256B: Algebraic Geometry

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

CURVES

*Every person believes that he knows what a curve is until he has
learned so much mathematics that the countless possible
abnormalities confuse him.*

—Felix Klein, [Kle16]

1.1 January 18

Here we go.

1.1.1 House-Keeping

Here are some notes on the course.

- We will continue to use [Har77]. Note that [Vak17] is also popular, as is [SP].

- Office hours will probably be after class on Wednesday and Friday.

- There is a bCourses.

- In the course, we plan to cover curves, some coherent cohomology (and maybe on Zariski sheaves), and some surfaces if we have time.

- Grading will be homework and a term paper. Homework will be challenging, so collaboration is encouraged.

In this course, we will discuss coherent cohomology, but we will begin by talking about curves.

1.1.2 Serre Duality Primer

For the next few weeks, we will focus on non-singular curves over an algebraically closed field. Here is our definition.

> **Definition 1.1** (curve). Fix a field k. A k-*curve* is an integral, proper, normal scheme of dimension 1. Note that being normal is equivalent to being smooth, so we are requiring our curves to be smooth!

We will want to talk about genus a little. Here is a working definition.

Definition 1.2 (arithmetic genus). Fix a projective k-variety X. Then the *arithmetic genus* $p_a(X)$ is defined

Definition 1.3 (geometric genus). Fix an irreducible k-variety X. Then the *geometric genus* is $p_g(X) := \dim_k \Gamma(X, \omega_X)$, where ω_X is the canonical sheaf. Explicitly, ω_X is the top exterior power of the sheaf of differential forms on X.

In general, the above notions are not the same, but they will be for curves.

Proposition 1.4. Fix a k-curve X. Then $p_g(X) = p_a(X)$. We denote this genus by $g(X)$ or g when the curve is clear.

We would like to actually compute some genera, but this is a bit difficult. One goal of the class is to build a cohomology theory $H^i(X, \mathcal{F})$ for coherent sheaves $\mathcal{F}$ on X, and it turns out we can use these cohomology groups to compute the genus of X. Roughly speaking, we will derive (on the right) the left-exact functor $\Gamma(X, \cdot)$, so the cohomology will in some sense measure the difference between global sections and local sections. For example, flasque sheaves will have trivial cohomology.

For now, we will black-box various things. Here is an example of something we will prove.

Proposition 1.5. Fix a projective k-variety X, and let $\mathcal{F}$ be a coherent sheaf. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$, and $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces for all $i \geq 0$.

To show the Riemann–Roch theorem, we will black-box Serre duality, which we will prove much later. In the case of curves, it says the following.

Theorem 1.6 (Serre duality). Fix a k-curve X. Then, for any vector bundle $\mathcal{L}$ on X, there is a duality

$$H^i(X, \mathcal{L}^\vee \otimes \omega_X) \otimes_k H^{1-i}(X, \mathcal{L}) \to k,$$

where $i \in \{0, 1\}$.

Remark 1.7. Notably, we see $p_g(X) = \dim_k \Gamma(X, \omega_X) = \dim_k H^0(X, \omega_X) = \dim_k H^1(X, \mathcal{O}_X)$.

We will also want the following fact.

Proposition 1.8. Fix a closed embedding $i \colon X \to Y$ of schemes. Given a sheaf $\mathcal{F}$ of abelian groups on Y, then
$$H^i(X, i_*\mathcal{F}) = H^i(Y, \mathcal{F}).$$

1.1.3 Divisors Refresher

We also want to recall a few facts about divisors. We begin with Weil divisors.

Definition 1.9 (Weil divisor). Fix an irreducible k-variety X. A *Weil divisor* $\mathrm{Div}(X)$ are $\mathbb{Z}$-linear combinations of codimension-1 irreducible closed subschemes. Then the *principal divisors* are the image of the map $\mathrm{div} \colon K(X) \to \mathrm{Div}(X)$, where div takes rational functions to poles. The *class group* $\mathrm{Cl}\, X$ is the quotient.

More generally, we have Cartier divisors.

Definition 1.10 (Cartier divisor)**.** Fix a scheme X. A *Cartier divisor* in $\operatorname{CaDiv} X$ is a global section of $\Gamma(X, \mathcal{K}^{\times}/\mathcal{O}_X^{\times})$, where $K^{\times}$ is the sheafification of the presheaf $U \mapsto \operatorname{Frac} \mathcal{O}_X(U)$. The *principal divisors* are the image of $\Gamma(X, \mathcal{K}^{\times})$, and the *class group* $\operatorname{CaCl} X$ is the quotient.

Notably, if X is an integral sheaf, then $\mathcal{K}$ is the constant sheaf $K(X)$. Then a global section is given by the pair $(\{U_i\}, \{f_i\})$ where the U_i cover X, and $f_i \in K(X)^{\times}$ so that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^{\times}$. (The coherence condition allows the Cartier divisors to glue.) Notably, each $f \in K(X)$ grants a principal divisor $(\{X\}, \{f\})$, which are exactly the principal divisors.

Here is the main result on these divisors.

Proposition 1.11. If X is an integral, separated, Noetherian, and locally factorial (notably, regular in codimension 1), then Weil divisors are in canonical isomorphism with Cartier divisors. Further, the principal divisors are in correspondence, and so the class groups are also isomorphic.

Example 1.12. Non-singular k-curves have all the required adjectives. Namely, codimension-1 means we are looking at points, and being smooth implies being regular, so all the local rings are dimension-1 regular local rings, which are discrete valuation rings. Notably, discrete valuation rings are

Yet another connection to divisors comes from invertible sheaves. Namely, for integral schemes X, the group of invertible sheaves $\operatorname{Pic} X$ is isomorphic to $\operatorname{CaCl} X$. The point here is that invertible shaves can be embedded into $\mathcal{K}^{\times}$ when X is integral.

We will be interested in some special divisors.

Definition 1.13 (effective)**.** Fix a k-curve X. Then an *effective Weil divisor* is a $\mathbb{Z}_{\geq 0}$ linear combination of closed points of X. Note that the collection of effective Weil divisors forms a submonoid of $\operatorname{Div} X$. We might be interested in knowing how many effective divisors are equivalent to some given divisor; the set of these is denoted $|D|$.

When our schemes X have enough adjectives, we note that the above correspondences tell us that there is a way to send a Cartier divisor $(\{U_i\}, \{f_i\})$ to a line bundle $\mathcal{L}$ embedded in $\mathcal{K}^{\times}$. Explicitly, we build $\mathcal{L}(D)$ by $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_X|_{U_i} \subseteq \mathcal{K}$, where the last isomorphism is by sending $1 \mapsto f_i^{-1}$. Notably, if D is effective, then the global section 1 of $\mathcal{K}^{\times}$ can be pulled back along to a nonzero global section on $\mathcal{L}(D)$ which is f_i on each U_i.

1.2   January 20

We continue moving towards Riemann–Roch.

1.2.1   Linear Systems

Let's discuss linear systems. Let X be a non-singular projective irreducible variety over a field k, and let D be a divisor of X.

Recall that a Cartier divisor $D = \{(U_i, f_i)\}$ on X is associated to the line bundle $\mathcal{L}(D)$ which is locally trivial on each U_i, given as $f_i^{-1}\mathcal{O}_X|_{U_i}$. Conversely, suppose that $\mathcal{L}$ is a line bundle on X. Then we pick up some nonzero global section $\Gamma(X, \mathcal{L})$. Give $\mathcal{L}$ a trivializing open cover $\{U_i\}$, where we are given isomorphisms $\varphi_i \colon \mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$. Setting $f_i := \varphi_i(s)$ recovers an (effective) Cartier divisor $\{(U_i, f_i)\}$ on X. We call this line bundle $\operatorname{div}(\mathcal{L}, s)$.

This thinking gives the following result.

> **Proposition 1.14.** Let X be a non-singular projective integral variety over a field k. Given a Cartier divisor D_0, and let $\mathcal{L} := \mathcal{L}(D_0)$ be the corresponding line bundle.
>
> (a) For each nonzero section $s \in \Gamma(X, \mathcal{L})$, the divisor $\mathrm{div}(\mathcal{L}, s)$ is an effective divisor linearly equivalent to D_0.
>
> (b) Every effective divisor linearly equivalent tot D_0 is obtained in this way.
>
> (c) If k is algebraically closed, we have $\mathrm{div}(\mathcal{L}, s) = \mathrm{div}(\mathcal{L}, s')$ if and only if s and s' differ by a scalar in $k^\times$.

The above result essentially says that we can study $\Gamma(X, \mathcal{L})$ as a k-vector space instead of trying to understand linear equivalence of divisors. For example, if $\Gamma(X, \mathcal{L}) = 0$, then D is not equivalent to any effective divisor!

Proof. We go one at a time.

(a) Embed $\mathcal{L} \subseteq \mathcal{K}_X$ as usual. Then $s \in \Gamma(X, \mathcal{L})$ becomes a rational function in $K(X)$. By the construction of $\mathcal{L}$, we have an open cover $\{U_i\}$ and some f_i so that $\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_X|_U$. Because we have a global section, we may write $\varphi_i(s) = f_i f$ for some fixed f, and then tracking through our Cartier divisor, we get
$$\mathrm{div}(\mathcal{L}, s) = D_0 + \mathrm{div}(f),$$
as needed.

(b) Suppose D is an effective divisor with $D = D_0 + \mathrm{div}(f)$. Then we see $(f) \geq -D_0$, so f determines a nonzero global section of $\mathcal{L}\mathcal{L}(D_0)$ by tracking through the above constructions: namely, set $s|_{U_i} = f_i^{-1}f$ and glue. (In particular, $(f) \geq -D_0$ means $f/f_i \in \mathcal{O}_X(U_i)$ for each i.) So we see $D = \mathrm{div}(\mathcal{L}, s)$.

(c) One can see directly that $s = cs'$ for $c \in k^\times$ will have $\mathrm{div}(\mathcal{L}, s) = \mathrm{div}(\mathcal{L}, s)$. Conversely, if $\mathrm{div}(\mathcal{L}, s) = \mathrm{div}(\mathcal{L}, s')$, then under the embedding $\mathcal{L} \subseteq \mathcal{K}_X$, we may correspond s and s' to $f, f' \in K(X)^\times$. Thus, $f/f' \in \Gamma(X, \mathcal{O}_X^\times)$. But because k is algebraically closed and X is proper over k, we have $\Gamma(X, \mathcal{O}_X) = k$, so we are done. ∎

> **Remark 1.15.** More generally, we have the following: let k be a field, and let X be a proper, geometrically reduced scheme over k. Then $\Gamma(X, \mathcal{O}_X) = k$ if and only if X is geometrically reduced.

So we have the following.

> **Corollary 1.16.** Let X be a non-singular projective integral variety over a field k. The set $|D_0|$ of effective divisors linearly equivalent to a given divisor D_0 is in natural bijection with $(\Gamma(X, \mathcal{L}(D)) \setminus \{0\})/k^\times$.

With this in mind, we set the following notation.

> **Notation 1.17.** Let X be a non-singular projective integral variety over a field k. Given a divisor D_0 of X, we define $\ell(D_0) := \dim_k \Gamma(X, \mathcal{L}(D))$ and $\dim D_0 := \ell(D_0) - 1$.

The Riemann–Roch theorem is interested in the values of $\ell(D_0)$. Here is a quick lemma.

> **Lemma 1.18.** Let X be a non-singular projective integral variety over a field k. Fix a divisor D of X.
>
> (a) If $\ell(D) \neq 0$, then $\deg D \geq 0$.
>
> (b) If $\ell(D) \neq 0$ and $\deg D = 0$, then D is linearly equivalent to 0.

Proof. Note $\ell(D) \neq 0$ enforces $D \sim D_0$ for some effective divisor D, so $\deg D = \deg D_0 \geq 0$, which shows (a). Then for (b), we note $\deg D_0 = 0$ forces $D_0 = 0$. ∎

1.2.2 Riemann–Roch for Curves

We now force $\dim X = 1$, meaning that X is a curve. Let $\Omega_{X/k}$ denote the sheaf of differentials, which is equal to the canonical sheaf $\omega_X = \bigwedge^{\dim X} \Omega_{X/k}$. Any divisor linearly equivalent to $\Omega_{X/k}$ will be denoted K and is called the "canonical divisor." Note that the canonical divisor is really a canonical divisor class.

> **Theorem 1.19** (Riemann–Roch). Let D be a divisor on a k-curve X, and let g be the genus of X. Further, suppose k is algebraically closed. Then
> $$\ell(D) - \ell(K - D) = \deg D + 1 - g.$$

Proof. Set $\mathcal{L} := \mathcal{L}(D)$ for brevity. Note $\mathcal{L}(K - D) \cong \omega_X \otimes \mathcal{L}^\vee$, so Serre duality implies
$$\ell(K - D) = \dim_k \Gamma(\omega_X \otimes \mathcal{L}^\vee) = \dim_k H^1(X, \mathcal{L}).$$
Thus, our left-hand side is $\chi(\mathcal{L}) := \dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L})$.[1] Quickly, note $D = 0$ can be seen directly by
$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = \dim k - g = 1 - g,$$
which is what we wanted.

We now perturb D by a point. We show the formula holds for D if and only if the formula holds for $D + p$, where $p \in X$ is some closed point. Note we have a short exact sequence
$$0 \to \mathcal{L}(-p) \to \mathcal{O}_X \to k(p) \to 0,$$
where $k(p)$ refers to the skyscraper sheaf which is the structure sheaf about p. Tensoring with $\mathcal{L}(D + p)$, we get
$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + p) \to k(p) \to 0.$$
Now, χ is additive in short exact sequences by using the long exact sequence in cohomology, so
$$\chi(\mathcal{L}(D)) = \chi(\mathcal{L}(D + p)) + \chi(k(P)),$$
but $\chi(k(p)) = \dim_k \Gamma(X, k(p)) = \dim_k k = 1$ because k is algebraically closed. The conclusion now follows because $\deg(D + p) = \deg D + 1$. ∎

1.3 January 23

Today we apply the Riemann–Roch theorem.

> **Remark 1.20.** Here is a quick hint for the homework: fix a Weil divisor $D = \sum_P n_P P$ on a k-curve X, where k is algebraically closed. Then $\Gamma(X, \mathcal{O}_X(D))$ can be described as space of rational functions f on X such that $D + \operatorname{div}(f)$ is effective. In other words, for each point $P \in X$, we see f has a pole of order at worse n_P at P.

1.3.1 Applications of Riemann–Roch

Let's give a few applications of Theorem 1.19.

> **Example 1.21.** Fix a k-curve X, where k is algebraically closed. Further, let g be the genus of X and K the canonical divisor. We can compute $\deg K$ as follows: plugging into Theorem 1.19, we see
> $$g - 1 = \ell(K) - \ell(0) = \deg K - 1 + g,$$
> so $\deg K = 2g - 2$.

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[1] This is the Euler characteristic of $\mathcal{L}(D)$ because our higher cohomology groups vanish.

> **Remark 1.22.** More generally, we can see that plugging in $K - D$ into Theorem 1.19 is only able to deduce $\deg K = 2g - 2$.

> **Example 1.23.** Let D be a divisor on a k-curve X, where k is algebraically closed. Further, let g be the genus of X and K the canonical divisor. We would like to study $\dim |nD| = \ell(nD) - 1$ for $n \in \mathbb{Z}^+$. We have the following cases.
>
> - If $\deg D < 0$, then $\deg(nD) < 0$ still, so $\ell(nD) = 0$, so $\dim |nD| = -1$ always.
>
> - If $\deg D = 0$, then there are two possibilities. Namely, if nD is linearly equivalent to 0, then $\ell(nD) = 1$, so $\dim |nD| = 0$; otherwise, D will not be linearly equivalent to any effective divisor (the only effective divisor with degree 0 is 0), so $\dim |nD| = -1$.
>
> - If $\deg D > 0$, then for n large enough, we see $\deg(K - nD) < 0$, so $\ell(K - nD) = 0$, so Theorem 1.19 implies $\ell(nD) = n \deg D + 1 - g$, so $\dim |nD| = n \deg D - g$. Here, "n large enough" is just $n > \deg K / \deg D$.

Here is a more interesting corollary.

> **Lemma 1.24.** Let X be a k-curve, where k is algebraically closed. Suppose that two distinct closed points P and Q produce linearly equivalent Weil divisors. Then $X \cong \mathbb{P}_k^1$.

Proof. We are given that $\operatorname{div}(f) = P - Q$ for some $f \in K(X)$. Thus, we induce a map $k(t) \to K(X)$ given by $t \mapsto f$, where we view $k(t)$ as the fraction field of $\mathbb{P}_k^1$. Notably, t has a zero at 0 and a pole at ∞, and f has a zero at Q and a pole at P. This will induce a finite map $g \colon X \to \mathbb{P}^1$, which we can compute to have degree 1 by the following discussion (notably, the pull-back of the divisor $[0]$ is $[P]$), so g is a birational map and hence an isomorphism.

Now, for any finite map of curves $g \colon X \to Y$, recall there is a map on divisors $g^* \colon \operatorname{Cl}(Y) \to \operatorname{Cl}(X)$ as follows: given point $P \in Y$ inside an affine open subscheme $V \subseteq Y$, we can take the pre-image to X to produce a Weil divisor.[2] More formally, we send P to

$$g^*(P) := \sum_{Q \in g^{-1}(\{P\})} v_Q(t) Q,$$

where t is a uniformizer parameter for $Q_{X,P}$, and $v_Q(t)$ is its valuation at the local ring $\mathcal{O}_{X,Q}$.

What?

In fact, we showed the following last semester, which we used in the proof above.

> **Proposition 1.25.** Let $g \colon X \to Y$ be a finite map of k-curves. For any divisor D on Y, we have $\deg g^* D = (\deg g)(\deg D)$.

Proof. Let's recall the proof: it suffices to show this in the case where $D = P$ is a point. Plugging into the definition of g^*, we are showing that

$$\sum_{Q \in g^{-1}(\{P\})} v_Q(t) = \deg g^* P \overset{?}{=} \deg g.$$

This statement is local at P, so we may assume that $Y = \operatorname{Spec} B$, whereupon taking the pre-image along g enforces $X = \operatorname{Spec} A$ for some A. For dimension-theory reasons, we see that g is dominant, so the induced map $B \to A$ is injective.

Localizing, we set $A' := A \otimes_B \mathcal{O}_{Y,P}$, so we are really interested in the map $\mathcal{O}_{Y,P} \to A'$, which is still injective. It follows that A' is a finite (by g) torsion-free (by this injectivity argument) module over $\mathcal{O}_{Y,P}$. But

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[2] Alternatively, one can view this operation as the pullback $g^* \colon \operatorname{Pic} Y \to \operatorname{Pic} X$ and then recall that each element of the class group corresponds to an isomorphism class in Pic.

$\mathcal{O}_{Y,P}$ is a principal ideal domain, so we may appeal to the structure theorem. Namely, we want to compute the rank of A' over $\mathcal{O}_{Y,P}$, for which it suffices to take fraction fields everywhere and instead compute

$$\operatorname{rank}_{\mathcal{O}_{Y,P}} A' = [\operatorname{Frac} A' : \operatorname{Frac} \mathcal{O}_{Y,P}] = [\operatorname{Frac} A : \operatorname{Frac} B].$$

On the other hand, given uniformizer t of $\mathcal{O}_{X,P}$, we can compute the corresponding rank of A'/tA' over $k = k(P)$ is $\deg g$. However, $\operatorname{Spec} A'/tA'$ is the pre-image of P, so we go ahead and note A'/tA' is a product of local Artinian rings which are quotients corresponding to points in $g^{-1}(\{P\})$. In particular, for each $Q \in g^{-1}(P)$, we see $v_Q(t) = m$ means that $\mathcal{O}_{X,Q}$ appears in A'/tA' as $\mathcal{O}_Q/(\varpi_Q)^m$, where ϖ_Q is a uniformizer at Q. So we can write

$$A'/tA' = \prod_{Q \in g^{-1}(\{P\})} \mathcal{O}_{X,Q}/(\varpi_Q)^{v_Q(t)}.$$

But the k-rank of this is $\deg g^*(\{P\})$ by definition of g^*, which must equal the $\mathcal{O}_{Y,P}$-rank of A', so we are done. ∎

This proposition finishes justifying the first paragraph of the proof. ∎

> **Corollary 1.26.** Let X be a k-curve of genus 0, where k is algebraically closed. Then X is isomorphic to $\mathbb{P}^1_k$.

Proof. As an aside, we note that $\mathbb{P}^1_k$ is certainly a k-curve of genus 0.

Quickly, choose any two points P and Q on X. As such, we take $D := P - Q$ so that $\deg(K - D) = -2 < 0$, so $\ell(K - D) = 0$. Thus, Theorem 1.19 implies $\ell(D) = 1$. Thus, D is linearly equivalent to an effective divisor, but the only effective divisor with degree 0 is 0 itself, so we see that $P - Q$ is linearly equivalent to 0. This is enough to conclude that $X \cong \mathbb{P}^1_k$ by Lemma 1.24; note that X being a curve requires X to have infinitely many points and thus distinct points. ∎

Lastly, let's give a corollary for elliptic curves.

> **Definition 1.27** (elliptic)**.** A (proper) k-curve X is *elliptic* if and only if X has genus 1.

> **Corollary 1.28.** Let X be an elliptic k-curve, where k is algebraically closed. We give $X(k)$ a group law arising from $\operatorname{Pic} X$.

Proof. Let K be a canonical divisor for X, and we see $\deg K = 0$ by Example 1.21. However, $\ell(K) = 1$ is the genus, so K is linearly equivalent to some effective divisor, so as usual we note that K is linearly equivalent to 0.

Quickly, we note that the group structure on the Picard group $\operatorname{Pic} X$ of isomorphism classes of line bundles on $X(k)$ induces a group law on X. Indeed, fix some k-point $P_0 \in X$. We now claim that the map $X(k) \to \operatorname{Pic}^0 X$ given by

$$P \mapsto \mathcal{O}_X(P - P_0)$$

is a bijection. (Here, $\operatorname{Pic}^0 X$ is the subgroup of degree-0 line bundles.) This will give $X(k)$ a group law by stealing it from $\operatorname{Pic} X$.

Because we already know that $\operatorname{Pic} X$ is in bijection with divisors more generally, it's enough to show that any divisor D of degree 0 is linearly equivalent to a divisor of the form $P - P_0$ for $P \in X(k)$. Well, we use Theorem 1.19 with $D + P_0$, which yields

$$\ell(D + P_0) - \ell(K - D - P_0) = 1 + 1 - g = 1,$$

but $K - D - P_0$ has degree -1 and so $\ell(K - D - P_0) = 0$. Thus, $\dim |D + P_0| = 0$, so there is a unique effective divisor of degree 1 linearly equivalent to $D + P_0$. However, an effective divisor of degree 1 is just a point P, so we are done. ∎

1.4 January 25

Today we discuss ramification.

1.4.1 Ramification

Given a finite morphism $f\colon X \to Y$ of k-curves, there are some numbers we can attach to f. For example, we have the degree $\deg f := [K(X) : K(Y)]$. Additionally, for each $y \in Y$, we have a map

$$\mathcal{O}_{Y,y} \to \prod_{x \in f^{-1}(\{y\})} \mathcal{O}_{X,x}.$$

Each of these discrete valuation rings have residue field k, so when k is algebraically closed, this is pretty simple to understand. Indeed, this gives rise to "ramification" information.

> **Definition 1.29** (ramification). Let $f\colon X \to Y$ be a morphism of k-curves. Then for $x \in X$, we define the *ramification index* as $e_x := v_x(t_y)$, where $t_y \in \mathcal{O}_{Y,y}$ is a uniformizer for $\mathcal{O}_{Y,y}$, and $v_x(t_y)$ refers to the valuation of t_y embedded in $\mathcal{O}_{X,x}$.

Note $e_x > 0$ because $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a map of local rings.

When k is not algebraically closed, things get a little more complicated, and we will also want to keep track of the degree of the corresponding residue field extension.

> **Definition 1.30** (ramification). Let $\varphi\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a map of discrete valuation rings. Fix a uniformizer $\varpi \in \mathfrak{m}$, and define $e := v_B(\varphi(\varpi))$ and $f := [B/\mathfrak{n} : A/\mathfrak{m}]$ and $p := \mathrm{char}(A/\mathfrak{m})$.
>
> - φ is *unramified* if and only if $e = 1$ and $B/\mathfrak{n}$ is separable over $A/\mathfrak{m}$.
>
> - φ is *tamely ramified* if and only if $p \nmid e$ and $B/\mathfrak{n}$ is separable over $A/\mathfrak{m}$.
>
> - Otherwise, φ is *wildly ramified*.

> **Remark 1.31.** Algebraic number theory has a lot to say about how the above process works for local fields (or even just number rings).

When k is algebraically closed, we see that the extension $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k$, so this extension is of course separable, so Definition 1.30 simplifies somewhat in our situation.

> **Definition 1.32.** Let $f\colon X \to Y$ be a finite morphism of k-curves. Then we say that $x \in X$ is *unramified/tamely ramified/wildly ramified* if and only if the corresponding map $f^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is unramified/tamely ramified/wildly ramified.

Notably, last class we recalled that

$$f^*[y] = \sum_{x \in f^{-1}(\{y\})} e_x[x].$$

Now, even in our algebraically closed situation, we will want to care about separable extensions.

> **Definition 1.33** (separable). Let $f\colon X \to Y$ be a finite morphism of k-curves. Then f is *separable* if and only if the extension $K(Y) \subseteq K(X)$ is separable.

Now, we would like to keep track of our ramification information all at once.

Lemma 1.34. Let $f\colon X \to Y$ be a finite separable morphism of k-curves. Then

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

is an exact sequence of line bundles on X.

Proof. We know from last semester that this map is exact on the right, so we need the map $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. Well, we may check exactness of quasicoherent sheaves on affine open subschemes, so we may assume that $X = \operatorname{Spec} A$, where the map looks like $A \to A$. Letting I denote the kernel of this map, we get an embedding $A/I \subseteq A$, but if I is nontrivial, then this means that the map $A \to A$ is zero at the generic point.

Thus, we want to check that the map $A \to A$ is nonzero at the generic point. Everything is compatible with localization, so we are now looking at

$$f^*\Omega_{K(Y)/k} \to \Omega_{K(X)/k} \to \Omega_{K(X)/K(Y)} \to 0.$$

Thus, to show that the map on the left is nonzero, it suffices to show that $\Omega_{K(X)/K(Y)} = 0$. We now use the fact that $K(X)/K(Y)$ is separable, for which the statement is true. ∎

The point here is that $\Omega_{X/Y}$ precisely measures the "difference" between $\Omega_{Y/k}$ and $\Omega_{X/k}$.

In what follows, we fix the following notation. Let $f\colon X \to Y$ be a finite morphism of k-curves. Given $x \in X$ and $y := f(x)$, let ϖ_x be a uniformizer for $\mathcal{O}_{X,x}$ and ϖ_y be a uniformizer for $\Omega_{Y,y}$. Then we note $d\varpi_x$ generates $(\Omega_{X/k})_x$, and $d\varpi_y$ generates $(\Omega_{Y/k})_y$, and we have a map

$$(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x \simeq \mathcal{O}_{X,x}$$

by sending $f^*\colon d\varpi_y \mapsto d\varpi_y/d\varpi_x \cdot d\varpi_x$, where $d\varpi_y/d\varpi_x$ is an element of $\mathcal{O}_{X,x}$. (This is the definition of $d\varpi_y/d\varpi_x$.)

Proposition 1.35. Let $f\colon X \to Y$ be a finite separable morphism of k-curves.

(a) $\Omega_{X/Y}$ is supported on exactly the set of ramification points of f, so the set of ramified points is finite.

(b) For each $x \in X$, we have $(\Omega_{Y/X})_x$ is a principal $\mathcal{O}_{X,x}$-module of length $v_x(d\varpi_y/d\varpi_x)$.

(c) If f is tamely ramified at x, then the length of $(\Omega_{Y/X})_x$ is $e_x - 1$; if it's wildly ramified, then the length is larger.

Proof. We show these one at a time. As a warning, all uniformizers might be swapped here.

(a) Recall that $\Omega_{Y/X}$ is generically zero. Now, $(\Omega_{Y/X})_p = 0$ if and only if the map

$$(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x$$

is an isomorphism, which means that a uniformizer for $\mathcal{O}_{Y,f(x)}$ is going to a uniformizer of $\mathcal{O}_{X,x}$, which is equivalent to f being unramified at x. The point here is that the set of ramified points correspond to some dimension-zero subset and is therefore finite.

(b) The length of $(\Omega_{Y/X})_p$ is its k-dimension, which we can compute as

$$\operatorname{length}(\Omega_{X/k})_x - \operatorname{length}(f^*\Omega_{Y/K})_x,$$

which we can compute is e_x by hand. Notably, this has to do with how we identify $(\Omega_{X/k})_x$ with $\mathcal{O}_{X,x}$.

(c) Letting e denote our ramification index, we may set $\varpi_y = a\varpi_x^e$ where $a \in \mathcal{O}_{X,x}^\times$, which upon taking differentials reveals

$$d\varpi_y = ea\varpi_x^{e-1}d\varpi_x + \varpi_x^e da.$$

Now, if f is tamely ramified at x, we see $\operatorname{char} k \nmid e$, so we see that the valuation here is in fact $e_x - 1$. The statement for wild ramification follows similarly. ∎

> **Remark 1.36.** The length of the modules here coincides with the dimension as a k-vector space. This is because $\mathcal{O}_{X,x}$ is a discrete valuation ring with residue field k.

1.5 January 27

There is homework due tonight. Today we keep talking towards the Riemann–Hurwitz formula.

1.5.1 The Riemann–Hurwitz Formula

Throughout, $f\colon X \to Y$ is a finite separable morphism of k-curves, where k is algebraically closed.

> **Definition 1.37** (ramification divisor)**.** Let $f\colon X \to Y$ be a finite separable morphism of k-curves, where k is algebraically closed. Then the *ramification divisor* is the divisor
> $$R(f) := \sum_{x \in X} \operatorname{length}(\Omega_{Y/X})_x x.$$
> Note that there are only finitely many ramified points, so this is indeed a divisor.

In particular, in tame ramification, this length is in fact exactly our ramification.

> **Lemma 1.38.** Let $f\colon X \to Y$ be a finite separable morphism of k-curves, where k is algebraically closed. Further, let K_X and K_Y denote the canonical divisors. Then K_X is linearly equivalent to $f^*K_Y + R$.

Proof. We can see by hand that the structure sheaf $\mathcal{O}_R$ of R as a closed subscheme is exactly $\Omega_{Y/X}$, so the exact sequence
$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{Y/X} \to 0$$
can be tensored with $\Omega_{X/k}^{-1}$ to give

What?

$$0 \to f^*\Omega_{Y/k} \otimes \Omega_{X/k}^{-1} \to \mathcal{O}_X \to \mathcal{O}_R \to 0.$$

Now, we see $\mathcal{O}_R = \mathcal{O}_X(-R)$ by unwinding definitions, so $f^*\Omega_{Y/k} \otimes \Omega_{X/k}^{-1} = \mathcal{O}_X(-R)$, which is what we wanted. ∎

> **Theorem 1.39** (Hurwitz)**.** Let $f\colon X \to Y$ be a finite separable map of k-curves, where k is algebraically closed. Letting $n := \deg f$, we have
> $$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R(f).$$

Proof. Take degrees of Lemma 1.38. ∎

Let's derive some corollaries.

Definition 1.40 (étale). A morphism locally of finite presentation $f\colon X \to Y$ of schemes is *étale* at $x \in X$ if and only if the following conditions hold.

- f is flat at x. In other words, $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module.

- $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a field and separable over $k(f(x))$. Equivalently, we are requiring $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and for the residue field extension to be separable.

Then f is étale if and only if f is étale at all points.

Remark 1.41. The locus of points for which a morphism is étale is open essentially because this is true for both conditions individually.

Remark 1.42. If the morphism of k-curves $f\colon X \to Y$ is étale, then it is unramified because we are essentially saying $\operatorname{length} \Omega_{X/Y} = 0$. In fact, unramified morphisms are étale, which should roughly be our intuition. Flatness is a bit mysterious, but such is life.

Remark 1.43. Equivalently, a morphism $f\colon X \to Y$ is étale if and only if f is a smooth morphism of relative dimension 0. For example, if both X and Y are varieties, then we are asking for the varieties to have the same sheaf of differentials.

Geometrically, we imagine finite étale morphisms as covering space maps. This motivates the following definition.

Definition 1.44 (simply connected). A finite étale morphism $f\colon X \to Y$ is *trivial* if and only if X is a disjoint union of copies of Y, and f is a disjoint union of automorphisms. Then a curve Y is *simply connected* if and only if all finite étale morphisms $X \to Y$ are trivial.

Roughly speaking, we are trying to say that the fundamental group is trivial.

Proposition 1.45. Let k be an algebraically closed field. Then $\mathbb{P}_k^1$ is simply connected.

Example 1.46. We can see that the sphere $\mathbb{P}_{\mathbb{C}}^1$ is simply connected.

Proof. Fix some finite étale morphism $f\colon X \to Y$. By breaking this morphism into connected components, we may assume that X is connected. We show that $X = \mathbb{P}_k^1$. Because f is smooth, we see that the structure map $X \to Y \to k$ is smooth, so X is a (smooth) curve. (In particular, we can also see that X is irreducible.) Now, because f is étale, it is étale at the generic point, so f is also separable.

We are now ready to apply Theorem 1.39. Here, $R(f) = 0$ and $g(Y) = 0$, so we are left with

$$2g(X) - 2 = (\deg f)(0 - 2) = -2\deg f.$$

However, $g(X) \geq 0$ and $\deg f \geq 1$, so we must have $g(X) = 0$ and $\deg f = 1$, so f is in fact an isomorphism $X \cong \mathbb{P}_k^1$. $\blacksquare$

Remark 1.47. In characteristic 0, we will have $\mathbb{A}_k^1$ is simply connected. However, in positive characteristic, this is no longer true; indeed, $\pi_1^{\text{ét}}(\mathbb{A}_{\mathbb{F}_p}^1)$ is infinite.

1.6 January 30

Homework was assigned and still due on Friday, sadly.

Remark 1.48. Let k be an algebraically closed field of positive characteristic p. It turns out $G := \pi_1^{\text{ét}}(\mathbb{A}_k^1)$ is profinite but not topologically finitely generated—it's very large. In fact, one can show that any finite p-group arises as a quotient of $\pi_1^{\text{ét}}(\mathbb{A}_k^1)$. More generally, any finite quasi-p-group is a quotient, where a quasi-p-group is a finite group generated by its Sylow p-subgroups cover G.

1.6.1 Everything Is Frobenius

Thus far we roughly understand finite separable morphisms of curves. We now investigate the purely inseparable case. In particular, today k will be an algebraically closed field of positive characteristic p. Note there is a canonical embedding $\mathbb{F}_p \hookrightarrow k$, which gives rise to the Frobenius automorphism as follows.

Definition 1.49 (Frobenius). Let k be a field of characteristic $p > 0$. Given a k-scheme X, we define the *Frobenius automorphism* $F\colon X \to X$ as being the identity on topological spaces and being the pth-power map $F_U^\sharp\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ for all open $U \subseteq X$.

We can see that this map takes units of $\mathcal{O}_{X,x}$ to units of $\mathcal{O}_{X,x}$ for any $x \in X$, so we have defined a morphism of locally ringed spaces.

Example 1.50. Let $X = \operatorname{Spec} A$ be an $\mathbb{F}_p$-scheme. Then the ring homomorphism $F\colon A \to A$ given by the pth power map is the Frobenius $F\colon X \to X$. To show the map is the identity on the topological space, we note $a^p \in \mathfrak{p}$ is equivalent to $a \in \mathfrak{p}$ for $a \in A$, where we are using the primality of $\mathfrak{p}$. Thus, $F^{-1}(\mathfrak{p}) = \mathfrak{p}$ for any prime $\mathfrak{p} \in \operatorname{Spec} A$.

The Frobenius map defined above is not k-linear because it is the pth power map on k too. To make this k-linear, we essentially cheat.

Definition 1.51. Fix a scheme X over a field k of characteristic p. Then we define the k-scheme X_p which is equal to X as a scheme but whose structure morphism to $\operatorname{Spec} k$ is given by

$$X_p \to \operatorname{Spec} k \xrightarrow{F} \operatorname{Spec} k.$$

The point is that the diagram

$$\begin{array}{ccc} X_p & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{F} & \operatorname{Spec} k \end{array} \qquad (1.1)$$

commutes, so we do genuinely have a Frobenius morphism $F\colon X_p \to X$.

Remark 1.52. Explicitly, for each affine open $\operatorname{Spec} A \subseteq X$, we get an identical affine open $\operatorname{Spec} A_p \subseteq X_p$, but if the k-algebra structure on A is given by $g\colon k \to A$, then the k-algebra structure on A is given by $g_p(x) \cdot \alpha := g\left(x^p\right) \cdot \alpha$.

Namely, with everything being contravariant on functions, we see that

$$\begin{array}{ccc} A & \xleftarrow{F} & A \\ \uparrow & & \uparrow \\ k & \xleftarrow{F} & k \end{array}$$

commutes.

Remark 1.53. Certainly $X_p \cong X$ as schemes because they are literally the same data. If k is perfect, then $X_p \cong X$ is an isomorphism of k-schemes because the pth-power map on k is an isomorphism.

Why?

Example 1.54. Let k be a perfect field of characteristic $p > 0$. Set $X = \mathbb{A}_k^1 = \operatorname{Spec} k[t]$. Then $F \colon X_p \to X$ is given by the morphism $k[t_p] \to k[t]$ by $f \mapsto f^p$. Thus, to witness our isomorphism of k-schemes, we note that we can post-compose $k[t_p] \to k[t]$ with the morphism which extends $k \cong k$ by $a^p \mapsto a$ by $t_p \mapsto t_p$, so we have made a k-linear isomorphism. One can essentially extend this construction to work in general when k is perfect.

Example 1.55. Let k be a field of characteristic $p > 0$, and let X be an integral k-scheme. Then $F \colon X_p \to X$ is given by a morphism $K(X) \to K(X_p)$ by $\alpha \mapsto \alpha^p$. (Yes, this is k-linear because the k-action on $K(X_p)$ is by pth powers.) As such, we have defined an embedding of $K(X)$ into an algebraic extension $K(X_p)$: namely, every $\alpha \in K(X_p)$ is a root of the polynomial $t^p - \alpha^p = 0$ in $K(X)[t]$.

Conversely, $t^p - \beta \in K(X)[t]$ always has a root in $K(X_p)$ because $\beta \in K(X)$ embeds into $K(X_p)$ as β^p, so this polynomial "looks like" $t^p - \beta^p$ in $K(X_p)$, where β is our root. Thus,

$$K(X_p) = K(X)^{1/p}.$$

The point is that a k-curve X will have the Frobenius morphism $X_p \to X$ induced by the embedding $K(X) \to K(X)^{1/p}$.

Now, here is our main result.

Theorem 1.56. Let k be an algebraically closed field of characteristic $p > 0$. Further, let $f \colon X \to Y$ be a finite map of k-curves which induces a purely inseparable extension $f^\sharp \colon K(Y) \to K(X)$. Then f is some iterate of the Frobenius morphism. In particular, X and Y are isomorphic as schemes and thus have the same genus.

Proof. Note $\deg f = [K(X) : K(Y)]$ is a power of p, which we call p^ν. Being purely inseparable then enforces $K(X) \subseteq K(Y)^{1/p^\nu}$; namely, the minimal polynomial if all elements $\alpha \in K(X)$ must be the minimal polynomial of the form $x^{p^r} - \beta = 0$, for otherwise there is a separable subextension, violating our pure inseparability.

Now, consider iterated Frobenius morphisms

$$Y_{p^\nu} \to Y_{p^{\nu-1}} \to \cdots \to Y_p \to Y,$$

which corresponds to the inclusion of fields

$$K(Y) \subseteq K(Y)^{1/p} \subseteq \cdots \subseteq K(Y)^{1/p^{\nu-1}} \subseteq K(Y)^{1/p^\nu},$$

where the inclusions have reversed.

Thus, to conclude, we would like to show $K(X) = K(Y)^{1/p^\nu}$. By degree arguments, it's enough to conclude $\left[K(Y)^{1/p^\nu} : K(Y)\right] = p^\nu$. By induction, it's enough to show $\left[K(Y)^{1/p} : K(Y)\right] = p$. We now must use the fact that Y is a smooth k-curve, so we will push its proof into the following lemma.

Lemma 1.57. Let k be an algebraically closed field of characteristic $p > 0$. If Y is a smooth k-curve, then $\left[K(Y)^{1/p} : K(Y)\right] = p$.

Proof. Equivalently, we would like to show that $[K(Y) : K(Y)^p] = p$. Note that $\Omega_{K(Y)/k}$ is a $K(Y)$-vector space of dimension 1; we refer to Theorem II.8.6.A, where the point is that k being perfect tells us $K(Y)/k$ is separably generated, so $\dim_k \Omega_{K(Y)/k}$ is the transcendence degree of $K(Y)$ over k, which is 1.

We now note that dx generates $\Omega_{K(Y)/k}$ if and only if $x \in K(Y)$ yields a power basis $\{1, x, \ldots, x^{p-1}\}$ of $K(Y)$ over $K(Y)^p$, which completes the proof. ∎

Okay.

It remains to show that the genus does not change. Well, $g = \dim_k H^1(X, \mathcal{O}_X)$, and we see that the only difference between X and Y is the structure morphism, and this dimension does not change if we change the structure morphism. $\blacksquare$

> **Remark 1.58.** The above proof basically shows that the Frobenius morphism $F \colon X_p \to X$ in our setting is a finite morphism of degree p. In particular, it cannot be an isomorphism.

1.7 February 1

We continue discussing the Frobenius morphism.

1.7.1 Relative Frobenius

We begin class with a few remarks on the Frobenius automorphism when k is not perfect. Roughly speaking, the issue is that a Frobenius morphism $F \colon X_p \to X$ is not p. In general, one sees $[K(X) : kK(X)^p] = p$ (note we have taken the composite with p!), so the extension $[K(X) : K(X)^p] > p$.

The idea is to generalize (1.1). Namely, we construct our Frobenius $X^{(p)} \colon X$ by pullback in the following square.

Here $F_{X/k}$ is the relative Frobenius. Notably, when k is perfect, we see that the canonical projection $X^{(p)} \to X$ is an isomorphism of k-schemes because the Frobenius on the bottom is an isomorphism. One can even make this isomorphism explicit by arguing as in Example 1.54.

We quickly check that this roughly generalizes our earlier construction.

> **Example 1.59.** Set $X = \operatorname{Spec} A$ for a k-algebra A. We track the diagram in this case.

Proof. Then $X^{(p)} = \operatorname{Spec}(A \otimes_k k)$, where the k-action on k is given by the Frobenius $F \colon k \to k$. Thus, when k is perfect, we are indeed looking k acting on A by $f \otimes \alpha^p = \alpha f \otimes 1$, so we have an isomorphism $A \otimes_k k \cong A$, where the k-action is given by

$$g^{(p)}(\alpha)a = g(\alpha^{1/p})a,$$

where $g \colon k \to A$ is the structure morphism.

To understand the relative Frobenius $X^{(p)} \to X$, we note that it sends $f \otimes \alpha \mapsto \alpha f^p$ by tracking the diagram. As such, when k is perfect, we may think of our morphism as

$$\alpha^{1/p} f \mapsto \alpha f^p,$$

so the map is in fact k-linear. $\blacksquare$

> **Remark 1.60.** Thus, when k is not perfect, we see that $X^{(p)}$ need not be isomorphic to X, even absolutely. For example, take $A = k[x]/\left(x^2 - \alpha\right)$ for some $\alpha \in k$. In $A \otimes_k k$, we see that
>
> $$(x \otimes 1)^2 = \alpha \otimes 1 = 1 \otimes \alpha^p.$$
>
> Thus, $A \otimes_k k$, even though it is a two-dimensional k-algebra, is isomorphic to $k[x_p]/\left(x_p^2 - \alpha^p\right)$, which is not the same as A when k is not perfect!

Anyway, let's see the analogue of Theorem 1.56 in our setting.

> **Theorem 1.61.** Fix a finite morphism $f\colon X \to Y$ of (smooth, proper, integral) k-curves, where k is a field of characteristic $p > 0$. Further, suppose that $f^\sharp\colon K(Y) \to K(X)$ is purely inseparable of degree p. Then $Y \cong X^{(p)}$ for some r, and f is the relative Frobenius under this isomorphism.

1.7.2 Inseparability for Fun and Profit

We will not prove this, but it's fun to know.

> **Corollary 1.62.** Fix a finite morphism $f\colon X \to Y$ of k-curves, where k is algebraically closed. Then $g(X) \geq g(Y)$.

Proof. Factor the field extension $K(Y) \subseteq K(X)$ into a purely inseparable extension followed by a separable extension. Genus does not change when we are dealing with an inseparable extension by Theorem 1.56, so it suffices to show that the genus does not fall with separated morphisms. Well, by Theorem 1.39, we see

$$2g(X) - 2 = (\deg f)(2g(Y) - 2) + \deg R \geq 2g(Y) - 2,$$

so $g(X) \geq g(Y)$ follows. ∎

> **Remark 1.63.** For our equality cases, we see that one either has an isomorphism or an unramified finite morphism of elliptic curves. Such unramified maps of elliptic curves (which are not isomorphisms) do exist.

Classify them?

1.7.3 Embeddings of Curves

We now return to the case where k is algebraically closed. Our next goal is to show that every k-curve can be embedded into $\mathbb{P}^3_k$. As such, we are roughly speaking interested in showing that certain linear systems separate points (which means that we are base-point-free) and tangent vectors.

Let's begin by discussing being base-point-free.

> **Lemma 1.64.** Fix a divisor D on a k-curve X. Then $\mathcal{O}_X(D)$ is base-point-free if and only if
>
> $$\dim |D| = \dim |D - P| + 1.$$

Proof. We are essentially showing $\ell(D) = \ell(D - P) + 1$. Notably, we do always have $\ell(D) \leq \ell(D - P) + 1$ by staring at the short exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0$$

and then tensoring by $\mathcal{O}_X(D)$ to give

$$0 \to \mathcal{L}(D - P) \to \mathcal{L}(D) \to k(P) \to 0.$$

We now discuss the equality. Note $\ell(D) = \ell(D - P) + 1$ is just telling us that there is a global section in $\Gamma(X, \mathcal{O}_X(D)) \setminus \Gamma(X, \mathcal{O}_X(D - P))$. In other words, we have some $f \in K(X)$ such that $D + \operatorname{div}(f)$ is effective, but $D - P + \operatorname{div}(f)$ is not. Thus, we see that we are saying P is not in the support of f, so P is not a base-point for D. ∎

Next time we will make a similar dimension condition to be ample and very ample.

1.8 February 3

We continue discussing the theory of things which go bump in the night.

1.8.1 Line Bundle Review

We begin by recalling a couple of facts.

> **Proposition 1.65.** Fix an A-scheme X, where A is an affine scheme. We recall that a morphism $X \to \mathbb{P}_A^n$ has equivalent data to giving a line bundle $\mathcal{L}$ on X together with global sections $(s_0, \ldots, s_n)$ which generate $\mathcal{L}$.

Recall that the global sections generate $\mathcal{L}$ if and only if the induced map $\mathcal{O}_X^{n+1} \to \mathcal{L}$ is surjective, which means that the global sections generate all the talks of $\mathcal{L}$.

> **Remark 1.66.** If $A = k$ is a field, then defining a morphism $X \to \mathbb{P}_k^n$ can be defined (up to automorphism on $\mathbb{P}_k^n$), then it suffices to just provide a globally generated line bundle $\mathcal{L}$, and the precise choice of spanning set $(s_0, \ldots, s_n)$ merely adds an automorphism.

> **Remark 1.67.** The chosen global sections technically need not fully span $\Gamma(X, \mathcal{L})$.

> **Remark 1.68.** We remark that the pullback of the line bundle $\mathcal{O}_{\mathbb{P}_A^n}(1)$ under $X \to \mathbb{P}_A^1$ is (canonically) $\mathcal{L}$, and the pullback of the global section x_i is s_i. Explicitly, there is morphism $x_i \colon \mathcal{O}_{\mathbb{P}_A^n} \to \mathcal{O}_{\mathbb{P}_A^n}(1)$ which will pull back to $s_i \colon \mathcal{O}_X \to \mathcal{L}$ upon applying φ^*.

We might want to upgrade Proposition 1.65 to give a closed embedding into projective space. Here are the corresponding conditions.

> **Proposition 1.69.** Fix an algebraically closed field k and a k-variety X. Fix a morphism $\varphi \colon X \to \mathbb{P}_k^n$ corresponding to the line bundle $\mathcal{L}$ equipped with global sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Further, set $V = \operatorname{span}(s_0, \ldots, s_n)$. Then φ is a closed immersion if and only if V satisfies the following.
>
> - Separates points: for any distinct $x, x' \in X$, there is a section $s \in V$ such that $x \in \operatorname{supp} \operatorname{div}(\mathcal{L}, s)$ but $x' \notin \operatorname{supp} \operatorname{div}(\mathcal{L}, s)$ (i.e., $s \in \mathfrak{m}_x \mathcal{L}_x \setminus \mathfrak{m}_{x'} \mathcal{L}_{x'}$).
>
> - Separates tangent vectors: for every $x \in X$, the set
>
> $$\{s \in V : s \in \mathfrak{m}_x \mathcal{L}_x\}$$
>
> spans the Zariski tangent place $\mathfrak{m}_x \mathcal{L}_x / (\mathfrak{m}_x \mathcal{L}_x)^2$.

Notably, when X is a curve, our Zariski tangent space has dimension 1, so we just want some section to show up in there.

We will also want the following definitions.

> **Definition 1.70** (ver ample)**.** A line bundle $\mathcal{L}$ on a scheme X is *very ample* relative to a scheme Y if and only if there is a locally closed embedding $\iota \colon X \to \mathbb{P}_Y^n$ for some $n > 0$ such that $\mathcal{L} = \iota^* \mathcal{O}_{\mathbb{P}_Y^n}(1)$.

> **Remark 1.71.** In the case where X is a k-curve, because X is proper, any locally closed embedding $\iota \colon X \to \mathbb{P}_k^n$ is automatically closed. As such, in this situation we may as well talk about closed embeddings.

Definition 1.72 (ample)**.** A line bundle $\mathcal{L}$ on a scheme X is *ample* if each coherent sheaf $\mathcal{F}$ on X makes $\mathcal{F} \otimes \mathcal{L}^n$ globally generated for n sufficiently large.

Here is how these notions relate.

Proposition 1.73. Fix a scheme X of finite type over a Noetherian ring A. Then a line bundle $\mathcal{L}$ on X is ample if and only if $\mathcal{L}^n$ is very ample (relative to A) for some $n > 0$.

1.8.2 Projective Embeddings for Curves

We now return to talk about curves. We quickly extend our definition of ample.

Definition 1.74 (ample, very ample)**.** Fix a k-curve X. Then a divisor D on X is *ample* or *very ample* if and only if $\mathcal{O}_X(D)$ is as well.

So let's translate what we know about projective embeddings into our language of divisors.

Proposition 1.75. Fix a divisor D on a k-curve X.

(a) The complete linear system $|D|$ is base-point-free if and only if

$$\dim |D| = \dim |D - P| + 1$$

for all closed points $P \in X$.

(b) The divisor D is very ample if and only if

$$\dim |D| = \dim |D - P - Q| + 2$$

for all $P, Q \in X$.

Roughly speaking, (a) asks for us to separate points somehow, and (b) asks if we can separate points with the right multiplicity. Namely, we're asking for something to be in $\mathfrak{m}_P$ but not $\mathfrak{m}_P^2$.

Proof. We have two parts to show.

(a) Note $\dim |D| = \dim |D - P| + 1$ is equivalent to

$$\ell(D) = \ell(D - P) + 1,$$

which is equivalent to $\mathcal{O}_X(D)(X) \setminus \mathcal{O}_X(D - P)(X)$ being nonempty. (Recall that $\dim \mathcal{O}_X(D) \le \mathcal{O}_X(D - P) + 1$ at the very least.) Thus, we are saying there is $f \in K(X)$ such that $P \in \operatorname{supp} D$ but $P \in \operatorname{supp}(D + (f))$, so we have a global section f which does not vanish at P. Repeating this for all points P shows that $|D|$ is base-point-free, and running this argument in reverse gives us the converse implication.

(b) Certainly if D is very ample, then $\mathcal{L}(D)$ is base-point-free. Additionally, note that the conclusion of (b) implies the conclusion of (a) because

$$\dim |D - P - Q| + 2 \le \dim |D - P| + 1 \le \dim |D|,$$

so equalities must hold everywhere and in particular on the right inequality. In particular, we can assume that $|D|$ is base-point-free in either direction.

As such, in either direction, we already know that D determines a morphism to projective space, so we need to check that we have defined a closed embedding.

- Separate points: for every distinct $P, Q \in X$, some $s \in \Gamma(X, \mathcal{O}_X(D))$ has $P \in \operatorname{supp} \operatorname{div}(\mathcal{L}, s)$ but $Q \in \operatorname{supp} \operatorname{div}(\mathcal{L}, s)$ (namely, we vanish at P) is equivalent to $\operatorname{div}(\mathcal{L}, s) \in \Gamma(X, \mathcal{O}_X(D - P))$ but Q is not a base-point of $\mathcal{O}_X(D - P)$ at s (namely, we do not vanish at Q). Hitting this with (a) again, we are asking for
$$\dim |D - P| = \dim |D - P - Q| + 1.$$
  We already have (a), so we conclude $\dim |D| = \dim |D - P - Q| + 2$. Running this argument in reverse gets the other implication.

- Separates tangent vectors: for every $P \in X$, we are asking for $s \in \Gamma(X, \mathcal{O}_X(D))$ which vanishes at order 1 at P. (Indeed, this is saying $s \in \mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P$ is nonzero.) Arguing as above, we are asking for P to not be a base-point for $D - P$. So we can again hit this condition with (a) to say that we are asking for
$$\dim |D - P| = \dim |D - P - P| + 1$$
  and use the fact that D is base-point-free already to finish.

The above discussion completes the proof. ∎

> **Corollary 1.76.** Fix a divisor D on a k-curve X of genus g.
>
> (a) If $\deg D \geq 2g$, then D is base-point-free.
>
> (b) If $\deg D \geq 2g + 1$, then D is very ample.

Proof. We use Theorem 1.19. Recall $\deg K = 2g - 2$, where K is the canonical divisor for X.

(a) Note $\deg(K - D) \leq 0$ and $\deg(K - (D - P)) \leq 0$, so $\ell(K - D) = \ell(K - D - P) = 0$, so we conclude that
$$\ell(D) = \deg D + 1 - G = 1 + \deg(D - P) + 1 - g = 1 + \ell(D - P),$$
so we are done by Proposition 1.75.

(b) Here, we also get $\deg(K - (D - P - Q)) \leq 0$, so $\ell(K - (D - P - Q)) = 0$ again, so arguing as above completes the proof by Proposition 1.75. ∎

In particular, we see that every k-curve X has a closed embedding into projective space with a morphism of degree at most $2g + 1$.

1.9 February 6

As is to be expected, we sleep with one eye open.

> **Remark 1.77.** The next homework will be due Sunday night. The hope is that it is fun.

1.9.1 Small Projective Embeddings

We are still trying to embed curves into $\mathbb{P}_k^3$. Our main tool continues to be Proposition 1.75.

> **Corollary 1.78.** Fix a divisor D on a k-curve X is ample if and only if $\deg D > 0$.

Proof. Certainly if $\deg D \leq 0$, then $\deg(nD) = 0$ and thus $\ell(nD) = 0$ for all positive integers n, so nD is never very ample, so D is not ample. (Alternatively, if nD is very ample, then nD is the pull-back of $\mathcal{O}_{\mathbb{P}_k^1}(1)$ some closed embedding $f : X \to \mathbb{P}_k^1$ and hence has positive degree.) Conversely, if $\deg D > 0$, then $\deg(2g+1)D \geq 2g + 1$, so $(2g + 1)D$ is very ample by Corollary 1.76, so D is ample. ∎

Example 1.79. With $X = \mathbb{P}_k^1$, we note that $\mathcal{O}_{\mathbb{P}_k^1}(1)$ is very ample by the (identity) embedding $X \cong \mathbb{P}_k^1$. It follows that $\mathcal{O}_{\mathbb{P}_k^1}(n)$ are all very ample (and hence ample) for all $n > 0$. By Corollary 1.78, these are all the ample divisors of X.

Remark 1.80. It turns out that a very ample divisor D on X yielding a closed embedding $f\colon X \to \mathbb{P}_k^n$ has $\deg D = \deg f(X)$. Here, $\deg f(X)$ is defined using some intersection theory; for example, if $n = 2$, then this degree is the degree of the polynomial cutting out X. For example, if $g(X) = 1$ and D is a divisor of degree 3 (and hence very ample by Corollary 1.78). Further,

$$\ell(D) = \ell(K - D) + \deg D + (1 - g) = 0 + 3 + 0 = 3,$$

so we define a closed embedding to $\mathbb{P}_k^2$. We conclude that X has an embedding as a cubic curve in $\mathbb{P}_k^2$. Conversely, Exercise I.7.2(b) in [Har77] tells us that any cubic plane curve has genus $\frac{1}{2}(3-1)(3-2) = 1$. Note that adjusting the divisor's linear equivalence class can give us different embeddings to $\mathbb{P}_k^2$ (which are not the same up to an automorphism of $\mathbb{P}_k^2$).

Anyway, we are now almost ready to prove our main result.

Theorem 1.81. Fix a k-curve X. Then X has an embedding to $\mathbb{P}_k^3$.

> K alge-
> braically
> closed?

The outline here is as follows. To begin, fix some closed embedding $X \to \mathbb{P}_k^n$ for some $n > 0$. Then if $n > 3$, we will show that we can project down from $\mathbb{P}_k^n$ to $\mathbb{P}_k^{n-1}$ in a way which preserves us having a closed embedding. Inducting n downwards like this will complete the proof.

We know how to do this first step by Corollary 1.76, so we focus on the second step.

Proposition 1.82. Fix a k-curve X embedded into $\mathbb{P}_k^n$ for some positive integer n. Given $O \in \mathbb{P}_k^n \setminus X$, then the projection $\varphi\colon X \to \mathbb{P}_k^{n-1}$ from O is a closed embedding if and only if the following both hold.

- O does not belong on any secant line.

- O does not belong on any tangent line.

Note that the line we are projecting from O onto does not matter so much because it merely adjusts the map by an automorphism of the ambient space $\mathbb{P}_k^n$, which turns into an automorphism of $\mathbb{P}_k^{n-1}$ on the embedding.

Example 1.83. One can compute that the projection from $O = [0 : \cdots : 0 : 1]$ in $\mathbb{P}_k^n$ to $V(x_n)$ is given by the projection

$$[a_0 : \cdots : a_n] \mapsto [a_0 : \cdots : a_{n-1}].$$

Namely, the line connecting O and $[a_0 : \cdots : a_{n-1}]$ is parameterized by $[t_0 : t_1] \mapsto [t_0 a_0 : \cdots : t_0 a_{n-1} : t_1 a_n]$, which tells us what the intersection with $V(x_n)$ should be.

In our theory of linear systems, we note that the global sections $x_0, \ldots, x_{n-1}$ span some subspace $V \subseteq \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1))$, and we can see that the only base-point here is the point O because this is the only point of simultaneous vanishing. Thus, our theory grants us a morphism $(\mathbb{P}_k^n \setminus \{0\}) \to \mathbb{P}_k^{n-1}$ given by the above formula! Explicitly, on the affine chart $U_i \subseteq \mathbb{P}_k^n$ where $x_i \neq x_0$ doesn't vanish, we are looking at the ring map

$$k\left[\frac{y_0}{y_i}, \ldots, \frac{y_{n-1}}{y_i}\right] \to k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$$

given by $y_j/y_i \mapsto x_j/x_i$.

Proof. Roughly speaking, we don't want O to be on any secant line so that the projection doesn't send two

points to the same point. Additionally, we don't want O to be on any tangent line so that the closed embedding separates tangent vectors. We omit the remainder of the proof aside from this general intuition. ∎

We now turn to the proof of Theorem 1.81.

Proof of Theorem 1.81. By Corollary 1.76, we have some closed embedding $X \to \mathbb{P}_k^n$ for some n large enough. Now, if $n > 3$, we use Proposition 1.82 to project down to $\mathbb{P}_k^{n-1}$. Thus, it suffices to find a point O not on any secant line or tangent line. Well, the "secant variety" $\operatorname{Sec} X$ defined by the image of the obvious map

$$(X \times X \setminus \Delta_X) \times \mathbb{P}_k^1 \to \mathbb{P}_k^n$$

we can see has dimension of the image at most 3. Explicitly, on points, this map is given by

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n], [t_0 : t_1]) \mapsto [a_0 t_0 + b_0 t_1 : \cdots : a_n t_0 + b_n t_1].$$

Similarly, the "tangent variety" $\operatorname{Tan} X$ defined by the image of the obvious map

$$X \times \mathbb{P}_k^1 \to \mathbb{P}_k^n$$

we can see has dimension of the image at most 2. Thus, with $n > 3$, these closed subschemes are proper and so have dense complement, meaning that we can find can point O in the complement of these varieties. This completes the proof. ∎

1.10   February 8

Last class we showed that every curve can be embedded into $\mathbb{P}_k^3$.

> **Remark 1.84.** We will not show this, but one can show with a little more work that any curve is birational to a curve in the plane with at worst nodes as singularities.

1.10.1   The Canonical Embedding

Throughout, X is a k-curve, where k is algebraically closed. We will set $g := g(X)$ and let K denote the canonical divisor.

> **Example 1.85.** Notably, $g(X) = 0$ forces $\dim |K| = -1$, so $|K|$ is empty. (After all, $X \cong \mathbb{P}_k^1$.)

> **Example 1.86.** If $g(X) = 1$, then we are looking at $\dim |K| = 0$, and here $\deg K = 0$, so we are just looking at the mapping from X to a point.

In higher genus, things get more interesting.

> **Lemma 1.87.** Fix a k-curve X with $g(X) \geq 2$. Then the canonical divisor K is base-point-free.

Proof. By Proposition 1.75, it suffices to show that $\ell(K - P) = \ell(K) - 1 = g - 1$ for any $P \in X$. Now, because $g(X) \geq 2 > 0$, we know that $X \not\cong \mathbb{P}_k^1$, so $\dim |P| = 0$ is forced. Now, using Theorem 1.39, we solve

$$\ell(P) = \ell(K - P) + \deg(P) + (1 - g),$$

so $\ell(K - P) = g - 1$, which is what we wanted. ∎

What?

In our study of curves, the following curves will make a somewhat large class.

Definition 1.88 (hyperelliptic). A k-curve X is *hyperelliptic* if and only if it admits a degree-2 map to $\mathbb{P}^1_k$.

Remark 1.89. Given any divisor D on X with $\ell(D) = 2$ and $\deg D = 2$, one gets a rational map $X \dashrightarrow \mathbb{P}^1_k$ determined by D. (One can show that all such divisors are base-point-free using Proposition 1.75.) Notably, D is linearly equivalent to an effective divisor.

Remark 1.90. If $g(X) = 2$, then X is hyperelliptic. This was shown on the homework. In brief, K is base-point-free and has degree 2, so it determines a degree-2 map $X \to \mathbb{P}^1_k$.

One can improve Lemma 1.87 as follows.

Proposition 1.91. Fix a k-curve X with $g(X) \geq 2$. Then K is very ample if and only if X is not hyperelliptic.

Proof. By Proposition 1.75, we are interested in the condition

$$\ell(K - P - Q) = \ell(K) - 2 \overset{?}{=} g - 2$$

for any $P, Q \in X$. As such, by Theorem 1.39, we compute

$$\ell(P + Q) = \ell(K - P - Q) + \deg(P + Q) + (1 - g) = \ell(K - P - Q) + 3 - g,$$

so $\ell(K - P - Q) = g - 2$ is equivalent to

$$\ell(P + Q) = 1.$$

Now, if X is hyperelliptic, then one can find a divisor D with $\dim |D| = 1$ and $\deg D = 2$, so D linearly equivalent to an effective divisor which looks like $P + Q$. But then $\ell(P + Q) = 2 > 1$, which is a problem. In the other direction, if X is not hyperelliptic, then each effective divisor D with degree 2 must have $D = P + Q$, which forces $\ell(P + Q) < 2$ to retain being not hyperelliptic. ∎

Thus, we are interested in the embedding induced by this canonical divisor.

Definition 1.92 (canonical morphism). Fix a k-curve X of genus $g(X) \geq 2$. Then the canonical divisor K determines the *canonical morphism* $X \to \mathbb{P}^{g-1}_k$ by Lemma 1.87. If X is not hyperelliptic curves, then we in fact get a *canonical embedding* by Proposition 1.91.

Remark 1.93. If $\deg \mathcal{L} = d$, and $\mathcal{L}$ is very ample, then $\mathcal{L}$ grants a closed embedding of X to $\mathbb{P}^{r-1}_k$ where $r := \dim_k \Gamma(X, \mathcal{L})$ (using the canonical embedding), and this embedding retains X as a curve of degree d.

1.11   February 10

Last class, we were in the middle of describing the canonical morphism of hyperelliptic curves. I have moved the entire proof of today because we did more work on it today.

1.11.1   Hyperelliptic Curves

We will spend the rest of class understanding hyperelliptic curves. The following is our statement.

> **Theorem 1.94.** Fix a hyperelliptic curve X of genus $g(X) \geq 2$.
>
> (a) Then X has a unique divisor class g_2^1 yielding the double-cover $\pi\colon X \to \mathbb{P}_k^1$.
>
> (b) The canonical morphism $f\colon X \to \mathbb{P}^{g-1}$ can be written as
>
> $$X \to \mathbb{P}_k^1 \to \mathbb{P}_k^{g-1},$$
>
> where the last map is the $(g-1)$-uple embedding.
>
> (c) Every effective canonical divisor can be written as the sum of $(g-1)$ effective divisors linearly equivalent to g_2^1.

Proof. For brevity, let X' be the image $f(X)$, and we fix some g_2^1 on X yielding a double-cover $X \to \mathbb{P}_k^1$.

Now, for any effective divisor $P + Q$ linearly equivalent to g_2^1, then we claim Q is a base-point of $K - P$: indeed, we know $\ell(K - P) = g - 1$ because K is base-point-free by Lemma 1.87, but K is not very ample witnessed by P and Q by the proof of Proposition 1.91, so $\ell(K - P - Q) = \ell(K - P)$ is forced. Thus, Q is in fact a base-point.

We now claim that f is not birational. There are two cases.

- In the case where $P \neq Q$, we see that K does not separate the points P and Q. Explicitly, we see that $s \in \Gamma(X, \mathcal{O}_X(K - P))$ is equivalent to $s \in \Gamma(X, \mathcal{O}_X(K - P - Q))$ is equivalent to $s \in \Gamma(X, \mathcal{O}_X(K - Q))$ (by symmetry). In other words, we are saying that any divisor $K + \operatorname{div}(s)$ which retains P in its support will also retain Q in its support.

  It follows that $f(P) = f(Q)$: otherwise $f(P) \neq f(Q)$ lets us separate the two points in $\mathbb{P}_k^{g-1}$ and hence find a basis of $\Gamma(\mathbb{P}_k^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ separating them. But then we can pull back this basis to $\Gamma(X, \mathcal{O}_X)$ to find sections separating P and Q.

- Otherwise, we in the case where $P = Q$, we are now given that P is a base-point of $K - P$. Explicitly, we know that $s \in \Gamma(X, \mathcal{O}_X(K - P))$ is equivalent to $s \in \Gamma(X, \mathcal{O}_X(K - 2P))$, which now means that f does not separate tangent vectors at P—namely, the image of $\Gamma(X, \mathcal{O}_X(K))$ does not generate $\mathfrak{m}_P \mathcal{O}_X(K)_P / \mathfrak{m}_P / \mathcal{O}_X(K)_P$. However, $\mathcal{O}_{\mathbb{P}_k^{g-1}}(1)$ does separate these tangent spaces by hyperplanes, so f is again not a closed embedding at the local ring at P.

By adjusting $P + Q$ appropriately in its linear equivalence class, we see that f fails to be a closed embedding at infinitely many points, so f is not a birational morphism.

As such, $\mu := \deg f$ is at least 2.[3] However, letting d denote the degree of $X' \subseteq \mathbb{P}_k^{g-1}$, we claim

$$d\mu \overset{?}{=} 2g - 2.$$

For this, we must understand $d = \deg X'$, which we note is the number of intersections (counted with multiplicity) of a hyperplane section of $\mathbb{P}_k^{g-1}$ intersecting $Y = X' \setminus \operatorname{Sing} X'$, where $\operatorname{Sing} X'$ is the singular locus. Letting $H \cap Y$ denote such an intersection (with d points and hence degree d as a divisor), we restrict $f\colon X \to X'$ to $f\colon X \to \widetilde{X'}$, granting

$$\deg f^*(H \cap \widetilde{X'}) = (\deg f)d = \mu d$$

by explicitly writing down what does f^* does on points on the finite morphism of smooth curves $f|_{\widetilde{X'}}$. In total, we conclude that $d \leq g - 1$.

Now, the composite $\widetilde{X'} \to X' \subseteq \mathbb{P}_k^{g-1}$ is a projective morphism and so grants a divisor $\mathfrak{d}$ on $\widetilde{X'}$, of dimension $g - 1$. To compute the degree, we intersect $\widetilde{X'}$ with a generic hyperplane (for example, avoiding

---

[3] Alternatively, note that $\dim |g_2^1| = 1$ forces there to be infinitely many representatives of the form $P + Q$ with $P \neq Q$, for otherwise there are infinitely many representatives of the form $2P$, which would mean that our double-cover $X \to \mathbb{P}_k^1$ is ramified at infinitely many points, which is impossible.

the singular locus of X'), which still has d intersections after passing to the birational curve X'. As such, we see $\deg \mathfrak{d} = d$. But from the homework, we have the inequality

$$\dim \mathfrak{d} \leq \deg D \leq g - 1,$$

so we conclude that equalities must hold everywhere, which in turn requires $\mathfrak{d}$ to be a complete linear system. Further, the homework also tells us that the equality case forces $g(X') = 0$, so we must have $X \cong \mathbb{P}^1_k$ (note the degree of $\mathfrak{d}$ is positive!), so comparing degrees enforces $\mathfrak{d} = \mathcal{O}_{\mathbb{P}^1_k}(g-1)$. Notably, we also get $\deg \mu = 2$ here.

Now, the map $\widetilde{X'} \to X' \subseteq \mathbb{P}^{g-1}_k$ is given by the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(g-1)$ and g spanning sections, but there is a basis of the g monomials of degree $g - 1$ in the variables x_0 and x_1, so our morphism we can write out explicitly as

$$[x_0 : x_1] \mapsto \left[x_0^{g-1} : x_0^{g-2} x_1 : \cdots : x_0 x_1^{g-2} : x_1^{g-1} \right]$$

by expanding out our sections. (One can also show that this is a closed embedding.) This is exactly the $(g-1)$-uple embedding (or "Veronese" embedding). We conclude that the canonical morphism $X \to \mathbb{P}^{g-1}_k$ has been factored as

$$X \to \widetilde{X'} \cong \mathbb{P}^1_k \to \mathbb{P}^{g-1}_k,$$

where the map $\mathbb{P}^1_k \to \mathbb{P}^{g-1}_k$. In particular, the image of the $(g-1)$-uple embedding can be checked to be nonsingular, so X' is successfully nonsingular, and we have $\widetilde{X'} \cong X'$.

Roughly speaking, it remains to discuss the uniqueness of our double-cover $\pi \colon X \to \mathbb{P}^1_k$. Notably, for any $P + Q$ linearly equivalent to g^1_2, pulling back the divisor $f(P) \in X' \cong \mathbb{P}^1_k$ up to $P + Q$ enforces $f(P) = f(Q)$. Running the same argument with π, which comes from a particular representative of g^1_2, we conclude that f and π coincide for infinitely many points of X, so they must coincide because $\mathrm{eq}(f, \pi)$ has positive codimension in X and therefore must vanish. It follows that π and f are uniquely determined by representatives of g^1_2. And quickly, we note that g^1_2 is also unique: it factors through the canonical morphism $f \colon X \to \mathbb{P}^{g-1}_k$ (followed by an embedding), which we already know to be unique, so the precise morphism π is unique (of course, up to automorphism of $\mathbb{P}^1_k$).

As such, we have shown (a) and (b). To show (c), we note that all effective canonical divisors are realizes as pullbacks of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^{g-1}_k}(1)$ on $X' \subseteq \mathbb{P}^{g-1}_k$. But we can pull this back along f factored as

$$X \xrightarrow{\pi} \mathbb{P}^1_k \cong X' \subseteq \mathbb{P}^{g-1}_k.$$

Well, this hyperplane intersects X' at $g - 1$ points, and then each of those points in $\mathbb{P}^1_k$ will pull back to an effective divisor which represents g^1_2. ∎

1.11.2 Special Divisors

We pick up the following definition.

> **Definition 1.95** (special)**.** Fix a divisor D on a k-curve X. Then D is *special* if and only if $\ell(K - D) > 0$.

> **Remark 1.96.** If $\ell(K - D) = 0$, then Theorem 1.39 tells us
>
> $$\ell(D) = \deg D + 1 - g,$$
>
> so we understand the divisor pretty well.

We will show the following result.

> **Proposition 1.97** (Clifford)**.** Fix an effective special divisor D on the k-curve X. Then
>
> $$\dim |D| \leq \frac{1}{2} \deg D.$$

To show this, we have the following lemma.

> **Lemma 1.98.** Fix effective divisors D and E on a K-curve X. Then
> $$\dim |D| + \dim |E| \le \dim |D + E|.$$

Proof. Note that there is a map $\varphi \colon |D| \times |E| \to |D + E|$ by sending a pair (D', E') of effective divisors in the correct linear equivalence classes to $D' + E'$. Note that any effective divisor in $|D + E|$ can be written as a sum of effective divisors in only finitely many ways, so this map has finite fibers.

Going further, we can upgrade φ to a k-bilinear map of k-vector spaces given by

$$\Gamma(X, \mathcal{L}(D)) \times \Gamma(X, \mathcal{L}(E)) \to \Gamma(X, \mathcal{L}(D + E))$$

by sending $(f, g) \mapsto fg$. In particular, this grants us a map on projective spaces

$$\mathbb{P}\Gamma(X, \mathcal{L}(D)) \times \mathbb{P}\Gamma(X, \mathcal{L}(E)) \to \mathbb{P}\Gamma(X, \mathcal{L}(D + E))$$

agreeing with our earlier definition of φ by tracking through how $\mathbb{P}\Gamma(X, \mathcal{L}(D))$ agrees with $|D|$. Notably, this map still has finite fibers, so it is quasifinite, so taking dimensions (this morphism is basically finite) lets us conclude $\dim |D| + \dim |E| \le \dim |D + E|$. ■

We are now ready to show Proposition 1.97.

Proof of inequality in Proposition 1.97. Because D is effective and special, we know $\ell(K - D) \ne 0$, so we can find an effective divisor E linearly equivalent to $K - D$. As such, Lemma 1.98 grants

$$\dim |D| + \dim |K - D| = \dim |D| + \dim |K - D| \le \dim |K + E| = \dim |K| = g - 1,$$

but Theorem 1.39 tells us

$$\dim |D| - \dim |K - D| \le \deg D + 1 - g,$$

so rearranging grants the inequality. ■

> **Remark 1.99.** We can also check some equality cases, quickly. If $D = 0$, there is nothing to say. When $D = K$, we simply note $\dim |K| = g - 1$, but $\deg K = 2g - 2$, so we are done. Lastly, suppose X is hyperelliptic and $D \sim kg_2^1$ for some k. We only care about linear equivalence class, so we might as well assume $D = kg_2^1$ and proceed by induction on k. At $k = 1$, we note $\dim g_2^1 = 1$ and $\deg g_2^1 = 2$, so equality holds. Then for our induction, we note Lemma 1.98 gives
> $$\dim \left|(k + 1)g_2^1\right| \ge \dim \left|kg_2^1\right| + \dim \left|kg_2^1\right| = k + 1,$$
> but we just showed that $\dim \left|(k + 1)g_2^1\right| \le \frac{1}{2} \deg \left((g + 1)g_2^1\right) = k + 1$ above, so we must have equalities.

1.12   February 13

Today we finish discussing curves. The fourth assignment will be released later today.

1.12.1   Finishing Clifford's Theorem

We are interested in the equality cases of Proposition 1.97. Here is the result.

Theorem 1.100 (Clifford). Fix an effective special divisor D on the k-curve X. Then $\dim |D| = \frac{1}{2} \deg D$ if and only if $D = 0$, or $D = K$, or X is hyperelliptic where D is linearly equivalent to a multiple of X's g_2^1.

Proof. We showed each of these equality cases work in Remark 1.99. It remains to show that the equality implies one of these conditions.

Well, suppose that $\dim |D| = \frac{1}{2} \deg D$ and $D \neq 0$ and $D \neq K$; we will show that X is hyperelliptic, and D is a multiple of g_2^1. Note Theorem 1.39 tells us that

$$\dim |D| - \dim |K - D| \leq \deg D + 1 - g$$

must achieve equality, by the proof of Proposition 1.97 above. Now, we see $\deg D$ must be even. We will induct on $\deg D$. For example, if $\deg D = 2$ and $\ell(D) = 1$ forces D to induce a double-cover $X \to \mathbb{P}_k^1$ (meaning X is hyperelliptic), so $D \sim g_2^1$ follows from this uniqueness of this divisor from Theorem 1.94.

To finish the proof, suppose $\deg D \geq 4$ so that $\dim |D| \geq 2$. Because $\dim |K - D| \geq 0$ (recall D is special), we get an effective divisor $E \in |K - D|$. Without loss of generality, we may assume that $\deg E > 0$ (otherwise $D = K$), so we can find a point P in the support of E and a point Q not in the support of D. Without loss of generality, we can find an effective divisor linearly equivalent to D with support containing P and Q, which is doable because $\dim |D| \geq 2$: indeed, $|D|$ determines a projective morphism $f\colon X \to \mathbb{P}_k^n$ for some $n \geq 2$, and then one can pull back a hyperplane H of $\mathbb{P}_k^n$ containing $f(P)$ and $f(Q)$ so that $D \sim f^*H$ contains P and Q in its support.

We now set $D' := D \cap E$, which by definition is the maximal effective divisor such that $D' \leq D$ and $D' \leq E$. Note that $D' < D$ in fact because Q is in the support of D but not E, but $D' > 0$ because P is in the support of both D and E. We are going to want to use the inductive hypothesis on D'. Notably, we claim we have an exact sequence

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \oplus \mathcal{O}_X(E) \to \mathcal{O}_X(D + E - D') \to 0.$$

Here, $\mathcal{O}_X(D') \to \mathcal{O}_X(D) \oplus \mathcal{O}_X(E)$ is the diagonal map, and the second map is $(f, g) \mapsto (f - g)$. Notably, for any valuation v on $K(X)$, we see $v(a + b) \geq \min\{v(a), v(b)\}$ with equality provided $v(a) \neq v(b)$.

To see our exactness, let's work an example.

Example 1.101. Fix an open subscheme $U \subseteq X$, and take $D = P + 2Q$ and $E = 2P + Q$ on restriction to U so that $D' = P + Q$ here. Given some f with a pole of order 1 at P and order 2 at Q, and some g with a pole of order 2 at P and of order 1 at Q. Indeed, we see that $f - g$ has poles of order 2 at both P and Q, which tells us $f - g$ is in fact in $\mathcal{O}_X(D + E - D')$.

This example explains our exactness: by checking surjectivity at the stalks of some point R, we may assume that $D = aR$ and $E = bR$ where $a \geq b$ without loss of generality. Then the exact sequence looks roughly like

$$0 \to \mathcal{O}_X(bR)_R \to \mathcal{O}_X(aR) \oplus \mathcal{O}_X(E)_{bR} \to \mathcal{O}_X(aR) \to 0,$$

which is more clearly exact: we are injective on the left by definition, we are surjective on the right by taking elements of the form $(0, -g)$, and we are exact in the middle because the only elements which vanish in the right map are diagonal ones.

Now, taking global sections, we get the left-exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X(D')) \to \Gamma(X, \mathcal{O}_X(D)) \oplus \Gamma(X, \mathcal{O}_X(E)) \to \Gamma(X, \mathcal{O}_X(D + E - D')).$$

Taking dimensions everywhere, we see

$$\ell(D) + \ell(E) \leq \ell(D') + \ell(D + E + D'),$$

which implies

$$g - 1 = \dim |K| = \dim |D| + \dim |D - K| \leq \dim |D'| + \dim |K - D'| \leq \dim |K| = g - 1,$$

where we have used Lemma 1.98 repeatedly. Thus, equalities hold everywhere, and we are forced to have $\dim |D'| + \dim |K - D'| = g - 1$ as well, which means that D' also satisfies $\dim |D'| = \frac{1}{2} \deg D'$ by the argument of Proposition 1.97. However, $\deg D' < \deg D$, and $D' \neq 0, K$, so the inductive hypothesis now applies to D', telling us that X is hyperelliptic. It remains to show that D is a multiple of g_2^1.

The trick is to relate D to K and use Theorem 1.94 because we already know that K is a multiple of g_2^1. Well, set $r := \dim |D|$. Note that D being special implies that $\ell(K - D) > 0$, so $\deg D \leq 2g - 2$, so we go ahead and study like to show $S := D + (g - 1 - r)g_2^1$ is linearly equivalent to K, where the point is that $\deg S = 2g - 2$. We would like to compute $\dim |S|$. On one hand, we see

$$\dim |S| \geq \dim |D| \geq \dim |D| + (g - 1 - r) = g - 1.$$

On the other hand, Theorem 1.39 implies

$$|S| - |K - S| = (2g - 2) + (1 - g) = g - 1,$$

so $|K - S| \geq 0$. Thus, $K - S$ is linearly equivalent to an effective divisor of degree 0, which actually forces $K - S \sim 0$, so $S \sim K$ follows. However, this then implies that

$$D \sim K - (g - 1 - r)g_2^1,$$

which finishes because K is a multiple of g_2^1 by Theorem 1.94. This completes the proof. ∎

> **Remark 1.102.** This concludes our discussion of curves. We hope it motivates our discussion of cohomology, which will follow.

Cohomology

2.1 February 13

We now switch gears and talk about cohomology. Notably, this is not the usual algebraic topology version of cohomology from the topology on our scheme because the Zariski topology on a scheme is so terrible.

2.1.1 Some Starting Remarks

Fix a scheme X and an abelian sheaf $\mathcal{F}$ on X. In other words, $\mathcal{F}$ is a sheaf valued in an abelian category (though we personally will only request that $\mathcal{F}$ have sections which are modules). Later, we might ask for $\mathcal{F}$ to be an $\mathcal{O}_X$-module.

Roughly speaking, computing cohomology is about computing derived functors. Notably, it will turn out that all relevant categories have enough injectives, so we will be able to build an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^1 \to \mathcal{I}^2 \to \mathcal{I}^2 \to \cdots.$$

Now, $H^\bullet(X, \mathcal{F})$ is intended to be the right-derived functor of $\Gamma(X, -)$, so we will want to take global sections to get a complex

$$0 \xrightarrow{d^{-1}} \Gamma(X, \mathcal{I}^1) \xrightarrow{d^0} \Gamma(X, \mathcal{I}^2) \xrightarrow{d^1} \cdots,$$

which is no longer exact. It then turns out by derived functor magic that we can compute cohomology from these groups. Namely, by definition, we will take

$$H^i(X, \mathcal{F}) := \frac{\ker d^i}{\operatorname{im} d^{i-1}},$$

which turns out to be exactly the cohomology groups we desire.

> **Remark 2.1.** The above recipe will work fine for any "Grothendieck topology," such as the étale cohomology. We won't see that in this class, but it is a useful general machine.

> **Remark 2.2.** Classically, cohomology was computed using more general "acyclic" resolutions (most notably for Čech cohomology), but it turns out that injective resolutions are better-behaved for our abstract arguments. Nonetheless, we will see these acyclic resolutions in this class because they are useful; in particular, we will be building resolutions using flasque sheaves in this class.

2.2 February 15

Let's begin our discussion of cohomology.

2.2.1 Abelian Categories

Let's be somewhat formal with our definition of an abelian category.

> **Definition 2.3** (pre-additive)**.** A category $\mathcal{C}$ is *pre-additive* if and only if it is enriched over the category of abelian groups. In particular, for objects $X, Y.Z \in \mathcal{C}$, the set $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group, and for any $f\colon X \to Y$, the maps
>
> $$(- \circ f)\colon \mathrm{Mor}_{\mathcal{C}}(Y, Z) \to \mathrm{Mor}_{\mathcal{C}}(X, Z) \qquad \text{and} \qquad (f \circ -)\colon \mathrm{Mor}_{\mathcal{C}}(Z, X) \to \mathrm{Mor}_{\mathcal{C}}(Z, Y)$$
>
> are group homomorphisms.

> **Definition 2.4** (additive)**.** A pre-additive category $\mathcal{C}$ is *additive* if and only if it has finite products.

> **Remark 2.5.** In a pre-additive category, finite products and coproducts are canonically isomorphic. In other words, given objects $X, Y \in \mathcal{C}$, if $X \oplus Y$ exists, then so does $X \times Y$ (and conversely), and there is a canonical isomorphism between them.

> **Remark 2.6.** By Remark 2.5, the empty product and coproduct coincide, so we have an object which is both initial and final, which we see is our zero object. Thus, additive categories have a zero object.

Let's explain Remark 2.5 quickly. Roughly speaking, it turns out that both universal properties are equivalent to having a given object Z fit into the following commutative diagram.

Here, we also require that $\pi_Y \iota_X = 0$ and $\pi_X \iota_Y = 0$ and $\mathrm{id}_Z = \iota_X \pi_X + \iota_Y \pi_Y$. It is not too hard to see that any object Z with the above data is both a product (Z, π_X, π_Y) and a coproduct (Z, ι_X, ι_Y). For example, given an object A with maps $\varphi_X\colon X \to A$ and $\varphi_Y\colon Y \to A$, we claim that having $\varphi\colon A \to Z$ with $\pi_X \varphi = \varphi_X$ and $\pi_Y \varphi = \varphi_Y$ is equivalent to

$$\varphi = \iota_X \varphi_X + \iota_Y \varphi_Y,$$

so our $\varphi\colon A \to Z$ and is unique. Indeed, this φ works, and it is forced because

$$\varphi = \mathrm{id}_Z \circ \varphi = (\iota_X \pi_X + \iota_Y \pi_Y) \circ \varphi = \iota_X \varphi_X + \iota_Y \varphi_Y.$$

Showing that Z is a coproduct follows similarly.

Conversely, if we (for example) have a coproduct (Z, ι_X, ι_Y), we can build the projection map $\pi_X\colon Z \to X$ by applying to the universal property in the following diagram.

Defining π_Y similarly, we see that $\pi_Y \iota_X = 0$ and $\pi_X \iota_Y = 0$ for free, and then one can show that $\mathrm{id}_Z = \iota_X \pi_X + \iota_Y \pi_Y$.

We now continue our march towards abelian categories.

> **Definition 2.7** (pre-abelian)**.** An additive category $\mathcal{C}$ is *pre-abelian* if and only if it admits kernels and cokernels.

> **Remark 2.8.** Given $f, g \colon X \to Y$, we see that $\mathrm{eq}(f,g) = \ker(f-g)$, so in fact we see we admit equalizers. Because we also admit coequalizers for similar reasons, we see that pre-abelian categories have finite limits and colimits.

> **Definition 2.9** (abelian)**.** A pre-abelian category $\mathcal{C}$ is *abelian* if and only if every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

Here, a monomorphism $f \colon X \to Y$ merely means that the induced maps $\mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ is injective for any object Z, and epimorphisms are defined dually.

We can combine the above definitions into the following cleaner one.

> **Proposition 2.10.** Fix a pre-abelian category $\mathcal{C}$. Then $\mathcal{C}$ is abelian if and only if the following hold.
>
>   (a) Every monomorphism is the kernel of its cokernel.
>
>   (b) Every epimorphism is the cokernel of its kernel.
>
>   (c) Further, any morphism can be factored as an epimorphism followed by a monomorphism.

Proof. Omitted. This more or less follows from expanding out our definitions. ∎

> **Remark 2.11.** Freyd's embedding theorem says that any locally small abelian category has a faithful embedding to Mod_R for some ring R. As such, we will usually just reason in Mod_R instead of general abelian categories.

2.2.2   Complexes

Throughout, we fix an abelian category $\mathcal{C}$. We now discuss complexes.

> **Definition 2.12** (complex)**.** Fix an abelian category $\mathcal{C}$. Then a *complex* is the data $(K^\bullet, d^\bullet)$ of objects $\{K^i : i \in \mathbb{Z}\}$ and maps $d^i \colon K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for each i. We will often notate this as
> $$\cdots \to K^{-2} \xrightarrow{d^{-2}} K^{-1} \xrightarrow{d^{-1}} K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2 \to \cdots .$$

> **Definition 2.13** (cohomology)**.** Fix a complex $(K^\bullet, d^\bullet)$ in an abelian category $\mathcal{C}$, the *ith cohomology* for $i \in \mathbb{Z}$ is the object
> $$H^i(K^\bullet) := \frac{\ker d^i}{\mathrm{im}\, d^{i-1}},$$
> where we have embedded in K^i. Note that there is a map $\mathrm{im}\, d^{i-1} \to \ker d^i$ because $d^i \circ d^{i-1} = 0$.

We also have morphisms.

Definition 2.14 (complex morphism)**.** Fix an abelian category $\mathcal{C}$. Then a *morphism of complexes* $(K^\bullet, d_K^\bullet) \to$ $(L^\bullet, d_L^\bullet)$ is a collection of maps $f^\bullet\colon K^\bullet \to L^\bullet$ making the following diagram commute.

$$\cdots \longrightarrow K^{-2} \longrightarrow K^{-1} \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow \cdots$$
$$\downarrow f^{-2} \qquad \downarrow f^{-1} \qquad \downarrow f^0 \qquad \downarrow f^1 \qquad \downarrow f^2$$
$$\cdots \longrightarrow L^{-2} \longrightarrow L^{-1} \longrightarrow L^0 \longrightarrow L^1 \longrightarrow L^2 \longrightarrow \cdots$$

Remark 2.15. A morphism $(f^\bullet)\colon (K^\bullet, d_K^\bullet) \to (L^\bullet, d_L^\bullet)$ gives rise (functorially) to maps

$$H^i(f^\bullet)\colon H^i(K^\bullet) \to H^i(L^\bullet)$$

for any $i \in \mathbb{Z}$. This is some technical argument dealing with the commutativity of the following diagram.

$$\begin{array}{ccccc} K^{i-1} & \xrightarrow{d_K^{i-1}} & K^i & \xrightarrow{d_K^i} & K^{i+1} \\ \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} \\ L^{i-1} & \xrightarrow[d_L^{i-1}]{} & L^i & \xrightarrow[d_L^i]{} & L^{i+1} \end{array}$$

Quickly, we see that an element of the image of d_K^{i-1} does get mapped to the image of d_L^i because $f^i \circ d_K^{i-1} = d_K^i \circ f^{i-1}$; similarly, the kernel gets mapped to the kernel. As such, our map on cohomology is well-defined, and we get to define our morphism. Functoriality checks are routine.

And we even have a notion of exactness.

Definition 2.16 (exact)**.** Fix an abelian category $\mathcal{C}$. Given a sequence

$$(A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet) \to (C^\bullet, d_C^\bullet),$$

this is *exact* at $B^\bullet$ if and only if the corresponding maps on each degree i are exact at B^i.

Here is our central result for cohomology here.

Proposition 2.17. Fix an abelian category $\mathcal{C}$ and an exact sequence

$$0 \to (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet) \to (C^\bullet, d_C^\bullet) \to 0$$

of complexes. Then there is a long exact sequence in cohomology as follows.

$$\cdots \longrightarrow H^{i-1}(A^\bullet) \longrightarrow H^{i-1}(B^\bullet) \longrightarrow H^{i-1}(C^\bullet)$$
$$H^i(A^\bullet) \longleftarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow \cdots$$

Proof. The argument of Remark 2.15 grants us a morphism of exact sequences as follows.

$$\begin{array}{ccccccc} A^i/\operatorname{im} d_A^{i-1} & \longrightarrow & B^i/\operatorname{im} d_B^{i-1} & \longrightarrow & C^i/\operatorname{im} d_C^{i-1} & \longrightarrow & 0 \\ \downarrow d_A^i & & \downarrow d_B^i & & \downarrow d_C^i & & \\ 0 \longrightarrow \ker d_A^{i+1} & \longrightarrow & \ker d_B^{i+1} & \longrightarrow & \ker d_C^{i+1} & & \end{array}$$

Hitting this with the snake lemma gives us the desired exact sequence. Zippering our exact sequences together creates the full long exact sequence. ∎

We might be interested when two complexes give the same cohomology. This is typically done using a chain homotopy.

Definition 2.18 (chain homotopy). Fix an abelian category $\mathcal{C}$ and complex morphisms $(f^\bullet), (g^\bullet)\colon (K^\bullet, d_K^\bullet) \to (L^\bullet, d_L^\bullet)$. Then a *chain homotopy* is a collection of maps $h^i\colon K^i \to L^{i+1}$ such that $f - g = d_L h + h d_K$. Here, this arithmetic is taking place in the following squares.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & K^{-2} & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & K^2 & \longrightarrow & \cdots \\
& & \downarrow & \swarrow_{h^{-1}} & \downarrow & \swarrow_{h^0} & \downarrow & \swarrow_{h^1} & \downarrow & \swarrow_{h^2} & \downarrow & & \\
\cdots & \longrightarrow & L^{-2} & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \longrightarrow & L^1 & \longrightarrow & L^2 & \longrightarrow & \cdots
\end{array}$$

Note that this diagram does not commute.

Here is the main result.

Proposition 2.19. Fix an abelian category $\mathcal{C}$ and complex morphisms $(f^\bullet), (g^\bullet)\colon (K^\bullet, d_K^\bullet) \to (L^\bullet, d_L^\bullet)$ such that we have a chain homotopy $h^\bullet$ between these morphisms. Then $f^\bullet$ and $g^\bullet$ induce the same morphism on cohomology.

Proof. We will show that $dh + hd$ induces the zero map on cohomology. This is now some diagram-chase: fixing some index i, we pick an element $a \in \ker d^i$ which we want to show has $(dh + hd)(a) \in \operatorname{im} d^{i-1}$. Well, we have $dha \in \operatorname{im} d^{i-1}$ automatically and $hda = 0$ by construction of a, so we are done. ∎

We will start talking about resolutions next class.

Definition 2.20 (injective). Fix an abelian category $\mathcal{C}$. Then an object $I \in \mathcal{C}$ is *injective* if and only if any monomorphism $A \hookrightarrow B$ can extend any morphism $A \to I$ to a morphism $B \to I$.

Remark 2.21. We will show on the homework that Mod_R and $\operatorname{Mod}_{\mathcal{O}_X}$ both have enough injectives.

2.3   February 17

We continue talk about cohomology.

2.3.1   Functors for Abelian Categories

We might want to discuss functors between our categories, so we want them to respect some of our internal data.

Definition 2.22 (additive). A functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories is *additive* if and only if $A, A' \in \mathcal{A}$ implies

$$F\colon \operatorname{Hom}_{\mathcal{A}}(A, A') \to \operatorname{Hom}_{\mathcal{B}}(FA, FA')$$

is a group homomorphism.

Remark 2.23. Note that additive functors preserve direct sums/products. Indeed, we saw last class that a direct sum/product of X and Y is an object Z with maps $\iota_X \colon X \to Z$ and $\iota_Y \colon Y \to Z$ and $\pi_X \colon Z \to X$ and $\pi_Y \colon Z \to Y$ satisfying the equations

$$\pi_X \iota_X = \mathrm{id}_X, \qquad \pi_Y \iota_Y = \mathrm{id}_Y, \qquad \mathrm{id}_Z = \iota_X \pi_X + \iota_Y \pi_Y.$$

These equations are preserved by an additive functor, so FZ is a direct sum of FX and FY.

Definition 2.24 (exact). An additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is *left-exact* if and only if it sends left-exact sequences

$$0 \to A' \to A \to A''$$

to left-exact sequences

$$0 \to FA' \to FA \to FA''.$$

The definition of *right-exact* is entirely dual, and the F is *exact* if and only if it is both left-exact and right-exact.

Remark 2.25. In fact, left-exact functors preserve all finite limits. Directly from the definition, left-exact functors preserve all kernels, so they also preserve equalizers because equalizers are just kernels in abelian categories. Further, because the functor is additive, it preserves finite products, so because all finite limits are an equalizer of some morphism of finite products, our functor does indeed preserve all finite limits.

Warning 2.26. Our definition of left-exact requires the functor to be additive.

2.3.2 Injective Resolutions

To make cohomology work, we will want the following definition.

Definition 2.27 (enough injectives). An abelian category $\mathcal{A}$ has *enough injectives* if and only any object $A \in \mathcal{A}$ has a monomorphism $A \hookrightarrow I$ where I is an injective object.

Having enough injectives grant us injective resolutions.

Definition 2.28. Fix an abelian category $\mathcal{A}$. Given an object $A \in \mathcal{A}$, an *injective resolution* for A is a complex $0 \to I^\bullet$ of injective objects such that $H^0(I^\bullet) = A$ but $H^i(I^\bullet) = 0$ for $i > 0$.

Proposition 2.29. Fix an abelian category $\mathcal{A}$ with enough injectives. For any $A \in \mathcal{A}$, there are injective modules $I^0, I^1, I^2, \dots$ fitting into an (augmented) injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots.$$

Proof. Because $\mathcal{A}$ has enough injectives, we can embed $A \hookrightarrow I^0$ for some injective module I^0. This gives us a short exact sequence

$$0 \to A \to I^0 \to I^0/A \to 0.$$

Now, because $\mathcal{A}$ has enough injectives, we can embed I^0/A into some injective module I^1, so we get to extend our injective resolution $0 \to A \to I^0 \to I^1$. Running this argument inductively, we get the exact augmented injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots,$$

as desired. ∎

As an aside, Given an augmented injective resolution $A \to I^\bullet$, we note that there is a morphism of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \big\| & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}
$$

which is an isomorphism on cohomology. This motivates the following definition.

> **Definition 2.30.** Fix an abelian category $\mathcal{A}$. Then a morphism $A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if and only if it induces an isomorphism on cohomology.

It turns out that any two injective resolutions for A are quasi-isomorphic, but this takes some care.

> **Proposition 2.31.** Fix an abelian category $\mathcal{A}$. Given injective resolutions $0 \to I^\bullet$ and $0 \to J^\bullet$ of an object $A \in \mathcal{A}$, they are chain homotopic.

Proof. The main difficulty is constructing some morphism $I^\bullet \to J^\bullet$. This is done using the injectivity. Consider the following diagram.

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \\
& & \big\| & & & & & & & & \\
0 & \longrightarrow & A & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots
\end{array}
$$

Because J^0 is injective, the inclusion $A \subseteq I^0$ allows us to extend $A \to J^0$ to a map $I^0 \to J^0$ commuting, so we get to extend this as follows.

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \\
& & \big\| & & \big\downarrow & & & & & & \\
0 & \longrightarrow & A & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots
\end{array}
$$

We now continue this process. Namely, we have an inclusion $I^0/A \subseteq I^1$, so we can extend the map $I^0/A \to J^1$ to a map $I^1 \to J^1$ to make the following diagram commute.

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \\
& & \big\| & & \big\downarrow & & \big\downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots
\end{array}
$$

We can now continue this process inductively.

Repeating this process, we also get a map $J^\bullet \to I^\bullet$ again induced by $\mathrm{id}_A \colon A \to A$. Composing our two morphisms of complexes and subtracting out the identity complex morphism, we want to show that any map of complexes looking like

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{d^{-1}} & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \xrightarrow{d^2} & \cdots \\
& & {\scriptstyle 0}\big\downarrow & & {\scriptstyle f^0}\big\downarrow & & {\scriptstyle f^1}\big\downarrow & & {\scriptstyle f^2}\big\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{d^{-1}} & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \xrightarrow{d^2} & \cdots
\end{array}
$$

will have a chain homotopy with 0. We now go construct our chain homotopy. Well, we simply define $h^0 = 0$, and then we define $h^1 \colon I^1 \to I^0$ by extending the map $I^0/A \to J^0$ by the inclusion $I^0/A \to I^1$.

We now continue the process inductively. For example, having h^1, we extend the map $I^1/I^0 \to I^1$ given by $f^1 - dh^1$ by the inclusion $I^1/I^0 \to I^2$ to give a map $h^2 \colon I^2 \to I^1$ by our injectivity. This process will now work in general to complete the definition of our chain homotopy, which shows that we are in fact inducing the zero morphism on cohomology. ∎

2.3.3 Right-Derived Functors

We are now reedy to define right-derived functors.

> **Definition 2.32** (right-derived functor)**.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. Then we define the *right-derived functor* $R^iF\colon \mathcal{A} \to \mathcal{B}$ by taking any object $A \in \mathcal{A}$, forming an injective resolution $A \to I^\bullet$ and computing $R^iF \coloneqq H^i(FI^\bullet)$.

One ought to check that this construction is functorial and so on, but we won't bother with these checks. Here are some facts about right-derived functors, which we mostly won't prove.

> **Proposition 2.33.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. Fix $A \in \mathcal{A}$.
>
> (a) For $i < 0$, we have $R^iF = 0$.
>
> (b) The functors R^iF are additive, and they are independent of the choice of injective resolution up to some sufficiently canonical isomorphism.
>
> (c) We have $R^0F = F$.
>
> (d) If I is injective, then $R^iF = 0$ for $i > 0$.

Proof. Here we go.

(a) This is by definition of the complex we are taking cohomology of.

(b) The fact that the injective resolutions do not change R^iF is by Proposition 2.31. Additivity follows from an argument about extending the addition of two morphisms, as discussed in Proposition 2.31.

(c) Applying F to
$$0 \to A \to I^0 \to I^1$$
grants
$$0 \to FA \to FI^0 \to FI^1,$$
so $R^0F(A) = H^0(FI^\bullet) = A$.

(d) Use the injective resolution
$$0 \to I \to I \to 0 \to 0 \to \cdots$$
of I, so we are computing the cohomology of
$$0 \to FI \to 0 \to 0 \to \cdots,$$
so we do indeed vanish in higher degree. ∎

Lastly, here is our main result.

> **Theorem 2.34.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. Given an exact sequence
> $$0 \to A' \to A \to A'' \to 0,$$
> there is a long exact sequence in cohomology given as follows.
>
> $$0 \longrightarrow R^0F(A') \longrightarrow R^0F(A) \longrightarrow R^0F(A'')$$
> $$R^1F(A') \longleftarrow R^1F(A) \longrightarrow R^1F(A'') \longrightarrow \cdots$$
>
> In fact, this construction is functorial in the short exact sequence.

Proof. It turns out that one can turn our short exact sequence into a short exact sequence of injective reso-
lutions by some technical lemma and then take cohomology, using Proposition 2.17. Lastly, one can check
that a morphism of short exact sequences induces a morphism of the long exact sequences by using the
functoriality of the Snake lemma, using Proposition 2.17 once more.   ∎

Injective objects are somewhat hard to handle, so we introduce acyclic resolutions.

> **Definition 2.35** (acyclic)**.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough
> injectives. An object $I \in \mathcal{A}$ is *acyclic* if and only if $R^i F(I) = 0$ for all $i > 0$.

> **Proposition 2.36.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injec-
> tives. For $A \in \mathcal{A}$, we can compute cohomology using an acyclic resolution of A instead of an injective
> one.

> **Remark 2.37.** Theorem 2.34 essentially says that $\left\{ R^i F \right\}_{i \geq 0}$ is a δ-functor. In fact, our right-derived
> functors are "universal," which means that any other δ-functor $\left(T^i, \delta^i \right)$ with a natural transformation
> $R^0 F \Rightarrow T^0$ can extend this natural transformation uniquely to a full morphism of the δ-functors.

2.4   February 22

Another homework will be assigned probably later today.

2.4.1   Universal δ-Functors

Quickly, we state a useful abstract nonsense result.

> **Definition 2.38** (effaceable)**.** Fix an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories. Then F is *efface-*
> *able* if and only if each $A \in \mathcal{A}$ has some monomorphism $u\colon A \hookrightarrow M$ such that $F(u) = 0$. We define
> *coeffaceable* dually.

Of course, no left-exact functor will be effaceable, but we do have the following.

> **Theorem 2.39** (Grothendeick)**.** Fix a δ-functor $\left\{ T^i, \delta^i \right\}_{i \geq 0} : \mathcal{A} \to \mathcal{B}$ between abelian categories. If each
> T^i is effaceable, then $\left\{ T^i, \delta^i \right\}_{i \geq 0}$ is universal.

The point here is that we can show two δ-functors are the same by showing that one is universal and that
the other is effaceable.

2.4.2   Sheaf Cohomology

On the homework, we established the following results.

> **Theorem 2.40.** Fix a ring A. Then Mod_A has enough injectives.

> **Theorem 2.41.** Fix a ringed space $(X, \mathcal{O}_X)$. Then $\mathrm{Mod}_{\mathcal{O}_X}$ has enough injectives.

> **Corollary 2.42.** Fix a topological space X. Then the category Ab_X of abelian sheaves on X has enough injectives.

Proof. Turn X into a ringed space $(X, \mathcal{O}_X)$ by $\mathcal{O}_X = \underline{\mathbb{Z}}$. Then Theorem 2.41 completes the proof. ∎

In particular, we can use our abstract nonsense on the left-exact functor $\Gamma(X, -) \colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$. So we may define sheaf cohomology.

> **Definition 2.43.** Fix a topological space. The right-derived functors of the functor $\Gamma(X, -) \colon \mathrm{Ab}_X \to \mathrm{Ab}$ exist and are denoted by $H^i(X, -)$.

> **Remark 2.44.** If $\mathcal{F}$ is also a sheaf of $\mathcal{O}_X$-modules, then notably our abstract nonsense is forgetting about this extra structure. Nonetheless, we will shortly see that these cohomology groups are the same.

We are going to want to compute some cohomology groups. It will be helpful to be able to more easily build some acyclic sheaves.

> **Definition 2.45** (flasque). Fix a topological space X. A sheaf $\mathcal{F}$ on X is *flasque* if and only if the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are always surjective.

Roughly speaking, it will turn out that these flasque sheaves are acyclic for our cohomology. Here is a quick result which suggests something along these lines.

> **Proposition 2.46.** Fix a ringed space $(X, \mathcal{O}_X)$. Then every injective $\mathcal{O}_X$-module $\mathcal{F}$ is flasque.

Proof. Fix an open subset $i \colon U \subseteq X$. Then we note $\mathcal{O}_U$ is $i_! i^* \mathcal{O}_X = i_! \mathcal{O}_U$. The point is that $\mathrm{Hom}_X(\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_U(\mathcal{O}_U, \mathcal{F}|_U)$, which is $\mathcal{F}(U)$ because $\mathcal{F}$ is an $\mathcal{O}_U$-module. However, for any other open subset $j \colon V \subseteq U$, then we note our embedding $\mathcal{O}_V = j_! j^* \mathcal{O}_U \hookrightarrow \mathcal{O}_U$ allows us to push forward a section $s \in \mathcal{F}(V)$ inducing a morphism $\mathcal{O}_V \to \mathcal{F}$ to a section $s' \in \mathcal{F}(U)$ inducing a morphism $\mathcal{O}_U \to \mathcal{F}$ by the injectivity of $\mathcal{F}$. Unwinding, we see that our restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. ∎

What?

We quickly note the following properties of flasque sheaves.

> **Lemma 2.47.** Fix an exact sequence
> $$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
> of sheaves on a topological space X. If $\mathcal{F}'$ and $\mathcal{F}$ is flasque, then so is $\mathcal{F}''$ is flasque. Additionally, if $\mathcal{F}'$ is flasque, then
> $$0 \to \Gamma(U, \mathcal{F}'0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'') \to 0$$
> is exact for any open $U \subseteq X$.

Proof. We showed these last class. Notably, the first statement follows from the second by hitting ourselves with the Five lemma on the following diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\mathcal{F}', U) & \longrightarrow & \Gamma(\mathcal{F}, U) & \longrightarrow & \Gamma(\mathcal{F}'', U) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(\mathcal{F}', V) & \longrightarrow & \Gamma(\mathcal{F}, V) & \longrightarrow & \Gamma(\mathcal{F}'', V) & \longrightarrow & 0
\end{array}
$$

Indeed, the left two morphisms being surjective forces the right morphism to also be surjective. ∎

> **Proposition 2.48.** Fix a flasque sheaf $\mathcal{F}$ of abelian groups on a topological space X. Then $\mathcal{F}$ is acyclic for the functor $\Gamma(X, -)$.

Proof. Embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$. Taking quotients, we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0,$$

so the previous lemma does tell us that $\mathcal{G}$ must be flasque. We now finish by dimension-shifting: by the long exact sequence, we see that $H^i(X, \mathcal{G}) = H^{i+1}(X, \mathcal{F})$. Furthermore, our long exact sequence has

$$\Gamma(X, \mathcal{I}) \twoheadrightarrow \Gamma(X, \mathcal{G}) \to H^1(X, \mathcal{F}) \to \underbrace{H^1(X, \mathcal{I})}_{0},$$

so we see $H^1(X, \mathcal{F}) = 0$ follows. But then $H^1(X, \mathcal{G}) = 0$ because $\mathcal{G}$ is flasque, so $H^2(X, \mathcal{F}) = 0$ also, and we can continue our induction upwards. $\blacksquare$

> **Corollary 2.49.** The right-derived functors of $\Gamma(X, -) \colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ coincide with $H^\bullet(X, -)$.

Proof. Take an injective resolution of some $\mathcal{O}_X$-module $\mathcal{F}$. But this is an acyclic resolution for $\Gamma(X, -) \colon \mathrm{Ab}_X \to \mathrm{Ab}$, so we are done by Proposition 2.36. $\blacksquare$

> **Remark 2.50.** Notably, if X actually has the structure of a ringed space $(X, \mathcal{O}_X)$, then deriving with $\Gamma(X, -) \colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ grants $H^\bullet(X, -)$ a module structure over $\Gamma(X, \mathcal{O}_X)$. For example, if X is a k-variety, this tells us that the $H^\bullet(X, -)$ should be k-vector spaces.

2.4.3 Dimension-Bounding Cohomology

While we're here, we pick up the following vanishing result.

> **Theorem 2.51.** Fix a Noetherian topological space X of dimension n. For any abelian sheaf $\mathcal{F}$, we have $H^i(X, \mathcal{F}) = 0$ for $i > n$.

We are going to want to pick up a few starting facts.

> **Lemma 2.52.** Fix a continuous map $f \colon X \to Y$ of topological spaces. Given a flasque abelian sheaf $\mathcal{F}$ on X, then $f_*\mathcal{F}$ is flasque on Y.

Proof. For any open subsets $V \subseteq U \subseteq Y$. Then the restriction map $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ is simply the restriction map $\mathcal{F}\left(f^{-1}(U)\right) \to \mathcal{F}\left(f^{-1}(V)\right)$, which is surjective because $\mathcal{F}$ is flasque. $\blacksquare$

> **Corollary 2.53.** Fix a closed embedding $j \colon Z \to X$ of schemes. Given an abelian sheaf $\mathcal{F}$ on Z, we have $H^\bullet(X, j_*\mathcal{F}) = H^\bullet(Z, \mathcal{F})$.

Proof. Note that $j \colon Z \to X$ being a closed embedding means that j_* is just extension by zero and is thus exact by checking everything at stalks—in particular, $(j_*\mathcal{F})_x = 0$ for $x \notin j(Z)$. However, j_* also preserves flasque sheaves by Lemma 2.52, so we can push forward a flasque resolution of $\mathcal{F}$ on Z to a flasque resolution of $j_*\mathcal{F}$ on X.

Computing cohomology with these two resolutions completes the proof. Let's be more explicit: give $\mathcal{F}$ an injective resolution as follows

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots.$$

Hitting this with j_* gives us an exact resolution of flasque sheaves

$$0 \to j_*\mathcal{F} \to j_*\mathcal{I}^0 \to j_*\mathcal{I}^1 \to j_*\mathcal{I}^2 \to \cdots.$$

Taking global sections of these two resolutions will give the same complex

$$0 \to \mathcal{I}^0(Z) \to \mathcal{I}^1(Z) \to \mathcal{I}^2(Z) \to \cdots,$$

so they yield the same cohomology groups. ∎

We also want to know how cohomology compares with directed colimits so that we can take about stalks. This is the content of the next lemma.

> **Definition 2.54** (directed set). A partially ordered set $(\lambda, \le)$ is *directed* if and only if any two $a, b \in \lambda$ have some $c \in \lambda$ such that $a, c \le c$.

> **Lemma 2.55.** Fix a Noetherian topological space X. If $\{\mathcal{F}_\alpha\}_{\alpha \in \lambda}$ is a collection of sheaves indexed by a directed set, then
> $$H^\bullet(X, \operatorname{colim} \mathcal{F}) = \operatorname{colim} H^\bullet(X, \mathcal{F}_i).$$

This result is useful because it more or less bounds the size of $H^\bullet(X, \operatorname{colim} \mathcal{F})$ by the various $H^\bullet(X, \mathcal{F}_i)$s. Roughly speaking, we are going to use this result to prove Theorem 2.51 by replacing $\mathcal{F}$ with some kind of directed colimit of finitely generated subsheaves and work there instead.

2.5 February 24

Today we finish the proof of Theorem 2.51.

2.5.1 Finishing Dimension-Bounding Cohomology

We begin by proving the following result.

> **Lemma 2.55.** Fix a Noetherian topological space X. If $\{\mathcal{F}_\alpha\}_{\alpha \in \lambda}$ is a collection of sheaves indexed by a directed set, then
> $$H^\bullet(X, \operatorname{colim} \mathcal{F}) = \operatorname{colim} H^\bullet(X, \mathcal{F}_i).$$

For this we will want the following result.

> **Lemma 2.56.** There are functorial injective resolutions $\operatorname{Inj}\colon \operatorname{Ab}_X \to \operatorname{Kom}(\operatorname{Ab}_X)$.

Proof. We have shown that every object produces some injective resolution. The difficulty here lies in the functoriality of the morphisms between our injective resolutions. Roughly speaking, if one goes back through the proof of having enough injectives, then everything is functorial at every step, so we are okay building these resolutions in a functorial manner. ∎

> **Lemma 2.57.** The categories Ab_X and $\operatorname{Mod}_{\mathcal{O}_X}$ are Grothendieck abelian categories. In other words, directed colimits exist and are exact.

Proof. Roughly speaking, we must show that a directed system of exact sequences of sheaves

$$0 \to \mathcal{F}'_\alpha \to \mathcal{F}_\alpha \to \mathcal{F}''_\alpha \to 0$$

produces an exact sequence

$$0 \to \operatorname{colim} \mathcal{F}'_\alpha \to \operatorname{colim} \mathcal{F}_\alpha \to \operatorname{colim} \mathcal{F}''_\alpha \to 0.$$

I think one can check this stalk-locally because colimits commute with taking stalks. ∎

> **Lemma 2.58.** Fix a Noetherian topological space X. If $\{\mathcal{F}_\alpha\}$ is a directed system of flasque sheaves, then the colimit is also flasque.

Proof. One can just check this directly. One can check that the presheaf

$$\mathcal{L} \colon U \mapsto \operatorname{colim} \mathcal{F}_\alpha(U)$$

is in fact a sheaf because our colimit is directed, and our space is directed. Thus, because colimits commute with cokernels, the surjectivity of the restriction maps commutes with the colimit.

   Let's show that our presheaf $\mathcal{L}$ is in a sheaf.

- Identity: fix sections $a, b \in \mathcal{L}(U)$ and an open cover $\{U_\beta\}$ of U such that $a|_{U_\beta} = b|_{U_\beta}$ vanishes for each U_β. We must show that a vanishes. Well, X is Noetherian, so every open subset is quasicompact, so we may replace our open cover with a finite one $\{U_i\}_{i=1}^n$.

   Now, $a|_{U_i} = b|_{U_i}$ for each i. In fact, because we are in the colimit, we can find some α_i such that $a|_{U_i} = 0$ in merely $\mathcal{F}_{\alpha_i}(U)$. Taking some α exceeding all the α_i, we see that $a|_{U_i} = 0$ for each i in $\mathcal{F}_\alpha(U)$ (again, because are just in a colimit), so $a = 0$ in $\mathcal{F}_\alpha(U)$ because $\mathcal{F}_\alpha$ is a sheaf.

- The gluability condition now proceeds as usual. Namely, given some local conditions, we can glue up in some fixed $\mathcal{F}_\alpha$ and then use the above identity axiom to check that this works on all local conditions. We omit the details. ∎

> **Remark 2.59.** One can use the above result to show that the infinite direct sum of flasque sheaves is a flasque sheaf by taking a direct colimit over the finite direct sums.

We are now ready to show Lemma 2.55.

Proof of Lemma 2.55. Let the directed set by λ so that $\mathcal{F}_\bullet$ is really a functor $\lambda \to \operatorname{Ab}_X$. Then we get injective resolutions

$$\lambda \xrightarrow{\mathcal{F}_\bullet} \operatorname{Ab}_X \xrightarrow{\operatorname{Inj}} \operatorname{Kom}(\operatorname{Ab}_X).$$

Taking the colimit of the produced injective resolution term-wise produces still an exact flasque resolution: it's exact because taking directed colimits is exact, and it's flasque because the directed colimit of flasque sheaves remains flasque. Computing cohomology with this injective resolution will complete the proof because taking global sections commutes with taking directed colimits.

   Let's see this more explicitly. For each $\alpha \in \lambda$, we get some injective resolution

$$0 \to \mathcal{F}_\alpha \to \mathcal{I}^0_\alpha \to \mathcal{I}^1_\alpha \to \mathcal{I}^2_\alpha \to \cdots,$$

and then we produced an injective resolution

$$0 \to \operatorname{colim} \mathcal{F}_\alpha \to \operatorname{colim} \mathcal{I}^0_\alpha \to \operatorname{colim} \mathcal{I}^1_\alpha \to \operatorname{colim} \mathcal{I}^2_\alpha \to \cdots.$$

Now, taking the global sections of these sheaves after colimits is okay because we showed that directed colimits commutes with taking sections above. Thus, we see global sections has

$$0 \to \operatorname{colim} \mathcal{F}_\alpha(U) \to \operatorname{colim} \mathcal{I}^0_\alpha(U) \to \operatorname{colim} \mathcal{I}^1_\alpha(U) \to \operatorname{colim} \mathcal{I}^2_\alpha(U) \to \cdots,$$

and we can now compute cohomology here to finish. ∎

   We now show Theorem 2.51.

> **Theorem 2.51.** Fix a Noetherian topological space X of dimension n. For any abelian sheaf $\mathcal{F}$, we have $H^i(X, \mathcal{F}) = 0$ for $i > n$.

Proof. We show this in steps.

1. To begin, we reduce to the case where X is irreducible, by induction. As such, suppose that $X = Z \cup Y$ where Z is irreducible and closed, where we know the statement for Y. The point is that we can set $U := X \setminus Z$ and use extension by zero. Let $j\colon U \subseteq X$ and $i\colon Z \subseteq X$ be the embeddings, and then any abelian sheaf $\mathcal{F}$ produces an exact sequence

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0,$$

   which we can see is exact by checking stalk-locally. (The important point is that each $x \in X$ lies in exactly one of Z or U.) We now want to hit this with the long exact sequence and use Lemma 2.55. The main claim is that

$$H^\bullet(X, j_! j^* \mathcal{F}) \stackrel{?}{=} H^\bullet(\overline{U}, j_! j^* \mathcal{F}),$$

   which will complete the proof because $\overline{U}$ has fewer irreducible components than X, and we can use the long exact sequence above to finish the needed vanishing. Well, to see the claim, let $k\colon \overline{U} \subseteq X$ be the inclusion, and it suffices to show that $k_* k^*$ sends injective resolutions to flasque resolutions so that we can compute cohomology in the usual way via our injectives.

   Well, taking k_* is certainly sending flasque sheaves to flasque sheaves, so it suffices to check that k^* sends flasque sheaves to flasque sheaves. For this, we see that

$$k^* \mathcal{I}(V) = \operatorname*{colim}_{V' \supseteq V} \mathcal{I}(V')$$

   because we know that taking colimits here is safe because this is a directed colimit. Thus, we can check that this is flasque more locally on our global sections, which is fine because taking colimits preserves surjectivity.

2. We now induct on dimension, assuming that X is Noetherian. If $\dim X = 0$, then being irreducible forces X to be a single point, so taking global sections induces an equivalence of categories $\mathrm{Ab}_X \to \mathrm{Ab}$. This is exact, so all higher cohomology vanishes (namely, strictly after degree 0), as needed.

   Now for our induction, take $\dim X > 0$, and fix $\mathcal{F} \in \mathrm{Ab}_X$. Set

$$B := \bigcup_{U \subseteq X} \mathcal{F}(U),$$

   and we let $\mathcal{A}$ be the set of finite subsets of B; note $\mathcal{A}$ is directed by inclusion. Now, for each $\alpha \in \lambda$, we let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the finitely many sections in α. More formally, for each $a \in B$, we see a belongs to some $U_a \subseteq X$ and so induces a map

$$\mathbb{Z}_{U_a} \to \mathcal{F}$$

   by sending 1 to a; the image here is the subsheaf generated by a. We can generalize this construction to arbitrary $\alpha \in \mathcal{A}$. Now, it suffices to show that each $\mathcal{F}_\alpha$ has vanishing cohomology by taking colimits over $\mathcal{A}$ to get up to $\mathcal{F}$.

   Notably, for any α, α' with $\alpha' \subseteq \alpha$, we have an exact sequence

$$0 \to \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \overline{\mathcal{F}}_{\alpha \setminus \alpha'} \to 0,$$

   where $\overline{\mathcal{F}}_{\alpha \setminus \alpha'}$ is some quotient of $\mathcal{F}_{\alpha \setminus \alpha'}$. Thus, by inducting on $|\alpha|$ (note that the quotient of $\mathcal{F}_{\alpha \setminus \alpha'}$ still has fewer generators), we may assume that $|\alpha| = 1$.

   Thus, we see that we may assume $\mathcal{F}$ is generated by a single section $a \in \mathcal{F}(U)$, which means that we have an exact sequence

$$0 \to \mathcal{K} \to \mathbb{Z}_U \to \mathcal{F} \to 0.$$

Notably, for each $x \in U$, we have an embedding $\mathcal{K}_x \subseteq \mathbb{Z}$ by taking stalks. Now, if $\mathcal{K}_x$ vanishes for all $x \in U$, then $\mathcal{K}$ itself will have to vanish because $\mathcal{K}$ is a sheaf. Otherwise, we can find the least positive integer d such that $\mathcal{R}_x$ embeds into $d\mathbb{Z} \subseteq \mathbb{Z} = (\mathbb{Z}_U)_x$ for all x.

We claim that this d is actually achieved. Well, we can find some small V where $d \in \mathbb{Z}$ will extend to a section on $\mathcal{R}$, so we claim that $\mathcal{R}|_V = d\mathbb{Z}$. However, all the other stalks over U are either vanishing or a subgroup of $d\mathbb{Z}$, so we are forced to have equality by the minimality of d.

In total, our exact sequence now reads

$$0 \to \mathcal{R}_v \to \mathcal{R} \to \mathcal{R}_{U \setminus V} \to 0,$$

but irreducibility tells us that $\mathcal{R}_{U \setminus V}$ will have vanishing cohomology because it has smaller dimension. Thus, we are reduced to showing that $\mathbb{Z}_U$ are going to have vanishing cohomology. But then we have the exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_{X \setminus U} \to 0.$$

The inductive hypothesis deals with $\mathbb{Z}_{X \setminus U}$ as having smaller dimension, and $\mathbb{Z}$ is flasque because it is a constant sheaf on an irreducible space! (Namely, all locally constant functions $V \to \mathbb{Z}$ must be constant because V is irreducible.) Thus, $\mathbb{Z}$ is going to have vanishing cohomology, so $\mathbb{Z}_U$ has vanishing cohomology in large enough dimension, so we are done. ∎

2.6 February 27

We continue.

2.6.1 Cohomology of Quasicoherent Sheaves

We are going to show that quasicoherent sheaves over an affine scheme have vanishing cohomology. This is analogous to the result that Stein spaces have vanishing cohomology.

Theorem 2.60. Fix a Noetherian ring A and affine scheme $X = \operatorname{Spec} A$. If $\mathcal{F}$ is a quasicoherent sheaf, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

The main input to this result is the following result.

Proposition 2.61. Fix a Noetherian ring A and an injective A-module I. Then the quasicoherent sheaf $\widetilde{I}$ on $\operatorname{Spec} A$ is flasque.

Let's see how Proposition 2.61 implies Theorem 2.60.

Proof of Theorem 2.60. We may write $\mathcal{F} := \widetilde{M}$ for some A-module M, and we give M an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots.$$

Now, the functor $\widetilde{\cdot}$ is exact (this can be checked on stalks because localization is flat), so we get the resolution

$$0 \to \widetilde{M} \to \widetilde{I^0} \to \widetilde{I^1} \to \widetilde{I^2} \to \cdots,$$

which is a flasque resolution by Proposition 2.61. Now, computing cohomology with this flasque resolution, we recover the original exact injective resolution for A. In particular, we see that we remain exact and therefore have vanishing higher cohomology. ∎

Thus, we turn to the proof of Proposition 2.61. We require the following algebraic ingredient.

Theorem 2.62 (Krull)**.** Fix a Noetherian ring A. Given an ideal $\mathfrak{a} \subseteq A$ and finite A-modules $M \subseteq N$, then the $\mathfrak{a}$-adic topology on M is induced by that on N. In other words, for any $n > 0$, there exists $n' > 0$ such that $\mathfrak{a}^n M \supseteq (\mathfrak{a}^{n'} N) \cap M$.

Note that $(\mathfrak{a}^{n'} N) \cap M \supseteq \mathfrak{a}^{n'} M$ automatically, so our system of fundamental neighborhoods yielding topology on A to in fact produce the same topology on M.

To show Theorem 2.62, we will need a lemma.

Definition 2.63 (sheaf with support)**.** Given a topological space X and a closed subset $Z \subseteq X$, we define for any sheaf $\mathcal{F}$ of abelian groups the set

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) : s|_p = 0 \text{ for } p \notin Z\}.$$

This is the *sheaf of sections with support contained in Z*.

Definition 2.64. Fix a ring A and an ideal $\mathfrak{a}$. Then we define

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M : a^n m = 0 \text{ for some } n > 0 \text{ for all } a \in \mathfrak{a}\}.$$

Notably, $U \mapsto \Gamma_{Z \cap U}(U, \mathcal{F})$ defines a sheaf on $\mathcal{F}$, which is left exact and produces the cohomology groups $H_Z^i(X, \mathcal{F})$.

In our affine case, we have the following.

Lemma 2.65. Fix a Noetherian ring A. Then for $Z = V(\mathfrak{a})$,

$$\Gamma_Z(X, \widetilde{M}) := \{m \in M : a^n m = 0 \text{ for some } n > 0 \text{ for all } a \in \mathfrak{a}\}.$$

Proof. Quickly, let the right-hand set be $\Gamma_{\mathfrak{a}}(M)$.

In one direction, for $m \in \Gamma_{\mathfrak{a}}(M)$, we see $\mathfrak{a}^n m = 0$ for some $n > 0$ because A is Noetherian (so that $\mathfrak{a}$ is finitely generated, allowing us to find n large enough so that all generators to some sufficiently higher power n lie in $\operatorname{Ann} m$). Thus, if $p \notin Z$, then $\mathfrak{a} \not\subseteq p$, so we see there is $f \in \mathfrak{a} \setminus p$ which is invertible in M_p. In particular, $f^n m = 0$, so $m|_p = 0$, so we see that p is not in the support of m. Thus, the support of m is contained in Z.

In the other direction, suppose $m \in M$ has support in Z. Now, for $p \notin Z$, we have $m|_p = 0$ by hypothesis, so there is $f \in A \setminus p$ such that $f^{n_f} m = 0$ for some $n_f > 0$. In particular, it follows that $\operatorname{Ann} m \not\subseteq p$ as witnessed by f. In other words, we see that $\operatorname{Ann} m \subseteq p$ implies that $p \in Z$ and so $\mathfrak{a} \subseteq p$, so we conclude that

$$\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a} = \bigcap_{p \supseteq \mathfrak{a}} p \subseteq \bigcap_{p \supseteq \operatorname{Ann} m} p = \operatorname{rad} \operatorname{Ann} m.$$

Because $\mathfrak{a}$ is finitely generated (as A is Noetherian), we conclude $\mathfrak{a}^n \subseteq \operatorname{Ann} m$ for some $n > 0$ by looking at each generator. It follows that $m \in \Gamma_{\mathfrak{a}}(M)$. ∎

Remark 2.66. Thus, we would like to show that $H_Z^0(X, \mathcal{F})$ is quasicoherent when $\mathcal{F}$ is quasicoherent; notably, from the above, we will see that $H_Z^0(X, \mathcal{F}) = \widetilde{\Gamma_{\mathfrak{a}}(M)}$ by taking global sections.

Well, we note that we have an exact sequence

$$0 \to H_Z^0(X, \mathcal{F}) \to \mathcal{F} \to j_* \mathcal{F}|_{X \setminus Z},$$

where $j \colon (X \setminus Z) \subseteq X$ is the inclusion. Indeed, exactness here can be checked at stalks. But now this is a scheme-theoretic result: the kernel of a morphism of quasicoherent sheaves is in fact quasicoherent, so it will be enough to see that $j_* \mathcal{F}|_{X \setminus Z}$ is quasicoherent. But in our setting above, the direct image sheaf of a quasicoherent sheaf is still quasicoherent when the relevant scheme morphism is separated and quasicompact (when the source is Noetherian), but of course all these hypotheses are satisfied.

We now investigate this functor $\Gamma_{\mathfrak{a}}$ further.

Lemma 2.67. Fix a Noetherian ring A and an ideal $\mathfrak{a} \subseteq M$. Then the functor $\Gamma_{\mathfrak{a}}$ preserves injectives.

Proof. Fix an injective I, and set $J := \Gamma_{\mathfrak{a}}(I)$. By the homework, it is enough to show that any ideal $\mathfrak{b} \subseteq A$ with a morphism $\varphi \colon \mathfrak{b} \to J$ can be extended to a morphism $A \to J$.

Now, for each $b \in \mathfrak{b}$, we note that $\mathfrak{a}^n \varphi(b) = 0$ for some $n > 0$. In particular, $\mathfrak{a}^n b \subseteq \ker \varphi$. By choosing generators for $\mathfrak{b}$, we may actually assume that $\mathfrak{a}^n \mathfrak{b} \subseteq \ker \varphi$. However, by Theorem 2.62, we may thus find n' such that $\mathfrak{a}^{n'} \cap \mathfrak{b} \subseteq \mathfrak{a}^n \mathfrak{b}$. In total, we have the following diagram.

$$
\begin{array}{ccc}
A & \twoheadrightarrow & \dfrac{A}{\mathfrak{a}^{n'}} \\
\uparrow & & \uparrow \\
\mathfrak{b} & \twoheadrightarrow & \dfrac{\mathfrak{b}}{\mathfrak{a}^{n'} \cap \mathfrak{b}} \longrightarrow J \longrightarrow I
\end{array}
$$

Well, we would like to show that the induced map $A/\mathfrak{a}^{n'} \to I$ factors through J, but this can just be checked directly from the construction. Indeed, the element $x \in A/\mathfrak{a}^{n'}$ in I must be killed by $\mathfrak{a}^{n'}$, so the image of 1 does in fact live in $\Gamma_{\mathfrak{a}}(I) = J$. ∎

We now move towards showing Proposition 2.61.

Lemma 2.68. Fix a Noetherian ring A. Given an injective A-module I, any $f \in A$ has the localization map $I \to I_f$ surjective.

Proof. For each $n > 0$, we set $\mathfrak{b}_n := \operatorname{Ann} f^n$. Notably, this grants the ascending chain of ideals

$$\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \mathfrak{b}_3 \subseteq \cdots,$$

so this must stabilize at some $\mathfrak{b}_r$. Setting $\theta \colon I \to I_f$ to be the localization map, pick any $y/f^n \in I_f$ for $y \in I$, which we would like to be in the image of I.

Now, note there is a map $(f^{n+r}) \to I$ by $f^{n+r} \mapsto f^r y$. This is well-defined by construction of r. In particular, if $af^{n+r} = a'f^{n'+r}$, we can check that $af^r y = a'f^r y$ by computing some annihilators. Thus, the injectivity of I lifts this map to some

$$\psi \colon A \to I$$

such that $\psi\left(f^{n+r}\right) = f^r y$. Notably, setting $z := \psi(1)$, we see $f^{n+r} z = \psi\left(f^{n+r}\right) = f^r y$. Rearranging, we see that z gets sent to y/f^n in I_f, which is what we wanted. ∎

We will continue the proof next class.

2.7 March 1

We continue talking about quasicoherent sheaves.

2.7.1 Finishing Cohomology of Quasicoherent Sheaves

Here is our result.

> **Proposition 2.61.** Fix a Noetherian ring A and an injective A-module I. Then the quasicoherent sheaf $\widetilde{I}$ on $\operatorname{Spec} A$ is flasque.

Proof. We use Noetherian induction on the closed set $Y(I) := \overline{\operatorname{Supp} \widetilde{I}}$. As our base case, if $Y(I)$ is a singleton $\{x\}$, then the closed embedding $i \colon \{x\} \to X$ tells us that

$$\widetilde{I} = i_* i^* \widetilde{I}$$

by checking the obvious map at stalks. And now one can see that any skyscraper sheaf will be flasque by simply looking at it.

In general, we would like to show that $\Gamma(X, \mathcal{I}) \to \Gamma(U, \mathcal{I})$ is surjective by simply always lifting to a global section before restricting. Now, if $Y \cap U = \varnothing$, then $\Gamma(U, \mathcal{I}) = 0$, so there is nothing to say; as such, we may assume that $Y \cap U \neq \varnothing$. Using the standard base on the affine scheme, we may find $f \in A$ such that $D(f) \subseteq U$ and $D(f) \cap Y \neq \varnothing$.

Continuing, we set $Z := V(f)$ so that we have the following commutative diagram.

$$
\begin{array}{ccc}
\Gamma_Z(X, \widetilde{I}) & \longrightarrow & \Gamma_{Z \cap U}(U, \widetilde{I}) \\
\downarrow & & \downarrow \\
\Gamma(X, \widetilde{I}) & \longrightarrow \Gamma(U, \widetilde{I}) \longrightarrow & \Gamma(D(f)), \widetilde{I}
\end{array}
$$

Now, the bottom composite is surjective by Lemma 2.68.

We now proceed directly. Given $t \in \Gamma(U, \widetilde{I})$, we see that we can lift $t|_{D(f)}$ to some $s \in \Gamma(X, \mathcal{I})$ by this surjectivity. But then $s|_U - t$ will have support in $X \setminus D(f) = Z$. As such, if we could just show that the map

$$\Gamma_Z(X, \widetilde{I}) \to \Gamma_{Z \cap U}(U, \widetilde{I})$$

is surjective, then we could lift $s|_U - t$ to some $r \in \Gamma_Z(X, \widetilde{I})$ so that $(s - r)|_{D(f)} = s|_{D(f)} = t$, but $(s - r)|_U - t = (s|_U - t) - r|_U = 0$. Thus, we want to show that the above map is surjective, which amounts to showing that the sheaf $\Gamma_Z(-, \mathcal{I}) = \widetilde{\Gamma_{(f)}(I)}$ is flasque, where this equality is by Remark 2.66.

However, the support of $\widetilde{\Gamma_{(f)}(I)}$ is contained in $Z \cap Y$, where it is contained in Y by construction of I and contained in Z by construction of (f). But now we are done by Noetherian induction because $Z \cap Y$ is a smaller closed subset. Explicitly, we may suppose the result for all strictly smaller closed subsets than Y, for otherwise we would be able to build some infinite descending chain of closed sets in $\operatorname{Spec} A$, which is not possible because A is Noetherian. ∎

> **Corollary 2.69.** Fix a Noetherian scheme X. Then any quasicoherent sheaf $\mathcal{F}$ can be embedded into a flasque quasicoherent sheaf.

Proof. Over affine opens, we can embed into an injective module, which is flasque by the above. Taking the direct image sheaf to X and then taking the product produces the desired flasque sheaf and embedding. ∎

> **Remark 2.70.** Professor Barrett is unsure if injectives in the category of quasicoherent sheaf are actually acyclic for $\Gamma(X, -)$.

2.7.2 Serre's Criterion for Affines

Let's give a cohomological criterion for being affine.

> **Theorem 2.71** (Serre). Fix a Noetherian scheme X. Then the following are equivalent.
>
>   (a) X is affine.
>
>   (b) $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and quasicoherent sheaves $\mathcal{F}$ on X.
>
>   (c) $H^1(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ of ideals.

For this, we want the following lemma.

> **Lemma 2.72.** A scheme X is affine if and only if there exist global sections $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ which generate $\Gamma(X, \mathcal{O}_X)$ such that all the subschemes $D(f_i)$ are affine.

Proof. This is by Exercise II.2.17 of [Har77]. We did this for homework last semester. ∎

Now let's show the theorem.

Proof of Theorem 2.71. We show our implications in sequence. Note (a) implies (b) by Theorem 2.60, and (b) implies (c) with no work, so the hard part is showing (c) implies (a). We would like to use Lemma 2.72. Well, for each $p \in X$, we let $p \in U$ be some affine open neighborhood and set $Y \coloneqq X \setminus U$. Then we get some exact sequence

$$0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(p) \to 0,$$

where we can check exactness at stalks. Now, by the long exact sequence we see

$$\Gamma(X, \mathcal{I}_Y) \to k(p) \to H^1(X, \mathcal{I}_{Y \cup \{p\}}),$$

where the last cohomology group vanishes by hypothesis. In total, our surjectivity produces $f \in \Gamma(X, \mathcal{I}_Y) \subseteq \Gamma(X, \mathcal{O}_X)$ such that $f|_p = 1$ in $k(p)$. Further, f has support contained in Y, so $X_f \subseteq U$, so $X_f = X_f \cap U = U_f$ is an affine open neighborhood of U.

    Thus, we have produced an affine open cover $\{X_f\}$ of X. Because X is Noetherian, we see that X is quasicompact, so we can reduce this to a finite affine open cover $\{X_{f_i}\}_{i=1}^r$. By Lemma 2.72, it suffices to show that the f_i generate $\Gamma(X, \mathcal{O}_X)$. Well, each f_i produces some map $\mathcal{O}_X \to \mathcal{O}_X$, so we get a map

$$\pi \colon \mathcal{O}_X^r \to \mathcal{O}_X$$

given by $(a_1, \ldots, a_r) \mapsto a_1 f_1 + \cdots + a_r f_r$. Notably, because the X_{f_i} cover X, we see that this map is surjective by checking at stalks because we do not have $f_i|_p \in \mathfrak{m}_p$ for all i at any fixed point p. Letting $\mathcal{F} \coloneqq \ker \pi$, we get the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^r \xrightarrow{\pi} \mathcal{O}_X \to 0,$$

so it now suffices to show that $H^1(X, \ker \pi)$ because this will give us surjectivity on the right upon taking global sections. Well, we consider the filtration

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}_X^r \supseteq \mathcal{F} \cap \mathcal{O}_X^{r-1} \supseteq \cdots \supseteq \mathcal{F} \cap \mathcal{O}_X \supseteq 0.$$

Note that coherent sheaves are closed under taking kernels and finite products, so each of these intersections remains a coherent sheaf. Well, we do this by induction. Note $\mathcal{F} \cap \mathcal{O}_X$ is a coherent sheaf of ideals, so its H^1 vanishes by hypothesis; this is our base case. To go up, we look at the following short exact sequence.

$$0 \to \mathcal{F} \cap \mathcal{O}_X^n \to \mathcal{F} \cap \mathcal{O}_X^{n+1} \to \mathcal{C}_n \to 0.$$

Thus, it now suffices to show that $\mathcal{C}_n$ has vanishing H^1 by the long exact sequence. Well, we note that we have a morphism of exact sequences.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} \cap \mathcal{O}_X^n & \longrightarrow & \mathcal{F} \cap \mathcal{O}_X^{n+1} & \longrightarrow & \mathcal{C}_n & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^n & \longrightarrow & \mathcal{O}_X^{n+1} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}
$$

In particular, we see that $\mathcal{C}_n$ is coherent by the top, and it embeds into $\mathcal{O}_X$ by the Snake lemma, so it is a coherent sheaf of ideals and therefore has vanishing H^1 by hypothesis. $\blacksquare$

COMPUTING COHOMOLOGY

3.1 March 3

Here we go.

3.1.1 Introducing Čech Cohomology

One point of Čech cohomology is for computations, but we will see a little more. Fix a topological space X and open cover $\mathcal{U} := \{U_\alpha\}_{\alpha \in \lambda}$, and we fix a total order $\leq$ on λ. Now, given a sheaf of abelian groups $\mathcal{F}$ on X, we set

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0, \ldots, \alpha_p}),$$

where

$$U_{\alpha_0, \ldots, \alpha_p} := \bigcap_{i=0}^{p} U_{\alpha_i}.$$

In other words, an element of $C^p(\mathfrak{U}, \mathcal{F})$ is an element $s_{\alpha_0, \ldots, \alpha_p} \in \mathcal{F}(U_{\alpha_0, \ldots, \alpha_p})$ for each $\alpha_0 < \cdots < \alpha_p$. For our differential, we set

$$(d^p s)_{\alpha_0 < \cdots < \alpha_{p+1}} := \sum_{q=0}^{p+1} (-1)^k s_{\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{q+1}},$$

where the hat denotes omission. Here, we have implicitly restricted all sections to $U_{\alpha_0, \ldots, \alpha_{p+1}}$.

> **Example 3.1.** Give X an open covering by $\{U_0, U_1, U_2\}$. Then $C^1(\mathfrak{U}, \mathcal{F})$ is a choice of sections $s_{01} \in \mathcal{F}(U_0 \cap U_1)$ and $s_{02} \in \mathcal{F}(U_0 \cap U_2)$ and $s_{12} \in \mathcal{F}(U_1 \cap U_2)$. Here, we have $(d^1 s)_{012} = s_{12} - s_{02} + s_{01}$. Note that all these sections make sense over U_{012}.

> **Remark 3.2.** Notably, if $p > \#\lambda$, then our chain complex term vanishes.

One can check that $d^2 = 0$ by hand. Roughly speaking, this amounts to the fact that there are two ways to remove two fixed indices i and j from the tuple $(\alpha_0, \ldots, \alpha_{p+2})$, which produce different signs. This grants the following definition.

> **Definition 3.3** (Čech cohomology). Fix everything as above. The *Čech cohomology* $\check{H}(\mathfrak{U}, \mathcal{F})$ groups are the cohomology groups of the complex $C(\mathfrak{U}, \mathcal{F})$.

Remark 3.4. These cohomology functors do not give a δ-functor. For example, if our open cover is given by $\mathfrak{U} = \{X\}$, then the complex is given by

$$\Gamma(X, \mathcal{F}) \to 0 \to 0 \to \cdots ,$$

whose cohomology does not give a long exact sequence.

Exercise 3.5. We compute the cohomology of $X = \mathbb{P}^1_k$ where the sheaf is $\mathcal{F} = \Omega^1_X$.

Proof. Let our open cover be given by $\mathfrak{U} = \{U_0, U_\infty\}$ be the standard affine charts around 0 and ∞.

- Note $C^0(\mathfrak{U}, \mathcal{F})$ consists of a choice of a section in $\mathcal{F}(U_0) = k[x] \, dx$ and a section in $\mathcal{F}(U_\infty) = k[y] \, dy$.

- Note $C^1(\mathfrak{U}, \mathcal{F})$ consists of a choice of a section in $\mathcal{F}(U_0 \cap U_1) = k[x, 1/x] \, dx$.

In total, we want to compute the cohomology of the sequence

$$0 \to k[x] \, dx \oplus k[y] \, dy \to k[x, 1/x] \, dx \to 0,$$

where the middle map is given by $(f(x) \, dx, g(y) \, dy) \mapsto (-g(1/x)/x^2 \, dx - f(x) \, dx)$.

- Notably, $-g(1/x)/x^2$ is a polynomial of negative degree, so there is no way for cancellation to occur, so $\check{H}^0(\mathfrak{U}, \mathcal{F}) = 0$.

- By letting f and g vary, we note that everything in $k[x, 1/x] \, dx$ is hit except for the terms of the form $c/x \, dx$ for $c \in k^\times$, so $\check{H}^1(\mathfrak{U}, \mathcal{F}) = k$. ∎

Exercise 3.6. We compute the cohomology of S^1 where the sheaf is $\mathcal{F} = \mathbb{Z}$.

Proof. Use the open cover $\mathfrak{U} = \{U_0, U_1\}$ of two large arcs of S^1 fully covering S^1.

- We see that $C^0(\mathfrak{U}, \mathbb{Z})$ is a choice of a section on U_0 and a section on U_1, both of which are connected, so we are looking at $\mathbb{Z} \oplus \mathbb{Z}$.

- Similarly, $C^0(\mathfrak{U}, \mathbb{Z})$ is a choice of a section of $U_0 \cap U_1$, which consists of two disconnected arcs, so this is also $\mathbb{Z} \oplus \mathbb{Z}$.

In total, we want the cohomology of the complex given by

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0,$$

where the middle map is given by $(a, b) \mapsto (b - a, b - a)$. As such, we can see $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \mathbb{Z}$ and $\check{H}^1(\mathfrak{U}, \mathcal{F}) = \mathbb{Z}$ by direct computation. ∎

3.1.2 Abstract Remarks

To understand what Čech cohomology is doing, we want the following result.

Theorem 3.7. Fix a topological space X and open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$ where λ is total-ordered. Further, suppose $\mathcal{F}$ is a sheaf on X which is acyclic on the restrictions U_α for $\Gamma(U_\alpha, -)$. Then

$$\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) = H^\bullet(X, \mathcal{F}).$$

> **Example 3.8.** If X is a Noetherian scheme and $\mathcal{F}$ is a quasicoherent sheaf, then we can let $\mathfrak{U}$ be any affine open cover by Theorem 2.60.

Let's give some more general remarks. Roughly speaking, there is a spectral sequence

$$E_2^{p,q} := \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

where $\underline{H}^p(\mathcal{F})(V) = H^q(V, \mathcal{F})$ is a presheaf. Now, Čech cohomology is a tool for computing some cosimplicial limits: set Δ to be the "simplex" category whose objects are $[n] := \{0, \dots, n\}$, and we consider the order preserving maps in our category of simplexes. For example, there are two order-preserving maps $[0] \to [1]$, and more generally there are some canonical maps $[n] \to [n+1]$ depending on where we choose to pause.

> **Remark 3.9.** We have seen this in some context already. Namely, we are more or less considering ways to go back and forth along $U^{n+1} \to U^n$.

There is some "Dold–Kan" correspondence, which gives an equivalence of between cosimplicial objects $\operatorname{CoSimp}(\mathcal{A})$ of an abelian category $\mathcal{A}$ and the cochains of nonnegative degree in $\operatorname{CoCh}^{\geq 0}(\mathcal{A})$.

Let's be more concrete about this discussion to see where our cohomology comes from. Let $\{U_\alpha\}_{\alpha \in \lambda}$ be an open cover of X. Then we get a cosimplicial object in the category of abelian groups given as follows.

$$\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U} \times \mathcal{U}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathcal{F}(\mathcal{U} \times \mathcal{U} \times \mathcal{U}) \longrightarrow \cdots$$

Here, $\mathcal{U}$ is the disjoint union of the open sets in our open cover $\{U_\alpha\}_{\alpha \in \lambda}$. In particular, under the Dold–Kan correspondence, the above "cosimplicial abelian group" goes to the Čech complex, and the limit of these is precisely $\mathcal{F}(X)$ because $\mathcal{F}$ is a sheaf. However, $\mathcal{F}$ is a sheaf, so we can actually identify $\mathcal{F}(X)$ with the equalizer of the maps

$$\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U} \times \mathcal{U}).$$

One can now upgrade the above ideas to give the desired equality in Theorem 3.7, for example replacing our limit with a homotopy limit.

3.2 March 6

Today we prove Theorem 3.7. As an aside, our goal is to compute the cohomology of projective space.

3.2.1 Čech Cohomology is Correct

Throughout, X will be a topological space, $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$ will be an open cover where λ is total-ordered, and $\mathcal{F}$ is a sheaf on X. Recall our main result, which we will not prove in full generality but instead just in our special case.

> **Theorem 3.7.** Fix a topological space X and open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$ where λ is total-ordered. Further, suppose $\mathcal{F}$ is a sheaf on X which is acyclic on the restrictions U_α for $\Gamma(U_\alpha, -)$. Then
> $$\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) = H^\bullet(X, \mathcal{F}).$$

We begin with the following lemma.

> **Lemma 3.10.** Fix a topological space X and open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$ where λ is total-ordered. Then $\check{H}^0(\mathfrak{U}, \mathcal{F}) = H^0(X, \mathcal{F})$.

Proof. Let $C^\bullet(\mathfrak{U}, \mathcal{F})$ denote the Čech complex. Then $\check{H}^0(\mathfrak{U}, \mathcal{F})$ is the kernel of the map $d^0 \colon C^0(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{F})$. Well, $s \in C^0(\mathfrak{U}, \mathcal{F})$ lives in the kernel of this map if and only if each $\alpha < \beta$ in λ has

$$0 = (ds)_{\alpha\beta} = s_\alpha|_{U_\alpha \cap U_\beta} - s_\beta|_{U_\alpha \cap U_\beta},$$

so we are looking at exactly the coherence condition for our $\{s_\alpha\}_{\alpha \in \lambda}$ to assemble into a global section in $\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$. $\blacksquare$

Next we want to show that our Čech complex is a resolution we can use to compute cohomology, but we have to sheafify our complex first. Fix X, $\mathcal{F}$, and $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$ as usual. We then define

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} j_*(\mathcal{F}|_{U_{\alpha_0, \dots, \alpha_p}}),$$

where throughout j will denote the open embedding of the obvious open set. (Yes, the notation is ambiguous, but we already have too many indices floating around.) Note that this is a sheaf because restrictions, direct images, and products of sheaves produce sheaves. Notably, for any open $V \subseteq U$, we have

$$\Gamma(V, \mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = C^p(\{U_\alpha \cap V\}_\alpha, \mathcal{F}),$$

by construction, so we can define our differential as the one coming from $C^p(\{U_\alpha \cap V\}_\alpha, \mathcal{F})$. This makes us a complex by checking locally on open sets.

Now here is our resolution result.

> **Lemma 3.11.** Fix everything as above. Then
>
> $$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$
>
> is an exact resolution of $\mathcal{F}$ with identity map chain homotopic to 0 on stalks.

The game plan is to show that the identity map on the complex is chain homotopic to 0, implying that we have an acyclic complex, which lets us compute cohomology from this. Anyway, let's prove the lemma.

Proof. This exact in degree 0 by the previous lemma.

Now, we may check exactness for positive degrees on stalks; it is enough to show that we have a chain homotopy between the identity and 0 because we already have a complex, so this establishes that every element of our kernel will arise from image. Thus, for our $x \in X$, we find β such that $x \in U_\beta$. Then for each index p, we define $h \colon \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x \to \mathcal{C}^{p-1}(\mathfrak{U}, \mathcal{F})_x$ as follows: with $s_x \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$, we may say $s_x = s|_x$ for some s defined in an open neighborhood V of x contained in U_β. As such, we define

$$h(s_x)_{\alpha_0, \dots, \alpha_{p-1}} := s_{\beta, \alpha_0, \dots, \alpha_{p-1}},$$

where we are defining $s_{\alpha_0, \dots, \alpha_p}$ to vanish if any of the indices fail to be distinct and to equal $(\operatorname{sgn} \sigma) s_{\sigma\alpha_0, \dots, \sigma\alpha_p}$ if the indices are distinct, where $\sigma \in S_{p+1}$ is the permutation placing $(\alpha_0, \dots, \alpha_p)$ so that $\sigma\alpha_0 < \cdots < \sigma\alpha_{p+1}$. Note that the above definition is compatible with restriction of s, so h is in fact defined on stalks.

It remains to show that $dh + hd = \operatorname{id}$, where d is our differential. As such, given $s_x \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$, we want to show that

$$(dh + hd)(s)_{\alpha_0, \dots, \alpha_p} \overset{?}{=} 0.$$

There are two cases.

- Suppose β is one of the indices. On one hand, $hd(s)_{\alpha_0, \dots, \alpha_p}$ will vanish automatically because we are adding in β to our index list, so we have a repeat index, so this vanishes. On the other hand, we want to see that $dh(s)_{\alpha_0, \dots, \alpha_p}$ vanishes. Well, we compute

$$dh(s)_{\alpha_0, \dots, \alpha_p} = \sum_{k=0}^p (-1)^k h(s)_{\alpha_0, \dots, \widehat{\alpha_k}, \dots, \alpha_p} = \sum_{k=0}^p (-1)^k s_{\beta, \alpha_0, \dots, \widehat{\alpha_k}, \dots, \alpha_p}.$$

  However, all these terms vanish unless with removed α_k index was the one equal to β (of which there is exactly one), so transposing appropriately, we see that this last sum compresses down to just $s_{\alpha_0, \dots, \alpha_p}$. Thus, $dh + hd = \operatorname{id}$ here.

- Suppose β is none of the indices. Then we must do some computation. On one hand, we see

$$hd(s)_{\alpha_0,\ldots,\alpha_p} = d(s)_{\beta,\alpha_0,\ldots,\alpha_p} = \sum_{k=0}^{p+1}(-1)^k s_{\beta,\alpha_0,\ldots,\widehat{\alpha_\bullet},\ldots,\alpha_p},$$

  where the removed term in the last sum depends on where β is placed among the remaining $\alpha_\bullet$s. On the other hand, we see

$$dh(s)_{\alpha_0,\ldots,\alpha_p} = \sum_{k=0}^{p}(-1)^k h(s)_{\alpha_0,\ldots,\widehat{\alpha_k},\ldots,\alpha_p} = \sum_{k=0}^{p-1} s_{\beta,\alpha_0,\ldots,\widehat{\alpha_k},\ldots,\alpha_p}.$$

  Summing the above two expressions, the signs perfectly cancel except there is an extra term of the previous sum, which namely is where β is removed, totaling to $s_{\alpha_0,\ldots,\alpha_p}$, as needed.

Thus, we have $dh + hd$ here, completing the proof of exactness at our stalk. Explicitly, for any element $s \in \ker d$, we see $s = dh(s) + hd(s) = dh(s)$, so $s \in \operatorname{im} d$. ∎

Our next step is to build some commonly acyclic sheaves.

> **Lemma 3.12.** Fix everything as above. If $\mathcal{F}$ is flasque, then $\mathcal{F}$ is acyclic for $\check{H}^\bullet(\mathfrak{U}, -)$.

Proof. Note that restriction sends flasque sheaves to flasque sheaves, as do j_* and products, so we see that each $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ is flasque by definition. As such,

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$

is a flasque resolution of $\mathcal{F}$. So taking global sections and computing cohomology with our usual sheaf cohomology, we see

$$0 = H^i(X, \mathcal{F}) = H^p\Gamma(X, \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})) = H^p C^\bullet(\mathfrak{U}, \mathcal{F}) = \check{H}^p(X, \mathcal{F}),$$

which is what we wanted. ∎

To continue, we recall the following fact.

> **Remark 3.13.** Given objects A and B in an abelian category equipped with injective resolutions $A \to I^\bullet$ and $B \to J^\bullet$, then a morphism $\varphi \colon A \to B$ extends to a map on the injective resolutions unique up to homotopy. However, this proof really only requires $J^\bullet$ to be injective!

From Remark 3.13, we note that the identity on $\mathcal{F}$ induces maps on our resolutions as follows, where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 & \longrightarrow & \cdots \\
& & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) & \longrightarrow & \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) & \longrightarrow & \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) & \longrightarrow & \cdots
\end{array}
$$

As such, we induce a unique map on cohomology $\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) \to H^\bullet(X, \mathcal{F})$.

> **Theorem 3.14.** Fix everything as above, and suppose we have a separated Noetherian scheme X and a quasicoherent sheaf $\mathcal{F}$ on X. Then the above maps are isomorphisms.

3.3   March 8

We continue discussing Čech cohomology.

3.3.1   Finishing Čech Cohomology's Correctness

Here is the precise statement we will show.

> **Theorem 3.15.** Fix a Noetherian separated scheme X with affine open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$. For any quasicoherent sheaf $\mathcal{F}$ on X, we have $\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ for any index $p \geq 0$.

Proof. Because X is separated, the intersection of any two affine open subschemes U and V is still an affine open subscheme because $U \cap V = \Delta^{-1}(U \times V)$ where $\Delta \colon X \times X \to X$ is the diagonal map. (Notably, Δ is an affine morphism because it is a closed embedding.)

Now, we apply dimension-shifting. By checking affine-locally, we may embed $\mathcal{F}$ into a flasque quasicoherent sheaf $\mathcal{G}$; letting $\mathcal{R}$ denote the cokernel, we have the exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0.$$

Now, for any indices $\alpha_0 < \cdots < \alpha_p$, we get the exact sequence

$$0 \to \mathcal{F}(U_{\alpha_0,\ldots,\alpha_p}) \to \mathcal{G}(U_{\alpha_0,\ldots,\alpha_p}) \to \mathcal{R}(U_{\alpha_0,\ldots,\alpha_p}) \to 0$$

because $U_{\alpha_0,\ldots,\alpha_p}$ is affine and everything is a quasicoherent sheaf. Taking the product over all such $(p+1)$-tuples of indices and looping over all indices, we get an exact sequence of Čech cohomology

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{G}) \to C^\bullet(\mathfrak{U}, \mathcal{R}) \to 0.$$

This exact sequence of complexes now produces a long exact sequence in cohomology as follows.

$$\cdots \longrightarrow \check{H}^p(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{H}^p(\mathfrak{U}, \mathcal{G}) \longrightarrow \check{H}^p(\mathfrak{U}, \mathcal{R})$$

$$\check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) \longleftarrow \check{H}^{p+1}(\mathfrak{U}, \mathcal{G}) \longrightarrow \check{H}^{p+1}(\mathfrak{U}, \mathcal{R}) \longrightarrow \cdots$$

Using the transition maps $\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) \to H^\bullet(X, \mathcal{F})$ given last class, we get the following morphism of exact sequences.

$$
\begin{array}{ccccccccc}
\check{H}^p(\mathfrak{U}, \mathcal{F}) & \longrightarrow & \check{H}^p(\mathfrak{U}, \mathcal{G}) & \longrightarrow & \check{H}^p(\mathfrak{U}, \mathcal{R}) & \longrightarrow & \check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) & \longrightarrow & 0 \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow & & \\
H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{G}) & \longrightarrow & H^p(X, \mathcal{R}) & \longrightarrow & H^{p+1}(X, \mathcal{F}) & \longrightarrow & 0
\end{array}
$$

(Note that $\check{H}^{p+1}(\mathfrak{U}, \mathcal{G}) = 0$ by Lemma 3.12. Additionally, we are natural in the $\mathcal{R} \to \mathcal{F}$ boundary map because the transition maps we defined actually come from a complex, so we are saying that a morphism of complexes is giving rise to a morphism of long exact sequences, which we know.) By the inductive hypothesis (note that our base case is degree-0 by Lemma 3.10), the three morphisms on the left are isomorphisms. Thus, the desired morphism $\check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) \to H^{p+1}(X, \mathcal{F})$ is an isomorphism by the Five lemma or something, which completes our induction. ∎

3.3.2   Review of Projective Space

For a Noetherian ring A, we will be computing the cohomology of $\mathbb{P}_A^r := \operatorname{Proj} A[x_0, \ldots, x_r]$. We will be using the usual facts about the Proj construction. In particular, for a graded ring S, for any homogeneous $f \in S_+$ of positive degree, we have the distinguished open set $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$, which has structure sheaf given by $\operatorname{Spec}(S_f)_0$.

Example 3.16. We work with $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_r]$. If we take $f = x_i$ for some index i, then the distinguished open subscheme is given by

$$D_+(x_i) = \operatorname{Spec}(A[x_0, \ldots, x_n]_{x_i})_0 = \operatorname{Spec} \frac{A[x_{0/i}, \ldots, x_{r/i}]}{(x_{i/i} - 1)}$$

where the isomorphism is given by $x_j/x_i \mapsto x_{j/i}$.

Now, for a graded S-module M, then one can follow the above construction almost verbatim to produce a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Proj} S$. Namely, for homogeneous $f \in S_+$ of positive degree, we set $\widetilde{M}|_{D_+(f)} := \widetilde{(M_f)_0}$. As such, one can check that any $\mathfrak{p} \in \operatorname{Proj} S$ yields

$$\widetilde{M}_{\mathfrak{p}} := (M_{\mathfrak{p}})_0.$$

Here, the localization $M_{\mathfrak{p}}$ only permits denominators which are homogeneous elements outside $\mathfrak{p}$.

Remark 3.17. If S is Noetherian, and M is finitely generated, then one can check that $\widetilde{M}$ is coherent.

At the end of the day, we care about the quasicoherent sheaves $\mathcal{O}(n) := \widetilde{S(n)}$, where $S(n)$ is the S-module S with grading given by $S(n)_i = S_{n+i}$.

Example 3.18. Notably, in our case of $S = A[x_0, \ldots, x_n]$, we see that $\mathcal{O}(n)$ is a line bundle by checking affine-locally, where $\mathcal{O}(n)$ just looks like $\operatorname{Spec} k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$ on the distinguished open subscheme $D_+(x_i)$.

Our goal is to compute the cohomology of these quasicoherent sheaves $\mathcal{O}(n)$. Here is a start.

Proposition 3.19. Fix a ring A. We compute $H^0\left(\mathbb{P}^r_A, \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(n)\right) = S$.

Proof. Fix some n, and we will compute $H^0(\mathbb{P}^r_A, \mathcal{O}(n)) = \Gamma(\mathbb{P}^r_A, \mathcal{O}(n))$; because cohomology commutes with direct sums, this will complete the proof. Checking affine-locally, we note that a global section $t_i \in \Gamma(D_+(x_i), \mathcal{O}(n))$ is simply of an element which is homogeneous of degree n in S_{x_i} by the definition of $\mathcal{O}(n)$ as $\widetilde{S(n)}$. Now to glue our $\{t_i\}_{i=1}^n$ together, we are requiring that

$$t_i|_{D_+(x_i x_j)} = t_j|_{D_+(x_i x_j)}$$

which means that t_i and t_j have the same image in $S_{x_i x_j}$. Patching these together, we are saying that we want our global section to live in the intersection

$$\bigcap_{i=1}^n S_{x_i},$$

which means that we want an element of total degree n with nonnegative degree in each coordinate. Thus, we are just looking at the degree-n homogeneous polynomials in S. Thus, $H^0(\mathbb{P}^r_A, \mathcal{O}(n))$ consists of the degree-n homogeneous polynomials in $A[x_0, \ldots, x_n]$. ∎

3.4   March 10

Today we talk about cohomology of projective space.

3.4.1 Cohomology of Projective Space

Fix a Noetherian ring A, and we work with $\mathbb{P}_A^r = \operatorname{Proj} A[x_0, \ldots, x_r]$. Here is our result; note that we are looking at our cohomology as A-modules, which we know is legal because we can take flasque resolutions and value everything in Mod_A instead of merely Ab.

> **Theorem 3.20.** Fix a Noetherian ring A, and set $X := \mathbb{P}_A^r$ with $r > 0$. Further, set $S := A[x_0, \ldots, x_n]$ and $\mathcal{F} := \bigoplus_{r \in \mathbb{Z}} \mathcal{O}(n)$.
>
> (a) We have $H^0(X, \mathcal{F}) = S$. In other words, $H^0(X, \mathcal{O}(n)) = S_n$ for any n.
>
> (b) For any $0 < i < r$, we have $H^i(X, \mathcal{F}) = 0$.
>
> (c) We have $H^r(X, \mathcal{O}(-r-1)) \cong A$.
>
> (d) For all n, there is a perfect A-pairing
> $$H^0(X, \mathcal{O}(n)) \otimes_A H^r(X, \mathcal{O}(-n-r-1)) \to H^r(X, \mathcal{O}(-r-1)) = A.$$
> In other words, $H^0(X, \mathcal{O}(n)) \cong \operatorname{Hom}_A(H^r(X, \mathcal{O}(-n-r-1)), A)$.

> **Remark 3.21.** Because $\mathbb{P}_A^r$ can be covered by $r+1$ affine open subschemes in the standard affine open cover $\mathfrak{U}$, we see
> $$H^i(X, \mathcal{F}) = \check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$$
> for $i > r$ by definition of Čech cohomology. Thus, we have no higher cohomology than given in Theorem 3.20.

> **Remark 3.22.** To properly state the perfect pairing, we note that a global section $s \in H^0(X, \mathcal{O}(n))$ produces a map $s \colon \mathcal{O}(-n-r-1) \to \mathcal{O}(-r-1)$ by multiplication and checking degrees, which by functoriality produces a map $s \colon H^r(X, \mathcal{O}(-n-r-1)) \to H^r(X, \mathcal{O}(-r-1))$. The perfect pairing is asserting that the induced map
> $$H^0(X, \mathcal{O}(n)) \cong \operatorname{Hom}_A(H^r(X, \mathcal{O}(-n-r-1)), A)$$
> is an isomorphism. Notably, by adjunction, we are really talking about a map $s \otimes t \mapsto st$, which is what is included in the statement.

Proof of Theorem 3.20. We showed (a) last class in Proposition 3.19. Now, we use Čech cohomology. Set $U_i := D_+(x_i) \cong \operatorname{Spec} A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$ so that $\mathfrak{U} := \{U_i\}$ is an open cover of X. Notably, we see that
$$U_{i_0 \cdots i_p} = D_+(x_{i_0} \cdots x_{i_p}).$$
Computing our sections now, we see
$$\Gamma(U_{i_0 \cdots i_p}, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(U_{i_0 \cdots i_p}, \mathcal{O}(n)) = \bigoplus_{})n \in \mathbb{Z} \Gamma(D_+(x_{i_0} \cdots x_{i_p}), \mathcal{O}(n)).$$

Now, accounting for our shifting, $\Gamma(D_+(x_{i_0} \cdots x_{i_p}), \mathcal{O}(n))$ is asking for degree-0 elements of $(S_n)_{x_{i_0} \cdots x_{i_p}}$, which are really the degree-n elements of $S_{x_{i_0} \cdots x_{i_p}}$. As such,
$$\Gamma(U_{i_0 \cdots i_p}, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} (S_{x_{i_0} \cdots x_{i_p}})_n = S_{x_{i_0} \cdots x_{i_p}}.$$

As such, our Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ looks like
$$0 \to \prod_{i=0}^r S_{x_i} \xrightarrow{d^0} \prod_{0 < i_0 < i_1 \le r} S_{x_{i_0} x_{i_1}} \xrightarrow{d^1} \cdots \to \prod_{0 < i_0 < \cdots < i_r \le r} S_{x_{i_0} \cdots x_{i_r}} \xrightarrow{d^{r-1}} S_{x_0 \cdots x_r} \to 0. \tag{3.1}$$

For example, $\operatorname{im} d^{r-1}$ is the span of the monomials $x_0^{\nu_1} \cdots x_r^{\nu_r}$ where at least one of the $\nu_\bullet$ is nonnegative because it must arise from some $S_{x_0 \cdots \widehat{x_k} \cdots x_r}$, which means that x_k is not used. Now, we can compute

$$\check{H}^r(\mathfrak{U}, \mathcal{F}) = \frac{S_{x_0 \cdots x_r}}{\operatorname{im} d^{r-1}}.$$

In particular, the numerator is a free A-module generated by monomials $x_0^{\nu_1} \cdots x_r^{\nu_r}$, but the denominator is a free A-module generated by monomials with at least one positive coordinate. Thus, by taking the quotient appropriately, $\check{H}^r(\mathfrak{U}, \mathcal{F})$ is a free A-module generated by $x_0^{\nu_0} \cdots x_r^{\nu_r}$ where all the terms are (strictly) negative!

For example, if we insist on looking at $\check{H}^r(\mathfrak{U}, \mathcal{O}(-r-1))$, then we are looking at an A-module freely generated by $x_0^{\nu_0} \cdots x_r^{\nu_r}$ where each $\nu_\bullet$ is negative, and $\nu_0 + \cdots + \nu_r = -r-1$. However, the only way to do this is for everything to be -1, so we see

$$H^r(X, \mathcal{O}(-r-1)) = \check{H}^r(\mathfrak{U}, \mathcal{O}(-r-1)) = A x_0^{-1} \cdots x_r^{-1}.$$

This is (c).

One can see (d) similarly. In particular, $H^0(X, \mathcal{O}(n))$ has an A-basis of monomials $x_0^{\nu_0'} \cdots x_r^{\nu_r'}$ with $\nu_\bullet' \geq 0$ and total degree n; on the other hand, $H^r(X, \mathcal{O}(-n-r-1))$ has an A-basis of monomials $x_0^{\nu_0} \cdots x_r^{\nu_r}$ with $\nu_\bullet \leq 0$ and total degree $-n-r-1$, so we can simply multiply these homogeneous polynomials to produce a homogeneous polynomial of degree $-r$, which we can put into $H^r(X, \mathcal{O}(-r-1))$. In other words, we are sending

$$x_0^{\nu_0'} \cdots x_r^{\nu_r'} \cdot x_0^{\nu_0} \cdots x_r^{\nu_r} \mapsto x_0^{\nu_0'+\nu_0} \cdots x_r^{\nu_r'+\nu_r} \quad (\operatorname{mod} \operatorname{im} d^{r-1}).$$

It remains to show that we are producing a perfect pairing in this way. In other words, given some morphism $H^r(X, \mathcal{O}(-n-r-1)) \to H^r(X, \mathcal{O}(-r-1))$, it must arise from some unique element of S via the above multiplication; it suffices to check this on our basis of monomials. One can see this from a direct computation: given some monomial $x_0^{\nu_0} \cdots x_r^{\nu_r}$ satisfying

$$\sum_{i=0}^{r} (-\ell_i - 1) = n,$$

we need to produce a monomial $x_0^{\nu_0'} \cdots x_{\nu_r'}$ such that the corresponding products land us in degree $-r-1$. However, we see that we must have $0 \leq \nu_i \leq -\nu_i - 1$ everywhere to ensure that we land in a monomial with negative degrees everywhere (up to some residue class—anything with a nonnegative degree somewhere will vanish entirely). But then equalities must hold everywhere to get the correct total degree, which makes our chosen monomial unique.

Lastly, we must show (b). Let's compute $H^i(X, \mathcal{F})_{x_r}$. Here, our Čech complex (3.1) can be localized by x_r to compute $H^i(\mathfrak{U}, \mathcal{F})_{x_r}$ because localization is an exact functor and therefore preserves quotients. However, we can also view this localization as restricting the open cover $\mathfrak{U}$ by intersecting everything to $D_+(x_r)$, so we are really looking at $H^i(X, \mathcal{F})_{x_r} = H^i(\mathfrak{U}, \cap D_+(x_r), \mathcal{F}|_{D_+(x_r)})$, which vanishes by Theorem 2.60.

However, we claim that

$$H^i(X, \mathcal{F}) \xrightarrow{x_r} H^i(X, \mathcal{F})$$

is an isomorphism, which will complete the proof by embedding into the localization (which vanishes); we show this by induction. Well, we note that we have an exact sequence of graded S-modules given by

$$0 \to S(-1) \xrightarrow{x_r} S \to S/(x_r) \to 0. \tag{3.2}$$

Taking $\widetilde{\ }$ produces an exact sequence of sheaves by

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \iota_* \mathcal{O}_H \to 0,$$

where H is a hyperplane $\iota \mathbb{P}_A^{r-1} \subseteq \mathbb{P}_A^r$. Twisting and summing appropriately, we get the exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \iota_* \mathcal{F}_H \to 0$$

where $\mathcal{F}_H = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_H(n)$. Now, for $1 < i < r-1$, the long exact sequence provides

$$0 \to H^i(X, \mathcal{F}(-1)) \xrightarrow{x_r} H^1(X, \mathcal{F}) \to 0 \tag{3.3}$$

by induction (inducting on r noting that H has dimension $r-1$), finishing. It remains to deal with degree 0 and degree $r-1$. In degree 0, we note that we already have exactness

$$0 \to \Gamma(X, \mathcal{F}(-1)) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \iota_* \mathcal{O}_H) \to 0$$

because this is just (3.2) on global sections. Thus, we still get (3.3) in degree 0, so we are safe.

Lastly, in degree r, we have

$$0 \to H^{r-1}(X, \mathcal{F}(-1)) \to H^{r-1}(X, \mathcal{F}) \to H^{r-1}(X, \mathcal{F}_H) \xrightarrow{\delta} H^r(X, \mathcal{F}(-1)) \to H^r(X, \mathcal{F}) \to 0.$$

We want to show that δ is injective to recover (3.3) in degree $r-1$. The main idea is that Čech cohomology produces an exact sequence of complexes: because $\mathcal{F}(-1)$ is quasicoherent, any sequence of indices has the exact sequence

$$0 \to \mathcal{F}(-1)(U_{i_0 \cdots i_p}) \to \mathcal{F}(U_{i_0 \cdots i_p}) \to \mathcal{F}_H(U_{i_0 \cdots i_p}) \to 0,$$

so summing over everything gives

$$0 \to \underbrace{\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}(-1))}_{A} \to \underbrace{\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})}_{B} \to \underbrace{\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}_H)}_{C} \to 0.$$

Now, we want to compute the connecting homomorphism δ_{r-1}. Well, it is induced by the Snake lemma in the following diagram.

$$
\begin{array}{ccccccc}
A^{r-1}/\operatorname{im} d_A^{r-2} & \longrightarrow & B^{r-1}/\operatorname{im} d_B^{r-2} & \longrightarrow & C^{r-1}/\operatorname{im} d_C^{r-2} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \ker d_A^r & \longrightarrow & \ker d_B^r & \longrightarrow & \ker d_A^r & &
\end{array}
$$

Well, an element of $C^{r-1}/\operatorname{im} d_C^{r-2}$ is

$$\prod_{k=0}^r (S/(x_r))_{x_0 \cdots \widehat{x_k} \cdots x_r},$$

but this is just $(S/(x_r))_{x_0 \cdots x_{r-1}}$ and is therefore given by a monomial $x_0^{\ell_0} \cdots x_{r-1}^{\ell_{r-1}}$ with nonnegative degree in each coordinate. We can lift this monomial directly to a class in

$$B^{r-1} = \prod_{k=0}^r S_{x_0 \cdots \widehat{x_k} \cdots x_r},$$

where the terms with $k \neq r$ we'll just take to vanish. As such, applying d_B^{r-1} to our monomial, we compute

$$d_B^{r-1}\left(x_0^{\ell_0} \cdots x_{r-1}^{\ell_{r-1}}\right) = (-1)^r \left[x_0^{\ell_0} \cdots x_{r-1}^{\ell_{r-1}}\right]$$

because all other terms of our alternating sum vanish. Now pulling this back to $\ker d_A^r$, we have to divide by x_r and thus produce the class represented by the monomial $x_0^{\ell_0} \cdots x_{r-1}^{\ell_{r-1}} x_r^{-1}$. Thus, our connecting homomorphism δ is simply division by x_r, which is indeed injective. Indeed, it has "inverse" given by the next map $x_r \colon H^r(X, \mathcal{F}(-1)) \to H^r(X, \mathcal{F})$ in our long exact sequence because then we are dealing with a monomial of nonnegative degree, which vanishes. ∎

3.5   March 13

Term paper topics have been posted. We began class by completing the proof of Theorem 3.20. I have edited directly into those notes for continuity.

3.5.1 Cohomology of Projective Varieties

As an application of our computation, we have the following result.

> **Theorem 3.23** (Serre)**.** Suppose X is a proper variety over a Noetherian ring A, and suppose we have a very ample sheaf $\mathcal{O}_X(1)$. Further, let $\mathcal{F}$ be a coherent sheaf on X.
>
> (a) The A-modules $H^i(X, \mathcal{F})$ are finitely generated over A.
>
> (b) We have $H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X(1)^n\right) = 0$ for n sufficiently large (depending on $\mathcal{F}$).

Proof. Because X is proper, the locally closed embedding arising from the very ample sheaf is actually a closed embedding $\iota\colon X \to \mathbb{P}_A^r$ (by taking the base-change of the structure morphism). Then we can compute

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}_A^r, \iota_* \mathcal{F}).$$

Thus, we can replace X with $\mathbb{P}_A^r$ without tears. To continue, we pick up the following lemma.

> **Lemma 3.24.** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}_A^r$. Then we can find some $\mathcal{E} = \bigoplus_{i=0}^k \mathcal{O}(n_i)$ with a surjection $\mathcal{E} \to \mathcal{F}$.

Proof. This is [Har77, Corollary II.5.18]. ∎

As such, to show the result, we have an exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0.$$

This gives the exact sequence

$$H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R}).$$

The statements (a) and (b) are certainly true for i large enough because everything vanishes. As such, we can apply descending induction on i, where we know that the left and right terms are finite (the left by the previous computation and the right by our induction), so we conclude that the middle term is also finite.

  For (b), we can check the statements are true for $\mathcal{E}$ by our previous computation: upon high enough twisting, we see that $H^i(X, \mathcal{O}(n))$ vanishes on our projective space, so $\mathcal{E}$ on upon high enough twisting will also have vanishing cohomology. As such, we see that $H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{R}(n))$ is an injection for n large enough by our exactness, but our descending induction gives us vanishing for n large enough, which is what we wanted. ∎

3.6 March 15

We continue discussing line bundles with cohomology.

3.6.1 A Criterion for Being Ample

As another application, we will be able to give a cohomological criterion for a sheaf to ample.

> **Theorem 3.25.** Fix a proper scheme X over a Noetherian ring A. For any line bundle $\mathcal{L}$, the following are equivalent.
>
> (a) $\mathcal{L}$ is ample.
>
> (b) For any coherent $\mathcal{F}$, we have $H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}\right) = 0$ for n sufficiently large, for any i.
>
> (c) For any coherent $\mathcal{F}$, we have $H^1\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}\right) = 0$ for n sufficiently large.

Proof. We show our implications independently.

- Suppose that $\mathcal{L}$ is ample. Then $\mathcal{M} := \mathcal{L}^{\otimes m}$ is very ample for some m. Now, for a fixed very ample sheaf $\mathcal{F}$ and index i, for each $k \in [0, m)$, we may find some n_k such that

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^k \otimes \mathcal{M}^n\right) = 0$$

  for $n \geq n_k$. Letting N be the maximum of all the n_i, we see that we have looped over all our residue classes, so for $n > m + N$, we see that $H^i\left(X, \mathcal{F} \otimes \mathcal{L}^n\right) = 0$, which is what we wanted.

- Conversely, suppose (b). Given a coherent sheaf $\mathcal{F}$, we want to know that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for n sufficiently large. Well, fix some closed point p. This grants us the exact sequence

$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to k(p) \to 0,$$

  where $\mathcal{I}_p$ is the corresponding ideal sheaf. Tensoring with $\mathcal{F}$, we get the right-exact sequence

$$\mathcal{I}_p \otimes \mathcal{F} \to \mathcal{F} \to\to \mathcal{F} \otimes k(p) \otimes \mathcal{F} \to 0.$$

  Now, the image of $\mathcal{I}_p \otimes \mathcal{F}$ in $\mathcal{F}$ is simply sections of $\mathcal{F}$ multiplied by sections of $\mathcal{I}_p$, so we are looking at

$$0 \to \mathcal{I}_p\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes k(p) \to 0.$$

  Notably, because $\mathcal{L}^{\otimes n}$ is flat (it's locally trivial), we get the exact sequence

$$0 \to \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F} \otimes k(p) \otimes \mathcal{L}^{\otimes n} \to 0$$

  for any n, so because cohomology vanishes on the left for n large enough, we get a surjection on global sections

$$\Gamma\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}\right) \to \Gamma\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes k(p)\right).$$

  Notably, we are seeing that global sections are surjecting onto $\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes k(p) = \mathcal{F}_p \otimes \mathcal{L}_p^n \otimes \mathcal{O}_{X,p}/\mathfrak{m}_p$, which by Nakayama's lemma tells us that it must lift to a surjection onto the stalk $\mathcal{F}_p \otimes \mathcal{L}_n^p$; in particular, we are looking over the local ring $\mathcal{O}_{X,p}$ with residue field $\mathcal{O}_{X,p}/\mathfrak{m}_p$, so a basis will lift to a generating set. So we are indeed globally generated at closed points for n large enough, say larger than n_p.

  Now, our fixed closed point p, we can pick up our global sections to build a surjection

$$\mathcal{O}_X^{\oplus k} \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^{\oplus n},$$

  so the cokernel has vanishing stalk at p, so the cokernel must vanish in an open neighborhood of p, so this map is surjective on an open neighborhood $U_{p,n}$ of p (for n larger than n_p).

  We would like for $U_{p,n}$ to not depend on n. Taking $\mathcal{F} = \mathcal{O}_X$, for each $p \in X$, we may find some m_p and an open neighborhood U_p so that $\mathcal{L}^{\otimes m_p}$ is generated by global sections on U_p. Refining this to a finite open cover and taking the product of all the relevant $m_\bullet$ produces some m such that $\mathcal{L}^{\otimes m}$ is globally generated.[1]

  We now apply our residue class trick again. For each $i \in \{0, 1, \ldots, m-1\}$, and $p \in U$, we find an open neighborhood U_i of p upon which $\mathcal{F} \otimes \mathcal{L}^{\otimes(i+n_i m)}$ is globally generated for some n_i. By taking the intersection of all these U_i, we produce an open neighborhood U_p such that

$$\mathcal{F} \otimes \mathcal{L}^{\otimes i}$$

  is globally generated for all i large enough (say, larger than N_p), where the point is that multiplying through by the globally generated sheaf $\mathcal{L}^{\otimes m}$ is safe.

  To finish, we use the compactness trick one more time. Because X is quasicompact, we only need to care about finitely many of the U_p and N_p, so we can take the maximum of all the relevant N_p producing an open cover to finish the proof. ∎

---

[1] We are using the fact that the tensor product of globally generated sheaves remains globally generated. Indeed, if $\mathcal{F}$ and $\mathcal{G}$ are globally generated, then we get surjections $\mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{F}$ and $\mathcal{O}^{\oplus m} \twoheadrightarrow \mathcal{G}$, so we get a surjection $\mathcal{O}^{\oplus(n+m)} \twoheadrightarrow \mathcal{F} \otimes \mathcal{G}$.

3.6.2 Ext Sheaves

We are going to want a few different kinds of cohomology before we continue.

> **Definition 3.26** (Ext)**.** Fix a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-module $\mathcal{F}$.
>
> - The functor $\mathrm{Hom}(\mathcal{F}, -)\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ is left-exact, so we define $\mathrm{Ext}^i(\mathcal{F}, -)$ to be the right-derived functors.
>
> - The functor $\mathcal{H}om(\mathcal{F}, -)\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_X}$ is left-exact, so we define $\mathcal{E}xt^i(\mathcal{F}, -)$ to be the right-derived functors.

Here are our usual basic facts.

> **Lemma 3.27.** Fix a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-module $\mathcal{F}$.
>
> - For any $\mathcal{O}_X$-module $\mathcal{G}$, we have $\mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$.
>
> - For any $\mathcal{O}_X$-module $\mathcal{G}$, we have $\mathcal{E}xt^0(\mathcal{F}, \mathcal{G}) = \mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proof. This follows directly from our discussion of derived functors. ∎

> **Example 3.28.** Because $\mathcal{H}om(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$ for any $\mathcal{G}$, we note that this functor is the identity and hence exact, so $\mathcal{E}xt^i(\mathcal{O}_X, -)$ is the zero functor for $i > 0$.

> **Example 3.29.** Note that $\mathrm{Hom}(\mathcal{O}_X, \mathcal{G}) = \Gamma(X, \mathcal{G})$ is global sections, so $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$ for any i.

We are going to want the following abstract nonsense result.

> **Lemma 3.30.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. If the exact functor $f\colon \mathcal{A} \to \mathcal{G}$ is left adjoint to the functor $g\colon \mathcal{G} \to \mathcal{F}$, then g preserves injectives.

Proof. Fix an injective object $I \in \mathcal{B}$. We want to show that $g(I)$ is injective. Well, suppose we that we have a monomorphism $M \hookrightarrow N$ and a map $M \to g(I)$ so that we want to extend this map to $N \to g(I)$ as follows.

$$\begin{array}{ccc} M & \lhook\joinrel\longrightarrow & N \\ & \searrow & \big\downarrow \\ & & g(I) \end{array}$$

Well, hitting this with the adjunction, we really want a morphism filling in the following triangle.

$$\begin{array}{ccc} f(M) & \lhook\joinrel\longrightarrow & f(N) \\ & \searrow & \big\downarrow \\ & & I \end{array}$$

However, I is now injective, so we can fill in this arrow: f is exact, so the $f(M) \hookrightarrow f(N)$ remains injective. Reflecting back with the adjunction finishes. ∎

For our purposes, we will be interested in using the fact that f^* and f_* are adjoints here, and restriction to an open set is left adjoint to extension by 0.

Proposition 3.31. Fix a ringed space $(X, \mathcal{O}_X)$. For an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0,$$

any $\mathcal{O}_X$-module $\mathcal{G}$ produces the long exact sequences as follows.

$$0 \to \operatorname{Hom}(\mathcal{F}'', \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}', \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \cdots$$
$$0 \to \mathcal{H}om(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^1(\mathcal{F}'', \mathcal{G}) \to \cdots$$

Proof. Provide $\mathcal{G}$ with an augmented injective resolution $0 \to \mathcal{G} \to \mathcal{I}^\bullet$. Then the functor $\operatorname{Hom}(-, \mathcal{I})$ is exact for any injective $\mathcal{I}$—this is direct from the definition of injectivity—so we get the exact sequence of complexes

$$0 \to \operatorname{Hom}(\mathcal{F}'', \mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}, \mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}', \mathcal{I}^\bullet) \to 0.$$

Taking the long exact sequence arising from cohomology here produces our first long exact sequence. For the sheaf $\mathcal{H}om$, we see that we can check the exactness of

$$0 \to \mathcal{H}om(\mathcal{F}'', \mathcal{I}^\bullet) \to \mathcal{H}om(\mathcal{F}, \mathcal{I}^\bullet) \to \mathcal{H}om(\mathcal{F}', \mathcal{I}^\bullet) \to 0$$

by restricting to some open set U and evaluating at global sections (which allows us to check at stalks), where the point is that $\operatorname{Hom}(-, \mathcal{I}|_U)$ remains exact because $\mathcal{I}|_U$ remains injective because restriction to an open subset is an exact left adjoint to extension by 0. ∎

3.7   March 17

We are pleased to be joined by a goose today. The homework has been extended to next Friday.

3.7.1   Review of δ-Functors

We are going to want to talk about universal δ-functors quite a bit.

Definition 3.32 (δ-functor)**.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then a δ-*functor* is a collection of functors $T^i \colon \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ and connecting morphisms δ^i (for $i \geq 0$) such that a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

gives rise to a long exact sequence

$$0 \to T^0 A' \to T^0 A \to T^0 A'' \xrightarrow{\delta^0} T^1 A' \to T^1 A \to T^1 A'' \xrightarrow{\delta^1} T^2 A' \to \cdots$$

natural in the short exact sequence.

Definition 3.33. A *morphism of δ-functors* $\{T^\bullet, \delta^\bullet\}, \{S^\bullet, \varepsilon^\bullet\} \colon \mathcal{A} \to \mathcal{B}$ of abelian categories is a collection of natural transformations $\eta^i \colon T^i \to S^i$ such that short exact sequences $0 \to A' \to A \to A'' \to 0$ give rise to the following commuting square.

$$
\begin{array}{ccc}
T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\
\downarrow{\scriptstyle \eta^i} & & \downarrow{\scriptstyle \eta^{i+1}} \\
S^i(A'') & \xrightarrow{\varepsilon^i} & S^{i+1}(A')
\end{array}
$$

> **Definition 3.34** (universal)**.** Fix a δ-functor $\{T^\bullet, \delta^\bullet\}\colon \mathcal{A} \to \mathcal{B}$ of abelian categories. Then $\{T^\bullet, \delta^\bullet\}$ is *universal* if and only if, for any other δ-functor $\{S^\bullet, \varepsilon^\bullet\}$ and natural transformation $\eta^0\colon T^0 \to S^0$, then η^0 extends to a full unique morphism $\eta^\bullet\colon \{T^\bullet, \delta^\bullet\} \to \{S^\bullet, \varepsilon^\bullet\}$.

> **Theorem 3.35.** Fix abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then any universal δ-functors $\{T^\bullet, \delta^\bullet\}$ and $\{S^\bullet, \varepsilon^\bullet\}$ equipped with a natural isomorphism $T^0 \cong S^0$ are uniquely isomorphic.

Proof. The usual dimension-shifting argument goes through. Roughly speaking, use the definition of universal to construct morphisms going both ways and then show they will give our isomorphism. ∎

We might want to identify when particular δ-functors are universal.

> **Definition 3.36** (effaceable)**.** A covariant additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is *effaceable* if and only if each $A \in \mathcal{A}$ has a monomorphism $u\colon A \hookrightarrow A'$ such that $Fu = 0$.

> **Theorem 3.37.** Fix a δ-functor $\{T^\bullet, \delta^\bullet\}\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, and the functors T^i are effaceable for $i > 0$, then $\{T^\bullet, \delta^\bullet\}$ is universal.

> **Corollary 3.38.** Derived functors are universal δ-functors.

Proof. Explicitly, we have a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories and are examining the δ-functor given by $R^\bullet F$. Well, using Theorem 3.37, we note that any object $A \in \mathcal{A}$ can be embedded into an injective object I, but then the embedding $u\colon A \to I$ is monic even though $R^i F(u) = 0$ for $i > 0$ because $R^i F(I) = 0$ for $I > 0$. ∎

Frequently in life, we are going to want to show that two δ-functors are isomorphic, where one is derived. So if we can just show it at degree-0, and check that each is effaceable (e.g., it helps if one is a derived functor), we will be done immediately.

> **Remark 3.39.** Everything in the above discussion dualizes. For example, there is a notion of contravariant δ-functors, coeffaceability, etc.

3.7.2 Computing Ext

Throughout, $(X, \mathcal{O}_X)$ is a ringed space.

> **Proposition 3.40.** Fix a ringed space $(X, \mathcal{O}_X)$, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module equipped with a resolution of locally free sheaves of finite rank given by $\mathcal{L}^\bullet \to \mathcal{F} \to 0$. Then
> $$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = H^i(\mathcal{H}om(\mathcal{L}^\bullet, \mathcal{G}))$$
> for any $\mathcal{O}_X$-module $\mathcal{G}$.

Proof. We have three checks.

- Both are δ-functors in $\mathcal{G}$. On the left, we are derived. On the right, we check this by hand. Suppose we have an exact sequence
$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0.$$

Then because the $\mathcal{L}^\bullet$ is a complex of locally free sheaves, we see that we get an exact sequence of complexes

$$0 \to \mathcal{H}om(\mathcal{L}^\bullet, \mathcal{G}') \to \mathcal{H}om(\mathcal{L}^\bullet, \mathcal{G}) \to \mathcal{H}om(\mathcal{L}^\bullet, \mathcal{G}'') \to 0.$$

Indeed, we may check this on stalks, but $\mathcal{H}om(\mathcal{L}, \mathcal{H})_x = \mathrm{Hom}(\mathcal{L}_x, \mathcal{H}_x)$ (for example, use the fact that $\mathcal{L}$ is locally free here), so our exactness follows because $\mathcal{L}_x$ is free over $\mathcal{O}_{X,x}$.

- We check that both are universal. On the left, this follows from Corollary 3.38. On the right, we check effaceable using Theorem 3.37: we take any $\mathcal{O}_X$-module $\mathcal{G}$, embed it into an injective module $\mathcal{I}$, and we see that $H^i(\mathcal{H}om(\mathcal{L}^\bullet, \mathcal{I}))$ will end up vanishing because $\mathcal{H}om(-, \mathcal{I})$ is injective.

- Lastly, we agree in degree-0 because both are $\mathcal{H}om(\mathcal{F}, -)$. ∎

Lemma 3.41. Fix a ringed space $(X, \mathcal{O}_X)$. If $\mathcal{L}$ is locally free, then $\mathcal{L} \otimes -$ preserves injectives.

Proof. Note that $\mathcal{L} \otimes -$ is left adjoint to $\mathcal{H}om(\mathcal{L}, -)$, which is exact because $\mathcal{L}$ is locally free. So we are done by Lemma 3.30. ∎

Proposition 3.42. Fix a ringed space $(X, \mathcal{O}_X)$. Given a locally free sheaf $\mathcal{L}$ and an $\mathcal{O}_X$-module $\mathcal{F}$, we have

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) = \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \qquad \text{and} \qquad \mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) = \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G})$$

for any $\mathcal{O}_X$-module $\mathcal{G}$.

Proof. We use the lemma. Fix an injective resolution of $\mathcal{G}$ given by $0 \to \mathcal{G} \to \mathcal{I}^\bullet$. Then we see $\mathrm{Ext}^\bullet(\mathcal{F} \otimes \mathcal{L}, \mathcal{G})$ is the cohomology of the complex

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{I}^\bullet) = \mathrm{Hom}(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{I}^\bullet),$$

but by Lemma 3.41, this last complex is an injective resolution of $\mathcal{L}^\vee \otimes \mathcal{G}$. ∎

We can now have some fun.

Proposition 3.43. Fix a Noetherian scheme X and a coherent sheaf $\mathcal{F}$ on X. Then for any $\mathcal{O}_X$-module $\mathcal{G}$, we show

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x = \mathrm{Ext}^i(\mathcal{F}_x, \mathcal{G}_x)$$

for any $x \in X$.

Proof. Everything is local on X, so we may assume that $X = \mathrm{Spec}\, A$ for some ring A. Then write $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. Thus, we may give M a free resolution $L^\bullet \to M \to 0$, so Proposition 3.40 lets us compute

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x = H^i(\mathcal{H}om(\widetilde{L}^\bullet, \widetilde{N}))_x = H^i\left(\mathrm{Hom}(L_x^\bullet, \mathcal{G}_x)\right)$$

by passing the stalks all the way through, where we are using the fact that exactness may be checked on stalks. However, we now see that we have a free resolution $L_x^\bullet \to F_x \to 0$, so this last cohomology is simply $\mathrm{Ext}^i(\mathcal{F}_x, \mathcal{G}_x)$, thus finishing the proof. ∎

Proposition 3.44. Fix a projective scheme X over a Noetherian ring A. Now, let $\mathcal{O}_X(1)$ be the very ample line bundle, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves. Then for each i, we have

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) = \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n)))$$

for n sufficiently large.

Proof. We proceed in steps.

1. If $i = 0$, we are just looking at global sections everywhere, so we are done.

2. If $\mathcal{F} = \mathcal{O}_X$, then we note that the left-hand side is merely $H^i(X, \mathcal{G}(n))$ vanishes for n large enough by Theorem 3.25. Similar holds for the right-hand side by checking at stalks and using Proposition 3.43.

3. If $\mathcal{F}$ is locally free, one uses Proposition 3.42 to move $\mathcal{F}$ over to the right coordinate and finish by the previous step.

4. To finish, we use dimension-shifting. Namely, we find a locally free sheaf $\mathcal{E}$ projecting onto $\mathcal{F}$, and we let $\mathcal{R}$ denote the kernel. This gives us a short exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

to dimension-shift with. Namely, for our induction, we see that we have natural maps as follows.

$$\begin{array}{ccccccc}
\operatorname{Ext}^i(\mathcal{E}, \mathcal{G}(n)) & \longrightarrow & \operatorname{Ext}^i(\mathcal{R}, \mathcal{G}(n)) & \longrightarrow & \operatorname{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n)) & \longrightarrow & \operatorname{Ext}^{i+1}(\mathcal{E}, \mathcal{G}(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(X, \mathcal{E}xt^i(\mathcal{E}, \mathcal{G}(n))) & \longrightarrow & \Gamma(X, \mathcal{E}xt^i(\mathcal{R}, \mathcal{G}(n))) & \longrightarrow & \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))) & \longrightarrow & \Gamma(X, \mathcal{E}xt^i(\mathcal{E}, \mathcal{G}(n)))
\end{array}$$

The rightmost terms vanish because $\mathcal{E}$ is locally free, and the top row is exact for free. Further, the two left maps are isomorphisms by the inductive hypothesis, so we will be done if we can show that the bottom row is exact with vanishing rightmost term in high enough twisting. Well, large twisting kills cohomology by Theorem 3.25, so for n large enough, taking global sections will be exact, which is what we wanted. ∎

> **Remark 3.45.** Using spectral sequences, one may instead simply use the spectral sequence
> $$H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F}, \mathcal{G}).$$
> However, by twisting $\mathcal{G}$ high enough, we kill these cohomology objects for n large enough because these $\mathcal{E}xt$ sheaves vanish (see Lemma 3.46 below and Theorem 3.25) which provides the needed equality.

> **Lemma 3.46.** Fix a Noetherian scheme X. Given coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, the sheaves $\mathcal{E}xt^\bullet(\mathcal{F}, \mathcal{G})$ are coherent.

Proof. The question is local on X, so we may assume that $X = \operatorname{Spec} A$ by taking an affine open cover. Now, we provide $\mathcal{F} = \widetilde{M}$ with a resolution by free $\mathcal{O}_X$-modules of finite rank $\mathcal{L}_\bullet \to \mathcal{F} \to 0$. We now recall from Proposition 3.40 that
$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = H^i(\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G})).$$

Because each $\mathcal{G}$ is coherent (and in particular finitely presented), we see that each $\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G})$ is also coherent, so taking cohomology remains coherent. Explicitly, write $\mathcal{L}_i = \widetilde{L}_i$ for each i, and then $\mathcal{H}om(\mathcal{L}_\bullet, \mathcal{G}) = \operatorname{Hom}(L_i, \mathcal{G}(X))$ because X is affine, which is a finitely generated module because each of L_i and $\mathcal{G}(X)$ are. ∎

3.8   March 20

Drew has joined us for class today, along with the goose.

3.8.1 Serre Duality for $\mathbb{P}_k^n$

Serre duality is the most technical result we will prove in this course. Let's first discuss the case for $\mathbb{P} := \mathbb{P}_A^n$. The statement of Serre duality requires the notion of a dualizing sheaf, which in our case is $\omega_\mathbb{P} := \Omega_{\mathbb{P}/A}^{\wedge n}$. One can show (using Theorem 3.20) that $\omega = \mathcal{O}_\mathbb{P}(-n-1)$: indeed, the short exact sequence

$$0 \to \Omega \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0$$

induces the short exact sequence

$$0 \to \Omega^{\wedge(n+1)} \to \mathcal{O}(-n-1) \to \Omega^{\wedge n} \to 0$$

by taking the standard filtration associated to the exterior power. Now, the statement of Serre duality amounts to isomorphisms

$$\mathrm{Ext}^i(\mathcal{F}, \omega_\mathbb{P}) \cong H^{n-i}(X, \mathcal{F})^\vee,$$

where $(-)^\vee$ refers to the dual as an A-module.

In general, we will have a locally closed embedding $i \colon X \to \mathbb{P}_A^n$, where X has codimension r. To generalize the above statement of Serre duality, the issue is to replace the dualizing sheaf $\omega_\mathbb{P}$ with some other sheaf ω_X. (Indeed, we know how to replace all other objects here.) Well, we will define

$$\omega_X^\circ := R^r i^! \omega_\mathbb{P},$$

where $i^!$ is the sheaf with supports in Z, where $i^! := \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(i_*\mathcal{O}_X, -)$. In particular, we can see visually that this is a left-exact functor, and we can check locally on open sets that

$$\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(i_*\mathcal{O}_X, -) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, i^*-)$$

by the adjunction, so $i^!$ naturally produces $\mathcal{O}_X$-modules.

Anyway, here is our statement of Serre duality for $\mathbb{P}_k^n$.

> **Lemma 3.47.** Fix a projective scheme X over a field. Then $\mathrm{Ext}^\bullet(-, \mathcal{G})$ is co-effaceable for any coherent sheaf $\mathcal{G}$.

Proof. Fix some coherent sheaf $\mathcal{F}$, and we may use [Har77, Corollary II.5.18] to find a surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$ where $\mathcal{E} = \bigoplus_{k=1}^N \mathcal{O}(-q_k)$ for various $q_k > 0$, and we may make the q_k as large as we please. Now, we compute

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{G}) = \bigoplus_{k=1}^N \mathrm{Ext}^i(\mathcal{O}(-q_k), \mathcal{G}) = \bigoplus_{k=1}^N H^i(X, \mathcal{G}(q_k)).$$

For $i > n$, this vanishes by Theorem 2.51. Then by Theorem 3.23, we can ensure that the $q_\bullet$ are each large enough so that we can kill all the finitely many cohomology groups $H^n(X, \mathcal{G}(q_k))$. This completes our check. $\blacksquare$

> **Theorem 3.48** (Serre duality for $\mathbb{P}_k^n$)**.** Fix a field k, and set $X := \mathbb{P}_k^n$. Define $\omega := \Omega_{X/k}^{\wedge n}$.
>
> (a) We have $H^n(X, \omega) \cong k$.
>
> (b) For any coherent sheaf $\mathcal{F}$, we have a perfect pairing
>
> $$\mathrm{Ext}^i(\mathcal{F}, \omega) \times H^{n-i}(X, \mathcal{F}) \to H^n(X, \Omega) \cong k$$
>
> for each $0 \leq i \leq n$.

Proof. We know (a) because $\omega = \mathcal{O}(-n-1)$, so this is simply Theorem 3.20. For (b), we start in index 0. We remark that the pairing in index 0 is induced because a morphism $\mathcal{F} \to \omega$ will induce a morphism $H^n(X, \mathcal{F}) \to H^n(X, \omega)$. For our proof, we have the following cases.

1. Suppose $\mathcal{F} = \mathcal{O}(q)$ for some $q \in \mathbb{Z}$. Remembering that we are looking in index 0, the statement is
$$\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-q-n-1)) \times H^n(X, \mathcal{O}(q)) \to H^n(X, \mathcal{O}(-n-1)) \cong k.$$
   However, this follows directly from Theorem 3.20.

2. In general, we know from Corollary II.5.18 in [Har77] that we can write $\mathcal{F}$ as the cokernel in an exact sequence
$$\mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0,$$
   where $\mathcal{E}'$ and $\mathcal{E}$ are sums of line bundles $\mathcal{O}(q)$ for various q. Using the left-exactness of $\operatorname{Hom}(\mathcal{F}, -)$, we produce the short exact sequence
$$0 \to \operatorname{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) \to \operatorname{Hom}(\mathcal{E}, \mathcal{O}(-n-1)) \to \operatorname{Hom}(\mathcal{E}', \mathcal{O}(-n-1)).$$
   But now we already have a natural map $\operatorname{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) \to H^n(X, \mathcal{F})^\vee$ when we described the pairing above, so we produce the following morphisms.

$$
\begin{array}{ccccc}
0 \longrightarrow & \operatorname{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) & \longrightarrow & \operatorname{Hom}(\mathcal{E}, \mathcal{O}(-n-1)) & \longrightarrow & \operatorname{Hom}(\mathcal{E}', \mathcal{O}(-n-1)) \\
& \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
0 \longrightarrow & H^n(X, \mathcal{F})^\vee & \longrightarrow & H^n(X, \mathcal{E})^\vee & \longrightarrow & H^n(X, \mathcal{E}')^\vee
\end{array}
$$

   The goal now is to show that the bottom sequence is exact, which will finish the proof. This requires some care. By dualizing, it is enough to show that we have a right-exact sequence
$$H^n(X, \mathcal{E}') \to H^n(X, \mathcal{E}) \to H^n(X, \mathcal{F}) \to 0.$$
   Well, we let $\mathcal{R}$ denote the kernel of the map $\mathcal{E} \twoheadrightarrow \mathcal{F}$, and let $\mathcal{R}'$ denote the kernel of the map $\mathcal{E}' \twoheadrightarrow \mathcal{R}$. We now have two checks.

   - The long exact sequence of $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$ produces the exact sequence
$$H^n(X, \mathcal{R}) \to H^n(X, \mathcal{E}) \to H^n(X, \mathcal{F}) \to 0.$$
     We want to replace $H^n(X, \mathcal{R})$ with $H^n(X, \mathcal{E}')$ here.
   - The long exact sequence of $0 \to \mathcal{R}'' \to \mathcal{E}' \to \mathcal{R} \to 0$ produces the surjection $H^n(X, \mathcal{E}') \twoheadrightarrow H^n(X, \mathcal{R})$, so plugging this in grants the exact sequence
$$H^n(X, \mathcal{E}') \to H^n(X, \mathcal{E}) \to H^n(X, \mathcal{F}) \to 0.$$

   The above checks complete the proof in degree 0.

It remains to extend past $i = 0$. In fact, we will exhibit a natural isomorphism
$$\operatorname{Ext}^i(-, \omega) \simeq H^{n-i}(X, -)^\vee.$$

We are going to make some kind of δ-functor argument. In particular, we can see automatically that both sides are contravariant δ-functors (for example, the left-hand side is a contravariant δ-functor by Proposition 3.31). Further, we have natural isomorphisms in degree 0 by the above argument.

It remains to show that these functors are universal, for which by Theorem 3.37 it is enough to show that these functors are co-effaceable. This has two checks; fix some coherent sheaf $\mathcal{F}$, and we may use [Har77, Corollary II.5.18] to find a surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$ where $\mathcal{E} = \bigoplus_{k=1}^N \mathcal{O}(-q_k)$ for various $q_k > 0$.

- We check that $\operatorname{Ext}^i(-, \omega)$ is co-effaceable. This is exactly Lemma 3.47.

- We check that $H^{n-i}(X, -)^\vee$ is co-effaceable. Again, we compute
$$H^{n-i}(X, \mathcal{E})^\vee \cong \bigoplus_{k=1}^N H^{n-i}(X, \mathcal{O}(-q_k))^\vee.$$

  Again, for $0 < i < n$, these factors vanish automatically for Theorem 3.20; for $i = n$, we still have $H^0(X, \mathcal{O}(-q_k)) = 0$ because $q_k > 0$. Thus, the surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$ witnesses that $\operatorname{Ext}^i(-, \omega)$ is co-effaceable.

In total, the uniqueness of our universal (contravariant) δ-functors completes the proof. $\blacksquare$

3.9   March 22

We continue our discussion of Serre duality. Our goal for today is to discuss the dualizing sheaf.

3.9.1   The Dualizing Sheaf

Serre duality arises from the dualizing sheaf. On $\mathbb{P}_k^n$, our dualizing sheaf was $\omega := \Omega\mathbb{P}_k^n/k^{\vee n}$, where the "dualizing" refers to Theorem 3.48. Let's make rigorous what a dualizing sheaf should be.

> **Definition 3.49** (dualizing sheaf). Fix a proper n-dimensional k-scheme X. Then a *dualizing sheaf* is a coherent sheaf ω_X° on X equipped with a trace morphism $t\colon H^n(X, \omega_X^\circ) \to k$ such that the induced pairing
> $$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k$$
> is a perfect pairing; here, $\varphi\colon \mathcal{F} \to \omega_X^\circ$ is sent to the natural map $H^n(\varphi)\colon H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ)$.

Here is a quick remark that allows us to say "the dualizing sheaf."

> **Proposition 3.50.** Fix a proper n-dimensional k-scheme X. If the dualizing sheaf (ω_X°, t) exists, then it is unique up to unique isomorphism.

Proof. Suppose that we have dualizing sheaves (ω, t) and (ω', t'). We show there is a unique isomorphism $\omega \cong \omega'$ such that the induced composite

$$H^n(X, \omega) \cong H^n(X, \omega') \xrightarrow{t'} k$$

is simply t.

Well, there's not much that we can do here. Because ω' is a dualizing sheaf, we see that

$$\operatorname{Hom}(\omega, \omega') \cong H^n(X, \omega)^\vee,$$

so there is a unique morphism $\varphi'\colon \omega \to \omega'$ such that the composite $t' \circ H^n(\varphi')$ is equal to t. Thus, we see that the map $\omega \to \omega'$ satisfying the requirement on our composite above is unique, so it remains to show that this unique map is actually an isomorphism.

Well, reversing the roles of ω and ω' in this argument, there is also a unique morphism $\varphi\colon \omega' \to \omega$ such that $t \circ H^n(\varphi) = t'$. We now claim that φ and φ' are our canonical isomorphisms. By symmetry, it is enough to check that $\varphi \circ \varphi' = \operatorname{id}_\omega$. Well, we note that

$$t \circ H^n(\varphi \circ \varphi') = t \circ H^n(\varphi) \circ H^n(\varphi') = t' \circ H^n(\varphi') = t$$

by construction of φ and φ'. However, there is a unique map $\psi\colon \omega \to \omega$ such that $t \circ H^n(\psi) = t$ because ω is a dualizing sheaf, and this map ψ must be id_ω, so we conclude that $\varphi \circ \varphi' = \operatorname{id}_\omega$. This completes the proof.   ∎

It remains to construct a dualizing sheaf for projective schemes. Roughly speaking, this arises from the more general framework of Grothendieck duality. Namely, the more natural statement is that we have to look at a "dualizing complex" as an adjunction statement like

$$Rf_*(R\mathcal{H}om(\mathcal{F}, Rf^!\mathcal{G})) = \mathcal{H}om(Rf_*\mathcal{F}, \mathcal{G}).$$

Here, one has to look at complexes instead of sheaves, which is slightly problematic for our statement of Serre duality, but we will be able to show that this dualizing complex is in fact concentrated in a single degree, under nice enough hypotheses.

Explicitly, our dualizing sheaf is going to be $\omega_X = Ri^!\omega_\mathbb{P}$, where $i\colon X \to \mathbb{P}$ is our closed embedding, and it turns out that most of this complex vanishes. Here is our result.

> **Proposition 3.51.** Set $\mathbb{P} := \mathbb{P}^n_k$ for a field k, and fix a closed subscheme $i \colon X \to \mathbb{P}$ of codimension r. Then $\mathcal{E}xt^i(i_*\mathcal{O}_X, \omega^\circ_\mathbb{P}) = 0$ for $i < r$.

Proof. For brevity, set $\mathcal{F}^i := \mathcal{E}xt^i(i_*\mathcal{O}_X, \omega^\circ_\mathbb{P}) = 0$. Because $\mathcal{F}^i$ is coherent, it is generated by global sections after large enough twisting (this is the definition of being ample), so it is enough to show that

$$\Gamma(X, \mathcal{F}^i(q)) = 0$$

for q sufficiently large. Well, for q sufficiently large, Proposition 3.44 (and Proposition 3.42) tells us that

$$\Gamma(X, \mathcal{F}^i(q)) = \mathrm{Ext}^i(i_*\mathcal{O}_X, \omega^\circ_\mathbb{P}(q)) = \mathrm{Ext}^i(i_*\mathcal{O}_X(-q), \omega^\circ_\mathbb{P}).$$

We can now use Serre duality on projective space: by Theorem 3.48 (and Corollary 2.53), we see

$$\mathrm{Ext}^i(i_*\mathcal{O}_X(-q), \omega^\circ_\mathbb{P}) = H^{n-i}(\mathbb{P}, i_*\mathcal{O}_X(-q))^\vee = H^{n-i}(X, \mathcal{O}_X(-q))^\vee.$$

However, for $i < r$, we see $n - i > r$, so these cohomology groups $H^{n-i}(X, \mathcal{O}_X(-q))$ vanishes by Theorem 2.51, so we are done. $\blacksquare$

We've been talking about $i^!$ for some time now, but we haven't given it a formal definition, so we should do that now. Roughly speaking, topologically, this should mean cohomology with supports, but we need to add in our algebraic geometry information of nilpotents for everything to go through. So here is our definition.

> **Definition 3.52.** Fix a closed embedding $i \colon Y \to X$ of schemes given by the quasicoherent ideal sheaf $\mathcal{I}$. Then for an abelian sheaf $\mathcal{F}$ on X, we define
>
> $$i^!(\mathcal{F}) := i^* \mathcal{H}om_X(i_*\mathcal{O}_Y, \mathcal{F}).$$
>
> Explicitly, $\mathcal{H}om_X(i_*\mathcal{O}_X, \mathcal{F})$ is naturally a module over $\mathcal{O}_X/\mathcal{I}$, so we have defined a sheaf on Y.

By the definition, we see that $i^!$ is left-exact, and we have an adjunction

$$\mathrm{Hom}(i_*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(i_*\mathcal{F} \otimes i_*\mathcal{O}_Y, \mathcal{G}) = \mathrm{Hom}(i_*\mathcal{F}, \mathcal{H}om(i_*\mathcal{O}_Y, \mathcal{G})).$$

But now $\mathcal{H}om(i_*\mathcal{O}_Y, \mathcal{G})$ is supported in Y as noted above, so nothing changes if we hit with i_*i^* to do nothing, and so because i_* is fully faithful, we see

$$\mathrm{Hom}(i_*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, i^*\mathcal{H}om(i_*\mathcal{O}_Y, \mathcal{G})) = \mathrm{Hom}(\mathcal{F}, i^!\mathcal{G}).$$

Similarly, there is a natural isomorphism $\mathcal{H}om(i_*\mathcal{F}, \mathcal{G}) \to i_*\mathcal{H}om(\mathcal{F}, i^!\mathcal{G})$. Applying the adjunction (i_*, i^*), it is enough to provide a natural isomorphism

$$\mathcal{H}om(i_*\mathcal{F}, \mathcal{G}) \otimes i_*\mathcal{F} \cong i_*(i^*\mathcal{H}om(i_*\mathcal{F}, \mathcal{G}) \otimes \mathcal{F}).$$

Applying the adjunction again, it is enough to provide the map $i^*i_*\mathcal{F} \to \mathcal{F}$ and then take the tensor product with $i^*\mathcal{H}om(i_*\mathcal{F}, \mathcal{G})$.

Anyway, because $i^!$ is left-exact (it's basically $\mathcal{H}om_X(i_*\mathcal{O}_Y, -)$), we can right-derive it and so on.

> **Lemma 3.53.** Set $\mathbb{P} := \mathbb{P}^n_k$ for a field k, and fix a closed subscheme $i \colon X \to \mathbb{P}$ of codimension r. Then set $\omega^\circ_X := R^r i^! \omega_\mathbb{P}$. Then $\omega^\circ_X = \mathcal{E}xt^r(i_*\mathcal{O}_X, \omega_\mathbb{P})$, and for any $\mathcal{O}_X$-module $\mathcal{F}$, we have
>
> $$\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) \cong \mathrm{Ext}^r_\mathbb{P}(i_*\mathcal{F}, \omega_\mathbb{P}).$$

Proof. We provide an abstract proof. Let τ denote the truncation of a complex. The point is that applying $R\operatorname{Hom}$ to the exact sequence

$$0 \to \tau_{\leq 0}\omega_X^\circ \to \omega_X^\circ \to \tau_{>0}\omega_X^\circ \to 0$$

and hit it with $R\operatorname{Hom}(\mathcal{F}, -)$ to produce a distinguished triangle

$$0 \to R\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \to R\operatorname{Hom}(\mathcal{F}, Ri^!\omega_{\mathbb{P}}[r]) \to R\operatorname{Hom}(\mathcal{F}, \tau_{>0}Ri^!\omega_{\mathbb{P}}[r]) \to 0,$$

so the long exact sequence in degree 0 produces

$$0 \to \operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \to \operatorname{Ext}^r(i_*\mathcal{F}, \omega_{\mathbb{P}}) \to 0,$$

which is the result that we wanted in conclusion. (Notably, the rightmost term vanishes in degree 0 because it has been truncated to positive degrees.)

For the other claim, we use the previous adjunction to write

$$R\mathcal{H}om(i_*\mathcal{O}_X, \omega_{\mathbb{P}}) = i_*R\mathcal{H}om_Y(\mathcal{O}_X, Ri^!\omega_{\mathbb{P}}) = i_*Ri^!\omega_{\mathbb{P}},$$

so taking cohomology in degree r completes the proof. ∎

3.10 March 24

Spring break is upon us. There will not be a problem set over spring break.

3.10.1 Stating Serre Duality

Today we would like to finish proving Serre duality. Given a projective scheme $i\colon X \to \mathbb{P}_k^n$, we would like to show that $\omega_X^\circ := R^r i^!\omega_{\mathbb{P}_k^n}$ is in fact a dualizing sheaf.

> **Proposition 3.54.** Set $\mathbb{P} := \mathbb{P}_k^N$ for a field k, and fix a closed subscheme $i\colon X \to \mathbb{P}$ of codimension r. Then set $\omega_X^\circ := R^r i^!\omega_{\mathbb{P}}$. Then ω_X° is a dualizing sheaf on X.

Proof. For brevity, set $n := N - r$, which is $\dim X$. To begin, we require a trace morphism $t\colon H^n(X, \omega_X^\circ) \to k$ inducing the desired perfect pairing

$$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \to H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k.$$

Now, Lemma 3.53 and applying Theorem 3.48 tells us that

$$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \simeq \operatorname{Ext}_{\mathbb{P}}^r(i_*\mathcal{F}, \omega_{\mathbb{P}}) \simeq H^{N-r}(\mathbb{P}, i_*\mathcal{F})^\vee \simeq H^n(X, \mathcal{F})^\vee \tag{3.4}$$

for any coherent sheaf $\mathcal{F}$, and these isomorphisms are natural in $\mathcal{F}$.

Thus, we appear to have a perfect pairing, but we need to figure out what the corresponding trace map was supposed to be. Well, the identity id in $\operatorname{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ by the above computation produces some map $t \in H^n(X, \mathcal{F})^\vee$, which we claim is our trace map. Indeed, for some $\alpha\colon \mathcal{F} \to \omega_X^\circ$, we have the following commutative diagram.

$$
\begin{array}{ccccc}
\operatorname{Hom}(\omega_X^\circ, \omega_X^\circ) & \longrightarrow & \operatorname{Ext}^r(i_*\omega_X^\circ, \omega_{\mathbb{P}}) & \longrightarrow & H^n(X, \omega_X^\circ)^\vee \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) & \longrightarrow & \operatorname{Ext}^r(i_*\mathcal{F}, \omega_{\mathbb{P}}) & \longrightarrow & H^n(X, \mathcal{F})^\vee
\end{array}
$$

Tracking around $\operatorname{id}_{\omega_X^\circ}$, we have the following.

$$
\begin{array}{ccc}
\operatorname{id} & \longmapsto & t \\
\downarrow & & \downarrow \\
\alpha & \longmapsto & \psi_\alpha = t \circ H^n(\alpha)
\end{array}
$$

So we conclude that the natural isomorphism in (3.4) exactly sends α to $t \circ H^n(\alpha)$, which is exactly what we needed for t to be a trace map. ∎

And here is Serre duality.

> **Theorem 3.55** (Serre duality)**.** Set $\mathbb{P} := \mathbb{P}_k^N$ for a field k, and fix a closed subscheme $i \colon X \to \mathbb{P}$ of codimension r. Further, set $n := n - r = \dim X$ and $\omega_X^\circ := R^r i^! \omega_{\mathbb{P}}$ to be our dualizing sheaf with trace map t. For any coherent sheaf $\mathcal{F}$ on X and index $i \geq 0$, there are natural maps
>
> $$\theta^i \colon \operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$$
>
> is induced by the trace map of our dualizing sheaf. In fact, following are equivalent.
>
> (a) X is Cohen−Macaulay and equidimensional.
>
> (b) For every locally free sheaf $\mathcal{F}$ and $a < n$, we have $H^a(X, \mathcal{F}(-q)) = 0$ for q sufficiently large.
>
> (c) The maps θ^i are isomorphisms for all $i \geq 0$ and $\mathcal{F}$ coherent.
>
> (d) We have $R^p i^! \omega_{\mathbb{P}} = 0$ for $p \neq r$.

Proof of existence. We have θ^0 defined as an isomorphism at Proposition 3.54. Now, $\operatorname{Ext}^i(-, \omega_X^\circ)$ is a contravariant δ-functor by Proposition 3.31, as is $H^{n-i}(X, -)^\vee$ because higher-dimensional cohomology vanishes by Theorem 2.51. Further, $\operatorname{Ext}^i(-, \omega_X^\circ)$ is universal because it is coeffaceable by Lemma 3.47 and hence universal by Theorem 3.37, and this induces the desired θ^i maps by definition of being universal. ∎

3.10.2 Depth and All That

To continue, we must discuss Cohen−Macaulay rings.

> **Definition 3.56** (regular sequence)**.** Fix a ring A and an A-module M. Then a *regular sequence* for M is a sequence of elements $x_1, \ldots, x_n \in A$ such that the x_i are not zero-divisors of M, and x_i is not a zero-divisor on $M/(x_1, \ldots, x_{i-1})M$ for each i.

> **Definition 3.57** (depth, Cohen−Macaulay)**.** Fix a Noetherian local ring A. The *depth* of an A-module M is the length of a maximal regular sequence for M. The ring A is *Cohen−Macaulay* if and only if $\dim A = \operatorname{depth} A$.

It is a lemma that the depth of M is well-defined; we will not show it here.

> **Remark 3.58.** One can show that any regular local ring is Cohen−Macaulay. This is essentially following directly from the definition of regular.

> **Remark 3.59.** In fact, any localization of a Cohen−Macaulay ring at a prime ideal is Cohen−Macaulay.

Let's discuss depth in some detail.

> **Definition 3.60** (projective resolution, length)**.** Fix a ring A. A *projective resolution* of an A-module M is a complex $P_\bullet$ of projective modules exact in all degrees except having cohomology isomorphic to M in degree 0. The *length* of $P_\bullet$ is the smallest nonnegative integer n such that $P_k = 0$ for $k > n$.

> **Definition 3.61** (projective dimension)**.** Fix a ring A. The *projective dimension* $\mathrm{pd}(M)$ of an A-module M is the least length of a projective resolution for M or is ∞ if M has no finite projective resolution.

> **Proposition 3.62.** Fix a regular local ring A. For a finitely generated A-module M, we have $\mathrm{pd}(M) \leq n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for $i > n$.

Proof. We explain the easy direction. If $\mathrm{pd}(M) \leq n$, we have a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Computing $\mathrm{Ext}^i(M, A)$ with this projective resolution (namely, by applying $\mathrm{Hom}(-, A)$ here), we see that terms higher than n vanish. ∎

> **Proposition 3.63.** Fix a module M over a ring A. Then $\mathrm{pd}\, M \leq n$ if and only if $\mathrm{Ext}^i(M, N) = 0$ for $i > n$ and any A-module N.

Proof. Similar to the previous bound. ∎

> **Proposition 3.64.** Fix a regular local Noetherian ring A. Then for each A-module M, we have
> $$\mathrm{pd}\, M + \mathrm{depth}\, M = \dim A.$$

3.11 April 3

Today we complete our discussion of Serre duality.

3.11.1 Finishing Serre Duality

Recall our statement.

> **Theorem 3.55** (Serre duality)**.** Set $\mathbb{P} := \mathbb{P}_k^N$ for a field k, and fix a closed subscheme $i \colon X \to \mathbb{P}$ of codimension r. Further, set $n := n - r = \dim X$ and $\omega_X^\circ := R^r i^! \omega_{\mathbb{P}}$ to be our dualizing sheaf with trace map t. For any coherent sheaf $\mathcal{F}$ on X and index $i \geq 0$, there are natural maps
> $$\theta^i \colon \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$$
> is induced by the trace map of our dualizing sheaf. In fact, following are equivalent.
>
> (a) X is Cohen–Macaulay and equidimensional.
>
> (b) For every locally free sheaf $\mathcal{F}$ and $a < n$, we have $H^a(X, \mathcal{F}(-q)) = 0$ for q sufficiently large.
>
> (c) The maps θ^i are isomorphisms for all $i \geq 0$ and $\mathcal{F}$ coherent.
>
> (d) We have $R^p i^! \omega_{\mathbb{P}} = 0$ for $p \neq r$.

Last class we showed the existence of the θ^i.

Proof of equivalences. Here are our implications.

- We show that (a) implies (b) and (d). Well, fix some locally free sheaf $\mathcal{F}$ on X. For any closed point $x \in X$, we note $\mathcal{F}_x$ has depth as an $\mathcal{O}_{X,x}$-module is equal to $n = \dim X$ by Proposition 3.64 because X is Cohen–Macaulay and equidimensional. Now, we quickly note that we have a surjection

$$\mathcal{O}_{\mathbb{P},x} \twoheadrightarrow \mathcal{O}_{X,x}.$$

Further, $\mathbb{P}$ is regular, so $\mathcal{O}_{\mathbb{P},x}$ is regular local of dimension N, so by pulling back a regular sequence from $\mathcal{O}_{X,x}$ we see that $\mathrm{depth}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{F}_x = n$ as well. (Indeed, pulling back the regular sequence gives $\geq n$, but we can also push forward a regular sequence to get $\leq n$; the point is that the $\mathcal{O}_{\mathbb{P},x}$-action on $\mathcal{F}_x$ is happening through the quotient, so an element is a zero-divisor if and only if it is a zero-divisor after projected to $\mathcal{O}_{X,x}$.) It follows from Proposition 3.64 that $\mathrm{pd}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{F}_x = N - n$, so

$$\mathcal{E}xt^a_{\mathcal{O}_\mathbb{P}}(\mathcal{F},\mathcal{G})_x = \mathcal{E}xt^a_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{F}_x,\mathcal{G}_x)$$

vanishes for $a > N - n$ and quasicoherent sheaf $\mathcal{G}$. We already showed that this vanishes for $a < r$ by Proposition 3.51 (again, we use that $\mathcal{F}$ is locally free here), so we only have cohomology in $a = r$; taking $\mathcal{F} = \omega_X^\circ$ produces (d).

It remains to show (b). Well, Theorem 3.48 tells us that

$$H^a(X,\mathcal{F}(-q))^\vee = H^a(\mathbb{P},\mathcal{F}(-q))^\vee = \mathrm{Ext}^{N-i}_{\mathcal{O}_\mathbb{P}}(\mathcal{F}(-q),\omega_\mathbb{P}) = \mathrm{Ext}^{N-i}(\mathcal{F},\omega(q)).$$

(Note the first equality is legal: $X \hookrightarrow \mathbb{P}$ is a closed embedding, so Corollary 2.53 says we're okay; the last equality is okay by Proposition 3.42.) Now, by Proposition 3.44, we see that this is

$$\Gamma\left(\mathbb{P},\mathcal{E}xt^{N-a}_{\mathcal{O}_\mathbb{P}}(\mathcal{F},\omega_\mathbb{P}(q))\right)$$

for q large enough. But this dies for $N - a > r$ (which means $a < n$) by Theorem 2.51, where we are moving cohomology back along our closed embedding.

- We show that (b) implies (d) implies (a). Define $\mathcal{E}^{N-a} := \mathcal{E}xt^{N-a}_{\mathcal{O}_\mathbb{P}}(\mathcal{F},\omega_\mathbb{P})$. Note that building some kind of locally free resolution and then computing $\mathcal{E}xt$ from that lets us move twists in and out of or $\mathcal{E}xt$, so we see

$$\mathcal{E}^{N-a}(q) = \mathcal{E}xt^{N-a}_{\mathcal{O}_\mathbb{P}}(\mathcal{F},\omega_\mathbb{P}(q)).$$

Now, $\mathcal{E}^{N-a}(q)$ is globally generated for q large enough because $\mathcal{O}(1)$ is ample, but this vanishes for q large enough by (b), so we conclude that $\mathcal{E}^{N-i}(q) = 0$ and thus $\mathcal{E}^{N-i} = 0$ by twisting back. By using $\mathcal{F} = \mathcal{O}_X$, we conclude that $\mathcal{E}^a(\mathcal{O}_X,\omega_\mathbb{P}) = 0$ for $a > N - n$, from which (d) follows by Lemma 3.53. (Again, we note that we vanish for $a < r$ already by Proposition 3.51.)

It remains to show (a) from (d). Now, taking stalks from what we just had, we have concluded that

$$\mathcal{E}xt^a_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x},\omega_{\mathbb{P},x}) = 0$$

for any closed point $x \in X$ and any $a > r$. Notably, $\omega_\mathbb{P} = \mathcal{O}_\mathbb{P}(-N-1)$, so $\omega_{\mathbb{P},x} = \mathcal{O}_{\mathbb{P},x}$. It follows by Proposition 3.62 that $\mathrm{pd}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x} \leq r$, so Proposition 3.64 tells us that $\mathrm{depth}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x} \geq n$. But then it follows that $\mathrm{depth}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x} = n$ because the depth cannot exceed dimension, so we conclude that

$$\mathrm{depth}\,\mathcal{O}_{X,x}\mathcal{O}_{X,x} = n$$

again by doing some argument about regular sequences in quotient rings. Because each closed point has the same depth equal to the dimension, we conclude that X is equidimensional and Cohen–Macaulay.

- We show that (ii) implies (iii). It is enough to show that $H^{n-a}(X,-)^\vee$ is co-effaceable (and thus universal), using the fact that θ^0 is an isomorphism already. Well, for our coherent sheaf, we pick up some surjection

$$\bigoplus_{i=1}^k \mathcal{O}(-q_i) \to \mathcal{F},$$

where we can make the q_i as large as we please by twisting. But these $\mathcal{O}(-q_i)$ have vanishing cohomology for q_i large enough by (ii), so we get co-effaceable.

- We show that (iii) implies (ii). Fix $\mathcal{F}$ locally free. Well, using our duality, we see

$$H^{n-a}(X, \mathcal{F}(-q))^{\vee} = \operatorname{Ext}^a(\mathcal{F}(-q), \omega_X^{\circ}).$$

Using Proposition 3.42, this is

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^{\vee} \otimes \omega_X^{\circ}(q)) = H^i(X, \mathcal{F}^{\vee} \otimes \omega_X^{\circ}(q)),$$

but this dies after high enough twisting by Theorem 3.23. ∎

3.12 April 5

Today we compute the dualizing sheaf.

3.12.1 Some Cohen–Macaulay Facts

We pick up a few more facts about Cohen–Macaulay rings whose proofs we omit.

Lemma 3.65. Fix a Noetherian Cohen–Macaulay local ring A. Then a sequence $x_1, \ldots, x_r \in A$ forms a regular sequence if and only if

$$\dim \frac{A}{(x_1, \ldots, x_r)} = \dim A - r.$$

Lemma 3.66. Fix a Noetherian Cohen–Macaulay local ring A. Then any regular sequence $x_1, \ldots, x_r$ of elements in A yields a Cohen–Macaulay ring $A/(x_1, \ldots, x_r)$.

Lemma 3.67. Fix a Noetherian Cohen–Macaulay local ring A. Given a regular sequence $x_1, \ldots, x_r$, set $I := (x_1, \ldots, x_r)$. Then I/I^2 is a free (A/I)-module of rank r.

Today we will also want the following definition.

Definition 3.68 (local complete intersection). Fix a closed subscheme $Y \subseteq X$, where X is a nonsingular k-variety. Then Y is a *local complete intersection* if and only if the ideal sheaf $\mathcal{I}_Y$ of Y is locally generated by $\operatorname{codim}_X Y$ elements.

Remark 3.69. By checking locally at each point, we see that local complete intersections are equidimensional. Namely, the dimension must be fully equal to $\dim Y$ at each closed point.

Note that the Principal ideal theorem says that $\mathcal{I}_Y$ cannot be locally generated by any fewer than $\operatorname{codim}_X Y$ elements.

Proposition 3.70. Fix a local complete intersection Y of a nonsingular k-variety X. Then Y is Cohen–Macaulay.

Proof. At each point $p \in X$, the local ring $\mathcal{O}_{X,p}$ is regular local. Then $\mathcal{O}_{Y,p} = \mathcal{O}_{X,p}/\mathcal{I}_{Y,p}$ for each $p \in Y$, so because Y is a local complete intersection, we note that the generators of $\mathcal{I}_{Y,p}$ must form a regular sequence in $\mathcal{O}_{X,p}$ by Lemma 3.65, so the quotient is Cohen–Macaulay by Lemma 3.66. ∎

3.12.2 Computing the Dualizing Sheaf

Before continuing, here's a quick corollary of Serre duality.

> **Corollary 3.71.** Fix a Cohen–Macaulay projective k-scheme X of equidimension n. For any locally free sheaf $\mathcal{F}$ of finite rank on X, we have
> $$H^i(X, \mathcal{F}) \cong H^{n-i}\left(\mathcal{F}^\vee \otimes \omega_X^\circ\right),$$
> where ω_X° is the dualizing sheaf.

Proof. By Theorem 3.55, observe that
$$H^i(X, \mathcal{F}) = \operatorname{Ext}_X^{n-i}(\mathcal{F}, \omega_X^\circ)^\vee = \operatorname{Ext}_X^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ)^\vee = H^{n-i}(\mathcal{F}^\vee \otimes \omega_X^\circ)^\vee,$$
which is what we wanted. ∎

This is a surprise tool which will help us later. We now work with nonsingular projective varieties.

> **Proposition 3.72.** Fix a closed subscheme X of $\mathbb{P}_k^N$ which is a local complete intersection of codimension r; let $\mathcal{I}$ be the ideal sheaf of X. Then
> $$\omega_X^\circ \cong \omega_{\mathbb{P}} \otimes \left(\bigwedge^r \mathcal{I}/\mathcal{I}^2\right)^\vee.$$
> Thus, ω_X° is an invertible sheaf on X.

Proof. Note that (locally) $\mathcal{I}/\mathcal{I}^2$ is free of rank r by Lemma 3.67, so the right-hand side of the desired expression is in fact locally free of rank 1. So the last sentence does follow from the desired isomorphism.

It remains to compute ω_X°. Recall that this is $\mathcal{E}xt^r(\mathcal{O}_X, \omega_{\mathbb{P}}) = R^r i^! \omega_{\mathbb{P}}$ by definition. Now, to compute $\mathcal{E}xt$, we would like a locally free resolution of $\mathcal{O}_X$, for which we use the Koszul complex. To be explicit, because $\mathcal{I}$ is locally generated by r elements, so for any point $p \in X$, we may find r germs $f_1, \ldots, f_r$ generating locally; taking the intersection of the open sets upon which the $f_\bullet$ are defined, we get to work with an affine open subscheme U.

Thus, setting $A := \Gamma(U, \mathcal{O}_U)$ and $I := \Gamma(U, \mathcal{I})$, we let $\varepsilon \colon J \to A$ (where $J := A^{\oplus r}$) be the map given by $(f_1, \ldots, f_r)$ so that we have the Koszul complex
$$0 \to J^{\wedge r} \xrightarrow{d} \cdots J^{\wedge 2} \xrightarrow{d} J^{\wedge 1} \xrightarrow{\varepsilon} J^{\wedge 0} \to A/I \to 0.$$
To be explicit, the maps d are given by
$$d(j_1 \wedge \cdots \wedge j_k) := \sum_{i=1}^k (-1)^{i+1} \varepsilon(j_i)(j_1 \wedge \cdots \wedge \widehat{j_i} \wedge \cdots \wedge j_k).$$

On the homework, we showed that this is exact when the sequence $f_1, \ldots, f_r$ is regular, the above sequence is exact; for example, this will be exact when localized at our point p. Because localization commutes with taking homology, we see that the Koszul homology is killed upon localizing at p; however, we can instead localize by precisely the elements needed to kill homology (which exist because they are in the denominators of A_p) and then replace $U = \operatorname{Spec} A$ by the distinguished affine open subscheme given by localizing these elements. In total, by shrinking U a little further, we may assume that our Koszul complex is fully exact.

Now, we can take $\widetilde{\ }$ everywhere to produce a locally free resolution $\widetilde{K}_\bullet$ of $\mathcal{O}_X|_U$, so we see
$$\mathcal{E}xt_U^r(\mathcal{O}_X, \omega_{\mathbb{P}}) = H^r \mathcal{H}om(\widetilde{K}_\bullet, \omega_{\mathbb{P}}) = H^r\left(\omega_{\mathbb{P}} \otimes \widetilde{K}_\bullet^\vee\right).$$

We now have to actually compute what these maps are given by. Writing everything out at degree r, we are looking at the following.

$$
\begin{array}{ccccc}
\omega_{\mathbb{P}} \otimes \left(\widetilde{\wedge^{r-1}J}\right)^{\vee} & \longrightarrow & \omega_{\mathbb{P}} \otimes \left(\widetilde{\wedge^{r}J}\right)^{\vee} & \longrightarrow & 0 \\
\| & & \| & & \\
\displaystyle\bigoplus_{i=1}^{r}\omega_{\mathbb{P}} & \longrightarrow & \omega_{\mathbb{P}} & \longrightarrow & 0
\end{array}
$$

We can compute out what this dualized map does, but the point is that it will send the ith copy of $\omega_{\mathbb{P}}$ to $f_i\omega_{\mathbb{P}}$, so our cohomology here is given by $\omega_{\mathbb{P}}/(f_1, \ldots, f_r)\omega_{\mathbb{P}} = \omega_{\mathbb{P}} \otimes \mathcal{O}_X|_U$, as desired.

We would like to glue this local computation into a global computation, but this is a little tricky. Namely, the concern is that we made a choice in choosing the f_i, so we need to consider what happens when we choose some different f_i given by f_i'. Then we have some transition function $T\colon J \to J$ sending the basis of the f_i to the basis of the f_i'. Notably, taking determinants produces the map $(\det T)\colon J^{\wedge r} \to J^{\wedge r}$ corresponding to the determinant of our change-of-basis matrix. In particular, we can fix our non-canonical isomorphism

$$
\mathcal{E}xt_U^r(\mathcal{O}_X, \omega_{\mathbb{P}}) \cong \omega_{\mathbb{P}} \otimes \mathcal{O}_X|_U
$$

into a more canonical isomorphism

$$
\mathcal{E}xt_U^r(\mathcal{O}_X, \omega_{\mathbb{P}}) \cong (\omega_{\mathbb{P}} \otimes \mathcal{O}_X|_U) \otimes \left(\left(\mathcal{I}/\mathcal{I}^2\right)^{\wedge r}\right)^{\vee}|_U.
$$

Namely, over U, the change of coordinates arising from $\mathcal{O}_X$ will cause $\mathcal{O}_X$ to change by $\det T$, but then the isomorphism $\wedge^r \mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X$ will end up changing by exactly $\det T$ as well, so dualizing fixes this change. Thus, we do have a canonical isomorphism as stated. ∎

Corollary 3.73. Fix an integral projective nonsingular k-variety X. Then $\omega_X^{\circ} = \omega_X$.

Proof. Because X is nonsingular, it is regular and hence a local complete intersection. Thus, fixing an embedding $i\colon X \hookrightarrow \mathbb{P}_k^N$ giving X codimension r, we see

$$
\omega_X^{\circ} = \omega_{\mathbb{P}} \otimes \left(\bigwedge^r \mathcal{I}/\mathcal{I}^2\right)^{\vee}.
$$

However, this is indeed ω_X. Namely, being smooth grants us the short exact sequence

$$
0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}} \otimes \mathcal{O}_X \to \Omega_X \to 0.
$$

Then we can take determinants everywhere to produce

$$
\underbrace{\Omega_X^{\wedge(N-r)}}_{\omega_X} \otimes \left(\mathcal{I}/\mathcal{I}^2\right)^{\wedge r} \cong \underbrace{\Omega_{\mathbb{P}}^{\wedge N}}_{\omega_{\mathbb{P}}},
$$

which is what we wanted after combining with Proposition 3.72. ∎

3.13   April 7

Today is Friday. We're finally done with Serre duality. It's going to be a good day.

3.13.1 Higher Direct Images

Fix a continuous map $f\colon X \to Y$ of topological spaces. This then produces a functor $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$, which is our direct-image functor. This functor is left-exact because taking sections is left-exact, and we can check left-exactness on the level of sections. To be explicit, given a left-exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'',$$

we get the left-exact sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

for any open subset $U \subseteq X$. Thus, we have the left-exact sequence

$$0 \to f_*\mathcal{F}'(V) \to f_*\mathcal{F}(V) \to f_*\mathcal{F}''(V)$$

for any open subset $V \subseteq Y$, so we conclude that $f_*\mathcal{F}' = \ker(f_*\mathcal{F}) \to f_*\mathcal{F}''$) by definition of the kernel. Anyway, the point is that we can right-derive.

> **Definition 3.74.** Fix a continuous map $f\colon X \to Y$ of topological spaces. Then we define $R^i f_*$ to be the ith right-derived functor of the left-exact functor $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$.

Now, these sheaves are nice because they turn out to be essentially sheafifications of cohomology.

> **Proposition 3.75.** Fix a continuous map $f\colon X \to Y$ of topological spaces and sheaf $\mathcal{F}$ on X. Then for each $i \geq 0$, $R^i f_*\mathcal{F}$ is the sheafification of the presheaf (on Y) given by
> $$V \mapsto H^i\left(f^{-1}V, \mathcal{F}\right).$$

Proof. If $i = 0$, we are looking at $f_*\mathcal{F}(V) = \Gamma\left(f^{-1}V, \mathcal{F}\right) = H^0\left(f^{-1}V, \mathcal{F}\right)$. So it remains to show that each functor is universal. The right-derived functor is a universal δ-functor because it is derived.

Next, define $\mathcal{H}^i$ to be the functor defined in the proposition. One can see that R^i is a right-derived functor and hence universal. Before sheafification, we send short exact sequences of presheaves to long exact sequences of presheaves, but sheafification is exact: sheafification is a left adjoint of the forgetful functor, so it preserves finite colimits. But also sheafification preserves finite products (which are finite coproducts), so it remains to check that it preserves kernels. This can be checked directly.

Anyway, the point is that $\mathcal{H}^i$ is a δ-functor by taking our long exact sequence of presheaves and taking sheafification. Next, to be effaceable, we note that our category has enough injectives, so we may embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$, but then the map

$$H^i\left(f^{-1}V, \mathcal{F}\right) \to H^i\left(f^{-1}V, \mathcal{I}\right)$$

is the zero map because $\mathcal{I}$ is injective (and will remain flasque and hence acyclic on any open subset $f^{-1}V \subseteq X$). Taking sheafification will retain this as the zero map (indeed, $\mathcal{H}^i(\mathcal{I}) = 0$ still), so we get that $\mathcal{H}^i$ is effaceable. $\blacksquare$

> **Corollary 3.76.** Fix a continuous map $f\colon X \to Y$ of topological spaces and sheaf $\mathcal{F}$ on X. If $V \subseteq Y$ is open, then $\left(R^i f_*\mathcal{F}\right)|_V = R^i f'_*\left(\mathcal{F}|_{f^{-1}V}\right)$, where $f'\colon f^{-1}V \to V$ is the restricted map.

Proof. Sheafification commutes with restriction (just look directly at the construction in sheafification as taking compatible stalks), so we see that $\left(R^i f_*\mathcal{F}\right)|_V$ and $R^i f'_*\left(\mathcal{F}|_{f^{-1}V}\right)$ are both sheafifications of the presheaf

$$V' \mapsto H^i\left(f^{-1}V', \mathcal{F}\right)$$

by Proposition 3.75. $\blacksquare$

> **Corollary 3.77.** Fix a continuous map $f\colon X \to Y$ of topological spaces and sheaf $\mathcal{F}$ on X. If $\mathcal{F}$ is flasque, then $R^i f_* \mathcal{F} = 0$ for any $i > 0$.

Proof. By Proposition 3.75, we see that $R^i f_* \mathcal{F}$ is the sheafification of the presheaf

$$V \mapsto H^i\left(f^{-1}V, \mathcal{F}\right).$$

However, the right-hand side here always vanishes because $\mathcal{F}$ is flasque (and remains so upon restriction to $f^{-1}V$), so we are looking at the zero presheaf, which yields the zero sheaf upon sheafification. ∎

> **Corollary 3.78.** Fix a continuous map $f\colon X \to Y$ of ringed spaces and sheaf $\mathcal{F}$ on X. Then $R^i f_*$ is the ith right-derived of $f_*\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$.

Proof. We certainly have $R^0 f_* = f_*$ as needed, and both functors can use the same acyclic resolutions to compute cohomology, namely flasque resolutions by Corollary 3.77. This is what we wanted. ∎

> **Proposition 3.79.** Fix a Noetherian scheme X, and let $f\colon X \to Y$ be a map to an affine scheme Y. For a quasicoherent sheaf $\mathcal{F}$ on X, we have $R^i f_* \mathcal{F} = \widetilde{H^i(X, \mathcal{F})}$.

Proof. In degree 0, we know that f_* sends quasicoherent sheaves to quasicoherent sheaves because X is Noetherian. Thus, we use that we are quasicoherent to observe that

$$f_* \mathcal{F} = \Gamma\widetilde{(Y, f_* \mathcal{F})} = \Gamma\widetilde{(X, \mathcal{F})} = \widetilde{H^0(X, \mathcal{F})},$$

so we have our isomorphism in degree-0. Thus, it remains to show that we have universal δ-functors. On one hand, $R^i f_*$ is a right-derived functor and hence a universal δ-functor.

On the other hand, $\widetilde{H^i(X, -)}$ does send short exact sequences to first the long exact sequence for $H^i(X, -)$ and then to a long exact sequence for $\widetilde{H^i(X, -)}$ by taking the exact functor $\widetilde{}$ everywhere. Lastly, to be effaceable, we embed $\mathcal{F}$ into an injective quasicoherent sheaf $\mathcal{I}$, and then the morphism

$$\widetilde{H^i(X, \mathcal{F})} \to \widetilde{H^i(X, \mathcal{I})}$$

is the zero map because the map $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{I})$ is the zero map because $H^i(X, \mathcal{I}) = 0$ because $\mathcal{I}$ is injective. ∎

> **Corollary 3.80.** Fix a Noetherian scheme X, and let $f\colon X \to Y$ be a map to an affine scheme Y. For a quasicoherent sheaf $\mathcal{F}$ on X, $R^i f_* \mathcal{F}$ remains quasicoherent.

Proof. For any affine open subscheme V, we use Corollary 3.76 to see that

$$(R^i f_* \mathcal{F})|_V = R^i f_*\left(\mathcal{F}|_{f^{-1}V}\right).$$

Now, $\mathcal{F}|_{f^{-1}V}$ is quasicoherent, and now $f'\colon f^{-1}V \to V$ is a map to an affine scheme, so we use Proposition 3.79 to write $R^i f_*\left(\mathcal{F}|_{f^{-1}V}\right)$ as $\widetilde{M}$ for some module M on $\mathcal{O}_V(V)$. ∎

3.13.2 Higher Direct Images for Projective Morphisms

Recall the following definition.

> **Definition 3.81** (projective). A scheme morphism $f\colon X \to Y$ is *projective* if and only if it factors through a closed immersion into $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ followed by the projection to Y.

And here is our result.

> **Theorem 3.82.** Fix a projective morphism $f\colon X \to Y$ of Noetherian schemes; let $\mathcal{O}(1)$ be a very ample line bundle on X, and let $\mathcal{F}$ be a coherent sheaf on X.
>
> (a) The map $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for n large enough.
>
> (b) $R^i f_* \mathcal{F}$ are coherent sheaves.
>
> (c) For any $i > 0$, we have $R^i f_* \mathcal{F}(n) = 0$ for n large enough.

Proof. Here we go.

- To show (b) and (c), we note that Corollary 3.76 lets us reduce to an affine open subscheme $\operatorname{Spec} A$ of Y and then pull it back to an affine open subscheme in the pre-image in X. Then with $Y = \operatorname{Spec} A$, we see
$$R^i f_* \mathcal{F} = \widetilde{H^i(X, \mathcal{F})}.$$
Now, by Theorem 3.23, we see that $H^i(X, \mathcal{F})$ is in fact finitely generated over A and vanishes in high enough twisting. This gives (b) automatically, and (c) follows by twisting far enough to kill cohomology on each affine open subscheme of some fixed finite affine open cover of the quasicompact scheme X.

- To show (a), we again may assume that Y is affine because surjectivity may be checked at stalks in X and is thus local on X and thus on Y. So set $Y = \operatorname{Spec} B$, and on some affine open subscheme $U = \operatorname{Spec} A$ of X, we see
$$(f^* f_* \mathcal{F}(n))(U) = \Gamma(U, \mathcal{F}(n)) \otimes_B A$$
by computing global sections. The point is that for n large enough we are looking at the map
$$\Gamma(U, \mathcal{F}(n)) \otimes_B A \to \Gamma(U, \mathcal{F}(n)),$$
which is indeed surjective for n large enough, which I guess can be checked directly. ∎

What?

Continuing, we note that we can even compute our higher direct images via some version of Čech cohomology.

> **Proposition 3.83.** Fix a map $f\colon X \to Y$ of separated Noetherian schemes. Fix a quasicoherent sheaf $\mathcal{F}$ on X, and equip X with affine open cover $\mathfrak{U} := \{U_\alpha\}_{\alpha \in \lambda}$ so that we have the Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$. Let $C^\bullet(\mathfrak{U}, \mathcal{F})^{\mathrm{sh}}$ be the natural sheafification. Then
> $$R^p f_* \mathcal{F} = H^p \left(f_* C^\bullet(\mathfrak{U}, \mathcal{F})^{\mathrm{sh}} \right).$$

Proof. To be explicit, on an open subset $U \subseteq X$, one has
$$C^p(\mathfrak{U}, \mathcal{F})^{\mathrm{sh}} = \prod_{i_0 < \cdots < i_p} j_* \left(\mathcal{F}|_{U_{i_0 \cdots i_p}} \right),$$
where j is the relevant extension by zero. (Note that $U_{i_0 \cdots i_p}$ is affine because X is separated.) The point is that these remain to be quasicoherent sheaves because the restriction and then extension by zero (from a

Noetherian scheme) of a quasicoherent sheaf remain quasicoherent. For example, if Y is affine, then we use the fact that

$$H^p(X, \mathcal{F}) = H^p C^\bullet(\mathfrak{U}, \mathcal{F})$$

from our discussion of Čech cohomology, so it follows that

$$R^p f_* \mathcal{F} = \widetilde{H^p(X, \mathcal{F})} = \widetilde{H^p C^\bullet(\mathfrak{U}, \mathcal{F})},$$

which gives the desired result after comparing with the quasicoherent complex $f_* C^\bullet(\mathfrak{U}, \mathcal{F})^{\mathrm{sh}}$. (Explicitly, one should compute that everything matches on global sections and then use the fact that our sheaves are quasicoherent.) It remains to reduce to the case where Y is affine, but this is not hard: for affine open subscheme $V \subseteq Y$, we see that $f^{-1}V \cap U_{i_0 \cdots i_p}$ is also affine. (Namely, a separated morphism out of an affine scheme is affine, so the map out of $U_{i_0 \cdots i_p} \to Y$ is affine, so taking the pre-image of V will give an affine open subscheme.) ∎

3.14   April 10

There are many cookies to be eaten. The homework is now due on Friday.

3.14.1   Flatness for Rings

Today we're talking about flat morphisms. Our goal today is to prove flat base-change. The point here is that the fibers are supposed to communicate with each other. Anyway, our definition comes from commutative algebra.

> **Definition 3.84** (flat)**.** Fix a ring A. Then an A-module M is *flat* if and only if the functor $M \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_A$ is exact.

> **Remark 3.85.** Because $M \otimes_A -$ is already right-exact automatically, the functor $M \otimes_A -$ is flat if and only if injections $N' \hookrightarrow N$ remain injections $M \otimes_A N' \hookrightarrow M \otimes_A N$.

> **Remark 3.86.** There is also a notion of "faithfully flat" which means that $M \otimes_A -$ also "reflects" exactness. Namely, we require that a sequence
> $$0 \to N' \to N \to N'' \to 0$$
> is exact if and only if
> $$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$
> is exact.

Here are some properties of flat morphisms.

> **Lemma 3.87.** Fix a ring A. Then an A-module M is flat if and only if the morphism $I \otimes_A M \to M$ is injective for any finitely generated A-ideal I. In other words, we require the natural map $I \otimes_A M \to IM$ to be an isomorphism.

Proof. The forward direction simply applies flatness to the injection $I \hookrightarrow A$. The backward direction is just a bunch of reductions; we omit the details. ∎

Lemma 3.88 (Base extension). Fix a B-algebra A. If an A-module M is flat, then $M \otimes_A B$ is flat over B.

Proof. Fix an injection $N' \hookrightarrow N$ of B-modules. Then we have the commutative diagram

$$
\begin{array}{ccc}
(M \otimes_A B) \otimes_B N' & \longrightarrow & (M \otimes_A B) \otimes N \\
\downarrow \simeq & & \downarrow \simeq \\
M \otimes_A N' & \longrightarrow & M \otimes_A N
\end{array}
$$

whose vertical arrows are isomorphisms. The bottom row is injective because M is flat over A, so the top row is also injective. ∎

Lemma 3.89 (Transitivity). Fix a flat A-algebra B. If M is flat as a B-module, then it is also flat as an A-module.

Proof. As usual, fix an injection of A-modules $N' \hookrightarrow N$. Then the map $N' \otimes_A B \to N \otimes_A B$ is injective because B is a flat A-algebra, so

$$N' \otimes_A B \otimes_B M \to N \otimes_A B \otimes_B M$$

is injective because M is flat over B. But the above map is isomorphic to $N' \otimes_A M \to N \otimes_A M$, so we are done. ∎

Lemma 3.90. Fix a ring A. Then an A-module M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec} A$.

Proof. The forward direction is from Lemma 3.88 because $M_{\mathfrak{p}} \simeq M \otimes_A A_{\mathfrak{p}}$. For the backward direction, this is the statement that exactness can be checked on the level of stalks combined with the natural isomorphism $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \simeq (M \otimes_A N)_{\mathfrak{p}}$. ∎

Lemma 3.91. Fix a ring A. For any multiplicatively closed subset $S \subseteq A$, the A-algebra $A\left[S^{-1}\right]$ is flat.

Proof. Fix an inclusion $N' \hookrightarrow N$, and we want to show that the map $N'\left[S^{-1}\right] \to N\left[S^{-1}\right]$ is still injective. But given $\frac{x'}{s}$ going to 0 in $N\left[S^{-1}\right]$, we see that $s'x' = 0$ for some $s' \in 0$, but this means that $\frac{x'}{s'} = 0$ back in $N'\left[S^{-1}\right]$. ∎

Lemma 3.92. Fix an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-modules. If M' and M'' is flat, them M is flat. Also, if M and M'' are flat, then M' is flat.

Proof. We use Lemma 3.87. Fix an A-ideal I. Then we have the following commutative diagram.

$$
\begin{array}{ccccccc}
I \otimes_A M' & \longrightarrow & I \otimes_A M & \longrightarrow & I \otimes_A M'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}
$$

For the first claim, we simply note that the left and right vertical arrows are injective, so the middle vertical arrow is injective, which completes the proof. For the second claim, I think it follows directly from the Tor characterization of flatness. ∎

Lemma 3.93. Fix a Noetherian local ring A. If M is a finitely generated flat A-module, then M is free.

Proof. Omitted. ∎

Let's see some examples.

Example 3.94. For any A-ideal I, the completion

$$\widehat{A} := \lim_n A/I^n$$

is flat. We will not show this; it is mildly technical.

Example 3.95. Fix a principal ideal domain A. Then we claim that M is flat if and only if M is torsion-free. Indeed, by Lemma 3.87, we are asking for the map $(a) \otimes_A M \to M$ to be injective for any $a \in A$, which is equivalent to a not being a zero-divisor on M. Looping over all $a \in A$ says that this is equivalent to M being torsion-free.

3.14.2   Flatness for Sheaves

We now translate our lemmas above to algebraic geometry.

Definition 3.96 (flat)**.** Let $f\colon X \to Y$ be a morphism of schemes. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at a point* $x \in X$ if and only if the stalk $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,f(x)}$-module. Then $\mathcal{F}$ is *flat* if and only if it is flat at each point $x \in X$. Lastly, f is *flat* if and only if $\mathcal{O}_X$ is flat.

Remark 3.97. If $\mathcal{F}$ is a quasicoherent sheaf over X, then we may check flatness on affine open subschemes by Lemma 3.90. Explicitly, given an A-algebra B and a B-module M, then M is flat over A if and only if $\widetilde{M}$ is flat over $\mathrm{Spec}\, A$.

Here are the usual checks.

Lemma 3.98. An open embedding $f\colon X \to Y$ is flat.

Proof. Stalks don't change in open embeddings, so there is nothing to say. ∎

Lemma 3.99 (Base change)**.** Fix a morphism of schemes $f\colon X \to Y$, and let $g\colon X' \to X$ be some flat morphism. If $\mathcal{F}$ is a flat $\mathcal{O}_X$-module, then $g^*\mathcal{F}$ is flat over X'.

Proof. Check at stalks. ∎

Lemma 3.100 (Transitivity)**.** Fix scheme morphisms $f\colon X \to Y$ and $g\colon Y \to Z$. Given an $\mathcal{O}_X$-module $\mathcal{F}$ flat over Y, then $\mathcal{F}$ is also flat over Z.

Proof. Check at stalks and use transitivity. ∎

> **Lemma 3.101.** Fix a Noetherian scheme X. Then a coherent sheaf $\mathcal{F}$ is flat over X if and only if $\mathcal{F}$ is locally free.

Proof. Check at stalks and use Lemma 3.93. ∎

3.14.3   Flat Base-Change

Anyway, here is our main result.

> **Theorem 3.102.** Fix a flat finite-type separated morphism $f\colon X \to Y$ of Noetherian schemes. Further, let $u\colon Y' \to Y$ be a flat morphism of Noetherian schemes, and let $v\colon X \times_Y Y' \to X$ be the base-change of u and $g\colon X \times_Y Y' \to Y$ be the base-change of g. Given a quasicoherent sheaf $\mathcal{F}$ on X, we have natural isomorphisms
> $$u^* R^i f_* \mathcal{F} \to R^i g_* v^* \mathcal{F}.$$

Proof. Here is our pullback square.

$$
\begin{array}{ccc}
X' & \xrightarrow{\;v\;} & X \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\;u\;} & Y
\end{array}
$$

Everything is local on Y and Y', so we may replace those with affine open subschemes, where the statement reduces to a statement about bona fide cohomology. The statement essentially reduces to showing an isomorphism
$$H^i(X, \mathcal{F}) \otimes_A A' \cong H^i(X \times_Y Y', v^* \mathcal{F}).$$

Now, using Čech cohomology via to compute cohomology on $X \times_Y Y'$, we want some equivalence of Čech complexes, which we can compute directly. ∎

3.15   April 12

For the next few lectures, we are going to lecture from Mumford.

3.15.1   A Corollary of Flat Base-Change

Last class we roughly proved Theorem 3.102. Here is a nice corollary.

> **Corollary 3.103.** Fix a finite-type separated morphism $f\colon X \to Y$ of Noetherian schemes. Further, suppose that Y is affine, and y is a point in Y, and we let $X_y := X \times_Y k(y)$ denote the fiber of f over $y \in Y$. Letting $\underline{k(y)}$ be the constant sheaf on $\overline{\{y\}}$, we have
> $$H^i(X_y, \mathcal{F}_y) \cong H^i(X, \mathcal{F} \otimes_k \underline{k(y)}).$$

Proof. Set $Y' := \overline{\{y\}}$ be the reduced closed subscheme in Y. Let $i\colon Y' \to Y$ and $j\colon X' \to X$ be the corresponding embeddings. Then we see that both sides of our equation depend only on $\mathcal{F}_y = i^* \mathcal{F}$ and not on all of $\mathcal{F}$; namely, we can compute that
$$H^i(X, \mathcal{F} \otimes \underline{k(y)}) = H^i(X, i_*(i^* \mathcal{F} \otimes \underline{k(y)})) = H^i(X', i^* \mathcal{F} \otimes \underline{k(y)})$$

because of the support of $\underline{k(y)}$ is zero outside Y'; namely, we are using the fact that the closed embedding $X' \to X$ will not change our cohomology. Thus, we may replace X with X' and $\mathcal{F}$ with $i^*(\mathcal{F} \otimes k(y))$ and Y

with Y'. In particular, we are now assuming that Y is integral with generic point given by y. Observe that Y continues to be affine because Y' is the closed subscheme of the affine scheme Y.

The point is that $u\colon \operatorname{Spec} k(y) \to Y$ is a flat map (namely, we are embedding an integral domain into its fraction field). Now, let $Y = \operatorname{Spec} A$, and we compute

$$u^* R^i f_* \mathcal{F} = H^i(X, \widetilde{\mathcal{F}}) \otimes_A k(y) \quad \text{and} \quad R^i f'_*(u')^* \mathcal{F} = H^i(\widetilde{X_y, \mathcal{F}_y}) = H^i(X_y, \widetilde{\mathcal{F} \otimes_A k(y)}).$$

Now, by our reduction replacing $\mathcal{F}$ with $i^*(\mathcal{F} \otimes k(y))$, we see that $\mathcal{F}$ is in fact a sheaf of $k(y)$-modules, so the tensor product on both sides does not really do anything. So we finish by Theorem 3.102. ∎

Example 3.104. If y is a closed point, then we can just directly compute

$$H^i(X_y, \mathcal{F}_y) = H^i(X_y, i^*\mathcal{F}) = H^i(X, i_* i^*\mathcal{F}) = H^i(X, \mathcal{F} \otimes \underline{k(y)}),$$

where we have used the fact that closed embeddings do not change cohomology.

3.15.2 Associated Primes

We are going to define associated points for sheaves of modules. The rough idea is that the associated points are "generic points" for the sheaf. For example, the associated points of the structure sheaf of a closed subscheme are going to be the generic points of the irreducible components of the closed subscheme. Our goal, for now, is to prove the following result.

Proposition 3.105. Fix a morphism $f\colon X \to Y$ of Noetherian schemes. If the coherent sheaf $\mathcal{F}$ is flat over Y, then associated points of $\mathcal{F}$ get mapped by f to associated points of $\mathcal{O}_Y$.

Anyway, here is our definition of associated point.

Definition 3.106 (associated point)**.** Fix a coherent sheaf $\mathcal{F}$ on a Noetherian scheme X. Then the set of *associated points* of $\mathcal{F}$ is

$$A(\mathcal{F}) := \{x \in X : \text{there is } s \in \mathcal{F}_x \text{ annihilated by } I \subseteq \mathcal{O}_{X,x} \text{ where } \operatorname{rad} I = \mathfrak{m}_x\}.$$

Remark 3.107. Quickly, $x \in A(\mathcal{F})$ is equivalent to having an open neighborhood $U \subseteq X$ of x and section $s \in \mathcal{O}_X(U)$ such that $\operatorname{supp} s = \overline{\{x\}}$. Indeed, having $s \in \mathcal{F}_x$ annihilated with annihilator which is not $\mathfrak{m}_x$-primary—i.e., $\operatorname{Ann} s$ is not contained in $\mathfrak{m}_x$—means that s is supported on a closed subscheme of $\operatorname{Spec} \mathcal{O}_{X,x}$, which means that the support of s is smaller than $\overline{\{x\}}$. On the other hand, if s is not supported at x, then it vanishes on the stalk.

Remark 3.108. The generic points of $\operatorname{Supp} \mathcal{F}$ will thus be contained in $A(\mathcal{F})$, but the containment might be strict. However, if $\mathcal{F} = \mathcal{O}_Z$ for some reduced closed subscheme $Z \subseteq X$, then $A(\mathcal{O}_Z)$ will indeed be the generic points of Z.

Remark 3.109. We also note that $A(\mathcal{F})$ is always finite.

Associated points also relate to depth.

Lemma 3.110. Fix a Noetherian scheme Y. Then $y \in A(\mathcal{O}_Y)$ if and only if $\operatorname{depth} \mathcal{O}_{Y,y} = 0$.

Proof. Saying that $\operatorname{depth} \mathcal{O}_{Y,y} = 0$ is really requiring that all non-units in $\mathcal{O}_{Y,y}$ to fail to be able to be in a regular sequence, which means that all non-units are zero-divisors. In one direction, if $y \in A(\mathcal{O}_Y)$, then we may find $s \in \mathcal{O}_{Y,y}$ annihilated by $I \subseteq \mathcal{O}_{X,x}$ with $\operatorname{rad} I = \mathfrak{m}_x$. In other words, for any $f \in \mathfrak{m}_y$, we have $f^n \in I$ for some n large enough has $f^n s = 0$, meaning that f is a zero-divisor. We omit the proof of the other direction, for now. ∎

In commutative algebra, we also have an analogous notion of associated primes.

> **Definition 3.111** (associated prime). Fix an R-module M. Then a prime $\mathfrak{p}$ is *associated to* M if and only if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{Ann} m$. The set of all associated primes will be denoted $A(M)$.

> **Remark 3.112.** If R is Noetherian and M is finitely generated over R, then we can show that $A(M)$ is finite. Additionally, $A(M)$ still contains the generic points of $\operatorname{Supp} M$.

We are going to want the following results.

> **Lemma 3.113.** Fix a Noetherian ring R. Then a finitely generated module M is nonzero if and only if $A(M) \neq \varnothing$.

Proof. Of course, if M is zero, then $A(M) = \varnothing$. However, if M is nonzero, then $A(M)$ will contain the generic points of $\operatorname{Supp} M$. Proving this previous claim is a little technical. Roughly speaking, the idea is to build a filtration of M as

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M,$$

where M_i/M_{i+1} is $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. (Roughly speaking, one can do this by an induction.) This filtration tells us what the associated primes should be. ∎

3.16 April 14

We continue discussing flatness. Throughout, X and Y are Noetherian schemes, and $\mathcal{F}$ is a coherent sheaf on X.

3.16.1 Flatness and Associated Primes

We would like to show the opposite direction of Lemma 3.110. In the process, we are building some notion of associated primes. We have the following proposition.

> **Proposition 3.114.** Let M be a finitely generated module M over a Noetherian ring R. Then
>
> $$\bigcup_{\mathfrak{q} \in A(M)} \mathfrak{q}$$
>
> is the set of zero-divisors on M.

The key point of the above proposition is that $A(M)$ is finite, so we are taking a finite union!

Proof. Of course, if $x \in \mathfrak{q}$ for $\mathfrak{q} \in A(M)$, then we can find $m_{\mathfrak{q}} \in M$ with annihilator given by $\mathfrak{q}$, so $x \cdot m_{\mathfrak{q}} = 0$, so x is a zero-divisor on M.

Conversely, if $x \in R$ is a zero-divisor on M, define

$$N := \{m \in M : xm = 0\}.$$

But now N is nonzero by construction of x, so it has an associated prime $\mathfrak{p}$. Thus, we can embed

$$R/\mathfrak{p} \hookrightarrow N \subseteq M,$$

so $\mathfrak{p}$ is an associated prime of M. Lastly, we note $x \in \mathfrak{p}$ because $\mathfrak{p}$ is the annihilator of some $m \in N$, but $x \in \operatorname{Ann} m$ because $m \in N$, so $x \in \mathfrak{p}$ follows. ∎

And here is our opposite direction.

Proof of converse of Lemma 3.110. Suppose that $\mathfrak{m}_y$ consists only of zero-divisors. Then we see

$$\mathfrak{m}_y = \bigcup_{\mathfrak{q} \in \operatorname{Ass} \mathcal{O}_{Y,y}} \mathfrak{q},$$

so $\mathfrak{m}_y$ is a finite union of primes. Thus, by prime avoidance, we see that $\mathfrak{m}_y$ must actually be contained in one of these primes, so because $\mathfrak{m}_y$ is a maximal ideal, we conclude that $\mathfrak{m}_y = \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass} \mathcal{O}_{Y,y}$. Thus, $\mathfrak{m}_y$ is associated, so $y \in A(\mathcal{O}_Y)$. ∎

We are now ready to prove Proposition 3.105.

> **Proposition 3.105.** Fix a morphism $f\colon X \to Y$ of Noetherian schemes. If the coherent sheaf $\mathcal{F}$ is flat over Y, then associated points of $\mathcal{F}$ get mapped by f to associated points of $\mathcal{O}_Y$.

Proof of Proposition 3.105. Fix $x \in A(\mathcal{O}_X)$ and $y := f(x)$. Suppose for contradiction that $f(x) \notin A(\mathcal{O}_Y)$. By Lemma 3.110, we see that there is a non-zero-divisor $a \in \mathfrak{m}_y$. Thus, the embedding $a\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}$ remains injective after tensoring with $\mathcal{F}_x$, so $a\colon \mathcal{F}_x \to \mathcal{F}_x$ is injective. Inductively, we note that any $n > 0$ has

$$a^n \colon \mathcal{F}_x \to \mathcal{F}_x$$

also has trivial kernel and is therefore injective. We are now essentially done. Indeed, we show there is no $s \in \mathcal{F}_x$ killed by an ideal primary to $\mathfrak{m}_x$, for we can see that any s is annihilated by $I \subseteq \mathcal{O}_{X,x}$, but this ideal ought to be contained in $\mathfrak{m}_x$ and hence maps to $\mathfrak{m}_y$ but will then contain a pre-image of a power of a. Thus, $\operatorname{rad} I \neq \mathfrak{m}_x$ because no power of $a \in \mathfrak{m}_x$ can land in I. ∎

> **Remark 3.115.** Fix a Noetherian affine scheme $X = \operatorname{Spec} A$. Then we claim that $A(\mathcal{O}_X) = A(A)$. Of course $A(M) \subseteq A(\widetilde{M})$ because having a prime actually annihilate an element $a \in M$ yields our section $a_\mathfrak{p} \in A_\mathfrak{p}$ killed by exactly the maximal ideal $\mathfrak{p}A_\mathfrak{p} \subseteq A_\mathfrak{p}$. Conversely, if $x \in A(\mathcal{O}_X)$, then Lemma 3.110, we note $\mathfrak{m}_x A_x$ consists of zero-divisors on A, so $\mathfrak{m}_x$ is in $\operatorname{Ass}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$. This implies that $x \in \operatorname{Ass}_A A$ as soon as we note that
> $$\operatorname{Ass}_A M \cap \operatorname{Spec} A\left[S^{-1}\right] = \operatorname{Ass}_A M\left[S^{-1}\right].$$

3.16.2 Flatness Facts

As an aside, there is a strengthening of Proposition 3.105.

> **Proposition 3.116.** Fix a morphism $f\colon X \to Y$ of Noetherian schemes. If the coherent sheaf $\mathcal{F}$ is flat over Y, then $x \in A(\mathcal{F})$ if and only if $f(x) \in A(\mathcal{O}_Y)$ and $x \in A(\mathcal{F} \otimes k(f(x)))$.

We will not prove this.

> **Remark 3.117.** In fact, under the same set-up, one can be a little more precise: one can compute
> $$\operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \operatorname{depth} \mathcal{O}_{Y,f(x)} + \operatorname{depth}_{\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}}(\mathcal{F}_x \otimes k(y))$$

Here is a last result.

> **Theorem 3.118** (Miracle flatness)**.** Fix a homomorphism $\varphi\colon R \to S$ of local Noetherian rings. If R is regular, and S is Cohen–Macaulay, and $\dim S = \dim R + \dim S/\mathfrak{m}_R S$, then the map φ is flat.

We will also not prove this. Here are two useful points about flatness.

> **Proposition 3.119.** Fix a scheme morphism $f\colon X \to Y$ is a morphism of schemes. Then the "flat locus" is open: if f is flat at $x \in X$, then f is flat on an open neighborhood of x. Further, if f is flat and locally of finite presentation, then f is open.

Proof. Again, omitted, even though this is quite a nice result. The first result is a spreading our argument, and the second result uses going-down for flat ring maps. ∎

Here is another strengthening of Proposition 3.105.

> **Proposition 3.120.** Fix a morphism $f\colon X \to Y$ of Noetherian schemes, where Y is a regular curve. Further, fix a coherent sheaf $\mathcal{F}$ on X. If $x \in A(\mathcal{F})$ implies $f(x)$ is the generic point of Y, then f is flat.

Proof. The point is to apply force based on what we know about curves. Suppose that $\mathcal{F}$ is not flat over the scheme Y so that $\mathcal{F}$ is not flat over the point $x \in X$. Namely, $\mathcal{F}_x$ is not flat over $\mathcal{O}_{Y,y}$. Because Y is a regular scheme, $\mathcal{O}_{Y,y}$ is a field or discrete valuation ring, but all modules over fields are free and hence flat, so $\mathcal{O}_{Y,y}$ must be a discrete valuation ring. In particular, y is not the generic point.

We now try to find a point in $A(\mathcal{F})$ which is in the fiber over y, which will complete the proof. By using the classification of modules over a principal ideal domain, we know that $\mathcal{F}_x$ is the sum of torsion modules and a free part, but the free part is flat over $\mathcal{O}_{Y,y}$, so we see that $\mathcal{F}_x$ must have torsion. Namely, letting $\pi \in \mathcal{O}_{Y,y}$ be a uniformizer, we have $s \in \mathcal{F}_x$ killed by a power of π; say $\pi^n s = 0$. By adjusting s to a power, we may assume that $n = 1$.

Continuing, we set $I \coloneqq \operatorname{Ann}_{\mathcal{O}_{X,x}}(s)$, and we let $\mathfrak{p}$ be a minimal prime of $\mathcal{O}_{X,x}$ containing I. This prime $\mathfrak{p}$ corresponds to x' such that $x \in \overline{\{x'\}}$. In total, we have composite

$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x'}.$$

Label this composite by φ. Notably, $\varphi^{-1}\mathfrak{m}_{x'}$ contains I and therefore will contain π, so $\varphi^{-1}\mathfrak{m}_{x'} = \mathfrak{m}_y$, which is what $f(x') = y$ means according to our scheme morphism.

Now, we claim that $I\mathcal{O}_{X,x'}$ is primary for $\mathfrak{m}_{x'}$ and kills s. Then $x' \in A(\mathcal{F})$ by definition associated points, so we will finish. To prove this claim, choose $\mathfrak{p}$ minimal over I, but then

$$\mathcal{O}_{X,x'}/I\mathcal{O}_{X,x'} = (\mathcal{O}_{X,x})_\mathfrak{p}/I(\mathcal{O}_{X,x})_\mathfrak{p},$$

meaning that n consists of nilpotents, so $\sqrt{I\mathcal{O}_{X,x}} = \mathfrak{m}_{x'}$ follows. ∎

3.17 April 17

Today we would like to finish flatness.

3.17.1 Flatness via Locally Free

Let's finally prove that our two notions of associated primes coincide.

> **Proposition 3.121.** Fix a nonzero finitely generated module M over a Noetherian ring R. Then $A(\widetilde{M}) = A(M)$.

Proof. Certainly we have $A(M) \subseteq A(\widetilde{M})$. On the other hand, $x \in A(\widetilde{M})$ means that we have some section $s \in M$ annihilated by an ideal primary to $\mathfrak{m}_x$. It follows that every element of $\mathfrak{m}_x$ is a zero-divisor on M by taking powers, so we can write

$$\mathfrak{m}_x = \bigcup_{\mathfrak{q} \in A_{\mathcal{O}_{X,x}}(M)} \mathfrak{q},$$

so prime avoidance enforces $\mathfrak{m}_x$ to live in $A_{\mathcal{O}_{X,x}}(M_x)$, so undoing the localization confirms $x \in A(M)$. Namely, it is true more generally that

$$A_{R[S^{-1}]}M\left[S^{-1}\right] = A_R(M) \cap \operatorname{Spec} R\left[S^{-1}\right]$$

for any multiplicative subset $S \subseteq R$, which completes the proof. ∎

To solidify our idea that flatness means behaved fibers, we pick up two results.

> **Proposition 3.122.** Fix a finite morphism $f\colon X \to Y$ of Noetherian schemes. Then a coherent sheaf $\mathcal{F}$ on X is flat over Y if and only if $f_*\mathcal{F}$ is locally free.

Proof. Flatness is a stalk-local property, so we may assume that Y is affine, whereupon taking the pre-image along f forces X to be affine. Thus, we let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ so that $f\colon A \to B$ makes B into a finite A-algebra. In algebra, $\mathcal{F}$ becomes some B-module M flat over A, and $f_*\mathcal{F}$ is the corresponding A-module (under restriction).

Now, $\mathcal{F}_x$ is a localization of $(f_*\mathcal{F})_y$ by tracking through the definitions (namely, f might not be injective). Thus, in the backward direction, we note that $\mathcal{F}_x$ being flat over $\mathcal{O}_{Y,y}$ ensures that $\mathcal{F}_x$ is free over $\mathcal{O}_{Y,y}$, which does not change upon a further localization.

In the forward direction, we are given that M is flat over A, which means $M_\mathfrak{p}$ is flat over $A_\mathfrak{p} = \mathcal{O}_{Y,y}$. Now, finitely generated modules over Noetherian local rings being flat implies that they are free, so $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$, which is what we wanted. To be explicit, one can find a surjection $\mathcal{O}_V^{\oplus n} \to \mathcal{F}$ (for an open subset $V \subseteq Y$) which is an isomorphism at $y \in V$, so the kernel and cokernel vanish at the point $y \in Y$, so this will be an isomorphism in some open subset of V, so we are indeed locally free. ∎

> **Proposition 3.123.** Fix a finite morphism $f\colon X \to Y$ of Noetherian schemes; further, suppose that Y is integral. Then a coherent sheaf $\mathcal{F}$ on X is flat over Y if and only if the function
>
> $$y \mapsto \dim_{k(y)}((f_*\mathcal{F})_y \otimes k(y))$$
>
> is constant.

Proof. Again, everything is being checked stalk-locally, so we may reduce and adopt notation as in the previous proof. As a sanity check, note that the point here is that

$$(f_*\mathcal{F})_y \otimes k(y) = M \otimes_A k(y)$$

is a finite-dimensional $k(y)$-vector space, so at the very least it makes sense to discuss its dimension.

The forward direction is given by the previous proposition because $\mathcal{F}$ being flat implies locally free, which means locally constant rank. In the reverse direction, we are trying to show the following algebraic statement: fix a Noetherian local integral domain A. Then a finitely generated A-module M is free if and only if

$$\dim_{\operatorname{Frac} A} M \otimes_A \operatorname{Frac} A = \dim_{A/\mathfrak{m}} M \otimes_A (A/\mathfrak{m}).$$

Notably, we are comparing the generic point of A to the closed point, which will give our constancy. Anyway, the forward direction here has nothing to say. For the backward direction, fix a basis of $M/\mathfrak{m}M$ (over

$A/\mathfrak{m}$), and choosing lifts to M yields by Nakayama's lemma a surjection $\pi\colon A^d \to M$ which reduce to an isomorphism $\pmod{\mathfrak{m}}$.[2] At the current moment, we have an exact sequence

$$0 \to \ker \pi \to A^d \to M \to 0.$$

But now we can localize at the generic point to receive the exact sequence

$$0 \to K \otimes_A \ker \pi \to K^d \to K \otimes_A M \to 0$$

where $K := \operatorname{Frac} A$. However, the right-hand map is a surjection of two K-vector spaces of the same dimension (by hypothesis!), so it is an isomorphism, so $K \otimes_A \ker \pi$ vanishes, so $\ker \pi$ vanishes because it is torsion-free (as the submodule of the torsion-free module A^d). Thus, $A^d \cong M$ follows.   ∎

Let's see some examples.

> **Example 3.124.** Fix an algebraically closed field k, and set $X := \operatorname{Spec} k[x]$ and $Y := \operatorname{Spec} k[y]$, and let $f\colon X \to Y$ be the map given by $y \mapsto x^2$. Then f is flat. There are three cases.
>
> - To check that $\mathcal{O}_X$ is flat at the point $(y - \alpha)$ for $\alpha \neq 0$, we write $\alpha = \beta^2$ and compute
>
> $$\dim_{k(y)} \left(\frac{k[x]}{(x^2 - \alpha)} \otimes k(y) \right) = 2.$$
>
> - At the point (y), then we are looking at $k[x]/\left(x^2\right)$, which still has dimension 2 over residue fields k.
>
> - At the generic point, we are looking at $k(y)[x]/\left(x^2 - y\right)$, which still has dimension 2 over the residue field $k(y)$.

> **Remark 3.125.** For varieties, it should be enough to just check flatness at closed points by some kind of upper semi-continuity result. Alternatively, recall that the flat locus is open.

> **Example 3.126.** Fix an algebraically closed field k, and set $X := \operatorname{Spec} k[x, y]$ and $Y := \operatorname{Spec} k\left[x^2, xy, y^2\right]$ with the natural map $X \to Y$ given by the inclusion $k\left[x^2, xy, y^2\right] \subseteq k[x, y]$. Then f is not going to be flat. For example, at the closed point $(x - \alpha, y - \beta) \in X$, so the image in Y is the maximal ideal $\mathfrak{q} := \left(x^2 - \alpha^2, xy - \alpha\beta, y^2 - \beta^2\right)$. Reversing, we can compute our fiber as
>
> $$\frac{k[x, y]}{\mathfrak{p}k[x, y]} = \frac{k[x, y]}{(x - \alpha, y - \beta)} \oplus \frac{k[x, y]}{(x + \alpha, y + \beta)},$$
>
> so the fiber over p has the closed points (α, β) and $(-\alpha, -\beta)$, provided that α and β are nonzero. However, in the case that $\alpha = \beta = 0$, our fiber essentially collapses down a dimension.

3.18   April 19

Today we start talking about smooth morphisms.

---

[2] To see this application, recall that Nakayama's lemma asserts that $IM = M$ for I contained in the maximal ideal of our local ring A implies that $M = 0$. Then the point is that our surjection $A^d \to M$ has cokernel vanishing when reduced $\pmod{\mathfrak{m}}$, so the cokernel must fully vanish by Nakayama's lemma.

3.18.1 Unramified Morphisms

Morally speaking, a morphism $f \colon X \to S$ locally of finite presentation[3] is smooth at a point $x \in X$ if and only if we can find an open neighborhood U of x so that $f|_U$ factors as

$$U \to \mathbb{A}_S^n \to S$$

where the map $U \to \mathbb{A}_S^n$ is étale. Here, étale morphisms are flat and unramified, where unramified is a word we have not defined. Roughly speaking, we are trying to say that f is locally the projection from affine space, up to some étale map (which should be thought of as almost an open embedding). Then f is smooth if and only if it is smooth everywhere.

To turn this into a less clunky definition, we will want to use the relative differentials. The unramified map $U \to \mathbb{A}_S^n$ preserves relative differentials, so we will want to use the "free-ness" of the relative differentials $\Omega_{\mathbb{A}_S^n/S}$.

Anyway, let's begin our definitions.

> **Definition 3.127** (relative dimension)**.** A scheme morphism $f \colon X \to S$ locally of finite presentation has *relative dimension* d if and only if each fiber $f^{-1}(f(x))$ is equidimensional and has
> $$\dim f^{-1}(f(x)) = d$$
> for any $x \in X$. In other words, we are saying that all fibers have dimension d.

For example, étale maps will be smooth of relative dimension 0.

> **Definition 3.128** (unramified)**.** A scheme morphism $f \colon X \to S$ locally of finite presentation is *unramified at* $x \in X$ if and only if $(\Omega_{X/S})_x = 0$. Thus, the morphism f is unramified if and only if $\Omega_{X/S} = 0$.

> **Remark 3.129.** Checking affine-locally, because f is locally of finite presentation, at $x \in X$, we can place $x \in \operatorname{Spec} A \subseteq X$ and $f(U) \subseteq \operatorname{Spec} B \subseteq S$. Because A is finitely presented over B, we may factor $f|_U$ here as
> $$U \hookrightarrow \mathbb{A}_B^n \to B$$
> where $U \hookrightarrow \mathbb{A}_B^n$ is a closed embedding, say given by the finitely generated ideal $I \subseteq B[x_1, \ldots, x_n]$. Thus, we see we have the right-exact sequence
> $$\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{A}_S^n/S} \to \Omega_{X/S} \to 0,$$
> by gluing these pieces together. Bringing this together, we see that being unramified is equivalent to asking for the map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{A}_S^n/S}^1$ to be surjective.

As usual, one can check that unramified morphisms have open locus (because the definition is stalk-local on a sheaf), and we can check on stalks that we are stable under composition. Because the pullback of the sheaf of differentials behaves in squares, we also see that unramified morphisms are stable under base-change.

It will be easier to describe being unramified in terms of the diagonal.

---

[3] Recall that a morphism of schemes $f \colon X \to S$ is locally of finite presentation if and only if each $x \in X$ has an affine open neighborhood $\operatorname{Spec} A \subseteq X$ which goes to an affine open neighborhood $\operatorname{Spec} B \subseteq S$ so that A is finitely presented as an A-algebra.

> **Proposition 3.130.** Fix a morphism $f\colon X \to S$ locally of finite presentation. Then the following are equivalent.
>
> (a) f is unramified at a point $x \in X$.
>
> (b) The diagonal $\Delta_f\colon X \to X \times_S X$ is an open embedding in an open neighborhood of $x \in X$.
>
> (c) We can check on fibers: X_s is unramified over $k(s)$ at $x \in X$.
>
> (d) Letting $\mathfrak{m}_s \subseteq \mathcal{O}_{S,s}$ be the maximal ideal for some $s \in S$ where $s := f(x)$, then $\mathfrak{m}_s \mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)$ thus becomes a finite separable extension of $k(s)$.

Proof. Let's have fun with our adjectives.

- We show that (a) and (b) are equivalent. Let $\mathcal{I}$ be the ideal sheaf of the diagonal in $X \times_S X$. Then by definition, we have $\Omega_{X/S} = \Delta^* \left(\mathcal{I}/\mathcal{I}^2 \right)$. This essentially completes the proof. For example, if $(\Omega_{X/S})_x = 0$, then $\left(\mathcal{I}/\mathcal{I}^2 \right)_x = 0$, so $\mathcal{I}_x = 0$ by Nakayama's lemma. (Note we are using the fact that $\mathcal{I}_x$ is finitely generated here!) In the other direction, if $\mathcal{I}_x = 0$, then we can read back $(\Omega_{X/S})_x = 0$, which is what we wanted.

- We note that (a) implies (c) by base-change to a fiber.

- We show (c) implies (a). Let $s\colon k(s) \to S$ be the corresponding bound. We have the following pullback square.

$$
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow \\
k(s) & \longrightarrow & S
\end{array}
$$

Now, $s^* \Omega_{X/S} = \Omega_{X_s/k(s)}$, so we are being given that $\left(s^* \Omega_{X/S} \right)_x = 0$. Computing this pullback, we see

$$
\left(s^* \Omega_{X/S} \right)_x = (\Omega_{X/S})_x \otimes_{\mathcal{O}_{S,s}} k(s) = (\Omega_{X/S})_x / \mathfrak{m}_s (\Omega_{X/S})_x = 0,
$$

so Nakayama's lemma again tells us that $\left(\Omega_{X/S} \right)_x = 0$.

- We show that (d) implies (a). Note that (c) allows us to pullback along $s\colon k(s) \to \operatorname{Spec} S$ so that we can check being unramified $k(s)$. In other words, we may assume that $\mathcal{O}_{S,s}$ is a field, whereupon looking locally we are assuming that $S = \operatorname{Spec} k$ for some field $k := k(s)$. (One should check that the hypotheses in (d) are also preserved by this base-change, which is true.) Now, we may compute that

$$
\left(\Omega_{X/S} \right)_x = \Omega_{k(x)/k(s)}
$$

will vanish because $k(x)/k(s)$ is a finite separable extension.

- We show that (b) implies (d). As usual, by taking the base-change to the fiber, we may assume that $S = \operatorname{Spec} k$ for a field $k := k(s)$. (One should check that $X_s \to X_s \times_k X_s$ remains an open embedding, but the point is that we are just computing some base-change of $X \to X \times_S X$.) Our goal now is to show that $X = X_s$ is a finite disjoint union of spectra of finite separable extensions of k. By checking locally, we may assume that $X = \operatorname{Spec} A$.

  Continuing, we may base-change by $k \to \overline{k}$ to allow us to assume that k is algebraically closed. Thus, we now want to show that A is a finite product of fields isomorphic to k because k is algebraically closed. (Notably, we will have shown that A before the base-change to the algebraic closure is a product of fields, and if A has any factor which was inseparable, then we would obtain nilpotents in this base-change, so we are actually showing that A before this base-change only has factors which are finite separable extensions of k.)

93

Now, the map $X \to X \times_k X$ is a closed embedding and an open embedding, so it will follow that X is some finite union of closed points. Well, for a closed point $z \in X$, we can take the pre-image of $\Delta(X)$ along the map

$$X \to \{z\} \to X$$

to see that $\{z\}$ is open. So closed points are open in X, from which it follows that X is a finite union of closed points.

It remains to show that we have no nilpotents. By shrinking X more, we may assume that X is a single point. However, the diagonal becomes the map $A \otimes_k A \to A$ which must be an isomorphism as both an open and closed embedding, so comparing dimensions enforces $\dim_k A = 1$. Thus, we see that A has no nilpotents. ∎

> **Remark 3.131.** The above proposition explains why f being unramified implies being relative dimension 0. Indeed, I think this follows more or less from (d) because we are essentially checking things on fibers.

3.19 April 21

Today we continue moving towards smoothness, which we will continue to do until the end of the semester.

3.19.1 Smooth Morphisms

Fix a morphism $f\colon X \to S$ locally of finite presentation so that we can factor f as

$$X \xrightarrow{j} \mathbb{A}_S^n \to S$$

where $j\colon X \to \mathbb{A}_S^n$ is some locally closed embedding. Working affine-locally, we may suppose that, in the neighborhood of $j(x) \in \mathbb{A}_S^n$ for some $x \in X$, the scheme X is cut out by some $n-r$ equations $g_{r+1}, \ldots, g_n$ in $\mathcal{O}_{\mathbb{A}_S^n}$. Further, suppose that

$$dg_{r+1}(j(x)), \ldots, dg_n(j(x))$$

are linearly independent over the $k(j(x))$-vector space $\Omega^1_{\mathbb{A}_S^n/S} \otimes k(j(x))$. We now claim that $f^{-1}(f(x))$ is equidimensional with dimension r.

For this, require the following lemma.

> **Lemma 3.132.** Fix a local ring $(A, \mathfrak{m})$. Given elements $f_1, \ldots, f_i \in \mathfrak{m}$ which generate an i-dimensional subspace in the $(A/\mathfrak{m})$-vector space $\mathfrak{m}/\mathfrak{m}^2$. Then $A/(f_1, \ldots, f_i)$ is a regular local ring of dimension $\dim A - i$.

Proof. The point is that $(f_1, \ldots, f_i)$ produces a regular sequence, so modding out by a regular sequence produces for us a regular local ring with the required dimension. ∎

To continue our story, recall that any locally closed embedding of finite presentation is going to be unramified; indeed, one can check this by (d) in Proposition 3.130. However, closed embeddings are not flat, so for flatness we add a new word.

> **Definition 3.133** (étale). A scheme morphism $f\colon X \to S$ locally of finite presentation is *étale* if and only if it is flat and unramified.

Notably, closed embeddings tend not to be flat (the fibers are far from continuous). Explicitly, one can check that étale locally closed embeddings are just open embeddings.

Remark 3.134. Finite étale maps are open and closed (because proper), so finite étale maps are able to play the role of covering spaces in algebraic geometry. Namely, this allows us to discuss the étale fundamental group via these maps.

We can now define smoothness and explain how it relates to our existing notions.

Definition 3.135 (smooth)**.** Fix a morphism $f\colon X \to S$ locally of finite presentation. Then f is *smooth at $x \in X$ of relative dimension r* if and only if there is an open neighborhood U of x and a factorization of $f|_U$ by

$$U \xrightarrow{j} \mathbb{A}_S^n \to S$$

satisfying the following constraints.

- For each $x \in X$, the sheaf of ideals $\mathcal{I}$ defining the locally closed embedding $j\colon U \to \mathbb{A}_S^n$ is generated by $(n - r)$ sections $g_{r+1}, \ldots, g_n \in \mathcal{O}_{\mathbb{A}_S^n}$.

- The differentials $dg_{r+1}(j(x)), \ldots, dg_n(j(x))$ in the $k(j(x))$-vector space $\Omega_{\mathbb{A}_S^n/S} \otimes k(j(x))$ are linearly independent.

The full morphism f is *smooth* if and only if it is smooth at all points in X.

Remark 3.136. It turns out that f is étale if and only if f is smooth of relative dimension 0. We will not prove this here.

Remark 3.137. We will shortly see that smooth morphisms are flat and hence open.

Remark 3.138. One can show that a morphism is smooth if and only if it is flat and has smooth fibers; this is nice because it is easier to check smoothness at fibers.

Remark 3.139. As with flatness, the smooth locus is open. Similar holds for being unramified and for being étale.

Remark 3.140. One can show that a smooth morphism of relative dimension r (at all points) has relative dimension r.

Let's pick up some facts.

Proposition 3.141. Fix a smooth morphism $f\colon X \to Y$ of S-schemes.

(a) $\Omega_{X/Y}$ is locally free with rank equal to the relative dimension.

(b) There is an exact sequence

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

is exact and locally split (namely, on affine open subschemes).

Proof. Here we go.

(a) We may assume that Y and then X is affine. Now, choose a factorization $U \to \mathbb{A}_Y^n \to Y$ promised by smoothness, where $j\colon U \to \mathbb{A}_Y^n$ has the required properties. Let $X \subseteq \mathbb{A}_Y^n$ be locally cut out by the

sheaf of ideals $\mathcal{I}$. We now note that we have the right-exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to j^*\Omega_{\mathbb{A}^n_Y/Y} \to \Omega_{X/Y} \to 0.$$

Now, by hypothesis, $\mathcal{I}$ is generated by $n-r$ sections where r is the relative dimension, which we label as $g_{r+1}, \ldots, g_n$. Further, we note that $\Omega_{\mathbb{A}^n_Y/Y}$ is free of rank n over Y, so its pullback to X along j is as well.

We now note that the $dg_{r+1}(j(x)), \ldots, dg_n(j(x))$ are linearly independent in $\Omega_{\mathbb{A}^n_Y/Y} \otimes k(j(x))$, so we can extend this to a basis $m_1, \ldots, m_r, \ldots, dg_{r+1}, \ldots, dg_r$ looking in the stalk $(\Omega_{\mathbb{A}^n_Y/Y})_{j(x)}$. By Nakayama's lemma, these germs generate, but $\Omega_{\mathbb{A}^n_Y/Y}$ is free of rank n, so the germs $m_1, \ldots, m_r, \ldots, dg_{r+1}, \ldots, dg_r$ are actually a basis of $(\Omega_{\mathbb{A}^n_Y/Y})_{j(x)}$.

Now, extend all our germs to be in some open neighborhood V of $j(x)$. Now, taking the m_i to $\Omega_{X/Y}|_V$, we see that they generate (because we have a surjection), and they generate with no relations from V by the construction of the m_i, so we conclude that $\Omega_{X/Y}|_V$ is in fact free of rank r. In fact, we have seen that

$$0 \to \mathcal{I}/\mathcal{I}^2 \to j^*\Omega_{\mathbb{A}^n_Y/Y} \to \Omega_{X/Y} \to 0$$

is a locally split exact sequence of locally free modules, of ranks $n-r$, n, and r, respectively.

(b) We have right-exactness in general, so it remains to discuss left-exactness. From part (a), everything involved is locally free of known dimensions (by computing relative dimensions), and we can compute ranks to get our injectivity by checking at stalks and using Nakayama's lemma.

Explicitly, denote $\alpha\colon f^*\Omega_{Y/S} \to \Omega_{X/S}$ by the left map. Now, for each $x \in X$, we see $(f^*\omega_{Y/S})_x$ that embeds into $\ker \alpha_x \oplus \operatorname{im} \alpha_x$ because $(\Omega_{X/S})_x$ embeds into $\operatorname{im}\alpha_x \oplus (\Omega_{X/Y})_x$, so computing ranks enforces $(f^*\omega_{Y/S})_x = \ker \alpha_x$. ∎

3.20 April 24

Today we talk about formal smoothness.

3.20.1 Formally Unramified and Friends

Here is our result for today.

> **Theorem 3.142.** Fix a scheme morphism $f\colon X \to S$ locally of finite presentation. The following are equivalent.
>
> (a) f is unramified/smooth/étale.
>
> (b) For all affine S-schemes Y and closed subschemes $Y' \subseteq Y$ defined by an ideal sheaf $\mathcal{I}$ such that $\mathcal{I}^2 = 0$, the canonical map
> $$\operatorname{Hom}_S(Y, X) \to \operatorname{Hom}_S(Y', X)$$
> is injective/surjective/bijective.

For example, for étale morphisms, we are drawing a square

$$
\begin{array}{ccc}
Y' & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
Y & \longrightarrow & S
\end{array}
$$

and claiming that there exists a unique map $Y' \to X$ making the diagram commute. We will prove the statement for smoothness for time reasons.
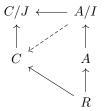
Proof. It turns out that (b) can have everything reduce to affine open subschemes. The concern here is mostly that we need the maps to glue reasonably, but we will not say more. Certainly the statement in (a) is defined at stalks and is thus local, so everything is local, so we may assume that $S = \operatorname{Spec} R$ and $X = \operatorname{Spec} B$ is a closed subscheme of an open subscheme $\operatorname{Spec} A$ of $\mathbb{A}_S^n$, using the fact that f is locally of finite presentation; namely, we may say $B := A/I$ where I is finitely generated.

- Now, in one direction, suppose that f is smooth. Here, $\Omega_{\mathbb{A}_S^n/S}$ is free of rank n. Because we are working locally, we may use the definition of smoothness to find $g_1, \ldots, g_n \in A$ such that $dg_1, \ldots, dg_n$ generate the A-module $\Omega_{A/R} = \Omega_{\mathbb{A}_S^n/S}|_{\operatorname{Spec} A}$ and $g_{r+1}, \ldots, g_n$ generate $I \subseteq A$. Notably, the B-module I/I^2 is generated by $\overline{g}_{r+1}, \ldots, \overline{g}_n$. Now, we claim that

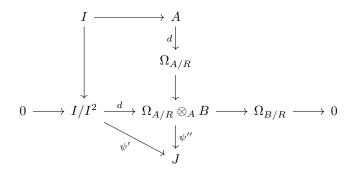$$0 \to I/I^2 \to \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \to 0$$

is a split exact sequence. Indeed, this is just a computation: the generators $\overline{g}_{r+1}, \ldots, \overline{g}_n$ go to the free generators $dg_{r+1}, \ldots, dg_n$ in $\Omega_{A/R} \otimes_A B$. Thus, we get exactness on the left, and we can compute the kernel of the right map is exactly generated by these elements by examining the map $\Omega_{A/R} \otimes_A B \to \Omega_{B/R}$. So we get what we want. (Note that we have left-exactness of this sequence even without f being smooth!)

We now approach our proof directly. Fix $Y := \operatorname{Spec} C$ where C is an A-algebra, and let $Y' := \operatorname{Spec} C/J$ with $J \subseteq C$ an ideal with $J^2 = 0$. Now, our affine diagram is as follows.

$$
\begin{array}{ccc}
C/J & \longleftarrow & A/I \\
\uparrow & \nwarrow & \uparrow \\
C & & A \\
& \nwarrow & \uparrow \\
& & R
\end{array}
$$

Explicitly, given some $\overline{\varphi}$, we want to fill in the dotted line φ. Now, A is a free algebra over a localization R_f of R; say $\operatorname{Spec} A \subseteq \operatorname{Spec} R[x_1, \ldots, x_n]$. Then we can track the image of $x_\bullet$ in C/J, choose some arbitrary lift to C, and then we have constructed a map $R[x_1, \ldots, x_n] \to C$ making the above diagram commute. In fact, we are defining a map $\psi \colon A \to C$, which is safe because $\operatorname{Spec} C/J \cong \operatorname{Spec} C$; notably, being homeomorphic allows us to restrict appropriately to $\operatorname{Spec} A \subseteq \mathbb{A}_S^n$ because we started with a map $\operatorname{Spec} C/J \to \operatorname{Spec} A$.

Now, $\psi(I) \subseteq J$ because $A \to C \to C/J$ factors as $A \to A/I \to C/J$, which vanishes on I; notably, this provides a B-module map $\psi' \colon I/I^2 \to J$. We now want to use the universal property of $\Omega_{A/R}$ as universal among derivations $d \colon A \to \Omega_{A/R}$. Using the splitting of the injection $I/I^2 \to \Omega_{A/R} \otimes_A B$, we produce a map ψ'' fitting in the following diagram.

$$
\begin{array}{ccc}
I & \longrightarrow & A \\
& & \downarrow{\scriptstyle d} \\
& & \Omega_{A/R} \\
& & \downarrow \\
0 \longrightarrow I/I^2 & \overset{d}{\longrightarrow} \Omega_{A/R} \otimes_A B & \longrightarrow \Omega_{B/R} \longrightarrow 0 \\
& {\scriptstyle \psi'}\searrow & \downarrow{\scriptstyle \psi''} \\
& & J
\end{array}
$$

A direct computation shows that the rectangle commutes; notably, we are applying the derivation d once each way around the rectangle. Now, ψ'' in the above diagram is producing an R-derivation $\delta \colon A \to J$ by composing with $d \colon A \to \Omega_{A/R}$ as above, which we see from the above commutativity has $\delta|_I = \psi|_I$.

97

Quickly, observe that the map $A \xrightarrow{\delta} C \to C/J$ vanishes because $\operatorname{im} \delta \subseteq J$. Additionally, we note that $\psi - \delta$ vanishes on I because $\delta|_I = \psi|_I$. In total, $\psi - \delta$ produces a map

$$\varphi \colon A/I \to C.$$

Quickly, ψ is a map of R-algebras, and δ is an R-derivation, so φ is in fact a map of R-algebras. It remains to check that

$$
\begin{array}{ccc}
C/J & \xleftarrow{\ \overline{\varphi}\ } & A/I \\
\uparrow & \nearrow{\scriptstyle\varphi}\ \ & \uparrow \\
C & \xleftarrow[\psi]{} & A
\end{array}
$$

commutes. Well, because $A \to A/I$ is surjective and thus epic, it suffices to show that the composite $A \to A/I \to C \to C/J$ and $A \to A/I \to C/J$ are equal. However, we can see that these are equal by the construction of φ. Namely, the bottom triangle commutes by construction of φ (as discussed: δ does no change on A/I), and the outer square commutes by construction of ψ.

- We sketch the converse. As usual, we may assume that X is a closed subscheme of $\mathbb{A}_S^n$ defined by the sheaf of ideals $\mathcal{I}$. It suffices to check that

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{A}_S^n/S} \otimes \mathcal{O}_X \to \Omega_{X/S} \to 0$$

is locally split-exact. Indeed, if split-exact at some $x \in X$, then the inclusion $\mathcal{I}_x \subseteq \mathfrak{m}_x$ grants us some generators $\overline{g}_{r+1}, \ldots, \overline{g}_n$ of $\left(\mathcal{I}/\mathcal{I}^2\right)_x$, whose lifts will generate $\mathcal{I}_x$; split-exactness tells us that we are surjecting onto a free submodule $dg_{r+1}, \ldots, dg_n$ in $\Omega_{\mathbb{A}_S^n/S}$, which provide the needed sections for smoothness.

We now continue with the notation previously established. We now build the following diagram.

$$
\begin{array}{ccc}
A/I & =\!=\!= & A/I \\
\uparrow & \nwarrow\ \ \ & \uparrow \\
A/I^2 & & A \\
& \nwarrow\quad & \uparrow \\
& & R
\end{array}
$$

In other words, the formal splitting grants us an R-algebra map $\varphi \colon A/I \to A/I^2$ making the above diagram commutes. One can see that this means that φ produces a splitting of

$$0 \to I/I^2 \to A/I^2 \xrightarrow{\nu} A/I \to 0,$$

so the splitting here produces a section $\tau \colon A/I^2 \to I/I^2$ as $\operatorname{id}_{A/I^2} - \varphi \circ \nu$. One can compute from here that τ is an R-derivation upon writing it out; thus, our universal property of $\Omega_{A/R}$ produces an A-module map $q \colon \Omega_{A/R} \to I/I^2$, which one can check is the desired splitting. ∎

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LIST OF DEFINITIONS