737: Weil II for Curves

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1

Review of Étale Theory

1.1 January 23

This is a pretty small class, so it will be rather informal. This course is going to assume some basic étale theory, roughly speaking up to the construction of the derived functors and some of their fundamental properties. We will also freely black-box the difficult theorems of the theory, most notably the Grothendieck–Lefschetz trace formula.

In this course, we are interested in proving the Weil conjectures, but we will be modest and focus on curves. Historically, the proof of the Weil conjectures for curves is much older than Weil II, but part of our goal will be to introduce the important relative techniques. For example, there should be a notion of weights attached to sheaves on a variety, known already from Hodge theory. However, we will require a way to see this purely from algebraic geometry; in fact, one expects the notion of weight to be motivic.

1.1.1 The Zeta Function

Let's begin by setting some notation which will be in place for the entire course. We take k to be a finite field $\mathbb{F}_q$ of characteristic p, embedded in a fixed algebraic closure $\overline{k} = \overline{\mathbb{F}_p}$; we write $q = p^n$. For brevity, we may write $k_m = \mathbb{F}_{q^m}$ for each $m \geq 1$. Then we let X be a smooth, projective, geometrically connected variety over the field k; we set $d \coloneqq \dim X$.

Definition 1.1 (zeta function)**.** Let X be a variety over $\mathbb{F}_q$. Then we define the *zeta function* as the generating function

$$\zeta_X(T) \coloneqq \exp\left(\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| \frac{T^m}{m}\right).$$

In order to do algebraic geometry to $\zeta_X(T)$, we would like to have a different description for $X(\mathbb{F}_{q^m})$. For this, we need to discuss closed points.

Definition 1.2 (closed point)**.** Let X be a variety over k. Then a point $x \in X$ is *closed* if and only if $\dim \overline{\{x\}} = 0$. Its *degree* $\deg x$ is the degree $[k(x) : k]$, where $k(x)$ is the minimal field of definition.

We now see that

$$X(\mathbb{F}_{q^m}) = \operatorname{Mor}_{\mathbb{F}_q}\left(\operatorname{Spec} \mathbb{F}_{q^m}, X\right).$$

For example, we see that this consists of the collection of closed points $x \in X$ of degree dividing m, counted with a certain multiplicity.

Now, to read off fields of definition, we introduce some Frobenius morphisms.

Definition 1.3. Fix a scheme X over $k = \mathbb{F}_q$. Then there is a *Frobenius morphism* $\operatorname{Frob}_X \colon X \to X$ defined as being an identity on the underlying topological space and the q-power map on $\mathcal{O}_X$. We may write $\operatorname{Frob}_{X,q}$ for Frob_X if we want to remember the power. We may also extend scalars and write $\operatorname{Frob}_{X_{\overline{k}},q} = \operatorname{Frob}_{X,q} \times \mathrm{id}_{\overline{k}}$, which we note is a morphism of schemes over $\overline{k}$ by its construction.

Remark 1.4. Fix a morphism $f \colon X \to Y$ of schemes over $\mathbb{F}_q$. Then we see $\operatorname{Frob}_Y \circ f = f \circ \operatorname{Frob}_X$, which can be checked directly: both sides are f on the topological spaces, and both sides are the same on the level of sheaves.

Example 1.5. On $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, our Frobenius map may be defined as the k-algebra endomorphism of $k[x_1, \ldots, x_n]$ which sends $x_i \mapsto x_i^q$. Thus, on points, we see that $(p_1, \ldots, p_n) \in \mathbb{A}^n_k(\overline{k})$ has

$$F_{\mathbb{A}^n_k}(p_1, \ldots, p_n) = \left(\operatorname{Frob}_q^{-1} p_1, \ldots, \operatorname{Frob}_q^{-1} p_n\right).$$

Remark 1.6. We now see that we can think about $X(\mathbb{F}_q)$ as the subset of $X(\overline{k})$ fixed by F_{X,q^m}. Thus, we note that one can realize $X(k_m)$ as the set of closed points of the scheme $(\Gamma_{F_{X,q^m}} \cap \Delta)$, where $\Delta \colon X \times X \to X$ is the diagonal map.

Definition 1.7 (arithmetic Frobenius)**.** The *arithmetic Frobenius* Frob_k is the q-power automorphism of $\overline{k}$.

Definition 1.8 (geometric Frobenius)**.** Let X be a scheme over k. Then we define the *geometric Frobenius* of $X_{\overline{k}}$ as $F_X := \mathrm{id}_{X_{\overline{k}}} \times \operatorname{Frob}_{\overline{k}}^{-1}$. It fits in the following commutative diagram.

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r] \arrow[d] \arrow[dr, dashed, "F_X"] & X \arrow[dr, equals] & \\
\operatorname{Spec} \overline{k} \arrow[dr] & X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} \overline{k} \arrow[r, "\operatorname{Frob}_{\overline{k}}^{-1}"'] & \operatorname{Spec} k
\end{tikzcd}$$

Definition 1.9 (absolute Frobenius)**.** Let X be a scheme over k. One can check that F_X commutes with $\operatorname{Frob}_{X,q}$. We then define the *absolute Frobenius* as the composite $F_X \circ \operatorname{Frob}_{X_{\overline{k}},q}$.

Remark 1.10. It turns out that the absolute Frobenius is the identity on the level of étale cohomology.

We now return to our zeta function. To be able to undo the exponential, we note

$$\log\left(\frac{1}{1-T^d}\right) = \sum_{m \geq 1} d \cdot \frac{T^{nd}}{dn}.$$

Thus,

$$\sum_{m \geq 1} |X(\mathbb{F}_{q^m})| \frac{T^m}{m} = \sum_{\text{closed } x \in X} \log\left(\frac{1}{1-T^{\deg x}}\right),$$

so taking the exponential reveals

$$\zeta_X(T) = \prod_{\text{closed } x\in X} \frac{1}{1-T^{\deg x}},$$

and now this Euler product appears similar to the usual Euler products we expect.

1.2 January 28

Today we do something with cohomology.

1.2.1 The Rationality Conjecture

We would like to relate our zeta function to cohomology. It turns out that the key input is the following result.

Theorem 1.11 (Grothendieck–Lefschetz trace formula)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$. Then

$$X(\mathbb{F}_{q^m}) = \sum_{i\ge 0} (-1)^i \operatorname{tr}\left(\operatorname{Frob}^m_{X_{\overline{k}}}; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

Here, recall that

$$\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \varprojlim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}/\ell^\bullet\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Namely, this is our Weil cohomology (over the field $\mathbb{Q}_\ell$) produced by étale cohomology.

Remark 1.12. It is the goal of Weil II (and thus of the course) to be able to work more general local systems than the "constant" sheaf $\mathbb{Q}_\ell$.

To relate this to ζ_X, we recall the following result from linear algebra.

Lemma 1.13. Fix an endomorphism φ of a finite-dimensional vector space V (over a field K). Then we have an equality of power series

$$\exp\left(\sum_{m\ge 1} \operatorname{tr}(\varphi^m; V)\frac{T^m}{m}\right) = \det(1-\varphi T; V)^{-1}.$$

Proof. It is enough to check the equality after base-changing to the algebraic closure, so we may assume that K is algebraically closed. Then we may give V a basis so that φ is upper-triangular.

Let $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of φ. Then we are tasked with showing

$$\exp\left(\sum_{m\ge 0}\sum_{i=1}^d \lambda_i^m \cdot \frac{T^m}{m}\right) \stackrel{?}{=} \prod_{i=1}^d \frac{1}{1-\lambda_i T}.$$

Well, we may move the sum on the left-hand side outside so that we see we are interested in showing

$$\exp\left(\sum_{m\ge 1} \frac{(\lambda T)^m}{m}\right) = \frac{1}{1-\lambda T}$$

for any eigenvalue λ of φ. The result now follows by considering the Taylor expansion $-\log(1-x) = \sum_{m\ge 1} x^m/m$. ■

Here is the punchline: we are able to prove the rationality conjecture.

Proposition 1.14 (Rationality)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Then there are polynomials $P_0, \ldots, P_{2d} \in \mathbb{Q}_\ell[T]$ such that

$$Z_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}.$$

Proof. By Theorem 1.11, we see that

$$Z_X(T) = \prod_{i=0}^{2d} \exp\left(\sum_{m \geq 1} \operatorname{tr}\left(\operatorname{Frob}_{X_{\overline{k}}}^m ; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) \right) \frac{T^m}{m} \right)^{(-1)^i}.$$

We now define

$$P_i(T) := \det\left(1 - \operatorname{Frob}_{X_{\overline{k}}} T; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

The result now follows from Lemma 1.13. ■

Remark 1.15. In fact, we see that $P_i(T)$ has degree $\dim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$. This fact can be combined with the comparison theorem to Betti cohomology.

Remark 1.16. We thus see that $Z_X(T) \in \mathbb{Q}_\ell(T)$, so because we already know $Z_X(T) \in \mathbb{Q}[[T]]$, we see $Z_X(T) \in \mathbb{Q}(T)$.

Remark 1.17. It turns out that $P_i(T) \in \mathbb{Z}[T]$ and is independent of ℓ, but the proof above does not show this.

Example 1.18. At $i = 0$, we see that the Frobenius acts trivially on $\mathrm{H}^0_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, so $P_0(T) = 1 - T$. Using Poincaré duality, we can similarly compute $P_{2d}(T) = 1 - q^d T$.

Remark 1.19. There is also a functional equation for $Z_X(T)$, which is purely formal from the above expression for Z_X when combined with Poincaré duality for étale cohomology.

1.2.2 The Riemann Hypothesis

This course will be interested in the following conjecture.

Conjecture 1.20 (Riemann hypothesis)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Fix an index $i \in \{0, \ldots, 2d\}$.

(a) The eigenvalues of $\operatorname{Frob}_{X_{\overline{k}}}$ on $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ are algebraic integers of magnitude $q^{i/2}$.

(b) The characteristic polynomial $P_i(T)$ of this Frobenius action is in $\mathbb{Z}[T]$ and is independent of ℓ.

Remark 1.21. Part (a) can be viewed as a Riemann hypothesis: substituting $T = q^{-s}$ into ζ_X, we see that we are requiring our zeroes (and poles) of $\zeta_X(q^{-s})$ to live on the vertical lines

$$\{s : \operatorname{Re} s = i/2\}$$

as i varies over $\{1, \ldots, 2\dim X\}$.

The condition in (a) is interesting enough to deserve a name.

Definition 1.22 (q-Weil)**.** An algebraic integer $\alpha \in \overline{\mathbb{Q}}$ is q-*Weil of weight* i if and only if $|\iota(\alpha)| = q^{i/2}$ for all embeddings $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Example 1.23. The number $\sqrt{2}$ is a 2-Weil number. The number $1+\sqrt{2}$ is not a q-Weil number for any q.

In general, we find that the eigenvalues of a Frobenius action on a local system will still be q-Weil numbers of prescribed weight.

To be precise, the goal of this course will be to prove the following generalization of the above Riemann hypothesis.

Theorem 1.24 (Deligne)**.** Let $f\colon X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X that is mixed of weights at most n. Then $R^i f_! \mathcal{F}$ is also mixed of weights at most $w+i$.

We will define the notion of weights shortly. The idea intuitively comes from Hodge theory: the cohomology groups on a complex Kähler manifold naturally have a weight filtration, which then lifts to sheaves by taking a suitable compactification and studying differential forms suitably. Weights in our context will come from reading off q-Weil numbers.

Remark 1.25. Issues with compactification explain why we are forced to merely deal with mixed weights instead of upgrading this result to one on pure weights. Already this can be seen in Hodge theory.

This course will not prove Theorem 1.24 in full. Instead, we will focus on the case where f has fibers of dimension 1; it turns out that the general case follows from this from some argument involving fibering by curves and using the Leray spectral sequence.

Corollary 1.26. Let X be a scheme of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X.

(a) If $\mathcal{F}$ is mixed of weights at most n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at most $n+i$.

(b) If $\mathcal{F}$ is mixed of weights at least n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at least $n+i$.

(c) Assume that X is smooth and that $\mathcal{F}$ is pure of weight n. Then the image of the canonical map $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

(d) Assume that X is smooth and proper and that $\mathcal{F}$ is pure of weight n. Then $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

Proof. Here, (a) is direct from Theorem 1.24. Then (b) will follow from (a) via Poincaré duality as soon as we know that the duality given by Poincaré duality inverts the weights. Now, (c) follows from combining (a) and (b), and (d) follows from (c). ■

Remark 1.27. One can then prove the result for sheaves over $\mathbb{Q}_\ell$ by base-changing up to the algebraic closure.

The moral of the story is that we are going to use weights to significant profit in this course. Next class we will define weights.

1.3 January 30

Today we review some étale cohomology.

1.3.1 The Étale Site

For completeness, here is the definition of étale.

Definition 1.28 (étale)**.** Fix a scheme morphism $\varphi\colon X \to Y$.

(a) φ is *locally of finite presentation* if and only if $\mathcal{O}_X$ is finitely presented as a $\varphi^{-1}\mathcal{O}_Y$-module (say, on Zariski open neighborhoods or on stalks).

(b) φ is *flat* if and only if the pushforward $\varphi_*\mathcal{O}_X$ is flat over $\mathcal{O}_Y$ (say, on Zariski open neighborhoods or on stalks).

(c) φ is *unramified* if and only if $\Omega_{X/Y} = 0$.

(d) φ is *étale* if and only if it is locally of finite presentation, flat, and unramified.

Example 1.29. One can check that open embeddings are étale.

Remark 1.30. Note that the unramified condition adds some separability, which is a rough explanation for where Galois representations enter the story.

And here is the relevant site.

Definition 1.31 (étale site)**.** Given a scheme S, the *étale site* Ét_S is the category of étale morphisms to S. This site comes with a notion of covering: a collection of morphisms $\{U_i \to U\}$ in Ét_S is a *covering* if and only if the whole covering is surjective on the underlying topological spaces.

Remark 1.32. Technically, we have defined the "small" étale topos.

An advantage of working with étale cohomology is that our points gain automorphism groups arising from Galois information.

Definition 1.33 (geometric point)**.** Fix a scheme S. A *geometric point* $\overline{x} \hookrightarrow S$ is a morphism of schemes from an algebraically closed field; abusing notation, we may write $\overline{x}$ as $\operatorname{Spec} K$ or as the morphism $\overline{x}\colon \operatorname{Spec} K \to S$.

Remark 1.34. We do not require that our geometric points have closed image in S.

Remark 1.35. Requiring that we have a morphism of schemes amounts to requiring that the algebraically closed field K contains the residue field of the image $x \in S$ of $\overline{x}$. In other words, the data of the morphism $\overline{x}\colon \operatorname{Spec} K \hookrightarrow S$ amounts to the choice of a point $x \in S$ and an embedding $\kappa(x) \hookrightarrow K$.

Definition 1.36 (étale neighborhood)**.** Fix a scheme S. Then an *étale neighborhood* $(U, \overline{u})$ of a geometric point $\overline{x} \hookrightarrow S$ is an étale morphism $\pi\colon U \to S$ equipped with a geometric point $\overline{u} \hookrightarrow U$ together with an embedding $\overline{x} \hookrightarrow U$ over S. A morphism of étale neighborhoods is a morphism of étale covers of S preserving the basepoint.

1.3.2 Sheaves on the Étale Site

With a site, one wants sheaves.

Definition 1.37 (sheaf)**.** Fix a small category $\mathcal{C}$ and a scheme S. An *étale presheaf* $\mathcal{F}$ of $\mathcal{C}$ on S is a contravariant functor $\text{Ét}_S^{\text{op}} \to \mathcal{C}$. An *étale sheaf* is a presheaf $\mathcal{F}$ such that any $U \in \text{Ét}_S$ equipped with a covering $\{U_i \to U\}$ makes $\mathcal{F}(U)$ equal the equalizer

$$\mathcal{F}(U) = \operatorname{eq}\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)\right).$$

Remark 1.38. As in the Zariski topology, there is a sheafification functor $(-)^{\text{sh}} \colon \text{PSh}(S) \to \text{Sh}(S)$ sending the category of presheaves to sheaves. It is a left adjoint to the forgetful functor. The construction of this functor is rather technical, so we will only mention the key property that the sheafification functor is an isomorphism on stalks. For example, one can use sheafification to show that the category $\text{Sh}(S)$ is abelian.

With sheaves, one has stalks.

Definition 1.39. Fix a scheme S and a geometric point $\overline{x} \hookrightarrow S$. For a presheaf $\mathcal{F}$ on S, we define the *stalk* $\mathcal{F}_{\overline{x}}$ as

$$\mathcal{F}_{\overline{x}} := \varinjlim_{(U,\overline{u})} \mathcal{F}(U),$$

where the direct limit is taken over étale neighborhoods of $\overline{x}$.

Let's give some examples.

Proposition 1.40. Fix a scheme S. For any étale scheme V over S, the presheaf $\underline{V}$ given by $\underline{V}(U) := \operatorname{Hom}_S(U, V)$ is an étale sheaf.

Proof. This follows from some descent argument. ■

Example 1.41. Using the identity map $S \to S$ reveals that $\mathcal{O}_S$ is an étale sheaf. The stalk is

$$\mathcal{O}_{S,\overline{x}} = \varinjlim_{(U,\overline{u})} \Gamma(U, \mathcal{O}_U).$$

It turns out that this is a strictly Henselian ring.

Example 1.42. Fix a positive integer $n \geq 1$. Define the S-scheme μ_n as $\operatorname{Spec} \mathbb{Z}[T]/(T^n - 1) \times_{\operatorname{Spec} \mathbb{Z}} S$. If the multiplication map $n \colon \mathcal{O}_S \to \mathcal{O}_S$ is an isomorphism, then μ_n is étale over S, so this is an étale sheaf.

Example 1.43. For a finite set Σ, we may define the S-scheme Σ given by $\Sigma \times S$ (namely, a disjoint union of Σ-many copies of S). This then produces an étale sheaf $\underline{\Sigma}$.

It is too hard to work with all sheaves. Roughly speaking, we will be interested in "local systems." Here is the version of this notion in algebraic geometry.

Definition 1.44 (locally constant, constructible)**.** Fix an étale sheaf $\mathcal{F}$ on a scheme S and valued in a category $\mathcal{C}$.

(a) $\mathcal{F}$ is *locally constant* if and only if there is a finite étale covering $\{U_i \to S\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to a constant sheaf (still valued in $\mathcal{C}$).

(b) $\mathcal{F}$ is *constructible* if and only if there is a finite stratification $\{S_i\}$ of S into locally closed subsets such that $\mathcal{F}|_{S_i}$ is a locally constant sheaf of finite type.

Remark 1.45. The notion of "finite type" changes depending on $\mathcal{C}$. For example, if $\mathcal{C}$ is the category of abelian groups, then one wants to consider finite abelian groups. If $\mathcal{C}$ is a category of vector spaces, then one wants to consider finite-dimensional vector spaces.

Example 1.46. The constant sheaf $\underline{A}$ of an abelian group A is a locally constant constructible sheaf.

Example 1.47. If S is a variety over $\mathbb{C}$, and $\pi\colon X \to S$ is an étale covering, then the pushforward $\pi_*\underline{\mathbb{Z}}$ is locally constant and constructible.

Example 1.48. If $\pi\colon X \to S$ is an étale covering of schemes, then the sheaves $R^i\pi_*\mathbb{Z}_\ell$ (suitably interpreted) is locally constant and constructible.

Remark 1.49. As in topology, it turns out that one can think about locally constant constructible étale sheaves are representations of a fundamental group $\pi_1^{\text{ét}}(S, \overline{x})$.

1.4 February 6

We didn't have class on Tuesday. We will continue doing a little review of étale cohomology. In particular, we will talk about the étale fundamental group.

1.4.1 Galois Theory of Schemes

We would like to bring the notion of the topological fundamental group to the étale fundamental group. Roughly speaking, the topological fundamental group can be realized as the group of automorphisms of a universal cover, and this universal cover can be seen as some limit of finite covers. In algebraic geometry, we do not have access to this limit of finite covers, but we do have access to finite covers: they are étale coverings.

Before saying anything of substance, we recall some properties of étale morphisms.

Lemma 1.50. Fix scheme morphisms $f\colon X \to Y$ and $g\colon Y \to X$.

(a) Composition: if f and g are étale, then $g \circ f$ is étale.

(b) Cancellation: if $g \circ f$ and g are étale, then f is étale.

Lemma 1.51. Étale morphisms are preserved by base-change. More precisely, fix a pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes. If f is étale, then so is f'.

Lemma 1.52. Fix a scheme morphism $f\colon X \to Y$. Then f is unramified if and only if the diagonal $\Delta_f\colon X \to X \times_Y X$ is an open embedding.

The following result should be understood as a version of "unique path-lifting."

Proposition 1.53 (Rigidity). Fix finite étale schemes X and Y over S. If X is connected, and Y is separated, then any geometric point $\overline{x} \hookrightarrow X$ (where $\overline{x} = \operatorname{Spec} \overline{k}$) induces an injection

$$\operatorname{Mor}_S(X, Y) \to Y(\overline{k})$$

given by $f \mapsto f(\overline{x})$.

Sketch. Suppose we have two morphisms $f, g\colon X \to Y$ over S with $f(\overline{x}) = g(\overline{x})$. Because X and Y are both étale over S, we see that f and g are étale. Thus, f and g are local isomorphisms, so the connectivity of X tells us that they are determined by the image of $\overline{x}$. ■

Corollary 1.54. Suppose S is a connected scheme and that $X \to S$ is a finite étale morphism. Choose a geometric point $\overline{s} \hookrightarrow S$, and choose some $\overline{x} \hookrightarrow X$ in the fiber. Then the map

$$\operatorname{Aut}_S(X) \to X_{\overline{s}}$$

given by $f \mapsto f(\overline{x})$ is injective.

Proof. Set $X = Y$ in Proposition 1.53. ■

Remark 1.55. If X and S are points, then we are looking at some finite separable field extension L/K, and the corollary amounts to asserting that the following map is an injective: define

$$\operatorname{Aut}_K(L) \hookrightarrow \operatorname{Hom}_K(L, \overline{K})$$

by sending $\sigma\colon L \to L$ to the embedding $\iota \circ \sigma$, where $\iota\colon L \hookrightarrow \overline{K}$ is a chosen embedding. Note that L/K is a Galois extension if and only if this map is an isomorphism!

The above remark motivates the following definition.

Definition 1.56 (Galois). Fix a connected scheme S. A finite étale morphism $f\colon X \to S$ is *Galois* if and only if $\# \operatorname{Aut}_S(X) = \# X_{\overline{s}}$ for some (or equivalently, any) geometric point $\overline{s} \hookrightarrow S$.

We are now motivated to build a "Galois theory" of schemes. Here are some statements, which we will not prove.

Proposition 1.57. Fix a connected scheme S, and choose a Galois covering $X \to S$.

(a) There is a map sending finite étale subcovers $Y \to S$ of $X \to S$ to the subgroup of $\operatorname{Aut}_Y(X) \subseteq \operatorname{Aut}_S(X)$.

(b) There is a map sending subgroups $H \subseteq \operatorname{Aut}_S(X)$ to finite étale subcovers $X^H \to S$ of $X \to Y$.

(c) The two maps of (a) and (b) are order-preserving mutually inverse isomorphisms.

Remark 1.58. There is some difficulty in constructing the quotient X^H. Roughly speaking, one can take fixed points (as expected) on the level of affine schemes, and then one uses the fact that our morphisms are finite and étale to show that this is good enough for the general case.

Proposition 1.59. Fix a connected scheme S. For any finite étale covering $X \to S$, there is a unique (up to isomorphism) connected finite étale covering $X' \to X$ satisfying the following conditions.

(i) The composite covering $X' \to S$ is Galois.

(ii) Any other Galois covering $X'' \to S$ which factors through $X \to S$ will also factor uniquely through $X'' \to X'$.

So we have a Galois correspondence and a construction of normal closures.

1.4.2 The Étale Fundamental Group

Let's now turn to the absolute Galois group, extending our discussion of Galois theory of schemes.

Definition 1.60 (étale fundamental group)**.** Fix a connected scheme S and a geometric point $\overline{s} \hookrightarrow S$. Then we define the *étale fundamental group* $\pi_1^{\text{ét}}(S, \overline{s})$ as the automorphism group of the fiber functor $\operatorname{Fib}_{\overline{s}}$ sending finite étale covers $X \to S$ to the set $\operatorname{Hom}_S(\overline{s}, X)$.

Remark 1.61. It is a theorem that Fib upgrades to an equivalence of categories between the category of finite étale covers $X \to S$ and the category of finite $\pi_1^{\text{ét}}(S, \overline{s})$-sets.

Example 1.62. Let k be an algebraically closed field, and take $S = \operatorname{Spec} k$ with geometric points $S = S$. Then finite étale covers of S all look like $S \sqcup \cdots \sqcup S$, which is a unique map down to S, so the fiber functor has no nontrivial automorphisms. We conclude that $\pi_1^{\text{ét}}(S, \overline{s}) = 1$.

Remark 1.63. Intuitively, the automorphism group of the fiber functor should grow in size if there are fewer finite étale covers.

One is even able to recover some kind of universal cover, but it is not exactly a scheme: it is a limit of schemes.

Proposition 1.64. Fix a connected scheme S and a geometric point $\overline{s} \hookrightarrow S$, and let $\{(X_\alpha, \overline{x}_\alpha)\}$ be the inverse system of all finite Galois covers of $(S, \overline{s})$ preserving the (geometric) basepoint.

(a) For any finite étale cover $Y \to S$, we have

$$\operatorname{Fib}_{\overline{s}}(Y) = \varinjlim \operatorname{Hom}_S(X_\bullet, Y).$$

(b) We have

$$\pi_1^{\text{ét}}(S, \overline{s}) = \varprojlim \operatorname{Aut}_S(X_\bullet)^{\text{op}}.$$

Example 1.65. Let k be any field, and take $S = \operatorname{Spec} k$ with geometric point given by a chosen embedding $k \hookrightarrow \overline{k}$. Then the proposition tells us

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, \operatorname{Spec} \overline{k}) = \varprojlim_{\text{Galois } L/k} \operatorname{Gal}(L/k) = \operatorname{Gal}(k^{\text{sep}}/k).$$

Remark 1.66. For two basepoints $\overline{s}_1, \overline{s}_2 \hookrightarrow S$, one can produce a natural isomorphism of the relevant fiber functors, so the étale fundamental groups become isomorphic by some isomorphism coming from conjugation.

Remark 1.67. Choose a connected scheme W of finite type over $\mathbb{Z}$, and choose a closed point $w \in W$, which comes from a morphism $\operatorname{Spec} k(w) \hookrightarrow W$. A choice of algebraic closure of $k(w)$ then gives a geometric point $\overline{w} \hookrightarrow W$. There is thus a map

$$\pi_1^{\text{ét}}(\operatorname{Spec} k(w), \overline{w}) \to \pi_1^{\text{ét}}(W, \overline{w})$$

by base-changing automorphisms of the fiber functor for W to merely the closed point w. Now, $k(w)$ should be a finite field, so the left-hand group is topologically generated by a Frobenius element; note that one typically takes the geometric Frobenius to topologically generate $\pi_1^{\text{ét}}(\operatorname{Spec} k(w), \overline{w})$.

Changing the embedding $k(w) \hookrightarrow \overline{k(w)}$ tells us that this Frobenius element in $\pi_1^{\text{ét}}(W, \overline{w})$ should really be only defined up to conjugacy. In this way, we produce a canonical conjugacy class in $\pi_1^{\text{ét}}(W, \overline{w})$ for each closed point in W.

As a last statement, we recall the homotopy exact sequence.

Theorem 1.68. Fix a geometrically connected and quasicompact scheme S over a field k, and choose a basepoint $\overline{x} \hookrightarrow S_{\text{k}^{\text{sep}}}$. Then there is a short exact sequence

$$1 \to \pi_1^{\text{ét}}(S_{k^{\text{sep}}}, \overline{s}) \to \pi_1^{\text{ét}}(S, \overline{s}) \to \pi_1^{\text{ét}}(\operatorname{Spec} k) \to 1.$$

Remark 1.69. We won't prove the theorem, but we will remark that the sequence of morphisms is induced by functoriality from the morphisms

$$\overline{s} \hookrightarrow S \to \operatorname{Spec} k.$$

1.5 February 11

Today we continue.

1.5.1 Torsion Sheaves

Recall that the affine scheme $\mu_{n,S} = \operatorname{Spec} \mathbb{Z}[T]/(T^n - 1) \times S$ is a group scheme over S. This allows to build an étale sheaf, sending finite étale covers $X \to S$ to the group

$$\underline{\mu_n}_S(X) = \operatorname{Hom}_S(\mu_{n,S}, X).$$

Now, we can use this sheaf to say something about the equivalence of categories of finite étale covers of S and finite sets with action by $\pi_1^{\text{ét}}(S, \overline{s})$ (for S connected).

Corollary 1.70. Fix a connected scheme S and geometric point $\overline{s} \in S$. Then there is an equivalence of categories between locally constant constructible sheaves on S with finite continuous $\pi_1^{\text{ét}}(S, \overline{s})$-sets.

Proof. This equivalence of categories takes such a sheaf $\mathcal{F}$ to the stalk $\mathcal{F}_{\overline{s}}$. The fact that étale covers are producing the required sheaves is formal. ■

Remark 1.71. This equivalence is known as the finite monodromy correspondence.

Remark 1.72. If one works with the abelian category of locally constant constructible sheaves valued in an abelian category (frequently valued in finite abelian groups), then our stalks and $\pi_1^{\text{ét}}(S, \overline{s})$ inherit this extra structure.

Let's be explicit about the sheaves we are interested in.

Definition 1.73. An étale sheaf $\mathcal{F}$ of abelian groups on a scheme S is *torsion* if and only if either of the following conditions hold.

(i) All stalks of $\mathcal{F}$ are torsion abelian groups.

(ii) One has that $\mathcal{F}(U)$ is a torsion abelian group for all étale open subsets $U \to S$.

Remark 1.74. The embedding of $\mathcal{F}(U)$ into a stalk explains why (i) implies (ii). The fact that any element in a stalk comes from some étale open neighborhood explains why (i) implies (ii).

Let's make a few more remarks about these sheaves.

Lemma 1.75. Fix a scheme S.

(a) A torsion étale sheaf $\mathcal{F}$ on a scheme S is Noetherian if and only if it is locally constant constructible.

(b) Every torsion sheaf is the (filtered) direct limit of its constructible subsheaves.

Sketch. For (a), the idea is to consider descending chains of the required filtrations. For (b), the idea is to show that any element in any stalk of the torsion sheaf can be found in some constructible subsheaf. ■

Remark 1.76. It follows that locally constant constructible sheaves form an abelian category!

Eventually, we will have control over torsion sheaves. However, one would still like coefficients in characteristic 0, which we do by taking a limit.

Definition 1.77. Fix a scheme S and a prime ℓ. An *ℓ-adic sheaf* $\mathcal{F}$ on S is a projective system $\{\mathcal{F}_n\}_{n\geq 0}$ of constructible sheaves of abelian groups such that $\mathcal{F}_0 = 0$ and $\ell^1\mathcal{F}_n = 0$ for $n \geq 0$, and there are isomorphisms

$$\mathcal{F}_{n+1}/\ell^{n+1}\mathcal{F}_{n+1} \to \mathcal{F}_n$$

for each $n \geq 0$. We say that $\mathcal{F}$ is *smooth* if and only if each $\mathcal{F}_n$ is locally constant constructible. There is a similar definition to have coefficients in a finite extension E of $\mathbb{Q}_\ell$; then an étale $\overline{\mathbb{Q}}_\ell$-sheaf is one with coefficients in some finite extension E of $\mathbb{Q}_\ell$.

Remark 1.78. As before, there is an equivalence of categories (given by taking stalks) between étale $\overline{\mathbb{Q}}_\ell$-sheaves and continuous representations of $\pi_1(S,\overline{s})$ on $\overline{\mathbb{Q}}_\ell$-vector spaces. The sheaf is smooth if and only if the representation is finite-dimensional.

1.5.2 Back to Frobenius Morphisms

Let's begin to do something with these sheaves. As usual, we take $k = \mathbb{F}_q$. Here are some Frobenius elements.

- There is an arithmetic Frobenius automorphism $\sigma \in \operatorname{Gal}(\overline{k}/k)$ given by $\sigma\colon x \mapsto x^q$, and we recall that it is a topological generator of $\operatorname{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}}$. It generates the cyclic "Weil" group $W_{\overline{k}/k} \cong \mathbb{Z}$.
- The inverse $F := \sigma^{-1}$ is the geometric Frobenius element.
- Then for any $\mathbb{F}_q$-scheme Y, there is an absolute Frobenius $\sigma_{Y/k}\colon Y \to Y$. It is the identity on the topological sheaves, and it acts by qth powers on the level of structure sheaves. Technically, this is not a morphism of schemes over $\overline{k}$, so we may have occasion to write the target as $Y^{(q)}$.

Now, let X be a scheme over k of finite type, and let $\mathcal{G}_0$ be an étale sheaf on X; then we let $\mathcal{G}$ denote the pullback sheaf on $X_{\overline{k}}$. By functoriality of the absolute Frobenius, one gets a diagram as follows.

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r, dashed] \arrow[dr, "\sigma_{X,\overline{k}}"] \arrow[dd] & X_{\overline{k}} \arrow[r] \arrow[d, "\mathrm{id}_{X_{\overline{k}}}\times\sigma"] & X \arrow[d, equal] \\
& X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}\overline{k} \arrow[r, "\sigma"] & \operatorname{Spec}\overline{k} \arrow[r] & \operatorname{Spec}k
\end{tikzcd}$$

We claim that the dashed arrow exists and is unique. Indeed, it is enough to note that the top-right square is Cartesian and then checking some commutativity using the relevant functoriality.

We are now able to call the dashed map $\operatorname{Fr}_{X_{\overline{k}}}$, which we note satisfies

$$(\mathrm{id}_{X_{\overline{k}}}\times\sigma)\circ\operatorname{Fr}_{X_{\overline{k}}} = \sigma_{X_{\overline{k}}}$$

by construction. Let's see an example.

Example 1.79. Take $X = \mathbb{A}^1$. Then one has the following diagram.

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[r, dashed, "\mathrm{Fr}"] \arrow[dr, "\sigma_{\mathbb{A}^1}"'] & \mathbb{A}^1 \arrow[d, "\mathrm{id}\times\sigma"] \\
& \mathbb{A}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{k}[t] \arrow[r] \arrow[dr] & \overline{k}[t] \arrow[d] \\
& \overline{k}[t]
\end{tikzcd}
\qquad
\begin{tikzcd}
at \arrow[r, mapsto, dashed] \arrow[dr, mapsto] & at^q \arrow[d, mapsto] \\
& a^qt^q
\end{tikzcd}$$

Thus, we see on the structure sheaf that we have managed to construct Fr as a morphism of schemes over $\overline{k}$, but it is still more or less applying a q-power to points.

Now, let $\overline{x} \in X$ be a geometric point, where its image $x \in X$ is closed. Recall that sheaves on x are essentially abelian groups, which one can see by taking the stalk at $\overline{x}$; of course, we do see that these abelian groups must come with an action by $\pi_1(x, \overline{x}) = \operatorname{Gal}(\overline{k}/\kappa(x))$, which we will call G, so taking this stalk produces a discrete abelian G-module.

Let $\mathcal{G}$ be an étale $\overline{\mathbb{Q}}_\ell$-sheaf on X, and we see that the stalk $\mathcal{G}_{\overline{x}}$ will admit an action by the geometric Frobenius element F, which we will call $F_x \colon \mathcal{G}_{\overline{x}} \to \mathcal{G}_{F\overline{x}}$. (This is some formality of scheme morphisms, and we are implicitly using that F is an isomorphism of the topological spaces.) Now, $\mathcal{G}$ is really some étale E-sheaf for an extension E of $\mathbb{Q}_\ell$. Further, $\mathcal{G}$ is really some inverse system $\{\mathcal{G}_i\}_{i\ge 1}$ of finite étale E-sheaves. Now, each $\mathcal{G}_i$ comes from a bona fide étale scheme G_i over $X_{\overline{k}}$. Then the diagram

$$\begin{tikzcd} G_i \arrow[dr, dashed] \arrow[drr, bend left, "F_{G_i}"] \arrow[ddr, bend right] & & \\ & F^*_{X_{\overline{k}}} G_i \arrow[r] \arrow[d] & G_i \arrow[d] \\ & X_{\overline{k}} \arrow[r, "F_{X_{\overline{k}}}"'] & X_{\overline{k}} \end{tikzcd}$$

is able to produce a relative Frobenius morphism $G_i \to F_{X_{\overline{k}}} \to G_i$, so we produce a genuine map of sheaves

$$\mathcal{G} \to F^*_{X_{\overline{k}}} \mathcal{G}$$

upon taking the limit (in the category of étale sheaves) again.

We claim there is a canonical isomorphism $F^*_{\mathcal{G}} \colon F^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}$, which describes the étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ as being $\operatorname{Gal}(\overline{k}/k)$-equivariant sheaf. (Intuitively, one looks at the top horizontal map in the last diagram and takes a limit.) This claim is rather intricate, so we won't prove it: it comes down to explicating what the Galois action should be.

But now the consequence is that we receive a composite

$$\mathcal{G} \to F^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}.$$

By construction of the last isomorphism, one finds that this is the Frobenius morphism $\mathcal{G}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ at a geometric point $\overline{x} \hookrightarrow X$. This allows to define the L-function for $\mathcal{G}$.

Definition 1.80. Fix an étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$, as above. Then we define

$$L(X, \mathcal{G}, T) := \prod_{\text{closed } x \in X} \det\left(1 - T^{\deg x} F_x; \mathcal{G}_{\overline{x}}\right)^{-1}.$$

By using the trace formula, one can prove that this L-series has a cohomological interpretation.

Theorem 1.81. Fix an étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on a scheme X over $\mathbb{F}_q$. Then

$$L(X, \mathcal{G}; T) = \prod_{i=0}^{2\dim X} \det\left(1 - tF^*; \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G})\right)^{(-1)^{i+1}}.$$

Now, as before, we note that one can construct a map $\mathcal{G} \to \mathrm{Fr}^*_{X_{\overline{k}}}\mathcal{G}$, and one can show that there is a canonical isomorphism $\mathrm{Fr}_{\mathcal{G}} \colon \mathrm{Fr}^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}$. Thus, we gain a diagram

$$\begin{tikzcd} \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G}) \arrow[r, "\mathrm{Fr}^*_{X_{\overline{k}}}"] \arrow[dr, "F^*"'] & \mathrm{H}^i_c(X_{\overline{k}}, \mathrm{Fr}^*_{X_{\overline{k}}}\mathcal{G}) \arrow[d, "\mathrm{Fr}^*_{\mathcal{G}}"] \\ & \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G}) \end{tikzcd}$$

for which we can use to prove the above theorem.

1.6 February 13

Here we go.

1.6.1 Weil Sheaves

We would like to generalize our sheaves somewhat. Note that we had an action of the geometric Frobenius element on $X_{\overline{k}}$ from the start. In particular, our étale sheaves have a full action by $\pi_1(X,\overline{x})$, but we really only need to know about Frobenius action.

Definition 1.82 (Weil group)**.** Fix a finite field $k := \mathbb{F}_q$. The *Weil group* $W(\overline{k}/k)$ is the cyclic group generated by the Frobenius.

Now, note $W(\overline{k}/k)$ acts on $X_{\overline{k}} = X \times_k \overline{k}$, so any étale sheaf $\mathcal{G}$ on X gains an action of $W(\overline{k}/k)$ after base-changing to $X_{\overline{k}}$. Because this sheaf comes from X, it should be $W(\overline{k}/k)$-equivariant, which amounts to providing the data of an isomorphism $F^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}$. We are now ready to make the following definition, which generalizes our étale sheaves.

Definition 1.83 (Weil sheaf)**.** A *Weil sheaf* $\mathcal{G}_0$ on X consists is a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on $X_{\overline{k}}$ together with an isomorphism $F^*\colon F^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}$. This Weil sheaf $\mathcal{G}_0$ is *smooth of rank* r if and only if the same is true of $\mathcal{G}$ on $X_{\overline{k}}$.

Notably, this Weil sheaf is not required to actually come from a sheaf on X! (This is the whole point!) This extra flexibility of not having to actually come from a scheme over $\mathbb{F}_q$ will be helpful for us later.

Let's give a few remarks about the category of Weil sheaves.

Remark 1.84. The category of Weil sheaves on X form an abelian category. This category contains the category of étale sheaves on X, embedded as described above.

Remark 1.85. If $k' \subseteq \overline{k}$ is a finite extension of k, then the category of Weil sheaves on $X_{k'}$ is the same as the category of Weil sheaves on X. Namely, restriction of scalars from $X_{k'}$ to X defines the required equivalence.

Remark 1.86. The category of Weil sheaves has the six operations (e.g., pullbacks, derived direct images, and direct image with compact support). The point is that one can do these operations on $X_{\overline{k}}$, and then one just needs to carry around the extra data of the isomorphism F^*.

We now remark that a Weil sheaf $\mathcal{G}_0$ on X is still going to come with an isomorphism $F\colon \mathrm{H}^i_c(X_{\overline{k}},\mathcal{G}) \to \mathrm{H}^i_c(X_{\overline{k}},\mathcal{G})$, so one can define $L(X,\mathcal{G};T)$ as before. There is still a cohomological interpretation, but we will omit its proof.

These Weil sheaves appear to form a Tannakian category, so we are allowed to ask for the corresponding reductive group. To see this, we recall that the short exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}},\overline{x}) \to \pi_1^{\text{ét}}(X,\overline{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

The inverse image of $W(\overline{k}/k) \subseteq \mathrm{Gal}(\overline{k}/k)$ is some Weil group $W(X,\overline{x}) \subseteq \pi_1^{\text{ét}}(X,\overline{x})$. It turns out that the induced quotient $W(X,\overline{x})/\pi_1(X_{\overline{k}},\overline{x})$ is $W(\overline{k}/k) \cong \mathbb{Z}$. We are now ready to compare these Tannakian categories.

- Recall that the category of étale sheaves is equivalent to the category of continuous representations of $\pi_1^{\text{ét}}(X,\overline{x})$ on vector spaces over $\overline{\mathbb{Q}}_\ell$.

- This then restricts to an equivalence between the category of smooth étale sheaves and the category of finite-dimensional continuous representations of $\pi_1^{\text{ét}}(X, \overline{x})$.
- However, the category of Weil sheaves can be seen as the category of continuous representations of $W(X, \overline{x})$ of $\overline{\mathbb{Q}}_\ell$-vector spaces. To see this, we note that we are essentially removing some data from being a sheaf over $\pi_1^{\text{ét}}(X, \overline{x})$.

  The adjective of smoothness adds a finite-dimensional requirement.

1.7 February 18

1.7.1 Weil Sheaves from Étale Sheaves

Fix a scheme X over a finite field $\mathbb{F}_q$, and choose a geometric point $\overline{x} \hookrightarrow X$. Last time, we explained that Weil sheaves on X correspond to continuous representations of the Weil group $W(X, \overline{x})$. Thus, we are interested in the representation theory of $W(X, \overline{x})$. The easiest representations (and indeed, the easiest sheaves) have rank 1, so let's write these down.

Proposition 1.87. Weil sheaves on $\operatorname{Spec} \mathbb{F}_q$ of rank 1 are in bijection with $\overline{\mathbb{Q}}_\ell^\times$.

Proof. A Weil sheaf of rank 1 is known to correspond to a representation

$$W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell).$$

The left-hand side is $\mathbb{Z}$, so the result follows. ■

Notation 1.88. For $b \in \overline{\mathbb{Q}}_\ell^\times$, we let $\mathcal{L}_b$ denote the corresponding Weil sheaf on $\operatorname{Spec} \mathbb{F}_q$.

These sheaves help explain when Weil sheaves come from étale sheaves.

Theorem 1.89. Let X be a scheme over $\mathbb{F}_q$, and let $\mathcal{G}_0$ be a Weil sheaf on X. Suppose further that X is normal and geometrically connected and that $\mathcal{G}_0$ is irreducible (as a representation) and smooth of rank r. Then $\mathcal{G}_0$ is an étale $\overline{\mathbb{Q}}_\ell$-sheaf if and only if $\wedge^r \mathcal{G}_0$ is an étale sheaf on X.

Here are some applications of Theorem 1.89.

Corollary 1.90. Fix a smooth irreducible Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected scheme X over $\mathbb{F}_q$. Then there is $b \in \overline{\mathbb{Q}}_\ell^\times$ and an étale sheaf $\mathcal{F}_0$ on X such that

$$\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b.$$

Proof. We will prove this later, along with the theorem. ■

Corollary 1.91. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected scheme X over $\mathbb{F}_q$. Then there is a filtration of subsheaves

$$0 = \mathcal{G}_0^{(0)} \subseteq \mathcal{G}_0^{(1)} \subseteq \cdots \subseteq \mathcal{G}_0^{(r)} = \mathcal{G}_0$$

such that $\mathcal{G}_0^{(j)}/\mathcal{G}_0^{(j-1)}$ is isomorphic to some tensor $\mathcal{F}_0^{(j)} \otimes \mathcal{L}_{b_j}$ with $\mathcal{F}_0^{(j)}$ a smooth étale $\overline{\mathbb{Q}}_\ell$-sheaf and $b_j \in \overline{\mathbb{Q}}_\ell^\times$.

Proof. This follows by using filtrations of sheaves of finite rank and applying the example. ■

Remark 1.92. One can think of the twisting $-\otimes\mathcal{L}_b$ as some kind of ampleness condition.

As another application, we note that this machinery lets us write down an L-series for Weil sheaves. The idea is that filtrations decompose a Weil sheaf $\mathcal{G}_0$ into some extensions of irreducible sheaves by irreducible sheaves, allowing us to reduce to irreducible Weil sheaves. Then irreducible Weil sheaves can be controlled because there are merely étale sheaves twisted by an explicit character. Let's write this out.

Notation 1.93. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a scheme X over $\mathbb{F}_q$. Then we define its L-function by

$$L(X,\mathcal{G},t) := \prod_{\text{closed } x\in X} \det\left(1-t^{\deg x}F_x^*;\mathcal{G}_{\overline{x}}\right)^{-1}.$$

Corollary 1.94. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected scheme X over $\mathbb{F}_q$. Then

$$L(X,\mathcal{G},t) = \prod_{i=0}^{2\dim X} \det\left(1-tF^*;\mathrm{H}_c^i(X_{\overline{k}},\mathcal{G})\right)^{(-1)^{i+1}}.$$

Again, we will write this out in more detail next week.

1.7.2 Weights

Let's say something about weights. The motivation is that one can control the location of poles and zeroes of the L-function. Throughout, we will fix an isomorphism $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$.

Definition 1.95 (pure)**.** Fix a scheme X over $\mathbb{F}_q$, and choose an isomorphism $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$. We say that a smooth Weil sheaf $\mathcal{G}_0$ is *ι-pure of weight* β for some $\beta>0$ if and only if all eigenvalues of $\alpha\in\overline{\mathbb{Q}}_\ell$ of $F_x\colon\mathcal{G}_{0,\overline{x}}\to\mathcal{G}_{0,\overline{x}}$ has

$$|\iota(\alpha)|^2 = \#\kappa(x)^\beta.$$

We say that $\mathcal{G}_0$ is *pure of weight* β if and only if it is ι-pure of weight β for all chosen ι.

The sheaves one comes across in practice may not be pure on the nose but instead have some pure part that we can remove and then handle via some induction.

Definition 1.96 (mixed)**.** Fix a scheme X over $\mathbb{F}_q$, and choose an isomorphism $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$. We say that a smooth Weil sheaf $\mathcal{G}_0$ is *ι-mixed* if and only if there is a finite filtration

$$0=\mathcal{G}_0^{(0)}\subseteq\mathcal{G}_0^{(1)}\subseteq\cdots\subseteq\mathcal{G}_0^{(r)}=\mathcal{G}_0$$

such that the quotients $\mathcal{G}_0^{(i)}/\mathcal{G}_0^{(i-1)}$ are ι-pure of some weight. We say that $\mathcal{G}_0$ is *mixed* if and only if it is ι-mixed for all chosen ι.

A motivation of Weil II is that we would like these weights to be found as some geometric invariant. For example, we expect that the weights of sheaves to be preserved under some controlled morphisms.

Proposition 1.97. Suppose that $\pi \colon X \to Y$ is a morphism of schemes over $\mathbb{F}_q$. Given a Weil sheaf $\mathcal{G}_0$ on Y.

(a) If $\mathcal{G}_0$ is ι-pure of weight β, then $f^*\mathcal{G}_0$ is ι-pure of weight β.

(b) If f is surjective and $f^*\mathcal{G}_0$ is ι-pure of weight β, then $\mathcal{G}_0$ is ι-pure of weight β.

(c) If f is finite and $\mathcal{G}_0$ is ι-pure of weight β, then $f_*\mathcal{G}_0$ is ι-pure of weight β.

(d) If $\mathcal{G}_0$ is a Weil sheaf on X, then $\mathcal{G}_0$ on X is ι-pure of weight β if and only if $\mathcal{G}_0$ on $X_{\mathbb{F}_{q^r}}$ is ι-pure of weight β.

Sketch. Here, (a) and (b) are essentially formal because the fibers of $f^*\mathcal{G}_0$ are the same as the fibers of $\mathcal{G}_0$. For (d), this is again purely formal from computing how the Frobenius and the degree simultaneously adjust on extension of the base field. Alternatively, (d) can be derived from (a)–(c) because the canonical morphism $X_{\mathbb{F}_{q^r}} \to X_{\mathbb{F}_q}$ is surjective and finite. We won't say anything about (c), but it should also follow from some fiber-wise computations. ■

For inductive applications, it may be useful to have some largest weight.

Definition 1.98. Fix a scheme X over $\mathbb{F}_q$, and let $\mathcal{G}_0$ be a Weil sheaf on X, and choose some isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Then we define

$$w(\mathcal{G}_0) \coloneqq \sup_{\text{closed } x \in X} \sup_{\text{eigenvalue } \alpha} \frac{\log |\iota(\alpha)|^2}{\log \#\kappa(x)}$$

if $\mathcal{G}_0$ is nontrivial, and $w(0) = -\infty$.

Lemma 1.99. Fix a scheme X over $\mathbb{F}_q$, and let $\mathcal{G}_0$ be a Weil sheaf on X, and choose some isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Suppose $w(\mathcal{G}_0) \le \beta$ for some $\beta > 0$. Then the L-series

$$\iota L(X, \mathcal{G}, t) \coloneqq \prod_{\text{closed } x \in X} \iota \det\left(1 - t^{\deg x} F_x^*; \mathcal{G}_{\overline{x}}\right)^{-1}$$

converges for all $t \in \mathbb{C}$ with $|t| < q^{-\beta/2 - \dim X}$.

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List of Definitions