250B: Commutative Algebra

Or, Eisenbud With Details

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Spring 2022

Contents

Theme 1: Applications of Local Study

he could climb to it, if he climbed alone, and once there he could suck on the pap of life, gulp down the incomparable milk of wonder.

—Francis Scott Key Fitzgerald

1.1 February 17

Here we go.

1.1.1 The Nullstellensatz, Special Case

Today we prove Hilbert's Nullstellensatz. Here is the statement.

Theorem 1.1 (Nullstellensatz)**.** Fix k an algebraically closed field.

(a) There is a bijection between algeraic sets $X \subseteq \mathbb{A}^n(k)$ and radical ideals $J \subseteq k[x_1, \ldots, x_n]$ by taking

$$X \mapsto I(X) := \{f \in k[x_1, \ldots, x_n] : f(p) = 0 \text{ for all } p \in X\},$$

and

$$J \mapsto Z(J) := \{p \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } p \in J\}.$$

In particular, $I(Z(J)) = J$ and $Z(I(X)) = X$.

(b) Points p of X are in bijection with maximal ideals of $A(X) := k[x_1, \ldots, x_n]/I(X)$, which are in bijection with maximal ideals of $k[x_1, \ldots, x_n]$ containing $I(X)$.

Proof of Theorem 1.1 for uncountable fields. We start with a proof where k is an uncountable field; in other words, one should read $k = \mathbb{C}$ into the following proof. We start with part (b). We have the following lemma.

Lemma 1.2. Fix k an uncountable field and F/k a field extension with $\#[F : k] = \#\mathbb{N}$. Then the extension F/k is algebraic.

Proof. The main point is that $\#k > \#[F : k]$. Indeed, we work by contrapositive. Suppose that F/k is not algebraic so that we are promised some element $x \in F$ which is not algebraic over k, and we show $\#[F : k] \geq \#k$. But then $k(x) \subseteq F$, so the set

$$\left\{\frac{1}{x-a} : a \in k\right\}$$

is an uncountable k-linearly independent set in F, forcing $\#[F : k] \geq \#k$. ■

Corollary 1.3. Fix k an uncountable field. Then, for any maximal ideal $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$, the field extension

$$\frac{k[x_1, \ldots, x_n]}{\mathfrak{m}} \supseteq k$$

is algebraic.

Proof. The degree of the extension is countable and hence algebraic. ■

Thus, when k is algebraic and $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ is maximal, we have that

$$\frac{k[x_1, \ldots, x_n]}{\mathfrak{m}}$$

is an algebraic extension of k, which gives part (b) because k is algebraically closed. In particular, one can track where each of the $x_\bullet$ go to in the isomorphism $k[x_1, \ldots, x_n]/\mathfrak{m} \cong k$ (say, $x_\bullet \mapsto a_\bullet$) and use this to describe $\mathfrak{m}$ (as $(x_i - a_i)_{i=1}^n$).

Now we attack part (a).

Lemma 1.4. Fix k an algebraically closed field, and let $R := k[x_1, \ldots, x_n]$. Then any prime ideal $\mathfrak{p} \subseteq R$ is the intersection of the maximal ideals containing $\mathfrak{p}$.

Proof. If $\mathfrak{p}$ is maximal, then there is nothing to say. So take $\mathfrak{p}$ prime but not maximal so that $R/\mathfrak{p}$ is a domain but not a field. Of course it is true that

$$\mathfrak{p} \subseteq \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}.$$

The other inclusion is harder. For the other inclusion, we pick up $b \notin \mathfrak{p}$, and we need to find some maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$ but not b.

For this, we work in $R/\mathfrak{p}\left[b^{-1}\right]$. We note that $R/\mathfrak{p}\left[b^{-1}\right]$ is not a field: if $R/\mathfrak{p}\left[b^{-1}\right]$ is a field, then because it has countable degree over k, we see that

$$k \subseteq R/\mathfrak{p}\left[b^{-1}\right]$$

is an algebraic extension. But with k algebraically closed, this implies that b^{-1} is algebraic over k, so in particular, b^{-1} is the root of some monic polynomial in $k[x]$, so b^{-1} is integral over k and hence over $R/\mathfrak{p}$, so $R/\mathfrak{p}$ is a field because $R/\mathfrak{p}\left[b^{-1}\right]$ is an integral extension over $R/\mathfrak{p}$. This is our contradiction.

So because $R/\mathfrak{p}\left[b^{-1}\right]$ is not a field, we will have some maximal ideal which we can pull back to a maximal ideal $\mathfrak{m} \subseteq R$ which contains $\mathfrak{p}$ but not b. ■

Now we show part (a). Because I is a radical ideal, we can use **??** to write

$$I = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p} = \bigcap_{\mathfrak{p} \supseteq I} \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m},$$

where we have used Lemma 1.4. However, we remark that we have classified our maximal ideals! A maximal ideal containing I is the same as an ideal $(x_1 - a_1, \ldots, x_n - a_n)$ where $(a_1, \ldots, a_n) \in Z(I)$. Thus, $I = I(Z(I))$, and we are done. (That $Z(I(X)) = X$ is easier, and we have showed it before.) ■

1.1.2 The Nullstellensatz, General Proof

We now provide an alternative, more general proof.

General proof of Theorem 1.1. We have the following definition.

Jacobson

Definition 1.5 (Jacobson)**.** A ring R is *Jacobson* if and only if any prime ideal is the intersection of some maximal ideals. In other words, by pulling back from $R/\mathfrak{p}$, for each prime $\mathfrak{p}$, we have $\operatorname{rad} R/\mathfrak{p} = (0)$.

Example 1.6. The ring $\mathbb{Z}$ is Jacobson because all nonzero primes are maximal, and

$$(0) = \bigcap_{p \neq 0} (p).$$

Example 1.7. For the same reason, $k[x]$ is Jacobson.

Non-Example 1.8. A local domain which is not a field is not Jacobson; e.g., $\mathbb{Z}_2$ is not local. The issue is that being local implies that there is only one maximal ideal, but it is not (0) because we are not in a field, but (0) is some prime because we are in a domain.

We will want to show that $k[x_1, \ldots, x_n]$ is Jacobson, akin to Lemma 1.4.

Lemma 1.9. Fix R a domain but not a field. Then $\operatorname{rad} R = (0)$ if and only if $R\left[b^{-1}\right]$ is not a field for any $b \in R \setminus \{0\}$.

Proof. The main point is that ideals of $R\left[b^{-1}\right]$ are in one-to-one correspondence with ideals of R which avoid b. To be explicit, if $R\left[b^{-1}\right]$ is a field for some $b \in R \setminus \{0\}$, then all maximal ideals will have to pull back from (0) and hence contain b. In the other direction, if

$$\bigcap_{\mathfrak{m}} \mathfrak{m} = 0,$$

then any maximal ideal avoiding b will witness $R\left[b^{-1}\right]$ having a nonzero proper ideal. ■

Corollary 1.10. Fix R a ring. Then R is Jacobson if and only if each prime but not maximal prmie $\mathfrak{p}$ has $R/\mathfrak{p}\left[b^{-1}\right]$ not a field for each $b \notin \mathfrak{p}$.

Proof. Use the lemma on $R/\mathfrak{p}$. ■

And here is our main result.

Theorem 1.11. Fix R a Jacobson ring and S a finitely generated R-algebra.

(a) Then S is a Jacobson ring.

(b) For each maximal ideal $\mathfrak{m} \subseteq S$, we have $\mathfrak{m} \cap R$ maximal in R, and

$$\frac{R}{\mathfrak{m} \cap R} \subseteq \frac{S}{\mathfrak{m}}$$

is a finite extension.

Theorem 1.1 will follow from this, essentially using the same argument from before. Explicitly, (b) above gives (b) for Theorem 1.1, and (a) above combined with the argument from the uncountable case will prove (a).

Proof of Theorem 1.11. By induction, it will suffice to show the case where S is generated by a single element over R. Namely, if n is generated by $x_1, \ldots, x_n$, induction can get us up to $R[x_1, \ldots, x_{n-1}]$, and then we add x_n and mod out by the necessary statement. So let $S = R[t]/J$ for some $J \subseteq R[t]$.

We begin with (a). The main point is to use Corollary 1.10. Well, fix $\mathfrak{p}$ a prime of R which is not maximal. Now, the main point is that we have an integral extension

$$\underbrace{\frac{R}{R \cap \mathfrak{p}}}_{R'} \subseteq \underbrace{\frac{S}{\mathfrak{p}}}_{S'}.$$

Using the same t and polynomial I, we see that S' is still generated over R' by a single element, so we write $S' = R'[t]/I$. We have two cases; it suffices to show that $S'\left[b^{-1}\right]$ is not a field for any $b \neq 0$.

- Take $I = 0$. Then, for any $b \neq 0$ in R', we are easily not going to have $R'[t]\left[b^{-1}\right]$ not a field because $R'[t]$ was not a field.

- Otherwise, take our $b \in S'$. Because S' is finitely generated over R', we note that t must satisfy some polynomial equation
$$a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0 = 0,$$
where $a_\bullet \in R$. Now, $R'\left[a_0^{-1}\right]$ is not a field because R' is Jacobson (!). Continuing, we write
$$c_n b^n + c_{n-1}b^{n-1} + \cdots + c_0 = 0,$$
where $c_\bullet \in R$. But now we see that
$$R'\left[c_m^{-1}a_0^{-1}\right] \subseteq S'\left[c_m^{-1}a_0^{-1}, b^{-1}\right]$$
will be an integral extension, so the right-hand ring cannot be a field because it would force the left-hand side to give a field by integrality.

Lastly, we note that part (b) is the case that we just proved in the case where I is maximal. ■

The above theorem finishes the proof, as discussed. So we are done. ■

Remark 1.12. The midterm will only include up to Nakayama's lemma because we have not done homework on the other content.

1.1.3 Example Problems

Let's do some example problems, to review.

Exercise 1.13. Fix a field k. We work in $k^{n\times n}$. We show that the ideal

$$\det \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}.$$

We show that $(\det X)$ is a prime ideal in $k[x_{ij}]$. Note that it suffices to show $\det X$ is irreducible.

Example 1.14. In the case of $n=2$, we are showing $\det X = x_{11}x_{22} - x_{12}x_{21}$ is irreducible. Well, for any x_{ij}, if we could write

$$\det X = f(X)g(X),$$

then we must have $\deg_{x_{ij}} f = 0$ or $\deg_{x_{ij}} g = 0$. In particular, we have two cases.

- We might have $(x_{11}x_{22} - b)c$ for some b and c. But then this forces $c = 1$.
- We might have $(x_{11} - b)(x_{22} - c)$ for some b and c. But then cx_{11} would have to live in the polynomial, so $cx_{11} = 0$, and similar for bx_{22}, causing everything to collapse.

Proof of Exercise 1.13. Use expansion by minors to write

$$\det X = x_{11} \det X_{11} - q,$$

where X_{11} is X without the top and left row. By induction, we may assume that $\det X_{11}$ is irreducible.

Now we attempt to factor $\det X = fg$. By looking at the degree of x_{11}, we see that exactly one of f or g will have the linear term x_{11}, which means we factor into one of the following two forms.

- We might have $\det X = (x_{11} + b)(\det X_{11} + c)$. This cannot be because $x_{11} \det X_{11}$ contains all terms with an x_{11}, so $c = 0$ is forced. But then $\det X_{11} \mid \det X$, which does not make sense. For example, running the above argument again for x_{12} shows that the analogously defined X_{12} has $\det X_{12} \mid \det X$, but $\det X_{11}$ and $\det X_{12}$ are distinct irreducibles and hence coprime, which forces

  $$\deg \det X \geq \deg \det X_{11} + \deg \det X_{12},$$

  which does not make sense.
- We might have $\det X = (x_{11}X_{11} + b)c$. But by degree arguments, we see that c is constant, which means that c is a unit already.

In particular, no other factorizations are possible because they would require factoring $\det X_{11}$, which is irreducible. ■

Exercise 1.15. Fix $R = k[x,y]$ and $M = k[x,y]/\left(x^2, xy\right)$.

- We compute $\operatorname{Ass} M$.
- We compute $\operatorname{Supp} M$.
- We compute $H_M(s)$.

Proof. We start with $H_M(s)$. Let's tabulate.

- $H_M(0) = 1$, with 1.
- $H_M(1) = 2$, with x and y.
- $H_M(2) = 1$ with y^2.
- In fact, $H_M(s) = 1$ for $s > 1$ with y^s because all other monomials have xy killed.

The point is that M looks like $k[x,y]/(x)$, which is $x = 0$.

Now we compute $\operatorname{Supp} M$. Because M is a finitely generated module over a Noetherian ring, the support consists of the primes $\mathfrak{p} \subseteq k[x,y]$ containing $\operatorname{Ann} M = \left(x^2, xy\right)$. Well, any such prime must contain (x), and then we can have any larger prime, which look like $(x, y-a)$ for any $a \in k$. These are the only primes containing (x) by considering $k[x]/(x)$.

Lastly, we compute $\operatorname{Ass} M$. Well, we note $\operatorname{Ann} y = (x)$ and $\operatorname{Ann} x = (x,y)$. However, the other elements of the support take the form $(x, y-a)$, which will not be annihilators. ■

Remark 1.16. Professor Serganova recommends doing exercises 2.19, 2.22, and 4.11 from Eisenbud.