

215A: Algebraic Topology

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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THEME 1

INTRODUCTION

1.1 August 24

It begins.

1.1.1 Logistics

Here are the logistical notes.

- The professor is Ian Agol, whose office is Evans 921. Office hours are Tuesdays after class, Monday at 3PM, Wednesday at 9AM, or by appointment.
- There is a [bCourses](#).
- Homework will be weekly, and it will make up the entire grade.
- The prerequisites are Math 113 and 202A or equivalent. From point-set topology in particular we will want notions of compactness, connectedness, metric spaces, and a few topologies like the identification topology with respect to a continuous map.

1.1.2 Overview

We will cover chapters 0–3 of [Hat01].

- Chapter 0 consists of “geometric notions.” Particularly important are the notion of homotopy and CW complexes.
- Chapter 1 is on fundamental groups.
- Chapter 2 is on homology. This is an abelian extension of fundamental groups.
- Chapter 3 is on cohomology. Poincaré duality relates cohomology with homology.

Chapter 4 is typically covered in Math 215B, on homotopy theory.

Let’s talk a bit about the interests of the course. Topology as a whole is interested in “spaces up to deformation.” In this class, deformation will mean homotopy mostly, but there are finer notions of interest like homeomorphism. As for the spaces, we will focus on spaces which are locally homogeneous in some sense, like manifolds (which are locally homeomorphic to \mathbb{R}^n). These notions come up naturally throughout mathematics; for example, integrals of holomorphic functions are roughly independent of path chosen. Poincaré himself was interested in differential equations, whose configuration spaces could be manifolds.

In this class, we will attach invariants to our topological spaces to be able to understand how to differentiate between our spaces (up to deformation). We focus on the following invariants.

- Fundamental groups and covering spaces. This has a close tie to Galois theory, an analogy made process by the étale fundamental group in algebraic geometry.
- Cohomology. The origins are from complex analysis and Stokes's theorem, but cohomology itself has vast generalizations and manifestations throughout mathematics, leading to the field of homological algebra. However, there are applications to algebraic geometry, number theory, and so on. The most notable application here is the proof of the Weil conjectures.
- Higher homotopy groups. Our approach will not begin with this viewpoint, but it is possible.

1.1.3 Homotopy and Homotopy Type

Let's jump in chapter 0.

Notation 1.1. We set $I := [0, 1]$ for convenience.

Definition 1.2 (deformation retract). Fix a subspace A of a topological space X . Then a *deformation retract* is a family of functions $f_\bullet: X \times I \rightarrow X$ where $f_0 = \text{id}_X$ and $\text{im } f_1 = A$ and $f_t|_A = \text{id}_A$ for all $t \in I$.

Example 1.3 (mapping cylinder). Fix a continuous function $f: X \rightarrow Y$. Then the *mapping cylinder* M_f is the space $(X \times I) \sqcup Y$ quotiented by $(x, 1) \sim f(x)$. Then M_f has a deformation retraction to Y by $f_t(x) := (x, t)$. Visually, we have attached Y to a thickening of X .

Example 1.4. Define $f: S^1 \rightarrow S^1$ by $f(z) := z^2$. Then M_f has S^1 on one domain side and S^1 covered twice on the target side. With a little deformation, this is a Möbius strip. Approximately speaking, one should cut the cylinder in half and then rearrange. One can see that the Möbius strip deformation retracts to S^1 by squishing the width of the cylinder to the central line.

A deformation retract is a special case of a homotopy. Here is the definition of a homotopy.

Definition 1.5 (homotopy). Two continuous maps $f_0, f_1: X \rightarrow Y$ are *homotopic* if and only if there is a continuous function $F_\bullet: X \times I \rightarrow Y$ such that $F_0 = f_0$ and $F_1 = f_1$. Here, F is called a *homotopy*, and we write $f_0 \sim f_1$.

Example 1.6. A subspace $A \subseteq X$ has a deformation retract if and only if id_X is homotopic to some $r: X \rightarrow X$ with $\text{im } r = A$ and $r|_A = \text{id}_A$. Indeed, the deformation retract is exactly the needed homotopy.

Example 1.7. Suppose $f, g: X \rightarrow Y$ are equal maps. Then define $h: X \times I \rightarrow Y$ by $h_t = f = g$ for all t . We see that h is continuous ($h^{-1}(V) = f^{-1}(U) \times I$ for any open $V \subseteq Y$), so it provides a homotopy from f and g .

It should not be surprising that homotopy is an equivalence relation.

Lemma 1.8. Fix topological spaces X and Y . Then \sim is an equivalence relation on continuous functions $X \rightarrow Y$.

Proof. We have the following checks.

- Reflexive: this is direct from Example 1.7.

- Symmetric: if $f \sim g$, then we have $F_\bullet: X \times I \rightarrow Y$ with $F_0 = f$ and $F_1 = g$. We now define $G_\bullet: X \times I \rightarrow Y$ by $G_t := F_{1-t}$. Then G is continuous by the continuity of $t \mapsto 1-t$ and F , and $G_0 = g$ and $G_1 = f$, so G witnesses $g \sim f$.
- Transitive: if $f \sim g$ and $g \sim h$, find $F_\bullet: X \times I \rightarrow Y$ and $G_\bullet: X \times I \rightarrow Y$ with $F_0 = f$ and $F_1 = g$ and $G_0 = g$ and $G_1 = h$. Then we define $H_\bullet: X \times I \rightarrow Y$ by

$$H_t := \begin{cases} F_{2t} & \text{if } 0 \leq t \leq 1/2, \\ G_{2t-1} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Note that this is well-defined at $t = 1/2$ because $F_1 = g = G_0$. Note H will witness $f \sim h$ once we show that it is continuous, which is what we do now.

By looking locally at F or G , we see that H is continuous at any point not of the form $(x, 1/2)$. Then for any point of the form $(x, 1/2)$ and open subset $V \subseteq Y$ containing $H_{1/2}(x)$, continuity of F gives an open subset $U_F \times (1/2 - \varepsilon, 1/2]$ mapping to V , and continuity of G gives an open subset $U_G \times [1/2, 1/2 + \varepsilon)$ mapping to V , so $(U_F \cap U_G) \times (1/2 - \varepsilon, 1/2 + \varepsilon)$ will suffice. ■

Homotopy also behaves well with composition.

Lemma 1.9. Fix topological spaces X, Y, Z , and let $f_0, f_1: X \rightarrow Y$ and $g_0, g_1: Y \rightarrow Z$ be homotopic maps. Then $(g_0 \circ f_0) \sim (g_1 \circ f_1)$.

Proof. Fix a homotopy $F_\bullet: X \times I \rightarrow Y$ with $F_0 = f_0$ and $F_1 = f_1$ and a homotopy $G_\bullet: Y \times I \rightarrow Z$ with $G_0 = g_0$ and $G_1 = g_1$. Then we define $H_\bullet: X \times I \rightarrow Z$ by

$$H_t(x) := G_t(F_t(x)).$$

Then $H_0 = g_0 \circ f_0$ and $H_1 = g_1 \circ f_1$, so we will be done if we can show H is continuous. Well, H_\bullet is the composite map

$$X \times I \xrightarrow{(F, \text{id}_I)} Y \times I \xrightarrow{G} Z,$$

which we can see is the composite of continuous maps. ■

Homotopy allows us to define homotopy equivalence.

Definition 1.10 (homotopy equivalence). A continuous map $f: X \rightarrow Y$ is a *homotopy equivalence* if and only if there is a continuous map $g: Y \rightarrow X$ such that $(g \circ f) \sim \text{id}_X$ and $(f \circ g) \sim \text{id}_Y$. We then say that X and Y have the same *homotopy type* and write $X \simeq Y$.

Remark 1.11. It is not enough to merely require $(g \circ f) \sim \text{id}_X$. For example, let $X := \{x\}$ be a point. Then any map $f: \{x\} \rightarrow Y$ can use the unique map $g: Y \rightarrow \{x\}$ so that $(g \circ f) = \text{id}_X$.

Here is a quick sanity check.

Lemma 1.12. Ignoring size issues, homotopy equivalence provides an equivalence relation on topological spaces.

Proof. We have the following checks. Fix topological spaces X, Y, Z .

- Reflexive: we show $X \simeq X$. Indeed, use the maps $\text{id}_X, \text{id}_X: X \rightarrow X$ so that $\text{id}_X \circ \text{id}_X = \text{id}_X$ is homotopic to id_X by Example 1.7.
- Symmetric: we show $X \simeq Y$ implies $Y \simeq X$. Indeed, let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be the promised maps so that $(f \circ g) \sim \text{id}_Y$ and $(g \circ f) \sim \text{id}_X$. Reading these data in reverse tell us that $Y \simeq X$.

- Transitive: suppose $X \simeq Y$ and $Y \simeq Z$, and we show $X \simeq Z$. Thus, we have maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ and $f': Y \rightarrow Z$ and $g': Z \rightarrow Y$ such that $(f \circ g) \sim \text{id}_Y$ and $(g \circ f) \sim \text{id}_X$ and $(f' \circ g') \sim \text{id}_Z$ and $(g' \circ f') \sim \text{id}_Y$. We now claim that $(f' \circ f): X \rightarrow Z$ and $(g \circ g'): Z \rightarrow X$ are the desired maps to witness $X \simeq Z$. Well, using Lemma 1.9, we compute

$$(f' \circ f) \circ (g \circ g') = f' \circ (f \circ g) \circ g' \sim f' \circ \text{id}_Y \circ g' = f' \circ g' \sim \text{id}_Z,$$

and similar for the other direction. ■

Remark 1.13. One can check directly that \sim is an equivalence relation on spaces. The main check here is that one can compose homotopies.

We will often find that our algebraic invariants are only able to detect homotopy equivalence, which is why homotopy equivalence will be so important to us.

Example 1.14. Example 1.4 shows that the Möbius strip is homotopic to S^1 .

More generally, one can show that a deformation retract is a homotopy equivalence.

Lemma 1.15. Fix a subspace A of a topological space X . Then a deformation retract witnesses a homotopy between the inclusion $i: A \hookrightarrow X$ and the identity $\text{id}_X: X \rightarrow X$. In particular, it follows that i is a homotopy equivalence.

Proof. This is a matter of unraveling the definitions. Fix a deformation retract $f_\bullet: X \times I \rightarrow X$, and let $r := f_1$ so that $\text{im } r = A$. We now claim that i and r are inverse homotopy equivalences.

- We show that $(r \circ i) \sim \text{id}_A$. Indeed, $r(i(a)) = a$ for any $a \in A$ by hypothesis on r , so in fact $r \circ i = \text{id}_A$.
- We show that $(i \circ r) \sim \text{id}_X$. The relevant homotopy is just f_\bullet : we have $f_0 = \text{id}_X$ and $f_1 = (i \circ r)$, so $\text{id}_X \sim (i \circ r)$ by Lemma 1.8. ■

Example 1.16 (dunce cap). Take the disc D^2 and glue the edges together as follows: mark three points A, B , and C , and glue AB to AC to CB (in those orientations). Then the resulting space is homotopic to a point.

We have a special name for being homotopic to a point.

Definition 1.17 (contractible). A topological space X is *contractible* if and only if it is homotopic to a point.

These notions allow us to define a homotopy category, whose objects are homotopy classes of topological spaces and morphisms are continuous maps. In some sense, our algebraic invariants are trying to distinguish between objects in this category. It turns out that this category is not concrete, meaning that there is no way to realize its objects as sets reasonably. Approximately speaking, this means that there can be no canonical representing topological space for each homotopy class, but topologists try anyway.

Remark 1.18. There are a number of results called “topological rigidity” theorems which give homeomorphism $X \cong Y$ given merely $X \simeq Y$ and some extra hypotheses. For example, this holds for closed surfaces by a classification result.

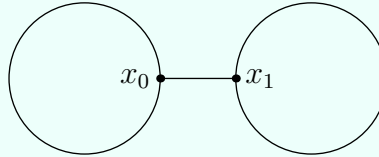
Example 1.19. Attach two S^1 s by a line to make a space X , and attach them along an edge to make a space Y . These spaces are homotopic, but they are not homeomorphic (removing a point from X may disconnect it, but this is not the case for Y).

1.1.4 CW Complexes

Here is our definition.

Definition 1.20 (CW complex). Let X^0 be a discrete set of points, and define X^n inductively by $X^{n+1} := X^n \cup \{e_\alpha^{n+1}\}$, where $\varphi_\alpha: \partial e_\alpha^{n+1} \rightarrow X^n$ is a homeomorphism telling us how to union. Here, e_α^{n+1} is a copy of the n -ball B^n , so the φ_α are explaining how to identify the edges.

Example 1.21. Here is a CW complex.



Namely, $X^0 = \{x_0, x_1\}$, and X^1 is the edges.

Example 1.22. Take a point $\{*\}$ for X^0 , and define φ_n to be some loop based on $\{*\}$. Then the resulting space is some infinite union of circles intersecting at $\{*\}$. Notably, this space is not compact and in fact should not even be embedded into the plane or \mathbb{R}^3 because such an embedding is unlikely to be a homeomorphism.

Example 1.23. The sphere $S^n := D^n / \partial D^n$ is a CW structure with only two cells: it is $e^0 \cup e^n$. Notably, the CW structure here has $X^0 = X^1 = \dots = X^{n-1}$.

Example 1.24. Alternatively, one can define S^n inductively as follows: take S^0 to be two points, and define S^{n+1} to be S^n as an equator unioned with two $(n+1)$ -cells making hemispheres attached to the equator. One can then define S^∞ to be the union of all the S^n where we identify $S^n \hookrightarrow S^{n+1}$ via the equator. This is a CW complex of infinite dimension. It turns out that S^∞ is contractible, though S^n is not for any finite n .

Example 1.25. Define real projective space \mathbb{RP}^n as the set of vectors $x \in \mathbb{R}^{n+1} \setminus \{0\}$ where we identify x with λx for any $\lambda \in \mathbb{R}^\times$. Notably, by setting the last coordinate equal to 0, we expect to get \mathbb{RP}^{n-1} . But if the last coordinate is equal to zero, we can scale it uniquely to 1, and then the remaining coordinates may vary arbitrarily. In total, we find

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup \mathbb{R}^{n-1}.$$

Thus, we get the cell structure $\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$.

Remark 1.26. The CW structure is not unique. For example, one can separate out edges by putting a point in the middle of them.

One can show that the CW complex is compact if and only if it has finitely many cells.

1.2 August 29

Last time we discussed homotopies, homotopy equivalence, and CW complexes. To review, the goal of algebraic topology is to define (algebraic) invariants of topological spaces and then perhaps figure out when

two spaces are equivalent (for suitable definition of equivalent). In theory, our invariants would be able to entirely classify some subset of spaces we are looking at, but it is rather rare. To execute this plan, we need a source of spaces (mostly CW complexes and ways to combine them) and then methods to tell if spaces are equivalent.

1.2.1 Operations on Spaces

Let's discuss how to make new spaces from old ones. Thankfully, our operations will send CW complexes to CW complexes, though there is something to check.

Definition 1.27 (product). Fix CW complexes X and Y . Then we form the *product* $X \times Y$ (at the level of CW complexes) using as $(n + m)$ -cells $e_\alpha^m \times f_\beta^n$ where e_α^m is an m -cell of X and f_β^n is an n -cell of Y . Notably, the n -skeleton is

$$(X \times Y)^n = \bigcup_{k+\ell=n} X^k \times Y^\ell,$$

and one can attach in the obvious way. This produces a CW structure.

Remark 1.28. It is possible that $X \times Y$ with its CW structure need not be the same as the product topology. There is an example in the appendix of [Hat01], but we won't care so much for this course.

Definition 1.29 (subcomplex). Fix a CW complex X . Then a *subcomplex* is a closed subspace $A \subseteq X$ which is a union of cells of X and also a CW complex.

Definition 1.30 (quotient). Fix a subcomplex A of a CW complex X . Then X/A is also a CW complex. Here, the definition of X/A is somewhat technical: its cells are the cells of $X \setminus A$ and then a 1-cell from A , and one attaches in the obvious way (inductively) via the quotient map $X^{n-1} \rightarrow X^{n-1}/A^{n-1}$.

Definition 1.31 (suspension). Fix a CW complex X . Then the *suspension* is the quotient

$$SX := \frac{X \times I}{\{(0, x) \sim (0, x') \text{ and } (1, x) \sim (1, x')\}}.$$

Example 1.32. Take $X = S^0$, which is two points. Then $X \times I$ is two lines, and we then identify the endpoints of the two lines accordingly to produce a circle S^1 . More generally, $SS^n = S^{n+1}$ essentially by just gluing two S^n s onto the equator of S^{n+1} .

Definition 1.33 (join). Fix CW complexes X and Y . Then the *join* $X * Y$ is the product $X \times Y \times I$ (as CW complexes) modded out by the equivalence relation identifying $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$.

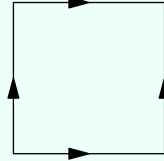
Example 1.34 (simplex). Consider $X = Y = I = \Delta^1$. Then $X * Y$ is the cube modded out by crushing Y on one end and crushing X on the other end, forming a tetrahedron, which is Δ^3 . More generally, $\Delta^n * \Delta^m = \Delta^{n+m+1}$.

Example 1.35. One has $S^0 * S^0 = S^1$, and more generally $S^n * X = SX$. Essentially, we are gluing two copies of X onto an equator, which is the suspension.

Definition 1.36 (wedge product). Fix CW complexes X and Y and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then we form the wedge product $X \vee Y$ as $X \sqcup Y$ identifying $x_0 \sim y_0$.

Definition 1.37 (smash product). Fix CW complexes X and Y and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then the smash product is $(X \times Y)/(X \vee Y)$, where $X \vee Y$ is embedded into $X \times Y$ as $x \mapsto (x, y_0)$ and $y \mapsto (y, x_0)$.

Example 1.38. One can check that $S^1 \times S^1$ is a torus. To form the smash product, we are crushing the boundary of the square as follows.



More generally, $S^m \wedge S^n = S^{m+n}$.

Definition 1.39 (attach). Fix a subcomplex A of a CW complex X_1 and a map $f: A \rightarrow X_0$ to another CW complex X_0 . Then $X_0 \sqcup_f X_1$ is the space $X_0 \sqcup X_1$ modded out by the equivalence relation $a \sim f(a)$ for all $a \in A$.

Example 1.40. An attaching map $\varphi_\alpha: \partial D^n \rightarrow X^{n-1}$ of a CW complex are attachments $X^{n-1} \sqcup_{\varphi_\alpha} D^n$ in the above sense.

1.2.2 Homotopy Extension

We are going to, over time, prove the following results. To begin, quotients preserve homotopy type.

Proposition 1.41. Fix a subcomplex A of a CW complex X . If A is contractible, then the quotient map $X \rightarrow X/A$ is a homotopy equivalence.

Example 1.42. Fix a connected graph X , which is a one-dimensional CW complex. Fix a spanning tree $T \subseteq X$, which is contractible (any tree can be contracted one edge at a time), so $X \rightarrow X/T$ is a homotopy equivalence. Then X/T becomes a wedge of loops corresponding (roughly) to the number of “independent” cycles. Notably, this collapsing is far from canonical, essentially unique up to choosing the spanning tree and then an order of edges. In some sense, because the homotopy group of a wedge of loops is a free group, we are able to study automorphisms of the free group in this way.

Proposition 1.43. Fix a subcomplex A of a CW complex X_1 . Given homotopic maps $f, g: A \rightarrow X_0$, then $X_0 \sqcup_f X_1 = X_0 \sqcup_g X_1$.

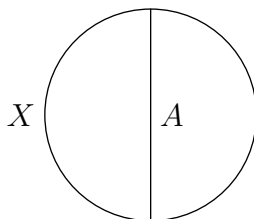
The idea of the above result is that if we can move the attaching maps f and g around, we should not really be adjusting the homotopy type.

To prove these results, we want access to the homotopy extension property.

Definition 1.44 (homotopy extension property). Fix a subspace A of a topological space X . Then the pair (X, A) has the *homotopy extension property* if and only if all $F_0: X \rightarrow Y$ and small homotopy $f_\bullet: A \times I \rightarrow Y$ with $F_0|_A = f_0$, then there is an extended homotopy $F_\bullet: X \times I \rightarrow Y$ where $F_t|_A = f_t$ for all $t \in I$.

It will turn out that a subcomplex A of a CW complex X makes (X, A) have the homotopy extension property, but this will take some work to prove.

By way of example, make Y the following “theta graph,” and the left edge is X , and A is the middle interval.



Here, $A \subseteq X$ is going to have the homotopy extension property. For example, one can contract A to a point and imagine dragging neighborhoods of $A \cap X$ in X (and in fact all of Y) along for the ride.

One way to think about the homotopy extension property is that we have a map $X \cup (A \times I) \rightarrow Y$ (by taking the union F_0 and f_\bullet), and we want to extend it to a full map $X \times I \rightarrow Y$. With this in mind, we would thus like to have a retraction $r: (X \times I) \rightarrow (X \cup (A \times I))$ and then composing. By taking $Y = X \times I$, one sees that having such a retraction r is in fact equivalent to the homotopy extension property.

So we want to find the retraction $r: (X \times I) \rightarrow (X \cup (A \times I))$.

Lemma 1.45. Fix a subspace A of a topological space X . Then (X, A) has the homotopy extension property if and only if A has a “mapping cylinder neighborhood.” In other words, there is a space B and map $f: B \rightarrow A$ such that M_f is homeomorphic to a neighborhood of A .

Approximately speaking, what’s going on here is that the mapping cylinder allows us some squishing region through which to extend homotopies. Then the above criteria can be checked for CW pairs (X, A) by tracking through attachments. Namely, a reparameterization of the attaching map has mapping cylinder which has the property needed above. Rigorously, one inducts on the n -skeleton of a CW complex X , using the homotopy extension property for cells of X not in A (and not caring about cells already in A).

Read
Hatcher

1.3 August 31

We now shift gears and talk about our first algebraic invariant: the fundamental group.

1.3.1 The Fundamental Group

Let’s start with an example.

Example 1.46. Fix a loop $\gamma: S^1 \rightarrow (\mathbb{C} \setminus \{0\})$ which is continuously differentiable. Then complex analysis tells us that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} dz$$

counts the number of times that γ “winds” around the integer. We might call this the “linking number” of γ . Notably, one can check that continuously varying γ does not adjust the linking number, so this linking number is homotopy invariant.

The fundamental group is a generalization of this notion.

Definition 1.47 (fundamental group). Let X be a topological space, and fix a basepoint $x_0 \in X$. Then the *fundamental group* $\pi_1(X, x_0)$ is the set of homotopy equivalence classes

$$\pi_1(X, x_0) := \{[f] \text{ such that } f: I \rightarrow X \text{ has } f(0) = f(1) = x_0\}.$$

We will give $\pi_1(X, x_0)$ a group structure below.

Remark 1.48. There is also a $\pi_0(X)$, which consists of homotopy classes of points $[x]$ for $x \in X$, where $[x]$ denotes the path-connected component of X . If we let $\Omega(X, x_0)$ denote the topological space of loops $f: I \rightarrow X$ such that $f(0) = f(1) = x_0$, then we find $\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$.

Remark 1.49. If we don't want to care about basepoints, one can look at $C(S^1, X)$, which is the set of maps $S^1 \rightarrow X$. This can be given a topology via the compact-open topology. Approximately speaking, these will correspond to conjugacy classes in $\pi_1(X, x_0)$ provided that X is path-connected. For example, the homotopy class of a constant loop $S^1 \rightarrow X$ consists of the contractible loops in X ; note there is something to check here in that one wants to know that a contractible loop (not relative to the basepoints) is in fact contractible relative to the basepoint.

Example 1.50. Let $X = \{x_0\}$ be a point. Then $\pi_1(X, x_0) = 1$ because there is only path $I \rightarrow X$.

Example 1.51. Let X be a convex subset of \mathbb{R}^n for some $n > 0$. Then for any $x_0 \in X$ has $\pi_1(X, x_0) = 1$. Indeed, use the convex hypothesis to shrink any path down to the constant path.

We can give $\pi_1(X, x_0)$ a product via composition.

Definition 1.52 (composition). Let X be a topological space, and fix a basepoint $x_0 \in X$. Given paths $f, g: I \rightarrow X$ such that $f(1) = g(0)$, we define the path $(f \cdot g): I \rightarrow X$ via

$$(f \cdot g)(t) := \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2, \\ g(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Note that $f \cdot g$ is well-defined at $t = 1/2$ because $f(1) = g(0)$.

The point of the above definition is to “squish” a path to do both f and g in the interval I , but at twice the speed. One has the following checks.

- The class $[f \cdot g]$ does not depend on the choice of representatives f and g . Essentially, if $f_1 \sim f_2$ and $g_1 \sim g_2$, then one can use these two homotopies to glue together to make a new homotopy $(f_1 \cdot g_1) \sim (f_2 \cdot g_2)$.
- We have $[(f \cdot g) \cdot h] = [f \cdot (g \cdot h)]$, so composition associates. The point is that these are basically reparameterizations of each other.
- There is an identity path given by $e_{x_0}(t) := x_0$. The identity check is done again by some idea of reparameterization.
- For a given path $f: I \rightarrow X$, we can define $\bar{f}: I \rightarrow X$ by $\bar{f}(t) := f(1 - t)$ and then check that

$$f \cdot \bar{f} \sim e_{f(0)},$$

so $[\bar{f}]$ provides the inverse path for $[f]$ in $\pi_1(X, x_0)$. The point is that $f \cdot \bar{f}$ is

$$(f \cdot \bar{f})(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2, \\ f(2 - 2t) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

One can then provide a homotopy by

$$h_s(t) := \begin{cases} f(2t) & \text{if } 0 \leq t \leq s/2, \\ f(s) & \text{if } s/2 \leq t \leq 1 - s/2, \\ f(2 - 2t) & \text{if } 1 - s/2 \leq t \leq 1, \end{cases}$$

so $h_0 = e_{f(0)}$ and $h_1 = f \cdot \bar{f}$.

For these checks, it is helpful to have lemmas establishing continuity of piecewise functions and establishing that reparameterization does not affect homotopy class.

Remark 1.53. Staring hard at our definition of composition, one sees that our reparameterization business is really just choosing various piecewise affine maps $I \rightarrow I$ with slopes in $2^{\mathbb{Z}}$ and breaks at the dyadic rationals $2^{\mathbb{Z}}\mathbb{Z} \subseteq \mathbb{Q}$. These maps form a group called the Thompson group.

Remark 1.54 (fundamental groupoid). Fix a topological space X , and define a category where the objects are points $x \in X$, and the morphisms $x \rightarrow y$ are paths (up to homotopy fixing endpoints). The above checks now show that this is in fact a category, where each morphism has an inverse. This category is called the *fundamental groupoid*. Modding out by isomorphism, our objects are now path components in X , and choosing a particular component produces the fundamental group in its endomorphisms.

Remark 1.55. Verifying that $\pi_1(X, x_0)$ only required reparameterization. So as in Remark 1.54, we could also look at the category where paths are only considered up to reparameterization, and the above checks still go through. This is related to the notion of “thin homotopy.”

Lemma 1.56. Fix a topological space X . Further, fix a path $p: I \rightarrow X$. Then $f \mapsto (\bar{p} \cdot f \cdot p)$ provides an isomorphism $\pi_1(X, p(1)) \rightarrow \pi_1(X, p(0))$.

Proof. This is well-defined because $f_1 \sim f_2$ implies $\bar{p} \cdot f_1 \sim \bar{p} \cdot f_2$ implies $\bar{p} \cdot f_1 \cdot p \sim \bar{p} \cdot f_2 \cdot p$. This is a group homomorphism because

$$\bar{p} \cdot f \cdot g \cdot p \sim \bar{p} \cdot f \cdot p \cdot \bar{p} \cdot g \cdot p.$$

Lastly, this is an isomorphism because \bar{p} provides the inverse map. ■

Remark 1.57. The above result roughly says that we can indeed look at the fundamental groupoid only in terms of the path-connected components.

Thus, we see that $\pi_1(X, x_0)$ is well-defined up to base-point provided that X is path-connected. However, the isomorphism between base-points is only defined up to path between those basepoints! Roughly speaking, the problem is that elements of $\pi_1(X, x_0)$ should really only be thought of up to inner automorphism because we can pre- and post-compose by some loop at x_0 .

Lemma 1.58. If X is homeomorphic to Y by $\varphi: X \rightarrow Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$ for any $x_0 \in X$.

Proof. Use φ . ■

1.3.2 The Fundamental Group of S^1

Here is our result.

Theorem 1.59. Fix any $x \in S^1$. Then $\pi_1(S^1, x) \cong \mathbb{Z}$. In fact, there is an isomorphism $\Phi: \mathbb{Z} \rightarrow \pi_1(S^1, x)$ given by

$$n \mapsto [t \mapsto (\cos 2\pi nt, \sin 2\pi nt)].$$

Sketch without covering spaces. We show injectivity and surjectivity independently.

- Think of S^1 as embedded in \mathbb{C} as $\{z : |z| = 1\}$ and take a smooth path $f: I \rightarrow S^1$, lift it to a map $\tilde{f}: I \rightarrow \mathbb{R}$ via

$$\tilde{f}(t) := \int_0^t d\theta,$$

where $d\theta$ is some differential form S^1 (say, $x dy - y dx$). Then $\tilde{f}(1)$ is intuitively contained in $2\pi\mathbb{Z}$ and is homotopy invariant. Now, f is not smooth, then we can use some small homotopy to make f smooth and then use the above argument. This provides an inverse map to Φ and thus shows that Φ is injective.

- For surjectivity, one can use uniform continuity of any path $f: I \rightarrow S^1$ and the compactness of S^1 in order to divide up I into intervals on which f can be written as a composition of well-behaved paths, which eventually allows us to force f to make piecewise linear. Once f is piecewise linear, we go interval-by-interval and fix f to be constant speed. Eventually f becomes one of the $\Phi(n)$ for some n . ■

For the covering space approach, the point is that we understand the fundamental group of \mathbb{R} well, and we have a fairly well-behaved “covering map” $p: \mathbb{R} \rightarrow S^1$ given by $p(\theta) := (\cos 2\pi\theta, \sin 2\pi\theta)$. The main claim, then, is that any path $\omega: I \rightarrow S^1$ has a unique lift $\tilde{\omega}: I \rightarrow \mathbb{R}$ such that $\tilde{\omega}(0) = \omega(0)$ and $p \circ \tilde{\omega} = \omega$. The point is that once we lift, we can use a homotopy up in \mathbb{R} (fixing the endpoints of $\tilde{\omega}$), which will then go back down to a homotopy on S^1 if we are careful. Anyway, this lifting process can essentially be done as described in the surjectivity check above.

1.4 September 5

Today we actually prove $\pi_1(S^1) \cong \mathbb{Z}$.

1.4.1 Eckmann–Hilton Argument

Because it is fun, we begin with some nonsense.

Proposition 1.60 (Eckmann–Hilton). Let X be a set equipped with the binary operations \circ and $*$ such that the following hold.

- (a) Identity: there are elements $1_\circ, 1_* \in X$ such that $1_\circ \circ a = a \circ 1_\circ = a$ and $1_* * a = a * 1_* = a$ for all $a \in X$.
- (b) Distribution: we have $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$ for all $a, b, c, d \in X$.

Then \circ and $*$ are the same operation and in fact are both commutative and associative.

Proof. This is purely formal. We proceed in steps.

1. We show that $1_\circ = 1_*$. Indeed, note

$$1_* = 1_* * 1_* = (1_* \circ 1_\circ) * (1_\circ \circ 1_*) = (1_* * 1_\circ) \circ (1_\circ * 1_*) = 1_\circ \circ 1_\circ = 1_\circ.$$

From now on, we use the symbol 1 to denote our identity $1_\circ = 1_*$.

2. We show that $a * b = a \circ b$. Indeed, note

$$a * b = (a \circ 1) * (1 \circ b) = (a * 1) \circ (1 * b) = a \circ b.$$

Thus, our operations are the same, and we will use the symbol $*$ to denote both of them now. Notably, our distribution law is $(a * b) * (c * d) = (a * c) * (b * d)$.

3. We show that $*$ is commutative. Indeed, for any $a, b \in X$, we see

$$a * b = (1 * a) * (b * 1) = (1 * b) * (a * 1) = b * a.$$

4. We show that $*$ is associative. Indeed,

$$(a * b) * c = (a * b) * (1 * c) = (a * 1) * (b * c) = a * (b * c),$$

for any $a, b, c \in X$. ■

As an application, we have the following result.

Corollary 1.61. Let G be a topological group with identity $e \in G$. Then $\pi_1(G, e)$ is abelian.

Proof. Let \cdot denote the usual concatenation operation on $\pi_1(G, e)$. The point is to give another binary operation to $\pi_1(G, e)$ and then apply Proposition 1.60.

Well, let $*$ denote the group operation on G , and for paths $f, g: I \rightarrow G$ based at e , we define the path $(f * g): I \rightarrow G$ by $(f * g)(t) := f(t) * g(t)$. Here are the necessary checks for our purposes.

- Note $f * g$ is a continuous map because it is the composite of the continuous maps

$$I \xrightarrow{(\text{id}_I, \text{id}_I)} I \times I \xrightarrow{(f, g)} G \times G \xrightarrow{*} G.$$

- We show $[f * g]$ does not depend on the choice of homotopy classes $[f]$ and $[g]$, so we may view $*$ as a binary operation on $\pi_1(G, e)$. Suppose $f \sim f'$ and $g \sim g'$ by the homotopies F_\bullet and G_\bullet , respectively. We want to show that $f * g \sim f' * g'$. Well, define $H_\bullet: G \times I \rightarrow G$ by $H_t(x) := F_t(x) * G_t(x)$ for all $t \in I$ and $x \in G$. Then we see that $H_0 = F_0 * G_0 = f * g$ and $H_1 = F_1 * G_1 = f' * g'$, and H_\bullet is continuous because it is the composite

$$G \times I \xrightarrow{(F_\bullet, G_\bullet)} G \times G \xrightarrow{*} G.$$

- Note that $*$ has an identity element given by the constant path $c(t) := e$ for all $t \in I$. Indeed, for any $[f] \in \pi_1(G, e)$, we see that $(f * c)(t) = f(t) * c(t) = f(t)$ for all $t \in I$, so $[f] * [c] = [f * c] = [f]$.
- Fix $[a], [b], [c], [d] \in \pi_1(G, e)$. We claim that

$$([a] \cdot [b]) * ([c] \cdot [d]) \stackrel{?}{=} ([a] * [c]) \cdot ([b] * [d]).$$

Removing all the homotopy classes, it is enough to show that $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$. Well, for any $t \in I$, we compute

$$((a \cdot b) * (c \cdot d))(t) = (a \cdot b)(t) * (c \cdot d)(t) = \begin{cases} a(t) * c(t) & \text{if } t \leq 1/2, \\ b(t) * d(t) & \text{if } t \geq 1/2, \end{cases}$$

and

$$((a * c) \cdot (b * d))(t) = \begin{cases} (a * c)(t) & \text{if } t \leq 1/2, \\ (b * d)(t) & \text{if } t \geq 1/2, \end{cases}$$

which is the same path.

Now, Proposition 1.60 shows that $*$ and \cdot must be the same operation on $\pi_1(G, e)$ and that \cdot is commutative, which is what we wanted. ■

1.4.2 Covering Spaces

Our computation is going to use the notion of a covering space.

Definition 1.62 (covering space). Fix a topological space X . Then a *covering space* is a topological space \tilde{X} together with a projection map $p: \tilde{X} \rightarrow X$ such that each $x \in X$ has an open neighborhood $U \subseteq X$ containing x such that $p^{-1}(U) = \bigsqcup_{\alpha \in \lambda} U_\alpha$ where U_α is open and $p: U_\alpha \rightarrow U$ is a homeomorphism. In this set up, the open set $U \subseteq X$ is said to be *evenly covered*.

The fact we will require about covering spaces is the following “fibration property.”

Proposition 1.63. Fix a topological space X and covering space $p: \tilde{X} \rightarrow X$. Further, suppose we have maps $F: Y \times I \rightarrow X$ and $\tilde{F}: Y \times \{0\} \rightarrow \tilde{X}$ such that $p \circ \tilde{F}|_{Y \times \{0\}} = F|_{Y \times \{0\}}$. Then there is a unique extension $\tilde{F}: Y \times I \rightarrow \tilde{X}$ such that $p \circ \tilde{F} = F$.

Proof. We proceed in steps. Say that a subset $U \subseteq X$ is “evenly covered” if and only if $p^{-1}(U) = \bigsqcup_{\alpha \in \lambda} U_\alpha$ and $p: U_\alpha \rightarrow U$ is a homeomorphism. Note that making an evenly covered open subset smaller will retain it being evenly covered using the fact that the maps $p: U_\alpha \rightarrow U$ is a homeomorphism.

1. To set us up, given $y \in Y$, we claim that there we can find an open neighborhood V of y and a finite open cover \mathcal{U} of I such that $F(V \times U)$ is contained in an evenly covered subset of X for any $U \in \mathcal{U}$. The point is to use compactness to shrink an evenly covered subset containing $F(V \times I)$ sufficiently. Well, for each $t \in I$, we may find an evenly covered subset $U_t \subseteq X$ containing $F(y, t)$ and then find $\varepsilon_t > 0$ and an open neighborhood V_t of y such that $V_t \times (t - \varepsilon_t, t + \varepsilon_t) \subseteq F^{-1}(U_t)$.

Now, by compactness, we may choose finitely many t labeled $\{t_1, \dots, t_n\}$ and set $\varepsilon_i := \varepsilon_{t_i}$ and $V_i := V_{t_i}$ and $U_i := U_{t_i}$ such that the intervals $(t_i - \varepsilon_i, t_i + \varepsilon_i)$ covers I and $F(V_i \times (t_i - \varepsilon_i, t_i + \varepsilon_i)) \subseteq F^{-1}(U_i)$. Now, set

$$V := \bigcap_{i=1}^n V_i$$

so any $t \in I$ lives in some $(t_i - \varepsilon_i, t_i + \varepsilon_i)$ has $F(V \times (t_i - \varepsilon_i, t_i + \varepsilon_i)) \subseteq U_i$.

2. We prove uniqueness. It is enough to show this in the case where Y is a point. Namely, fix suppose we have two lifts \tilde{F}_1 and \tilde{F}_2 of F which agree with \tilde{F} . Then, fixing some $y \in Y$, we see that $\tilde{F}_1(y)$ and $\tilde{F}_2(y)$ are maps $I \rightarrow \tilde{X}$ lifting $F(y): I \rightarrow X$ which equal $\tilde{F}(y, 0)$ at 0. In this setting, we want to show that $\tilde{F}_1(y, t) = \tilde{F}_2(y, t)$ for all $t \in I$. As such, we suppress the point $y \in Y$ in the argument which follows.

The previous step promises us a finite open cover \mathcal{U} of I such that $F(U)$ is contained in an evenly covered open subset of X for each $U \in \mathcal{U}$. Ordering the endpoints of \mathcal{U} , we produce a partition $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0, 1]$ such that $F([t_i, t_{i+1}])$ is covered in an evenly covered subset of U_i for each i .

We are now ready to show our uniqueness. We show that $\tilde{F}_1(t) = \tilde{F}_2(t)$ for each $t \in [0, t_i]$ by induction on i . At $i = 0$, there is nothing to say because $\tilde{F}_1(0) = \tilde{F}(0) = \tilde{F}_2(0)$. Now, for the induction, we are given that $\tilde{F}_1(t_i) = \tilde{F}_2(t_i)$. The point is that $F([t_i, t_{i+1}])$ is contained in an evenly covered subset $U_i \subseteq X$, so $\tilde{F}_1([t_i, t_{i+1}])$ lands in one of the disjoint copies of U_i of $p^{-1}(U_i)$, and it lands in exactly one because $[t_i, t_{i+1}]$ is connected; let \tilde{U}_i be the corresponding disjoint copy. The same statement holds for \tilde{F}_2 , and in fact $\tilde{F}_2([t_i, t_{i+1}]) \subseteq \tilde{U}_i$ because $\tilde{F}_2([t_i, t_{i+1}])$ needs to land in the same copy of U_i containing $\tilde{F}_1(t_i) = \tilde{F}_2(t_i)$.

We are now done. Note $p: \tilde{U}_i \rightarrow U_i$ is injective, so

$$p \circ \tilde{F}_1 = p \circ \tilde{F}_2$$

for $t \in [t_i, t_{i+1}]$ forces equality after removing t .

3. Fix some $y \in Y$. We will extend locally: we construct some open neighborhood V of y and a lift $\tilde{F}: V \times I \rightarrow \tilde{X}$ of $F|_{V \times I}$. The point is to “spread out” from $\{y\} \times I$ using the previous step.

As before, the first step promises us an open neighborhood V of y and a finite open cover \mathcal{U} of I such that $F(V \times U)$ is contained in an evenly covered subset for each $U \in \mathcal{U}$. Arranging the endpoints of the open sets in \mathcal{U} , we may say that we have a partition $0 = t_0 < t_1 < \dots < t_n = 1$ such that $F(V \times [t_i, t_{i+1}])$ is contained in an evenly covered open subset $U_i \subseteq X$ for each i .

We now extend F to \tilde{F} on $[0, t_i]$ inductively. For $i = 0$, there is nothing to do because $\tilde{F}|_{Y \times \{0\}}$ is already fixed. Now, suppose we have a definition of \tilde{F} on $V \times [0, t_i]$. Say $F(V \times [t_i, t_{i+1}]) \subseteq U_i$, and select the copy of U_i named $\tilde{U}_i \subseteq p^{-1}(U_i)$ by requiring it to contain $\tilde{F}(y, t_i)$. Now, shrink V so that $V \times \{t_i\}$ contains y still but now is contained in \tilde{U}_i . Now, define \tilde{F} on $V \times [t_i, t_{i+1}]$ by pre-composing with the homeomorphism

$$p^{-1}: U_i \rightarrow \tilde{U}_i,$$

and we produce a continuous map because we have agreed on the seam at $V \times \{t_i\}$.¹ This completes the lifting to a neighborhood V of y .

4. We can now glue the lifts \tilde{F} constructed in the previous step, and the gluing is well-defined because they must agree on intersections by the uniqueness of the second step. This completes the proof. ■

And now here is our result.

Theorem 1.64. For any $x \in S^1$, we have $\pi_1(S^1, x) \cong \mathbb{Z}$.

Proof. For brevity, embed S^1 into \mathbb{C} as $S^1 = \mathbb{R}/\mathbb{Z}$, and let our basepoint be $0 \in S^1$. We now abbreviate our fundamental group to $\pi_1(S^1)$.

Now, we note that we have the continuous (in fact, holomorphic) path $\omega_n: [0, 1] \rightarrow S^1$ given by $t \mapsto nt$. A reparameterization argument can show that $[\omega_n] \cdot [\omega_m] = [\omega_{m+n}]$ for any $m, n \in \mathbb{Z}$, so we have defined a homomorphism $\varphi: \mathbb{Z} \rightarrow \pi_1(S^1)$. We would like to show that this map is an isomorphism. We will use Proposition 1.63, for which we note that $p: \mathbb{R} \rightarrow S^1$ given by $p(t) := t$ is a covering space map. Indeed, for each $t \in S^1$, choose the neighborhood $(t - 0.1, t + 0.1)$ so that

$$p^{-1}((t - 0.1, t + 0.1)) = (t - 0.1, t + 0.1) + \mathbb{Z} = \bigsqcup_{n \in \mathbb{Z}} (t + n - 0.1, t + n + 0.1).$$

We now show that φ is an isomorphism.

- **Surjective:** let $f: I \rightarrow S^1$ be a loop, and we want to show that $f \sim \omega_n$ for some $n \in \mathbb{Z}$. By Proposition 1.63 applied with Y being a point, we get a path $\tilde{f}: I \rightarrow \mathbb{R}$ such that $f = p \circ \tilde{f}$. Now, set $n := \tilde{f}(1)$, which is indeed an integer, and we claim $\tilde{f} \sim \tilde{\omega}_n$, where $\tilde{\omega}_n(t) := nt$; this will finish after composing with the projection p as it shows that $f \sim \omega_n$ by Lemma 1.9.

To see this, we define the map $h: I \times I \rightarrow \mathbb{R}$ by

$$h_t(s) := (1 - t)\tilde{f}(s) + t\tilde{\omega}_n(s).$$

Then h is continuous because it is the composite

$$I \xrightarrow{(\text{id}, 1 - \text{id}, \tilde{f}, \omega_n)} I \times I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$$

where the last map is taking a linear combination. Now, $h_0 = \tilde{f}$ and $h_1 = \tilde{\omega}_n$, so $\tilde{f} \sim \tilde{\omega}_n$ follows.

- **Injective:** suppose $\omega_n \sim \omega_0$, and we want to show that $n = 0$. Then we have a homotopy $h_\bullet: I \times I \rightarrow X$ such that $h_0 = \omega_n$ and $h_1 = \omega_0$. Then Proposition 1.63 produces a unique lift $\tilde{h}_\bullet: I \times I \rightarrow \tilde{X}$ of h such that $\tilde{h}_t(0) = 0$ for each $t \in I$. Now, the map $t \mapsto \tilde{h}_t(1)$ is continuous, and $h_t(1) = 0$ for each $t \in I$, so the map $t \mapsto \tilde{h}_t(1)$ maps to the discrete space \mathbb{Z} . It follows that $\tilde{h}_0(1) = \tilde{h}_1(1)$, so $0 = n$ because of how ω_0 and ω_n lift to \mathbb{R} . ■

¹ To avoid this annoyance at the seam, one can allow the partition to overlap a bit so that we only ever glue continuous maps along open sets, which is legal. I won't write this out.

1.4.3 The Fundamental Group Functor

Let's do some nonsense checks, for fun.

Definition 1.65 (based topological space). A based topological space (X, x_0) is a topological space X together with a basepoint $x_0 \in X$. A map of based topological spaces $\varphi: (X, x_0) \rightarrow (Y, y_0)$ is a continuous map $\varphi: X \rightarrow Y$ such that $\varphi(x_0) = y_0$. The category with these objects and morphisms is Top_* .

We won't bother to check that Top_* is a category. Here is the main point of this subsection.

Proposition 1.66. We have a functor $\pi_1: \text{Top}_* \rightarrow \text{Grp}$.

Proof. We already know that $\pi_1(X, x_0)$ is a group for each based topological space (X, x_0) , so we really only have to check the functoriality properties.

Fix a map $\varphi: (X, x_0) \rightarrow (Y, y_0)$ of based topological spaces. We need to define a group homomorphism $\pi_1(\varphi): \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$. Well, given a loop $f: I \rightarrow X$ based at x_0 , we note that $(\varphi \circ f): I \rightarrow Y$ is a loop based at $y_0 = \varphi(x_0)$, so we hope that our desired map is $(\varphi \circ -)$. Here are our checks.

- Well-defined: if $f \sim f'$, we need to show that $\varphi \circ f \sim \varphi \circ f'$. This is simply Lemma 1.9.
- Group homomorphism: we need to show that $(\varphi \circ f) \cdot (\varphi \circ g) \sim \varphi \circ (f \cdot g)$ for loops $f, g: I \rightarrow X$ based at x_0 . In fact, these paths are equal: for $t \in I$, we compute

$$((\varphi \circ f) \cdot (\varphi \circ g))(t) = \begin{cases} \varphi(f(2t)) & \text{if } t \leq 1/2, \\ \varphi(g(2t-1)) & \text{if } t \geq 1/2, \end{cases} = (\varphi \circ (f \cdot g))(t).$$

We now prove functoriality of π_1 .

- Identity: note that $\text{id}_X: (X, x_0) \rightarrow (X, x_0)$ has $\text{id}_X \circ f = f$ for any path $f: I \rightarrow X$, so $\pi_1(\text{id}_X)([f]) = [f]$ for any $[f] \in \pi_1(X, x_0)$.
- Composition: given maps $\varphi: (X, x_0) \rightarrow (Y, y_0)$ and $\psi: (Y, y_0) \rightarrow (Z, z_0)$ and a loop $f: I \rightarrow X$ based at x_0 , we see that

$$\pi_1(\psi \circ \varphi)([f]) = [\psi \circ \varphi \circ f] = \pi_1(\psi)([\varphi \circ f]) = (\pi_1(\psi) \circ \pi_1(\varphi))([f]),$$

which finishes. ■

Of course, just being a functor is not terribly interesting. Here is a nice property.

Proposition 1.67. Fix based topological spaces (X, x_0) and (Y, y_0) . Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Let $p_X: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0)$ and $p_Y: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0)$ denote the projections. Now, note that we have a map

$$(\pi_1(p_X), \pi_1(p_Y)): \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

which we claim is an isomorphism. For brevity, let this morphism be φ . Of course, φ is a homomorphism because π_1 is a functor (see Proposition 1.66).

- Surjective: fix loops $f_X: I \rightarrow X$ and $f_Y: I \rightarrow Y$ based at x_0 and y_0 respectively. Then the map $f(t) := (f_X(t), f_Y(t))$ defines a loop $I \rightarrow X \times Y$ based at (x_0, y_0) , and by construction $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$, so

$$\varphi(f) = ([p_X \circ f], [p_Y \circ f]) = ([f_X], [f_Y]).$$

- **Injective:** suppose $\varphi([f]) = \varphi([g])$, and we want to show that $[f] = [g]$. Well, we have homotopies $h_{X\bullet}: I \times I \rightarrow X$ and $h_{Y\bullet}: I \times I \rightarrow Y$ such that $h_{X0} = p_X \circ f$ and $h_{X1} = p_X \circ g$ and $h_{Y0} = p_Y \circ f$ and $h_{Y1} = p_Y \circ g$. Then we define $h_\bullet: I \times I \rightarrow X \times Y$ by

$$h_t(s) := (h_{Xt}(s), h_{Yt}(s)).$$

Note h_t is continuous because it is continuous in each coordinate. To finish, we see $h_0 = f$ and $h_1 = g$ by checking after applying the projections p_X and p_Y , so $f \sim g$ follows. ■

Remark 1.68. More precisely, the above proof has shown that π_1 preserves products.

Example 1.69. We have $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ by Proposition 1.67 and Theorem 1.64.

Example 1.70. We show that there is no retraction $r: D^2 \rightarrow S^1$. Let $i: S^1 \rightarrow D^2$ be the inclusion. If there is a retraction r , then we see that $r \circ i = \text{id}_{S^1}$, so functoriality of π_1 means that the composite

$$\pi_1(S^1) \xrightarrow{i} \pi_1(D^2) \xrightarrow{r} \pi_1(S^1)$$

is an isomorphism. In particular, i is injective. However, $\pi_1(S^1) \cong \mathbb{Z}$ by Theorem 1.64, and $\pi_1(D^2) = 0$ because D^2 is convex and hence contractible.

Remark 1.71. One can use Example 1.70 to show Brouwer's fixed point theorem: we show that any continuous map $h: D^2 \rightarrow D^2$ has a fixed point. Well, suppose h has no fixed point. Then there is a continuous map sending $x \in D^2$ to the point on S^1 which intersects with the ray starting at $h(x)$ and then going through x . Then $h: D^2 \rightarrow S^1$ defines a retraction, contradicting Example 1.70.

1.5 September 7

Today we prove the van Kampen theorem.

1.5.1 Free Products of Groups

We will be somewhat brief on this because this is not an algebra class.

Definition 1.72 (free product). Let $\{G_\alpha\}_{\alpha \in \lambda}$ be a collection of groups. Then we form the free product $\ast_{\alpha \in \lambda} G_\alpha$ as having underlying set given by strings of words whose letters are in the G_α , modded out by the relations $g_\alpha \cdot h_\alpha = g_\alpha h_\alpha$ whenever $g_\alpha, h_\alpha \in G_\alpha$ for some $\alpha \in \lambda$.

Perhaps one should check that this forms a group, so we will sketch what one should do.

1. Let W be the set of finite strings (i.e., words) whose letters are g or g^{-1} where $g \in G_\alpha$ for some $\alpha \in \lambda$. Then we build \overline{W} by allowing combining $g_\alpha \cdot h_\alpha$ into a single character $g_\alpha h_\alpha$ provided that g_α and h_α belong to the same group G_α . We will realize our desired group as a subgroup of $\text{Aut}(W)$.
2. For each $g \in W$, define the function $L_g: W \rightarrow W$ by left concatenation. One should show that L_g is in fact a well-defined function, which depends on the equivalence relation defining W , but in short, one can show that having two words w and w' with $w \sim w'$ enforces $g \cdot w \sim g \cdot w'$ by using the same concatenation rules on both sides. A rigorous argument would need to use an induction, which we won't bother to write out.

3. Note that L_e (where e denotes the empty string) is the identity on W , and $L_{g^{-1}}$ is the inverse of L_g . Thus, the image of L_\bullet in W is a subgroup of $\text{Aut}(W)$, and we call this subgroup $*_{\alpha \in \lambda} G_\alpha$. One realizes this group as the free product described above by identifying L_g with $L_g(e)$. The point of introducing L_\bullet at all is to make the various group law checks easier.

One has the following universal property, which we will not prove, again because this is not an algebra class.

Proposition 1.73. Let $\{G_\alpha\}_{\alpha \in \lambda}$ be a collection of groups. Given homomorphisms $\varphi_\alpha: G_\alpha \rightarrow H$ to a target group H , there is a unique homomorphism $\varphi: *_{\alpha \in \lambda} G_\alpha \rightarrow H$ such that the following diagram commutes.

$$\begin{array}{ccc} G_\alpha & & \\ \downarrow \iota_\alpha & \searrow \varphi_\alpha & \\ *_{\alpha \in \lambda} G_\alpha & \xrightarrow{\varphi} & H \end{array}$$

Here, $\iota_\alpha: G_\alpha \rightarrow *_{\alpha \in \lambda} G_\alpha$ is the inclusion.

Proof. Let's sketch the proof. We begin by showing uniqueness of φ . Given a word $g_{\alpha_1} \cdots g_{\alpha_n}$ in $*_{\alpha \in \lambda}$, we see that the commutativity of the diagram enforces

$$\begin{aligned} \varphi(g_{\alpha_1} \cdots g_{\alpha_n}) &= \varphi(g_{\alpha_1}) \cdots \varphi(g_{\alpha_n}) \\ &= \varphi(\iota_{\alpha_1}(g_{\alpha_1})) \cdots \varphi(\iota_{\alpha_n}(g_{\alpha_n})) \\ &= \varphi_{\alpha_1}(g_{\alpha_1}) \cdots \varphi_{\alpha_n}(g_{\alpha_n}). \end{aligned}$$

Thus, φ is uniquely determined by the φ_α . It remains to show that the above formula in fact defines a group homomorphism, which follows roughly speaking by the minimal construction of $*_{\alpha \in \lambda}$. Namely, we have thus far defined a function $\varphi: W \rightarrow H$ where W is the set of all words, so one needs to check that we are still safe after modding out by the requisite equivalence relation on W . We will not do this, but in short, one can use induction on the various generators of the group presentation of $*_{\alpha \in \lambda} G_\alpha$. ■

In the discussion that follows, we will frequently use group presentations, which is an expression of the form

$$\langle a_1, a_2, \dots, w_1, w_2, \dots \rangle,$$

where the a_\bullet are generators for words giving the group and w_\bullet are words intended to produce relations for the group, by default of the form $w_\bullet = 1$.

Example 1.74. The group $\langle a \rangle$ gives \mathbb{Z} . Namely, the group consists of the elements

$$\{\dots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \dots\}.$$

Example 1.75. The group $\langle a : a^2 \rangle$ gives $\mathbb{Z}/2\mathbb{Z}$. Namely, our isomorphism is by sending $k \in \mathbb{Z}/2\mathbb{Z}$ to a^k . This is well-defined because $2 \mapsto a^2$, and a^2 is the identity of the group.

1.5.2 van Kampen's Theorem

In this subsection, we state and prove the van Kampen theorem. Let's explain the idea. Suppose we can decompose X into path-connected open subsets $\{A_\alpha\}_{\alpha \in \lambda}$. Then the inclusions $i_\alpha: A_\alpha \hookrightarrow X$ induce maps $\pi_1(A_\alpha) \rightarrow \pi_1(X)$, which by the nature of the free product induces a map

$$*_{\alpha \in \lambda} \pi_1(A_\alpha) \rightarrow \pi_1(X).$$

It is not too hard to see that this map is surjective.

Lemma 1.76. Fix a topological space X which is the union of path-connected open subsets $\{A_\alpha\}_{\alpha \in \lambda}$ each containing a basepoint $x_0 \in X$. For any loop $\gamma: I \rightarrow X$ based at x_0 , there are loops $\gamma_{\alpha_1}, \dots, \gamma_{\alpha_n}$ based at x_0 such that

$$\gamma \sim \gamma_{\alpha_1} \cdot \dots \cdot \gamma_{\alpha_n}$$

and γ_{α_n} is a path connected in A_{α_n} for each α_n .

Proof. For each $\alpha \in \lambda$, decompose $\gamma^{-1}(A_\alpha) \subseteq I$ into a collection of intervals \mathcal{I}_α . Then

$$I = \gamma^{-1}(X) = \bigcup_{\alpha \in \lambda} \gamma^{-1}(A_\alpha) = \bigcup_{\alpha \in \lambda} \bigcup_{I' \in \mathcal{I}_\alpha} I'.$$

Now, I is compact, so this open cover can be turned into a finite subcover $\{(a_k, b_k)\}_{k=1}^n$ where $\gamma((a_k, b_k)) \subseteq A_{\alpha_k}$ for some $\alpha_k \in \lambda$. Ordering the (a_k, b_k) , we produce a partition $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ such that $\gamma([t_k, t_{k+1}]) \subseteq A_{\alpha_k}$ for some perhaps different n and $\alpha_k \in \lambda$.

We are now ready to finish. For each $1 \leq k \leq n-1$, we recall that A_{α_k} is path-connected, so we can find a path η_k from $\gamma(t_k)$ to x_0 . Then we see that

$$\begin{aligned} \gamma &\sim \gamma|_{[t_0, t_1]} \cdot \gamma|_{[t_1, t_2]} \cdot \dots \cdot \gamma|_{[t_{n-2}, t_{n-1}]} \cdot \gamma|_{[t_{n-1}, t_n]} \\ &\sim \underbrace{\gamma|_{[t_0, t_1]} \cdot \eta_1}_{\gamma_0 :=} \cdot \underbrace{\bar{\eta}_1 \cdot \gamma|_{[t_1, t_2]} \cdot \eta_2}_{\gamma_1 :=} \cdot \dots \cdot \underbrace{\eta_{n-2} \cdot \bar{\eta}_{n-2} \cdot \gamma|_{[t_{n-2}, t_{n-1}]} \cdot \eta_{n-1}}_{\gamma_{n-2} :=} \cdot \underbrace{\bar{\eta}_{n-1} \cdot \gamma|_{[t_{n-1}, t_n]}}_{\gamma_{n-1} :=}. \end{aligned}$$

The above expression provides the desired factorization. ■

Corollary 1.77. Fix a topological space X which is the union of path-connected open subsets $\{A_\alpha\}_{\alpha \in \lambda}$ each containing a basepoint $x_0 \in X$. Then the map induced map

$$\ast_{\alpha \in \lambda} \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$$

is surjective.

Proof. This is direct from Lemma 1.76. ■

We would now like to compute its kernel of our induced map. Well, if $A_\alpha \cap A_\beta$ is path-connected, then we let $i_{\alpha\beta}: A_\alpha \cap A_\beta \rightarrow A_\alpha$ denote the inclusion, and we note that

$$\begin{array}{ccc} A_\alpha \cap A_\beta & \xrightarrow{i_{\alpha\beta}} & A_\alpha \\ i_{\beta\alpha} \downarrow & & \downarrow i_\alpha \\ A_\beta & \xrightarrow{i_\beta} & X \end{array}$$

commutes, so

$$\begin{array}{ccc} \pi_1(A_\alpha \cap A_\beta) & \xrightarrow{\pi_1(i_{\alpha\beta})} & \pi_1(A_\alpha) \\ \pi_1(i_{\beta\alpha}) \downarrow & & \downarrow \pi_1(i_\alpha) \\ \pi_1(A_\beta) & \xrightarrow{\pi_1(i_\beta)} & \pi_1(X) \end{array}$$

also commutes. Thus, for any $\gamma \in \pi_1(A_\alpha \cap A_\beta)$, we see that $\pi_1(i_\alpha)(\pi_1(i_{\alpha\beta})([\gamma])) = \pi_1(i_\beta)(\pi_1(i_{\beta\alpha})([\gamma]))$, which produces a relation belonging to the kernel of our surjection $\ast_{\alpha \in \lambda} \pi_1(A_\alpha) \rightarrow \pi_1(X)$. Under favorable circumstances, van Kampen's theorem tells us that this is the entire kernel.

1.6 September 12

Let's wrap up some loose ends. People are doing pretty well on the homeworks, but please cite theorems and so on to be rigorous.

1.6.1 The Fundamental Group of a Torus Knot

Let's give a few applications of van Kampen's theorem.

Example 1.78. Let $K \subseteq \mathbb{R}^n$ be a compact subset for $n \geq 3$. Embed $\mathbb{R}^n \subseteq S^n$ by stereographic projection, and van Kampen shows that

$$\pi_1(\mathbb{R}^n \setminus K) \cong \pi_1(S^n \setminus K).$$

More precisely, we have S^n sitting inside $S^{n-1} \times \mathbb{R}$ (place K inside a large ball, and we can continuously deform any loop in $\mathbb{R}^n \setminus K$ into this large ball), and the π_1 arising from this \mathbb{R} cannot help you.

Example 1.79 (torus knots). Fix positive integers $m, n \in \mathbb{Z}$ bigger than 1 with $\gcd(m, n) = 1$. Define the torus knot $K_{m,n} \subseteq T^2$ (where $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$) as the image of the line $my = nx$; alternatively, it is the image of the map $t \mapsto (mt, nt)$. For example, here is $K_{3,2}$ sitting inside the square $\mathbb{R}^2/\mathbb{Z}^2$.



We compute $\pi_1(\mathbb{R}^3 \setminus K_{m,n})$.

Proof. Professor Agol seems to prefer the “Clifford torus” thought of as

$$T^2 = \{(z_1, z_2) : |z_1| = |z_2| = 1/\sqrt{2}\}.$$

This sits inside $S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$. Anyway, we begin by giving us some breathing room. Define the “thickening” of K as

$$A := \left\{ (z_1, z_2) : |z_1| < \frac{1}{\sqrt{2}} + \varepsilon \right\} \setminus \left\{ (rz^m, \sqrt{1-r^2}z^n) : z \in S^1, \frac{1}{\sqrt{2}} \leq r \leq \frac{1}{\sqrt{2}} + \varepsilon \right\}$$

(namely, A is the torus thickened in such a way that it carries the subtraction of $K_{m,n}$) and in the other way as

$$B := \left\{ (z_1, z_2) : |z_2| < \frac{1}{\sqrt{2}} + \varepsilon \right\} \setminus \left\{ (\sqrt{1-r^2}z^m, rz^n) : \frac{1}{\sqrt{2}} \leq r \leq \frac{1}{\sqrt{2}} + \varepsilon \right\}.$$

Intersecting, we see that $A \cap B$ is $(S^1 \times S^1) \setminus K_{m,n}$ thickened by $(-\varepsilon, \varepsilon)$, which we note can be deformed to $(S^1 \times \mathbb{R}) \times (-\varepsilon, \varepsilon)$, which has fundamental group \mathbb{Z} . Notably, $\pi_1(A) \cong \mathbb{Z}$ and $\pi_1(B) \cong \mathbb{Z}$ by deforming them carefully to the circle S^1 , so our fundamental group is going to be $(\mathbb{Z} * \mathbb{Z})/\mathbb{Z}$ by van Kampen.

However, we need to compute the image of $\pi_1(A \cap B)$ in $\pi_1(A) * \pi_1(B)$. In the retraction of A down to a circle says that the image in $\pi_1(A)$ is by multiplication by n , and similarly going to B is multiplication by m . We conclude that our fundamental group is

$$\langle a, b : a^n = b^m \rangle.$$

As an aside, we note that $S^3 \setminus K_{m,n}$ will have a deformation retract back to $K_{m,n}$ shifted upwards by some amount (for example, see the diagram and imagine a copy of $K_{m,n}$ shifted up by some small $\varepsilon > 0$). Anyway, for $m, n > 1$ we can see that the center of the above group is $\langle a^n \rangle$, so modding out by the center yields $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. In total, we are able to distinguish the torus knots $S^3 \setminus K_{m,n}$ from each other.² To deal with the signs of m and n , we need a notion of isotopy to distinguish a knot from its “mirror image.” ■

² Alternatively, the abelianization $\pi_1(S^3 \setminus K_{m,n})$ is the free group with $(m-1)(n-1)$ generators, and the abelianization of $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$ is mn , from which we can read off m and n .

1.6.2 The Fundamental Group of a CW Complex

Let's move on from knots and compute the fundamental group of some cell complexes.

Example 1.80. Let X be a connected graph (i.e., a 1-dimensional CW complex), then $\pi_1(X)$ is homotopy equivalent to a wedge of circles, which has fundamental group \mathbb{Z}^{*r} for some r , which is the free group on r letters.

Now, if Y is a connected CW complex, then $\pi_1(Y^1)$ is a free group. Then $\pi_1(Y^2)$ might be complicated, but let's imagine computing $\pi_1(Y^3)$. The point is that we take some ball $e_\alpha^3 \cong D^2$ and attach it via some $\varphi_\alpha: \partial D^3 \rightarrow Y^2$.

To compute the fundamental group of this, we cover $Y^2 \sqcup_{\varphi_\alpha} e_\alpha^3$ by $A := Y^2 \cup_{\varphi_\alpha} (e_\alpha^3 \setminus \{x\})$ and $B = (e_\alpha^3)^\circ$ (Here, x is some point in the interior.) Notably, the intersection is simply $S^2 \times \mathbb{R}$, which is trivial, so we conclude that the attachment e_α^3 did nothing to our fundamental group by van Kampen. Applying this argument inductively (perhaps transfinitely), we see that $\pi_1(Y^3) = \pi_1(Y^2)$. One can continue upwards to conclude that $\pi_1(Y) = \pi_1(Y^2)$.³

Now, let's say that we actually want to compute $\pi_1(Y^2)$. To do so, we note that we have a surjection $\pi_1(Y^1) \rightarrow \pi_1(Y^2)$ given by the inclusion (any loop can be deformed off the 2-skeleton to the 1-skeleton). Now, for each 2-cell e_α^2 attached via $\varphi_\alpha: \partial e_\alpha^2 \rightarrow Y^1$, we choose a path $\gamma_\alpha: I \rightarrow Y^1$ so that $\gamma_\alpha(0) = y$ and $\gamma_\alpha(1) = \varphi_\alpha(0)$ and then find that

$$\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha$$

ought to be in the kernel of our projection. An argument shows that these elements will generate the needed kernel. One can show this by an analogous argument to the above: the point is that the attachment of e_α^2 kills basically exactly the loop given above and nothing else, so we can use an inductive argument to conclude.

Remark 1.81. One can use this result to show that any group G arises as the fundamental group of a CW complex of dimension 2. Roughly speaking, the point is that any group is the quotient of a free group, and the above argument allows us to dictate relations, provided that we are sufficiently careful.

Example 1.82. Fix a positive integer g . Define S_g by starting with a $4g$ -gon and attaching the edges. Namely, for $n < 3$, an $n \pmod{4}$ edge is identified with the next over $n + 2 \pmod{4}$ edge in the opposite direction. Roughly speaking, after some manipulation, one finds that S_g ought to be a g -hole torus. Using the above argument, one finds that $\pi_1(S_g)$ is generated by $2g$ generators $a_1, \dots, a_g, b_1, \dots, b_g$ modded out by the relations $a_i b_i a_i^{-1} b_i^{-1}$ for each i . In particular, the abelianization of $\pi_1(S_g)$ has all the commutators, so we get \mathbb{Z}^{2g} . Thus, π_1 distinguishes our surfaces.

Example 1.83 (projective space). We note $\pi_1(\mathbb{RP}^\infty) \cong \pi_1(\mathbb{RP}^2)$ because the higher cells cannot help you in the fundamental group. Further, we see $\pi_1(\mathbb{RP}^2)$ is a disk with semicircles identified in the opposite direction, which we can see from the above argument is simply $\mathbb{Z}/2\mathbb{Z}$.

Example 1.84 (lens space). Fix positive integers p and q with $\gcd(p, q) = 1$. Take S^2 and divide the equator into p circles, and we glue the top hemisphere to the bottom hemisphere by gluing after a $2\pi p/q$ rotation. The space has fundamental group $\mathbb{Z}/p\mathbb{Z}$. Indeed, our 1-skeleton is the equator, and the a^p comes from how we attached our disks together.

Next time we will talk about covering spaces.

³ One can do this without transfinite induction by working with a single loop and arguing about homotopy equivalence. The point is that a single loop (and in fact a single homotopy) can be compact and therefore only cares about finitely many cells.

1.7 September 14

We're talking about covering spaces today.

1.7.1 Examples of Covering Spaces

Our goal is to generalize the method we used to compute $\pi_1(S^1)$. Let's recall our definition.

Definition 1.62 (covering space). Fix a topological space X . Then a *covering space* is a topological space \tilde{X} together with a projection map $p: \tilde{X} \rightarrow X$ such that each $x \in X$ has an open neighborhood $U \subseteq X$ containing x such that $p^{-1}(U) = \bigsqcup_{\alpha \in \lambda} U_\alpha$ where U_α is open and $p: U_\alpha \rightarrow U$ is a homeomorphism. In this set up, the open set $U \subseteq X$ is said to be *evenly covered*.

Example 1.85. The map $p: \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi i t}$ is a covering space map. Here, we are viewing S^1 as $\{z \in \mathbb{C} : |z| = 1\}$. The point is that, for any $e^{2\pi i \theta} \in S^1$, we have

$$p^{-1}(S^1 \setminus \{e^{2\pi i \theta}\}) = \bigsqcup_{n \in \mathbb{Z}} (\theta, \theta + 2\pi).$$

Non-Example 1.86. The map $p: (0, 2) \rightarrow S^1$ given by $t \mapsto e^{2\pi i t}$ is not a covering space map. For example, any open interval U around $1 \in S^1$ will have pre-image by p looking like $(0, \varepsilon) \sqcup (1 - \varepsilon, 1 + \varepsilon) \sqcup (2 - \varepsilon, 2)$, and $(0, \varepsilon)$ is not mapped homeomorphically to our $U \subseteq S^1$.

Example 1.87. The map $f: \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ given by $f: z \mapsto z^n$ for a positive integer n is a covering space map. Roughly speaking, for any ray ℓ through the origin in \mathbb{C} , one can define $\log: (\mathbb{C} \setminus \ell) \rightarrow \mathbb{C}$, which allows us to define an n th root $\sqrt[n]{w} := \exp(\frac{1}{n} \log w)$ on $\mathbb{C} \setminus \ell$; this makes $\mathbb{C} \setminus \ell$ into an evenly covered subset, so we are a covering space upon letting ℓ vary.

Example 1.88. Fix a topological space X and a discrete set E . Then of course $p: X \times E \rightarrow X$ is a covering space: indeed, X is an evenly covered subset. In fact, if $p: \tilde{X} \rightarrow X$ is a covering space map where X is evenly covered, then the definition of p requires $\tilde{X} \cong \bigsqcup_{\alpha \in \lambda} X_\alpha$

Example 1.89. Map $p: S^\infty \rightarrow \mathbb{RP}^\infty$ by sending $x \in S^\infty$ to the corresponding line in \mathbb{RP}^∞ . More precisely, embed some $S^n \subseteq S^\infty$ into \mathbb{R}^{n+1} and then take lines down to \mathbb{RP}^n . Notably, $p(x) = p(-x)$ for each x (and conversely $p(x) = p(y)$ if and only if $\mathbb{R}x = \mathbb{R}y$ if and only if $x = \pm y$), so p is 2-to-1. One can check that p is a covering space map by looking on the level of cell complexes: the pre-image of the interior of the unique n -cell $(e^n)^\circ \subseteq \mathbb{RP}^\infty$ is the disjoint union of the interior of the two n -cells of S^∞ . More precisely, the n -cell e_i^n inside \mathbb{RP}^n given by

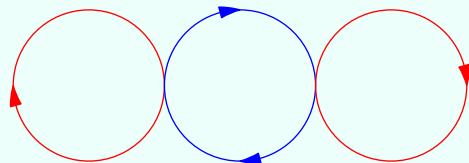
$$\{[x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] : x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \mathbb{R}\}$$

is evenly covered in the map $S^n \rightarrow \mathbb{RP}^n$. One can extend this idea up to \mathbb{RP}^∞ to conclude: let e_i be the above subset except we do not terminate at x_n , and then e_i is covered by the open subsets $e_{i,\pm} \subseteq S^\infty$ defined as

$$e_{i,\pm} = \{(x_0, x_1, \dots) \in S^\infty : x_i \text{ has sign } \pm\}.$$

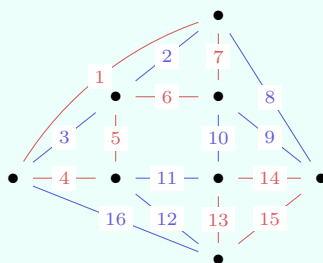
Let's do a few examples on $S^1 \vee S^1$.

Example 1.90. We examine 2-fold (i.e., 2-to-1) covers of $S^1 \vee S^1$. There is the trivial one with two copies of $S^1 \vee S^1$. As another example, note that $S^1 \vee S^1 \vee S^1$ loop around $S^1 \vee S^1$ twice: the first S^1 goes around the first S^1 , then half of the second S^1 goes around the second S^1 , then the third S^1 goes around the first S^1 around. Here is an image.

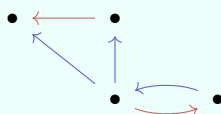


It turns out that, with one more, these are all the 2-to-1 covering maps, which can be seen by finding index-2 subgroups of $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, as we will soon see.

Example 1.91. Consider the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$. This is then a covering space of $S^1 \vee S^1$ by sending the $\mathbb{Z} \times \mathbb{R}$ to traverse one of the circles S^1 and the $\mathbb{R} \times \mathbb{Z}$ to traverse the other circle of S^1 . More generally, it turns out that covering spaces are exactly graphs where every vertex has degree 4, which we can see by coloring the edges red and blue so that each vertex has exactly two red edges and two blue edges; then choosing an Euler cycle provides the needed covering space. The previous example is one way to do this. Here is another example of such a graph, with marked Euler cycle.



Example 1.92. We can take a subgroup of $\mathbb{Z} * \mathbb{Z}$ to produce a covering space of $S^1 \vee S^1$. As an example, take the subgroup generated by ab and $b^{-1}ab$. Reading off these generators produces a graph as follows.



In general, we basically fold edges together to make relations. For example, the multiple outgoing blue edges should be folded together.

Example 1.93. There is an infinite tree where each vertex has degree 4. A coloring of the edges produces a "Cayley graph" C_2 , which will turn out to be the universal covering space once we define such a notion. It turns out to be maximal in the sense that it covers any path-connected cover of $S^1 \vee S^1$.

1.7.2 Lifting with Covering Spaces

We will want the following result.

Proposition 1.94. Covering spaces have the homotopy lifting property. In other words, given a covering space $p: \tilde{X} \rightarrow X$, a “homotopy” $f_\bullet: Y \times I \rightarrow X$ with a given lift $\tilde{f}_0: Y \rightarrow \tilde{X}$ will lift uniquely to $\tilde{f}_\bullet: Y \times I \rightarrow \tilde{X}$ agreeing with X .

Proof. This is direct from Proposition 1.63. ■

Corollary 1.95. Fix a covering space $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$. Then $\pi_1(p): \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.

Proof. Fix some loop $\tilde{f}_0: I \rightarrow \tilde{X}$ in the kernel of $\pi_1(p)$. Then there is a homotopy $f_\bullet: I \times I \rightarrow X$ from \tilde{f}_0 to the constant path, which by Proposition 1.94 will lift uniquely to a homotopy $\tilde{f}_\bullet: I \times I \rightarrow \tilde{X}$ agreeing on \tilde{f}_0 . Now $p \circ \tilde{f}_1$ is constant, so looking locally at \tilde{x}_0 , we conclude that \tilde{f}_1 is constant, so \tilde{f}_0 is homotopic to the constant map and hence vanishes in $\pi_1(\tilde{X}, \tilde{x}_0)$. ■

1.8 September 19

Today we continue discussing covering spaces.

1.8.1 Using Path-Lifting

Last time we showed that covering space maps $(\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ induce subgroups $\pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$. Note this subgroup can then communicate information about the covering space.

Proposition 1.96. Fix a covering space $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ of path-connected spaces. Then the number of sheets covering an evenly covered neighborhood of x_0 is the index

$$\left[\pi_1(X, x_0) : \pi_1(\tilde{X}, \tilde{x}_0) \right],$$

where we have implicitly embedded $\pi_1(\tilde{X}, \tilde{x}_0) \hookrightarrow \pi_1(X, x_0)$.

Remark 1.97. Because X is connected, the number of sheets of the covering space map is well-defined. Indeed, for any positive integer n , the set of $x \in X$ such that there is an n -sheeted evenly covered open neighborhood $U_x \subseteq X$ is open. So we produce a continuous map $X \rightarrow \mathbb{N}$ sending x to the number of sheets, so connectedness of X forces the number of sheets to be constant.

Proof. We roughly describe the idea. Let $\Omega(Y, y_1, y_2)$ denote the set of homotopy classes of paths from y_1 to y_2 . The point is that $\Omega(X, x_0, x_0)$ is in bijection with

$$\bigsqcup_{\tilde{x} \in p^{-1}(\{x_0\})} \Omega(\tilde{X}, \tilde{x}_0, \tilde{x})$$

by lifting paths. Now, $\pi_1(\tilde{X}, \tilde{x}_0)$ acts on $\Omega(\tilde{X}, \tilde{x}_0, \tilde{x})$ for each \tilde{x} , and each orbit will correspond to a coset of our quotient. ■

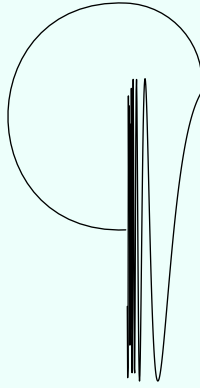
Remark 1.98. Proposition 1.96 can help us check that the covers of Example 1.90 are 2-to-1. For example, the subgroup corresponding to the shown covering space is $\langle a, b^2, bab \rangle$. Note that we have produced the free group with free generators as a subgroup of the free group with two generators.

We would like to go in the other direction, from subgroups back to covering space maps. This requires some technical hypotheses.

Definition 1.99 (locally path-connected). A topological space X is *locally path-connected* if and only if each open neighborhood $U \subseteq X$ of a point $x \in X$ has some perhaps smaller open neighborhood $U' \subseteq U$ of $x \in X$ which is path-connected.

Example 1.100. CW complexes are locally path-connected.

Non-Example 1.101. The topologist's sin curve is not locally path-connected at the origin $(0, 0)$.



Being locally path-connected allows us to lift covering spaces.

Proposition 1.102. Fix a path-connected, locally path-connected topological space Y with basepoint $y_0 \in Y$. For a covering space $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ and continuous map $f: (Y, y_0) \rightarrow (X, x_0)$, there is a lift $\tilde{f}: (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ making the following diagram commute if and only if $\pi_1(f)(\pi_1(Y, y_0)) \subseteq \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}_0))$.

$$\begin{array}{ccc} (Y, y_0) & \xrightarrow{\tilde{f}} & (\tilde{X}, \tilde{x}_0) \\ & \searrow f & \downarrow p \\ & & (X, x_0) \end{array}$$

Proof. The backwards direction follows from functoriality of π_1 because we are asking for $\pi_1(f) = \pi_1(p) \circ \pi_1(\tilde{f})$. For any $y \in Y$, composition with f defines a composite

$$\bigsqcup_{y \in Y} \Omega(Y, y_0, y) \rightarrow \bigsqcup_{x \in X} \Omega(X, x_0, x) \rightarrow \bigsqcup_{\tilde{x} \in \tilde{X}} \Omega(\tilde{X}, \tilde{x}_0, \tilde{x})$$

where the last map is by path-lifting $\tilde{\cdot}$. Then for any path $\gamma \in \Omega(Y, y_0, y)$, we simply define $\tilde{f}(y) := \tilde{f \circ \gamma}(1)$. To see that this is well-defined, the point is that choosing a different path $\gamma' \in \Omega(Y, y_0, y)$ produces a loop $\gamma \cdot \gamma'^{-1}$ based at y , which is able to lift to a loop upstairs in \tilde{X} , so the value of $\tilde{f}(y)$ does not move.

For continuity, we will need to use that Y is locally path-connected. Fix $y \in Y$, and we will show that \tilde{f} is continuous at y . Set $x := f(y)$, and let U be an evenly covered neighborhood of x , and lift it to $\tilde{U} \subseteq \tilde{X}$ where $p: \tilde{U} \rightarrow U$ is a homeomorphism, and $\tilde{f}(y) \in \tilde{U}$. We now may choose a path-connected open subset $V \subseteq f^{-1}(U)$ containing y and check continuity using \tilde{U} , where $\tilde{f}(y')$ for any $y' \in V$ can be somewhat easily defined because V is path-connected. ■

In fact, we have uniqueness of this lifting.

Proposition 1.103. Fix a connected topological space Y , and fix a covering space map $p: \tilde{X} \rightarrow X$ and a map $f: Y \rightarrow X$. Given lifts $\tilde{f}_1, \tilde{f}_2: Y \rightarrow \tilde{X}$ such that $p \circ \tilde{f}_1 = f = p \circ \tilde{f}_2$ and \tilde{f}_1 and \tilde{f}_2 agree at a single point, we have $\tilde{f}_1 = \tilde{f}_2$.

Proof. Define the subsets

$$E := \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\} \quad \text{and} \quad N := \{y \in Y : \tilde{f}_1(y) \neq \tilde{f}_2(y)\}.$$

One can use the covering space decomposition (by looking locally at $f(y)$ for some $y \in Y$) to show that both E and N are open, but they are disjoint with E nonempty, so connectedness of Y forces $Y = E$. ■

1.8.2 Classifying Covering Spaces

Our goal, roughly speaking, is to construct universal covers.

Definition 1.104 (universal cover). A covering space map $(\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a *universal cover* if and only if \tilde{X} is simply-connected (i.e., path-connected and $\pi_1(\tilde{X}, \tilde{x}_0) = 1$).

Remark 1.105. Proposition 1.102 tells us that a universal cover \tilde{X} will cover any covering space of X .

We will want yet another definition.

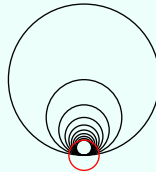
Definition 1.106 (semilocally simply-connected). A space X is *semilocally simply-connected* if and only if each $x \in X$ has an open neighborhood U of x such that the induced inclusion $\pi_1(U, x) \rightarrow \pi_1(X, x)$ is the trivial map.

Remark 1.107. Let's explain this condition. Suppose $(\tilde{X}, \tilde{x}_0) \rightarrow (X, x)$ is a simply connected and path-connected covering space. Then any evenly covered subset $U \subseteq X$ with lift \tilde{U} , then the inclusion $\pi_1(U) \rightarrow \pi_1(X)$ decomposes as

$$\pi_1(U) \rightarrow \pi_1(\tilde{U}) \rightarrow \pi_1(\tilde{X}) \rightarrow \pi_1(X),$$

which must be the trivial map because $\pi_1(\tilde{X}) = 1$. In other words, we have checked that X is semilocally simply-connected.

Example 1.108. The earring space is not semilocally simply-connected at the origin because any neighborhood at the origin will have circles inside.



Being semilocally simply-connected is basically, then, the right hypothesis to have a universal cover.

Theorem 1.109. Let X be a topological space which is path-connected, locally path-connected, and semi-locally simply-connected. Then X has a simply-connected covering space $\tilde{X} \rightarrow X$ which is unique up to isomorphism of pointed topological spaces over X .

Proof. Uniqueness follows from Proposition 1.102 because the corresponding lifts we write down must be local homeomorphisms.

It remains to show existence. Fix a basepoint $x_0 \in X$. We simply define

$$\tilde{X} := \{[\gamma] : \gamma \text{ is a path } I \rightarrow X \text{ with } \gamma(0) = x_0\}.$$

The point is that paths should lift uniquely up to \tilde{X} already, so we might as well define \tilde{X} in this way. We may define the function $p: \tilde{X} \rightarrow X$ by sending $[\gamma] \mapsto \gamma(1)$. It remains to show that \tilde{X} is a simply-connected topological space and that p is a covering space map.

Let's produce a topology on \tilde{X} . Using our hypotheses on X , each $x \in X$ has a path-connected open neighborhood $V \subseteq X$ such that $\pi_1(V) \rightarrow \pi_1(X)$ is trivial. We then use V to define a subset around $[\gamma]$ with $\gamma(0) = x_0$ and $\gamma(1) = x$ by

$$\tilde{V} := \{[\gamma \cdot \gamma'] : \gamma' \text{ is a path } I \rightarrow V \text{ such that } \gamma'(0) = x_0 \text{ and } \gamma'(1) = y\}.$$

Now, \tilde{V} is in bijection with V by p , so we make the restricted map $p: \tilde{V} \rightarrow V$ a homeomorphism. One can check that the topology is well-defined and that p becomes a covering space map from this. ■

One can now use the universal cover to produce any covering space.

Theorem 1.110. Let X be a pointed topological space which is path-connected, locally path-connected, and semilocally simply-connected, and let $x_0 \in X$ be a basepoint. Then there is a bijection between pointed path-connected covering spaces $(Y, y_0) \rightarrow (X, x_0)$ and subgroups of $\pi_1(X, x_0)$. Unpointed covering space maps correspond to conjugacy classes of subgroups.

To produce the desired covering space given a subgroup, one repeats the proof of Theorem 1.109 by taking a quotient of the produced \tilde{X} . Then one shows that this is a bijection with some work.

Remark 1.111. One can also use permutations of the pre-image of a basepoint in order to describe our covering spaces. Namely, if $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering space, then any loop $[\alpha] \in \pi_1(X, x_0)$ will lift to a permutation of $p^{-1}(\{x_0\})$. Conversely, such automorphisms are able to produce an automorphism of the universal covering space $\tilde{X} \rightarrow \tilde{X}$. (On the level of paths, we send $[\gamma] \in \tilde{X}$ to $[\gamma \cdot \alpha]$. One can check that this is continuous with continuous inverse and thus a homeomorphism.)

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