256A: Algebraic Geometry

Nir Elber

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Theme 1
Introduction

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 24

A feeling of impending doom overtakes your soul.

1.1.1 Administrative Notes

Here are housekeeping notes.

- Here are some housekeeping notes. There is a syllabus on `bCourses`.
- We hope to cover Chapter II of [Har77], mostly, supplemented with examples from curves.
- There are lots of books.
  - We use [Har77] because it is short.
  - There is also [Vak17], which has more words.
  - The book [Liu06] has notes on curves.
  - There are more books in the syllabus. Professor Tang has some opinions on these.
- Some proofs will be skipped in lecture. Not all of these will appear on homework.
- Some examples will say lots of words, some of which we won't have good definitions for until later. Do not be afraid of words.

Here are assignment notes.

- Homework is 70% of the class.
- Homework is due on noon on Fridays. There will be 6–8 problems, which means it is pretty heavy. The lowest homework score will be dropped.

- Office hours exist. Professor Tang also answers emails.
- The term paper covers the last 30% of the grade. They are intended to be extra but interesting topics we don't cover in this class.

1.1.2 Motivation

We're going to talk about schemes. Why should we care about schemes? The point is that schemes are "correct."

Example 1.1. In algebraic topology, there is a cup product map in homology, which is intended to algebraically measure intersections. However, intersections are hard to quantify when we aren't dealing with, say manifolds.

Here is an example of algebraic geometry helping us with this rigorization.

Theorem 1.2 (Bézout)**.** Let C_1 and C_2 be curves in $\mathbb{P}^2(k)$, for some algebraically closed k, where C_1 and C_2 are defined by homogeneous polynomials f_1 and f_2. Then the "intersection number" between the curves C_1 and C_2 is $(\deg f_1)(\deg f_2)$.

This is a nice result, for example because it automatically accounts for multiplicities, which would be difficult to deal with directly using (say) geometric techniques. Schemes will help us with this.

Example 1.3. Moduli spaces are intended to be geometric objects which represent a family of geometric objects of interest. For example, we might be interested in the moduli space of some class of elliptic curves.

It turns out that the correct way to define these objects is using schemes as a functor; we will see this in this class.

Remark 1.4. One might be interested in when a functor is a scheme. We will not cover this question in this class in full, but it is an interesting question, and we will talk about this in special cases.

1.1.3 Elliptic Curves

For the last piece of motivation, let's talk about elliptic curves, over a field k.

Definition 1.5 (Elliptic curve)**.** An *elliptic curve* over k is a smooth projective curve of genus 1, with a marked k-rational point.

Remember that we said that we not to be afraid of words. However, we should have some notion of what these words mean: being a curve means that we are one-dimensional, being smooth is intuitive, and having genus 1 roughly means that base-changing to a complex manifold has one hole. Lastly, the k-rational point requires defining a scheme as a functor.

Here's another (more concrete) definition of an elliptic curve.

Definition 1.6 (Elliptic curve)**.** An *elliptic curve* over k is an affine variety in $\mathbb{A}^2(k)$ cut out be a polynomial of the form

$$y^2 + a_1xy + a_3y^2 = x^3 + a_2x^2 + a_4x + a_6$$

with nonzero discriminant plus a point $\mathcal{O}$ at infinity.

Remark 1.7. Why are these equivalent? Well, the Riemann–Roch theorem approximately lets us take a smooth projective curve of genus 1 and then write it as an equation; the marked point goes to $\mathcal{O}$. In the reverse direction, one merely needs to embed our affine curve into projective space and verify its smoothness and genus.

Instead of working with affine varieties, we can also give a concrete description of an elliptic curve using projective varieties.

Definition 1.8 (Elliptic curve)**.** An *elliptic curve* over k is a projective variety in $\mathbb{P}^2(k)$ cut out be a polynomial of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with nonzero discriminant.

We get the equivalence of the previous two definitions via the embedding $\mathbb{A}_2(k) \hookrightarrow \mathbb{P}^2(k)$ by $(x, y) \mapsto [x : y : 1]$; the point at infinity $\mathcal{O}$ is $[0 : 1 : 0]$.

1.1.4 Crackpot Varieties

In order to motivate schemes, we should probably mention varieties, so we will spend some time in class discussing affine and projective varieties. For convenience, we work over an algebraically closed field k.

Definition 1.9 (Affine space)**.** Given a field k, we define *affine n-space over k*, denoted $\mathbb{A}^n(k)$. An *affine variety* is a subset $Y \subseteq \mathbb{A}^n(k)$ of the form

$$Y = V(S) := \{p \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } f \in S\},$$

where $S \subseteq k[x_1, \ldots, x_n]$.

Remark 1.10. The set $S \subseteq k[x_1, \ldots, x_n]$ in the above definition need not be finite or countable. In certain cases, we can enforce this condition; for example, if $n = 1$, then $k[x]$ is a principal ideal domain, so we may force $\#S = 1$.

Note that we have defined vanishing sets $V(S)$ from subsets $S \subseteq k[x_1, \ldots, x_n]$. We can also go from vanishing sets to subsets.

Definition 1.11. Fix a field k and subset $Y \subseteq \mathbb{A}^n(k)$. Then we define the ideal

$$I(Y) := \{f \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } p \in Y\}.$$

Remark 1.12. One should check that this is an ideal, but we won't bother.

So we've defined some geometry. But we're in an algebraic geometry class; where's the algebra?

Theorem 1.13 (Hilbert's Nullstellensatz)**.** Fix an algebraically closed field k and ideal $J \subseteq k[x_1, \ldots, x_n]$. Then

$$I(V(J)) = \operatorname{rad} I,$$

where $\operatorname{rad} I$ is the radical of I.

Remark 1.14. The Nullstellensatz has no particularly easy proof.

The point of this result is that it ends up giving us a contravariant equivalence of posets of radical ideals and affine varieties.

Why do we care? In some sense, we prefer to work with ideals because it "remembers" more information than merely the points on the variety. To see this, note that elements $f \in k[x_1, \ldots, x_n]$ we are viewing as giving functions on $\mathbb{A}^n(k)$. However, when we work on a variety $Y \subseteq \mathbb{A}^n(k)$, then sometimes two functions will end up being identical on Y. So the correct ring of functions on Y is

$$k[x_1, \ldots, x_n]/I(Y),$$

so indeed keeping track of the (algebraic) ideal $V(Y)$ gets us some extra (geometric) information.

We will use this discussion as a jumping-off point to discuss affine schemes and then schemes. Affine schemes will have the following data.

- A commutative ring A, which we should think of as the ring of functions on a variety.
- A topological space $\operatorname{Spec} A$, which has more information than merely points on the variety.
- A structure sheaf of functions on $\operatorname{Spec} A$.

Remark 1.15. Our topological space $\operatorname{Spec} A$ will contain more points than just the points on the variety. In some sense, these extra points make the topology more apparent.

Remark 1.16. Going forward, one might hope to remove requirements that the field k is algebraically closed (e.g., to work with a general ring) or talk about ideals which are not radical. This is the point of scheme theory.

1.2 August 26

Let's finish up talking varieties, and then we'll move on to affine schemes.

1.2.1 Projective Varieties

We're going to briefly talk about projective varieties. Let's start with projective space.

Definition 1.17 (Projective space)**.** Given a field k, we define *projective n-space over k*, denoted $\mathbb{P}^n(k)$ as

$$\frac{k^{n+1} \setminus \{(0, \ldots, 0\}}{\sim},$$

where $\sim$ assigns two points being equivalent if and only if they span the same 1-dimensional subspace of k^{n+1}. We will denote the equivalence class of a point $(a_0, \ldots, a_n)$ by $[a_0 : \ldots : a_n]$.

To work with varieties, we don't quite cut out by general polynomials but rather by homogeneous polynomials.

Definition 1.18 (Projective variety)**.** Given a field k and a set of some homogeneous polynomials $T \subseteq k[x_1, \ldots, x_n]$, we define the *projective variety* cut out by T as

$$V(T) := \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in T\}.$$

Example 1.19. The elliptic curve corresponding to the affine algebraic variety in $\mathbb{A}^2(k)$ cut out by $y^2 - x^3 - 1$ becomes the projective variety in $\mathbb{P}^2(k)$ cut out by

$$Y^2Z - X^3 - Z^3 = 0.$$

Remark 1.20. One can give projective varieties some Zariski topology as well, which we will define later in the class.

What to remember about projective varieties is that we can cover $\mathbb{P}^2(k)$ (say) by affine spaces as

$$\begin{aligned}\mathbb{P}^2(k) &= \{[X:Y:Z] : X,Y,Z \in k \text{ not all } 0\} \\ &= \{[X:Y:Z] : X,Y,Z \in k \text{ and } X \neq 0\} \\ &\quad \cup \{[X:Y:Z] : X,Y,Z \in k \text{ and } Y \neq 0\} \\ &= \{[1:y:z] : y,z \in k\} \\ &\quad \cup \{[x:1:z] : x,z \in k\} \\ &\simeq \mathbb{A}^2(k) \cup \mathbb{A}^2(k).\end{aligned}$$

The point is that we can decompose $\mathbb{P}^2(k)$ into an affine cover.

Example 1.21. Continuing from Example 1.19, we can decompose $Z\left(Y^2Z - X^3 - Z^3\right)$ into having an affine open cover by

$$\underbrace{\left\{(x,y) : y^2 - x^3 - 1 = 0\right\}}_{z\neq 0} \cup \underbrace{\left\{(x,z) : z - x^3 - z^3 = 0\right\}}_{y\neq 0} \cup \underbrace{\left\{(y,z) : y^2z - 1 - z^3 = 0\right\}}_{x\neq 0}.$$

Notably, we get almost everything from just one of the affine chunks, and we get the point at infinity by taking one of the other chunks.

Remark 1.22. It is a general fact that we only need two affine chunks to cover our projective curve.

1.2.2 The Spectrum

The definition of a(n affine) scheme requires a topological space and its ring of functions. We will postpone talking about the ring of functions until we discuss sheaves, so for now we will focus on the space.

Definition 1.23 (Spectrum)**.** Given a ring A, we define the *spectrum*

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}.$$

Example 1.24. Fix a field k. Then $\operatorname{Spec} k = \{(0)\}$. Namely, non-isomorphic rings can have homeomorphic spectra.

Exercise 1.25. Fix a field k. We show that

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(\pi) : \pi \text{ is monic, irred.}, \deg \pi > 0\}.$$

Proof. To begin, note that (0) is prime, and (π) is prime for irreducible non-constant polynomials π because irreducible elements are prime in principal ideal domains. Additionally, we note that all the given primes are distinct: of course (0) is distinct from any prime of the form (π), but further, given monic non-constant irreducible polynomials α and β, having

$$(\alpha) = (\beta)$$

forces $\alpha = c\beta$ for some $c \in k[x]^\times$. But $k[x]^\times = k^\times$, so $c \in k^\times$, so $c = 1$ is forced by comparing the leading coefficients of α and β.

It remains to show that all prime ideals $\mathfrak{p} \subseteq k[x]$ take the desired form. Well, $k[x]$ is a principal ideal domain, so we may write $\mathfrak{p} = (\pi)$ for some $\pi \in k[x]$. If $\pi = 0$, then we are done. Otherwise, $\deg \pi \geq 0$, but $\deg \pi > 0$ because $\deg \pi = 0$ implies $\pi \in k[x]^\times$. By adjusting by a unit, we may also assume that π is monic. And lastly, note that (π) is prime means that π is prime, so π is irreducible. ■

Example 1.26. If k is an algebraically closed field, then the only nonconstant irreducible polynomials are linear (because all nonconstant polynomials have a root and hence a linear factor), and of course any linear polynomial is irreducible. Thus,

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(x - \alpha) : \alpha \in k\}.$$

Set $\mathfrak{m}_\alpha := (x - \alpha)$ so that $\alpha \mapsto \mathfrak{m}_\alpha$ provides a natural map from $\mathbb{A}^1_k$ to $\operatorname{Spec} k[x]$. In this way we can think of $\operatorname{Spec} k[x]$ as $\mathbb{A}^1_k$ with an extra point (0).

Remark 1.27. Continuing from Example 1.26, observe that we can also recover function evaluation at a point $\alpha \in \mathbb{A}^1_k$: given $f \in k[x]$, the value of $f(\alpha)$ is the image of f under the canonical map

$$k[x] \twoheadrightarrow \frac{k[x]}{\mathfrak{m}_\alpha} \cong k,$$

where the last map is the forced $x \mapsto \alpha$. Observe running this construction at the point $(0) \in \operatorname{Spec} k[x]$ makes the "evaluation" map just the identity.

Example 1.28. Similar to $k[x]$, we can classify $\operatorname{Spec} \mathbb{Z}$: all ideals are principal, so our primes look like (p) where $p = 0$ or is a rational prime. Namely, essentially the same proof gives

$$\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) : p \text{ prime}, p > 0\}.$$

The condition $p > 0$ is to ensure that all the points on the right-hand side are distinct; certainly we can write all nonzero primes $(p) \subseteq \mathbb{Z}$ for some nonzero (p), and we can adjust p by a unit to ensure $p > 0$. Conversely, $(p) = (q)$ with $p, q > 0$ forces $p \mid q$ and $q \mid p$ and so $p = q$.

We might hope to have a way to view $\operatorname{Spec} k[x]$ as points even when k is not algebraically closed.

Example 1.29. Set $k = \mathbb{Q}$. There is a map sending a nonconstant monic irreducible polynomial $\pi \in \mathbb{Q}[x]$ to its roots in $\overline{\mathbb{Q}}$, and note that this map is injective because one can recover a polynomial from its roots. Further, all the roots of π are Galois conjugate because π is irreducible, and a Galois orbit S_α of a root α corresponds to the polynomial

$$\pi(x) = \prod_{\beta \in S_\alpha} (x - \beta),$$

where $\pi(x) \in \mathbb{Q}[x]$ because its coefficients are preserved the Galois action. Thus, there is a bijection between the nonconstant monic irreducible polynomials $\pi \in \mathbb{Q}[x]$ and Galois orbits of elements in $\overline{\mathbb{Q}}$.

So far, all of our examples have been "dimension 0" (namely, a field k) or "dimension 1" (namely, $\mathbb{Z}$ and $k[x]$). Here is an example in dimension 2.

Exercise 1.30. Let k be algebraically closed. Any $\mathfrak{p} \in \operatorname{Spec} k[x, y]$ is one of the following types of prime.

- Dimension 2: $\mathfrak{p} = (0)$.
- Dimension 1: $\mathfrak{p} = (f(x, y))$ where f is nonconstant and irreducible.
- Dimension 0: $\mathfrak{p} = (x - \alpha, y - \beta)$, where $\alpha, \beta \in k$.

Proof. We follow [Vak17, Exercise 3.2.E]. If $\mathfrak{p} = (0)$, then we are done. If $\mathfrak{p}$ is principal, then we can write $\mathfrak{p} = (f)$ where $f \in k[x, y]$ is a prime element and hence irreducible. Observe that if f is irreducible, then f is also a prime element because $k[x, y]$ is a unique factorization domain.

Lastly, we suppose that $\mathfrak{p}$ is not principal. We start by finding $f, g \in \mathfrak{p}$ with no nonconstant common factors. Because $\mathfrak{p} \neq 0$, we can find $f_0 \in \mathfrak{p} \setminus \{0\}$, and assume that (f_0) is maximal with respect to this (namely, $f_0 \notin (f_0')$ for any $f_0' \in \mathfrak{p}$). Because $\mathfrak{p}$ is not principal, we can find $g_0 \in \mathfrak{p} \setminus (f_0)$. Now, we can use unique prime factorization of f_0 and g_0 to find some $d \in k[x, y]$ such that

$$f_0 = fd \qquad \text{and} \qquad g_0 = gd$$

where f and g share no common factors. (Namely, $\nu_\pi(d) = \min\{\nu_\pi(f_0), \nu_\pi(g_0)\}$ for all irreducible factors $\pi \in k[x, y]$.) Note $d \notin \mathfrak{p}$ by the maximality of f_0, so $f, g \in \mathfrak{p}$ is forced.

Continuing, embedding f and g into $k(x)[y]$ and using the Euclidean algorithm there, we can write

$$af + bg = 1$$

where $a, b \in k(x)[y]$, because f and g have no common factors in $k(x)[y]$. (Any common factor would lift to a common factor in $k[x, y]$.[1]) Clearing denominators, we see that we can find $h(x) \in k[x] \cap \mathfrak{p}$, but by factoring $h(x)$ using the fact that k is algebraically closed, we see that we can actually enforce $(x - \alpha) \in \mathfrak{p}$ for some $\alpha \in k$.

By symmetry, we can force $(y - \beta) \in \mathfrak{p}$ for some $\beta \in \mathfrak{p}$ as well, so $(x - \alpha, y - \beta) \subseteq \mathfrak{p}$. However, we see that $(x - \alpha, y - \beta)$ is maximal because of the isomorphism

$$\frac{k[x, y]}{(x - \alpha, y - \beta)} \to k$$

by $x \mapsto \alpha$ and $y \mapsto \beta$. Thus, $\mathfrak{p} = (x - \alpha, y - \beta)$ follows. ∎

Remark 1.31. The intuition behind Exercise 1.30 is that the prime ideal $(x - \alpha, y - \beta)$ "cuts out" the zero-dimensional point $(\alpha, \beta) \in \mathbb{A}_k^2$. Then the prime ideal (f) cuts out some one-dimensional curve in $\mathbb{A}_k^2$, and the prime ideal (0) cuts out the entire two-dimensional plane. We have not defined dimension rigorously, but hopefully the idea is clear.

Remark 1.32. It is remarkable that the number of equations we need to cut out a variety of dimension d is $2 - d$. This is not always true.

The point is that we seem to have recovered $\mathbb{A}_k^1$ by looking at $\operatorname{Spec} k[x]$ and $\mathbb{A}_k^2$ by looking at $\operatorname{Spec} k[x, y]$, so we can generalize this to arbitrary rings cleanly, realizing some part of Remark 1.16.

Definition 1.33 (Affine space)**.** Given a ring R, we define *affine n-space over R* as

$$\mathbb{A}_R^n \coloneqq \operatorname{Spec} R[x_1, \ldots, x_n].$$

So far all the rings we've looked at so far have been integral domains, but it's worth pointing out that working with general rings allows more interesting information.

Example 1.34. We classify $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$. Notably, all prime ideals here must correspond to prime ideals of $k[\varepsilon]$ containing $\left(\varepsilon^2\right)$ and hence contain $\operatorname{rad}\left(\varepsilon^2\right) = (\varepsilon)$, which allows only the prime (ε). (We will make this correspondence precise later.) So $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ has a single point.

[1] If $d(x, y)/e(x)$ divides both f and g in $k(x)[y]$, where d and e share no common factors, then $d \mid fe, ge$ in $k[x, y]$. Unique prime factorization now forces $d \mid f, g$ in $k[x, y]$.

Remark 1.35. In some sense, $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ will be able to let us talk about differential information algebraically: ε is some very small nonzero element such that $\varepsilon^2 = 0$. So we can study a "function" $f \in k[x]$ locally at a point p by studying $f(p+\varepsilon)$. Rigorously, $f(x) = \sum_{i=0}^d a_i x^i$ has

$$f(x+\varepsilon) = \sum_{i=0}^d a_i (x+\varepsilon)^i = \sum_{i=0}^d a_i x^i + \sum_{i=1}^d i a_i x^{i-1}\varepsilon = f(x) + f'(x)\varepsilon.$$

One can recover more differential information by looking at $k[\varepsilon]/\left(\varepsilon^n\right)$ for larger n.

1.2.3 The Zariski Topology

Thus far we've defined our space. Here's our topology.

Definition 1.36 (Zariski topology)**.** Fix a ring A. Then, for $S \subseteq A$, we define the *vanishing set*

$$V(S) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\}$$

Then the *Zariski topology* on $\operatorname{Spec} A$ is the topology whose closed sets are the $V(S)$.

Intuitively, we are declaring A as the (continuous) functions on $\operatorname{Spec} A$, and the evaluation of the function $f \in A$ at the point $\mathfrak{p} \in \operatorname{Spec} A$ is $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27). Then the vanishing sets of a continuous function must be closed, and without easy access to any other functions on $\operatorname{Spec} A$, we will simply declare that these are all of our closed sets.

In the affine case, we can be a little more rigorous.

Example 1.37. Set $A \coloneqq k[x_1, \ldots, x_n]$, where k is algebraically closed. Then, given $f \in k[x_1, \ldots, x_n]$, we want to be convinced that $V(\{f\})$ matches up with the affine k-points $(a_1, \ldots, a_n)$ which vanish on f. Well, $(a_1, \ldots, a_n)$ corresponds to the prime ideal $(x_1 - a_1, \ldots, x_n - a_n) \in \operatorname{Spec} A$, and

$$\{f\} \subseteq (x_1 - a_1, \ldots, x_n - a_n)$$

is equivalent to f vanishing in the evaluation map

$$k[x_1, \ldots, x_n] \twoheadrightarrow \frac{k[x_1, \ldots, x_n]}{(x_1 - a_1, \ldots, x_n - a_n)} \to k,$$

which is equivalent to $f(a_1, \ldots, a_n) = 0$. So indeed, f vanishes on $(a_1, \ldots, a_n)$ if and only if the corresponding maximal ideal is in $V(\{f\})$.

With intuition out of the way, we should probably check that the sets $V(S)$ make a legitimate topology. To begin, here are some basic properties.

Lemma 1.38. Fix a ring A.

(a) If subsets $S, T \subseteq A$ have $S \subseteq T$, then $V(T) \subseteq V(S)$.

(b) A subset $S \subseteq A$ has $V(S) = V((S))$.

(c) An ideal $I \subseteq A$ has $V(I) = V(\operatorname{rad} I)$.

Proof. We go in sequence.

(a) Note $\mathfrak{p} \in V(T)$ implies that $T \subseteq \mathfrak{p}$, which implies $S \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(S)$.

(b) Surely $S \subseteq (S)$, so $V((S)) \subseteq V(S)$. Conversely, if $\mathfrak{p} \in V(S)$, then $S \subseteq \mathfrak{p}$, but then the generated ideal (S) must also be contained in $\mathfrak{p}$, so $\mathfrak{p} \in V((S))$.

(c) Surely $I \subseteq \operatorname{rad} I$, so $V(\operatorname{rad} I) \subseteq V(I)$. Conversely, if $\mathfrak{p} \in V(I)$, then $\mathfrak{p} \subseteq I$, so

$$\mathfrak{p} \subseteq \bigcap_{\mathfrak{q} \supseteq I} \mathfrak{q} = \operatorname{rad} I,$$

so $\mathfrak{p} \in V(\operatorname{rad} I)$. ■

And here are our checks.

Lemma 1.39. Fix a ring A.

(a) $V(A) = \varnothing$ and $V((0)) = \operatorname{Spec} A$.

(b) Given ideals $I, J \subseteq A$, then $V(I) \cup V(J) = V(IJ)$.

(c) Given a collection of ideals $\mathcal{I} \subseteq \mathcal{P}(A)$, we have

$$\bigcap_{I \in \mathcal{I}} V(I) = V\left(\sum_{I \in \mathcal{I}} I\right).$$

Proof. We go in sequence.

(a) All primes are proper, so no prime $\mathfrak{p}$ has $A \subseteq \mathfrak{p}$, so $V(A) = \varnothing$. Also, 0 is an element of all ideals, so all $\mathfrak{p} \in \operatorname{Spec} A$ have $(0) \subseteq \mathfrak{p}$, so $V((0)) = \operatorname{Spec} A$.

(b) Note $IJ \subseteq I, J$, so $V(I) \cup V(J) \subseteq V(IJ)$ follows. Conversely, take $\mathfrak{p} \in V(IJ)$, and suppose $\mathfrak{p} \notin V(I)$ so that we need $\mathfrak{p} \in V(J)$. Well, $\mathfrak{p} \notin V(I)$ implies $I \not\subseteq \mathfrak{p}$, so we can find $a \in I \setminus \mathfrak{p}$. Now, for any $b \in J$, we see

$$ab \in IJ \subseteq \mathfrak{p},$$

so $a \notin \mathfrak{p}$ forces $b \in \mathfrak{p}$. Thus, $J \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(J)$.

(c) Certainly any $J \in \mathcal{I}$ has $J \subseteq \sum_{I \in \mathcal{I}} I$, so $V\left(\sum_{I \in \mathcal{I}} I\right) \subseteq \bigcap_{I \in \mathcal{I}} V(I)$ follows.

Conversely, suppose $\mathfrak{p} \in \bigcap_{I \in \mathcal{I}} V(I)$. Then $I \subseteq \mathfrak{p}$ for all $I \in \mathcal{I}$, so $\sum_{I \in \mathcal{I}} I \subseteq \mathfrak{p}$ follows. Thus, $\mathfrak{p} \in V\left(\sum_{I \in \mathcal{I}} I\right)$. ■

It follows that the collection of vanishing sets is closed under finite union and arbitrary intersection, so they do indeed specify the closed sets of a topology.

1.2.4 Easy Nullstellensatz

While we're here, let's also generalize Definition 1.11.

Definition 1.40. Fix a ring A. Then, given a subset $Y \subseteq \operatorname{Spec} A$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

And here is our nice version of Theorem 1.13.

Proposition 1.41. Fix a ring A and an ideal $\mathfrak{a} \subseteq A$ and a subset $Y \subseteq \operatorname{Spec} A$.

(a) The ideal $I(Y)$ is radical.

(b) We have that
$$I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a} \qquad \text{and} \qquad V(I(Y)) = \overline{Y},$$
where $\overline{Y}$ means the closure in the Zariski topology (i.e., the "Zariski closure").

(c) There is a one-to-one inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. Omitted. ■

Here are some more quick remarks.

Remark 1.42. A ring homomorphism $\varphi\colon A \to B$ will induce a continuous map
$$\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A.$$
We will state this more concretely later.

Example 1.43. Fix a ring A and ideal $\mathfrak{a} \subseteq A$. Then the surjection $A \twoheadrightarrow A/\mathfrak{a}$ induces a natural homeomorphism
$$\operatorname{Spec} A/\mathfrak{a} \cong V(\mathfrak{a}) = \{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{a}\}$$
because primes of $A/\mathfrak{a}$ are approximately primes of A containing $\mathfrak{a}$. This gives a description of the closed sets of $\operatorname{Spec} A$ as coming from other spectra.

Example 1.44. A ring homomorphism $A \to k$, where k is a field, induces a map from the single closed point of $\operatorname{Spec} k$ to $\operatorname{Spec} A$. We call the images of these maps the "k-points."

Example 1.45. Given a ring A and $f \in A$, the sets
$$D(f) := (\operatorname{Spec} A) \setminus V(f) = \{\mathfrak{p} : f \notin \mathfrak{p}\}$$
form a base of the open sets in $\operatorname{Spec} A$. To see this, we can see directly that
$$(\operatorname{Spec} A) \setminus V(T) = \bigcup_{f \in T} D(f).$$
To turn $D(f)$ into a spectrum, we have $D(f) \cong \operatorname{Spec} A_f$, where A_f refers to the localization of A at $f^{\mathbb{N}}$. Notably, the prime ideals of $\operatorname{Spec} A_f$ are the primes of A which are disjoint from $\{f\}$. Notably, the homeomorphism
$$\operatorname{Spec} A_f \cong D(f)$$
is given by the localization map $A \to A_f$.

Remark 1.46. Not every open set is of the form $D(f)$ or even $D(S)$ where S is a multiplicative set. For example,
$$\mathbb{A}^2_k \setminus \{(0,0)\} \subseteq \mathbb{A}^2_k$$
is an open set.

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List of Definitions