

# 185: Introduction to Complex Analysis

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# THEME 1: INTRODUCTION

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## 1.1 January 19

It is reportedly close enough to start.

### 1.1.1 Logistics

We are online for the first two weeks, as with the rest of Berkeley. We will be using bCourses a lot, so check it frequently. There is also a website. There is a homework due on Friday, but do not worry about it.

Here are some syllabus things.

- Our main text is *Complex Variables and Applications, 8th Edition* because it is the version that Professor Morrow used. There is a free copy online.
- Homeworks are readings (for each course day) and weekly problem sets. Late homeworks are never accepted.
- Lowest two homework scores are dropped.
- There are two midterms and a final. The final is cumulative, as usual. The final can replace one midterm if the score is higher.
- Regrade requests can be made in GradeScope within one week of being graded.
- The class is curved but usually only curved at the end. The average on exams is expected to be 80%–83%.

### 1.1.2 Complex Numbers

Welcome to complex analysis. What does that mean?



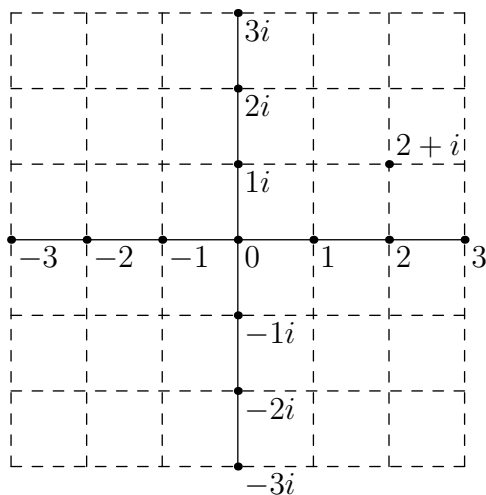
**Idea 1.1.** In complex analysis, we study functions  $f : \mathbb{C} \rightarrow \mathbb{C}$ , usually analytic to some extent.

There are two pieces here: we should study  $\mathbb{C}$  in themselves and then we will study the functions.

Complex  
numbers

**Definition 1.2** (Complex numbers). The set of complex numbers  $\mathbb{C}$  is  $\{a + bi : a, b \in \mathbb{R}\}$ , where  $i^2 = -1$ .

Hopefully  $\mathbb{R}$  is familiar from real analysis. As an aside, we see  $\mathbb{R} \subseteq \mathbb{C}$  because  $a = a + 0i \in \mathbb{C}$  for each  $a \in \mathbb{R}$ . The complex numbers have an inherent geometry as a two-dimensional plane.



The point is that  $\mathbb{C}$  looks like the real plane  $\mathbb{R}^2$ . More precisely,  $\mathbb{C} \cong \mathbb{R}^2$  as an  $\mathbb{R}$ -vector space, where our basis is  $\{1, i\}$ .

We would like to understand  $\mathbb{C}$  geometrically, “as a space.” The first step here is to create a notion of size.

Norm on  $\mathbb{C}$

**Definition 1.3** (Norm on  $\mathbb{C}$ ). We define the **norm map**  $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$  by  $|z| := \sqrt{z\bar{z}}$ . In other words,

$$|a + bi| := \sqrt{a^2 + b^2}.$$

Note that this agrees with the absolute value on  $\mathbb{R}$ : for  $a \in \mathbb{R}$ , we have  $\sqrt{a^2} = |a|$ .

Norm functions, as in the real case, give us a notion of distance.

Metric on  $\mathbb{C}$

**Definition 1.4** (Metric on  $\mathbb{C}$ ). We define the *metric on  $\mathbb{C}$*  to be  $d_{\mathbb{C}}(z_1, z_2) := |z_1 - z_2|$ .

One can check that this is in fact a metric, but we will not do so here.

**Remark 1.5.** The distance in  $\mathbb{C}$  is defined to match the distance in  $\mathbb{R}^2$  under the basis  $\{1, i\}$ .

Again as we discussed in real analysis, having a metric gives us a metric topology by open balls. Lastly it is this topology that our geometry will follow from: we have turned  $\mathbb{C}$  into a topological space.

### 1.1.3 Complex Functions

There are lots of functions on  $\mathbb{C}$ , and lots of them are terrible. So we would like to focus on functions with some structure. We’ll start with *continuous functions*, which are more or less the functions that respect topology.

Then from continuous functions, we will be able to define *holomorphic functions*, which are complex differentiable. This intended to be similar to being real differentiable, but complex differentiable turns out to be a very strong condition. Nevertheless, everyone’s favorite functions are holomorphic.

**Example 1.6.** Polynomials,  $\exp$ ,  $\sin$ , and  $\cos$  are all holomorphic.

To make concrete that complex differentiability is stronger than real differentiability, the Cauchy–Riemann equations which provides a partial differential equation to test complex differentiability.

From here we define *analytic functions*, which essentially are defined as taking the form

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Analytic functions are super nice in that we have an ability to physically write them down, so the following theorem is amazing.

**Theorem 1.7.** Holomorphic functions on  $\mathbb{C}$  are analytic.

To prove this, we will need the following result, which is what Professor Morrow calls the most fundamental result in complex analysis, the *Cauchy integral formula*.

In short, the Cauchy integral formula lets us talk about the value of holomorphic functions (and derivatives) at a point in terms of integrals around the point. This will essentially let us build the power series for a holomorphic function by hand. But as described, we will need a notion of complex (path) integration to even be able to talk about the Cauchy integral formula.

The Cauchy integral formula has lots of applications; for example, *Liouville's theorem* on holomorphic functions and the *Fundamental theorem of algebra*.

**Remark 1.8.** It is very hard to spell Liouville.

Additionally, we remark that our study of holomorphic functions, via the Cauchy integral formula, will boil down to a study of complex path integrals. So we will finish out our story with the *Residue theorem*, which provides a very convenient way to compute such integrals.

Then as a fun addendum, we talk about automorphisms of the complex numbers.

**Definition 1.9** (Automorphisms of  $\mathbb{C}$ ). A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is an *automorphism of  $\mathbb{C}$*  if it is bijective and both  $f$  and  $f^{-1}$  are holomorphic.

What is amazing is that all of these functions have a concrete description in terms of *Möbius transformations*.

### 1.1.4 Why Care?

Whenever taking a class, it is appropriate to ask why one should care. Here are some reasons to care.

- Algebraic geometry in its study of complex analytic spaces uses complex analysis.
- Analytic number theory (e.g., the Prime number theorem) makes heavy use of complex analysis.
- Combinatorics via generating functions can use complex analysis.
- Physics uses complex analysis.

The first two Professor Morrow is more familiar with, the last two less so.

## 1.2 January 21

We're reviewing set theory today.

### 1.2.1 Set Theory Notation

We have the following definitions.

- $\emptyset$  means the empty set.
- $a \in X$  means that  $a$  is an element of the set  $X$ .
- $A \subseteq B$  means that  $A$  is a subset of  $B$ .
- $A \subsetneq B$  means that  $A$  is a proper subset of  $B$ .

Automor-  
phisms of  $\mathbb{C}$

- $A \cup B$  consists of the elements which are in at least one of  $A$  or  $B$ .
- $A \cap B$  consists of the elements which are in both  $A$  and  $B$ .
- $A \setminus B$  consists of the elements of  $A$  which are not in  $B$ .
- Two sets  $A$  and  $B$  are *disjoint* if and only if  $A \cap B = \emptyset$ .
- Given a set  $X$ , we define  $\mathcal{P}(X)$  to be the set of all subsets of  $X$ .
- $|X| = \#X$  is the cardinality of  $X$ , or (roughly speaking) the number of elements of  $X$ .

As an example of unwinding notation, we have the following.

**Proposition 1.10** (De Morgan's Laws). Fix  $\mathcal{S} \subseteq \mathcal{P}(X)$  a collection of subsets of a set  $X$ . Then

$$X \setminus \bigcap_{S \in \mathcal{S}} S = \bigcup_{S \in \mathcal{S}} (X \setminus S) \quad \text{and} \quad X \setminus \bigcup_{S \in \mathcal{S}} S = \bigcap_{S \in \mathcal{S}} (X \setminus S).$$

*Proof.* We take these one at a time.

- Note  $a \in X \setminus \bigcap_{S \in \mathcal{S}} S$  if and only if  $a \in X$  and  $a \notin \bigcap_{S \in \mathcal{S}} S$ . However,  $a \notin \bigcap_{S \in \mathcal{S}} S$  is merely saying that  $a$  is not in all of the sets  $S \in \mathcal{S}$ , which is equivalent to saying  $a \notin S$  for one of the  $S \in \mathcal{S}$ .  
Thus, this is equivalent to saying  $a \in X$  while  $a \notin S$  for some  $S \in \mathcal{S}$ , which is equivalent to  $a \in \bigcup_{S \in \mathcal{S}} (X \setminus S)$ .
- Note  $a \in X \setminus \bigcup_{S \in \mathcal{S}} S$  if and only if  $a \in X$  and  $a \notin \bigcup_{S \in \mathcal{S}} S$ . However,  $a \notin \bigcup_{S \in \mathcal{S}} S$  is merely saying that  $a$  is not in any of the sets  $S \in \mathcal{S}$ , which is equivalent to saying  $a \notin S$  for each of the  $S \in \mathcal{S}$ .  
Thus, this is equivalent to saying  $a \in X$  while  $a \notin S$  for each  $S \in \mathcal{S}$ , which is equivalent to  $a \in \bigcap_{S \in \mathcal{S}} (X \setminus S)$ . ■

### 1.2.2 Some Conventions

In this class, we take the following names of standard sets.

- $\mathbb{N} = \{0, 1, 2, \dots\}$  is the set of natural numbers. Importantly,  $0 \in \mathbb{N}$ .
- $\mathbb{N}^+ = \{1, 2, 3, \dots\}$  is the set of positive integers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  is the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$  is the set of rationals.
- $\mathbb{R}$  is the set of real numbers. We will not specify a construction here; see any real analysis class.
- $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq 0\}$  is the nonzero real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$  is the positive real numbers.
- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$  is the nonnegative real numbers.
- $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$  is the nonpositive real numbers.
- $\mathbb{C}$  is the complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$  is the set of nonzero complex numbers.

### 1.2.3 Relations

Let's review some set theory definitions.

Cartesian  
product

**Definition 1.11** (Cartesian product). Given two sets  $A$  and  $B$ , we define the *Cartesian product*  $A \times B$  to be the set of ordered pairs  $(a, b)$  such that  $a \in A$  and  $b \in B$ .

Binary  
relation

**Definition 1.12** (Binary relation). A *binary relation* on  $A$  is any subset  $R \subseteq A^2 := A \times A$ . We may sometimes notate  $(x, y) \in R$  by  $xRy$ , read as " $x$  is related to  $y$ ."

**Example 1.13.** Equality is a binary relation on any set  $A$ ; namely, it is the subset  $\{(a, a) : a \in A\}$ .

The best relations are equivalence relations.

Equivalence  
relation

**Definition 1.14** (Equivalence relation). An *equivalence relation* on  $A$  is a binary relation  $R$  satisfying the following three conditions.

- Reflexive: each  $x \in A$  has  $(x, x) \in R$ .
- Symmetric: each  $x, y \in A$  has  $(x, y) \in R$  implies  $(y, x) \in R$ .
- Transitive: each  $x, y, z \in A$  has  $(x, y) \in R$  and  $(y, z) \in R$  implies  $(x, z) \in R$ .

Equivalence relations are nice because they allow us to partition the set into "equivalence classes."

Equivalence  
class

**Definition 1.15** (Equivalence class). Fix  $A$  a set and  $R \subseteq A^2$  an equivalence relation. Then, for given  $x \in A$ , we define

$$[x]_R := \{y \in A : (x, y) \in R\}$$

to be the *equivalence class* of  $x$ .

The hope is that equivalence classes partition the set. What is a partition?

Partition

**Definition 1.16** (Partition). A *partition* of a set  $A$  is a collection of nonempty subsets  $S \subseteq \mathcal{P}(A)$  of  $A$  such that any two distinct  $S_1, S_2 \in \mathcal{S}$  are disjoint while  $A = \bigcup_{S \in \mathcal{S}} S$ .

And now let's manifest our hope.

**Lemma 1.17.** Equivalence relations are in one-to-one correspondence with partitions of  $A$ .

*Proof.* Given an equivalence relation  $R$ , we define the collection

$$\mathcal{S}(R) = \{[x]_R : x \in A\}.$$

We claim that  $R \mapsto \mathcal{S}(R)$  is our needed bijection. We have the following checks.

- Well-defined: observe that  $\mathcal{S}(R)$  does partition  $A$ : if we have  $[x]_R, [y]_R \in \mathcal{S}$ , then  $[x]_R \cap [y]_R \neq \emptyset$  implies there is some  $z$  with  $(x, z) \in R$  and  $(z, y) \in R$ , so  $x \in [y]_R$  and then  $[x]_R \subseteq [y]_R$  follows. So by symmetry,  $[y]_R \subseteq [x]_R$  as well, so we finish the disjointness check.

Further, we see that

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} [x]_R \subseteq A$$

because  $x \in [x]_R$ , so indeed the equivalence classes cover  $A$ .

- **Injective:** suppose  $R_1$  and  $R_2$  have  $\mathcal{S}(R_1) = \mathcal{S}(R_2)$ . We show that  $R_1 \subseteq R_2$ , and  $R_2 \subseteq R_1$  will follow by symmetry, finishing.

We notice that, for any  $\mathcal{S}$  partitioning  $A$ , being a partition, will have exactly one subset which contains  $x$ . But for  $\mathcal{S}(R)$  for an equivalence relation  $R$ , we see  $x \in [x]_R \in \mathcal{S}(R)$ , so this equivalence class must be the one.

So because  $[x]_{R_1}$  and  $[x]_{R_2}$  are the only subsets of  $\mathcal{S}(R_1)$  and  $\mathcal{S}(R_2)$  containing  $x$  (respectively), we must have  $[x]_{R_1} = [x]_{R_2}$ . Thus,  $(x, y) \in R_1$  implies  $y \in [x]_{R_1} = [x]_{R_2}$  implies  $(x, y) \in R_2$ .

- **Surjective:** suppose that  $\mathcal{S}$  is a partition of  $A$ . As noted above, each  $x \in A$  is a member of exactly one set  $S \in \mathcal{S}$ , which we call  $[x]$ . Then we define  $R \subseteq A^2$  by  $(x, y) \in R$  if and only if  $y \in [x]$ . One can check that this is an equivalence relation, which we will not do here in detail.<sup>1</sup>

The point is that

$$[x]_R = \{y : (x, y) \in R\} = \{y : y \in [x]\} = [x],$$

so  $\mathcal{S}(R) = \mathcal{S}$ . So our mapping is surjective. ■

We continue our discussion.

Quotient set

**Definition 1.18** (Quotient set). Given an equivalence relation  $R \subseteq A^2$ , we define the *quotient set*  $A/R$  is the set of equivalence classes of  $R$ . In other words,

$$A/R = \{[x]_R : x \in A\}.$$

Intuitively, the quotient set is the set where we have gone ahead and identified the elements which are "similar" or "related."

We would like a more concrete way to talk about equivalence classes, for which we have the following.

Representatives

**Definition 1.19** (Representatives). Given an equivalence relation  $R \subseteq A^2$ , we say that  $C \subseteq A$  is a *set of representatives of  $R$ -equivalence classes of  $A$*  if and only if  $C$  consists of exactly one element from each equivalence class in  $A/R$ .

## 1.2.4 Functions

To finish off, we discuss functions.

Functions

**Definition 1.20** (Functions). A *function*  $f : X \rightarrow Y$  is a relation  $f \subseteq X \times Y$  satisfying the following.

- For each  $x \in X$ , there is some  $y \in Y$  such that  $(x, y) \in f$ . Intuitively, each  $x \in X$  goes somewhere.
- For each  $x \in X$  and given some  $y_1, y_2 \in Y$  such that  $(x, y_1), (x, y_2) \in f$ , then  $y_1 = y_2$ . Intuitively, each  $x \in X$  goes to at most one place.

We will write  $f(x) = y$  as notational sugar for  $(x, y) \in f$ . Note this equality is legal because the value  $y$  with  $(x, y) \in f$  is uniquely given.

We would like to create new functions from old. Here are two ways to do this.

Restriction

**Definition 1.21** (Restriction). Given a function  $f : X \rightarrow Y$  and a subset  $A \subseteq X$ , we define

$$f|_A = \{(x, y) \in f : x \in A\} \subseteq A \times Y$$

to be a function  $f|_A : A \rightarrow Y$ .

<sup>1</sup> Note  $x \in [x]$  by definition of  $[x]$ . If  $y \in [x]$ , then note  $y \in [y]$  as well, so  $[x] = [y]$  is forced by uniqueness, so  $x \in [y]$ . If  $y \in [x]$  and  $z \in [y]$ , then again by uniqueness  $[x] = [y] = [z]$ , so  $z \in [x]$  follows.



We will not check that  $f|_A$  is actually a function; it is, roughly speaking inherited from  $f$ .

**Definition 1.22.** Given two functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , we define the *composition* of  $f$  and  $g$  to be some function  $g \circ f : X \rightarrow Z$  defined by

$$(g \circ f)(x) := g(f(x)).$$

Again, we will not check that this makes a function; it is.

Functions can also help create new sets.

Image

**Definition 1.23** (Image). Given a function  $f : X \rightarrow Y$ , we define the *image* of  $f$  to be

$$\text{im } f = f(X) := \{y \in Y : \text{there is } x \in X \text{ such that } f(x) = y\}.$$

Namely,  $\text{im } f$  consists of all elements hit by someone in  $X$  hit by  $f$ .

Fiber,  
pre-image

**Definition 1.24** (Fiber, pre-image). Given a function  $f : X \rightarrow Y$  and some  $y \in Y$ , we define the *fiber* of  $f$  over  $y$  to be

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

In general, we define the *pre-image* of a subset  $A \subseteq Y$  to be

$$f^{-1}(A) := \{x \in X : f(x) \in A\} = \bigcup_{a \in A} \{x \in X : f(x) = a\} = \bigcup_{a \in A} f^{-1}(a).$$

Some functions have nicer properties than others.

Inj-, sur-,  
bijective

**Definition 1.25** (Inj-, sur-, bijective). Fix a function  $f : X \rightarrow Y$ . We have the following.

- Then  $f$  is *injective* or *one-to-one* if and only if, given  $x_1, x_2 \in X$ ,  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ .
- Then  $f$  is *surjective* or *onto* if and only if  $\text{im } f = Y$ . In other words, for each  $y \in Y$ , there exists  $x \in X$  with  $f(x) = y$ .
- Then  $f$  is *bijective* if and only if it is both injective and surjective.

Here is an example.

Identity

**Definition 1.26** (Identity). For a given set  $X$ , the function  $\text{id}_X : X \rightarrow X$  defined by  $\text{id}_X(x) := x$  is called the *identity function*.

For completeness, here are the checks that  $\text{id}_X$  is bijective.

- Injective: given  $x_1, x_2 \in X$ , we see  $\text{id}_X(x_1) = \text{id}_X(x_2)$  implies  $x_1 = \text{id}_X(x_1) = \text{id}_X(x_2) = x_2$ .
- Surjective: given  $x \in X$ , we see that  $x \in \text{im } \text{id}_X$  because  $x = \text{id}_X(x)$ .

We leave with some lemmas, to be proven once in one's life.

**Lemma 1.27.** Fix a finite sets  $X$  and  $Y$  such that  $\#X = \#Y$ . Then a function  $f : X \rightarrow Y$  is bijective if and only if it is injective or surjective.

*Proof.* Certainly if  $f$  is bijective, then it is both injective and surjective, so there is nothing to say.

The reverse direction is harder. We proceed by induction on  $\#X = \#Y$ . If  $\#X = \#Y = 0$ , then  $X = Y = \emptyset$ , and all functions  $f : \emptyset \rightarrow \emptyset$  are vacuously bijective: for injective, note that any  $x_1, x_2 \in \emptyset$  have  $x_1 = x_2$ ; for surjective, note that any  $x \in \emptyset$  has  $f(x) = x$ .

Otherwise  $\#X = \#Y > 0$ . We have two cases.

- Take  $f$  injective; we show  $f$  is surjective. In this case,  $\#X > 0$ , so choose some  $a \in X$ . Note that  $x \in X$  with  $x \neq a$  will have  $f(x) \neq f(a)$  by injectivity, so we may define the restriction

$$f|_{X \setminus \{a\}} : X \setminus \{a\} \rightarrow Y \setminus \{f(a)\}.$$

Observe that  $f|_{X \setminus \{a\}}$  is injective because  $f$  is: if  $x_1, x_2 \in X \setminus \{a\}$  have

$$f(x_1) = f|_{X \setminus \{a\}}(x_1) = f|_{X \setminus \{a\}}(x_2) = f(x_2),$$

then  $x_1 = x_2$  follows.

Now,  $\#(X \setminus \{a\}) = \#(Y \setminus \{f(a)\}) = \#X - 1$ , so by induction  $f|_{X \setminus \{a\}}$  will be bijective because it is injective. In particular,  $f$  by way of  $f|_{X \setminus \{a\}}$  fully hits  $Y \setminus \{f(a)\}$  in its image, so because  $f(a) \in \text{im } f$  as well, we conclude  $\text{im } f = Y$ . So  $f$  is surjective.

- Take  $f$  surjective; we show  $f$  is injective. Define a function  $g : Y \rightarrow X$  as follows: for each  $y \in Y$ , the surjectivity of  $f$  promises some  $x \in X$  such that  $f(x) = y$ , so choose any such  $x$  and define  $g(y) := x$ .<sup>2</sup> Observe that  $f(g(y)) = y$  by construction.

Now, we notice that  $g$  is injective: if  $y_1, y_2 \in Y$  have  $g(y_1) = g(y_2)$ , then  $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$ . So the previous case tells us that  $g$  is in fact bijective.

So now choose any  $x_1, x_2 \in X$  such that  $f(x_1) = f(x_2)$ . The surjectivity of  $f$  promises some  $y_1, y_2 \in Y$  such that  $g(y_1) = x_1$  and  $g(y_2) = x_2$ , so we see that

$$x_1 = g(y_1) = g(f(g(y_1))) = g(f(x_1)) = g(f(x_2)) = g(f(g(y_2))) = g(y_2) = x_2,$$

proving our injectivity. ■

**Lemma 1.28.** Fix  $f : X \rightarrow Y$  a bijective function. Then there is a unique function  $g : Y \rightarrow X$  such that  $f \circ g = \text{id}_Y$  and  $g \circ f = \text{id}_X$ .

*Proof.* We show existence and uniqueness separately.

- We show existence. Note that, because  $f : X \rightarrow Y$  is surjective, each  $y \in Y$  has some  $x \in X$  such that  $f(x) = y$ . In fact, this  $x \in X$  is uniquely defined because  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ , so we may define  $g(y)$  as the value  $x$  for which  $f(x) = y$ .

By construction,  $f(g(y)) = y$ , so  $f \circ g = \text{id}_Y$ . Additionally, we note that, given any  $x \in X$ , the value  $x_0$  for which  $f(x) = f(x_0)$  is  $x = x_0$  by the injectivity, so  $g(f(x)) = x$ . Thus,  $g \circ f = \text{id}_X$ , as claimed.

- We show uniqueness. Suppose that we have two functions  $g_1, g_2 : Y \rightarrow X$  which satisfy

$$f \circ g_1 = f \circ g_2 = \text{id}_Y \quad \text{and} \quad g_1 \circ f = g_2 \circ f = \text{id}_X.$$

Then we see that

$$g_1 = g_1 \circ \text{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \text{id}_X \circ g_2 = g_2,$$

where we have used the fact that function composition associates. This finishes. ■

<sup>2</sup> Technically we are using the Axiom of Choice here. One can remove this with an induction because all sets are finite, but I won't bother.