250B: Commutative Algebra
For the Morbidly Curious

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Theme 1
Introduction to Dimension

In this sense the algebraic geometers have never left paradise: There is no snake (that is, Peano curve) in the garden.

—David Eisenbud

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We continue.

1.1.1 Fractional Ideals

We continue discussing fractional ideals. Last time we showed the following results.

Lemma 1.1. Fix a Noetherian domain R. Then M is invertible if and only if M is isomorphic to some nonzero fractional ideal.

We were also in the middle of the following proof, which we will finish today.

Lemma 1.2. Fix a Noetherian domain R. If I and J are nonzero fractional ideals, then

$$IJ \cong I \otimes_R J \qquad \text{and} \qquad I^{-1}J \cong \operatorname{Hom}(I.J).$$

Proof. The map
$$I \otimes_R J \to IJ$$
is by $a \otimes b \mapsto ab$. This is of course surjective, so we just need injectivity. It suffices to show injectivity upon localizing by any prime $\mathfrak{p}$. But now we are looking at the map

$$I_\mathfrak{p} \otimes_{R_\mathfrak{p}} J_\mathfrak{p} \to (IJ)_\mathfrak{p},$$

which is injective because $I_\mathfrak{p}$ and $J_\mathfrak{p}$ are both free $R_\mathfrak{p}$-modules (by definition), so we get the injection here automatically because $R_\mathfrak{p}$ is an integral domain.

The map
$$I^{-1}J \to \operatorname{Hom}_R(I, J)$$

is by sending t to $\mu_t : x \mapsto tx$. This map is injective because $\mu_{t_1} = \mu_{t_2}$ implies they are equal on $a \in I$ (say), so $t_1 = t_2$ because R is an integral domain. In fact, we can even say that μ_t is injective for each nonzero t.

It remains to show surjectivity. Well, pick up some R-module homomorphism $\varphi : I \to J$. Now, for some $f \in I \setminus \{0\}$, suppose $\varphi(f) = g$ so that we may consider

$$\mu_{g/f}(f) = g,$$

and we can check that $\mu_{g/f} = \varphi$ everywhere by some computation. ■

Here is another helpful result.

Lemma 1.3. Fix a Noetherian domain R. Then $I \subseteq K(R)$ is an invertible fractional ideal if and only if $I^{-1}I = R$.

Proof. On one hand, we see that I being invertible implies that $I^{-1}I \cong \operatorname{Hom}_R(I, I) \cong R$. On the other hand, suppose $I^{-1}I = R$. Localizing gives us

$$I_\mathfrak{p} I_\mathfrak{p}^{-1} = R_\mathfrak{p}.$$

But then $vI_\mathfrak{p} \not\subseteq \mathfrak{p}R_\mathfrak{p}$ for some $v \in I^{-1}$, so we can conclude $vI_\mathfrak{p} = R_\mathfrak{p}$, so I is indeed locally free. ■

As such, we are able to build the following group.

Definition 1.4 (Cartier divisors)**.** Fix a Noetherian domain R. Then a *Cartier divisor* is an invertible fractional ideal.

From the above results, the Cartier divisors are an abelian group with respect to multiplication, which we all $C(R)$.

Now, we note that we have a homomorphism

$$C(R) \to \operatorname{Pic} R$$

by $I \mapsto [I]$. Notably, Lemma 1.1 tells us that this homomorphism is surjective, and its kernel consists of ideals I such that $[I] = [R]$, which means $I \cong R$ (as R-modules), which means I is principal, generated by some element of $K(R)$. Thus, we have the exact sequence

$$K(R)^\times \to C(R) \to \operatorname{Pic} R \to 0.$$

We would like to make this have 0s on the end, so we note that $a \in K(R)^\times$ will have $(a) = R$ if and only if $a \in R^\times$, so we get to write

$$0 \to R^\times \to K(R)^\times \to C(R) \to \operatorname{Pic} R \to 0.$$

As such, we have a way to measure $\operatorname{Pic} R$ by objects only internal to $K(R)$.

To make this behave a little better, we pick up the following lemma.

Lemma 1.5. The group $C(R)$ is generated by invertible ideals $I \subseteq R$.

Proof. The point is to multiply an arbitrary invertible ideal from $K(R)$ to R. Indeed, any invertible fractional ideal $I \in C(R)$ will at least live in $K(R)$. Picking up any nonzero $a \in I \times R$, we note that

$$I = \left(a^{-1}\right) \cdot (aI),$$

and $aI \subseteq R$ by construction of a. So we are indeed able to generate $C(R)$ as an R-module by these invertible ideals. ■

Let's see some examples.

Example 1.6. Fix a principal ideal domain R. Then every ideal is principal and hence isomorphic to R, so $\operatorname{Pic} R = 0$. Namely, $C(R)$ only consists of principal ideals.

Exercise 1.7. We discuss $\operatorname{Pic} \mathbb{Z}[\sqrt{-5}]$.

Proof. Fix the Noetherian domain $R = \mathbb{Z}[\sqrt{-5}]$. This is normal because it is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-5})$, as we showed on the homework. This is dimension 1 because R is integral over $\mathbb{Z}$, and $\dim \mathbb{Z} = 1$. However, R is not a principal ideal domain because it is not factorial, as

$$\left(1+\sqrt{-5}\right)\left(1-\sqrt{-5}\right) = 2 \cdot 3$$

shows. In particular, the ideal $\mathfrak{p} := \left(2, 1+\sqrt{-5}\right)$ is not principal. In fact, $\mathbb{Z}/\mathfrak{p} = \mathbb{Z}/2\mathbb{Z}$ is a field,[1] so $\mathfrak{p}$ is maximal.

We will take on faith that $\mathfrak{p}$ is not principal because just look at it. To show that $\mathfrak{p}$ is invertible, we note that localizing at any prime which is not $\mathfrak{p}$ will automatically trivialize, so we have left to study

$$\mathfrak{p}R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}.$$

But in $R_{\mathfrak{p}}$, we see that

$$2 = \frac{1}{3} \cdot \left(1-\sqrt{-5}\right)\left(1+\sqrt{-5}\right),$$

so

$$\mathfrak{p}R_{\mathfrak{p}} = \left(1+\sqrt{-5}\right),$$

which is indeed principal.

Thus, we have a nontrivial element of $\operatorname{Pic} \mathbb{Z}[\sqrt{-5}]$. We can also compute

$$\mathfrak{p}^2 = \left(2, 1+\sqrt{-5}\right)^2 = \left(4, 2+2\sqrt{-5}, -4+2\sqrt{-5}\right) = \left(4, 2+2\sqrt{-5}, -6\right) = \left(2, 2+2\sqrt{-5}, -6\right) = (2),$$

so this is indeed principal. So $\mathfrak{p}$ is of order 2 in $\operatorname{Pic} \mathbb{Z}[\sqrt{-5}]$. In fact, this is an isomorphism, which one can see by taking Math 254A. ■

Example 1.8. Fix $R := k[x,y]/\left(y^2 - x^3\right) \cong k\left[t^2, t^3\right]$ so that $k[t]$ is the normalization of R. Now, any ideal of $k[t]$ is principal, so $\operatorname{Pic} k[t] = 0$. However, for any invertible ideal I of $k[t]$, then $I \cap k\left[t^2, t^3\right]$ will remain invertible by tracking through the definition. For example, if we take $1 + at$ as a varies over k, we have a map

$$k \to \operatorname{Pic} R$$

by $a \mapsto (1 + at)$, which turns out to be an isomorphism. For more, see exercises 11.15 and 11.16.

Remark 1.9. It is not technically necessary for R to be a domain in the above results, but the proofs are more annoying. Namely, instead of using the fraction field $K(R)$, one should use the total quotient $K(R)$.

1.1.2 Divisors

We now talk about divisors a little more generally. We pick up the following definition.

Definition 1.10 (Pure codimension)**.** Fix a Noetherian domain R. Then $I \subseteq R$ has *pure codimension* 1 if and only if every prime associated to I has codimension 1.

[1] Track through the map $\mathbb{Z} \to \mathbb{Z}[\sqrt{-5}]/\mathfrak{p}$, and we can note that it is surjective and has kernel (2).

Theorem 1.11. Fix a Noetherian domain R such that $R_\mathfrak{m}$ is factorial for each maximal ideal $\mathfrak{m}$. Then the following are true.

(a) An ideal $I \subseteq R$ is invertible if and only I has pure codimension 1.

(b) An invertible fractional ideal I can be written uniquely as

$$I = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_n^{m_n},$$

for distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of codimension 1.

We will prove this momentarily, but let's talk about some consequences.

Corollary 1.12. Fix a Noetherian domain R. Then $C(R)$ is a free abelian group generated by prime ideals $\mathfrak{p}$ of codimension 1.

Proof. This follows directly from part (b) of the theorem. ■

Here is the case that number theorists care about.

Definition 1.13 (Dedekind)**.** A *Dedekind domain* is a Noetherian normal domain of dimension 1.

Notably, in a Dedekind domain, all primes of codimension 1 are maximal, which are all now invertible by (a) of the theorem. In particular, $R_\mathfrak{m}$ is indeed factorial for all maximal ideals $\mathfrak{m}$ because we showed last class that a Noetherian domain being normal is equivalent to all the primes $\mathfrak{p}$ associated to a principal ideal has $\mathfrak{p}R_\mathfrak{p} \subseteq R_\mathfrak{p}$ principal, which makes $R_\mathfrak{p}$ a discrete valuation ring and in particular factorial.

We now prove our theorem.

Proof of Theorem 1.11. We go one at a time.

(a) Fix I an invertible fractional ideal. Then $R_\mathfrak{m}$ is factorial, so we showed a while ago that this implies $\mathfrak{m}R_\mathfrak{m}$ (which is a codimension-1 prime) must be principal, so we are done.

We now show the other direction. Well, if $\mathfrak{p}$ is a prime of codimension 1, then place $\mathfrak{p}$ in some maximal ideal $\mathfrak{m}$, and we see that $\mathfrak{p}_\mathfrak{m}$ is principal and hence codimension 1 in the factorial ring $R_\mathfrak{m}$. This finishes this direction.

(b) Fix an invertible fractional ideal I. Then we know that any prime $\mathfrak{p}$ associated to I has codimension 1, by part (a). To start, we show that I is a finite product of primes. Well, otherwise we could find an ideal I of R maximal with respect to not being a product of primes, and place I in a maximal ideal $\mathfrak{m}$. Of course, $I \subsetneq \mathfrak{m}$ because $\mathfrak{m}$ is its own factorization, so we look at

$$\mathfrak{m}^{-1}I \subsetneq R.$$

Notably, $\mathfrak{m}^{-1}I \supsetneq I$ would imply that $\mathfrak{m}^{-1}I$ would have a factorization into primes, giving I a factorization into primes.

So we have left to show $I \subsetneq \mathfrak{m}^{-1}I$ require using that R is normal. In particular, $\mathfrak{m}^{-1}I = I$ would imply, by the Cayley–Hamilton theorem, we have that every element $x \in \mathfrak{m}^{-1}$ is integral over R and hence is in R, so $\mathfrak{m}^{-1} = R$, which does not make sense.

Lastly, we show uniqueness. Well, if

$$\prod_{k=1}^{m} \mathfrak{p}_k = \prod_{\ell=1}^{n} \mathfrak{q}_\ell,$$

we pick up some $\mathfrak{q}_n$, and by the product, we can say that some $\mathfrak{p}_k$ contains $\mathfrak{q}_1$. But $\mathfrak{p}_k$ has codimension 1, so $\mathfrak{p}_k = \mathfrak{q}_1$, so we can cancel from both sides and then induct downwards. ■

With the above in mind, we see that we are justified in only caring about the primes of codimension 1. This gives us the following definition.

Definition 1.14 (Divisor)**.** Fix a Noetherian domain R. Then the group of *divisors* $\operatorname{Div} R$ is the free abelian group generated by all primes of codimension 1 (as letters).

Notably, there is a good homomorphism

$$\varphi : C(R) \to \operatorname{Div} R,$$

though they are not the same. To see this, take an invertible ideal $I \in C(R)$ and then set

$$\varphi(I) := \sum_{\mathfrak{p}} \ell(R_{\mathfrak{p}}/I_{\mathfrak{p}})[\mathfrak{p}].$$

Notably, the length $\ell(R_{\mathfrak{p}}/I_{\mathfrak{p}})$ is finite because $\dim R_{\mathfrak{p}}/I_{\mathfrak{p}} = 0$ (making $R_{\mathfrak{p}}/I_{\mathfrak{p}}$ Artinian) by the principal ideal theorem: we get $\dim R_{\mathfrak{p}} = 1$ and $\dim I_{\mathfrak{p}} = 1$, so we bound $\dim R_{\mathfrak{p}}/I_{\mathfrak{p}}$ down to 0. It requires some work to show that φ is a homomorphism. Namely, we have to show that

$$\ell(R_{\mathfrak{p}}/(IJ)_{\mathfrak{p}}) \stackrel{?}{=} \ell(R_{\mathfrak{p}}/I_{\mathfrak{p}}) + \ell(R_{\mathfrak{p}}/I_{\mathfrak{p}}).$$

We are able to force $I_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ to be principal by using our theory of modules of finite length, so by replacing R with $R_{\mathfrak{p}}$, we are showing

$$\ell(R/(IJ)) \stackrel{?}{=} \ell(R/I) + \ell(R/J),$$

where $I = (a)$ and $J = (b)$. But then we can build the filtration for $R/(IJ)$ by hand by zippering the filtrations for R/I and R/J together.

Remark 1.15. The homomorphism φ is in general not injective, but it will be injective when R is also normal. The main idea is that, if R is normal, then $R_{\mathfrak{p}}$ will be factorial and in particular a discrete valuation ring, so $\ell(R_{\mathfrak{p}}/I_{\mathfrak{p}})$ vanishing everywhere forces I to vanish.

List of Definitions