18.906: Algebraic Topology II

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
∞-Categories

Language turns us all into jesters.

—Savannah Brown, [Bro24]

1.1 September 4

Here are some administrative notes.

- Office hours will be on Tuesday and Thursday immediately after class in 2-374.
- The syllabus will be posted to the course website later.
- The syllabus will contain some recommended textbooks, which are some free online texts that contain supersets of our class material.
- The grade will be 20% from a fifty-minute exam and 80% coming from problem sets. The exam will probably occur shortly before the drop deadline.

We hope to cover simplicial sets, ∞-categories, homotopy theory, Eilenberg–MacLane spaces, Postnikov towers, the Serre spectral sequence, and a little on vector bundles and characteristic classes. In particular, we see that the first part of the class is some purely formal nonsense, which we then use to set up the ∞-category of spaces, which is the natural setting for homotopy theory.

1.1.1 Category Theory

Here are some examples of categories to keep in mind for this course. We refer to Appendix A for the definitions.

Example 1.1. Any category $\mathcal{C}$ gives rise to a core groupoid $\operatorname{Core}\mathcal{C}$, which is the subcategory with the same objects but only taking the morphisms which are isomorphisms. One can check that this is in fact a subcategory.

In mathematics, one frequently encounters a category $\mathcal{C}$, and we are then interested in classifying the objects up to isomorphism.

Example 1.2. If $\mathcal{C} = \mathrm{Set}$, then isomorphisms are bijections, so sets "up to bijection" are simply given by their cardinalities.

Example 1.3. A commutative ring R gives rise to a category Mod_R of (left) R-modules. If R is a field, then this is a category of vector spaces, and objects up to isomorphism are given by their dimensions.

Example 1.4. One can consider the category Top of topological spaces, whose morphisms are continuous maps. (We will frequently restrict our category of topological spaces with some nicer subcategories, such as CW complexes or manifolds.) It is rather hard to classify objects up to isomorphism (here, isomorphisms are homeomorphisms), but there are some tools. For example, there are homology functors

$$\mathrm{H}_i(-;R)\colon \mathrm{Top} \to \mathrm{Mod}_R.$$

Because functors preserve isomorphisms, homeomorphic spaces must have isomorphic homology.

The definition of homology finds itself focused on continuous maps $|\Delta^n| \to X$, where $|\Delta^n|$ is the (topological) n-simplex.

Definition 1.5 (n-simplex)**.** The *(topological) n-simplex* $|\Delta^n|$ is the subspace

$$|\Delta^n| := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} t_i = 1 \right\}.$$

We will soon upgrade this topological n-simplex $|\Delta^n|$, which explains why we are writing $|\Delta^n|$ instead of Δ^n.

1.1.2 Homotopy Types, Intuitively

The homology functors $\mathrm{H}_i(-;R)\colon \mathrm{Top} \to \mathrm{Mod}_R$ factors through the homotopy category $\mathrm{Ho}(\mathrm{Top})$, which is obtained from Top by declaring homotopic maps to be equal.

We are going to be homotopy theorists for the most part, which means that we will be interested in understanding invariants of the category $\mathrm{Ho}(\mathrm{Top})$. One may complain that only studying spaces up to homotopy is an over-simplification. However, there are good reasons to only be interested in these "homotopy types" because such things also come up in other areas of mathematics.

Approximately speaking, a homotopy type is a collection of objects and morphisms between them. To ensure some level of homogeneity, one may require that any pair $f, g\colon A \to B$ of morphisms has a collection of "2-isomorphisms" $f \Rightarrow g$. Furthermore, there should be "3-isomorphisms" between these 2-isomorphisms, and this thinking continues inductively.

Example 1.6. Given two objects A and B with an isomorphism $f\colon A \to B$, one may think about these objects as being identified. Similarly, if we have a third isomorphism $g\colon B \to C$, then we can canonically identify all three objects. Here is diagram for this situation.

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad \simeq \qquad A$$

Example 1.7. Given two objects A and B, there may be two isomorphisms $f, g\colon A \to B$. One may want to identify these two objects via either isomorphism, but then we don't want to forget about the other isomorphism, so perhaps we are thinking about an object with an automorphism. Here is a diagram for this situation.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd} \qquad \simeq \qquad \begin{tikzcd} A \arrow[loop above, "g"] \end{tikzcd}$$

If one wanted to identify f and g, then there should be a "2-isomorphism" identifying f and g.

Example 1.8. We can think about a set as a homotopy type where all isomorphisms, 2-isomorphisms, and so on are all just the identity maps.

Example 1.9. We can think about a groupoid as a homotopy type where all 2-isomorphisms, 3-isomorphisms, and so on are all just the identity maps.

The above two examples will let us think about $\operatorname{Ho}(\operatorname{Top})$ as a category of ∞-groupoids. This is more or less why homotopy types are relevant to other areas of mathematics: one is frequently interested in not just isomorphisms between objects but also the uniqueness of those isomorphisms, and also the uniqueness of the isomorphisms identifying the isomorphisms, and so on.

Of course, we started with topological spaces, so let's explain how then make some ∞-groupoid.

Example 1.10. Given a topological space X, we can build a corresponding ∞-groupoid as follows.

- The points provide objects in our ∞-groupoids; these are functions $\left|\Delta^0\right| \to X$.
- The maps between points are given by paths; these are functions $\left|\Delta^1\right| \to X$.
- The maps between paths are given by homotopies of paths; these are functions $\left|\Delta^2\right| \to X$. Technically speaking, $\left|\Delta^2\right|$ gives two paths f and g whose composite should be homotopic to h. Thus, the structure of such a map tells us something about how composition should behave!

1.1.3 Simplices

After building up some intuition, we are now forced to do some combinatorics in order to get ourselves off of the ground.

Notation 1.11. For each integer $n \geq 0$, we define the category $[n]$ whose objects are the elements of $\{0, 1, \ldots, n\}$ and whose morphisms are given by

$$\operatorname{Hom}_{[n]}(i, j) := \begin{cases} \varnothing & \text{if } i < k, \\ * & \text{if } i \leq j, \end{cases}$$

where $*$ simply refers to some one-element set.

We remark that identities and the composition maps are then all uniquely defined (because everything is unique in the one-element set $*$); similarly, the coherence checks of identity and associativity have no content because everything is equal in $*$.

Remark 1.12. Combinatorially, $[n]$ is the poset category given by the totally ordered set

$$0 \le 1 \le 2 \le \cdots \le n.$$

Definition 1.13 (simplex)**.** The *simplex category* Δ has objects given by the categories $[n]$, and the morphisms are given by the collection of functors between any two such categories.

Remark 1.14. Combinatorially, we see that a functor $F\colon [n] \to [m]$ amounts to the data of an increasing map. Indeed, whenever $i \le j$ in $[n]$, which is equivalent to having a morphism $i \to j$, we see that there is a morphism $Fi \to Fj$, which is equivalent to the requirement $Fi \le Fj$.

Example 1.15. There are six morphisms $[1] \to [2]$, as follows.

- If $0 \mapsto 0$, then $1 \in [1]$ can go anywhere.
- If $0 \mapsto 1$, then 1 maps to 1 or 2 in $[2]$.
- If $0 \mapsto 2$, then 1 maps to 2.

Example 1.16. For each nonnegative integer n, there is a unique map $[n] \to [0]$ for each n. Indeed, everything must go to 0.

Remark 1.17. There is an important functor $F\colon \Delta \to \mathrm{Top}$ given by sending $[n] \mapsto |\Delta^n|$. Let's explain what this functor is on morphisms: given an increasing map $f\colon [n] \to [m]$, then we need to provide a continuous map $Ff\colon |\Delta^n| \to |\Delta^m|$. Well, we may identify $[n]$ and $[m]$ with bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, so f is now a function on bases, so it upgrades uniquely to a linear map $\mathbb{R}^n \to \mathbb{R}^m$ given by

$$Ff\left(\sum_{i=0}^n t_ie_i\right) := \sum_{i=0}^n t_ie_{f(i)}.$$

Thus, we see that Ff does restrict to a map $|\Delta^n| \to |\Delta^m|$. Functoriality follows by the uniqueness of the construction of Ff: given two increasing maps $f\colon [n] \to [n']$ and $g\colon [n'] \to [n'']$, we see $Fg \circ Ff$ and $F(g \circ f)$ definitionally are both defined as $g \circ f$ on the basis of $\mathbb{R}^n$. (Of course, we should mention $\mathrm{id}_{[n]}\colon [n] \to [n]$ defines the identity on $\mathbb{R}^n$.)

1.1.4 Simplicial Sets

The following is the first important definition of this course.

Definition 1.18 (simplicial set)**.** A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. We let sSet denote the category of such functors. In other words, $\mathrm{sSet} = \mathrm{PSh}(\Delta)$.

Note that this "functor category" is in fact a category by Lemma A.13. Here are some examples of simplicial sets.

Example 1.19 ($\mathrm{Sing}(X)$)**.** There is a functor $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$ such that

$$\mathrm{Sing}(X)\colon [n] \mapsto \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X).$$

Proof. We have many checks to do, which we handle in sequence.

- We define $\operatorname{Sing}(X)$ on morphisms. Well, given an increasing map $f\colon [n] \to [m]$, the functor F of Remark 1.17 provides a continuous map $Ff\colon |\Delta^m| \to |\Delta^n|$, so there is a map
$$(- \circ Ff)\colon \operatorname{Mor}_{\mathrm{Top}}(|\Delta^n|, X) \to \operatorname{Mor}_{\mathrm{Top}}(|\Delta^m|, X).$$
- We check that $\operatorname{Sing}(X)$ is a functor. First, the identity morphism $\mathrm{id}_{[n]}\colon [n] \to [n]$ goes to the map
$$(- \circ F\mathrm{id}_{[n]})\colon \operatorname{Mor}_{\mathrm{Top}}(|\Delta^n|, X) \to \operatorname{Mor}_{\mathrm{Top}}(|\Delta^n|, X),$$
which is the identity because $F\mathrm{id}_{[n]} = \mathrm{id}_{|\Delta^n|}$. Second, given increasing maps $f\colon [n] \to [n']$ and $g\colon [n'] \to [n'']$, we need to check that
$$(- \circ F(g \circ f)) = (- \circ Ff) \circ (- \circ Fg),$$
which is true because $F(g \circ f) = Fg \circ Ff$.
- We define Sing on morphisms. Well, given a continuous map $f\colon X \to Y$, we use the map
$$(f \circ -)\colon \operatorname{Mor}_{\mathrm{Top}}(|\Delta^n|, X) \to \operatorname{Mor}_{\mathrm{Top}}(|\Delta^n|, Y).$$
- We check that Sing is a functor. First, the identity $\mathrm{id}_X\colon X \to X$ goes to the map $(\mathrm{id}_X \circ -)$, which is just the identity composition. Second, given continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$, we note that
$$(g \circ -) \circ (f \circ -) = ((g \circ f) \circ -)$$
by the associativity of composition. ■

Remark 1.20. It turns out that not all simplicial sets arise from this construction. In particular, it turns out that the image of Sing has many nice properties.

Remark 1.21. It will turn out that the homotopy type of X is uniquely determined by $\operatorname{Sing}(X)$. This is remarkable because one expects Top to be a difficult category, even taken up to homotopy, but sSet just looks like some combinatorial data.

We are also interested in generalizing categories, so we pick up the following example.

Example 1.22 (nerve). Fix a category $\mathcal{C}$. Then there is a "nerve" functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ such that
$$N(\mathcal{C})\colon [n] \mapsto \operatorname{Fun}([n], \mathcal{C}).$$
The proof of this claim is exactly the same as in Example 1.19 (note that $\operatorname{Fun}([n], \mathcal{C}) = \operatorname{Mor}_{\mathrm{Cat}}([n], \mathcal{C})$), except now there is no need for the auxiliary functor $F\colon \Delta \to \mathrm{Top}$ because Δ is already a category. (Being brazen, one can copy the same proof but erasing all Fs, replacing Top with Cat throughout, and replacing $|\Delta^\bullet|$s with $[\bullet]$s throughout.)

Remark 1.23. As in Remark 1.20, nerves of categories have some nice properties which prevent them from producing all simplicial sets. It turns out that ∞-categories will be some kind of simultaneous generalization of Sings and nerves.

1.1.5 Simplicial Sets by Combinatorics

Even though we will avoid doing so as much as possible in the sequel, it can be worthwhile to have a purely combinatorial description of a simplicial set. Let's begin by classifying increasing maps. We will get some utility out of the following lemma, which allows us to think about increasing maps f in terms of the multi-set $\operatorname{im} f$.

Lemma 1.24. Let $f, g\colon [n] \to [m]$ be increasing maps. Suppose that

$$\#f^{-1}(\{k\}) = \#g^{-1}(\{k\})$$

for all $k \in [m]$. Then $f = g$.

Proof. We proceed by induction on n. If $n = 0$, then $[n]$ is a singleton, so there is a unique $k \in [m]$ for which $f^{-1}(\{k\})$ and $g^{-1}(\{k\})$ are nonempty, namely $f(0)$ and $g(0)$ respectively, so the result follows.

For the induction, we are given two increasing maps $f, g\colon [n+1] \to m$. There are two steps.

1. The main claim is that $f(n+1) = g(n+1)$. To show this, note that $\operatorname{im} f = \operatorname{im} g$ because these sets are just the $k \in [m]$ with nonempty fibers. Thus, because $n+1$ is the maximum of $[n+1]$, we see that $f(n+1)$ and $g(n+1)$ are maximal elements of $\operatorname{im} f$ and $\operatorname{im} g$, respectively, so $f(n+1) = g(n+1)$ follows.

2. We now complete the proof. Note that

$$\#f|_{[n]}^{-1}(\{k\}) = \begin{cases} \#f^{-1}(\{k\}) & \text{if } f(n+1) \neq k, \\ \#f^{-1}(\{k\}) - 1 & \text{if } f(n+1) = k, \end{cases}$$

and similar for g, so $f|_{[n]}$ and $g|_{[n]}$ have fibers of the same cardinality, so $f|_{[n]} = g|_{[n]}$ by the induction, so $f = g$ follows because they are already equal on $n+1$. ■

Let's now classify injective maps.

Definition 1.25 (face maps)**.** Given some $i \in [n]$, we define the *face map* $\delta^i\colon [n-1] \to [n]$ to be the embedding which omits i by sending the set $\{0, \ldots, i-1\}$ to itself and sending the set $\{i, \ldots, n\}$ to one more than each element.

Lemma 1.26. Every injective increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite

$$f = \delta^{i_1} \circ \cdots \circ \delta^{i_r},$$

where $i_1 > i_2 > \cdots > i_r$.

Proof. We proceed in steps.

1. Given a decreasing sequence $i_1 > i_2 > \cdots > i_r$, we claim that the map

$$(\delta^{i_1} \circ \cdots \circ \delta^{i_r})\colon [n] \to [n+r]$$

avoids the set $\{i_1, \ldots, i_r\}$. We proceed by induction on r; for $r = 0$, the statement is vacuous. For the induction, we are given a decreasing sequence $i_1 > i_2 > \cdots > i_r > i_{r+1}$. By the induction, the map

$$(\delta^{i_2} \circ \cdots \circ \delta^{i_{r+1}})\colon [n] \to [n+r]$$

already avoids the set $\{i_2, \ldots, i_{r+1}\}$. Then δ^{i_1} preserves the set $\{0, \ldots, i_1 - 1\}$ (and in particular preserves the omitted set $\{i_2, \ldots, i_{r+1}\}$) while going on to omit i_1, so the total composite $\delta^{i_1} \circ \cdots \circ \delta^{i_{r+1}}$ successfully omits $\{i_1, \ldots, i_{r+1}\}$.

2. We show that any injective f is a composite of $\delta^\bullet$s as given. Well, given an injective increasing map $f\colon [n] \to [m]$, set $I := [m] \setminus \operatorname{im} f$, and arrange the elements of I as $\{i_1, \ldots, i_r\}$ in decreasing order. Then $\delta^{i_1} \circ \cdots \circ \delta^{i_r}$ is another injective increasing map which omits I by the previous step, so it equals f by Lemma 1.24.

3. We show that two composites of decreasing $\delta^\bullet$s are equal if and only if the indices are equal. More precisely, suppose that
$$\delta^{i_1} \circ \cdots \delta^{i_r} = \delta^{i'_1} \circ \cdots \delta^{i'_{r'}}$$
as maps $[n] \to [m]$, and the sequence of indices are both strictly decreasing; denote this map by f for brevity. By the first step, the size of the fibers of f can be read off of the indices $i_\bullet$ or $i'_\bullet$ (an index is present exactly when not in $\operatorname{im} f$), so these sequences must be equal. ■

Remark 1.27. It follows that δ^i is the unique injection $[n] \to [n+1]$ omitting a given element of $[n+1]$.

Remark 1.28. The requirement that the indices are strictly decreasing is necessary for the uniqueness. Indeed, if $i \le j$, then $\delta^i \circ \delta^j$ avoids i and $j+1$, so it equals $\delta^{j+1} \circ \delta^i$.

Analogously, we have should handle surjective increasing maps.

Definition 1.29 (degeneracy maps)**.** Given some $j \in [n+1]$, we define the *degeneracy map* $\sigma^j\colon [n+1] \to [n]$ to be the surjection which hits j twice by sending the set $\{0, \ldots, j\}$ to itself and sending $\{j+1, \ldots, n+1\}$ to one less than each element.

Lemma 1.30. Every surjective increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite
$$f = \sigma^{j_1} \circ \cdots \circ \sigma^{j_r},$$
where $j_1 \ge j_2 \ge \cdots \ge j_r$.

Proof. The structure of this proof is similar to Lemma 1.26, but the technical core requires a couple modifications.

1. Given a decreasing sequence $j_1 \ge j_2 \ge \cdots \ge j_r$, we claim that the map
$$(\sigma^{j_1} \circ \cdots \circ \sigma^{j_r})\colon [n] \to [n-r]$$
has fiber over $k \in [n-r]$ of size equal to $1 + \#\{t : j_t = k\}$. We proceed by induction on r; for $r = 0$, the statement is vacuous. For the induction, we are given a decreasing sequence $j_1 \ge \cdots \ge j_{r+1}$. By induction, we know that the fiber of $\sigma := \sigma^{j_2} \circ \cdots \circ \sigma^{j_{r+1}}$ over k is $1 + \{2 \le t \le r+1 : j_t = k\}$.

   Now, $\sigma^{j_1} \circ \sigma$ has the same-size fibers over any $k < j_1$ as σ because σ^{j_1} preserves $\{0, \ldots, j_1\}$. For $k > j_1$, we note that the fibers of σ over each such k is 1 because $k > j_i$ for each i (by the induction), so $\sigma^{j_1} \circ \sigma$ also has fiber of size 1 over this k. Lastly, for $k = j_1$, we see that the fiber increases in size by 1 because σ^{j_1} sends $j_1 + 1$ (whose fiber has size 1 for σ) to j_1. This casework completes the proof.

2. We show that any surjective f is a composite of $\sigma^\bullet$s as given. Well, let J be the multi-subset of $[m]$ hit multiple times by f, counted with multiplicity, and we may arrange the elements of J as $\{j_1, \ldots, j_r\}$ in decreasing order. Then $\sigma^{j_1} \circ \cdots \circ \sigma^{j_r}$ and f have fibers of the same size by the previous step, so they are equal functions by Lemma 1.24.

3. We show that two composites of decreasing $\sigma^\bullet$s are equal if and only if the indices are equal. More precisely, suppose that
$$\sigma^{j_1} \circ \cdots \circ \sigma^{j_r} = \sigma^{j'_1} \circ \cdots \circ \sigma^{j'_{r'}}$$

as maps $[n] \to [m]$, and the sequences of indices are both decreasing; denote this map by f. Well, the fibers of f can be read off the indices $\{j_\bullet\}$ or $\{j'_\bullet\}$ by the first step, so these sequences must be equal. ■

Remark 1.31. As in Remark 1.27, we note that the requirement that the indices are decreasing is necessary for the uniqueness. Indeed, if $i < j$, then $\sigma^i \circ \sigma^j$ hits $j-1$ twice and i twice (counted with multiplicity), so $\sigma^i \circ \sigma^j = \sigma^{j-1} \circ \sigma^i$.

We are now ready to classify general maps.

Lemma 1.32. Every increasing map $f\colon [n] \to [m]$ can be written uniquely as a composite

$$f = (\delta^{i_1} \circ \cdots \circ \delta^{i_r}) \circ (\sigma^{j_1} \circ \cdots \sigma^{j_s}),$$

where $i_1 > \cdots > i_r$ and $j_1 \geq \cdots \geq j_s$.

Proof. The main point is to show that any increasing map f admits a unique decomposition as $\delta \circ \sigma$ where $\delta\colon [k] \to [m]$ is injective and $\sigma\colon [n] \to [k]$ is surjective. The existence and uniqueness of the required decomposition now follows by the existence and uniqueness of the decomposition $f = \delta \circ \sigma$ with Lemmas 1.26 and 1.30. For example, to get the uniqueness, if

$$\delta^{i_1} \circ \cdots \circ \delta^{i_r} \circ \sigma^{j_1} \circ \cdots \sigma^{j_s} = \delta^{i'_1} \circ \cdots \circ \delta^{i'_{r'}} \circ \sigma^{j'_1} \circ \cdots \sigma^{j'_{s'}},$$

then the composites of the $\delta^\bullet$s and of the $\sigma^\bullet$s must each be equal (because those are injections and surjections, respectively), and then the equalities of the indices follows from using Lemmas 1.26 and 1.30, respectively.

It remains to show the main claim. We show existence and uniqueness separately.

- Existence: note $\operatorname{im} f \subseteq [m]$ is some totally ordered subset, so we let its cardinality be $k+1$. By suitably ordering the elements of $\operatorname{im} f$, we receive a totally ordered bijection $[k] \to \operatorname{im} f$. Then we see that f decomposes into

$$\underbrace{[n] \xrightarrow{f} \operatorname{im} f \leftarrow [k]}_{\sigma} = \underbrace{[k] \to \operatorname{im} f \subseteq [m]}_{\delta},$$

  as required.

- Uniqueness: suppose we have two equal decompositions $f = \delta \circ \sigma = \delta' \circ \sigma'$ where $\sigma\colon [n] \to [k]$ and $\sigma'\colon [n] \to [k']$ and $\delta\colon [k] \to [m]$ and $\delta\colon [k'] \to [m]$. To begin, note that the injectivity of δ and δ' implies that $k+1$ and $k'+1$ are both the cardinality of $\operatorname{im} f$, so $k = k'$ follows. Now, because δ and δ' have the same image, and both are injective, it follows that all their fibers from $[m]$ have the same size (as either 0 or 1)! Thus, $\delta = \delta'$ follows from Lemma 1.24. The injectivity of δ now shows that $\delta \circ \sigma = \delta \circ \sigma'$ implies $\sigma = \sigma'$. ■

Remark 1.33. As in Remarks 1.27 and 1.31, we note that putting δs before σs is important for the uniqueness. Suppose we have some $\sigma^j \circ \delta^i$, and then we have the following cases.

- If $j > i$, then σ^j fixes $\{0, \ldots, i+1\}$, so $\sigma^j \circ \delta^i$ avoids i and hits j twice. This is the same as $\delta^i \circ \sigma^{j-1}$.
- If $j = i$ or $j = i-1$, then $\sigma^j \circ \delta^i$ fixes $\{0, \ldots, i-1\}$ throughout, and the elements at least i get $+1$ from δ^i and -1 from σ^j. Thus, $\sigma^j \circ \delta^i = \mathrm{id}$.
- If $j < i-1$, then $\sigma^j \circ \delta^i$ avoids $i-1$ (-1 from σ^j) and hits j twice. This is the same as $\delta^{i-1} \circ \sigma^j$.

Having access to generators of these maps and some relations between them allows us to provide a combinatorial definition of a simplicial set.

Definition 1.34. A *combinatorial simplicial set* is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ equipped with face maps $d_0, \ldots, d_n \colon X_n \to X_{n-1}$ and degeneracy maps $s_0, \ldots, s_n \colon X_n \to X_{n+1}$ (for each n) satisfying the following simplicial identities

$$\begin{cases} d_j d_i = d_i d_{j+1} & \text{if } i \le j, \\ s_j s_i = s_i s_{j-1} & \text{if } i < j, \end{cases} \qquad \text{and} \qquad \begin{cases} d_i s_j = s_{j-1} d_i & \text{if } i < j, \\ d_i s_j = \mathrm{id} & \text{if } i = j \text{ or } i = j+1, \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

A morphism $f \colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets is a function $f_n \colon X_n \to Y_n$ for each n commuting with the face and degeneracy maps; i.e., $f_{n-1} \circ d_n = d_n \circ f_n$ and $f_{n+1} \circ s_n = s_n \circ f_n$.

Remark 1.35. One can check that there is a category of combinatorial simplicial sets. In particular, the identity is given by $(\mathrm{id}_X)_n := \mathrm{id}_{X_n}$, and composition is defined by $(g \circ f)_n := g_n \circ f_n$ (which commutes with the face and degeneracy maps because g and f do).

Proposition 1.36. There is an isomorphism of categories from the category of simplicial sets to the category of combinatorial simplicial sets by sending $X \in \mathrm{sSet}$ to a combinatorial simplicial set given by

$$\begin{cases} X_n := X([n]), \\ d_\bullet := X(\delta^\bullet) & \text{for each } n \in \mathbb{N}, \\ s_\bullet := X(\sigma^\bullet) & \text{for each } n \in \mathbb{N}. \end{cases}$$

Proof. We run our many checks in sequence.

- To check that $X \in \mathrm{sSet}$ is sent to a combinatorial simplicial set $\{X_n\}$, we just need to check that the $d_\bullet$s and $s_\bullet$s satisfy the simplicial identities. This follows from the functoriality of X and Remarks 1.27, 1.31 and 1.33.
- We define $X \mapsto \{X_n\}$ on morphisms. Well, a functor $f \colon X \Rightarrow Y$ of simplicial sets defines maps $f_{[n]} \colon X([n]) \to Y([n])$, which we claim assembles into a morphism $f \colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets by $f_n := f_{[n]}$. To check this, we need to check compatibility with the face and degeneracy maps. Well, $f_{n-1} \circ d_n = d_n \circ f_n$ and $f_{n+1} \circ s_n = s_n \circ f_n$ follow by naturality of f because these amount to requiring
$$f_{[n-1]} \circ X(\delta^n) = Y(\delta^n) \circ f_{[n]} \qquad \text{and} \qquad f_{[n+1]} \circ X(\sigma^n) = Y(\sigma^n) \circ f_{[n]}.$$
- We show that $X \mapsto \{X_n\}$ is functorial. To begin, note $\mathrm{id} \colon X \Rightarrow X$ goes to the identity maps $\mathrm{id}_n \colon X_n \to X_n$. Then given $f \colon X \Rightarrow Y$ and $g \colon Y \Rightarrow Z$, we see that the composite $(g \circ f) \colon \{X_n\} \to \{Z_n\}$ is given by $(g \circ f)_n = (g \circ f)_{[n]} = g_{[n]} \circ f_{[n]} = g_n \circ f_n$, as required.
- We define a map from combinatorial simplicial sets back to simplicial sets. Well, given a combinatorial simplicial set $\{X_n\}$, we begin defining our functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ by $X([n]) := X_n$. On morphisms $f \colon [n] \to [m]$, we need to define some map $Xf \colon X_m \to X_n$. For this, we note that Lemma 1.32 allows us to write f uniquely as a composite
$$f = (\delta^{i_1} \circ \cdots \circ \delta^{i_r}) \circ (\sigma^{j_1} \circ \cdots \circ \sigma^{j_s}),$$
where $i_\bullet$ is strictly decreasing and $j_\bullet$ is decreasing. Thus, we define
$$Xf := (s_{j_s} \circ \cdots \circ s_{j_1}) \circ (d_{i_r} \circ \cdots \circ d_{i_1}).$$
For example, $f = \mathrm{id}_{[n]}$ is equal to the empty composite everywhere, so $X\mathrm{id}_{[n]} = \mathrm{id}_{X_n}$.

To complete our functoriality check, because any morphisms can be written as a composite of $\delta^\bullet$s and $\sigma^\bullet$s, it is enough to check functoriality for such morphisms. Namely, we have to check that

$$\begin{cases} X(\delta_i\delta_j) = X(\delta_j)X(\delta_i), \\ X(\sigma_i\sigma_j) = X(\sigma_j)X(\sigma_i), \\ X(\delta_i\sigma_j) = X(\sigma_j)X(\delta_i), \\ X(\sigma_j\delta_i) = X(\delta_i)X(\sigma_j). \end{cases}$$

For the first, this is by definition when $i > j$ and follows from the simplicial identities otherwise; the second is similar. The third is also automatic, and the last follows from the simplicial identities again.

- We define our map on morphisms. Well, given a morphism $F\colon \{X_n\} \to \{Y_n\}$ of combinatorial simplicial sets, we already have our component morphisms $F_n\colon X_n \to Y_n$ which will become our morphisms $F_{[n]}\colon X([n]) \to Y([n])$. It remains to check the naturality of $F\colon X \Rightarrow Y$. Well, let $f\colon [n] \to [m]$ be an increasing map, we should check that $Xf \circ F_m = F_n \circ Yf$. Because f can be written as a composite of $\delta^\bullet$s and $\sigma^\bullet$s (by Lemma 1.32), it is enough to check this for $f \in \{\delta^\bullet, \sigma^\bullet\}$, which now follows because F started its life as a morphism of combinatorial simplicial sets.

- We show that $\{X_n\} \mapsto X$ is functorial. To begin, note $\mathrm{id}\colon \{X_n\} \to \{X_n\}$ goes to the identity maps $\mathrm{id}_{[n]}\colon X([n]) \to X([n])$. Then given $f\colon \{X_n\} \to \{Y_n\}$ and $g\colon \{Y_n\} \to \{Z_n\}$, we see that the composite $(g \circ f)\colon X \Rightarrow Z$ is given by $(g\circ f)_{[n]} = (g \circ f)_n = g_n \circ f_n = g_{[n]} \circ f_{[n]}$.

- We complete the check that we have defined inverse equivalences. For concreteness, let $A\colon \{X_n\} \mapsto X$ and $B\colon X \mapsto \{X_n\}$ be our functors.

  Let's check $BA = \mathrm{id}$. On an object $\{X_n\}$, we see that $BA\{X_n\}$ has $(BA\{X_n\})_n = A\{X_n\}([n]) = X_n$ and simplicial maps d_i and s_j given by $A(\{X_n\})(\delta^i)$ and $A(\{X_n\})(\sigma^j)$ which are d_i and s_j, respectively. On morphisms, we see $BAf = f$ because $(BAf)_n = Af_{[n]} = f_n$ for each n.

  Lastly, let's check $AB = \mathrm{id}$. On an object X, we analogously see that $ABX([n]) = BX_n = X([n])$; further, to check that $ABX(f) = X(f)$ for an increasing map f, we note that Lemma 1.32 reduces this check to $\delta^\bullet$ and $\sigma^\bullet$ by functoriality, which similarly follows by construction of A and B (which turns $\delta^\bullet$s and $\sigma^\bullet$s to $d_\bullet$s and $s_\bullet$s and vice versa). Lastly, on morphisms, we see $(ABf)_{[n]} = Bf_n = f_{[n]}$ for each n. ■

Remark 1.37. In light of Proposition 1.36, we will occasionally identify simplicial sets and combinatorial simplicial sets. In particular, the term "combinatorial simplicial set" will not appear again.

Remark 1.38. There is also a notion of "semi-simplicial set" where we remove all the data associated to the $s_\bullet$s. This notion is sufficient to work with homology, but because we are now homotopy theorists, we work with simplicial sets.

1.2 September 9

The first problem set will be posted in about a day.

1.2.1 Some Simplicial Sets

We are now allowed to remove the absolute value bars from our Δ^n.

Definition 1.39 (simplex)**.** For each $n \geq 0$, we define the n-*simplex* Δ^n as the simplicial set $よ([n])$.

Remark 1.40. As in Remark A.18, we see $\Delta^n([m])$ is $\operatorname{Mor}_{\mathrm{sSet}}([m], \Delta^n)$, which is $よ_{[n]}([m])$ or $\operatorname{Fun}([m],[n])$. Of course, this is just the collection of order-preserving maps $[m] \to [n]$. Then given an increasing map $f\colon [m] \to [m']$, we see that $\Delta^n(f)\colon \Delta^n([m']) \to \Delta^n([m])$ is

$$よ_{[n]}(f)\colon \operatorname{Mor}_{\mathrm{sSet}}([m'], \Delta^n) \to \operatorname{Mor}_{\mathrm{sSet}}([m], \Delta^n),$$

which of course is just $(- \circ f)$. All the prior identifications chain to show that we are still looking at $(- \circ f)$ on the level of order-preserving maps.

Example 1.41. We see $(\Delta^2)_0$ has three elements, and $(\Delta^2)_1$ has six elements. Here are three of the elements of $(\Delta^2)_1$.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

Remark 1.42. On the other hand, for a simplicial set X, Remark A.18 tells us that $\operatorname{Mor}_{\mathrm{sSet}}(\Delta^n, X)$ is in bijection with X_n, and this bijection takes $\varphi\colon \Delta^n \to X$ to $\varphi_{[n]}(\mathrm{id}_{[n]}) \in X_n$. We even know that the inverse map takes some $x \in X$ and outputs a natural transformation $\Delta^n \to X$ defined by sending $f\colon [m] \to [n]$ in $\Delta^n(m)$ to $Xf(x) \in X_m$.

Example 1.43. The maps $\delta^i\colon [n-1] \to [n]$ and $\sigma^i\colon [n+1] \to [n]$ induce maps $よ\delta^i\colon \Delta^{n-1} \to \Delta^n$ and $よ\sigma^i\colon \Delta^{n+1} \to \Delta^n$ given by $(- \circ \delta^i)$ and $(- \circ \sigma^i)$, respectively. We continue to label our face maps by $d_i\colon \Delta^{n-1} \to \Delta^n$ and degeneracy maps by $s_i\colon \Delta^{n+1} \to \Delta^n$.

Here is the sort of thing that this language allows us to prove.

Lemma 1.44. A map $f\colon X \to Y$ of simplicial sets is monic if and only if $f_n\colon X_n \to Y_n$ is injective for all $n \geq 0$.

Proof. In one direction, if f_n is injective for all $n \geq 0$, then one can see directly that f is monic: given maps $g_1, g_2\colon Y \to Z$, we see that $f \circ g_1 = f \circ g_2$ implies that $g_1 = g_2$, as we can see by passing to n-simplices for each n.

In the reverse direction, suppose that f is monic. Then f must induce injections

$$\operatorname{Mor}_{\mathrm{sSet}}(\Delta_n, X) \to \operatorname{Mor}_{\mathrm{sSet}}(\Delta_n, Y).$$

However, as described in Remark 1.42, this is (naturally) isomorphic to the map $f_n\colon X_n \to Y_n$, which is therefore also injective! ■

Here are two more important simplicial sets.

Definition 1.45 (boundary)**.** For each $n \geq 0$, we define the *boundary* $\partial\Delta^n \in \mathrm{sSet}$ to be the subfunctor of Δ^n with $\partial\Delta^n(m)$ given by the non-surjective maps $[m] \to [n]$.

Remark 1.46. There are canonical inclusions $\partial\Delta^n(m) \subseteq \Delta^n(m)$ for each m, which we claim upgrades into an embedding of simplicial sets. Well, for each $g\colon [m] \to [m']$ and $f \in \partial\Delta^n(m)$, we see that $(f \circ g)$ continues to not be surjective, so $(f \circ g) \in \partial\Delta^n(m')$, so the claim follows from Remark A.16.

Definition 1.47 (horn)**.** For each $i \in [n]$, we define the *ith horn* $\Lambda_i^n \in \mathrm{sSet}$ to be the subfunctor of Δ^n with Λ_n^i given by the maps $[m] \to [n]$ which avoid an element not equal to i. We say that Λ_i^n is an *inner horn* if and only if $0 < i < n$; otherwise, Λ_i^n is an *outer horn*.

Remark 1.48. As in Remark 1.46, there are canonical inclusions $\Lambda_n^i(m) \subseteq \partial\Delta^n(m)$ for each i and n and m with $0 \le i \le n$. We check that this upgrades to an inclusion of simplicial sets with Remark A.16: we have to check that any maps $g\colon [m] \to [m']$ and $f \in \partial\Delta^n(m)$ has $(f \circ g) \in \partial\Delta^n(m)$, which is true because $i \notin \operatorname{im} f$.

Example 1.49. Intuitively, Λ_i^n deletes the face opposite i. For example, here is Λ_0^2.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd}$$

One can similarly draw Λ_1^2 (which omits $0 \to 2$) and Λ_2^2 (which omits $0 \to 1$).

1.2.2 Dimension

We take a moment to introduce a complexity measure of simplicial sets, which will be helpful in the sequel when we are making inductive arguments.

Definition 1.50 (dimension)**.** Fix a simplicial set X. Then X has *dimension* d if and only if d is the smallest nonnegative integer for which all simplices in X_k are in the image of the degeneracy maps for $k > d$. (We may say that such a simplex is *degenerate*.) If there is no such nonnegative integer d, we say that X is infinite-dimensional. We may write the dimension as $\dim X$.

Remark 1.51. Given a morphism $f\colon X \to Y$, if $x \in X_n$ is degenerate, then so is $f(x) \in Y_n$. Indeed, we are provided with $x' \in X_{n-1}$ for which $x = f(x')$, so the claim follows from the commutativity of the following diagram.

$$\begin{tikzcd} X_{n-1} \arrow[r, "f"] \arrow[d, "s_i"'] & Y_{n-1} \arrow[d, "s_i"'] & & x' \arrow[r, mapsto] \arrow[d, mapsto] & f(x') \arrow[d, mapsto] \\ X_n \arrow[r, "f"] & Y_n & & x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

Example 1.52. We claim that $\dim \Delta^n = n$. In fact, we will show that a map $[m] \to [n]$ in $\Delta^n(m)$ is non-degenerate if and only if it is injective, which completes the proof of the claim ($\mathrm{id}_{[n]}$ is injective, but nothing will be injective for $m > n$). In one direction, for any $m \ge 0$, the degeneracy map $s_i\colon X_m \to X_{m+1}$ sends an increasing map $[m] \to [n]$ to the composite $[m+1] \twoheadrightarrow [m] \to [n]$; in particular, the composite necessarily fails to be injective. Conversely, any non-injective map sends two consecutive inputs i and $i+1$ to the same output because it is still increasing. Thus, $[k] \to [n]$ factors through $\sigma_i\colon [k] \to [k-1]$ and is therefore degenerate!

Example 1.53. We claim that $\dim \Lambda_i^n = n - 1$. As in Example 1.52, it is enough to show that a map $[m] \to [n]$ in $\Lambda_i^n(m)$ is non-degenerate if and only if it is injective, which completes the proof of the claim (δ_i is injective, but nothing will be injective for $m \ge n$). The arguments are now exactly the same as in Example 1.52, with the caveat that the converse argument must ensure that we avoid an element away from i in the image at all times.

Intuitively, we expect that a simplicial set X can be (canonically) built out of its k-simplices where k varies over all nonnegative integers bounded by $\dim X$. Here, "built" means that we are gluing via colimits.

Lemma 1.54. Let $F\colon \mathcal{I} \to \mathrm{sSet}$ be a functor with colimit X. Then

$$\dim A \le \sup_{i\in\mathcal{I}} F(i).$$

Proof. Let the given supremum be d. We have to show that $d \le n$ implies that $\dim A \le n$ for any n. To show that $\dim A \le n$, we have to show that any element of A_k for $k > n$ is degenerate. Because colimits are computed pointwise, we know that

$$A_k = \operatorname*{colim}_{i\in I} F(i),$$

so the construction of colimits in Set implies that a is in the image of one of the canonical maps $F(i)_k \to A_k$. But all elements in $F(i)_k$ are degenerate, and the fact that the elements in $F(i)_k$ are degenerate implies that the elements of A_k by Remark 1.51. ■

Now, we will show that simplicial sets can be built out of their k-simplices for $k \le \dim X$ in steps. We will require the following technical lemma.

Lemma 1.55. Fix some map $\sigma\colon \Delta^n \to X$ of simplicial sets. Then σ factors uniquely into a composite

$$\Delta^n \twoheadrightarrow \Delta^m \hookrightarrow X,$$

where the map $\Delta^n \to \Delta^m$ is induced by a surjection $[n] \twoheadrightarrow [m]$ and the map $\Delta^m \hookrightarrow X$ maps to a non-degenerate m-simplex.

Proof. It is easier to show the uniqueness of the factorization: simply set m to be the smallest positive integer for which σ factors into

$$\Delta^n \overset{\pi}{\twoheadrightarrow} \Delta^m \overset{\iota}{\hookrightarrow} X.$$

We have now two checks.

- Surjective: suppose for the sake of contradiction, suppose that the map $[n] \to [m]$ induced by π fails to be surjective. Then we can factor the map $[n] \to [m]$ into some composite $[n] \to [m-1] \to [m]$, so π can be factored into a composite $\Delta^n \to \Delta^{m-1} \to \Delta^m$, which violates the minimality of m.
- Non-degenerate: suppose that the image of ι is not a non-degenerate m-simplex. Then x is in the image of some degeneracy map $X_{m-1} \to X_m$; say $x = s_i(x')$. Thus, the commutative diagram in Remark 1.51 shows that ι factors as some composite $\Delta^{m-1} \overset{s_i}{\to} \Delta^m \to X$, which again violates the minimality of m.

It remains to show the uniqueness of the given factorization. Well, suppose that we have another such factorization

$$\Delta^n \overset{\pi'}{\twoheadrightarrow} \Delta^{m'} \overset{\iota'}{\hookrightarrow} X$$

such that π' is induced by a surjection $[n] \twoheadrightarrow [m']$, and ι' maps $[m']$ to a non-degenerate m'-simplex.

In fact, the main claim will be that π factors through π'. Let's explain why this is enough. Say that $\pi = \pi''\pi'$, where $\pi''\colon \Delta^{m'} \to \Delta^m$ is some map. We will end up showing that π'' is the identity. Now, we choose a section α' of π' (so that $\pi'\alpha' = \mathrm{id}_{\Delta^{m'}}$), and $\iota\pi''\pi' = \iota'\pi'$ implies that $\iota\pi''\pi'\alpha = \iota'\pi'\alpha$ and so

$$\iota\pi'' = \iota'.$$

We now see that having π'' is the identity implies that $m = m'$ and $\pi = \pi'$ and $\iota = \iota'$, so we focus our efforts on showing π'' is the identity. Because π is induced by a surjective map $[n] \twoheadrightarrow [m]$, we conclude that π'' is induced by a surjective map $[m'] \twoheadrightarrow [m]$. It remains to show that $m \le m'$, which will then force π'' to be induced by the identity map. Well, if π'' fails to be injective, then π'' has degenerate image, so $\iota\pi''$ has degenerate image, so ι' will have degenerate image! This contradicts the construction of ι', so we are done.

We now turn our attention to showing the main claim, which is some hands-on combinatorics. To show that π factors through π', we will identify these maps with their induced maps $[n] \twoheadrightarrow [m]$ and $[n] \twoheadrightarrow [m']$. Then π factors through π' if and only if $\pi'(i) = \pi'(j)$ implies $\pi(i) = \pi(j)$. Assume this is not the case for the sake of contradiction. In other words, we have some i and j for which $\pi'(i) = \pi'(j)$ while $\pi(i) \neq \pi(j)$; note that i and j must be distinct. Now, find a section $\alpha\colon [m] \hookrightarrow [n]$ of π whose image includes both i and j, which is possible because $\pi(i) \neq \pi(j)$. Then $\iota\pi = \iota'\pi'$ implies that $\iota'\pi'\alpha = \iota$, but $\pi'\alpha$ is not injective (because $\pi'(i) = \pi'(j)$), so $\iota'\pi'\alpha$ does not map to a non-degenerate simplex, which contradicts the construction of ι. ■

Proposition 1.56. Fix a simplicial set X and a nonnegative integer d. Then the following are equivalent.

(a) $\dim X \leq d$.

(b) X is the colimit of some functor $F\colon \mathcal{I} \to \mathrm{sSet}$, where $\dim F(i) \leq d$ for all i.

(c) X is the colimit of some functor $F\colon \mathcal{I} \to \mathrm{sSet}$, and each i has some $k \leq d$ for which $F(i) = \Delta^k$.

(d) Let $\mathcal{I}$ be the category of pairs $([k], x)$, where $k \leq d$ and $x \in X_k$ is some map, where morphisms are increasing maps $[k] \to [k']$ for which the induced map $X_k \to X_{k'}$ sends x to x'. Then X is the colimit of the natural functor $F\colon \mathcal{I} \to \mathrm{sSet}$ given by $F(([k], x)) = \Delta^k$.

Proof. Note that (d) implies (c) with no effort, and (c) implies (b) with no effort as soon as we recall Example 1.52. Similarly, (b) implies (a) is exactly Lemma 1.54.

It remains to show that (a) implies (d), for which we will use Lemma 1.55. Quickly, we remark that the natural functor $F\colon \mathcal{I} \to \mathrm{sSet}$ simply sends the pair $([k], x)$ to the element $x \in X_k$. Note that there is a natural map

$$f\colon \operatorname*{colim}_{i \in \mathcal{I}} F(i) \to X$$

induced by the maps $F(([k], x)) \to X$ defined by $x\colon \Delta^k \to X_k$. (Indeed, we have induced a map from the colimit because the maps $F(i) \to X$ automatically commute with the internal maps $F(i) \to F(j)$ for each $i \to j$ in $\mathcal{I}$ by the definition of the category $\mathcal{I}$.) We would like to show that f is an isomorphism.

It is enough to show that f is injective and surjective; then the inverse can be constructed on the level of the simplices by hand. Quickly, we show that f is surjective. This uses the existence assertion in Lemma 1.55, which shows that $\operatorname{im} f$ contains every non-degenerate m-simplex of X for each $m \leq d$. We would like to show that these simplices generate X (via the degeneracy maps). Well, for any $x \in X_n$, either x is non-degenerate or in the image of a degeneracy map. Continuing inductively, we conclude that x equals $\sigma(x')$ where $x' \in X_m$ is non-degenerate, and σ is some composite of degeneracy maps. Note then that $m \leq d$ because $\dim X \leq d$, so $x' \in \operatorname{im} f$. Thus, $x \in \operatorname{im} f$ as well because f commutes with the degeneracy maps!

We will now show that f is injective, which will complete the proof. We will use the uniqueness assertion in lemma 1.55. Well, suppose that we have two elements y and y' in the colimit for which $f(y) = f(y')$; we would like to show that $y = y'$. We may assume that y and y' are both ℓ-simplices for the same ℓ because they both must map to the same graded piece of X.

We now apply some reductions to simplify the presentation of y and y'. Set $x \coloneqq f(y) = f(y')$ for brevity.

- Because colimits are computed pointwise, we see that y arises as an ℓ-simplex of some $F(([k], x)) = \Delta^k$. This means that we have an increasing map $[\ell] \to [k]$, and $f(y)$ is induced by the map $\Delta^\ell \to \Delta^k \to X$. By factoring $[\ell] \to [k]$ into a surjection and an injection via Lemma 1.32, we may as well assume that the map $[\ell] \to [k]$ is a surjection (simply by changing the pair $([k], x)$). One similarly finds y' in the image of some $F([k'], x')$, where the promised map $[\ell'] \to [k']$ is surjective.

- We reduce to the case where x is non-degenerate. Note $x\colon \Delta^k \to X$ factors as

$$\Delta^k \twoheadrightarrow \Delta^m \hookrightarrow X,$$

where $\Delta^k \twoheadrightarrow \Delta^m$ is induced by a surjection, and $\Delta^m \to X$ goes to a non-degenerate m-simplex. We can now replace k with m so that $F(([k], x))$ also has x non-degenerate. One can do something similar for y' to assume that x' is non-degenerate.

Now, the moral is that $f(y) = f(y')$ implies that the composites $\Delta^\ell \twoheadrightarrow \Delta^k \hookrightarrow X$ and $\Delta^\ell \twoheadrightarrow \Delta^{k'} \hookrightarrow X$ produces the same ℓ-simplex, but this factorization is unique by Lemma 1.55. This shows that $([k], x) = ([k'], x')$, and in fact we see that $y = y'$ because y and y' amount to the data of the map $\Delta^\ell \to \Delta^k$. ■

The category $\mathcal{I}$ may look like it admits a rather complicated definition, but it can be used effectively in practice. Indeed, it is essentially a combinatorial gadget which tells us how to glue simplices together.

Example 1.57. We write Λ^2_1 as a colimit of Δ^0s and Δ^1s. By Example 1.53, the non-degenerate m-simplices are given by injective maps $[m] \to [n]$ avoiding an element away from 1, so of course $m \in \{0, 1\}$. In particular, our 0-simplices are 0, 1, and 2; our 1-simplices are $0 \le 1$ and $1 \le 2$. Keeping track of the maps between these simplices, we see that Λ^2_1 is the colimit of the following diagram, where we are identifying an m-simplex of Λ^2_1 with Δ^m.

$$\begin{tikzcd}
& (0 \le 1) \arrow[dl, shift left, "\delta^1"] \arrow[dr, shift left, "\delta^0"] & & (1 \le 2) \arrow[dl, shift left, "\delta^2"] \arrow[dr, shift left, "\delta^1"] & \\
(0) \arrow[ur, shift left, "\sigma^0"] & & (1) \arrow[ul, shift left, "\sigma^1"] \arrow[ur, shift left, "\sigma^1"] & & (2) \arrow[ul, shift left, "\sigma^2"]
\end{tikzcd}$$

In this way, we see that a map $\Lambda^2_1 \to X$ amounts to the data of three objects $e_0, e_1, e_2 \in X$ and two morphisms $f_{01}, f_{12} \in X$ such that $d_1 f_{01} = e_0$, $d_0 f_{01} = d_1 f_{12} = e_1$, and $d_0 f_{12} = e_2$. In fact, one can throw away the data of e_0 and e_2 because it is determined by f_{01} and f_{12} by the face maps.

1.2.3 More on Nerves

It is worthwhile to explain nerves a little more. In this section, we will characterize which simplicial sets appear as nerves of categories.

Exercise 1.58. Fix a category $\mathcal{C}$. We work our $N(\mathcal{C})_i$ for $i \in \{0, 1, 2\}$.

Proof. Here we go.

- We see $N(\mathcal{C})_0$ consists of functors from the category $\{\bullet\}$, which are just objects of $\mathcal{C}$.
- Similarly, $N(\mathcal{C})_1$ consists of functors from the category $\{\bullet \to \bullet\}$ to $\mathcal{C}$, which are just morphisms of $\mathcal{C}$.
- Lastly, we note $N(\mathcal{C})_2$ consists of functors from the category $\{\bullet \to \bullet \to \bullet\}$ to $\mathcal{C}$, which amounts to the data of a diagram

  $$\begin{tikzcd}
  \bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] \\
  & \bullet
  \end{tikzcd}
  \qquad \Rightarrow \qquad
  \begin{tikzcd}
  c \arrow[r, "f"] \arrow[dr, "(g \circ f)"'] & c' \arrow[d, "g"] \\
  & c''
  \end{tikzcd}$$

  so that the nerve is required to know something about composition!

We can also describe N on some easy morphisms. For example, there is a unique map $[n] \to [0]$ for any $n \ge 0$: it sends all objects of $[n]$ to 0 and all morphisms to id_0. For example, the corresponding map $N(\mathcal{C})_0 \to N(\mathcal{C})_1$ needs to send the object $c \in \mathcal{C}$, which corresponds to the constant functor $[0] \to \mathcal{C}$, to the identity map $\mathrm{id}_c \colon c \to c$, which indeed is the image of the morphism in $[1]$ when passed through the composite $[1] \to [0] \xrightarrow{c} \mathcal{C}$. ■

Remark 1.59. More generally, an element of $N(\mathcal{C})_m$ is a functor $F\colon [m] \to \mathcal{C}$. Because $[m]$ is a totally ordered set, this amounts to having objects $\{F(0), \ldots, F(m)\}$ and morphisms $F(i) \to F(i+1)$ for each $i \in [m-1]$; from here, we see that the morphism $i \to i+j$ equals the composite $i \to i+1 \to \cdots \to i+j$ and thus goes to the composite

$$F(i) \to F(i+1) \to \cdots \to F(i+j)$$

by functoriality. The construction of these maps provides functoriality automatically. (We should send the identity maps to identity maps, of course.)

Remark 1.60. Given an element $F \in N(\mathcal{C})_m$, it is possible to use naturality to extract out the objects $F(i)$ and morphisms $F(i) \to F(j)$ (where $i \le j$).

- There is a morphism $\varepsilon_i\colon [0] \to [m]$ defined by $0 \mapsto i$, which then induces a map $N(\mathcal{C})\varepsilon_i\colon N(\mathcal{C})_m \to N(\mathcal{C})_0$ by functoriality. This map sends F to $(F \circ \varepsilon_i)\colon [0] \to \mathcal{C}$, which is the functor which picks out the object $F(i)$.
- There is a morphism $\varepsilon_{ij}\colon [1] \to [m]$ defined by $0 \mapsto i$ and $1 \mapsto j$, which then induces a map $N(\mathcal{C})\varepsilon_{ij}\colon N(\mathcal{C})_m \to N(\mathcal{C})_1$ by functoriality. This map sends F to $(F \circ \varepsilon_{ij})\colon [1] \to \mathcal{C}$, which is the functor which picks out the morphism $F(i) \to F(j)$.

We will get some utility out of the following lemmas. Roughly speaking, the point is that a map to a nerve is determined by what it does on the level of morphisms.

Lemma 1.61. Fix a category $\mathcal{C}$. Suppose that we have a simplicial set X and two maps $\varphi, \psi\colon X \to N(\mathcal{C})$. If $\varphi_0 = \psi_0$ and $\varphi_1 = \psi_1$, then $\varphi = \psi$.

Proof. The main point is to use Remark 1.60. We need to show that $\varphi_m = \psi_m$ for all m. Note that there is nothing to do for $m \in \{0, 1\}$, so we may assume that $m \ge 2$. Thus, we begin by choosing some $x \in X_m$. We claim that the value of $\varphi_m(x)$ only depends on $\varphi_0(X\varepsilon_i(x))$s and $\varphi_1(X\varepsilon_{ij}(x))$s; an analogous claim holds for ψ by symmetry, so the proof will be complete.

Well, as in Remark 1.59, $\varphi_m(x)$ amounts to the data of some chain

$$c_0 \to \cdots \to c_m$$

of morphisms in $\mathcal{C}$. As explained in Remark 1.60, we see that $c_i = N(\mathcal{C})\varepsilon_i(\varphi_m(x))$ for each i, which is simply $\varphi_1(X\varepsilon_i(x))$ by naturality. Similarly, the map $c_i \to c_j$ for $i \le j$ is $N(\mathcal{C})\varepsilon_{ij}(\varphi_m(x))$, which is simply $\varphi_1(X\varepsilon_{ij}(x))$ by naturality again! ■

We even remark that we have the following existence result.

Lemma 1.62. Fix a category $\mathcal{C}$ along with $n+1$ objects $\{c_0, \ldots, c_n\}$ and morphisms $f_{i,i+1}\colon c_i \to c_{i+1}$ for each i. Suppose we have a simplicial subset $S \subseteq \Delta^n$ such that S_1 contains all maps $\varepsilon_i\colon [1] \to [n]$ of the form $x \mapsto (i+x)$ for each $0 \le i < n$. Then there is a map $\varphi\colon S \to N(\mathcal{C})$ for which $\varphi_0(x) = c_x$ for each x and $\varphi_1(\varepsilon_i) = f_{i,i+1}$ for each i.

Proof. Let's begin with existence so that we know what we are expecting. For the existence argument, we may as well assume that $S = \Delta^n$ because the problem only becomes harder with a larger simplicial set. With Remark 1.59 in mind, we define $f_{i,i+j}$ for any $j \ge 0$ as being the composite $f_{i+j-1,i+j} \circ \cdots \circ f_{i,i+1}$. Then we send any increasing map $g\colon [m] \to [n]$ in $\Delta^n(m)$ to the functor $[m] \to \mathcal{C}$ defined by

$$c_{g(0)} \to c_{g(1)} \to \cdots \to c_{g(m)},$$

where the intermediate morphisms are $f_{g(i),g(i+1)}\colon c_{g(i)} \to c_{g(i+1)}$. To show that this is natural, we choose a map $h\colon [m] \to [m']$ and observe that the diagram

$$\begin{tikzcd} \Delta^n(m') \arrow[r] \arrow[d, "(-\circ h)"'] & \operatorname{Fun}([m'],\mathcal{C}) \arrow[d, "(-\circ h)"] \\ \Delta^n(m) \arrow[r] & \operatorname{Fun}([m],\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto] \arrow[d, mapsto] & c_{g(0)} \to \cdots \to c_{g(m')} \arrow[d, mapsto] \\ (g \circ h) \arrow[r, mapsto] & c_{g(h(0))} \to \cdots \to c_{g(h(m))} \end{tikzcd}$$

commutes. ■

Non-Example 1.63. One does not expect any map $\Lambda_0^2 \to N(\mathcal{C})$ to always extend to Δ^2. Indeed, Λ_0^2 only has the maps $0 \to 1$ and $0 \to 2$, but there is no obvious way to then produce a map $1 \to 2$ in the nerve!

Lemma 1.64. The functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful.

Proof. We will show this directly.

- Faithful: given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ such that $NF = NG$, we need to check that $F = G$. Well, for any object $c \in \mathcal{C}$, we see that $c \in N(\mathcal{C})_0$, and by construction $(NF)_0(c) = Fc$ and $(NG)_0(c) = Gc$. (Explicitly, $(NF)_0$ is the composite functor $[0] \xrightarrow{c} \mathcal{C} \to \mathcal{D}$, which simply picks out the object Fc.) Similarly, for any morphism $f\colon c \to c'$, we note that $f \in N(\mathcal{C})_1$, whereupon we find that the construction of the nerve has $NF(f) = Ff$ and $NG(f) = Gf$. (Again, one explicitly has that $(NF)_1$ is the composite functor $[1] \xrightarrow{f} \mathcal{C} \to \mathcal{D}$.)

- Full: fix a map $\varphi\colon N\mathcal{C} \to N\mathcal{D}$ of simplicial sets, and we need to go back and define a functor $F\colon \mathcal{C} \to \mathcal{D}$. As in the previous point, we see that $\varphi_0\colon (N\mathcal{C})_0 \to (N\mathcal{D})_0$ is a map of objects, so we define $Fc := \varphi_0(c)$ for all $c \in \mathcal{C}$. Similarly, we see that $\varphi_1\colon (N\mathcal{C})_1 \to (N\mathcal{D})_1$ is a map of morphisms, so we define $Ff := \varphi_1(f)$ for all morphisms $f\colon c \to c'$.

  It remains to check that these data assemble into a functor F and that $NF = \varphi$. For example, as in Exercise 1.58, the identity map $\mathrm{id}_c\colon c \to c$ is the image of c along the canonical map $N(\mathcal{C})_0 \to N(\mathcal{C})_1$. Because φ is a natural transformation, we see $\varphi_1(\mathrm{id}_c)$ must then be the image of $\varphi_0(c)$ along the canonical map $N(\mathcal{D})_0 \to N(\mathcal{D})_1$, which of course is $\mathrm{id}_{\varphi_0(c)}$; we conclude $F\mathrm{id}_c = \mathrm{id}_{Fc}$.

  Next, we need to check associativity. Choose maps $f_{01}\colon c_0 \to c_1$ and $f_{12}\colon c_1 \to c_2$, and we need to check $F(f_{12} \circ f_{01}) = Ff_{12} \circ Ff_{01}$; set $f_{02}\colon c_0 \to c_2$ for brevity. Well, consider the functor $G\colon [2] \to \mathcal{C}$ given by $c_0 \to c_1 \to c_2$ (as in Remark 1.59); this will go to some element $\varphi_2(G)\colon [2] \to \mathcal{D}$ which we would like to describe. This is a diagram of the form $\bullet \to \bullet \to \bullet$, and as described in Remark 1.60, we can extract out the ith object as $N(\mathcal{D})\varepsilon_i(\varphi_2(G))$ and the morphism between the ith and jth object as $N(\mathcal{D})\varepsilon_{ij}(\varphi_2(G))$. By naturality, we see that $N(\mathcal{D})\varepsilon_i(\varphi_2(G)) = \varphi_1(c_i) = Fc_i$ for each i. Similarly, we see and that $N(\mathcal{D})\varepsilon_{ij}(\varphi_2(G)) = \varphi(f_{ij}) = Ff_{ij}$ for each $i \le j$. In total, $\varphi_2(G)$ is a diagram of the form

$$\begin{tikzcd} Fc_0 \arrow[r, "Ff_{01}"] \arrow[dr, "Ff_{02}"'] & Fc_1 \arrow[d, "Ff_{12}"] \\ & Fc_2 \end{tikzcd}$$

  thereby completing the argument because the triangle must commute.

  Lastly, we have to check that $NF = \varphi$. But these are equal in degrees 0 and 1 by construction, so we are done by Lemma 1.61. ■

These horns allow us to state a special property of nerves.

Proposition 1.65. Fix a category $\mathcal{C}$. Then any map $\Lambda_i^n \to N(\mathcal{C})$ from an inner horn Λ_i^n extends uniquely to a map $\Delta^n \to N(\mathcal{C})$, as in the following diagram.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed, "!"] \end{tikzcd}$$

Proof. If $n \geq 3$, then any map $[1] \to [n]$ avoids at least 2 elements, so $(\Lambda_i^n)_0 = (\Delta^n)_0$ and $(\Lambda_i^n)_1 = (\Delta^n)_1$. Thus, uniqueness follows from Lemma 1.61, and existence follows from Lemma 1.62. We note that the same argument even works for $n = 2$ because we are only worried about the inner horn Λ_1^2, which still has the morphisms $0 \to 1$ and $1 \to 2$. Lastly, there are no inner horns for $n \in \{0, 1\}$, so the statement is vacuous, and we are done. ■

Proposition 1.66. Fix a simplicial set N. Suppose that every map $\Lambda_i^n \to N$ from an inner horn extends uniquely to a map $\Delta^n \to N$. Then there is a category $\mathcal{C}$ such that $N \cong N(\mathcal{C})$.

Proof. We proceed in steps.

1. We define the objects and morphisms of the category $\mathcal{C}$. Indeed, simply take the objects to be N_0 and the collection of all morphisms to be N_1. More specifically, for any $x, y \in \mathcal{C}$, we use Remark 1.60 to motivate the definition

   $$\operatorname{Mor}_{\mathcal{C}}(x, y) := \{f \in N_1 : N\varepsilon_0(f) = x \text{ and } N\varepsilon_1(f) = y\}.$$

   For example, the canonical map $\sigma_0 \colon [1] \to [0]$ induces a special morphism $s_0(x)$ for each $x \in N_0$ such that $N\varepsilon_0(s_0(x)) = N\varepsilon_1(s_0(x)) = x$. Accordingly, we define $\mathrm{id}_x := s_0(x)$.

2. We define composition in the category of $\mathcal{C}$. Well, suppose we have two morphisms $f_{01} \colon c_0 \to c_1$ and $f_{12} \colon c_1 \to c_2$ which we would like to compose.

   The point is that these data assemble into a map $\Lambda_1^2 \to N$ which maps $i \mapsto c_i$ in degree 0 and maps $(i \to j)$ to f_{ij} (for $(i, j) \in \{(0, 1), (1, 2)\}$) in degree 1; this in fact gives a map $\Lambda_2^1 \to N$ by Example 1.57. Thus, the hypothesis on N gives us a unique $f_{12} \odot f_{01} \in N$ such that $d_2(f_{12} \odot f_{01}) = f_{01}$ and $d_0(f_{12} \odot f_{01}) = f_{12}$, so we define

   $$f_{12} \circ f_{01} := d_1(f_{12} \odot f_{01}).$$

   By construction, we see that $(f_{12} \circ f_{01}) \colon c_0 \to c_2$.

3. We check that composition is unital. Well, start with some map $f \colon c_0 \to c_1$, and we need to show that $f \circ \mathrm{id}_{c_0} = f$ and $\mathrm{id}_{c_1} \circ f = f$. We will content ourselves with showing that $f \circ \mathrm{id}_{c_0} = f$ because the argument is symmetric for the other identity. For this, we claim that $s_0(f) = (f \odot \mathrm{id}_{c_0})$, which will complete the proof because then $d_1(s_0(f)) = f$ by the simplicial identities, thereby implying that $f \circ \mathrm{id}_{c_0} = f$. Now, to check that $s_0(f) = (f \odot \mathrm{id}_{c_0})$, we use the uniqueness of our lifting: we must check that $d_2(s_0(f)) = \mathrm{id}_{c_0}$ and $d_0(s_0(f)) = f$, which both follow from the simplicial identities.

4. We check that composition is associative. This will be a little technical because it requires us to work with inner horns of Δ^3. We are given three morphisms $f_{01} \colon c_0 \to c_1$ and $f_{12} \colon c_1 \to c_2$ and $f_{23} \colon c_2 \to c_3$, and we want to check that $f_{23} \circ (f_{12} \circ f_{01}) = (f_{23} \circ f_{12}) \circ f_{01}$. We go ahead and set $f_{02} := f_{12} \circ f_{01}$ and $f_{13} := f_{23} \circ f_{12}$.

   Next, we build some 2-simplices. By hypothesis on N, it is enough to define a map from an inner horn Λ_i^3. As in Example 1.57, it is enough to provide three suitable elements of N_2. Well, for any $i < j < k$ in $\{0, 1, 2, 3\}$, we define the 2-simplex $f_{ijk} := f_{jk} \odot f_{ij}$, which we note is always well-defined because $\{k - j, j - i\} \subseteq \{1, 2\}$.

We now build a 3-simplex to do our bidding. We can build a map $\Lambda^3_1 \to N$ via Proposition 1.56. To start, we send $i \mapsto c_i$ for each i and $i \le j$ to f_{ij} for each i and j. Lastly, on 2-simplices, we will glue f_{012}, f_{013}, and f_{123}, which cohere in the colimit because we only have to check that the edges $0 \le 1$ and $1 \le 2$ and $1 \le 3$ and $2 \le 3$ agree in Λ^3_1. Thus, Λ^3_1 now extends to a unique 3-simplex $e_1 \in N_3$.

Now, on one hand, we note that $d_1(e_1)$ is a 2-simplex with $d_2(d_1(e_1)) = f_{23}$ and $d_0(d_1(e_1)) = f_{02}$, so $d_1(e_1) = f_{23} \odot f_{02}$, so $d_1(d_1(e_1)) = f_{23} \circ f_{02}$. In the same way, we calculate that $d_2(e_1) = f_{13} \odot f_{01}$, so $d_1(d_2(e_1)) = f_{13} \circ f_{01}$.

However, $(d_1 \circ d_1)$ and $(d_1 \circ d_2)$ are the same map $N_3 \to N_1$, so the claim follows.

5. We define a natural transformation $\eta\colon N \to N(\mathcal{C})$. Thus, for each n, we must define a map $N_n \to N(\mathcal{C})_n$, so each $x \in N_n$ must produce some functor $\eta_n(x)\colon [n] \Rightarrow \mathcal{C}$.

   Well, on objects, we should send $i \in [n]$ to $(x \circ \varepsilon_i) \in N_0$, and on morphisms, we should send $i \to j$ to the map $(x\varepsilon_{ij}) \in N_1$; note that $(x\varepsilon_{ij})\colon (x\varepsilon_i) \to (x\varepsilon_j)$ by construction of $\mathcal{C}$. The construction of the identities in $\mathcal{C}$ shows that $(x\varepsilon_{ii}) = s_i(x\varepsilon_i) = \mathrm{id}_{x\varepsilon_i}$. Lastly, given two maps $\varepsilon_{ij}\colon i \to j$ and $\varepsilon_{jk}\colon j \to k$, we need to check that $x\varepsilon_{jk} \circ x\varepsilon_{ij} = x\varepsilon_{ik}$. Well, there is a map $\varepsilon_{ijk}\colon \Delta^3 \to \Delta^n$ given by $i \le j \le k$, which we notice has

   $$d_2(x\varepsilon_{ijk}) = x\varepsilon_{ij} \qquad \text{and} \qquad d_2(x\varepsilon_{ijk}) = x\varepsilon_{jk}$$

   by an explicit calculation, so we see that $x\varepsilon_{ijk} = x\varepsilon_{jk} \odot x\varepsilon_{ik}$ and thus

   $$x\varepsilon_{jk} \circ x\varepsilon_{ij} = d_1(x\varepsilon_{ijk}),$$

   which is $x\varepsilon_{ik}$, as required.

   We now must check that the map $\eta_n\colon N_n \to N(\mathcal{C})_n$ is natural. Well, for any increasing map $f\colon [m] \to [n]$ and $x \in N_n$, we need to check that $\eta_m(f(x)) = f(\eta_n(x))$ in $N(\mathcal{C})_m$. As explained in Remark 1.59, the data of a functor $[m] \to N(\mathcal{C})_m$ amounts to the data in degrees 0 and 1, so we should check the equalities against ε_is and ε_{ij}s. Unwinding all the abuse of notation, we see that $\eta_m(f(x))$ on the morphism ε_{ij} is $Nf(x) \circ \varepsilon_{ij}$, which is $x \circ f \circ \varepsilon_{ij}$ or $x\varepsilon_{f(i),f(j)}$. On the other hand, $f(\eta_n(x))$ on $i \le j$ is $\eta_n(x)$ on $f(i) \le f(j)$ (by the functoriality described in Remark 1.59), which is $x\varepsilon_{f(i),f(j)}$ again.

6. We show that η is an isomorphism of simplicial sets. Because we have already described a morphism of simplicial sets, we must merely check that η_n is a bijection for each n. We will do this by induction, where the cases $n \in \{0,1\}$ have no content. For $n \ge 2$, we may choose some i with $0 < i < n$, and then we see that the inclusion defines a natural map

   $$(-)_n = \mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, -) \to \mathrm{Mor}_{\mathrm{sSet}}\left(\Lambda^n_i, -\right),$$

   which is an isomorphism for both N and $N(\mathcal{C})$ by the uniqueness of the lifting. Now, Proposition 1.56 (combined with Example 1.53) shows that Λ^n_i is the colimit of some functor $F\colon \mathcal{I} \to \mathrm{sSet}$, where $F(i) = \Delta^{k_i}$ for some $k_i < n$ for each i. Thus, we see that there is a natural map

   $$(-)_n \to \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathrm{sSet}}(\Delta^{k_i}, -) = \lim_{i \in \mathcal{I}} (-)_{k_i},$$

   which is still an isomorphism for both N and $N(\mathcal{C})$. Thus, the map $\eta\colon N \to N(\mathcal{C})$ fits into a commuting diagram

   $$\begin{tikzcd} N_n \arrow[r] \arrow[d, "\eta_n"'] & \lim\limits_{i \in \mathcal{I}} N_{k_i} \arrow[d, "\lim \eta"] \\ N(\mathcal{C})_n \arrow[r] & \lim\limits_{i \in \mathcal{I}} N(\mathcal{C})_{k_i} \end{tikzcd}$$

   where we know that the horizontal arrows are isomorphisms as described above. But the right arrow is an isomorphism by the inductive hypothesis, so the left arrow is as well, so we are done. ■

1.2.4 More on Sing

We now turn to Sing.

Proposition 1.67. The functor $\mathrm{Sing}\colon \mathrm{sSet} \to \mathrm{Top}$ admits a left adjoint $|\cdot| : \mathrm{Top} \to \mathrm{sSet}$. In fact, $|\Delta^n|$ is defined to be the topological n-simplex.

Proof. We use Proposition A.33. Recall that Remark 1.17 provides us with a functor $\Delta \to \mathrm{Top}$ which sends $[n]$ to $|\Delta^n|$. Then Proposition A.33 provides a unique continuous extension of this functor $|\cdot| : \mathrm{sSet} \to \mathrm{Top}$ for which $|\Delta^n|$ is the topological n-simplex $|\Delta^n|$ described previously.

Additionally, Proposition A.33 admits a right adjoint, which we claim is Sing, which will complete the proof. Indeed, the right adjoint provided by Proposition A.33 is given by sending a topological space X to $\mathrm{Mor}_{\mathrm{Top}}(|\cdot|, X)$, which is exactly Sing! ■

We are now able to give the important property for the image of Sing, akin to Proposition 1.65

Proposition 1.68. Fix a topological space Y. Then any map $\Lambda^n_i \to \mathrm{Sing}\, Y$ (for any $i \in [n]$) admits a lift to a map $\Delta^n \to \mathrm{Sing}\, Y$.

Proof. We are asking for the map

$$\mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, \mathrm{Sing}\, Y) \to \mathrm{Mor}_{\mathrm{sSet}}(\Lambda^n_i, \mathrm{Sing}\, Y)$$

to be surjective. By the adjunction of Proposition 1.67, it is enough to show that the map

$$\mathrm{Mor}_{\mathrm{sSet}}(|\Delta^n|, Y) \to \mathrm{Mor}_{\mathrm{sSet}}(|\Lambda^n_i|, Y)$$

to be surjective. For this, it is enough to find a retraction $r\colon |\Delta^n| \to |\Lambda^n_i|$ of the inclusion $i\colon |\Lambda^n_i| \to |\Delta^n|$. Indeed, this will show that any map $f\colon |\Lambda^n_i| \to Y$ is of the form $f \circ r \circ i$ and therefore factors through $|\Delta^n|$.

We are now forced to unwind the definition of a $|\cdot|$. Observe that Λ^n_i is the union of the images of the maps $\Delta^{n-1} \to \Delta^n$ which map onto an increasing sequence $[n-1] \to [n]$ which does not avoid i. Because unions are colimits (in Set and therefore also in $\mathrm{PSh}(\Delta)$ by Example A.26), we see that Λ^n_i is the colimit of these maps to Δ^n. Now, $|\cdot|$ preserves colimits, so we see that $|\Lambda^n_i|$ is also the colimit of these $|\Lambda^n_i|$. The maps $|\Delta^{n-1}| \to |\Delta^n|$ simply map onto the faces which are not opposite e_i. Thus, by taking the union, we see that

$$|\Lambda^n_i| = \bigcup_{\substack{0 \le j \le n \\ j \ne i}} \{(x_0, \ldots, x_n) : x_0 + \cdots + x_n = 1, x_j = 0\}.$$

We now see that $|\Lambda^n_i|$ only depends on i up to a rearrangement of the coordinates, so we may as well assume that $i = 0$ for simplicity.

We are now ready to define our retract $r\colon |\Delta^n| \to |\Lambda^n_0|$. Let v be the vector which points from the center of the face opposite e_0 to e_0; namely $v = \frac{1}{n}(e_1 + \cdots + e_n) - e_0$. Now, for every $x \in \Delta^n$, we define $r(x)$ to be the element of $|\Lambda^n_0|$ on the ray $\{x + tv : t \ge 0\}$. More precisely, we define

$$r(x) := x - \min\{x_j : j \ne 0\} \cdot nv.$$

Taking x_j to be this smallest coordinate, we see that $r(x)$ continues to have nonnegative coordinates, and the coordinates still sum to 1 because the sum of the coordinates of v is zero. But now $r(x)_j = 0$, so $r(x) \in \Lambda^n_0$. Note that taking minimums is continuous, so r is still continuous. Lastly, we note that $r \circ i = \mathrm{id}$ because any $x \in \Lambda^n_0$ already has some $j \ne 0$ with $x_j = 0$, so $r(x) = x$ follows. ■

Proposition 1.68 motivates the following definition.

Definition 1.69 (Kan complex)**.** A *Kan complex* is a simplicial set X in which every $\Lambda^n_i \to X$ admits a lift to a map $\Delta^n \to X$.

Example 1.70. By Proposition 1.68, we see that $\operatorname{Sing} Y$ is always a Kan complex.

At long last, we may define ∞-categories, which is intended to simultaneously generalize nerves and Kan complexes.

Definition 1.71 (∞-category, quasicategory)**.** An *∞-category* or *quasicategory* is a simplicial set X for which every inner horn $\Lambda^n_i \to X$ admits a lift to $\Delta^n \to X$. We may call X_0 the *objects*, call X_1 the *morphisms*, and call X_n the n-morphisms for $n \geq 1$. More concretely, for any $E \in \mathcal{C}_2$, we may say that $d_1 E$ exhibits a 2-*isomorphism* between $d_0 E$ and $d_2 E$.

Definition 1.72 (homotopic)**.** Two maps $f, g \colon X \to Y$ are *homotopic* if and only if there is a map $h \colon X \times \Delta^1 \to Y$ such that the composites with $d_0 \colon X \times \Delta^0 \to X \times \Delta^1$ and $d_1 \colon X \times \Delta^0 \to X \times \Delta^1$ are g and f, respectively.

Remark 1.73. It turns out that being homotopic is an equivalence relation; the symmetry check uses the fact that Y is a Kan complex.

Definition 1.74 (homotopy equivalent)**.** Two Kan complexes X and Y are homotopy equivalent if and only if there are maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identities.

We will make use of the following hard(!) theorem.

Theorem 1.75 (Quillen)**.** If X is a CW complex, then $|\operatorname{Sing} X|$ is homotopy equivalent to X. Similarly, if X is a Kan complex, then $\operatorname{Sing} |X|$ is homotopy equivalent to X.

Corollary 1.76. The homotopy category of topological spaces is equivalent to the homotopy category of Kan complexes.

This theorem is a purely motivational statement: it allows us to pass from topological spaces to just Kan complexes.

Remark 1.77 (Jeremy Hahn)**.** I do not believe in point-set topology, at least for the purposes of this class.

1.3 September 11

We continue discussing our quasicategories.

1.3.1 More on Kan Complexes

Let's say a little more about Kan complexes.

Remark 1.78. Fix a Kan complex C and a morphism $f \colon c \to c'$ in C_1. Then one can construct an inverse for f as follows: the outer horn

$$\begin{tikzcd} & c' \\ c \arrow[ur, "f"] \arrow[rr, equal] & & c \end{tikzcd}$$

to C must fill out to a map $\Delta^2 \to C$, which provides us with the data of some map $g \colon B \to A$. Then the composite $g \circ f$ can be seen to be homotopic to the identity via the map $\Delta^2 \to C$.

Here is a more general notion.

Definition 1.79 (Kan fibration)**.** A morphism $X \to Y$ of simplicial sets is a *Kan fibration* if and only if, for any n and $i \in [n]$, any two maps $\Lambda^n_i \to X$ and $\Delta^n \to Y$ admits a lifting map $\Delta^n \to X$ making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Example 1.80. A simplicial set X has a canonical map to the terminal object Δ^0, so the map $X \to \Delta^0$ is a Kan fibration if and only if X is a Kan complex.

For today, we will be interested in what sorts of maps $A \to B$ admit lifts against Kan fibrations. Of course, this includes the horn inclusions $\Lambda^n_i \to \Delta^n$, but there are more ways to generate such morphisms.

Definition 1.81 (saturated)**.** A nonempty class Σ of morphisms of simplicial sets is *saturated* if and only if it is closed under pushouts, retracts, coproducts, composition, and transfinite composition.

We will explain the terms in this definition shortly, but let's start by explaining why we should care.

Definition 1.82 (anodyne)**.** The smallest saturated class of maps containing all horn inclusions is the class of *anodyne* maps. The smallest saturated class of maps containing all inner horn inclusions is the class of *inner anodyne* maps.

Proposition 1.83. Suppose that a map $f\colon A \to B$ is anodyne. Then any Kan fibration $g\colon X \to Y$ fitting into the solid square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

will admit a dashed map $B \to X$ making the diagram commute.

Remark 1.84. It turns out that a morphism $A \to B$ is anodyne if and only if it is monic and the induced map $|A| \to |B|$ is a homotopy equivalence. We will not prove this, so we will not use it.

Let's now explain what it means for a class $\mathcal{P}$ of morphisms to be saturated.

Definition 1.85. Let $\mathcal{P}$ be a class of morphisms in a category. If $f\colon A \to B$ is in $\mathcal{P}$, and we are given any map $A \to A'$ living in a pushout diagram as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

Then $\mathcal{P}$ is *closed under pushouts* if we always have $f' \in \mathcal{P}$ for all such diagrams.

Example 1.86. We claim that the class of anodyne maps is closed under pushouts. Indeed, given any map $A \to A'$ producing a pushout map $f' \colon A' \to B'$, we would like to know if we can always fill in for the dashed arrow.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "f'"] & X \arrow[d] \\ B \arrow[r] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Well, we are granted a map $B \to X$ making the diagram commute, so there is a map $B' \to X$ making the diagram commute because B' is a pushout.

Definition 1.87 (retract)**.** A map $f' \colon A' \to B'$ is a *retract* of a map $f \colon A \to B$ if and only if it fits into a diagram as follows.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] \arrow[rr, bend left, equal] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B' \arrow[r] \arrow[rr, bend right, equal] & B \arrow[r] & B' \end{tikzcd}$$

Definition 1.88 (coproduct)**.** Given a collection of maps $f_i \colon A_i \to B_i$ (as i varies over an index set I), then the coproduct map is defined as the induced map

$$\coprod_{i \in I} f_i \colon \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i.$$

Definition 1.89 (transfinite composition)**.** Suppose that there is a diagram

$$A_0 \to A_1 \to A_2 \to \cdots$$

of maps. Then the transfinite composition is the colimit of the diagram

$$A_0 \to \operatorname*{colim}_i A_i.$$

One can check that the class of anodyne maps has all the above closure properties.

1.3.2 Pushout Products

Here is are a few more useful facts about anodyne maps.

Remark 1.90. All anodyne maps are monomorphisms.

Next, we will want to prove that anodyne maps are closed under products. For this, we pick up the following definition.

Definition 1.91 (pushout product)**.** Given two maps $f \colon A \to B$ and $g \colon C \to D$ in a category $\mathcal{C}$, the *pushout product* is the induced map

$$(B \times C) \coprod_{A \times C} (A \times D) \to (B \times D).$$

We may write this map as $f \boxtimes g$.

Example 1.92. The pushout product of a map $f\colon A \to B$ and the initial map $\varnothing \to C$ is the induced map

$$A \times C \to B \times C.$$

Remark 1.93. Pushout products are associative in a suitable way, which we will not prove in general.

Theorem 1.94. Fix a monomorphism $f\colon A \to B$ of simplicial sets and an anodyne (respectively, inner anodyne) map $g\colon C \to D$. Then the pushout product $f \boxtimes g$ is anodyne (respectively, inner anodyne).

The rest of this class will be spent proving this result. We will focus on the anodyne case; the proof of the inner anodyne case is basically the same.

Remark 1.95 (Jeremy Hahn)**.** If you try to visualize this with geometric realizations, then you will recover some exercise that is in Hatcher somewhere.

Let's give the main claim for Theorem 1.94.

Lemma 1.96. The class of anodyne maps is the smallest saturated class containing all maps of the form $f \boxtimes i$, where $f\colon A \to B$ is monic, and $i\colon \Delta^0 \to \Delta^1$ is some map.

Proof of Theorem 1.94 from Lemma 1.96. Fix a monomorphism i, and we consider the class $\mathcal{P}$ of maps j such that $i \boxtimes j$ is anodyne. We want to check that $\mathcal{P}$ contains all anodyne maps. One can check that this class is saturated, so by Lemma 1.96, it remains to prove that $\mathcal{P}$ contains the maps $i \boxtimes (j \boxtimes k)$ where $j\colon A \to B$ is monic, and $k\colon \Delta^0 \to \Delta^1$ is some map. Well,

$$i \boxtimes (j \boxtimes k) = (i \boxtimes j) \boxtimes k,$$

and $i \boxtimes j$ continues to be monic, so this is anodyne by Lemma 1.96! ■

It remains to prove Lemma 1.96. Let $\mathcal{P}$ be the saturated class of maps in the statement, and we would like to show that it coincides with the saturated class of anodyne maps. This will be done via two inclusions, so it suffices to show that each class contains the other's generators.

Lemma 1.97. Let $\mathcal{P}$ be the smallest saturated class containing all maps of the form $f \boxtimes i$, where $f\colon A \to B$ is monic, and $i\colon \Delta^0 \to \Delta^1$ is some map. Then every anodyne map is in $\mathcal{P}$.

Proof. It is enough to show that the horn inclusions $\Lambda^n_i \to \Delta^n$ are in $\mathcal{P}$, which amounts to some explicit combinatorics. By symmetry, we may assume that $i < n$. We now claim that the diagram

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, "i"'] & (\Lambda^n_i \times \Delta^1) \coprod_{\Lambda^n_i \times \Delta^0} (\Delta^n \times \Delta^0) \arrow[r] \arrow[d, "i \boxtimes \varepsilon_0"'] & \Lambda^n_i \arrow[d] \\
\Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n
\end{tikzcd}$$

is a retract diagram, which will complete the proof. Because the vertical maps are all monic, we only have to construct the bottom maps, and then one needs to check that one can induce the top maps accordingly. Well, the first map $\Delta^n \to \Delta^n \times \Delta^1$ is just $\mathrm{id}_{\Delta^n} \times \varepsilon_1$. The second map requires some more work to define. In a picture, it is defined via the n-simplex

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \cdots \arrow[r] & i-1 \arrow[r] \arrow[d] & i \arrow[r] \arrow[d] & i \arrow[r] \arrow[d] & \cdots \arrow[r] & i \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & i-1 \arrow[r] & i \arrow[r] & i+1 \arrow[r] & \cdots \arrow[r] & n
\end{tikzcd}$$

of $\Delta^n \times \Delta^1$. ■

For the second inclusion, we will use the following result.

Proposition 1.98. Suppose $A \to B$ is a monomorphism of simplicial sets. Then there is a canonical sequence of morphisms

$$A \to A_0 \to A_1 \to \cdots$$

with $A = A_{-1}$ such that B is the colimit of this diagram, and there are pushout squares as follows.

$$\begin{tikzcd} \coprod_{I_i} \partial\Delta^i \arrow[r] \arrow[d] & A_{i-1} \arrow[d] \\ \coprod_{I_i} \Delta^i \arrow[r] & A_i \end{tikzcd}$$

Proof. Intuitively, we are attaching i-simplices to A_{i-1} to produce B. More precisely, one sets A_i to be the smallest simplicial subset of B containing A and for which the maps $A_i(j) \to B(j)$ are isomorphisms for $j \leq i$. ■

Lemma 1.99. Let $\mathcal{P}$ be the smallest saturated class containing all maps of the form $f \boxtimes i$, where $f \colon A \to B$ is monic, and $i \colon \Delta^0 \to \Delta^1$ is some map. Then every map in $\mathcal{P}$ is anodyne.

Proof. Given any monomorphism $f \colon A \to B$ and map $i \colon \Delta^0 \to \Delta^1$, we must check that $f \boxtimes i$ is anodyne. By Proposition 1.98, we reduce to the case where f is the inclusion $\partial\Delta^n \to \Delta^n$. This is rather complicated, so we will content ourselves with $n = 1$. Here, we are looking at the following map.

$$(\partial\Delta^1 \times \Delta^1) \coprod_{\partial\Delta^1 \times \Delta^0} (\Delta^1 \times \Delta^0) \to (\Delta^1 \times \Delta^1).$$

The right-hand side is a square, and the left-hand side is the following boundary.

$$\begin{tikzcd} \bullet & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

To show that this is anodyne, we note that we can fill in the lower-right triangle via a pushout against $\Lambda^2_1 \to \Delta^2$, and then we can fill in the upper-left triangle by pushing out against $\Lambda^2_0 \to \Delta^2$. Thus, our inclusion is anodyne. ■

This concludes the proof of Lemma 1.96 and thus the proof of Theorem 1.94.

1.3.3 Internal Mor

Let's explain an application.

Definition 1.100. Fix simplicial sets X and Y. Then we define the simplicial set $\underline{\mathrm{Mor}}(X, Y)$ as having n-simplices

$$\underline{\mathrm{Mor}}(X, Y)_n := \mathrm{Mor}_{\mathrm{sSet}}(X \times \Delta^n, Y).$$

Remark 1.101. This gives sSet "internal Mors," making it a Cartesian closed category. Namely, one can see that we have a natural isomorphism

$$\mathrm{Mor}_{\mathrm{sSet}}(A, \underline{\mathrm{Mor}}(B, C)) \simeq \mathrm{Mor}_{\mathrm{sSet}}(A \times B, C).$$

To motivate our definition, we note that requiring this would require

$$\underline{\mathrm{Mor}}(X, Y)_n = \mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, \underline{\mathrm{Mor}}(X, Y)) = \mathrm{Mor}_{\mathrm{sSet}}(\Delta^n \times X, Y),$$

which is the given definition up to commutativity of the product.

Proposition 1.102. Fix simplicial sets X and Y such that Y is a Kan complex. Then $\underline{\mathrm{Mor}}(X, Y)$ is a Kan complex. Similarly, if Y is a quasicategory, then $\underline{\mathrm{Mor}}(X, Y)$ is a quasicategory.

Proof. We show the first statement because the proof of the second is almost identical. Given any map $\Lambda^n_i \to \underline{\mathrm{Mor}}(X, Y)$, we need to exhibit a lifted map $\Delta^n \to \underline{\mathrm{Mor}}(X, Y)$. By Remark 1.101, we need to exhibit a lift in the following diagram.

$$\begin{tikzcd} \Lambda^n_i \times X \arrow[r] \arrow[d] & Y \\ \Delta^n \times X \arrow[ur] & \end{tikzcd}$$

However, the left-hand map is anodyne by Theorem 1.94, so we are done! ■

Remark 1.103. This is an incarnation of the fact that the category of Kan complexes is much nicer than Top: we have internal Mors, which is much harder to come by for topological spaces.

Let's keep working with Kan complexes.

Definition 1.104 (isomorphism)**.** Two objects in a Kan complex are *isomorphic* if and only if there is a map between them.

Definition 1.105 (equivalence)**.** A map $f\colon X \to Y$ of Kan complexes is a *homotopy equivalence* if and only if there is a map $g\colon Y \to X$ such that $f \circ g$ is isomorphic to id_Y in $\underline{\mathrm{Mor}}(Y, Y)$ and that $g \circ f$ is isomorphic to id_X in $\underline{\mathrm{Mor}}(X, X)$.

Definition 1.106. Fix a Kan complex X. Then $\pi_0(X)$ is the set of isomorphism classes of objects in X.

1.4 September 16

There is a grader now.

1.4.1 Some Fibrations

Just as Kan fibrations admit lifts against anodyne maps, we may define inner fibrations.

Definition 1.107. A map $f\colon X \to Y$ of simplicial sets is an *inner fibration* if and only if every inner horn $\Lambda_i^n \to \Delta^n$ with a map $\Lambda_i^n \to X$ and $\Delta^n \to Y$ (commuting with f) admits a lifting making the following diagram commute.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Remark 1.108. Not every anodyne map is inner anodyne. For example, arbitrary ∞-categories are not expected to be ∞-groupoids.

Example 1.109. A simplicial set X is an ∞-category if and only if the canonical map $X \to \Delta^0$ is an inner fibration.

Example 1.110. Let J be the nerve of the category with two isomorphic points. Then the inclusion $\Delta^1 \to J$ is anodyne but not inner anodyne, which we can see because arbitrary nerves are not expected to receive lifts of the map $\Delta^1 \to J$.

Remark 1.111. Every monomorphism is in the saturated class containing the boundary maps $\partial\Delta^n \to \Delta^n$. This is because any monomorphism can be written as a (transfinite) composition of cell attachments.

Definition 1.112 (trivial fibration)**.** A map $f\colon X \to Y$ of simplicial sets is a *trivial fibration* if and only if every boundary horn $\partial\Delta^n \to \Delta^n$ with maps $\partial\Delta^n \to X$ and $\Delta^n \to Y$ (commuting with f) admits a lifting making the following diagram commute.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Remark 1.113. Not every anodyne map is inner anodyne. For example, arbitrary ∞-categories are not expected to be ∞-groupoids.

1.4.2 Common Names for Fibrations

Here are some definitions for quasicategories.

Definition 1.114 (functor)**.** A *functor* of quasicategories $\mathcal{C}$ and $\mathcal{D}$ is a map of the underlying simplicial sets.

Definition 1.115 (natural transformation)**.** A *natural transformation* of quasicategories $\mathcal{C}$ and $\mathcal{D}$ is a morphism in $\underline{\mathrm{Mor}}(\mathcal{C}, \mathcal{D})$, which has the same data as a map $\mathcal{C} \times \Delta^1 \to \mathcal{D}$. If this morphism factors through the two-element category J with two isomorphic elements, then we say that the morphism is a *natural isomorphism*

Definition 1.116 (isomorphism)**.** A morphism f in a quasicategory $\mathcal{C}$ is an *isomorphism* if and only if the induced map $\Delta^1 \to \mathcal{C}$ factors through the two-element category J with two isomorphic elements.

Definition 1.117 (equivalence)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ of quasicategories is an *equivalence* if and only if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ for which FG and GF are both isomorphic to the identities in $\underline{\mathrm{Mor}}(\mathcal{D},\mathcal{D})$ and $\underline{\mathrm{Mor}}(\mathcal{C},\mathcal{C})$, respectively.

Proposition 1.118. Fix a functor $F\colon \mathcal{C} \to \mathcal{D}$. If F is a trivial fibration, then F is an equivalence.

Proof. Note that the canonical map $\varnothing \to \mathcal{D}$ is monic, so we gain a functor $G\colon \mathcal{D} \to \mathcal{C}$ making the diagram

$$\begin{tikzcd} \varnothing \arrow[r, hook] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D} \arrow[r, equal] \arrow[ur, dashed, "G"] & \mathcal{D} \end{tikzcd}$$

commute. By construction, we see that FG is equal to the identity in $\underline{\mathrm{Mor}}(\mathcal{D},\mathcal{D})$. To compute GF, we draw the diagram

$$\begin{tikzcd} (\{0\}\times\mathcal{C}) \sqcup (\{1\}\times\mathcal{C}) \arrow[rr, "{(GF,\mathrm{id}_\mathcal{C})}"] \arrow[d, hook] & & \mathcal{C} \arrow[d, "F"] \\ \Delta^1 \times \mathcal{C} \arrow[r, "\mathrm{pr}_2"'] \arrow[urr, dashed, "\eta"] & \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

where the outer square commutes because $FG = \mathrm{id}_\mathcal{D}$, so we get to induce a lift $\eta\colon \Delta^1 \times \mathcal{C} \to \mathcal{C}$, which is a natural isomorphism from GF to $\mathrm{id}_\mathcal{C}$. To check that η is actually a natural isomorphism, we note that the commutativity of the outer square in

$$\begin{tikzcd} \Delta^1 \times \mathcal{C} \arrow[rr, "\eta"] \arrow[d, hook] & & \mathcal{C} \arrow[d, "F"] \\ J \times \mathcal{C} \arrow[r, "\mathrm{pr}_2"'] \arrow[urr, dashed] & \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

shows that we get a lifting map $J \times \mathcal{C} \to \mathcal{C}$, witnessing that η is a natural isomorphism. ■

Proposition 1.119. Fix a map $f\colon X \to Y$ of Kan complexes. If f is anodyne, then f is an equivalence.

Proof. We will show that the induced map $\underline{\mathrm{Mor}}(Y,Z) \to \underline{\mathrm{Mor}}(X,Z)$ is a trivial fibration for each Kan complex Z. Plugging in $Z \in \{X,Y\}$ shows that $\underline{\mathrm{Mor}}(Y,Y) \to \underline{\mathrm{Mor}}(X,Y)$ and $\underline{\mathrm{Mor}}(Y,Z) \to \underline{\mathrm{Mor}}(X,X)$ are trivial fibrations, so they are equivalence of Kan complexes. To check this, we need to show that we have lifts in a square of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \underline{\mathrm{Mor}}(Y,Z) \arrow[d] \\ \Delta^n \arrow[r] & \underline{\mathrm{Mor}}(X,Z) \end{tikzcd}$$

which is equivalent to finding a lift in the square

$$\begin{tikzcd} (\partial\Delta^n \times Y) \displaystyle\coprod_{\partial\Delta^n \times X} (\Delta^n \times X) \arrow[r] \arrow[d] & Z \\ \Delta^n \times Y \arrow[ur, dashed] & \end{tikzcd}$$

by using Remark 1.101. But this exists because the left map is inner anodyne by Lemma 1.96. ■

Proposition 1.120. Fix a map $f\colon X \to Y$ of quasicategories. If f is inner anodyne, then f is an equivalence.

Proof. The proof is the same as in Proposition 1.119. ■

1.4.3 Homotopy Groups

We now begin doing some algebraic topology. We are interested in constructing (and computing) algebraic invariants of topological spaces, which for this class amount to Kan complexes. For example, a topological space X with basepoint $x \in X$ has a fundamental group $\pi_1(X, x)$ of loops in X based at x (up to homotopy), and there is a group operation given by path concatenation.

To do this for simplicial sets, we pick up the following notation.

Notation 1.121. Fix morphisms $A \to B$ and $X \to Y$ of simplicial sets. Then we define the simplicial set $\underline{\mathrm{Mor}}((B, A), (Y, X))$ as the pullback in the following diagram.

$$\begin{tikzcd} \underline{\mathrm{Mor}}((B,A),(Y,X)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Mor}}(B,Y) \arrow[d] \\ \underline{\mathrm{Mor}}(A,X) \arrow[r] & \underline{\mathrm{Mor}}(A,Y) \end{tikzcd}$$

Remark 1.122. If $A \to B$ and $X \to Y$ are monomorphisms of Kan complexes, then $\underline{\mathrm{Mor}}((B, A), (Y, X))$ is a Kan complex. This more or less follows from Theorem 1.94.

Definition 1.123 (fundamental group)**.** Fix a Kan complex X and a point $x \in X_0$. Then we define the *fundamental group* $\pi_1(X, x)$ is the isomorphism classes of

$$\underline{\mathrm{Mor}}((\Delta^1, \{0\} \sqcup \{1\}), (X, \{x\})).$$

To go to higher homotopy groups, it will be technically convenient to have some more simplicial sets.

Notation 1.124. For each nonnegative integer n, we define $\square^n$ as the n-fold product $\left(\Delta^1\right)^{\times n}$ and its boundary $\partial\square^n$ as the union

$$\bigcup_{j=0}^{n-1} \left(\Delta^1\right)^{\times j} \times (\{0\} \sqcup \{1\}) \times \left(\Delta^1\right)^{\times(n-j-1)}.$$

Example 1.125. Here is a picture of $\partial\square^1 \subseteq \Delta^1$.

$$\bullet \qquad \bullet \qquad \hookrightarrow \qquad \bullet \longrightarrow \bullet$$

Here is a picture of $\partial\square^2$.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

We note that $\square^2$ would also include the inner 2-simplex.

We are now able to define higher homotopy groups.

Definition 1.126 (homotopy groups)**.** Fix a Kan complex x and a point $x \in X_0$. For each nonnegative integer n, we define the *homotopy group* $\pi_n(X, x)$ is the isomorphism classes of

$$\underline{\mathrm{Mor}}((\square^n, \partial\square^n), (X, \{x\})).$$

Remark 1.127. One could get away with using the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ instead of the squares, which we will prove later.

Let's check that we have actually defined groups.

Lemma 1.128. Fix a Kan complex X and a point $x \in X_0$. Then there is a unital multiplication

$$\pi_n(X,x) \to \pi_n(X,x) \to \pi_n(X,x)$$

by lifting as follows.

$$\begin{tikzcd} & \Lambda^2_1 \times \square^{n-1} \arrow[r, "{(f,g)}"] \arrow[d] & X \\ \Delta^1 \times \square^{n-1} \arrow[r, "d_1"] & \Delta^2 \times \square^{n-1} \arrow[ur, dashed] & \end{tikzcd}$$

Proof. The motivation here is that the map $\Lambda^2_1 \times \square^{n-1} \to X$ amounts to stacking the two square maps $f\colon \square^n \to X$ and $g\colon \square^n \to X$ on top of each other. Then the whole gluing process basically extends ■

There are actually n different unital multiplications given by how we glue our boxes together. The fact that there are at least different multiplications for $n \geq 2$ implies that these multiplications are all equal, associative, and commutative. This follows from the Eckmann–Hilton argument.

Lemma 1.129 (Eckmann–Hilton)**.** Fix a set A with two binary operations $\cdot_1$ and $\cdot_2$ which are equipped with two-sided units 1_1 and 1_2. Further, suppose that

$$(a \cdot_1 b) \cdot_2 (c \cdot_1 d) = (a \cdot_2 c) \cdot_1 (b \cdot_2 d).$$

Then $\cdot_1 = \cdot_2$ and $1_1 = 1_2$. In fact, $\cdot_1$ and $\cdot_2$ are associative and commutative.

Proof. We will write $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ for $a \cdot_2 b$ and $\left[\begin{smallmatrix} a & b \end{smallmatrix}\right]$ for $a \cdot_1 b$. The given identity amounts to saying that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is well-defined.

Let's start by checking that the units are the same. For this, we write

$$\begin{aligned} 1_2 &= \begin{bmatrix} 1_2 \\ 1_2 \end{bmatrix} \\ &= \begin{bmatrix} 1_2 & 1_1 \\ 1_1 & 1_2 \end{bmatrix} \\ &= \begin{bmatrix} 1_1 & 1_1 \end{bmatrix} \\ &= 1_1. \end{aligned}$$

Thus, we may just write 1 for our unit. For the commutativity, we write

$$\begin{aligned} \begin{bmatrix} a & b \end{bmatrix} &= \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \\ &= \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \\ &= \begin{bmatrix} b & a \end{bmatrix}. \end{aligned}$$

Note that this has also shown that the two operations are the same. Lastly, to check associativity, we write

$$\begin{aligned}\begin{bmatrix}a & \begin{bmatrix}b & c\end{bmatrix}\end{bmatrix} &= \begin{bmatrix}a & \begin{bmatrix}b \\ c\end{bmatrix}\end{bmatrix} \\ &= \begin{bmatrix}a & b \\ 1 & c\end{bmatrix} \\ &= \begin{bmatrix}\begin{bmatrix}a & b\end{bmatrix} \\ c\end{bmatrix} \\ &= \begin{bmatrix}\begin{bmatrix}a & b\end{bmatrix} & c\end{bmatrix}.\end{aligned}$$

This finishes the proof! ■

Remark 1.130. Let's say a bit about motivation for these homotopy groups. One reason, of course, is that many meaningful invariants in mathematics turn out to be related to homotopy groups.

1.5 September 18

Today we prove Whitehead's theorem. I have moved all discussion of Whitehead's theorem to the present class.

1.5.1 Whitehead's theorem

For this class, the following explains why we should care about homotopy.

Theorem 1.131 (Whitehead)**.** Fix a map $f\colon X \to Y$ of Kan complexes, and choose a basepoint $x \in X$. Then f is a homotopy equivalence if and only if the functorial maps

$$\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,f(x))$$

are bijections for $n = 0$ and group isomorphisms for $n \geq 1$.

Let's begin with the first lemma.

Definition 1.132 (pointed Kan complex)**.** A *pointed Kan complex* is a pair (X,x) of a Kan complex X equipped with a point $x \in X_0$.

Remark 1.133. Viewing x as a morphism $x\colon \Delta^0 \to X$, we are allowed to make sense of the Kan complex

$$\underline{\operatorname{Mor}}((X,x),(Y,y))$$

where (X,x) and (Y,y) and pointed Kan complexes. An object in this simplicial set is a morphism of pointed Kan complexes; and a morphism in this simplicial set is a pointed homotopy. These notions produce a definition of a pointed homotopy equivalence.

Remark 1.134. For each $n \geq 0$, we now receive functors π_n on the category of pointed Kan complexes. If $n = 0$, we output a pointed set, and if $n \geq 1$, then we output a group; in fact, if $n \geq 2$, then we output an abelian group.

While we're here, we note that there is a pointed version of Whitehead's theorem.

Theorem 1.135 (Whitehead, pointed)**.** Fix a map $f\colon (X,x) \to (Y,y)$ of pointed connected Kan complexes. Then f is an equivalence if and only if the induced map $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,y)$ is an isomorphism for all $n \geq 1$.

Let's start proving some lemmas. We pick up the following definition.

Definition 1.136 (fiber)**.** Fix a map $f\colon (X,x) \to (Y,y)$ of pointed Kan complexes. Further, suppose that $X \to Y$ is a Kan fibration. Then the *fiber* of f is the pullback F fitting into the following diagram.

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd}$$

Note that the map $x\colon \Delta^0 \to X$ extends via pullback to a unique map $x\colon \Delta^0 \to F$, so (F,x) is a pointed Kan complex.

Lemma 1.137. Fix a map $f\colon (X,x) \to (Y,y)$ of pointed Kan complexes. Further, suppose that $X \to Y$ is a Kan fibration, and let (F,x) be the fiber of f. Then there is an exact sequence

$$\pi_1(Y,y) \to \pi_0 F \to \pi_0 X \to \pi_0 Y$$

of pointed sets.

Proof. We will check exactness at $\pi_0 F$ and $\pi_0 X$ separately. We proceed in steps.

1. Exact at $\pi_0 X$: for each $e \in X_0$ for which $f(e)$ is in the path component of y, we need to show that e is in the same path component of some element in the fiber $F \subseteq X$. For this, we note let $p\colon \Delta^1 \to Y$ be a path connecting y and $f(e)$, and we note that we have a diagram

$$\begin{tikzcd} \{1\} \arrow[r, "e"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^1 \arrow[r] & Y \end{tikzcd}$$

   which commutes, so the Kan fibration $X \to Y$ admits a lift $\Delta^1 \to X$. This path connected e to an element of the fiber, so we are done!

2. We define the map $\pi_1(Y,y) \to \pi_0 F$. Given some loop $[\gamma] \in \pi_1(Y,y)$, we can select a representation $\gamma\colon \Delta^1 \to Y$ for which $\gamma(0) = \gamma(1) = y$. We now get a diagram

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^1 \arrow[r, "\gamma"] & Y \end{tikzcd}$$

   which commutes and therefore induces a map $\Delta^1 \to X$. The image of 1 along the lift $\Delta^1 \to X$ still maps down to $y \in Y$ along f, so we have produced an element of the fiber F.

3. Exact at $\pi_0 F$: suppose that we have some $x' \in F$ which is in the same connected component of $x \in X$. Then we want to find a loop in Y which lifts to a path in X connecting x and x'. Well, the path connecting x to x' in X goes down to the desired loop in Y. ∎

Remark 1.138. All the path-lifting we do in this proof shows that we can think of Kan fibrations as covering spaces.

Remark 1.139. In fact, there is a long exact sequence

$$\cdots \to \pi_{n+1}(F,x) \to \pi_{n+1}(X,x) \to \pi_{n+1}(Y,y) \to \pi_n(F,x) \to \cdots.$$

We now start moving towards Whitehead's theorem. Here is the first lemma, which provides the forward direction.

Lemma 1.140. Fix a homotopy equivalence $f\colon X \to Y$ of Kan complexes. Then for each $x \in X$, there f induces a pointed homotopy equivalence $f\colon (X,x) \to (Y,f(x))$.

Proof. Set $y := f(x)$ for brevity. We will show that the induced map

$$\pi_0\underline{\mathrm{Mor}}((Y,y),(Z,z)) \to \pi_0\underline{\mathrm{Mor}}((X,x),(Z,z))$$

is surjective for all pointed Kan complexes (Z,z). To see that this is enough, we note that we may plug in $(Z,z) = (X,x)$, and then we are granted a map $g\colon (Y,y) \to (X,x)$ for which $gf \sim 1_{(X,x)}$. But because f is a(n unpointed) homotopy equivalence, we conclude that g is as well, so we get a map h for which $hg \sim 1_{(Y,y)}$. We are now allowed to control that $f = h$, so f is a pointed map with pointed homotopic inverse g, so we are done.

It remains to show our surjectivity, for which we will use Lemma 1.137. Observe that the fiber of the map

$$\mathrm{ev}_z\colon \underline{\mathrm{Mor}}(X,Z) \to Z$$

has fiber over z given by $\underline{\mathrm{Mor}}((X,x),(Z,z))$. Doing the same for (Y,y), we produce a diagram as follows.

$$\begin{tikzcd}
\pi_1(Z,z) \arrow[r] \arrow[d, equal] & \pi_0\underline{\mathrm{Mor}}((X,x),(Z,z)) \arrow[r] & \pi_0\underline{\mathrm{Mor}}(X,Z) \arrow[r] & \pi_0 Z \arrow[d, equal] \\
\pi_1(Z,z) \arrow[r] & \pi_0\underline{\mathrm{Mor}}((Y,y),(Z,z)) \arrow[r] \arrow[u] & \pi_0\underline{\mathrm{Mor}}(Y,Z) \arrow[r] \arrow[u] & \pi_0 Z
\end{tikzcd}$$

The third map is actually a surjection because X and Y are homotopic, so it follows by a diagram-chase that the second map is surjective, as required. ■

This completes the proof of the forward direction of Theorem 1.131, which requires two lemmas.

Lemma 1.141. Fix a Kan fibration $f\colon X \to Y$ of Kan complexes. If $\pi_0 f$ is a bijection, and $\pi_n f$ is an isomorphism for $n \geq 1$, then f is a trivial fibration.

Proof. Let's begin with a special case: if X is a connected Kan complex for which $\pi_n(X,x) = 0$ for all $n \geq 1$, then we claim that the map $X \to \Delta^0$ is a trivial fibration. (In other words, we will try to show that X is contractible.)

For this, we claim that $\pi_n(X,x)$ is isomorphic (as sets) to $\pi_0\underline{\mathrm{Mor}}((\Delta^n,\partial\Delta^n),(X,x))$. The novelty here is that we are working with the n-simplex instead of the cube. We will only work out the case $n=2$. Let C be the boundary with one 2-simplex added. Then the quotient $\square^2/C$ is isomorphic to $\Delta^2/\partial\Delta^2$ as pointed simplicial sets, so we get a chain

$$\underline{\mathrm{Mor}}((\square^2,\partial\square^2),(X,x)) = \underline{\mathrm{Mor}}((\square^2/\partial\square^2,\Delta^0),(X,x)) \leftarrow \underline{\mathrm{Mor}}((\square^2/C,*),(X,x)) = \underline{\mathrm{Mor}}((\Delta^2,\partial\Delta^2),(X,x)).$$

It remains to check that the map $\leftarrow$ is an isomorphism on π_0, which is true because it is actually an equivalence.

We now return to the assumption that $\pi_n(X,x) = 0$ for all $n \geq 1$. We will want another intermediate claim: suppose that we have an inclusion $A \subseteq B$ of simplicial sets, a map $f\colon B \to X$, and a homotopy $\eta\colon \Delta^1 \times A \to X$ from $f|_A$ to the constant map $x\colon A \to X$. Then we claim that η extends to a homotopy

$\Delta^1 \times B \to X$ from f to the constant map $x\colon B \to X$. Well, by an induction, we are allowed to merely consider the case $A = \partial\Delta^n$ and $B = \Delta^n$. For this, we draw the diagram

$$\begin{tikzcd} (\{0\}\times\Delta^n) \coprod_{\{0\}\times\partial\Delta^n} (\Delta^1\times\partial\Delta^n) \arrow[r, "(f,\eta)"] \arrow[d] & X \arrow[d] \\ \Delta^1\times\Delta^n \arrow[r] & \Delta^0 \end{tikzcd}$$

from which we note that an extension $\Delta^1 \times \Delta^n \to X$ will produce the desired homotopy. It turns out that this extension exists because $\pi_n(X, x) = 0$, though the reason is not totally clear to the professor.

We now finally show that X is a trivial fibration, so choose a map $\partial\Delta^n \to X$ which we would like to fill in. The previous paragraph applied to $(A, B) = (\varnothing, \partial\Delta^n)$ grants a homotopy $\eta\colon \Delta^1 \times \partial\Delta^n \to X$. Well, we basically want to fill in the diagram

$$\begin{tikzcd} (\Delta^1\times\partial\Delta^n) \coprod_{\{1\}\times\partial\Delta^n} (\{1\}\times\Delta^n) \arrow[r] \arrow[d] & X \\ \Delta^1\times\Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

(to supply a map $\{0\} \times \Delta^n \to \{1\} \times \Delta^n \to X$). But this lifting exists because the left map is anodyne.

By the long exact sequence for homotopy groups (stated in Remark 1.139), one finds that the fiber of $X \to Y$ is contractible by the above special case, which eventually implies that it is enough to consider the case of Y being a point. ■

Thus, we see that if we can get a Kan fibration, then we will be able to conclude by Proposition 1.118. Our last lemma explains how to produce this Kan fibration.

Lemma 1.142 (Quillen's small object argument)**.** Every map of simplicial sets is the composite of an anodyne map and a Kan fibration.

Proof. Fix a map $f\colon X \to Y$ of simplicial sets. If we are given lots of lifting problems

$$\begin{tikzcd} \coprod \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod \Delta^n \arrow[r] & Y \end{tikzcd}$$

that we want to solve; we may as well take the coproduct of all such lifting problems we could ever want to solve. Perhaps we cannot solve them, but we can form the pushout of the upper-left triangle P_0, and it follows that the map $X \to P_0$ which is anodyne. It follows that any of our lifting problems $\Lambda^n_i \to P_0$ and $\Delta^n \to Y$ which happen to factor through X. Thus, we pick up the remaining lifting problems we want to solve, take a pushout, and we form P_1. Continuing this process inductively, we receive a diagram

$$X \to P_0 \to P_1 \to \cdots,$$

and we let P be the colimit. We now claim that $P \to Y$ is a Kan fibration, which will complete the proof because the map $X \to P$ continues to be anodyne. Indeed, any lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & P \arrow[d] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

has Λ^n_i with only finitely many non-degenerate simplices, so the map $\Lambda^n_i \to P$ will have to factor as $\Lambda^n_i \to P_k$ for some k. But then we know that we can product a lift $\Delta^n \to P_{k+1} \to P$. ■

Remark 1.143. This is called the small object argument because it crucially depends on the fact that Λ^n_i is a "small" object.

We are now able to complete the proof of Theorem 1.131: factor $X \to Y$ as $X \to Z \to Y$ of an anodyne map followed by a Kan fibration. Because $Z \to Y$ is a Kan fibration, we conclude that Z is a Kan complex. We are now basically done because anodyne maps are automatically homotopy equivalences.

Theme 2
Computing Homotopy

What we didn't do is make the construction at all usable in practice! This time we will remedy this.

—Kiran S. Kedlaya, [Ked21]

2.1 September 23

It turns out that I have basically given up adding in details for this class. Theorem 1.131 has convinced us to compute some homotopy groups.

Remark 2.1 (Jeremy Hahn)**.** If you compute some homotopy groups, then you end up being a professor here.

2.1.1 The Simplicial Set Spaces

We are going to want to be able to talk about the ∞-category of spaces. There are three ways to build this ∞-category. The first two both boil down to the following idea.

Idea 2.2 (Homotopy coherent mathematics)**.** We never require objects be equal: we only require that there is a specified homotopy between them.

For the first way, we may consider Kan to be a full subcategory of sSet whose objects are Kan complexes, and we can take its nerve to produce a simplicial set. Now, to take a quotient by homotopy equivalences, we form the pushout

$$\begin{tikzcd} \coprod_{\text{homotopy equiv}} \Delta^1 \arrow[r] \arrow[d, two heads] & \mathrm{Kan} \arrow[d] \\ \coprod_{\text{homotopy equiv}} J \arrow[r] & \overline{\mathrm{Kan}} \end{tikzcd}$$

to build some $\overline{\mathrm{Kan}}$. Now, the map $\overline{\mathrm{Kan}} \to \Delta^0$ can be expanded into a composite of an inner anodyne map and a Kan fibration, and we let Spaces fit into the sequence

$$\overline{\mathrm{Kan}} \to \mathrm{Spaces} \to \Delta^0.$$

Here is a result apparent from this construction.

Proposition 2.3. Fix some ∞-category $\mathcal{C}$. Then $\underline{\mathrm{Mor}}(\mathrm{Spaces}, \mathcal{C})$ is equivalent to the full subcategory of $\underline{\mathrm{Mor}}(\mathrm{Kan}, \mathcal{C})$ such that the homotopy equivalences go to isomorphisms.

Corollary 2.4. The projection $\mathrm{Kan} \twoheadrightarrow \mathrm{ho}(\mathrm{Kan})$ factors through Spaces.

For the second way, we will construct Spaces as an explicit simplicial set.

0. The set of 0-simplices consists of all Kan complexes.

1. The set of 1-simplices has all morphisms of Kan complexes. (Thus far, this just looks like the nerve of Kan.)

2. The 2-simplices are given by the data of three complexes $\{X_0, X_1, X_2\}$ and maps $f_{ij} \colon X_i \to X_j$ whenever $0 \le i < j \le 2$ and the data of a map $\Delta^1 \to \mathrm{Mor}(X_0, X_1)$ which witnesses a homotopy from $(f_{12} \circ f_{01})$ to f_{02}. (Note Kan_2 requires $f_{12} \circ f_{01} = f_{02}$, which is a requirement instead of a piece of data!)

3. The 3-simplices consist of four complexes $\{X_0, X_1, X_2, X_3\}$ and maps $f_{ij} \colon X_i \to X_j$ whenever $0 \le i < j \le 2$, homotopies for any non-degenerate 2-simplex in Δ^3, and a map $\Delta^1 \times \Delta^1 \to \underline{\mathrm{Mor}}(X_0, X_3)$ satisfying some additional conditions. Namely, this last map should fill in the diagram

$$\begin{tikzcd} f_{13}f_{01} \arrow[r] & f_{23}f_{12}f_{01} \\ f_{03} \arrow[u] \arrow[r] \arrow[ur] & f_{23}f_{03} \arrow[u] \end{tikzcd}$$

   where the edges are given by the 2-simplices described prior. (This is an approximation of the associativity of composition, but we use a homotopy.)

4. The higher simplices are defined similarly. For example, 4-simplices involves a map from $\Delta^1 \times \Delta^1 \times \Delta^1$.

Remark 2.5. A square in Spaces does not need to actually commute in Kan: it only commutes up to some specified homotopies. Accordingly, one should imagine that the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

is "filled in" by given homotopies. We will frequently ignore the middle map $A \to D$.

2.1.2 Homotopy Pullbacks

It is worthwhile to note that we can kind of form pullbacks.

Definition 2.6 (homotopy pullback)**.** A square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

in Spaces is a *pullback* if and only if any Kan complex X has

$$\pi_0\underline{\mathrm{Mor}}(X, P) \to \pi_0\underline{\mathrm{Mor}}\left((\Delta^1 \times \Delta^1, \Delta^0 \sqcup \Lambda^2_2), (\mathrm{Spaces}, (X, A \to B \leftarrow C)\right).$$

Sometimes we will call this a *homotopy pullback square* to emphasize that the square is filled in with a homotopy.

In other words, up to homotopy, maps $X \to P$ are the same as squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

considered up to specified homotopy.

Lemma 2.7. Given maps $f\colon A \to C$ and $g\colon B \to C$, the homotopy pullback is given by the Kan complex which is the pullback X of

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Mor}}\left(\Delta^1, C\right) \arrow[d] \\ A \times B \arrow[r] & C \times C \end{tikzcd}$$

notably considered in the category sSet.

Example 2.8. We can see that X_0 consists of the data of two points in A and B and the specified data of a path between their images in C. Something similar is true for the higher simplices.

Remark 2.9. If A, B, and C are all pointed, then the homotopy pullback is now seen to be pointed as well by putting the basepoints everywhere.

Here is our application for today.

Example 2.10 (loop space)**.** Fix a pointed Kan complex X. Then the *loop space* ΩX is the homotopy pullback of the basepoint map $\Delta^0 \to X$ considered twice. For example, the 0-simplices are given by the data of two points in X connected by a specified path. Note that the pullback in sSet is just Δ^0!

Remark 2.11. One can check on the level of simplices of our explicit representatives that

$$\Omega X \simeq \underline{\mathrm{Mor}}\left((\Delta^1, \partial\Delta^1), (X, \{x\})\right).$$

Thus, $\pi_0\Omega X = \pi_1(X, x)$. In fact, $\pi_n\Omega X = \pi_{n+1}X$ for all $n \geq 0$: the data of an n-cube $\square^n \to \Omega X$ produces an $(n+1)$-cube $\square^{n+1} \to \Omega X$.

Example 2.12. Fix a group G, which we can view as the one-object groupoid BG, which in turn is a Kan complex. Then $\Omega BG = G$ by construction of BG (namely, the loops in BG are the elements of G), so

$$\pi_i BG \cong \begin{cases} 0 & \text{if } i \neq 1, \\ G & \text{if } i = 1. \end{cases}$$

Remark 2.13. It is a little difficult to take a group G and produce a CW complex representing the homotopy type of BG. For example, it turns out that $B(\mathbb{Z}/2\mathbb{Z})$ is homotopic to $\mathbb{RP}^\infty$.

Here is a special kind of pullback.

Definition 2.14 (homotopy fiber)**.** Fix a map $f\colon A \to B$ of pointed Kan complexes. Then the *homotopy fiber* of f is the homotopy pullback of the diagram $\Delta^0 \to B \leftarrow A$. We may write this fiber as $\operatorname{hfiber}(f)$.

Remark 2.15. We see that the 0-simplices of $\operatorname{hfiber}(f)$ may be described as given by pairs of points of A and a map from their image along f to the basepoint of B.

Lemma 2.16. Fix a map $f\colon A \to B$ of pointed Kan complexes. Then the homotopy fiber of the induced map $\operatorname{hfiber}(f) \to A$ is ΩB.

Proof. Let the induced map be $g\colon \operatorname{hfiber}(f) \to A$. Then we stack our squares as

$$\begin{tikzcd} \operatorname{hfiber}(g) \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\ \operatorname{hfiber}(f) \arrow[r, "g"] \arrow[d] & A \arrow[d, "f"] \\ \Delta^0 \arrow[r] & B \end{tikzcd}$$

so that the result follows because a stack of (homotopy) pullback squares continues to be a (homotopy) pullback square. ■

Theorem 2.17. If $f\colon A \to B$ is a morphism of pointed Kan complexes with homotopy fiber F, then we can take the induced map $\Omega B \to F$ and form a "long exact sequence"

$$\Omega^2 B \to \Omega F \to \Omega A \to \Omega B \to F \to A \to B.$$

Taking π_0 produces a sequence of maps

$$\pi_2 B \to \pi_1 F \to \pi_1 B \to \pi_1 A \to \pi_0 F \to \pi_0 A \to \pi_0 B.$$

This is an exact sequence of pointed sets.

Proof. At the end, the argument can be done by hand as in Lemma 1.137, and the rest follows from functoriality at every step. For example, exactness at $\pi_1 B$ follows because $\Omega B \to F \to A$ is also a homotopy fiber! ■

Theorem 2.18. Fix a Kan fibration $f\colon A \to B$ which is also a morphism of pointed Kan complexes. Then the inclusion of the fiber F into A is an equivalence.

Proof. By Theorem 1.131, it is enough to show that this inclusion induces an isomorphism on $\pi_\bullet$s, which can be checked by hand. ■

We are now allowed to provide a third construction of Spaces.

Remark 2.19. We expect Spaces to have (co)limits. In our setting, a diagram is a map $D \to \mathrm{Spaces}$ from a quasicategory D. For example, any Kan complex X admits a map to Spaces_0 and thus a map $X \to \Delta^0 \to \mathrm{Spaces}$, and it turns out that X is the colimit of this morphism. This is a homotopy-coherent version of the statement that any set S can be viewed as a discrete category and the induced functor $S \to \mathrm{Set}$ has colimit S.

This intuition grants a universal property for Spaces.

Theorem 2.20. Let C be a quasicategory with all colimits. Then a functor $F\colon \mathrm{Spaces} \to \mathcal{C}$ preserving colimits has the same data as an object of $\mathcal{C}$.

Remark 2.21. This is a homotopy-coherent version of the statement that a colimit preserving functor $\mathrm{Set} \to \mathcal{C}$ has the same data as an object of $\mathcal{C}$. Indeed, this is true because any set is a colimit of points.

2.2 September 25

Today we build some spaces with homotopy groups which are easy to compute.

2.2.1 Simplicial Modules

It is useful to know something about limits and colimits in Spaces.

Remark 2.22. The functor $\mathrm{Kan} \to \mathrm{Spaces}$ preserves pullbacks of the form

$$\begin{tikzcd} & A \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

where the map $A \to B$ is a Kan fibration.

Remark 2.23. The functor $\mathrm{Kan} \to \mathrm{Spaces}$ preserves pushouts of the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \\ C & \end{tikzcd}$$

where $A \hookrightarrow B$ is monic. For example, we preserve coproducts.

We now say something about

Definition 2.24 (simplicial module)**.** Fix a commutative ring R. Then a *simplicial R-module* is a functor $\Delta^{\mathrm{op}} \to \mathrm{Mod}_R$. The category is denoted $\mathrm{sSet}(R)$.

Remark 2.25. A simplicial R-module is a simplicial group, which we can see by composing with the forgetful functor $\mathrm{Mod}_R \to \mathrm{Ab}$. Accordingly, it turns out that simplicial R-modules are Kan complexes.

Remark 2.26. If X is a simplicial R-module, then we can form a chain complex $C_\bullet(X)$ with $C_n(X) \coloneqq X_n$, and the differential $\partial\colon C_n(X) \to C_{n-1}(X)$ is given by

$$\partial\sigma \coloneqq \sum_{i=0}^{n}(-1)^i d_i\sigma.$$

For example, we can check that ∂^2 is

$$\sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}d_jd_i\sigma,$$

which vanishes because $d_id_j = -d_jd_i$ when $0 \le i < j \le n$ by the simplicial identities.

Definition 2.27 (free module). Fix a simplicial set X. Then we define $\mathrm{Free}(X) \in \mathrm{sSet}(R)$ to be the simplicial R-module which at level n is the free R-module with basis X_n, and the simplicial maps are induced by functoriality. We will write

$$C_\bullet(X;R) \coloneqq C_\bullet(\mathrm{Free}\,X).$$

For a Kan complex X, we define $C_n(X;R) \coloneqq C_\bullet(\mathrm{Free}\,X)$.

Example 2.28. If X is a topological space, then $C_\bullet(\mathrm{Sing}X;R)$ is the usual chain complex for singular homology.

Example 2.29. We see that $C_\bullet\left(\Delta^0;R\right)$ is

$$\cdots \xrightarrow{1} R \xrightarrow{1} R \xrightarrow{0} R \to 0 \to 0,$$

where the rightmost R is in degree 0.

The above example is a little dishonest because we expect Δ^0 to have nothing interesting in positive degrees. To fix this, we need to get rid of the degenerate simplices, which we do by taking a quotient.

Notation 2.30. Fix a simplicial R-module A. Then we define $D_\bullet(A) \subseteq C_n(A)$ to be the chain complex where $D_n(A)$ is spanned by the degenerate n-simplices.

Remark 2.31. For this definition to make sense, we need to know that R-linear combinations of degenerate simplices in A_n get sent by ∂ to R-linear combinations of degenerate simplices. This amounts to checking that $\sum_i(-1)^id_is_j\sigma$ is degenerate for any j. The interesting thing here is that

$$(-1)^id_is_i + (-1)^{i+1}d_{i+1}s_i = (-1)^i\mathrm{id} + (-1)^{i+1}\mathrm{id} = 0.$$

All other j in the sum make d_is_j manifestly degenerate by the simplicial identities.

Notation 2.32. Fix a simplicial R-module A. Then we define

$$N_\bullet(A) := \frac{C_\bullet(A)}{D_\bullet(A)}.$$

For a Kan complex X, we define $N_n(X;R) := N_\bullet(\operatorname{Free} X)$. Thus, $N_n(A)$ is spanned by the non-degenerate simplices in A_n.

Example 2.33. We see that $N_\bullet\left(\Delta^0;R\right)$ is supported in degree 0.

Example 2.34. We see that $N_\bullet\left(\Delta^2;R\right)$ is

$$\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to 0 \to \cdots.$$

Remark 2.35. If X is a pointed Kan complex, then we can further reduce $N_\bullet(X;R)$ to $\widetilde{N}_\bullet(X;R)$ by taking the cokernel of the embedding

$$N_\bullet\left(\Delta^0;R\right) \to N(X;R).$$

2.2.2 The Dold–Kan Correspondence

Perhaps we should check that we did not lose anything by passing to $N_\bullet(A)$.

Theorem 2.36. Fix a simplicial R-module A. Then there is an explicit chain homotopy equivalence $C_\bullet(A) \to N_\bullet(A)$.

Theorem 2.37 (Dold–Kan correspondence)**.** Consider the functor $N_\bullet$ sending simplicial R-modules to chain complexes of R-modules supported in nonnegative degrees. Then $N_\bullet$ is an equivalence of categories. One frequently denotes the inverse functor by $K_\bullet$.

Remark 2.38. One should not expect $C_\bullet$ to be an equivalence because it does not need to know anything about the degeneracy maps.

Remark 2.39. There are nontrivial enhancements of Theorem 2.37 to other categories beyond Mod_R, such as Grp or Ring.

Example 2.40. Let A be a simplicial abelian group. Then $\pi_n A$ is $\pi_0 \underline{\operatorname{Mor}}\left(\Delta^n/\partial\Delta^n, A\right)$, which via the free–forgetful adjunction is

$$\pi_0 \underline{\operatorname{Hom}}(\operatorname{Free}\Delta^n/\partial\Delta^n, A),$$

and then we may pass to chain complexes as

$$\pi_0 \underline{\operatorname{Hom}}\left(\widetilde{N}_\bullet(\Delta^n/\partial\Delta^n;\mathbb{Z}), \widetilde{N}_\bullet(A)\right),$$

where we are using $\widetilde{N}$ because these things are pointed. Now, $\widetilde{N}_\bullet(\Delta^n/\partial\Delta^n;\mathbb{Z})$ is concentrated in degree n, so after unraveling definitions, one finds that $\pi_n A$ is just n-cycles modulo n-boundaries of $\widetilde{N}(A)$!

Thus, if we are handed a chain complex with known homology, we can go in the reverse direction to produce a Kan complex A (which is in fact a simplicial R-module) with exactly the homotopy groups given by that homology!

Definition 2.41 (Eilenberg–MacLane spaces)**.** Fix an abelian group A and a nonnegative integer $n \geq 0$, we define the *Eilenberg–MacLane space* $K(A, n)$ to be the simplicial abelian group associated to the chain complex

$$\cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots$$

via the Dold–Kan correspondence; here, A sits in degree n.

Remark 2.42. By construction, we see that

$$\pi_i K(A, n) = \begin{cases} A & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

Example 2.43. We can calculate that $K(A, 0)$ has n-simplices given by

$$\mathrm{Mor}_{\mathrm{sSet}}(\Delta^n, K(A, 0)) = \mathrm{Hom}_{\mathrm{sSet}(\mathbb{Z})}(\mathrm{Free}\,\Delta^n, K(A, 0)),$$

which by the Dold–Kan correspondence consists of the morphisms from the chain complex $N_\bullet(\Delta^n)$ to the chain complex corresponding to A (supported in degree 0). Because the target is so simple, one can calculate this is just A for all $n \geq 0$. Thus, we can see that we should have

$$K(A, 0) = \bigsqcup_A \Delta^0.$$

Example 2.44. By construction, we also know that $K(A, n) = \Omega K(A, n + 1)$.

Example 2.45. One can check that $K(A, 1)$ has n-simplices consisting of elements g_{ij} for $0 \leq i < j \leq n$ satisfying the "cocycle condition" $g_{ij} + g_{jk} = g_{ik}$ whenever applicable. It turns out that $K(A, 1) = BA$, which is the nerve of the one-object category corresponding to A.

Example 2.46. On the homework, we will show that $K(\mathbb{Z}, 1)$ is $\mathrm{Sing}\, S^1$.

Let's generalize these constructions.

Notation 2.47. If X is a pointed set, we define $K(X, 0) := \bigsqcup_X \Delta^0$. If G is a group, define $K(G, 1) := BG$.

There are a few more spaces one might be familiar with.

Example 2.48. One has that $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathrm{Sing}\, \mathbb{RP}^\infty$ and $K(\mathbb{Z}, 2) = \mathrm{Sing}\, \mathbb{CP}^\infty$. This explains the prevalence of projective space in algebraic topology courses.

Remark 2.49. Given a Kan complex X and an abelian group A with nonnegative integer $n \geq 0$, the Dold–Kan correspondence tells us that $\underline{\text{Mor}}(X, K(A,n))$ consists of maps $N_*(X)$ to the chain complex of A supported in degree n. It turns out that this corresponds to n-cocycles valued in A, which we denote $N^\bullet(X; A)$. Further, taking π_0, it turns out that taking π_0 produces cohomology, so we will write

$$\mathrm{H}^n(X;A) = \pi_0 \underline{\text{Mor}}(X, K(A,n)).$$

Thus, the Yoneda functor given by $K(A,n)$ recovers cohomology! This characterizes $K(A,n)$ up to homotopy equivalence by the Yoneda lemma.

Remark 2.50. There is a square

$$\begin{tikzcd} \text{Kan} \arrow[r, "\text{Free}"] \arrow[d] & \text{sSet}(\mathbb{Z}) \arrow[d] \\ \text{Spaces} \arrow[r] & D(\mathbb{Z})_{\geq 0} \end{tikzcd}$$

where the right-hand side formally inverts maps of simplicial abelian groups which are equivalences (as maps of simplicial sets). One can think of the bottom arrow as the unique functor which preserves colimits and sends Δ^0 to the chain complex of $\mathbb{Z}$ supported in degree 0; this functor is basically homology (before taking the actual quotients).

2.3 September 30

We will discuss spectral sequences today.

2.3.1 The Serre Spectral Sequences

Today, we do a quick crash course on how to compute the cohomology of some Kan complexes.

Proposition 2.51. Given a short exact sequence

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

of chain complexes, there is a long exact sequence

$$\cdots \to \mathrm{H}_i(A_\bullet) \to \mathrm{H}_i(B_\bullet) \to \mathrm{H}_i(C_\bullet) \to \mathrm{H}_{i-1}(A_\bullet) \to \cdots.$$

Proof. We omit this; it is a standard result in homological algebra. ∎

Intuitively, we may view this as a tool to decompose the homology of $B_\bullet$ via its subspace $A_\bullet$ and the quotient $C_\bullet$.

We may wonder what happens if we have a filtration

$$0 \subseteq (C_1)_\bullet \subseteq (C_2)_\bullet \subseteq \cdots$$

of chain complexes with union $C_\bullet$. Then one can upgrade the long exact sequence into a spectral sequence, which explains how to compute $H_\bullet(C)$ in terms of $H_\bullet(C_i/C_{i-1})$s.

Let's explain roughly how this works. Morally speaking, a class in $\mathrm{H}_i(C_\bullet)$ should arise from some class in $(C_j)_\bullet$; letting j be the smallest such j, then we expect the class to not vanish in the quotient C_j/C_{j-1} by the minimal of j. Thus, we are left to try to take a class in some $H_\bullet(C_i/C_{i-1})$ and ask if we can lift it up to $\mathrm{H}_\bullet(C_i)$, which is measured by the homology of C_{i-1}. Of course, we don't currently have access to the homology of C_{i-1}, but we can ask if the obstruction vanishes in C_{i-1}/C_{i-2}, and now we can inductively push this process inductively backwards.

Definition 2.52 (spectral sequence). A Serre graded, homological *spectral sequence* is a sequence of bi-graded abelian groups $\{E_{pq}^r\}_{i\geq 1}$ (called "pages") equipped with maps $d_r\colon E_{pq}^r \to E_{p-r,q+r-1}^r$ for which $d_r^2 = 0$ and an identification of E^{r+1} with the homology of d^r. In this situation, we define E^∞ as the colimit of E^r. We have *cohomological grading*, writing E_r^{pq}, if our differentials go $d_r\colon E_r^{pq} \to E_r^{p+r,q-r-1}$.

Example 2.53. Continue with our filtration

$$0 \subseteq (C_1)_\bullet \subseteq (C_2)_\bullet \subseteq \cdots$$

of chain complexes with union $C_\bullet$. Then the E_1 page consists of the homology of the quotients C_i/C_{i-1}, and there is a canonical way to extend this into a full spectral sequence.

Here is the main spectral sequence we will be using for this course.

Theorem 2.54 (Serre spectral sequence). Let $F \to E \to B$ be a homotopy fiber sequence of pointed Kan complexes. Suppose further that B is simply connected, meaning that $\pi_0 B$ and $\pi_1 B$ are trivial. Then there is a spectral sequence E with

$$E_{pq}^2 = \mathrm{H}_p(B; \mathrm{H}_q(F; R)).$$

Furthermore, this sequence converges to $\mathrm{H}_{p+q}(E; R)$, meaning that there is a filtration $F^\bullet$ of $\mathrm{H}_{p+q}(E; R)$ with

$$\frac{F^p \mathrm{H}_{p+q}(E; R)}{F^{p-1}\mathrm{H}_{p+q}(E; R)} \cong E_{pq}^\infty.$$

Sketch. Here is a quick construction of the spectral sequence. We may filter $\Delta^0 \to B$ by the skeletal filtration. Taking homotopy pullbacks produces the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E \arrow[d] \\
\Delta^0 \arrow[r] & \mathrm{sk}_0\, B \arrow[r] & \mathrm{sk}_1\, B \arrow[r] & \cdots \arrow[r] & B
\end{tikzcd}$$

where all squares are pullbacks. Then $C_\bullet(E; R)$ has a filtration by $C_\bullet(F_i; R)$, and this filtered complex produces a spectral sequence. We will give a more complete argument later; for example, it is rather non-obvious why we want $\pi_1 B = 0$. ■

Exercise 2.55. We compute $\mathrm{H}_\bullet\left(\Omega S^3; \mathbb{Z}\right)$, where we are thinking of S^3 as the Kan complex $\mathrm{Sing}_\bullet\, S^3$.

Proof. There is a homotopy fiber sequence $\Omega S^3 \to \Delta^0 \to S^3$ by Lemma 2.16. One can check that S^3 is simply connected by working with loops in S^3 directly, so Theorem 2.54 provides us with a spectral sequence E with $E_2 = \mathrm{H}_p\left(S^3; \mathrm{H}_q(\Omega S^3; \mathbb{Z})\right)$ converging to $\mathrm{H}_{p+q}(\Delta^0; \mathbb{Z})$. In the sequel, we will omit the coefficients $\mathbb{Z}$ as much as possible.

To use this, we should start by computing the homology of S^3, which one can compute to be

$$\mathrm{H}_i\left(S^3; A\right) \cong \begin{cases} A & \text{if } i \in \{0, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Our spectral sequence E now looks like

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathrm{H}_2(\Omega_0 S^3) & 0 & 0 & \mathrm{H}_2(\Omega_0 S^3) & 0 & \cdots \\
\mathrm{H}_1(\Omega_0 S^3) & 0 & 0 & \mathrm{H}_1(\Omega_0 S^3) & 0 & \cdots \\
{\color{red}\mathrm{H}_0(\Omega_0 S^3)} & 0 & 0 & \mathrm{H}_0(\Omega_0 S^3) & 0 & \cdots
\end{array}
$$

in the (first quadrant of the) second page; here the $(0,0)$ point has been colored red. The d_2 differential maps up 1 and left 2, so all these maps vanish, so $E^2 = E^3$. Then the d_3 differential maps up 2 and left 3, and all differentials afterward are going to have to vanish automatically. Thus, we see that we are going to converge at $E^\infty = E^4$.

For example, $\mathrm{H}_0(\Omega^0 S^3) = \mathrm{H}_0(\Delta_0;\mathbb{Z}) = \mathbb{Z}$ because the relevant d_3 maps vanish. Similarly, $\mathrm{H}_1(\Omega^0 S^3) = \mathrm{H}_1(\Delta_0;\mathbb{Z}) = 0$. Now, for $i \geq 0$, we see that

$$d_3\colon \mathrm{H}_i(\Omega S^3) \to \mathrm{H}_{i+2}(\Omega S^3)$$

must have no kernel or cokernel in order to have convergence to the (trivial!) homology of a point: $\mathrm{H}^i(\Omega S^3)$ lives at $(3,i)$ and so $\ker d_3$ contributes to $\mathrm{H}^{i+3}(\Delta^0) = 0$, and $\mathrm{H}^{i+2}(\Omega S_3)$ lives at $(0, i+2)$ and so $\operatorname{coker} d_3$ contributes to $\mathrm{H}^{i+2}(\Delta^0) = 0$. We conclude that $\mathrm{H}_i(\Omega S^3)$ is 2-peridioc. ■

Exercise 2.56. We compute $\mathrm{H}_\bullet(K(\mathbb{Z},2);\mathbb{Z})$.

Proof. Set $X := K(\mathbb{Z},2)$ for brevity. Because $K(\mathbb{Z},1) = \Omega K(\mathbb{Z},2)$, and $K(\mathbb{Z},1) = S^1$, there is a homotopy fiber sequence $S^1 \to \Delta^0 \to X$. Note X is simply connected because its homotopy groups vanish away from degree 2. Thus, Theorem 2.54 grants us a spectral sequence E with

$$E^2_{pq} = \mathrm{H}_p(X;\mathrm{H}_q(S^1;\mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(\Delta^0;\mathbb{Z}).$$

We will once again continue to omit the $\mathbb{Z}$s as much as possible.

To use this, we recall the homology $\mathrm{H}^i(S^1;A)$ is concentrated in degrees $i \in \{0,1\}$, where it is A. Our first quadrant of E_2 thus looks like the following.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots \\
\mathrm{H}_0(X) & \mathrm{H}_1(X) & \mathrm{H}_2(X) \arrow[ull] & \mathrm{H}_3(X) \arrow[ull] & \cdots \\
{\color{red}\mathrm{H}_0(X)} & \mathrm{H}_1(X) & \mathrm{H}_2(X) \arrow[ull] & \mathrm{H}_3(X) \arrow[ull] & \cdots
\end{tikzcd}
$$

We have once again highlighted $(0,0)$ in red. We have also drawn in the d_2s, and we can see that the differentials all vanish for higher degrees, so we will converge at E^3. For example, we must have $\mathrm{H}_0(X) \cong \mathrm{H}^0(\Delta^0) = \mathbb{Z}$, and so $\mathrm{H}_1(X) \cong 0$ because $\mathrm{H}_1(X)$ contributes directly to $\mathrm{H}_1(\Delta^0) = 0$. Continuing, for $i \geq 0$, the second differential

$$d_2\colon \mathrm{H}_{i+2}(X) \to \mathrm{H}_i(X)$$

must have trivial kernel and cokernel, which we know to contribute to $\mathrm{H}^{i+2}(\Delta^0)$ and $\mathrm{H}^{i+1}(\Delta^0)$, respectively, which both vanish! Thus, we see that $\mathrm{H}_i(X)$ is 2-periodic. ■

2.3.2 Another Construction

Let's sketch a better construction of the Serre spectral sequence. Recall that we have some ∞-category $D(\mathbb{Z})_{\geq 0}$, which is the category of chain complexes supported in nonnegative degrees, but we have inverted quasi-isomorphisms. (Formally, we proceed as we did to construct Spaces: pushout along the maps $\Delta^1 \to J$ inverting the quasi-isomorphisms to get some $\overline{\mathrm{Ch}(\mathbb{Z})_{\geq 0}}$, and then choose $D(\mathbb{Z})_{\geq 0}$ to be the Kan complex weakly equivalent to it chosen via Lemma 1.142.)

Now, any chain complex $C_\bullet$ admits a filtration

$$C_\bullet = \tau_{\geq 0} C_\bullet \supseteq \tau_{\geq 1} C_\bullet \supseteq \tau_{\geq 2} C_\bullet \supseteq \cdots,$$

where $\tau_{\geq p} C_\bullet$ is the chain complex where we force all terms of degree less than p to vanish, and we replace the pth degree term by the kernel of $C_p \to C_{p-1}$. For example, we see that

$$\mathrm{H}_i(\tau_{\geq p} C_\bullet) = \begin{cases} \mathrm{H}_i(C_\bullet) & \text{if } p \geq i, \\ 0 & \text{if } p < i. \end{cases}$$

We now receive a spectral sequence as follows: we take $E^1 = \bigoplus_i \mathrm{H}_i(C_\bullet)$ and all differentials to vanish, causing all pages to be the same.

Remark 2.57. We claim that $\tau_{\geq p} \colon D(\mathbb{Z})_{\geq 0} \to D(\mathbb{Z})_{\geq 0}$ is a functor. Indeed, it is certainly a functor on the level of chain complexes, and then its construction has ensured that it also sends quasi-isomorphisms to quasi-isomorphisms. In fact, it turns out that they are left adjoints to some suitable forgetful functor from $D(\mathbb{Z})_{\geq p}$.

Now, for our spectral sequence, we suppose that we have a Kan complex B and some functor $G \colon B \to D(\mathbb{Z})_{\geq 0}$, and we suppose that we would like to understand $\mathrm{H}_\bullet(\operatorname{colim} G)$. Well, $\operatorname{colim} G$ gets a filtration by taking $\operatorname{colim}(\tau_{\geq p} \circ G)$, so homological algebra provides us with a spectral sequence

$$E^1 = \mathrm{H}_p(\operatorname{colim} \tau_{\geq q} G / \operatorname{colim} \tau_{\geq q+1} G) \Rightarrow \mathrm{H}_\bullet(\operatorname{colim} G).$$

Now, because left adjoints commute with taking colimits, $\operatorname{colim} \tau_{\geq q} G / \operatorname{colim} \tau_{\geq q+1} G$ can be computed as the colimit of the functor $B \to \mathrm{D}(\mathbb{Z})_{\geq 0}$ which sends $b \mapsto \mathrm{H}_i(Gb; \mathbb{Z})$, where the target is the chain complex supported in degree q.

Note that this functor factors through the classical category Ab and therefore factors through the classical homotopy category $\mathrm{ho}(B)$, which is Δ^0 because B is simply connected! Thus, the functor $B \to \mathrm{D}(\mathbb{Z})_{\geq 0}$ is constant, so its colimit is particularly easy to compute: one gets

$$\mathrm{H}_i(Gb; \mathbb{Z}) \otimes \mathrm{H}_\bullet(B; \mathbb{Z}) = \mathrm{H}_\bullet(B; \mathrm{H}_i(Gb; \mathbb{Z})),$$

where the extra $\mathrm{H}_\bullet(B; \mathbb{Z})$ is present because it is present in the constant functor $B \to D(\mathbb{Z})_{\geq 0}$ placing $\mathbb{Z}$ in degree 0.

For the application to the Serre spectral sequence, we need the following.

Theorem 2.58 (Straightening correspondence)**.** If $F \to E \to B$ is a homotopy fiber sequence with B connected, then there is a functor $G \colon B \to \mathrm{Spaces}$ taking each $b \in B_0$ to F and such that

$$E = \operatorname{colim} G.$$

Remark 2.59. Grothendieck considered a version of this statement where B is an ordinary category, and Spaces is replaced by ordinary groupoids. Indeed, this makes a stack: it's a functor from rings to groupoids!

Remark 2.60. Professor Jeremy Hahn has many stories about people who only know that a scheme is a functor from rings to sets.

We now apply the functor $C_\bullet \colon \mathrm{Spaces} \to D(\mathbb{Z})_{\geq 0}$ to see that $C_\bullet E$ is the colimit of the functor

$$B \xrightarrow{G} \mathrm{Spaces} \xrightarrow{C} D(\mathbb{Z})_{\geq 0},$$

for which we can use the preceding discussion.

Remark 2.61. Of course, we have been working with E^1, but it turns out that one can shear E^1 and compute some differentials to pass to E^2, recovering Theorem 2.54.

2.4 October 2

The next problem set will be released shortly.

Remark 2.62. If B is connected, then changing basepoints of our pointed Kan complexes will not change the homotopy fiber F, though changing the choice of equivalence of the homotopy fibers depends on the choice of paths. We may say that this choice is "not contractible"; notably, even if B is simply connected, then the choice of path may be canonical, but it is still not a contractible choice!

Example 2.63. The most basic homotopy fiber sequence is $F \to F \times B \to B$. In this case, one finds that

$$\mathrm{H}_n(F \times B; R) = \bigoplus_{p+q=n} \mathrm{H}_p(B; \mathrm{H}_q(F; R))$$

by something like the Künneth formula. When $\mathrm{H}_q(F : R)$ is free over R (for example, if R is a field), then the right-hand summands are $\mathrm{H}_p(B; R) \otimes_R \mathrm{H}_q(F; R)$. This should be compared against Theorem 2.54 (and indeed, this can be proved with the Serre spectral sequence), where some higher differentials may cause the homology of a general E to be smaller!

2.4.1 Computing Cup Products

Quickly, we note that there is a Serre spectral sequence in cohomology, whose proof is totally dual.

Theorem 2.64. Fix a homotopy fiber sequence $F \to E \to B$ of Kan complexes. Then there is a cohomologically graded spectral sequence E with $E_2^{pq} = \mathrm{H}^p(B; \mathrm{H}^q(F; R))$ and $E_\infty^{pq} = \mathrm{H}^{p+q}(E; R)$.

Remark 2.65. If we have $\mathrm{H}^p(B; \mathrm{H}^q(F; R)) = \mathrm{H}^p(B; R) \otimes_R \mathrm{H}^q(F; R)$, then there is a cup product structure on E_2 given by taking

$$(x \otimes y) \cup (x' \otimes y') = (-1)^{\deg(y)\deg(x')}(x \cup x')(y \cup y').$$

It further turns out that $d_r(x \cup y) = d_r x \cup y + (-1)^{p+q} x \cup d_r y$ when x is in bidegree (p, q).

Exercise 2.66. As in Exercise 2.55, one can check that

$$\mathrm{H}^i\left(\Omega S^3;\mathbb{Z}\right)=\begin{cases}\mathbb{Z} & \text{if } i \text{ is even,}\\ 0 & \text{if } i \text{ is odd.}\end{cases}$$

We compute the cup products.

Proof. As discussed in Exercise 2.55, our spectral sequence looks like

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0
\end{tikzcd}$$

in both the second and third page; we have drawn in a single d_3 arrow, which we know must be an isomorphism in order to cause the $E_\infty = E_4$ page to give the correct answer. Note that $\mathrm{H}^\bullet\left(S^3;\mathbb{Z}\right)$ is $\mathbb{Z}[e]/\left(e^2\right)$ with e sitting in degree 3, which we can see directly from its cohomology.

Now, for each i, we define x_i to be a generator of $\mathrm{H}^{2i}\left(\Omega S^3;\mathbb{Z}\right)$. We will require that $d_3 x_{i+1} = x_i e$ for each i, where the product is the one described in Remark 2.65; notably, multiplication by e does give an isomorphism from bidegree $(0,2i)$ to $(2,2i)$ and shows that $x_i e$ is in fact a generator.

For example, x_1^2 is some multiple of x_2, so we compute

$$d_3\left(x_1^2\right)=d_3(x_1)x_1+x_1d_3(x_1)=2x_1.$$

(Note that everything in sight is commutative because at least one of the degrees is even at all times.) On the other hand, $d_3(x_2)=x_1$, so we conclude $x_1^2=2x_2$. Similarly,

$$d_3(x_1x_2)=d_3(x_1)x_2+x_1d_3(x_2)=ex_2+x_1(2x_1)=3x_2e,$$

so $x_1x_2=3x_2e$. In general, one has that $x_ix_j=\binom{i+j}{i}x_{i+j}$, which can be checked with a quick induction, so this is a divided power algebra. ■

Exercise 2.67. On the homework, we will show that

$$\mathrm{H}^\bullet\left(K(\mathbb{Z},2);\mathbb{Z}\right)=\mathbb{Z}\left[x\right],$$

where x lives in degree 2.

Exercise 2.68. We compute some parts of the ring $\mathrm{H}^\bullet(K(\mathbb{Z},3);\mathbb{Z})$.

Proof. There is a homotopy fiber sequence $K(\mathbb{Z},2)\to\Delta^0\to K(\mathbb{Z},3)$, so Theorem 2.54 provides us with a spectral sequence E with

$$E_2^{pq}=\mathrm{H}^p\left(K(\mathbb{Z},3);\mathrm{H}^q(K(\mathbb{Z},2);\mathbb{Z})\right).$$

Here are some example calculations.

- Note
$$E_2^{0q}=\mathrm{H}^0\left(K(\mathbb{Z},3);\mathrm{H}^q(K(\mathbb{Z},2);\mathbb{Z})\right)=\mathrm{H}^q\left(\mathbb{CP}^\infty;\mathbb{Z}\right),$$
  which we know is $\mathbb{Z}$ in even degrees and 0 otherwise.

- To compute E_2^{1q}, we note that it is the dual of H_1, which is the abelianization of π_1, which vanishes.
- Note E_2^{pq} vanishes for odd q because $\mathrm{H}^q(K(\mathbb{Z},2);\mathbb{Z}) = 0$.

Our E_2 page now looks like the following.

$$\begin{array}{cccccc}
0 & 0 & & & & \\
\mathbb{Z}x^2 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}x & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & & & &
\end{array}$$

For example, we now see that $E_2^{2,0}$ vanishes because no differentials can interact with this term, and we need to get the cohomology of Δ^0. This corresponds to computing $\mathrm{H}^2(K(\mathbb{Z},3);\mathbb{Z})$, so the entire column also vanishes. Similarly, the fourth column vanishes.

Continuing, the differential $d_3\colon E^{02} \to E^{30}$ shows that $\mathrm{H}^3(K(\mathbb{Z},3);\mathbb{Z})$ is $\mathbb{Z}$. This produces the following.

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & & \\
\mathbb{Z}x^2 & 0 & 0 & \mathbb{Z}x^2e & & \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}x & 0 & 0 & \mathbb{Z}xe & & \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}e & ? &
\end{array}$$

As a harder calculation, we note that $d_3\colon E^{04} \to E^{32}$ is the map $\mathbb{Z}x^2 \to \mathbb{Z}xe$, and we can calculate that x^2 maps to $2xe$. Accordingly, one can calculate that $E_\infty^{50} = 0$, so the fifth column also vanishes.

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}x^2 & 0 & 0 & \mathbb{Z}x^2e & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}x & 0 & 0 & \mathbb{Z}xe & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}e & 0 & 0 & ?
\end{array}$$

Because $d_3\colon \mathbb{Z}x^2 \to \mathbb{Z}xe$ is multiplication by 2, we see that $E_\infty^{60} = \mathbb{Z}/2\mathbb{Z}$, so we get the following.

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}x^2 & 0 & 0 & \mathbb{Z}x^2e & 0 & 0 & \mathbb{Z}2/\mathbb{Z}x^2f \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}x & 0 & 0 & \mathbb{Z}xe & 0 & 0 & \mathbb{Z}2/\mathbb{Z}xf \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}e & 0 & 0 & \mathbb{Z}/2\mathbb{Z}f
\end{array}$$

One can similarly see that the seventh column vanishes. But we can now calculate E_2^{80} is $\mathbb{Z}/3\mathbb{Z}$, so things seem to be getting complicated. ■

Remark 2.69. The presence of $\mathrm{H}^6(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ means that there is a map $K(\mathbb{Z},3) \to K(\mathbb{Z},6)$ given by the universal property. Applying the universal property again tells us that there is a natural map

$$\mathrm{H}^3(-;\mathbb{Z}) \to \mathrm{H}^6(-;\mathbb{Z}),$$

which turns out to be squaring (with the cup product). Because $\alpha \cup \alpha$ is always 2-torsion for $\alpha \in \mathrm{H}^3(X;\mathbb{Z})$ (because it is equal to its negative by the graded commutative), we find that this class is 2-torsion! Similarly, we see that there is no natural transformation $\mathrm{H}^3(-;\mathbb{Z}) \to \mathrm{H}^7(-;\mathbb{Z})$.

Remark 2.70. More generally, one sees that the cup product amounts to a map

$$K(\mathbb{Z},p) \times K(\mathbb{Z},q) \to K(\mathbb{Z},p+q).$$

Remark 2.71. The induced natural transformation $\mathrm{H}^3(-;\mathbb{Z}) \to \mathrm{H}^8(-;\mathbb{Z})$ is a Steenrod operation, which we will discuss later. The moral of our story is that the cohomology of $K(\mathbb{Z},n)$s tell us all natural transformations on cohomology.

2.5 October 7

Today we will continue using the Serre spectral sequence.

2.5.1 Hurewicz's Theorem

Throughout, $F \to E \to B$ is a homotopy fiber sequence of pointed Kan complexes with B simply connected. For any ring R, we then receive a Serre spectral sequence

$$E^2_{pq} = \mathrm{H}_p(B; \mathrm{H}_q(F;R)) \Rightarrow \mathrm{H}_{p+q}(E;R).$$

We will also assume that F is connected, which implies that $\mathrm{H}_0(F;R) = R$, so $E^2_{0q} = \mathrm{H}_q(F;R)$ and $E^2_{p0} = \mathrm{H}_p(B;R)$, which we visualize as follows.

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathrm{H}_2(F;R) & ? & ? & \cdots \\
\mathrm{H}_1(F;R) & ? & ? & \cdots \\
R & \mathrm{H}_1(B;R) & \mathrm{H}_2(B;R) & \cdots
\end{array}$$

The direction of the arrows implies that the E^∞ page will contain a quotient of $\mathrm{H}_q(F;R)$ and a subset of $\mathrm{H}_p(B;R)$.

Remark 2.72. It turns out that the aforementioned quotient of $\mathrm{H}_q(F;R)$ is the image from $\mathrm{H}_q(E;R)$, and the aforementioned subset of $\mathrm{H}_p(B;R)$ is the image of $\mathrm{H}_p(E;R)$. Both of these follow from unwinding the construction of the spectral sequence. For example, E^∞_{0q} should come from the filtered complex $C_\bullet(E;R)$ as the pieces which come from the zeroth part of the filtration, which is $C_\bullet(F;R)$.

Let's explain how we can try to compute some homotopy groups. Hopefully one can handle π_0 and π_1. (Perhaps π_1 is not easy, but at least its abelianization is H_1.) If we wanted to compute $\pi_2(X)$ for a simply connected Kan complex X, then we see that

$$\pi_2(X) = \pi_2(X)^{\mathrm{ab}} = \pi_1(\Omega X)^{\mathrm{ab}} = \mathrm{H}^1(\Omega X;\mathbb{Z}),$$

where we are using that the higher homotopy groups are abelian (via Lemma 1.129) and Remark 2.11. But now we can try to compute $\mathrm{H}_1(\Omega X;\mathbb{Z})$ using the homotopy fiber sequence $\Omega X \to \Delta^0 \to X$ of Lemma 2.16. Then the Serre spectral sequence (Theorem 2.54) produces the following E_2 page.

$$\begin{array}{cccc}
\mathrm{H}_3(\Omega X) & ? & ? & ? \\
\mathrm{H}_2(\Omega X) & ? & ? & ? \\
\mathrm{H}_1(\Omega X) & ? & ? & ? \\
\mathbb{Z} & \mathrm{H}_1(X) & \mathrm{H}_2(X) & \mathrm{H}_3(X)
\end{array} \tag{2.1}$$

Note $\mathrm{H}_1(X) = \pi_1(X)^{\mathrm{ab}} = 0$ because X is simply connected. We thus see that E_{01}^∞ needs to vanish (because it is computing homology of Δ^0), so $d_2\colon \mathrm{H}_2(X) \to \mathrm{H}_1(\Omega X)$ must be an isomorphism. We have thus proven the following.

Proposition 2.73. Suppose X is a simply connected pointed Kan complex. Then

$$\pi_2(X) = \mathrm{H}_2(X;\mathbb{Z}).$$

Here is the more general result.

Theorem 2.74 (Hurewicz)**.** Fix a pointed Kan complex X and a positive integer $n \geq 2$. If $\pi_k(X) = 0$ for $k < n$, then the natural map

$$\pi_n(X) \to \mathrm{H}_n(X;\mathbb{Z})$$

is an isomorphism.

Proof. The natural map sends a loop to the corresponding cycle; in other words, it sends the element $\Delta^n \to X$ to the image in $\mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(X;\mathbb{Z})$.

The proof is by an induction exactly as above. Suppose we have the statement at level n, and we would like to show it for $n+1$. In particular, we know that $\pi_k(X) = 0$ for $k < n+1$, so the induction tells us that $\mathrm{H}_k(X)$ for $k < n+1$. Now, we once again note that

$$\pi_{n+1}(X) = \pi_n(\Omega X) \cong \mathrm{H}_n(\Omega X),$$

and we would like to show that this right-hand group is isomorphic to $H_{n+1}(X)$. For example, we get to know that $\mathrm{H}_k(\Omega X) = 0$ for $k < n$. As usual, we will use the Serre spectral sequence (Theorem 2.54), which outputs the E_2 page (2.1). However, basically everything vanishes, and when we want to compute $\mathrm{H}_n(\Delta)$, we find that everything has vanished except $\mathrm{H}_n(\Omega X)$, which only admits the differential

$$d\colon \mathrm{H}_{n+1}(X) \to \mathrm{H}_n(\Omega X).$$

Thus, to vanish in E_∞, this must be an isomorphism, and the result follows. One can check that the various composites involved do produce the above natural map by unwinding the spectral sequence in this simple case. ■

Example 2.75. One can show that $\pi_0 S^3$ and $\pi_1 S^3$ are both trivial, so Theorem 2.74 tells us that

$$\pi_2 S^3 = \mathrm{H}^2\left(S^3;\mathbb{Z}\right) = 0.$$

Thus, Theorem 2.74 kicks in to tell us that

$$\pi_3 S^3 = \mathrm{H}^3\left(S^3;\mathbb{Z}\right) \cong \mathbb{Z}.$$

In general, one finds that $\pi_k S^n$ is trivial for $k < n$ by an induction. Then $\pi_n S^n \cong \mathrm{H}^n(S^n;\mathbb{Z}) = \mathbb{Z}$ for $n \geq 1$.

Example 2.76. We compute $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. There is a map $f\colon S^3 \to K(\mathbb{Z},3)$ given by choosing a generator of $\mathrm{H}^3(S^3;\mathbb{Z}) \cong \mathbb{Z}$. This is an isomorphism on H^i for $i \le 3$ (the groups are trivial at $i < 3$), so the Universal coefficient theorem tells us that it is also an isomorphism on H_i for $i \le 3$, and Theorem 2.74 then tells us that it is an isomorphism on π_* for $i \le 3$. Now, let F be the homotopy fiber of f, so the long exact sequence in homotopy (Remark 1.139) produces the exact sequence

$$\pi_{i+1}K(\mathbb{Z},3) \to \pi_i F \to \pi_i S^3 \to \pi_i K(\mathbb{Z},3),$$

so plugging in various vanishings shows that $\pi_i F$ is trivial for $i \in \{0,1,2,3\}$ and $\pi_4 F \cong \pi_4 S^3$.

But now we can use Theorem 2.74 to see that $\pi_4 F \cong \mathrm{H}_4(F;\mathbb{Z})$! We can now use the Serre spectral sequence. Comparing with the Universal coefficient theorem, one can compute some homology of $K(\mathbb{Z},3)$ via comparing with its cohomology as computed in Exercise 2.68. In particular, one finds that

$$\mathrm{H}_i(K(\mathbb{Z},3);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0,4,6\}, \\ 0 & \text{if } i \in \{1,2,4,6\}, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 5, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } i = 6. \end{cases}$$

(Note that there is some degree shift due to the presence of Ext groups in the Universal coefficient theorem.) In particular, $E^2_{pq} = \mathrm{H}_p(K(\mathbb{Z},3);\mathrm{H}_q(F))$ looks like the following.

$$\begin{array}{cccccc} \pi_4 S^3 & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \end{array}$$

Thus, we see that $\pi_4 S^3$ will only interact with the differential $d_5\colon \mathbb{Z}/2\mathbb{Z} \to \pi_4 S^3$, but in E^∞ these groups compute $\mathrm{H}^4\left(S^3\right) = \mathrm{H}^5\left(S^3\right) = 0$, so it follows that d_5 is an isomorphism, so we conclude that $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$. ■

Remark 2.77. Heuristically, we thus see that the homotopy groups of spheres "know about" the (co)-homology of Eilenberg–MacLane spaces. This is one reason to care about them.

2.5.2 The Whitehead Tower

Fix a pointed Kan complex X, and we will build a "tower" of spaces.

1. We define $\tau_{\ge 1}X$ to be the path component of the basepoint, which we can rigorously think of as the full ∞-subcategory generated by the objects isomorphic to the basepoint. Then we see that the inclusion $\tau_{\ge 1}X$ is a π_*-isomorphism in degrees bigger than 0, but $\pi_0\tau_{\ge 1}X = 0$.
2. Then let $\tau_{\ge 2}X$ be the universal cover of $\tau_{\ge 1}X$, and we see that $\tau_{\ge 2}X \to \tau_{\ge 1}X$ is a π_* isomorphism in degrees bigger than 1, but $\pi_1(\tau_{\ge 2}X)$ vanishes.
3. To continue, note that $\tau_{\ge 2}X$ is simply connected, so Theorem 2.74 tells us that $A := \pi_2(\tau_{\ge 2}X)$ is isomorphic to $\mathrm{H}_2(\tau_{\ge 2}X;\mathbb{Z}) \cong \mathrm{H}^2(\tau_{\ge 2}X;\mathbb{Z})$ and thus produces a map

$$\tau_{\ge 2}X \to K(A,2)$$

which is further an isomorphism on π_2. We then define $\tau_{\ge 3}X$ as the homotopy fiber of this map so that the inclusion $\tau_{\ge 3}X \to \tau_{\ge 2}X$ is a π_*-isomorphism in degrees at least 3, but $\pi_2(\tau_{\ge 3}X)$ vanishes.

Iterating the process in the last step, we see that we have produced a tower

$$\begin{tikzcd} \vdots \arrow[d] & \\ \tau_{\geq 3}X \arrow[r] \arrow[d] & K(\pi_2 X, 2) \\ \tau_{\geq 2}X \arrow[r] \arrow[d] & K(\pi_2 X, 2) \\ \tau_{\geq 1}X \arrow[d] & \\ X & \end{tikzcd} \tag{2.2}$$

where $\tau_{\geq n+1}X \to \tau_{\geq n}X \to K(\pi_n X, n)$ is a homotopy fiber sequence for each n, and this implies that

$$\pi_k(\tau_{\geq n}X) = \begin{cases} 0 & \text{if } k < n, \\ \pi_k(X) & \text{if } k \geq n. \end{cases}$$

Definition 2.78 (Whitehead tower)**.** Given a pointed Kan complex X, we define the *Whitehead tower* as the tower constructed in (2.2).

Remark 2.79. Because we have built these things as homotopy fiber sequences, the entire tower is unique up to homotopy.

Remark 2.80. The point is that the $\tau_{\geq k}X$ have lots of earlier vanishing homotopy groups, so one can hope to use the method of Example 2.76 to compute homotopy groups if only we understood the homology of the $\tau_{\geq k}$s well enough.

Remark 2.81. There is also a "Postnikov tower"

$$\cdots \to \tau_{\leq 2}X \to \tau_{\leq 1}X \to \tau_{\leq 0}X$$

for any unpointed Kan complex X; it turns out that X is the limit of this diagram. If X is pointed, then one can reconstruct $\tau_{\geq n+1}X$ as the homotopy fiber of the canonical map $X \to \tau_{\leq n}X$. In general, $\tau_{\leq 0}X = \coprod_{\pi_0 X} \Delta^0$ and $\tau_{\leq 1}X$ is the homotopy category of X; going further, $\tau_{\leq n}X$ should be the "homotopy n-category."

2.5.3 Whitehead's Theorem for Homology

We conclude with a theorem.

Theorem 2.82 (Whitehead, homology)**.** Fix a map $f\colon X \to Y$ of simply connected pointed Kan complexes. Then f is a homotopy equivalence if and only if $\mathrm{H}_*(f;\mathbb{Z})$ is an isomorphism of graded groups.

Proof. Let F be the homotopy fiber of f. To show that f is a homotopy equivalence, it is equivalent to show that $\pi_* f$ is an isomorphism in all degrees (by Theorem 1.131) which is equivalent to showing that $\pi_* F$ is trivial in all degrees. By Theorem 2.74, it is now enough to just check that $\pi_1 F = 0$ and $\mathrm{H}_n(F) = 0$ for all $n \geq 1$.

For one, $\pi_1(F) = 0$ by the homotopy long exact sequence (Remark 1.139), which works because X and Y are simply connected. This is a little involved, so let's explain. Note $\pi_0 F$ trivializes because of the exact sequence $\pi_1 Y \to \pi_0 F \to \pi_0 X$. For $\pi_1 F$, we write

$$\pi_2 X \to \pi_2 Y \to \pi_1 F \to \pi_1 X.$$

The last group vanishes because X is simply connected, so it remains to show that the map $\pi_2 X \to \pi_2 Y$ is surjective. Well, by Theorem 2.74, we see that the map $\pi_2 X \to \pi_2 Y$ is isomorphic to the map $\mathrm{H}_2(X;\mathbb{Z}) \to \mathrm{H}_2(Y;\mathbb{Z})$, which is an isomorphism by hypothesis on f.

It remains to show that $\mathrm{H}_n(F) = 0$ for $n \geq 1$, for which we use the Serre spectral sequence (Theorem 2.54). Well, Remark 2.72 has informed us that E^∞_{p0} is the image of $\mathrm{H}_p(X) \to \mathrm{H}_p(Y)$. But this is an isomorphism, so there cannot be any differentials outside this, and everything must vanish on E^∞ in order to get the right answer. One then inductively finds that $\mathrm{H}_n(F) = 0$ for $n \geq 1$ by studying how to get the Serre spectral sequence to never modify $E^\bullet_{p0}$s. ■

Remark 2.83. This is amazing, and suggests that we may as well spend out time studying homology. However, it turns out that other fields of mathematics find themselves caring about homotopy groups anyway, so it is worthwhile to have tools to compute them.

2.6 October 9

Today we continue computing homotopy groups.

2.6.1 Mod $\mathcal{C}$ Hurewicz's Theorem

Here is another application: we can characterize Eilenberg–MacLane spaces.

Proposition 2.84. Fix an abelian group A and a nonnegative integer $n \geq 2$. For a simply connected pointed Kan complex X such that

$$\pi_i X \cong \begin{cases} 0 & \text{if } i \neq n, \\ A & \text{if } i = n, \end{cases}$$

there is a homotopy equivalence $X \to K(A, n)$.

Proof. By Theorem 2.74, we know that $\mathrm{H}^n(X;\mathbb{Z}) = A$ (and $\mathrm{H}^i(X,\mathbb{Z}) = 0$ for $i < n$), so the universal property of $K(A,n)$ provides a map $f\colon X \to K(A,n)$ which is an isomorphism on $\mathrm{H}^n(-;\mathbb{Z})$. It follows by Theorem 2.74 that f is an isomorphism on π_n, and it is an isomorphism on π_i for $i \neq n$ because the relevant groups vanish, so the result follows from Theorem 1.131. ■

Remark 2.85. In fact, the homotopy equivalence $X \to K(A,n)$ is now unique up to "(-1)-connected" choice, meaning that it is canonical up to a choice of isomorphism $\pi_n X \cong A$.

Here is an application of the Whitehead tower: we will be able to show that homotopy groups are frequently finitely generated.

Notation 2.86. Throughout, $\mathcal{C}$ denotes a subclass of abelian groups which are either finitely generated or the subclass of finite abelian groups which are P-torsion for a set of primes P.

Example 2.87. If $\mathcal{P}$ is the set of all primes, then $\mathcal{C}$ becomes all finite abelian groups. But if $\mathcal{P} = \{2\}$, then these are 2-groups.

Lemma 2.88. Consider one of the full subcategories $\mathcal{C}$ of abelian groups.

(a) The category $\mathcal{C}$ admits finite limits and colimits.

(b) The category $\mathcal{C}$ is closed under extensions.

(c) For $A, B \in \mathcal{C}$, then $A \otimes_{\mathbb{Z}} B$ and $\operatorname{Tor}_1(A, B)$ are in $\mathcal{C}$.

Proof. These are all purely formal. ■

Lemma 2.89. Fix a homotopy fiber sequence $F \to E \to B$ of pointed Kan complexes with B simply connected. If any two of the families

$$\{\mathrm{H}_i(B;\mathbb{Z})\}_i\,, \qquad \{\mathrm{H}_i(E;\mathbb{Z})\}_i\,, \qquad \text{and} \qquad \{\mathrm{H}_i(F;\mathbb{Z})\}_i$$

are in $\mathcal{C}$, then so is the third.

Proof. We will only show one of the three implications because the other two are similar. In particular, suppose that $\widetilde{H}_i(B;\mathbb{Z})$ and $\widetilde{H}_i(F;\mathbb{Z})$ are in $\mathcal{C}$ for all i, and we want to show the same for $\widetilde{H}_i(E;\mathbb{Z})$. For this, we use the Serre spectral sequence (Theorem 2.54)

$$E^2_{pq} = \mathrm{H}_p(B; \mathrm{H}_q(F;\mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(E;\mathbb{Z}).$$

We can handle $\widetilde{H}^0(E;\mathbb{Z})$ directly by staring at the corner of the Serre spectral sequence, so we suppose that $i > 0$.

Now, by the filtration on E^∞, it is enough by Lemma 2.88 to merely show that the individual terms E^∞_{pq} are in $\mathcal{C}$. Because E^∞ is constructed from E^2 by taking some repeated kernels and quotients, it is enough by Lemma 2.88 to show that E^2_{pq} is in $\mathcal{C}$ when one of p or q is positive. One can handle $p = 0$ or $q = 0$ by hand, so we assume that p and q are positive. Then we note there is a short exact sequence

$$0 \to \mathrm{H}_p(B) \otimes \mathrm{H}_q(F) \to \mathrm{H}_p(B; \mathrm{H}_q(F)) \to \operatorname{Tor}_1(\mathrm{H}_{p-1}(B), \mathrm{H}_{q-1}(F)) \to 0$$

by the Universal coefficients theorem, so we are done by Lemma 2.88. ■

Lemma 2.90. Fix an abelian group A in $\mathcal{C}$. Then $\mathrm{H}_i(K(A,n);\mathbb{Z})$ is in $\mathcal{C}$ for all $i, n > 0$.

Proof. If $n \geq 2$, then the homotopy fiber sequence $K(A, n-1) \to \Delta^0 \to K(A,n)$ allows us to reduce to the case $n-1$ by Lemma 2.89, so we may assume that $n = 1$. (Note that $n \geq 2$ makes $K(A,n)$ simply connected!) We now have to study the homology of $K(A,1)$. Well, writing

$$K(A,1) = \mathbb{Z}^{\oplus r} \bigoplus_{p \text{ prime}} \bigoplus_{\nu \geq 0} (\mathbb{Z}/p^\nu\mathbb{Z})^{r_{p,\nu}}$$

for some nonnegative integers r and $r_{p,\nu}$. Now, homotopy commutes with taking products of spaces, so it follows that $K(A,1)$ splits as a finite product of $K(-,1)$s of the above spaces, so by the Künneth formula, it is enough to handle A equal to $\mathbb{Z}$ or some p-power cyclic group $\mathbb{Z}/p^\nu\mathbb{Z}$. For $A = \mathbb{Z}$, we have $K(\mathbb{Z},1) = S^1$, whose homology in positive degree is supported in degree 1, and it outputs $\mathbb{Z}$.

It remains to handle $A = \mathbb{Z}/p^\nu\mathbb{Z}$. Note that the map $p^n\colon \mathrm{H}^2(K(\mathbb{Z},2);\mathbb{Z}) \to \mathrm{H}^2(K(\mathbb{Z},2);\mathbb{Z})$ induces a map $p^n\colon K(\mathbb{Z},2) \to K(\mathbb{Z},2)$ by the universal property. Now, let F be the fiber of this latter p^n map, so the homotopy long exact sequence produces exact sequences

$$\pi_{i+1}K(\mathbb{Z},2) \stackrel{p^n}{\to} \pi_{i+1}K(\mathbb{Z},2) \to \pi_i F \to \pi_i K(\mathbb{Z},2) \stackrel{p^n}{\to} \pi_i K(\mathbb{Z},2),$$

so $\pi_i F$ is supported in degree 1, where it is isomorphic to A. Thus, Proposition 2.84 tells us F is $K(A,1)$, so we get a homotopy fiber sequence

$$K(\mathbb{Z},1) \to K(A,1) \to K(\mathbb{Z},2)$$

by taking loops and using Proposition 2.84. The result now follows by expanding the Serre spectral sequence to compute (co)homology. ■

Theorem 2.91. Fix a simply connected pointed Kan complex X, and choose a class $\mathcal{C}$ as in Notation 2.86. If $\pi_i X \in \mathcal{C}$ for $0 < i < n$, then the natural map $\pi_n X \to \mathrm{H}_n(X;\mathbb{Z})$ is an isomorphism modulo $\mathcal{C}$, meaning that the kernel and cokernel live in $\mathcal{C}$.

Proof. We use the Whitehead tower $\{\tau_{\geq i}X \to K(\pi_i X, i)\}$ for $i \in \{2,\ldots,n\}$. Because X is simply connected, we see that $X = \tau_{\geq 2}X$, so this will be enough. Because $\pi_n X = \mathrm{H}_n(\tau_{\geq n}X;\mathbb{Z})$ by construction of the Whitehead tower, we see that we need to show

$$\mathrm{H}^n(\tau_{\geq n}X;\mathbb{Z}) \to \mathrm{H}^n(\tau_{\geq 2}X;\mathbb{Z})$$

is an isomorphism modulo $\mathcal{C}$. We will do this by an induction, so it is enough to just do this in one step for $\tau_{\geq j}X \to \tau_{\geq j-1}X$, where $j-1 \geq 2$. Here, we have the homotopy fiber sequence $\tau_{\geq j}X \to \tau_{\geq j-1}X \to K(\pi_{j-1}X, j-1)$ by construction of the tower, so we get a Serre spectral sequence (Theorem 2.54) given by

$$E^2_{pq} = \mathrm{H}_p(K(\pi_{j-1}X, j-1); \mathrm{H}_q(\tau_{\geq j}X;\mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(\tau_{\geq j-1}X;\mathbb{Z}).$$

Now, $\mathrm{H}_p(K(\pi_{j-1}X, j-1);\mathbb{Z})$ is always in $\mathcal{C}$ by Lemma 2.90, so the result follows from staring hard at the relevant Serre spectral sequence, where the point is that the differentials only manage to go to and from things in $\mathcal{C}$. ■

Corollary 2.92. For any $i, n \geq 0$, the group $\pi_i(S^n)$ is finitely generated.

Proof. For $n \in \{0,1\}$, there is nothing to do. For $n \geq 2$, we know S^n is simply connected, so we may apply Theorem 2.91. Then let $\mathcal{C}$ be the class of finitely generated abelian groups, so we know that the least i for which $\pi_i X \notin \mathcal{C}$ has that the morphism $\pi_i X \to \mathrm{H}_i(X;\mathbb{Z})$ has kernel and cokernel in $\mathcal{C}$. But then $\mathrm{H}_i(X;\mathbb{Z})$ is always in $\mathcal{C}$ (for example, by using cellular homology), which implies that $\pi_i X$ is in $\mathcal{C}$, which is our contradiction. ■

Corollary 2.93. For $i > 3$, the group $\pi_i S^3$ is finite.

Proof. As in Example 2.76, there is a homotopy fiber sequence $F \to S^3 \to K(\mathbb{Z},3)$, where $S^3 \to K(\mathbb{Z},3)$ is given by an isomorphism $\mathrm{H}^3\left(S^3;\mathbb{Z}\right) \cong \mathbb{Z}$. Thus, $\pi_i S^3$ being finite for $i > 3$ is equivalent to showing that $\pi_i F$ is finite for all i (because the homotopy long exact sequence erases this "problem" at $i = 3$). Now, F is simply connected by a long exact sequence calculation, so we may apply Theorem 2.91 to see that it is equivalent to show that $\mathrm{H}_i(F;\mathbb{Z})$ is finite for all i.

However, Lemma 2.89 does do enough to tell us that $\mathrm{H}_i(F;\mathbb{Z})$ is finitely generated, so it is enough to show that $\mathrm{H}_i(F;\mathbb{Z})_{\mathbb{Q}} = 0$, which can be done via the Serre spectral sequence and calculating the rational homology of S^3 and $K(\mathbb{Z},3)$. ■

2.6.2 The Suspension

We are interested in understanding Spaces, which we note can be understood as nice topological spaces up to homotopy equivalence or Kan complexes up to homotopy equivalence or even simplicial sets up to weak equivalence. We have already said something about homotopy pullbacks, so let's discuss homotopy pushouts.

Definition 2.94 (homotopy pushout)**.** Given maps $C \leftarrow A \to B$ of simplicial sets, the *homotopy pushout* is the pushout of the diagram

$$\begin{tikzcd} A \sqcup A \arrow[r] \arrow[d] & B \sqcup C \\ A \times \Delta^1 & \end{tikzcd}$$

up to weak equivalence. For example, the map $A \times \Delta^1 \to B \times \{1\}$ is the product of the maps $A \to B$ and $\Delta^1 \twoheadrightarrow \{1\}$.

Remark 2.95. There is something subtle here that one may take the pushout and not receive something which is not a Kan complex in general!

Remark 2.96. There is also a pointed version, where we have to additionally produce a basepoint on the homotopy pushout, which we do by modding out by $\{a\} \times \Delta^1$.

Remark 2.97. Intuitively, we are making A into a cylinder as $A \times \Delta^1$, and one "face" of the cylinder is being glued onto C (via the map $A \to C$) and the other "face" of the cylinder is being glued onto B (via the map $A \to B$).

We will not need to work in full levels of generality for the homotopy pushout.

Definition 2.98 (suspension)**.** Fix a space X with basepoint x. Then we define the *suspension* ΣX is the quotient

$$\frac{X \times \Delta^1}{(X \times \{0,1\}) \cup (\{x\} \times \Delta^1)},$$

where the quotient means that we are collapsing $X \times \{0\}$ and $X \times \{1\}$ to a point, and we are collapsing $\{x\} \times [0,1]$ to a point.

Remark 2.99. Equivalently, ΣX is the homotopy pushout of $\Delta^0 \leftarrow X \to \Delta^0$ in pointed spaces. (The pointed-ness is why we have to collapse $\{x\} \times [0,1]$ in this definition.)

Remark 2.100. Comparing pullbacks and pushouts, there is an adjunction $\underline{\mathrm{Mor}}(\Sigma X, Y) \simeq \underline{\mathrm{Mor}}(X, \Omega Y)$ as pointed spaces. In fact, using our explicit constructions, there is an adjuction

$$\underline{\mathrm{Mor}}(\Sigma X, Y) \cong \underline{\mathrm{Mor}}(X, \Omega Y)$$

in simplicial sets!

Example 2.101. One can show that ΣS^n is homotopy equivalent to S^{n+1}: we basically take one copy of S^n and think of it is as a diameter of S^n, and the suspension adds two cones to the top and bottom of S^n, which become the two hemispheres of S^{n+1}! (Collapsing the basepoint at the end does not change the homotopy type.)

Example 2.102. In the category of pointed spaces, we see that $\underline{\mathrm{Mor}}_{\mathrm{Spaces}_*}(\Sigma S^n, X)$ is the pushout of the diagram

$$\begin{tikzcd} & \underline{\mathrm{Mor}}_{\mathrm{Spaces}_*}(\Delta^0, X) \arrow[d] \\ \underline{\mathrm{Mor}}_{\mathrm{Spaces}_*}(\Delta^0, X) \arrow[r] & \underline{\mathrm{Mor}}_{\mathrm{Spaces}_*}(S^n, X) \end{tikzcd}$$

in Spaces_*. But one can see that this is the same as the homotopy pullback of the diagram $\Delta^0 \to \Omega^n X \leftarrow \Delta^0$, which is $\Omega^{n+1}X = \underline{\mathrm{Mor}}_{\mathrm{Spaces}_*}\left(S^{n+1}, X\right)$. This allows us to compute $\Sigma S^n = S^{n+1}$ again!

2.7 October 14

Today we continue.

2.7.1 The Fruedenthal Suspension Theorem

For our next result, we note that the adjunction provides a canonical map $X \to \Omega\Sigma X$.

Theorem 2.103 (Fruedenthal suspension)**.** Fix a pointed space X for which $\pi_i X = 0$ for $i < n$. Then the canonical map

$$\pi_i X \to \pi_i(\Omega\Sigma X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i \leq 2n-2$ and a surjection for $i = 2n-1$.

Proof. There is a homotopy fiber sequence $\Omega\Sigma X \to \Delta^0 \to \Sigma X$. For $n \geq 1$, one can sow that ΣX is simply connected, so we get a Serre spectral sequence

$$E_2 = \mathrm{H}_p(\Sigma X; \mathrm{H}_q(\Omega\Sigma X; \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}\left(\Delta^0; \mathbb{Z}\right).$$

Now, by the Mayer–Vietoris sequence, we can calculate $\mathrm{H}_p(\Sigma X; \mathbb{Z}) = \mathrm{H}_{p-1}(X; \mathbb{Z})$, so Theorem 2.74 tells us that $\mathrm{H}_{p-1}(X; \mathbb{Z})$ vanishes for $0 < p-1 < n$. Thus, the E_2 page vanishes in columns $p \in \{1, 2, \ldots, n\}$. Similarly, Theorem 2.74 tells us that $\mathrm{H}_q(\Omega\Sigma X; \mathbb{Z})$ vanishes for small $q \in \{1, 2, \ldots, n-1\}$ because we can compare it to $\pi_q(\Omega\Sigma X) = \pi_{q+1}(\Sigma X)$.

Thus, because our E_∞ page needs to vanish except at the origin, we see that the only differentials which matter for small values are $E^{i0} \to E^{0,i-1}$ for $i < 2n+1$, which we see is

$$\mathrm{H}_i(\Sigma X) \to \mathrm{H}_{i-1}(\Omega\Sigma X).$$

Thus, these have to be isomorphisms.

We now argue as in Theorem 2.82 to turn isomorphisms in homology to isomorphisms in homotopy. Examining the Serre spectral sequence arising from the homotopy fiber sequence $F \to X \to \Omega\Sigma X$, we find that $\mathrm{H}_i(F; \mathbb{Z})$ needs to vanish for i less than about $2n$, so $\pi_i F$ needs to be trivial in the same region by Theorem 2.74, so the required isomorphism follows by taking the long exact sequence in homotopy. ■

2.7.2 Stable Homotopy Groups of Spheres

Here is an application.

Definition 2.104 (stable homotopy)**.** Fix some integer k. Then we define

$$\pi_k^{\mathrm{st}} := \lim_{n\to\infty} \pi_{n+k}\left(S^n\right).$$

Here, the limit is taken with respect to the canonical maps $\pi_{n+k}\left(S^n\right) \to \pi_{n+k+1}\left(S^{n+1}\right)$ defined in Theorem 2.103 (via Example 2.102). Note that this limit stabilizes for large n by Theorem 2.103.

Remark 2.105. In Example 2.76, we computed $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$, so $\pi_1^{\mathrm{st}} \cong \mathbb{Z}/2\mathbb{Z}$.

There is a lot known about stable homotopy groups of spheres. Here is a table of some known values.

i	0	1	2	3	4	5	6	7	8
π_i^{st}	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/240\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$

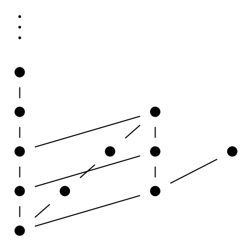
By using Serre's approach to compute homotopy groups by going up the Whitehead tower, one can show the following.

Theorem 2.106 (Adams spectral sequence)**.** For each prime p, there is a spectral sequence E for which

$$E_2^{st} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}^{\mathrm{st}} \otimes \mathbb{Z}_p,$$

where $\mathcal{A}$ is the graded Steenrod algebra, defined using the cohomology of Eilenberg–MacLane spaces $K(\mathbb{Z}/p\mathbb{Z}; n)$s.

Here is a picture of the Adams spectral sequence for $p = 2$.



(In this small region, one finds $E_2 = E_\infty$, but in general there will be a lot of cancellation.) The nth column communicates information about $\pi_n^{\mathrm{st}} \otimes \mathbb{Z}_p$. For example, the 0th column should be $\mathbb{Z}_2$, so it goes infinitely vertically. In particular, each dot produces a $\mathbb{Z}/2\mathbb{Z}$, and the vertical lines tell us how to assemble each $\mathbb{Z}/2\mathbb{Z}$ into a group. For example, one can read $\pi_3^{\mathrm{st}} =$.

Remark 2.107 (Lin–Wang–Xu)**.** One can show that there is a class h_6^2 in $\pi_{126}^{\mathrm{st}} \otimes \mathbb{Z}_p$. This is a very recent result!

Remark 2.108. One can see that there is a well-controlled subgroup of the stable homotopy groups defined as follows. Namely, for any n, there is a map $J \colon \mathrm{O}(n) \to \underline{\mathrm{Mor}}_{\mathrm{Spaces}_*}(S^n, S^n)$ by viewing S^n as the one-point compactification of $\mathbb{R}^n$. This latter group is $\Omega^n S^n$, so there are maps

$$\pi_k \, \mathrm{O}(n) \to \pi_k \Omega^n S^n = \pi_{k+n} S^n.$$

Thus, the image of J provides an interesting subgroup of π_k^{st}. This image is well-understood (meaning that they are understood combinatorially, apparently in terms of Bernoulli numbers). It turns out that they approximately correspond to parts of the Adams sequence which are in "high" filtration.

Here is one reason to care about stable homotopy groups.

Theorem 2.109 (Kervaire–Milnor)**.** For each $n \geq 5$, there is an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \to \frac{\pi_n^{\mathrm{st}}}{\mathrm{im}\, J} \to KI \to 0.$$

Here, $\mathrm{im}\, J \subseteq \pi_n^{\mathrm{st}}$ is constructed in Remark 2.108, and Θ_n is the group of exotic n-spheres up to diffeomorphism. Continuing, bP_{n+1} is given by the group of exotic n-spheres which are boundaries of framed manifolds, and it is a cyclic group understood combinatorially by denominators of Bernoulli numbers. Lastly, KI is an explicit group which is either 0 or $\mathbb{Z}/2\mathbb{Z}$.

Here is some of what it is known about KI.

- A theorem of Browder shows that KI is trivial unless n is 2 less than a power of 2, and Browder showed that it is $\mathbb{Z}/2\mathbb{Z}$ if and only if a certain class h_j^2 from the Adams spectral sequence (for the prime $p = 2$) survives to E_∞.
- Then Kervaire–Milnor showed that it is $\mathbb{Z}/2\mathbb{Z}$ if $n \in \{2, 6, 14\}$.
- Mahowald–Tangora–Jones showed that it is $\mathbb{Z}/2\mathbb{Z}$ for $n \in \{30, 62\}$.
- Lin–Wang–Xu recently proved that it is $\mathbb{Z}/2\mathbb{Z}$ when $n = 126$.
- Lastly, Hill–Hopkins–Ravenel showed that KI vanishes for $n > 126$. This result uses chromatic homotopy theory at height 4.

The moral is that differential topology in dimensions $n \geq 5$ can be turned into homotopy theory (up to combinatorial difficulties), and this moral has other incarnations.

Remark 2.110. A reasonable question to ask is how many dots are there on E_∞ of the Adams spectral sequence at $p = 2$ in the columns from 1 to n. The best known upper bound is $n^{\log(n)^2}$, which comes directly from counting dots on the E_2 page. The image of J provides a lower bound of $\sqrt{n}$; this was improved by Oka to n by producing explicit elements of the Adams spectral sequence. This was recently improved (by Jermey Hahn and some others) to $n \log n$.

2.8 October 16

Today is the last day which can appear on the exam. We will discuss the cohomology of the Eilenberg–MacLane spaces.

2.8.1 The Steenrod Square

We are going to compute $\mathrm{H}^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$. There is an analogue for everything over $\mathbb{F}_p$, but we will not discuss it.

Remark 2.111. Recall that $\mathrm{H}^i(K(\mathbb{F}_2, n); \mathbb{F}_2)$ consists of homotopy classes of maps

$$K(\mathbb{F}_2, n) \to K(\mathbb{F}_2, i)$$

by the universal property, which in turn are natural transformations $\mathrm{H}^n(-; \mathbb{F}_2) \Rightarrow \mathrm{H}^i(-; \mathbb{F}_2)$.

By Remark 2.111, we see that we are interested in defining some natural transformations.

Notation 2.112. For each $n \geq 0$, we define $i_n \in \mathrm{H}^n(K(\mathbb{F}_2, n); \mathbb{F}_2)$ to correspond by the universal property to the identity map $K(\mathbb{F}_2, n) \to K(\mathbb{F}_2, n)$.

Remark 2.113. Roughly speaking, to define a natural transformation, it basically amounts to define what it does for the "universal" elements i_n.

Proposition 2.114 (Steenrod square)**.** For each $i \geq 0$, there is a natural transformation $\operatorname{Sq}^i\colon \mathrm{H}^*(X;\mathbb{F}_2) \to \mathrm{H}^{*+i}(X;\mathbb{F}_2)$ satisfying the following. For example, we ought to have the following.

(a) Zero: $\operatorname{Sq}^0 = \mathrm{id}$.

(b) Top degree: $\operatorname{Sq}^i(x) = x \cup x$ if $i = \deg x$. In particular, $\operatorname{Sq}^n i_n = i_n^2$.

(c) Higher degree: $\operatorname{Sq}^i(x) = 0$ if $i > \deg x$.

(d) The natural transformation Sq^i is a homomorphism.

(e) We have
$$\operatorname{Sq}^k(ab) = \sum_{i+j=k} \operatorname{Sq}^i(a)\operatorname{Sq}^j(b).$$

(f) The map Sq^i commutes with the suspension isomorphism $\widetilde{\mathrm{H}}^k(X;\mathbb{F}_2) \to \widetilde{\mathrm{H}}^{k+1}(\Sigma X;\mathbb{F}_2)$.

Proof. It remains to describe $\operatorname{Sq}^n i_{n+k}$ for $k \geq 1$, for which we use the Serre spectral sequence E. Let's start with $k = 1$, where we work with the homotopy fiber sequence $K(\mathbb{F}_2, n) \to \Delta^0 \to K(\mathbb{F}_2, n+1)$, which has
$$E_2^{pq} = \mathrm{H}^p(K(\mathbb{F}_2, n+1);\mathbb{F}_2) \otimes \mathrm{H}^q(K(\mathbb{F}_2,n);\mathbb{F}_2) \Rightarrow \mathrm{H}^{p+q}\left(\Delta^0;\mathbb{F}_2\right).$$

Many of these columns vanish: by Theorem 2.74, we know that
$$\mathrm{H}_i(K(\mathbb{F}_2, n+1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } 0 < i < n+1, \\ \mathbb{F}_2 & \text{if } i = n+1. \end{cases}$$

By the Universal coefficients theorem, we thus find that E_2^{p0} vanishes in the region $0 < i < n+1$, and $E_2^{n+1,0} = \mathbb{F}_2$. The exact same calculation explains that $E_2^{0q} = 0$ for $0 < q < n$ and $E_2^{0n} = \mathbb{F}_2$. For example, we receive an isomorphism
$$d_n\colon E_2^{0n} \to E_2^{n+1,0}.$$

Here, E_2^{0n} is generated by i_n, so we are being told that $d_n(i_n) = i_{n+1}$. Similarly, we can calculate
$$d_n\left(i_n^2\right) = i_n(d_n i_n) + (di_n)i_n = 2i_n(di_n) = 0.$$

Thus, we see that the only other differential which can kill d_n must be
$$d_{2n}\colon E^{0,2n} \to E^{2n+1,0},$$

and we may define $\operatorname{Sq}^n i_{n+1} \coloneqq d_{2n}(i_n^2)$. The above construction can be iterated to define $\operatorname{Sq}^n i_{n+k}$ for each $k \geq 2$ by differentials in the Serre spectral sequence.

We will not check the last compatibilities. For (e), we remark that this corresponds to $(a+b)^2 = a^2 + b^2$ in degree i, so it is important that we are working over $\mathbb{F}_2$. ■

Example 2.115. It turns out that $\mathbb{RP}^\infty$ is a $K(\mathbb{F}_2, 1)$, so $\mathrm{H}^*(K(\mathbb{F}_2,1);\mathbb{F}_2)$ is $\mathbb{F}_2[x]$ where x lives in degree 1. We compute some Steenrod squares.

Proof. For example, we see that
$$\operatorname{Sq}^i x = \begin{cases} x & \text{if } i = 0, \\ x^2 & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

Similarly,

$$\mathrm{Sq}^i x^2 = \begin{cases} x^2 & \text{if } i = 0, \\ 0 & \text{if } i = 1 \text{ or } i > 2, \\ x^4 & \text{if } i = 2. \end{cases}$$

Here, the interesting calculation is $\mathrm{Sq}^1 x^2 = \mathrm{Sq}^0 x \cdot \mathrm{Sq}^1 x + \mathrm{Sq}^1 x \cdot \mathrm{Sq}^0 x = 0$ (in $\mathbb{F}_2$). We can also calculate $\mathrm{Sq}^1 x^3$ is

$$\mathrm{Sq}^1 x^2 \cdot \mathrm{Sq}^0 x + \mathrm{Sq}^0 x^2 \cdot \mathrm{Sq}^1 x = x^4.$$

In general, one can show that $\mathrm{Sq}^i x^k = \binom{k}{i} x^{k+i}$ after some combinatorics. ■

2.8.2 The Cohomology of $K(\mathbb{F}_2, n)$

Let's try to provide a more concrete description of this long differential. Fix a homotopy fiber sequence $F \to E \to B$ with B simply connected, so we get

$$E_2^{pq} = \mathrm{H}^p(B; \mathbb{F}_2) \otimes \mathrm{H}^q(F; \mathbb{F}_2) \Rightarrow \mathrm{H}^{p+q}(E; \mathbb{F}_2).$$

Remark 2.72 describes E_∞^{p0} as the image of $\mathrm{H}^p(B; \mathbb{F}_2) \to \mathrm{H}^p(E; \mathbb{F}_2)$ and describes E_∞^{0q} as the image of the map $\mathrm{H}^q(E; \mathbb{F}_2) \to \mathrm{H}^q(F; \mathbb{F}_2)$.

Definition 2.116 (transgressive)**.** Fix a homotopy fiber sequence $F \to E \to B$ with F connected and B simply connected. Then a pair $(x, y) \in \mathrm{H}^{i+1}(B; \mathbb{F}_2) \times \mathrm{H}^i(F; \mathbb{F}_2)$ is *transgressive* if and only if the image of $x \in \widetilde{\mathrm{H}}^{i+1}(B)$ and $y \in \mathrm{H}^i(F)$ having the same image in $\mathrm{H}^{i+1}(E, F)$, where $\mathrm{H}^{i+1}(E, F)$ is the homotopy pushout of the following.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \\ \Delta^0 & \end{tikzcd}$$

Remark 2.117. It turns out that (x, y) being transgressive is equivalent to y surviving to E_∞ and

$$d_{i+1}(y) = x.$$

Corollary 2.118 (Kudo transgression)**.** Fix a homotopy fiber sequence $F \to E \to B$ with F connected and B simply connected; let E be the associated cohomological Serre spectral sequence. Then if $y \in \mathrm{H}^{i+1}(F; \mathbb{F}_2)$ survives to E_{i+1} and $d_{i+1}(y) = x$, then $\mathrm{Sq}^n(y)$ survives to the E_{i+n+1} page and $d_{i+n+1}(y) = \mathrm{Sq}^n(x)$.

Proof. Track through the definition of the Steenrod squares, which we know already come from some transgressive pairs. ■

One can use Corollary 2.118 to fully compute the cohomology of $K(\mathbb{F}_2, 2)$.

Proposition 2.119 (Serre)**.** The cohomology ring $\mathrm{H}^*(K(\mathbb{F}_2, 2); \mathbb{F}_2)$ is the free polynomial algebra

$$\mathbb{F}_2 \left[i_2, \mathrm{Sq}^1 i_2, \mathrm{Sq}^2 \mathrm{Sq}^1 i_2, \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 i_2, \ldots \right].$$

Example 2.120. There is a cup product map $K(\mathbb{F}_2, 1) \times K(\mathbb{F}_2, 1) \to K(\mathbb{F}_2, 2)$. Namely, on cohomology, we find that $\mathrm{H}^*(K(\mathbb{F}_2, 1) \times K(\mathbb{F}_2, 1); \mathbb{F}_2) = \mathbb{F}_2[x, y]$ by the Künneth isomorphism (and Example 2.115). Thus, there is a ring homomorphism

$$\mathbb{F}_2\left[i_2, \mathrm{Sq}^1 i_2, \mathrm{Sq}^2 \mathrm{Sq}^1 i_2, \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 i_2, \ldots\right] \to \mathbb{F}_2[x, y].$$

For example, the ambient ring structure tells us that $i_2 \mapsto xy$. The rest can be calculated using the Steenrod squares: for example, $\mathrm{Sq}^1(i_2)$ goes to $\mathrm{Sq}^1(xy) = x^2y + xy^2$. It turns out that this ring homomorphism is injective in degree less than 4. For example, this injectivity could be used to calculate $\mathrm{Sq}^1 \mathrm{Sq}^1 i_2$: indeed, $\mathrm{Sq}^1 \mathrm{Sq}^1 xy = 0$ implies $\mathrm{Sq}^1 \mathrm{Sq}^1 i_2 = 0$. This identity $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$ must now hold in general!

One can extend these notions as follows.

Theorem 2.121 (Serre)**.** The ring $\mathrm{H}^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ is a polynomial ring with generators of the form

$$\mathrm{Sq}^{j_1} \cdots \mathrm{Sq}^{j_r} i_n$$

for which $j_k \geq 2j_{k+1}$ and $\sum_k (j_k - 2j_{k+1}) < n$.

Remark 2.122. This is a great theorem! It tells us that any natural transformation on cohomology is induced by some polynomial in the Steenrod squares.

Theorem 2.123 (Adem relations)**.** For any space X, for indices $i < 2j$, we have

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{i/2} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k.$$

Remark 2.124. Theorem 2.123 is proved using the technique of Example 2.120.

Example 2.125. One can calculate that $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^3$ and $\mathrm{Sq}^1\mathrm{Sq}^4 = \mathrm{Sq}^5$ and $\mathrm{Sq}^2\mathrm{Sq}^4 + \mathrm{Sq}^5\mathrm{Sq}^1 = \mathrm{Sq}^6$. In general, one finds that one can generate all Steenrod squares using just the Steenrod squares of powers of 2.

Remark 2.126. The Steenrod algebra $\mathcal{A}$ is the algebra generated by formal symbols Sq^i with the Adem relations. This is the Steenrod algebra $\mathcal{A}$ appearing in the Adams spectral sequence of Theorem 2.106. It turns out that this algebra also appears in the theory of formal group laws.

Theme 3
Off the Deep End

3.1 October 21

There will be a correction posted to the problem set.

3.1.1 $\mathbb{E}_1$-Spaces

Recall that we have embedded Set into the ∞-category Spaces, and we have embedded the category Mod_R into the ∞-category $D(R)_{\geq 0}$. Notably, it is hard to understand Spaces, but it is perhaps manageable to understand the analogous construction $D(R)_{\geq 0}$ because these are just chain complexes (by the Dold–Kan correspondence). One then uses $D(R)_{\geq 0}$ to understand Spaces by constructing some functors $\mathrm{Spaces} \to D(R)_{\geq 0}$ (for which it is enough to specify the image of Δ^0).

We may wish to weaken Mod_R in order to build more flexible functors out of Spaces. For this, we use the weakest algebraic structures we can find, which are monoids.

Definition 3.1 (discrete monoid)**.** A *discrete monoid* is a set X equipped with a multiplication $m\colon X \times X \to X$ which is unital and associative. We let Mon denote the category of monoids.

Example 3.2. There is a full subcategory Grp of Mon.

Example 3.3. Monoids are not required to have inverses, so $\mathbb{Z}_{\geq 0}$ is a monoid.

We are about to construct $\mathbb{E}_1$-spaces, which include monoids. Heuristically, they consist of a space X with some multiplication $m\colon X \times X \to X$ and an identity $e\colon \Delta^0 \to X$ for which any $x, y, z \in X$ admit a path $(xy)z \to x(yz)$ and a path $1x \to x$ and $x1 \to x$ and so on. There are also lots of different ways to associate, so we add in higher-order homotopies to encode these coherences.

Definition 3.4 ($\mathbb{E}_1$-space)**.** Given an ∞-groupoid X, an $\mathbb{E}_1$-structure on X is a functor $M\colon \Delta^{\mathrm{op}} \to \mathrm{Spaces}$ such that $M_0 = \Delta^0$ and $M_1 = X$ and the injections $[1] \to [n]$ (given by $i \leq i+1$) produce maps $M_n \to M_1^n$ which is an equivalence.

Remark 3.5. In particular, we see that $M_n = X^n$ for each n, and there is a specific map witnessing this identification.

Example 3.6. Given a classical monoid X, we can produce the functor M as factoring through Set. For example, the boundaries $X \to \Delta^0$ should be constants, and the face $\Delta^0 \to X$ should be the identity. More generally, the differential $d^i \colon X^{n+1} \to X^n$ should multiply in the ith coordinate, and the face maps include the identity at a particular index. In fewer words, this is the nerve of the one-object category produced from the monoid M.

Example 3.7. Consider a (nice) topological group G. Then the construction of Example 3.6 produces a functor $\Delta^{\mathrm{op}} \to \mathrm{Top}$ (via the same checks), which then produces a functor $\Delta^{\mathrm{op}} \to \mathrm{Spaces}$ by composing through Sing.

Example 3.8. Given a Kan complex X, we recall that we have a Kan complex $\underline{\mathrm{Mor}}(X, X)$, which is also an $\mathbb{E}_1$-space. The point is that we can compose two endomorphisms to produce a third endomorphism, which is our multiplication map.

Example 3.9. If X is a Kan complex, then ΩX is an $\mathbb{E}_1$-space. The point is that composition of loops is associative and unital, up to "coherent" homotopy. Namely, the choice of homotopies witnessing the associative and unital properties are more or less unique (up to some higher choices, which are unique up to higher choices, and so on). In fact, $\pi_0(\Omega X)$ is a group.

Definition 3.10 (group-like)**.** An $\mathbb{E}_1$-space is *group-like* if and only if $\pi_0 X$ is a group.

Remark 3.11. It turns out that $\mathbb{E}_1$-spaces form a full subcategory of $\underline{\mathrm{Mor}}(\Delta^{\mathrm{op}}, \mathrm{Spaces})$, where we are silently viewing Δ^{op} as a quasicategory. In particular, it may look like adding $\mathbb{E}_1$-structure looks like extra structure, but it does not restrict the morphisms. There is then the fully faithful subcategory of group-like $\mathbb{E}_1$-spaces. As such, being group-like is a property of an $\mathbb{E}_1$-space, not extra data.

Definition 3.12. Fix an $\mathbb{E}_1$-space X. Then BM is the colimit of $M \colon \Delta^{\mathrm{op}} \to \mathrm{Spaces}$.

Example 3.13. If G is a discrete group, then $BG = K(G, 1)$. Indeed, the colimit of this functor $\Delta^{\mathrm{op}} \to \mathrm{Spaces}$ factors through Set, so it will be given by the functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ which is the ordinary category BG. Similarly, if M is a discrete monoid, then $BM = BG$, where G is the group completion of M (given by adding formal inverses). Namely, we would expect to get the 1-object category given by M, but this is just BG up to weak equivalence.

Example 3.14. Consider a topological Lie group G. Then we receive an $\mathbb{E}_1$-space $\Delta^{\mathrm{op}} \to \mathrm{Spaces}$ (via Example 3.7), so we can define BG.

Definition 3.15 (vector bundle)**.** A *vector bundle* on a space X is a map $X \to B\,\mathrm{U}(n)$ of simplicial sets.

Remark 3.16. When X is a nice topological space, then this is analogous to defining a vector bundle as a representation of $\pi_1(X)$.

Remark 3.17. More generally, if G is a topological group, then $[X, BG]$ classifies principal G-bundles up to isomorphism.

It turns out that $\mathbb{E}_1$-spaces are complicated enough for most of our purposes.

Theorem 3.18. The functors B and Ω are inverse equivalences between the quasicategories of group-like $\mathbb{E}_1$-spaces and connected pointed spaces.

Remark 3.19. Intuitively, we are saying that 1-object groupoids (i.e., connected ∞-groupoids) are groups (i.e., group-like $\mathbb{E}_1$-spaces). Namely, one can recover a group G from BG by taking loops. Of course, the showing the inverse equivalence is hard!

Corollary 3.20 (van Kampen)**.** The functor π_1 on pointed connected spaces preserves pushouts.

Proof. We know $\pi_1 X = \pi_0 \Omega X$, so we are equivalently asking for π_0 to preserve pushouts on the category of group-like $\mathbb{E}_1$-spaces. ■

Corollary 3.21. Fix an $\mathbb{E}_1$-space M. Then ΩBM is the group completion of M.

Proof. For any group G, we see that

$$\operatorname{Hom}_{\mathbb{E}_1}(M, G) = \operatorname{Hom}_{\mathbb{E}_1}(\Omega BM, G),$$

but one can check directly that BM is insensitive to the group completion. As such, ΩBM must be the same as its group completion. ■

Corollary 3.22. The free group-like $\mathbb{E}_1$-space on a space X is $\Omega\Sigma X$. In other words, $\Omega\Sigma$ is left adjoint to the forgetful functor on $\mathbb{E}_1$-spaces.

Proof. We can calculate

$$\begin{aligned}
\operatorname{Hom}_{\mathbb{E}_1}(\Omega\Sigma X, G) &= \operatorname{Hom}_{\mathrm{ConSpaces}_*}(\Sigma X, BG) \\
&= \operatorname{Hom}_{\mathrm{Spaces}_*}(\Sigma X, BG) \\
&= \operatorname{Hom}_{\mathrm{Spaces}_*}(X, \Omega BG) \\
&= \operatorname{Hom}_{\mathrm{Spaces}_*}(X, G),
\end{aligned}$$

as required. ■

Example 3.23. If X is a group-like $\mathbb{E}_1$-space, then we can think of $\pi_n X$ as $\mathbb{E}_1$-space maps $\Omega S^{n+1} \to X$ (because $\Omega S^{n+1} = \Omega\Sigma S^n$). This roughly explains why we may care about ΩS^{n+1}.

3.2 October 23

Let's quickly say something about the homework.

Remark 3.24. Given a homotopy fiber sequence $F \to E \to B$, the functor $\operatorname{Mor}(X, -)$ preserves limits, so we get a homotopy fiber sequence

$$\operatorname{Mor}(X, F) \to \operatorname{Mor}(X, E) \to \operatorname{Mor}(X, B),$$

and taking the long exact sequence in homotopy produces a long exact sequence

$$[X, \Omega B] \to [X, F] \to [X, E] \to [X, B].$$

Accordingly, given a map $f: X \to E$, lifting it to $X \to F$ is exactly given by the data of a homotopy from $X \to E \to B$ and the constant. (This can be seen directly.) Furthermore, given two null-homotopies of the composite $X \to B$ glue together to a homotopy from the constant $X \to B$ to itself, which is given by an element of $\Omega \underline{\operatorname{Mor}}(X, B) = \underline{\operatorname{Mor}}(X, \Omega B)$. The point is that one can "subtract" two lifts in $[X, F]$ to an element in $[X, \Omega B]$, which amounts to saying that there is a group action of $\pi_1[X, B]$ on $[X, F]$.

Example 3.25. Here is an interesting example of a Wilson space. There is an embedding $\mathrm{U}(n) \to \mathrm{U}(n+1)$ given by $g \mapsto \operatorname{diag}(g, 1)$. This produces a sequence $\mathrm{U}(1) \to \mathrm{U}(2) \to \mathrm{U}(3) \to \cdots$ of group homomorphisms, so may take the colimit of

$$B\,\mathrm{U}(1) \to B\,\mathrm{U}(2) \to B\,\mathrm{U}(3) \to \cdots,$$

which produces some space $B\,\mathrm{U}$. This turns out to be a Wilson space.

3.2.1 $\mathbb{E}_\infty$-Algebras

Here is a starting definition.

Definition 3.26 ($\mathbb{E}_k$-space)**.** Fix some ∞-category $\mathcal{C}$ with products. Then an $\mathbb{E}_1$-*algebra* consists of the functors $F: \Delta^{\mathrm{op}} \to \mathcal{C}$ for which F_0 is the empty product, and the induced maps $F_n \to F_1^n$ are isomorphisms. Then, in general, for each $k \geq 1$, we define an $\mathbb{E}_{k+1}$-space to be an $\mathbb{E}_1$-algebra valued in $\mathbb{E}_k$-spaces.

Remark 3.27. One should check that $\mathbb{E}_k$-spaces has products, which it does.

Remark 3.28. Note that an $\mathbb{E}_2$-space has two forgetful functors to $\mathbb{E}_1$-spaces according to which operation we forget about. However, the Eckmann–Hilton argument applies in this case, so the two $\mathbb{E}_1$-structures turn out to be equivalent (although this equality is non-canonical).

Definition 3.29 (group-like)**.** An $\mathbb{E}_k$-space X is *group-like* if and only if $\pi_0 X$ is a group.

Theorem 3.30 (May recognition)**.** Let $\operatorname{Spaces}_{*,\geq k}$ be the full subcategory of pointed spaces with trivial homotopy groups vanish in degree less than k. Then

$$\Omega^k : \operatorname{Spaces}_{*,\geq k} \to \operatorname{Spaces}(\mathbb{E}_k)$$

is fully faithful and has image given by the group-like $\mathbb{E}_k$-spaces. The inverse functor is given by B^k.

Remark 3.31. It turns out the category of group-like $\mathbb{E}_k$-spaces is equivalent to the category of pointed spaces whose homotopy groups vanish before k. This is a direct generalization of Theorem 3.18.

The reason we have introduced $\mathbb{E}_k$-spaces is that we want to define $\mathbb{E}_\infty$-spaces.

Definition 3.32 ($\mathbb{E}_\infty$-space)**.** An $\mathbb{E}_\infty$*-space* is a space with a collection of compatible $\mathbb{E}_k$-space structure for all k. A *group-like* $\mathbb{E}_\infty$-space X is one where $\pi_0 X$ is a group.

Remark 3.33. A group-like $\mathbb{E}_\infty$ has the data of a sequence of spaces $\{X_i\}$ with equivalences $X_i = \Omega X_{i+1}$ and X_i has no nontrivial homotopy groups in degree less than i; in this situation, X_0 is the $\mathbb{E}_\infty$-space. Indeed, one can just iteratively apply Theorem 3.30. We may call such a sequence an infinite loop space.

Remark 3.34. Where $\mathbb{E}_1$-spaces generalize monoids and group-like $\mathbb{E}_1$-spaces generalize groups, we now say that $\mathbb{E}_\infty$-spaces generalize abelian groups. The point is that the infinite coherences remember the commutativity. What is amazing is that we have encoded commutativity as a piece of data instead of as a property!

Example 3.35. If A is a discrete abelian group, then

$$A = \Omega K(A,1) = \Omega K(A,2) = \Omega K(A,3) = \cdots ,$$

so A embeds as a discrete group-like $\mathbb{E}_\infty$-space. It turns out that a discrete abelian monoid upgrades to an $\mathbb{E}_\infty$-space.

Non-Example 3.36. Note that $\Omega^2 S^3$ is an $\mathbb{E}_2$-space, but it turns out that it is not an $\mathbb{E}_\infty$-space. On the other hand, it does admit an $\mathbb{E}_3$-structure because it is $\Omega^3 B\,\mathrm{SU}(2)$.

Example 3.37. Any symmetric monoidal classical groupoid is an $\mathbb{E}_\infty$-space. For example, FinSet (equipped with bijections for morphisms) is can be thought of as

$$\coprod_{n\geq 0} BS_n$$

because BS_n is the single object, but we are remembering that it "should" have n elements. Thus, this admits an $\mathbb{E}_\infty$-structure, where the operation is given by $\sqcup$ of the finite sets.

Example 3.38. Fix a classical commutative ring R. Then the category of finitely generated free R-modules (still with isomorphisms as morphisms) is also an $\mathbb{E}_\infty$-space with the operation given by $\oplus$.

Example 3.39. There is an $\mathbb{E}_\infty$-space

$$\coprod_{n\geq 0} B\,\mathrm{U}(n),$$

which is the groupoid of finite-dimensional complex vector spaces. The $\mathbb{E}_\infty$-space structure comes from $\oplus$.

Remark 3.40. The functor $M \mapsto \Omega BM$ is left adjoint to the forgetful functor from group-like $\mathbb{E}_\infty$-spaces to $\mathbb{E}_\infty$-spaces. However, this operation can be complicated (and interesting!).

Example 3.41. If $M = \coprod_{n\geq 0} B\,\mathrm{U}(n)$, then $\Omega BM = \mathrm{B}\,U \times \mathbb{Z}$, which we can think about as "formal differences" of vector spaces (i.e., "virtual" vector spaces).

Example 3.42. If M is given by the finitely generated free R-modules, then $\pi_i \Omega BM$ are called the algebraic K-groups $K_i(R)$ of R. For example, it turns out that the Kummer–Vandiver conjecture is equivalent to $K_{4i}(\mathbb{Z}) = 0$ for all $i \geq 0$.

Example 3.43. If M is given by the finite sets under disjoint union, then $\pi_i(\Omega BM)$ is the kth homotopy group of spheres.

Example 3.44. We will construct a group-like $\mathbb{E}_\infty$-space MO, whose objects are 0-dimensional manifolds, morphisms are cobordisms of objects, 2-morphisms are cobordisms of cobordisms, and so on. Disjoint union provides our $\mathbb{E}_\infty$-structure, where the basepoint is $\varnothing$; it turns out $\pi_k MO$ is closed k-manifolds up to cobordism. Lastly, to see that it is a group-like, we note that $* \sqcup *$ is equivalent to $\varnothing$ by an explicit cobordism, so $\pi_0 MO$ is a group (in fact, 2-torsion).

Example 3.45. Given any pointed space X, there are maps

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 \to \cdots .$$

The first map is induced by Theorem 2.103. The second map is Σ applied to the canonical map $\Sigma X \to \Omega\Sigma(\Sigma X)$, and one can continue this process. It follows that the colimit $\Omega^\infty\Sigma^\infty X$ is a group-like $\mathbb{E}_\infty$-space, where $\pi_i\Omega^\infty\Sigma^\infty X$ is the ith stable homotopy group of X (which makes sense by Theorem 2.103). It turns out that $\Omega\Sigma\mathrm{FinSet}$ is $\Omega^\infty\Sigma^\infty S^0$.

3.3 October 28

Today, we will continue discussing spectra.

3.3.1 Spectra

We can generalize our notion of group-like $\mathbb{E}_\infty$-spaces being thought of as infinite loop spaces.

Definition 3.46 (spectrum)**.** A *spectrum* is a sequence $(X_0, X_1, \ldots)$ of pointed spaces along with equivalences $X_i \to \Omega X_{i+1}$.

Remark 3.47. A morphism $X \to Y$ of spectra is a morphism $X_i \to Y_i$ of the underlying pointed spaces, along with homotopies of the equivalences $X_i \to \Omega X_{i+1}$. Diagrammatically, we are providing vertical arrows in the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \Omega X_1 \arrow[r] \arrow[d] & \Omega^2 X_2 \arrow[r] \arrow[d] & \cdots \\ Y_0 \arrow[r] & \Omega Y_1 \arrow[r] & \Omega^2 Y_2 \arrow[r] & \cdots \end{tikzcd}$$

along with "square homotopies" in each of the given squares.

Example 3.48. Any group-like $\mathbb{E}_\infty$-space produces a spectrum.

Example 3.49. As an instance of Example 3.48, we recall that a discrete abelian group A gives rise to the spectrum

$$A = \Omega K(A,1) = \Omega^2 K(A,2) = \cdots .$$

In fact, Example 3.35 shows that this is already an $\mathbb{E}_\infty$-space.

Example 3.50. Given a space X, we may follow Example 3.45 to construct a group-like $\mathbb{E}_\infty$-space $\Sigma^\infty X$ from a pointed space X. Indeed, we define $\Sigma^\infty X$ as the colimit

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots ,$$

where the internal maps arise from the adjunction of Ω and Σ. Namely, we define X_0 as the above colimit, and X_i in general is the colimit of $\Sigma^i X \to \Omega\Sigma^{i+1}X \to \Omega^2\Sigma^{i+2}X \to \cdots$. It follows that $X_i = \Omega X_{i+1}$ because $\Omega = \operatorname{Mor}\left(S^1, -\right)$ preserves filtered colimits (because S^1 is a compact object).

Example 3.51. As an instance of Example 3.50, we note that $\Delta^0 = \Omega\Delta^0$, so Δ^0 is a spectrum. We may call this the "zero" spectrum 0 because it is both initial and terminal. (Indeed, Δ^0 is already initial and terminal in pointed spaces.)

Remark 3.52. The functor $\Sigma^\infty \colon \mathrm{Spaces}_* \to \mathrm{Specta}$ admits a right adjoint Ω^∞ defined by $\Omega^\infty X := X_0$. Approximately speaking, Ω^∞ should be thought of as a forgetful functor removing all "higher" information.

Here is a nontrivial example.

Theorem 3.53 (Bott periodicity)**.** Let U be the infinite unitary group, which is the colimit of the finite unitary groups. It turns out that $\Omega(B\mathrm{U}\times\mathbb{Z}) \simeq \mathrm{U}$ and $\Omega U \simeq B\,\mathrm{U}\times\mathbb{Z}$.

Example 3.54. There is a spectrum $KU := (B\mathrm{U}\times\mathbb{Z}, \mathrm{U}, B\mathrm{U}\times\mathbb{Z}, \mathrm{U}, \ldots)$.

Remark 3.55. It is not a joke that both spectra and $\mathbb{E}_\infty$-spaces should be thought of as analogous to abelian groups. For a taste of how this works, we know that the Dold–Kan correspondence (Theorem 2.37) explains that the category of simplicial abelian groups is equivalent to the derived ∞-category of chain complexes supported in nonnegative degrees, so one can more or less view $\mathbb{E}_\infty$ as analogous this derived category. Then spectra are analogous to allowing chain complexes in all integral degrees.

3.3.2 Stability

The ∞-category of spectra will turn out to have many nice properties. The most important of these is stability, which we now explain.

Notation 3.56. Let $\mathcal{C}$ be an ∞-category with a 0 object and finite limits. Then we define the functor $\Omega_\mathcal{C} \colon \mathcal{C} \to \mathcal{C}$ by sending an object $x \in \mathcal{C}$ to the limit of the following diagram.

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & x \end{tikzcd}$$

If there is no risk of confusion, will abbreviate $\Omega_\mathcal{C}$ to Ω.

Example 3.57. The category of spectra has Ω arise from the usual loops: it sends $(X_0, X_1, X_2, \ldots)$ to $(\Omega X_0, X_0, X_1, \ldots)$. In fact, we see that Ω is an auto-equivalence on spectra. The inverse functor is given by

$$(X_0, X_1, \ldots) \mapsto (X_1, X_2, \ldots)$$

because $X_0 = \Omega X_1$. The point is that a spectrum can always be recovered from its "higher" terms.

Definition 3.58 (stable)**.** Let $\mathcal{C}$ be an ∞-category with a 0 object and finite limits. Then we say that $\mathcal{C}$ is *stable* if and only if $\Omega_{\mathcal{C}}$ is an equivalence. In this situation, we may write $\Sigma_{\mathcal{C}}$ for the inverse equivalence, which we will abbreviate to Σ whenever possible.

Example 3.59. Consider the derived category $\mathcal{D}(\mathbb{Z})$ of chain complexes of abelian groups. It turns out that Ω shifts a chain complex to the right by 1, so Ω is an auto-equivalence.

Remark 3.60. In Example 3.57, we checked that the category of spectra is stable. However, the corresponding functor Σ is not the usual suspension of pointed spaces.

The following result explains why stability is a nice notion.

Proposition 3.61. Fix a stable ∞-category $\mathcal{C}$.

(a) The ∞-category $\mathcal{C}$ admits finite colimits.

(b) The functor Σ sends an object x to the colimit of the diagram $0 \leftarrow c \rightarrow 0$.

(c) A square in $\mathcal{C}$ is a pullback if and only if it is a pushout.

Remark 3.62. One should think of (c) as analogous to the result for additive categories that products and coproducts of two objects exist and are the same.

Lemma 3.63. Fix a stable ∞-category $\mathcal{C}$. Let $\mathcal{P} \subseteq \underline{\mathrm{Hom}}\left(\Delta^1 \times \Delta^1, \mathcal{C}\right)$ be the full subcategory of pullback squares. Then the forgetful functor $\mathcal{P} \to \underline{\mathrm{Hom}}(\Lambda^2_0, \mathcal{C})$ is an equivalence.

Proof. We exhibit an inverse "up to loops." Namely, given some object $C \leftarrow A \rightarrow B$ in $\underline{\mathrm{Hom}}(\Lambda^2_0, \mathcal{C})$, we construct F, G, and X to fit into the diagram

$$\begin{tikzcd}
\Omega A \arrow[r] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \arrow[d] & \\
\Omega B \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & G \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \\
& 0 \arrow[r] & C &
\end{tikzcd}$$

where all squares are pullback squares (and as usual should be filled in with homotopies). Thus, we have functorially constructed X is the pullback of the maps $\Omega C \leftarrow \Omega A \rightarrow \Omega B$.

Now, let $\varphi \colon \mathcal{P} \to \underline{\mathrm{Hom}}(\Lambda^2_0, \mathcal{C})$ be the forgetful functor, and let $\psi \colon \underline{\mathrm{Hom}}(\Lambda^2_0, \mathcal{C})$ be the functor described in the previous paragraph. A quick calculation (and the uniqueness of pullbacks) shows that both composites $\varphi \circ \psi$ and $\psi \circ \varphi$ amount to taking Ω everywhere. Because $\mathcal{C}$ is stable, Ω admits an inverse, so we are done! ■

Proof of Proposition 3.61. This is a repeated application of Lemma 3.63. For example, to exhibit finite colimits, it is enough to exhibit pushouts of two elements (because we already have a 0 object). Well, given some diagram $C \leftarrow A \rightarrow B$, we see that maps to some test object T are equivalent to maps of diagrams to $T = T = T$. The equivalence of Lemma 3.63 then provides us with an object D fitting into a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

so that maps from this square to the square given by T are the same as maps from $C \leftarrow A \rightarrow B$ to T. We conclude that D is the pushout, which (a) follows.

Continuing, the above argument implies that the pushout of the diagram $0 \leftarrow \Omega D \rightarrow 0$ is just D, so (b) follows. Lastly, to show (c), we note that we have already shown that pushout squares are pullback squares by their construction. Conversely, a pullback square can forget about its lower-right corner, and then one can re-remember the lower-right corner is the pushout via the preceding paragraph. This shows that pullback squares are pushout squares. ■

Remark 3.64. One can unwind the proof of Lemma 3.63 to show that property (c) is equivalent to the stability of $\mathcal{C}$.

3.3.3 Stable Homotopy Groups

Here is our main definition.

Definition 3.65 (stable homotopy)**.** Fix a spectrum X and an integer $i \in \mathbb{Z}$. Then we define the *nth stable homotopy group* $\pi_i^{\text{st}}(X)$ as

$$\pi_i^{\text{st}}(X) := \pi_{i+j} X_j,$$

provided that $i + j \geq 0$. Whenever possible, we will abbreviate $\pi_i^{\text{st}}(X)$ to $\pi_i X$.

Remark 3.66. Note that the definition is independent of the choice of i and j because $\Omega X_{j+1} = X_j$, so $\pi_{i-1} X_{j+1} = \pi_{i-1} \Omega X_j = \pi_i X_j$.

Remark 3.67. A spectrum X arises from a group-like $\mathbb{E}_\infty$-space if and only if $\pi_n X = 0$ for all $n < 0$ because this means that X_i is i-connected for each i (namely, $\pi_j X_i = 0$ for $j < i$), so one can use Theorem 3.30 to recover $\mathbb{E}_\infty$-space structure.

As with pointed spaces, we have a Whitehead's theorem.

Theorem 3.68 (Whitehead for spectra)**.** Fix a morphism $f \colon X \to Y$ of spectra. Then f is an equivalence if and only if $\pi_i f \colon \pi_i X \to \pi_i Y$ is an isomorphism for all $i \in \mathbb{Z}$.

Proof. This immediately reduces to Whitehead's theorem for pointed spaces upon unwinding the definition of a morphism of spectra. ■

Remark 3.69. If X and Y are in fact group-like $\mathbb{E}_\infty$-spectra, then one knows that $\pi_i X = \pi_i Y$ for $i < 0$, so checking the conclusion of Theorem 3.68 only requires $i \geq 0$.

As with homotopy groups, we may hope for a long exact sequence.

Definition 3.70 (fiber sequence)**.** Fix a stable ∞-category $\mathcal{C}$. Then a sequence $F \to E \to B$ is a *fiber sequence* if and only if

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

is a pullback square. We may also call this a *cofiber sequence*.

Remark 3.71. We may identify the notions of fiber sequences and cofiber sequences because of Proposition 3.61.

Theorem 3.72. Fix a fiber sequence $F \to E \to B$ of spectra. Then there is a long exact sequence of stable homotopy groups

$$\cdots \to \pi_{i+1}B \to \pi_i F \to \pi_i E \to \pi_i B \to \pi_{i-1}F \to \cdots.$$

Proof. The same proof as in spaces shows that a fiber sequence $F \to E \to B$ gives rise to a fiber sequence $\Omega B \to F \to E$. Because Ω is now an auto-equivalence, we also receive a fiber sequence $E \to B \to \Sigma F$. Thus, we can see that it really amounts to checking exactness at π_0, which amounts to noting that Ω^∞ preserves limits (because it is a right adjoint) so that the fiber sequence $F \to E \to B$ of spectra goes to a fiber sequence $F_0 \to E_0 \to B_0$ of spaces. We now may apply the homotopy long exact sequence for pointed spaces. ■

Remark 3.73. Here is an amusing application: if $F \to E \to B$ is a cofiber sequence of pointed spaces, then $\Sigma^\infty F \to \Sigma^\infty E \to \Sigma^\infty B$ continues to be a cofiber sequence (Σ^∞ preserves colimits because it is a left adjoint), so we get a long exact sequence of stable homotopy groups

$$\cdots \to \pi_{i+1}^{\text{st}}B \to \pi_i^{\text{st}}F \to \pi_i^{\text{st}}E \to \pi_i^{\text{st}}B \to \pi_{i-1}^{\text{st}}F \to \cdots.$$

This is purely a statement about pointed spaces!

3.4 October 30

I have regained access to my computer. The exam will feature computations with the Serre spectral sequence.

3.4.1 Examples of Spectra

This class has been interested in the ∞-categories Spaces, Spaces_*, and $\mathrm{Spectra}$. There is a forgetful functor $\mathrm{Spaces}_* \to \mathrm{Spaces}$, which is adjoint to adding a disjoint basepoint $(-)_+ \colon \mathrm{Spaces} \to \mathrm{Spaces}_*$. There is are also adjoint functors $\Omega^\infty \colon \mathrm{Spectra} \to \mathrm{Spaces}_*$ and $\Sigma^\infty \colon \mathrm{Spaces}_* \to \mathrm{Spectra}$. We also recall that Σ^∞ outputs to the ∞-category of $\mathbb{E}_\infty$-spaces.

Example 3.74. If X is a pointed space, then $\Sigma^\infty X$, then we recall that

$$\pi_i \Sigma^\infty X = \pi_i^{\text{st}} X = \pi_i \left(\operatorname*{colim}_n \Omega^n \Sigma^n X \right).$$

Now, π_i passes through the filtered colimit because S^i is a compact object, so this is $\operatorname{colim}_n \pi_i \Omega^n \Sigma^n X$.

Example 3.75. As an instance of the above example, we set $\mathbb{S} := \Sigma^\infty S^0$ to be the sphere spectrum. It's called the sphere spectrum because

$$S^n = \Omega^\infty \Sigma^\infty S^n = \Omega^\infty \Sigma^n \mathbb{S},$$

To explain why it may be fundamental, recall $\Sigma^\infty \Delta^0 = 0$, so the suspension of just one point is uninteresting. But $\Sigma_+^\infty \Delta^0 = \Sigma^\infty \Delta_+^0 = \mathbb{S}$ has a chance of being interesting. In particular, it is the "free object on a point" in homotopy theory, so it holds the same interest as $\mathbb{Z}$ would for algebraists.

Recall that $\mathrm{Spectra}$ has many nice properties: it is stable (which implies that pushout squares and pullback squares are the same), admits all limits and colimits, and has a zero object.

Remark 3.76. It turns out that stability is in fact equivalent to pushout squares being the same as pullback squares.

It is also Cartesian: for spectra E and F, one has

$$\underline{\mathrm{Hom}}(E, F) = \Omega \underline{\mathrm{Hom}}(E, \Sigma F) = \cdots,$$

which can be unwound to give a definition of a spectrum. In other words, $\underline{\mathrm{Hom}}(E, F)$ has an underlying space given by viewing E and F as spaces. Then

$$\underline{\mathrm{Hom}}(E, F) = \underline{\mathrm{Hom}}(E, \Omega\Sigma F) = \Omega \underline{\mathrm{Hom}}(E, \Sigma F)$$

where Ω comes out of the limit because it is a limit. This process can be iterated to turn $\underline{\mathrm{Hom}}(E, F)$ into a spectrum: the nth term of the spectrum is the space $\underline{\mathrm{Hom}}(E, \Sigma^n F)$.

Example 3.77. We see $\underline{\mathrm{Hom}}(\mathbb{S}, E) = E$ by expanding everything out. Indeed, $\mathbb{S} = \Sigma^\infty S^0$, so this is $\underline{\mathrm{Hom}}(S^0, \Omega^\infty \Sigma^n E)$, which is $\Omega^\infty \Sigma^n E$ (because we are working with pointed spaces!).

Definition 3.78 (Eilenberg–MacLane spectrum)**.** If A is a discrete abelian group, then we let $\Sigma^n A$ be an *Eilenberg–MacLane spectrum.*

Example 3.79. By definition, we see that

$$\Omega^\infty \Sigma^n A = \begin{cases} K(A, n) & \text{if } n \geq 0, \\ \Delta^0 & \text{if } n < 0. \end{cases}$$

3.4.2 Generalized Cohomology

Here is our definition.

Definition 3.80 (generalized cohomology)**.** Fix a spectrum E. Then we define the *generalized reduced E-cohomology groups* $\widetilde{E}^n \colon \mathrm{Spaces}_* \to \mathrm{Ab}$ by

$$\widetilde{E}^n(X) := \pi_{-n} \underline{\mathrm{Hom}}_{\mathrm{Spectra}}(\Sigma^\infty X, E).$$

(This definition immediately extends to the case where $\Sigma^\infty X$ is replaced by a general spectrum.) We further define the *generalized E-cohomology* $E^n \colon \mathrm{Spaces}_* \to \mathrm{Ab}$ by $E^n(X) := \widetilde{E}^n(X_+)$.

The moral is that a spectrum gives rise to a generalized cohomology theory. Here are some instances of this.

Remark 3.81. Equivalently, we see $\widetilde{E}^n(X)$ is also

$$\pi_0\underline{\mathrm{Hom}}_{\mathrm{Spectra}}(\Sigma^\infty X, \Sigma^n E) = \pi_0\underline{\mathrm{Hom}}_{\mathrm{Spectra}}(X, \Omega^\infty\Sigma^n E).$$

For example, for the Eilenberg–MacLane spectrum A, we find that $\widetilde{E}^n(X)$ is $\mathrm{H}^n(X; A)$.

Remark 3.82. One can check that $E^n(X) = \widetilde{E}^n(X) \oplus E\left(\Delta^0\right)$, as one would expect from ordinary cohomology.

Remark 3.83. One sees that $E^n\left(\Delta^0\right)$ is $\widetilde{E}^n\left(S^0\right)$, which is $\pi_0\underline{\mathrm{Hom}}_{\mathrm{Spectra}}(S^n, E)$. One can further calculate this as $\pi_0\underline{\mathrm{Hom}}(\Sigma^n\mathbb{S}, E)$, which is $\pi_0\underline{\mathrm{Hom}}(\mathbb{S}, \Omega^n E)$, which is $\pi_0\underline{\mathrm{Hom}}_{\mathrm{Spaces}_*}(S^0, \Omega^\infty\Omega^n E)$, which is

$$\pi_0\underline{\mathrm{Hom}}_{\mathrm{Spaces}}(\Delta^0, \Omega^\infty\Omega^n E) = \pi_n E.$$

Thus, the sphere spectrum (and more generally, the cohomology theory) of E remembers the homotopy groups of E.

Lemma 3.84. Fix a group-like $\mathbb{E}_\infty$-space E. Then

$$E = \operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty\Omega^\infty\Sigma^n E.$$

Proof. The comparison maps are induced by moving the functor $\Sigma^{-1} = \Omega$ inside the limit $\Sigma^\infty\Omega^\infty$. (This also explains how we can construct a map from the colimit to E.) Note $\pi_i\Omega^\infty\Sigma^n E = \mathrm{Hom}(S^i, \Omega^\infty\Sigma^n E) = \mathrm{Hom}_{\mathrm{Spectra}}(S^i, \Sigma^n E) = \pi_{i-n}E$, so $\pi_i\Omega^\infty\Sigma^n E$ vanishes for $i < n$. Now, by the Freudenthal suspension theorem, we see that the stable homotopy groups of $\Omega^\infty\Sigma^n E$ agree with the homotopy groups up to degree $2n-1$. Thus, both sides of the colimit have the same homotopy groups up to degree n or so, so we are done by sending n to infinity. ■

Remark 3.85. The moral of Lemma 3.84 is that we can write a group-like $\mathbb{E}_\infty$-space E as a colimit of suspension spectra.

Example 3.86. View $E := \mathbb{F}_2$ as an Eilenberg–MacLane spectrum. The $\mathbb{F}_2$-cohomology of $\mathbb{F}_2$ is the Steenrod algebra.

Proof. By Lemma 3.84, we see that the cohomology is

$$\widetilde{E}^*(\mathbb{F}_2) = \widetilde{E}^*\left(\operatorname*{colim}_n \Sigma^{-n}\Sigma^\infty\Omega^\infty\Sigma^n\mathbb{F}_2\right) = \lim_n \widetilde{E}^*(\Sigma^{-n}\Sigma^\infty\underbrace{\Omega^\infty\Sigma^n\mathbb{F}_2}_{K(\mathbb{F}_2 n)}).$$

We can now compute this directly as the limit of $\pi_0\mathrm{Hom}_{\mathrm{Spectra}}(\Sigma^{-n}\Sigma^\infty K(\mathbb{F}_2, n), \Sigma^*\mathbb{F}_2)$, which is then the limit of $\pi_0\underline{\mathrm{Hom}}(K(\mathbb{F}_2, n), K(\mathbb{F}_2, n+*)) = \mathrm{H}^{*+n}(K(\mathbb{F}_2, n); \mathbb{F}_2)$ after undoing adjunctions. Now, Serre's theorems recasts this as the Steenrod algebra, where Sq^i lives in degree i. ■

Remark 3.87. One way to recast the algebra structure on the Steenrod algebra is that Sq^i is some map $\mathbb{F}_2 \to \Sigma^i\mathbb{F}_2$, which can be composed as follows: $\mathrm{Sq}^i \mathrm{Sq}^j$ is the composite

$$\mathbb{F}_2 \overset{\mathrm{Sq}^j}{\to} \Sigma^i\mathbb{F}_2 \overset{\Sigma^i \mathrm{Sq}^j}{\to} \Sigma^{i+j}\mathbb{F}_2.$$

Notably, the cohomology of spectra still has these Steenrod operations, but we do not have cup products!

Thus far, we have explained that the cohomology theory of a spectrum remembers interesting information about the spectrum. However, it also turns out that behaved cohomology theories all come from spectra.

Theorem 3.88 (Brown representability)**.** Fix a sequence of functors $\widetilde{h}^n \colon \mathrm{Spaces}_{*,\geq 1} \to \mathrm{Ab}$ satisfying Mayer–Vietoris. Then there is a spectrum E with $\widetilde{E}^n = \widetilde{h}^n$.

Remark 3.89. It is possible for different spectra to give the same cohomology theory. However, giving rise to the same cohomology theory is a very strong constraint; for example, the homotopy groups must agree by Remark 3.83. In general, there is a somewhat extensive theory of "phantom maps" explaining how such things happen.

Remark 3.90. Here is a fairly banal analogue of the homological Whitehead theorem: if $f \colon X \to Y$ induces isomorphisms $\widetilde{E}^n(X) \to \widetilde{E}^n(Y)$ for all spectra E and $n \in \mathbb{Z}$, then f induces an equivalence $\Sigma^\infty X \to \Sigma^\infty Y$. Indeed, one can just plug in $E = \Sigma^\infty X$ and apply the Yoneda lemma. However, we remark that it is possible for $X \ncong Y$ while $\Sigma^\infty X \cong \Sigma^\infty Y$.

3.4.3 The Tensor Product

We would like to form a tensor product $E \otimes F$ of spectra, which we do via adjunctions.

Definition 3.91 (tensor product)**.** Given spectra A and B, we define $A \otimes B$ by the Yoneda lemma via

$$\underline{\mathrm{Hom}}(A \otimes B, -) = \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(B, -)).$$

Existence follows from an adjoint functor theorem: the functor $- \otimes B$ is constructed as the left adjoint to $\underline{\mathrm{Hom}}(B, -)$. We may write this as $A \otimes_{\mathbb{S}} B$ for emphasis.

Remark 3.92. There are things to check about this tensor product, such as commutativity and associativity. Because we defined $- \otimes_{\mathbb{S}} B$, we also see that the tensor product commutes with colimits.

Example 3.93. We see

$$\underline{\mathrm{Hom}}(A \otimes \mathbb{S}, -) = \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(\mathbb{S}, -)) = \underline{\mathrm{Hom}}(A, -).$$

Thus, $A \otimes \mathbb{S} = A$.

Remark 3.94. If X and Y are unpointed spaces, then

$$\underline{\mathrm{Hom}}(X \times Y, -) = \underline{\mathrm{Hom}}(X, \underline{Hom}(Y, -)),$$

so $\Sigma^\infty_+(X \times Y) = \Sigma^\infty_+ X \otimes \Sigma^\infty_+ Y$. This explains how to compute tensor products of suspension spectra. Thus, if E and F are spectra with no negative homotopy groups, then the tensor product $E \otimes F$ arises as a colimit via Lemma 3.84. This explains why it may be commutative.

Example 3.95. If A and B are discrete abelian groups, then $A \otimes_{\mathbb{S}} B$ has no reason to be the discrete abelian group $A \otimes_{\mathbb{Z}} B$ (though they will agree in π_0). One could even do a derived tensor product $A \otimes^{\mathrm{L}}_{\mathbb{Z}} B$, which agrees in π_1 with $A \otimes_{\mathbb{S}} B$ but not higher.

Definition 3.96 (generalized homology)**.** Fix a spectrum E and a spectrum X. Then we define the *generalized E-homology* by

$$\widetilde{E}_*(X) := \pi_*(E \otimes_{\mathbb{S}} X).$$

If X is a pointed space, we may abbreviate $\widetilde{E}_*(\Sigma^\infty X)$ to $\widetilde{E}_*(X)$. If X is an unpointed space, we define $E_*(X) := \widetilde{E}_*(X_+)$.

Example 3.97. Taking $E = \mathbb{S}$ recovers stable homotopy groups.

Example 3.98. Because Σ^∞_+ and tensor products both preserve colimits, we see that we will be able to produce a long exact sequence in generalized E-homology.

Appendix A
Category Theory

In this appendix, we review some category theory as fast as possible.

Warning A.1. We will mostly ignore size issues. If it makes the reader feel better, we are willing to assume the existence of a countable ascending chain of inaccessible cardinals throughout the class.

A.1 Basic Definitions

Let's recall some starting notions of category theory.

Definition A.2 (category)**.** A *category* $\mathcal{C}$ is a collection of objects, a collection of morphisms $\operatorname{Mor}_{\mathcal{C}}(A,B)$ for each pair of objects, a distinguished identity element id_A in $\operatorname{Mor}_{\mathcal{C}}(A,A)$, and a composition law

$$\circ\colon \operatorname{Mor}_{\mathcal{C}}(B,C)\times\operatorname{Mor}_{\mathcal{C}}(A,B)\to\operatorname{Mor}_{\mathcal{C}}(A,C).$$

We then require the composition law to be associative and unital with respect to the identity maps.

Remark A.3. We will use the notation Hom for Mor whenever the category $\mathcal{C}$ is additive, meaning that these collections of morphisms are abelian groups, and the composition law is $\mathbb{Z}$-bilinear.

The following two examples are most important for this class.

Example A.4. There is a category Set of all sets. The morphisms are given by functions of sets, and the identity maps are compositions are all as usual. The fact that function composition is associative and unital implies that Set succeeds at being a category.

Example A.5. We let Δ to be the category whose objects are the nonnegative integers n, written $[n]\in\Delta$, where $\operatorname{Mor}_\Delta([m],[n])$ consists of the increasing maps $\{0,\ldots,m\}\to\{0,\ldots,n\}$. Composition of functions and identities are defined as for Set, allowing one to check that Δ is a category in exactly the same way.

The following example provides a useful technical tool.

Example A.6. Given a category $\mathcal{C}$, there is an opposite category $\mathcal{C}^{\mathrm{op}}$ whose objects are the same, but the morphisms are given by

$$\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(c, c') \coloneqq \mathrm{Mor}_{\mathcal{C}}(c', c).$$

We may write $f^{\mathrm{op}}\colon c' \to c$ for the morphism corresponding to $f\colon c \to c'$. Identities remain the same, but composition is defined by $f^{\mathrm{op}} \circ (f') = (f' \circ f)^{\mathrm{op}}$. The fact that $\mathcal{C}$ is a category makes $\mathcal{C}^{\mathrm{op}}$ into a category.

Definition A.7 (isomorphism)**.** A morphism $f\colon A \to B$ in a category $\mathcal{C}$ is an *isomorphism* if and only if there is a morphism $g\colon B \to A$ for which $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Definition A.8 (groupoid)**.** A *groupoid* is a category in which every morphism is an isomorphism.

It is worthwhile to have maps between categories as well.

Definition A.9 (functor)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the data of a map on the level of objects and maps

$$F\colon \mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathcal{D}}(Fc, Fc')$$

for any $c, c' \in \mathcal{C}$. Furthermore, we require $F\mathrm{id}_c = F\,\mathrm{id}_{Fc}$ and $F(f' \circ f) = Ff' \circ Ff$ for any $c \in \mathcal{C}$ and compose-able f and f'.

Here are some adjectives that a functor can have.

Definition A.10 (isomorphism)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *isomorphism* if and only if there is an inverse functor $G\colon \mathcal{D} \to \mathcal{C}$ for which FG and GF are both the identity functors.

Definition A.11 (full, faithful)**.** Fix a functor $F\colon \mathcal{C} \to \mathcal{D}$ and consider the maps

$$F\colon \mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathcal{D}}(Fc, Fc')$$

for any $c, c' \in \mathcal{C}$. We say that F is *full* if and only if these maps are always surjective, and we say that F is *faithful* if and only if these maps are always injective.

A.2 Natural Transformations

We are shortly going to get a lot of mileage out of the next example, so we spend some time to prove it in detail. We would like to define a category of functors between two given categories,[1] but this requires us to have a notion of morphism between functors.

Definition A.12 (natural transformation)**.** Given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta\colon F \Rightarrow G$ is the data of a morphism $\eta_A\colon FA \to GA$ for each object $A \in \mathcal{C}$. We further require that $Gf \circ \eta_A = \eta_B \circ Ff$ for any morphism $f\colon A \to B$. A *natural isomorphism* is a natural transformation η in which each morphism η_A is an isomorphism.

Diagrammatically, the equation $Gf \circ \eta_A = \eta_B \circ Ff$ amounts to the commutativity of the following square.

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"] & GB \end{tikzcd}$$

Anyway, here is our result.

[1] For those who are choosing to think about size issues, we remark that we will typically have one of $\mathcal{C}$ or $\mathcal{D}$ be locally small.

Lemma A.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then there is a functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ where the objects are functors $\mathcal{C} \to \mathcal{D}$ and the morphisms are natural transformations.

Proof. We have explained our objects and morphisms, but we still have to provide identities and composition laws and check that everything works.

- Identities: given a functor $F\colon \mathcal{C} \to \mathcal{D}$, there is an identity natural transformation $\mathrm{id}_F\colon F \to F$ given by $(\mathrm{id}_F)_A := \mathrm{id}_{FA}$; checking that this is a natural transformation amounts to noting that $Ff \circ \mathrm{id}_{FA} = \mathrm{id}_{FB} \circ Ff$ for any morphism $f\colon A \to B$.

- Composition: given two natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, we define the composite natural transformation $(\beta \circ \alpha)\colon F \Rightarrow H$ by $(\beta \circ \alpha)_A := \beta_A \circ \alpha_A$ for each $A \in \mathcal{A}$. Checking that this is a natural transformation amounts to checking the commutativity of the outer rectangle of

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"'] \arrow[d, "Ff"'] \arrow[rr, bend left, "(\beta\circ\alpha)_A"] & GA \arrow[r, "\beta_A"'] \arrow[d, "Gf"'] & HA \arrow[d, "Hf"] \\ FB \arrow[r, "\alpha_B"] \arrow[rr, bend right, "(\beta\circ\alpha)_B"'] & GB \arrow[r, "\beta_B"] & HB \end{tikzcd}$$

  which indeed commutes: the top and bottom triangles commute by definition of $\beta \circ \alpha$, and the two inner squares commute by naturality of α and β.

- Identities: given a natural transformation $\eta\colon F \Rightarrow G$, we need to check that $\mathrm{id}_G \circ \eta = \eta \circ \mathrm{id}_F = \eta$. Well, for any object A, we see that

$$(\mathrm{id}_G \circ \eta)_A = (\mathrm{id}_G \circ \eta)_A = \mathrm{id}_{G(A)} \circ \eta_A = \eta_A,$$

  and

$$(\eta \circ \mathrm{id}_F)_A = \eta_A \circ \mathrm{id}_{FA} = \eta_A.$$

- Associativity: given natural transformations α, β, and γ with appropriate domains and codomains, we must check that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$. Well, for any object A, we see that

$$((\alpha \circ \beta) \circ \gamma) = (\alpha_A \circ \beta_A) \circ \gamma_A = \alpha_A \circ (\beta_A \circ \gamma_A) = (\alpha \circ (\beta \circ \gamma))_A,$$

  as required. ∎

Example A.14. For any $c \in \mathcal{C}$, there is a functor $\mathrm{ev}_c\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ given by $\mathrm{ev}_c(F) := Fc$ on objects. On morphisms, we send a natural transformation $\eta\colon F \Rightarrow F'$ to the morphism $\mathrm{ev}_c(\eta) := \eta_c$. For example, $\mathrm{ev}_c(\mathrm{id}_F) = (\mathrm{id}_F)_c = \mathrm{id}_{Fc}$. Lastly, to check functoriality, we pick up two natural transformations $\eta\colon F \Rightarrow F'$ and $\eta'\colon F' \Rightarrow F''$, and we note that $\mathrm{ev}_c(\eta'\eta) = (\eta'\eta)_c = \eta'_c\eta_c = \mathrm{ev}_c(\eta')\,\mathrm{ev}_c(\eta)$.

A.3 The Yoneda Lemma

We are going to get some mileage out of using presheaf categories.

Definition A.15 (presheaf)**.** Fix a category $\mathcal{C}$. Then a *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Accordingly, the presheaf category $\mathrm{PSh}(\mathcal{C})$ is the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$.

Remark A.16. It will be worthwhile to have a way to build "subpresheaves." Given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, suppose that we have a collection of sets $\{Gc\}_{c\in\mathcal{C}}$ such that $Gc \subseteq Fc$ for each $c \in \mathcal{C}$. Furthermore, suppose that each map $f\colon c \to c'$ has $Ff(Gc) \subseteq Gc'$. Then we can define $Gf := Ff|_{Gc}$, and it follows that G is a functor because F is a functor, and the inclusions $Gc \subseteq Fc$ now assemble into a natural transformation.

Here is our main result.

Theorem A.17 (Yoneda lemma)**.** Fix a category $\mathcal{C}$. Then there is a functor $\text{よ}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ which is defined on objects by

$$\text{よ}_c := \mathrm{Mor}_{\mathcal{C}}(-, c).$$

Furthermore, よ is fully faithful.

Proof. This is purely formal. We proceed with our checks in sequence.

- For $c \in \mathcal{C}$, we show that $\text{よ}_c\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a functor. To review, on morphisms $g\colon d \to d'$, we define $\text{よ}(g)\colon \mathrm{Mor}_{\mathcal{C}}(d', c) \to \mathrm{Mor}_{\mathcal{C}}(d, c)$ by $\text{よ}_c(g) := (- \circ g)$. For example, $\text{よ}_c(\mathrm{id}_d) = (- \circ \mathrm{id}_d)$, which is just the identity map. Lastly, for maps $g\colon d \to d'$ and $g'\colon d' \to d''$, we see that $\text{よ}_c(g' \circ g)$ and $\text{よ}_c(f) \circ \text{よ}_c(f')$ both equal
$$(- \circ (g' \circ g)) = (- \circ g) \circ (- \circ g')$$
by the associativity of composition.

- We construct よ on morphisms. Given $f\colon c \to c'$, we need a natural transformation $\text{よ}_f\colon \text{よ}_c \Rightarrow \text{よ}_{c'}$. Well, for any object $d \in \mathcal{D}$, we define the component map $(\text{よ}_f)_d\colon \mathrm{Mor}_{\mathcal{C}}(d, c) \to \mathrm{Mor}_{\mathcal{C}}(d, c')$ by $(f \circ -)$. To see that よ_f assembles into a natural transformation, we see that any morphism $g\colon d \to d'$ makes the following diagram commute.
$$\begin{tikzcd} \text{よ}_c(d') \arrow[r, "\text{よ}_c(g)"] \arrow[d, "(\text{よ}_f)_{d'}"'] & \text{よ}_c(d) \arrow[d, "(\text{よ}_f)_d"] \\ \text{よ}_{c'}(d') \arrow[r, "\text{よ}_{c'}(g)"] & \text{よ}_c(d') \end{tikzcd} \qquad \begin{tikzcd} h \arrow[r, mapsto] \arrow[d, mapsto] & h \circ g \arrow[d, mapsto] \\ f \circ h \arrow[r, mapsto] & f \circ h \circ g \end{tikzcd}$$

- We show that よ is a functor. For example, よ_{id_c} is the identity natural transformation because $(\text{よ}_{\mathrm{id}_c})_d$ is just $(\mathrm{id}_c \circ -)$ for each $d \in \mathcal{C}$, which is in fact the identity. Continuing, for any $f\colon c \to c'$ and $f'\colon c' \to c''$, we see that any object $d \in \mathcal{C}$ makes both $(\text{よ}_{f' \circ f})_d$ and $(\text{よ}_{f'} \circ \text{よ}_f)_d$ equal to
$$((f' \circ f) -) = (f' \circ -) \circ (f \circ -)$$
by associativity.

- Fix any functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. We define an injective map $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(\text{よ}_c, F) \to Fc$. To define our map, we send a natural transformation $\eta\colon \text{よ}_c \Rightarrow F$ to $\eta_c(\mathrm{id}_c) \in Fc$. Let's check that this is injective: if η and η' have $\eta_c(\mathrm{id}_c) = \eta'_c(\mathrm{id}_c)$, then we need to show that $\eta_d(h) = \eta'_d(h)$ for any $d \in \mathcal{C}$ and $h\colon d \to c$. Well, the commutativity of the diagram
$$\begin{tikzcd} \text{よ}_c(c) \arrow[r, "\eta_c"] \arrow[d, "\text{よ}_c(h)"'] & Fc \arrow[d, "Fh"] \\ \text{よ}_c(d) \arrow[r, "\eta_d"] & Fd \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_c \arrow[r, mapsto] \arrow[d, mapsto] & \eta_c(\mathrm{id}_c) \arrow[d, mapsto] \\ h \arrow[r, mapsto, dashed] & Fh(\eta_c(\mathrm{id}_c)) \end{tikzcd}$$
reveals that $\eta_d(h) = Fh(\eta_c(\mathrm{id}_c))$. A similar argument holds for η', so we conclude that $\eta_d(h) = \eta'_d(h)$.

- We show that the map of the previous point is in fact surjective. The end of the argument informs our construction: for any $x \in Fc$, we define $\eta\colon \text{よ}_c \Rightarrow F$ by $\eta_d(h) := Fh(x)$ for any $d \in \mathcal{C}$ and $h\colon d \to c$. This η of course satisfies $\eta_c(\mathrm{id}_c) = x$, so it only remains to check that η is actually a natural transformation. Well, for any morphism $g\colon d \to d'$, we see that the following diagram commutes.

$$\begin{tikzcd} \text{よ}_c(d') \arrow[r, "\eta_d"] \arrow[d, "\text{よ}_c(g)"'] & Fd' \arrow[d, "Fg"] \\ \text{よ}_c(d) \arrow[r, "\eta_{d'}"] & Fd \end{tikzcd} \qquad \begin{tikzcd} h \arrow[r, mapsto] \arrow[d, mapsto] & Fh(x) \arrow[d, mapsto] \\ h \circ g \arrow[r, mapsto] & Fg(Fh(x)) \end{tikzcd}$$

- We complete the proof. We need to show that よ gives bijections $\text{よ}\colon \mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathrm{PSh}}(\text{よ}_c, \text{よ}_{c'})$. Well, the two previous points shows that the target is in bijection with $\text{よ}_{c'}(c)$ via $\eta \mapsto (\text{よ}_{c'})_c(\mathrm{id}_c)$. But the total composite
$$\mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathrm{PSh}}(\text{よ}_c, \text{よ}_{c'}) \to \text{よ}_{c'}(c)$$
sends a map $f\colon c \to c'$ to the natural transformation $(f \circ -)$ and then back to the map f. We conclude that よ is the inverse bijection for the map of the previous two points. ■

Remark A.18. It is worth noting that the previous point provides us with a bijection
$$\mathrm{Mor}_{\mathrm{PSh}}(\text{よ}_c, F) \to Fc$$
by $\eta \mapsto \eta_c(\mathrm{id}_c)$. In fact, we also exhibited an inverse map by sending $x \in Fc$ to the natural transformation η defined by $\eta_d(h) := Fh(x)$ for any $h\colon d \to c$.

Motivated by algebraic geometry, one has the following definition.

Definition A.19 (representable)**.** A presheaf $\mathcal{F}$ on a category $\mathcal{C}$ is *representable* if and only if there is an object $c \in \mathcal{C}$ for which $\mathcal{F}$ is isomorphic to $\text{よ}(c)$.

A.4 Limits and Colimits

Some objects in categories can be characterized by their special properties. Limits and colimits provide a convenient language for this.

Definition A.20 (limit)**.** Fix a functor $F\colon \mathcal{I} \to \mathcal{C}$. An object $c \in \mathcal{C}$ is a *limit* of F if and only if c is "universal" with respect to having morphisms $Fi \to c$ for each $i \in \mathcal{I}$ making the diagrams

$$\begin{tikzcd} c \arrow[r] \arrow[dr] & Fi \arrow[d] \\ & Fi' \end{tikzcd}$$

commute for any $i \to i'$. In other words, for any other object c' equipped with such morphisms, there is a unique map $c' \to c$ commuting with these morphisms. A *colimit* of F is the same notion, but the maps from F go into c instead of out of c.

Example A.21 (product)**.** A product is a limit of a functor $F\colon \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ has no non-identity morphisms. The colimit is called the coproduct.

Example A.22 (equalizer)**.** An equalizer is a limit of a functor $F\colon \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ has exactly two objects and one morphism between them. The colimit is called the co-equalizer.

Lemma A.23. Fix a functor $F\colon \mathcal{I} \to \mathcal{C}$. Then any two limits of F are isomorphic.

Proof. Let c and c' be limits of F. Then there are unique maps $c \to c'$ and $c' \to c$ commuting with the morphisms from $F(\mathcal{I})$ by definition of a limit. But then there is also a unique morphism $c \to c$ commuting with these morphisms, which must be the identity, so the composite

$$c \to c' \to c$$

must also be the identity. Similarly, we see that $c' \to c \to c'$ is the identity, thereby completing the proof. ■

The above lemma allows us to give some notation for our limits: we may write the limit of F as $\lim_{\mathcal{I}} F$ (when it exists!) and the colimit as $\operatorname{colim}_{\mathcal{I}} F$.

Remark A.24. Lemma A.23 also holds for colimits, which we can see by passing to opposite categories.

Example A.25. The category Set admits all limits and colimits of functors $F\colon \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ has at most κ many objects and morphisms for some cardinal κ.

Proof. We explicitly construct the limit and colimit of such a functor F.

- We handle the limit. Define the set

$$L := \left\{ (x_i) \in \prod_{i \in \mathcal{I}} Fi : Ff(x_i) = x_j \text{ for all } f\colon i \to j \right\}.$$

  We claim that L is the limit. Because L is a subset of the product, there are projection maps $\mathrm{pr}_i\colon L \to Fi$ for each $i \in \mathcal{I}$. Furthermore, for any $f\colon i \to j$ and $(x_i) \in L$, we see that $\mathrm{pr}_j(x_i) = f(\mathrm{pr}_i(x_i))$ by construction of L.

  It remains to check that L is universal. Well, if L' is any object equipped with such maps $\varphi_i\colon L' \to Fi$ for each i, then we must construct a unique map $\varphi\colon L' \to L$ commuting with everything. For the uniqueness, we see that this commuting requires $\mathrm{pr}_i \varphi(x') = \varphi_i(x')$ for all $i \in \mathcal{I}$, so we are forced to define

$$\varphi(x') := (\varphi_i(x'))$$

  for all $i \in \mathcal{I}$. It remains to check that this map works. We already know that it commutes with all the given maps, so we only have to check that φ is well-defined (i.e., outputs to L). Namely, for any $f\colon i \to j$, we need to check that $Ff(\varphi_i(x')) = \varphi_j(x')$, which is true by hypothesis on the $\varphi_\bullet$s.

- We handle the colimit, which is similar. We will only give the construction. Define the set

$$C := \left(\bigsqcup_{i \in \mathcal{I}} F_i \right) /\sim,$$

  where $\sim$ is the equivalence relation generated by having $x_i \in Fi$ and $x_j \in Fj$ similar if and only if there is a map $f\colon i \to j$ for which $Ff(x_i) = x_j$. Because L is a quotient of a disjoint union, there are inclusion maps $\iota_i\colon F_i \to L$ for each $i \in \mathcal{I}$. Furthermore, for any $f\colon i \to j$ and $\iota_i(x_i) \in L$, we see that $\iota_j(Ff(x_i)) = \iota_i(x_i)$ by definition of the equivalence relation.

  It remains to check that C is universal, which we omit. ■

Example A.26. Fix categories $\mathcal{C}$ and $\mathcal{D}$, and suppose that $\mathcal{D}$ admits all colimits from categories of cardinality at most κ. Then $\operatorname{Fun}(\mathcal{C},\mathcal{D})$ admits all limits and colimits of functors $F\colon \mathcal{I} \to \operatorname{Fun}(\mathcal{C},\mathcal{D})$, where the category $\mathcal{I}$ of cardinality at most κ.

Proof. We explicitly construct the colimit "pointwise." Namely, we need to define some functor A to be the colimit of F, so we ought to construct $Ac \in \mathcal{D}$ for each $c \in \mathcal{C}$. With this in mind, we set

$$Ac \coloneqq \operatorname*{colim}_{i\in\mathcal{I}} F(i)(c),$$

which exists as an element of $\mathcal{D}$ because $\mathcal{D}$ admits a colimit of the functor

$$\mathcal{I} \xrightarrow{F} \operatorname{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\mathrm{ev}_c} \mathcal{D}.$$

We will let $(\eta_i)_c\colon Fi(c) \to Ac$ be the induced morphism. Now, for each morphism $f\colon c \to c'$, we need to define a morphism $Af\colon Ac \to Ac'$. Well, we claim that the maps $F(i)(c) \to F(i)(c')$ assemble into a morphism

$$\operatorname*{colim}_{i\in\mathcal{I}} F(i)(c) \to \operatorname*{colim}_{i\in\mathcal{I}} F(i)(c').$$

Well, to map out of $\operatorname{colim}_{i\in\mathcal{I}} F(i)(c)$, we need to check that the maps $F(j)(c) \to F(j)(c') \to \operatorname{colim} F(i)(c')$ commute with the internal maps of $\mathcal{I}$, which amounts to the commutativity of the following diagram.

$$\begin{tikzcd}
Fj(c) \arrow[r, "Fj(f)"] \arrow[d, "Ff(c)"'] & Fj(c') \arrow[r] \arrow[d, "Ff(c')"] & \operatorname*{colim}_{i\in\mathcal{I}} Fi(c') \\
Fj'(c) \arrow[r, "Fj'(f)"] & Fj'(c') \arrow[ur] &
\end{tikzcd}$$

Here, the square commutes because the internal map $j \to j'$ goes to a natural transformation $Fj \Rightarrow Fj'$. The point is that Af is the unique map making the diagram

$$\begin{tikzcd}
Fj(c) \arrow[r, "Fj(f)"] \arrow[d] & Fj(c') \arrow[d] \\
Ac \arrow[r, "Af"] & Ac'
\end{tikzcd}$$

commute for every $j \in \mathcal{I}$; note that this is equivalent to saying that $\eta_j\colon Fj \Rightarrow A$ is a natural transformation for each $j \in \mathcal{J}$. For example, if $f = \mathrm{id}_c$, then certainly having $Af = \mathrm{id}_{Ac}$ will make the diagram commute. Similarly, given two maps $f\colon c \to c'$ and $f'\colon c' \to c''$, the commutativity of the diagram

$$\begin{tikzcd}
Fj(c) \arrow[r, "Fj(f)"'] \arrow[rr, bend left, "Fh(f'f)"] \arrow[d] & Fj(c') \arrow[r, "Fj(f')"'] \arrow[d] & Fj(c'') \arrow[d] \\
Ac \arrow[r, "Af"] \arrow[rr, bend right, "A(f'f)"'] & Ac' \arrow[r, "Af'"] & Ac''
\end{tikzcd}$$

for all $j \in \mathcal{I}$ implies that $A(f'f) = Af' \circ Af$ by the uniqueness of a map making the outer rectangle commute.

We thus see that we have constructed a functor $A\colon \mathcal{C} \to \mathcal{D}$. It remains to actually show that A is the colimit. Well, suppose we have an object $B\colon \mathcal{C} \to \mathcal{D}$ equipped with natural transformations $\varphi_i\colon Fi \Rightarrow B$ for each $i \in \mathcal{I}$ commuting with the induced maps from $\mathcal{I}$. We would like to induce a unique map $\varphi\colon A \Rightarrow B$ such that $\varphi\eta_i = \varphi_i$ for all $i \in \mathcal{I}$.

Let's show uniqueness by showing why φ is forced. Having $\varphi\eta_i = \varphi_i$ implies that $\varphi_c \circ (\eta_i)_c = (\varphi_i)_c$ for each $c \in \mathcal{C}$. On the other hand, these maps $(\eta_i)_c$ cause the diagram

$$\begin{tikzcd}
Fi(c) \arrow[r, "(\eta_i)_c"] \arrow[d] & Bc \\
Fj(c) \arrow[ur, "(\eta_j)(c)"'] &
\end{tikzcd}$$

to commute for each map $i \to j$ (by applying ev_c to the commutativity relation for the η_is). Thus, there is a unique map $\varphi_c \colon Ac \to Bc$ satisfying $\varphi_c \circ (\eta_i)_c = (\varphi_i)_c$. This proves uniqueness.

It remains to show that the data of these maps $\varphi_c \colon Ac \to Bc$ actually assemble into a natural transformation $\varphi \colon A \Rightarrow B$; this would directly imply that $\varphi \circ \eta_i = \varphi_i$ for each $i \in \mathcal{I}$ because we already know this to be true at each $c \in \mathcal{C}$. For our naturality check, we choose some map $f \colon c \to c'$, and we draw the following diagram.

$$\begin{tikzcd} Fi(c) \arrow[r, "(\eta_i)_c"'] \arrow[rr, bend left, "(\varphi_i)_c"] \arrow[d, "Fi(f)"'] & Ac \arrow[r, "\varphi_c"'] \arrow[d, "Af"] & Bc \arrow[d, "Bf"] \\ Fi(c') \arrow[r, "(\eta_i)_{c'}"] \arrow[rr, bend right, "(\varphi_i)_{c'}"'] & Ac' \arrow[r, "\varphi_{c'}"] & Bc' \end{tikzcd}$$

The left square and outer rectangle commute by naturality. Now, by the same colimit argument executed in the previous paragraph, there is at most one map $Ac \to Bc'$ factoring through the maps $Fi(c) \to Bc \to Bc'$ for all $i \in \mathcal{I}$; however, both $Bf \circ \varphi_c$ and $\varphi_{c'} \circ Af$ satisfy this property, so we conclude! ■

Remark A.27. There is an analogous statement if $\mathcal{D}$ admits all limits, whose proof is the same. (Indeed, one can recover the limit version by passing from $\mathcal{D}$ to $\mathcal{D}^{\mathrm{op}}$.)

In order to use the Yoneda lemma, it will be useful to have the following characterization of limits.

Lemma A.28. Fix a category $\mathcal{C}$ and a diagram $F \colon \mathcal{I} \to \mathcal{C}$ for which the colimit $\operatorname{colim}_{i\mathcal{I}} Fi$ exists. Then there is a natural isomorphism

$$\mathrm{Mor}_{\mathcal{C}}\left(\operatorname*{colim}_{i \in \mathcal{I}} Fi, -\right) \Rightarrow \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(Fi, -).$$

Proof. This follows from the construction of limits in $\mathrm{PSh}(\mathcal{C})$ given in Example A.26 (which are given pointwise) followed by their explicit construction in Example A.25, which tells us that

$$\lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(Fi, c) = \left\{ (f_i) \in \prod_{i \in \mathcal{I}} : f_i = f_j \circ Ff \text{ for each } f \in \mathrm{Mor}_{\mathcal{I}}(i, j) \right\}.$$

On the other hand, the universal property implies that a map $\operatorname{colim}_{i \in \mathcal{I}} Fi \to c$ is in unique bijection with collections of maps $f_i \colon Fi \to c$ for which $f_i = f_j \circ Ff$ whenever $f \colon i \to j$, which is exactly the limit given above!

To check naturality, we need to make our isomorphism more explicit: fix inclusions $g_i \colon Fi \to \operatorname{colim}_{i \in \mathcal{I}} Fi$, and then our map is given by

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}\left(\operatorname*{colim}_{i \in \mathcal{I}} Fi, c\right) &\to \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(Fi, -) \\ f \quad &\mapsto \quad (f \circ g_i)_i \end{aligned}$$

for which naturality follows by noticing that the square

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}\left(\operatorname*{colim}_{i \in \mathcal{I}} Fi, c\right) \arrow[r] \arrow[d] & \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(Fi, c) \arrow[d] & f \arrow[r, mapsto] \arrow[d, mapsto] & (fg_i)_i \arrow[d, mapsto] \\ \mathrm{Mor}_{\mathcal{C}}\left(\operatorname*{colim}_{i \in \mathcal{I}} Fi, c'\right) \arrow[r] & \lim_{i \in \mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(Fi, c') & hf \arrow[r, mapsto] & (hfg_i)_i \end{tikzcd}$$

commutes for any morphism $h \colon c \to c'$ ■

Here is an application of the Yoneda lemma.

Lemma A.29. Fix a category $\mathcal{C}$ and presheaf $A \in \mathrm{PSh}(\mathcal{C})$. Construct a category $\mathcal{I}$ of pairs (c, η), where $c \in \mathcal{C}$, and $\eta\colon \text{よ}(c) \Rightarrow A$. Then the canonical map

$$\operatorname*{colim}_{(c,\eta)\in\mathcal{I}} \text{よ}(c) \to A$$

is an isomorphism.

Proof. In other words, we would like to show that the given maps $\eta\colon \text{よ}(c) \to A$ for each pair (c, η) makes into A a colimit of the functor $F\colon \mathcal{I} \to \mathrm{PSh}(\mathcal{C})$. We run many checks.

1. We check that F is actually a functor. On objects, we are sending $F((c, \eta)) := \text{よ}(c)$. On morphisms, we send a morphism $f\colon (c, \eta) \to (c', \eta')$ to the morphism $\text{よ}(f)\colon \text{よ}(c) \to \text{よ}(c')$. Because $\text{よ}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ is already functorial, we see that F becomes functorial.

2. We note that each pair $(c, \eta) \in \mathcal{I}$ has $\eta\colon \text{よ}(c) \Rightarrow A$. For A to be a candidate colimit of F, we need to check that any map $f\colon (c, \eta) \to (c', \eta')$ makes the diagram

$$\begin{tikzcd} \text{よ}(c) \arrow[r, "\eta"] \arrow[d, "\text{よ}(f)"'] & A \\ \text{よ}(c') \arrow[ur, "\eta'"'] & \end{tikzcd}$$

commute, which of course is true by definition of the morphisms in $\mathcal{I}$.

3. We run the uniqueness part of the universality check for A. Indeed, suppose that we have some B with maps $\varphi_{(c,\eta)}\colon \text{よ}(c) \Rightarrow B$ for each $(c, \eta) \in \mathcal{I}$ such that $\text{よ}(f)\varphi_{(c,\eta)} = \varphi_{(c',\eta')}$ for each $f\colon (c, \eta) \to (c', \eta')$ in $\mathcal{I}$. Then we show that there is at most one map $\varphi\colon A \to B$ making the diagram

$$\begin{tikzcd} \text{よ}(c) \arrow[r, "\eta"] \arrow[dr, "\varphi_{(c,\eta)}"'] & A \arrow[d, dashed, "\varphi"] \\ & B \end{tikzcd}$$

commute for each pair $(c, \eta) \in \mathcal{I}$. Well, fix some object $c \in \mathcal{C}$ and element $a \in Ac$, and we need to show that $\varphi(a)$ has at most one value. Well, $a \in Ac$ has equivalent data to some natural transformation $\eta\colon \text{よ}(c) \to A$ satisfying $\eta_c(\mathrm{id}_c) \in Ac$ by Remark A.18, so we have a pair $(c, \eta) \in \mathcal{I}$. Similarly, the natural transformation $\varphi_{(c,\eta)}\colon \text{よ}(c) \to B$ has equivalent data to the element $(\varphi_{(c,\eta)})_c(\mathrm{id}_c)$. But now the commutativity of the above diagram

$$\begin{tikzcd} \text{よ}(c) \arrow[r, "\eta"] \arrow[dr, "\varphi_{(c,\eta)}"'] & A \arrow[d, dashed, "\varphi"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_c \arrow[r, mapsto] \arrow[dr, mapsto] & a \arrow[d, mapsto] \\ & \eta_c(\mathrm{id}_c) \end{tikzcd}$$

requires that $\varphi(a) = \eta_c(\mathrm{id}_c)$. Thus, there is at most one map φ.

4. We show that the recipe for $\varphi\colon A \to B$ defined in the previous step is actually a morphism of presheaves. Thus far, we have defined maps $\varphi_c\colon Ac \to Bc$ for each $c \in \mathcal{C}$, and we remark that these maps are well-defined by the uniqueness properties of Remark A.18. It now remains to show naturality. Well, fix a morphism $f\colon c \to c'$ and some $a' \in Ac'$, and we want to show that

$$Bf(\varphi_{c'}(a')) \stackrel{?}{=} \varphi_c(Af(a')).$$

Well, $a \in Ac$ corresponds (via Remark A.18) to some natural transformation $\eta\colon \text{よ}(c') \Rightarrow A$ such that $\eta_{c'}(\mathrm{id}_{c'}) = a'$, so we have a pair $(c', \eta') \in \mathcal{I}$; namely, for any $h\colon d \to c'$, we have $\eta'_d(h) = Ah(a')$. Similarly, $Af(a') \in Ac'$ corresponds to some natural transformation $\eta\colon \text{よ}(c) \Rightarrow A$ given by $\eta_d(h) =$

$Ah(Af(a))$ for any $h\colon d \to c$. By definition of よ, we also have a morphism of pairs $f\colon (c,\eta) \to (c',\eta')$. We are now able to chase around the following diagram.

$$\begin{tikzcd}
Ac' \arrow[r, "Af"] & Ac & \\
\text{よ}(c')(c') \arrow[u, "\eta'_{c'}"] \arrow[r, "\text{よ}(c')(f)"] \arrow[d, "(\varphi_{(c',\eta')})_{c'}"'] & \text{よ}(c')(c) \arrow[u, "\eta'_c"] \arrow[d, "(\varphi_{(c',\eta')})_c"'] & \text{よ}(c)(c) \arrow[l, "\text{よ}(f)(c)"'] \arrow[ul, "\eta_c"'] \arrow[dl, "(\varphi_{(c,\eta)})_c"] \\
Bc' \arrow[r, "Bf"] & Bc &
\end{tikzcd}
\qquad
\begin{tikzcd}
a' \arrow[rr, mapsto] & & Af(a) \\
\mathrm{id}_{c'} \arrow[u, mapsto] \arrow[d, mapsto] & & & \mathrm{id}_c \arrow[ul, mapsto] \arrow[dl, mapsto] \\
\varphi_{c'}(a') \arrow[rr, mapsto, dashed, "Bf"] & & \varphi_c(Af(a))
\end{tikzcd}$$

The triangles commute because we have morphisms of pairs. The squares are naturality squares. The $\varphi_{c'}(a')$ and $\varphi_c(Af(a))$ appear at the bottom by construction of $\varphi_{c'}$ and φ_c by definition of those maps. We are now done by commutativity! ■

A.5 Adjoints

Here is the main definition.

Definition A.30 (adjoint)**.** Fix two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. Then F and G are *adjoint functors* if and only if there are bijections

$$\operatorname{Mor}_{\mathcal{D}}(Fc,d) = \operatorname{Mor}_{\mathcal{C}}(c,Gd)$$

for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ which are functorial in c and d. In this situation, we say that F is the *left adjoint* and that G is the *right adjoint*.

Here is the main result on adjoints.

Proposition A.31. Fix adjoint functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. Then F preserves colimits.

Proof. We may as well check this directly, though there are more optimal ways. Suppose that $a \in \mathcal{C}$ is the colimit of some functor $A\colon \mathcal{I} \to \mathcal{C}$. Then we would like to check that $Fa \in \mathcal{D}$ is the colimit of the composite functor $FA\colon \mathcal{I} \to \mathcal{C}$. (We will not bother to check that FA is actually a functor.)

To show that Fa is a candidate colimit, we need to build maps $FAi \to Fa$ for each $i \in \mathcal{I}$. Indeed, a being the colimit of A implies that we get maps $\eta_i\colon Ai \to a$ for each $i \in \mathcal{I}$ for which any map $f\colon i \to j$ implies $\eta_j \circ Af = \eta_i$. Thus, we have maps $F\eta_i\colon FAi \to Fa$ for each $i \in \mathcal{I}$, and we note that any map $f\colon i \to j$ implies $F\eta_j \circ FAf = F\eta_i$ by functoriality.

It remains to show that Fa is universal, for which we use the adjunction. Suppose that we have $b \in \mathcal{D}$ equipped with maps $\varphi_i\colon GAi \to b$ for which any $f\colon i \to j$ yields $\varphi_j \circ GAf = \varphi_i$. We need to show that there is a unique map $\varphi\colon Fa \to b$ commuting with the $\eta_\bullet$s and $\varphi_\bullet$s. To see this, we recall that we have a functorial bijection

$$\operatorname{Mor}_{\mathcal{D}}(Fa,b) = \operatorname{Mor}_{\mathcal{C}}(a,Gb),$$

so the functoriality implies that there is a bijection between maps $\varphi\colon Fa \to b$ for which $\varphi_i = \varphi \circ F\eta_i$ for each i and maps $\psi\colon a \to Gb$ for which $\psi_i = \psi \circ \eta_i$ (where $\psi_i\colon Ai \to Gb$ is the map induced from $\varphi_i\colon FAi \to b$ by the adjunction). In a few more words, functoriality implies that the square

$$\begin{tikzcd}
\operatorname{Mor}_{\mathcal{D}}(Fa,b) \arrow[r, equal] \arrow[d, "(-\circ F\eta_i)"'] & \operatorname{Mor}_{\mathcal{C}}(a,Gb) \arrow[d, "(-\circ\eta_i)"] \\
\operatorname{Mor}_{\mathcal{D}}(FAi,b) \arrow[r, equal] & \operatorname{Mor}_{\mathcal{C}}(Ai,Gb)
\end{tikzcd}$$

commutes, so any $\varphi\colon Fa \to b$ on the left satisfies $\varphi_i = \varphi \circ F\eta_i$ (for a given $i \in \mathcal{I}$) if and only if φ goes to some map $\psi\colon a \to Gb$ for which $\psi \circ \eta_i = \psi_i$ (for a given $i \in \mathcal{I}$). We take a moment to remark that the condition

$\varphi_j \circ GAf = \varphi_i$ for any map $f\colon i \to j$ now translates to $\psi_j \circ Af = \psi_i$ by the same sort of functoriality argument.

But we are now basically done: because we have $\psi_j \circ Af = \psi_i$ for any map $f\colon i \to j$, we see that a being the colimit implies that there is a unique map $\psi\colon a \to Gb$ for which $\eta_i \circ \psi = \psi_i$ for all $i \in \mathcal{I}$. This completes the proof! ■

Remark A.32. Similarly, G preserves limits; one can see this by passing to the opposite category everywhere.

Here is an application to presheaves.

Proposition A.33. Fix a functor $F\colon \mathcal{C} \to \mathcal{D}$. If $\mathcal{D}$ admits all colimits, then there is a unique functor $\widetilde{F}\colon \operatorname{PSh}(\mathcal{C}) \to \mathcal{D}$ which preserves colimits and satisfies

$$\widetilde{F} \circ よ \simeq F.$$

In fact, $\widetilde{F}$ is a left adjoint to the functor $G\colon \mathcal{D} \to \operatorname{PSh}(\mathcal{C})$ given by $Gd := \operatorname{Mor}_{\mathcal{D}}(F(-), d)$.

Proof. Let's start with the uniqueness of $\widetilde{F}$. For each presheaf $A \in \operatorname{PSh}(\mathcal{C})$, we define the category $\mathcal{I}_A$ of pairs (c, η) where $c \in C$ and $\eta\colon よ(c) \Rightarrow A$; note that η has equivalent data to an element of Ac. By Lemma A.29, we see that the canonical map

$$\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} よ(c) \to A$$

is an isomorphism. Thus, if $\widetilde{F}$ preserves colimits, we see that we must have

$$\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} Fc \xrightarrow{\sim} \widetilde{F}A.$$

Thus, $\widetilde{F}$ is uniquely defined on objects. On a morphism $f\colon A \to B$, we claim that there is a canonical morphism

$$\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} よ(c) \to \operatorname*{colim}_{(c,\eta)\in\mathcal{I}_B} よ(c).$$

For this, we note that there is a functor $\mathcal{I}_f\colon \mathcal{I}_A \to \mathcal{I}_B$ given by sending the pair $(c, \eta) \in \mathcal{I}_A$ to $(c, f\eta)$; on morphisms, we send a map $(c, \eta) \to (c', \eta')$ to the map $(c, f\eta) \to (c', f\eta')$, which makes sense as the data of the map $c \to c'$ and (the requirement that the induced map $よ(c) \to よ(c') \overset{f\eta'}{\to} B$ is the same as $f\eta$). Thus, the above canonical morphism is simply induced by sending the $よ(c)$ in entry (c, η) of the colimit identically back to $よ(c)$ in entry $(c, f\eta)$ of the colimit. Because this canonical map is just made of identities, we see that the square

$$\begin{tikzcd}
\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} よ(c) \arrow[r] \arrow[d] & A \arrow[d, "f"] & \mathrm{id}_c \arrow[r, mapsto] \arrow[d, mapsto] & \eta \arrow[d, mapsto, "f"] \\
\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_B} よ(c) \arrow[r] & B & \mathrm{id}_c \arrow[r, mapsto] & f\eta
\end{tikzcd}$$

commutes. (Of course, there is a unique map on the left making this commute because the horizontal maps are isomorphisms!) Thus, we may this diagram with $\widetilde{F}$ to see that

$$\begin{tikzcd}
\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} Fc \arrow[r] \arrow[d] & \widetilde{F}A \arrow[d, "Ff"] \\
\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_B} Fc \arrow[r] & \widetilde{F}B
\end{tikzcd}$$

where the left map continues to be induced by identities. In this way, we see that $\widetilde{F}$ is uniquely determined up to natural isomorphism because the above diagram witnesses this natural isomorphism!

The uniqueness check tells us exactly what we should set to be $\widetilde{F}$ on objects and morphisms. It remains to check that these data actually assemble into a functor preserving colimits. For example, on an identity $\mathrm{id}_A\colon A \to A$, we see that the map $\widetilde{F}A \to \widetilde{F}A$ is induced by hitting the canonical map

$$\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} \text{よ}(c) \to \operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} \text{よ}(c)$$

with F, but of course this canonical map is just the identity, so $\widetilde{F}\mathrm{id}_A = \mathrm{id}_{\widetilde{F}A}$ follows. Similarly, given two maps $f\colon A \to B$ and $g\colon B \to C$, we see that the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} \text{よ}(c) \arrow[rr] \arrow[dr] \arrow[dd] & & A \arrow[dr, "f"] \arrow[dd, "gf"'] & \\
& \operatorname*{colim}_{(c,\eta)\in\mathcal{I}_B} \text{よ}(c) \arrow[rr] \arrow[dl] & & B \\
\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_C} \text{よ}(c) \arrow[rr] & & C \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes, where the left arrows are the canonical maps; namely, all squares commute as discussed above, and the horizontal maps are isomorphisms, so this enforces the commutativity on the left triangle. Hitting the leftward triangle with F now shows that $\widetilde{F}(gf) = \widetilde{F}g \circ \widetilde{F}f$, as required.

Lastly, we should check that $\widetilde{F}$ preserves colimits and is a left adjoint to the functor G. In light of Proposition A.31, it is enough to show that $\widetilde{F}$ is a left adjoint to G. Quickly, let's check that G is a functor: for each $d \in \mathcal{D}$, we see that $Gd = \text{よ}(d)\circ F$ is in fact a presheaf, so G is well-defined on objects. On morphisms, we see that a morphism $f\colon d \to d'$ will induce a map $\text{よ}(f)\colon \text{よ}(d) \to \text{よ}(d')$ and therefore a map $\text{よ}(f)\colon Gd \to Gd'$ by composing with F. Because よ is functorial, it follows that G is functorial.

It remains to check that $\widetilde{F}$ is a left adjoint to G. We will do this check directly. For this, we use Lemma A.29 to see that

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(A, G-) &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}\left(\operatorname*{colim}_{(c,\eta)\in\mathcal{I}_A} \text{よ}(c), G-\right) \\
&= \lim_{(c,\eta)\in\mathcal{I}_A} \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}\left(\text{よ}(c), G-\right) \\
&= \lim_{(c,\eta)\in\mathcal{I}_A} G-(c) \\
&= \lim_{(c,\eta)\in\mathcal{I}_A} \mathrm{Mor}_{\mathcal{D}}(Fc, -),
\end{aligned}$$

which unwinds back to $\mathrm{Mor}_{\mathcal{D}}(\widetilde{F}A, -)$ after plugging in the colimit definition of $\widetilde{F}$ and using Lemma A.28. To show that we have an adjunction, it remains to check that these natural isomorphisms are also natural in A. The first equality is natural in A as discussed previously. Lemma A.28 is natural in A by its definition as just composing morphisms on the right (note the canonical natural maps in A composes on the left, so these operations commute). The remaining equalities displayed are all for fixed limit, so they continue to be natural in A, and the unwinding back to $\mathrm{Mor}_{\mathcal{D}}(\widetilde{F}A)$ continues to be natural in A by the aforementioned naturality of Lemma A.28. ■

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List of Definitions