258: Harmonic Analysis

Nir Elber

Fall 2023

Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

1.1 August 28

Why am I here?

1.1.1 Logistics

Here are the usual logistics notes.

- The professor is Ruixiang Zhang.
- There will be three assignments, which determine the grade. They will be rather hard.
- Office hours are on Wednesday, during 10:30AM–11:30AM, 2PM–3PM, and 3PM–4PM.

1.1.2 Convergence of Fourier Series

The point of the course is to study differentiable functions on a space which has an action by a group. Last class we proved the following result.

Theorem 1.1 (Riemann localization principle)**.** Fix a 1-periodic function $f \in L^1(\mathbb{R}/\mathbb{Z})$ which vanishes in a neighborhood of $x \in \mathbb{R}$. Then

$$\lim_{N\to\infty} S_N f(x) = 0.$$

Here,

$$S_N f(x) \coloneqq \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x},$$

where

$$\hat{f}(k) \coloneqq \int_0^1 f(x) e^{-2\pi i k x}\, dx.$$

Anyway, here is a quick sketch.

Sketch of Theorem 1.1. One can show that $\hat{f}(n) \to 0$ as $|n| \to \infty$ by approximating $f \in L^1(\mathbb{R}/\mathbb{Z})$ by simple integrable functions. Then one uses a geometric series style argument to get cancellation, writing

$$S_N f(x) = \int_0^1 \frac{\sin(N+1)\pi t}{\sin \pi t} \cdot f(x-t)\, dt$$

and then expressing the integral as a sum of Fourier coefficients of functions in $L^1(\mathbb{R}/\mathbb{Z})$. ■

We are now ready to show Dini's criterion.

Theorem 1.2 (Dini's criterion)**.** Fix a function $f \in L^1(\mathbb{R}/\mathbb{Z})$ and $x \in \mathbb{R}$. Then suppose that

$$\int_{|t|<\delta} \left| \frac{f(x+t)-f(x)}{t} \right| dt < \infty$$

for all $\delta > 0$. Then $S_N f(x) \to f(x)$ as $N \to \infty$.

Proof. We take $\delta < 1/2$. Using the Dirichlet kernel

$$D_N(x) := \sum_{|k|\le N} e^{2\pi i k x} = \frac{\sin(2N+1)\pi x}{\sin \pi x},$$

one has

$$\begin{aligned} S_N f(x) - f(x) &= \int_{-1/2}^{1/2} f(x-t) D_N(t)\, dt - f(x) \\ &= \int_{-1/2}^{1/2} (f(x-t) - f(x)) D_N(t)\, dt \\ &= \underbrace{\int_{|t|<\delta} (f(x-t)-f(x)) D_N(t)\, dt}_{I_1} + \underbrace{\int_{\delta \le |t| \le 1/2} (f(x-t)-f(x)) D_N(t)\, dt}_{I_2}. \end{aligned}$$

The argument of Theorem 1.1 establishes that $I_2 \to 0$ as $N \to \infty$, so it is safe, or one can directly see that we have essentially constructed a function which vanishes on an interval around x and took its Fourier transform. For I_1, we bound by absolute value, we see

$$|I_1| \le \int_{|t|<\delta} \left| \frac{f(x-t)-f(x)}{\sin \pi t} \right| dt \ll \int_{|t|<\delta} \left| \frac{f(x-t)-f(x)}{t} \right| dt,$$

which disappears as we take δ small. Namely, taking $\delta' \le \delta$, the hypothesis tells us that

$$\int_{|t|<\delta'} \left| \frac{f(x-t)-f(x)}{t} \right| dt < \infty,$$

so finiteness of the integral at $\delta = \delta'$ enforces it to go to 0 as $\delta' \to 0^+$. ■

It is not clear what the hypothesis in Theorem 1.2 is good for, but we will use it shortly; as an example application, Hölder continuous functions satisfy the condition. But notably, continuity is not good enough to give us convergence. Anyway, here is another criterion.

Theorem 1.3 (Jordan's criterion)**.** Fix a function $f \in L^1(\mathbb{R}/\mathbb{Z})$ and $x \in \mathbb{R}$. Further, suppose that f is of bounded variation in $(x-\delta, x+\delta)$ for some $\delta > 0$. Then

$$\lim_{N\to\infty} S_n f(x) = \frac{f(x_-) + f(x_+)}{2},$$

where $f(x_\pm)$ denotes the value of $f(a)$ as $a \to x^\pm$.

Proof. Being bounded variation here roughly means that it is the difference of two monotonic functions. Again, we take $\delta < 1/2$. Then Theorem 1.1, we may also assume that f vanishes outside $(x-\delta, x+\delta)$.

(Namely, the convergence is local to x, so we can subtract out $g(t) \coloneqq f(t)1_{|t-x|>\delta}(t)$.) Now,

$$\begin{aligned} S_N f(x) &= \int_{-1/2}^{1/2} f(x-t) D_N(t)\,dt \\ &= \int_0^{1/2} (f(x+t)+f(x-t)) D_N(t)\,dt. \end{aligned}$$

We now set $g(t) \coloneqq f(x+t)+f(x-t)$, essentially fixing x, so we want to show

$$\lim_{N\to\infty} \int_0^{1/2} g(t) D_N(t)\,dt = \frac{1}{2} g(0_+).$$

Subtracting f by $\frac{1}{2}g(0_+)$, we may assume that $g(0_+)=0$. Also, f is the difference of two monotonic functions, and the above condition is linear, so we may as well assume that g is monotonic.

As before, take $\delta' < \delta$, and we split the integral into two parts, writing

$$\int_0^{1/2} g(t) D_N(t)\,dt = \underbrace{\int_0^{\delta'} g(t) D_N(t)\,dt}_{I_1 \coloneqq} + \underbrace{\int_{\delta'}^{\delta} g(t) D_N(t)\,dt}_{I_2 \coloneqq}.$$

Theorem 1.1 tells us that $I_2 \to 0$ as $N \to \infty$ because we are away from 0. Using a Mean value theorem argument, one finds

$$\int_0^{\delta'} g(t) D_N 9t)\,dt = g(\delta'_-) \int_v^{\delta'} D_N(t)\,dt$$

for some $v \in [0,\delta']$. To get convergence as $N \to \infty$, one needs to use cancellation within D_N. Well, we find

$$\int_v^{\delta'} D_N(t)\,dt = \int_v^{\delta'} \frac{\sin(2N+1)\pi t}{\sin \pi t}\,dt.$$

One would like to replace $\sin \pi t$ with t so that dt/t is the multiplicative Haar measure on $\mathbb{R}^\times$. Explicitly,

$$\left| \int_v^{\delta'} D_N(t)\,dt \right| = \left| \int_v^{\delta'} \sin(2N+1)\pi t \cdot \left(\frac{1}{\sin \pi t} - \frac{1}{\pi t} \right) dt \right| + \left| \int_v^{\delta'} \frac{\sin(2N+1)\pi t}{t}\,dt \right|.$$

We now see $\frac{1}{\sin \pi t} - \frac{1}{\pi t}$ is bounded by a constant in $[v,\delta']$, so the entire integral is also bounded by a constant; notably, this constant vanishes as $\delta' \to 0^+$. Applying a change of variables to the second term, we see that it is bounded by

$$\sup_{0<c_1<c_2<\delta'} \left| \int_{c_1}^{c_2} \frac{\sin \pi t}{t}\,dt \right|,$$

which also vanishes as $\delta' \to 0^+$, completing the proof. ■

1.2 August 30

I continue to not know why I am here.

1.2.1 Non-Convergence of Fourier Series

For sufficiently strong continuity, the Fourier series will converge pointwise to the function, say by Theorem 1.2. For general continuous functions, we are not so lucky.

Theorem 1.4. There exists a continuous function f on $\mathbb{R}/\mathbb{Z}$ whose Fourier series diverges at 0.

To show the above theorem, we want the following lemma.

Lemma 1.5 (uniform boundedness principle)**.** Let X and Y be Banach spaces, and let $\{T_\alpha\}_{\alpha\in\lambda}$ be a family of bounded linear operators $T_\alpha\colon X\to Y$. If

$$\sup_{\alpha\in\lambda}\|T_\alpha\|=+\infty,$$

then there is a point $x\in X$ such that $\sup_{\alpha\in\lambda}\|T_\alpha x\|=+\infty$.

Proof. This is a standard result in functional analysis. ■

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We work with $X:=C(\mathbb{R}/\mathbb{Z})$ and $Y:=\mathbb{C}$; we take $\|\cdot\|_X$ to be $\|\cdot\|_\infty$. Now, take the partial sum operators $T_N\colon f\mapsto S_Nf(0)$, but the operator norms can be become arbitrarily large, so it follows that there is a continuous function $f\in C(\mathbb{R}/\mathbb{Z})$ with $\sup_{N\in\mathbb{N}}\|S_Nf\|=+\infty$, meaning that the Fourier series diverges. Explicitly, it turns out that

$$\|T_N\|=\|D_N\|,$$

which is left as an exercise; roughly speaking, one chooses a continuous function f with $\|f\|_\infty=1$ and tries to make $f(x)=1$ whenever $D_N(x)\geq 0$ and $f(x)=-1$ whenever $D_N(x)<0$. Rigorizing this is somewhat annoying, so we won't bother.

Continuing, one can actually compute

$$L_N\stackrel{?}{=}\frac{4}{\pi^2}\log N+O(1),$$

so sending $N\to\infty$ will complete the proof by Lemma 1.5 as discussed above. To show the above equality, we integrate. Attempting to break up our integral into periods,

$$\begin{aligned}L_N&=2\int_0^{1/2}\left|\frac{\sin(2N+1)\pi t}{\sin\pi t}\right|\,dt\\&=\frac{2}{\pi}\int_0^{1/2}\left|\frac{\sin(2N+1)\pi t}{\pi t}\right|\,dt+O(1)\\&=\frac{2}{\pi}\int_0^{N+1/2}\left|\frac{\sin\pi t}{t}\right|\,dt+O(1)\\&=\frac{2}{\pi}\sum_{k=0}^{N-1}\int_k^{k+1}\left|\frac{\sin\pi t}{t}\right|\,dt+O(1),\end{aligned}$$

where we have been pretty fast and loose with our $O(1)$ term. The point now is that the internal integral is approximately a $1/k$, so we are going to pick up a $\log N$ from some harmonic series argument; this intuition is good enough to produce divergence. TO be more precise, we note that we can adjust to

$$L_N=\frac{2}{\pi}\sum_{k=0}^{N-1}\int_k^{k+1}\frac{|\sin\pi t|}{k+1}\,dt+O(1),$$

where this movement is okay because the difference between $\frac{1}{t}$ and $\frac{1}{k+1}$ is on the order of $\frac{1}{k^2}$, which sums to $O(1)$. We now compute

$$L_N=\frac{2}{\pi}\sum_{k=0}^{N-1}\frac{1}{k+1}\int_k^{k+1}|\sin\pi t|\,dt+O(1)=\frac{4}{\pi^2}\log N+O(1),$$

which is what we wanted. ■

1.2.2 Convergence in Norm

To set us up, fix $p \in [1, \infty)$. For now, there are two questions of interest.

1. Convergence in norm: given $f \in L^p(\mathbb{R}/\mathbb{Z})$, do we have
$$\lim_{N \to \infty} \|S_N f - f\|_p = 0?$$
2. Convergence almost everywhere: given $f \in L^p(\mathbb{R}/\mathbb{Z})$, do we have $S_N f \to f$ almost everywhere?

For the first question, we get the following lemma, which converts convergence problems to boundedness problems.

Lemma 1.6. Fix $p \in [1, \infty)$. Then we have $S_N f \to f$ in p-norm for all f if and only if there is some constant C_p such that $\|S_N f\|_p \le C_p \|f\|_p$ for all f.

Proof. Necessity is from Lemma 1.5 because otherwise we can get some f with arbitrarily large values of $\|S_N f - f\|_p$ (say). Sufficiency arises because trigonometric functions are then dense in $L^p(\mathbb{R}/\mathbb{Z})$, so we get the result essentially by continuity. (We will show this more carefully later.) To be explicit, one can use the inequality $\|S_N f\|_p \le C_p \|f\|_p$ in order to show that all $\varepsilon > 0$ has some trigonometric polynomial g such that $\|f - g\|_p < \varepsilon$. Then for N large enough, we know $S_N g = g$ on the nose, so
$$\|S_N f - f\|_p \le \|S_N(f - g)\|_p + \underbrace{\|S_N g - g\|_p}_{0} + \|g - f\|_p \le (C_p + 1)\|f - g\|_p = (C_p + 1)\varepsilon,$$
so sending $\varepsilon \to 0^+$ (or replace ε with $\varepsilon/(C_p + 1)$, which notably does not depend on N) completes the proof. ■

For Lemma 1.6, we will want to tell when $\|S_N f\|_p \le C_p \|f\|_p$. It turns out that this is okay for $p > 1$, but it will be some time before we get there. One can check that the condition does fail for $p = 1$; as a hint, try something like a Dirac function.

1.3 September 1

I continue to not why I am here.

1.3.1 Remarks on Convergence in Norm

We are interested in checking Lemma 1.6 for various p. For $p = 2$, we have that $L^2(\mathbb{R}/\mathbb{Z})$ becomes a Hilbert space, and the point is that Parseval produces
$$\|f\|^2_{L^2(\mathbb{R}/\mathbb{Z}} = \sum_{k \in \mathbb{Z}} \left|\hat{f}(k)\right|^2,$$
which is enough for our check.

Remark 1.7. Checking convergence $S_N f \to f$ almost everywhere is much harder than convergence in norm. However, the answer is known: for $p = 1$, the answer is no due to Kolmogov, but we have almost everywhere convergence for $p > 1$. The result for $p = 2$ was shown by Carleson and then generalized to $p > 1$ by Hunt.

1.3.2 Cesàro Summation

In order to help our convergence, we will want a different way to sum.

Definition 1.8 (Cesàro sum)**.** Fix a function f. Then the N*th Cesàro sum* is the average

$$\sigma_N f(x) := \frac{1}{N+1}\sum_{k=0}^{N} S_k f(x).$$

This σ_N is going to behave much better than S_N. Indeed, we see

$$\sigma_N f(x) = \int_0^1 f(t)\cdot \underbrace{\frac{1}{N+1}\sum_{k=0}^{N} D_k(x-t)}_{F_N(x-t):=}\,dt.$$

Here, F_N is the "Fejér kernel," which we can compute as

$$\begin{aligned} F_N(t) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(t) \\ &= \frac{1}{N+1}\cdot\frac{1}{\sin\pi t}\sum_{k=0}^{N}\sin(2k+1)\pi t \\ &= \frac{1}{N+1}\cdot\frac{1}{\sin\pi t}\sum_{k=0}^{N}\frac{e^{i(2k+1)\pi t}-e^{-i(2k+1)\pi t}}{2i} \\ &= \frac{1}{N+1}\left(\frac{\sin(N+1)\pi t}{\sin\pi t}\right)^2, \end{aligned}$$

where we have summed the geometric series and simplified in the last step. Importantly, $F_N(t)$ is non-negative. Because $\int_0^1 D_K(t)\,dt = 1$ always (write out the $\sin$s as exponentials and integrate), we see that $\int_0^1 F_N(t)\,dt = 1$ as well, so F_N can be thought of as a redistribution of mass. A direct computation is able to show that

$$\lim_{N\to\infty}\int_{\delta<|t|<1/2} F_N(t) \overset{?}{=} 0 \tag{1.1}$$

for any fixed $\delta > 0$. Indeed, we see $F_N(t) \le \frac{1}{N+1}(\sin\pi t)^{-2}$, but $(\sin\pi t)^{-2}$ is bounded on $[\delta, 1/2]$, so we can upper-bound $F_N(t) \le M/(N+1)$ for some M, which achieves the result upon sending $N\to\infty$.

The point of introducing σ_N is the following result.

Theorem 1.9. Fix a function f. Then

$$\lim_{N\to\infty}\|\sigma_N f - f\|_p = 0$$

for $f\in L^p(\mathbb{R}/\mathbb{Z})$ where $1\le p<\infty$ or for $f\in C(\mathbb{R}/\mathbb{Z})$ where $p=\infty$.

Proof. We omit the second case because the proof is similar. As for the first case, we write

$$\|\sigma_N f - f\|_p = \left\| x\mapsto \int_{-1/2}^{1/2} F_N(t)(f(x-t)-f(x))\,dt\right\|_p.$$

Approximately speaking, the integral is some kind of continuous weighted average of functions. For t small, the function $f(x-t)-f(x)$ is small, and for t large, one can use (1.1) to do our bounding. ■

Bibliography

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions