737: Weil II for Curves

Nir Elber

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Review of Étale Theory

1.1 January 23

This is a pretty small class, so it will be rather informal. This course is going to assume some basic étale theory, roughly speaking up to the construction of the derived functors and some of their fundamental properties. We will also freely black-box the difficult theorems of the theory, most notably the Grothendieck–Lefschetz trace formula.

In this course, we are interested in proving the Weil conjectures, but we will be modest and focus on curves. Historically, the proof of the Weil conjectures for curves is much older than Weil II, but part of our goal will be to introduce the important relative techniques. For example, there should be a notion of weights attached to sheaves on a variety, known already from Hodge theory. However, we will require a way to see this purely from algebraic geometry; in fact, one expects the notion of weight to be motivic.

1.1.1 The Zeta Function

Let's begin by setting some notation which will be in place for the entire course. We take k to be a finite field $\mathbb{F}_q$ of characteristic p, embedded in a fixed algebraic closure $\overline{k} = \overline{\mathbb{F}_p}$; we write $q = p^n$. For brevity, we may write $k_m = \mathbb{F}_{q^m}$ for each $m \geq 1$. Then we let X be a smooth, projective, geometrically connected variety over the field k; we set $d \coloneqq \dim X$.

Definition 1.1 (zeta function)**.** Let X be a variety over $\mathbb{F}_q$. Then we define the *zeta function* as the generating function

$$\zeta_X(T) \coloneqq \exp\left(\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| \frac{T^m}{m}\right).$$

In order to do algebraic geometry to $\zeta_X(T)$, we would like to have a different description for $X(\mathbb{F}_{q^m})$. For this, we need to discuss closed points.

Definition 1.2 (closed point)**.** Let X be a variety over k. Then a point $x \in X$ is *closed* if and only if $\dim \overline{\{x\}} = 0$. Its *degree* $\deg x$ is the degree $[k(x) : k]$, where $k(x)$ is the minimal field of definition.

We now see that

$$X(\mathbb{F}_{q^m}) = \operatorname{Mor}_{\mathbb{F}_q}\left(\operatorname{Spec} \mathbb{F}_{q^m}, X\right).$$

For example, we see that this consists of the collection of closed points $x \in X$ of degree dividing m, counted with a certain multiplicity.

Now, to read off fields of definition, we introduce some Frobenius morphisms.

Definition 1.3. Fix a scheme X over $k = \mathbb{F}_q$. Then there is a *Frobenius morphism* $\mathrm{Frob}_X \colon X \to X$ defined as being an identity on the underlying topological space and the q-power map on $\mathcal{O}_X$. We may write $\mathrm{Frob}_{X,q}$ for Frob_X if we want to remember the power. We may also extend scalars and write $\mathrm{Frob}_{X_{\overline{k}},q} = \mathrm{Frob}_{X,q} \times \mathrm{id}_{\overline{k}}$, which we note is a morphism of schemes over $\overline{k}$ by its construction.

Remark 1.4. Fix a morphism $f \colon X \to Y$ of schemes over $\mathbb{F}_q$. Then we see $\mathrm{Frob}_Y \circ f = f \circ \mathrm{Frob}_X$, which can be checked directly: both sides are f on the topological spaces, and both sides are the same on the level of sheaves.

Example 1.5. On $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, our Frobenius map may be defined as the k-algebra endomorphism of $k[x_1, \ldots, x_n]$ which sends $x_i \mapsto x_i^q$. Thus, on points, we see that $(p_1, \ldots, p_n) \in \mathbb{A}^n_k(\overline{k})$ has

$$F_{\mathbb{A}^n_k}(p_1, \ldots, p_n) = \left(\mathrm{Frob}_q^{-1} p_1, \ldots, \mathrm{Frob}_q^{-1} p_n\right).$$

Remark 1.6. We now see that we can think about $X(\mathbb{F}_q)$ as the subset of $X(\overline{k})$ fixed by F_{X,q^m}. Thus, we note that one can realize $X(k_m)$ as the set of closed points of the scheme $(\Gamma_{F_{X,q^m}} \cap \Delta)$, where $\Delta \colon X \times X \to X$ is the diagonal map.

Definition 1.7 (arithmetic Frobenius)**.** The *arithmetic Frobenius* Frob_k is the q-power automorphism of $\overline{k}$.

Definition 1.8 (geometric Frobenius)**.** Let X be a scheme over k. Then we define the *geometric Frobenius* of $X_{\overline{k}}$ as $F_X := \mathrm{id}_{X_{\overline{k}}} \times \mathrm{Frob}_{\overline{k}}^{-1}$. It fits in the following commutative diagram.

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r] \arrow[d] \arrow[dr, dashed, "F_X"] & X \arrow[dr, equal] & \\
\operatorname{Spec} \overline{k} \arrow[dr] & X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} \overline{k} \arrow[r, "\mathrm{Frob}_{\overline{k}}^{-1}"'] & \operatorname{Spec} k
\end{tikzcd}$$

Definition 1.9 (absolute Frobenius)**.** Let X be a scheme over k. One can check that F_X commutes with $\mathrm{Frob}_{X,q}$. We then define the *absolute Frobenius* as the composite $F_X \circ \mathrm{Frob}_{X_{\overline{k}},q}$.

Remark 1.10. It turns out that the absolute Frobenius is the identity on the level of étale cohomology.

We now return to our zeta function. To be able to undo the exponential, we note

$$\log\left(\frac{1}{1-T^d}\right) = \sum_{m\geq 1} d \cdot \frac{T^{nd}}{dn}.$$

Thus,

$$\sum_{m\geq 1} |X(\mathbb{F}_{q^m})| \frac{T^m}{m} = \sum_{\text{closed } x\in X} \log\left(\frac{1}{1-T^{\deg x}}\right),$$

so taking the exponential reveals

$$\zeta_X(T) = \prod_{\text{closed } x \in X} \frac{1}{1 - T^{\deg x}},$$

and now this Euler product appears similar to the usual Euler products we expect.

1.2 January 28

Today we do something with cohomology.

1.2.1 The Rationality Conjecture

We would like to relate our zeta function to cohomology. It turns out that the key input is the following result.

Theorem 1.11 (Grothendieck–Lefschetz trace formula)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$. Then

$$X\left(\mathbb{F}_{q^m}\right) = \sum_{i \geq 0} (-1)^i \operatorname{tr}\left(\operatorname{Frob}^m_{X_{\overline{k}}}; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

Here, recall that

$$\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \varprojlim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}/\ell^\bullet\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Namely, this is our Weil cohomology (over the field $\mathbb{Q}_\ell$) produced by étale cohomology.

Remark 1.12. It is the goal of Weil II (and thus of the course) to be able to work more general local systems than the "constant" sheaf $\mathbb{Q}_\ell$.

To relate this to ζ_X, we recall the following result from linear algebra.

Lemma 1.13. Fix an endomorphism φ of a finite-dimensional vector space V (over a field K). Then we have an equality of power series

$$\exp\left(\sum_{m \geq 1} \operatorname{tr}\left(\varphi^m; V\right) \frac{T^m}{m}\right) = \det\left(1 - \varphi T; V\right)^{-1}.$$

Proof. It is enough to check the equality after base-changing to the algebraic closure, so we may assume that K is algebraically closed. Then we may give V a basis so that φ is upper-triangular.

Let $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of φ. Then we are tasked with showing

$$\exp\left(\sum_{m \geq 0} \sum_{i=1}^d \lambda_i^m \cdot \frac{T^m}{m}\right) \stackrel{?}{=} \prod_{i=1}^d \frac{1}{1 - \lambda_i T}.$$

Well, we may move the sum on the left-hand side outside so that we see we are interested in showing

$$\exp\left(\sum_{m \geq 1} \frac{(\lambda T)^m}{m}\right) = \frac{1}{1 - \lambda T}$$

for any eigenvalue λ of φ. The result now follows by considering the Taylor expansion $-\log(1-x) = \sum_{m \geq 1} x^m/m$. ■

Here is the punchline: we are able to prove the rationality conjecture.

Proposition 1.14 (Rationality)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Then there are polynomials $P_0, \ldots, P_{2d} \in \mathbb{Q}_\ell[T]$ such that

$$Z_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}.$$

Proof. By Theorem 1.11, we see that

$$Z_X(T) = \prod_{i=0}^{2d} \exp\left(\sum_{m\geq 1} \operatorname{tr}\left(\operatorname{Frob}_{X_{\overline{k}}}^m; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right) \frac{T^m}{m}\right)^{(-1)^i}.$$

We now define

$$P_i(T) := \det\left(1 - \operatorname{Frob}_{X_{\overline{k}}} T; \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right).$$

The result now follows from Lemma 1.13. ■

Remark 1.15. In fact, we see that $P_i(T)$ has degree $\dim \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$. This fact can be combined with the comparison theorem to Betti cohomology.

Remark 1.16. We thus see that $Z_X(T) \in \mathbb{Q}_\ell(T)$, so because we already know $Z_X(T) \in \mathbb{Q}[[T]]$, we see $Z_X(T) \in \mathbb{Q}(T)$.

Remark 1.17. It turns out that $P_i(T) \in \mathbb{Z}[T]$ and is independent of ℓ, but the proof above does not show this.

Example 1.18. At $i = 0$, we see that the Frobenius acts trivially on $\mathrm{H}^0_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$, so $P_0(T) = 1 - T$. Using Poincaré duality, we can similarly compute $P_{2d}(T) = 1 - q^d T$.

Remark 1.19. There is also a functional equation for $Z_X(T)$, which is purely formal from the above expression for Z_X when combined with Poincaré duality for étale cohomology.

1.2.2 The Riemann Hypothesis

This course will be interested in the following conjecture.

Conjecture 1.20 (Riemann hypothesis)**.** Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$ of dimension d. Fix an index $i \in \{0, \ldots, 2d\}$.

(a) The eigenvalues of $\operatorname{Frob}_{X_{\overline{k}}}$ on $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ are algebraic integers of magnitude $q^{i/2}$.

(b) The characteristic polynomial $P_i(T)$ of this Frobenius action is in $\mathbb{Z}[T]$ and is independent of ℓ.

Remark 1.21. Part (a) can be viewed as a Riemann hypothesis: substituting $T = q^{-s}$ into ζ_X, we see that we are requiring our zeroes (and poles) of $\zeta_X(q^{-s})$ to live on the vertical lines

$$\{s : \operatorname{Re} s = i/2\}$$

as i varies over $\{1, \ldots, 2\dim X\}$.

The condition in (a) is interesting enough to deserve a name.

Definition 1.22 (q-Weil)**.** An algebraic integer $\alpha \in \overline{\mathbb{Q}}$ is q-*Weil of weight* i if and only if $|\iota(\alpha)| = q^{i/2}$ for all embeddings $\iota \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Example 1.23. The number $\sqrt{2}$ is a 2-Weil number. The number $1+\sqrt{2}$ is not a q-Weil number for any q.

In general, we find that the eigenvalues of a Frobenius action on a local system will still be q-Weil numbers of prescribed weight.

To be precise, the goal of this course will be to prove the following generalization of the above Riemann hypothesis.

Theorem 1.24 (Deligne)**.** Let $f \colon X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X that is mixed of weights at most n. Then $R^i f_! \mathcal{F}$ is also mixed of weights at most $w+i$.

We will define the notion of weights shortly. The idea intuitively comes from Hodge theory: the cohomology groups on a complex Kähler manifold naturally have a weight filtration, which then lifts to sheaves by taking a suitable compactification and studying differential forms suitably. Weights in our context will come from reading off q-Weil numbers.

Remark 1.25. Issues with compactification explain why we are forced to merely deal with mixed weights instead of upgrading this result to one on pure weights. Already this can be seen in Hodge theory.

This course will not prove Theorem 1.24 in full. Instead, we will focus on the case where f has fibers of dimension 1; it turns out that the general case follows from this from some argument involving fibering by curves and using the Leray spectral sequence.

Corollary 1.26. Let X be a scheme of finite type over $\mathbb{F}_q$. Fix an index i and a locally constant constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X.

(a) If $\mathcal{F}$ is mixed of weights at most n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at most $n+i$.

(b) If $\mathcal{F}$ is mixed of weights at least n, then $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is mixed of weights at least $n+i$.

(c) Assume that X is smooth and that $\mathcal{F}$ is pure of weight n. Then the image of the canonical map $\mathrm{H}^i_{c,\text{ét}}(X_{\overline{k}}, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

(d) Assume that X is smooth and proper and that $\mathcal{F}$ is pure of weight n. Then $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathcal{F})$ is pure of weight $n+i$.

Proof. Here, (a) is direct from Theorem 1.24. Then (b) will follow from (a) via Poincaré duality as soon as we know that the duality given by Poincaré duality inverts the weights. Now, (c) follows from combining (a) and (b), and (d) follows from (c). ■

Remark 1.27. One can then prove the result for sheaves over $\mathbb{Q}_\ell$ by base-changing up to the algebraic closure.

The moral of the story is that we are going to use weights to significant profit in this course. Next class we will define weights.

1.3 January 30

Today we review some étale cohomology.

1.3.1 The Étale Site

For completeness, here is the definition of étale.

Definition 1.28 (étale)**.** Fix a scheme morphism $\varphi\colon X \to Y$.

(a) φ is *locally of finite presentation* if and only if $\mathcal{O}_X$ is finitely presented as a $\varphi^{-1}\mathcal{O}_Y$-module (say, on Zariski open neighborhoods or on stalks).

(b) φ is *flat* if and only if the pushforward $\varphi_*\mathcal{O}_X$ is flat over $\mathcal{O}_Y$ (say, on Zariski open neighborhoods or on stalks).

(c) φ is *unramified* if and only if $\Omega_{X/Y} = 0$.

(d) φ is *étale* if and only if it is locally of finite presentation, flat, and unramified.

Example 1.29. One can check that open embeddings are étale.

Remark 1.30. Note that the unramified condition adds some separability, which is a rough explanation for where Galois representations enter the story.

And here is the relevant site.

Definition 1.31 (étale site)**.** Given a scheme S, the *étale site* Ét_S is the category of étale morphisms to S. This site comes with a notion of covering: a collection of morphisms $\{U_i \to U\}$ in Ét_S is a *covering* if and only if the whole covering is surjective on the underlying topological spaces.

Remark 1.32. Technically, we have defined the "small" étale topos.

An advantage of working with étale cohomology is that our points gain automorphism groups arising from Galois information.

Definition 1.33 (geometric point)**.** Fix a scheme S. A *geometric point* $\overline{x} \hookrightarrow S$ is a morphism of schemes from an algebraically closed field; abusing notation, we may write $\overline{x}$ as $\operatorname{Spec} K$ or as the morphism $\overline{x}\colon \operatorname{Spec} K \to S$.

Remark 1.34. We do not require that our geometric points have closed image in S.

Remark 1.35. Requiring that we have a morphism of schemes amounts to requiring that the algebraically closed field K contains the residue field of the image $x \in S$ of $\overline{x}$. In other words, the data of the morphism $\overline{x}\colon \operatorname{Spec} K \hookrightarrow S$ amounts to the choice of a point $x \in S$ and an embedding $\kappa(x) \hookrightarrow K$.

Definition 1.36 (étale neighborhood)**.** Fix a scheme S. Then an *étale neighborhood* $(U, \overline{u})$ of a geometric point $\overline{x} \hookrightarrow S$ is an étale morphism $\pi\colon U \to S$ equipped with a geometric point $\overline{u} \hookrightarrow U$ together with an embedding $\overline{x} \hookrightarrow U$ over S. A morphism of étale neighborhoods is a morphism of étale covers of S preserving the basepoint.

1.3.2 Sheaves on the Étale Site

With a site, one wants sheaves.

Definition 1.37 (sheaf)**.** Fix a small category $\mathcal{C}$ and a scheme S. An *étale presheaf* $\mathcal{F}$ of $\mathcal{C}$ on S is a contravariant functor $\text{Ét}_S^{\text{op}} \to \mathcal{C}$. An *étale sheaf* is a presheaf $\mathcal{F}$ such that any $U \in \text{Ét}_S$ equipped with a covering $\{U_i \to U\}$ makes $\mathcal{F}(U)$ equal the equalizer

$$\mathcal{F}(U) = \operatorname{eq}\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)\right).$$

Remark 1.38. As in the Zariski topology, there is a sheafification functor $(-)^{\text{sh}}\colon \mathrm{PSh}(S) \to \mathrm{Sh}(S)$ sending the category of presheaves to sheaves. It is a left adjoint to the forgetful functor. The construction of this functor is rather technical, so we will only mention the key property that the sheafification functor is an isomorphism on stalks. For example, one can use sheafification to show that the category $\mathrm{Sh}(S)$ is abelian.

With sheaves, one has stalks.

Definition 1.39. Fix a scheme S and a geometric point $\overline{x} \hookrightarrow S$. For a presheaf $\mathcal{F}$ on S, we define the *stalk* $\mathcal{F}_{\overline{x}}$ as

$$\mathcal{F}_{\overline{x}} \coloneqq \varinjlim_{(U,\overline{u})} \mathcal{F}(U),$$

where the direct limit is taken over étale neighborhoods of $\overline{x}$.

Let's give some examples.

Proposition 1.40. Fix a scheme S. For any étale scheme V over S, the presheaf $\underline{V}$ given by $\underline{V}(U) \coloneqq \operatorname{Hom}_S(U, V)$ is an étale sheaf.

Proof. This follows from some descent argument. ∎

Example 1.41. Using the identity map $S \to S$ reveals that $\mathcal{O}_S$ is an étale sheaf. The stalk is

$$\mathcal{O}_{S,\overline{x}} = \varinjlim_{(U,\overline{u}} \Gamma(U, \mathcal{O}_U).$$

It turns out that this is a strictly Henselian ring.

Example 1.42. Fix a positive integer $n \geq 1$. Define the S-scheme μ_n as $\operatorname{Spec} \mathbb{Z}[T]/\left(T^n - 1\right) \times_{\operatorname{Spec} \mathbb{Z}} S$. If the multiplication map $n\colon \mathcal{O}_S \to \mathcal{O}_S$ is an isomorphism, then μ_n is étale over S, so this is an étale sheaf.

Example 1.43. For a finite set Σ, we may define the S-scheme Σ given by $\Sigma \times S$ (namely, a disjoint union of Σ-many copies of S). This then produces an étale sheaf $\underline{\Sigma}$.

It is too hard to work with all sheaves. Roughly speaking, we will be interested in "local systems." Here is the version of this notion in algebraic geometry.

Definition 1.44 (locally constant, constructible)**.** Fix an étale sheaf $\mathcal{F}$ on a scheme S and valued in a category $\mathcal{C}$.

(a) $\mathcal{F}$ is *locally constant* if and only if there is a finite étale covering $\{U_i \to S\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to a constant sheaf (still valued in $\mathcal{C}$).

(b) $\mathcal{F}$ is *constructible* if and only if there is a finite stratification $\{S_i\}$ of S into locally closed subsets such that $\mathcal{F}|_{S_i}$ is a locally constant sheaf of finite type.

Remark 1.45. The notion of "finite type" changes depending on $\mathcal{C}$. For example, if $\mathcal{C}$ is the category of abelian groups, then one wants to consider finite abelian groups. If $\mathcal{C}$ is a category of vector spaces, then one wants to consider finite-dimensional vector spaces.

Example 1.46. The constant sheaf $\underline{A}$ of an abelian group A is a locally constant constructible sheaf.

Example 1.47. If S is a variety over $\mathbb{C}$, and $\pi\colon X \to S$ is an étale covering, then the pushforward $\pi_*\underline{\mathbb{Z}}$ is locally constant and constructible.

Example 1.48. If $\pi\colon X \to S$ is an étale covering of schemes, then the sheaves $R^i\pi_*\mathbb{Z}_\ell$ (suitably interpreted) is locally constant and constructible.

Remark 1.49. As in topology, it turns out that one can think about locally constant constructible étale sheaves are representations of a fundamental group $\pi_1^{\text{ét}}(S, \overline{x})$.

1.4 February 6

We didn't have class on Tuesday. We will continue doing a little review of étale cohomology. In particular, we will talk about the étale fundamental group.

1.4.1 Galois Theory of Schemes

We would like to bring the notion of the topological fundamental group to the étale fundamental group. Roughly speaking, the topological fundamental group can be realized as the group of automorphisms of a universal cover, and this universal cover can be seen as some limit of finite covers. In algebraic geometry, we do not have access to this limit of finite covers, but we do have access to finite covers: they are étale coverings.

Before saying anything of substance, we recall some properties of étale morphisms.

Lemma 1.50. Fix scheme morphisms $f\colon X \to Y$ and $g\colon Y \to X$.

(a) Composition: if f and g are étale, then $g \circ f$ is étale.

(b) Cancellation: if $g \circ f$ and g are étale, then f is étale.

Lemma 1.51. Étale morphisms are preserved by base-change. More precisely, fix a pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

of schemes. If f is étale, then so is f'.

Lemma 1.52. Fix a scheme morphism $f\colon X \to Y$. Then f is unramified if and only if the diagonal $\Delta_f\colon X \to X \times_Y X$ is an open embedding.

The following result should be understood as a version of "unique path-lifting."

Proposition 1.53 (Rigidity)**.** Fix finite étale schemes X and Y over S. If X is connected, and Y is separated, then any geometric point $\overline{x} \hookrightarrow X$ (where $\overline{x} = \operatorname{Spec} \overline{k}$) induces an injection

$$\operatorname{Mor}_S(X, Y) \to Y(\overline{k})$$

given by $f \mapsto f(\overline{x})$.

Sketch. Suppose we have two morphisms $f, g\colon X \to Y$ over S with $f(\overline{x}) = g(\overline{x})$. Because X and Y are both étale over S, we see that f and g are étale. Thus, f and g are local isomorphisms, so the connectivity of X tells us that they are determined by the image of $\overline{x}$. ■

Corollary 1.54. Suppose S is a connected scheme and that $X \to S$ is a finite étale morphism. Choose a geometric point $\overline{s} \hookrightarrow S$, and choose some $\overline{x} \hookrightarrow X$ in the fiber. Then the map

$$\operatorname{Aut}_S(X) \to X_{\overline{s}}$$

given by $f \mapsto f(\overline{x})$ is injective.

Proof. Set $X = Y$ in Proposition 1.53. ■

Remark 1.55. If X and S are points, then we are looking at some finite separable field extension L/K, and the corollary amounts to asserting that the following map is an injective: define

$$\operatorname{Aut}_K(L) \hookrightarrow \operatorname{Hom}_K(L, \overline{K})$$

by sending $\sigma\colon L \to L$ to the embedding $\iota \circ \sigma$, where $\iota\colon L \hookrightarrow \overline{K}$ is a chosen embedding. Note that L/K is a Galois extension if and only if this map is an isomorphism!

The above remark motivates the following definition.

Definition 1.56 (Galois)**.** Fix a connected scheme S. A finite étale morphism $f\colon X \to S$ is *Galois* if and only if $\# \operatorname{Aut}_S(X) = \# X_{\overline{s}}$ for some (or equivalently, any) geometric point $\overline{s} \hookrightarrow S$.

We are now motivated to build a "Galois theory" of schemes. Here are some statements, which we will not prove.

Proposition 1.57. Fix a connected scheme S, and choose a Galois covering $X \to S$.

(a) There is a map sending finite étale subcovers $Y \to S$ of $X \to S$ to the subgroup of $\operatorname{Aut}_Y(X) \subseteq \operatorname{Aut}_S(X)$.

(b) There is a map sending subgroups $H \subseteq \operatorname{Aut}_S(X)$ to finite étale subcovers $X^H \to S$ of $X \to Y$.

(c) The two maps of (a) and (b) are order-preserving mutually inverse isomorphisms.

Remark 1.58. There is some difficulty in constructing the quotient X^H. Roughly speaking, one can take fixed points (as expected) on the level of affine schemes, and then one uses the fact that our morphisms are finite and étale to show that this is good enough for the general case.

Proposition 1.59. Fix a connected scheme S. For any finite étale covering $X \to S$, there is a unique (up to isomorphism) connected finite étale covering $X' \to X$ satisfying the following conditions.

(i) The composite covering $X' \to S$ is Galois.

(ii) Any other Galois covering $X'' \to S$ which factors through $X \to S$ will also factor uniquely through $X'' \to X'$.

So we have a Galois correspondence and a construction of normal closures.

1.4.2 The Étale Fundamental Group

Let's now turn to the absolute Galois group, extending our discussion of Galois theory of schemes.

Definition 1.60 (étale fundamental group)**.** Fix a connected scheme S and a geometric point $\overline{s} \hookrightarrow S$. Then we define the *étale fundamental group* $\pi_1^{\text{ét}}(S, \overline{s})$ as the automorphism group of the fiber functor $\operatorname{Fib}_{\overline{s}}$ sending finite étale covers $X \to S$ to the set $\operatorname{Hom}_S(\overline{s}, X)$.

Remark 1.61. It is a theorem that Fib upgrades to an equivalence of categories between the category of finite étale covers $X \to S$ and the category of finite $\pi_1^{\text{ét}}(S, \overline{s})$-sets.

Example 1.62. Let k be an algebraically closed field, and take $S = \operatorname{Spec} k$ with geometric points $S = S$. Then finite étale covers of S all look like $S \sqcup \cdots \sqcup S$, which is a unique map down to S, so the fiber functor has no nontrivial automorphisms. We conclude that $\pi_1^{\text{ét}}(S, \overline{s}) = 1$.

Remark 1.63. Intuitively, the automorphism group of the fiber functor should grow in size if there are fewer finite étale covers.

One is even able to recover some kind of universal cover, but it is not exactly a scheme: it is a limit of schemes.

Proposition 1.64. Fix a connected scheme S and a geometric point $\overline{s} \hookrightarrow S$, and let $\{(X_\alpha, \overline{x}_\alpha)\}$ be the inverse system of all finite Galois covers of $(S, \overline{s})$ preserving the (geometric) basepoint.

(a) For any finite étale cover $Y \to S$, we have

$$\operatorname{Fib}_{\overline{s}}(Y) = \varinjlim \operatorname{Hom}_S(X_\bullet, Y).$$

(b) We have

$$\pi_1^{\text{ét}}(S, \overline{s}) = \varprojlim \operatorname{Aut}_S(X_\bullet)^{\text{op}}.$$

Example 1.65. Let k be any field, and take $S = \operatorname{Spec} k$ with geometric point given by a chosen embedding $k \hookrightarrow \overline{k}$. Then the proposition tells us

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, \operatorname{Spec} \overline{k}) = \varprojlim_{\text{Galois } L/k} \operatorname{Gal}(L/k) = \operatorname{Gal}(k^{\text{sep}}/k).$$

Remark 1.66. For two basepoints $\overline{s}_1, \overline{s}_2 \hookrightarrow S$, one can produce a natural isomorphism of the relevant fiber functors, so the étale fundamental groups become isomorphic by some isomorphism coming from conjugation.

Remark 1.67. Choose a connected scheme W of finite type over $\mathbb{Z}$, and choose a closed point $w \in W$, which comes from a morphism $\operatorname{Spec} k(w) \hookrightarrow W$. A choice of algebraic closure of $k(w)$ then gives a geometric point $\overline{w} \hookrightarrow W$. There is thus a map

$$\pi_1^{\text{ét}}(\operatorname{Spec} k(w), \overline{w}) \to \pi_1^{\text{ét}}(W, \overline{w})$$

by base-changing automorphisms of the fiber functor for W to merely the closed point w. Now, $k(w)$ should be a finite field, so the left-hand group is topologically generated by a Frobenius element; note that one typically takes the geometric Frobenius to topologically generate $\pi_1^{\text{ét}}(\operatorname{Spec} k(w), \overline{w})$.

Changing the embedding $k(w) \hookrightarrow \overline{k(w)}$ tells us that this Frobenius element in $\pi_1^{\text{ét}}(W, \overline{w})$ should really be only defined up to conjugacy. In this way, we produce a canonical conjugacy class in $\pi_1^{\text{ét}}(W, \overline{w})$ for each closed point in W.

As a last statement, we recall the homotopy exact sequence.

Theorem 1.68. Fix a geometrically connected and quasicompact scheme S over a field k, and choose a basepoint $\overline{x} \hookrightarrow S_{k^{\text{sep}}}$. Then there is a short exact sequence

$$1 \to \pi_1^{\text{ét}}(S_{k^{\text{sep}}}, \overline{s}) \to \pi_1^{\text{ét}}(S, \overline{s}) \to \pi_1^{\text{ét}}(\operatorname{Spec} k) \to 1.$$

Remark 1.69. We won't prove the theorem, but we will remark that the sequence of morphisms is induced by functoriality from the morphisms

$$\overline{s} \hookrightarrow S \to \operatorname{Spec} k.$$

1.5 February 11

Today we continue.

1.5.1 Torsion Sheaves

Recall that the affine scheme $\mu_{n,S} = \operatorname{Spec} \mathbb{Z}[T]/(T^n - 1) \times S$ is a group scheme over S. This allows to build an étale sheaf, sending finite étale covers $X \to S$ to the group

$$\underline{\mu_n}_S(X) = \operatorname{Hom}_S(\mu_{n,S}, X).$$

Now, we can use this sheaf to say something about the equivalence of categories of finite étale covers of S and finite sets with action by $\pi_1^{\text{ét}}(S, \overline{s})$ (for S connected).

Corollary 1.70. Fix a connected scheme S and geometric point $\overline{s} \in S$. Then there is an equivalence of categories between locally constant constructible sheaves on S with finite continuous $\pi_1^{\text{ét}}(S, \overline{s})$-sets.

Proof. This equivalence of categories takes such a sheaf $\mathcal{F}$ to the stalk $\mathcal{F}_{\overline{s}}$. The fact that étale covers are producing the required sheaves is formal. ∎

Remark 1.71. This equivalence is known as the finite monodromy correspondence.

Remark 1.72. If one works with the abelian category of locally constant constructible sheaves valued in an abelian category (frequently valued in finite abelian groups), then our stalks and $\pi_1^{\text{ét}}(S, \overline{s})$ inherit this extra structure.

Let's be explicit about the sheaves we are interested in.

Definition 1.73. An étale sheaf $\mathcal{F}$ of abelian groups on a scheme S is *torsion* if and only if either of the following conditions hold.

(i) All stalks of $\mathcal{F}$ are torsion abelian groups.

(ii) One has that $\mathcal{F}(U)$ is a torsion abelian group for all étale open subsets $U \to S$.

Remark 1.74. The embedding of $\mathcal{F}(U)$ into a stalk explains why (i) implies (ii). The fact that any element in a stalk comes from some étale open neighborhood explains why (i) implies (ii).

Let's make a few more remarks about these sheaves.

Lemma 1.75. Fix a scheme S.

(a) A torsion étale sheaf $\mathcal{F}$ on a scheme S is Noetherian if and only if it is locally constant constructible.

(b) Every torsion sheaf is the (filtered) direct limit of its constructible subsheaves.

Sketch. For (a), the idea is to consider descending chains of the required filtrations. For (b), the idea is to show that any element in any stalk of the torsion sheaf can be found in some constructible subsheaf. ∎

Remark 1.76. It follows that locally constant constructible sheaves form an abelian category!

Eventually, we will have control over torsion sheaves. However, one would still like coefficients in characteristic 0, which we do by taking a limit.

Definition 1.77. Fix a scheme S and a prime ℓ. An *ℓ-adic sheaf* $\mathcal{F}$ on S is a projective system $\{\mathcal{F}_n\}_{n\geq 0}$ of constructible sheaves of abelian groups such that $\mathcal{F}_0 = 0$ and $\ell^1\mathcal{F}_n = 0$ for $n \geq 0$, and there are isomorphisms

$$\mathcal{F}_{n+1}/\ell^{n+1}\mathcal{F}_{n+1} \to \mathcal{F}_n$$

for each $n \geq 0$. We say that $\mathcal{F}$ is *smooth* if and only if each $\mathcal{F}_n$ is locally constant constructible. There is a similar definition to have coefficients in a finite extension E of $\mathbb{Q}_\ell$; then an étale $\overline{\mathbb{Q}}_\ell$-sheaf is one with coefficients in some finite extension E of $\mathbb{Q}_\ell$.

Remark 1.78. As before, there is an equivalence of categories (given by taking stalks) between étale $\overline{\mathbb{Q}}_\ell$-sheaves and continuous representations of $\pi_1(S, \overline{s})$ on $\overline{\mathbb{Q}}_\ell$-vector spaces. The sheaf is smooth if and only if the representation is finite-dimensional.

1.5.2 Back to Frobenius Morphisms

Let's begin to do something with these sheaves. As usual, we take $k = \mathbb{F}_q$. Here are some Frobenius elements.

- There is an arithmetic Frobenius automorphism $\sigma \in \operatorname{Gal}(\overline{k}/k)$ given by $\sigma\colon x \mapsto x^q$, and we recall that it is a topological generator of $\operatorname{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}}$. It generates the cyclic "Weil" group $W_{\overline{k}/k} \cong \mathbb{Z}$.
- The inverse $F := \sigma^{-1}$ is the geometric Frobenius element.
- Then for any $\mathbb{F}_q$-scheme Y, there is an absolute Frobenius $\sigma_{Y/k}\colon Y \to Y$. It is the identity on the topological sheaves, and it acts by qth powers on the level of structure sheaves. Technically, this is not a morphism of schemes over $\overline{k}$, so we may have occasion to write the target as $Y^{(q)}$.

Now, let X be a scheme over k of finite type, and let $\mathcal{G}_0$ be an étale sheaf on X; then we let $\mathcal{G}$ denote the pullback sheaf on $X_{\overline{k}}$. By functoriality of the absolute Frobenius, one gets a diagram as follows.

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r, dashed] \arrow[dr, "\sigma_{X,\overline{k}}"] \arrow[dd] & X_{\overline{k}} \arrow[r] \arrow[d, "\mathrm{id}_{X_{\overline{k}}}\times\sigma"] & X \arrow[d, equal] \\
& X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}\overline{k} \arrow[r, "\sigma"] & \operatorname{Spec}\overline{k} \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

We claim that the dashed arrow exists and is unique. Indeed, it is enough to note that the top-right square is Cartesian and then checking some commutativity using the relevant functoriality.

We are now able to call the dashed map $\operatorname{Fr}_{X_{\overline{k}}}$, which we note satisfies

$$(\mathrm{id}_{X_{\overline{k}}} \times \sigma) \circ \operatorname{Fr}_{X_{\overline{k}}} = \sigma_{X_{\overline{k}}}$$

by construction. Let's see an example.

Example 1.79. Take $X = \mathbb{A}^1$. Then one has the following diagram.

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[r, dashed, "\mathrm{Fr}"] \arrow[dr, "\sigma_{\mathbb{A}^1}"'] & \mathbb{A}^1 \arrow[d, "\mathrm{id}\times\sigma"] \\
& \mathbb{A}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{k}[t] \arrow[r] \arrow[dr] & \overline{k}[t] \arrow[d] \\
& \overline{k}[t]
\end{tikzcd}
\qquad
\begin{tikzcd}
at \arrow[r, mapsto, dashed] \arrow[dr, mapsto] & at^q \arrow[d, mapsto] \\
& a^q t^q
\end{tikzcd}$$

Thus, we see on the structure sheaf that we have managed to construct Fr as a morphism of schemes over $\overline{k}$, but it is still more or less applying a q-power to points.

Now, let $\overline{x} \in X$ be a geometric point, where its image $x \in X$ is closed. Recall that sheaves on x are essentially abelian groups, which one can see by taking the stalk at $\overline{x}$; of course, we do see that these abelian groups must come with an action by $\pi_1(x, \overline{x}) = \operatorname{Gal}(\overline{k}/\kappa(x))$, which we will call G, so taking this stalk produces a discrete abelian G-module.

Let $\mathcal{G}$ be an étale $\overline{\mathbb{Q}}_\ell$-sheaf on X, and we see that the stalk $\mathcal{G}_{\overline{x}}$ will admit an action by the geometric Frobenius element F, which we will call $F_x \colon \mathcal{G}_{\overline{x}} \to \mathcal{G}_{F\overline{x}}$. (This is some formality of scheme morphisms, and we are implicitly using that F is an isomorphism of the topological spaces.) Now, $\mathcal{G}$ is really some étale E-sheaf for an extension E of $\mathbb{Q}_\ell$. Further, $\mathcal{G}$ is really some inverse system $\{\mathcal{G}_i\}_{i \geq 1}$ of finite étale E-sheaves. Now, each $\mathcal{G}_i$ comes from a bona fide étale scheme G_i over $X_{\overline{k}}$. Then the diagram

$$\begin{tikzcd}
G_i \arrow[dr, dashed] \arrow[drr, bend left, "F_{G_i}"] \arrow[ddr, bend right] & & \\
& F^*_{X_{\overline{k}}} G_i \arrow[r] \arrow[d] & G_i \arrow[d] \\
& X_{\overline{k}} \arrow[r, "F_{X_{\overline{k}}}"'] & X_{\overline{k}}
\end{tikzcd}$$

is able to produce a relative Frobenius morphism $G_i \to F_{X_{\overline{k}}} \to G_i$, so we produce a genuine map of sheaves

$$\mathcal{G} \to F^*_{X_{\overline{k}}} \mathcal{G}$$

upon taking the limit (in the category of étale sheaves) again.

We claim there is a canonical isomorphism $F^*_{\mathcal{G}} \colon F^*_{X_{\overline{k}}} \mathcal{G} \to \mathcal{G}$, which describes the étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ as being $\operatorname{Gal}(\overline{k}/k)$-equivariant sheaf. (Intuitively, one looks at the top horizontal map in the last diagram and takes a limit.) This claim is rather intricate, so we won't prove it: it comes down to explicating what the Galois action should be.

But now the consequence is that we receive a composite

$$\mathcal{G} \to F^*_{X_{\overline{k}}} \mathcal{G} \to \mathcal{G}.$$

By construction of the last isomorphism, one finds that this is the Frobenius morphism $\mathcal{G}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ at a geometric point $\overline{x} \hookrightarrow X$. This allows to define the L-function for $\mathcal{G}$.

Definition 1.80. Fix an étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$, as above. Then we define

$$L(X, \mathcal{G}, T) := \prod_{\text{closed } x \in X} \det\left(1 - T^{\deg x} F_x ; \mathcal{G}_{\overline{x}}\right)^{-1}.$$

By using the trace formula, one can prove that this L-series has a cohomological interpretation.

Theorem 1.81. Fix an étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on a scheme X over $\mathbb{F}_q$. Then

$$L(X, \mathcal{G}; T) = \prod_{i=0}^{2 \dim X} \det\left(1 - tF^* ; \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G})\right)^{(-1)^{i+1}}.$$

Now, as before, we note that one can construct a map $\mathcal{G} \to \mathrm{Fr}^*_{X_{\overline{k}}} \mathcal{G}$, and one can show that there is a canonical isomorphism $\mathrm{Fr}_{\mathcal{G}} \colon \mathrm{Fr}^*_{X_{\overline{k}}} \mathcal{G} \to \mathcal{G}$. Thus, we gain a diagram

$$\begin{tikzcd}
\mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G}) \arrow[r, "\mathrm{Fr}^*_{X_{\overline{k}}}"] \arrow[dr, "F^*"'] & \mathrm{H}^i_c(X_{\overline{k}}, \mathrm{Fr}^*_{X_{\overline{k}}} \mathcal{G}) \arrow[d, "\mathrm{Fr}^*_{\mathcal{G}}"] \\
& \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G})
\end{tikzcd}$$

for which we can use to prove the above theorem.

THEME 2

THE FORMALISM OF WEIL SHEAVES

2.1 February 13

Here we go.

2.1.1 Weil Sheaves

We would like to generalize our sheaves somewhat. Note that we had an action of the geometric Frobenius element on $X_{\overline{k}}$ from the start. In particular, our étale sheaves have a full action by $\pi_1(X,\overline{x})$, but we really only need to know about Frobenius action.

Definition 2.1 (Weil group)**.** Fix a finite field $k := \mathbb{F}_q$. The *Weil group* $W(\overline{k}/k)$ is the cyclic group generated by the Frobenius.

Now, note $W(\overline{k}/k)$ acts on $X_{\overline{k}} = X \times_k \overline{k}$, so any étale sheaf $\mathcal{G}$ on X gains an action of $W(\overline{k}/k)$ after base-changing to $X_{\overline{k}}$. Because this sheaf comes from X, it should be $W(\overline{k}/k)$-equivariant, which amounts to providing the data of an isomorphism $F^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}$. We are now ready to make the following definition, which generalizes our étale sheaves.

Definition 2.2 (Weil sheaf)**.** A *Weil sheaf* $\mathcal{G}_0$ on X consists is a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on $X_{\overline{k}}$ together with an isomorphism $F^*\colon F^*_{X_{\overline{k}}}\mathcal{G} \to \mathcal{G}$. This Weil sheaf $\mathcal{G}_0$ is *smooth of rank* r if and only if the same is true of $\mathcal{G}$ on $X_{\overline{k}}$.

Notably, this Weil sheaf is not required to actually come from a sheaf on X! (This is the whole point!) This extra flexibility of not having to actually come from a scheme over $\mathbb{F}_q$ will be helpful for us later.

Let's give a few remarks about the category of Weil sheaves.

Remark 2.3. The category of Weil sheaves on X form an abelian category. This category contains the category of étale sheaves on X, embedded as described above.

Remark 2.4. If $k' \subseteq \overline{k}$ is a finite extension of k, then the category of Weil sheaves on $X_{k'}$ is the same as the category of Weil sheaves on X. Namely, restriction of scalars from $X_{k'}$ to X defines the required equivalence.

Remark 2.5. The category of Weil sheaves has the six operations (e.g., pullbacks, derived direct images, and direct image with compact support). The point is that one can do these operations on $X_{\overline{k}}$, and then one just needs to carry around the extra data of the isomorphism F^*.

We now remark that a Weil sheaf $\mathcal{G}_0$ on X is still going to come with an isomorphism $F\colon \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G}) \to \mathrm{H}^i_c(X_{\overline{k}}, \mathcal{G})$, so one can define $L(X, \mathcal{G}; T)$ as before. There is still a cohomological interpretation, but we will omit its proof.

These Weil sheaves appear to form a Tannakian category, so we are allowed to ask for the corresponding reductive group. To see this, we recall that the short exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

The inverse image of $W(\overline{k}/k) \subseteq \mathrm{Gal}(\overline{k}/k)$ is some Weil group $W(X, \overline{x}) \subseteq \pi_1^{\text{ét}}(X, \overline{x})$. It turns out that the induced quotient $W(X, \overline{x})/\pi_1(X_{\overline{k}}, \overline{x})$ is $W(\overline{k}/k) \cong \mathbb{Z}$. We are now ready to compare these Tannakian categories.

- Recall that the category of étale sheaves is equivalent to the category of continuous representations of $\pi_1^{\text{ét}}(X, \overline{x})$ on vector spaces over $\overline{\mathbb{Q}}_\ell$.
- This then restricts to an equivalence between the category of smooth étale sheaves and the category of finite-dimensional continuous representations of $\pi_1^{\text{ét}}(X, \overline{x})$.
- However, the category of Weil sheaves can be seen as the category of continuous representations of $W(X, \overline{x})$ of $\overline{\mathbb{Q}}_\ell$-vector spaces. To see this, we note that we are essentially removing some data from being a sheaf over $\pi_1^{\text{ét}}(X, \overline{x})$.

  The adjective of smoothness adds a finite-dimensional requirement.

Remark 2.6. This Tannakian point of view explains one point of working with Weil sheaves instead of étale sheaves. Étale sheaves, viewed as representations of the (compact!) profinite fundamental group, are forced to have bounded eigenvalues when Frobenius acts continuously on a vector space. Weil sheaves do not have this requirement, so we are granted more flexibility.

2.2 February 18

Today we continue discussing Weil sheaves.

2.2.1 Weil Sheaves from Étale Sheaves

Fix a scheme X over a finite field $\mathbb{F}_q$, and choose a geometric point $\overline{x} \hookrightarrow X$. Last time, we explained that Weil sheaves on X correspond to continuous representations of the Weil group $W(X, \overline{x})$. Thus, we are interested in the representation theory of $W(X, \overline{x})$. The easiest representations (and indeed, the easiest sheaves) have rank 1, so let's write these down.

Proposition 2.7. Weil sheaves on $\operatorname{Spec} \mathbb{F}_q$ of rank 1 are in bijection with $\overline{\mathbb{Q}}_\ell^\times$.

Proof. A Weil sheaf of rank 1 is known to correspond to a representation

$$W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell).$$

The left-hand side is $\mathbb{Z}$, so the result follows. ■

Notation 2.8. For $b \in \overline{\mathbb{Q}}_\ell^\times$, we let $\mathcal{L}_b$ denote the corresponding Weil sheaf on $\operatorname{Spec} \mathbb{F}_q$.

These sheaves help explain when Weil sheaves come from étale sheaves.

Theorem 2.9. Let X be a scheme over $\mathbb{F}_q$, and let $\mathcal{G}_0$ be a Weil sheaf on X. Suppose further that X is normal and geometrically connected and that $\mathcal{G}_0$ is irreducible (as a representation) and smooth of rank r. Then $\mathcal{G}_0$ is an étale $\overline{\mathbb{Q}}_\ell$-sheaf if and only if $\wedge^r \mathcal{G}_0$ is an étale sheaf on X.

Here are some applications of Theorem 2.9.

Corollary 2.10. Fix a smooth irreducible Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected scheme X over $\mathbb{F}_q$. Then there is $b \in \overline{\mathbb{Q}}_\ell^\times$ and an étale sheaf $\mathcal{F}_0$ on X such that

$$\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b.$$

Proof. We will prove this later, along with the theorem. ■

Corollary 2.11. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected scheme X over $\mathbb{F}_q$. Then there is a filtration of subsheaves

$$0 = \mathcal{G}_0^{(0)} \subseteq \mathcal{G}_0^{(1)} \subseteq \cdots \subseteq \mathcal{G}_0^{(r)} = \mathcal{G}_0$$

such that $\mathcal{G}_0^{(j)}/\mathcal{G}_0^{(j-1)}$ is isomorphic to some tensor $\mathcal{F}_0^{(j)} \otimes \mathcal{L}_{b_j}$ with $\mathcal{F}_0^{(j)}$ a smooth étale $\overline{\mathbb{Q}}_\ell$-sheaf and $b_j \in \overline{\mathbb{Q}}_\ell^\times$.

Proof. This follows by using filtrations of sheaves of finite rank and applying the example. ■

Remark 2.12. One can think of the twisting $- \otimes \mathcal{L}_b$ as some kind of ampleness condition.

As another application, we note that this machinery lets us write down an L-series for Weil sheaves. The idea is that filtrations decompose a Weil sheaf $\mathcal{G}_0$ into some extensions of irreducible sheaves by irreducible sheaves, allowing us to reduce to irreducible Weil sheaves. Then irreducible Weil sheaves can be controlled because there are merely étale sheaves twisted by an explicit character. Let's write this out.

Notation 2.13. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a scheme X over $\mathbb{F}_q$. Then we define its L-function by

$$L(X, \mathcal{G}, t) := \prod_{\text{closed } x \in X} \det\left(1 - t^{\deg x} F_x^*; \mathcal{G}_{\overline{x}}\right)^{-1}.$$

Corollary 2.14. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a normal and geometrically connected scheme X over $\mathbb{F}_q$. Then

$$L(X, \mathcal{G}, t) = \prod_{i=0}^{2\dim X} \det\left(1 - tF^*; \mathrm{H}_c^i(X_{\overline{k}}, \mathcal{G})\right)^{(-1)^{i+1}}.$$

Again, we will write this out in more detail next week.

2.2.2 Weights

Let's say something about weights. The motivation is that one can control the location of poles and zeroes of the L-function. Throughout, we will fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$.

Definition 2.15 (pure)**.** Fix a scheme X over $\mathbb{F}_q$, and choose an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. We say that a smooth Weil sheaf $\mathcal{G}_0$ is *ι-pure of weight* β for some $\beta > 0$ if and only if all eigenvalues of $\alpha \in \overline{\mathbb{Q}}_\ell$ of $F_x\colon \mathcal{G}_{0,\overline{x}} \to \mathcal{G}_{0,\overline{x}}$ (for any geometric point $\overline{x} \hookrightarrow X$) has

$$|\iota(\alpha)|^2 = \#\kappa(x)^\beta.$$

We say that $\mathcal{G}_0$ is *pure of weight* β if and only if it is ι-pure of weight β for all chosen ι.

The sheaves one comes across in practice may not be pure on the nose but instead have some pure part that we can remove and then handle via some induction.

Definition 2.16 (mixed)**.** Fix a scheme X over $\mathbb{F}_q$, and choose an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. We say that a smooth Weil sheaf $\mathcal{G}_0$ is *ι-mixed* if and only if there is a finite filtration

$$0 = \mathcal{G}_0^{(0)} \subseteq \mathcal{G}_0^{(1)} \subseteq \cdots \subseteq \mathcal{G}_0^{(r)} = \mathcal{G}_0$$

such that the quotients $\mathcal{G}_0^{(i)}/\mathcal{G}_0^{(i-1)}$ are ι-pure of some weight. We say that $\mathcal{G}_0$ is *mixed* if and only if it is ι-mixed for all chosen ι.

A motivation of Weil II is that we would like these weights to be found as some geometric invariant. For example, we expect that the weights of sheaves to be preserved under some controlled morphisms.

Proposition 2.17. Suppose that $\pi\colon X \to Y$ is a morphism of schemes over $\mathbb{F}_q$. Given a Weil sheaf $\mathcal{G}_0$ on Y.

(a) If $\mathcal{G}_0$ is ι-pure of weight β, then $f^*\mathcal{G}_0$ is ι-pure of weight β.

(b) If f is surjective and $f^*\mathcal{G}_0$ is ι-pure of weight β, then $\mathcal{G}_0$ is ι-pure of weight β.

(c) If f is finite and $\mathcal{G}_0$ is ι-pure of weight β, then $f_*\mathcal{G}_0$ is ι-pure of weight β.

(d) If $\mathcal{G}_0$ is a Weil sheaf on X, then $\mathcal{G}_0$ on X is ι-pure of weight β if and only if $\mathcal{G}_0$ on $X_{\mathbb{F}_{q^r}}$ is ι-pure of weight β.

Sketch. Here, (a) and (b) are essentially formal because the fibers of $f^*\mathcal{G}_0$ are the same as the fibers of $\mathcal{G}_0$. For (d), this is again purely formal from computing how the Frobenius and the degree simultaneously adjust on extension of the base field. Alternatively, (d) can be derived from (a)–(c) because the canonical morphism $X_{\mathbb{F}_{q^r}} \to X_{\mathbb{F}_q}$ is surjective and finite. We won't say anything about (c), but it should also follow from some fiber-wise computations. ■

For inductive applications, it may be useful to have some largest weight.

Definition 2.18. Fix a scheme X over $\mathbb{F}_q$, and let $\mathcal{G}_0$ be a Weil sheaf on X, and choose some isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Then we define

$$w(\mathcal{G}_0) := \sup_{\text{closed } x \in X} \left(\sup_{\text{eigenvalue } \alpha} \frac{\log |\iota(\alpha)|^2}{\log \#\kappa(x)} \right)$$

if $\mathcal{G}_0$ is nontrivial, and $w(0) = -\infty$.

2.3 February 25

Today we will start moving towards a proof of Theorem 2.9, but this will require some preparation.

2.3.1 Finite Geometric Monodromy

Our first piece of preparation is the following result.

Theorem 2.19 (Grothendieck)**.** Fix a normal geometrically irreducible scheme X over $\mathbb{F}_q$, and choose a geometric point $\overline{x} \hookrightarrow X$. Further, fix a continuous character $\chi\colon W(X,\overline{x}) \to \overline{\mathbb{Q}}_\ell^\times$. Then $\chi\left(\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x})\right)$ is finite.

Roughly speaking, the above result will follow from geometric class field theory because we are looking at some degree-0 Picard group. We remark that this will enable us to prove Corollary 2.10 because one can use the character given above to define $\mathcal{F}_0$ and then twist by $\mathcal{L}_b$ in order to fix the action of the Frobenius (which is not seen in the geometric étale fundamental group).

Remark 2.20. Equivalently, one can view Theorem 2.19 as saying that χ^m is trivial on $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x})$ for sufficiently divisible m. As such, one may (without loss of generality) pass from X to some base-change and pass from χ to some finite power. Also, this point of view allows us to use Theorem 2.19 to factor χ into two characters $\chi_1\chi_2$ where χ_1 has finite order and χ_2 factors through the quotient $W(X,\overline{x}) \twoheadrightarrow W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Remark 2.21. Morally, the above remark is decomposing the character χ into the unramified piece χ_2 (vanishing on inertia) and the ramified piece χ_1, and we are being given control χ_1 by forcing it to be finite-order.

To visualize the quotient in the remark, we draw the morphism

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x}) \arrow[r] \arrow[d, equal] & \pi_1^{\text{ét}}(X,\overline{x}) \arrow[r] & \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x}) \arrow[r] & W(X,\overline{x}) \arrow[r] \arrow[u, phantom, "\cup\!\!\shortmid" description] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[u, phantom, "\cup\!\!\shortmid" description] & 1
\end{tikzcd}$$

of short exact sequences, which allows us to see what it means for the aforementioned second character χ_2 to factor through $W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \mathbb{Z}$.

Remark 2.22. Intuitively, the geometric étale fundamental group $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x})$ remembers topological information, notably forgetting the "arithmetic" information coming from having Frobenius elements in $\pi_1^{\text{ét}}(X,\overline{x})$.

We now proceed with the argument.

Proof of Theorem 2.19. We have the following steps.

1. Let's describe a few reductions, which we will use freely.
   - As discussed in Remark 2.20, we may base-change X, for example to ensure that it has a rational point. Additionally, we may pass from χ to a power.
   - Because χ is continuous, we know that it should factor through $E^\times \subseteq \overline{\mathbb{Q}}_\ell^\times$ for some finite extension E of $\mathbb{Q}_\ell$. By factoring $E^\times$ as a group, we find that by (passing to a power of χ), we may automatically assume that the image $\chi\left(\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x})\right)$ is a pro-ℓ group.

- Because X is already normal, any open nonempty subscheme $U \subseteq X$ containing $\overline{x}$ yields a surjection $\pi_1^{\text{ét}}(U,\overline{x}) \twoheadrightarrow \pi_1^{\text{ét}}(X,\overline{x})$; this allows us to prove the theorem by passing to open subsets of X.

  For example, by passing from X to the smooth locus (which is nonempty because X is already normal), we may assume that X is smooth. Certainly we may assume that X is quasiprojective by passing to small affine open subsets.

2. To get our engines running, we treat the case of a curve. We use the theory of valuations. Because X is normal and geometrically irreducible, one gets a surjection

$$\operatorname{Gal}(K^{\text{sep}}/K) \twoheadrightarrow \pi_1^{\text{ét}}(X,\overline{x}),$$

where K is the function field of X; intuitively, the idea is that one can take function fields of any étale covers of X to produce separable extensions of K.

Now that we are in the curve case, we would like to do something with (geometric) class field theory. Define I_K to fit into the short exact sequence

$$0 \to I_K \to W(X,\overline{x})^{\text{ab}} \overset{\deg}{\to} W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 0.$$

In particular, I_K is some kind of inertia subgroup.

We now have a large diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic}(X)^\circ(\mathbb{F}_q) \arrow[r] \arrow[d, equal] & \operatorname{Pic}(X)(\mathbb{F}_q) \arrow[r, "\deg"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & K^\times \backslash \mathbb{A}_K^{\times,1}/\widehat{\mathcal{O}}_K^\times \arrow[r] \arrow[d, "\sim"] & K^\times \backslash \mathbb{A}_K^{\times}/\widehat{\mathcal{O}}_K^\times \arrow[r, "\mathrm{N}"] \arrow[d, "\sim"] & q^{\mathbb{Z}} \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & I_K \arrow[r] & W(X,\overline{x})^{\text{ab}} \arrow[r, "\deg"] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & 0 \\
 & \pi_1^{\text{ét}}(\overline{X},\overline{x})^{\text{ab}} \arrow[r] \arrow[u, two heads] & \pi_1^{\text{ét}}(X,\overline{x})^{\text{ab}} \arrow[r] & \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 0
\end{tikzcd}$$

The isomorphism of the top two rows is formal algebraic geometry. The isomorphism of the second and third row is essentially geometric class field theory. The maps between the third and fourth rows are more or less formal coming from the various constructions of the various Weil groups.

Now, after extending X so that it admits a rational point, we see that $\operatorname{Pic}(X)^\circ$ is the Jacobian of X and hence projective (it's an abelian variety), so it will only have finitely many $\mathbb{F}_q$-rational points. We now note that χ factors through $W(X,\overline{x})^{\text{ab}}$ (because it's a character), so the commutative diagram

$$\begin{tikzcd}
\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x}) \arrow[r] \arrow[d, two heads] & W(X,\overline{x}) \arrow[r, "\chi"] \arrow[d] & \overline{\mathbb{Q}}_\ell^\times \\
\operatorname{Pic}(X)^\circ(\mathbb{F}_q) \arrow[r] & W(X,\overline{x})^{\text{ab}} \arrow[ur, dashed] &
\end{tikzcd}$$

allows us to use the finiteness of $\operatorname{Pic}(X)^\circ$ to conclude the proof.

3. We now complete the proof, assuming that X is smooth and quasiprojective. This requires a compactification. It is possible to produce a compactification X' of X which is normal and projective; say $X' \subseteq \mathbb{P}^N$ for some $N \geq 0$. Taking the regular locus of X' grants an open smooth subscheme $Y \subseteq X'$ containing X; say $Z \coloneqq X' \setminus Y$, which has codimension at least 2 by normality of X'.

We quickly handle the case where $X = Y$. If $X = Y$ so that $D = \varnothing$, then we choose a generic linear subspace $L \subseteq \mathbb{P}^N_{\overline{\mathbb{F}}_q}$ (via Bertini's theorem) of codimension $\dim X_{\overline{\mathbb{F}}_q} - 1$ such that $Z \cap L = \varnothing$ and $C \coloneqq$

$X_{\overline{\mathbb{F}}_q} \cap L$ is a smooth irreducible curve. By moving the geometric point $\overline{x}$ to C, it turns out that we admit a surjection $\pi_1(C, \overline{x}) \twoheadrightarrow \pi_1(X_{\overline{\mathbb{F}}_q}, \overline{x})$. So the result follows from the case of curves.

It remains to handle the case where $X \subsetneq Y$. Now, Chow's lemma tells us that $D := Y \setminus X$ is a locally closed subscheme which is pure of codimension 1 with smooth components; say $D = D_1 \cup \cdots \cup D_r$ is a decomposition into connected components, and extending the base field actually allows us to assume that these components are geometrically irreducible and admitting a rational point. (Namely, such connected components are irreducible already by smoothness.)

For each D_i, let $x_i \in D_i$ be an $\mathbb{F}_q$-rational point. Further, let $Y_i := \operatorname{Spec} \mathcal{O}_{Y,x_i}^{\mathrm{h}}$, where $(\cdot)^{\mathrm{h}}$ denotes Henselization; and let $X_{i,\overline{\mathbb{F}}_q}$ be the inverse image of $X_{\overline{\mathbb{F}}_q}$ in $Y_{i,\overline{\mathbb{F}}_q}$, where $Y_{i,\overline{\mathbb{F}}_q}$ should technically denote a strict Henselization. With this notation, each i produces a morphism of short exact sequences as follows.

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{\text{ét}}(X_{i,\overline{\mathbb{F}}_q}, \overline{x}) \arrow[r] \arrow[d] & W(X_i, \overline{x}) \arrow[r] \arrow[d] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \overline{x}) \arrow[r] & W(X, \overline{x}) \arrow[r] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1 \end{tikzcd}$$

(We are ignoring complications which come from basepoints.) Now, $\chi\left(\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \overline{x})\right)$ is already understood to be a pro-ℓ group, which we achieved because the image lives in $\overline{\mathbb{Q}}_\ell^\times$, so any subgroup can be made into a pro-ℓ group by passing to some power. Thus, the character χ is merely tamely ramified (note $\ell \neq p$!) along $D_{i,\overline{\mathbb{F}}_q}$ when restricted to $W(X_{i,\overline{\mathbb{F}}_q}, \overline{x})$. In fact, the same reasoning explains that χ actually factors through the abelianization of the ℓ-part of the tame ramification group, which turns out to be isomorphic to $\mathbb{Z}_\ell$. We denote this last group by J.

Now, let $\sigma \in W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the Frobenius, and we choose an inverse image $\widetilde{\sigma}$ in $W(X_i, \overline{x})$. Then the action of $W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $W(X, \overline{x})$ is given by $\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1} = \gamma^q$ for $\gamma \in J$. However, χ (still restricted to $W(X_{i,\overline{\mathbb{F}}_q}, \overline{x})$) needs to behave the same on γ and $\widetilde{\sigma}\gamma\widetilde{\sigma}^{-1}$, so we are forced to have χ factor through J/J^q, which is a finite group!

Looping over all i, we know that a power χ^m of χ will be trivial on each $W(X_{i,\overline{\mathbb{F}}_q}, \overline{x})$. We conclude that χ^m factors through $W(Y, \overline{x})$, and we are allowed to reduce to the case where $Y = X$. ■

We now achieve the following corollary.

Corollary 2.23. Fix a smooth Weil sheaf $\mathcal{G}_0$ of rank 1 on a normal geometrically irreducible scheme X over $\mathbb{F}_q$.

(a) There is an étale sheaf $\mathcal{F}_0$ (attached to a finite character) and $b \in \overline{\mathbb{Q}}_\ell^\times$ such that $\mathcal{G}_0 = \mathcal{F}_0 \otimes \mathcal{L}_b$.

(b) The sheaf $\mathcal{G}_0$ is ι-pure for $\iota = \log|\iota(b)|^2 / \log q$.

Proof. The sheaf $\mathcal{G}_0$ arises from a representation $\chi\colon W(X, \overline{x}) \to \overline{\mathbb{Q}}_\ell^\times$ for some χ. Then one factors χ as $\chi_1\chi_2$, where χ_1 has finite order and χ_2 factors through $W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \mathbb{Z}$. Then χ_1 produces $\mathcal{F}_0$, and χ_2 produces $\mathcal{L}_b$, which proves (a). To prove (a), we note that $\mathcal{F}_0$ has weight 0 (because its eigenvalues of Frobenius are roots of unity) and $\mathcal{L}_b$ has the described weight. ■

2.4 February 27

We spent class completing the proof of Theorem 2.19. I have edited there for continuity reasons.

2.5 March 4

As an aside, let's examine how the fundamental group changes when one passes to open subsets. Let E be an elliptic curve over $\mathbb{F}_q$. Any étale cover of E continues to be an elliptic curve, which allows one to compute the ℓ-primary component $\pi_1^{\text{ét}}(E_{\overline{\mathbb{F}}_q}, \overline{\eta})_\ell = T_\ell E$. Thus, we have a short exact sequence

$$1 \to \pi_1^{\text{ét}}(E_{\overline{\mathbb{F}}_q}, \overline{\eta}) \to \pi_1^{\text{ét}}(E, \overline{\eta}) \to \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1$$

whose ℓ-primary component reads as

$$0 \to \mathbb{Z}_\ell^2 \to \pi_1^{\text{ét}}(E, \overline{\eta}) \to \mathbb{Z}_\ell \to 0.$$

But now if we subtract out the point at infinity, we find

$$\pi_1^{\text{ét}}(E_{\overline{\mathbb{F}}_q} \setminus 0, \overline{\eta})$$

is basically $\widehat{\mathbb{Z}} \oplus \widehat{\mathbb{Z}} \rtimes \widehat{\mathbb{Z}}$ (again, with maybe some problems at p). The moral of the story is that removing a point allows some ramification at the point we removed.

Example 2.24. Philosophically, one can tell the same story for $\operatorname{Spec} \mathbb{Z}$, which we view as a quasiprojective subvariety of some Arakelov projectification. For example, if we remove a "second" point to work with $\operatorname{Spec} \mathbb{Z} \setminus \{p\}$ as étale fundamental group given by

$$\operatorname{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times.$$

2.5.1 Semisimple Monodromy

The previous theorem is all about image of our monodromy, so we are motivated to make the following definitions.

Definition 2.25 (monodromy group)**.** Fix a smooth Weil $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}_0$ on a normal geometrically irreducible scheme X over $\mathbb{F}_q$, and let $\rho\colon W(X, \overline{x}) \to \operatorname{GL}(V)$ be the corresponding representation where $V := \mathcal{G}_{0,\overline{x}}$.

- Then the *geometric monodromy group* is the image $\rho\left(\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \overline{x})\right)$, and the *arithmetic monodromy group* is $\rho\left(W(X, \overline{x})\right)$.
- We let G_{geom} be the Zariski closure of the geometric monodromy group in $\operatorname{GL}(V)_E$, where $E/\mathbb{Q}_\ell$ is the finite extension coming from $\mathcal{G}_0$, and we define

  $$G := G_{\text{geom}} \rtimes_\sigma W(\overline{\mathbb{F}}_q/\mathbb{F}_q),$$

  where σ is a chosen generator of the Weil group.

Remark 2.26. We note that any $g \in W(X, \overline{x})$ normalizes G_{geom} because $\rho(g)$ normalizing a subgroup amounts to an algebraic equation which must cut out some subvariety containing G_{geom}.

Remark 2.27. Note G is Zariski-locally a finitely generated group scheme. We also note that ρ induces the following morphism of short exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \overline{x}) \arrow[r] \arrow[d, "\rho"] & W(X, \overline{x}) \arrow[r] \arrow[d, "\rho"] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_{\text{geom}} \arrow[r] & G \arrow[r] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1
\end{tikzcd}$$

It will be helpful to know how the Weil group behaves in extensions.

Lemma 2.28. One has a morphism of

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_{q^r}}, \overline{x}) & \longrightarrow & W(X_{\mathbb{F}_{q^r}}, \overline{x}) & \longrightarrow & W(\overline{\mathbb{F}}_{q^r}/\mathbb{F}_{q^r}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \overline{x}) & \longrightarrow & W(X, \overline{x}) & \longrightarrow & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \end{array}$$

short exact sequences in which the right square is a pullback square.

Remark 2.29. Identifying $W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \mathbb{Z}$ in the usual way (with the Frobenius as the generator $1 \in \mathbb{Z}$), we find that the inclusion $W(\overline{\mathbb{F}}_q/\mathbb{F}_{q^r}) \subseteq W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is a subgroup of index m and must be identified with $m\mathbb{Z}$.

Lemma 2.30. Fix a connected étale cover $\pi\colon X' \to X$ of degree n with geometric basepoints $\pi(\overline{x}') = \overline{x}$. Then $W(X', \overline{x}')$ is canonically isomorphic to a subgroup of index n in $\pi_1(X_{\overline{\mathbb{F}}_q}, \overline{x}) \cap W(X', \overline{x}')$.

We omit the proofs of the previous two lemmas.

We are now ready to state our theorem.

Theorem 2.31. Fix a semisimple smooth Weil $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}_0$ on a normal geometrically irreducible scheme X over $\mathbb{F}_q$.

(a) The group G_{geom} and G_{geom}° are semisimple groups.

(b) The induced map $Z(G(\overline{\mathbb{Q}}_\ell)) \to W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ has finite kernel and cokernel.

Remark 2.32. Let's explain why we might want $\mathcal{G}_0$ to be semisimple. Let $\mathcal{G}_0$ be a semisimple smooth Weil $\overline{\mathbb{Q}}_\ell$-sheaf. Then the representation ρ of $W(X, \overline{x})$ and its restriction to $\pi_1(X_{\overline{\mathbb{F}}_q}, \overline{x})$ is semisimple, so the representation of G_{geom} on W is semisimple. Thus, if this representation is faithful (which is the case in our context), it follows that G_{geom} is reductive.

Remark 2.33. More concretely, (b) is asserting that the map $Z(G(\overline{\mathbb{Q}}_\ell)) \to W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is surjective with finite kernel after applying a finite extension of $\mathbb{F}_q$. The moral is that the center already knows everything about the Frobenius action.

2.6 March 6

Today, we prove Theorem 2.31.

2.6.1 Proving Semisimple Monodromy

We sketch the steps of the proof of (a). The idea is to reduce the proof to tori, whose representation theory is understood.

1. We begin by reducing to the case where G_{geom} is connected. This is done by replacing $\mathbb{F}_q$ with a finite extension and X with a finite étale cover. The moral is that the pre-image of G_{geom}° through the continuous Galois representation must give some finite-index subgroup of $W(X, \overline{x})$, and such a finite-index subgroup can be alternatively achieved by the aforementioned operations by Lemmas 2.28 and 2.30.

2. By passing to another finite extension, we may assume that the representation $\rho\colon W(X,\overline{x}) \to \mathrm{GL}(V)$ attached to $\mathcal{G}_0$ is faithful. Thus, G_{geom} is a connected reductive group; let $T \subseteq G_{\mathrm{geom}}$ be a maximal central torus. We will study the restriction of our representation to T.

   Choose $\sigma \in W(X,\overline{x})$ of degree 1, which acts by conjugation on G_{geom} and in particular fixing G_{geom}: indeed, this is true for elements of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x})$. Thus, the same is true for the $T \subseteq G_{\mathrm{geom}}$.

   Eventually we are going to need to understand automorphisms of G_{geom}, so we recall that $G_{\mathrm{geom},\overline{\mathbb{Q}}_\ell}$ has only finitely many outer automorphisms. Indeed, G_{geom} is reductive such things are parameterized by automorphisms of the finite Dynkin diagram. This finiteness will be eventually be dealt with by the same sort of operations done in Lemmas 2.28 and 2.30.

3. We apply produce some finite permutation of characters of T. Note that the action of $\rho(\sigma)$ permutes the finitely many characters which make a basis of $X^*(T)$. In particular, it permutes the finitely many characters which appear in the action of T on V, which we write as

$$V = \bigoplus_{\chi\in X^*(T)} V^\chi.$$

   (Because V is finite-dimensional, only finitely many V^χ are allowed to be nonzero.) So by replacing $\mathbb{F}_q$ with a finite extension we may assume that $\rho(\sigma)$ acts trivially.

4. We now get rid of the ambient outer automorphism. Because $\rho(\sigma)$ is fixing T, we use the finiteness of the outer automorphism group to know that some power of $\rho(\sigma)$ acts by inner automorphism on T. Namely, we are granted some $g \in G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$ such that

$$\rho(\sigma)h\rho(\sigma)^{-1} = g^{-1}hg$$

   for any h; here we have replaced σ with a power implicitly.

   We now pass to the Weil group of a larger field to force g to behave. Note now that $g\rho(\sigma)$ commutes with $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$ by construction, so $g\rho(\sigma) \in Z(G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell))$.

5. We now write

$$G(\overline{\mathbb{Q}}_\ell) = \bigcup_{j\in\mathbb{Z}} G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)(g\rho(\sigma))^j$$

   to permit a representation $\varphi\colon W(X,\overline{x}) \to G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$ given by the inclusion $W(X,\overline{x}) \to G(\overline{\mathbb{Q}}_\ell)$ followed by the projection down to $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$.

6. We are now ready to argue that G_{geom} is semisimple. This will require some geometric input. Failure to be semisimple produces a nontrivial central character $\chi\colon G_{\mathrm{geom}} \to \mathbb{G}_m$ which in particular must have dense image. However, Theorem 2.19 tells us that the restriction to $\pi_1(X_{\overline{\mathbb{F}}_q},\overline{x})$ has finite image, so no such algebraic character χ with dense image may exist. Semisimplicity follows.

2.7 March 11

We began class by finishing the proof of Theorem 2.31.

2.7.1 More on Semisimple Monodromy

It remains to prove (b) of Theorem 2.31. We proceed in steps. For brevity, set $Z \coloneqq Z(G(\overline{\mathbb{Q}}_\ell))$.

1. Note that an appropriate choice of the Frobenius (in degree 1) as in our proof of (a) tells us that $G(\overline{\mathbb{Q}}_\ell)$ splits into the direct product $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell) \times \mathbb{Z}$. Thus, we have a morphism

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q},\overline{x}) \arrow[r] \arrow[d] & W(X,\overline{x}) \arrow[r] \arrow[d] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell) \arrow[r] & G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)\times\mathbb{Z} \arrow[r] & W(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1
\end{tikzcd}$$

of short exact sequences, as soon as we pass to a finite extension of $\mathbb{F}_q$ in the bottom.

The diagram now defines our map $\psi\colon Z \to W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ (without using the splitting). Note $\ker\psi$ lives in $Z \cap G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$ by exactness, which is finite because G_{geom} is semisimple.

2. It remains to show that the cokernel is finite. Note that if we were allowed to pass to a finite extension of $\mathbb{F}_q$, then the short exact sequence defining ψ would split, so the result would have no content. Thus, the difficulty lives in dealing with the component group, which potentially makes the center smaller. Nonetheless, there is $\zeta \in G(\overline{\mathbb{Q}}_\ell)$ of positive degree by using the short exact sequence. In fact, we may even assume that ζ commutes with $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$ by using the splitting of the (second) short exact sequence in the previous step. We would like to show that a power of ζ lives in Z, which will complete the proof.

   To detect the center, we define the 1-cocycle $\varphi_g\colon \mathbb{Z} \to G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$
$$\varphi_g(n) \coloneqq g\zeta g^{-1}\zeta^{-n}.$$
   Because ζ commutes with $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$ already, we know that φ_g is in fact a homomorphism. Then one can compute that $\operatorname{im}\varphi_g$ lives in $Z(G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell))$ for all g, but this codomain is a finite group, so we can find n such that $\varphi_g(n) = 1$ for all g, which is the same as asserting that ζ^n commutes with $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$. This upgrades to $\zeta^n \in Z$ because $G(\overline{\mathbb{Q}}_\ell)$ is generated by some root of ζ and $G_{\mathrm{geom}}(\overline{\mathbb{Q}}_\ell)$, so we are done.

This completes the proof of Theorem 2.31.

Remark 2.34. Let's now sketch the idea of Theorem 2.9. The point is that étale sheaves should have controlled absolute values because we are looking at representations of the compact group $\pi_1^{\text{ét}}(X, \overline{x})$. Thus, our goal is to control eigenvalues when we are told that the determinant has controlled eigenvalues.

2.8 March 27

I missed last class because I was sick. We completed the proof of Theorem 2.9.

2.8.1 L-Functions of Weil Sheaves

Today we apply Theorem 2.9. Throughout, X continues to be a normal and geometrically connected scheme over $\mathbb{F}_q$, and $\mathcal{G}_0$ is a Weil sheaf on X. If $\mathcal{G}_0$ is further irreducible, we now know that there is a decomposition $\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b$, where $\mathcal{F}_0$ is étale and $\mathcal{L}_b$ is a line bundle. The moral is that we can adjust Weil sheaves (which are representations of the merely locally compact Weil group) to étale sheaves (which are representations of the compact étale fundamental group) at little cost. The ability to work with compact groups is beneficial from the point of view of representation theory, for example.

As another application, we get a trace formula.

Corollary 2.35. Fix a smooth Weil sheaf $\mathcal{G}_0$ on a normal and geometrically scheme X over $\mathbb{F}_q$. Then
$$L(X, \mathcal{G}_0, t) \coloneqq \prod_{\text{closed } x \in X} \det\left(1 - t^{\deg x} F_x^*; \mathcal{G}_{\overline{x}}\right)^{-1}$$
is a rational function in $\mathbb{Q}(t)$, admitting a decomposition
$$L(X, \mathcal{G}_0, t) = \prod_{i=0}^{2\dim X} \det\left(1 - tF^*; \mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}, \mathcal{G})\right)^{(-1)^{i+1}}.$$

Proof. By using filtrations (for which we note that the determinants and cohomology are suitable multiplicative), we may reduce to the case where $\mathcal{G}_0$ is irreducible. We use the decomposition $\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b$ granted by Corollary 2.10. Then

$$\det\left(1 - t^{\deg x} F_x^*; \mathcal{G}_{\overline{x}}\right) = \det\left(1 - (tb)^{\deg x} F_x^*; \mathcal{F}_{\overline{x}}\right),$$

and

$$\det\left(1 - tF^*; \mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}, \mathcal{G})\right) = \det\left(1 - (tb)F^*; \mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}, \mathcal{F})\right).$$

The result now follows by plugging in (tb) into the corresponding trace formula decomposition for $\mathcal{F}$. ■

For our next application, we use the notion of weight.

Lemma 2.36. Fix a normal scheme X of finite type over $\mathbb{F}_q$, and let $\mathcal{G}_0$ be a Weil sheaf on X, and choose some isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Suppose $w(\mathcal{G}_0) \leq \beta$ for some $\beta > 0$. Then the L-series

$$\iota L(X, \mathcal{G}, t) := \prod_{\text{closed } x \in X} \iota \det\left(1 - t^{\deg x} F_x^*; \mathcal{G}_{\overline{x}}\right)^{-1}$$

converges for all $t \in \mathbb{C}$ with $|t| < q^{-\beta/2 - \dim X}$.

Proof. By induction on dimension, we may reduce to the case where X is affine and irreducible (by covering X with irreducible or affine open subschemes), and we may assume that X is reduced because reduction does not affect our Frobenius actions. Now, we recall that

$$\exp\left(\sum_{n=1}^{\infty} \operatorname{tr}\left(\varphi^n; V\right) \frac{t^n}{n}\right) = \det\left(1 - \varphi t; V\right)^{-1}$$

for any endomorphism φ on a finite-dimensional vector space V, so we can compute that the logarithmic derivative of ιL is given by

$$\frac{\iota L'(X, \mathcal{G}, t)}{\iota L(X, \mathcal{G}, t)} = \sum_{\text{closed } x \in X} \left(\sum_{n=1}^{\infty} \deg(x) \operatorname{tr}\left(F_x^n; \mathcal{G}_{\overline{x}}\right) t^{\deg(x) n}\right).$$

All sums in sight are countable, and we will find that they absolutely converge with the given t, which will notably complete the proof upon undoing the logarithmic differentiation: checking that the infinite product converges is equivalent to checking that its logarithm converges, but the logarithm is some power series in t which must have the same radius of convergence as its derivative. So to check our absolute convergence, we may as well switch the order of the summation, writing

$$\frac{\iota L'(X, \mathcal{G}, t)}{\iota L(X, \mathcal{G}, t)} = \sum_{n=1}^{\infty} \left(\sum_{\substack{\text{closed } x \in X \\ \deg(x) \mid n}} \deg(x) \operatorname{tr}\left(F_x^{n/\deg x}; \mathcal{G}_{\overline{x}}\right)\right) t^{n-1}.$$

By the weight hypothesis, the trace is bounded in magnitude by $rq^{n\beta/2}$, where r is the maximum of the dimensions of the fibers $\mathcal{G}_{0,\overline{x}}$. We are left with wanting to show the absolute convergence of

$$\sum_{n=1}^{\infty} \# X(\mathbb{F}_q^n) r q^{n\beta/2} t^{n-1}.$$

Now, because X is affine and normal of finite type over a field, we may use Noether normalization to bound the point-counts $\# X(\mathbb{F}_{q^n})$ with $\# \mathbb{A}^{\dim X}(\mathbb{F}_{q^n})$ up to some explicit constant. Thus, we are left with the checking the absolute convergence of

$$\sum_{n=1}^{\infty} q^{n(\beta/2 + \dim X)} t^{n-1},$$

which is true by hypothesis on t. ■

We will continue this discussion next class.

2.9 April 1

Today is a serious class.

2.9.1 Semicontinuity of Weights

Lemma 2.36 merely requires an upper bound on the weight, so it is desirable to be able to produce such bounds. The following lemma achieves a "spreading out" result of this type.

Lemma 2.37. Fix a smooth irreducible curve X over $\mathbb{F}_q$, and let $j\colon U \subseteq X$ be a nonempty open subset with complement $S := X \setminus U$. Fix a Weil sheaf $\mathcal{G}_0$ on X, and suppose that $j^*\mathcal{G}_0$ is smooth and $\mathrm{H}^0_S(X, \mathcal{G}_0) = 0$ and $w(j^*\mathcal{G}_0) \le \beta$ for given β. Then $w(\mathcal{G}_0) \le \beta$.

Proof. Intuitively, we are trying to pass a bound on the weight from a dense open subset to the entire curve. We proceed in steps.

1. We do some reductions. By passing to a finite extension of $\mathbb{F}_q$, we may assume that X is geometrically irreducible. For brevity, we define $\mathcal{F}_0 := j^*\mathcal{G}_0$. Now, because $\mathrm{H}^0_S(X_{\overline{\mathbb{F}}_q}, \mathcal{G}_0) = 0$, it follows that there is an embedding $\mathcal{G}_0 \hookrightarrow j_*\mathcal{F}_0$, allowing us to basically replace $\mathcal{G}_0$ with $j_*\mathcal{F}_0$ so that $j_*j^*\mathcal{G}_0 = \mathcal{G}_0$. Continuing, by shrinking U, we may as well assume that X is affine.

   Now, the trace formula from Corollary 2.35 expands as

$$\begin{aligned}\iota L(X, \mathcal{G}_0, t) &= \iota L(U, j^*\mathcal{G}_0, t) \prod_{\text{closed } s \in S} \iota \det\left(1 - F_s t^{\deg s}; \mathcal{G}_{0,\overline{s}}\right)^{-1} \\ &= \frac{\iota \det\left(1 - Ft; \mathrm{H}^1_c(X_{\overline{\mathbb{F}}_q}; \mathcal{G})\right)}{\iota \det\left(1 - Ft; \mathrm{H}^2_c(X_{\overline{\mathbb{F}}_q}; \mathcal{G})\right)}.\end{aligned}$$

   We are interested in controlling eigenvalues of in the product over closed points $s \in S$, which more or less amounts to controlling the location of our poles.

2. Thus, we would like to compute a H^2_c, for which we write

$$\begin{aligned}\mathrm{H}^2_c(X_{\overline{\mathbb{F}}_q}, \mathcal{G}) &= \mathrm{H}^2_c(U_{\overline{\mathbb{F}}_q}, \mathcal{F}) \\ &= \mathrm{H}^0(U_{\overline{\mathbb{F}}_q}, \mathcal{F}^\vee)(-1) \\ &= V_{\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \overline{x})}(-1),\end{aligned}$$

   where $V = \mathcal{G}_{0,\overline{x}}$ is some fiber, and $V_{\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \overline{x})}$ refers to the co-invariants. Now, the poles of $\iota L(X, \mathcal{G}_0, t)$ may only occur at numbers of the form $\iota\left(\alpha^{-1}q^{-1}\right)$ for eigenvalues α of F acting on $V_{\pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \overline{x})}$. Certainly Lemma 2.36 tells us that we will find no singularities in the region $|t| < q^{-\beta/2-1}$.

3. Note $\dim S = 0$, so the product

$$\prod_{\text{closed } s \in S} \iota \det\left(1 - F_s t^{\deg s}; \mathcal{G}_{0,\overline{s}}\right)^{-1}$$

   is finite and cannot produce poles for $|t| < q^{-\beta/2-1}$. Comparing with the previous step, it follows that $|\iota\alpha|^2 \le q^{\deg(s)(\beta+2)}$ for all eigenvalues. If not for this $+2$, we would be done.

4. We now use the "tensor power trick," considering eigenvalues of the sheaves $j_*\mathcal{F}_0^{\otimes m}$. Namely, if α is an eigenvalue of F_s acting on $(j_*\mathcal{F}_0)_{\overline{s}}$, then α^m is an eigenvalue of F_s acting on the tensor power $(j_*\mathcal{F}_0^{\otimes m})_{\overline{s}}$. Note that this F_s further succeeds at being injective: we can rewrite this map as $j_*(\mathcal{F}_0)_{\overline{s}}^{\otimes m} \to (j_*\mathcal{F}_0^{\otimes m})_{\overline{s}}$. The point is that we may rerun the entire argument to now get an estimate $|\iota\alpha^m| \le q^{\deg(s)(m\beta+2)}$. Taking $m \to \infty$ completes the proof. ■

Continuing with our spreading out discussion, it may be desirable to control the representations outside a dense open subset.

Lemma 2.38. Fix a smooth irreducible curve X over $\mathbb{F}_q$, and let $\mathcal{F}_0$ be a smooth Weil sheaf given by a representation $\rho\colon W(X,\overline{x}) \to \mathrm{GL}(V)$, where $V = \mathcal{G}_{0,\overline{x}}$ for some x. As before, let $j\colon U \subseteq X$ be a nonempty open subset with complement $S := X \setminus U$. For $s \in S$, let $I_s \subseteq \pi_1^{\text{ét}}(U_{\overline{\mathbb{F}}_q}, \overline{x})$ be the ramification subgroup. Then

$$(j_*\mathcal{F})_{\overline{s}} = V^{I_s}.$$

Proof. Omitted. ■

Remark 2.39. In order to use the tensor power trick again later, we note that one thus has an inclusion

$$(j_*\mathcal{F})_{\overline{s}}^{\otimes m} = \left(V^{I_s}\right)^{\otimes m} \subseteq \left(V^{\otimes m}\right)^{I_s} = (j_*\mathcal{F}^{\otimes m})_{\overline{s}}.$$

Lemma 2.40. Let X be a normal irreducible scheme over $\mathbb{F}_q$, and choose an irreducible and smooth Weil sheaf $\mathcal{G}_0$. For any open dense subset $j\colon U \subseteq X$, the sheaf $j^*\mathcal{G}_0$ is still irreducible.

Proof. The normality of X implies that we have a surjection $\pi_1(U_{\overline{\mathbb{F}}_q}, \overline{x}) \twoheadrightarrow \pi_1(X_{\overline{\mathbb{F}}_q}, \overline{x})$ for any choice of geometric basepoint $\overline{x} \hookrightarrow U$. Thus, we similarly get a surjection $W(U,\overline{x}) \twoheadrightarrow W(X,\overline{x})$. The result now follows because any subrepresentation for $W(U,\overline{x})$ would also be a subrepresentation for $W(X,\overline{x})$. ■

We are now ready to state a main theorem on weights.

Theorem 2.41 (Semicontinuity of weights, I). Fix a smooth Weil sheaf $\mathcal{G}_0$ on a scheme X, and let $j\colon U \subseteq X$ be an open dense subscheme.

(a) $w(\mathcal{G}_0) = w(j_*\mathcal{G}_0)$.

(b) If $j^*\mathcal{G}_0$ is ι-pure of weight β, then $\mathcal{G}_0$ is ι-pure of weight of β.

Proof. Here we go.

(a) We may as well assume that X is irreducible by taking subsets, and we may as well replace X with the normalization of its reduction. Now, for $\dim X = 0$, there is nothing to say, and for $\dim X = 1$, we get the result from Lemma 2.37, where we note that smoothness for X is equivalent to being normal.

It remains to handle $\dim X > 1$. Because X is irreducible, any $s \in X \setminus U$ can be found in a curve $Y \subseteq X$ which intersects U nontrivially. The result now follows from the curve case.

(b) By (a), we see that $w(\mathcal{G}_0) = w(j_*\mathcal{G}_0)$, which gets an equality of the supremum of the possible eigenvalues. To get an equality of the infima, one simply reruns the argument with $\mathcal{G}_0^\vee$, which has inverse eigenvalues. ■

2.10 April 3

Today we continue talking about weights of Weil sheaves.

2.10.1 More on Semicontinuity of Weights

There is more we can say about semicontinuity.

Theorem 2.42 (Semicontinuity of weights, II)**.** Fix a smooth Weil sheaf $\mathcal{G}_0$ on a scheme X, and let $j \colon U \subseteq X$ be an open dense subscheme.

(a) If X is irreducible and normal, and $\mathcal{G}_0$ is irreducible, then $j^*\mathcal{G}_0$ being ι-mixed implies that $\mathcal{G}_0$ is ι-pure.

(b) If X is connected and $j^*\mathcal{G}_0$ is ι-mixed, then if $\mathcal{G}_0$ is ι-pure of weight β at any $\mathbb{F}_q$-rational point of X, then $\mathcal{G}_0$ is ι-pure of weight β.

Proof. We continue from Theorem 2.41.

(a) Because $j^*\mathcal{G}_0$ is ι-mixed, there is an open nonempty subscheme $V \subseteq X$ so that $j^*\mathcal{G}_0$ has a finite filtration by smooth sheaves with ι-pure quotients. We go ahead and shrink U to match V; by normality (using Lemma 2.40), this shrinking does not affect the irreducibility of $j^*\mathcal{G}_0$. We now see that $j^*\mathcal{G}_0$ must be ι-pure by irreducibility, so the result follows from Theorem 2.41.

(b) By connectivity and a little induction, it is enough to check the result for X irreducible. By passing to the reduction, we may assume that X is integral, and we may pass to a cover to assume that X is normal. By using a filtration of $\mathcal{G}_0$ as usual, we may assume further that $\mathcal{G}_0$ is irreducible. We now know that $\mathcal{G}_0$ is ι-pure of a given weight at a single point, which by (a) above spreads purity and thus the weight everywhere. ■

We take a moment to discuss real sheaves.

Definition 2.43 (real)**.** Fix an embedding $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. A Weil sheaf $\mathcal{G}_0$ on X is *ι-real* if $\iota \det\left(1 - F_x t; \mathcal{G}_{0,\overline{x}}\right)$ is in $\mathbb{R}[t]$ for all $\overline{x} \hookrightarrow X$.

The moral of the story is that our Frobenius eigenvalues are now required to come in complex conjugate pairs. The following lemma explains how to produce real sheaves.

Lemma 2.44. Choose an embedding $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Fix a smooth Weil sheaf $\mathcal{G}_0$ which is ι-pure of weight β. Then there is an ι-real and ι-pure sheaf $\mathcal{F}_0$ of weight β such that $\mathcal{G}_0$ is a direct summand of $\mathcal{F}_0$.

Proof. Define

$$\mathcal{F}_0 \coloneqq \left(\mathcal{G}_0^\vee \otimes \mathcal{L}_b\right) \oplus \mathcal{G}_0,$$

where we choose some $b \in \overline{\mathbb{Q}}_\ell^\times$ appropriately so that $\iota(b) = q^\beta$. A straightforward computation verifies that $\mathcal{F}_0$ is ι-real and ι-pure of weight β. ■

Remark 2.45. This is our second application of the "twisting Weil sheaves" $\mathcal{L}_b$, further motivating the place of Weil sheaves (over merely étale sheaves) in our theory.

2.10.2 The Sheaf-Function Correspondence

We now say a little about the sheaf-function correspondence.

Definition 2.46. Choose an embedding $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Fix a Weil sheaf $\mathcal{G}_0$ on a scheme X of finite type over $\mathbb{F}_q$. Given a positive integer m, we define the function $f^{\mathcal{G}_0}\colon X(\mathbb{F}_{q^m}) \to \mathbb{C}$ by

$$f^{\mathcal{G}_0}\colon \overline{x} \mapsto \iota \operatorname{tr}\left(F_x^{m/\deg x}; \mathcal{G}_{0,\overline{x}}\right)$$

These functions allow us to do some analysis. For example, one can define an inner product

$$(f,g)_m \coloneqq \sum_{y \in X(\mathbb{F}_{q^m})} f(y)\overline{g(y)},$$

yielding a square norm $\|f\|_m^2 \coloneqq (f,f)_m$.

Remark 2.47. With this definition, we find

$$\left(f^{\mathcal{G}_0}, 1\right)_m = \sum_{y \in X(\mathbb{F}_{q^m})} \iota \operatorname{tr}\left(F_x^{m/\deg y}; \mathcal{G}_{0,\overline{y}}\right) = \sum_{\substack{\text{closed } x \in X(\overline{\mathbb{F}}_q) \\ \deg(x) \mid m}} \deg(x) \iota \operatorname{tr}\left(F_x^{m/\deg x}; \mathcal{G}_{0,\overline{x}}\right),$$

so we find

$$\frac{\iota L'(t)}{\iota L(t)} = \sum_{m=1}^{\infty} \left(f^{\mathcal{G}_0}, 1\right)_m t^{m-1}.$$

We note that the proof of Lemma 2.36 now produces a bound $\left(f^{\mathcal{G}_0}, 1\right)_m = O\left(q^{n(w(\mathcal{G}_0)/2 + \dim X)}\right)$. Using similar techniques, one can show that $\left\|f^{\mathcal{G}_0}\right\|_m^2$ is bounded above by

$$\left(\max_{\text{closed } x \in X} \operatorname{rank} \mathcal{G}_{0,\overline{x}}\right)^2 \cdot \#X(\mathbb{F}_{q^m}) \cdot q^{mw(\mathcal{G}_0)} = O\left(q^{m(w(\mathcal{G}_0) + \dim X)}\right).$$

The above remark motivates the following definition.

Definition 2.48. Fix a Weil sheaf $\mathcal{G}_0$ on a scheme X of finite type over $\mathbb{F}_q$. Then

$$\|\mathcal{G}_0\| \coloneqq \sup\left\{\rho \in \mathbb{R} : \limsup_{m \to \infty} \frac{\left\|f^{\mathcal{G}_0}\right\|_m^2}{q^{m(\rho + \dim X)}} > 0\right\}.$$

The point of this definition is to read off the value of $w(\mathcal{G}_0)$ without an assumption that the Frobenius eigenvalues of $\mathcal{G}_0$ are actually controlled. Namely, we have recovered weights on the "function" side of the sheaf-function correspondence.

Remark 2.49. Alternatively, one can check that $q^{-\|\mathcal{G}_0\| - \dim X}$ is the radius of convergence of the power series

$$\varphi^{\mathcal{G}_0}(t) \coloneqq \sum_{m=1}^{\infty} \left\|f^{\mathcal{G}_0}\right\|_m^2 t^{m-1}.$$

In particular, one now sees that $\|\mathcal{G}_0\| \le w(\mathcal{G}_0)$ from (the proof of) Lemma 2.36.

Theme 3
From Sheaves to Functions

3.1 April 8

Today we continue talking about weights of Weil sheaves.

3.1.1 Weights of Weil Sheaves on Curves

For curves, one can say quite a bit about weights.

Definition 3.1. Fix an ι-mixed Weil sheaf $\mathcal{G}_0$ on a one-dimensional scheme X of finite type over $\mathbb{F}_q$. Then there is some open dense subset $j\colon U \subseteq X$ such that $j^*\mathcal{G}_0$ is smooth on U. Then we define

$$w_{\text{gen}}(\mathcal{G}_0) := w(j^*\mathcal{G}_0).$$

Theorem 3.2. Fix an ι-mixed Weil sheaf $\mathcal{G}_0$ on a scheme X of finite type over $\mathbb{F}_q$. Assume $\dim X \leq 1$, and let $j\colon U \subseteq X$ be an open subscheme consisting of all irreducible components of X of dimension equal to $\dim X$.

(a) We have $\|\mathcal{G}_0\| = \max\{w_{\text{gen}}(j^*\mathcal{G}_0), w(\mathcal{G}_0) - \dim X\}$. Here, $w_{\text{gen}}(j^*\mathcal{G}_0)$ is defined as follows.

(b) If X is a smooth curve with $\mathrm{H}^0_E(X, \mathcal{G}) = 0$ for all closed subsets $E \subseteq X$, then $\|\mathcal{G}_0\| = w(\mathcal{G}_0)$.

Remark 3.3. The -1 in (a) has to do with the fact that X may have disconnected points.

Proof. Note quickly that (a) implies (b). Indeed, in (b), X is smooth and in particular pure of dimension 1, so $X = U$. Then (a) allows us to compute

$$\|\mathcal{G}_0\| = \max(w_{\text{gen}}(j^*\mathcal{G}_0), w(\mathcal{G}_0) - 1),$$

and now $w_{\text{gen}}(j^*\mathcal{G}_0) = w(\mathcal{G}_0)$ by Lemma 2.37 (and some semicontinuity), so the result follows.

We now focus on (a). We proceed in steps.

1. We make some easy reductions. We may pass to the reduction of X with no harm, so we may assume that X is (geometrically) reduced. Further, we note that we may assume that X is connected. Indeed, viewing X as the disjoint union of some connected components, we see that

$$\varphi^{\mathcal{G}_0}(t) = \sum_{m=1}^{\infty} \left\|f^{\mathcal{G}_0}\right\|_m^2 t^{m-1}$$

can be decomposed into a finite sum over the connected components, so its radius of convergence is simply the maximum of the radii of convergence of the various sums given by each individual connected component. So having (a) for each connected component produces the result for the union X.

2. We quickly handle the case where X is a point. This is not so hard, but it does require us to unravel all the definitions involved. Because X is reduced and connected, it has a unique closed point s. The embedding $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ allows us to view $V := \mathcal{G}_{0,\overline{s}}$ as a vector space over $\mathbb{C}$, so as soon as we provide V with a basis, we may view the Frobenius $F_s\colon V \to V$ as given by a matrix A. We now recall that the logarithmic derivative of $\det\left(1 - A \otimes \overline{A} \cdot t^{\deg s}\right)^{-1}$ is
$$\sum_{n=1}^{\infty} \deg(s)\operatorname{tr}\left((A\otimes\overline{A})^n\right)t^{n\deg s-1} = \sum_{n=1}^{\infty}\deg(s)\left|\operatorname{tr}A^n\right|^2 t^{n\deg(s)-1}.$$
This is now seen to be $\varphi^{\mathcal{G}_0}(t)$, so we would like to compute the radius of convergence. We already have an interpretation in terms of $\det$, so we are basically looking for the smallest pole, so our radius of convergence comes out to be
$$\min_{\alpha,\beta}|\iota(\alpha)\iota(\beta)|^{-1/\deg s} = \min_{\alpha}|\iota(\alpha)|^{-2/\deg(s)},$$
where α and β vary over eigenvalues of F_s. This can be computed to be $q^{-w(\mathcal{G}_0)}$, which unravels into showing $\|\mathcal{G}_0\| = w(\mathcal{G}_0)$, as required.

3. Next, we handle the case where X is smooth and ι-pure on a smooth affine curve X. By a similar argument to the one in the first step, we may assume that A is geometrically irreducible (by passing to a large extension and dividing up the sum). Note that being able to write down $A\otimes\overline{A}$ was important to the previous computation, so we aim to construct such an object here as well. Set $\beta := w(\mathcal{G}_0)$ to be the weight of $\mathcal{G}_0$, and then we define
$$\overline{\mathcal{G}}_0 := \mathcal{G}_0^\vee \otimes \mathcal{L}_{\iota^{-1}q^\beta}$$
so that $w(\overline{\mathcal{G}}_0) = \beta$ as well. Now, because X is affine, the Grothendieck–Lefschetz trace formula yields
$$\iota L(X,\mathcal{G}_0\otimes\overline{\mathcal{G}_0}_0;t) = \frac{\iota\det\left(1-Ft;\mathrm{H}^1_c(X_{\overline{\mathbb{F}}_q}(\mathcal{G}\otimes\overline{\mathcal{G}}_0))\right)}{\iota\det\left(1-Ft;\mathrm{H}^2_c(X_{\overline{\mathbb{F}}_q}(\mathcal{G}\otimes\overline{\mathcal{G}}_0))\right)}.$$
We now make a few remarks.

   - To begin, we note that any zero of $\iota\det\left(1-Ft;\mathrm{H}^2_c(X_{\overline{\mathbb{F}}_q}(\mathcal{G}\otimes\overline{\mathcal{G}}_0))\right)$ satisfies $|\alpha| = q^{-\beta-1}$ by using Poincaré duality (and a little on weight considerations), so a pole of the L-function can only occur in this circle.
   - Being affine allows us to apply Lemma 2.36, which tells us that our L-function converges for $|t| < q^{-\beta-1}$. On the other hand, we can see that the local factors in the product
$$\prod_{\text{closed } x\in X}\iota\det\left(1-t^{\deg x}F_x^*;\mathcal{G}_{0,\overline{x}},\overline{\mathcal{G}}_{0,\overline{x}}\right)^{-1}$$
are some power series with real coefficients and leading coefficient 1.

   In total, we note that $\varphi^{\mathcal{G}_0}(t)$ is the logarithmic derivative of the above L-function, so the above inputs produce the result.

4. Then we can handle the case where $\mathcal{G}_0$ is merely ι-mixed while X remains smooth and affine.

5. We are now ready to complete the proof, which will go uncompleted in these notes. ■

3.2 April 17

Today, we will talk about the Fourier transform.

3.2.1 The Fourier Transform

Note that there is an Artin–Schreier covering $p\colon \mathbb{A}_k \to \mathbb{A}_k$ given by $x \mapsto x^q - x$, which is étale (we are in positive characteristic) and in fact Galois with Galois group $G := \mathbb{F}_q$. By the Tannakian formalism, we see that we are granted a surjection $\pi_1^{\text{ét}}\left(\mathbb{A}^1_{\overline{k}}, \overline{x}\right) \twoheadrightarrow G$, so any character $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ induces a character

$$\pi_1^{\text{ét}}(\mathbb{A}^1_{\overline{k}}, \overline{x}) \twoheadrightarrow G \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times.$$

Thus, we produce a line bundle, which we name $\mathcal{L}_0(\psi)$.

We now fix a nontrivial additive character $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$,[1] and we note that a different choice of $x \in \mathbb{F}_q$ gives rise to another character $\psi_x\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ given by $\psi_x(y) := \psi(xy)$. The mapping $x \mapsto \psi_x$ then defines an isomorphism from $\mathbb{F}_q$ to the group of characters: certainly it is a homomorphism of groups of the same order, and it is injective because $\psi_x = 1$ implies that $\psi(xy) = 1$ for all y, so $x = 0$ because ψ is nontrivial.

We now have two claims.

Lemma 3.4. The pushforward $p_*\overline{\mathbb{Q}}_\ell$ is a smooth sheaf on $\mathbb{A}_k$ of rank q.

Proof. This follows because p is étale and Galois of degree q. ∎

Lemma 3.5. The underlying representation of $p_*\overline{\mathbb{Q}}_\ell$ is given by the regular representation of $\overline{\mathbb{Q}}_\ell[\mathbb{F}_q]$.

Proof. Indeed, the point is that $p_*\overline{\mathbb{Q}}_\ell$ corresponds to a continuous map

$$\rho\colon \pi_1^{\text{ét}}(\mathbb{A}^1, \overline{x}) \to \mathrm{GL}_q(\overline{\mathbb{Q}}_\ell),$$

which on fibers is given by

$$(p_*\overline{\mathbb{Q}}_\ell)_{\overline{x}} \cong \mathrm{Mor}(G, \overline{\mathbb{Q}}_\ell).$$

In particular, a fiber of a geometric point is calculated to be a $\overline{\mathbb{Q}}_\ell$-vector space with action by G given by $(g\varphi)(h) = \varphi(g+h)$ by some computation on the fibers. ∎

Remark 3.6. In fact, by diagonalizing the regular representation, we see that

$$p_*\overline{\mathbb{Q}}_\ell = \bigoplus_{x \in \mathbb{F}_q} \mathcal{L}_0(\psi_x).$$

Remark 3.7. If we pass to a finite extension $\mathbb{F}_{q^m}$, then there is a trace map $\mathrm{tr}\colon \mathbb{F}_{q^m} \to \mathbb{F}_q$ which produces a nontrivial character $\psi_n\colon \mathbb{F}_{q^m} \to \mathbb{F}_q$ satisfying

$$\psi_x \circ \mathrm{tr} = (\psi \circ \mathrm{tr})_x.$$

Thus, we can calculate that extending $\mathcal{L}_0(\psi)$ and $\mathcal{L}_0(\psi_x)$ is compatible with such field extensions.

[1] For example, one convenient choice would be composing ι with the character $\mathbb{F}_q \to \mathbb{C}^\times$ given by $x \mapsto \exp(2\pi i\, \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)/p)$.

Remark 3.8. The Leray spectral sequence implies that

$$\mathrm{H}^1(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell) = \mathrm{H}^1(\mathbb{A}^1, p_*\overline{\mathbb{Q}}_\ell) = \mathrm{H}^1_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell) = \mathrm{H}^1_c(\mathbb{A}^1, p_*\overline{\mathbb{Q}}_\ell) = 0.$$

From this, one can calculate $\mathrm{H}^1(\mathbb{A}^1, \mathcal{L}(\psi_x)) = \mathrm{H}^1_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = 0$ by taking the sum.

We are now ready to define the Fourier transform.

Definition 3.9 (Fourier transform)**.** Fix a nontrivial character $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$, and consider the following diagram.

$$\begin{tikzcd} & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "m"] \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \mathbb{A}^1 \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{tikzcd}$$

Here, m is the multiplication map. Then we define $T_\psi\colon D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell) \to D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ to be given by

$$T_\psi(K_0) := \mathrm{R}\,\mathrm{pr}_{1!}\left(\mathrm{pr}_2^*\, K_0 \otimes m^*\mathcal{L}_0(\psi)\right)[1].$$

Remark 3.10 (Perverse sheaves)**.** The shift $[1]$ is included for technical reasons. In short, it ensures that our Fourier transform preserves perverse sheaves. Note that this shift $[1]$ is distinct from the Tate twist (1)! Perverse sheaves are the heart of the derived category of our complexes, and they have many desirable properties.

Remark 3.11. One can think about this Fourier transform as some kind of "intersection" of the complex K_0 with a cycle given by $\mathcal{L}(\psi)$, and this latter cycle is intimately related to the Frobenius because it arises from the Artin–Schreier covering.

Morally, we are "integrating" (which is $\mathrm{R}\pi_{1!}$) our complex K_0 twisted by the character m. It takes some work to see how taking trace of $\pi_{1!}$ is the desired integration.

We will want a way to calculate the Fourier transform.

Theorem 3.12. For $a \in \overline{\mathbb{F}}_q$ and $K_0 \in D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$, we have

$$(T_\psi K_0)_a = \mathrm{R}\Gamma_c(K \otimes \mathcal{L}(\psi_a))[1].$$

Proof. This follows directly from base-changing along $\{a\} \hookrightarrow \mathbb{A}^1$. ■

We've done a lot of discussion of sheaves, so let's now turn to the other side of the sheaf-function correspondence. Note that $K_0 \in D^c_b(X, \overline{\mathbb{Q}}_\ell)$ may define a function $f^{K_0}\colon X(\mathbb{F}_{q^m}) \to \mathbb{C}$ given by

$$f^{K_0}(\overline{x}) := \sum_v (-1)^v f^{\mathrm{H}^v(K_0)},$$

where $H^v(K_0)$ refers to the vth cohomology.

Example 3.13. Let's compute $f^{\mathcal{L}_0(\psi)}(x)$: our Frobenius is mapping $\alpha \in \overline{\mathbb{F}}_q$ which is a root of $T^{q^m} - T = x$ to a new solution $\beta := \alpha^{q^m}$. But then $\beta - \alpha = x$, so the Galois element we are looking at is simply given by $-x$. By diagonalizing the regular representation again, it follows that $f^{\mathcal{L}_0(\psi)}(x) = \psi(-x)$.

The above example allows us to compute the following.

Lemma 3.14. Choose $m \geq 1$ and a complex $K_0 \in D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$. Then one has

$$f^{T_\psi K_0}(t) = -\sum_{x \in \mathbb{F}_{q^m}} f^{K_0}(x)\overline{\psi}(xt).$$

Proof. This is a direct computation. We write

$$\begin{aligned}
f^{T_\psi K_0}(t) &= \sum_i (-1)^i \operatorname{tr}\left(F_t; \mathrm{H}^i\left(\mathrm{R}\operatorname{pr}_{1!}(\operatorname{pr}_2^* K_0 \otimes m^*\mathcal{L}_0(\psi))\right)[1]\right) \\
&= -\sum_i (-1)^i \operatorname{tr}\left(F_t; \mathrm{H}^i\left(\mathrm{R}\operatorname{pr}_{1!}(\operatorname{pr}_2^* K_0 \otimes m^*\mathcal{L}_0(\psi))\right)\right) \\
&\stackrel{*}{=} -\sum_i (-1)^i \sum_{x \in \mathbb{F}_{q^m}} \operatorname{tr}\left(F_{(t,x)}; \mathrm{H}^i(\pi_2^* K_0 \otimes m^*\mathcal{L}_0(\psi))\right) \\
&= -\sum_i (-1)^i \sum_{x \in \mathbb{F}_{q^m}} \operatorname{tr}\left(F_{(t,x)}; \mathrm{H}^i(\pi_2^* K_0)\right)\psi(-tx) \\
&= -\sum_i (-1)^i \sum_{x \in \mathbb{F}_{q^m}} \operatorname{tr}\left(F_x; \mathrm{H}^i(K_0)\right)\psi(-tx).
\end{aligned}$$

Here, $\stackrel{*}{=}$ is a special case of base-changing to a fiber, where the point is that taking a trace of $\mathrm{R}\operatorname{pr}_{1!}$ turns into summing along the fiber. The result now follows from a rearrangement. ∎

We are interested in developing some Fourier theory on the level of sheaves, so it is worth our time to recall how Fourier theory works for functions. In general, given a finite abelian group G, we let $\widehat{G}$ denote the character group. Then any function $f\colon G \to \mathbb{C}$ admits a Fourier transform $\hat{f}\colon \widehat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) \coloneqq \sum_{g \in G} f(g)\overline{\chi}(g).$$

Then there are two main statements: there is a Fourier inversion formula

$$f(g) = \frac{1}{|G|}\sum_{\chi \in \widehat{G}} \hat{f}(\chi)\chi(g),$$

and there is a Plancherel formula

$$\sum_{g \in G} |f(g)|^2 = \frac{1}{|G|}\sum_{\chi \in \widehat{G}} \left|\hat{f}(\chi)\right|^2.$$

We will lift these two results to sheaves next class, though we will not prove them.

3.3 April 22

Today, we finish our discussion of the Fourier transform.

3.3.1 Fourier Theory for Sheaves

We would like a version of the Plancherel formula and Fourier inversion for our sheaves. Because it is easier, we begin with the Plancherel formula.

Theorem 3.15 (Plancherel)**.** Choose $m \geq 1$ and a complex $K_0 \in D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$. Then

$$\left\|f^{T_\psi K_0}\right\|_m = q^{m/2}\left\|f^{K_0}\right\|_m.$$

Proof. This is a direct computation. By definition,

$$\|f^{T_\psi K_0}\|_m^2 = \sum_{x\in\mathbb{F}_{q^m}} f^{T_\psi K_0}(x)\overline{f^{T_\psi K_0}(x)}.$$

By Lemma 3.14, this expands to

$$\|f^{T_\psi K_0}\|_m^2 = \sum_{x,y,z\in\mathbb{F}_{q^m}} f^{K_0}(y)\overline{f^{K_0}(z)}\psi(-xy)\psi(xz).$$

We now isolate the sum over x, noting this internal sum is

$$\sum_{x\in\mathbb{F}_{q^m}} \psi(x(y-z)) = \begin{cases} q^m & \text{if } y=z, \\ 0 & \text{else.} \end{cases}$$

We conclude that

$$\|f^{T_\psi K_0}\|_m^2 = q^m \sum_{y\in\mathbb{F}_{q^m}} f^{K_0}(y)\overline{f^{K_0}(y)},$$

completing the proof. ■

Remark 3.16. Here is a more sheaf-theoretic implication: Theorem 3.2 more or less explains that $w(\mathcal{G}_0) = w(\mathcal{G}_0)$, so we are finding that the Fourier transform more or less preserves the weight. More precisely, suppose that K_0 has all the sheaves $\mathcal{H}^i(K_0)$ being ι-mixed. Then one may define

$$w(K_0) := \max_{i\in\mathbb{Z}}\{w(\mathcal{H}^i(K_0)) - i\}.$$

The character sum at the heart of the previous proof can be isolated into the following computation: Let's see a special case of this computation: the constant function 1 has Fourier transform

$$T_\psi 1(y) = \sum_{x\in\mathbb{F}_{q^m}} \psi(xy) = \begin{cases} q^m & \text{if } y=0, \\ 0 & \text{else.} \end{cases}$$

This motivates the following lemma.

Lemma 3.17. Let δ_0 be the skyscraper sheaf $i_{0*}\overline{\mathbb{Q}}_\ell$, where $i_0\colon \{0\}\hookrightarrow\mathbb{A}^1$ is the embedding. Then

$$T_\psi\overline{\mathbb{Q}}_\ell = \delta_0(-1)[-1].$$

Proof. We use the Leray spectral sequence on the Artin–Schreier cover $\pi\colon\mathbb{A}^1\to\mathbb{A}^1$ given by $\pi(x) := x^q - x$. Recall from Remark 3.6 that

$$p_*\overline{\mathbb{Q}}_\ell = \bigoplus_{x\in\mathbb{F}_q}\mathcal{L}_0(\psi_x).$$

Now, the Leray spectral sequence produces the fact

$$\mathrm{H}^i_c(\mathbb{A}^1_{\overline{\mathbb{F}}_q}, \mathcal{L}_0(\psi_x)) = \begin{cases} \overline{\mathbb{Q}}_\ell(-1) & \text{if } (x,i)=(0,2), \\ 0 & \text{else.} \end{cases}$$

We now compute from the definition that $T_\psi\overline{\mathbb{Q}}_\ell$ is $\mathrm{R}\pi_{1!}(m^*\mathcal{L}_0(\psi))[1]$, whose stalk at x is

$$\mathrm{R}\Gamma_c(\mathcal{L}(\psi_x))[1] = \delta_0(-1)[-1]$$

by using base-change. The result follows because having the same functions forces the same sheaves for the sheaf-function correspondence. ■

Remark 3.18. Let's compare weights. Note δ_0 has weight 0, so $\delta_0(-1)$ has weight 2, so $\delta_0(-1)[-1]$ has weight 1.

We now turn to Fourier inversion.

Theorem 3.19 (Fourier inversion)**.** For any complex $K_0 \in D^b_c(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$, we have

$$T_{\psi^{-1}} T_\psi K_0 = K_0(-1).$$

Proof. One more or less copies the proof of the usual Fourier inversion formula for $\mathbb{F}_q$, but every time one wants to exchange sums, one has to use some base-change formulae. ■

3.3.2 Reductions to $\mathbb{A}^1$

We are now ready to enter the proof of the main theorem.

Theorem 3.20. Let U be a smooth affine curve over $\mathbb{F}_q$, and let $\mathcal{F}_0$ be a smooth, ι-mixed Weil sheaf of weight at most w. Then the space $\mathrm{H}^i(X, j_*\mathcal{F}_0)$ is ι-mixed of weight at most $w+i$.

Remark 3.21. By dual considerations, we achieve the following: if $\mathcal{F}_0$ is ι-pure of weight w, then the space $\mathrm{H}^i(X, j_*\mathcal{F}_0)$ is ι-mixed of weight at most $w+i$.

Proof of reductions. There is a long series of reductions to reduce from the general case $U \subseteq X$ to the case of $\mathbb{A}^1 \subseteq \mathbb{P}^1$.

1. We may focus on the case $i = 1$.
   - For $i = 0$, we have two cases. If $\mathcal{F}_0$ is irreducible and non-constant, then $\mathrm{H}^0(X, j_*\mathcal{F}_0) = 0$. Namely, $\mathcal{F}_0$ has no global sections. Otherwise, one finds that $\mathrm{H}^0_c(U, \mathcal{F})$ is pure of weight w.
   - For $i = 2$, we see that Poincaré duality has

     $$\mathrm{H}^2_c(U, \mathcal{F}) = \mathrm{H}^0_c(U, \mathcal{F}^\vee)^\vee,$$

     so we reduce to the case $i = 0$ again.

2. Next, we note that one may shrink U. Let $V \subseteq U$ be some dense open subscheme. Because $V \subseteq U$ is open, one has $\mathrm{H}^1(U, j_!\mathcal{F}) = \mathrm{H}^1_c(V, \mathcal{F}|_V)$. Further, because $U \subseteq V$ is open, the natural map $\mathrm{H}^1_c(V, \mathcal{F}|_V) \to \mathrm{H}^1_c(U, \mathcal{F})$ is surjective: by Mayer–Vietoris, the cokernel is measured by some H^1 on $U \setminus V$, which is a zero-dimensional set!

3. Note that weight is essentially independent of extending the base field $\mathbb{F}_q$, so we may do so freely.

4. We may assume that $U \subseteq \mathbb{A}^1$. We use Noether normalization, which provides a finite map $\pi \colon U \to \mathbb{A}^1$. By restricting to the smooth locus of π (passing to some open subset of U), we may assume that π is smooth and hence étale. Now, let U' be the image of π, which is some open subset of $\mathbb{A}^1$, and we see that

   $$\mathrm{H}^1_c(U', \pi_*(\mathcal{F}|_{U'})) = \mathrm{H}^1_c(U, \mathcal{F}|_U),$$

   so we may as well replace U with U' and $\mathcal{F}|_U$ with $\pi_*(\mathcal{F}|_{U'})$. Do note that this process may make $\pi_*(\mathcal{F}|_{U'})$ larger because π may upset some monodromy, and in fact $\pi_*(\mathcal{F}|_{U'})$ is no longer required to be smooth on all $\mathbb{P}^1$!

5. We may assume that $\mathcal{F}_0$ is geometrically irreducible. Indeed, the functor $\mathrm{H}^1_c(U, -)$ is now left exact, so we may split $\mathcal{F}_0$ into its irreducible components (and extend the base field until we can see geometrically irreducible components).

6. We may assume that $\mathcal{F}_0$ is unramified at ∞. Indeed, choose any point $u \in U$, shrink U to avoid u, and then move u to ∞ be a Möbius transformation.

7. We handle the case where $\mathcal{F}_0$ is constant. Let $j \colon U \to \mathbb{P}^1$ be the embedding, and let $i \colon Z \to \mathbb{P}^1$ be the complement. Then one has an exact sequence

$$0 \to j_!\mathbb{Q}_\ell \to j_*\mathbb{Q}_\ell \to Q \to 0,$$

where $Q = i_*\mathbb{Q}_\ell$ by construction (taking stalks). In fact, note that $j_*\overline{\mathbb{Q}}_\ell$ is simply $\overline{\mathbb{Q}}_\ell$ because $(\mathbb{P}^1)^{\mathrm{sh}}_x \setminus \{x\}$ is connected.[2] Note this short exact sequence is Galois-invariant, so we can take cohomology

$$\underbrace{\mathrm{H}^0_c(U, j_*\mathbb{Q}_\ell)}_{\mathbb{Q}_\ell} \to \underbrace{\mathrm{H}^0_c(U, i_*\mathbb{Q}_\ell)}_{\mathbb{Q}_\ell^{\# Z_0}} \to \underbrace{\mathrm{H}^1_c(U, j_!\mathbb{Q}_\ell)}_{\mathrm{H}^1_c(U,\mathbb{Q}_\ell)} \to \underbrace{\mathrm{H}^1_c(U, j_*\mathbb{Q}_\ell)}_{\mathrm{H}^1(\mathbb{P}^1,\mathbb{Q}_\ell)},$$

which upon making these substitutions implies that the map $\mathbb{Q}_\ell^{\# Z_0 - 1} \to \mathrm{H}^1_c(U, \mathbb{Q}_\ell)$ is an isomorphism; notably, the last term above is $\mathrm{H}^1(\mathbb{P}^1, \mathbb{Q}_\ell) = 0$.

We will complete the proof next class. ■

3.4 April 24

Today, we finish the proof!

3.4.1 Completion of the Proof

Today, we complete the proof of Theorem 3.20. Last time, we reduced the proof to the case of $U \subseteq \mathbb{A}^1$. Here is what we will show today.

Theorem 3.22. Let $\mathcal{F}_0$ be a smooth, nonconstant, geometrically irreducible $\overline{\mathbb{Q}}_\ell$-sheaf on a smooth affine curve $U \subseteq \mathbb{A}^1$. Suppose that $\mathcal{F}_0$ is ι-mixed of weight at most w. Then $\mathrm{H}^1_c(U, \mathcal{F})$ has weight at most $w+1$.

Throughout, we work in the context of the theorem. There are three key points, which we state now.

Lemma 3.23. Let $\mathcal{G}_0 := j_!\mathcal{F}_0$ be a sheaf on $\mathbb{A}^1$, and fix an additive character $\psi \colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$.

(a) Sheaf: $T_\psi\mathcal{G}_0$ is a sheaf. Namely, the complex $T_\psi\mathcal{G}_0$ is concentrated in cohomological degree 0.

(b) Vanishing: $\mathrm{H}^0_c(\mathbb{A}^1, T_\psi\mathcal{G}_0) = 0$.

(c) Mixed: $T_\psi\mathcal{G}_0$ is ι-mixed.

Remark 3.24. Note that (a) fails if $\mathcal{G}_0$ is geometrically constant, which is why we dealt with this case separately in our reductions last class.

Let's go ahead and prove the theorem from the lemma.

Proof of Theorem 3.22 from Lemma 3.23. Let's explain how the Fourier transform enters the picture. Set $\mathcal{G}_0 := j_!\mathcal{F}_0$. Recall from Theorem 3.12 that

$$(T_\psi\mathcal{G}_0)_0 = \mathrm{R}\Gamma_c(\mathbb{A}^1, \mathcal{G})[1].$$

However, this is just $\mathrm{R}\Gamma_c(U, \mathcal{F})[1] = \mathrm{H}^1_c(U, \mathcal{F})$ by definition of $j_!$. Thus, we see that it is enough to show $w(T_\psi\mathcal{G}_0) \le w+1$.

[2] I am not sure what this sentence means.

We now study $T_\psi \mathcal{G}_0$. Well, $w(T_\psi \mathcal{G}_0)$ is known to be $\|T_\psi \mathcal{G}_0\|$ by Theorem 3.2, which we note needs (b) and (c) of Lemma 3.23 to make sense of.[3] On the other hand, we similarly know $w(\mathcal{G}_0) = \|\mathcal{G}_0\|$ (which is $w(\mathcal{F}_0)$ by Lemma 2.37), so it remains to check

$$\|T_\psi \mathcal{G}_0\| \overset{?}{\le} \|\mathcal{G}_0\| + 1,$$

which is exactly Theorem 3.15! ■

It remains to prove Lemma 3.23. We will show each part individually. The proofs of (a) and (b) are geometric; (c) will require some more work.

Proof of Lemma 3.23(a). By definition of the Fourier transform, We must show that $\mathrm{H}^0_c(\mathbb{A}^1, j_!\mathcal{F} \otimes \mathcal{L}(\psi_x))$ and $\mathrm{H}^2_c(\mathbb{A}^1, j_!\mathcal{F} \otimes \mathcal{L}(\psi_x))$ both vanish. (Note $\mathcal{L}(\psi_x)$ is smooth, so smoothness is being carried around.) Quickly, the H^0_c piece vanishes because U is affine: Poincaré duality relates this to some H^2, which vanishes for affine U.

For the H^2_c piece, we do a little representation theory. Let ρ be the underlying representation for $j_!\mathcal{F}$ so that we are interested in the representation $\rho \otimes \psi_x$ acting on some finite-dimensional vector space $(\mathcal{F}_0)_{\overline{a}} \otimes \mathcal{L}(\psi_x)_{\overline{a}}$. By Poincaré duality, we must show that

$$\mathrm{H}^2_c(U, \mathcal{F} \otimes \mathcal{L}(\psi_x)) = V(-1),$$

where V is the largest quotient of our underlying representation on which the representation acts trivially. To show that this vanishes, we have two cases.

- If $x = 0$, then we are simply looking at the irreducible representation $\rho\colon \pi_1^{\text{ét}}(U, \overline{a}) \to \mathrm{GL}(\mathcal{F}_{\overline{a}})$, which is nontrivial and irreducible. Thus, $V = 0$.
- If $x \neq 0$, we note that ρ and hence $\rho \otimes \psi_x$ is unramified at ∞, so $\rho \otimes \psi_x$ factors through $\pi_1^{\text{ét}}(\mathbb{P}^1, \overline{a}) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, which contradicts geometric irreducibility if $V \neq 0$. ■

We omit the proof of (b), which is less interesting than (c).

Proof of Lemma 3.23(c). We would like to show that $T_\psi \mathcal{G}_0$ is ι-mixed. For this, we use the fact (not included in these notes) that any subsheaf of a real sheaf is ι-mixed. In particular, consider the sheaf

$$\mathcal{H} := \Big(\underbrace{\left(\mathrm{pr}_2^*(j_!\mathcal{F}_0 \otimes m^*\mathcal{L}(\psi))\right)}_{\mathcal{A}:=} \oplus \underbrace{\left(\mathrm{pr}_2^*(j_!\mathcal{F}_0 \otimes m^*\mathcal{L}(\overline{\psi}))\right)}_{\mathcal{B}:=} \Big) \otimes \mathcal{L}_b$$

is a real sheaf for some b chosen to have $\iota(b) = q^{w(\mathcal{F}_0)}$. Note $\mathrm{R}^i\mathrm{pr}_{1!}\mathcal{A}$ and $\mathrm{R}^i\mathrm{pr}_{2!}\mathcal{B}$ are both concentrated in degree 1 by (a). As such, the Grothendieck trace formula yields

$$\prod_{y \in \mathrm{pr}_2^{-1}(\{x\})} \det\left(1 - tF_y^{\deg y}; \mathcal{H}_{\overline{y}}\right) = \det\left(1 - tF_x^{\deg x}; \mathrm{R}^1\mathrm{pr}_{1!}\mathcal{H}_{\overline{x}}\right).$$

We conclude that these factors have real coefficients, so $\mathrm{R}^1\mathrm{pr}_{1!}\mathcal{H}[-1]$ is a real sheaf. Summing appropriately, we see that $T_\psi \mathcal{G}_0$ is a factor of $\mathrm{R}^1\mathrm{pr}_{1!}\mathcal{H}[-1]$, so we are done. ■

Remark 3.25. We only achieved the main theorem for curves, but we remark that techniques of algebraic geometry (fibering by hyperplanes) are able to achieve the result for arbitrary dimension.

[3] This step is remarkable because it has moved our notion of weight from something "local" about Frobenius eigenvalues at stalks to a more "global" invariant in the sheaf-function correspondence.

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List of Definitions