250B: Commutative Algebra
For the Morbidly Curious

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Theme 1
Introduction to Dimension

In this sense the algebraic geometers have never left paradise: There is no snake (that is, Peano curve) in the garden.

—David Eisenbud

1.1 March 31

Welcome back from spring break.

1.1.1 Review

We start with a little review. Here was our central definition.

Definition 1.1. The *Krull dimension* of a ring R, denoted $\dim R$, is the supremum of the length r of a chain of distinct primes

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

Further, if $\mathfrak{p} \subseteq R$ is prime, we defined the codimension

$$\operatorname{codim} \mathfrak{p} := \dim R/\mathfrak{p}.$$

As an example of something we showed, $\dim R = 0$ if and only if R is Artinian. We also showed that integral extensions preserve dimension.

Proposition 1.2. Fix a ring homomorphism $\varphi : R \to S$ which makes S into an integral R-algebra. Then, for any $\mathfrak{p} \in \operatorname{Spec} R$ such that $\ker \varphi \subseteq \mathfrak{p}$, there exists $\mathfrak{q} \in \operatorname{Spec} S$ such that

$$\mathfrak{p} = \varphi^{-1}(\mathfrak{q}).$$

In fact, for any ideal $I \subseteq S$, we have $\dim S/I = \dim R/\varphi^{-1}(I)$.

And we also showed the principal ideal theorem.

Theorem 1.3. Fix a Noetherian ring R. Given $x \in R$, set $\mathfrak{p}$ to be a minimal prime over (x). Then

$$\operatorname{codim} \mathfrak{p} \leq 1.$$

We also generalized this to minimal primes over ideals I with s generators. In particular, if $\mathfrak{p} = (x_1, \ldots, x_r)$, then

$$\operatorname{codim} \mathfrak{p} \leq r.$$

We were even able to provide a converse.

Corollary 1.4. Fix $\mathfrak{p}$ a prime ideal of a ring R with codimension r. Then there are elements $x_1, \ldots, x_r$ such that $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_r)$.

Example 1.5. Given a polynomial ring $k[x_1, \ldots, x_n]$, fix $\mathfrak{p} := (x_1, \ldots, x_k)$. Then

$$\operatorname{codim} \mathfrak{p} = k,$$

which we could also see directly by constructing the chain

$$(0) \subsetneq (x_1) \subsetneq \cdots \subsetneq (x_1, \ldots, x_k) = \mathfrak{p}.$$

1.1.2 The Rank–Nullity Theorem

We now focus on local rings.

Proposition 1.6. Fix a local ring R with maximal ideal $\mathfrak{m}$. Then $\dim R$ is the minimal $d \in \mathbb{N}$ such that there exist generators $f_1, \ldots, f_d$ so that

$$\mathfrak{m}^n \subseteq (f_1, \ldots, f_d) \subseteq \mathfrak{m}$$

for some n.

Proof. Because $\mathfrak{m}$ is the unique maximal ideal, we have $\dim R = \operatorname{codim} \mathfrak{m}$. Now, in one direction, if

$$\mathfrak{m}^n \subseteq (f_1, \ldots, f_d) \subseteq \mathfrak{m},$$

for some n, we note that $\mathfrak{m}$ is the unique minimal prime over $(f_1, \ldots, f_d)$: indeed, $\mathfrak{m}/(f_1, \ldots, f_d)$ is a nilpotent ideal, so any prime between must be equal to $\mathfrak{m}$.

In the other direction, if $\mathfrak{m}$ is a minimal prime over $(f_1, \ldots, f_d)$, then the ring

$$R/(f_1, \ldots, f_d)$$

is local, and its only prime is $\mathfrak{m}/(f_1, \ldots, f_d)$ by the uniqueness, so $R/(f_1, \ldots, f_d)$ is Artinian, so the Jacobson radical

$$\mathfrak{m}/(f_1, \ldots, f_d)$$

is still nilpotent, so we get

$$\mathfrak{m}^n \subseteq (f_1, \ldots, f_d) \subseteq \mathfrak{m},$$

which is what we wanted. ■

Remark 1.7. There is a geometric meaning the discovered $(f_1, \ldots, f_d)$. Namely, if X is an algebraic set, and $P \in X$ is a point, then we can set $R := A(x)$ and $\mathfrak{p}$ to correspond to P. Then the functions $\{f_1, \ldots, f_d\}$ will be "local coordinates" in a neighborhood of P. Roughly speaking, these elements f_k have powers which go to 0, so they merely work in an infinitesimal neighborhood of P.

To set up our next result, we note that dimension is a somewhat general concept: for example, in Set, dimension should be the cardinality. As an example, given a map $\varphi : X \to B$, we can pick any point $p \in B$ and find that

$$\dim X \le \dim B + \dim \varphi^{-1}(p).$$

In our algebraic story, we are interested in families of varieties $X \to B$, which corresponds to ring maps $A(B) \to A(X)$. Still looking locally at the fiber at a point, we have the following.

Proposition 1.8. Fix two local rings R and S with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Now, given a map $\varphi : R \to S$ such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, we have

$$\dim S \le \dim R + \dim S/\mathfrak{n}S$$

In fact, if S is a flat as an R-module, then we have equality.

This is still approximately our geometric picture because $S/\mathfrak{m}S$ is the coordinate ring of the pre-image of the point $\mathfrak{m}$.

Proof. We use Proposition 1.6. For brevity, set $r := \dim R$ and $s := \dim S/\mathfrak{S}$. As such, Proposition 1.6 promises us elements $\{f_1, \ldots, f_r\} \subseteq \mathfrak{n}$ and some $p \in \mathbb{N}$ such that

$$\mathfrak{m}^p \subseteq (f_1, \ldots, f_r) \subseteq \mathfrak{m}.$$

Similarly, there exist $\{g_1, \ldots, g_s\} \subseteq \mathfrak{n}$ so that

$$\mathfrak{n}^q \subseteq (g_1, \ldots, g_s) + \mathfrak{m}S$$

so that

$$\mathfrak{n}^{pq} \subseteq \mathfrak{m}^p S + (g_1, \ldots, g_s) \subseteq (f_1, \ldots, f_r, g_1, \ldots, g_s) \subseteq \mathfrak{n}$$

by taking the pth power and applying the previous inequality. Thus, $\dim S = \operatorname{codim} \mathfrak{n} \le r + s$, which is what we wanted.

We now take S to be flat. Set $\mathfrak{q} \subseteq S$ to be a minimal prime over $\mathfrak{m}S$, and we note $s = \dim \mathfrak{q}$ because $\mathfrak{q}$ is minimal over $\mathfrak{m}S$. We do know that

$$\dim S \ge \dim \mathfrak{q} + \operatorname{codim} \mathfrak{q} = s + \operatorname{codim} \mathfrak{q},$$

so to get the result, we merely need to show $\operatorname{codim} \mathfrak{q} \ge r$. For this, we take the following lemma.

Lemma 1.9 (Going down)**.** Let S be a flat R-algebra by $\varphi : R \to S$. Further, fix $\mathfrak{p}' \subseteq \mathfrak{p}$ are prime ideals in R. If we have $\mathfrak{q} \subseteq S$ so that $\mathfrak{p} = \varphi^{-1}\mathfrak{q}$, then there is a $\mathfrak{q}'$ such that $\mathfrak{p}' = \varphi^{-1}\mathfrak{q}'$.

Pictorially, we are building $\mathfrak{q}'$ in the following diagram.

$$\begin{array}{ccccc} S & & \mathfrak{q}' & \subseteq & \mathfrak{q} \\ \uparrow{\scriptstyle\varphi} & & | & & | \\ R & & \mathfrak{p}' & \subseteq & \mathfrak{p} \end{array}$$

Anyways, here is the proof of the lemma.

Proof. For psychological reasons, we will replace R with $R/\mathfrak{p}'$ and S with $S/\mathfrak{p}'S$. We do note that $S/\mathfrak{p}'S := S \otimes_R R/\mathfrak{p}'$ must be flat over the R-algebra $R/\mathfrak{p}'$, as shown on the homework.

Thus, we may assume that R is a domain with $\mathfrak{p}' = (0)$. Our diagram is now as follows.

$$\begin{array}{ccccc} S & & \mathfrak{q}' & \subseteq & \mathfrak{q} \\ {\scriptstyle\varphi}\uparrow & & | & & | \\ R & & (0) & \subseteq & \mathfrak{p} \end{array}$$

As such, we define $\mathfrak{q}'$ to be a minimal prime over (0) inside $\mathfrak{q}$, which certainly exists by saying something about Zorn's lemma. It remains to show that

$$\varphi^{-1}(\mathfrak{q}) \stackrel{?}{=} (0).$$

Well, $\mathfrak{q}$ only has zero-divisors by minimality, but because S is flat (here is where we use the condition!), we have that the function $\mu_r : s \mapsto rs$ is injective for nonzero r, so $\varphi^{-1}(\mathfrak{q})$ had better not contain any nonzero element of R. So $\varphi^{-1}(\mathfrak{q}) = (0)$, finishing. ■

Now, note that the lemma finishes the theorem because $\operatorname{codim} \mathfrak{q}$ will be $\dim S/\mathfrak{q}$, which upon going down will be at least r by tracking a full chain of primes through. ■

Example 1.10. Consider $S := k[x,y,z,w]/(xw - yz)$ to be the ring of singular matrices, as an algebra over $R := k[x,y]$. Further, set $\mathfrak{n} := (x,y,z,w)$ so that $\mathfrak{q} = (x,y)$. We track through the dimensions of Proposition 1.8 at $\mathfrak{n}$.

Proof. Note that $\dim R_\mathfrak{m} = 2$ as a localization of a two-dimensional ring. Then we can compute

$$\dim S_\mathfrak{n}/\mathfrak{m}S_\mathfrak{n} = \dim k[z,w]_{(z,w)} = 2,$$

where we are using the fact that $(xw - yz)$ once localized will disappear as a condition. However, $\dim S = 3$, so $\dim S_\mathfrak{n} \le 3$, which is strictly less than $\dim R_\mathfrak{m} + \dim S_\mathfrak{n}/\mathfrak{m}S_\mathfrak{n} = 4$. ■

Remark 1.11. At a high level, we have codified the notion that flat means continuously varying fibers: the dimension of the fiber $S/\mathfrak{m}S$ must be $\dim S - \dim R$ and therefore is constant!

As a consequence, we show that dimension is preserved by completion.

Corollary 1.12. Fix a Noetherian local ring R with maximal ideal $\mathfrak{m}$. Then $\dim R = \dim \widehat{R}$.

Proof. We know that $\widehat{R}$ is flat as an R-algebra, via the natural map $R \to \widehat{R}$. Now, $\mathfrak{m}$ goes to $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ under this map, so we conclude from Proposition 1.8 that

$$\dim \widehat{R} = \dim R + \dim \widehat{R}/\widehat{\mathfrak{m}}.$$

But $\widehat{R}/\widehat{\mathfrak{m}} \cong R/\mathfrak{m}$ is a field, so it has dimension 0. This finishes. ■

And here is another corollary.

Proposition 1.13. Fix a Noetherian ring R. Then $\dim R[x] = \dim R + 1$.

Proof. We start by showing $\dim R[x] \geq \dim R + 1$. Well, taking $r := \dim R$, we pick up a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

Lifting these to $R[x]$, we get the chain

$$\mathfrak{p}_0 R[x] \subsetneq \mathfrak{p}_1 R[x] \subsetneq \cdots \subsetneq \mathfrak{p}_r R[x].$$

But now we can add $\mathfrak{p}_r R[x] + (x)$ to the end of this chain, which is prime by checking that the quotient $R[x]/(\mathfrak{p}_r R[x] + (x)) \cong R/\mathfrak{p}_r$ is an integral domain. In particular, $\mathfrak{p}_r R[x]$ contains no monic polynomials, so $\mathfrak{p}_r R[x] + (x)$ is indeed strictly bigger.

To finish, we show $\dim R[x] \leq \dim R + 1$. Well, pick up a maximal ideal $\mathfrak{n} \subseteq R[x]$ such that $\mathfrak{n}$ appears in the maximum chain, promising

$$\operatorname{codim} \mathfrak{n} = \dim R[x].$$

As such, we pull $\mathfrak{n}$ back to R as $\mathfrak{m} \subseteq R$. Localizing at $\mathfrak{m}$, we see

$$\dim R[x] = \dim R[x]_\mathfrak{n} \leq \dim R + \dim R[x]_\mathfrak{n}/\mathfrak{m}R[x]_\mathfrak{n},$$

where we have used Proposition 1.8. But now $\dim R[x]_\mathfrak{n}/\mathfrak{m}R[x]_\mathfrak{n} = 1$ because it is the dimension of a polynomial ring in one variable over a field. ■

Corollary 1.14. For a field k, we have $k[x_1, \ldots, x_n] = n$.

Proof. This follows from induction, starting with $\dim k = 0$ and incrementing by Proposition 1.13. ■

1.1.3 Dimension for Modules

We now add a notion of Krull dimension for modules.

Definition 1.15 (Krull dimension, modules)**.** Given an R-module M, we define the *dimension* of M as $\dim M := \operatorname{codim} \operatorname{Ann} M$.

This lets us intelligently talk about colength.

Definition 1.16 (Colength)**.** Fix an ideal $\mathfrak{q} \subseteq R$. Then $\mathfrak{q}$ *has finite colength* on a module M if and only if $M/\mathfrak{q}M$ has finite length.

We have the following check.

Lemma 1.17. Fix a local ring R with maximal ideal $\mathfrak{m}$. Then $\mathfrak{q}$ has finite colength if and only if there exists an integer n so that

$$\mathfrak{m}^n \subseteq \mathfrak{q} + \operatorname{Ann} M.$$

Proof. Note that $M/\mathfrak{m}^n M$ has finite length by Nakayama's lemma, so $M/\mathfrak{q}M$ having finite length needs to be measured as such somehow. ■

As usual, our notion of size behaves in short exact sequences.

Proposition 1.18. Fix a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0.$$

If $\mathfrak{q}$ has finite colength on A and C, then $\mathfrak{q}$ has finite colength on B.

Proof. Omitted. ■

And here are a few more results.

Proposition 1.19. Fix a local ring R with maximal ideal $\mathfrak{m}$ and an R-module M. Then $\dim M$ is equal to the minimal d such that $(f_1, \ldots, f_d)$ has finite colength on M.

Proof. Omitted. This follows from Lemma 1.17. ■

Proposition 1.20. Fix a local ring R with maximal ideal $\mathfrak{m}$ and an R-module M. Given $x \in \mathfrak{m}$, we have

$$\dim M/xM \geq \dim M - 1.$$

Proof. Use the previous proposition directly. Namely, if $d = \dim M/xM$, then Proposition 1.19 promises us an ideal $(f_1, \ldots, f_d)$ which has finite colength on M/xM. But then

$$(f_1, \ldots, f_d, x)$$

has finite colength over M, so $\dim M \leq \dim M/xM - 1$ by Proposition 1.19 again, which is what we wanted. ■

1.1.4 Regular Rings

We close class by defining regular.

Definition 1.21 (Regular)**.** Fix a local ring R with maximal ideal $\mathfrak{m}$, of dimension d. Then R is *regular* if and only if there exist elements $\{f_1, \ldots, f_d\} \subseteq R$ such that

$$\mathfrak{m} = (f_1, \ldots, f_d).$$

In particular, we know that $\mathfrak{m}$ is at least a minimal prime over such an ideal; regularity asserts us equality.

Remark 1.22. We have $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = d$ when R is regular, essentially by expanding out the definition and applying Nakayama's lemma.

Example 1.23. Fix $R := k[x_1, \ldots, x_d]_{\mathfrak{m}}$, where $\mathfrak{m} = (x_1, \ldots, x_d)$. Then R is regular.

Example 1.24. The completion $k[[x_1, \ldots, x_n]]$ is regular.

Example 1.25. Any local principal ideal domain R has dimension 1, and the maximal ideal also is generated by one element, so R is regular.

And here is a last result.

Proposition 1.26. A regular local ring is an integral domain.

Proof. We omit this proof. ■

List of Definitions