215A: Algebraic Topology

Nir Elber

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1

Introduction

1.1 August 24

It begins.

1.1.1 Logistics

Here are the logistical notes.

- The professor is Ian Agol, whose office is Evans 921. Office hours are Tuesdays after class, Monday at 3PM, Wednesday at 9AM, or by appointment.
- There is a bCourses.
- Homework will be weekly, and it will make up the entire grade.
- The prerequisites are Math 113 and 202A or equivalent. From point-set topology in particular we will want notions of compactness, connectedness, metric spaces, and a few topologies like the identification topology with respect to a continuous map.

1.1.2 Overview

We will cover chapters 0–3 of [Hat01].

- Chapter 0 consists of "geometric notions." Particularly important are the notion of homotopy and CW complexes.
- Chapter 1 is on fundamental groups.
- Chapter 2 is on homology. This is an abelian extension of fundamental groups.
- Chapter 3 is on cohomology. Poincaré duality relates cohomology with homology.

Chapter 4 is typically covered in Math 215B, on homotopy theory.

Let's talk a bit about the interests of the course. Topology as a whole is interested in "spaces up to deformation." In this class, deformation will mean homotopy mostly, but there are finer notions of interest like homeomorphism. As for the spaces, we will focus on spaces which are locally homogeneous in some sense, like manifolds (which are locally homeomorphic to $\mathbb{R}^n$). These notions come up naturally throughout mathematics; for example, integrals of holomorphic functions are roughly independent of path chosen. Poincaré himself was interested in differential equations, whose configuration spaces could be manifolds.

In this class, we will attach invariants to our topological spaces to be able to understand how to differentiate between our spaces (up to deformation). We focus on the following invariants.

- Fundamental groups and covering spaces. This has a close tie to Galois theory, an analogy made process by the étale fundamental group in algebraic geometry.
- Cohomology. The origins are from complex analysis and Stokes's theorem, but cohomology itself has vast generalizations and manifestations throughout mathematics, leading to the field of homological algebra. However, there are applications to algebraic geometry, number theory, and so on. The most notable application here is the proof of the Weil conjectures.
- Higher homotopy groups. Our approach will not begin with this viewpoint, but it is possible.

1.1.3 Homotopy and Homotopy Type

Let's jump in chapter 0.

Notation 1.1. We set $I := [0,1]$ for convenience.

Definition 1.2 (deformation retract)**.** Fix a subspace A of a topological space X. Then a *deformation retract* is a family of functions $f_\bullet\colon X \times I \to X$ where $f_0 = \mathrm{id}_X$ and $\operatorname{im} f_1 = A$ and $f_t|_A = \mathrm{id}_A$ for all $t \in I$.

Example 1.3 (mapping cylinder)**.** Fix a continuous function $f\colon X \to Y$. Then the *mapping cylinder* M_f is the space $(X \times I) \sqcup Y$ quotiented by $(x,1) \sim f(x)$. Then M_f has a deformation retraction to Y by $f_t(x) := (x,t)$. Visually, we have attached Y to a thickening of X.

Example 1.4. Define $f\colon S^1 \to S^1$ by $f(z) := z^2$. Then M_f has S^1 on one domain side and S^1 covered twice on the target side. With a little deformation, this is a Möbius strip. Approximately speaking, one should cut the cylinder in half and then rearrange. One can see that the Möbius strip deformation retracts to S^1 by squishing the width of the cylinder to the central line.

A deformation retract is a special case of a homotopy. Here is the definition of a homotopy.

Definition 1.5 (homotopy)**.** Two continuous maps $f_0, f_1\colon X \to Y$ are *homotopic* if and only if there is a continuous function $F_\bullet\colon X \times I \to Y$ such that $F_0 = f_0$ and $F_1 = f_1$. Here, F is called a *homotopy*, and we write $f_0 \sim f_1$.

Example 1.6. A subspace $A \subseteq X$ has a deformation retract if and only if id_X is homotopic to some $r\colon X \to X$ with $\operatorname{im} r = A$ and $r|_A = \mathrm{id}_A$. Indeed, the deformation retract is exactly the needed homotopy.

Example 1.7. Suppose $f, g\colon X \to Y$ are equal maps. Then define $h\colon X \times I \to Y$ by $h_t = f = g$ for all t. We see that h is continuous ($h^{-1}(V) = f^{-1}(U) \times I$ for any open $V \subseteq Y$), so it provides a homotopy from f and g.

It should not be surprising that homotopy is an equivalence relation.

Lemma 1.8. Fix topological spaces X and Y. Then $\sim$ is an equivalence relation on continuous functions $X \to Y$.

Proof. We have the following checks.

- Reflexive: this is direct from Example 1.7.

- Symmetric: if $f \sim g$, then we have $F_\bullet\colon X \times I \to Y$ with $F_0 = f$ and $F_1 = g$. We now define $G_\bullet\colon X \times I \to Y$ by $G_t := F_{1-t}$. Then G is continuous by the continuity of $t \mapsto 1-t$ and F, and $G_0 = g$ and $G_1 = f$, so G witnesses $g \sim f$.
- Transitive: if $f \sim g$ and $g \sim h$, find $F_\bullet\colon X \times I \to Y$ and $G_\bullet\colon X \times I \to Y$ with $F_0 = f$ and $F_1 = g$ and $G_0 = g$ and $G_1 = h$. Then we define $H_\bullet\colon X \times I \to Y$ by

$$H_t := \begin{cases} F_{2t} & \text{if } 0 \leqslant t \leqslant 1/2, \\ G_{2t-1} & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

  Note that this is well-defined at $t = 1/2$ because $F_1 = g = G_0$. Note H will witness $f \sim h$ once we show that it is continuous, which is what we do now.

  By looking locally at F or G, we see that H is continuous at any point not of the form $(x, 1/2)$. Then for any point of the form $(x, 1/2)$ and open subset $V \subseteq Y$ containing $H_{1/2}(x)$, continuity of F gives an open subset $U_F \times (1/2 - \varepsilon, 1/2]$ mapping to V, and continuity of G gives an open subset $U_G \times [1/2, 1/2 + \varepsilon)$ mapping to V, so $(U_F \cap U_G) \times (1/2 - \varepsilon, 1/2 + \varepsilon)$ will suffice. ■

Homotopy also behaves well with composition.

Lemma 1.9. Fix topological spaces X, Y, Z, and let $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$ be homotopic maps. Then $(g_0 \circ f_0) \sim (g_1 \circ f_1)$.

Proof. Fix a homotopy $F_\bullet\colon X \times I \to Y$ with $F_0 = f_0$ and $F_1 = f_1$ and a homotopy $G_\bullet\colon Y \times I \to Z$ with $G_0 = g_0$ and $G_1 = g_1$. Then we define $H_\bullet\colon X \times I \to Z$ by

$$H_t(x) := G_t(F_t(x)).$$

Then $H_0 = g_0 \circ f_0$ and $H_1 = g_1 \circ f_1$, so we will be done if we can show H is continuous. Well, $H_\bullet$ is the composite map

$$X \times I \stackrel{(F, \mathrm{id}_I)}{\to} Y \times I \stackrel{G}{\to} Z,$$

which we can see is the composite of continuous maps. ■

Homotopy allows us to define homotopy equivalence.

Definition 1.10 (homotopy equivalence)**.** A continuous map $f\colon X \to Y$ is a *homotopy equivalence* if and only if there is a continuous map $g\colon Y \to X$ such that $(g \circ f) \sim \mathrm{id}_X$ and $(f \circ g) \sim \mathrm{id}_Y$. We then say that X and Y have the same *homotopy type* and write $X \simeq Y$.

Remark 1.11. It is not enough to merely require $(g \circ f) \sim \mathrm{id}_X$. For example, let $X := \{x\}$ be a point. Then any map $f\colon \{x\} \to Y$ can use the unique map $g\colon Y \to \{x\}$ so that $(g \circ f) = \mathrm{id}_X$.

Here is a quick sanity check.

Lemma 1.12. Ignoring size issues, homotopy equivalence provides an equivalence relation on topological spaces.

Proof. We have the following checks. Fix topological spaces X, Y, Z.

- Reflexive: we show $X \simeq X$. Indeed, use the maps $\mathrm{id}_X, \mathrm{id}_X\colon X \to X$ so that $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$ is homotopic to id_X by Example 1.7.
- Symmetric: we show $X \simeq Y$ implies $Y \simeq X$. Indeed, let $f\colon X \to Y$ and $g\colon Y \to X$ be the promised maps so that $(f \circ g) \sim \mathrm{id}_Y$ and $(g \circ f) \sim \mathrm{id}_X$. Reading these data in reverse tell us that $Y \simeq X$.

- Transitive: suppose $X \simeq Y$ and $Y \simeq Z$, and we show $X \simeq Z$. Thus, we have maps $f\colon X \to Y$ and $g\colon Y \to X$ and $f'\colon Y \to Z$ and $g'\colon Z \to Y$ such that $(f \circ g) \sim \mathrm{id}_Y$ and $(g \circ f) \sim \mathrm{id}_X$ and $(f' \circ g') \sim \mathrm{id}_Z$ and $(g' \circ f') \sim \mathrm{id}_Y$. We now claim that $(f' \circ f)\colon X \to Z$ and $(g \circ g')\colon Z \to X$ are the desired maps to witness $X \simeq Z$. Well, using Lemma 1.9, we compute
$$(f' \circ f) \circ (g \circ g') = f' \circ (f \circ g) \circ g' \sim f' \circ \mathrm{id}_Y \circ g' = f' \circ g' \sim \mathrm{id}_Z,$$
and similar for the other direction. ■

Remark 1.13. One can check directly that $\sim$ is an equivalence relation on spaces. The main check here is that one can compose homotopies.

We will often find that our algebraic invariants are only able to detect homotopy equivalence, which is why homotopy equivalence will be so important to us.

Example 1.14. Example 1.4 shows that the Möbius strip is homotopic to S^1.

More generally, one can show that a deformation retract is a homotopy equivalence.

Lemma 1.15. Fix a subspace A of a topological space X. Then a deformation retract witnesses a homotopy between the inclusion $i\colon A \hookrightarrow X$ and the identity $\mathrm{id}_X\colon X \to X$. In particular, it follows that i is a homotopy equivalence.

Proof. This is a matter of unraveling the definitions. Fix a deformation retract $f_\bullet\colon X \times I \to X$, and let $r := f_1$ so that $\operatorname{im} r = A$. We now claim that i and r are inverse homotopy equivalences.

- We show that $(r \circ i) \sim \mathrm{id}_A$. Indeed, $r(i(a)) = a$ for any $a \in A$ by hypothesis on r, so in fact $r \circ i = \mathrm{id}_A$.
- We show that $(i \circ r) \sim \mathrm{id}_X$. The relevant homotopy is just $f_\bullet$: we have $f_0 = \mathrm{id}_X$ and $f_1 = (i \circ r)$, so $\mathrm{id}_X \sim (i \circ r)$ by Lemma 1.8. ■

Example 1.16 (dunce cap)**.** Take the disc D^2 and glue the edges together as follows: mark three points A, B, and C, and glue AB to AC to CB (in those orientations). Then the resulting space is homotopic to a point.

We have a special name for being homotopic to a point.

Definition 1.17 (contractible)**.** A topological space X is *contractible* if and only if it is homotopic to a point.

These notions allow us to define a homotopy category, whose objects are homotopy classes of topological spaces and morphisms are continuous maps. In some sense, our algebraic invariants are trying to distinguish between objects in this category. It turns out that this category is not concrete, meaning that there is no way to realize its objects as sets reasonably. Approximately speaking, this means that there can be no canonical representing topological space for each homotopy class, but topologists try anyway.

Remark 1.18. There are a number of results called "topological rigidity" theorems which give homeomorphism $X \cong Y$ given merely $X \simeq Y$ and some extra hypotheses. For example, this holds for closed surfaces by a classification result.

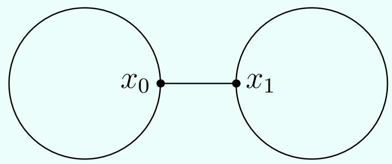
Example 1.19. Attach two S^1s by a line to make a space X, and attach them along an edge to make a space Y. These spaces are homotopic, but they are not homeomorphic (removing a point from X may disconnect it, but this is not the case for Y).

1.1.4 CW Complexes

Here is our definition.

Definition 1.20 (CW complex)**.** Let X^0 be a discrete set of points, and define X^n inductively by $X^{n+1} := X^n \cup \{e_\alpha^{n+1}\}$, where $\varphi_\alpha \colon \partial e_\alpha^{n+1} \to X^n$ is a homeomorphism telling us how to union. Here, e_α^{n+1} is a copy of the n-ball B^n, so the φ_α are explaining how to identify the edges.

Example 1.21. Here is a CW complex.



Namely, $X^0 = \{x_0, x_1\}$, and X^1 is the edges.

Example 1.22. Take a point $\{*\}$ for X^0, and define φ_n to be some loop based on $\{*\}$. Then the resulting space is some infinite union of circles intersecting at $\{*\}$. Notably, this space is not compact and in fact should not even be embedded into the plane or $\mathbb{R}^3$ because such an embedding is unlikely to be a homeomorphism.

Example 1.23. The sphere $S^n := D^n/\partial D^n$ is a CW structure with only two cells: it is $e^0 \cup e^n$. Notably, the CW structure here has $X^0 = X^1 = \cdots = X^{n-1}$.

Example 1.24. Alternatively, one can define S^n inductively as follows: take S^0 to be two points, and define S^{n+1} to be S^n as an equator unioned with two $(n+1)$-cells making hemispheres attached to the equator. One can then define S^∞ to be the union of all the $S^\bullet$ where we identify $S^n \hookrightarrow S^{n+1}$ via the equator. This is a CW complex of infinite dimension. It turns out that S^∞ is contractible, though S^n is not for any finite n.

Example 1.25. Define real projective space $\mathbb{RP}^n$ as the set of vectors $x \in \mathbb{R}^{n+1} \setminus \{0\}$ where we identify x with λx for any $\lambda \in \mathbb{R}^\times$. Notably, by setting the last coordinate equal to 0, we expect to get $\mathbb{RP}^{n-1}$. But if the last coordinate is equal to zero, we can scale it uniquely to 1, and then the remaining coordinates may vary arbitrarily. In total, we find

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup \mathbb{R}^{n-1}.$$

Thus, we get the cell structure $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$.

Remark 1.26. The CW structure is not unique. For example, one can separate out edges by putting a point in the middle of them.

One can show that the CW complex is compact if and only if it has finitely many cells.

1.2 August 29

Last time we discussed homotopies, homotopy equivalence, and CW complexes. To review, the goal of algebraic topology is to define (algebraic) invariants of topological spaces and then perhaps figure out when two spaces are equivalent (for suitable definition of equivalent). In theory, our invariants would be able to entirely classify some subset of spaces we are looking at, but it is rather rare. To execute this plan, we need a source of spaces (mostly CW complexes and ways to combine them) and then methods to tell if spaces are equivalent.

1.2.1 Operations on Spaces

Let's discuss how to make new spaces from old ones. Thankfully, our operations will send CW complexes to CW complexes, though there is something to check.

Definition 1.27 (product)**.** Fix CW complexes X and Y. Then we form the *product* $X \times Y$ (at the level of CW complexes) using as $(n+m)$-cells $e_\alpha^m \times f_\beta^n$ where e_α^m is an m-cell of X and f_β^m is an n-cell of Y. Notably, the n-skeleton is

$$(X \times Y)^n = \bigcup_{k+\ell=n} X^k \times Y^\ell,$$

and one can attach in the obvious way. This produces a CW structure.

Remark 1.28. It is possible that $X \times Y$ with its CW structure need not be the same as the product topology. There is an example in the appendix of [Hat01], but we won't care so much for this course.

Definition 1.29 (subcomplex)**.** Fix a CW complex X. Then a *subcomplex* is a closed subspace $A \subseteq X$ which is a union of cells of X and also a CW complex.

Definition 1.30 (quotient)**.** Fix a subcomplex A of a CW complex X. Then X/A is also a CW complex. Here, the definition of X/A is somewhat technical: its cells are the cells of $X\backslash A$ and then a 1-cell from A, and one attaches in the obvious way (inductively) via the quotient map $X^{n-1} \to X^{n-1}/A^{n-1}$.

Definition 1.31 (suspension)**.** Fix a CW complex X. Then the *suspension* is the quotient

$$SX := \frac{X \times I}{\{(0,x) \sim (0,x') \text{ and } (1,x) \sim (1,x')\}}.$$

Example 1.32. Take $X = S^0$, which is two points. Then $X \times I$ is two lines, and we then identify the endpoints of the two lines accordingly to produce a circle S^1. More generally, $SS^n = S^{n+1}$ essentially by just gluing two S^ns onto the equator of S^{n+1}.

Definition 1.33 (join)**.** Fix CW complexes X and Y. Then the *join* $X * Y$ is the product $X \times Y \times I$ (as CW complexes) modded out by the equivalence relation identifying $(x,y,0) \sim (x,y',0)$ and $(x,y,1) \sim (x',y,1)$.

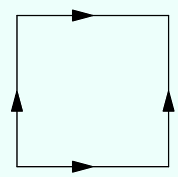
Example 1.34 (simplex)**.** Consider $X = Y = I = \Delta^1$. Then $X * Y$ is the cube modded out by crushing Y on one end and crushing X on the other end, forming a tetrahedron, which is Δ^3. More generally, $\Delta^n * \Delta^m = \Delta^{n+m+1}$.

Example 1.35. One has $S^0 * S^0 = S^1$, and more generally $S^n * X = SX$. Essentially, we are gluing two copies of X onto an equator, which is the suspension.

Definition 1.36 (wedge product)**.** Fix CW complexes X and Y and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then we form the *wedge product* $X \vee Y$ as $X \sqcup Y$ identifying $x_0 \sim y_0$.

Definition 1.37 (smash product)**.** Fix CW complexes X and Y and points $x_0 \in X^0$ and $y_0 \in Y^0$. Then the *smash product* is $(X \times Y)/(X \vee Y)$, where $X \vee Y$ is embedded into $X \times Y$ as $x \mapsto (x, y_0)$ and $y \mapsto (y, x_0)$.

Example 1.38. One can check that $S^1 \times S^1$ is a torus. To form the smash product, we are crushing the boundary of the square as follows.



More generally, $S^m \wedge S^n = S^{m+n}$.

Definition 1.39 (attach)**.** Fix a subcomplex A of a CW complex X_1 and a map $f: A \to X_0$ to another CW complex X_0. Then $X_0 \sqcup_f X_1$ is the space $X_0 \sqcup X_1$ modded out by the equivalence relation $a \sim f(a)$ for all $a \in A$.

Example 1.40. An attaching map $\varphi_\alpha : \partial D^n \to X^{n-1}$ of a CW complex are attachments $X^{n-1} \sqcup_{\varphi_\alpha} D^n$ in the above sense.

1.2.2 Homotopy Extension

We are going to, over time, prove the following results. To begin, quotients preserve homotopy type.

Proposition 1.41. Fix a subcomplex A of a CW complex X. If A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

Example 1.42. Fix a connected graph X, which is a one-dimensional CW complex. Fix a spanning tree $T \subseteq X$, which is contractible (any tree can be contracted one edge at a time), so $X \to X/T$ is a homotopy equivalence. Then X/T becomes a wedge of loops corresponding (roughly) to the number of "independent" cycles. Notably, this collapsing is far from canonical, essentially unique up to choosing the spanning tree and then an order of edges. In some sense, because the homotopy group of a wedge of loops is a free group, we are able to study automorphisms of the free group in this way.

Proposition 1.43. Fix a subcomplex A of a CW complex X_1. Given homotopic maps $f, g: A \to X_0$, then $X_0 \sqcup_f X_1 = X_0 \sqcup_g X_1$.

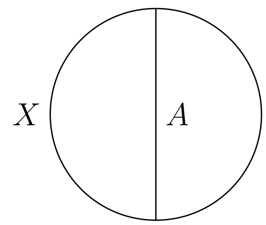
The idea of the above result is that if we can move the attaching maps f and g around, we should not really be adjusting the homotopy type.

To prove these results, we want access to the homotopy extension property.

Definition 1.44 (homotopy extension property)**.** Fix a subspace A of a topological space X. Then the pair (X, A) has the *homotopy extension property* if and only if all $F_0 : X \to Y$ and small homotopy $f_\bullet : A \times I \to Y$ with $F_0|_A = f_0$, then there is an extended homotopy $F_\bullet : X \times I \to Y$ where $F_t|_A = f_t$ for all $t \in I$.

It will turn out that a subcomplex A of a CW complex A makes (X, A) have the homotopy extension property, but this will take some work to prove.

By way of example, make Y the following "theta graph," and the left edge is X, and A is the middle interval.



Here, $A \subseteq X$ is going to have the homotopy extension property. For example, one can contract A to a point and imagine dragging neighborhoods of $A \cap X$ in X (and in fact all of Y) along for the ride.

One way to think about the homotopy extension property is that we have a map $X \cup (A \times I) \to Y$ (by taking the union F_0 and $f_\bullet$), and we and to extend it to a full map $X \times I \to Y$. With this in mind, we would thus like to have to retract $r\colon (X \times I) \to (X \cup (A \times I))$ and then composing. By taking $Y = X \times I$, one sees that having such a retraction r is in fact equivalent to the homotopy extension property.

So we want to find the retraction $r\colon (X \times I) \to (X \cup (A \times I))$.

> **Lemma 1.45.** Fix a subspace A of a topological space X. Then (X, A) has the homotopy extension property if and only if A has a "mapping cylinder neighborhood." In other words, there is a space B and map $f\colon B \to A$ such that M_f is homeomorphic to a neighborhood of A.

Approximately speaking, what's going on here is that the mapping cylinder allows us some squishing region through which to extend homotopies. Then the above criteria can be checked for CW pairs (X, A) by tracking through attachments. Namely, a reparameterization of the attaching map has mapping cylinder which has the property needed above. Rigorously, one inducts on the n-skeleton of a CW complex X, using the homotopy extension property for cells of X not in A (and not caring about cells already in A).

Read Hatcher

THEME 2

THE FUNDAMENTAL GROUP

2.1 August 31

We now shift gears and talk about our first algebraic invariant: the fundamental group.

2.1.1 The Fundamental Group

Let's start with an example.

Example 2.1. Fix a loop $\gamma\colon S^1 \to (\mathbb{C}\backslash\{0\})$ which is continuously differentiable. Then complex analysis tells us that

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz$$

counts the number of times that γ "winds" around the integer. We might call this the "linking number" of γ. Notably, one can check that continuously varying γ does not adjust the linking number, so this linking number is homotopy invariant.

The fundamental group is a generalization of this notion.

Definition 2.2 (fundamental group)**.** Let X be a topological space, and fix a basepoint $x_0 \in X$. Then the *fundamental group* $\pi_1(X, x_0)$ is the set of homotopy equivalence classes

$$\pi_1(X, x_0) := \{[f] \text{ such that } f\colon I \to X \text{ has } f(0) = f(1) = x_0\}.$$

We will give $\pi_1(X, x_0)$ a group structure below.

Remark 2.3. There is also a $\pi_0(X)$, which consists of homotopy classes of points $[x]$ for $x \in X$, where $[x]$ denotes the path-connected component of X. If we let $\Omega(X, x_0)$ denote the topological space of loops $f\colon I \to X$ such that $f(0) = f(1) = x_0$, then we find $\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$.

Remark 2.4. If we don't want to care about basepoints, one can look at $C\left(S^1, X\right)$, which is the set of maps $S^1 \to X$. This can be given a topology via the compact-open topology. Approximately speaking, these will correspond to conjugacy classes in $\pi_1(X, x_0)$ provided that X is path-connected. For example, the homotopy class of a constant loop $S^1 \to X$ consists of the contractible loops in X; note there is something to check here in that one wants to know that a contractible loop (not relative to the basepoints) is in fact contractible relative to the basepoint.

Example 2.5. Let $X = \{x_0\}$ be a point. Then $\pi_1(X, x_0) = 1$ because there is only path $I \to X$.

Example 2.6. Let X be a convex subset of $\mathbb{R}^n$ for some $n > 0$. Then for any $x_0 \in X$ has $\pi_1(X, x_0) = 1$. Indeed, use the convex hypothesis to shrink any path down to the constant path.

We can give $\pi_1(X, x_0)$ a product via composition.

Definition 2.7 (composition)**.** Let X be a topological space, and fix a basepoint $x_0 \in X$. Given paths $f, g\colon I \to X$ such that $f(1) = g(0)$, we define the path $(f \cdot g)\colon I \to X$ via

$$(f \cdot g)(t) := \begin{cases} f(2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ g(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Note that $f \cdot g$ is well-defined at $t = 1/2$ because $f(1) = g(0)$.

The point of the above definition is to "squish" a path to do both f and g in the interval I, but at twice the speed. One has the following checks.

- The class $[f \cdot g]$ does not depend on the choice of representatives f and g. Essentially, if $f_1 \sim f_2$ and $g_1 \sim g_2$, then one can use these two homotopies to glue together to make a new homotopy $(f_1 \cdot g_1) \sim (f_2 \sim g_2)$.
- We have $[(f \cdot g) \cdot h] = [f \cdot (g \cdot h)]$, so composition associates. The point is that these are basically reparameterizations of each other.
- There is an identity path given by $e_{x_0}(t) := x_0$. The identity check is done again by some idea of reparameterization.
- For a given path $f\colon I \to X$, we can define $\overline{f}\colon I \to X$ by $\overline{f}(t) := f(1-t)$ and then check that

$$f \cdot \overline{f} \sim e_{f(0)},$$

so $\left[\overline{f}\right]$ provides the inverse path for $[f]$ in $\pi_1(X, x_0)$. The point is that $f \cdot \overline{f}$ is

$$\left(f \cdot \overline{f}\right)(t) = \begin{cases} f(2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ f(2-2t) & \text{if } 0 \leqslant t \leqslant 1/2. \end{cases}$$

One can then provide a homotopy by

$$h_s(t) := \begin{cases} f(2t) & \text{if } 0 \leqslant t \leqslant s/2, \\ f(s) & \text{if } s/2 \leqslant t \leqslant 1 - s/2, \\ f(2-2t) & \text{if } 1 - s/2 \leqslant t \leqslant 1, \end{cases}$$

so $h_0 = e_{f(0)}$ and $h_1 = f \cdot \overline{f}$.

For these checks, it is helpful to have lemmas establishing continuity of piecewise functions and establishing that reparameterization does not affect homotopy class.

Remark 2.8. Staring hard at our definition of composition, one sees that our reparameterization business is really just choosing various piecewise affine maps $I \to I$ with slopes in $2^{\mathbb{Z}}$ and breaks at the dyadic rationals $2^{\mathbb{Z}}\mathbb{Z} \subseteq \mathbb{Q}$. These maps form a group called the Thompson group.

Remark 2.9 (fundamental groupoid)**.** Fix a topological space X, and define a category where the objects are points $x \in X$, and the morphisms $x \to y$ are paths (up to homotopy fixing endpoints). The above checks now show that this is in fact a category, where each morphism has an inverse. This category is called the *fundamental groupoid*. Modding out by isomorphism, our objects are now path components in X, and choosing a particular component produces the fundamental group in its endomorphisms.

Remark 2.10. Verifying that $\pi_1(X, x_0)$ only required reparameterization. So as in Remark 2.9, we could also look at the category where paths are only considered up to reparameterization, and the above checks still go through. This is related to the notion of "thin homotopy."

Lemma 2.11. Fix a topological space X. Further, fix a path $p\colon I \to X$. Then $f \mapsto (\overline{p} \cdot f \cdot p)$ provides an isomorphism $\pi_1(X, p(1)) \to \pi_1(X, p(0))$.

Proof. This is well-defined because $f_1 \sim f_2$ implies $\overline{p} \cdot f_1 \sim \overline{p} \cdot f_2$ implies $\overline{p} \cdot f_1 \cdot p \sim \overline{p} \cdot f_2 \cdot p$. This is a group homomorphism because

$$\overline{p} \cdot f \cdot g \cdot p \sim \overline{p} \cdot f \cdot p \cdot \overline{p} \cdot g \cdot p.$$

Lastly, this is an isomorphism because $\overline{p}$ provides the inverse map. ■

Remark 2.12. The above result roughly says that we can indeed look at the fundamental groupoid only in terms of the path-connected components.

Thus, we see that $\pi_1(X, x_0)$ is well-defined up to base-point provided that X is path-connected. However, the isomorphism between base-points is only defined up to path between those basepoints! Roughly speaking, the problem is that elements of $\pi_1(X, x_0)$ should really only be thought of up to inner automorphism because we can pre- and post-compose by some loop at x_0.

Lemma 2.13. If X is homeomorphic to Y by $\varphi\colon X \to Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$ for any $x_0 \in X$.

Proof. Use φ. ■

2.1.2 The Fundamental Group of S^1

Here is our result.

Theorem 2.14. Fix any $x \in S^1$. Then $\pi_1\left(S^1, x\right) \cong \mathbb{Z}$. In fact, there is an isomorphism $\Phi\colon \mathbb{Z} \to \pi_1\left(S^1, x\right)$ given by

$$n \mapsto [t \mapsto (\cos 2\pi n t, \sin 2\pi n t)]\,.$$

Sketch without covering spaces. We show injectivity and surjectivity independently.

- Think of S^1 as embedded in $\mathbb{C}$ as $\{z : |z| = 1\}$ and take a smooth path $f\colon I \to S^1$, lift it to a map $\widetilde{f}\colon I \to \mathbb{R}$ via

$$\widetilde{f}(t) := \int_0^t d\theta,$$

where $d\theta$ is some differential form S^1 (say, $x\,dy - y\,dx$). Then $\widetilde{f}(1)$ is intuitively contained in $2\pi\mathbb{Z}$ and is homotopy invariant. Now, f is not smooth, then we can use some small homotopy to make f smooth and then use the above argument. This provides an inverse map to Φ and thus shows that Φ is injective.

- For surjectivity, one can use uniform continuity of any path $f\colon I \to S^1$ and the compactness of S^1 in order to divide up I into intervals on which f can be written as a composition of well-behaved paths, which eventually allows us to force f to make piecewise linear. Once f is piecewise linear, we go interval-by-interval and fix f to be constant speed. Eventually f becomes one of the $\Phi(n)$ for some n. ■

For the covering space approach, the point is that we understand the fundamental group of $\mathbb{R}$ well, and we have a fairly well-behaved "covering map" $p\colon \mathbb{R} \to S^1$ given by $p(\theta) := (\cos 2\pi\theta, \sin 2\pi\theta)$. The main claim, then, is that any path $\omega\colon I \to S^1$ has a unique lift $\widetilde{\omega}\colon I \to \mathbb{R}$ such that $\widetilde{\omega}(0) = \omega(0)$ and $p \circ \widetilde{\omega} = \omega$. The point is that once we lift, we can use a homotopy up in $\mathbb{R}$ (fixing the endpoints of $\widetilde{\omega}$), which will then go back down to a homotopy on S^1 if we are careful. Anyway, this lifting process can essentially be done as described in the surjectivity check above.

2.2 September 5

Today we actually prove $\pi_1^(S^1) \cong \mathbb{Z}$.

2.2.1 Eckmann–Hilton Argument

Because it is fun, we begin with some nonsense.

Proposition 2.15 (Eckmann–Hilton)**.** Let X be a set equipped with the binary operations $\circ$ and $*$ such that the following hold.

(a) Identity: there are elements $1_\circ, 1_* \in X$ such that $1_\circ \circ a = a \circ 1_\circ = a$ and $1_* * a = a * 1_* = a$ for all $a \in X$.

(b) Distribution: we have $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$ for all $a, b, c, d \in X$.

Then $\circ$ and $*$ are the same operation and in fact are both commutative and associative.

Proof. This is purely formal. We proceed in steps.

1. We show that $1_\circ = 1_*$. Indeed, note
$$1_* = 1_* * 1_* = (1_* \circ 1_\circ) * (1_\circ \circ 1_*) = (1_* * 1_\circ) \circ (1_\circ * 1_*) = 1_\circ \circ 1_\circ = 1_\circ.$$
From now on, we use the symbol 1 to denote our identity $1_\circ = 1_*$.

2. We show that $a * b = a \circ b$. Indeed, note
$$a * b = (a \circ 1) * (1 \circ b) = (a * 1) \circ (1 * b) = a \circ b.$$
Thus, our operations are the same, and we will use the symbol $*$ to denote both of them now. Notably, our distribution law is $(a * b) * (c * d) = (a * c) * (b * d)$.

3. We show that $*$ is commutative. Indeed, for any $a, b \in X$, we see
$$a * b = (1 * a) * (b * 1) = (1 * b) * (a * 1) = b * a.$$

4. We show that $*$ is associative. Indeed,
$$(a * b) * c = (a * b) * (1 * c) = (a * 1) * (b * c) = a * (b * c),$$
for any $a, b, c \in X$. ■

As an application, we have the following result.

Corollary 2.16. Let G be a topological group with identity $e \in G$. Then $\pi_1(G, e)$ is abelian.

Proof. Let $\cdot$ denote the usual concatenation operation on $\pi_1(G, e)$. The point is to give another binary operation to $\pi_1(G, e)$ and then apply Proposition 2.15.

Well, let $*$ denote the group operation on G, and for paths $f, g \colon I \to G$ based at e, we define the path $(f * g) \colon I \to G$ by $(f * g)(t) := f(t) * g(t)$. Here are the necessary checks for our purposes.

- Note $f * g$ is a continuous map because it is the composite of the continuous maps
$$I \overset{(\mathrm{id}_I, \mathrm{id}_I)}{\to} I \times I \overset{(f,g)}{\to} G \times G \overset{*}{\to} G.$$
- We show $[f * g]$ does not depend on the choice of homotopy classes $[f]$ and $[g]$, so we may view $*$ as a binary operation on $\pi_1(G, e)$. Suppose $f \sim f'$ and $g \sim g'$ by the homotopies $F_\bullet$ and $G_\bullet$, respectively. We want to show that $f * g \sim f' * g'$. Well, define $H_\bullet \colon G \times I \to G$ by $H_t(x) := F_t(x) * G_t(x)$ for all $t \in I$ and $x \in G$. Then we see that $H_0 = F_0 * G_0 = f * g$ and $H_1 = F_1 * G_1 = f' * g'$, and $H_\bullet$ is continuous because it is the composite
$$G \times I \overset{(F_\bullet, G_\bullet)}{\to} G \times G \overset{*}{\to} G.$$
- Note that $*$ has an identity element given by the constant path $c(t) := e$ for all $t \in I$. Indeed, for any $[f] \in \pi_1(G, e)$, we see that $(f * c)(t) = f(t) * c(t) = f(t)$ for all $t \in I$, so $[f] * [c] = [f * c] = [f]$.
- Fix $[a], [b], [c], [d] \in \pi_1(G, e)$. We claim that
$$([a] \cdot [b]) * ([c] \cdot [d]) \overset{?}{=} ([a] * [c]) \cdot ([b] * [d]).$$
Removing all the homotopy classes, it is enough to show that $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d)$. Well, for any $t \in I$, we compute
$$((a \cdot b) * (c \cdot d))(t) = (a \cdot b)(t) * (c \cdot d)(t) = \begin{cases} a(t) * c(t) & \text{if } t \leqslant 1/2, \\ b(t) * d(t) & \text{if } t \geqslant 1/2, \end{cases}$$
and
$$((a * c) \cdot (b * d))(t) = \begin{cases} (a * c)(t) & \text{if } t \leqslant 1/2, \\ (b * d)(t) & \text{if } t \geqslant 1/2, \end{cases}$$
which is the same path.

Now, Proposition 2.15 shows that $*$ and $\cdot$ must be the same operation on $\pi_1(G, e)$ and that $\cdot$ is commutative, which is what we wanted. ■

2.2.2 Covering Spaces

Our computation is going to use the notion of a covering space.

Definition 2.17 (covering space)**.** Fix a topological space X. Then a *covering space* is a topological space $\widetilde{X}$ together with a projection map $p \colon \widetilde{X} \to X$ such that each $x \in X$ has an open neighborhood $U \subseteq X$ containing x such that $p^{-1}(U) = \bigsqcup_{\alpha \in \lambda} U_\alpha$ where U_α is open and $p \colon U_\alpha \to U$ is a homeomorphism. In this set up, the open set $U \subseteq X$ is said to be *evenly covered*.

The fact we will require about covering spaces is the following "fibration property."

Proposition 2.18. Fix a topological space X and covering space $p\colon \widetilde{X} \to X$. Further, suppose we have maps $F\colon Y \times I \to X$ and $\widetilde{F}\colon Y \times \{0\} \to \widetilde{X}$ such that $p \circ \widetilde{F}|_{Y\times\{0\}} = F|_{Y\times\{0\}}$. Then there is a unique extension $\widetilde{F}\colon Y \times I \to \widetilde{X}$ such that $p \circ \widetilde{F} = F$.

Proof. We proceed in steps. Say that a subset $U \subseteq X$ is "evenly covered" if and only if $p^{-1}(U) = \bigsqcup_{\alpha\in\lambda} U_\alpha$ and $p\colon U_\alpha \to U$ is a homeomorphism. Note that making an evenly covered open subset smaller will retain it being evenly covered using the fact that the maps $p\colon U_\alpha \to U$ is a homeomorphism.

1. To set us up, given $y \in Y$, we claim that there we can find an open neighborhood V of y and a finite open cover $\mathcal{U}$ of I such that $F(V \times U)$ is contained in an evenly covered subset of X for any $U \in \mathcal{U}$. The point is to use compactness to shrink an evenly covered subset containing $F(V \times I)$ sufficiently. Well, for each $t \in I$, we may find and evenly covered subset $U_t \subseteq X$ containing $F(y,t)$ and then find $\varepsilon_t > 0$ and an open neighborhood V_t of y such that $V_t \times (t - \varepsilon_t, t + \varepsilon_t) \subseteq F^{-1}(U_{t_i})$.

   Now, by compactness, we may choose finitely many t labeled $\{t_1, \ldots, t_n\}$ and set $\varepsilon_i := \varepsilon_{t_i}$ and $V_i := V_{t_i}$ and $U_i := U_{t_i}$ such that the intervals $(t_i - \varepsilon_i, t_i + \varepsilon_i)$ covers I and $F(V_i \times (t_i - \varepsilon_i, t_i + \varepsilon_i)) \subseteq F^{-1}(U_i)$. Now, set
   $$V := \bigcap_{i=1}^{n} V_i$$
   so any $t \in I$ lives in some $(t_i - \varepsilon_i, t_i + \varepsilon_i)$ has $F(V \times (t_i - \varepsilon_i, t_i + \varepsilon_i)) \subseteq U_i$.

2. We prove uniqueness. It is enough to show this in the case where Y is a point. Namely, fix suppose we have two lifts $\widetilde{F}_1$ and $\widetilde{F}_2$ of F which agree with $\widetilde{F}$. Then, fixing some $y \in Y$, we see that $\widetilde{F}_1(y)$ and $\widetilde{F}_2(y)$ are maps $I \to \widetilde{X}$ lifting $F(y)\colon I \to X$ which equal $\widetilde{F}(y,0)$ at 0. In this setting, we want to show that $\widetilde{F}_1(y,t) = \widetilde{F}_2(y,t)$ for all $t \in I$. As such, we suppress the point $y \in Y$ in the argument which follows.

   The previous step promises us a finite open cover $\mathcal{U}$ of I such that $F(U)$ is contained in an evenly covered open subset of X for each $U \in \mathcal{U}$. Ordering the endpoints of $\mathcal{U}$, we produce a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that $F([t_i, t_{i+1}])$ is covered in an evenly covered subset of U_i for each i.

   We are now ready to show our uniqueness. We show that $\widetilde{F}_1(t) = \widetilde{F}_2(t)$ for each $t \in [0, t_i]$ by induction on i. At $i = 0$, there is nothing to say because $\widetilde{F}_1(0) = \widetilde{F}(0) = \widetilde{F}_2(0)$. Now, for the induction, we are given that $\widetilde{F}_1(t_i) = \widetilde{F}_2(t_i)$. The point is that $F([t_i, t_{i+1}])$ is contained in an evenly covered subset $U_i \subseteq X$, so $\widetilde{F}_1([t_i, t_{i+1}])$ lands in one of the disjoint copies of U_i of $p^{-1}(U_i)$, and it lands in exactly one because $[t_i, t_{i+1}]$ is connected; let $\widetilde{U}_i$ be the corresponding disjoint copy. The same statement holds for $\widetilde{F}_2$, and in fact $\widetilde{F}_2([t_i, t_{i+1}]) \subseteq \widetilde{U}_i$ because $\widetilde{F}_2([t_i, t_{i+1}])$ needs to land in the same copy of U_i containing $\widetilde{F}_1(t_i) = \widetilde{F}_2(t_i)$.

   We are now done. Note $p\colon \widetilde{U}_i \to U_i$ is injective, so
   $$p \circ \widetilde{F}_1 = p \circ \widetilde{F}_2$$
   for $t \in [t_i, t_{i+1}]$ forces equality after removing t.

3. Fix some $y \in Y$. We will extend locally: we construct some open neighborhood V of y and a lift $\widetilde{F}\colon V \times I \to \widetilde{X}$ of $F|_{V\times I}$. The point is to "spread out" from $\{y\} \times I$ using the previous step.

   As before, the first step promises us an open neighborhood V of y and a finite open cover $\mathcal{U}$ of I such that $F(V \times U)$ is contained in an evenly covered subset for each $U \in \mathcal{U}$. Arranging the endpoints of the open sets in $\mathcal{U}$, we may say that we have a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $F(V \times [t_i, t_{i+1}])$ is contained in an evenly covered open subset $U_i \subseteq X$ for each i.

   We now extend F to $\widetilde{F}$ on $[0, t_i]$ inductively. For $i = 0$, there is nothing to do because $\widetilde{F}|_{Y\times\{0\}}$ is already fixed. Now, suppose we have a definition of $\widetilde{F}$ on $V \times [0, t_i]$. Say $F(V \times [t_i, t_{i+1}]) \subseteq U_i$, and select the copy of U_i named $\widetilde{U}_i \subseteq p^{-1}(U_i)$ by requiring it to contain $\widetilde{F}(y, t_i)$. Now, shrink V so that $V \times \{t_i\}$

contains y still but now is contained in $\widetilde{U}_i$. Now, define $\widetilde{F}$ on $V \times [t_i, t_{i+1}]$ by pre-composing with the homeomorphism

$$p^{-1}\colon U_i \to \widetilde{U}_i,$$

and we produce a continuous map because we have agreed on the seam at $V \times \{t_i\}$.[1] This completes the lifting to a neighborhood V of y.

4. We can now glue the lifts $\widetilde{F}$ constructed in the previous step, and the gluing is well-defined because they must agree on intersections by the uniqueness of the second step. This completes the proof. ■

And now here is our result.

Theorem 2.19. For any $x \in S^1$, we have $\pi_1\left(S^1, x\right) \cong \mathbb{Z}$.

Proof. For brevity, embed S^1 into $\mathbb{C}$ as $S^1 = \mathbb{R}/\mathbb{Z}$, and let our basepoint be $0 \in S^1$. We now abbreviate our fundamental group to $\pi_1\left(S^1\right)$.

Now, we note that we have the continuous (in fact, holomorphic) path $\omega_n\colon [0,1] \to S^1$ given by $t \mapsto nt$. A reparameterization argument can show that $[\omega_n] \cdot [\omega_m] = [\omega_{m+n}]$ for any $m, n \in \mathbb{Z}$, so we have defined a homomorphism $\varphi\colon \mathbb{Z} \to \pi_1(S^1)$. We would like to show that this map is an isomorphism. We will use Proposition 2.18, for which we note that $p\colon \mathbb{R} \to S^1$ given by $p(t) := t$ is a covering space map. Indeed, for each $t \in S^1$, choose the neighborhood $(t-0.1, t+0.1)$ so that

$$p^{-1}((t-0.1, t+0.1)) = (t-0.1, t+0.1) + \mathbb{Z} = \bigsqcup_{n \in \mathbb{Z}} (t+n-0.1, t+n+0.1).$$

We now show that φ is an isomorphism.

- Surjective: let $f\colon I \to S^1$ be a loop, and we want to show that $f \sim \omega_n$ for some $n \in \mathbb{Z}$. By Proposition 2.18 applied with Y being a point, we get a path $\widetilde{f}\colon I \to \mathbb{R}$ such that $f = p \circ \widetilde{f}$. Now, set $n := \widetilde{f}(1)$, which is indeed an integer, and we claim $\widetilde{f} \sim \widetilde{\omega}_n$, where $\widetilde{\omega}_n(t) := nt$; this will finish after composing with the projection p as it shows that $f \sim \omega_n$ by Lemma 1.9.

  To see this, we define the map $h\colon I \times I \to \mathbb{R}$ by

  $$h_t(s) := (1-t)\widetilde{f}(s) + t\widetilde{\omega}_n(s).$$

  Then h is continuous because it is the composite

  $$I \stackrel{(\mathrm{id}, 1-\mathrm{id}, \widetilde{f}, \omega_n)}{\to} I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

  where the last map is taking a linear combination. Now, $h_0 = \widetilde{f}$ and $h_1 = \widetilde{\omega}_n$, so $\widetilde{f} \sim \widetilde{\omega}_n$ follows.

- Injective: suppose $\omega_n \sim \omega_0$, and we want to show that $n = 0$. Then we have a homotopy $h_\bullet\colon I \times I \to X$ such that $h_0 = \omega_n$ and $h_1 = \omega_0$. Then Proposition 2.18 produces a unique lift $\widetilde{h}_\bullet\colon I \times I \to \widetilde{X}$ of h such that $\widetilde{h}_t(0) = 0$ for each $t \in I$. Now, the map $t \mapsto \widetilde{h}_t(1)$ is continuous, and $h_t(1) = 0$ for each $t \in I$, so the map $t \mapsto \widetilde{h}_t(1)$ maps to the discrete space $\mathbb{Z}$. It follows that $\widetilde{h}_0(1) = \widetilde{h}_1(1)$, so $0 = n$ because of how ω_0 and ω_n lift to $\mathbb{R}$. ■

2.2.3 The Fundamental Group Functor

Let's do some nonsense checks, for fun.

[1] To avoid this annoyance at the seam, one can allow the partition to overlap a bit so that we only ever glue continuous maps along open sets, which is legal. I won't write this out.

Definition 2.20 (based topological space)**.** A *based topological space* (X, x_0) is a topological space X together with a basepoint $x_0 \in X$. A map of based topological spaces $\varphi\colon (X, x_0) \to (Y, y_0)$ is a continuous map $\varphi\colon X \to Y$ such that $\varphi(x_0) = y_0$. The category with these objects and morphisms is Top_*.

We won't bother to check that Top_* is a category. Here is the main point of this subsection.

Proposition 2.21. We have a functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$.

Proof. We already know that $\pi_1(X, x_0)$ is a group for each based topological space (X, x_0), so we really only have to check the functoriality properties.

Fix a map $\varphi\colon (X, x_0) \to (Y, y_0)$ of based topological spaces. We need to define a group homomorphism $\pi_1(\varphi)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Well, given a loop $f : I \to X$ based at x_0, we note that $(\varphi \circ f)\colon I \to Y$ is a loop based at $y_0 = \varphi(x_0)$, so we hope that our desired map is $(\varphi \circ -)$. Here are our checks.

- Well-defined: if $f \sim f'$, we need to show that $\varphi \circ f \sim \varphi \circ f'$. This is simply Lemma 1.9.
- Group homomorphism: we need to show that $(\varphi \circ f) \cdot (\varphi \circ g) \sim \varphi \circ (f \cdot g)$ for loops $f, g\colon I \to X$ based at x_0. In fact, these paths are equal: for $t \in I$, we compute

$$((\varphi \circ f) \cdot (\varphi \circ g))(t) = \begin{cases} \varphi(f(2t)) & \text{if } t \leqslant 1/2, \\ \varphi(g(2t-1)) & \text{if } t \geqslant 1/2, \end{cases} = (\varphi \circ (f\dot{g}))(t).$$

We now prove functoriality of π_1.

- Identity: note that $\mathrm{id}_X : (X, x_0) \to (X, x_0)$ has $\mathrm{id}_X \circ f = f$ for any path $f\colon I \to X$, so $\pi_1(\mathrm{id}_X)([f]) = [f]$ for any $[f] \in \pi_1(X, x_0)$.
- Composition: given maps $\varphi\colon (X, x_0) \to (Y, y_0)$ and $\psi\colon (Y, y_0) \to (Z, z_0)$ and a loop $f\colon I \to X$ based at x_0, we see that

$$\pi_1(\psi \circ \varphi)([f]) = [\psi \circ \varphi \circ f] = \pi_1(\psi)([\varphi \circ f]) = (\pi_1(\psi) \circ \pi_1(\varphi))([f]),$$

  which finishes. ■

Of course, just being a functor is not terribly interesting. Here is a nice property.

Proposition 2.22. Fix based topological spaces (X, x_0) and (Y, y_0). Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Let $p_X : (X \times Y, (x_0, y_0)) \to (X, x_0)$ and $p_Y : (X \times Y, (x_0, y_0)) \to (Y, y_0)$ denote the projections. Now, note that we have a map

$$(\pi_1(p_X), \pi_2(p_Y))\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

which we claim is an isomorphism. For brevity, let this morphism be φ. Of course, φ is a homomorphism because π_1 is a functor (see Proposition 2.21).

- Surjective: fix loops $f_X : I \to X$ and $f_Y : I \to Y$ based at x_0 and y_0 respectively. Then the map $f(t) := (f_X(t), f_Y(t))$ defines a loop $I \to X \times Y$ based at (x_0, y_0), and by construction $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$, so

$$\varphi(f) = ([p_X \circ f], [p_Y \circ f]) = ([f_X], [f_Y]).$$

- Injective: suppose $\varphi([f]) = \varphi([g])$, and we want to show that $[f] = [g]$. Well, we have homotopies $h_{X\bullet}: I \times I \to X$ and $h_{Y\bullet}: I \times I \to Y$ such that $h_{X0} = p_X \circ f$ and $h_{X1} = p_X \circ g$ and $h_{Y0} = p_Y \circ f$ and $h_{Y1} = p_Y \circ g$. Then we define $h_\bullet: I \times I \to X \times Y$ by

$$h_t(s) := (h_{Xt}(s), h_{Yt}(s)).$$

Note h_t is continuous because it is continuous in each coordinate. To finish, we see $h_0 = f$ and $h_1 = g$ by checking after applying the projections p_X and p_Y, so $f \sim g$ follows. ■

Remark 2.23. More precisely, the above proof has shown that π_1 preserves products.

Example 2.24. We have $\pi_1\left(S^1 \times S^1\right) \cong \mathbb{Z}^2$ by Proposition 2.22 and Theorem 2.19.

Example 2.25. We show that there is no retraction $r: D^2 \to S^1$. Let $i: S^1 \to D^2$ be the inclusion. If there is a retraction r, then we see that $r \circ i = \mathrm{id}_{S^1}$, so functoriality of π_1 means that the composite

$$\pi_1\left(S^1\right) \xrightarrow{i} \pi_1\left(D^2\right) \xrightarrow{r} \pi_1\left(S^1\right)$$

is an isomorphism. In particular, i is injective. However, $\pi_1\left(S^1\right) \cong \mathbb{Z}$ by Theorem 2.19, and $\pi_1\left(D^2\right) = 0$ because D^2 is convex and hence contractible.

Remark 2.26. One can use Example 2.25 to show Brouwer's fixed point theorem: we show that any continuous map $h: D^2 \to D^2$ has a fixed point. Well, suppose h has no fixed point. Then there is a continuous map sending $x \in D^2$ to the point on S^1 which intersects with the raw starting at $h(x)$ and then going through x. Then $h: D^2 \to S^1$ defines a retraction, contradicting Example 2.25.

2.3 September 7

Today we prove the van Kampen theorem.

2.3.1 Free Products of Groups

We will be somewhat brief on this because this is not an algebra class.

Definition 2.27 (free product)**.** Let $\{G_\alpha\}_{\alpha\in\lambda}$ be a collection of groups. Then we form the free product $*_{\alpha\in\lambda}G_\alpha$ as having underlying set given by strings of words whose letters are in the G_α, modded out by the relations $g_\alpha \cdot h_\alpha = g_\alpha h_\alpha$ whenever $g_\alpha, h_\alpha \in G_\alpha$ for some $\alpha \in \lambda$.

Perhaps one should check that this forms a group, so we will sketch what one should do.

1. Let W be the set of finite strings (i.e., words) whose letters are g or g^{-1} where $g \in G_\alpha$ for some $\alpha \in \lambda$. Then we build $\overline{W}$ by allowing combining $g_\alpha \cdot h_\alpha$ into a single character $g_\alpha h_\alpha$ provided that g_α and h_α belong to the same group G_α. We will realize our desired group as a subgroup of $\mathrm{Aut}(W)$.
2. For each $g \in W$, define the function $L_g: W \to W$ by left concatenation. One should show that L_g is in fact a well-defined function, which depends on the equivalence relation defining W, but in short, one can show that having two words w and w' with $w \sim w'$ enforces $g \cdot w \sim g \cdot w'$ by using the same concatenation rules on both sides. A rigorous argument would need to use an induction, which we won't bother to write out.

3. Note that L_e (where e denotes the empty string) is the identity on W, and $L_{g^{-1}}$ is the inverse of L_g. Thus, the image of $L_\bullet$ in W is a subgroup of $\operatorname{Aut}(W)$, and we call this subgroup $\ast_{\alpha\in\lambda}G_\alpha$. One realizes this group as the free product described above by identifying L_g with $L_g(e)$. The point of introducing $L_\bullet$ at all is to make the various group law checks easier.

One has the following universal property, which we will not prove, again because this is not an algebra class.

Proposition 2.28. Let $\{G_\alpha\}_{\alpha\in\lambda}$ be a collection of groups. Given homomorphisms $\varphi_\alpha\colon G_\alpha \to H$ to a target group H, there is a unique homomorphism $\varphi\colon \ast_{\alpha\in\lambda} G_\alpha \to H$ such that the following diagram commutes.

$$\begin{tikzcd} G_\alpha \arrow[d, hook, "\iota_\alpha"'] \arrow[dr, "\varphi_\alpha"] & \\ \ast_{\alpha\in\lambda}G_\alpha \arrow[r, "\varphi"'] & H \end{tikzcd}$$

Here, $\iota_\alpha\colon G_\alpha \to \ast_{\alpha\in\lambda}G_\alpha$ is the inclusion.

Proof. Let's sketch the proof. We begin by showing uniqueness of φ. Given a word $g_{\alpha_1}\cdots g_{\alpha_n}$ in $\ast_{\alpha\in\lambda}$, we see that the commutativity of the diagram enforces

$$\begin{aligned} \varphi\left(g_{\alpha_1}\cdots g_{\alpha_n}\right) &= \varphi(g_{\alpha_1})\cdots\varphi(g_{\alpha_n}) \\ &= \varphi(\iota_{\alpha_1}(g_{\alpha_1}))\cdots\varphi(\iota_{\alpha_n}(g_{\alpha_n})) \\ &= \varphi_{\alpha_1}(g_{\alpha_1})\cdots\varphi_{\alpha_n}(g_{\alpha_n}). \end{aligned}$$

Thus, φ is uniquely determined by the φ_α. It remains to show that the above formula in fact defines a group homomorphism, which follows roughly speaking by the minimal construction of $\ast_{\alpha\in\lambda}$. Namely, we have thus far defined a function $\varphi\colon W \to H$ where W is the set of all words, so one needs to check that we are still safe after modding out by the requisite equivalence relation on W. We will not do this, but in short, one can use induction on the various generators of the group presentation of $\ast_{\alpha\in\lambda}G_\alpha$. ■

In the discussion that follows, we will frequently use group presentations, which is an expression of the form

$$\langle a_1, a_2, \ldots, : w_1, w_2, \ldots\rangle,$$

where the $a_\bullet$ are generators for words giving the group and $w_\bullet$ are words intended to produce relations for the group, by default of the form $w_\bullet = 1$.

Example 2.29. The group $\langle a\rangle$ gives $\mathbb{Z}$. Namely, the group consists of the elements

$$\left\{\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots\right\}.$$

Example 2.30. The group $\langle a : a^2\rangle$ gives $\mathbb{Z}/2\mathbb{Z}$. Namely, our isomorphism is by sending $k \in \mathbb{Z}/2\mathbb{Z}$ to a^k. This is well-defined because $2 \mapsto a^2$, and a^2 is the identity of the group.

2.3.2 van Kampen's Theorem

In this subsection, we state and prove the van Kampen theorem. Let's explain the idea. Suppose we can decompose X into path-connected open subsets $\{A_\alpha\}_{\alpha\in\lambda}$. Then the inclusions $i_\alpha\colon A_\alpha \hookrightarrow X$ induce maps $\pi_1(A_\alpha) \to \pi_1(X)$, which by the nature of the free product induces a map

$$\ast_{\alpha\in\lambda}\pi_1(A_\alpha) \to \pi_1(X).$$

It is not too hard to see that this map is surjective.

Lemma 2.31. Fix a topological space X which is the union of path-connected open subsets $\{A_\alpha\}_{\alpha\in\lambda}$ each containing a basepoint $x_0 \in X$. For any loop $\gamma\colon I \to X$ based at x_0, there are loops $\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}$ based at x_0 such that

$$\gamma \sim \gamma_{\alpha_1} \cdot \cdots \cdot \gamma_{\alpha_n}$$

and γ_{α_n} is a path connected in A_{α_n} for each α_n.

Proof. For each $\alpha \in \lambda$, decompose $\gamma^{-1}(A_\alpha) \subseteq I$ into a collection of intervals $\mathcal{I}_\alpha$. Then

$$I = \gamma^{-1}(X) = \bigcup_{\alpha\in\lambda} \gamma^{-1}(A_\alpha) = \bigcup_{\alpha\in\lambda} \bigcup_{I'\in\mathcal{I}_\alpha} I'.$$

Now, I is compact, so this open cover can be turned into a finite subcover $\{(a_k, b_k)\}_{k=1}^n$ where $\gamma((a_k, b_k)) \subseteq A_{\alpha_k}$ for some $\alpha_k \in \lambda$. Ordering the (a_k, b_k), we produce a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma([t_k, t_{k+1}]) \subseteq A_{\alpha_k}$ for some perhaps different n and $\alpha_k \in \lambda$.

We are now ready to finish. For each $1 \leqslant k \leqslant n-1$, we recall that A_{α_k} is path-connected, so we can find a path η_k from $\gamma(t_k)$ to x_0. Then we see that

$$\begin{aligned}\gamma &\sim \gamma|_{[t_0,t_1]} \cdot \gamma|_{[t_1,t_2]} \cdot \cdots \cdot \gamma|_{[t_{n-2},t_{n-1}]} \cdot \gamma|_{[t_{n-1},t_n]} \\ &\sim \underbrace{\gamma|_{[t_0,t_1]} \cdot \eta_1}_{\gamma_0:=} \cdot \underbrace{\overline{\eta}_1 \cdot \gamma|_{[t_1,t_2]} \cdot \eta_2}_{\gamma_1:=} \cdot \overline{\eta}_2 \cdot \cdots \cdot \eta_{n-2} \cdot \underbrace{\overline{\eta}_{n-2} \cdot \gamma|_{[t_{n-2},t_{n-1}]} \cdot \eta_{n-1}}_{\gamma_{n-2}:=} \cdot \underbrace{\overline{\eta}_{n-1} \cdot \gamma|_{[t_{n-1},t_n]}}_{\gamma_{n-1}:=} .\end{aligned}$$

The above expression provides the desired factorization. ■

Corollary 2.32. Fix a topological space X which is the union of path-connected open subsets $\{A_\alpha\}_{\alpha\in\lambda}$ each containing a basepoint $x_0 \in X$. Then the map induced map

$$\ast_{\alpha\in\lambda}\pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

Proof. This is direct from Lemma 2.31. ■

We would now like to compute its kernel of our induced map. Well, if $A_\alpha \cap A_\beta$ is path-connected, then we let $i_{\alpha\beta}\colon A_\alpha \cap A_\beta \to A_\alpha$ denote the inclusion, and we note that

$$\begin{tikzcd} A_\alpha \cap A_\beta \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & A_\alpha \arrow[d, "i_\alpha"] \\ A_\beta \arrow[r, "i_\beta"] & X \end{tikzcd}$$

commutes, so

$$\begin{tikzcd} \pi_1(A_\alpha \cap A_\beta) \arrow[r, "\pi_1(i_{\alpha\beta})"] \arrow[d, "\pi_1(i_{\beta\alpha})"'] & \pi_1(A_\alpha) \arrow[d, "\pi_1(i_\alpha)"] \\ \pi_1(A_\beta) \arrow[r, "\pi_1(i_\beta)"] & \pi_1(X) \end{tikzcd}$$

also commutes. Thus, for any $\gamma \in \pi_1(A_\alpha \cap A_\beta)$, we see that $\pi_1(i_\alpha)(\pi_1(i_{\alpha\beta})([\gamma])) = \pi_1(i_\beta)(\pi_1(i_{\beta\alpha})([\gamma]))$, which produces a relation belonging to the kernel of our surjection $\ast_{\alpha\in\lambda}\pi_1(A_\alpha) \to \pi_1(X)$. Under favorable circumstances, van Kampen's theorem tells us that this is the entire kernel.

2.4 September 12

Let's wrap up some loose ends. People are doing pretty well on the homeworks, but please cite theorems and so on to be rigorous.

2.4.1 The Fundamental Group of a Torus Knot

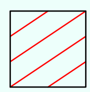
Let's give a few applications of van Kampen's theorem.

Example 2.33. Let $K \subseteq \mathbb{R}^n$ be a compact subset for $n \geqslant 3$. Embed $\mathbb{R}^n \subseteq S^n$ by stereographic projection, and van Kampen shows that

$$\pi_1\left(\mathbb{R}^n \backslash K\right) \cong \pi_1\left(S^n \backslash K\right).$$

More precisely, we have S^n sitting inside $S^{n-1} \times \mathbb{R}$ (place K inside a large ball, and we can continuously deform any loop in $\mathbb{R}^n \backslash K$ into this large ball), and the π_1 arising from this $\mathbb{R}$ cannot help you.

Example 2.34 (torus knots)**.** Fix positive integers $m, n \in \mathbb{Z}$ bigger than 1 with $\gcd(m, n) = 1$. Define the torus knot $K_{m,n} \subseteq T^2$ (where $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$) as the image of the line $my = nx$; alternatively, it is the image of the map $t \mapsto (mt, nt)$. For example, here is $K_{3,2}$ sitting inside the square $\mathbb{R}^2/\mathbb{Z}^2$.



We compute $\pi_1\left(\mathbb{R}^3 \backslash K_{m,n}\right)$.

Proof. Professor Agol seems to prefer the "Clifford torus" thought of as

$$T^2 = \left\{(z_1, z_2) : |z_1| = |z_2| = 1/\sqrt{2}\right\}.$$

This sits inside $S^3 = \left\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\right\}$. Anyway, we begin by giving us some breathing room. Define the "thickening" of K as

$$A := \left\{(z_1, z_2) : |z_1| < \frac{1}{\sqrt{2}} + \varepsilon\right\} \backslash \left\{\left(rz^m, \sqrt{1 - r^2} z^n\right) : z \in S^1, \frac{1}{\sqrt{2}} \leqslant r \leqslant \frac{1}{\sqrt{2}} + \varepsilon\right\}$$

(namely, A is the torus thickened in such a way that it carries the subtraction of $K_{m,n}$) and in the other way as

$$B := \left\{(z_1, z_2) : |z_2| < \frac{1\sqrt{2}}{+}\varepsilon\right\} \backslash \left\{(\sqrt{1 - r^2} z^m, rz^n) : \frac{1}{\sqrt{2}} \leqslant r \leqslant \frac{1}{\sqrt{2}} + \varepsilon\right\}.$$

Intersecting, we see that $A \cap B$ is $(S^1 \times S^1) \backslash K_{m,n}$ thickened by $(-\varepsilon, \varepsilon)$, which we note can be deformed to $(S^1 \times \mathbb{R}) \times (-\varepsilon, \varepsilon)$, which has fundamental group $\mathbb{Z}$. Notably, $\pi_1(A) \cong \mathbb{Z}$ and $\pi_1(B) \cong \mathbb{Z}$ by deforming them carefully to the circle S^1, so our fundamental group is going to be $(\mathbb{Z} * \mathbb{Z})/\mathbb{Z}$ by van Kampen.

However, we need to compute the image of $\pi_1(A \cap B)$ in $\pi_1(A) * \pi_1(B)$. In the retraction of A down to a circle says that the image in $\pi_1(A)$ is by multiplication by n, and similarly going to B is multiplication by m. We conclude that our fundamental group is

$$\langle a, b : a^n = b^m \rangle.$$

As an aside, we note that $S^3 \backslash K_{m,n}$ will have a deformation retract back to $K_{m,n}$ shifted upwards by some amount (for example, see the diagram and imagine a copy of $K_{m,n}$ shifted up by some small $\varepsilon > 0$). Anyway, for $m, n > 1$ we can see that the center of the above group is $\langle a^n \rangle$, so modding out by the center yields $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. In total, we are able to distinguish the torus knots $S^3 \backslash K_{m,n}$ from each other.[2] To deal with the signs of m and n, we need a notion of isotopy to distinguish a knot from its "mirror image." ■

[2] Alternatively, the abelianization $\pi_1(S^3 \backslash K_{m,n})$ is the free group with $(m-1)(n-1)$ generators, and the abelianization of $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$ is mn, from which we can read off m and n.

2.4.2 The Fundamental Group of a CW Complex

Let's move on from knots and compute the fundamental group of some cell complexes.

Example 2.35. Let X be a connected graph (i.e., a 1-dimensional CW complex), then $\pi_1(X)$ is homotopy equivalent to a wedge of circles, which has fundamental group $\mathbb{Z}^{*r}$ for some r, which is the free group on r letters.

Now, if Y is a connected CW complex, then $\pi_1(Y^1)$ is a free group. Then $\pi_1(Y^2)$ might be complicated, but let's imagine computing $\pi_1(Y^3)$. The point is that we take some ball $e^3_\alpha \cong D^2$ and attach it via some $\varphi_\alpha\colon \partial D^3 \to Y^2$.

To compute the fundamental group of this, we cover $Y^2 \sqcup_{\varphi_\alpha} e^3_\alpha$ by $A := Y^2 \cup_{\varphi_\alpha} (e^3_\alpha \backslash \{x\})$ and $B = (e^\alpha_3)^\circ$ (Here, x is some point in the interior.) Notably, the intersection is simply $S^2 \times \mathbb{R}$, which is trivial, so we conclude that the attachment e^3_α did nothing to our fundamental group by van Kampen. Applying this argument inductively (perhaps transfinitely), we see that $\pi_1(Y^3) = \pi_1(Y^2)$. One can continue upwards to conclude that $\pi_1(Y) = \pi_1(Y^2)$.[3]

Now, let's say that we actually want to compute $\pi_1(Y^2)$. To do so, we note that we have a surjection $\pi_1(Y^1) \to \pi_1(Y^2)$ given by the inclusion (any loop can be deformed off the 2-skeleton to the 1-skeleton). Now, for each 2-cell e^2_α attached via $\varphi_\alpha\colon \partial e^2_\alpha \to Y^1$, we choose a path $\gamma_\alpha\colon I \to Y^1$ so that $\gamma_\alpha(0) = y$ and $\gamma_\alpha(1) = \varphi_\alpha(0)$ and then find that

$$\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$$

ought to be in the kernel of our projection. An argument shows that these elements will generate the needed kernel. One can show this by an analogous argument to the above: the point is that the attachment of e^2_α kills basically exactly the loop given above and nothing else, so we can use an inductive argument to conclude.

Remark 2.36. One can use this result to show that any group G arises as the fundamental group of a CW complex of dimension 2. Roughly speaking, the point is that any group is the quotient of a free group, and the above argument allows us to dictate relations, provided that we are sufficiently careful.

Example 2.37. Fix a positive integer g. Define S_g by starting with a $4g$-gon and attaching the edges. Namely, for $n < 3$, an $n \pmod 4$ edge is identified with the next over $n + 2 \pmod 4$ edge in the opposite direction. Roughly speaking, after some manipulation, one finds that S_g ought to be a g-hole torus. Using the above argument, one finds that $\pi_1(S_g)$ is generated by $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ modded out by the relations $a_i b_i a_i^{-1} b_i^{-1}$ for each i. In particular, the abelianization of $\pi_1(S_g)$ has all the commutators, so we get $\mathbb{Z}^{2g}$. Thus, π_1 distinguishes our surfaces.

Example 2.38 (projective space)**.** We note $\pi_1(\mathbb{RP}^\infty) \cong \pi_1(\mathbb{RP}^2)$ because the higher cells cannot help you in the fundamental group. Further, we see $\pi_1(\mathbb{RP}^2)$ is a disk with semicircles identified in the opposite direction, which we can see from the above argument is simply $\mathbb{Z}/2\mathbb{Z}$.

Example 2.39 (lens space)**.** Fix positive integers p and q with $\gcd(p, q) = 1$. Take S^2 and divide the equator into p circles, and we glue the top hemisphere to the bottom hemisphere by gluing after a $2\pi p/q$ rotation. The space has fundamental group $\mathbb{Z}/p\mathbb{Z}$. Indeed, our 1-skeleton is the equator, and the a^p comes from how we attached our disks together.

Next time we will talk about covering spaces.

[3] One can do this without transfinite induction by working with a single loop and arguing about homotopy equivalence. The point is that a single loop (and in fact a single homotopy) can is compact and therefore only cares about finitely many cells.

2.5 September 14

We're talking about covering spaces today.

2.5.1 Examples of Covering Spaces

Our goal is to generalize the method we used to compute $\pi_1(S^1)$. Let's recall our definition.

Definition 2.17 (covering space)**.** Fix a topological space X. Then a *covering space* is a topological space $\widetilde{X}$ together with a projection map $p\colon \widetilde{X} \to X$ such that each $x \in X$ has an open neighborhood $U \subseteq X$ containing x such that $p^{-1}(U) = \bigsqcup_{\alpha\in\lambda} U_\alpha$ where U_α is open and $p\colon U_\alpha \to U$ is a homeomorphism. In this set up, the open set $U \subseteq X$ is said to be *evenly covered*.

Example 2.40. The map $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$ is a covering space map. Here, we are viewing S^1 as $\{z \in \mathbb{C} : |z| = 1\}$. The point is that, for any $e^{2\pi i\theta} \in S^1$, we have

$$p^{-1}\left(S^1 \backslash \left\{e^{2\pi i\theta}\right\}\right) = \bigsqcup_{n\in\mathbb{Z}} (\theta, \theta + 2\pi).$$

Non-Example 2.41. The map $p\colon (0,2) \to S^1$ given by $t \mapsto e^{2\pi i t}$ is not a covering space map. For example, any open interval U around $1 \in S^1$ will have pre-image by p looking like $(0,\varepsilon) \sqcup (1-\varepsilon, 1+\varepsilon) \sqcup (2-\varepsilon, 2)$, and $(0,\varepsilon)$ is not mapped homeomorphically to our $U \subseteq S^1$.

Example 2.42. The map $f\colon \mathbb{C}^\times \to \mathbb{C}^\times$ given by $f\colon z \mapsto z^n$ for a positive integer n is a covering space map. Roughly speaking, for any ray ℓ through the origin in $\mathbb{C}$, one can define $\log\colon (\mathbb{C}\backslash\ell) \to \mathbb{C}$, which allows us to define an nth root $\sqrt[n]{w} := \exp\left(\frac{1}{n}\log w\right)$ on $\mathbb{C}\backslash\ell$; this makes $\mathbb{C}\backslash\ell$ into an evenly covered subset, so we are a covering space upon letting ℓ vary.

Example 2.43. Fix a topological space X and a discrete set E. Then of course $p\colon X \times E \to X$ is a covering space: indeed, X is an evenly covered subset. In fact, if $p\colon \widetilde{X} \to X$ is a covering space map where X is evenly covered, then the definition of p requires $\widetilde{X} \cong \bigsqcup_{\alpha\in\lambda} X_\alpha$

Example 2.44. Map $p\colon S^\infty \to \mathbb{RP}^\infty$ by sending $x \in S^\infty$ to the corresponding line in $\mathbb{RP}^\infty$. More precisely, embed some $S^n \subseteq S^\infty$ into $\mathbb{R}^{n+1}$ and then take lines down to $\mathbb{RP}^n$. Notably, $p(x) = p(-x)$ for each x (and conversely $p(x) = p(y)$ if and only if $\mathbb{R}x = \mathbb{R}y$ if and only if $x = \pm y$), so p is 2-to-1. One can check that p is a covering space map by looking on the level of cell complexes: the pre-image of the interior of the unique n-cell $(e^n)^\circ \subseteq \mathbb{RP}^\infty$ is the disjoint union of the interior of the two n-cells of S^∞. More precisely, the n-cell e_i^n inside $\mathbb{RP}^n$ given by

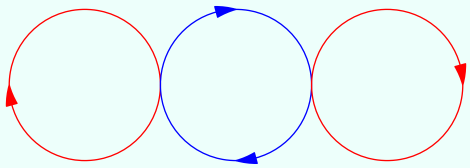
$$\{[x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] : x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbb{R}\}$$

is evenly covered in the map $S^n \to \mathbb{RP}^n$. One can extend this idea up to $\mathbb{RP}^\infty$ to conclude: let e_i be the above subset except we do not terminate at x_n, and then e_i is covered by the open subsets $e_{i,\pm} \subseteq S^\infty$ defined as

$$e_{i,\pm} = \{(x_0, x_1, \ldots) \in S^\infty : x_i \text{ has sign } \pm\}.$$

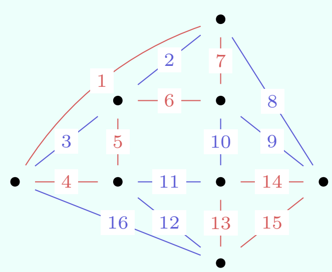
Let's do a few examples on $S^1 \vee S^1$.

Example 2.45. We examine 2-fold (i.e., 2-to-1) covers of $S^1 \vee S^1$. There is the trivial one with two copies of $S^1 \vee S^1$. As another example, note that $S^1 \vee S^1 \vee S^1$ loop around $S^1 \vee S^1$ twice: the first S^1 goes around the first S^1, then half of the second S^1 goes around the second S^1, then the third S^1 goes around the first S^1 around. Here is an image.

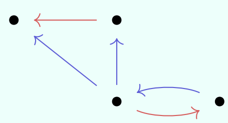


It turns out that, with one more, these are all the 2-to-1 covering maps, which can be seen by finding index-2 subgroups of $\pi_1\left(S^1 \vee S^1\right) = \mathbb{Z} * \mathbb{Z}$, as we will soon see.

Example 2.46. Consider the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$. This is then a covering space of $S^1 \vee S^1$ by sending the $\mathbb{Z} \times \mathbb{R}$ to traverse one of the circles S^1 and the $\mathbb{R} \times \mathbb{Z}$ to traverse the other circle of S^1. More generally, it turns out that covering spaces are exactly graphs where every vertex has degree 4, which we can see by coloring the edges red and blue so that each vertex has exactly two red edges and two blue edges; then choosing an Euler cycle provides the needed covering space. The previous example is one way to do this. Here is another example of such a graph, with marked Euler cycle.



Example 2.47. We can take a subgroup of $\mathbb{Z} * \mathbb{Z}$ to produce a covering space of $S^1 \vee S^1$. As an example, take the subgroup generated by ab and $b^{-1}ab$. Reading off these generators produces a graph as follows.



In general, we basically fold edges together to make relations. For example, the multiple outgoing blue edges should be folded together.

Example 2.48. There is an infinite tree where each vertex has degree 4. A coloring of the edges produces a "Cayley graph" C_2, which will turn out to be the universal covering space once we define such a notion. It turns out to be maximal in the sense that it covers any path-connected cover of $S^1 \vee S^1$.

2.5.2 Lifting with Covering Spaces

We will want the following result.

Proposition 2.49. Covering spaces have the homotopy lifting property. In other words, given a covering space $p\colon \widetilde{X} \to X$, a "homotopy" $f_\bullet\colon Y \times I \to X$ with a given lift $\widetilde{f}_0\colon Y \to \widetilde{X}$ will lift uniquely to $\widetilde{f}_\bullet\colon Y \times I \to \widetilde{X}$ agreeing with X.

Proof. This is direct from Proposition 2.18. ■

Corollary 2.50. Fix a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Then $\pi_1(p)\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x)$ is injective.

Proof. Fix some loop $\widetilde{f}_0\colon I \to \widetilde{X}$ in the kernel of $\pi_1(p)$. Then there is a homotopy $f_\bullet\colon I \times I \to X$ from $\widetilde{f}_0$ to the constant path, which by Proposition 2.49 will lift uniquely to a homotopy $\widetilde{f}_\bullet\colon I \times I \to \widetilde{X}$ agreeing on $\widetilde{f}_0$. Now $p \circ \widetilde{f}_1$ is constant, so looking locally at $\widetilde{x}_0$, we conclude that $\widetilde{f}_1$ is constant, so $\widetilde{f}_0$ is homotopic to the constant map and hence vanishes in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. ■

2.6 September 19

Today we continue discussing covering spaces.

2.6.1 Using Path-Lifting

Last time we showed that covering space maps $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induce subgroups $\pi_1(\overline{X}, \overline{x}_0) \to \pi_1(X, x_0)$. Note this subgroup can then communicate information about the covering space.

Proposition 2.51. Fix a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of path-connected spaces. Then the number of sheets covering an evenly covered neighborhood of x_0 is the index

$$\left[\pi_1(X, x_0) : \pi_1(\widetilde{X}, \widetilde{x}_0)\right],$$

where we have implicitly embedded $\pi_1(\widetilde{X}, \widetilde{x}_0) \hookrightarrow \pi_1(X, x_0)$.

Remark 2.52. Because X is connected, the number of sheets of the covering space map is well-defined. Indeed, for any positive integer n, the set of $x \in X$ such that there is an n-sheeted evenly covered open neighborhood $U_x \subseteq X$ is open. So we produce a continuous map $X \to \mathbb{N}$ sending x to the number of sheets, so connectedness of X forces the number of sheets to be constant.

Proof. We roughly describe the idea. Let $\Omega(Y, y_1, y_2)$ denote the set of homotopy classes of paths from y_1 to y_2. The point is that $\Omega(X, x_0, x_0)$ is in bijection with

$$\bigsqcup_{\widetilde{x} \in p^{-1}(\{x_0\})} \Omega(\widetilde{X}, \widetilde{x}_0, \widetilde{x})$$

by lifting paths. Now, $\pi_1(\widetilde{X}, \widetilde{x}_0)$ acts on $\Omega(\widetilde{X}, \widetilde{x}_0, \widetilde{x})$ for each $\widetilde{x}$, and each orbit will correspond to a coset of our quotient. ■

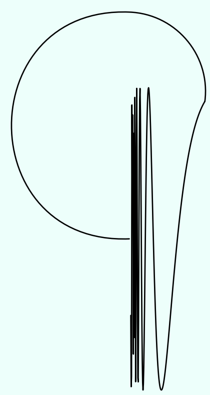
Remark 2.53. Proposition 2.51 can help us check that the covers of Example 2.45 are 2-to-1. For example, the subgroup corresponding to the shown covering space is $\langle a, b^2, bab \rangle$. Note that we have produced the free group with free generators as a subgroup of the free group with two generators.

We would like to go in the other direction, from subgroups back to covering space maps. This requires some technical hypotheses.

Definition 2.54 (locally path-connected)**.** A topological space X is *locally path-connected* if and only if each open neighborhood $U \subseteq X$ of a point $x \in X$ has some perhaps smaller open neighborhood $U' \subseteq U$ of $x \in X$ which is path-connected.

Example 2.55. CW complexes are locally path-connected.

Non-Example 2.56. The topologist's $\sin$ curve is not locally path-connected at the origin $(0, 0)$.



Being locally path-connected allows us to lift covering spaces.

Proposition 2.57. Fix a path-connected, locally path-connected topological space Y with basepoint $y_0 \in Y$. For a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and continuous map $f\colon (Y, y_0) \to (X, x_0)$, there is a lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ making the following diagram commute if and only if $\pi_1(f)(\pi_1(Y, y_0)) \subseteq \pi_1(p)(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

$$\begin{tikzcd} (Y, y_0) \arrow[r, "\widetilde{f}"] \arrow[dr, "f"'] & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ & (X, x_0) \end{tikzcd}$$

Proof. The backwards direction follows from functoriality of π_1 because we are asking for $\pi_1(f) = \pi_1(p) \circ \pi_1(\widetilde{f})$. For any $y \in Y$, composition with f defines a composite

$$\bigsqcup_{y \in Y} \Omega(Y, y_0, y) \to \bigsqcup_{x \in X} \Omega(X, x_0, x) \to \bigsqcup_{\widetilde{x} \in \widetilde{X}} \Omega(\widetilde{X}, \widetilde{x}_0, \widetilde{x})$$

where the last map is by path-lifting $\widetilde{\cdot}$. Then for any path $\gamma \in \Omega(Y, y_0, y)$, we simply define $\widetilde{f}(y) := \widetilde{f \circ \gamma}(1)$. To see that this is well-defined, the point is that choosing a different path $\gamma' \in \Omega(Y, y_0, y)$ produces is able to lift to basically a loop upstairs in $\widetilde{X}$, so the value of $\widetilde{f}(y)$ does not move.

For continuity, we will need to use that Y is locally path-connected. Fix $y \in Y$, and we will show that $\widetilde{f}$ is continuous at y. Set $x := f(y)$, and let U be an evenly covered neighborhood of x, and lift it to $\widetilde{U} \subseteq \widetilde{X}$ where $p\colon \widetilde{U} \to U$ is a homeomorphism, and $\widetilde{f}(y) \in \widetilde{U}$. We now may choose a path-connected open subset $V \subseteq f^{-1}(U)$ containing y and check continuity using $\widetilde{U}$, where $\widetilde{f}(y')$ for any $y' \in V$ can be somewhat easily defined because V is path-connected. ■

In fact, we have uniqueness of this lifting.

Proposition 2.58. Fix a connected topological space Y, and fix a covering space map $p\colon \widetilde{X} \to X$ and a map $f\colon Y \to X$. Given lifts $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ such that $p \circ \widetilde{f}_1 = f = \circ \widetilde{f}_2$ and $\widetilde{f}_1$ and $\widetilde{f}_2$ agree at a single point, we have $\widetilde{f}_1 = \widetilde{f}_2$.

Proof. Define the subsets

$$E := \left\{y \in Y : \widetilde{f}_1(y) = \widetilde{f}_2(y)\right\} \qquad \text{and} \qquad N := \left\{y \in Y : \widetilde{f}_1(y) \neq \widetilde{f}_2(y)\right\}.$$

One can use the covering space decomposition (by looking locally at $f(y)$ for some $y \in Y$) to show that both E and N are open, but they are disjoint with E nonempty, so connectedness of Y forces $Y = E$. ■

2.6.2 Classifying Covering Spaces

Our goal, roughly speaking, is to construct universal covers.

Definition 2.59 (univesal cover). A covering space map $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a *universal cover* if and only if $\widetilde{X}$ is simply-connected (i.e., path-connected and $\pi_1(\widetilde{X}, \widetilde{x}_0) = 1$).

Remark 2.60. Proposition 2.57 tells us that a universal cover $\widetilde{X}$ will cover any covering space of X.

We will want yet another definition.

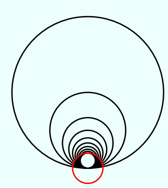
Definition 2.61 (semilocally simply-connected). A space X is *semilocally simply-connected* if and only if each $x \in X$ has an open neighborhood U of x such that the induced inclusion $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map.

Remark 2.62. Let's explain this condition. Suppose $(\widetilde{X}, \widetilde{x}_0) \to (X, x)$ is a simply connected and path-connected covering space. Then any evenly covered subset $U \subseteq X$ with lift $\widetilde{U}$, then the inclusion $\pi_1(U) \to \pi_1(X)$ decomposes as

$$\pi_1(U) \to \pi_1(\widetilde{U}) \to \pi_1(\widetilde{X}) \to \pi_1(X),$$

which must be the trivial map because $\pi_1(\widetilde{X}) = 1$. In other words, we have checked that X is semilocally simply-connected.

Example 2.63. The earring space is not semilocally simply-connected at the origin because any neighborhood at the origin will have circles inside.



Being semilocally simply-connected is basically, then, the right hypothesis to have a universal cover.

Theorem 2.64. Let X be a topological space which is path-connected, locally path-connected, and semilocally simply-connected. Then X has a simply-connected covering space $\widetilde{X} \to X$ which is unique up to isomorphism of pointed topological spaces over X.

Proof. Uniqueness follows from Proposition 2.57 because the corresponding lifts we write down must be local homeomorphisms.

It remains to show existence. Fix a basepoint $x_0 \in X$. We simply define

$$\widetilde{X} := \{[\gamma] : \gamma \text{ is a path } I \to X \text{ with } \gamma(0) = x_0\}.$$

The point is that paths should lift uniquely up to $\widetilde{X}$ already, so we might as well define $\widetilde{X}$ in this way. We may define the function $p\colon \widetilde{X} \to X$ by sending $[\gamma] \mapsto \gamma(1)$. It remains to show that $\widetilde{X}$ is a simply-connected topological space and that p is a covering space map.

Let's produce a topology on $\widetilde{X}$. Using our hypotheses on X, each $x \in X$ has a path-connected open neighborhood $V \subseteq X$ such that $\pi_1(V) \to \pi_1(X)$ is trivial. We then use V to define a subset around $[\gamma]$ with $\gamma(0) = x_0$ and $\gamma(1) = x$ by

$$\widetilde{V} := \{[\gamma \cdot \gamma'] : \gamma' \text{ is a path } I \to V \text{ such that } \gamma'(0) = x_0 \text{ and } \gamma'(1) = y\}.$$

Now, $\widetilde{V}$ is in bijection with V by p, so we make the restricted map $p\colon \widetilde{V} \to V$ a homeomorphism. One can check that the topology is well-defined and that p becomes a covering space map from this. ■

One can now use the universal cover to produce any covering space.

Theorem 2.65. Let X be a pointed topological space which is path-connected, locally path-connected, and semilocally simply-connected, and let $x_0 \in X$ be a basepoint. Then there is a bijection between pointed path-connected covering spaces $(Y, y_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$. Unpointed covering space maps correspond to conjugacy classes of subgroups.

To produce the desired covering space given a subgroup, one repeats the proof of Theorem 2.64 by taking a quotient of the produced $\widetilde{X}$. Then one shows that this is a bijection with some work.

Remark 2.66. One can also use permutations of the pre-image of a basepoint in order to describe our covering spaces. Namely, if $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space, then any loop $[\alpha] \in \pi_1(X, x_0)$ will lift to a permutation of $p^{-1}(\{x_0\})$. Conversely, such automorphisms are able to produce an automorphism of the universal covering space $\widetilde{X} \to \widetilde{X}$. (On the level of paths, we send $[\gamma] \in \widetilde{X}$ to $[\gamma \cdot \alpha]$. One can check that this is continuous with continuous inverse and thus a homeomorphism.)

2.7 September 21

We continue to cover spaces.

2.7.1 Deck Transformations

Let X be a path-connected, locally path-connected, and semilocally simply-connected space with universal cover $\widetilde{X} \to X$. We would like to use the universal cover to produce intermediate covering maps.

Definition 2.67 (deck transformation)**.** Let X be a path-connected, locally path-connected, and semilocally simply-connected space with cover $p\colon \widetilde{X} \to X$. A homeomorphism $f\colon \widetilde{X} \to \widetilde{X}$ such that $p = p \circ f$ is called a *deck transformation*.

In our set-up, let G be the group of deck transformations of the universal cover $\widetilde{X} \to X$. Then $G \cong \pi_1(X)$. Let's explain why. Fix a basepoint $\widetilde{x}_0 \in \widetilde{X}$ lying over $x_0 \in X_0$. The point is that a deck transformation is uniquely determined by where it sends $\widetilde{x}_0$ by how path-lifting works. So a deck transformation $f\colon \widetilde{X} \to \widetilde{X}$ produces a path from $\widetilde{x}_0$ to $f(\widetilde{x}_0)$ (which is unique up to homotopy class because $\widetilde{X}$ is simply-connected), and then mapping this down to p produces an element of $\pi_1(G, x_0)$. And conversely a loop in $\pi_1(X, x_0)$ lifts to a path up in $\widetilde{X} \to \widetilde{X}$ sending $\widetilde{x}_0 \mapsto f(\widetilde{x}_0)$, and there is a unique automorphism $f\colon \widetilde{X} \to \widetilde{X}$ sending $\widetilde{x}_0$ to the right place.[4]

Remark 2.68. More generally, if $(Y, y_0) \to (X, x_0)$ is any covering space, one has a bijection between $\pi_1(X, x_0)/\pi_1(Y, y_0)$ and points in the fiber of x_0.

Extending the above discussion, we have the following result.

Theorem 2.69. Fix a path-connected covering space $p\colon (Y, yx_0) \to (X, x_0)$, and let G be the group of deck transformations. Then X is homeomorphic to Y/G in the natural way if and only if $\pi_1(Y, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$. In this case, $G \cong \pi_1(X)/\pi_1(Y)$.

Proof. Let $(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal cover. Then the universal property allows us to factor as follows.

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{x}_0) \arrow[r, "r"] \arrow[dr, "q"'] & (Y, y_0) \arrow[d, "p"] \\
& (X, x_0)
\end{tikzcd}$$

Now, for all $y \in p^{-1}(\{x_0\})$, we see that $r^{-1}(\{y\})$ will correspond to a coset of $\pi_1(\widetilde{X})$ in $\pi_1(X)$ via the discussion with the universal cover; looping over y, we produce a bijection with points in the fiber of $q^{-1}(\{x_0\})$.
What? Normality of the subgroup then follows because the action of G here is trying to act on cosets. ■

A less careful version of this discussion lets us work with more general subgroups.

Proposition 2.70. Let X be a path-connected, locally path-connected, and semilocally simply connected space with universal cover $p\colon \widetilde{X} \to X$. For any subgroup $H \subseteq \pi_1(X, x_0)$, the quotient space $\widetilde{X}/H$ is a covering space of X and has fundamental group H.

Proof. Track through the above discussion without focusing on the group being normal. ■

2.7.2 Attempts for Universal Covers

We are interested in the universal covering space construction having the lifting property. For our purposes, we will assume that our topological space (X, x_0) which is locally path-connected, and we can still just try to define $\widetilde{X}$ as the set of homotopy classes of paths starting at x_0. Then the topology is defined by building a sub-base as follows: for open path-connected subsets $V \subseteq X$, one defines an open set around $[\gamma]$ with $\gamma(0) = x_0$ and $\gamma(1) = x$ by

$$\widetilde{V} := \{[\gamma \cdot \gamma'] : \gamma' \text{ is a path } I \to V \text{ such that } \gamma'(0) = x_0 \text{ and } \gamma'(1) = y\}.$$

Let's see some examples.

Example 2.71. Let's apply this to the earring E. One can show that this construction produces an open map $p\colon \widetilde{E} \to E$, but it is not a covering space. Nonetheless, $\widetilde{E}$ is path-connected, locally path-connected, and simply-connected, and it has the unique path-lifting property. Indeed, for any locally path-connected map $f\colon Y \to E$ where $f_*\pi_1(Y) \subseteq \pi_1(X)$ is trivial, the map f factors through p uniquely.

One might want to try to draw $\widetilde{E}$, but this is hard: for example, with $e \in E$ the vertex of the earring, one has $p^{-1}(\{e\})$ uncountable, and E is an $\mathbb{R}$-tree, meaning any two points has a unique path connecting them.

[4] At this point, it is perhaps clearer to use the direct construction of $\widetilde{X}$ as homotopy classes of paths starting at x_0.

Example 2.72. Let $X := \prod_{i\in\mathbb{N}} S^1$. By how the product topology works, this remains path-connected (as the product of path-connected spaces) but is not semilocally simply-connected because any open set contains at least one S^1, which fails to be simply-connected. Nonetheless, the map $\mathbb{R} \to S^1$ remains continuous, so there is a map

$$\prod_{i\in\mathbb{N}} \mathbb{R} \to \prod_{i\in\mathbb{N}} S^1$$

which behaves like a covering space.

Example 2.73. CW-complexes X are locally contractible and hence locally path-connected and locally simply-connected. Thus, our construction provides a universal covers for connected CW complexes. For simplicity, we work with the 2-skeleton $X^{(2)}$, which encodes all π_1-information anyway, and we will focus on constructing $\widetilde{X}$. One can show that the covering space of a CW-complex remains a CW-complex because one can lift sufficiently small evenly covered cells to produce a CW-structure on the covering space. Looking at how $X^{(2)}$ is constructed by adding 2-cells to produce quotients, we see that $\widetilde{X}^{(1)}$ corresponds to the kernel of $\pi_1(X^{(1)}) \to \pi_1(X)$, which by van Kampen is the normal subgroup generated by 2-cells as $\pi_1(\partial e^2_\bullet)$ for the various $e^2_\bullet$.

Example 2.74. Fix coprime positive integers p and q, and construct the lens space $L(p,q)$ by taking the quotient of D^3 by dividing an equator S^1 into q pieces and then gluing the top and bottom hemisphere after rotating by $2\pi q/p$. Equivalently, one can view this as $S^3/(\mathbb{Z}/p\mathbb{Z})$, where the action is given by $k \cdot (z_1, z_2) := \left(\zeta_p^k z_1, \zeta_{qk} z_2\right)$. One sees that $L(p,q)$ is a CW-complex with 1-skeleton given by S^1 and two-skeleton by attaching D^2 and identifying z with $\zeta_q z$ for each z.

- $\widetilde{L(p,q)}^{(1)}$ is S^1 again, but it is viewed as the p-fold cover of S^1.
- $\widetilde{L(p,q)}^{(2)}$ is p disks glued at their boundaries.
- $\widetilde{L(p,q)}$ fills in these disks with 3-balls.

Example 2.75. Suppose $X^{(1)} = \bigvee_S S^1$, then $\pi_1\left(X^{(1)}\right)$ is the free group on S as letters. Each attached 2-cell to $X^{(1)}$ gives a relation for $G := \pi_1\left(X^{(2)}\right)$. Now, $\widetilde{X^{(2)}}^{(1)}$ turns out to be Cayley graph of G, and its 0-skeleton is in bijection with G, where edges are given by group elements in the natural way.

Let's be more explicit: for any generating set $S \subseteq G$, let N be the kernel of the surjection $F(S) \twoheadrightarrow G$, and then we can view our Cayley graph via some covering space quotient.

For the next few examples, we have the following definition.

Definition 2.76. We say that a CW-complex X is $K(G,1)$ if and only if it has fundamental group G and has contractible $\widetilde{X}$.

It turns out that $K(G,1)$ is unique up to homotopy equivalence, so it allows us to talk more canonically about the group G via topology. Here are some examples.

Example 2.77. Note $K(\mathbb{Z},1) = S^1$ because $\widetilde{S^1} = \mathbb{R}$ is contractible.

Example 2.78. Note $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^\infty$ because $S^\infty = \widetilde{\mathbb{RP}^\infty}$ is contractible.

Example 2.79. We see $S^1 \times S^1 = K(\mathbb{Z}^2, 1)$ because the universal cover of $S^1 \times S^1$ is the contractible space $\mathbb{R}^2$. Of course, we can take arbitrary powers and products like this.

2.8 September 26

Today we discuss free groups and graphs.

2.8.1 Spanning Tree

For technical reasons, it will be helpful to rigorize give graphs a CW topology.

Definition 2.80 (graph)**.** A *graph* is a 1-dimensional CW complex X built as follows: the vertices are X^0, and the edges are built by taking two vertices $v_1, v_2 \in X^0$ and connecting them by an edge e_α with $\partial e_\alpha = \{v_1, v_2\}$.

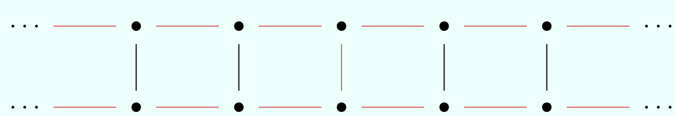
Remark 2.81. A graph X with a vertex $v \in X$ of infinite degree fails to be locally compact. Indeed, any open neighborhood of v will intersect infinitely many edges, which is not contained in any compact set because one can build an open cover with an open set from each of the individual edges, from which no finite subcover is possible to construct.

Definition 2.82 (subgraph)**.** A *subgraph* is a closed CW subcomplex of a graph.

Trees are the simplest graphs.

Definition 2.83 (tree)**.** A *tree* is a contractible graph. A subtree T of a graph X is *maximal* or *spanning* if and only if $T^0 = X^0$.

Example 2.84. The highlighted edges make a maximal subtree of the following graph.



We have the following result on trees.

Proposition 2.85. Any connected graph X contains a maximal tree. In fact, any subtree can be extended to a maximal tree.

Proof. We begin by fixing some subtree $X_0 \subseteq X$. Then to construct X_{n+1} from X_n, we look at the set of vertices adjacent to a vertex in X_n, and we add exactly one edge to X_{n+1} to add in all these vertices. Each added edge maintains being contractible, and adding them all in at once will continue to be contractible; explicitly, X_{n+1} has a deformation retract back to X_n and will therefore be contractible by induction.

Eventually the union T of $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ will hit every vertex: note X is connected and locally path-connected hence path-connected, so it follows that any two vertices can be connected by a path, which may only hit finitely many vertices and edges along its path by compactness of the interval.[5] Thus, T is the desired subtree. ■

[5] Hitting infinitely many vertices or edges implies that the image of $[0, 1]$ has an infinite discrete closed subset (choose a single point from each vertex and from each hit edge), violating compactness.

Remark 2.86. One needs some form of the axiom of choice to achieve the above result because we may be making infinitely many choices in the construction of X_{n+1} from X_n.

2.8.2 Fundamental Groups of Graphs

Having spanning trees allows us to compute fundamental groups. Fix a spanning tree $T \subseteq X$. Fix a basepoint $x_0 \in T$. Then each edge e_α of $X \setminus T$ produces a loop based at x_0: if e_α connects v_1 and v_2, then we have a loop going from x_0 to v_1 (through T) to v_2 (through e_α) and back to x_0 (through T again). These loops generate the fundamental group.

Proposition 2.87. Fix a connected graph X with spanning tree T. Then $\pi_1(X)$ is a free group with basis $[e_\alpha]$ where e_α is an edge of $X \setminus T$.

Proof. The quotient map $X \twoheadrightarrow X/T$ is a homotopy equivalence because T is contractible (it's a tree). However, X/T now only has a single vertex x_0, and we see that each edge e_α of $X \setminus T$ then goes down to a loop at x_0. Thus, X/T is S^1 wedged with itself once for each edge in $X \setminus T$, so the result follows. ■

Our work allows us the following application.

Lemma 2.88. Every covering space of a graph X is itself a graph whose vertices and edges as pre-images.

Proof. Let $p \colon \widetilde{X} \to X$ be a covering space. Set vertices of $\widetilde{X}$ to be $p^{-1}\left(X^0\right)$, and our edges are similarly given by pre-images because p is locally a homeomorphism, we see that $\widetilde{X}$ has the desired topology. ■

Theorem 2.89. Any subgroup of a free group is free.

Proof. A free group F generated by κ generators is the fundamental group of the graph $X := \left(S^1\right)^\kappa$. Then any subgroup $F' \subseteq F$ arises from the fundamental group of the covering space $p \colon \widetilde{X} \to X$, and the lemma tells us that $\widetilde{X}$ is a graph, so its fundamental group is in fact also free by Proposition 2.87. ■

The above result is quite nice: it is quite non-obvious that this result should be true purely from the algebra, but the topology makes it easier to attack.

Remark 2.90. There is an algorithm (due to Reidemeister–Schreier) to find a generating set for finite-index subgroups of a free group.

2.8.3 $K(G,1)$s

We have the following definition.

Definition 2.91 ($K(G,1)$)**.** Fix a group G. A path-connected topological space X is a $K(G,1)$ if and only if $\pi_1(X) \cong G$, and X has a contractible universal cover.

It turns out that $K(G,1)$ is unique up to homotopy equivalence. Here are some examples.

Example 2.92. The space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$. The fundamental group can be computed by seeing that the universal cover is $S^\infty \twoheadrightarrow \mathbb{RP}^\infty$. Let's see that S^∞ is in fact contractible: the map $(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$ defines an embedding $i\colon S^\infty \to S^\infty$. However, i has a linear homotopy to id given by

$$f_t(x_1, x_2, \ldots) := \frac{(1-t)(x_1, x_2, \ldots) + t(0, x_1, \ldots)}{\|(1-t)(x_1, x_2, \ldots) + t(0, x_1, \ldots)\|},$$

and then i has a linear homotopy to a constant map by

$$g_t(x_1, x_2, \ldots) := \frac{(1-t)(1, 0, \ldots) + t(0, x_1, \ldots)}{\|(1-t)(1, 0, \ldots) + t(0, x_1, \ldots)\|}.$$

(Note we needed the inclusion i because the linear combination $(1-t)(1, 0, \ldots) + t(x_1, x_2, \ldots)$ goes through the origin if we use the point $(x_1, x_2, \ldots) = (-1, 0, \ldots)$.)

Example 2.93. The space $S^\infty/(\mathbb{Z}/m\mathbb{Z})$ is a $K(\mathbb{Z}/m\mathbb{Z}, 1)$. Here, $\mathbb{Z}/m\mathbb{Z}$ acts on S^∞ by having $1 \in \mathbb{Z}/m\mathbb{Z}$ be pointwise multiplication by $e^{2\pi i/m}$. The covering space is still S^∞, which is contractible by the previous example.

Example 2.94. Fix a closed, connected subspace $K \subseteq S^3$ (thought of as a knot). If $G := \pi_1\left(S^3 \setminus K\right)$, then $S^3 \setminus K$ is a $K(G, 1)$; this is a result to Papakyriakopoulos (yes, this name is hard to spell). Note that having S^3 is important; otherwise, if K is bounded, we could just place a large box around $K \subseteq \mathbb{R}^3$, and it is not possible to contract this box in $\mathbb{R}^3 \setminus K$. Instead, we want to contract it in S^3 by passing to the point at infinity.

Example 2.95. Let X_G be a $K(G, 1)$, and let X_H be a $K(H, 1)$, and we assume that both are CW complexes. Then $X_G \times X_H$ (given the product topology!) becomes a $K(G \times H, 1)$ because the universal cover of $X_G \times X_H$ is the product of the universal covers, which will then remain contractible.

Taking a product of $K(\mathbb{Z}/m\mathbb{Z}, 1)$s, we see that there is a $K(G, 1)$ for a finitely generated abelian group G. One can in fact give a $K(G, 1)$ for any group G, though this trickier. Let's see this. The following notions will be helpful.

Definition 2.96 (simplex)**.** An n-*simplex* is constructed by taking affinely linearly independent vectors $v_0, \ldots, v_n \in \mathbb{R}^m$ (i.e., the set $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent—note that this condition is independent of rearranging the $v_\bullet$) and setting

$$[v_0, v_1, \ldots, v_n] := \left\{\sum_{i=0}^n t_i v_i : 0 \leqslant t_i \text{ for each } i \text{ and } \sum_{i=1}^n t_i = 1\right\}.$$

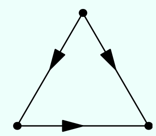
Namely, $[v_0, v_1, \ldots, v_n]$ is the convex hull of the $v_\bullet$; a *face* of this n-simplex is an $(n-1)$-simplex of the form $[v_0, \ldots, \widehat{v}_i, \ldots, v_n]$ attained by deleting one of the vertices v_i. Then the boundary of the n-simplex is

$$\partial[v_0, \ldots, v_n] := \bigsqcup_{i=1}^n [v_0, \ldots \widehat{v}_i, \ldots, v_n],$$

and the interior is defined in the obvious way.

Definition 2.97 (Δ-complex)**.** A Δ-*complex* is a CW complex X such that the cells e^n_α are homeomorphic to $(\Delta^n)^\circ$, where we require that the attaching maps $\varphi_\alpha\colon \partial\Delta^n \to X^{n-1}$ restricts to a face $\varphi_\alpha|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]}$ is an attaching map $\varphi_\beta\colon \Delta^{n-1} \to X^{n-1}$ for some β.

Example 2.98 (dunce cap)**.** Glue the following 2-simplex to a 1-simplex following the arrows.



This is weird, but we allow it.

We now describe $K(G,1)$ for a general group G. We begin by constructing the universal cover EG, which will be a Δ-complex. The vertices of EG are elements of G. Then the n-simplices of EG (for $n \geqslant 1$) are simply $[g_0, \ldots, g_n]$ attached to the $(n-1)$-simplices $[g_0, \ldots, \widehat{g}_i, \ldots, g_n]$ in the obvious way.

Example 2.99. Take G to be the trivial group. Then we have a single n-simplex $[e, \ldots, e]$ for each n. For example, the two-simplex $[e,e]$ is attaching at its ends to a single vertex. Then the $[e,e,e]$ is attaching its edges to the loops as in Example 2.98.

Example 2.100. Take G to be $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$. Then we have 2^{n+1} total n-simplices.

Note that G acts freely on EG by multiplication of the vertices, so we produce a covering space $EG \to BG$, where $BG := EG/G$. We claim that EG is contractible, which will complete our construction with BG as our $K(G,1)$. Indeed, inside any n-simplex $[g_0, \ldots, g_n]$, we embed it into $[e, g_0, \ldots, g_n]$ and then use the linear homotopy to the identity e. This will be well-defined with respect to our gluing, so we have indeed produced contraction.

2.9 September 28

Today we talk about graphs of groups.

Remark 2.101. Problem 1.B.9 on the homework needs to assume that the edge maps are injective.

2.9.1 Using Classifying Spaces

Given a group G, last time we constructed a contractible Δ-complex EG, and from there we built $BG := EG/G$, and we argued that BG is a $K(G,1)$ because the action of G on EG was free, making $\pi_1(BG) = \pi_1(EG/G) = G$. Though huge, the EG and BG construction are nice because they are functorial: a homomorphism $\varphi\colon G \to H$ of groups produces a continuous map $E\varphi\colon EG \to EH$ by moving the vertices (which continuously will send simplices to simplices), and this commutes with the group actions on both spaces, so we produce a map $BG\colon BG \to BH$. Explicitly, $B\varphi([g]) = [\varphi(g)]$, so

$$B\varphi([g_1, \ldots, g_n]) = [\varphi(g_1), \ldots, \varphi(g_n)],$$

and this map is preserved by the group actions because

$$B\varphi(g \cdot [g']) = B\varphi([gg']) = [\varphi(gg')] = \varphi(g) \cdot [\varphi(g')] = \varphi(g) \cdot B\varphi([g']),$$

so there is a quotient down to a map $E\varphi\colon EG \to EH$.

One might now hope that we can produce a map $K(\varphi,1)\colon K(G,1) \to K(H,1)$, but for this to make sense, we need to know that $K(G,1)$ is well-defined in some sense.

Theorem 2.102. The homotopy type of a CW-complex $K(G,1)$ is uniquely determined by G.

The main input to the theorem is the following functoriality result.

Proposition 2.103. Fix a connected CW-complex X, and let Y be a $K(G,1)$. Then any homomorphism $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is induced by a map $\Phi\colon (X,x_0) \to (Y,y_0)$ which is unique up to homotopy (relative to basepoints).

Proof. We construct $X \to Y$ inductively. Map X^0 to y_0. As in our discussion of graphs, choose a spanning tree T of X^1, and we see that each edge e of $X^1 \backslash T$ determines a generator $[e]$ of $\pi_1(X^1)$, and we map these down to the corresponding generator in $\pi_1(Y,y_0)$ as required by φ.

By way of example, we can take $X = S^1 \times S^1$ to be the torus, mapping the two generators of π_1 to $1 \in \mathbb{Z}$. Then may extend Φ on the vertices to X^2 linearly via this triangulation (check up in the covering space to be told how to do this), viewing things as simplices. One can then keep going up to higher X^n by continuing to go linearly, noting that the effect on the fundamental group is now not doing anything.

For the uniqueness, suppose we have two maps $\Phi, \Psi\colon (X,x_0) \to (Y,y_0)$. This will essentially follow from the homotopy extension property. If they induce the same map $\pi_1(\Phi) = \pi_1(\Psi)$, then we move them up to the universal cover, and the convex combinations as described in the previous paragraph are forced and homotopic (linearly), where we are essentially using contractability of our universal cover. One needs to do this by induction on the skeletons: there is a homotopy on the 0-skeleton by moving, there is a homotopy on the 1-skeleton because they have the same π_1, there is a homotopy on the 2-skeleton because the relations are the same, and from here one inducts upwards. ■

We can now prove Theorem 2.102.

Proof of Theorem 2.102. One has identities relating fundamental groups on two $K(G,1)$s, so one produces maps in both directions by the proposition, and then the composition of these maps (in both directions) are homotopy equivalent to identity maps by uniqueness of these maps up to homotopy. ■

Classifying spaces allow one to classify principal bundles with fiber given by a particular group. For example, the annulus A provides a double-cover of the Möbius strip M, so we see that this double-cover corresponds to $2\mathbb{Z} \subseteq \mathbb{Z}$. (Note the Möbius strip has a deformation retraction to S^1, so the fundamental groups are the same.) Now, each fiber has a $(\mathbb{Z}/2\mathbb{Z})$-action, and mapping $M \to \mathbb{RP}^\infty$ (given by the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ and using the $K(\mathbb{Z}/2\mathbb{Z},1)$ universal property), we see that the composite $A \to M \to \mathbb{RP}^\infty$ is now trivial on π_1, so we induce a map making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, two heads] & S^\infty \arrow[d, two heads] \\ M \arrow[r] & \mathbb{RP}^\infty \end{tikzcd}$$

Namely, this map is given by tracking fibers through on the map $M \to \mathbb{RP}^\infty$.

More generally, if we have a covering space $\widetilde{X} \to X$, where G acts freely and transitively (as deck transformations), then $G = \pi_1(X)/\operatorname{im}\pi_1(p)$, so maps $\pi_1(X) \twoheadrightarrow G$ will be given by maps $X \to K(G,1)$ via the above construction. So $K(G,1)$ in some sense allows us to classify these covering spaces $\widetilde{X} \to X$, which is of interest. Indeed, one can go the other direction: given a map $\varphi\colon X \to K(G,1)$, we pull back the bundle $p\colon EG \to K(G,1)$ to X to produce the necessary covering space. Namely, set

$$\widetilde{X} := \{(x,y) \in X \times EG : \varphi(x) = p(y)\} \subseteq X \times EG,$$

and one can check that the induced map $\widetilde{X} \to EG$ is continuous, and the map $\widetilde{X} \to X$ is a covering space map where G is acting on the fibers via EG.

2.9.2 Graphs of Groups

Fix a connected directed graph Γ, and for each vertex $v \in \Gamma^0$, we place a group G_v, and for each edge $e \in \Gamma^1$ connecting v to w, we place a homomorphism $\varphi_e\colon G_v \to G_w$. This will be our set-up for this subsection.

We are going to build a classifying space $B\Gamma$ for this graph by putting a classifying space BG_v (which is a CW-complex) at each vertex and attaching these along vertices with the mapping cylinders $MB\varphi_e$ for each $B\varphi_e \colon BG_v \to BG_w$. Notably, $B\varphi_e$ can always be constructed by Proposition 2.103. We will be interested in $\pi_1(B\Gamma)$. Note that $\pi_1(B\Gamma)$ does not depend on the choices of $BG - v$ and $B\varphi_e$ because these things are all well-defined up to homotopy.

Example 2.104. Consider the following graph.

$$\begin{tikzcd} G_{v_1} & & G_{v_2} \\ & e \arrow[ul] \arrow[ur] \arrow[d] & \\ & G_{v_3} & \end{tikzcd}$$

Now, $K(e,1)$ is just a point, so the corresponding $B\Gamma$ is just a wedge product, so van Kampen tells us that this is $G_{v_1} * G_{v_2} * G_{v_3}$.

Example 2.105. Consider the following graph.

$$\mathbb{Z} \xleftarrow{q} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$$

Applying van Kampen to the resulting $B\Gamma$, we get a group presentation of $\langle a, b : a^p = b^q \rangle$. If $p = q = 2$, one can squint very hard and see a Klein bottle as we are in some sense attaching two Möbius strips.

Example 2.106. Consider the following graph.

$$\begin{tikzcd} C \arrow[r, bend left, "\varphi_1"] \arrow[r, bend right, "\varphi_2"'] & A \end{tikzcd}$$

This looks like $\pi_1(B\Gamma) = \langle A, t : t\varphi_2(c)t^{-1} = \varphi_1(c) \text{ for } c \in C \rangle$, again by some van Kampen argument.

Anyway, here is our main theorem.

Theorem 2.107. Fix everything as above, and further assume that the φ_e maps are injective. Then $B\Gamma$ is a $K(G,1)$ where $G := \pi_1(B\Gamma)$, and the maps $\pi_1(BG_v) \to \pi_1(B\Gamma)$ are injective.

Proof. Start with a specific edge $B\varphi_e \colon BG_v \to BG_w$. Then the $MB\varphi_e$ connecting the two will lift to connect EG_v and EG_w by checking each "end" of this cylinder. We now build upwards via a tree to slowly encompass the entire graph. Being path-connected implies that this inductive process will union out to give us a legitimate "tree of spaces" connecting all the groups. Now, each vertex group G_v successfully acts on EG_v and then goes on to act on the mapping cylinders adjacent, so we have the right fundamental group. And we can see by reversing the inductive constructive process that we can deformation retract our mapping cylinders away to show that our covering space is contractible. ∎

Theme 3

Homology

3.1 October 3

The homeworks will now get a little longer.

3.1.1 Homology for Δ-Complexes

Let's recall our construction of Δ-complexes.

Definition 3.1 (simplex)**.** We define the n-simplex

$$\Delta^n := \left\{ (t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : \sum_{k=0}^{n} t_k = 1 \right\}.$$

The ith face $\Delta_i^{n-1} \subseteq \Delta^n$ consists of the points with $t_i = 0$. An *orientation* of the simplex consists of an ordering of the vertices modulo the action of A_{n+1} on the vertices $\{0, 1, \ldots, n\}$.

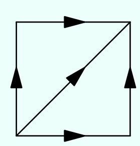
The orientation basically indicates which vertices are "small" and which are "large."

Definition 3.2 (Δ-complex)**.** A Δ-*complex* is a CW-complex X with maps $\sigma_\alpha \colon \Delta^n \to X$ satisfying the following properties.

- Interiors: the map σ_α is injective on the interior of Δ^n.
- Faces: the map σ_α restricted to the face Δ_i^{n-1} is simply another map $\sigma_\beta \colon \Delta^{n-1} \to X$.
- Continuity: if $A \subseteq X$ is open, then $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

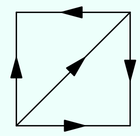
Given a Δ-complex X, orientations will tend to extend uniquely to X.

Example 3.3. We provide an orientation on the torus T^2.



Note that the diagonal arrow cannot go the other way to have an orientation because this would create a loop!

Example 3.4. We provide an orientation on the projective plane $\mathbb{P}^2$.



We would like to define homology. For this, we have a notion of a chain.

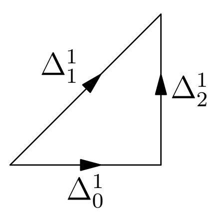
Definition 3.5 (chain). Fix a Δ-complex X with maps $\sigma_\alpha\colon \Delta^n \to X$. Then we define *chains* $\Delta_n(X)$ to be the formal sums

$$\Delta_n(X) := \left\{ \sum_\alpha n_\alpha \sigma_\alpha : n_\alpha \in \mathbb{Z} \right\},$$

and then we define the *chain map* $\partial_n\colon \Delta_n(X) \to \Delta_{n-1}(X)$ given by

$$\partial_n(\sigma_\alpha) := \sum_{i=0}^n (-1)^i \sigma_\alpha|_{\Delta_i^{n-1}}.$$

The point of the signs in the definition of ∂_n is to have the correct orientation. For example, suppose we want to go "around" Δ^2 as in this diagram.



One now has the following check.

Proposition 3.6. Fix a Δ-complex X. For any positive integer n, we have $\partial_{n-1} \circ \partial_n = 0$.

Proof. Direct computation. It suffices to show this for Δ^n because $\Delta_n(X)$ is freely generated by images of this Δ^n. And for Δ^n, the point is that our signs are going to cancel:

$$\begin{aligned}(\partial_{n-1} \circ \partial_n)(\Delta^n) &= \partial_{n-1}\left(\sum_{i=0}^n (-1)^i \Delta_i^{n-1} \right) \\ &= \sum_{i=0}^n (-1)^i \partial_{n-1}\left(\Delta_i^{n-1}\right).\end{aligned}$$

Now, for some notation, writing out the vertices Δ^n as $\{0, 1, \ldots, n\}$, we write $\Delta^n = [0, 1, \ldots, n]$ so that $\Delta_i^{n-1} = [0, \ldots, \widehat{i}, \ldots, n]$, so w are looking at

$$\begin{aligned}(\partial_{n-1} \circ \partial_n)(\Delta^n) &= \sum_{i=0}^n (-1)^i \partial_{n-1}([0, \ldots, \widehat{i}, \ldots, n]) \\ &= \sum_{i=0}^n \left(\sum_{j=0}^{i-1} (-1)^i (-1)^j [0, \ldots, \widehat{j}, \ldots, \widehat{i}, \ldots, n] + \sum_{j=i+1}^n (-1)^i (-1)^{j+1} [0, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, n] \right) \\ &= \sum_{j<i} (-1)^{i+j} [0, \ldots, \widehat{j}, \ldots, \widehat{i}, \ldots, n] - \sum_{i<j} (-1)^{i+j} [0, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, n] \\ &= 0,\end{aligned}$$

as desired. ■

We are now ready to define homology.

Definition 3.7 (simplicial homology)**.** Fix a Δ-complex X. Then we define $\Delta(X)$ to be the graded module $\bigoplus_{n=0}^{\infty} \Delta_n(X)$, and we define the n*th homology group* as

$$H_n^{\Delta}(X) = H_n(\Delta(X)) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

For notation, we set $Z_n(X) := \ker \partial_n$ to be n-*cycles* and $B_n(X) := \operatorname{im} \partial n + 1$ to be n-*boundaries*. Then $H_n(\Delta(X)) = Z_n(X)/B_n(X)$, so we are measuring cycles which are not boundaries, which approximately is finding holes.

Note that we have not shown that $H_\bullet$ does not depend on the choice of Δ-structure, which is why we are marking our H_n^{Δ} by Δ, but we will do this in due time.

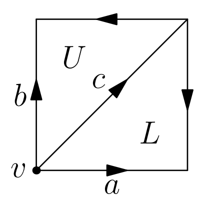
Example 3.8. Give S^1 a Δ-complex structure by attaching both endpoints of Δ^1 together at some vertex v as an edge e.

- We see $H_0^{\Delta}\left(S^1\right)$ is $\ker \partial_0 / \operatorname{im} \partial_1$, but $\operatorname{im} \partial_1 = 0$ because we are looking at $\partial_1(e) = v - v = 0$. However, $\ker \partial_0$ is simply all $\mathbb{Z}v$, so we have $\mathbb{Z}$.
- We see $H_1^{\Delta}\left(S^1\right)$ is $\ker \partial_1 / \operatorname{im} \partial_2$, and then $\partial_1 = \mathbb{Z}e$ as shown in the previous point, but $\operatorname{im} \partial_2 = 0$ because there is nothing to map, so we have $\mathbb{Z}$.

We note that all the higher homology groups vanish because there is nothing to compute.

Example 3.9. Give T^2 the Δ-complex as described earlier. We expect to have a two-dimensional hole and two one-dimensional holes. We compute some homology.

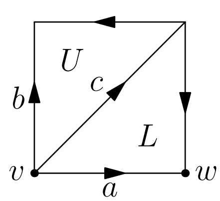
Proof.



Now, $\partial_2(U) = b - c + a$ and $\partial_2(L) = a - c + b$, which is the same, so $\ker \partial_2$ is generated by $U - L$. Now, $\operatorname{im} \partial_3 = 0$ (there is nothing to compute), so $H_2^{\Delta}\left(T^2\right) \cong \mathbb{Z}$. As for H_1^{Δ}, we note that ∂_1 is identically zero because there is only a single vertex, so $\ker \partial_1 = \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c$, so $H_1^{\Delta}\left(T^2\right) = \ker \partial_1 / \operatorname{im} \partial_2 \cong \mathbb{Z}a \oplus \mathbb{Z}b$. ■

Example 3.10. Give $\mathbb{P}^2$ the Δ-complex as described earlier. We compute some homology.

Proof. Here is our structure.



Here are our computations.

- We see $H_0^{\Delta}\left(\mathbb{P}^2\right) = \mathbb{Z}v \oplus \mathbb{Z}w/(\mathbb{Z}(v - w)) \cong \mathbb{Z}$, where the point is that $\partial_1(c) = 0$ and $\partial_1(a) = w - v$ and $\partial_1(b) = v - w$.

- Next up, we compute $\partial_2(U) = b-a+c$ and $\partial_2(L) = a-b+c$, so ∂_2 is injective, so $H_2^\Delta\left(\mathbb{P}^2\right) = 0$. Further, we note $\ker\partial_1 = \mathbb{Z}c \oplus \mathbb{Z}(a-b)$, and we have $\partial_2(U+L) = 2c$ and $\partial_2(U-L) = 2a-2b$, so we have

$$H_1^\Delta\left(\mathbb{P}^2\right) = \frac{\mathbb{Z}c \oplus \mathbb{Z}(a-b)}{\mathbb{Z}(2c) \oplus \mathbb{Z}(2a-2b) \oplus \mathbb{Z}(a-b+c)} \cong \frac{\mathbb{Z}}{2\mathbb{Z}},$$

finishing. ■

Example 3.11. We note that $\partial\Delta^{n+1} \cong S^n = \Delta^n$, so we can give S^n a natural Δ-complex structure. Then we can compute that $H_n^\Delta\left(\partial\Delta^{n+1}\right) \cong \mathbb{Z}$, where the point is that $\partial_{n+1}\left(\Delta^{n+1}\right)$ does provide a cycle, and all cycles are generated in this way.

3.1.2 Singular Homology

Let's define singular homology now.

Definition 3.12 (singular simplex)**.** Fix a topological space X. A *singular n-simplex* is simply a map $\sigma\colon \Delta^n \to X$ to a topological space, with no other requirements. We define our n-chains $C_n(X)$ to be the $\mathbb{Z}$-linear formal sums of such σs, and we define our chain maps $\partial_n\colon C_n(X) \to C_{n-1}(X)$ given in the usual way by

$$\partial_n(\sigma) \coloneqq \sum_{i=0}^{n} (-1)^i \sigma|_{\Delta_i^{n-1}}.$$

As before, one can do the exact same proof to show that $\partial_n \circ \partial_{n+1} = 0$, and so we may define homology.

Definition 3.13 (singular homology)**.** Fix a topological space X. Then we define $S(X)$ to be the Δ-complex with exactly one $n$simplex Δ_σ^n for each singular n-simplex $\sigma\colon \Delta^n \to X$, attached via faces. Then we define $H_n(X)$ to be the nth homology on $S(X)$ of the chain

$$\cdots \to C_{n+1}(X) \to C_n(X) \to C_{n-1}(X) \to \cdots.$$

Dealing with $S(X)$ is a little annoying. By allowing for repetitions, we may assume that all our $\mathbb{Z}$-coefficients are actually 1. For $n=1$, one can realize these as oriented loops, and for $n=2$, we can think of these as maps of oriented surfaces.

3.2 October 5

We continue our discussion of homology.

3.2.1 Basic Homology Facts

Let's continue working with our singular homology because it is a little more canonical. To begin, it suffices to look at path-connected spaces.

Proposition 3.14. Fix a topological space X with path-connected components X_α for $\alpha \in \pi_0(X)$. Then

$$H_n(X) \cong \bigoplus_{\alpha\in\pi_0(X)} H_n(X_\alpha)$$

Proof. Note that

$$C_\bullet(X) \cong \bigoplus_{\alpha\in\pi_0(X)} C_\bullet(X_\alpha)$$

because any map $\Delta^n \to X$ must land in a single path-connected component. We can see that this provides an isomorphism of chain complexes, so the isomorphism in homology follows. ■

Proposition 3.15. Fix a nonempty path-connected topological space X. Then $H_0(X) \cong \mathbb{Z}$.

Proof. Let $\varepsilon\colon C_0(X) \to \mathbb{Z}$ be the map given by sending

$$\sum_\sigma \alpha_\sigma \sigma \mapsto \sum_\sigma \alpha_\sigma.$$

Intuitively, some $\sigma\colon \Delta^0 \to X$ is just marking a point of X. Now, when X is path-connected, we see that $\operatorname{im}\partial_1 = \ker\varepsilon$. Note that $\ker\varepsilon$ is generated by differences $p-q$ for points $p, q \in X$. So to get these differences, note that for any two points $p, q \in X$, we have a path $f\colon \Delta^1 \to X$ with $f(0) = q$ and $f(1) = p$, so $\partial_1(f) = p-q$, as needed. So we see that

$$H^0(X) \cong \frac{C_0(X)}{\operatorname{im}\partial_1} = \frac{C_0(X)}{\ker\varepsilon} \cong \mathbb{Z},$$

as needed. ■

Remark 3.16. The above points we are checking go under the "Eilenberg–Steenrod axioms."

Proposition 3.17. If X is a point, then $H_n(X) = 0$ for $n > 0$.

Proof. We do this computation by hand. Notably, for each n, there is a unique n-simplex $\sigma_n\colon \Delta^n \to X$ sending everyone to the point. Then we note

$$\partial\sigma_n = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{is odd}, \\ \sigma_{n-1} & \text{if } n \text{ is even}. \end{cases}$$

Thus, our chain complex looks like

$$\cdots \cong \underbrace{C_3(X)}_{\mathbb{Z}\sigma_3} \xrightarrow{0} \underbrace{C_2(X)}_{\sigma_2} \cong \underbrace{C_1(X)}_{\mathbb{Z}\sigma_1} \xrightarrow{0} \underbrace{C_0(X)}_{\mathbb{Z}\sigma_0} \to 0.$$

At odd degrees $2n+1$, we have $\ker\partial_{2n+1} = C_{2n+1}(X) = \operatorname{im}\partial_{2n+2}$, so homology vanishes; at even degrees $\ker\partial_{2n} = 0 = \operatorname{im}\partial_{2n+1}$, so homology still vanishes. ■

The following technical definition will be helpful, mostly for functoriality reasons.

Definition 3.18 (reduced homology)**.** Fix a topological space X, and let $\varepsilon\colon C_0(X) \to \mathbb{Z}$ be the augmentation map. Then we define

$$\widetilde{H}_0(X) = \frac{\ker\varepsilon}{\operatorname{im}\partial_1},$$

and $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$. In particular, $\widetilde{H}_0(\{p\}) = 0$.

3.2.2 Functoriality of Homology

Note that H_n is in fact a functor.

Proposition 3.19. Fix a continuous map $f\colon X \to Y$. Then there is an induced map $H_\bullet(f)\colon H_\bullet(X) \to H_\bullet(Y)$.

Proof. Post-composition will send some $\sigma\colon \Delta^n \to X$ to some $(f \circ \sigma)\colon \Delta^n \to Y$. This extends to a map of chain complexes

$$C_\bullet(f)\colon C_\bullet(X) \to C_\bullet(Y),$$

so we induce a map on homology. Rigorously, one notes that $(f \circ -)$ commutes with ∂: one checks that

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n^X"] \arrow[d, "(f\circ -)"'] & C_{n-1}(X) \arrow[d, "(f\circ -)"] \\ C_n(Y) \arrow[r, "\partial_n^Y"] & C_{n-1}(Y) \end{tikzcd} \qquad \begin{tikzcd} \sigma \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{i=0}^n \sigma|_{\Delta_{n-1}^i} \arrow[d, mapsto] \\ (f\circ\sigma) \arrow[r, mapsto] & \sum_{i=0}^n (f\circ\sigma)|_{\Delta_{n-1}^i} \end{tikzcd}$$

commutes, and this is enough to induce a map on the homology upon checking what lives in what kernels and images. Let's explain this: to begin, we note that $C_n(f)$ maps $\ker \partial_n^X \to \ker \partial_n^Y$ because $\partial_n^Y(C_n(f)(\alpha)) = C_n(f)(\partial_n^X(\alpha)) = 0$. Similarly, we note that $C_n(f)$ maps $\operatorname{im} \partial_{n+1}^X \to \operatorname{im} \partial_{n+1}^Y$ because $C_n(f)(\partial_{n+1}^X(\alpha)) = \partial_{n+1}^Y(C_n(f)(\alpha))$. Thus, we get to produce a map

$$H_n(f)\colon \underbrace{\frac{\ker \partial_n^X}{\operatorname{im} \partial_{n+1}^Y}}_{H_n(X)} \to \underbrace{\frac{\ker \partial_n^Y}{\operatorname{im} \partial_{n+1}^Y}}_{H_n(Y)},$$

as needed. ■

Remark 3.20. As usual, one can check the usual functoriality checks such as that $H_\bullet(f \circ g) = H_\bullet(f) \circ H_\bullet(g)$ and $H_\bullet(\mathrm{id}_X) = \mathrm{id}_{H_\bullet(X)}$. These facts follow directly from the definition of $H_\bullet$.

More generally, the above proof establishes the following result.

Proposition 3.21. Fix a homomorphism $f\colon (C, \partial^C) \to (D, \partial^D)$ of chain complexes such that $\partial^C \circ f = f \circ \partial^D$. Then f induces a natural map on homology.

Proof. This is the last half of the proof of the above proposition. ■

We are now ready to show homotopy invariance. This will follow from the following result.

Theorem 3.22. Fix homotopic maps $f, g\colon X \to Y$ of topological spaces. Then $H_n(f) = H_n(g)$.

Proof. The point is to construct a "chain homotopy" between the maps $H_n(f)$ and $H_n(g)$. Let $F_\bullet\colon X \times I \to Y$ be the needed homotopy from f to g with $F_0 = f$ and $F_1 = g$. Then any singular simplex $\sigma\colon \Delta^n \to X$ will induce a map $(F_\bullet \circ \sigma)\colon \Delta^n \times I \to Y$ with $(F_0 \circ \sigma) = (f \circ \sigma)$ and $(F_1 \circ \sigma) = (g \circ \sigma)$. Technically, $F \circ \sigma$ is not a singular chain, but it is somewhat close.

The goal is as follows: for any chain $[c] \in C_n(X)$, we would like to produce a chain $[d] \in C_{n+1}(X)$ such that $[\partial d] = [f(c)] - [g(c)]$, and this will show that $H_n(f) = H_n(g)$. For this, we would like to make $\Delta^n \times I$

more like a simplex, so we triangulate it in a way which will be compatible with restricting to faces (and hence compatible with ∂).

As a warm-up, let's explain how to triangulate $I^{n+1} = [0,1]^{n+1}$. This is a cube with vertices of the form $(x_0, \ldots, x_n)$ where $x_\bullet\{0,1\}$ for each $x_\bullet$. Now, for each $\sigma \in S_{n+1}$, we choose the $(n+1)$-simplex given by

$$\Delta_\sigma := \{(x_0, \ldots, x_n) : x_{\sigma(0)} \leqslant x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}.$$

Notably, every face will be homeomorphic to I^n, and we roughly respect rearranging the coordinates (it just moves simplices around), though reflections will reverse the orientation of the simplex; also, there are $(n+1)!$ total simplices. Summing, we see that I^n is triangulated as

$$\sum_{\sigma \in S_{n+1}} (-1)^{\operatorname{sgn}\sigma} \Delta_\sigma.$$

Now, each simplex contains $(0, \ldots, 0)$ to $(1, \ldots, 1)$, and one can read off σ by noting the simplex has a unique monotonic path along the vertices of the cube from $(0, \ldots, 0)$ to $(1, \ldots, 1)$.

We now return to note that $\Delta^n \times I = \Delta^n \times \Delta^1$ embeds into $(\Delta^1)^n = I^n$, so we may triangulate $\Delta^n \times I$ as a Δ-subcomplex. Explicitly, we see that we are essentially choosing our monotonic path as having its first $i+1$ vertices in $\Delta^n \times \{0\}$ and its last $n-i+1$ vertices in $\Delta^n \times \{1\}$. Anyway, for this chosen Δ-complex structure on $\Delta^n \times I$, there is a "prism operator," we get something

$$\rho_n := \sum_i (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_n],$$

where the vertices of $\Delta^n \times \{0\}$ are given by $v_0, \ldots, v_n$, and the vertices of $\Delta^n \times \{1\}$ are given by $w_0, \ldots, w_n$. Taking faces, we see that

$$\partial \rho_n = [v_0, \ldots, v_n] - [w_0, \ldots, w_n] + \sum_i (-1)^i F_i \circ \rho_{n-1},$$

where F_i corresponds to the ith face. But by construction of $\rho_\bullet$ and our Δ-complex structure, it follows that this summation is merely $\rho_{n-1} \circ \partial$, so we get the inductive equation

$$\partial \rho_n = [v_0, \ldots, v_n] - [w_0, \ldots, w_n] + \rho_{n-1}\partial.$$

Applying F, we get the needed chain homotopy: given a singular simplex $\sigma \colon \Delta^n \to X$, we define

$$P(\sigma) := (F \circ \sigma)(\rho_n),$$

which is a map $P \colon C_n(X) \to C_{n+1}(Y)$, and the relation tells us that

$$\partial \circ P = C_\bullet(g) - C_\bullet(f) - P \circ \partial,$$

so upon going down to homology, we are done. ■

Remark 3.23. Here is an intuitive argument, using the notation of the first paragraph of the above proof. As a reduction step, we let $i_0 \colon X \to X \times I$ and $i_1 \colon X \to X \times I$ be the embeddings so that $i_t(a) := (a, t)$. Now, $f = F \circ i_0$ and $g = F \circ i_1$, so by functoriality, it is enough to check that $H_n(i_0) = H_n(i_1)$. Thus, we may as well assume that Y is $X \times I$ and that f and g are i_0 and i_1 respectively. At this point, the result is somewhat intuitive because one should be able to continuously deform $i_0 \circ \sigma$ to $i_1 \circ \sigma$ for any $\sigma \colon \Delta^n \to X$. However, it is mildly difficult to make this argument precise.

Corollary 3.24. Fix a homotopy equivalence $f \colon X \to Y$. Then $H_n(f) \colon H_n(X) \to H_n(Y)$ is an isomorphism.

Proof. This follows from functoriality. Let $g \colon Y \to X$ be the inverse homotopy equivalence for f. Then

$$H_n(f) \circ H_n(g) = H_n(f \circ g) \overset{*}{=} H_n(\mathrm{id}_Y) = \mathrm{id}_{H_n(Y)},$$

where $\overset{*}{=}$ follows from Theorem 3.22. A symmetric argument shows that $H_n(g) \circ H_n(f) = \mathrm{id}_{H_n(X)}$, so $H_n(f)$ is an isomorphism with inverse given by $H_n(g)$. ■

3.3 October 10

We would like to compute homology groups. The main tool for π_1 was van Kampen's theorem, which essentially allowed us to compute $\pi_1(A \cup B)$ from $\pi_1(A)$ and $\pi_1(B)$. Our goal is to build a similar computation for homology. To do this, we will require a little more homological algebra.

3.3.1 The Mayer–Vietoris Sequence

Let's discuss chain complexes on their own terms.

Definition 3.25 (chain complex)**.** Fix a ring R, and fix a sequence of maps of R-modules

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n_1} \to \cdots.$$

This is a *chain complex* if and only if $\operatorname{im} \alpha_{n+1} \subseteq \ker \alpha_n$ for each n; it is *exact* or *acyclic* if equality holds. We may write this chain complex as $(A_\bullet, \alpha_\bullet)$. A *morphism* of chain complexes $(\varphi_\bullet)\colon (A_\bullet, \alpha_\bullet) \to (B_\bullet, \beta_\bullet)$ is a sequence of maps $\varphi_\bullet\colon A_\bullet \to B_\bullet$ commuting with the boundaries.

Definition 3.26 (homology group)**.** Given a chain complex $(A_\bullet, \alpha_\bullet)$ of R-modules, we define the n*th homology group* to be

$$H_n(A_\bullet) := \frac{\ker \alpha_n}{\operatorname{im} \alpha_{n+1}}.$$

Example 3.27. Given a topological space X, we have shown that

$$\cdots \to C_{n+1}(X) \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_1(X) \to C_0(X) \to 0$$

is a chain complex.

Example 3.28. The sequence $0 \to A \to B$ is exact if and only if $A \to B$ is injective.

Example 3.29. The sequence $A \to B \to 0$ is exact if and only if $A \to B$ is surjective.

Example 3.30. The sequence $0 \to A \to B \to 0$ if and only if $A \to B$ is an isomorphism.

Example 3.31. The sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is a short exact sequence.

To compute our homology groups, it will help to have the following terminology.

Definition 3.32. A *good pair* of spaces (X, A) is a topological space X along with a closed subspace $A \subseteq X$ such that A is a deformation retract of some open subset $U \subseteq X$ containing A.

Example 3.33. If A is a CW-subcomplex of a CW-complex X, then (X, A) is a good pair by very slightly expanding the CW cells around $A \subseteq X$.

And now here is our result.

Theorem 3.34 (Mayer–Vietoris)**.** Fix a good pair (X, A). Then there is a long exact sequence as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_n(A) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(X/A) \arrow[dll, "\partial"] & \\
& \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X) \arrow[r] & \widetilde{H}_{n-1}(X/A) \arrow[r] & \cdots
\end{tikzcd}$$

Here, the maps $\widetilde{H}_n(A) \to \widetilde{H}_n(X)$ are given by inclusion $A \subseteq X$, and the maps $\widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$ are given by the quotient map $X \twoheadrightarrow X/A$. Note that we have not currently defined the boundary map ∂.

It will take us a while to prove Theorem 3.34. Here is an application.

Example 3.35. We show that

$$\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

Proof. Note that $S^{n-1} \subseteq D^n$ makes a good pair, and D^n is contractible, so $\widetilde{H}^\bullet(D^n) = 0$ always. Thus, for each i, we find

$$\underbrace{\widetilde{H}_i(D^n)}_{0} \to \widetilde{H}_i(S^n) \to \widetilde{H}_{i-1}\left(S^{n-1}\right) \to \underbrace{\widetilde{H}_i(D^n)}_{0},$$

so the result follows by induction, where the base case is given by $\widetilde{H}_0\left(S^0\right) \cong \mathbb{Z}$ and $\widetilde{H}_i\left(S^0\right) \cong 0$ for $i > 0$, which can be checked directly because S^0 is just two points. ■

3.3.2 Building Long Exact Sequences

The proof of Theorem 3.34 will make use of "relative homology groups."

Definition 3.36 (relative homology)**.** Fix a subspace $A \subseteq X$. We define the *relative chains* by $C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A)$. Then the boundary maps $\partial^X : C_\bullet(X) \to C_\bullet(X)$ and $\partial^A : C_\bullet(A) \to C_\bullet(A)$ induce a boundary map $\partial : C_\bullet(X, A) \to C_\bullet(X, A)$, granting us a chain complex

$$\cdots \to C_{n+1}(X, A) \to C_n(X, A) \to C_{n-1}(X, A) \to \cdots.$$

From here, the *relative homology groups* are the homology groups of the above chain complex.

In particular, we see that some $[\alpha] \in H_n(X, A)$ has $\alpha \in C_n(X)$, where $[\alpha]$ will vanish only when $\alpha = \partial\beta + \gamma$ where $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$. Namely, $H_n(X, A)$ is $\subseteq \partial_n^X \subseteq C_n(X)$ upon taking a quotient by $\operatorname{im} \partial_{n+1}^X$ and by $C_n(A)$.

We are now equipped to show a long exact sequence close to Theorem 3.34.

Proposition 3.37. Fix a subspace $A \subseteq X$. Then there is a long exact sequence as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_n(A) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(X/A) \arrow[dll, "\partial"] & \\
& \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X) \arrow[r] & \widetilde{H}_{n-1}(X/A) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. By construction, we have a short exact sequence of chain complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0.$$

Explicitly, for each $n \geqslant 1$, the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "\partial"] & C_n(X) \arrow[r] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0 \end{tikzcd}$$

As such, the result follows directly from the following proposition. ■

Proposition 3.38. Fix a short exact sequence

$$0 \to (A_\bullet, \alpha_\bullet) \xrightarrow{\varphi_\bullet} (B_\bullet, \beta_\bullet) \xrightarrow{\psi_\bullet} (C_\bullet, \gamma_\bullet) \to 0$$

of chain complexes of R-modules; i.e., this is a short exact sequence at each fixed index. Then there is a long exact sequence in homology as follows.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, "\partial"] \\ & H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

Proof. Let's describe the boundary map $\partial\colon H_n(C) \to H_{n-1}(A)$, which is really the only interesting thing. Well, given $[z] \in H_n(C)$ with $z \in \ker \gamma_n$, we can lift it up to some $y \in B_n$ such that $\varphi_n(y) = z$. Then take $\beta_n(y)$, which we see lives in the kernel of φ_n, so exactness finds some $x \in A_{n-1}$ such that $\psi_{n-1}(x) = \beta_n(y)$. We can check that $\alpha_{n-1}(x) = 0$ by construction, so it follows that x represents some class in $H_{n-1}(A)$, which is the desired class.

For completeness, we describe why this is well-defined. The content is in explaining why the choice of lift y does not affect our element in $H_{n-1}(A)$. Well, choosing a separate element y' in B_n will have $y - y'$ in the image of A_n by exactness, say equal to $\alpha_n(x_0)$. Then choosing $x, x' \in A_{n-1}$ such that $\varphi_{n-1}(x) = \beta_n(y)$ and $\varphi_{n-1}(x') = \beta_n(y')$, we claim that $x - x' = \alpha_n(x_0)$. For this, it is enough to check after applying the injective map φ_{n-1}, which is true by construction of x_0.

Let's quickly sketch some exactness arguments.

- Exact at $H_n(A)$: on one hand, we note that any $[z] \in H_{n+1}(C)$ will have $\varphi_n(\partial([z])) = 0$ by construction of the boundary map. Explicitly, $\varphi_n(\partial(z))$ (suitably defined) will live in the image of β_{n+1}, which is what vanishing means.

  On the other hand, given $[x] \in H_n(A)$ which vanishes under φ_n, meaning that $\varphi_n(x) = \beta_{n+1}(y')$ for some y', allowing us to set $z' := \psi_{n+1}(y')$. The construction of the boundary maps shows $\partial([z']) = [x]$, as needed.

- Exact at $H_n(B)$: on one hand, we note that any $[x] \in H_n(A)$ has $\psi_n(\varphi_n([x])) = 0$ because $\psi_n \circ \varphi_n = 0$.

  On the other hand, given $[y] \in H_n(A)$ which vanishes under ψ_n, we see that $\psi_n(y)$ must be in $\operatorname{im} \gamma_{n+1}$, so write $\psi_n(y) = \gamma_{n+1}(z')$, but then ψ_{n+1} is surjective, so $\psi_n(y) = \gamma_{n+1}(\psi_{n+1}(y')) = \psi_n(\beta_{n+1}(y'))$, so replacing y with $y - \beta_{n+1}(y')$ (which is in the same class) provides $\psi_n(y) = 0$. Thus, exactness grants $y \in \operatorname{im} \varphi_n$, as needed.

- Exact at $H_n(C)$: on one hand, we note that any $[y] \in H_n(B)$ has $\partial(\psi_n([y])) = 0$ by construction of the boundary map: $\psi_n([y])$ has a lift in B_n given by y itself, which by definition of $H_n(B)$ will vanish upon applying β_n.

  On the other hand, given $[z] \in H_n(C)$, going down to 0 in $H_{n-1}(A)$ implies that means that there is a lift $y \in C_n(B)$ of z such that $\beta_n(y) = 0$. But then $[y]$ is a class in $H_n(B)$ mapping to $[z]$, exhibiting our exactness.

That's enough for me. ■

Remark 3.39. One can define the boundary map $H_n(X, A) \to H_{n-1}(A)$ more explicitly by taking some class $[z] \in H_n(X, A)$ and then viewing z as a class of objects in $C_n(X)$, we can literally take its boundary as a chain in X and note that ∂z must then vanish in $C_{n-1}(X)/C_{n-1}(A)$ by construction of the reduced homology, so we produce a chain in $C_{n-1}(A)$. This is essentially the above construction where we have described our objects topologically.

More generally, the above arguments are able to prove the following result.

Lemma 3.40 (Snake). Fix a "snake" (commutative) diagram as follows.

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

The following are true.

(a) There is an exact sequence

$$\ker a \xrightarrow{f} \ker b \xrightarrow{g} \ker c \xrightarrow{\delta} \operatorname{coker} a \xrightarrow{f'} \operatorname{coker} b \xrightarrow{g'} \operatorname{coker} c,$$

where $\ker x \xrightarrow{h} \ker y$ is restriction, δ is the connecting morphism, and $\operatorname{coker} x \xrightarrow{h'} \operatorname{coker} y$ is induced by h' by modding out.

(b) If f is injective, then $\ker a \xrightarrow{f} \ker b$ is injective.

(c) If g' is surjective, then $\operatorname{coker} b \xrightarrow{g'} \operatorname{coker} c$ is surjective.

Proof. Analogous to the last half of the proof of Proposition 3.38. Namely, the construction of the boundary map δ is exactly what we constructed: pull back along g, push through b, and then pull back along f'. ■

Anyway, let's see an example.

Example 3.41. Analogous to Example 3.35, we see that Proposition 3.37 produces in the long exact sequence the exact sequence

$$\underbrace{\widetilde{H}_i\left(D^n\right)}_{0} \to \widetilde{H}_i\left(D^n, S^{n-1}\right) \to \widetilde{H}_{i-1}\left(S^{n-1}\right) \to \underbrace{\widetilde{H}_{i-1}\left(D^n\right)}_{0}.$$

Thus, the middle map is an isomorphism.

3.3.3 Excision

We close class by stating excision, which is a primary tool to compute homology groups.

Theorem 3.42 (excision). Fix subspaces $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A$. Then the inclusion $(X \backslash Z, A \backslash Z) \subseteq (X, A)$ induces isomorphisms $H_n(X \backslash Z, A \backslash Z) \to H_n(X, A)$.

Of course, we see that there is a map $C_n(X\backslash Z, A\backslash Z) \to C_n(X, A)$ given by the inclusions $C_n(X\backslash Z) \subseteq C_n(X)$ and $C_n(A\backslash Z) \subseteq C_n(A)$. The main content, then, is in going the other way. Approximately speaking, the idea is to take some $\alpha \in C_n(X, A)$ and then attempt to throw out the parts of α that live in Z. But for this to make sense, we must subdivide $X\backslash Z$ in order to make sure that we are going to get a chain at the end of this process.

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List of Definitions