174: Category Theory

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Theme 1: Introduction

Category theory is much easier once you realize that it is designed to formalize and abstract things you already know.

—Ravi Vakil

1.1 January 19

Reportedly there is a lot of material that Bryce would like to cover today.

1.1.1 Our Definition

We're doing category theory, so let's define what a category is.

Category

Definition 1.1 (Category). A category $\mathcal{C}$ is a pair of objects and morphisms $(\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C})$ satisfying the following.

- $\operatorname{Ob}\mathcal{C}$ is a collection of *objects*. By abuse of notation, when we write $c \in \mathcal{C}$
- $\operatorname{Mor}\mathcal{C}$ is a collection of *morphisms*. Morphisms might also be called arrows or maps or functions or continuous functions or similar.

  A morphism is written $f : x \to y$ where $x, y \in \operatorname{Ob}\mathcal{C}$. Here, x is the *domain*, and y is the *codomain*.

These morphisms have a little extra structure.

- For each $x \in \mathcal{C}$, there is a morphism $\operatorname{id}_x : x \to x$.
- Given any pair of morphisms $f : x \to y$ and $g : y \to z$, there exists a *composition* $gf : x \to z$. Importantly, the codomain of f is the domain of g.

Additionally, morphisms satisfy the following coherence conditions.

- Associativity: for any morphisms $f : a \to b$ and $g : b \to c$ and $h : c \to d$, we have that $h(gf) = (hg)f$.
- Identity: given any morphism $f : a \to b$, we have $\operatorname{id}_b f = f$ and $f \operatorname{id}_a = f$.

Yes, this is a long definition. For reference, it is on page 3 of Riehl.

The intuition to have here is that we have objects to be thought of as points a whole bunch of morphisms which are to be thought of arrows between them. Here is an example of some morphisms in a category.

$$
\begin{tikzcd}
\bullet \arrow[loop left] \arrow[r] \arrow[d] & \bullet \arrow[loop above] \arrow[dl] \arrow[dr] & \\
\bullet \arrow[loop below] \arrow[rr] & & \bullet \arrow[loop below]
\end{tikzcd}
$$

The loops are identity morphisms. As an aside, it is reasonable to think that definition of a category is overly abstract. Most of the time we will be thinking about some concrete category.

Before continuing, we bring in the following definition.

Hom-sets

Definition 1.2 (Hom-sets)**.** Fix a category $\mathcal{C}$. Then, given objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ or $\operatorname{Hom}_{\mathcal{C}}(x, y)$ or $\operatorname{Hom}(x, y)$ or $\operatorname{Mor}(x, y)$ for the set of morphisms $f : x \to y$. I personally prefer $\operatorname{Mor}(x, y)$.

Note that two objects need not have a morphism between them. For example, the following is a category even though the two objects have a morphism between them.

$$\begin{tikzcd} \bullet \arrow[loop left] & \bullet \arrow[loop right] \end{tikzcd}$$

As a less contrived example, there is no morphism between $\mathbb{F}_2$ and $\mathbb{F}_3$ in the category of fields.

1.1.2 Examples

Let's talk about examples.

Example 1.3. The category Set has objects which are all sets and its morphisms are the functions between sets.

Example 1.4. The category Grp has objects which are all groups and its morphisms are group homomorphisms. Similarly, Ab has abelian groups.

Example 1.5. The category Ring has objects which are all rings (with identity) and its morphisms are group homomorphisms.

Example 1.6. The category Field has objects which are all fields and its morphisms are field/ring homomorphisms.

Example 1.7. The category Vec_k has objects which are all k-vector spaces and its morphisms are k-linear transformations.

Those are the good examples. We like them because they are with familiar objects.

Here are some weirder examples.

Example 1.8 (Walking arrow)**.** The diagram

$$\bullet \longrightarrow \bullet$$

indues a category with a single non-identity morphism.

Note that we will stop writing down all of the identity morphisms and all induced morphisms because they're annoying to write out.

Example 1.9 (Walking isomorphism)**.** The diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

induces a category with two non-identity morphisms. We declare that any composition of the two non-identity morphisms is the identity.

There are also such things as a poset category, but for this we should define a poset first.

Poset

Definition 1.10 (Poset)**.** A *poset* $(\mathcal{P}, \leq)$ is a set $\mathcal{P}$ and a relation $\leq$ on $\mathcal{P}$ which satisfies the following; let $a, b, c \in \mathcal{P}$.

- Reflexive: $a \leq a$.
- Antisymmetric: $a \leq b$ and $b \leq a$ implies $a = b$.
- Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$.

Now, it turns out that all posets induce a category.

Example 1.11 (Poset category)**.** Given any poset $(\mathcal{P}, \leq)$, we can define the poset category as follows.

- The objects are elements of $\mathcal{P}$.
- For $x, y \in \mathcal{P}$, there is a morphism $x \to y$ if and only if $x \leq y$, and there is only one morphism.

Checking that the poset category is in fact a category is not very interesting. The identity law comes from reflexivity, where id_a witnesses $a \leq a$.

Additionally, transitivity defines our composition: if $a \leq b$ and $b \leq c$, then $a \leq c$, and the morphism representing $a \leq c$ is unambiguous because there is at most one moprhism $a \to c$. This uniqueness is in fact crucial for our composition: if $f : a \to b$ and $g : b \to c$ and $h : c \to d$ are morphisms, then $h(gf) = (hg)f$ because they are both morphisms $a \to d$, of which there is at most one.

We continue with our examples. We will not check that these are actually categories formally; perhaps the reader can do the checks on their own time.

Example 1.12 (Groups)**.** Given a group G, we can define the category bG to have one object $*$ and morphisms $g : * \to *$ given by group elements $g \in G$. Composition in the category is group multiplication; the identity morphism id_* needed is the identity element of G; and the associativity check comes from associativity in G.

Example 1.13 (Pointer sets)**.** We define the category of pointed sets Set_* to consist of objects which are orderd pairs (X, x) where X is a set and $x \in X$ is an element. Then morphism are "based maps" $f : (X, x) \to (Y, y)$ to consist of the data of a function $f : X \to Y$ such that $f(x) = y$.

Example 1.14. Given any set S, we can define a category consisting of objects which are elements of S and morphisms which are only the required identity morphisms.

This last example generalizes.

Discrete, indiscrete

Definition 1.15 (Discrete, indiscrete)**.** Fix a category $\mathcal{C}$. Then $\mathcal{C}$ is *discrete* if and only if the only morphisms are identity morphisms. Additionally, $\mathcal{C}$ is *indiscrete* if and only if $\mathrm{Mor}(x, y)$ has exactly one element for each pair of objects (x, y).

Warning 1.16. A total order with more than one element is not a category. Namely, if we have distinct objects x and y, then we cannot have both $x \leq y$ and $y \leq x$, so not both $\mathrm{Mor}(x, y)$ and $\mathrm{Mor}(y, x)$ inhabited.

1.1.3 Size Issues

Let's briefly talk about why we are calling $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ "collections." In short, we cannot have a set that contains all sets, but we would still like a category which contiains all categories. There are a few ways around this; here are two.

- Grothendeik inaccessible categories: we essentially upper-bound the size of our sets and then let Set contain all of our sets.
- Proper classes: we add in things called "classes" to foundational mathematics we are allowed to be bigger than sets.

We will avoid doing anything like this in this course, so here is a definition making our avoidance concrete.

Small, locally small

Definition 1.17 (Small, locally small)**.** Fix $\mathcal{C}$ a category. Then $\mathcal{C}$ is *small* if and only if $\mathrm{Mor}\,\mathcal{C}$ is a set. Alternatively, $\mathcal{C}$ is *locally small* if and only if $\mathrm{Mpr}(x,y)$ is a set.

Example 1.18. The category Set is locally small, but it is not small. To see that it is not small, note that $S \mapsto \mathrm{Mor}(\{*\}, S)$ is an injective map, so $\mathrm{Mor}\,\mathrm{Set}$ must be at least as big as Set.

It turns out that most of our categories will be locally small. It is a very nice property to have.

1.1.4 Isomorphism

In algebra (e.g., group theory), we are interested in when two objects are the same. In category theory, we focus on the morphisms between objects, so we need to be careful how we define this. Here is our definition.

Isomorphism

Definition 1.19 (Isomorphism)**.** Fix a category $\mathcal{C}$. Then a morphism $f: x \to y$ is an *isomorphism* if and only if there is a morphism $g: y \to x$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. We call g the *inverse* of f and often notate it f^{-1}.

This is fairly intuitive: isomorphisms are those morphisms with a way to reverse them.

Observe that we called g "the" inverse of f, and we may do so because inverses are unique.

Proposition 1.20. Fix a category $\mathcal{C}$. Inverses of morphisms, if they exist, are unique.

Proof. Fix $f: x \to y$ some isomorphism, and suppose that we have found two inverse morphisms $g, h: y \to x$. Then

$$g = g\,\mathrm{id}_y = g(fh) = (gf)h = \mathrm{id}_x\,h = h,$$

so indeed the inverse morphisms that we found are the same. ■

Anyways, here are some examples.

Example 1.21. In Set, the isomorphisms are the bijective maps. For this we would have to show that bijective maps have inverse maps, which is not too hard to show.

Example 1.22. In Grp, the isomorphisms are group isomorphisms. Similarly, isomorphisms in Ring are ring isomorphisms.

As a warning, we will say now that lots of categories do not have a good categorial notion of injectivity or surjectivity, so we will not be able to say that isomorphisms are merely "bijective" morphisms.

1.2 January 21

By the way, this course is being run by Bryce (interested in category theory, homological algebra, and algebraic topology) and Chris (interested in representation theory and category theory).

1.2.1 Small Correction

Last class we discussed trying to a total order $(\mathcal{P}, \leq)$ into an indiscrete category. One way to do this is to say to give a morphism between two objects $a, b \in \mathcal{P}$ if and only if one of $a < b$ or $b < a$ or $a = a$ is true. Observe that the order does not actually matter here because any two objects have exactly one morphism anyways.

1.2.2 Groupoids

Reportedly, there will usually not be a lecture to begin out our discussion sections, but here is a lecture to begin out our first discussion section.

Last time we left off talking about indiscrete categories. Here is a nice fact.

Proposition 1.23. Fix $\mathcal{C}$ an indiscrete category. Then all maps are isomorphisms.

Proof. Fix any morphism $f : x \to y$. There is also a morphism $g : y \to x$, and we see that $gf \in \operatorname{Mor}(x, x)$. But $\operatorname{id}_x \in \operatorname{Mor}(x, x)$ as well, so we are forced to have $gf = \operatorname{id}_x$ by uniqueness of morphisms. Similar shows that $fg = \operatorname{id}_y$, finishing the proof. ■

Remark 1.24. This statement is also true for discrete categories but only because all identity morphisms are isomorphisms immediately.

The property of the proposition is nice enough to deserve a definition.

Groupoid

Definition 1.25 (Groupoid)**.** A category in which all morphisms are isomorphisms is called a *groupoid*.

Example 1.26. Viewing groups as one-element categories, we see that groups are groupoids because all elements (i.e., morphisms of the one-object set) have inverses and hence are isomorphisms.

Intuitively, a groupoid is a group but more "spread out."

1.2.3 Arrow Words

We close out with some miscellaneous definitions for our morphisms.

Endo-, automorphism

Definition 1.27 (Endo-, automorphism)**.** Fix a category $\mathcal{C}$. A morphism $f : x \to y$ is an *endomorphism* if and only if $x = y$. A morphism $f : x \to y$ is an *autormophism* if and only if it is an isomorphism and an endomorphism.

Example 1.28. In the category of abelian groups, the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 is an endomorphism but not an automorphism.

Monic, epic

Definition 1.29 (Monic, epic)**.** Fix a category $\mathcal{C}$ and a morphism $f : x \to y$.

- We say f is a *monomorphism* (or is *monic*) if and only if $fg = fh$ implies $g = h$ for any morphisms $g, h : c \to x$. In other words, the map

$$\operatorname{Mor}(c, x) \stackrel{f\circ -}{\to} \operatorname{Mor}(c, y)$$

is injective. (This map is called "post-composition.") We might write $f : x \hookrightarrow y$ for emphasis.

- We say f is an *epimorphism* (or is *epic*) if and only if $gf = hf$ implies $g = h$ for any morphisms $g, h : y \to c$. In other words, the map

$$\operatorname{Mor}(y, c) \stackrel{-\circ f}{\to} \operatorname{Mor}(x, c)$$

is injective. (This map is called "pre-composition.") We might write $f : x \twoheadrightarrow y$ for emphasis.

Intuitively, the monomorphism condition looks like the injectivity condition (namely, $f(x) = f(y)$ implies $x = y$), so monic is supposed to be a generalization for injective.

Example 1.30. In the category of sets, monic is equivalent to injective, and epic is equivalent to surjective. Then it happens that being monic and epic implies being an isomorphic. We will not fill in the details here.

Warning 1.31. It is not always true that being monic and epic implies being isomorphic. It is true in $\mathrm{Set}, \mathrm{Ab}, \mathrm{Grp}$ but not in, say, Ring as the below example shows.

Example 1.32. The inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring is both epic and monic but not an isomorphism. We run some of the checks.

- We show monic. Suppose $g, h : R \to \mathbb{Z}$ are morphisms with $fg = fh$. We claim $g = h$. Well, for any $r \in R$, we see $g(r) = f(g(r))$ and $h(r) = f(h(r))$ because f is merely an inclusion, so $g(r) = h(r)$ follows.

- We show epic. Suppose $g, h : \mathbb{Q} \to R$ are morphisms with $gf = hf$. We claim $g = h$. We start by noting any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}$ will have

$$g\left(n/m\right) \cdot g(m) = g(n)$$

and similar for h. However, $g(m) = g(f(m)) = h(f(m)) = h(m)$ and $g(n) = h(n)$ for the same reason, so $g\left(\frac{n}{m}\right) = g(n)/g(m) = h(n)/h(m) = h\left(\frac{n}{m}\right)$, and we are done because any rational can be expressed as some $\frac{n}{m}$.

- Lastly, f is not an isomorphism because $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. For example, $2x - 1$ has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

And now discussion begins.