-1} \right) = \prod_{i < j} (X_i - X_j).$$

Note that the right-hand side is precisely the determinant of the matrix

$$\begin{bmatrix} 1 & X_1 & \cdots & X_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_n & \cdots & X_n^{n-1} \end{bmatrix}.$$

By Theorem 1.116, we could attempt to compute  $\dim L_\lambda$  by plugging in  $h = 0$  to the polynomial.

**Theorem 1.121 (Weyl dimension formula).** Fix a semisimple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$ , and let  $\Phi^+$  be a set of positive roots. Then for any dominant weight  $\lambda$ ,

$$\dim L_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\alpha^\vee, \lambda + \rho)}{(\alpha^\vee, \rho)}.$$

*Proof.* This is slightly complicated because the given denominator vanishes at  $h = 0$ , so we are left to compute the limit

$$\dim L_\lambda = \lim_{h \rightarrow 0} \frac{\sum_{w \in W} \det(w) e^{(w(\lambda + \rho), h)}}{\prod_{\alpha \in \Phi_+} (e^{(\alpha, h)/2} - e^{-(\alpha, h)/2})}.$$

This limit looks difficult to compute, but we may merely compute it on the line  $h := 2th_\rho$ , where  $h_\rho$  is the dual element for  $\rho \in h^*$ . In particular, it turns out that the numerator will factor in this case! To see this, note  $\alpha(h_\rho) = (\alpha, \rho)$ , so

$$\sum_{w \in W} \det(w) e^{(w(\lambda + \rho), 2th_\rho)} = \sum_{w \in W} \det(w) e^{2t(w(\lambda + \rho), \rho)},$$

which upon moving the  $w$  around and plugging into Example 1.119 gives

$$\prod_{\alpha \in \Phi_+} (e^{t(\alpha, \lambda + \rho)} - e^{-t(\alpha, \lambda + \rho)}).$$

The denominator is now

$$\prod_{\alpha \in \Phi_+} (e^{(\alpha, h)/2} - e^{-(\alpha, h)/2}) = \prod_{\alpha \in \Phi_+} (e^{t(\alpha, \rho)} - e^{-t(\alpha, \rho)}).$$

Sending  $t \rightarrow 0$  (and cancelling out a factor of two) proves the result. ■

### 1.2.5 Representation Theory of $\mathfrak{sl}_n(\mathbb{C})$

The dominant weights  $\omega_i$  are the “simplest,” so the simplest interesting representations will be the  $L_{\omega_i}$ s. For example, we may hope to find all other representations inside them.

**Proposition 1.122.** Fix a semisimple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$ , and let  $\Pi = \{\alpha_1, \dots, \alpha_r\}$  be an ordered set of simple roots. Further, let  $\{\omega_1, \dots, \omega_r\}$  be the set of dominant weights. Then the category finite-dimensional representations of  $\mathfrak{g}$  is  $\otimes$ -generated by the representations  $\{L_{\omega_i}\}_{i=1}^r$ .

*Proof.* Because any finite-dimensional representation is a direct sum of the irreducible representations of the form  $L_\lambda$ , where  $\lambda$  is a dominant weight, it is enough to find these representations inside some tensor product of  $L_{\omega_i}$ s. Well,  $\lambda$  is dominant, so we may write  $\lambda = \sum_{i=1}^r k_i \omega_i$  for some nonnegative integers  $\{k_1, \dots, k_r\}$ . We then consider the representation

$$V := \bigotimes_{i=1}^r L_{\omega_i}^{k_i}.$$

For any dominant weight  $\mu$ , recall that  $\mu$  is the highest weight of  $L_\mu$ , meaning that the weights of  $L_\mu$  are concentrated in  $\mu - P^+$ , where  $P^+$  consists of the dominant weights. Additionally, the nature of the  $\mathfrak{g}$ -action on the tensor product implies that  $W[\mu] \otimes W'[\mu'] = (W \otimes W')[\mu + \mu']$  for any  $W$  and  $W'$  and  $\mu$  and  $\mu'$ . Thus, we can see that the highest weight of  $V$  is  $\lambda = \sum_{i=1}^r k_i \omega_i$ , and it has multiplicity 1. It follows that there is a map  $M_\lambda \rightarrow V$ , which must descend to an embedding  $L_\lambda \hookrightarrow V$ , as desired. ■

**Remark 1.123.** The proof also shows that the other  $L_\mu$ s appearing in the representation  $V$  have  $\mu < \lambda$ . Indeed, one can simply calculate the weights appearing in  $V/L_\lambda$  and note that they are all strictly smaller than  $\lambda$ . The multiplicities of the various  $L_\mu$  appearing in  $V$  can be interesting to calculate.

**Remark 1.124.** In fact, we can see that  $L_\lambda$  is the irreducible subrepresentation generated by the highest weight vector

$$v_\lambda = v_{\omega_1}^{\otimes k_1} \otimes \cdots \otimes v_{\omega_r}^{\otimes k_r}$$

in  $V$ . Indeed, the image of  $L_\lambda$  in  $V$  certainly contains this vector because this is the unique line with weight  $\lambda$ , so it follows that the generated subrepresentation must be exactly the irreducible representation  $L_\lambda$ .

Let's work out our general theory in an interesting example.

**Example 1.125.** Let  $V_{\text{std}}$  be the standard representation of  $\mathfrak{sl}_n(\mathbb{C})$ . There is a choice of ordered simple roots of  $\mathfrak{sl}_n(\mathbb{C})$  for which

$$L_{\omega_i} \cong \wedge^i V_{\text{std}}.$$

*Proof.* We proceed in steps.

1. We set up notation. Here,  $\mathfrak{h}$  is the set of diagonal matrices in  $\mathfrak{sl}_n(\mathbb{C})$ , which we embed into  $\mathbb{C}^n$  as the subspace with sum zero. Let  $\{e_1, \dots, e_n\}$  be the natural basis of  $\mathbb{C}^n$ , so we find that

$$\Phi = \{e_i - e_j : i \neq j\}.$$

Indeed, the elementary matrix  $E_{ij}$  has weight  $e_i - e_j$ . By choosing the vector  $t := (n, n-1, n-2, \dots, 1, 0)$  in  $\mathbb{C}^n$ , we see that  $\Phi^+$  consists of those  $e_i - e_j$  with  $i < j$ . Then the simple roots are given by  $\alpha_i := e_i - e_{i+1}$ , which we can check is in fact a basis.

For example, we see that this implies that the positive spaces in  $\mathfrak{g}$  are given by

$$\mathfrak{n}_+ := \bigoplus_{p < q} \mathbb{C} E_{pq}.$$

2. We calculate the fundamental weights. Now, the dual space  $\mathfrak{h}^*$  is then the quotient of  $\mathbb{C}^n$  by the diagonal subspace  $\mathbb{C}(1, \dots, 1)$ . To compute our fundamental weights  $\omega_i$ , we see that we are on the hunt for some vectors  $\omega_i$  for which

$$(\omega_i, e_j - e_{j+1}) = 1_{i=j}.$$

Well,  $\omega_i = (1, \dots, 1, 0, \dots, 0)$  works, where there are  $i$  total 1s. Accordingly, our dominant weights are the  $\mathbb{Z}_{\geq 0}$ -linear combinations of the  $\omega_i$ , so they are the decreasing sequences of nonnegative integers.

3. We claim that the representation  $\wedge^i V_{\text{std}}$  is irreducible. Well, let  $W \subseteq \wedge^i V_{\text{std}}$  be some nonzero subrepresentation, and we want to show that  $W = \wedge^i V_{\text{std}}$ . Give  $V_{\text{std}}$  the standard basis  $\{v_1, \dots, v_n\}$ , and we note that this basis is an eigenbasis for the  $\mathfrak{h}$ -action on  $V_{\text{std}}$ . Thus,

$$\{v_{n_1} \wedge \cdots \wedge v_{n_i} : n_1 < \cdots < n_i\}$$

is an eigenbasis for the  $\mathfrak{h}$ -action on  $\wedge^i V_{\text{std}}$ . Because  $W \subseteq V$  is a  $\mathfrak{g}$ -subrepresentation, we conclude that it is an  $\mathfrak{h}$ -subrepresentation, so  $W$  must contain some eigenvector  $v_{n_1} \wedge \cdots \wedge v_{n_i}$ .

Now, for any  $p \neq q$ , we see that  $E_{pq} \in \mathfrak{sl}_n(\mathbb{C})$  acts by zero on all basis vectors  $v_\bullet$  except it sends  $v_q \mapsto v_p$ . Thus, we see that we may apply such elementary matrices to any given eigenvector  $v_{n_1} \wedge \cdots \wedge v_{n_i}$  to move it to  $v_1 \wedge \cdots \wedge v_i$  (namely, apply  $E_{1n_1}$ , then  $E_{2n_2}$ , and so on), and one can apply the process in reverse to move any  $v_1 \wedge \cdots \wedge v_i$  to any other eigenvector. We conclude that  $W$  being stable under  $\mathfrak{g}$  forces  $W = \wedge^i V$ .

4. We are now ready to show that  $L_{\omega_i} \cong \wedge^i V_{\text{std}}$ . Because  $\wedge^i V_{\text{std}}$  is already irreducible, it is enough to check that its highest weight is  $\omega_i$ . Well, we claim that the highest weight vector is

$$w := v_1 \wedge \cdots \wedge v_i.$$

Well, for some  $p < q$ , we can see that  $E_{pq}w = 0$  because  $E_{pq}$  can only ever send a basis vector  $v_q$  to the “earlier” basis vector  $v_p$ , which either causes  $w$  to vanish outright (if  $p > q$ ) or introduces a multiplicity into  $w$ , still causing  $E_{pq}w = 0$ . Thus, we do indeed see that  $\mathfrak{n}_+w = 0$ . As for the weight, one merely needs to calculate

$$\text{diag}(x_1, \dots, x_n)w = (x_1 + \dots + x_i)w,$$

which is precisely the action by  $\omega_i$ ! ■

**Remark 1.126.** The moral is the story is that the category  $\text{Rep } \mathfrak{sl}_n(\mathbb{C})$  is  $\otimes$ -generated by the representations  $\wedge^i V_{\text{std}}$ . In fact, because  $\wedge^i V_{\text{std}}$  is a quotient of  $V_{\text{std}}^{\otimes i}$ , it follows that the category is  $\otimes$ -generated by  $V_{\text{std}}$ .

**Remark 1.127.** There is a perfect pairing

$$\wedge^i V_{\text{std}} \otimes \wedge^{n-i} V_{\text{std}} \rightarrow \wedge^n V_{\text{std}} = \mathbb{C}$$

given simply by  $w \otimes w' \mapsto w \wedge w'$ ; indeed, we can see this is a perfect pairing where a basis vector  $v_{n_1} \wedge \dots \wedge v_{n_i}$  has dual basis vector  $v_{m_1} \wedge \dots \wedge v_{m_{n-i}}$ , where  $\{n_1, \dots, n_i\} \cup \{m_1, \dots, m_{n-i}\} = \{1, \dots, n\}$ . Thus, we see that  $L_{\omega_i}^* = L_{\omega_{n-i}}$  for each  $i$ .

Here is another interesting family of representations.

**Example 1.128.** Let  $V_{\text{std}}$  be the standard representation of  $\mathfrak{sl}_n(\mathbb{C})$ . There is a choice of ordered simple roots of  $\mathfrak{sl}_n(\mathbb{C})$  for which

$$L_{m\omega_1} \cong \text{Sym}^m V_{\text{std}}.$$

*Proof.* We continue from Example 1.125.

1. We claim that the representation  $\text{Sym}^m V_{\text{std}}$  is irreducible. As in Example 1.125, we see that

$$\{v_{i_1} \cdots v_{i_m} : i_1 \leq \dots \leq i_m\}$$

is an eigenbasis for the action of  $\mathfrak{h}$  on  $V_{\text{std}}$ . Thus, any nonzero subrepresentation  $W$  of  $V_{\text{std}}$  contains such an eigenvector. However, given one such eigenvector  $v_{i_1} \cdots v_{i_m}$ , one can iteratively apply  $E_{1i_1}$ , then  $E_{1i_2}$ , and so on, eventually proving that  $v_1^m \in W$ . Moving this basis vector around, we see that  $v_i^m \in W$  for each  $i$ .

We now claim that each basis vector  $v_{i_1} \cdots v_{i_m}$  lives in  $W$ . Indeed, by hitting  $v_1^m$  with  $E_{11} - E_{22} + \sum_{i=2}^n c_i E_{1i}$ , we see that

$$\left( v_1 + \sum_{i=2}^n c_i v_2 \right)^m \in W.$$

Each coefficient varies with a different multivariate polynomial in the  $c_i$ s, so by choosing specializations carefully (and generically), we see that each monomial lives in  $W$ . Formally, one can inductively allow more and more of the  $c_i$ s to be nonzero, one at a time.

2. We complete the proof. In light of the previous step, it is enough to show that  $v_1^m$  is the highest weight vector of  $\text{Sym}^m V_{\text{std}}$  and has weight  $m\omega_1$ . Certainly  $E_{pq}$  kills  $v_1^m$  for each  $p < q$  because  $q > 1$ . As for the weight calculation, we note that

$$\text{diag}(x_1, \dots, x_n) \cdot v_1^m = mx_1 \cdot v_1^m,$$

as desired. ■

**Remark 1.129.** Another way to check that  $\text{Sym}^m V_{\text{std}}$  is irreducible would be to use the Weyl dimension formula (Theorem 1.121) to compute that  $\dim L_{m\omega_1} = \dim \text{Sym}^m V_{\text{std}}$ . Then the second step provides a canonical map  $L_{m\omega_1} \rightarrow V_{\text{std}}$ , which must be injective and hence an isomorphism.

## 1.3 February 9

Today we discuss Schur–Weyl duality.

### 1.3.1 Representations of $\mathrm{GL}_n(\mathbb{C})$

Here is our result.

**Proposition 1.130.** Irreducible representations of  $\mathrm{GL}_n(\mathbb{C})$  are parameterized by pairs  $(m, \lambda)$ , where  $m \in \mathbb{Z}$  and  $\lambda = (\lambda_1, \dots, \lambda_n)$  is a decreasing sequence of nonnegative integers with  $\lambda_n = 0$ , and

$$m = \sum_i \lambda_i + nr$$

for some integer  $r$ . The highest weight of the corresponding irreducible representation  $L_{m,\lambda}$  is  $(\lambda_1 + r, \dots, \lambda_n + r)$ .

*Proof.* It is not quite the case that  $\mathrm{GL}_n(\mathbb{C})$  is  $\mathbb{G}_m \times \mathrm{SL}_n(\mathbb{C})$ , but the natural covering map

$$\mathbb{G}_m(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$$

has finite kernel  $\mu_n \subseteq \mathbb{C}^\times$ . Thus, the irreducible representations of  $\mathrm{GL}_n(\mathbb{C})$  consist of the irreducible representations of  $\mu_n \times \mathrm{SL}_n(\mathbb{C})$  for which  $\mu_n$  acts by the identity. Notably, such an irreducible representation can be described as a tensor product of a representation of  $\mathbb{G}_m(\mathbb{C})$  and a representation of  $\mathrm{SL}_n(\mathbb{C})$ , with some condition on the diagonal.

However, irreducible representations of  $\mathbb{C}^\times$  are one-dimensional. Passing to the Lie algebra, we see that we are basically given the data of an operator  $h$  for which  $e^{2\pi ih} = 1$ , which means that the operator admits integer eigenvalues, so we see that the corresponding representations of  $\mathbb{C}^\times$  are given by the characters  $\chi_m(z) := z^m$ .

Thus, irreducible representations of  $\mathbb{C}^\times \mathrm{SL}_n(\mathbb{C})$  can be described as tensor products

$$L_{m,\lambda} := \chi_m \otimes L_\lambda,$$

where  $\lambda$  is some weight of  $\mathfrak{sl}_n(\mathbb{C})$ . Recall that such weights can be described as decreasing sequences of  $n$  integers with  $\lambda_n = 0$ . Notably,  $\mu_n$  acts on  $L_{m,\lambda}$  by sending the scalar  $z$  to  $z^{-m+\sum_i \lambda_i}$ , which needs to be divisible by  $n$  for our representation to descend to  $n$ . We can compute the highest weight of  $L_{m,\lambda}$  as  $(\lambda_1 + r, \dots, \lambda_n + r)$ . ■

**Remark 1.131.** Equivalently, we may say that representations of  $\mathrm{GL}_n(\mathbb{C})$  are parameterized by decreasing sequences of integers  $\lambda = (\lambda_1, \dots, \lambda_n)$ , which is the highest weight.

**Example 1.132.** We can compute that there is now a fundamental weight  $(1, \dots, 1)$ , which is the determinant representation.

Here is an important class of our representations.

**Definition 1.133 (polynomial).** An irreducible representation  $L_\lambda$  of  $\mathrm{GL}_n(\mathbb{C})$  is *polynomial* if and only if  $\lambda_n \geq 0$ .

**Remark 1.134.** We can think about the sequence  $\lambda$  as instead being a partition

$$\mu := (\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n)$$

of  $\lambda_1$ . We may then write  $L_\mu := L_\lambda$ .

**Remark 1.135.** Partitions are labeled by Young diagrams, which we notably have no use for.

**Remark 1.136.** One can check that  $L_\lambda$  is polynomial if and only if  $L_\lambda$  embeds into  $V^{\otimes N}$  for some  $N$ , where  $V$  is the standard representation. In particular, we see that  $L_\lambda$  is contained in  $V^{\otimes |\mu|}$ , which we can see by the same argument as in Proposition 1.122. The point is that the determinant representation can still be found in  $V^{\otimes n}$ , but the inverse of the determinant cannot!

**Remark 1.137.** One can check that polynomial representations are precisely those which extend continuously to  $M_n(\mathbb{C})$ ; equivalently, we may say that these representations are those for which the matrix coefficients are polynomial (not just rational). Again, the point is that the determinant representation has polynomial matrix coefficients, but the inverse of the determinant does not.

### 1.3.2 Schur–Weyl Duality

Having said something about the representations of  $\mathrm{GL}_n(\mathbb{C})$ , we are ready to relate them to the symmetric group. Set  $V := \mathbb{C}^n$ . Then we note that both  $\mathrm{GL}_n(\mathbb{C})$  and  $S_N$  act on  $V^{\otimes N}$ , where notably  $S_N$  acts by permuting the coordinates.

**Theorem 1.138 (Schur–Weyl duality).** Fix positive integers  $n$  and  $N$ , and set  $V := \mathbb{C}^n$ . Let  $A$  be the image of the natural map  $U\mathfrak{gl}(V) \rightarrow \mathrm{End}_{\mathbb{C}} V^{\otimes N}$ , and let  $B$  be the image of the natural map  $\mathbb{C}[S_N] \rightarrow \mathrm{End}_{\mathbb{C}} V^{\otimes N}$ .

(a) The algebras  $A$  and  $B$  are centralizers of each other.

(b) There is a decomposition

$$V^{\otimes N} = \bigotimes_{\lambda \vdash N} L_\lambda \otimes \pi_\lambda$$

of  $V^{\otimes N}$  into irreducible representations of  $A \times B$ ; the indexing consists of partitions of  $N$  with at most  $n$  parts.

(c) As  $\lambda$  varies over partitions of at most  $n$  parts, then the  $\pi_\lambda$ s are pairwise non-isomorphic irreducible representations of  $S_N$ .

(d) If  $n \geq N$ , then the  $\pi_\lambda$ s exhaust all irreducible representations of  $S_N$ .

**Remark 1.139.** Note that  $n = N$  implies that all permutations of  $N$  have at most  $n$  parts, so both (c) and (d) apply.

**Remark 1.140.** The existence of the decomposition in (b) should not be a surprise: for any group  $G$  and completely reducible representation  $Y$  of  $G$ , the natural map

$$\bigoplus_{W \in \mathrm{IrRep}(G)} W \otimes \mathrm{Hom}_G(W, Y) \rightarrow Y$$

is an isomorphism; indeed, to prove this, one simply decomposes  $Y$  and checks that the statement is true on irreducibles, where it follows from Schur's lemma. For (b), we see that we can simply let  $\pi_\lambda$  be the space  $\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$ .

**Definition 1.141 (Schur algebra).** The algebra  $A$  in Theorem 1.138 is the *Schur algebra*, denoted  $\mathcal{S}_{n,N}$ . The algebra  $B$  is then justly called the *centralizer algebra*, denoted  $C_{n,N}$ .

We are going to need a few lemmas. Our first goal is to show that  $A$  is the centralizer of  $B$ .

**Lemma 1.142.** Let  $U$  be a complex vector space. Then  $S^N U$  is spanned by the pure tensors

$$\{x \otimes \cdots \otimes x : x \in U\}.$$

*Proof.* Any  $u \in S^N U$  has finitely many terms, so by taking the finite number of vectors contained therein, we may reduce to the finite-dimensional case. Now, the span of our pure tensors forms a nonzero subrepresentation of  $S^N U$  (for the group  $\mathrm{GL}(U)$ ), but this representation is already known to be irreducible by Example 1.128, so the span must cover everything. ■

**Lemma 1.143.** Let  $R$  be an algebra over  $\mathbb{C}$ , and let  $S^N R$  be the  $N$ th symmetric power, which we note is also an algebra. Now, define  $\Delta: R \rightarrow S^N R$  by

$$\Delta(x) := (x \otimes 1 \otimes \cdots \otimes 1) + (1 \otimes x \otimes \cdots \otimes 1) + \cdots + (1 \otimes 1 \otimes \cdots \otimes x).$$

Then  $R$  is generated as an algebra by elements of the form  $\Delta(x)$ .

*Proof.* By Newton's identities, there is a polynomial  $P_N$  (with rational coefficients) for which

$$z_1 \cdots z_N = P_N \left( \sum_{i=1}^N z_i^1, \dots, \sum_{i=1}^N z_i^N \right).$$

(This is an identity that takes place in  $\mathbb{Q}[z_1, \dots, z_N]$ .) It follows that

$$x \otimes \cdots \otimes x = P_N (\Delta(x), \dots, \Delta(x^N))$$

for any  $x \in R$ , so we may conclude by Lemma 1.142. ■

**Lemma 1.144.** In the context of Theorem 1.138, the algebra  $A$  is the centralizer of  $B$ .

*Proof.* Note that certainly  $A$  and  $B$  commute with each other with no effort because  $B$  only permutes the factors of  $V$ . On the other hand, note that the centralizer  $Z_B$  of  $B$  is

$$\begin{aligned} Z_B &= \mathrm{End}(V^{\otimes N})^{S_N} \\ &\stackrel{*}{=} (\mathrm{End}_{\mathbb{C}}(V)^{\otimes N})^{S_N} \\ &= S^N \mathrm{End}_{\mathbb{C}}(V). \end{aligned}$$

Here,  $\stackrel{*}{=}$  holds because there is a natural map going upwards, and it can be seen to be an isomorphism on a basis. Thus, by Lemma 1.143, it is enough to show that the elements  $\Delta(\varphi)$  are in  $A$  for each  $\varphi \in \mathrm{End}_{\mathbb{C}}(V)$ , which is true by definition of  $A$ . ■

The rest of Schur–Weyl duality follows from the following piece of algebra.

**Lemma 1.145 (Double centralizer).** Fix a finite-dimensional vector space  $V$  over any field  $k$ , and choose two algebras  $A, B \subseteq \text{End}_k V$ . Suppose that  $B$  is a sum of matrix algebras, and  $A$  is the centralizer of  $B$ .

- (a) The algebra  $A$  is isomorphic to a sum of matrix algebras.
- (b) The algebra  $B$  is the centralizer of  $A$ .
- (c) There is a decomposition

$$V = \bigoplus_{i \in I} (X_i \otimes Y_i),$$

where the lists  $\{X_i\}_{i \in I}$  and  $\{Y_i\}_{i \in I}$  are exactly lists of irreducible representations of  $A$  and  $B$ , respectively.

*Proof.* Start by letting  $\{Y_i\}_{i \in I}$  be the list of irreducible representations of  $B$  so that we have a decomposition

$$V = \bigoplus_{i \in I} (X_i \otimes Y_i),$$

where  $X_i := \text{Hom}_B(Y_i, V)$ . Because  $B$  embeds in  $\text{End}_k V$ , we see that  $Y_i$  does admit an embedding into  $V$ : note that  $B$  is a finite-dimensional semisimple algebra, so one can see from the structure theory that any faithful representation contains every irreducible representation. Now,  $A$  is the centralizer of  $B$ , which is  $\text{End}_B(V)$ , but this is

$$\text{End}_B(V) = \bigoplus_{i \in I} \text{End}_k(W_i)$$

because the  $V_i$ s cannot map to distinct other representations. Then (a) is immediate, and (b) and (c) follow by a computation reversing everything in sight to compute decompositions over  $A$ . Namely, one will find that  $B = \bigoplus_{i \in I} \text{End}_k Y_i$ . ■

*Proof of Theorem 1.138.* Note that  $B$  is a quotient of  $\mathbb{C}[S_N]$  and is therefore a sum of matrix algebras by the representation theory of finite groups. We now apply Lemma 1.145, which we may do by Lemma 1.144. The decomposition in (b) follows immediately because the irreducible representations of  $A$  are labeled by partitions  $\lambda$  of at most  $n$  parts, and we see that (c) also immediately follows.

It remains to show (d). It is enough to show that the projection  $\mathbb{C}[S_N] \rightarrow B$  is injective so that irreducible representations of  $B$  are irreducible representations of  $S_N$ . Well, let  $\{e_1, \dots, e_n\}$  be a basis of  $V$ , and then we see that  $\mathbb{C}[S_N]$  acts faithfully on  $e_1 \otimes \dots \otimes e_N$  because the  $S_N$ -action produces linearly independent vectors! Thus, the map  $\mathbb{C}[S_N] \rightarrow \text{End}_{\mathbb{C}} V^{\otimes N}$  is injective, and we are done. ■

**Remark 1.146.** Theorem 1.138 provides a parameterization of irreducible representations of  $S_N$  by partitions of  $N$  for every  $n \geq N$ . This parameterization does not depend on the choice of  $n$ , which one can see via the natural diagonal embedding  $\text{GL}_n(\mathbb{C}) \rightarrow \text{GL}_{n+1}(\mathbb{C})$ . This is not too interesting, so we will not write it out.

### 1.3.3 Schur Functors

We would now like to describe the representations  $L_\lambda$  of  $\text{GL}_n(\mathbb{C})$  more explicitly.

**Definition 1.147 (Schur functor).** Fix a partition  $\lambda$  of a positive integer  $N$ . Then we define the *Schur functor* by

$$S^\lambda := \text{Hom}_{S_N} (\pi_\lambda, (-)^{\otimes N}).$$

**Example 1.148.** If  $\lambda = (N)$ , then  $L_\lambda = S^N V$ , so  $\pi_\lambda$  is the trivial representation, so  $S^\lambda V = S^N V$ .

**Example 1.149.** If  $\lambda = (1, \dots, 1)$ , then  $L_\lambda = \wedge^N V$  (which is the determinant representation when  $\dim V = N$ ), so  $\pi_\lambda$  is the sign representation, so  $S^\lambda V = \wedge^N V$ .

**Remark 1.150.** In general, Theorem 1.138 has given us a decomposition

$$V^{\otimes N} = \bigoplus_{\lambda \vdash N} S^\lambda V \otimes \pi_\lambda.$$

**Example 1.151.** We have a decomposition

$$V^{\otimes 2} = (S^2 V \otimes \mathbb{C}_{\text{triv}}) \oplus (\wedge^2 V \otimes \mathbb{C}_{\text{sgn}}).$$

Note that we have silently applied Examples 1.148 and 1.149.

**Example 1.152.** We have a decomposition

$$V^{\otimes 3} = (S^3 V \otimes \mathbb{C}_{\text{triv}}) \oplus (\wedge^3 V \otimes \mathbb{C}_{\text{sgn}}) \oplus (S^{(2,1)} V \otimes \pi_{(2,1)}).$$

By referencing a character table of  $S_3$ , we see that  $\pi_{(2,1)}$  must be the standard representation of  $S_3$ , which is its action on the trace-zero hyperplane in  $\mathbb{C}^3$ . For example, if we restrict to the group  $\text{GL}_3(\mathbb{C})$ , then we have a decomposition

$$V^{\otimes 3} = S^3 V \oplus \wedge^3 V \oplus S^{(2,1)} V^{\oplus 2}.$$

On the other hand, the left-hand side is  $V^{\otimes 2} \otimes V = (S^2 V \otimes V) \oplus (\wedge^2 V \otimes V)$ . The former factor has a copy of  $S^3 V$ , and the latter factor has a copy of  $\wedge^3 V$ ; none of these inclusions are equalities, so it follows that each has one copy of  $S^{(2,1)} V$ . This allows us to describe  $S^{(2,1)} V \subseteq S^2 V \otimes V$  as the subset whose total symmetrization is zero. (There is a similar description of  $S^{(2,1)} V$  as embedded in  $\wedge^2 V \otimes V$ .)

While it is a little annoying to describe the  $S^\lambda V$ s explicitly (though Example 1.152 gives a method), there is a formula for their dimension.

**Proposition 1.153.** Fix  $V := \mathbb{C}^n$ . Then

$$\dim S^\lambda V = \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j) + (j - i)}{j - i}.$$

*Proof.* Recall that  $\rho = (N-1, N-2, \dots, 0)$  because the positive roots are  $e_i - e_j$  where  $i < j$ . Thus, we see that  $(\rho, e_i - e_j) = (j - i)$ , so  $(\lambda + \rho, e_i - e_j) = (\lambda_i - \lambda_j) + (j - i)$ , and Theorem 1.121 gives the result. ■

We take a moment to give some simplifications. If  $\lambda$  has only  $k$  parts, then we could split the product into

$$\prod_{1 \leq i < j \leq k} \frac{(\lambda_i - \lambda_j) + (j - i)}{j - i} \cdot \prod_{1 \leq i \leq k < j \leq N} \frac{\lambda_i + (j - i)}{j - i}.$$

The latter product now telescopes after  $\lambda_i$  terms, allowing us to rewrite this as

$$\prod_{1 \leq i < j \leq k} \frac{(\lambda_i - \lambda_j) + (j - i)}{j - i} \cdot \prod_{i=1}^k \frac{(N+1-i) \cdots (N+\lambda_i-i)}{(k+1-i) \cdots (k+\lambda_i-i)}.$$

For example, it follows that the dimension is a polynomial  $P_\lambda$  with rational coefficients, whose roots are integers contained in  $[1 - \lambda, \dots, k - 1]$ .

## BIBLIOGRAPHY

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[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

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