256B: Algebraic Geometry

Nir Elber

Spring 2023

CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1
CURVES

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

1.1 January 18

Here we go.

1.1.1 House-Keeping

Here are some notes on the course.

- We will continue to use [Har77]. Note that [Vak17] is also popular, as is [SP].
- Office hours will probably be after class on Wednesday and Friday.
- There is a `bCourses`.
- In the course, we plan to cover curves, some coherent cohomology (and maybe on Zariski sheaves), and some surfaces if we have time.
- Grading will be homework and a term paper. Homework will be challenging, so collaboration is encouraged.

In this course, we will discuss coherent cohomology, but we will begin by talking about curves.

1.1.2 Serre Duality Primer

For the next few weeks, we will focus on non-singular curves over an algebraically closed field. Here is our definition.

Definition 1.1 (curve)**.** Fix a field k. A k-*curve* is an integral, proper, normal scheme of dimension 1. Note that being normal is equivalent to being smooth, so we are requiring our curves to be smooth!

We will want to talk about genus a little. Here is a working definition.

Definition 1.2 (arithmetic genus)**.** Fix a projective k-variety X. Then the *arithmetic genus* $p_a(X)$ is defined

Definition 1.3 (geometric genus)**.** Fix an irreducible k-variety X. Then the *geometric genus* is $p_g(X) := \dim_k \Gamma(X, \omega_X)$, where ω_X is the canonical sheaf. Explicitly, ω_X is the top exterior power of the sheaf of differential forms on X.

In general, the above notions are not the same, but they will be for curves.

Proposition 1.4. Fix a k-curve X. Then $p_g(X) = p_a(X)$.

We would like to actually compute some genera, but this is a bit difficult. One goal of the class is to build a cohomology theory $H^i(X, \mathcal{F})$ for coherent sheaves $\mathcal{F}$ on X, and it turns out we can use these cohomology groups to compute the genus of X. Roughly speaking, we will derive (on the right) the left-exact functor $\Gamma(X, \cdot)$, so the cohomology will in some sense measure the difference between global sections and local sections. For example, flasque sheaves will have trivial cohomology.

For now, we will black-box various things. Here is an example of something we will prove.

Proposition 1.5. Fix a projective k-variety X, and let $\mathcal{F}$ be a coherent sheaf. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$, and $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces for all $i \geq 0$.

To show the Riemann–Roch theorem, we will black-box Serre duality, which we will prove much later. In the case of curves, it says the following.

Theorem 1.6 (Serre duality)**.** Fix a k-curve X. Then, for any vector bundle $\mathcal{L}$ on X, there is a duality

$$H^i(X, \mathcal{L}^\vee \otimes \omega_X) \otimes_k H^{1-i}(X, \mathcal{L}) \to k,$$

where $i \in \{0, 1\}$.

Remark 1.7. Notably, we see $p_g(X) = \dim_k \Gamma(X, \omega_X) = \dim_k H^0(X, \omega_X) = \dim_k H^1(X, \mathcal{O}_X)$.

We will also want the following fact.

Proposition 1.8. Fix a closed embedding $i\colon X \to Y$ of schemes. Given a sheaf $\mathcal{F}$ of abelian groups on Y, then

$$H^i(X, i_*\mathcal{F}) = H^i(Y, \mathcal{F}).$$

1.1.3 Divisors Refresher

We also want to recall a few facts about divisors. We begin with Weil divisors.

Definition 1.9 (Weil divisor)**.** Fix an irreducible k-variety X. A *Weil divisor* $\operatorname{Div}(X)$ are $\mathbb{Z}$-linear combinations of codimension-1 irreducible closed subschemes. Then the *principal divisors* are the image of the map $\operatorname{div}\colon K(X) \to \operatorname{Div}(X)$, where div takes rational functions to poles. The *class group* $\operatorname{Cl} X$ is the quotient.

More generally, we have Cartier divisors.

Definition 1.10 (Cartier divisor)**.** Fix a scheme X. A *Cartier divisor* in $\operatorname{CaDiv} X$ is a global section of $\Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$, where $K^\times$ is the sheafification of the presheaf $U \mapsto \operatorname{Frac} \mathcal{O}_X(U)$. The *principal divisors* are the image of $\Gamma(X, \mathcal{K}^\times)$, and the *class group* $\operatorname{CaCl} X$ is the quotient.

Notably, if X is an integral sheaf, then $\mathcal{K}$ is the constant sheaf $K(X)$. Then a global section is given by the pair $(\{U_i\}, \{f_i\})$ where the U_i cover X, and $f_i \in K(X)^\times$ so that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^\times$. (The coherence condition allows the Cartier divisors to glue.) Notably, each $f \in K(X)$ grants a principal divisor $(\{X\}, \{f\})$, which are exactly the principal divisors.

Here is the main result on these divisors.

Proposition 1.11. If X is an integral, separated, Noetherian, and locally factorial (notably, regular in codimension 1), then Weil divisors are in canonical isomorphism with Cartier divisors. Further, the principal divisors are in correspondence, and so the class groups are also isomorphic.

Example 1.12. Non-singular k-curves have all the required adjectives. Namely, codimension-1 means we are looking at points, and being smooth implies being regular, so all the local rings are dimension-1 regular local rings, which are discrete valuation rings. Notably, discrete valuation rings are

Yet another connection to divisors comes from invertible sheaves. Namely, for integral schemes X, the group of invertible sheaves $\operatorname{Pic} X$ is isomorphic to $\operatorname{CaCl} X$. The point here is that invertible shaves can be embedded into $\mathcal{K}^\times$ when X is integral.

We will be interested in some special divisors.

Definition 1.13 (effective)**.** Fix a k-curve X. Then an *effective Weil divisor* is a $\mathbb{Z}_{\geq 0}$ linear combination of closed points of X. Note that the collection of effective Weil divisors forms a submonoid of $\operatorname{Div} X$. We might be interested in knowing how many effective divisors are equivalent to some given divisor; the set of these is denoted $|D|$.

When our schemes X have enough adjectives, we note that the above correspondences tell us that there is a way to send a Cartier divisor $(\{U_i\}, \{f_i\})$ to a line bundle $\mathcal{L}$ embedded in $\mathcal{K}^\times$. Explicitly, we build $\mathcal{L}(D)$ by $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_X|_{U_i} \subseteq \mathcal{K}$, where the last isomorphism is by sending $1 \mapsto f_i^{-1}$. Notably, if D is effective, then the global section 1 of $\mathcal{K}^\times$ can be pulled back along to a nonzero global section on $\mathcal{L}(D)$ which is f_i on each U_i.

1.2 January 20

We continue moving towards Riemann–Roch.

1.2.1 Linear Systems

Let's discuss linear systems. Let X be a non-singular projective irreducible variety over a field k, and let D be a divisor of X.

Recall that a Cartier divisor $D = \{(U_i, f_i)\}$ on X is associated to the line bundle $\mathcal{L}(D)$ which is locally trivial on each U_i, given as $f_i^{-1}\mathcal{O}_X|_{U_i}$. Conversely, suppose that $\mathcal{L}$ is a line bundle on X. Then we pick up some nonzero global section $\Gamma(X, \mathcal{L})$. Give $\mathcal{L}$ a trivializing open cover $\{U_i\}$, where we are given isomorphisms $\varphi_i \colon \mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$. Setting $f_i \coloneqq \varphi_i(s)$ recovers an (effective) Cartier divisor $\{(U_i, f_i)\}$ on X. We call this line bundle $\operatorname{div}(\mathcal{L}, s)$.

This thinking gives the following result.

Proposition 1.14. Let X be a non-singular projective integral variety over a field k. Given a Cartier divisor D_0, and let $\mathcal{L} := \mathcal{L}(D_0)$ be the corresponding line bundle.

(a) For each nonzero section $s \in \Gamma(X, \mathcal{L})$, the divisor $\operatorname{div}(\mathcal{L}, s)$ is an effective divisor linearly equivalent to D_0.

(b) Every effective divisor linearly equivalent tot D_0 is obtained in this way.

(c) If k is algebraically closed, we have $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s')$ if and only if s and s' differ by a scalar in $k^\times$.

The above result essentially says that we can study $\Gamma(X, \mathcal{L})$ as a k-vector space instead of trying to understand linear equivalence of divisors. For example, if $\Gamma(X, \mathcal{L}) = 0$, then D is not equivalent to any effective divisor!

Proof. We go one at a time.

(a) Embed $\mathcal{L} \subseteq \mathcal{K}_X$ as usual. Then $s \in \Gamma(X, \mathcal{L})$ becomes a rational function in $K(X)$. By the construction of $\mathcal{L}$, we have an open cover $\{U_i\}$ and some f_i so that $\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_X|_U$. Because we have a global section, we may write $\varphi_i(s) = f_i f$ for some fixed f, and then tracking through our Cartier divisor, we get
$$\operatorname{div}(\mathcal{L}, s) = D_0 + \operatorname{div}(f),$$
as needed.

(b) Suppose D is an effective divisor with $D = D_0 + \operatorname{div}(f)$. Then we see $(f) \geq -D_0$, so f determines a nonzero global section of $\mathcal{L}\mathcal{L}(D_0)$ by tracking through the above constructions: namely, set $s|_{U_i} = f_i^{-1} f$ and glue. (In particular, $(f) \geq -D_0$ means $f/f_i \in \mathcal{O}_X(U_i)$ for each i.) So we see $D = \operatorname{div}(\mathcal{L}, s)$.

(c) One can see directly that $s = cs'$ for $c \in k^\times$ will have $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s)$. Conversely, if $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s')$, then under the embedding $\mathcal{L} \subseteq \mathcal{K}_X$, we may correspond s and s' to $f, f' \in K(X)^\times$. Thus, $f/f' \in \Gamma(X, \mathcal{O}_X^\times)$. But because k is algebraically closed and X is proper over k, we have $\Gamma(X, \mathcal{O}_X) = k$, so we are done. ■

Remark 1.15. More generally, we have the following: let k be a field, and let X be a proper, geometrically reduced scheme over k. Then $\Gamma(X, \mathcal{O}_X) = k$ if and only if X is geometrically reduced.

So we have the following.

Corollary 1.16. Let X be a non-singular projective integral variety over a field k. The set $|D_0|$ of effective divisors linearly equivalent to a given divisor D_0 is in natural bijection with $(\Gamma(X, \mathcal{L}(D)) \setminus \{0\})/k^\times$.

With this in mind, we set the following notation.

Notation 1.17. Let X be a non-singular projective integral variety over a field k. Given a divisor D_0 of X, we define $\ell(D_0) := \dim_k \Gamma(X, \mathcal{L}(D))$ and $\dim D_0 := \ell(D_0) - 1$.

The Riemann–Roch theorem is interested in the values of $\ell(D_0)$. Here is a quick lemma.

Lemma 1.18. Let X be a non-singular projective integral variety over a field k. Fix a divisor D of X.

(a) If $\ell(D) \neq 0$, then $\deg D \geq 0$.

(b) If $\ell(D) \neq 0$ and $\deg D = 0$, then D is linearly equivalent to 0.

Proof. Note $\ell(D) \neq 0$ enforces $D \sim D_0$ for some effective divisor D, so $\deg D = \deg D_0 \geq 0$, which shows (a). Then for (b), we note $\deg D_0 = 0$ forces $D_0 = 0$. ■

1.2.2 Riemann–Roch for Curves

We now force $\dim X = 1$, meaning that X is a curve. Let $\Omega_{X/k}$ denote the sheaf of differentials, which is equal to the canonical sheaf $\omega_X = \bigwedge^{\dim X} \Omega_{X/k}$. Any divisor linearly equivalent to $\Omega_{X/k}$ will be denoted K and is called the "canonical divisor." Note that the canonical divisor is really a canonical divisor class.

Theorem 1.19 (Riemann–Roch)**.** Fix everything as above. Further, suppose k is algebraically closed. Then

$$\ell(D) - \ell(K - D) = \deg D - 1 + g,$$

where g is the genus.

Proof. Set $\mathcal{L} \coloneqq \mathcal{L}(D)$ for brevity. Note $\mathcal{L}(K - D) \cong \omega_X \otimes \mathcal{L}^\vee$, so Serre duality implies

$$\ell(K - D) = \dim_k \Gamma(\omega_X \otimes \mathcal{L}^\vee) = \dim_k H^1(X, \mathcal{L}).$$

Thus, our left-hand side is $\chi(\mathcal{L}) \coloneqq \dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L})$.[1] Quickly, note $D = 0$ can be seen directly by

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = \dim k - g = 1 - g,$$

which is what we wanted.

We now perturb D by a point. We show the formula holds for D if and only if the formula holds for $D + p$, where $p \in X$ is some closed point. Note we have a short exact sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{O}_X \to k(p) \to 0,$$

where $k(p)$ refers to the skyscraper sheaf which is the structure sheaf about p. Tensoring with $\mathcal{L}(D + p)$, we get

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + p) \to k(p) \to 0.$$

Now, χ is additive in short exact sequences by using the long exact sequence in cohomology, so

$$\chi(\mathcal{L}(D)) = \chi(\mathcal{L}(D + p)) + \chi(k(P)),$$

but $\chi(k(p)) = \dim_k \Gamma(X, k(p)) = \dim_k k = 1$ because k is algebraically closed. The conclusion now follows because $\deg(D + p) = \deg D + 1$. ■

[1] This is the Euler characteristic of $\mathcal{L}(D)$ because our higher cohomology groups vanish.

Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977.

[Kle16] Felix Klein. *Elementary Mathematics from a Higher Standpoint*. Trans. by Gert Schubring. Vol. II. Springer Berlin, Heidelberg, 2016.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Vak17] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. 2017. URL: http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.

[SP] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.

List of Definitions