Dimension Theory for the Impatient

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Abstract

This document collects a variety of dimension-computing results from Eisenbud's *Commutative Algebra: with a View Toward Algebraic Geometry*. All references are to this book.

Contents

1 Definitions

1.1 Kinds of Dimension

Definition (Dimension). The *Krull dimension* of a ring R, denoted $\dim R$, is the supremum of the length r of a chain of distinct primes

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

Definition (Dimension, ideals). Fix a ring R and an ideal $I \subseteq R$. Then we define the *dimension* of an ideal I to be $\dim I := \dim R/I$.

Lemma. Fix an ideal I of a ring R. Then $\dim I$ is equal to the length of the longest chain of primes

$$I \subseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

in R.

Definition (Codimension). Fix I a proper ideal of a ring R.

- If $I = \mathfrak{p}$ is a prime ideal of R, then we define the *codimension* as $\operatorname{codim} \mathfrak{p} := \dim R_\mathfrak{p}$.
- More generally, we define the *codimension* as

  $$\operatorname{codim} I := \min_{\mathfrak{p} \subseteq I} \operatorname{codim} \mathfrak{p},$$

  where the minimum is over all prime ideals $\mathfrak{p}$ containing I.

Lemma. Fix a prime ideal $\mathfrak{p}$ of a ring R. Then $\operatorname{codim} \mathfrak{p}$ is equal to the length of the longest chain of primes

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{p},$$

where $\mathfrak{p}$ is included in the chain; i.e., $\operatorname{codim} \mathfrak{p} = d$ here.

Definition (Dimension, modules). Given an R-module M, we define the *dimension* of M as $\dim M := \dim R/\operatorname{Ann} M$.

1.2 Kinds of Rings

Definition (Regular). Fix a local ring R of dimension $d := \dim R$. Further, let $\mathfrak{m}$ be the maximal ideal of R. Then R is *regular* if and only if there exist elements $\{f_1, \ldots, f_d\} \subseteq R$ such that

$$\mathfrak{m} = (f_1, \ldots, f_d).$$

Remark (Corollary 10.14). Regular local rings are integral domains.

Definition (Discrete valuation ring). A *discrete valuation ring* is an integral domain R equipped with a valuation $\nu\colon K(R)^\times \to \mathbb{Z}$.

Proposition (Proposition 11.1)**.** Fix a Noetherian ring R. The following are equivalent.

- R is a discrete valuation ring.
- R is a field or regular local ring of dimension 1.

Definition (Dedekind)**.** A *Dedekind domain* is a Noetherian normal domain of dimension 1.

2 Theorems

2.1 First Results

Proposition. Fix I an ideal of a ring R. Then

$$\dim I + \operatorname{codim} I \le \dim R.$$

Remark. Equality for the above holds when R is an affine domain, by Corollary 13.4.

Lemma. Fix ideals I and J in a ring R. If $I \subseteq J$, then

$$\dim I \ge \dim J \qquad \text{and} \qquad \operatorname{codim} I \le \operatorname{codim} J.$$

Similarly, if $\mathfrak{p}$ and $\mathfrak{q}$ are primes with $\mathfrak{p} \subseteq \mathfrak{q}$, then $\operatorname{codim} \mathfrak{p} \le \operatorname{codim} \mathfrak{q}$, with equality if and only if $\mathfrak{p} = \mathfrak{q}$.

2.2 Localization, Completion, Polynomials

Theorem. Fix a ring R. Then

$$\dim R = \max_{\mathfrak{p} \in \operatorname{Spec} R} \dim R_{\mathfrak{p}}.$$

In other words, dimension is a local quantity.

Corollary (Corollary 10.12)**.** Fix a local Noetherian ring R and a finitely generated module M. Then $\dim R = \dim \widehat{R}$.

Remark. Corollary 10.12 is strictly weaker than Corollary 12.15.

Corollary (Corollary 10.13)**.** Fix a Noetherian ring R with finite dimension. Then $\dim R[x] = \dim R + 1$.

2.3 Comparing Rings

Proposition (Proposition 9.2)**.** Fix a ring homomorphism $\varphi\colon R \to S$ which makes S into an integral R-algebra. Then, for any $\mathfrak{p} \in \operatorname{Spec} R$ such that $\ker \varphi \subseteq \mathfrak{p}$, there exists $\mathfrak{q} \in \operatorname{Spec} S$ such that

$$\mathfrak{p} = \varphi^{-1}(\mathfrak{q}).$$

In fact, for any ideal $I \subseteq S$, we have $\dim S/I = \dim R/\varphi^{-1}(I)$. In particular, if $\varphi\colon R \to S$ is injective, then $\dim R = \dim S$.

Lemma. Fix a ring R and a multiplicatively closed subset $U \subseteq R$. Further, set $S := R\left[U^{-1}\right]$ with the natural map $\varphi : R \to S$. Then, for any prime $\mathfrak{p} \subseteq R\left[U^{-1}\right]$, we have

$$\operatorname{codim} \varphi^{-1}(\mathfrak{p}) = \operatorname{codim} \mathfrak{p}.$$

Theorem (Theorem 10.10). Fix two local rings R and S with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Given a map $\varphi : R \to S$ of local rings so that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, we have

$$\dim S \leq \dim R + \dim S/\mathfrak{m}S$$

In fact, if S is a flat as an R-module, then we have equality.

2.4 Generating Elements

Theorem (Theorem 10.2). Fix a Noetherian ring R. Given an ideal $(x_1, \ldots, x_s) \in R$, suppose $\mathfrak{p}$ is a minimal prime over $(x_1, \ldots, x_s)$. Then

$$\operatorname{codim} \mathfrak{p} \leq s.$$

Corollary (Corollary 10.5). Fix a prime ideal $\mathfrak{p}$ of a Noetherian ring R with codimension r. Then there are elements $x_1, \ldots, x_r$ such that $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_r)$, and in fact $\operatorname{codim}(x_1, \ldots, x_r) = r$.

Proposition (Proposition 10.8). Fix a local ring R with maximal ideal $\mathfrak{m}$. Then $\dim R$ is the minimal $d \in \mathbb{N}$ such that there exist generators $f_1, \ldots, f_d$ so that

$$\mathfrak{m}^n \subseteq (f_1, \ldots, f_d) \subseteq \mathfrak{m}$$

for some n.

Corollary. Let R be a Noetherian regular local ring with maximal ideal $\mathfrak{m}$. Then

$$\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim R.$$

2.5 Dimension for Modules

Proposition (Proposition 10.8). Fix a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and an R-module M. Then $\dim M$ is equal to the minimal d such that there is some proper ideal $(f_1, \ldots, f_d) \subseteq R$ with finite colength on M.

Corollary (Corollary 10.9). Fix a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and an R-module M. Given $x \in \mathfrak{m}$, we have

$$\dim M/xM \geq \dim M - 1.$$

Corollary (Corollary 12.5). Fix a Noetherian local ring R. Given a finitely generated R-module M with parameter ideal $\mathfrak{q}$,

$$\dim M = \dim \widehat{M}_{\mathfrak{q}} = \dim(\operatorname{gr}_{\mathfrak{q}} M)_{\mathfrak{P}}.$$

Here, $\mathfrak{P} \subseteq \operatorname{gr}_{\mathfrak{q}} M$ is the irrelevant ideal.

2.6 The Hilbert Function

Definition (Hibert–Samuel function)**.** Fix a local Noetherian ring R with finitely generated R-module M and some prime of finite colength $\mathfrak{q}$. Then we define the *Hilbert–Samuel function* by

$$H_{\mathfrak{q},M}(n) := \ell\left(\mathfrak{q}^n M/\mathfrak{q}^{n+1}\right).$$

Lemma. Fix a local Noetherian ring R with maximal ideal $\mathfrak{m}$. Further suppose that there is a map $k \hookrightarrow R$ such that the composite

$$k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$$

is an isomorphism. Then, for any finitely generated R-module M of finite length,

$$\ell_R(M) = \dim_k M.$$

Theorem (Theorem 12.4)**.** Fix a local Noetherian ring R with unique maximal ideal $\mathfrak{m}$. Further, take a finitely generated module M and an ideal $\mathfrak{q}$ of finite colength on M. Then

$$\dim M = 1 + \deg P_{\mathfrak{q},M}.$$

Corollary (Corollary 13.7)**.** Fix a Noetherian graded ring $R := R_0 \oplus R_1 \oplus \cdots$. Then $\dim R$ is the supremum of $\dim R_{\mathfrak{p}}$ for all homogeneous prime ideals $\mathfrak{p}$.

Thus, if R_0 is a field, then

$$\dim R = 1 + \deg P_R,$$

where P_R is the Hilbert polynomial for R.

2.7 Affine Domains

Theorem (Theorem 13.3)**.** Fix an affine ring R of dimension d. Given a chain

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq R$$

with $d_j := \dim I_j$ such that $\{d_j\}_{j=0}^m$ is strictly decreasing and $d_m > 0$. Then there is a subring $S \subseteq R$ such that

(a) $S \cong k[x_1, \ldots, x_d]$,

(b) R is finite over S, and

(c) any ideal I_j has $S \cap I_j = (x_{d_j+1}, \ldots, x_d)$.

Theorem (Theorem A)**.** Fix an affine domain R over a field k. Then

$$\dim R = \text{transcendence degree}_k R.$$

Corollary (Corollary 13.4)**.** Fix an affine domain R. Given an ideal $I \subseteq R$, we have

$$\dim I + \operatorname{codim} I = \dim R.$$

Corollary (Corollary 13.5)**.** Suppose that we have an inclusion $R \subseteq T$ of affine domains over k. Then

$$\dim T = \dim R + \dim K(R) \otimes_R T.$$

Corollary (Corollary 13.11)**.** Fix an affine domain R. If $f \in R \setminus \{0\}$ is not a unit, then

$$\dim R/(f) = \dim R - 1.$$