18.786: Automorphic Forms

Nir Elber

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Contents **2**

1 Tate's Thesis **4**
1.1 February 2 . . . 4
1.1.1 Logistic Notes . . . 4
1.1.2 Places of Global Fields . . . 4
1.1.3 Adéles . . . 6
1.1.4 Characters on the Adéles . . . 7
1.2 February 4 . . . 8
1.2.1 Adelic Quotients . . . 8
1.2.2 Idelic Quotients . . . 10
1.2.3 Pontryagin Duality . . . 11
1.2.4 Fourier Theory . . . 13
1.3 February 9 . . . 13
1.3.1 Duality for Local Fields . . . 13
1.3.2 Duality for the Adéles . . . 15
1.3.3 Duality for Function Fields . . . 17
1.4 February 11 . . . 19
1.4.1 Multiplicative Measures . . . 19
1.4.2 Our L-functions . . . 20
1.4.3 Functional Equations . . . 22
1.5 February 17 . . . 23
1.5.1 More General L-functions . . . 23
1.5.2 Proof of the Global Functional Equation . . . 24
1.5.3 A Little Geometric Class Field Theory . . . 25
1.5.4 Local Theory . . . 26

2 Representation Theory **28**
2.1 February 18 . . . 28
2.1.1 Overview . . . 28

2.1.2 Totally Disconnected Groups . . . 28
2.1.3 Smooth Representations . . . 29
2.1.4 Categorical Properties . . . 30
2.1.5 Hecke Algebra . . . 31
2.2 February 25 . . . 33
2.2.1 Representations of the Hecke Algebra . . . 33
2.2.2 Parabolic Induction . . . 35

Bibliography **38**

List of Definitions **39**

Theme 1
Tate's Thesis

1.1 February 2

Here we go.

1.1.1 Logistic Notes

Here are some logistic notes.

- The book *Automorphic Forms and Representations* by Bump [Bum97] will be our main reference. We will focus on the third chapter.

- There are two quizzes, which will count as about 30% of the final grade. The rest of the grade will come from the homework.

- The course requires knowledge of number theory and some representation theory. Most notably, we need control of Lie groups and Lie algebras.

The story of automorphic forms begins with modular forms. Roughly speaking, a modular form is a function on the upper-half plane which is symmetric for $\mathrm{SL}_2(\mathbb{C})$. There is an exposition in the last chapter of [Ser12]. However, our story will start with GL_1 instead of GL_2. The perspective we take is Tate's thesis, who used Fourier analysis to reprove the analytic properties of the relevant (automorphic) L-functions.

1.1.2 Places of Global Fields

To do our Fourier analysis, we need to decompose our number field at each place, for which we will need the ring of adéles.

Definition 1.1 (global field)**.** A *global field* F is either a number field or a function field. Here, a number field is a finite extension of $\mathbb{Q}$, and a function field refers to the function field of a smooth, projective, geometrically connected curve X over a finite field $\mathbb{F}_q$.

Example 1.2. The field $\mathbb{F}_q(t)$ is a global field.

Definition 1.3 (place)**.** Fix a global field F. Then a *place* is an equivalence class of multiplicative absolute values of F.

- If F is a number field, then the *finite* (or *nonarchimedean*) places are those in bijection with $\mathcal{O}_F$, and the *infinite places* (or *archimedean*) are those in bijection with the embeddings $F \hookrightarrow \mathbb{C}$ (up to conjugation).
- If F is the function field of a curve X, then the places are in bijection with the closed points of the curve X.

We let $V(F)$ be the set of all places, and we let $V(F)_\infty$ denote the set of infinite places.

Example 1.4. For $F = \mathbb{Q}$, the place at a finite prime p is represented by $|q|_p := p^{-\nu_p(q)}$, where $\nu_p(q)$ is the number of ps appearing in the prime factorization of q.

Example 1.5. For $F = \mathbb{F}_q(t)$, we have the curve $X = \mathbb{P}^1$, so there is one place at infinity, and the rest of the points come from $\mathbb{A}^1$. The places in $\mathbb{A}^1$ are parameterized by the monic irreducible polynomials of $\mathbb{F}_q[t]$.

Notation 1.6. Fix a global field F. For each place v, we let F_v be the completion of F along a norm represented by v. We let $\mathcal{O}_v$ denote the elements with norm at most 1; we let $\mathcal{O}_v$ denote the elements with norm 1, and we let $\mathfrak{p}_v$ denote the elements with norm less than !.

Remark 1.7. If v is nonarchimedean, then it turns out that $\mathcal{O}_v$ is a discrete valuation ring with maximal ideal $\mathfrak{p}_v$. It also turns out that there is an exact sequence

$$1 \to \mathcal{O}_v^\times \to F_v^\times \to \mathbb{Z} \to 0,$$

where the map $F_v^\times \to \mathbb{Z}$ is the valuation map.

It is helpful to normalize our absolute values. Let's start with the global fields.

Notation 1.8. Fix a place v of a global field F. We normalize a choice of absolute value $|\cdot|_v$ as follows.

- For $F = \mathbb{Q}$, each prime p produces the absolute value $|q|_p := p^{-\nu_p(q)}$. The infinite place ∞ produces the absolute value $|x|_\infty$ which is the usual one (in $\mathbb{R}$).
- For a finite extension F of $\mathbb{Q}$, say that v lies over v_0 of $\mathbb{Q}$, and we define

$$|x|_v := \left| \mathrm{N}_{F_v/\mathbb{Q}_{v_0}}(x) \right|_{v_0}.$$

Example 1.9. For $F_v = \mathbb{C}$, we see that $|x|_v = |x\overline{x}|_\mathbb{R}$ is the square of the usual absolute value on $\mathbb{C}$. Note that this norm does not obey the triangle inequality.

For a function field $\mathbb{F}_q(X)$, there is not a canonical embedding $\mathbb{F}_q(t)$ into $\mathbb{F}_q(X)$, so it does not seem suitable to proceed as above by taking norms. Instead, we normalize directly.

Notation 1.10. Fix a function field F of a smooth, projective, geometrically connected curve X over $\mathbb{F}_q$, and choose a place $v \in X$. Then the completion F_v is isomorphic to $k_v((t))$, where $t \in \mathcal{O}_v$ is a choice of uniformizer and $k_v/\mathbb{F}_q$ is a finite extension. Then we normalize our norm $|\cdot|_v$ by $|t|_v := (\#k_v)^{-1}$.

These choices of normalization obey a product formula.

Proposition 1.11. Fix a global field F. For each $x \in F$,

$$\prod_{v \in V(F)} |x|_v = 1.$$

Sketch. This is included in a standard first course in number theory, so we will be brief. For number fields, this is checked directly by passing to $\mathbb{Q}$, where it is a consequence of unique prime factorization. For function fields $\mathbb{F}_q(X)$, we may think of $f \in \mathbb{F}_q(X)$ as a rational function X, and $|f|_v = q^{\deg(v) \cdot - \operatorname{ord}_v(f)}$, where ord_v is the order of vanishing. Thus, the product formula more or less amounts to the statement that the sum of the zeroes and poles of f all cancel out (over the algebraic closure). ■

1.1.3 Adéles

We now define the adéles by gluing together our localizations.

Definition 1.12 (adéles)**.** Fix a global field F. Then the ring of *adéles* $\mathbb{A}_F$ is defined as the restricted product

$$\mathbb{A}_F \coloneqq \prod_{v \in V(F)} (F_v, \mathcal{O}_v),$$

meaning that $\mathbb{A}_F$ consists of sequences of elements in F_v which are in $\mathcal{O}_v$ for all but finitely many v.

Remark 1.13. By construction, we see that

$$\mathbb{A}_F = \bigcup_{\substack{\text{finite } S \subseteq V(F) \\ S \supseteq V(F)_\infty}} \left(\prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} F_v \right).$$

Thus, $\mathbb{A}_F$ is a colimit of (product) topological rings, so $\mathbb{A}_F$ is a topological ring.

Remark 1.14. A basis neighborhood basis of $0 \in \mathbb{A}_F$ is given as follows: for any choice of finite $S \subseteq V(F)$ containing $V(F)_\infty$, choose open neighborhoods $U_v \subseteq \mathcal{O}_v$ of 0, and then we have the open subset

$$\prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} U_v.$$

One can further require that the open subsets U_v take the form $\mathfrak{p}_v^{m_v}$, where m_v is some integer.

Tate's thesis is about $\mathrm{GL}_1(\mathbb{A}_F)$, so the following group will be important to us.

Definition 1.15. Fix a global field F. Then the group of *idéles* $\mathbb{A}_F^\times$ is defined as the restricted product

$$\mathbb{A}_F^\times \coloneqq \prod_{v \in V(F)} (F_v^\times, \mathcal{O}_v^\times),$$

meaning that $\mathbb{A}_F^\times$ consists of sequences of elements in $F_v^\times$ which are in $\mathcal{O}_v^\times$ for all but finitely many v.

Notably, $\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$.

Remark 1.16. This is not the set of nonzero elements in $\mathbb{A}_F$ because we require the inverse to also be an adéle!

Remark 1.17. One should not give the subset $\mathbb{A}_F^\times \subseteq \mathbb{A}_F$ the subspace topology. Instead, the topology should be given by the restricted product, whose open subsets can be smaller. Thus, an element of the neighborhood basis of $1 \in \mathbb{A}_F^\times$ can be described as follows: for any choice of finite $S \subseteq V(F)$ containing $V(F)_\infty$, choose open neighborhoods $U_v \subseteq \mathcal{O}_v$ of 0, and then we have the open subset

$$\prod_{v \notin S} \mathcal{O}_v^\times \times \prod_{v \in S} U_v.$$

One can further require that the open subsets U_v take the form $\mathfrak{p}_v^{m_v}$, where m_v is some integer.

Later in the course, we will even want to study groups like $\mathrm{GL}_2(\mathbb{A}_F)$ or $\mathrm{GL}_n(\mathbb{A}_F)$. Let's be explicit about what this notation means.

Definition 1.18 (general linear group)**.** Fix a ring R. Then we define $\mathrm{GL}_n(R)$ to be the group of invertible $n \times n$ matrices. Explicitly, this can be described as the group of $n \times n$ matrices with entries in R whose determinant is invertible.

Remark 1.19. Fix a global field F. One can check that

$$\mathrm{GL}_n(\mathbb{A}_F) = \prod_{v \in V(F)} (\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_v)),$$

which also tells us what the topology should be.

Remark 1.20. Here is another way to construct the topology: GL_n can embed (as a scheme) as a closed subspace of (n^2+1)-dimensional space A, where the embedding sends $g \in \mathrm{GL}_n(R)$ to the tuple of coordinates follows by the inverse of the determinant. This is a closed embedding, essentially by definition of GL_n. Then we can give $\mathrm{GL}_n(\mathbb{A}_F)$ the natural topology given as a closed subspace of $A(\mathbb{A}_F)$. It is not too hard (but rather annoying) to check that these definitions agree.

Example 1.21. The determinant map $\det\colon \mathrm{GL}_n(\mathbb{A}_F) \to \mathbb{A}_F^\times$ is continuous. One can see this via Remark 1.20 because the determinant and its inverse are both continuous maps to $\mathbb{A}_F$. But the topology on $\mathbb{A}_F^\times$ is given as a closed subspace of $\mathbb{A}_F \times \mathbb{A}_F$ (where the embedding is given by $x \mapsto (x, 1/x)$).

This course is interested in the representation theory of $\mathrm{GL}_n(\mathbb{A}_F)$, focusing on the cases $n \in \{1,2\}$. If we think about such representations appropriately, it turns out that such a representation π will decompose into a tensor product $\bigotimes_v' \pi_v$, where π_v is a representation of $\mathrm{GL}_n(F_v)$. More than half of the course will thus be interested in the representation theory of $\mathrm{GL}_n(F_v)$ because we will want to study the finite and infinite places separately.

1.1.4 Characters on the Adéles

We will need more structure theory of the adéles.

Proposition 1.22. Fix a global field F. The diagonal embedding $F \hookrightarrow \mathbb{A}_F$ embeds F as a discrete subgroup.

Proof. Fix distinct $a, b \in F$. By examining the open subsets we have access to, we need to show that $|a-b|_v \geq 1$ for some v, which follows from Proposition 1.11. ■

Corollary 1.23. Fix a global field F. The diagonal embedding $F^\times \hookrightarrow \mathbb{A}_F^\times$ embeds F as a discrete subgroup.

Proof. For each $a \in F^\times$, we need to know that there is an open subset U of $\mathbb{A}_F^\times$ for which $U \cap F^\times = \{a\}$. But there is such an open subset of $\mathbb{A}_F$, which continues to be open in $\mathbb{A}_F^\times$. ■

We will be interested in characters on $\mathbb{A}_F^\times$.

Notation 1.24. Fix a place v of a global field F. Given a continuous character $\chi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$, we let $\chi_v\colon F_v^\times \to \mathbb{C}^\times$ denote the induced character.

Remark 1.25. The continuity of χ forces $\chi_v|_{\mathcal{O}_v^\times} = 1$ for all but finitely many v. Conversely, given a family $\{\chi_v\}_{v\in V(F)}$ of continuous characters for which $\chi_v|_{\mathcal{O}_v^\times} = 1$ for all but finitely many v, one can check that there is a unique continuous character χ on $\mathbb{A}_F^\times$ gluing them together.

The previous remark motivates the following definition.

Definition 1.26 (unramified)**.** Fix a place v of a global field F. Then a character $\chi_v\colon F_v^\times \to \mathbb{C}^\times$ is *unramified* if and only if $\chi_v|_{\mathcal{O}_v^\times} = 1$.

Example 1.27. By definition, χ_v factors through $F_v^\times/\mathcal{O}_v^\times \cong \mathbb{Z}$. Thus, χ_v can be described as $\chi_v = |\cdot|_v^s$ for some $s \in \mathbb{C}$.

Not all characters are interesting to us because we want our characters χ_v to talk to each other.

Definition 1.28 (Hecke character)**.** Fix a global field F. A *Hecke character* is a continuous character $\chi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ which vanishes on $F^\times$.

Remark 1.29. It is equivalent to ask for χ to be continuous on $F^\times\backslash\mathbb{A}_F^\times$ by Corollary 1.23.

1.2 February 4

Here we go.

1.2.1 Adelic Quotients

Thus, it will be worthwhile to know something about the quotient $F^\times\backslash\mathbb{A}_F^\times$. Let's start with the additive group.

Theorem 1.30 (approximation)**.** Fix a number field F. Then

$$\mathbb{A}_F = F + \prod_{v\notin V(F)_\infty} \mathcal{O}_v + \prod_{v\in V(F)_\infty} F_v.$$

Proof. Given an adéle $(a_v)_v \in \mathbb{A}_F$, we see that we may ignore the infinite places. Then we are asked to find $a \in F$ for which $a \equiv a_v \pmod{\mathcal{O}_v}$ for all v. After multiplying out some denominators, this amounts to the Chinese remainder theorem for $\mathcal{O}_F$. ■

Here is an analog for function fields.

Proposition 1.31. Fix a function field $F = \mathbb{F}_q(X)$. Then one has

$$F\backslash\mathbb{A}_F \Big/ \prod_{v \in V(F)} \mathcal{O}_v \cong \mathrm{H}^1(X; \mathcal{O}_X).$$

Proof. The idea is to use the "two-step complex" $F \to \mathbb{A}_F / \prod_v \mathcal{O}_v$ to compute the cohomology of $\mathcal{O}_X$. Note that $\mathbb{A}_F / \prod_v \mathcal{O}_v$ is the restricted product

$$\prod_{v \in V(F)} \left(\frac{F_v}{\mathcal{O}_v}, \frac{\mathcal{O}_v}{\mathcal{O}_v} \right) = \bigoplus_{v \in V(F)} F_v/\mathcal{O}_v.$$

Now, there is a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{K} \to \bigoplus_{v \in V(F)} i_{v*}(F_v/\mathcal{O}_v) \to 0,$$

where $\mathcal{K}$ is the constant sheaf of rational functions. Taking the long exact sequence in cohomology produces an exact sequence

$$F \to \bigoplus_{v \in V(F)} F_v/\mathcal{O}_v \to \mathrm{H}^1(X; \mathcal{O}_X) \to \mathrm{H}^1(X; \mathcal{K}).$$

Because X is irreducible, the constant sheaf $\mathcal{K}$ is flasque, so $\mathrm{H}^1(X; \mathcal{K}) = 0$. The result now follows. ■

Remark 1.32. As an application, the right-hand side will frequently have more than one element: it has dimension over $\mathbb{F}_q$ equal to the genus of X, so the cohomology group has one element if and only if X is $\mathbb{P}^1_{\mathbb{F}_q}$!

Remark 1.33. If one expands one of the $\mathcal{O}_v$s to F_vs, then it turns out that the quotient is trivial.

Remark 1.34. One can check that the stabilizer of a double coset of the F-action on a double coset is exactly $\mathbb{F}_q = \mathrm{H}^0(X; \mathcal{O}_X)$.

Returning to number fields, we see that Theorem 1.30 grants us a surjection

$$F \otimes_{\mathbb{Q}} \mathbb{R} \twoheadrightarrow F\backslash\mathbb{A}_F \Big/ \prod_{v \notin V(F)} \mathcal{O}_v.$$

The kernel is exactly given by the elements $x \in F$ for which $v \in \mathcal{O}_v$ for all v, which is exactly $\mathcal{O}_F$. Thus, there is an isomorphism

$$\frac{F \otimes_{\mathbb{Q}} \mathbb{R}}{\mathcal{O}_F} \to F\backslash\mathbb{A}_F \Big/ \prod_{v \notin V(F)} \mathcal{O}_v$$

of topological groups. Here, the left-hand side is a torus of dimension $[F : \mathbb{Q}]$: it is isomorphic as a topological group to $\mathbb{R}^d/\mathbb{Z}^d$.

1.2.2 Idelic Quotients

Of course, we are more interested in $\mathbb{A}_F^\times$, so let's turn our attention there. As usual, arguing with function fields is easier.

Proposition 1.35. Fix a function field $F := \mathbb{F}_q(X)$. Then one has

$$F^\times \backslash \mathbb{A}_F^\times \Big/ \prod_{v \in V(F)} \mathcal{O}_v^\times \cong \operatorname{Pic} X.$$

Proof. Once again, we use the "two-step complex" $F^\times \to \mathbb{A}_F^\times / \prod_v \mathcal{O}_v^\times$. Here, $\mathbb{A}_F^\times / \prod_{v \in V(F)} \mathcal{O}_v^\times$ is the restricted product

$$\prod_{v \in V(F)} \left(\frac{F_v^\times}{\mathcal{O}_v^\times}, \frac{\mathcal{O}_v^\times}{\mathcal{O}_v^\times} \right) \cong \bigoplus_{v \in V(F)} \mathbb{Z}.$$

Here, the last isomorphism occurs by taking valuations. Now, this latter group is isomorphic to $\operatorname{Div}(X)$, and we can see that $F^\times$ embeds via these isomorphisms as the principal divisors. The result follows. ■

Remark 1.36. It turns out that Pic upgrades into a group scheme Pic_X with a connected component Pic_X^n for each degree. The Jacobian $\operatorname{Jac} X$ is exactly Pic_X^0. Thus, $\operatorname{Pic}(X)$ is infinite, but the degree-zero part $\operatorname{Jac} X(\mathbb{F}_q)$ is some group, which has about q^g points by the Weil conjectures.

Remark 1.37. The kernel of the map $F^\times \to \operatorname{Div} X$ is exactly the constant functions $\mathbb{F}_q^\times$. This reflects the fact that line bundles have some action by $\mathbb{F}_q^\times$.

And now we move to number fields. Here is a starting result.

Lemma 1.38. Fix a number field F. The map

$$\prod_{v \mid \infty} F_v^\times \to F^\times \backslash \mathbb{A}_F^\times \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

has cokernel isomorphic to the class group of F.

Proof. The cokernel is

$$F^\times \backslash \mathbb{A}_{F,f}^\times \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times,$$

where $\mathbb{A}_{F,f}^\times$ is the ring of finite adéles. The right-hand quotient is $\bigoplus_{v \nmid \infty} F_v^\times / \mathcal{O}_v^\times$, which is isomorphic to the group of fractional ideals (or equivalently, $\operatorname{Div}(\operatorname{Spec} \mathcal{O}_F)$). Taking a further quotient by $F^\times$ shows that the cokernel is the class group. ■

Remark 1.39. The kernel of the map is exactly the elements of $F^\times$ which are units at every place, which is exactly $\mathcal{O}_F^\times$. It follows that we have an exact sequence

$$1 \to \mathcal{O}_F^\times \to (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \to F^\times \backslash \mathbb{A}_F^\times \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times \to \operatorname{Cl} F \to 0.$$

To continue cutting down the size of the quotient, note that both $F^\times$ and $\prod_v \mathcal{O}_v^\times$ have global norm in $\mathbb{A}_F^\times \to \mathbb{R}^+$ equal to 1.

Notation 1.40. Fix a number field F. Then we define $\mathbb{A}_F^{\times,1}$ to be the subset of elements with global norm 1.

Remark 1.41. The map

$$F^\times \backslash \mathbb{A}_F^\times \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times \to F^\times \backslash \mathbb{A}_{F,f}^\times \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

continues to be surjective because we can always choose the archimedean part of an adéle so that the adéle has norm 1.

Thus, our exact sequence now looks like

$$1 \to \mu(F) \to \mathcal{O}_F^\times \to (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times,1} \to F^\times \backslash \mathbb{A}_F^{\times,1} \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times \to \operatorname{Cl} F \to 0,$$

where $(F \otimes_{\mathbb{Q}} \mathbb{R})_1$ refers to the subgroup whose product is 1. Taking $\log|\cdot|_v$ (with $|\cdot|_v$ chosen as before) maps $(F \otimes_{\mathbb{Q}} \mathbb{R})^{\times,1}$ into a Euclidean space isomorphic to $\mathbb{R}^{r_1+r_2-1}$, where (r_1, r_2) is the signature of F. Note that the kernel of $\log|\cdot|_v$ is given by the elements of archimedean norm 1, which when restricted to $\mathcal{O}_F^\times$ is exactly the group $\mu(F)$ of roots of unity.

Now, by Dirichlet's unit theorem, we see that $\mathcal{O}_F^\times$ embeds as a lattice of full rank into $\mathbb{R}^{r_1+r_2-1}$, so the quotient is a compact torus. We have thus proven the following result.

Theorem 1.42. Fix a number field F with signature (r_1, r_2) The double quotient

$$F^\times \backslash \mathbb{A}_F^{\times,1} \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

is isomorphic to an extension of $(\mathbb{R}/\mathbb{Z})^{r_1+r_2-1}$ by the class group $\operatorname{Cl} F$. In particular, it is compact.

Remark 1.43. Thus, we see that

$$F^\times \backslash \mathbb{A}_F^\times \Big/ \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

is not compact, but it is an extension of a compact abelian group by $\mathbb{R}^+$.

1.2.3 Pontryagin Duality

Our next task is to do some Fourier analysis on $\mathbb{A}_F$ and $\mathbb{A}_F^\times$. Let's first recall generalities of Fourier analysis on locally compact abelian topological groups.

Definition 1.44 (Pontryagin dual)**.** Fix a locally compact abelian group X. Then its *Pontryagin dual* X^* is the set of homomorphisms $X \to S^1$, equipped with the compact open topology.

Remark 1.45. There is a functoriality as follows: for any homomorphism $f\colon X \to Y$, we have a homomorphism $f^*\colon Y^* \to X^*$ given by pre-composition.

Here are some theorems about this construction.

Theorem 1.46 (Duality)**.** There is a natural isomorphism $\operatorname{id} \Rightarrow (-)^{**}$. For a given group G, it is given by sending $g \in G$ to the character $\operatorname{ev}_g\colon G^* \to S^1$ defined by $\operatorname{ev}_g\colon \chi \mapsto \chi(g)$.

Theorem 1.47 (Exact)**.** The functor $(-)^*$ is exact.

Let's see some examples.

Example 1.48. If $X = \mathbb{Z}$, then its Pontryagin dual is just S^1. On the other hand, all continuous homomorphisms $S^1 \to S^1$ take the form $z \mapsto z^n$, so $(S^1)^* = \mathbb{Z}$.

Example 1.49. Homomorphisms $\mathbb{R} \to S^1$ all take the form $\chi_\xi \colon t \mapsto e^{i\xi t}$, where $\xi \in \mathbb{R}$ is some real number. Thus, $\mathbb{R}^* = \mathbb{R}$.

Example 1.50. In general, given a finite-dimensional real vector space V, we may identify the dual V^* with the Pontryagin dual, where one sends $\varphi \colon V \to \mathbb{R}$ to the character $v \mapsto e^{i\varphi(v)}$.

Example 1.51. Homomorphisms $\mathbb{Z}/n\mathbb{Z} \to S^1$ are uniquely determined by where they send 1, so

$$(\mathbb{Z}/n\mathbb{Z})^* \cong \mu_n.$$

Conversely, all homomorphisms $\mu_n \to \mu_n$ are given by $z \mapsto z^k$ for some k, so $\mu_n^* \cong \mathbb{Z}/n\mathbb{Z}$. In particular, we see that $(\mathbb{Z}/n\mathbb{Z})^*$ is non-canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$, but the isomorphism $\mathbb{Z}/n\mathbb{Z} \to (\mathbb{Z}/n\mathbb{Z})^{**}$ is canonical! In general, for any finite abelian group G, we see that G^* is non-canonically isomorphic to G.

Example 1.52. Exactness of the functor $(-)^*$ implies that

$$\mathbb{Z}_p^* = (\lim \mathbb{Z}/p^\bullet\mathbb{Z})^* = \operatorname{colim} \mu_{p^\bullet} = \mu_{p^\infty}.$$

Example 1.53. Once again, exactness of the functor $(-)^*$ implies that

$$\mathbb{Q}_p^* = \left(\operatorname{colim}\left(\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \cdots\right)\right)^* = \lim\left(\mu_{p^\infty} \xrightarrow{p} \mu_{p^\infty} \xrightarrow{p} \cdots\right).$$

Thus, this limit is some kind of coherent sequence of taking pth roots, which is then isomorphic to $\mathbb{Q}_p$. Indeed, μ_{p^∞} is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$, which we see by sending $a/p^n \in \mathbb{Q}_p/\mathbb{Z}_p$ to $\exp(2\pi i a/p^n)$. In fact, it turns out that the exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

dualizes to an isomorphism

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_p \arrow[r] \arrow[d] & \mathbb{Q}_p \arrow[r] \arrow[d] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\ 1 \arrow[r] & \lim \mu_{p^n} \arrow[r] & \mathbb{Q}_p^* \arrow[r] & \mu_{p^\infty} \arrow[r] & 1 \end{tikzcd}$$

sending $x \in \mathbb{Z}_p$ to the sequence $\{\zeta_{p^n}^x\}_n$.

Thus, we see that all local fields are identified with their Pontryagin duals. In fact, all of our constructions amount to identifying a space with its dual upon choosing a single character.

Remark 1.54. Explicitly, given a choice of nontrivial character $\psi_p \in \mathbb{Q}_p^*$, there is a map $\mathbb{Q}_p \to \mathbb{Q}_p^*$ given by taking x to the character $y \mapsto \psi_p(xy)$. It turns out that this map is an isomorphism, so we have more or less defined a non-degenerate bilinear form $\mathbb{Q}_p \times \mathbb{Q}_p \to S^1$. This procedure also works for $\mathbb{R}$!

Remark 1.55. A similar argument shows that each local field F_v has $F_v^* \cong F_v$, again given by a choice of character $\psi_v \in F_v^*$. Namely, there is a map $F_v \to F_v^*$ given by sending $x \in F_v$ to the character $y \mapsto \psi_p(xy)$.

1.2.4 Fourier Theory

To do Fourier analysis, we need a notion of measure.

Theorem 1.56 (Haar)**.** Fix a locally compact group X. Then there is a left-invariant Radon measure dx on X which is unique up to scalar.

Now, here is our Fourier transform.

Definition 1.57 (Fourier transform)**.** Fix a locally compact abelian group X. The *Fourier transform* sends a function $f \in L^1(X)$ to the function $\widehat{f} \colon X^* \to \mathbb{C}$ given by

$$\widehat{f}(\xi) = \int_X f(x)\overline{\xi}(x)\,dx.$$

It is a large theorem that there is an inversion.

Theorem 1.58 (Fourier inversion)**.** Fix a locally compact abelian group X, and let dx be a Haar measure on X. Then there is a Haar measure $d\chi$ on X^* such that

$$f(x) = \int_{X^*} \widehat{f}(\chi)\chi(x)\,d\chi$$

for any $f \in L^1(X)$ for which $\widehat{f} \in L^1(X^*)$.

Remark 1.59. It turns out that the Fourier transform extends to an isomorphism $L^2(G) \to L^2(G^*)$.

Remark 1.60. Equivalently, we see that the double Fourier transform of f is $f(-x)$.

Remark 1.61. If X admits an isomorphism $X \cong X^*$, then the Haar measure $d\chi$ is not necessarily equal to the Haar measure dx because it might be off by a scalar: indeed, replacing dx with $c\,dx$ replaces $\widehat{f}$ with $c\widehat{f}$, and so we see that we end up replacing $d\chi$ with $c^{-1}\,d\chi$. Thus, there is a unique measure dx on X (even up to scalar!) which is “Fourier self-dual.”

1.3 February 9

Here we go.

1.3.1 Duality for Local Fields

In light of the previous remark, it is useful to fix some characters.

Notation 1.62. Fix a place v of $\mathbb{Q}$.

- If $v = p$ is finite, then we define the character $\psi_p\colon \mathbb{Q}_p \to S^1$ by the composite $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \mu_{p^\infty}$, where the second isomorphism sends a/p^n to $e^{2\pi i a/p^n}$.
- If $v = \infty$ is infinite, then we define the character $\psi_\infty\colon \mathbb{R} \to S^1$ by $\psi_\infty(x) := e^{-2\pi i x}$.

In general, for a place w of a number field F lying over place v of $\mathbb{Q}$, we define $\psi_v := \psi_p \circ \operatorname{tr}_{F_v/\mathbb{Q}_p}$.

Remark 1.63. The character ψ_p has the property that it is trivial on $\mathbb{Z}_p$ but nontrivial on $p^{-1}\mathbb{Z}_p$. In general, for finite places v of a field F, one finds that $\psi_v|_{\mathcal{O}_v} = 1$, but $\mathcal{O}_v$ may not be the largest subgroup with this property, though this is true for all but finitely many v.

Remark 1.64. The choice of ψ_∞ is done so that the assembled character $\psi_\mathbb{Q}\colon \mathbb{A}_\mathbb{Q} \to S^1$ vanishes on $\mathbb{Q}$. In fact, the character $\psi_F := \prod_v \psi_v$ also multiplies to 1 because we can just take a trace of everything to $\mathbb{Q}$.

Self-duality allows us to define Fourier transforms sending functions on F_v to functions on F_v. Let's be careful about what sorts of functions we want to integrate.

Definition 1.65 (Schwartz)**.** Fix a local field F_v.

- If F_v is nonarchimedean, then a *Schwartz function* is a locally constant, compactly supported function $F_v \to \mathbb{C}$.
- If F_v is archimedean, then a *Schwartz function* is a smooth function, all of whose derivatives vanish faster than a polynomial at ∞.

The space of all Schwartz functions is denoted $\mathcal{S}(F_v)$.

Notation 1.66. Fix a Schwartz f on a local field F_v. Then we define the *Fourier transform* by

$$\widehat{f}(y) := \int_{F_v} f(x)\psi_v(xy)\, d_v x.$$

Remark 1.67. One can check that the Fourier transform of a Schwartz function is Schwartz. We could of course integrate any function in $L^2(F_v)$, but we will not have a reason to.

Remark 1.68. By Remark 1.61, there is a unique choice of Haar measure $d_v x$ so that the second Fourier transform of a function f is $f(-x)$. For $F_v = \mathbb{R}$, this turns out to be the Lebesgue measure, and for $F_v = \mathbb{C}$, this turns out to be twice the Lebesgue measure. One can check this by plugging in the function f to be a Gaussian, which turns out to be self-dual for the given measures.

The previous remark (in the case of $\mathbb{C}$) has indicated that the self-dual Haar measure is potentially interesting. Let's explain what we receive for nonarchimedean local fields.

Example 1.69. For $\mathbb{Q}_p$, one can check that $\int_{\mathbb{Z}_p} d_px = 1$. Indeed, suppose that $V = \int_{\mathbb{Z}_p} d_px$, and consider the indicator function $1_{\mathbb{Z}_p}$. Then the Fourier transform

$$\widehat{1}_{\mathbb{Z}_p}(y) = \int_{\mathbb{Z}_p} \psi_p(xy)\, d_px.$$

Now, for $y \in \mathbb{Z}_p$, this integral is constant, so we receive V. For $y \notin \mathbb{Z}_p$, we see that this character $x \mapsto \psi_p(xy)$ is nontrivial on $\mathbb{Z}_p$, so the integral vanishes. (To be explicit, say $y = a/p^n$ where $\gcd(a,p) = 1$ and $n \geq 1$, and then $\int_{\mathbb{Z}_p} \psi_p(ax/p^n)\, d_px = \int_{\mathbb{Z}_p} \psi_p(a(x+1)/p^n)\, d_px = \psi_p(a/p^n) \int_{\mathbb{Z}_p} \psi_p(ax/p^n)\, d_px$. Because $\psi_p(a/p^n) \neq 0$, so the integral vanishes.) Thus, $\widehat{1}_{\mathbb{Z}_p} = V1_{\mathbb{Z}_p}$, so the second Fourier transform is $V^2 1_{\mathbb{Z}_p}$. So for d_px to be self-dual, we are required to have $V = 1$. As a by-product, we note that we have also shown that the function $1_{\mathbb{Z}_p}$ is self-dual!

Remark 1.70. On the homework, we will compute $\int_{\mathcal{O}_v} d_vx$ for places v lying over p, which is done by a similar procedure. Namely, we find that

$$\widehat{1}_{\mathcal{O}_v} = \int_{\mathbb{Z}_p} \psi_p(xy)\, d_px$$

is V times an indicator of the inverse different ideal

$$\mathcal{D}_v^{-1} := \{y \in F_v : \operatorname{tr} y\mathcal{O}_v \subseteq \mathbb{Z}_p\}.$$

This contains $\mathcal{O}_v$, but it may in general be bigger. If $F_v/\mathbb{Q}_p$ is unramified, then $\mathcal{D}_v^{-1} = \mathbb{Z}_p$, so in fact our self-dual measure should give $\int_{\mathcal{O}_v} d_vx = 1$. Otherwise, one finds that setting $\int_{\mathcal{O}_v} d_vx = 1$ need not be self-dual: one needs to multiply or divide by some square root of the index of $\mathcal{D}_v$.

1.3.2 Duality for the Adéles

We are now ready to put a measure on $\mathbb{A}_F$.

Definition 1.71 (Schwartz)**.** Fix a number field F. A function $f\colon \mathbb{A}_F \to \mathbb{C}$ is *Schwartz* if and only if it lives in the restricted tensor product

$$\mathcal{S}(\mathbb{A}_F) := \bigotimes_v (\mathcal{S}(F_v), 1_{\mathcal{O}_v}),$$

where it is restricted in the sense that almost all factors of a pure tensor are equal to $1_{\mathcal{O}_v}$.

Definition 1.72. Fix a number field F. Then we define a measure dx on $\mathbb{A}_F$ to be the product $\prod_v d_vx$, where the measurable subsets consist of finite unions of basic open sets (except at the infinite places).

Remark 1.73. This measure is well-defined because any subset in the Borel algebra will produce a factor $\mathcal{O}_v$ at all but finitely many places v, and all but finitely many of those places have $\int_{\mathcal{O}_v} d_vx = 1$, so the entire product turns out to be finite on any Borel subset.

Remark 1.74. Let's be more precise about this construction. There is a unique Haar measure dx on $\mathbb{A}_F$ such that the measure of $\prod_{v\nmid\infty} \mathcal{O}_v \times \prod_{v\mid\infty} B_v(0,1)$ is the expected product of the given measures. The character $\psi_F\colon \mathbb{A}_F \to S^1$ produces an isomorphism $\mathbb{A}_F \to \mathbb{A}_F^*$ (by gluing together the local isomorphisms), and the corresponding self-dual measure can be checked to be dx by computing the Fourier transform of the indicator of $\bigotimes_v f_v$, where $f_v = 1_{\mathcal{O}_v}$ at finite places and the Gaussian at infinite places.

The reason we want to be able to work globally is that there is a Poisson summation formula for the subgroup $F \subseteq \mathbb{A}_F$.

Theorem 1.75 (Poisson summation)**.** Fix a number field F. For $f \in \mathcal{S}(\mathbb{A}_F)$,

$$\sum_{x\in F} f(x) = \sum_{y\in F} \widehat{f}(y).$$

Let's explain the general story here.

Definition 1.76 (cocompact)**.** Fix a topological abelian group X. A closed subgroup $\Gamma \subseteq X$ is *cocompact* if and only if the quotient $\Gamma\backslash X$ is compact.

Remark 1.77. It turns out that X is compact if and only if X^* is discrete. Thus, if $\Gamma \subseteq X$ is discrete and cocompact, then the dual subgroup

$$\Gamma^\perp := \{\chi \in X^* : \chi|_\Gamma = 1\}$$

is a discrete cocompact subgroup of X^*.

Example 1.78. The subgroup $\mathbb{Z} \subseteq \mathbb{R}$ is discrete and cocompact because $\mathbb{R}/\mathbb{Z}$ is the circle group. Upon identifying $\mathbb{R}^*$ with $\mathbb{R}$ via the character ψ_∞, the dual subgroup $\mathbb{Z}^*$ of $\mathbb{R}$ is exactly $\mathbb{Z}$: indeed, we are asking for $x \in \mathbb{R}$ for which the character $y \mapsto \psi_\infty(xy)$ is trivial on $\mathbb{Z}$, which is equivalent to having $x \in \mathbb{Z}$ because $\psi_\infty(xy) = e^{-2\pi i xy}$.

Theorem 1.79 (Poisson summation)**.** Fix a locally compact topological group X, and let $\Gamma \subseteq X$ be a discrete cocompact subgroup. For any $f \in L^2(X)$, we have

$$\operatorname{vol}(\Gamma\backslash X; dx) \sum_{x\in\Gamma} f(x) = \sum_{y\in\Gamma^\perp} \widehat{f}(y)$$

provided that the left-hand side converges absolutely and uniformly. Here, X^* has been given the dual measure of Theorem 1.58.

Proof. The idea is to consider the function

$$\varphi(x) = \sum_{\gamma\in\Gamma} f(x+\gamma).$$

Provided convergence, this function descends to a function on $\Gamma\backslash X$. This then has a Fourier transform $\widehat{\varphi}$, which is a function on $(\Gamma\backslash X)^* = \Gamma^\perp$. Fourier inversion now provides the equality

$$\varphi(0) = \sum_{y\in\Gamma^\perp} \widehat{\varphi}(y).$$

We will be done as soon as we can check that $\widehat{\varphi}(y) = \widehat{f}(y)$, which is some explicit calculation. Namely, the self-duality $X \cong X^*$ comes from a pairing $X \times X \to S^1$, which gives our Fourier transform the form

$$\begin{aligned}\widehat{f}(y) &= \int_X f(x)\langle x,y\rangle\,dx \\ &\overset{*}{=} \int_{\Gamma\backslash X} f(x)\langle x,y\rangle\,dx \\ &= \widehat{\varphi}(y),\end{aligned}$$

where $\overset{*}{=}$ holds because the inner product $\langle x, y\rangle$ only depends on the class in $\Gamma\backslash X$ (because $y \in \Gamma^\perp$). ■

Remark 1.80. If we have an isomorphism $X \cong X^*$ which sends Γ to $\Gamma^\perp$, and we choose a self-dual Haar measure under the isomorphism, then one can check that the induced volume of $\Gamma\backslash X$ is 1 by plugging in f and its Fourier transform into the Poisson summation formula!

Example 1.81. For the application to $F \subseteq \mathbb{A}_F$, one needs to check that $F^\perp \subseteq \mathbb{A}_F$ is identified with F in the isomorphism $\mathbb{A}_F \cong \mathbb{A}_F^*$ given by the character ψ_F. Certainly $F \subseteq F^\perp$ because $\psi_F(a) = 1$ for all $a \in F$. For the other inclusion, we see that $F^\perp \subseteq \mathbb{A}_F$ is a discrete subgroup of $\mathbb{A}_F$ including F, which we show must be F on the homework. (Here, $F^\perp$ is discrete because its dual is the compact quotient $\mathbb{A}_F/F$.)

Remark 1.82. One should also check that the function

$$y \mapsto \sum_{x\in F} f(x+y)$$

converges absolutely and uniformly for any $f \in \mathcal{S}(\mathbb{A}_F)$. Well, we may descend to a function of the form $\prod_v f_v$, so we are looking at some indicator on a set which is a product of compacts. Using the finite places, we see that we are requiring some bounded valuation at every place, so we are summing over a fractional ideal. But f is Schwartz at the infinite places, so the desired converges follows.

1.3.3 Duality for Function Fields

Let's say something about duality for function fields $F = \mathbb{F}_q(X)$. Then the correct dual object for $\mathbb{A}_F$ turns out to be differential forms.

Notation 1.83. Fix a function field F. Then we define

$$\mathbb{A}_\omega := \prod_v \left(\omega_{F_v}, \omega_{\mathcal{O}_v}\right).$$

Here, ω is the F-bundle of 1-forms on X, ω_v is the stalk at v, $\omega_{\mathcal{O}_v}$ is its completion, and ω_{F_v} is the base-change $\omega_{\mathcal{O}_v} \otimes_{\mathcal{O}_v} F_v$.

Let's see how this produces duality.

Definition 1.84 (residue)**.** Fix a closed point v of a smooth projective geometrically connected curve X. Then we define the *residue* $\omega_{F_v} \to \mathbb{F}_q(v)$ defined as follows: for a differential form θ, choose a local coordinate dt and expand

$$\theta = \sum_{n\in\mathbb{Z}} a_n t^n \, dt.$$

Then the residue is $\operatorname{res}_v \theta := a_{-1}$.

Remark 1.85. It turns out that the residue is independent of the choice of coordinate t. Indeed, any other coordinate is of the form $s = ut$ for a unit u, and we see that $dt/t = ds/s$.

Here is the local duality.

Proposition 1.86. Fix a function field $F = \mathbb{F}_q(X)$, and choose a nontrivial character $\psi\colon \mathbb{F}_p \to S^1$. Then the pairing $F_v \times \omega_{F_v} \to S^1$ defined by

$$\langle f, \theta\rangle_v := \psi(\operatorname{tr}_{\mathbb{F}_q(v)/\mathbb{F}_p} f\theta)$$

realizes an isomorphism $F_v^* \cong \omega_{F_v}$. Furthermore, $\mathcal{O}_v^\perp$ is identified with $\omega_{\mathcal{O}_v}$.

Here is the global duality, which we prove on the homework.

Proposition 1.87. Fix a function field $F = \mathbb{F}_q(X)$, and choose a nontrivial character $\psi\colon \mathbb{F}_p \to S^1$. Then the product pairing $\mathbb{A}_F \times \mathbb{A}_{\omega_F} \to S^1$ defined by

$$(a, \theta) \mapsto \prod_v \langle a_v, \theta_v\rangle_v$$

produces an isomorphism $\mathbb{A}_F^* \to \mathbb{A}_{\omega_F}$.

Remark 1.88. A choice of $\theta \in \omega_F$ allows us to identify $\mathbb{A}_{\omega_F}$ with $\mathbb{A}_F$ and thus identify $\mathbb{A}_F$ with itself. This is analogous to choosing a full character of $\mathbb{Q}_p$ for each p.

Remark 1.89. On the homework, we will show that $F^\perp = \omega_F$. The inclusion $\omega_F \subseteq F^\perp$ follows from the residue theorem: for any $\theta \in \omega_F$, we have

$$\sum_v \operatorname{tr}_{\mathbb{F}_q(v)/\mathbb{F}_q} \operatorname{res}_v \theta = 0.$$

The other inclusion uses a compactness argument.

Now that we are refusing to write down any self-dualities, our choices of Haar measure now depend on a scalar. Instead, we will take by convention that $\operatorname{vol}(\mathcal{O}_v, d_v x) = 1$ for each $v \in X$, and we take the product to produce a measure on $\mathbb{A}_F$.

Example 1.90. Recall that there is an exact sequence

$$\mathbb{F}_q \to \prod_v \mathcal{O}_v \to F\backslash\mathbb{A}_F \to \mathrm{H}^1(X; \mathcal{O}_X) \to 0.$$

This shows that the measure of $F\backslash\mathbb{A}_F$ is q^{g-1}.

Example 1.91. Analogously, one can show that there is an identification of

$$\omega_F\backslash\mathbb{A}_{\omega_F} \Big/ \prod_v \mathcal{O}_v$$

with $\mathrm{H}^1(X; \mathcal{O}_X)$, and there is a stabilizer of $\mathrm{H}^0(X; \mathcal{O}_X)$. Thus, $\omega_F\backslash\mathbb{A}_{\omega_F}$ has measure q^{1-g}.

Remark 1.92. It turns out that the corresponding Poisson summation formula is an incarnation of the Riemann–Roch formula, if one plugs in a special choice of function.

1.4 February 11

Here we go.

1.4.1 Multiplicative Measures

We are now ready to move from adéles to idéles. Let's start by trying to fix a measure.

Remark 1.93. Fix some local field F_v. By definition of $|\cdot|_v$, we see that $d_vx/|\cdot|_v$ is a Haar measure on $F_v^\times$.

Now, for a global field F, one may attempt to put a measure on $\mathbb{A}_F^\times$ by multiplying together all the local measures. However, we are going to want to integrate indicators on basic open subsets of $\mathbb{A}_F^\times$. For example, we could try to integrate $\prod_{v\nmid\infty} 1_{\mathcal{O}_v}\prod_{v\mid\infty} 1_{B_v(0,1)}$, whose integral will be a scalar times

$$\prod_{v\nmid\infty}\int_{\mathcal{O}_v^\times} d_v^\times x.$$

But this integral is $\mu_v(\mathcal{O}_v)-\mu_v(\varpi_v\mathcal{O}_v)=(1-|\varpi_v|_v)\,\mu_v(\mathcal{O}_v)$. For all but finitely many v, we see that $\mu_v(\mathcal{O}_v)=1$, so we see that the above product has all but finitely many of its factors not equal to 1! In fact, it vanishes for $F=\mathbb{Q}$. Thus, this normalization is not suitable for our purposes. Instead, we divide out by this factor $1-|\varpi_v|_v$.

Definition 1.94. Fix a nonarchimedean local field F_v. Then we define

$$d_v^\times x := \frac{1}{1-q_v^{-1}}\cdot\frac{d_vx}{|x|_v},$$

and we define $d^\times x$ on $\mathbb{A}_F^\times$ as the product measure.

Remark 1.95. It follows from the preceding calculation that $d^\times x$ is a finite product on basic open subsets.

An interesting question is to calculate the volume of $F^\times\backslash\mathbb{A}_F^{\times,1}$ according to the measure $d^\times x$. For number fields F, recall from the proof of Theorem 1.42 that we have an exact sequence

$$1\to\mu(F)\to\mathcal{O}_F^\times\to(F\otimes_\mathbb{Q}\mathbb{R})^{\times,1}\to F^\times\backslash\mathbb{A}_F^{\times,1}/\prod_{v\nmid\infty}\mathcal{O}_v^\times\to\operatorname{Cl}F\to 0.$$

One eventually finds the following.

Proposition 1.96. Fix a number field F with signature (r_1,r_2). Then the volume of $F^\times\backslash\mathbb{A}_F^{\times,1}$ is

$$\frac{h_F\operatorname{Reg}_F}{w}\cdot\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|\operatorname{disc}\mathcal{O}_F|}}.$$

Here, h_F is the class number, Reg_F is the regulator (which is an appropriately measured covolume of $\mathcal{O}_F^\times$ sitting in $(F\otimes_\mathbb{Q}\mathbb{R})^{\times,1}$), and w is the number of roots of unity.

Remark 1.97. The second factor basically arises from how we have chosen our additive measures.

We will shortly see that the volume of $F^\times\backslash\mathbb{A}_F^{\times,1}$ is also related to the Dedekind ζ_F-function, thereby proving the analytic class number formula.

Definition 1.98 (Dedekind ζ-function)**.** Fix a global field F. Then we define the *Dedekind* ζ_F*-function* as

$$\zeta_F(s) \coloneqq \sum_{I \subseteq \mathcal{O}_F} \frac{1}{\mathrm{N}(I)}.$$

This sum converges absolutely and uniformly on compacts for $\operatorname{Re} s > 1$.

Remark 1.99. Unique prime factorization of ideals produces an Euler product

$$\zeta_F(s) = \prod_{v \nmid \infty} \frac{1}{1 - q_v^s}.$$

This is remarkable because it looks like "$\zeta_F(1)$" is the scale factor between the failed "product" measure $dx/|x|$ on $\mathbb{A}_F^\times$ and our successful measure $d^\times x$.

1.4.2 Our L-functions

The goal of Tate's thesis is to reprove some general results on the functional equations of L-functions. For example, we will be able to reprove the functional equation of the Dirichlet L-functions, defined by

$$L(s, \chi) \coloneqq \sum_{\substack{n=1 \\ \gcd(n,N)=1}}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi \colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is some character.

Example 1.100. Taking $\chi = 1$ (and $N = 1$) recovers Riemann's ζ-function.

Roughly speaking, our functional equation will equate $L(s,\chi)$ and $L(1-s,\overline{\chi})$, after we "fix" these L-functions slightly.

We will be able to work over general global fields. Let's start by relating the discussion of the previous example with our adélic discussion.

Example 1.101. It turns out that $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \times \mathbb{R}^+ \times \widehat{\mathbb{Z}}^\times$ (recall $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$), which one can see by taking some successive quotients and appealing to Theorem 1.42. The moral is that a Dirichlet character $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ can be viewed as a continuous character $\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ which "happens" to factor through $\widehat{\mathbb{Z}}^\times$.

Definition 1.102 (idéle class character)**.** Fix a global field F. Then an *idéle class character* is a character $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. It is *unitary* if its image is contained in S^1.

Remark 1.103. The unitary characters (by definition) live in the Pontryagin dual of $F^\times \backslash \mathbb{A}_F^\times$.

Example 1.104. Note that there is a norm character $|\cdot| : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{R}^+$. In fact, for any $s \in \mathbb{R}$, we receive a character $|\cdot|^s$.

Remark 1.105. Suppose F is a number field. Then $|\cdot| : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{R}^+$ is surjective, so we can take $\operatorname{Hom}(-,\mathbb{C}^\times)$ of the short exact sequence

$$1 \to F^\times\backslash\mathbb{A}_F^{\times,1} \to F^\times\backslash\mathbb{A}_F^\times \to \mathbb{R}^+ \to 1$$

to see that $\left(F^\times\backslash\mathbb{A}_F^\times\right)^*$ is an extension of $\operatorname{Hom}(\mathbb{R}^+,\mathbb{C}^\times) = \mathbb{C}$ by a discrete group $\left(F^\times\backslash\mathbb{A}_F^{\times,1}\right)^*$. (We have used the compactness of $F^\times\backslash\mathbb{A}_F^{\times,1}$ to show that any map to $\mathbb{C}^\times$ factors through S^1. Note that exactness holds on the right after the duality because already $(-)^*$ is exact.) We are thus able to conclude that the space of idéle class characters inherits a topology of a complex manifold.

Remark 1.106. Suppose F is a function field $\mathbb{F}_q(X)$. Then the norm $|\cdot| : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{R}^+$ surjects instead onto the discrete group $q^{\mathbb{Z}}$. We thus see that $\operatorname{Hom}\left(F^\times\backslash\mathbb{A}_F^\times,\mathbb{C}^\times\right)$ is an extension of $\operatorname{Hom}(\mathbb{Z},\mathbb{C}^\times) = \mathbb{C}^\times$ by a discrete group. The space of idéle class characters continues to be a complex manifold by the same argument.

Remark 1.107. By continuity of χ, we see that $\chi|_{\mathcal{O}_v^\times} = 1$ for almost all v. Indeed, this follows from a "no small subgroups" argument applied to continuous maps on the group $\prod_{v\nmid\infty}\mathcal{O}_v^\times$.

Definition 1.108 (unramified)**.** An idéle class character χ is *unramified* at a finite place v if and only if $\chi|_{\mathcal{O}_v} = 1$; otherwise, we see that it is *ramified* at v.

We are now ready to define our L-functions.

Definition 1.109. Fix an idéle class character χ of a global field F, and choose a finite place v. Then we define

$$L_v(\chi_v) := \begin{cases} \frac{1}{1-\chi_v(\varpi_v)} & \text{if } \chi_v|_{\mathcal{O}_v^\times} = 1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$L(\chi) := \prod_{v\nmid\infty} L_v(s,\chi_v).$$

We may write $L_v(s,\chi_v)$ and $L(s,\chi)$ for $L_v\left(\chi_v\,|\cdot|_v^s\right)$ and $L\left(\chi\,|\cdot|^s\right)$, respectively.

Remark 1.110. We can expand $L(\chi)$ out as a sum

$$L(\chi) = \sum_{I\subseteq\mathcal{O}_F} \frac{\chi(I)}{\mathrm{N}(I)},$$

where $\chi(I)$ means $\prod_v \chi(\varpi_v)^{\nu_v(I)}$, and $\chi(\varpi_v)$ in this expression means 0 if χ is ramified at v.

Remark 1.111. If χ is unitary, then the function $s\mapsto L(s,\chi)$ can be checked to converge absolutely and uniformly on compacts in the region $\operatorname{Re}s > 1$.

Example 1.112. Taking $\chi = 1$ recovers Dedekind ζ-functions.

1.4.3 Functional Equations

Tate's main global result is a duality statement.

Definition 1.113 (global integral)**.** Fix an idéle class character χ on a number field F. For $f \in \mathcal{S}(\mathbb{A}_F)$, we define

$$Z(\chi, f) := \int_{\mathbb{A}_F^\times} \chi(a) f(a)\, d^\times a.$$

We may also write $Z(s, \chi, f) := Z(\chi\, |\cdot|^s, f)$.

Remark 1.114. Of course, there is redundancy in our notation: indeed, $Z(s+t, \chi, f) = Z(s, \chi\, |\cdot|^t, f)$.

Theorem 1.115 (Tate)**.** Fix a global field F and some $f \in \mathcal{S}(\mathbb{A}_F)$.

(a) The function $\chi \mapsto Z(\chi, f)$ admits a meromorphic continuation and a functional equation

$$Z(\chi, f) = Z\left(\chi^{-1}\, |\cdot|\, , \widehat{f}\right).$$

(b) If χ is nontrivial on $\mathbb{A}_F^{\times,1}$, then $s \mapsto Z(s, \chi, f)$ is holomorphic.

(c) The function $s \mapsto Z(s, 1, f)$ is holomorphic everywhere except for simple poles at $s \in \{0, 1\}$ with residue $-f(0)\operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}; d^\times x\right)$ at $s = 0$.

Let's explain how this relates to our L-functions.

Theorem 1.116. Fix a number field F. Then there is a functional equation relating $\zeta_F(s)$ and $\zeta_F(1-s)$.

Proof. The idea is to make f and $\widehat{f}$ the same at almost all places. Define a function (f_v) in $\mathcal{S}(\mathbb{A}_F)$ as follows.

- For finite v, we define $f_v = 1_{\mathcal{O}_v}$. In particular, at unramified places v, we recall that $1_{\mathcal{O}_v}$ is self-dual.
- If $F_v = \mathbb{R}$ and $\chi_v = 1$, then the function $e^{-\pi x^2}$ is self-dual. The case where $\chi_v = \operatorname{sgn}$ takes the function $xe^{-\pi x^2}$.
- If $F_v = \mathbb{C}$ and $\chi_v = 1$, then the function $e^{-\pi |z|^2}$ is self-dual. If χ_v is more general, then some slightly different recipe is used.

One can now compute

$$Z(s, 1, f) = \prod_v \int_{F_v^\times} |a_v|^s f_v(a_v)\, d_v^\times a.$$

If v is finite, then we are being asked to compute

$$\begin{aligned}\int_{\mathcal{O}_v \setminus \{0\}} |a|^s \, d_v^\times a &= \sum_{n \geq 0}^{\infty} \int_{\varpi^n \mathcal{O}_v} |a|^s \, d_v^\times a \\ &= \sum_{n \geq 0} q_v^{-ns} \operatorname{vol}\left(\mathcal{O}_v^\times\right) \\ &= \frac{\operatorname{vol}\left(\mathcal{O}_v^\times\right)}{1 - q_v^{-s}}.\end{aligned}$$

If v is unramified over $\mathbb{Q}_p$, then the volume on top is 1; if it is ramified, then we are computing some square root of the norm of the different (which is the discriminant). Thus, up to these contributions from rational factors, we find that $Z(s,1,f)$ is

$$\zeta_F(s) \prod_{\text{real } v} \pi^{-s/2}\Gamma(s/2) \prod_{\text{complex } v} 2(2\pi)^{1-s}\Gamma(s).$$

We are thus able to produce a functional equation for $\zeta_F(s)$. We did not work this out in class, and I do not have time to do it on my own currently. ■

For function fields $F = \mathbb{F}_q(X)$, we have been careful to avoid identifying $\mathbb{A}_F$ with itself, so we don't have a self-duality.

Example 1.117. In this case, note that the Euler product of $\zeta_F(s)$ expands into

$$\sum_{\text{effective } D\subseteq X} q^{-s\deg D} = \sum_{d\geq 0} q^{-ds}\cdot \#X^{(d)}(\mathbb{F}_q),$$

where $X^{(d)}(\mathbb{F}_q)$ refers to the number of effective divisors on D of degree d defined over $\mathbb{F}_q$. We have used the notation $X^{(d)}$ to indicate that this could be thought of as the stack X^d/Σ_d.

Example 1.118. Suppose that χ is an unramified idéle class character, meaning that it factors through $F^\times\backslash\mathbb{A}_F^\times/\prod_v \mathcal{O}_v^\times$, which we recall is $\operatorname{Pic} X$. One can calculate as before that

$$L(s,\chi) = \sum_{D\subseteq X} q^{-s\deg D}\chi(\mathcal{O}_X(D)).$$

Further suppose that $\chi|_{\operatorname{Pic}^0 X} \neq 1$. Then we claim that $L(s,\chi)$ is a polynomial in q^{-s}. Indeed, in light of the above expansion, it is enough to show that all line bundles of large degree d appear an equal number of times as $\mathcal{O}_X(D)$ as D varies over effective divisors of d. But this is true because $d > 2g-2$ makes $X^{(d)}$ a $\mathbb{P}^{d-g}$-bundle over $\operatorname{Pic}^d X$: by an argument with linear systems (and the Riemann–Roch theorem), the fibers are all copies of $\mathbb{P}^{d-g} = \mathbb{P}\mathrm{H}^0(X;\mathcal{O}_X(D))$.

1.5 February 17

Today we prove Tate's theorem.

1.5.1 More General L-functions

To start us off, let's do a local calculation.

Definition 1.119 (local integral)**.** Fix a local field F and a continuous character $\chi\colon F^\times \to \mathbb{C}^\times$. For $f \in \mathcal{S}(\mathbb{A}_F)$, we define

$$Z(\chi,f) \coloneqq \int_{F^\times} \chi(a)f(a)\,d^\times a.$$

We may also write $Z(s,\chi,f) \coloneqq Z(\chi\left|\cdot\right|^s, f)$.

Proposition 1.120. Fix a nonarchimedean local field F, and suppose χ is unramified, in the sense that $\chi|_{\mathcal{O}_v^\times} = 1$. Then

$$Z(s,\chi,1_{\mathcal{O}}) = \frac{1}{\chi(\varpi)q^{-s}}\cdot \operatorname{vol}(\mathcal{O}^\times; d^\times a).$$

Proof. Using the translation-invariance of $d^\times a$, we find that

$$\int_{\mathcal{O}^\times} \chi(a)\,|a|^s\ d^\times a = \sum_{n\geq 0} \chi(\varpi^n a) q^{-ns} \int_{\mathcal{O}^\times} \chi(a)\,d^\times a,$$

so the result follows. ■

Thus, we see that there is a finite set S for which

$$Z(s,\chi,f) = \prod_{v\notin S} L_v(s,\chi_v) \cdot \prod_{v\in S} Z_v(s,\chi_v,f_v).$$

Indeed, one can just take S to contain the archimedean places, the places where $\operatorname{vol}(\mathcal{O}_v^\times; d_v^\times x) \neq 1$, and the places where f_v is not $1_{\mathcal{O}_v}$. Now, the finite product is relatively easy to understand, and an explicit calculation shows that $Z_v(s,\chi_v,f_v)$ admits a meromorphic calculation. For nonarchimedean places, one can argue as above by turning the integral into a geometric series; for archimedean places, one needs to do some analysis with the Schwartz hypothesis.

The point is that meromorphic continuation for $Z(s,\chi,f)$ directly implies meromorphic continuation for the L-function $L(s,\chi)$. In fact, one can argue as in Theorem 1.116 to see that Theorem 1.115 implies that $L(s,\chi)$ admits meromorphic continuation, functional equation, and it has prescribed poles.

1.5.2 Proof of the Global Functional Equation

We are now ready to prove Theorem 1.115.

Proof of Theorem 1.115. Define

$$Z_\pm(s,\chi,f) \coloneqq \int_{|a|^{\pm1}>1} f(a)\chi(a)\,|a|^s\ d^\times a.$$

Because $|a| > 1$, we see that $Z_+(s,\chi,f)$ converges everywhere: both Z_+ and Z_- already converges for $\operatorname{Re} s > 1$, and Z_+ will only get smaller as $\operatorname{Re} s$ gets smaller.

The idea to prove (a) is to relate $Z_-(s,\chi,f)$ with $Z_+(1-s,\chi^{-1},\widehat{f})$. Indeed, because $\chi\,|\cdot|^s$ factors through $F^\times\backslash\mathbb{A}_F^\times$, we may "unfold" our integral, writing

$$\begin{aligned} Z_-(s,\chi,f) &= \int_{\substack{a\in\mathbb{A}_F^\times \\ |a|<1}} f(a)\chi(a)\,|a|^s\ d^\times a \\ &= \int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|<1}} \left(\sum_{\gamma\in F^\times} f(\gamma a)\right)\chi(a)\,|a|^s\ d^\times a. \end{aligned}$$

We would like to apply Poisson summation, but we need to add back in $0 \in F$ for this to make sense. To this end, define $f_a \in \mathcal{S}(\mathbb{A})$ by $f_a(x) \coloneqq f(ax)$. Then

$$Z_-(s,\chi,f) = \int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|<1}} \left(\sum_{\gamma\in F} f_a(\gamma)\right)\chi(a)\,|a|^s\ d^\times a - f(0)\int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|<1}} \chi(a)\,|a|^s\ d^\times a.$$

We now apply Theorem 1.75. Note $\widehat{f_a}(x) = |a|^{-1}\,\widehat{f}(x/a)$, so we see

$$Z_-(s,\chi,f) = \int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|<1}} \left(\sum_{\gamma\in F} \widehat{f}(\gamma/a)\right)\chi(a)\,|a|^{s-1}\ d^\times a - f(0)\int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|<1}} \chi(a)\,|a|^s\ d^\times a,$$

which by sending a to a^{-1} gives

$$Z_-(s,\chi,f) = \int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|>1}} \left(\sum_{\gamma\in F} \widehat{f}(\gamma a)\right)\chi^{-1}(a)\,|a|^{1-s}\ d^\times a - f(0)\int_{\substack{a\in F^\times\backslash\mathbb{A}_F^\times \\ |a|<1}} \chi(a)\,|a|^s\ d^\times a.$$

It may look like we should send $d^\times a$ to $-d^\times a$, but this sign is absorbed into the orientation: we start integrating $(0,1]$ and want to end integrating $[1,\infty)$.

Let's spend a moment to simplify the right-hand term. By fixing $|a|$, this integral is

$$\int_0^1 \left(\int_{F^\times \backslash \mathbb{A}_F^{\times,1}} \chi(at_\infty) t^s \, d^\times a \right) d^\times t.$$

Here, t_∞ is some idéle supported at a single place, chosen so that $|t_\infty| = t$; it is found basically by splitting the norm map $|\cdot| : \mathbb{A}_F^\times \to \mathbb{R}^+$.[1] Now, if $\chi|_{\mathbb{A}_F^{\times,1}} \neq 1$, then the internal integral vanishes; otherwise, if $\chi = |\cdot|^t$, then the inner integral is $\operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}; d^\times x\right)$, and the outer integral then gives $\int_0^1 t^{s+t} \frac{dt}{t} = \frac{1}{s+t}$. Thus,

$$Z_-(s,\chi,f) = \int_{\substack{a \in F^\times \backslash \mathbb{A}_F^\times \\ |a|>1}} \left(\sum_{\gamma \in F} \widehat{f}(\gamma a) \right) \chi^{-1}(a) \, |a|^{1-s} \, d^\times a - \frac{f(0) \operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}\right)}{s+t} 1_{\chi(\mathbb{A}_F^{\times,1})=1}.$$

We now take out the $\gamma = 0$ term from the internal sum and repeat the procedure of the previous paragraph. (There are some nasty sign problems here. The difficulty is that the case of $\chi|_{\mathbb{A}_F^{\times,1}} = 1$ receives an integral of $\int_0^1 t^{1-s-t} \, dt/t = -1/(1-s-t)$.) Being careful with our factors, we are left with

$$Z_-(s,\chi,f) + \frac{f(0) \operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}\right)}{s+t} 1_{\chi(\mathbb{A}_F^{\times,1})=1} = Z_+(1-s,\chi^{-1},\widehat{f}) + \frac{\widehat{f}(0) \operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}\right)}{s-1-t} 1_{\chi(\mathbb{A}_F^{\times,1})=1}.$$

The right-hand side now has good analytic properties, so we receive our meromorphic continuation, and the location (and type) of the poles follows from the expression as well. Sending $f \mapsto \widehat{f}$ and $s \mapsto (1-s)$ and $\chi \mapsto \chi^{-1}$ and summing completes the proof of the functional equation. Indeed, we find

$$Z_+(s,\chi,f) - \frac{f(0) \operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}\right)}{s+t} 1_{\chi(\mathbb{A}_F^{\times,1})=1} = Z_-(1-s,\chi^{-1},\widehat{f}) - \frac{\widehat{f}(0) \operatorname{vol}\left(F^\times \backslash \mathbb{A}_F^{\times,1}\right)}{s-1-t} 1_{\chi(\mathbb{A}_F^{\times,1})=1},$$

so summing gives functional equation. ■

Remark 1.121. Basically the same proof works for function fields as soon as we choose an isomorphism $\mathbb{A}_F \to \mathbb{A}_F^*$, which amounts to the data of an isomorphism $\mathbb{A}_F \to \mathbb{A}_{\omega_F}$, which is the data of a nonzero meromorphic differential form θ. Even though this identification depends on the choice of θ, it turns out that the self-dual Haar measure does not. One checks this by comparing the self-dual Haar measure for θ and some $f\theta$, for any $f \in K(X)^\times$. Alternatively, one can show that the volume of $\prod_v \mathcal{O}_v^\times$ with respect to the self-dual Haar measure only depends on the genus of X.

1.5.3 A Little Geometric Class Field Theory

Let's give a few remarks about the argument for function fields.

Remark 1.122. One can show that Theorem 1.115 for $\chi = 1$ and function fields $\mathbb{F}_q(X)$ amounts to the functional equation for ζ_X. This is on the homework.

One may be interested in what happens for nontrivial χ. For example, if χ is unramified, then it factors through $F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_v^\times = \operatorname{Pic} X$. To think about such characters geometrically, we need some class field theory. Indeed, class field theory grants a reciprocity map $\operatorname{Art}_F \colon F^\times \backslash \mathbb{A}_F^\times \to \operatorname{Gal}(F^{\mathrm{sep}}/F)^{\mathrm{ab}}$, which turns out to fit into a pullback square as follows.

$$\begin{tikzcd} F^\times \backslash \mathbb{A}_F^\times \arrow[r, "\operatorname{Art}_F"] \arrow[d, "\deg"'] & \operatorname{Gal}(F^{\mathrm{sep}}/F)^{\mathrm{ab}} \arrow[d, two heads] \\ \mathbb{Z} \arrow[r, "\operatorname{Frob}_q"] & \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \end{tikzcd}$$

[1] For number fields, we can split by choosing any archimedean place. For function fields, I think something similar is possible.

Now, note that finite étale covers of X correspond to everywhere unramified field extensions of F, so we see that $\operatorname{Gal}(F^{\mathrm{sep}}/F) = \pi_1^{\text{ét}}(X)$. The quotient on the other side is $F^\times\backslash\mathbb{A}_F^\times/\prod_v \mathcal{O}_v^\times = \operatorname{Pic} X$. Taking the kernel with respect to degree, we may track around the following diagram.

$$\begin{tikzcd}
 & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q})^{\mathrm{ab}} \arrow[d, two heads] \\
F^\times\backslash\mathbb{A}_F^{\times,1}/\prod_v \mathcal{O}_v^\times \arrow[r] \arrow[d, hook] & \ker \arrow[d, hook] \\
F^\times\backslash\mathbb{A}_F^{\times}/\prod_v \mathcal{O}_v^\times \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(X)^{\mathrm{ab}} \arrow[d] \\
\operatorname{Pic} X \arrow[r, "\deg"] & \widehat{\mathbb{Z}}
\end{tikzcd}$$

Indeed, we see that $\operatorname{Pic}^0 X$ maps to the kernel of $\pi_1^{\text{ét}}(X) \to \widehat{\mathbb{Z}}$, which at least admits a surjection from $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q})^{\mathrm{ab}}$.[2] Thus, χ produces a homomorphism

$$\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \to \mathbb{C}^\times,$$

which is the data of a local system $\mathcal{L}_\chi$ on $X_{\overline{\mathbb{F}}_q}$ of rank 1. (This map even extends to $\pi_1^{\text{ét}}(X)$ by the same argument, so we receive a local system on X.)[3] It now turns out that

$$L(s,\chi) = \prod_{i=0}^{2} \det\left(1 - \operatorname{Frob}_q q^{-s}; \mathrm{H}^i(X_{\overline{\mathbb{F}_q}}; \mathcal{L}_\chi)\right)^{(-1)^{i+1}},$$

which basically follows from the Lefschetz trace formula. For example, if χ vanishes on $\operatorname{Pic}^0 X$, then the $i=0$ term is $1-\chi(a)q^{-s}$ (for some $a \in \mathbb{A}_F^\times$ with $\deg a = 1$), and the $i=2$ term is $1-\chi(a)q^{1-s}$. But if χ is nontrivial on $\operatorname{Pic}^0 X$, then $\mathrm{H}^0(X;\mathcal{L}_\chi)$ vanishes, so H^2 also vanishes by duality, so we only have H^1. The total degree should remain the same no matter what χ is (by some Euler characteristic calculation), so we see that

$$L(s,\chi) = \det\left(1 - \operatorname{Frob}_q q^{-s}; \mathrm{H}^1(X_{\overline{\mathbb{F}}_q}; \mathcal{L}_\chi)\right)$$

is some polynomial of degree $2g-2$. Then one can see that Theorem 1.115 yields

$$L(1-s,\chi^{-1}) = q^{(2g-2)s} \det\left(\operatorname{Frob}_q; \mathrm{H}^*(X_{\overline{\mathbb{F}_q}}; \mathcal{L}_\chi)\right) L(s,\chi).$$

1.5.4 Local Theory

Let's say a few sentences about the local theory.

Definition 1.123 (local integral)**.** Fix a local field F. For any $f \in \mathcal{S}(F)$ and continuous $\chi\colon F^\times \to \mathbb{C}^\times$, we define

$$Z(s,\chi,f) := \int_{F^\times} f(a)\chi(a)\,|a|^s\, d^\times a.$$

Remark 1.124. A direct calculation shows that $Z(s,\chi,f)$ is some Laurent polynomial in q^{-s} (which depends on f) with the controlled denominator $L(s,\chi)$, which is the local L-factor.

[2] It turns out that the kernel is exactly the Frobenius co-invariants of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q})$, which is a finite group.

[3] It is occasionally convenient to pass from $\pi_1^{\text{ét}}(X)$ to the Weil group.

Theorem 1.125. Fix a local field F and a nontrivial character $\psi\colon F \to \mathbb{C}^\times$. Then

$$Z\big(1-s,\chi^{-1},\widehat{f}\big) = \gamma(s,\chi,\psi) Z(s,\chi,f)$$

for some function $\gamma(s,\chi,\psi)$ which is independent of f.

Proof. Omitted. ■

Remark 1.126. One can decompose γ further into an ε-factor, which is

$$\frac{Z\big(1-s,\chi^{-1},\widehat{f}\big)/L\left(1-s,\chi^{-1}\right)}{Z(s,\chi,f)/L(s,\chi)}.$$

Remark 1.127. Combining the local and global functional equations reveals that there is some $\varepsilon(s,\chi)$ for which $L(s,\chi) = \varepsilon(s,\chi)L\left(1-s,\chi^{-1}\right)$, and

$$\varepsilon(s,\chi) = \prod_v \varepsilon_v(s,\chi_v,\psi_v).$$

This factorization is fairly surprising! For example, in the function field situation, we have factored some determinant of Frobenius action on cohomology into a product of local terms.

THEME 2
REPRESENTATION THEORY

2.1 February 18

Today we say something about representation theory.

2.1.1 Overview

Approximately speaking, given a reductive group G over a global field F, we will be interested in "automorphic representations," which are roughly speaking those representations of $G(\mathbb{A}_F)$ which appear in $L^2(G(F)\backslash G(\mathbb{A}_F))$. Such representations turn out to have poor categorical properties, so there are some technicalities involved in the definition of "automorphic."

Because G is an affine group scheme over F, one finds that $G(\mathbb{A}_F)$ is the restricted direct product

$$G(\mathbb{A}_F) = \prod_v (G(F_v), G(\mathcal{O}_v)),$$

where we have given G an integral model which extends over all but finitely many places of F. Thus, we should start by discussing some "local" representation theory. We'll start with p-adic groups, whose representations are understood via a Hecke algebra; then we will turn to real groups, whose representations are understood via $(\mathfrak{g}, K)$-modules.

Once we have control over local representation theory, we can find automorphic representations of $G(\mathbb{A}_F)$ to be given as a (restricted) tensor product of local representations. Eventually, for $G = \mathrm{GL}_2$, we will be able to relate all this to the classical theory of modular forms.

2.1.2 Totally Disconnected Groups

For now, F will be a nonarchimedean local field with ring of integers $\mathcal{O}$ and residue field k.

Definition 2.1. An *algebraic group* G over a field F is an affine group scheme of finite type over F.

Proof. It turns out that any such G admits a closed embedding into GL_n for some n. This amounts to the statement that G admits a faithful representation, which can be found by taking a suitable regular representation. ■

When F is a local field (or more generally, a Hausdorff topological ring), one can give $G(F)$ a topology so that closed embeddings of schemes (over F) go to closed embeddings of topological spaces, and the topology of $\mathbb{A}^n(F) = F^n$ is the obvious one.

Example 2.2. The group GL_n admits a closed embedding into $\mathbb{A}^{n+1}$ by $g \mapsto \big((g_{ij}), \det g^{-1}\big)$.

Example 2.3. If F is a nonarchimedean local field, then $\det\colon \mathrm{GL}_n(F) \to F^\times$ is already continuous, so in fact the map

$$\mathrm{GL}_n(F) \to M_n(F)$$

is an open embedding. Indeed, by construction of the topology, an open neighborhood basis of $\mathrm{GL}_n(F)$ are the set

$$K_{a,b} := \{A \in \mathrm{GL}_n(F) : A - 1 \in \mathfrak{p}^a M_n(\mathcal{O}), \det A \in (1+\mathfrak{p}^b)\},$$

but $\det^{-1}$ is continuous as a map from the subspace of $n \times n$ invertible matrices to $F^\times$ (indeed, $\det^{-1}$ is a rational function), so we may always expand a to satisfy the b condition.

Remark 2.4. Set

$$K_m := \{A \in \mathrm{GL}_n(\mathcal{O}_F) : (A-1) \in \mathfrak{p}^n M_n(\mathcal{O})\}.$$

These are compact open subgroups of $\mathrm{GL}_n(\mathcal{O}_F)$. By definition, K_m is the kernel fitting in the exact sequence

$$1 \to K_m \to \mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^m).$$

Example 2.5. Fix a quadratic space V over a field F. One can now describe the topology on a classical group $\mathrm{O}(V)(F)$ as the closed subspace topology from $\mathrm{GL}(V)$.

Having so many compact open subgroups is a remarkable property.

Definition 2.6 (totally disconnected)**.** A topological group G is *totally disconnected* if and only if G is Hausdorff, second countable, and admits an open neighborhood basis of the identity consisting of compact open subgroups.

Example 2.7. For any affine algebraic group G over a local field F, we can embed $G(F) \subseteq \mathrm{GL}_n(F)$. Then the countable open compact neighborhood basis for $\mathrm{GL}_n(F)$ makes $G(F)$ totally disconnected.

2.1.3 Smooth Representations

We will be able to do quite a bit with general totally disconnected groups.

Definition 2.8 (smooth)**.** Fix a totally disconnected group G. Then a complex representation V of G is *smooth* if and only if the action map

$$G \times V \to V$$

is continuous, where V has been given the discrete topology. A *homomorphism* of smooth representations is a morphism of the underlying representations of G.

Remark 2.9. Equivalently, we are asking for each $v \in V$ to have open pre-image in $G \times V$. Because V has been given the discrete topology, this simply means that $\mathrm{Stab}_G(v) \subseteq G$ is open for all $v \in V$. Note that it is enough for the subgroup to merely contain an open neighborhood U of 1 because then $\mathrm{Stab}_G(v)$

Example 2.10. Fix a nonarchimedean local field F. Any continuous homomorphism $F^\times \to \mathbb{C}^\times$ admits some open subgroup $1 + \ker \mathfrak{p}^\bullet$ in its kernel, by a "no small subgroups" argument.

Sometimes our representations are not smooth, but we can find smooth representations within.

Definition 2.11 (smooth)**.** Fix a complex representation V of a totally disconnected group G. A vector $v \in V$ is *smooth* if and only if $\operatorname{Stab}_G(v)$ is open. We let V^{sm} denote the collection of smooth vectors.

Remark 2.12. By taking intersections of open subgroups, we see that linear combinations of smooth vectors continue to be smooth. It follows that V^{sm} is a subspace, and we can see that it is even G-invaraint.

Example 2.13 (smooth functions)**.** Let V be the space of all functions $G \to \mathbb{C}$, and let G act on V by $(gf)(x) := f(xg)$. (One could also restrict to those functions with compact support.) However, functions $f\colon G \to \mathbb{C}$ are rarely smooth: this amounts to requiring that $f(xg) = f(x)$ for g contained in an open subgroup of G, meaning that f descends to a quotient G/K for an open subgroup K of G. Equivalently, we may write this representation as

$$\bigcup_{m\geq 1} \operatorname{Mor}(G/K_m, \mathbb{C}),$$

where $\{K_m\}$ is some countable compact open neighborhood basis of the identity.

These smooth functions will be useful to us.

Notation 2.14. Fix a totally disconnected group G. We let $C_c^\infty(G)$ denote the collection of locally constant functions on G of compact support.

Remark 2.15. It turns out that functions in $f \in C_c^\infty(G)$ admit an open neighborhood K for which

$$f(kxk') = f(x)$$

for any $k, k' \in K$. Indeed, f can be written as a finite linear combination of indicators of compact open subsets. By translating, we may further reduce to the case where f is the indicator of some compact open subgroup K', for which we can take $K := K'$.

Example 2.16 (induction)**.** Given a subgroup $H \subseteq G$, we can let G act by right translation on functions $H\backslash G \to \mathbb{C}$. Then $\operatorname{Mor}(H\backslash G, \mathbb{C})$ is the induction of the trivial representation from H to G.

2.1.4 Categorical Properties

Let's make a few remarks about functoriality.

Notation 2.17. Fix a totally disconnected group G. The category of smooth representations of G will be denoted $\operatorname{Rep}_{\mathbb{C}}^{\text{sm}}(G)$.

Remark 2.18. By definition, $\operatorname{Rep}_{\mathbb{C}}^{\text{sm}}(G)$ is a full subcategory of $\operatorname{Rep}_{\mathbb{C}}(G)$.

Remark 2.19. One can check that the direct sum, subrepresentation, and quotients of smooth representations all continue to be smooth. It follows that $\operatorname{Rep}_{\mathbb{C}}^{\text{sm}}(G)$ is an abelian category, and it admits limits and colimits.

There is also a dual, but we must be careful because functionals have no reason to be smooth.

Definition 2.20 (contragredient)**.** Fix a smooth representation V of a totally disconnected group G. Then the *contragredient* $V^\vee$ consists of the smooth vectors in V^*.

Remark 2.21. Explicitly, V^* consists of those functionals $\varphi\colon V \to \mathbb{C}$ for which there is an open subgroup $K \subseteq G$ for which

$$\varphi(kv) = \varphi(v)$$

for all $v \in V$ and $k \in K$.

Remark 2.22. By construction, we see that $(V^\vee)^K = \operatorname{Mor}(V_K, \mathbb{C})$, where V_K denotes the "coinvariants."

The category of smooth representations is quite flexible, but smooth representations will occasionally be "too big" for our purposes. We don't want to require that our representations are fully finite-dimensional, but it will be nice to have some finiteness.

Definition 2.23 (admissible)**.** Fix a smooth representation V of a totally disconnected group G. Then V is *admissible* if and only if

$$\dim V^K < \infty$$

for any compact open subgroup $K \subseteq G$.

Remark 2.24. Note that V being smooth means that $V = \bigcup_{K \subseteq G} V^K$, so admissibility means that V is a union of finite-dimensional vector spaces.

Non-Example 2.25. If G fails to be compact, then the right regular representation $\operatorname{Mor}(G, \mathbb{C})^{\mathrm{sm}}$ is not admissible. Indeed, for any open compact subgroup $K \subseteq G$, we see that $\operatorname{Mor}(G, \mathbb{C})^K = \operatorname{Mor}(K\backslash G, \mathbb{C})$, which is infinite-dimensional because $K\backslash G$ is not compact.

Example 2.26. Even if G fails to be compact, we may be able to find a closed subgroup $H \subseteq G$ such that $H\backslash G$ is compact. Then for any compact open subgroup $K \subseteq G$, we see that $H\backslash G/K$ is finite (because K is open), so

$$\dim \operatorname{Mor}(H\backslash G, \mathbb{C})^K < \infty.$$

It follows that the induction $\operatorname{Mor}(H\backslash G, \mathbb{C})^{\mathrm{sm}}$ is admissible.

Remark 2.27. One can check that the natural map $V \to (V^\vee)^\vee$ is an isomorphism if V is admissible. This is shown on the homework.

Remark 2.28. For a fixed compact open subgroup $K \subseteq G$, we will show on the homework that there is a canonical isomorphism

$$\bigoplus_{\rho \in \operatorname{Irr} K} V_\rho \otimes \operatorname{Hom}_K(V_\rho, V) \to V$$

for any smooth V. It turns out that V is admissible if and only if the multiplicity spaces $\operatorname{Hom}_K(V_\rho, V)$ are finite-dimensional.

2.1.5 Hecke Algebra

Our next task is to realize our category of representations as modules over a convenient algebra.

Example 2.29. For any group G, we know that $\operatorname{Rep}_{\mathbb{C}}(G) \cong \operatorname{Mod}(\mathbb{C}[G])$: indeed, a G-action on a vector space extends uniquely (linearly) to an action by $\mathbb{C}[G]$.

Remark 2.30. It turns out to be more convenient to view $\mathbb{C}[G]$ (dually) as the functions $G \to \mathbb{C}$ with finite support. In this case, the multiplication is given by convolution: one has

$$(f_1 * f_2)(x) \coloneqq \sum_{y \in G} f_1(y) f_2\left(y^{-1}x\right).$$

Indeed, we can see that $1_{g_1} * 1_{g_2} = 1_{g_1 g_2}$, so this is the multiplication that we expect.

Motivated by the above remark, we find the following.

Definition 2.31 (Hecke algebra)**.** Fix a totally disconnected group G. Then we define the *Hecke algebra* $\mathcal{H}_G$ to be $C_c^\infty(G)$. If no confusion is possible, we will write $\mathcal{H}$ for $\mathcal{H}_G$. Upon fixing a left Haar measure dg, we may define a convolution operation by

$$(f_1 * f_2)(x) \coloneqq \int_G f_1(g) f_2\left(g^{-1}x\right)\, dg.$$

The convolution operation is distributive. To see that it is associative, note that

$$((f_1 * f_2) * f_3)(x) = \int_G \int_G f_1(h) f_2\left(h^{-1}g\right) f_3\left(g^{-1}x\right)\, dh\, dg,$$

but

$$(f_1 * (f_2 * f_3))(x) = \int_G \int_G f_1(g) f_2(h) f_3\left(h^{-1}g^{-1}x\right)\, dh\, dg.$$

These integrals are seen to be the same by sending $h \mapsto g^{-1}h$.

Remark 2.32. There was some concern about if dg should be left- or right-invariant. If $G(F)$ is the F-points of a reductive algebraic group, then it turns out that $G(F)$ is unimodular (which one sees by an explicit construction of this measure by choosing a trivialization of $\wedge^{\dim G} TG$), so this doesn't matter!

However, the algebra is not unital.

Remark 2.33. If G is discrete, then we can normalize dg to be the counting measure. Then $\mathcal{H}$ admits a unit δ_1 given by the indicator at the identity: one can directly compute $f * \delta_1 = f$. As such, we do not expect $\mathcal{H}$ to be unital.

Remark 2.34. The algebra $\mathcal{H}$ admits many idempotents: for any compact open subgroup $K \subseteq G$, note that $1_K * 1_K = \operatorname{vol}(K; dg) 1_K$, so $e_K \coloneqq \operatorname{vol}(K; dg)^{-1} 1_K$ is an idempotent element! In fact, these elements form an approximate identity: for any function f, one can find K small enough so that $e_K * f = f$.

It is somewhat inconvenient that the convolution structure depends on the choice of dg. Let's fix this.

Remark 2.35. It is actually more convenient to view $\mathcal{H}$ as locally constant, compactly supported measures on G, which have an easier time acting on representations. In other words, we are looking at measures of the form $f(g)\, dg$, where $f \in C_c^\infty(G)$. The reason this is convenient is that the multiplication now becomes canonical: given two measures μ_1 and μ_2, note that the product measure $\mu_1 \boxtimes \mu_2$, which continues to have compact support and be locally constant. Thus, there is a pushforward measure $m_*(\mu_1 \boxtimes \mu_2)$, which is again locally constant of compact support. One can unwind the definitions to see that $m_*(f_1\, dg \boxtimes f_2\, dg) = (f_1 * f_2)\, dg$, so this operation agrees with the previous one.

Example 2.36. For a compact open subgroup $K \subseteq G$, the elements $e_K := \operatorname{vol}(K; dg)^{-1} 1_K \, dg$ are idempotent: indeed, one can check that $e_K * e_K = e_K$ directly. In fact, if $K' \supseteq K$, then $e_K * e_{K'} = e_{K'} * e_K = e_{K'}$. For example,

$$(e_K * e_{K'})(x) = \frac{1}{\operatorname{vol}(K; dg) \operatorname{vol}(K'; dg)} \int_G e_K(g) e_{K'} \left(g^{-1}x\right) \, dg$$

restricts immediately to those $g \in G$ with $g \in K$ and $g^{-1}x \in K'$. In other words, we are integrating over $K \cap xK'$, which is simply an indicator for K times the measure of K'.

Warning 2.37. In the sequel, we may work with the Hecke algebra $\mathcal{H}$ as $C_c^\infty(G) \, dg$ instead of $C_c^\infty(G)$.

2.2 February 25

Today we continue our nonarchimedean representation theory.

2.2.1 Representations of the Hecke Algebra

Let's take a moment to explain how $\mathcal{H}$ acts on representations.

Lemma 2.38. Fix a totally disconnected group G. There is a fully faithful embedding $\operatorname{Rep}^{\mathrm{sm}}_{\mathbb{C}}(G) \to \operatorname{Mod}(\mathcal{H})$.

Proof. For (a), the action is given by

$$\mu \cdot v := \int_G gv \, d\mu(g).$$

Note that the integral always converges because μ has compact support. By an explicit calculation with $f\,dg$s, one can check that this action makes V into an $\mathcal{H}$-module. This functor is of course faithful (because linear maps are just sent to themselves).

It remains to check that our functor is full. Fix a linear map φ between two smooth representations V and W, and suppose that φ preserves the $\mathcal{H}$-module structure. We would like to show that $g\varphi(v) = \varphi(gv)$ for any $g \in G$ and $v \in V$. Well, for any sufficiently small compact open subgroup K of g, we see that $e_K v = v$, so the given equation is equivalent to φ preserving the action by the translated measure ge_K. ■

Remark 2.39. Further, by shrinking K, we may assume that $v \in V^K$ and that $\mu = f \, dg$ where f is K-invariant. It follows that

$$\mu \cdot v = \sum_{g \in G/K} f(g) \operatorname{vol}(K; dg)(gv)$$

by factoring the integral.

We use this to give a definition of the "smooth" $\mathcal{H}$-modules.

Definition 2.40 (smooth)**.** Fix a totally disconnected group G. Then an $\mathcal{H}$-module M is *smooth* if and only if

$$M = \bigcup_{K \subseteq G} e_K M.$$

Remark 2.41. If V is a smooth representation of G, then one can check that e_K projects V onto V^K. Indeed, e_K certainly fixes V^K, and for any $k \in K$ and $v \in V$, we see that $k(e_K v) = e_K v$ by pushing the linear map k through the integral

$$e_K v = \frac{1}{\operatorname{vol}(K; dg)} \int_K gv \, dg.$$

Thus, smooth representations of G go to smooth $\mathcal{H}$-modules.

Remark 2.42. Conversely, given a smooth $\mathcal{H}$-module M, we can write

$$M = \operatorname*{colim}_{K \subseteq G} e_K M.$$

Then each $e_K M$ upgrades from an $\mathcal{H}$-module to a representation of G/K and thus by inflation to a representation of G. Thus, the colimit upgrades to a smooth representation of G because the category of smooth representations is closed under such colimits. This construction provides an inverse functor to the one in Lemma 2.38, thereby providing an equivalence of categories from smooth representations to smooth modules.

It is notable that all of our discussion can also be made to work globally for function fields.

Example 2.43. If G is an affine algebraic group over a function field F, then $\mathbb{A}_F$ is totally disconnected, so $G(\mathbb{A}_F)$ is a totally disconnected. For example, it turns out that

$$\operatorname{GL}_n(\mathbb{A}_F) = \prod_v (\operatorname{GL}_n(F_v), \operatorname{GL}_n(\mathcal{O}_v)),$$

which one can see directly from the construction of the topology on $G(\mathbb{A}_F) \subseteq \mathbb{A}_F^{n^2+1}$. The basic open subsets of $\operatorname{GL}_n(\mathbb{A}_F)$ can thus be indexed by effective divisors D on X: indeed, given such a $D = \sum_v n_v v$, we can define K_D as the kernel of the natural projection

$$\prod_v \operatorname{GL}_n(\mathcal{O}_v) \twoheadrightarrow \prod_{n_v \neq 0} \operatorname{GL}_n(\mathcal{O}_v/\varpi_v^{n_v}).$$

In particular, because this map is surjective, we see that K_D has finite index in $\prod_v \operatorname{GL}_n(\mathcal{O}_v)$, where the index is given by the size of the right-hand group.

Remark 2.44. Fix an affine algebraic group G over a global field F. Then one may want to write down a decomposition $G(\mathbb{A}_F)$ as a restricted product, but this requires us to make sense of $G(\mathcal{O}_v)$ for all but finitely many places v. Luckily, this is not too hard: one simply can choose any model $\mathcal{G}$ over $\operatorname{Spec} \mathcal{O}_F$, and then we pass to the open subset of $\operatorname{Spec} \mathcal{O}_F$ where G remains an affine algebraic group. Because any two models $\mathcal{G}$ and $\mathcal{G}'$ are generically isomorphic to G (namely, over F), one can spread out the composite isomorphism $\mathcal{G}_F \to G \leftarrow \mathcal{G}'_F$ to show that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic at all but finitely many places. Thus, the restricted product $\prod_v (G(F_v), \mathcal{G}(\mathcal{O}_v))$ does not depend on the choice of model $\mathcal{G}$.

Thus, for an affine algebraic group G over a function field F, one has a Hecke algebra $\mathcal{H}_{G(\mathbb{A}_F)}$. We will later want a Hecke algebra also for number fields, which requires us to find a Hecke algebra for archimedean groups. We will return to this later.

Let's do some more representation theory.

Proposition 2.45 (Schur's lemma)**.** Fix admissible irreducible representations V_1 and V_2 of a totally disconnected group G. Then
$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V_1, V_2) = 1_{V_1 \cong V_2}.$$

Proof. Note that any nonzero map $V_1 \to V_2$ has neither full kernel nor full cokernel, so the irreducibility implies that any nonzero map is an isomorphism. Thus, if V_1 and V_2 are not isomorphic, $\dim \operatorname{Hom}_G(V_1, V_2) = 0$.

It remains to work in the case $V_1 \cong V_2$. We will write V for both representations, for brevity, so we want to show that $\operatorname{End}_G(V) = \mathbb{C}$. The previous paragraph has shown that any nonzero element of $D := \operatorname{End}_G(V)$ is invertible, so this is a division algebra. Furthermore, D contains the scalars in $\mathbb{C}$, so D is a division algebra over $\mathbb{C}$.

The rest of the argument will be by counting, using the bound that V has countable dimension (because it is a countable union of finite subspaces).

- On one hand, suppose for the sake of contradiction that $D \setminus \mathbb{C}$ is nonempty. Because $\mathbb{C}$ is algebraically closed, this means that D has a transcendental element T over $\mathbb{C}$, so D contains $\mathbb{C}(T)$. But the elements $\{1/(T-a) : a \in \mathbb{C}\}$ are linearly independent, so we conclude that $\dim_{\mathbb{C}} D$ is uncountable.
- On the other hand, $\dim_{\mathbb{C}} D$ is countable because $\dim V$ is countable: indeed, any nonzero vector $v \in V$ generates V under the G-action, so a G-equivariant map $V \to V$ is uniquely determined by the image of v. Thus, the map $\operatorname{ev}_v \colon D \to V$ is injective, so $\dim_{\mathbb{C}} D$ is countable.

The above two points produce contradiction if $D \neq \mathbb{C}$, so we conclude that $D = \mathbb{C}$. ■

Remark 2.46. Suppose that G is a reductive group over a nonarchimedean local field F. It turns out that all irreducible smooth representations $G(F)$ are automatically admissible. This is difficult to prove, so we will not.

Remark 2.47. Again, suppose that G is a reductive group over a nonarchimedean local field F. Then Proposition 2.45 remains true for representations over $\overline{\mathbb{Q}}$, but a different argument is required. In short, one can still make a size argument by using the fact that the relevant universal enveloping algebra of the Lie algebra is filtered and has finitely generated associated graded algebra.

2.2.2 Parabolic Induction

We now fix a reductive group G over a nonarchimedean local field F.

Definition 2.48 (parabolic)**.** Fix a reductive group G over a field F. Then a subgroup $P \subseteq G$ is *parabolic* if and only if G/P is proper.

Remark 2.49. It is equivalent for $P_{\overline{F}}$ to contain a Borel subgroup $B_{\overline{F}}$, which is a maximal connected solvable subgroup of G. For example, $G_{\overline{F}}/B_{\overline{F}}$ can be seen to be proper because this is some sort of flag variety. Indeed, $G_{\overline{F}}/B_{\overline{F}}$ is the space of Borel subgroups, which are all conjugate to each other.

Remark 2.50 (Levi decomposition)**.** Given a parabolic subgroup P with unipotent radical U, the quotient $L := P/U$ is reductive. It turns out that the quotient map $P \twoheadrightarrow L$ admits a (non-canonical) splitting, so we may view L as a subgroup of P.

Example 2.51. A Borel subgroup B of GL_n is given by the upper-triangular matrices, and this is the only one up to conjugation. Thus, one can see that parabolic subgroups of GL_n look like

$$(\mathrm{GL}_{n_1} \times \cdots \mathrm{GL}_{n_d})\, B,$$

where $(n_1, \ldots, n_d)$ is some sequence of positive integers with sum n.

Let's attempt to define parabolic induction. Choose a smooth representation V of the Levi quotient L. Then we may inflate V into a representation of P via the quotient map $P \twoheadrightarrow L$. We could attempt to define the induction to G as consisting of those functions $f\colon G \to V$ such that

$$f(px) = pf(x),$$

where the G-action is given by $(gf)(x) := f(xg)$. This is certainly a well-defined representation, but it has a couple defects, which we now fix.

- This representation has no reason to be smooth. To fix this, we will want to work with those functions $f\colon G \to V$ admitting an open compact subgroup $K \subseteq G$ for which $kf = f$ for all $k \in K$.[1]
- It is useful to normalize by the modular character.

We thus have the following definition.

Definition 2.52 (parabolic induction)**.** Fix a reductive group G over a nonarchimedean field F, and let P be a parabolic subgroup with Levi quotient L. Given a smooth representation V of $L(F)$, we define the *parabolic induction* $\operatorname{Ind}_P^G V$ to consist of those functions $f\colon G(F) \to V$ with the following two properties.

- Smooth: there is an open compact subgroup $K \subseteq G(F)$ such that $f(xk) = f(x)$ for all $x \in G(F)$ and $k \in K$.
- Induction: for all $p \in P(F)$ and $x \in G(F)$, we have $f(px) = \delta_P(p)^{1/2}\tau(p)f(x)$, where δ_P is the modular character.

The G-action on $\operatorname{Ind}_P^G V$ is given by $(gf)(x) := f(xg)$.

Let's explain the presence of the modular character δ_P, and thereby also recall its definition. (The following discussion works for all local fields F.) Note that $P(F)\backslash G(F)$ is a (nonarchimedean) manifold, so it admits an integration theory. Indeed, any manifold X of dimension n admits an integration theory as follows: if a top differential form $\theta \in \omega_X$ is supported on a single chart U with local coordinates $\theta = f\,dx_1 \wedge \cdots \wedge dx_n$, then we can define

$$\int_X |\theta| := \int_U |f|\ dx_1 \ldots dx_n.$$

This expression is independent of the choice of chart, which one can check by some Jacobian calculation. From here, one can use a partition of unity to compute $\int_X |\theta|$ even if θ is not supported on a single chart. (The expression will further be independent of a choice of partition of unity, which we can see by further refining the partition of unity given two such choices.)

We now apply our integration theory to $X := P(F)\backslash G(F)$. The action of P on G is free, so we see that $T_e(P(F)\backslash G(F)) = \mathfrak{p}\backslash\mathfrak{g}$. It follows that

$$TX = (\mathfrak{p}\backslash\mathfrak{g}) \times^P G.$$

In particular, TX admits a global frame (by choosing a basis of $\mathfrak{p}\backslash\mathfrak{g}$), so the line bundle of top differential forms is trivial, which we see by writing

$$\omega_X = \wedge^{\dim X}(\mathfrak{p}\backslash\mathfrak{g})^* \times^P G.$$

[1] This really is extracting those locally constant maps on $P(F)\backslash G(F)$ because $P(F)\backslash G(F)$ is a compact space.

But now P acts on $\wedge^{\text{top}}(\mathfrak{p}\backslash\mathfrak{g})^*$ by the scalar

$$\det(p; (\mathfrak{p}\backslash\mathfrak{g})^*) = \frac{\det(p; \mathfrak{p})}{\det(p; \mathfrak{g})}.$$

The absolute value of this is denoted $\delta_P(p)$.

The relevance of δ_P to our integration theory is as follows: our discussion above explained how to integrate expressions of the form $|\theta|$ where $\theta \in \omega_X$. But we can merely view $|\omega_X|$ as some real line bundle. By choosing some top differential form, we can then identify sections of $|\omega_X|$ with functions $f\colon G(F) \to \mathbb{R}$ such that

$$f(px) = \delta(p)f(x)$$

for all $p \in P$ and $x \in G$. Thus, $\operatorname{Ind}_P^G V$ consists of sections of the line bundle $|\omega_X|^{1/2}$, which are known as "half-densities." The relevance of this for us is that two half-densities $f, g \in \operatorname{Ind}_P^G V$ will have a product which can be integrated. Eventually, this will imply that the parabolic induction of a unitary representation remains unitary.

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List of Definitions

adéles, 6
admissible, 31

cocompact, 16
contragredient, 31

Dedeking ζ-function, 20

Fourier transform, 13

general linear group, 7
global field, 4
global integral, 22

Hecke algebra, 32
Hecke character, 8

idéle class character, 20

local integral, 23, 26

parabolic, 35
parabolic induction, 36
place, 5
Pontryagin dual, 11

residue, 17

Schwartz, 14, 15
smooth, 29, 30, 33

totally disconnected, 29

unramified, 8, 21