

# 202A: Introduction to Topology and Analysis

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# THEME 1

# METRIC SPACES

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*My personal view on spaces is that every space I ever work with is either metrizable or is the Zariski topology.*

—Evan Chen, [Che22]

## 1.1 August 24

Good morning everyone. This is my first class of the semester.

### 1.1.1 Administrative Notes

Here are some housekeeping remarks.

- The webpage for this class is [math.berkeley.edu/~rieffel/202AannF22.html](https://math.berkeley.edu/~rieffel/202AannF22.html).
- The midterm date is negotiable. We will have a vote on Friday. The possible dates are Friday 14 October, Monday 17 October, or Wednesday 19 October.
- There will be no vote on the final exam. It is on 15 December at 7PM.
- Homework will be due Fridays by midnight, approximately every week.
- There is no particular text for this course, and any given text covers more than we have time for. That said, we will (very) loosely follow [Lan12], but it is helpful to have a number of different expositions around.
- Please wear a mask during lectures and office hours.

Here is a summary of the course.

- We will spend the next couple of lectures talking about metric spaces.
- We will then spend the first half of the course on general topology. The second half of the course will be on measure and integration.
- Throughout we will see a little on functional analysis.

### 1.1.2 Metric Spaces

Hopefully we remember something about metric spaces. Here's the definition.

**Definition 1.1 (Metric).** A metric  $d$  on a set  $X$  is a function  $d: X \times X \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following rules for any  $x, y, z \in X$ .

- (a) Zero:  $d(x, x) = 0$ .
- (b) Zero:  $d(x, y) = 0$  implies  $x = y$ .
- (c) Symmetry:  $d(x, y) = d(y, x)$ .
- (d) Triangle inequality:  $d(x, y) + d(y, z) \geq d(x, z)$ .

We call  $(X, d)$  a metric space.

We will want some "almost" metrics as well. Here are their names.

**Definition 1.2 (Semi-metric).** A semi-metric  $d$  on a set  $X$  satisfies (a), (c), and (d) of Definition 1.1. We call  $(X, d)$  a semi-metric space.

**Definition 1.3 (Extended metric).** An extended metric  $d$  on a set  $X$  is a function  $d: X \times X \rightarrow \mathbb{R}_{\geq 0}^{\infty}$  satisfying (a)–(d) of Definition 1.1. We call  $(X, d)$  an extended metric space.

Intuitively, we might want extended metrics if we have points that we never want to be able to get to from other ones.

We can turn spaces with a semi-metric into a space with a metric.

**Lemma 1.4.** Fix a semi-metric space  $(X, d)$ , and define the relation  $\sim$  on  $X$  by  $x \sim y$  if and only if  $d(x, y) = 0$ . Then  $\sim$  is an equivalence relation.

*Proof.* We run these checks by hand. Fix any  $x, y, z \in X$ .

- Reflexive:  $d(x, x) = 0$  means that  $x \sim x$ .
- Symmetry: if  $x \sim y$ , then  $d(x, y) = 0$ , so  $d(y, x) = 0$ , so  $y \sim x$ .
- Transitive: if  $x \sim y$  and  $y \sim z$ , then

$$0 \leq d(x, z) \leq d(x, y) + d(y, z) = 0,$$

so  $d(x, z) = 0$ , so  $x \sim z$ . ■

As such, given a semi-metric space  $(X, d)$ , we may look at the set of equivalence classes under  $\sim$ , which we will denote  $X/\sim$ .<sup>1</sup>

**Proposition 1.5.** Fix a semi-metric space  $(X, d)$  and define  $\sim$  as in Lemma 1.4. Then  $d$  naturally descends to a metric  $\tilde{d}$  on  $X/\sim$ .

*Proof.* Let  $[x]$  denote the equivalence class of  $x \in X$  under  $\sim$ . We claim that the function

$$\tilde{d}([x], [y]) := d(x, y)$$

is a well-defined metric. We have the following checks; fix any  $x, y, z \in X$ .

<sup>1</sup> The notation of  $/\sim$  is intended to make us think of quotients.

- Well-defined: if  $x \sim x'$  and  $y \sim y'$ , then note that

$$d(x, y) \leq d(x, x') + d(x', y) = d(x', y) \leq d(x', y') + d(y', y) = d(x', y').$$

By symmetry, we also have  $d(x', y') \leq d(x, y)$ , so equality follows. So  $d$  does descent properly to the quotient  $X/\sim$ .

- Zero: note that  $\tilde{d}([x], [y]) = 0$  if and only if  $d(x, y) = 0$  if and only if  $x \sim y$  if and only if  $[x] = [y]$ .
- Symmetry: note that

$$\tilde{d}([x], [y]) = d(x, y) = d(y, x) = \tilde{d}([y], [x]).$$

- Triangle inequality: note that

$$\tilde{d}([x], [z]) = d(x, z) \leq d(x, y) + d(y, z) = \tilde{d}([x], [y]) + \tilde{d}([y], [z]),$$

which finishes. ■

Here are some examples of metric spaces.

**Example 1.6.** Given a connected graph  $G = (V, E)$  with a weighting function  $w: E \rightarrow \mathbb{R}_{\geq 0}$ , we can build a metric as follows: define the “shortest-path” function  $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$  sending two vertices  $v, w \in V$  to the length of the shortest path. If the graph  $G$  is not connected, we merely have an extended metric.

**Example 1.7 (Euclidean metric).** The function  $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

is a metric.

Observe that it is not completely obvious that [Example 1.7](#) satisfies the triangle inequality, but this will follow from the theory of the next subsections.

### 1.1.3 Norms on Vector Spaces

Norms provide convenient ways to build metrics.

**Definition 1.8 (Norm).** Fix a vector space  $V$  over  $\mathbb{R}$  or  $\mathbb{C}$ . A norm  $\|\cdot\|: V \rightarrow \mathbb{R}_{\geq 0}$  is a function satisfying the following, for any  $r \in \mathbb{R}$  and  $v, w \in V$ .

- (a) Zero:  $\|v\| = 0$  if and only if  $v = 0$ .
- (b) Scaling:  $\|rv\| = |r| \cdot \|v\|$ .
- (c) Triangle inequality:  $\|v + w\| \leq \|v\| + \|w\|$ .

**Remark 1.9.** We can probably work with a more general normed field instead of “merely”  $\mathbb{R}$  or  $\mathbb{C}$ .

And here is our result.

**Proposition 1.10.** Given a metric space  $V$  with a norm  $\|\cdot\|: V \rightarrow \mathbb{R}_{\geq 0}$ , then the function

$$d(v, w) := \|v - w\|$$

defines a metric on  $V$ .

*Proof.* We run the checks directly. Let  $x, y, z \in V$  be points.

- Zero: note that  $d(x, y) = 0$  if and only if  $\|x - y\| = 0$  if and only if  $x - y = 0$  if and only if  $x = y$ .
- Symmetry: note that

$$d(x, y) = \|x - y\| = |-1| \cdot \|y - x\| = 1 \cdot \|y - x\| = d(y, x).$$

- Triangle inequality: note that

$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z),$$

which finishes the check. ■

Here are the usual examples.

**Example 1.11.** Set  $V := \mathbb{R}^n$  or  $V := \mathbb{C}^n$ . Then the following are norms on  $V$ .

- $\|(x_1, \dots, x_n)\|_2 := (\sum_{i=1}^n |x_i|^2)^{1/2}$ .
- $\|(x_1, \dots, x_n)\|_1 := \sum_{i=1}^n |x_i|$ .

Here are some more esoteric examples.

**Example 1.12.** Set  $V := \mathbb{R}^n$  or  $V := \mathbb{C}^n$ . Then

$$\|(x_1, \dots, x_n)\|_\infty := \sup\{|x_1|, \dots, |x_n|\}$$

provides a norm on  $V$ .

**Example 1.13.** Set  $V := \mathbb{R}^n$  or  $V := \mathbb{C}^n$ . Then, given  $p \geq 1$ ,

$$\|(x_1, \dots, x_n)\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

provides a norm on  $V$ .

**Remark 1.14.** Taking the limit as  $p \rightarrow \infty$  of  $\|f\|_p$  gives  $\|f\|_\infty$ . This justifies the notation.

**Remark 1.15.** Despite having lots of examples, all of these norms are equivalent in a topological sense.

These normed vector spaces actually allow us to define a metric on any subset.

**Proposition 1.16.** Given a metric space  $(X, d)$  and a subset  $Y \subseteq X$ , the restriction of  $d$  to  $Y \times Y$  is a metric.

*Proof.* All the requirements for  $d$  on  $Y \times Y$  are satisfied for any points in  $X$ , so we are done by doing no work. ■

**Example 1.17.** Any subset  $X \subseteq \mathbb{R}^n$  has an induced metric by restricting the (say) Euclidean metric.

### 1.1.4 A Hint of $L^p$ Spaces

Here is a more complicated example of a metric.

**Example 1.18.** Define  $V := C([0, 1])$  to be the  $\mathbb{R}$ -vector space of  $\mathbb{R}$ -valued (or  $\mathbb{C}$ -valued) continuous functions on  $[0, 1]$ . The following are norms.

- $\|f\|_\infty := \sup\{|f(x)| : x \in [0, 1]\}$ .
- $\|f\|_1 := \int_0^1 |f(t)| dt$ .
- $\|f\|_2 := \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}$ .
- More generally, given  $p \geq 1$

$$\|f\|_p := \left(\int_0^1 |f(t)|^p dt\right)^{1/p}.$$

These integrals are finite because  $[0, 1]$  is compact, forcing  $f$  to achieve a finite maximum on  $[0, 1]$ .

**Remark 1.19.** We can tell the same story for  $C(X)$ , for any measurable compact space  $X$ .

**Remark 1.20.** Note the analogy of [Example 1.18](#) with [Example 1.13](#). To see this more rigorously, set  $X$  to be the finite set  $\{1, \dots, n\}$  so that  $C(X) = \mathbb{R}^n$ .

We should probably justify the claims of this subsection, so here is our result.

**Proposition 1.21.** Define  $V := C([0, 1])$  to be the vector space of  $\mathbb{R}$ -valued (or  $\mathbb{C}$ -valued) continuous functions on  $[0, 1]$ . Then, given  $p \geq 1$ , the function  $\|\cdot\|_p : C \rightarrow \mathbb{R}_{\geq 0}$  by

$$\|f\| := \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$$

is a norm.

*Proof.* We run the checks directly.

- Zero: if  $f = 0$ , then of course  $\int_0^1 |f(t)|^p dt = 0$ .
- Zero: suppose that  $f \in C([0, 1])$  has  $f(t_0) \neq 0$  for any  $t_0 \in [0, 1]$ ; set  $y := f(t_0)$ . Then  $f^{-1}((y/2, 3y/2))$  is a nonempty open subset of  $X$  and hence contains a nonempty open interval  $(a, b)$  with  $a < b$ . As such,

$$\int_X |f(t)|^p dt \geq \int_a^b |f(t)|^p dt \geq \int_a^b |y/2|^p dt > 0,$$

so we are done.

- Scaling: given  $f \in C([0, 1])$  and a scalar  $r$ , we have

$$\|rf\| = \left(\int_0^1 |rf(t)|^p dt\right)^{1/p} = \left(|r|^p \int_0^1 |f(t)|^p dt\right)^{1/p} = |r| \cdot \|f\|.$$

- Triangle inequality: we borrow from [Tao09]. Given  $f, g \in C([0, 1])$ , for psychological reasons we will assume that  $f$  and  $g$  are nonzero (else this is clear); then  $\|f\|, \|g\| \neq 0$ , so we may scale everything so that  $\|f\| + \|g\|$ . In fact, we may again use scaling to find  $a, b \in V$  such that

$$f = (1 - \theta)a \quad \text{and} \quad g = \theta b$$

where  $\theta \in (0, 1)$  and  $\|a\| = \|b\| = 1$ . Now, the triangle inequality translates into showing

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p dt = \|(1-\theta)a + \theta b\|_p^p \stackrel{?}{\leq} \left( \|(1-\theta)a\|_p + \|\theta b\|_p \right)^p = 1.$$

Well, because  $p \geq 1$ , the function  $t \mapsto t^p$  is convex, so we get to write

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p dt \leq (1-\theta) \int_0^1 |a(t)|^p dt + \theta \int_0^1 |b(t)|^p dt,$$

which is what we wanted.

The above checks complete the proof; note that the proof of the triangle inequality was nontrivial. ■

**Remark 1.22.** Now, to show [Remark 1.20](#), replace all  $\int_0^1$  with  $\sum_{i=1}^n$  and adjust all the language accordingly. The point is that “integrating over  $[0, 1]$ ” is analogous to “integrating over  $\{1, \dots, n\}$ .” A more thorough understanding of measure theory will allow us to rigorize this.

Next class we will talk about completeness.

## 1.2 August 26

Today we’re talking about completeness of metric spaces.

### 1.2.1 Isometries

In mathematics, we are interested in objects not in isolation but as they relate to each other. Namely, we are interested also in the maps between our objects.

The philosophy here comes from category theory, where one is really most interested in the “morphisms” between “objects” instead of the objects themselves. For concreteness, here is a definition of a category.

**Definition 1.23 (Category).** A category  $\mathcal{C}$  consists of a class of objects  $\text{Ob } \mathcal{C}$  and class of morphisms  $\text{Mor } \mathcal{C}$  such that any two objects  $A, B \in \text{Ob } \mathcal{C}$  have a morphism class  $\text{Mor}(A, B)$ . This data satisfy the following properties.

- Composition: given objects  $A, B, C \in \text{Ob } \mathcal{C}$ , there is a binary composition operation

$$\circ: \text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C).$$

Explicitly, given  $f \in \text{Mor}(A, B)$  and  $g \in \text{Mor}(B, C)$ , there is a composition  $(g \circ f) \in \text{Mor}(A, C)$ .

- Given  $A \in \text{Ob } \mathcal{C}$ , there is an identity morphism  $\text{id}_A \in \text{Mor}(A, A)$ .
- Identity: any  $f \in \text{Mor}(A, B)$  has  $f \circ \text{id}_A = f = \text{id}_B \circ f$ .
- Associativity: any  $f \in \text{Mor}(A, B)$  and  $g \in \text{Mor}(B, C)$  and  $h \in \text{Mor}(C, D)$  has  $(h \circ g) \circ f = h \circ (g \circ f)$ .

**Example 1.24.** There is a category of groups, where the morphisms are group homomorphisms. The identity function gives the identity morphism, and composition of functions gives the required composition.

For completeness, we check that composition is well-defined: given homomorphisms  $f: A \rightarrow B$  and  $g: B \rightarrow C$ , we need  $(g \circ f): A \rightarrow C$  to be a group homomorphism. Well,

$$(g \circ f)(a \cdot a') = g(f(a \cdot a')) = g(f(a) \cdot f(a')) = g(f(a)) \cdot g(f(a')) = (g \circ f)(a) \cdot (g \circ f)(a').$$



In our discussion of metric spaces, there are many possible kinds of morphisms for us to consider. Here is the strongest type.

**Definition 1.25 (Isometry).** Given metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , an *isometry* is a function  $f: X \rightarrow Y$  preserving the metric as

$$d_Y(f(x), f(x')) = d_X(x, x').$$

**Example 1.26.** The  $90^\circ$  rotation  $r: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $r(x, y) \mapsto (y, -x)$  is an isometry, where  $\mathbb{R}^2$  is given the Euclidean metric. Indeed, any  $(x, y), (x', y') \in \mathbb{R}^2$  have

$$\begin{aligned} d(r(x, y), r(x', y')) &= d((y, -x), (y', -x')) \\ &= \sqrt{(y - y')^2 + (-x - -x')^2} \\ &= \sqrt{(x - x')^2 + (y - y')^2} \\ &= d((x, y), (x', y')). \end{aligned}$$

**Notation 1.27.** Fix two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ . Given a function  $f: X \rightarrow Y$  with extra structure respecting some aspect of the metric, we might write  $f: (X, d_X) \rightarrow (Y, d_Y)$  to emphasize this.

To show that isometries are valid morphisms, we need to check that the identity function  $\text{id}_X: X \rightarrow X$  is an isometry (which of course it is) and that the composition of two isometries is an isometry. We check this last one in a quick lemma.

**Lemma 1.28.** Given two isometries  $f: (X, d_X) \rightarrow (Y, d_Y)$  and  $g: (Y, d_Y) \rightarrow (Z, d_Z)$ , the composition  $g \circ f$  is an isometry.

*Proof.* Well, any two points  $x, x' \in X$  have

$$d_Z(g(f(x)), g(f(x'))) = d_Y(f(x), f(x')) = d_X(x, x'),$$

which is what we wanted. ■

One can restrict further to surjective isometries, where the main point is that (again) the composition of two surjective functions remains surjective. (Note that the identity is of course surjective.) The following is the reason why a surjective isometry is a good notion.

**Lemma 1.29.** A surjective isometry  $f: (X, d_X) \rightarrow (Y, d_Y)$  is bijective, and its inverse function is also an isometry.

*Proof.* To see that  $f$  is bijective, we only need to know that  $f$  is injective. Well, given  $x, x' \in X$ , note that  $f(x) = f(x')$  if and only if  $d_Y(f(x), f(x')) = 0$  if and only if  $d_X(x, x') = 0$  if and only if  $x = x'$ .<sup>2</sup>

Thus,  $f$  is indeed bijective; let  $g: Y \rightarrow X$  be its inverse. We now need to show that  $g$  is an isometry. Well, given  $y, y' \in Y$ , we may find  $x, x' \in X$  such that  $f(x) = y$  and  $f(x') = y'$ . Then

$$d_X(g(y), g(y')) = d_X((g \circ f)(x), (g \circ f)(x')) = d_X(x, x') \stackrel{*}{=} d_Y(f(x), f(x')) = d_Y(y, y'),$$

where in  $\stackrel{*}{=}$  we have used the fact that  $f$  is an isometry. ■

<sup>2</sup> In fact, this argument shows that all isometries are injective. We will shortly see that all actually Lipschitz continuous functions are injective.

**Remark 1.30.** The above result is somewhat subtle in its importance: the inverse function of a bijection is only an inverse in the category of sets. The above result is saying that this inverse morphism in the category of sets is lifting to an inverse morphism in the category of metric spaces with isometries as morphisms. In general, it is not always true that bijective morphisms are invertible, as we shall soon see.

### 1.2.2 Lipschitz Continuity

Isometries are somewhat restrictive, so we might weaken this as follows.

**Definition 1.31** (Lipschitz continuous). Given metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a function  $f: X \rightarrow Y$  is a *Lipschitz continuous* if and only if there is a constant  $c \in \mathbb{R}$  such that

$$d_Y(f(x), f(x')) \leq c \cdot d_X(x, x').$$

**Remark 1.32.** Equivalently, we are asking for the ratio

$$\frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

to be uniformly bounded above for all  $x \neq x'$ . Notably, the inequality is trivially satisfied whenever  $x = x'$ , or equivalently whenever  $d(x, x') = 0$ .

**Example 1.33.** Any isometry  $f: (X, d_X) \rightarrow (Y, d_Y)$  is Lipschitz continuous: indeed, set  $c := 1$  so that, for any  $x, x' \in X$ ,

$$d_Y(f(x), f(x')) = d_X(x, x') \leq 1 \cdot d_X(x, x').$$

**Example 1.34.** Provide  $\mathbb{R}$  and  $\mathbb{R}^2$  their usual Euclidean metrics. Then the projection  $\pi: \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $\pi: (x, y) \mapsto x$  is Lipschitz continuous: indeed, set  $c := 1$  so that, for any  $(x, y), (x', y') \in \mathbb{R}^2$ , we have

$$d_{\mathbb{R}^2}((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2} \geq \sqrt{(x - x')^2} = d_{\mathbb{R}}(x, x') = d_{\mathbb{R}}(\pi((x, y)), \pi((x', y'))).$$

Again, one can see that the identity function  $\text{id}_X: (X, d_X) \rightarrow (X, d_X)$  is Lipschitz continuous (with  $c := 1$ ), and here is our composition check.

**Lemma 1.35.** If  $f: (X, d_X) \rightarrow (Y, d_Y)$  and  $g: (Y, d_Y) \rightarrow (Z, d_Z)$  are Lipschitz continuous, then the composition  $(g \circ f): (X, d_X) \rightarrow (Z, d_Z)$  is also Lipschitz continuous.

*Proof.* We are given constants  $c$  and  $d$  such that any  $x, x' \in X$  and  $y, y' \in Y$  have

$$d_Y(f(x), f(x')) \leq c \cdot d_X(x, x') \quad \text{and} \quad d_Z(g(y), g(y')) \leq d \cdot d_Y(y, y').$$

As such, we use the constant  $cd$  to witness our Lipschitz continuity: any  $x, x' \in X$  have

$$d_Z(g(f(x)), g(f(x'))) \leq d \cdot d_Y(f(x), f(x')) \leq cd \cdot d_X(x, x'),$$

which is what we wanted. ■

It will be shortly worth our time to talk about the constant  $c$  appearing in [Definition 1.31](#).

**Lemma 1.36.** Fix a Lipschitz continuous function  $f: (X, d_X) \rightarrow (Y, d_Y)$ . Then there exists a constant  $c_f$  (possibly  $-\infty$ ) such that any real number  $c \geq c_f$  is equivalent to the following property: any  $x, x' \in X$  have

$$d_Y(f(x), f(x')) \leq c \cdot d_X(x, x').$$

*Proof.* Let  $S$  denote the set of all constants  $c$  such that any  $x, x' \in X$  have

$$d_Y(f(x), f(x')) \leq c \cdot d_X(x, x').$$

Equivalently, using [Remark 1.32](#),  $S$  is the set of upper-bounds for

$$R := \left\{ \frac{d_Y(f(x), f(x'))}{d_X(x, x')} : x, x' \in X, x \neq x' \right\}.$$

Now,  $S$  is nonempty because  $f$  is Lipschitz continuity, so we set  $c_f := \sup R$  to be the least upper bound for  $R$ —observe that  $c_f = -\infty$  is permissible when  $X$  has one point. It is now pretty clear that  $S = [c_f, \infty)$ . ■

Note that  $c_f$  the property stated in the lemma automatically implies that  $c_f$  is the least possible constant and is unique. Being least is immediate (by the backwards direction), and being unique follows from being least. So because we have some uniqueness, we get a definition.

**Definition 1.37 (Lipschitz constant).** Given a Lipschitz continuous function  $f: (X, d_X) \rightarrow (Y, d_Y)$ , the *Lipschitz constant*  $c_f$  for  $f$  is the least real number  $c$  such that

$$d_Y(f(x), f(x')) \leq c \cdot d_X(x, x').$$

We could, as before, look at surjective Lipschitz continuous functions, but these need not be bijective anymore as shown by [Example 1.34](#). What's worse is that, as warned possible in [Remark 1.30](#), bijective Lipschitz continuous functions need not even have a Lipschitz continuous inverse.

**Exercise 1.38.** We exhibit a function between metric spaces which is bijective and Lipschitz continuous, but its inverse function is not Lipschitz continuous.

*Proof.* Set  $X := (0, 1)$  and  $Y := (1, \infty)$ , both metric spaces with the Euclidean (subspace) metric, and set  $f: (0, \infty) \rightarrow (0, \infty)$  by  $f: x \mapsto 1/x$ . Notably,  $x \in X$  implies  $f(x) \in Y$ , and  $y \in Y$  implies  $f(y) \in X$ .

- Note  $f|_Y$  is bijective with inverse  $f|_X$  because  $f(f(x)) = f(1/x) = x$  for all  $x \in (0, \infty)$ .
- Note  $f|_Y$  is Lipschitz continuous: set  $c := 1$  and note that any  $y, y' \in Y$  have

$$|f(y) - f(y')| = \left| \frac{1}{y} - \frac{1}{y'} \right| = \left| \frac{y - y'}{yy'} \right| \leq |y - y'|.$$

- But  $f|_X$  is not Lipschitz continuous: suppose for contradiction that  $f|_X$  is Lipschitz continuous, and use [Lemma 1.36](#) to recover the needed constant  $c_0$ . Then set  $c := \max\{c_0, 4\}$ , which must also work as a constant, and set  $x := 1/c$  and  $x' := 3/c$  so that

$$|f(x) - f(x')| = |c - 3c| = 2c > c \cdot |x - x'|.$$

This is a contradiction, so we are done. ■

**Remark 1.39 (Nir).** In some sense, the problem here is that the definition of Lipschitz continuity allows  $d_Y(f(x), f(x'))$  to be “too small,” which permits the inverse function to have distances which blow up.

In light of [Exercise 1.38](#), we introduce a new definition.

**Definition 1.40** (Lipschitz isomorphism). Give metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a function  $f: X \rightarrow Y$  is a *Lipschitz isomorphism* if and only if  $f$  is Lipschitz continuous and has an inverse function which is also Lipschitz continuous.

**Remark 1.41.** A good reason to care about this notion of continuity (and isomorphism) is that all normed  $\mathbb{R}$ -vector spaces of some finite dimension  $n$  are Lipschitz isomorphic.

### 1.2.3 Fun with Continuity

Here is yet a weaker notion of morphism.

**Definition 1.42** (Uniformly continuous). Given metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a function  $f: X \rightarrow Y$  is a *uniformly continuous* if and only if every  $\varepsilon > 0$  has some  $\delta > 0$  such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

for all  $x, x' \in X$ .

**Example 1.43.** Any Lipschitz continuous function  $f: (X, d_X) \rightarrow (Y, d_Y)$  is also uniformly continuous: indeed, for any  $\varepsilon > 0$ , set  $\delta := \max\{c_f, 1\}\varepsilon > 0$  (where  $c_f$  is the Lipschitz constant) so that

$$d_X(x, x') < \varepsilon \implies d_Y(f(x), f(x')) \leq c_f \cdot d(x, x') < \delta.$$

**Example 1.44.** Give  $[0, 1]$  the Euclidean (subspace) metric, and set  $f: [0, 1] \rightarrow [0, 1]$  by  $f(x) := \sqrt{x}$ .

- Note  $f$  is uniformly continuous because it is continuous on a compact set.
- However,  $f$  is not Lipschitz continuous: for any constant  $c > 0$ , set  $x = 1/(c+1)^2$  and  $x' = 0$  so that

$$\left| \frac{f(x) - f(x')}{x - x'} \right| = \left| \frac{1/(c+1)}{1/(c+1)^2} \right| = |c+1| > c,$$

so [Remark 1.32](#) tells us that we are not Lipschitz continuous.

By rearranging quantifiers, we get another useful (but weaker) notion.

**Definition 1.45** (Continuous). Given metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a function  $f: X \rightarrow Y$  is a *continuous* if and only if, at any  $x \in X$ , all  $\varepsilon > 0$  have some  $\delta_x > 0$  such that

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$

**Example 1.46.** All uniformly continuous functions  $f: (X, d_X) \rightarrow (Y, d_Y)$  are continuous. Indeed, at any  $x_0 \in X$  with  $\varepsilon > 0$ , uniform continuity promises  $\delta > 0$  so that

$$|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon$$

for all  $x, x' \in X$ . Setting  $x'$  to  $x_0$  recovers continuity.

**Example 1.47.** Give  $\mathbb{R}$  the usual Euclidean metric, and set  $f: \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) := x^2$ .

- Note  $f(x)$  is continuous because it is a polynomial.
- However,  $f(x)$  is not uniformly continuous: take  $\varepsilon = 1$ . Now, for any  $\delta > 0$ , set  $x = 1/\delta$  and  $x' = 1/\delta + \delta/2$  so that  $|x - x'| < \delta$ , but

$$|f(x) - f(x')| = \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2} = 1 + \frac{\delta^2}{4} > \varepsilon.$$

As usual, the identity function is uniformly continuous and continuous (it's an isometry), and these continuities are preserved by composition. We will have a different way to see that continuous functions remain continuous under composition later, so for now we will focus on uniform continuity.

**Lemma 1.48.** Fix uniformly continuous morphisms  $f: (X, d_X) \rightarrow (Y, d_Y)$  and  $g: (Y, d_Y) \rightarrow (Z, d_Z)$ . Then the function  $(g \circ f)$  is uniformly continuous.

*Proof.* For any  $\varepsilon > 0$ , the uniform continuity of  $g$  promises  $\delta_g > 0$  such that

$$d_Y(y, y') < \delta_g \implies d_Z(g(y), g(y')) < \varepsilon$$

for any  $y, y' \in Y$ . Continuing, the uniform continuity of  $f$  promises  $\delta_f > 0$  such that

$$d_X(x, x') < \delta_f \implies d_Y(f(x), f(x')) < \delta_g \implies d_Z(g(f(x)), g(f(x')) < \varepsilon$$

for any  $x, x' \in X$ , which is what we wanted. ■

**Remark 1.49.** In some sense, isometries and Lipschitz continuous functions have their definition fundamentally interrelated with the metric. In contrast, the weaker notion of continuity will readily generalize to general topological spaces. Uniform continuity also generalizes to “uniformities,” which is a different notion.

## 1.2.4 Completeness

To discuss completeness, we need to talk about convergence.

**Definition 1.50 (Converge).** Fix a metric space  $(X, d)$ . A sequence of points  $\{x_n\}_{n \in \mathbb{N}} \subseteq X$  converges to  $x \in X$  if and only if, for any  $\varepsilon > 0$ , we can find  $N > 0$  such that

$$n > N \implies d(x_n, x) < \varepsilon.$$

We might write this as  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

We would like a notion of convergence which only uses data internal to the sequence, and this leads to the following definition.

**Definition 1.51 (Cauchy).** Fix a metric space  $(X, d)$ . A sequence of points  $\{x_n\}_{n \in \mathbb{N}} \subseteq X$  is a *Cauchy sequence* if and only if, for any  $\varepsilon > 0$ , we can find  $N > 0$  such that

$$n, m > N \implies d(x_n, x_m) < \varepsilon.$$

One can check that all convergent sequences are Cauchy, and we in general hope that our Cauchy sequences will converge. As such, we have the following definition.

**Definition 1.52 (Complete).** A metric space  $(X, d)$  is *complete* if and only if every Cauchy sequence in  $X$  converges to a point in  $X$ .

We are sad when a metric space is not complete, so we hope to have a way to make it complete. The most natural way to do this is by using the notion of density.

**Definition 1.53 (Density).** Fix a metric space  $(X, d)$ . Then  $S \subseteq X$  is *dense* if and only if, given any  $x \in X$  and  $\varepsilon > 0$ , we may find  $x' \in S$  with  $d(x, x') < \varepsilon$ .

And here is our completion.

**Definition 1.54 (Completion).** A *completion* of the metric space  $(X, d)$  is a metric space  $(\bar{X}, \bar{d})$  equipped with an isometry  $\iota: X \rightarrow \bar{X}$  such that  $(\bar{X}, \bar{d})$  is complete and  $\text{im } \iota$  is dense in  $\bar{X}$ .

One can show that any metric space has a completion and that they are all isometric and therefore in some sense the same. The uniqueness result will appear on the homework, so for now we will discuss existence.

**Theorem 1.55.** Any metric space  $(X, d)$  has a completion.

*Sketch.* Let  $\tilde{X}$  denote the set of all Cauchy sequences in  $X$ . We hope to make  $\tilde{X}$  into our completion, but this requires a little care. To begin, we have the following lemma.

**Lemma 1.56.** Given a metric space  $(X, d)$  with two Cauchy sequences  $\{x_n\}_{n \in \mathbb{N}}$  and  $\{y_n\}_{n \in \mathbb{N}}$ , then the sequence

$$\{d(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$$

converges.

*Proof.* Omitted. ■

Thus, we define  $\tilde{d}: \tilde{X} \times \tilde{X} \rightarrow \mathbb{R}_{\geq 0}$  be this well-defined function. One can show that  $\tilde{d}$  is a semi-norm, and so we use [Proposition 1.5](#) to induce a metric  $\bar{d}$  on  $\bar{X} := \tilde{X}/\sim$ .

To finish the proof, one has to check that  $\bar{X}$  is in fact a completion. It is somewhat annoying to check that  $\bar{X}$  is complete (though it is not terribly tricky), and the required isometry  $\iota: X \rightarrow \bar{X}$  is given by

$$\iota(x) := \{x\}_{n \in \mathbb{N}},$$

which is certainly Cauchy. ■

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