737: Weil II for Curves

Nir Elber

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Review of Étale Theory

1.1 January 23

This is a pretty small class, so it will be rather informal. This course is going to assume some basic étale theory, roughly speaking up to the construction of the derived functors and some of their fundamental properties. We will also freely black-box the difficult theorems of the theory, most notably the Grothendieck–Lefschetz trace formula.

In this course, we are interested in proving the Weil conjectures, but we will be modest and focus on curves. Historically, the proof of the Weil conjectures for curves is much older than Weil II, but part of our goal will be to introduce the important relative techniques. For example, there should be a notion of weights attached to sheaves on a variety, known already from Hodge theory. However, we will require a way to see this purely from algebraic geometry; in fact, one expects the notion of weight to be motivic.

1.1.1 The Zeta Function

Let's begin by setting some notation which will be in place for the entire course. We take k to be a finite field $\mathbb{F}_q$ of characteristic p, embedded in a fixed algebraic closure $\overline{k} = \overline{\mathbb{F}_p}$; we write $q = p^n$. For brevity, we may write $k_m = \mathbb{F}_{q^m}$ for each $m \geq 1$. Then we let X be a smooth, projective, geometrically connected variety over the field k; we set $d \coloneqq \dim X$.

Definition 1.1 (zeta function)**.** Let X be a variety over $\mathbb{F}_q$. Then we define the *zeta function* as the generating function

$$\zeta_X(T) \coloneqq \exp\left(\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| \frac{T^m}{m}\right).$$

In order to do algebraic geometry to $\zeta_X(T)$, we would like to have a different description for $X(\mathbb{F}_{q^m})$. For this, we need to discuss closed points.

Definition 1.2 (closed point)**.** Let X be a variety over k. Then a point $x \in X$ is *closed* if and only if $\dim \overline{\{x\}} = 0$. Its *degree* $\deg x$ is the degree $[k(x) : k]$, where $k(x)$ is the minimal field of definition.

We now see that

$$X(\mathbb{F}_{q^m}) = \operatorname{Mor}_{\mathbb{F}_q}\left(\operatorname{Spec} \mathbb{F}_{q^m}, X\right).$$

For example, we see that this consists of the collection of closed points $x \in X$ of degree dividing m, counted with a certain multiplicity.

Now, to read off fields of definition, we introduce some Frobenius morphisms.

Definition 1.3. Fix a scheme X over $k = \mathbb{F}_q$. Then there is a *Frobenius morphism* $\operatorname{Frob}_X \colon X \to X$ defined as being an identity on the underlying topological space and the q-power map on $\mathcal{O}_X$. We may write $\operatorname{Frob}_{X,q}$ for Frob_X if we want to remember the power. We may also extend scalars and write $\operatorname{Frob}_{X_{\overline{k}},q} = \operatorname{Frob}_{X,q} \times \operatorname{id}_{\overline{k}}$, which we note is a morphism of schemes over $\overline{k}$ by its construction.

Remark 1.4. Fix a morphism $f \colon X \to Y$ of schemes over $\mathbb{F}_q$. Then we see $\operatorname{Frob}_Y \circ f = f \circ \operatorname{Frob}_X$, which can be checked directly: both sides are f on the topological spaces, and both sides are the same on the level of sheaves.

Example 1.5. On $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, our Frobenius map may be defined as the k-algebra endomorphism of $k[x_1, \ldots, x_n]$ which sends $x_i \mapsto x_i^q$. Thus, on points, we see that $(p_1, \ldots, p_n) \in \mathbb{A}^n_k(\overline{k})$ has

$$F_{\mathbb{A}^n_k}(p_1, \ldots, p_n) = \left(\operatorname{Frob}_q^{-1} p_1, \ldots, \operatorname{Frob}_q^{-1} p_n\right).$$

Remark 1.6. We now see that we can think about $X(\mathbb{F}_q)$ as the subset of $X(\overline{k})$ fixed by F_{X,q^m}. Thus, we note that one can realize $X(k_m)$ as the set of closed points of the scheme $(\Gamma_{F_{X,q^m}} \cap \Delta)$, where $\Delta \colon X \times X \to X$ is the diagonal map.

Definition 1.7 (arithmetic Frobenius)**.** The *arithmetic Frobenius* Frob_k is the q-power automorphism of $\overline{k}$.

Definition 1.8 (geometric Frobenius)**.** Let X be a scheme over k. Then we define the *geometric Frobenius* of $X_{\overline{k}}$ as $F_X \coloneqq \operatorname{id}_{X_{\overline{k}}} \times \operatorname{Frob}_{\overline{k}}^{-1}$. It fits in the following commutative diagram.

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r] \arrow[d] \arrow[dr, dashed, "F_X"] & X \arrow[dr, equal] & \\
\operatorname{Spec} \overline{k} \arrow[dr] & X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} \overline{k} \arrow[r, "\operatorname{Frob}_{\overline{k}}^{-1}"'] & \operatorname{Spec} k
\end{tikzcd}$$

Definition 1.9 (absolute Frobenius)**.** Let X be a scheme over k. One can check that F_X commutes with $\operatorname{Frob}_{X,q}$. We then define the *absolute Frobenius* as the composite $F_X \circ \operatorname{Frob}_{X_{\overline{k}},q}$.

Remark 1.10. It turns out that the absolute Frobenius is the identity on the level of étale cohomology.

We now return to our zeta function. To be able to undo the exponential, we note

$$\log\left(\frac{1}{1-T^d}\right) = \sum_{m\geq 1} d \cdot \frac{T^{nd}}{dn}.$$

Thus,

$$\sum_{m\geq 1} |X(\mathbb{F}_{q^m})| \frac{T^m}{m} = \sum_{\text{closed } x \in X} \log\left(\frac{1}{1-T^{\deg x}}\right),$$

so taking the exponential reveals

$$\zeta_X(T) = \prod_{\text{closed } x\in X} \frac{1}{1-T^{\deg x}},$$

and now this Euler product appears similar to the usual Euler products we expect.

1.2 January 28

Today we do something with cohomology.

1.2.1 The Rationality Conjecture

We would like to relate our zeta function to cohomology. It turns out that the key input is the following result.

Theorem 1.11 (Grothendieck–Lefschetz trace formula)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$. Then

$$X\left(\mathbb{F}_{q^m}\right) = \sum_{i\ge 0} (-1)^i \operatorname{tr}\left(\operatorname{Frob}^m_{X_{\overline{k}}}; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

Here, recall that

$$\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \varprojlim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}/\ell^\bullet\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Namely, this is our Weil cohomology (over the field $\mathbb{Q}_\ell$) produced by étale cohomology.

Remark 1.12. It is the goal of Weil II (and thus of the course) to be able to work more general local systems than the "constant" sheaf $\mathbb{Q}_\ell$.

To relate this to ζ_X, we recall the following result from linear algebra.

Lemma 1.13. Fix an endomorphism φ of a finite-dimensional vector space V (over a field K). Then we have an equality of power series

$$\exp\left(\sum_{m\ge 1} \operatorname{tr}\left(\varphi^m; V\right)\frac{T^m}{m}\right) = \det\left(1-\varphi T; V\right)^{-1}.$$

Proof. It is enough to check the equality after base-changing to the algebraic closure, so we may assume that K is algebraically closed. Then we may give V a basis so that φ is upper-triangular.

Let $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of φ. Then we are tasked with showing

$$\exp\left(\sum_{m\ge 0}\sum_{i=1}^d \lambda_i^m \cdot \frac{T^m}{m}\right) \stackrel{?}{=} \prod_{i=1}^d \frac{1}{1-\lambda_i T}.$$

Well, we may move the sum on the left-hand side outside so that we see we are interested in showing

$$\exp\left(\sum_{m\ge 1} \frac{(\lambda T)^m}{m}\right) = \frac{1}{1-\lambda T}$$

for any eigenvalue λ of φ. The result now follows by considering the Taylor expansion $-\log(1-x) = \sum_{m\ge 1} x^m/m$. ■

Here is the punchline: we are able to prove the rationality conjecture.

Proposition 1.14 (Rationality). Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Then there are polynomials $P_0, \ldots, P_{2d} \in \mathbb{Q}_\ell[T]$ such that

$$Z_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}.$$

Proof. By Theorem 1.11, we see that

$$Z_X(T) = \prod_{i=0}^{2d} \exp\left(\sum_{m \geq 1} \operatorname{tr}\left(\operatorname{Frob}_{X_{\overline{k}}}^m; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right) \frac{T^m}{m} \right)^{(-1)^i}.$$

We now define

$$P_i(T) := \det\left(1 - \operatorname{Frob}_{X_{\overline{k}}} T; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

The result now follows from Lemma 1.13. ■

Remark 1.15. In fact, we see that $P_i(T)$ has degree $\dim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$. This fact can be combined with the comparison theorem to Betti cohomology.

Remark 1.16. We thus see that $Z_X(T) \in \mathbb{Q}_\ell(T)$, so because we already know $Z_X(T) \in \mathbb{Q}[[T]]$, we see $Z_X(T) \in \mathbb{Q}(T)$.

Remark 1.17. It turns out that $P_i(T) \in \mathbb{Z}[T]$ and is independent of ℓ, but the proof above does not show this.

Example 1.18. At $i = 0$, we see that the Frobenius acts trivially on $\mathrm{H}^0_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, so $P_0(T) = 1 - T$. Using Poincaré duality, we can similarly compute $P_{2d}(T) = 1 - q^d T$.

Remark 1.19. There is also a functional equation for $Z_X(T)$, which is purely formal from the above expression for Z_X when combined with Poincaré duality for étale cohomology.

1.2.2 The Riemann Hypothesis

This course will be interested in the following conjecture.

Conjecture 1.20 (Riemann hypothesis). Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Fix an index $i \in \{0, \ldots, 2d\}$.

(a) The eigenvalues of $\operatorname{Frob}_{X_{\overline{k}}}$ on $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ are algebraic integers of magnitude $q^{i/2}$.

(b) The characteristic polynomial $P_i(T)$ of this Frobenius action is in $\mathbb{Z}[T]$ and is independent of ℓ.

Remark 1.21. Part (a) can be viewed as a Riemann hypothesis: substituting $T = q^{-s}$ into ζ_X, we see that we are requiring our zeroes (and poles) of $\zeta_X(q^{-s})$ to live on the vertical lines

$$\{s : \operatorname{Re} s = i/2\}$$

as i varies over $\{1, \ldots, 2\dim X\}$.

The condition in (a) is interesting enough to deserve a name.

Definition 1.22 (q-Weil)**.** An algebraic integer $\alpha \in \overline{\mathbb{Q}}$ is q-*Weil of weight* i if and only if $|\iota(\alpha)| = q^{i/2}$ for all embeddings $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Example 1.23. The number $\sqrt{2}$ is a 2-Weil number. The number $1+\sqrt{2}$ is not a q-Weil number for any q.

In general, we find that the eigenvalues of a Frobenius action on a local system will still be q-Weil numbers of prescribed weight.

To be precise, the goal of this course will be to prove the following generalization of the above Riemann hypothesis.

Theorem 1.24 (Deligne)**.** Let $f\colon X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X that is mixed of weights at most n. Then $R^i f_! \mathcal{F}$ is also mixed of weights at most $w+i$.

We will define the notion of weights shortly. The idea intuitively comes from Hodge theory: the cohomology groups on a complex Kähler manifold naturally have a weight filtration, which then lifts to sheaves by taking a suitable compactification and studying differential forms suitably. Weights in our context will come from reading off q-Weil numbers.

Remark 1.25. Issues with compactification explain why we are forced to merely deal with mixed weights instead of upgrading this result to one on pure weights. Already this can be seen in Hodge theory.

This course will not prove Theorem 1.24 in full. Instead, we will focus on the case where f has fibers of dimension 1; it turns out that the general case follows from this from some argument involving fibering by curves and using the Leray spectral sequence.

Corollary 1.26. Let X be a scheme of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X.

(a) If $\mathcal{F}$ is mixed of weights at most n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at most $n+i$.

(b) If $\mathcal{F}$ is mixed of weights at least n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at least $n+i$.

(c) Assume that X is smooth and that $\mathcal{F}$ is pure of weight n. Then the image of the canonical map $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

(d) Assume that X is smooth and proper and that $\mathcal{F}$ is pure of weight n. Then $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

Proof. Here, (a) is direct from Theorem 1.24. Then (b) will follow from (a) via Poincaré duality as soon as we know that the duality given by Poincaré duality inverts the weights. Now, (c) follows from combining (a) and (b), and (d) follows from (c). ■

Remark 1.27. One can then prove the result for sheaves over $\mathbb{Q}_\ell$ by base-changing up to the algebraic closure.

The moral of the story is that we are going to use weights to significant profit in this course. Next class we will define weights.

1.3 January 30

Today we review some étale cohomology.

1.3.1 The Étale Site

For completeness, here is the definition of étale.

Definition 1.28 (étale)**.** Fix a scheme morphism $\varphi\colon X \to Y$.

(a) φ is *locally of finite presentation* if and only if $\mathcal{O}_X$ is finitely presented as a $\varphi^{-1}\mathcal{O}_Y$-module (say, on Zariski open neighborhoods or on stalks).

(b) φ is *flat* if and only if the pushforward $\varphi_*\mathcal{O}_X$ is flat over $\mathcal{O}_Y$ (say, on Zariski open neighborhoods or on stalks).

(c) φ is *unramified* if and only if $\Omega_{X/Y} = 0$.

(d) φ is *étale* if and only if it is locally of finite presentation, flat, and unramified.

Example 1.29. One can check that open embeddings are étale.

Remark 1.30. Note that the unramified condition adds some separability, which is a rough explanation for where Galois representations enter the story.

And here is the relevant site.

Definition 1.31 (étale site)**.** Given a scheme S, the *étale site* $\mathrm{\acute{E}t}_S$ is the category of étale morphisms to S. This site comes with a notion of covering: a collection of morphisms $\{U_i \to U\}$ in $\mathrm{\acute{E}t}_S$ is a *covering* if and only if the whole covering is surjective on the underlying topological spaces.

Remark 1.32. Technically, we have defined the "small" étale topos.

An advantage of working with étale cohomology is that our points gain automorphism groups arising from Galois information.

Definition 1.33 (geometric point)**.** Fix a scheme S. A *geometric point* $\overline{x} \hookrightarrow S$ is a morphism of schemes from an algebraically closed field; abusing notation, we may write $\overline{x}$ as $\operatorname{Spec} K$ or as the morphism $\overline{x}\colon \operatorname{Spec} K \to S$.

Remark 1.34. We do not require that our geometric points have closed image in S.

Remark 1.35. Requiring that we have a morphism of schemes amounts to requiring that the algebraically closed field K contains the residue field of the image $x \in S$ of $\overline{x}$. In other words, the data of the morphism $\overline{x}\colon \operatorname{Spec} K \hookrightarrow S$ amounts to the choice of a point $x \in S$ and an embedding $\kappa(x) \hookrightarrow K$.

Definition 1.36 (étale neighborhood)**.** Fix a scheme S. Then an *étale neighborhood* $(U, \overline{u})$ of a geometric point $\overline{x} \hookrightarrow S$ is an étale morphism $\pi\colon U \to S$ equipped with a geometric point $\overline{u} \hookrightarrow U$ together with an embedding $\overline{x} \hookrightarrow U$ over S. A morphism of étale neighborhoods is a morphism of étale covers of S preserving the basepoint.

1.3.2 Sheaves on the Étale Site

With a site, one wants sheaves.

Definition 1.37 (sheaf)**.** Fix a small category $\mathcal{C}$ and a scheme S. An *étale presheaf* $\mathcal{F}$ of $\mathcal{C}$ on S is a contravariant functor $\text{Ét}_S^{\text{op}} \to \mathcal{C}$. An *étale sheaf* is a presheaf $\mathcal{F}$ such that any $U \in \text{Ét}_S$ equipped with a covering $\{U_i \to U\}$ makes $\mathcal{F}(U)$ equal the equalizer

$$\mathcal{F}(U) = \operatorname{eq}\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)\right).$$

Remark 1.38. As in the Zariski topology, there is a sheafification functor $(-)^{\text{sh}} \colon \text{PSh}(S) \to \text{Sh}(S)$ sending the category of presheaves to sheaves. It is a left adjoint to the forgetful functor. The construction of this functor is rather technical, so we will only mention the key property that the sheafification functor is an isomorphism on stalks. For example, one can use sheafification to show that the category $\text{Sh}(S)$ is abelian.

With sheaves, one has stalks.

Definition 1.39. Fix a scheme S and a geometric point $\overline{x} \hookrightarrow S$. For a presheaf $\mathcal{F}$ on S, we define the *stalk* $\mathcal{F}_{\overline{x}}$ as

$$\mathcal{F}_{\overline{x}} \coloneqq \varinjlim_{(U,\overline{u})} \mathcal{F}(U),$$

where the direct limit is taken over étale neighborhoods of $\overline{x}$.

Let's give some examples.

Proposition 1.40. Fix a scheme S. For any étale scheme V over S, the presheaf $\underline{V}$ given by $\underline{V}(U) \coloneqq \operatorname{Hom}_S(U, V)$ is an étale sheaf.

Proof. This follows from some descent argument. ■

Example 1.41. Using the identity map $S \to S$ reveals that $\mathcal{O}_S$ is an étale sheaf. The stalk is

$$\mathcal{O}_{S,\overline{x}} = \varinjlim_{(U,\overline{u})} \Gamma(U, \mathcal{O}_U).$$

It turns out that this is a strictly Henselian ring.

Example 1.42. Fix a positive integer $n \geq 1$. Define the S-scheme μ_n as $\operatorname{Spec} \mathbb{Z}[T]/(T^n - 1) \times_{\operatorname{Spec} \mathbb{Z}} S$. If the multiplication map $n \colon \mathcal{O}_S \to \mathcal{O}_S$ is an isomorphism, then μ_n is étale over S, so this is an étale sheaf.

Example 1.43. For a finite set Σ, we may define the S-scheme Σ given by $\Sigma \times S$ (namely, a disjoint union of Σ-many copies of S). This then produces an étale sheaf $\underline{\Sigma}$.

It is too hard to work with all sheaves. Roughly speaking, we will be interested in "local systems." Here is the version of this notion in algebraic geometry.

Definition 1.44 (locally constant, constructible)**.** Fix an étale sheaf $\mathcal{F}$ on a scheme S and valued in a category $\mathcal{C}$.

(a) $\mathcal{F}$ is *locally constant* if and only if there is a finite étale covering $\{U_i \to S\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to a constant sheaf (still valued in $\mathcal{C}$).

(b) $\mathcal{F}$ is *constructible* if and only if there is a finite stratification $\{S_i\}$ of S into locally closed subsets such that $\mathcal{F}|_{S_i}$ is a locally constant sheaf of finite type.

Remark 1.45. The notion of "finite type" changes depending on $\mathcal{C}$. For example, if $\mathcal{C}$ is the category of abelian groups, then one wants to consider finite abelian groups. If $\mathcal{C}$ is a category of vector spaces, then one wants to consider finite-dimensional vector spaces.

Example 1.46. The constant sheaf $\underline{A}$ of an abelian group A is a locally constant constructible sheaf.

Example 1.47. If S is a variety over $\mathbb{C}$, and $\pi\colon X \to S$ is an étale covering, then the pushforward $\pi_*\underline{\mathbb{Z}}$ is locally constant and constructible.

Example 1.48. If $\pi\colon X \to S$ is an étale covering of schemes, then the sheaves $R^i\pi_*\mathbb{Z}_\ell$ (suitably interpreted) is locally constant and constructible.

Remark 1.49. As in topology, it turns out that one can think about locally constant constructible étale sheaves are representations of a fundamental group $\pi_1^{\text{ét}}(S, \overline{x})$.

Bibliography

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions