# 185: Introduction to Complex Analysis

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# Contents

# Theme 1: Introducing Complex Numbers

## 1.1 January 19

It is reportedly close enough to start.

### 1.1.1 Logistics

We are online for the first two weeks, as with the rest of Berkeley. We will be using `bCourses` a lot, so check it frequently. There is also a website. There is a homework due on Friday, but do not worry about it.

Here are some syllabus things.

- Our main text is *Complex Variables and Applications, 8th Edition* because it is the version that Professor Morrow used. There is a free copy online.
- Homeworks are readings (for each course day) and weekly problem sets. Late homeworks are never accepted.
- Lowest two homework scores are dropped.
- There are two midterms and a final. The final is culmulative, as usual. The final can replace one midterm if the score is higher.
- Regrade requests can be made in GradeScope within one week of being graded.
- The class is curved but usually only curved at the end. The average on exams is expected to be 80%–83%.

### 1.1.2 Complex Numbers

Welcome to complex analysis. What does that mean?

**Idea 1.1.** In complex analysis, we study functions $f : \mathbb{C} \to \mathbb{C}$, usually analytic to some extent.

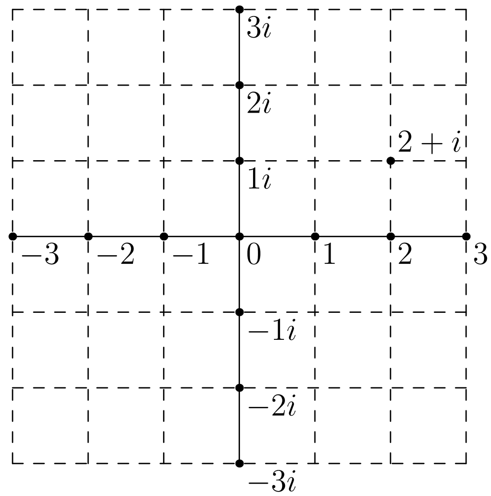
There are two pieces here: we should study $\mathbb{C}$ in themselves and then we will study the functions.

Complex numbers

**Definition 1.2** (Complex numbers)**.** The set of complex numbers $\mathbb{C}$ is $\{a + bi : a, b \in \mathbb{R}\}$, where $i^2 = -1$.

Hopefully $\mathbb{R}$ is familiar from real analysis. As an aside, we see $\mathbb{R} \subseteq \mathbb{C}$ because $a = a + 0i \in \mathbb{C}$ for each $a \in \mathbb{R}$.

The complex numbers have an inherent geometry as a two-dimensional plane.




The point is that $\mathbb{C}$ looks like the real plane $\mathbb{R}^2$. More precisely, $\mathbb{C} \cong \mathbb{R}^2$ as an $\mathbb{R}$-vector space, where our basis is $\{1, i\}$.

We would like to understand $\mathbb{C}$ geometrically, "as a space." The first step here is to create a notion of size.

Norm on $\mathbb{C}$

**Definition 1.3** (Norm on $\mathbb{C}$)**.** We define the **norm map** $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by $|z| := \sqrt{z\overline{z}}$. In other words,

$$|a + bi| := \sqrt{a^2 + b^2}.$$

Note that this agrees with the absolute value on $\mathbb{R}$: for $a \in \mathbb{R}$, we have $\sqrt{a^2} = |a|$.

Norm functions, as in the real case, give us a notion of distance.

Metric on $\mathbb{C}$

**Definition 1.4** (Metric on $\mathbb{C}$)**.** We define the *metric on* $\mathbb{C}$ to be $d_{\mathbb{C}}(z_1, z_2) := |z_1 - z_2|$.

One can check that this is in fact a metric, but we will not do so here.

**Remark 1.5.** The distance in $\mathbb{C}$ is defined to match the distance in $\mathbb{R}^2$ under the basis $\{1, i\}$.

Again as we discussed in real analysis, having a metric gives us a metric topology by open balls. Lastly it is this topology that our geometry will follow from: we have turned $\mathbb{C}$ into a topological space.

### 1.1.3 Complex Functions

There are lots of functions on $\mathbb{C}$, and lots of them are terrible. So we would like to focus on functions with some structure. We'll start with *continuous functions*, which are more or less the functions that respect topology.

Then from continuous functions, we will be able to define *holomorphic functions*, which are complex differentiable. This intended to be similar to being real differentiable, but complex differentiable turns out to be a very strong condition. Nevertheless, everyone's favorite functions are holomorphic.

**Example 1.6.** Polynomials, $\exp$, $\sin$, and $\cos$ are all holomorphic.

To make concrete that complex differentiability is stronger than real differentiability, the Cauchy–Riemann equations which provides a partial differential equation to test complex differentiability.

From here we define *analytic* functions, which essentially are defined as taking the form

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

Analytic functions are super nice in that we have an ability to physically write them down, so the following theorem is amazing.

**Theorem 1.7.** Holomorphic functions on $\mathbb{C}$ are analytic.

To prove this, we will need the following result, which is what Professor Morrow calls the most fundamental result in complex analysis, the *Cauchy integral formula*.

In short, the Cauchy integral formula lets us talk about the value of holomorphic functions (and derivatives) at a point in terms of integrals around the point. This will essentially let us build the power series for a holomorphic function by hand. But as described, we will need a notion of complex (path) integration to even be able to talk about the Cauchy integral formula.

The Cauchy integral formula has lots of applications; for example, *Liouville's theorem* on holomorphic functions and the *Fundamental theorem of algebra*.

**Remark 1.8.** It is very hard to speill Liouville.

Additionally, we remark that our study of holomorphic functions, via the Cauchy integral formula, will boil down to a study of complex path integrals. So we will finish out our story with the *Residue theorem*, which provides a very convenient way to compute such integrals.

Then as a fun addendum, we talk about automorphisms of the complex numbers.

Automorphisms of $\mathbb{C}$

**Definition 1.9** (Automorphisms of $\mathbb{C}$)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is an *automorphism of* $\mathbb{C}$ if it is bijective and both $f$ and $f^{-1}$ are holomorphic.

What is amazing is that all of these functions have a concrete description in terms of *Möbius transformations*.

### 1.1.4 Why Care?

Whenever taking a class, it is appropriate to ask why one should care. Here are some reasons to care.

- Algebraic geometry in its study of complex analytic spaces uses complex analysis.
- Analytic number theory (e.g., the Prime number theorem) makes heavy use of complex analysis.
- Combinatorics via generating functions can use complex analysis.
- Physics uses complex analysis.

The first two Professor Morrow is more familiar with, the last two less so.

## 1.2 January 21

We're reviewing set theory today.

### 1.2.1 Set Theory Notation

We have the following definitions.

- $\varnothing$ means the empty set.
- $a \in X$ means that $a$ is an element of the set $X$.
- $A \subseteq B$ means that $A$ is a subset of $B$.
- $A \subsetneq B$ means that $A$ is a proper subset of $B$.

- $A \cup B$ consists of the elements which are in at least one of $A$ or $B$.
- $A \cap B$ consists of the elements which are in both $A$ and $B$.
- $A \setminus B$ consits of the elements of $A$ which are not in $B$.
- Two sets $A$ and $B$ are *disjoint* if and only if $A \cap B = \varnothing$.
- Given a set $X$, we define $\mathcal{P}(X)$ to be the set of all subsets of $X$.
- $|X| = \#X$ is the cardinality of $X$, or (roughly speaking) the number of elements of $X$.

As an example of unwinding notation, we have the following.

**Proposition 1.10** (De Morgan's Laws)**.** Fix $\mathcal{S} \subseteq \mathcal{P}(X)$ a collection of subsets of a set $X$. Then

$$X \Bigg\backslash \bigcap_{S \in \mathcal{S}} S = \bigcup_{S \in \mathcal{S}} (X \setminus S) \qquad \text{and} \qquad X \setminus \bigcup_{S \in \mathcal{S}} S = \bigcap_{S \in \mathcal{S}} (X \setminus S).$$

*Proof.* We take these one at a time.

- Note $a \in X \setminus \bigcap \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcap \mathcal{S}$. However, $a \notin \bigcap \mathcal{S}$ is merely saying that $a$ is not in all of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for one of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for some $S \in \mathcal{S}$, which is equivalent to $a \in \bigcup_{S \in \mathcal{S}} (X \setminus S)$.

- Note $a \in X \setminus \bigcup \mathcal{S}$ if and only if $a \in X$ and $a \notin \bigcup \mathcal{S}$. However, $a \notin \bigcup \mathcal{S}$ is merely saying that $a$ is not in any of the sets $S \in \mathcal{S}$, which is equivalent to saying $a \notin S$ for each of the $S \in \mathcal{S}$.

  Thus, this is equivalent to saying $a \in X$ while $a \notin S$ for each $S \in \mathcal{S}$, which is equivalent to $a \in \bigcap_{S \in \mathcal{S}} (X \setminus S)$. ■

### 1.2.2 Some Conventions

In this class, we take the following names of standard sets.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. Importantly, $0 \in \mathbb{N}$.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}$ and $q \neq\}$ is the set of rationals.
- $\mathbb{R}$ is the set of real numbers. We will not specify a construction here; see any real analysis class.
- $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq\}$ is the nonzero real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ is the positive real numbers.
- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ is the nonnegative real numbers.
- $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$ is the nonpositive real numbers.
- $\mathbb{C}$ is the complex numbers.
- $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$ is the set of nonzero complex numbes.

### 1.2.3 Relations

Let's review some set theory definitions.

Cartesian product

**Definition 1.11** (Cartesian product). Given two sets $A$ and $B$, we define the *Cartesian product* $A \times B$ to be the set of orderd pairs $(a, b)$ such that $a \in A$ and $b \in B$.

Binary relation

**Definition 1.12** (Binary relation). A *binary relation* on $A$ is any subset $R \subseteq A^2 := A \times A$. We may sometimes notate $(x, y) \in R$ by $xRy$, read as "$x$ is related to $y$."

**Example 1.13.** Equality is a binary relation on any set $A$; namely, it is the subset $\{(a, a) : a \in A\}$.

The best relations are equivalence relations.

Equivalence relation

**Definition 1.14** (Equivalence relation). An *equivalence relation* on $A$ is a binary relation $R$ satisfying the following three conditions.

- Reflexive: each $x \in A$ has $(x, x) \in R$.
- Symmetric: each $x, y \in A$ has $(x, y) \in R$ implies $(y, x) \in R$.
- Transitive: each $x, y, z \in A$ has $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Equivalence relations are nice because they allow us to partition the set into "equivalence classes."

Equivalence class

**Definition 1.15** (Equivalence class). Fix $A$ a set and $R \subseteq A^2$ an equivalence relation. Then, for given $x \in A$, we define

$$[x]_R := \{y \in A : (x, y) \in R\}$$

to be the *equivalence class* of $x$.

The hope is that equivalence classes partition the set. What is a parition?

Parition

**Definition 1.16** (Parition). A *partition* of a set $A$ is a collection of nonempty subsets $\mathcal{S} \subseteq \mathcal{P}(A)$ of $A$ such that any two distinct $S_1, S_2 \in \mathcal{S}$ are disjoint while $A = \bigcup_{S \in \mathcal{S}} S$.

And now let's manifest our hope.

**Lemma 1.17.** Equivalence relations are in one-to-one correspondence with partitions of $A$.

*Proof.* Given an equivalence relation $R$, we define the collection

$$\mathcal{S}(R) = \{[x]_R : x \in A\}.$$

We claim that $R \mapsto \mathcal{S}(R)$ is our needed bijection. We have the following checks.

- Well-defined: observe that $\mathcal{S}(R)$ does partition $A$: if we have $[x]_R, [y]_R \in \mathcal{S}$, then $[x]_R \cap [y]_R \neq \varnothing$ implies there is some $z$ with $(x, z) \in R$ and $(z, y) \in R$, so $x \in [y]_R$ and then $[x]_R \subseteq [y]_R$ follows. So by symmetry, $[y]_R \subseteq [x]_R$ as well, so we finish the disjointness check.

  Further, we see that
  $$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} [x]_R \subseteq A$$
  because $x \in [x]_R$, so indeed the equivalence classes cover $A$.

- Injective: suppose $R_1$ and $R_2$ have $\mathcal{S}(R_1) = \mathcal{S}(R_2)$. We show that $R_1 \subseteq R_2$, and $R_2 \subseteq R_1$ will follow by symmetry, finishing.

  We notice that, for any $\mathcal{S}$ paritioning $A$, being a partition, will have exactly one subset which contains $x$. But for $\mathcal{S}(R)$ for an equivalence relation $R$, we see $x \in [x]_R \in \mathcal{S}(R)$, so this equivalence class must be the one.

  So because $[x]_{R_1}$ and $[x]_{R_2}$ are the only subsets of $\mathcal{S}(R_1)$ and $\mathcal{S}(R_2)$ containing $x$ (respectively), we must have $[x]_{R_1} = [x]_{R_2}$. Thus, $(x, y) \in R_1$ implies $y \in [x]_{R_1} = [x]_{R_2}$ implies $(x, y) \in R_2$.

- Surjective: suppose that $\mathcal{S}$ is a partition of $A$. As noted above, each $x \in A$ is a member of exactly one set $S \in \mathcal{S}$, which we call $[x]$. Then we define $R \subseteq A^2$ by $(x, y) \in R$ if and only if $y \in [x]$. One can check that this is an equivalence relation, which we will not do here in detail.[1]

  The point is that
  $$[x]_R = \{y : (x, y) \in R\} = \{y : y \in [x]\} = [x],$$
  so $\mathcal{S}(R) = \mathcal{S}$. So our mapping is surjective. ■

We continue our discussion.

Quotient set

**Definition 1.18** (Quotient set)**.** Given an equivalence relation $R \subseteq A^2$, we define the *quotient set* $A/R$ is the set of equivalence classes of $R$. In other words,

$$A/R = \{[x]_R : x \in A\}.$$

Intuitively, the quotient set is the set where we have gone ahead and identified the elements which are "similar"or "related."

We would like a more concrete way to talk about equivalence classes, for which we have the following.

Representatives

**Definition 1.19** (Representatives)**.** Given an equivalence relation $R \subseteq A^2$, we say that $C \subseteq A$ is a *set of representatives of $R$-equivalence classes of $A$* if and only if $C$ consists of exactly one element from each equivalence class in $A/R$.

### 1.2.4 Functions

To finish off, we discuss functions.

Functions

**Definition 1.20** (Functions)**.** A *function* $f : X \to Y$ is a relation $f \subseteq X \times Y$ satisfying the following.

- For each $x \in X$, there is some $y \in Y$ such that $(x, y) \in f$. Intuitively, each $x \in X$ goes somewhere.
- For each $x \in X$ and given some $y_1, y_2 \in Y$ such that $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$. Intuitively, each $x \in X$ goes to at most one place.

We will write $f(x) = y$ as notational sugar for $(x, y) \in f$. Note this equality is legal because the value $y$ with $(x, y) \in f$ is uniquely given.

We would like to create new functions from old. Here are two ways to do this.

Restriction

**Definition 1.21** (Restriction)**.** Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define

$$f|_A = \{(x, y) \in f : x \in A\} \subseteq A \times Y$$

to be a function $f|_A : A \to Y$.

[1] Note $x \in [x]$ by definition of $[x]$. If $y \in [x]$, then note $y \in [y]$ as well, so $[x] = [y]$ is forced by uniqueness, so $x \in [y]$. If $y \in [x]$ and $z \in [y]$, then again by uniqueness $[x] = [y] = [z]$, so $z \in [x]$ follows.

We will not check that $f|_A$ is actually a function; it is, roughly speaking inherited from $f$.

**Definition 1.22.** Given two functions $f : X \to Y$ and $g : Y \to Z$, we define the *composition* of $f$ and $g$ to be some functino $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) := g(f(x)).$$

Again, we will not check that this makes a function; it is.

Functions can also help create new sets.

Image

**Definition 1.23** (Image). Given a function $f : X \to Y$, we define the *image* of $f$ to be

$$\operatorname{im} f = f(X) := \{y \in Y : \text{there is } x \in X \text{ such that } f(x)y\}.$$

Namely, $\operatorname{im} f$ consists of all elements hit by someone in $X$ hit by $f$.

Fiber, pre-image

**Definition 1.24** (Fiber, pre-image). Given a functino $f : X \to Y$ and some $y \in Y$, we define the *fiber* of $f$ over $y$ to be

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

In general, we define the *pre-image* of a subset $A \subseteq X$ to be

$$f^{-1}(A) := \{x \in A : f(x) \in A\} = \bigcup_{a \in A} \{x \in A : f(x) = a\} = \bigcup_{a \in A} f^{-1}(a).$$

Some functions have nicer properties than others.

Inj-, sur-, bijective

**Definition 1.25** (Inj-, sur-, bijective). Fix a function $f : X \to Y$. We have the following.

- Then $f$ is *injective* or *one-to-one* if and only if, given $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- Then $f$ is *surjective* or *onto* if and only if $\operatorname{im} f = Y$. In other words, for each $y \in Y$, there exists $x \in X$ with $f(x) = y$.
- Then $f$ is *bijective* if and onlt if it is both injective and surjective.

Here is an example.

Identity

**Definition 1.26** (Identity). For a given set $X$, the function $\operatorname{id}_X : X \to X$ defined by $\operatorname{id}_X(x) := x$ is called the *identity function*.

For completeness, here are the checks that $\operatorname{id}_X$ is bijective.

- Injective: given $x_1, x_2 \in X$, we see $\operatorname{id}_X(x_1) = \operatorname{id}_X(x_2)$ implies $x_1 = \operatorname{id}_X(x_1) = \operatorname{id}_X(x_2) = x_2$.
- Surjective: given $x \in X$, we see that $x \in \operatorname{im} \operatorname{id}_X$ because $x = \operatorname{id}_X$.

We leave with some lemmas, to be proven once in one's life.

**Lemma 1.27.** Fix a finite sets $X$ and $Y$ such that $\#X = \#Y$. Then a function $f : X \to Y$ is bijective if and only if it is injective or surjective.

*Proof.* Certainly if $f$ is bijective, then it is both injective and surjective, so there is nothing to say.

The reverse direction is harder. We proceed by induction on $\#X = \#Y$. If $\#X = \#Y = 0$, then $X = Y = \varnothing$, and all functions $f : \varnothing \to \varnothing$ are vacuously bijective: for injective, note that any $x_1, x_2 \in \varnothing$ have $x_1 = x_2$; for surjective, note that any $x \in \varnothing$ has $f(x) = x$.

Otherwise $\#X = \#Y > 0$. We have two cases.

- Take $f$ injective; we show $f$ is surjective. In this case, $\#X > 0$, so choose some $a \in X$. Note that $x \in X$ with $x \neq a$ will have $f(x) \neq f(a)$ by injectivity, so we may define the restriction

$$f|_{X \setminus \{a\}} : X \setminus \{a\} \to Y \setminus \{f(a)\}.$$

  Observe that $f|_{X\setminus\{a\}}$ is injective because $f$ is: if $x_1, x_2 \in X \setminus \{a\}$ have

$$f(x_1) = f|_{X\setminus\{a\}}(x_1) = f|_{X\setminus\{a\}}(x_2) = f(x_2),$$

  then $x_1 = x_2$ follows.

  Now, $\#(X \setminus \{a\}) = \#(Y \setminus \{f(a)\}) = \#X - 1$, so by induction $f|_{X\setminus\{a\}}$ will be bijective because it is injective. In particular, $f$ by way of $f|_{X\setminus\{a\}}$ fully hits $Y \setminus \{f(a)\}$ in its image, so because $f(a) \in \operatorname{im} f$ as well, we conclude $\operatorname{im} f = Y$. So $f$ is surjective.

- Take $f$ surjective; we show $f$ is injective. Define a functiono $g : Y \to X$ as follows: for each $y \in Y$, the surjectivity of $f$ promises some $x \in X$ such that $f(x) = y$, so choose any such $x$ and define $g(y) := x$.[2] Observe that $f(g(y)) = y$ by construction.

  Now, we notice that $g$ is injective: if $y_1, y_2 \in Y$ have $g(y_1) = g(y_2)$, then $y_1 = f(g(y_1)) = f(g(y_2)) = y_2$. So the previous case tells us that $g$ is in fact bijective.

  So now choose any $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. The surjectivity of $f$ promises some $y_1, y_2 \in Y$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$, so we see that

$$x_1 = g(y_1) = g(f(g(y_1))) = g(f(x_1)) = g(f(x_2)) = g(f(g(y_2))) = g(y_2) = x_2,$$

  proving our injectivity. ■

**Lemma 1.28.** Fix $f : X \to Y$ a bijective function. Then there is a unique function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

*Proof.* We show existence and uniqueness separately.

- We show existence. Note that, because $f : X \to Y$ is surjective, each $y \in Y$ has some $x \in X$ such that $f(x) = y$. In fact, this $x \in X$ is uniquely defined because $f(x_1) = f(x_2)$ implies $x_1 = x_2$, so we may define $g(y)$ as the value $x$ for which $f(x) = y$.

  By construction, $f(g(y)) = y$, so $f \circ g = \mathrm{id}_Y$. Additionally, we note that, given any $x \in X$, the value $x_0$ for which $f(x) = f(x_0)$ is $x = x_0$ by the injectivity, so $g(f(x)) = x$. Thus, $g \circ f = \mathrm{id}_X$, as claimed.

- We show uniqueness. Suppose that we have two functions $g_1, g_2 : Y \to X$ which satisfy

$$f \circ g_1 = f \circ g_2 = \mathrm{id}_Y \qquad \text{and} \qquad g_1 \circ f = g_2 \circ f = \mathrm{id}_X\,.$$

  Then we see that

$$g_1 = g_1 \circ \mathrm{id}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathrm{id}_X \circ g_2 = g_2,$$

  where we have used the fact that function composition associates. This finishes. ■

## 1.3 January 24

Good morning everyone.

[2] Technically we are using the Axiom of Choice here. One can remove this with an induction because all sets are finite, but I won't bother.

### 1.3.1 Algebraic Structure

Today we are reviewing the complex numbers (reportedly, “some basics”). Or at least it is hopefully mostly review. Here is our main character this semester.

Complex numbers

**Definition 1.29** (Complex numbers)**.** The set $\mathbb{C}$ of *complex numbers* is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Here $i$ is some symbol such that $i^2 = -1$ formally.

In particular, two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

The complex numbers also have some algebraic structure.

$+$ and $\times$ in $\mathbb{C}$

**Definition 1.30** ($+$ and $\times$ in $\mathbb{C}$)**.** Given complex numbers $a_1 + b_1 i, a_2 + b_2 i \in \mathbb{C}$, we define

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i,$$

and

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i,$$

defined essentially by direct expansion, upon recalling $i^2 = -1$.

Here is the corresponding algebraic structure.

**Proposition 1.31.** The set $\mathbb{C}$ with the above operations is a two-dimensional $\mathbb{R}$-vector space with basis $\{1, i\}$.

*Proof.* The elements $\{1, i\}$ span $\mathbb{C}$ because all complex numbers in $\mathbb{C}$ can be written as $a + bi = a \cdot 1 + b \cdot i$ by definition.

To see that these elements are linearly independent, suppose $a + bi = 0$. If $b = 0$, then $a = 0$ follows, and we are done. Otherwise, take $b \neq 0$, but then we see $(-a/b) = i$, so

$$(-a/b)^2 = -1 < 0,$$

which does not make sense for real numbers. This finishes. ■

**Proposition 1.32.** The set $\mathbb{C}$ with the above operations is a field.

*Proof.* We have the following checks.

- The element $0 + 0i$ is our additive identity. Indeed, one can check that $(0 + 0i) + (a + bi) = (a + bi) + (0 + 0i) = a + bi$.
- The element $1 + 0i$ is our multiplicative identity. Indeed, one can check that $(1 + 0i)(a + bi) = (a + bi)(1 + 0i) = a + bi$.
- Commutativity of addition and multiplication follow from by expansion.
- The distributive laws can again be checked by expansion.
- The additive inverse of $a + bi$ is $(-a) + (-b)i$.
- The multiplicative inverse of $a + bi$ can be found by wishing really hard and writing

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

Then one can check this works. ■

Sometimes we would like to extract our coefficients from our basis.

$\operatorname{Re}$ and $\operatorname{Im}$

**Definition 1.33** ($\operatorname{Re}$ and $\operatorname{Im}$)**.** Given $z := a + bi \in \mathbb{C}$, we define the operations

$$\operatorname{Re} z := a \qquad \text{and} \qquad \operatorname{Im} z := b.$$

Importantly, $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$.

Because we are merely doing basis extraction, it makes sense that these operations will preserve some (additive) structure.

**Proposition 1.34.** Fix $z = a + bi$ and $w = c + di$. Then the following.

(a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$.

(b) $\operatorname{Im}(z + w) = \operatorname{Im} z + \operatorname{Im} w$.

*Proof.* We proceed by direct expansion. Observe

$$\operatorname{Re}(z + w) = \operatorname{Re}((a + c) + (b + d)i) = a + c = \operatorname{Re} z + \operatorname{Re} w,$$

and

$$\operatorname{Im}(z + w) = \operatorname{Im}((a + c) + (b + d)i) = b + d = \operatorname{Im} z + \operatorname{Im} w.$$

This finishes. ■

It also turns out that the complex numbers have a very special transformation.

Conjugate

**Definition 1.35** (Conjugate)**.** Given $z := a + bi \in \mathbb{C}$, we define the *complex conjugate* to be $\overline{z} := a - bi \in \mathbb{C}$.

We promised conjugation would be special, so here are some special things.

**Proposition 1.36.** Fix $z = a + bi \in \mathbb{C}$. Then the following.

(a) $z + \overline{z} = 2 \operatorname{Re} z$.

(b) $z - \overline{z} = 2i \operatorname{Im} z$.

(c) $\overline{\overline{z}} = z$.

*Proof.* We take these one at a time.

(a) Write $a + bi + \overline{a + bi} = a + bi + a - bi = 2a$.

(b) Write $a + bi - \overline{a + bi} = a + bi - (a - bi) = 2bi$.

(c) Write $\overline{\overline{a + bi}} = \overline{a - bi} = a + bi$. ■

In fact, more is true.

**Proposition 1.37.** Fix $z = a + bi \in \mathbb{C}$ and $w = c + di \in \mathbb{C}$. Then the following.

(a) $\overline{z + w} = \overline{z} + \overline{w}$.

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$.

*Proof.* We take these one at a time.

- Write
$$\overline{z+w} = (a+c) - (b+d)i = (a-bi) + (c-di) = \overline{z} + \overline{w}.$$
- Write
$$\begin{aligned}\overline{z}\cdot\overline{w} &= (a-bi)(c-di)\\ &= (ac-bd) - (ad+bc)i\\ &= \overline{(ac-bd)+(ad+bc)i}\\ &= \overline{zw}.\end{aligned}$$

This finishes. ■

### 1.3.2 Defining Distance

Complex conjugation actually gives rise to a notion of size.

Norm on $\mathbb{C}$

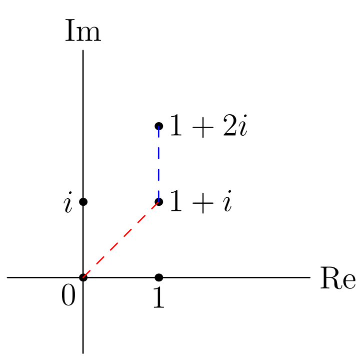
**Definition 1.38** (Norm on $\mathbb{C}$)**.** Given $z := a+bi$, we define the *norm function on* $\mathbb{C}$ by
$$|z| := \sqrt{a^2+b^2}.$$

Size actually gives distance.

Distance on $\mathbb{C}$

**Definition 1.39** (Distance on $\mathbb{C}$)**.** Given complex numbers $z = a+bi$ and $w = c+di$, we define the *distance* between $z$ and $w$ to be
$$|z-w| = \sqrt{(a-c)^2+(b-d)^2}.$$

Here are some examples.




One can ask what is the distance between $0+0i$ and $1+1i$, and we can compute directly that this is $\sqrt{1+1} = \sqrt{2}$. Similarly, the distance between $1+2i$ and $1+i$ is $|(1+2i)-(1+i)| = |i| = 1$. It should agree with our geometric intuition.

We mentioned complex conjugation is involved here, so we have the following lemma.

**Lemma 1.40.** Fix $z, w \in \mathbb{C}$. The following are true.

(a) $|z|^2 = z\overline{z}$.

(b) $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.

(c) $|z| = |\overline{z}| = |-z|$.

(d) $|z| = 0$ if and only if $z = 0$.

(e) $|zw| = |z| \cdot |w|$.

*Proof.* We take these one at a time. Set $z = a + bi$.

(a) We have
$$|z|^2 = a^2 + b^2 = (a+bi)(a-bi) = z\overline{z}.$$
Here we have used subtraction of two squares, which one can see when writing $a^2 + b^2 = a^2 - (ib)^2$.

(b) We have $a^2 \le a^2 + b^2$ and $b^2 \le a^2 + b^2$ by the Trivial inequality, so
$$|\operatorname{Re} z| = |a| \le \sqrt{a^2 + b^2} = |z|,$$
and similarly,
$$|\operatorname{Im} z| = |b| \le \sqrt{a^2 + b^2} = |z|.$$

(c) Note
$$|\overline{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|,$$
and
$$|-z| = |-a - bi| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(d) From (b), we know that $|\operatorname{Re} z|, |\operatorname{Im} z| \le |z|$, but $|z| = 0$ then forces $\operatorname{Re} z = \operatorname{Im} z = 0$, so $z = 0$.

(e) From (a), we can write $|zw|^2 = zw \cdot \overline{zw}$, which will expand out into
$$z \cdot w \cdot \overline{z} \cdot \overline{w}.$$
We can collect this into $z\overline{z} \cdot w\overline{w} = |z|^2|w|^2$. Thus, by (a) again, $|zw|^2 = |z|^2|w|^2$.But because all norms must be nonnegative real numbers, we may take square roots to conclude $|zw| = |z| \cdot |w|$. ■

**Remark 1.41.** Norms are actually more general constructions. For example, the requirement $|zw| = |z| \cdot |w|$ makes $|\cdot|$ into a "multiplicative" norm.

To finish off, we actually show that our distance function is good: we show the triangle inequality.

**Lemma 1.42** (Triangle inequality)**.** For every $x, y, z \in \mathbb{C}$, we claim
$$|z - x| \le |z - y| + |y - z|.$$

This claim should be familiar from real analysis. Intuitively, it means that travelling between $z$ and $x$ cannot be made into a shorter trip by taking a detour to some other point $y$ first.

*Proof.* Let $a := z - y$ and $b := y - z$ so that $a + b = z - x$. Thus, we are showing that
$$|a + b| \overset{?}{\le} |a| + |b|,$$

which is nicer because it only has two letters. For this, because everything is a nonnegative real numbers, it suffices to show the square of this requirement; i.e., we show

$$(|a|+|b|)^2-|a+b|^2 \overset{?}{\geq} 0.$$

Fully expanding, it suffices to show

$$|a|^2+|b|^2+2|a|\cdot|b|-|a+b|^2 \overset{?}{\geq} 0.$$

Expanding out $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$, we are showing

$$a\overline{a}+b\overline{b}+2|a|\cdot|b|-(a+b)(\overline{a}+\overline{b}) \overset{?}{\geq} 0.$$

This is nice because the expansion of the rightmost term will induce some cancellation: it expands into $a\overline{a}+a\overline{b}+\overline{a}b+b\overline{b}$, so we are left with showing

$$2|a|\cdot|b|-(a\overline{b}+b\overline{a}) \overset{?}{\geq} 0.$$

Note that $\overline{a}b = \overline{a\overline{b}}$, so we can collect the final term as $2\operatorname{Re}(a\overline{b})$. Similarly, we can write $|a|\cdot|b| = |a|\cdot|\overline{b}| = |a\overline{b}|$, so we are showing

$$2|a\overline{b}|-2\operatorname{Re}(a\overline{b}) \geq 0,$$

which is true because the real part does exceed the norm. This finishes. ■