Measure Theory for the Impatient

Nir Elber

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Abstract

This document collects a variety of definitions and results from measure theory.

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1 Definitions

1.1 Rings and Friends

Definition 1 (Prering)**.** Fix a set X. A *prering* of a set X is a nonempty collection $\mathcal{P} \subseteq \mathcal{P}(X)$ satisfying the following.

- Intersection: if $E, F \in \mathcal{P}$, then $E \cap F = \mathcal{P}$.
- Decomposition: if $E, F \in \mathcal{P}$, then we can write

$$E \setminus F = \bigsqcup_{i=1}^{n} G_i$$

for some finite disjoint union on the right-hand side with $G_i \in \mathcal{P}$ for each i.

Definition 2 (Ring)**.** Fix a set X. A ring is a nonempty collection $\mathcal{R} \subseteq \mathcal{P}(X)$ with the following properties.

- Union: if $E, F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$.
- Subtraction: if $E, F \in R$, then $E \setminus F \in \mathcal{R}$.

Definition 3 (σ-ring)**.** Fix a set X. Then a ring $\mathcal{R} \subseteq \mathcal{P}(X)$ is a *σ-ring* if and only if R is closed under countable unions.

Definition 4 (σ-algebra)**.** Fix a set X. Then a ring $\mathcal{R} \subseteq \mathcal{P}(X)$ is a *σ-algebra* if and only if R is a σ-ring and contains X.

Example 5. Given a topological space $(X, \mathcal{T})$, the σ-algebra generated by $\mathcal{T}$ make the σ-algebra of *Borel subsets* of X.

Definition 6 (Hereditary)**.** Fix a set X and nonempty family $\mathcal{G} \subseteq \mathcal{P}(X)$. Then $\mathcal{G}$ is *hereditary* if and only if $A \in \mathcal{G}$ and $A' \subseteq A$ implies $A' \in \mathcal{G}$.

Definition 7 (Hereditary σ-ring)**.** Fix a set X and nonempty family $\mathcal{F} \subseteq \mathcal{P}(X)$. Then the *hereditary σ-ring* $\mathcal{H}(F)$ generated by $\mathcal{F}$ consists of all subsets $E \subseteq X$ such that there exists a countable subcollection $\{F_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i.$$

1.2 Measures and Friends

Definition 8 (Finitely additive measure)**.** Fix a set X and ring $\mathcal{R} \subseteq \mathcal{P}(X)$. Then a *finitely additive measure* is a function $\mu \colon \mathcal{R} \to [0, \infty]$ such that any disjoint $E, F \in \mathcal{R}$ have

$$\mu(E \sqcup F) = \mu(E) + \mu(F)$$

Definition 9 (Countably additive)**.** Fix a set X and a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(X)$. A function $\mu\colon \mathcal{C} \to [0,\infty]$ is *countably additive* if and only if any pairwise disjoint subcollection $\{E_i\}_{i=1}^\infty \subseteq \mathcal{C}$ with $\bigsqcup_{i=1}^\infty E_i \in \mathcal{C}$ has

$$\mu\left(\bigsqcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i).$$

Notably, we are allowed to have the right-hand side diverge to ∞ if the left-hand side is ∞.

Definition 10 (Premeasure)**.** Fix a set X and a prering $\mathcal{P} \subseteq \mathcal{P}(X)$. A *premeasure* on $\mathcal{P}$ is a countably additive function $\mu\colon \mathcal{P} \to [0,\infty]$.

Definition 11 (Measure)**.** Fix a set X and σ-ring $\mathcal{S}$. Then a *measure* on $\mathcal{S}$ is a function $\mu\colon \mathcal{S} \to [0,\infty]$ which is countably additive. We call the triple $(X,\mathcal{S},\mu)$ a *measure space*.

Example 12. Fix a left-continuous, increasing function $\alpha\colon \mathbb{R} \to \mathbb{R}$, and let $\mathcal{P} \subseteq \mathcal{P}(\mathbb{R})$ as the prering of half-open intervals $[a,b)$ for $a < b$. Then

$$\mu_\alpha([a,b)) := \alpha(b) - \alpha(a)$$

is a premeasure on $\mathcal{P}$.

Notation 13. Fix a set X and nonempty family $\mathcal{F} \subseteq \mathcal{P}(X)$. Then give $\mu\colon \mathcal{F} \to [0,\infty]$, we will define $\mu^*\colon \mathcal{H}(\mathcal{F}) \to [0,\infty]$ by

$$\mu^*(E) := \inf\left\{\sum_{i=1}^\infty \mu(E_i) : \{E_i\}_{i=1}^\infty \subseteq \mathcal{F} \text{ and } E \subseteq \bigcup_{i=1}^\infty E_i\right\}.$$

Definition 14 (Lebesgue–Stieltjes measure)**.** Let $\mathcal{P}$ be the prering of right-half-open intervals, and fix a left-continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$. Then the measure $\mu_\alpha^*|_{\mathcal{M}(\mu_\alpha^*)}$ from the premeasure of Example 12 is the *Lebesgue–Stieltjes measure*. The *Lebesgue measure* is the measure coming from $\alpha(t) = t$.

1.3 Adjectives for Measures

Definition 15 (Monotone)**.** Fix a collection $\mathcal{F}$ of subsets of a set X. A function $\mu\colon \mathcal{F} \to [0,\infty]$ is *monotone* if and only if any $E, F \in \mathcal{F}$ with $E \subseteq F$ have $\mu(E) \le \mu(F)$.

Definition 16 (Countably subadditive)**.** Fix a set X and a collection $\mathcal{F} \subseteq \mathcal{P}(X)$. A function $\mu\colon \mathcal{F} \to [0,\infty]$ is *countably subadditive* if and only if

$$E \subseteq \bigcup_{i=1}^\infty E_i \implies \mu(E) \le \sum_{i=1}^\infty \mu(E_i)$$

for any $E \in \mathcal{F}$ and $\{E_i\}_{i=1}^\infty \subseteq \mathcal{F}$.

Definition 17. Fix a set X and a hereditary σ-ring $\mathcal{H}$ on X, and fix an outer measure $\nu\colon \mathcal{H} \to [0,\infty]$. Then a set $E \subseteq \mathcal{H}$ is *ν-measurable* if and only if

$$\nu(A) = \nu(A \cap E) + \nu(A \setminus E)$$

for any $A \in \mathcal{H}$. We will let $\mathcal{M}(\nu)$ denote the set of ν-measurable sets.

Definition 18 (Compelete)**.** Fix a set X and a family $\mathcal{F} \subseteq \mathcal{P}(X)$. Then a function $\nu\colon \mathcal{F} \to [0,\infty]$ is *complete* if and only if any $E \in \mathcal{F}$ with $F \subseteq E$ and $\nu(E) = 0$ must have $F \in \mathcal{F}$ and $\nu(F) = 0$.

Example 19. If ν is an outer measure on a hereditary σ-ring $\mathcal{H}$, then $\nu|_{\mathcal{M}(\nu)}$ is complete when $\mathcal{M}(\nu)$ is nonempty.

1.4 Measurable Functions and Friends

Definition 20 (Simple measurable function)**.** Fix a ring $\mathcal{S}$ on a set X and a normed vector space B. Then a *simple $\mathcal{S}$-measurable B-valued function* is a function $f\colon X \to B$ such that $\operatorname{im} f$ is finite and $f^{-1}(\{y\}) \in \mathcal{S}$ for any $y \in B \setminus \{0\}$.

Remark 21. Any simple $\mathcal{S}$-measurable function $f\colon X \to S$ can be written as

$$f = \sum_{y \in (\operatorname{im} f)\setminus\{0\}} y 1_{f^{-1}(\{y\})}.$$

Definition 22 (Measurable function)**.** Fix a set X and a σ-ring $\mathcal{S}$ on X. Given a normed vector space B, an *$\mathcal{S}$-measurable function* is a function $f\colon X \to B$ such that there is a sequence of simple $\mathcal{S}$-measurable functions $\{f_n\}_{n\in\mathbb{N}}$ which converge to f pointwise.

Definition 23 (Separable)**.** A topological space M is *separable* if and only if there is a countable dense subset of M. As such, a subset $A \subseteq M$ is *separable* if and only if A is separable with the restricted metric; in other words, $A \subseteq M$ is separable if and only if there is a countable subset $B \subseteq A$ such that $A \subseteq \overline{B}$.

Theorem 24. Fix a normed vector space B and a set X with a σ-ring $\mathcal{S}$ on X. Then a function $f\colon X \to B$ is $\mathcal{S}$-measurable if and only if

(i) $\operatorname{im} f$ is separable, and

(ii) for any open $U \subseteq B$, we have $f^{-1}(U \setminus \{0\}) \in \mathcal{S}$.

1.5 All the Convergences

Definition 25 (Null set)**.** Fix a set X and a σ-ring $\mathcal{S}$ on X equipped with a measure μ. A *null set* is a subset $N \subseteq X$ such that there is some $E \in \mathcal{S}$ such that $N \subseteq E$ and $\mu(N) = 0$.

Definition 26 (Almost everywhere)**.** Fix a set X and a σ-ring $\mathcal{S}$ on X equipped with a measure μ. A property $P(x)$ for points $x \in X$ holds *almost everywhere* if and only if $\{x \in X : \neg P(x)\}$ is a null set.

Definition 27 (Converge in measure)**.** Fix a measure space $(X, \mathcal{S}, \mu)$ and normed vector space $(B, \|\cdot\|)$. Then a sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}$-measurable functions *converges in measure* to an $\mathcal{S}$-measurable function f if and only if all $\varepsilon > 0$ have

$$\lim_{n\to\infty} \mu(\{x \in X : \|f(x) - f_n(x)\| \geq \varepsilon\}) = 0.$$

1.6 Integration

Definition 28 (Simple integrable function)**.** Fix a ring $\mathcal{S}$ on a set X and a metric space B. Further, let μ be a finitely additive measure μ on $\mathcal{S}$. Then a function $f\colon X \to B$ is a *simple $\mathcal{S}$-integrable function* if and only if $\operatorname{im} f$ is finite, and $f^{-1}(\{y\}) \in \mathcal{S}$ has finite measure for each $y \in (\operatorname{im} f) \setminus \{0\}$.

Definition 29 (Integral)**.** Fix a ring $\mathcal{S}$ on a set X and a metric space B. Further, let μ be a finitely additive measure μ on $\mathcal{S}$. Given a simple μ-integrable function f, we define the *integral*

$$\int_X f\,d\mu \coloneqq \sum_{y\in(\operatorname{im} f)\setminus\{0\}} \mu\left(f^{-1}(\{y\})\right) y.$$

Note this is a finite sum with $\mu\left(f^{-1}(\{y\})\right)$ finite, so $\int_X f\,d\mu$ is finite.

Notation 30. Fix a normed vector space B and a ring $\mathcal{S}$ on a set X equipped with a finitely additive measure μ. Given a (simple) μ-integrable function $f\colon X \to B$, we define

$$\|f\|_1 \coloneqq \int_X \|f\|\,d\mu.$$

Note $\|f\|$ is in fact simple μ-integrable.

2 Lemmas and Results

2.1 Checks on Measures

Lemma 31 (Monotone)**.** Fix a prering $\mathcal{P}$ on X and a finitely additive function $\mu\colon \mathcal{P} \to [0,\infty]$. Given $E, F \in \mathcal{P}$, then $\mu(E) \geq \mu(E \cap F)$. In particular, if $E \supseteq F$, then $\mu(E) \geq \mu(F)$.

Lemma 32 (Countably subadditive)**.** Fix a prering $\mathcal{P}$ on a set X, and let μ be a premeasure on $\mathcal{P}$. Then μ is countably subadditive.

Lemma 33. Fix a set X and nonempty family $\mathcal{F} \subseteq \mathcal{P}(X)$. Further, fix some $\mu\colon \mathcal{F} \to [0,\infty]$. Then we have the following.

(a) $\mu^*(E) \leq \mu(E)$ for any $E \in \mathcal{F}$.

(b) μ^* is monotone.

(c) μ^* is countably subadditive.

Lemma 34. Fix a set X and a prering $\mathcal{P}$ on X equipped with a premeasure μ on $\mathcal{P}$. Then $\mu^*(E) = \mu(E)$ for any $E \in \mathcal{P}$.

Theorem 35. Fix a set X and a hereditary σ-ring $\mathcal{H}$ on X, and fix an outer measure $\nu\colon \mathcal{H} \to [0,\infty]$. If nonempty, $\mathcal{M}(\nu)$ is a σ-ring, and $\nu|_{\mathcal{M}(\nu)}$ is a measure.

Theorem 36. Fix a set X and a prering $\mathcal{P}$ on X equipped with a premeasure μ on $\mathcal{P}$. Then $\mathcal{P} \subseteq \mathcal{M}(\mu^*)$.

Theorem 37. Fix a set X and a prering $\mathcal{P}$ on X equipped with a σ-finite premeasure μ on X. Then, for some σ-ring $\mathcal{S} \subseteq \mathcal{M}(\mu^*)$, our $\mu^*|_{\mathcal{S}}$ is the unique extension of μ to a measure on $\mathcal{S}$.

Proposition 38. Fix a σ-ring $\mathcal{S}$ on a set X equipped with a measure μ on $\mathcal{S}$. A collection $\{E_i\}_{i=1}^\infty \subseteq \mathcal{S}$ such that $E_n \subseteq E_{n+1}$ for each i will have

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{i=1}^\infty E_i\right).$$

Corollary 39. Fix a σ-ring $\mathcal{S}$ on a set X equipped with a measure μ on $\mathcal{S}$. Suppose we have a collection $\{E_i\}_{i=1}^\infty \subseteq \mathcal{S}$ such that $\mu(E_1) < \infty$ and $E_n \supseteq E_{n+1}$ for each i. Then we have

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{i=1}^\infty E_i\right).$$

2.2 Checks on Measurable Functions

Lemma 40. Fix a measure space $(X, \mathcal{S}, \mu)$ and a Banach space $(B, \|\cdot\|)$ over a normed field k. Given $a, b \in k$ and $E \in \mathcal{S}$, if f and g are (simple $\mathcal{S}$-measurable, $\mathcal{S}$-measurable, simple μ-integrable, μ-integrable), then $af + b$ and $\|f\|$ and $f1_E$ and $f1_{X\setminus E}$ are as well.

Lemma 41. Convergence statement.