250B: Commutative Algebra

For the Morbidly Curious

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CONTENTS

THEME 1
INTRODUCTION TO DIMENSION

In this sense the algebraic geometers have never left paradise: There is no snake (that is, Peano curve) in the garden.

—David Eisenbud

1.1 April 12

We continue.

1.1.1 The Hilbert Function

Today we are discussing the Hilbert–Samuel function and its relation to dimension. Here is the main result for today.

Definition 1.1 (Hilbert function)**.** Fix a local Noetherian ring R with unique maximal ideal $\mathfrak{m}$. Let $\kappa := R/\mathfrak{m}$ be the residue class field. Then the function

$$H_R(n) := \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

is called the *Hilbert function* of R.

Theorem 1.2. Fix a local Noetherian ring R with unique maximal ideal $\mathfrak{m}$. Let $\kappa := R/\mathfrak{m}$ be the residue class field. Then the Hilbert function of R agrees with a polynomial $P_R(n)$ for sufficiently large n, and $\dim R = 1 + \deg P_R$.

As such, we have the following definition.

Definition 1.3 (Hilbert polynomial)**.** Fix a local Noetherian ring R. Then the polynomial P_R which agrees with the Hilbert function H_R is called the *Hilbert polynomial*.

Remark 1.4. If R is not local, then we could get different dimensions out of $H_R(M)$, but we want to talk about $\dim R$ "globally." As such, we need the ring to be local.

Example 1.5. Fix $R := k[x_1, \ldots, x_r]_{(x_1,\ldots,x_r)}$ which has maximal ideal $\mathfrak{m} := (x_1, \ldots, x_r)$. Then

$$H_R(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

Another way to view this as is $\operatorname{gr}_\mathfrak{m} R$ is isomorphic to $k[x_1, \ldots, x_r]$. As such, we are counting the number of monomials of degree n in r variables, which is

$$P_R(n) := \binom{n+r-1}{r-1}$$

by a counting argument: from $n+r-1$ slots, choose $r-1$ dividers, which uniquely determines a tuple of nonnegative integers which sum to n. As such, we see that $\deg P_R(n) = r-1 = \dim R - 1$, which is what we wanted.

Example 1.6. If $\dim R = 0$, then R is Artinian, so the filtration

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \cdots$$

must stabilize, so $\mathfrak{m}^{n+1} = \mathfrak{m}^n$ for sufficiently large n. As such, we see that $H_R(n) = 0$ for sufficiently large n, so $P_R \equiv 0$. As such, we set by convention $\deg P_R = \deg 0 = -1$ to agree with $\dim R = 0$.

1.1.2 The Hilbert–Samuel Function

To prove Theorem 1.2, we work in higher generality. First, we will replace R with a finitely generated module; second, we will replace $\mathfrak{m}$ by a more arbitrary ideal. To start, recall the following definitions.

Definition 1.7 (Krull dimension, modules)**.** Fix a finitely generated module M over a Noetherian ring R. Then we define the *dimension*

$$\dim M := \dim R/\operatorname{Ann} M.$$

Definition 1.8 (Finite colength)**.** Fix a finitely generated module M over a Noetherian ring R. Then an ideal $\mathfrak{q} \subseteq R$ is of *finite colength* if and only if $\ell(M/\mathfrak{q}M) < \infty$.

For example, if M is a faithful module (i.e., with trivial annihilator), then there exists d such that

$$\mathfrak{m}^d \subseteq \mathfrak{q} \subseteq \mathfrak{m},$$

where $\mathfrak{q}$ can be generated by $\dim M$ total elements, by the Principal ideal theorem. More generally, if we first mod out R by $\operatorname{Ann} M$, we can say that

$$\mathfrak{m}^d \subseteq \mathfrak{q} + \operatorname{Ann} M \subseteq \mathfrak{m}.$$

As such, we take the following definition.

Definition 1.9 (Hibert–Samuel function)**.** Fix a local Noetherian ring R and a finitely generated R-module M with $\mathfrak{q}$ some prime of finite colength. Then we define the *Hilbert–Samuel function* by

$$H_{\mathfrak{q},M}(n) := \ell\left(\mathfrak{q}^n M/\mathfrak{q}^{n+1}\right).$$

We start by checking that this is well-defined.

Lemma 1.10. The value $\ell\left(\mathfrak{q}^n M/\mathfrak{q}^{n+1} M\right)$ is finite.

Proof. Without loss of generality, we know immediately that M is faithful (by first modding out by $\operatorname{Ann} M$). We start by noting that $M/\mathfrak{q}M$ has finite length by hypothesis on $\mathfrak{q}$. Now, $R/\mathfrak{q}$ embeds into $\operatorname{End}_R(M/\mathfrak{q}M)$, the latter of which is finite length because $M/\mathfrak{q}M$ is of finite length, so we conclude that $R/\mathfrak{q}$ is of finite length and in particular Artinian. It follows that $\mathfrak{q}^n M/\mathfrak{q}^{n+1}M$, which is finitely generated over $R/\mathfrak{q}$, must also be Artinian and in particular of finite length because everything involved is Noetherian. ■

The point is that, provided that our module is faithful, we see that we get to replace $\mathfrak{m}$ with any ideal containing some power of $\mathfrak{m}$.

Remark 1.11. We can replace R with $\operatorname{gr}_{\mathfrak{q}} R$ and M with $\operatorname{gr}_{\mathfrak{q}} M$.

1.1.3 Finite Differences

We now have a short digression into finite differences.

Definition 1.12 (Discrete derivative)**.** Given a function $f : \mathbb{N} \to \mathbb{C}$, we define the *discrete derivative*

$$\delta(f) := f(n+1) - f(n).$$

We have the following result.

Lemma 1.13. Suppose that we have some $f : \mathbb{N} \to \mathbb{C}$ such that $\delta(f)$ is a polynomial of degree d, for sufficiently large n. Then f is a polynomial of degree $d+1$, for sufficiently large n.

Proof. By shifting, we may assume that $\delta(f)$ is a polynomial of degree d. Now, note that the functions

$$\binom{n}{k}$$

form a basis of the set of polynomials $\mathbb{N} \to \mathbb{C}$. In fact,

$$\delta\left(\binom{n}{k}\right) = \binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1},$$

so δ is very well-behaved here. As such, writing

$$\delta(f)(n) = \sum_{k=0}^{d} a_k \binom{n}{k},$$

we can use our evaluation of δ on the binomials to read back the coefficients of f. ■

1.1.4 The Hilbert–Samuel Polynomial

And so ends our intermission. Here is a proposition.

Proposition 1.14. Fix a finitely generated module M over a local Noetherian ring R. Given an ideal $\mathfrak{q} = (x_1, \ldots, x_r)$ of finite colength on M, we have the following.

(a) The function $H_{\mathfrak{q},M}(n)$ agrees with a polynomial $P_{\mathfrak{q},M}$ for sufficiently large n.

(b) $\deg P_{\mathfrak{q},M} \leq r$.

Proof. We induct on r. The point is that we can apply an inductive hypothesis to M/x_1M so that $\mathfrak{q}' := (x_2, \ldots, x_r)$ has finite colength on M/x_1M. As such, we have the following exact sequence.

$$0 \to \ker x_1 \to M \stackrel{x_1}{\to} M(1) \to (\operatorname{coker} x_1)(1) \to 0.$$

Notably, we are using $M(1)$ (which is the twist of M by $M(1)_n := M_{n+1}$) by reducing to the graded case where $\operatorname{gr}_\mathfrak{q} M \mapsto M$ and $\operatorname{gr}_\mathfrak{q} R \mapsto R$. Taking the length everywhere in the nth component, we find that

$$H_{\mathfrak{q},\ker x_1}(n) - H_{\mathfrak{q},M}(n) + H_{\mathfrak{q},M(1)}(n) - H_{\mathfrak{q},\operatorname{coker} x_1(1)}(n) = 0.$$

Applying the shifting, we see that

$$\delta(H_{\mathfrak{q},M})(n) = H_{\mathfrak{q},M}(n+1) - H_{\mathfrak{q},M}(n) = H_{\mathfrak{q},\operatorname{coker} x_1}(n+1) - H_{\mathfrak{q},\ker x_1}(n).$$

Now, both $\operatorname{coker} x_1 = M/x_1M$ and $H_{\mathfrak{q},\ker x_1}$ will have degree at most $r-1$ by the inductive hypothesis, so we are done by Lemma 1.13. ∎

To prove Theorem 1.2, we will need to be a little more careful in the above argument. We start by keeping track of the degree in short exact sequences.

Lemma 1.15. Fix a local Noetherian ring R. Given a short exact sequence of finitely generated modules

$$0 \to A \to B \to C \to 0,$$

Then

$$P_{\mathfrak{q},B}(n) = P_{\mathfrak{q},A}(n) + P_{\mathfrak{q},C}(n) - F,$$

where F is some polynomial of degree strictly less than $\deg P_{\mathfrak{q},A}(n)$. In fact, the coefficients of F are all positive.

Remark 1.16. The main idea here is to generalize the fact that we get an exact equality when we are looking at just lengths.

Proof. We construct an auxiliary function

$$L_{\mathfrak{q},M}(n) := \ell\left(M/\mathfrak{q}^nM\right) = \sum_{i=0}^{n-1} H_{\mathfrak{q},M}(i)$$

to more easily keep track of the length in our filtration. In particular, $\delta(L_{\mathfrak{q},M}) = H_{\mathfrak{q},M}$, so $\deg L_{\mathfrak{q},M} = 1 + \deg H_{\mathfrak{q},M}$, assuming things are nonzero. Now, we would like to quotient our short exact sequence by $\mathfrak{q}^nB$, but we cannot do that because that doesn't preserve exactness. So we instead write

$$0 \to (A \cap \mathfrak{q}^nB)/\mathfrak{q}^nA \to A/\mathfrak{q}^nA \to B/\mathfrak{q}^nB \to C/\mathfrak{q}^nC \to 0.$$

As such, we see that

$$L_{\mathfrak{q},B}(n) = L_{\mathfrak{q},A}(n) + L_{\mathfrak{q},C}(n) - \ell\left(\frac{A \cap \mathfrak{q}^nM}{\mathfrak{q}^nA}\right).$$

We would like to understand the object $\frac{A\cap\mathfrak{q}^nM}{\mathfrak{q}^nA}$, for which we use the Artin–Rees lemma. Recall the statement.

Theorem 1.17. Fix R a Noetherian ring and $I \subseteq R$ an ideal with M a finitely generated R-module granted a stable I-filtration $\mathcal{J}$ denoted by

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots.$$

Then given a submodule $M' \subseteq M$, the induced filtration by $M'_k := M_k \cap M'$ is also a stable I-filtration.

In particular, we see that the $\mathfrak{q}$-filtration on B induces a $\mathfrak{q}$-stable filtration on A. In other words, there is an m so that $n \geq m$ will have

$$A \cap \mathfrak{q}^n B = \mathfrak{q}^{n-m}\left(A \cap \mathfrak{q}^n B\right) = \mathfrak{q}^{n-m} A,$$

so the length

$$\ell\left(\frac{A \cap \mathfrak{q}^n M}{\mathfrak{q}^n A}\right) \leq L_{\mathfrak{q},A}(n) - L_{\mathfrak{q},A}(n-m),$$

which agrees with a polynomial of smaller degree, so we are done because F is a polynomial for free as it is the difference of polynomials. ■

And here is our theorem.

Theorem 1.18. Fix a local Noetherian ring R with unique maximal ideal $\mathfrak{m}$. Further, take a finitely generated module M and an ideal $\mathfrak{q}$ of finite colength on M. Then

$$\dim M = 1 + \deg P_{\mathfrak{q},M}.$$

Proof. The proof, like the original Star Wars, comes in three parts.

1. We show that $\deg P_{\mathfrak{q},M}$ does not depend on $\mathfrak{q}$. Being finite colength means that we can write

   $$\mathfrak{m}^d \subseteq \mathfrak{q} + \operatorname{Ann} M \subseteq \mathfrak{m}$$

   for each d. This implies that

   $$H\mathfrak{m}, M(n) \leq H_{\mathfrak{q},M}(n) \leq H\mathfrak{m}, M(dn),$$

   but the Hilbert polynomials on the left and right have the same degree.

2. We show $1 + \deg P_{\mathfrak{q},M} \leq \dim M$. By modding out by $\operatorname{Ann} M$ everywhere, we may assume that M is faithful, meaning $\dim M = \dim R$. For brevity, set $\dim M := r$ so that we can choose $\mathfrak{q}$ so that

   $$\mathfrak{m}^d \subseteq \mathfrak{q} \subseteq \mathfrak{m}$$

   to have r generators, by the Principal ideal theorem. So we are done by Proposition 1.14.

3. We show $1 + \deg P_{\mathfrak{q},M} \geq \dim M$. Again, modding out by $\operatorname{Ann} M$ everywhere lets us assume that M is faithful, giving $\dim M = \dim R$.

   Now, choose $\mathfrak{p}$ to be a prime associated to M so that $\dim M = \dim R/\mathfrak{p}$. In practice, this means that $\mathfrak{p}$ is minimal over (0) to minimize $\dim R/\mathfrak{p}$. Now, if M has dimension zero, then we are done by Example 1.6. Otherwise, $\mathfrak{q} \supsetneq \mathfrak{p}$, so we may find $x \in \mathfrak{q} \notin \mathfrak{p}$.

   Further, note that x can be chosen to not be a zero-divisor, yielding

   $$0 \to M \xrightarrow{x} M \to M/xM \to 0.$$

   In particular, Lemma 1.15 tells us that

   $$P_{\mathfrak{q},M} = P_{\mathfrak{q},M} + P_{\mathfrak{q},M/xM} - F.$$

   We now appeal to the following lemma to give $\deg P_{\mathfrak{q},M/xM} < \deg P_{\mathfrak{q},M} \leq \dim M$ exactly.

Lemma 1.19. If M is a finitely generated R-module with $x \in \mathfrak{m}$, then

$$\dim M/xM \geq \dim M - 1.$$

Thus, we get $\deg P_{\mathfrak{q},M/xM} = \dim M - 1$, so we are done by an induction on M, from this last statement.

The above steps finish the proof. ■

Corollary 1.20. Fix a local Noetherian ring R and a finitely generated module M. Then $\dim M = \dim \widehat{M}$. In particular, $\dim R = \dim \widehat{R}$.

Proof. This follows from the fact that $P_R = P_{\widehat{R}}$ because $H_R = H_{\widehat{R}}$ because

$$\operatorname{gr}_{\mathfrak{m}} R = \operatorname{gr}_{\widehat{\mathfrak{m}}} \widehat{R},$$

so we are done. ■

1.1.5 An Example

We close class with an example.

Exercise 1.21 (Eisenbud 12.2)**.** Consider the ideal $I \subseteq k[x, y, z, w]$ generated by the 2×2 minors of

$$\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

In particular, $I := (xz - y^2, yw - z^2, xw - yz)$. We work out the Hilbert polynomial $P_{R,\mathfrak{q}}$ for $R = k[x, y, z, w]_{\mathfrak{m}}/I_{\mathfrak{m}}$, where $\mathfrak{m} = (x, y, z, w)$ and $\mathfrak{q} = (x, w)$.

Proof. We start by checking that $\mathfrak{q}$ is in fact of finite colength on R. Indeed, we are computing $\ell(R/\mathfrak{q})$, in which case (after taking the completion), we find

$$\ell(R/\mathfrak{q}) = \ell\left(\widehat{k[y,z]}/\left(y^2, z^2, yz\right)\right)$$

by sending z and w to 0. Because we can make three monomials, we see that this length is $3 < \infty$.

So we do indeed have a legitimate Hilbert function $H_{\mathfrak{q},R}$. The trick is to inject

$$R \to \widehat{k[s,t]}$$

by $x \mapsto s^3$ and $y \mapsto s^2 t$ and $z \mapsto st^2$ and $w \mapsto t^3$. We can check that this is an embedding. It follows that the image is all polynomials of degree divisible by 3, which for sufficiently large n agrees with a polynomial of degree 2 because we can compute directly as $3m + 1$ different monomials of prescribed degree. So our dimension comes out to be $\boxed{2}$. ■

LIST OF DEFINITIONS