18.757: Representation Theory of Lie Groups

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Fall 2025

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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Introduction

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Welcome to the class.

1.1.1 Administrative Notes

Here are some administrative notes.

- There will be problem sets every two weeks, due on Fridays. They are not expected to be too time-consuming.
- Technically, this course is a sequel to 18.745–18.755, but one can get away with a bit less. In particular, we will assume familiarity with some basic notions in Lie theory, things about simple complex Lie algebras (as related to compact Lie groups), the theory of roots and weights, and this theory of finite-dimensional representations. For example, things like the Peter–Weyl theorem may come up.
- We will largely follow Etingof's lecture notes.

This course is about the representations of Lie groups, especially those which are not necessarily compact. For example, we may focus on real reductive Lie groups such as $\mathrm{SL}_n(\mathbb{R})$, and there is a new feature here that we must care about infinite-dimensional representations.

One of our motivations comes from quantum physics, where one finds groups acting on infinite-dimensional Hilbert spaces. Another motivation is number-theoretic: one uses this theory to set up the archimedean theory of automorphic forms.

1.1.2 Finite Groups

Let's recall some background. As one does, let's begin with the representation theory of finite groups. We split this into a few theorems.

Theorem 1.1 (Maschke)**.** Let G be a finite group. Then every finite-dimensional complex representation is unitary (by averaging any given Hermitian form) and semisimple.

Theorem 1.2 (Peter–Weyl)**.** Let G be a finite group. Then there is a decomposition

$$\mathbb{C}[G] = \bigoplus_{V \in \mathrm{IrRep}(G)} \mathrm{End}_{\mathbb{C}}(V)$$

of $\mathbb{C}$-algebras. It follows that the characters $\{\mathrm{tr}_V\}_{V\in\mathrm{IrRep}(G)}$ form an orthonormal basis of the class functions $G \to \mathbb{C}$.

Remark 1.3. Let G be a finite group. For $f \in \mathbb{C}[G]$, there is a dimension formula

$$f(1) = \frac{1}{|G|} \sum_{V \in \mathrm{IrRep}(G)} \dim V_i \cdot \mathrm{tr}_{V_i} f.$$

Indeed, this follows from writing $\langle \varphi, \psi \rangle = \sum_V \mathrm{tr}_V(\varphi\psi')$ (where $\psi' \colon g \mapsto \psi(g^{-1})$) and then noting that $\langle \varphi, \psi' \rangle = \frac{1}{|G|}(\varphi * \psi')(1)$.

1.1.3 Compact Groups

We now move up to the representation theory of compact connected Lie groups. Here is a generalization of Maschke's theorem.

Theorem 1.4 (Weyl's unitarian trick)**.** Let G be a compact Lie group. Then the representations of G are semisimple, and the irreducible representations of G are finite-dimensional and unitary.

In order to compute the representations of G, one wants to pass to the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. One needs to be slightly careful about this.

Proposition 1.5. Fix a Lie group G with Lie algebra $\mathfrak{g}$. Consider the functor $F \colon \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$ sending a representation ρ to the differential representation $d\rho_1$.

(a) If G is connected, then F is fully faithful.

(b) If G is connected and simply connected, then F is also essentially surjective and hence an equivalence.

Example 1.6. Take $G = \mathrm{U}(1)$. Because G is abelian, all irreducible representations are one-dimensional. These representations $\mathrm{U}(1) \to \mathbb{C}^\times$ are indexed by $n \in \mathbb{Z}$, given by $z \mapsto z^n$. Note that these are not in bijection with the representations of $\mathrm{Lie}\, G$ because G is not simply connected!

Example 1.7. Take $G = \mathrm{SU}(n)$ so that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. A weight is a character of the maximal torus T, for which we can take to be the subgroup of diagonal matrices. Explicitly,

$$T = \{\mathrm{diag}(z_1, \ldots, z_n) : z_1 \cdots z_n = 1\},$$

so the weight lattice is $\mathbb{Z}^n/\mathbb{Z}(1,\ldots,1)$, and a weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is dominant when the entries are increasing.

Example 1.8. For example, with $\mathrm{SU}(2)$, we have an isomorphism $\mathbb{Z}^2/\mathbb{Z}(1,1) \to \mathbb{Z}$ given by $(\lambda_1, \lambda_2) \mapsto \lambda_2 - \lambda_1$, and the dominant weights are the nonnegative integers. One finds that weight n corresponds to the nth symmetric power of the standard representation of $\mathrm{SU}(2)$.

Here is a generalization of the Peter–Weyl theorem.

Theorem 1.9 (Peter–Weyl)**.** Let G be a compact Lie group. The canonical map

$$\bigoplus_{\text{dominant } \lambda} V_\lambda \otimes V_\lambda^* \to C^\infty(G)$$

is an embedding with dense image; here V_λ refers to the irreducible representation corresponding to the dominant weight λ.

Example 1.10 (Fourier analysis)**.** With $G = \mathrm{U}(1)$, then one can calculate that $V_n \otimes V_n^* \to C^\infty(G)$ has image given by the nth power map $\mathrm{U}(1) \to \mathrm{U}(1)$. Thus, we are asserting that the collection of such polynomials are dense in the collection of all smooth functions $\mathrm{U}(1) \to \mathbb{C}$. If we identify $\mathrm{U}(1)$ with $\mathbb{R}/\mathbb{Z}$ via the exponential map, then this is asserting that the exponentials $z \mapsto e^{2\pi i n z}$ have dense span in the collection of all smooth functions $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$.

One can characterize the image of the Peter–Weyl map.

Definition 1.11 (finite)**.** A function $f \in C^\infty(G)$ is G-*finite* if and only if the span of $\{gf : g \in G\}$ is finite-dimensional. We let $C_{\text{fin}}(G)$ be the space of G-finite functions.

Remark 1.12. It turns out that the map

$$\bigoplus_{\text{dominant } \lambda} V_\lambda \otimes V_\lambda^* \to C^\infty(G)$$

has image given by the space of G-finite functions. (This is often proven as an input to the proof of the Peter–Weyl theorem; it is much easier to show!)

Remark 1.13. The vector space $C_{\text{fin}}(G)$ has two ring structures: there is pointwise multiplication (in $\mathbb{C}$) and also convolution given by

$$(\varphi * \psi) \colon g \mapsto \int_G \varphi(x)\psi\left(x^{-1}g\right)\, dx,$$

where dx is a Haar measure for G normalized so that $\int_G dx = 1$. (Because G is compact, dx is bi-invariant.)

Remark 1.14. The convolution algebra is non-unital, so one sometimes upgrades to the algebra of distributions, where we have the unit δ_1. Similar remarks hold for $C^\infty(G)$ and even $C(G)$.

Remark 1.15 (algebraic groups)**.** In fact, $C_{\text{fin}}(G)$ also has a comultiplication given by pulling back along the multiplication map $m \colon G \times G \to G$. Namely, the comultiplication is the composite

$$C_{\text{fin}}(G) \stackrel{m^*}{\to} C_{\text{fin}}(G \times G) = C_{\text{fin}}(G) \otimes C_{\text{fin}}(G).$$

Thus, we have a Hopf algebra, which allows us to associate a complex algebraic group G_{alg} to G, and it turns out that unitary representations of G all arise from algebraic representations of G_{alg}. Conversely, one can take a complex algebraic group G_{alg} and then find a maximal compact subgroup $G \subseteq G_{\text{alg}}(\mathbb{C})$ which is unique up to conjugacy.

1.1.4 Unitary Representations

We will be interested in the unitary representations of Lie groups G, which we no longer assume to be compact.

Remark 1.16. If G is simple and not compact, then all unitary representations are infinite-dimensional. Proceeding by contraposition, suppose that G is simple and admits a finite-dimensional unitary representation $\rho\colon G \to \mathrm{SU}(n)$. Because G is simple, this is an embedding, and because $\mathrm{SU}(n)$ is compact, we conclude that G must then also be compact.

Thus, we see that we will be interested in infinite-dimensional representations. Of course, one still must add in topologies everywhere, though this point is more technical now that our vector spaces are not finite-dimensional. For example, for unitary representations, we are looking for actions of G on Hilbert spaces, though we will find occasion to look at more general topological vector spaces.

Our main source of examples of representations arise from more general group actions.

Example 1.17. If G acts on a "geometric space" X, then we receive an induced action of G on classes of functions on X. For example, when X is a reasonably nice topological space, then we can think about G acting on $L^2(X)$; when X is a manifold, we can think about G acting on $C^\infty(X)$.

Example 1.18. The action of G on G itself by left multiplication gives rise to some "regular" representations.

Example 1.19. Many matrix groups such as $\mathrm{SL}(n,\mathbb{R})$ admit standard group action on a vector space. Note that this standard action may not be unitary!

Let's begin building our language.

Definition 1.20 (subrepresentation, irreducible)**.** Fix a group G and topological vector space V. If $\rho\colon G \to \mathrm{GL}(V)$ is a continuous representation, then a *subrepresentation* is a closed subspace $W \subseteq V$ which is G-invariant. We say that ρ is *irreducible* if and only if there are no proper nontrivial subspaces.

Note the hypothesis that $W \subseteq V$ is closed for our subrepresentations!

Example 1.21. The action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathbb{R}^n$ has no nontrivial proper subrepresentations and hence is irreducible. However, this representation is not unitary.

Remark 1.22. The action of G on itself makes $L^2(G)$ a representation of G. However, this representation frequently fails to be irreducible. For example, $L^2(G)$ has many automorphisms, so it cannot be irreducible by a suitable version of Schur's lemma. In some cases, we can see this more concretely: taking $G=\mathbb{R}$, then we know $\mathbb{R}$ is isomorphic to its dual, so $L^2(\mathbb{R}) \cong L^2\left(\mathbb{R}^{2\vee}\right)$, and this right-hand side has more obvious subrepresentations given by the subspace of functions which vanish on a given subset of positive measure.

1.2 September 9

Today, we will discuss some general nonsense of topological vector spaces.

1.2.1 Examples of Representations

Last time, we ended with the following example, which we recall here.

Example 1.23 (Heisenberg group)**.** Fix a positive integer $n \geq 1$. Then we define the Heisenberg group H_n as the matrix group

$$H_n := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{bmatrix} : a \in \mathbb{R}^{1\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R} \right\}.$$

It turns out that H_n admits a natural action on $L^2(\mathbb{R}^n)$, which is an irreducible representation. Quickly, the a-coordinate will act by translation on the $\mathbb{R}^n$, and the b-coordinate will act by a character $b \mapsto e^{2\pi i\langle b,-\rangle}$. One finds a similar action on $C^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$.

Check it

Remark 1.24. There is a finite analogue, where all the $\mathbb{R}$s are replaced with a finite field $\mathbb{F}_p$, and the character $b \mapsto e^{2\pi i\langle b,-\rangle}$ is replaced with $b \mapsto e^{2\pi i\langle b,-\rangle/p}$. Equivalently, we find $H_n(\mathbb{F}_p) = \mathbb{F}_p \times V \times V^\vee$ (where $V = \mathbb{F}_p^n$); then $H_n(\mathbb{F}_p)$ admits a natural action on V, where V acts by translation, and $\xi \in V^\vee$ acts by multiplying with the function $\psi_\xi(x) := e^{2\pi i\langle x,\xi\rangle/p}$. The central $\mathbb{F}_p \subseteq H$ now acts by scalar multiplication as $a \mapsto e^{2\pi i a/p}$.

Let's check that this representation is irreducible. It is enough to check that $\operatorname{End}_{H_n}(\mathbb{C}[V])$ is $\mathbb{C}$. Well, commuting with the V leaves us with

$$\operatorname{End}_V(\mathbb{C}[V]) = \operatorname{End}_{\mathbb{C}[V]}(\mathbb{C}[V]) = \mathbb{C}[V].$$

Further, one sees that commuting with the scalar multiplication by elements in $V^\vee$ restricts the possible endomorphisms all the way down to scalars.

Example 1.23 appears in the book, and the argument is not too different from the one given in the remark.

Example 1.25. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ and therefore has a unitary representation on $L^2(\mathbb{R}^2)$. This cannot possibly be irreducible because $\mathrm{SL}_2(\mathbb{R})$ commutes with the extra scalar action on $\mathbb{R}^2$, so $L^2(\mathbb{R}^2)$ has too many endomorphisms. To make this representation smaller, we can choose $s \in \mathbb{C}$, which produces a character on $\mathbb{R}^+$ by $\chi_s : t \mapsto t^s$; then we can define $L^2(\mathbb{R}^2, \chi_s)$ to be the functions with commute with this character (namely, $f(t^{-1}x) = t^s f(x)$). (Geometrically, this is basically the sections of a line bundle on $\mathbb{RP}^1$ given by the character.) We will soon see that almost all s produces an irreducible representation.

For example, for $s = 0$, then $L^2(\mathbb{R}^2, \chi_0)$ consists of the functions on $\mathbb{RP}^1$. This is not irreducible because it has a subrepresentation given by the constant functions. But $L^2(\mathbb{R}^2, \chi_0)/(\mathbb{C}\cdot 1)$ is still not irreducible: it turns out to be the sum of two irreducible representations L^+ and L^-, where L^+ is the closure of $z\mathbb{C}[z]$, and L^- is the closure of $z^{-1}\mathbb{C}[z^{-1}]$, where z is a standard coordinate on $\mathbb{RP}^1$. This can be related to Fourier series by embedding $\mathbb{RP}^1$ into $\mathbb{CP}^1$, which is basically a circle. We will prove all these claims later.

Remark 1.26. Here is an amusing way to view $L^2(\mathbb{R}^2, \chi_s)$: this amounts to sections of a line bundle on $\mathbb{RP}^1$, and after removing ∞, we see that we are looking at functions on $\mathbb{R}$. One can check that these are the functions which transform by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} f(z) = f\left(\frac{az+b}{cz+d}\right) |cz+d|^s.$$

Fix a Lie group G acting on an orientable manifold X. If G preserves a volume form dx on X, then $C_c^\infty(X)$ will have an invariant pairing

$$\langle \varphi, \psi \rangle := \int_X \varphi(x)\overline{\psi(x)}\, dx.$$

More generally, one can work with half-densities.

Definition 1.27 (density)**.** An s-density on a smooth manifold X is a section of a line bundle whose sections on an affine patch are just functions but which has transformations between coordinate charts $(x_n) \mapsto (x_n')$ given by

$$f(x_1, \ldots, x_n) \mapsto f(x_1', \ldots, x_n') \left| \det \left(\frac{\partial x_i'}{\partial x_j} \right) \right|^s.$$

The point of working with half-densities is that we can define the standard inner product between them in the usual way.

Example 1.28. Half-densities of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{RP}^1$ (in the obvious way) amounts to considering $L^2(\mathbb{R}^2, \chi_{-1})$.

1.2.2 Topological Vector Spaces

We spend a moment reviewing what we need about locally convex topological spaces; we refer to Appendix A for a more in-depth treatment.

Convention 1.29. All topological vector spaces are over $\mathbb{C}$ and are Hausdorff.

Definition 1.30 (locally convex)**.** A topological vector space V is *locally convex* if and only if 0 has an open neighborhood basis of convex sets.

Remark 1.31. Equivalently, by Corollary A.23 a topological vector space V if and only if its topology is generated by a collection of seminorms.

All representations in this class will actually be given by "Fréchet spaces."

Definition 1.32 (Fréchet)**.** A topological vector space V is *Fréchet* if and only if it is locally convex, has a countable basis of neighborhoods of 0, and is sequentially complete.

Remark 1.33. By Proposition A.27, having a countable basis of neighborhoods of 0 is equivalent to being metrizable. (In fact, one can choose the metric to be translation-invariant.) Once V is metrizable, being sequentially complete is equivalent to being complete.

We will also frequently take our vector spaces V to be separable.

Definition 1.34 (separable)**.** A topological space X is *separable* if and only if it admits a countable basis.

Convention 1.35. In this course, all Fréchet spaces are separable unless otherwise specified.

Non-Example 1.36. For $p < 1$, the space $L^p([0,1])$ fails to be locally convex. In fact, the only open nonempty convex subset is the whole space!

Example 1.37. For a topological space X, let $C(X)$ be the space of continuous functions, where $\{f_i\} \to f$ if and only if we have uniform convergence on compact sets. If X is (Hausdorff) compact, then $C(X)$ is a Banach space (given by $\|\cdot\|_\infty$). However, if X is merely a (Hausdorff, second countable) locally compact topological space, then $C(X)$ is merely a Fréchet space: write X as a countable union $\bigcup_i K_i$ of compact sets, and then we can use the seminorms $\|\cdot|_{K_i}\|_\infty$. We refer to Example A.55 for details.

Remark 1.38. We may also write $C_0(X)$ for $C(X)$, for no particular reason.

Example 1.39. If X is a manifold, then we can consider the topological space $C^k(X)$, where $\{f_i\} \to f$ converges if and only if the first k derivatives of these functions converge uniformly on compact sets. Similarly, we see that $C^k(X)$ is Banach when k is finite and X is compact; otherwise, it is merely Fréchet. (The seminorms are now given by $\|\partial^\alpha(\cdot)|_K\|_\infty$ as K varies over a covering collection of compact closed balls.) The argument is basically the same as the one in Example A.55, so we omit it. The same sort of argument shows that the space $S(\mathbb{R}^n)$ of Schwartz functions is Fréchet.

Non-Example 1.40. If X fails to be compact, then the subspace $C_c^\infty(X)$ of $C^\infty(X)$ may fail to be a Fréchet space because it fails to be complete.

Sometimes, we will find ourselves in a circumstance where we can restrict to a nice class of spaces.

Definition 1.41 (Banach)**.** A topological vector space V is a *Banach space* if and only if its topology is given by a norm, and it is complete with respect to that norm.

For example, Hilbert spaces are Banach spaces.

Here is one benefit of working with a Banach space.

Lemma 1.42. Fix a topological group G acting on a Banach space V. Then the action $G \times V \to V$ is continuous if and only if the induced map $\rho\colon G \to \operatorname{Aut}(V)$ is continuous in the strong topology, in which $\{E_i\} \to E$ if and only if $\{E_i v\} \to Ev$ converges for all $v \in V$.

Proof. The forward direction has little content: given a net $\{g_i\} \to g$, we know that $\rho(g_i)v \to \rho(g)v$ for each v by continuity, so it follows that $\rho(g_i) \to \rho(g)$.

The reverse direction follows from the Uniform boundedness principle, which claims that $|E_i v|$ being bounded for every v implies that $\|E_i\|$ is bounded. Namely, to show that $G \times V \to V$ is continuous, we choose a net $\{(g_i, v_i)\} \to (g, v)$ in $G \times V$, and we would like to show that $g_i v_i \to gv$. By translation, we may assume that $g = 1$. We are given that $\rho(g_i) \to 1$ in the strong topology, which implies that $\rho(g_i)w \to w$ for all w, so we may also assume that $v = 0$ by translation. Because now $v_i \to 0$, it is enough to show that $\|\rho(g_i)\|$ is bounded, which is what follows from the Uniform boundedness principle. ■

Remark 1.43. A Banach space V can alternatively give $\operatorname{End} V$ the norm topology, but then the map $G \to \operatorname{Aut} V$ need not be continuous with the norm topology. For example, the action of $\mathbb{R}$ on $L^2(\mathbb{R})$ by translation $T_a f(x) := f(x+a)$ fails to be continuous: as $a \to 0$, we see $T_a \to \mathrm{id}$, but $\|T_a - \mathrm{id}\| = 1$ for all $a \neq 0$.

1.3 September 11

Today, we work with measures.

1.3.1 Partition of Unity

It will be helpful to have the following incarnation of partition of unity.

Lemma 1.44 (Partition of unity). Let X be metric space. For any countable open cover $\{U_i\}_{i\in\mathbb{N}}$ of pre-compact open subsets of X, there are continuous functions $\{f_i\}_{i\in\mathbb{N}}$ from X to $[0,1]$ such that $f_i|_{X\setminus U_i}=0$ and

$$\sum_{i\in\mathbb{N}} f_i = 1.$$

Proof. Let's begin by proving a version of Urysohn's lemma: for any nonempty closed subset $A\subseteq X$, the function $d_A\colon X\to\mathbb{R}$ given by

$$d_A(x) := \inf_{a\in A} d(x,a)$$

is well-defined and (Lipschitz) continuous because $d_A(x)\le d_A(y)+d(x,y)$ for any $x,y\in X$. Furthermore, we see that $d_A(x)=0$ if and only if $x\in A$: if $d_A(x)=0$, then there is a sequence of points $\{x_i\}$ in A for which $d(x,x_i)\to 0$, but this implies that $x_i\to x$, so $x\in A$ because A is closed.

Now, for each i, we start with the function $g_i := d_{A_i}/(1+d_{A_i})$ where $A_i := X\setminus U_i$, so we see that $\operatorname{im} g_i\subseteq[0,1]$, and $g_i(x)=0$ if and only if $x\notin U_i$. We now define $G(x):=\sum_i 2^{-i}g_i$ and set $f_i := 2^{-i}g_i/G$. The function G is continuous by the Weierstrass M-test, and it is always positive because the U_is cover X; thus, the functions f_i are well-defined, positive, and continuous. Lastly, we see that

$$\sum_{i\in\mathbb{N}} f_i = \frac{1}{G}\sum_{i\in\mathbb{N}} 2^{-i}g_i,$$

which is 1 by construction of G. ∎

Because we will have occasion to retreat to topological spaces instead of manifolds, we pick up the following result.

Lemma 1.45. Let X be a locally compact second countable (Hausdorff) topological space X. Then X is metrizable.

Proof. By the Urysohn metrization theorem, it is enough to check that X is Hausdorff, second countable, and regular. However, regularity follows from being locally compact and Hausdorff. ∎

1.3.2 Measures on a Space

Let's say a bit about measures.

Definition 1.46 (measure). Fix a locally compact second countable topological space X. Then a *measure* is an element of the topological dual $C(X)^*$. In other words, it is a continuous linear functional on $C(X)$. Given $\mu\in C(X)^*$, we may write $\int_X f\,d\mu$ for $\mu(f)$. We denote this space by $\operatorname{Meas}_c(X)$, and give it the weak topology, where $\{\mu_i\}\to\mu$ if and only if $\{\mu_i(f)\}\to\mu(f)$ for all $f\in C(X)$.

Remark 1.47. In other words, the topology is given by the family of seminorms $\mu\mapsto|\mu(f)|$. Thus, $C(X)^*$ is Hausdorff and locally convex by Corollary A.23.

Example 1.48. Suppose that X is an orientable manifold with volume form ω. Then $C_c(X)$ embeds into $C(X)^*$ by sending $g \in C_c(X)$ to the functional

$$g\,\omega\colon f \mapsto \int_X fg\,\omega.$$

This is linear by the linearity of the integral, so it remains to check that is continuous; by Corollary A.42, it is enough to show that $|g\,\omega|$ is bounded above by a continuous seminorm. Let K be the support of g, which is compact, and we claim that $g\,\omega \le \left(\|g\|_\infty \int_K \omega\right) \|\cdot|_K\|_\infty$, which will be enough. Well, for any f, we see $g\,\omega(f)$ is bounded by

$$\left|\int_K fg\,\omega\right| \le \|g\|_\infty \int_K \omega \cdot \|f|_K\|_\infty .$$

Example 1.49. For any $x \in X$, there is a δ-distribution $\delta_x\colon C(X) \to \mathbb{C}$ given by $\delta_x(f) \coloneqq f(x)$. To show that this measure is continuous, we use Corollary A.42, which tells us that it is enough to check that $|\delta_x| \le \|\cdot|_K\|_\infty$ for some compact $K \subseteq X$. Well, taking $K = \{x\}$, we see $|\delta_x(f)| = |f(x)| = \|f|_K\|_\infty$ for any f.

Example 1.50. If $f\colon X \to Y$ is a continuous map, then there is a linear map $f^*\colon C(Y) \to C(X)$ given by $f^* \coloneqq (- \circ f)$. (It sends continuous maps to continuous maps because f is continuous.) Continuing, we get a map $f_*\colon C(X)^* \to C(Y)^*$ by sending $\mu \in C(X)^*$ to $(\mu \circ f^*) \in C(Y)^*$, where f^* is continuous by Example A.57. We may write $\mu \circ f^*$ as $f_*\mu$. Note f_* is continuous: if $\{\mu_i\} \to \mu$ in $C(X)^*$, then any $g \in C(Y)$ has $\{\mu_i(g \circ f)\} \to \mu(g \circ f)$, so $\{f_*\mu_i(g)\} \to f_*\mu_i(g)$.

Remark 1.51. If i is the inclusion $K \subseteq X$ of a compact set, then i_* is just the restriction map $C(K)^* \to C(X)^*$ given by $i_*\mu(g) \coloneqq \mu(g|_K)$.

Remark 1.52. If X is compact, then $C(X)^*$ has a natural (operator) norm given by

$$\|\mu\| \coloneqq \sup_{\|f\|_\infty = 1} |\mu(f)| .$$

However, $C(X)^*$ is usually not equipped with the norm topology: for example, if $\{x_i\} \to x$, then $\{\delta_{x_i}\} \to \delta_x$ in the weak topology because $\{f(x_i)\} \to f(x)$ for any continuous f, but $\|\delta_x - \delta_{x_i}\| = 2$ whenever $x \ne x_i$.

1.3.3 Supports

Here is our main definition.

Definition 1.53 (support)**.** Fix a locally compact second countable topological space X. Then the *support* $\operatorname{supp}\mu$ of a functional μ on $C(X)$ is defined by $x \notin \operatorname{supp}\mu$ if and only if there is an open neighborhood U of x for which any f satisfying $f|_{X\setminus U} = 0$ also has $\mu(f) = 0$.

Lemma 1.54. Fix a locally compact second countable topological space X. For any functional $\mu \in C(X) \to \mathbb{C}$, the subset $\operatorname{supp}\mu$ is closed. Furthermore, if μ is continuous, then $\operatorname{supp}\mu$ is compact.

Proof. To see that $\operatorname{supp}\mu$ is closed, we show that its complement is open. Indeed, note that any $x \notin \operatorname{supp}\mu$ admits an open neighborhood U of x satisfying some condition in the definition; but then any $y \in U$ still admits this open neighborhood, so $U \cap \operatorname{supp}\mu = \varnothing$.

It remains to show that $\operatorname{supp}\mu$ is compact. We will show that $\operatorname{supp}\mu$ failing to be compact implies that $\mu\colon C(X) \to \mathbb{F}$ fails to be continuous. For this, it is enough to check that there is a compact set K containing $\operatorname{supp}\mu$. Well, because X is locally compact and second countable, we can find a countable collection of precompact open subsets covering X, so upon taking suitable unions and closures, we see that there is a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact sets covering X. (This construction is also used in Example A.55.)

We now use the assumption that $\operatorname{supp}\mu$ fails to be compact, which means that it cannot be contained in any K_i. Then we are granted $x_i \in \operatorname{supp}\mu \setminus K_i$ for each i, which means that any open neighborhood U_i of x_i admits a function f_i such that $f_i|_{X\setminus U_i} = 0$ and $\mu(f_i) \neq 0$; we go ahead and shrink U_i to fully avoid K_i. Additionally, by adjusting f_i by a scalar, we may assume that $\mu(f_i) = 1$. Now, the point is to consider the sum

$$\sum_i f_i,$$

which converges uniformly on compact sets: as in Example A.55, it is enough to check uniform convergence on just the K_is, and we see that this is a finite sum when restricted to any K_n because f_i is nonzero only inside U_i, so $f_i|_{K_n} = 0$ for $i > n$.

On the other hand, we see that continuity of μ would imply that

$$\mu\left(\sum_i f_i\right) = \sum_i \mu(f_i)$$

is a finite number, which is false because $\mu(f_i) = 1$ for each i! Thus, μ fails to be continuous. ■

Remark 1.55. Lemma 1.54 explains why there is a c in our notation $\operatorname{Meas}_c(X)$.

Lemma 1.56. Fix a locally compact second countable topologically space X. For any $\mu \in C(X)^*$ and $f \in C(X)$ for which $f|_{\operatorname{supp}\mu} = 0$, we have $\mu(f) = 0$.

Proof. For each $x \notin \operatorname{supp}\mu$, we are granted an open neighborhood U_x of x such that any function g vanishes outside U has $\mu(g) = 0$. Because X is second countable, the open cover $\{U_x\}_{x\notin\operatorname{supp}\mu}$ of $X \setminus \operatorname{supp}\mu$ can be refined to a countable subcover. Let $\{U_i\}_{i\in\mathbb{N}}$ be this countable subcover, and we let $\{f_i\}_{i\in\mathbb{N}}$ be the partition of unity granted by Lemma 1.44 (which applies by Lemma 1.45). But now we see that

$$f = \sum_{i\in\mathbb{N}} f_i f.$$

Thus, $\mu(f) = 0$: it's enough to see that $\mu(f_i f) = 0$ for all $i \in \mathbb{N}$ because $f_i f$ vanishes outside U. ■

As an application, we are able to push forward measures!

Lemma 1.57. Fix a locally compact second countable topologically space X. Given $\mu \in C(X)^*$ such that $\operatorname{supp}\mu$ is contained in a compact set K, there is a unique measure $\nu \in C(K)^*$ such that

$$\nu(f) := \mu(\widetilde{f}),$$

whenever $\widetilde{f} \in C(X)^*$ satisfies $\widetilde{f}|_K = f$.

Proof. Let's begin with uniqueness. This will be accomplished as soon as we show that any function $f \in C(K)$ is the restriction of some function $\widetilde{f} \in C(X)$. This amounts to the Tietze extension theorem. To avoid issues with the normality hypothesis, we write X as an ascending union $\bigcup_{i\in\mathbb{N}} K_i$ of compact sets with $K_0 := K$. Then compact Hausdorff spaces are normal, so $f \in C(K)$ may inductively be extended to each K_n

via the Tietze extension theorem, so f admits an extension $\widetilde{f}$ to $C(X)$. We remark that the Tietze extension theorem allows this extension to be done with $\left\|\widetilde{f}\right\|_\infty = \|f\|_\infty$.

It remains to show existence. For this, we should show that the given formula for ν is well-defined, linear, and continuous.

- Well-defined: we use Lemma 1.56. Indeed, if $\widetilde{f}_1$ and $\widetilde{f}_2$ in $C(X)$ restrict to the same function $f \in C(K)$, then $\mu(\widetilde{f}_1 - \widetilde{f}_2) = 0$ by Lemma 1.56.
- Linear: for any functions f and g in $C(X)^*$, we see that $\nu(af|_K + bg|_K) = a\nu(f|_K) + b\nu(f|_K)$ because this check lifts to μ.
- Continuous: because μ is continuous, it is bounded by Lemma A.41, so there is an open neighborhood U of 0 for which $\sup_{f\in U} |\mu(f)| < \infty$. However, basic open neighborhoods of 0 look like finite intersections of open sets of the form

  $$\{f \in C(X) : \|f|_E\|_\infty < \varepsilon\}$$

  for some compact subset $E \subseteq X$ and some $\varepsilon > 0$. By taking the union of the relevant Es (and the minimum of the εs), we may assume that U actually has the above form. Additionally, because μ only depends on $f|_K$, we may as well assume that $E \subseteq K$.

  Now, the formula $\|f|_E\|_\infty < \varepsilon$ also cuts out an open subset of $C(K)$, and we claim that

  $$\sup_{f\in U} |\mu(f)| \stackrel{?}{=} \sup_{f\in U\cap C(K)} |\nu(f)|,$$

  which shows that ν is bounded and hence continuous by Lemma A.41. Well, any $f \in U \cap C(K)$ can be extended to some $\widetilde{f} \in U$ by our Tietze extension theorem argument, so the result follows by construction of ν. ■

Notation 1.58. Fix a locally compact second countable topologically space X, and let $i\colon K \to X$ be the inclusion of a compact subset. Given $\mu \in C(X)^*$ such that $\operatorname{supp}\mu$ is contained in K, we let $i^*\mu$ denote the measure constructed in Lemma 1.57. This is called the pullback measure.

Remark 1.59. We note that i^* is continuous. Well, if $\mu_n \to \mu$, we need to check that $i^*\mu_n \to i^*\mu$. Checking this on the weak topology is not hard: given any function $f \in C(K)$, we need to know that $i^*\mu_n(f) \to i^*\mu(f)$, which follows because $\mu_n(\widetilde{f}) \to \mu(\widetilde{f})$ for any lift $\widetilde{f} \in C(X)$ of f.

Lemma 1.60. Fix a locally compact second countable topologically space X, and let $i\colon K \to X$ be the inclusion of a compact subset.

(a) If $\mu \in C(X)^*$ has $\operatorname{supp}\mu \subseteq K$, then $i_*i^*\mu = \mu$.

(b) If $\mu \in C(K)^*$, then $\operatorname{supp} i_*\mu \subseteq K$ and $i^*i_*\mu = \mu$.

(c) The map $i_*\colon C(K)^* \to C(X)^*$ is a homeomorphism onto the space of measures μ with $\operatorname{supp}\mu \subseteq K$.

Proof. Note (c) follows from (a) and (b). To see (a), we note that any $f \in C(X)$ has $i_*i^*\mu(f) = i^*\mu(f|_K)$, which is $\mu(f)$ by definition of the pullback. To see (b), we start by assuming $\operatorname{supp} i_*\mu \subseteq K$. Then we may take any $f \in C(K)$, lift it to $\widetilde{f} \in C(X)$, and we compute that $i^*i_*\mu(f) = i_*\mu(\widetilde{f})$ is $\mu(\widetilde{f}|_K) = \mu(f)$.

It remains to show that $\operatorname{supp} i_*\mu \subseteq K$. Well, for any $x \notin K$, we need to show $x \notin \operatorname{supp} i_*\mu$, for which we use the open neighborhood $U := X \setminus K$. Indeed, if $f|_{X\setminus U} = 0$, then $f|_K = 0$, so $i_*\mu(f) = 0$ follows. ■

1.3.4 Sequential Completeness

We now turn towards showing that $C(X)^*$ is sequentially complete.

Lemma 1.61. Fix a locally compact second countable topologically space X. If $\{\mu_i\}$ is a Cauchy sequence in $C(X)^*$, then there is a compact set K such that $\operatorname{supp}\mu_i \subseteq K$ for all i.

Proof. This argument is similar to Lemma 1.54, but the bookkeeping is more painful.

Suppose there is no such compact set K, and we will show that $\{\mu_i\}$ has a subsequence which fails to be Cauchy. Before doing anything, note we are trying to see if $\bigcup_{i\in\mathbb{N}}\operatorname{supp}\mu_i$ is contained in a compact set. If we ever have $\operatorname{supp}\mu_i$ contained in $\bigcup_{j<i}\operatorname{supp}\mu_j$, then we may as well remove μ_i from the Cauchy sequence because it will not affect the final containment in a compact set.

Now, we will run an inductive construction as follows. Start with the compact set $K_0 := \varnothing$ and $\nu_0 := \mu_0$. Now, for every $n \geq 0$, we take (K_n, μ_n) and build a triple $(f_n, K_{n+1}, \nu_{n+1})$ as follows.

1. Observe that there is a point $x_n \in \operatorname{supp}\nu_n \setminus K_n$. Thus, we are granted a function f_n and a small open neighborhood U_n of x_n such that f_n vanishes outside U_n and $\mu_n(f_n) \neq 0$. By shrinking U_n, we may avoid K_n and assume that U_n is precompact. By rescaling, we may assume that $\mu_n(f_1 + \cdots + f_n) = n$.
2. Now, expand K_n (by $\overline{U_n}$) to a compact set K_{n+1} containing U_n and $\operatorname{supp}\nu_n$. Because $\bigcup_{i\in\mathbb{N}}\operatorname{supp}\mu_i$ is contained in no compact set, we may find ν_{n+1} such that $\operatorname{supp}\nu_{n+1} \setminus K_{n+1}$.

The above process continues indefinitely by the hypothesis on the $\mu_\bullet$s.

The point is that the series $\sum_n f_n$ converges uniformly on the compacts $\{K_n\}$ because $f_n|_{K_m} = 0$ for $n > m$. Letting this limiting function be f, we then see that $\nu_n(f)$ only has the nonzero terms $\nu_n(f_1+\cdots+f_n) = n$: the point is that f_m is only nonzero on U_m, so if $m > n$, then $\operatorname{supp}\nu_n$ is contained in K_{n+1}, so $\nu_n(f_m) = 0$. Thus, we see that

$$|\nu_n(f) - \nu_m(f)| = |n - m|$$

fails to go to 0 as $n, m \to \infty$, so the sequence $\{\nu_n\}$ fails to be Cauchy. ■

Remark 1.62. This argument must do something a little tricky because the statement is false if $\{\mu_i\}$ is replaced by a Cauchy net. Indeed, if that were true, then Remark 1.64 explains that $C(X)^*$ would be complete, which is not true in general (see Example 1.65).

Here is an application of our work with supports.

Lemma 1.63. Fix a locally compact second countable topological space X. The space $C(X)^*$ is sequentially complete.

Proof. We proceed in steps. The first half of the proof handles the case where X is compact by using the Uniform boundedness principle, and the second half of the proof reduces to this case by using Lemma 1.61.

1. Let's start by handling the case where X is compact. The general case will then follow by applying Lemma 1.61. Fix a Cauchy sequence $\{\mu_i\}_{i\in\mathbb{N}}$ for which we want to find a limit. Well, for each $f \in C(X)$, the nature of the weak topology makes $\{\mu_i(f)\}$ a Cauchy sequence in $\mathbb{R}$, so it admits a limiting value which we label
$$\mu(f) := \lim_{i\to\infty}\mu_i(f).$$
Thus, we have defined a function $\mu\colon C(X) \to \mathbb{C}$. Note linearity of the μ_is and taking the limit implies that μ is linear. Additionally, having $\mu_i(f) \to \mu(f)$ for all functions f implies that we will have $\mu_i \to \mu$ as soon as we actually check that $\mu \in C(X)^*$.

2. When X is compact, it still remains to check that μ as defined above is actually continuous. The point is that, because X is compact, $C(X)$ is a Banach space with norm given by $\|\cdot\|_\infty$ (see Example A.55). By Corollary A.42, it is enough to check that $\|\mu\|$ is bounded, for which we will use the Uniform boundedness principle. As such, we will be able to use the Uniform boundedness principle: because each sequence $\{\mu_i(f)\}$ is Cauchy for every f, we see that these sequences are always bounded, so the operator norms $\|\mu_i\|$ must also be bounded. In particular, there is a large constant C for which $|\mu_i(f)| \le C\|f\|_\infty$ for all f, so we see that $|\mu(f)| \le C\|f\|_\infty$ for all f, so $\|f\| \le C$.

3. We now turn to the general case. Once again, fix a Cauchy sequence $\{\mu_i\}_{i\in\mathbb{N}}$ for which we want to find a limit. We use Lemma 1.61, which allows us to produce a compact subset $K \subseteq X$ such that we may assume that $\operatorname{supp}\mu_i \subseteq K$ for each i. Let $j\colon K \to X$ be the inclusion.

   To reduce to the compact case, we note that Lemma 1.60 shows that the μ_is are in the image of the topological embedding $j_*\colon C(K)^* \to C(X)^*$, so it is enough to find a limit in $C(K)^*$. This is exactly the compact case! ■

Remark 1.64. The argument for X in the compact case applies to general Cauchy nets, so it follows that $C(X)^*$ is complete. More generally, the argument shows that if $\{\mu_i\}$ is a Cauchy net for which

$$\bigcup_i \operatorname{supp}\mu_i$$

is contained in a compact set, then the limit exists.

It is worthwhile to have an example where $C(X)^*$ is not complete.

Example 1.65. Give $X = \mathbb{N}$ the discrete topology. Then we show that $C(X)^*$ is not complete.

Proof. Here, $C(X)$ is just functions $\mathbb{N} \to \mathbb{C}$, which are sequences in $\mathbb{C}$. We let δ_n be the sequence $\{\delta_n(i)\}_{i\in\mathbb{N}}$.

Next, we calculate $C(X)^*$. For any $\mu \in C(X)^*$, we see that $\mu(f)$ only depends on $f|_{\operatorname{supp}\mu}$ by Lemma 1.56. But $\operatorname{supp}\mu$ is some compact set in $\mathbb{N}$, so it is some finite set $\{n_1, \ldots, n_r\}$. Thus, we see that $\mu(f)$ is

$$\mu(f|_K) = \sum_{i=1}^r f(n_i)\mu(\delta_{n_i}).$$

Thus, we see that μ only depends on the sequence $\{\mu(\delta_n)\}_{n\in\mathbb{N}}$, which has only finitely many nonzero terms. Conversely, any such sequence $\{\mu_n\}_{n\in\mathbb{N}}$ defines a measure by

$$\mu(f) := \sum_{n\in\mathbb{N}} f(n)\mu_n,$$

which is a finite sum because only finitely many of the $\mu_\bullet$s are nonzero. It is not hard to see that these constructions are linear and inverse, so our measures are in bijection with finitely supported sequences in this manner. (For example, any measure μ produces the sequence $\{\mu(\delta_n)\}$, which then produces the measure $f \mapsto \sum_n f(n)\mu(\delta_n)$, which is the original measure. The other inverse check is easier.)

To show that $C(X)^*$ fails to be complete, we will show that $C(X)^*$ is a dense subset of the dual space $C(X)^\vee$ (where $C(X)^\vee$ continues to have the weak topology). To explain why this completes this proof, note that $C(X)^* \ne C(X)^\vee$ for dimension reasons, so $C(X)^*$ cannot be a closed subset, so $C(X)^*$ will have some Cauchy net not admitting a limit, as required.

It remains to show that $C(X)^* \subseteq C(X)^\vee$ is dense, where $C(X)^\vee$ has the weak topology. Well, basic open subsets in the weak topology look like $\mu + U$ where $\mu \in C(X)^\vee$ and U has the form

$$\bigcap_{i=1}^n \underbrace{\{\nu : |\nu(f_i)| < \varepsilon_i\}}_{U_i :=}$$

for some finitely many functions $\{f_1, \ldots, f_n\}$ and small real numbers $\varepsilon_i > 0$. We may as well assume that these functions are linearly independent, or else we can make the list of functions smaller. We would like to find some element of $C(X)^*$ in $\mu + U$.

Because $\{f_1, \ldots, f_n\}$ are linearly independent, they span an n-dimensional subspace. We now claim that there is a subset $K \subseteq \mathbb{N}$ of size n for which the sequences $\{f_i(k)\}_{k\in K}$ continue to be linearly independent in $\mathbb{C}^K$. Well, the sequences $\{f_i(n)\}_{n\in\mathbb{N}}$ form an n-dimensional subspace of $\mathbb{C}^{\mathbb{N}}$, so one can apply Gaussian elimination to this $n \times |\mathbb{N}|$ matrix to put it into row-reduced Echelon form, from which the subset K can be read off as the pivots. (Row operations do not change the dimension of the span of any given collection of columns!)

Now, by rearranging, we may as well take $K = \{1, 2, \ldots, n\}$. Then we can find constants $\{c_1, \ldots, c_n\}$ solving the system of n equations

$$\mu(f_i) = \sum_{j=1}^{n} c_j f_i(j)$$

because the matrix $\{f_i(j)\}_{i,j}$ is invertible by the previous paragraph. These constants now define a measure ν in $C(X)^*$ satisfying $\mu(f_i) = \nu(f_i)$ for each i, so $\nu \in \mu + U$ follows. ■

Remark 1.66. What goes wrong in Example 1.65 is that $C(\mathbb{N})^*$ cannot be first countable, which we can also check directly. Indeed, suppose that we have an open basis of neighborhoods $\{U_i\}_{i\in\mathbb{N}}$ of 0. Now, each U_i is still absorbing by Example A.17, so we can find a real number $\varepsilon_i > 0$ for which $\varepsilon_i\delta_i \in U_i$ for each i. But now $\varepsilon_i\delta_i \to 0$ in $C(\mathbb{N})^*$ by definition of the weak topology while the function $f : \mathbb{N} \to \mathbb{C}$ given by $f(i) := \varepsilon_i^{-1}$ has $(\varepsilon_i\delta_i)(f) = 1$ for all i. This is a contradiction!

Here is the theoretical culmination of our work with measures.

Notation 1.67. Fix a locally compact second countable topological space X. Then we set $\operatorname{Meas}_c^0(X) \subseteq \operatorname{Meas}_c(X)$ to be the subspace spanned by $\{\delta_x\}_{x\in X}$.

Proposition 1.68. Fix a locally compact second countable topological space X. Then $\operatorname{Meas}_c^0(X)$ is sequentially dense in $\operatorname{Meas}_c(X)$ is sequentially dense.

Proof. Quickly, we reduce to the case where X is compact. Fix some $\mu \in \operatorname{Meas}_c(X)$ that we want to approximate, and set $K := \operatorname{supp}\mu$. Then Lemma 1.60 shows that μ is in the image of the topological embedding $j_* : \operatorname{Meas}_c(K) \to \operatorname{Meas}_c(X)$, where $j : K \to X$ is the embedding. Note further that j_* restricts to a linear map $\operatorname{Meas}_c^0(K) \to \operatorname{Meas}_c^0(X)$ by Example 1.50: it's enough to check this on generators, for which we note $j_*\delta_k = \delta_k$. Thus, for any open neighborhood $U \subseteq \operatorname{Meas}_c(X)$ of μ, it is enough to find ν in the intersection $U \cap \operatorname{Meas}_c^0(K)$ (meaning j^*U) and then apply j_* to complete the argument.

Thus, we may assume that X is compact. The idea is to basically approximate μ using Riemann sums. Because we only have access to continuous functions, our Riemann sums will have to use a partition of unity. To explain the idea, we suppose that we are given a finite open cover $\mathcal{U}$ of X. Then Lemma 1.44 (which applies by Lemma 1.45) grants us continuous nonnegative functions $\{f_U\}_{U\in\mathcal{U}}$ such that $\operatorname{supp} f_U \subseteq U$ for each $U \in \mathcal{U}$ and that $\sum_{U\in\mathcal{U}} f_U = 1$. We then define

$$\mu_{\mathcal{U}} := \sum_{U\in\mathcal{U}} \mu(f_U)\delta_{x_U} \in \operatorname{Meas}_c^0(X),$$

where $x_U \in U$ is some basepoint. (Technically, $\mu_{\mathcal{U}}$ also depends on the functions $f_\bullet$ and basepoints $x_\bullet$, but we will suppress this from the notation.) The hope is that if the open sets $\mathcal{U}$ are small enough, then we can show that μ and $\mu_{\mathcal{U}}$ are close. To get some idea of what we need, we note that any $f \in C(X)$ has $\mu(f) - \mu_{\mathcal{U}}(f)$

equal to

$$\begin{aligned}\mu(f)-\mu_{\mathcal{U}}(f)&=\sum_{U\in\mathcal{U}}\mu(ff_U)-\sum_{U\in\mathcal{U}}\mu(f_U)f(x_U)\\&=\mu\Bigg(\sum_{U\in\mathcal{U}}ff_U-f(x_U)f_U\Bigg),\end{aligned}$$

which one may hope to be small because the functions $ff_U-f(x_U)f_U$ should be small and have small support. Indeed, we now note that μ being continuous means that it is bounded, so because $C(X)$ is Banach with the norm $\|\cdot\|_\infty$ (recall X is compact!), we are able to write

$$\begin{aligned}|\mu(f)-\mu_{\mathcal{U}}(f)|&\le\|\mu\|\sup_{x\in X}\sum_{U\in\mathcal{U}}|f(x)-f(x_U)|\,f_U(x)\\&\le\|\mu\|\sup_{\substack{U\in\mathcal{U}\\x\in U}}|f(x)-f(x_U)|\,.\end{aligned}$$

To continue, we see that we are going to need to produce good, uniform approximations of f with our open covers.

We are now ready to define our open covers $\mathcal{U}$. Fix some $\delta>0$. Because X is a compact metric space, we may take $\mathcal{U}$ to be a collection of open balls of radius δ. Let μ_δ be the resulting measure, and we claim that $\{\mu_\delta\}\to\mu$ as $\delta\to0^+$. This will complete the proof because, for example, we also see that $\{\mu_{1/n}\}\to\mu$ as $n\to\infty$.

To check that $\{\mu_\delta\}\to\mu$, we fix a function $f\in C(X)$, and we must show that $\{\mu_\delta(f)\}\to\mu(f)$. Because X is compact(!), the function f is actually uniformly continuous, so for any $\varepsilon>0$, we can choose any $\delta>0$ small enough so that $d(x,y)<\delta$ implies that $|f(x)-f(y)|<\varepsilon$. Then we see that

$$\sup_{\substack{U\in\mathcal{U}\\x\in U}}|f(x)-f(x_U)|<\varepsilon,$$

so $|\mu(f)-\mu_{\mathcal{U}}(f)|<\varepsilon$. The claim follows. ■

1.3.5 The Algebra of Measures

The point of Proposition 1.68 is that we can define maps on $C(X)^*$ by just defining them on δ_xs and extending by continuity. Here is an application of this.

Lemma 1.69. Fix locally compact second countable spaces X and Y. Then there is a unique bilinear map

$$\boxtimes\colon\ \operatorname{Meas}_c(X)\times\operatorname{Meas}_c(Y)\to\operatorname{Meas}_c(X\times Y)$$

such that $\delta_x\boxtimes\delta_y=\delta_{(x,y)}$ and which is continuous.

Proof. Observe that the given relation implies that $\boxtimes$ is automatically uniquely defined on $\operatorname{Meas}_c^0(X)\times\operatorname{Meas}_c^0(Y)$ by linearity, which is dense in $\operatorname{Meas}_c(X)\times\operatorname{Meas}_c(Y)$ by Proposition 1.68, so there is certainly at most one map $\boxtimes$ satisfying the given relation (by Proposition A.30). Thus, we see that the main problem is existence.

Let's begin by handling the case where X and Y are compact. By Proposition A.30, it is enough to check that $\boxtimes\colon\ \operatorname{Meas}_c^0(X)\times\operatorname{Meas}_c^0(Y)\to\operatorname{Meas}_c(X\times Y)$ is continuous. Thus, fix Cauchy nets $\{\mu_i\}\to\mu$ and $\{\nu_j\}\to\nu$, and we want to check that $\{\mu_i\boxtimes\nu_j\}\to\mu\boxtimes\nu$ as $i,j\to\infty$.

The benefit of working with compact spaces is that we are able to use the Stone–Weierstrass theorem. Indeed, note that $C(X)\otimes C(Y)\subseteq C(X\times Y)$ is a subalgebra separating points, so it is dense. Thus, though we want to check that $\{\mu_i\boxtimes\nu_j(h)\}\to\mu\boxtimes\nu(h)$ for any $h\in C(X\times Y)$, we may find some $h'\in C(X)\otimes C(Y)$ for which $\|h-h'\|_\infty<\varepsilon$ and then observe that

$$|\mu\boxtimes\nu(h)-\mu_i\boxtimes\nu_j(h)|\le|\mu\boxtimes\nu(h')-\mu_i\boxtimes\nu_j(h')|+(\|\mu\|\cdot\|\nu\|+\|\mu_i\|\cdot\|\nu_j\|)\,\|h-h'\|_\infty\,.$$

All the norms are uniformly bounded by the Uniform boundedness principle, so we see that it is now enough to check that $\{\mu_i \boxtimes \nu_j(h')\} \to \mu \boxtimes \nu(h')$.

By linearity, we may further assume that $h' = fg$ for some $f \in C(X)$ and $g \in C(Y)$. In this case, we see that $\delta_x \boxtimes \delta_y(fg) = f(x)g(y)$ is a relation bilinear in both coordinates, so we see that $\mu' \boxtimes \nu'(fg) = \mu'(f)\nu'(g)$ for any $\mu' \in C(X)$ and $\nu' \in C(Y)$. Now, $\{\mu_i(f)\} \to \mu(f)$ and $\{\nu_j(g)\} \to \nu(g)$ are convergent sequences by the definition of the weak topology, so we conclude that $\{\mu_i(f)\nu_j(g)\} \to \mu(f)\nu(g)$ and so $\{\mu_i \boxtimes \nu_j(fg)\} \to \mu \boxtimes \nu(fg)$.

This completes the proof in the case where X and Y are compact. In the general case, we recall from Example A.78 that

$$C(X)^* \times C(Y)^* = \operatorname*{colim}_{\substack{K \subseteq X \\ L \subseteq Y}} (C(K)^* \times C(L)^*).$$

With this in mind, we will try to glue together the maps provided for us in the colimit. Well, for any inclusions $K \subseteq K'$ and $L \subseteq L'$ of compact subsets of X and Y, respectively, we note that the diagram

$$\begin{tikzcd} C(K)^* \times C(L)^* \arrow[r, "\boxtimes"] \arrow[d] & C(K \times L)^* \arrow[d] & (\delta_x, \delta_y) \arrow[r, mapsto] \arrow[d, mapsto] & \delta_{(x,y)} \arrow[d, mapsto] \\ C(K')^* \times C(L')^* \arrow[r, "\boxtimes"] & C(K' \times L')^* & (\delta_x, \delta_y) \arrow[r, mapsto] & \delta_{(x,y)} \end{tikzcd}$$

commutes because it commutes on the dense subspace spanned by the $\delta_\bullet$s (see Proposition 1.68). Thus, by passing to the colimits, we receive a continuous composite map

$$\boxtimes\colon C(X)^* \times C(Y)^* \to C(X \times Y)^*$$

such that $\delta_x \boxtimes \delta_y = \delta_{(x,y)}$. ■

Notation 1.70. Fix locally compact second countable spaces X and Y. Given $\mu \in \operatorname{Meas}_c(X)$ and $\nu \in \operatorname{Meas}_c(Y)$, we define $\mu \boxtimes \nu \in \operatorname{Meas}_c(X \times Y)$ via Lemma 1.69.

Intuitively,

$$\int_{X \times Y} h \, d(\mu \boxtimes \nu) = \int_X \int_Y h \, d\nu \, d\mu.$$

Here is the sort of thing that we can show with this definition. The main idea of all these proofs is to find a way to reduce to δs.

Lemma 1.71. Fix locally compact second countable spaces X, Y, and Z. Up to identifying $(X \times Y) \times Z$ and $X \times (Y \times Z)$, we have that $\boxtimes$ is associative.

Proof. The first part of the argument of Lemma 1.69 shows that there is a unique tri-linear, sequentially continuous map

$$\boxtimes\colon \operatorname{Meas}_c(X) \times \operatorname{Meas}_c(Y) \times \operatorname{Meas}_c(Z) \to \operatorname{Meas}_c(X \times Y \times Z)$$

such that $\delta_x \boxtimes \delta_y \boxtimes \delta_z = \delta_{(x,y,z)}$. (Namely, linearity extends this map to the Meas_c^0s, which are then sequentially dense.) However, the two maps $(\mu_X, \mu_Y, \mu_Z) \mapsto (\mu_X \boxtimes \mu_Y) \boxtimes \mu_Z$ and $(\mu_X, \mu_Y, \mu_Z) \mapsto \mu_X \boxtimes (\mu_Y \boxtimes \mu_Z)$ both satisfy this relation. ■

Notation 1.72. Fix a locally compact second countable topological group G. Then we define the binary operator $*$ on $\operatorname{Meas}_c(G)$ by

$$\mu * \nu := m_*(\mu \boxtimes \nu),$$

where $m\colon G \times G \to G$ is the multiplication map.

Example 1.73. For $g, h \in G$, we compute that $\delta_g * \delta_h$ is $m_*(\delta_g \boxtimes \delta_h) = \delta_{gh}$.

Remark 1.74. Because m_* and $\boxtimes$ are both sequentially continuous, we conclude that $*$ is also sequentially continuous.

Example 1.75. For any $\mu \in \operatorname{Meas}_c(G)$ and $f \in C(G)$, we claim that $\delta_g * \mu(f) = \mu(x \mapsto f(gx))$. Intuitively, this corresponds to the formula

$$\int_{G\times G} f(xy)\, d\delta_g(x)\, d\mu(y) = \int_G f(gy)\, d\mu(y).$$

Anyway, the map sending f to $g^*f \colon x \mapsto f(gx)$ is continuous by Example A.57, so we are checking for an equality of measures. Because $\mu \mapsto \delta_g * \mu$ is sequentially continuous, as is $\mu \mapsto g_*\mu$, it is enough to check this on the dense subspace $\operatorname{Meas}^0_c(X)$, so by linearity, it is enough to check this for $\mu = \delta_h$ for $h \in G$, where this now has no content.

The point of defining $\operatorname{Meas}_c(G)$ is that it provides some continuous extension of the group algebra. For example, the group algebra has a natural action on any representation.

Lemma 1.76. Fix a Fréchet representation V of a locally countable second countable topological group G. Then the natural map

$$\operatorname{Meas}^0_c(G) \times V \to V$$

is continuous.

Proof. We remark that the natural map is defined by linearly extending $(\delta_g, v) \mapsto \rho(g)v$, where $\rho \colon G \to \operatorname{Aut}(V)$ is the action map. We let $\rho \colon \operatorname{Meas}^0_c(G) \to \operatorname{End}(V)$ be the induced map.

Now, for two convergent nets $\{\mu_i\} \to \mu$ and $\{v_j\} \to v$, we would like to show that $\{\rho(\mu_i)v_j\} \to \rho(\mu)v$. Thus, for any seminorm p on V, we must show that $\{p(\rho(\mu_i)v_j - \rho(\mu)v)\} \to 0$. However, we note that

$$p(\rho(\mu_i)v_j - \rho(\mu)v) \le p(\rho(\mu_i - \mu)v) + p(\rho(\mu_i)(v_j - v)).$$

The action map $\rho(\mu_i) \colon V \to V$ is certainly continuous, so the right-hand term vanishes as $j \to \infty$.

It remains to deal with the left-hand term $p(\rho(\mu_i - \mu)v)$. By replacing $\mu_i - \mu$ with μ_i, we see that we may now assume that $\mu = 0$. Thus, given that $\{\mu_i\} \to 0$, we want to show that $\{p(\rho(\mu_i)v)\} \to 0$. We will do this by upper-bounding the map $\mu \mapsto p(\rho(\mu)v)$ by a continuous function vanishing at $\mu = 0$, which will complete the proof. We do this in steps.

1. For any $\mu \in \operatorname{Meas}^0_c(X)$, we note that we may expand $\mu = \sum_{g\in G} c_g\delta_g$, from which we define $|\mu| := \sum_{g\in G} |c_g|\, \delta_g$. One can check directly with the weak topology that $|\cdot|$ is continuous.

2. The same expansion $\mu = \sum_{g\in G} c_g\delta_g$ allows us to see

$$|p(\rho(\mu)v)| \le \sum_{g\in G} |c_g|\, p(\rho(g)v).$$

With this in mind, we define $f \in C(G)$ by $f(g) := p(\rho(g))$, and then $|p(\rho(\mu)v)| \le |\mu|\,(f)$. However, the function $\mu \mapsto |\mu|\,(f)$ is continuous by the construction of the weak topology, so we are done. ■

Remark 1.77. In this situation, we note that the action map $\rho\colon \operatorname{Meas}_c^0(G) \to \operatorname{End}(V)$ is continuous, where the target has been given the strong topology: this amounts to checking that $\{\mu_i\} \to \mu$ implies $\{\rho(\mu_i)v\} \to \rho(\mu)v$ for all v. By continuity (via a suitable modification of Proposition A.30), we then have a continuous map

$$\operatorname{Meas}_c(G) \to \operatorname{End}(V).$$

(Technically, we should check that $\operatorname{End}(V)$ is complete when given the strong topology; let us content ourselves by allowing non-continuous endomorphisms in $\operatorname{End}(V)$ for this statement, and then this is not so hard. This is also implied by the Uniform boundedness principle if V is Banach.) Note that this improves Lemma 1.42!

1.3.6 Dense Subspaces

In this subsection, we show that certain subspaces of controlled vectors are in fact dense. We will get some utility out of the following construction.

Definition 1.78 (approximate identity)**.** Fix a locally compact second countable topological group G with Haar measure dg. Then an *approximate identity* is a sequence of functions $\{\varphi_i\}$ in $C_c(G)$ satisfying the following two properties.

- We have $\int_G \varphi_i\,dg = 1$ for all n.
- For any open neighborhood U of 1, we have $\operatorname{supp}\varphi_i \subseteq U$ for all but finitely many i.

If G is a Lie group, we further require that the φ_i are smooth.

Remark 1.79. Let's explain why approximate identities exist. Let $\{U_i\}_{i\in\mathbb{N}}$ be a neighborhood basis for $1 \in G$, which is allowed to be countable because G is first countable. By shrinking the U_is, we may as well assume that it is a descending sequence of precompact open subsets. Then a suitable version of Urysohn's lemma (on the $\overline{U_i}$s) provides a nonnegative function φ_i in $C(U_i) \subseteq C_c(G)$, and scaling allows us to assume that $\int_G \varphi_n\,dg = 1$. (If G is a Lie group, then the U_ns can be precompact open balls, and we can define the φ_is as smooth bump functions.) Now, $\{\varphi_i\}$ is an approximate identity because the $\{U_i\}$s form a descending neighborhood basis of 1.

Remark 1.80. The key property of approximate identities is that $\{\varphi_i\,dg\} \to \delta_1$ in the weak topology. Indeed, for $f \in C(G)$, we need to show that $\int_G f\varphi_i\,dg \to f(1)$; by translation, we may assume that $f(1) = 0$. Well, by continuity of f, any $\varepsilon > 0$ has an open neighborhood U of 1 such that $|f(x)| < \varepsilon$ for all $x \in U$. Choosing U to be precompact, we see that any i large enough has $\operatorname{supp}\varphi_i \subseteq U$, so we may write

$$\left|\int_G f\varphi_i\,dg\right| \le \int_U \varepsilon\varphi_i\,dg = \varepsilon.$$

Here is our first class of well-behaved vectors.

Definition 1.81 (finite)**.** Let K be a compact second countable topological group. Fix a continuous representation V of K. Then a vector $v \in V$ is *K-finite* if and only if the vector space

$$\operatorname{span}\{gv : g \in K\}$$

is finite-dimensional. We let V_{fin} denote the space of K-finite vectors.

Remark 1.82. Note that V_{fin} is in fact a subspace of V. For example, certainly $0 \in V$, and if $v, w \in V_{\text{fin}}$, then

$$\operatorname{span}\{agv + bhw : a, b \in \mathbb{F}, g, h \in K\}$$

is finite-dimensional, so $\operatorname{span}\{v, w\} \subseteq V_{\text{fin}}$.

Remark 1.83. We claim that $v \in V$ is K-finite if and only if it is K°-finite. Indeed, K has only finitely many connected components (because K is compact), so we can find a finite subset $S \subseteq K$ representing K/K°. (Note also that K° is normal: for any $g \in K$, the conjugation automorphism is continuous while fixing the identity and therefore must send K° to itself.) Now, if v is K-finite, then of course it is K°-finite. On the other hand, if v is K°-finite, then the normality of $K^\circ \subseteq K$ implies that sv is still K°-finite for all $s \in S$, so the space

$$\operatorname{span}\{gv : g \in K\} = \operatorname{span}\{gsv : g \in K^\circ, s \in S\}$$

continues to be finite-dimensional.

Proposition 1.84. Fix a compact second countable topological group K. For any Fréchet representation V of K, the subspace V_{fin} is sequentially dense in V.

Proof. By Remark 1.83, we may assume that K is connected. Now, the trick is to convolve with an approximate identity. Let $\{\varphi_i\}$ be an approximate identity for K, which exists by Remark 1.79. For any $v \in V$, we note that $\{(\varphi_i\, dg)\} \to \delta_1$ (by Remark 1.80) implies that $\{(\varphi_i\, dg) \cdot v\} \to v$, where the action map is the one given by Remark 1.77.

It now remains to approximate $(\varphi_i\, dg) \cdot v$ by a K-finite vector. But this is easier because we know how to approximate functions. Indeed, by the Peter–Weyl theorem, we know that the K-finite vectors of $L^2(K)$ are given by the dense subspace

$$\bigoplus_{\rho \in \operatorname{IrRep}(K)} \rho \otimes \rho^*$$

of $L^2(K)$ and are in particular given by continuous functions. Thus, we may find a K-finite $\psi_i \in C(K)$ for which $\|\varphi_i - \psi_i\|_2 < 1/i$ for each i. It follows that $\varphi_i - \psi_i \to 0$ in $L^2(K)$, so we also have this limit in $L^1(K)$ because K is compact (via Cauchy–Schwarz), so $\varphi_i f - \psi_i f \to 0$ in $L^1(K)$ for any $f \in C(K)$, so $\varphi_i\, dg - \psi_i\, dg \to 0$ in $\operatorname{Meas}_c(K)$, so $\psi_i\, dg \to \delta_1$ in $\operatorname{Meas}_c(K)$.

Thus, we once again have $(\psi_i\, dg) \cdot v \to v$ as $i \to \infty$, and we can now show that $(\psi_i\, dg) \cdot v$ is K-finite. Indeed, for any $h \in G$, we note that

$$h \cdot (\psi_i\, dg \cdot v) = (\delta_h * \psi_i\, dg) \cdot v$$

by the associativity of our action (which, as usual, can be checked on δ-distributions). But now $\delta_h * \psi_i\, dg = h_*(\psi_i\, dg)$ by Example 1.75, which we claim is $h\psi_i\, dg$. Indeed, for any $f \in C(G)$, we see that

$$h_*(\psi_i\, dg)(f) = \int_G f(hg)\psi_i(g)\, dx = \int_G f(g)\psi_i\left(h^{-1}g\right)\, dg.$$

Thus, the span of the vectors $h \cdot (\psi_i\, dg \cdot v)$ equals the span of the vectors $(h\psi_i\, dg) \cdot v$, which is finite-dimensional because ψ_i is K-finite! ∎

Corollary 1.85. Fix a compact second countable topological group K. Any irreducible representation of K is finite-dimensional.

Proof. Proposition 1.84 shows that there is a nonzero K-finite vector, which spans a finite-dimensional subrepresentation. But then this subrepresentation must be the whole space by irreducibility! ∎

For our next application, we pass to Lie groups.

Definition 1.86 (smooth)**.** Fix a Lie group G. For any representation V of G, a vector $v \in V$ is *smooth* if and only if the function $g \mapsto gv$ is smooth. We let V_{sm} denote the collection of smooth vectors.

Remark 1.87. Suppose G is compact. If $V = C(G)$ (with G acting by either left or right multiplication), then $V_{\text{sm}} = C^\infty(G)$.

Proposition 1.88. Fix a Lie group G. Then V_{sm} is dense in G.

Proof. Apparently, this follows from the Peter–Weyl theorem because matrix coefficients are smooth. ■

Proposition 1.89. Fix a Lie group G and a compact Lie subgroup $K \subseteq G$. Then $V_{\text{fin}} \cap V_{\text{sm}}$ is sequentially dense in V. Here, V_{fin} refers to the K-finite vectors.

Proof. One uses a similar argument. Choose $\{\mu_i\}$ to be K-finite and $\{\mu_i'\}$ to be smooth. Then $\mu_i * \mu_i'$ is both K-finite and smooth. The same sort of convolution argument now goes through. ■

Let's preview what we will do next class. For any continuous representation V, one can expand

$$V_{\text{fin}} = \bigoplus_{L_i \in \operatorname{IrRep}(K)} V_i \otimes L_i,$$

where V_i is some finite-dimensional vector space. It turns out that there is also a natural action of $\mathfrak{g}$ on this space, so we are able to produce something called a $(\mathfrak{g}, K)$-module.

1.4 September 16

Today, we walk the roads laid by Harish-Chandra.

1.4.1 Admissible Representations

For today, G is a connected Lie group, $K \subseteq G$ is a compact Lie subgroup, and $\mathfrak{g}$ is the Lie algebra of G. We will soon take G to be reductive and K to be a maximal compact subgroup.

Notation 1.90. Fix a connected Lie group G with compact Lie subgroup K. Given an irreducible representation ρ of K, we may write

$$V_\rho := \operatorname{Hom}_K(\rho, V).$$

Remark 1.91. One finds that

$$V_{\text{fin}} = \bigoplus_{\rho \in \operatorname{IrRep}(K)} V_\rho \otimes \rho$$

as representations of K.

Remark 1.92. Fixing Haar measures, one finds that $V_\rho \otimes \rho$ can be realized as the image of the idempotent e_ρ given by sending 1_K through the composite

$$C^\infty(K) \hookrightarrow \operatorname{Meas}_c(K) \subseteq \operatorname{Meas}_c(G).$$

By choosing an approximate identity in the usual way, one can show that $V_\rho \otimes \rho \subseteq V_{\mathrm{sm}}$.

Admissibility is simply a size constraint on the size of the representation.

Definition 1.93 (admissible)**.** Fix a connected Lie group G with compact Lie subgroup K. A Fréchet representation V is K*-admissible* if and only if

$$\dim V_\rho < \infty$$

for each irreducible representation ρ of K. We may just say that V is *admissible* if K is understood from context.

Example 1.94. Take $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{O}(2)$. Let L_s be the s-densities of $\mathrm{SL}(2, \mathbb{R})$ acting on $\mathbb{RP}^1$. It turns out that the action of K on $\mathbb{RP}^1$ is transitive, and one can find that L_s is admissible because $\dim \operatorname{Hom}_K(\rho, L_s) \le 1$ for all irreducible representations ρ of K.

It turns out that the admissible representations are the ones that admit passage to algebra.

Lemma 1.95. Fix a connected Lie group G with compact Lie subgroup K. For any Fréchet representation V, we have $V_{\mathrm{fin}} \subseteq V_{\mathrm{sm}}$.

Proof. Matrix coefficients are smooth. ■

Of course, if we want to understand V, it is of course not just enough to restrict to K and look at V_{fin} on its own. To introduce more data, we note that there is an action of the Lie algebra $\mathfrak{g}$.

Notation 1.96. Fix a Fréchet representation V of a Lie group G with Lie algebra $\mathfrak{g}$. Then for each $X \in \mathfrak{g}$, we define an operator on V_{sm} by

$$Xv := \lim_{t \to 0} \frac{\gamma(y)v - v}{t},$$

where $\gamma \colon (-1, 1) \to G$ is some path with $\gamma(0) = 1$ and $\gamma'(0) = X$.

Remark 1.97. It turns out that this defines a Lie algebra representation.

Remark 1.98. One can check that V_{fin} is preserved by this $\mathfrak{g}$-action.

We are now ready to make the following definition.

Definition 1.99 (Harish-Chandra pair)**.** Fix a Lie algebra $\mathfrak{g}$ and compact Lie group K. Suppose that there is a map $\varphi \colon \operatorname{Lie} K \to \mathfrak{g}$ and an action of K on $\mathfrak{g}$ for which φ preserves the two actions. In this situation, we call $(\mathfrak{g}, K)$ a *Harish-Chandra pair*.

Definition 1.100 ($(\mathfrak{g}, K)$-module)**.** Fix a Lie algebra $\mathfrak{g}$ and compact Lie group K for which $(\mathfrak{g}, K)$ is a Harish-Chandra pair.

(a) A $(\mathfrak{g}, K)$-*module* is a vector space M with actions by K and $\mathfrak{g}$ for which M is a sum of finite-dimensional continuous representations of K and the two induced actions by $\operatorname{Lie} K$ coincide.

(b) A $(\mathfrak{g}, K)$-module M is *admissible* if and only if $\dim \operatorname{Hom}_K(\rho, M)$ is finite for all irreducible representations ρ of K.

(c) An admissible $(\mathfrak{g}, K)$-module M

Remark 1.101. Here are some motivational notes. Complex representations of $\mathfrak{g}$ more or less reduce to representations of $\mathfrak{g}_\mathbb{C}$. Furthermore, finite-dimensional representations of K are basically the algebraic ones, which also pass to $K_\mathbb{C}$. Thus, in our definition of $(\mathfrak{g}, K)$-module, we may as well make everything complex and require the K-action on M to be algebraic.

Example 1.102. Fix a compact connected real Lie group K, and we set $G := K_\mathbb{C}$, which we think of as a real Lie group via restriction of scalars. Then we set $\mathfrak{g}$ to be $\operatorname{Lie} G = \mathfrak{k} \otimes \mathbb{C}$, where $\mathfrak{k} := \operatorname{Lie} K$; note $\mathfrak{g}_\mathbb{C}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$. Using the diagonal action $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, we receive a Harish-Chandra pair. In this way, we see that a Harish-Chandra module is the same as a $(\mathfrak{g}, \mathfrak{g})$-module which is admissible for the diagonal copy of $\mathfrak{g}$.

Example 1.103. Consider the $\mathfrak{g}$-bimodule $U\mathfrak{g}$, where the first copy of $\mathfrak{g}$ acts by $X : Y \mapsto XY$, and the second copy of $\mathfrak{g}$ acts by $X : Y \mapsto -YX$. This turns out to not be admissible.

1.4.2 Weakly Analytic Vectors

We are going to want access to some weaker smoothness conditions.

Definition 1.104 (matrix coefficient)**.** Fix a Fréchet representation V of a Lie group G. For any $v \in V$ and $\ell \in V^*$, we define the *matrix coefficient* $G \to \mathbb{C}$ by

$$g \mapsto \ell(gv).$$

Definition 1.105 (weakly analytic)**.** Fix a Fréchet representation V of a Lie group G. A vector $v \in V$ is *weakly analytic* if and only if the matrix coefficient map $g \mapsto \ell(gv)$ is real analytic for all $\ell \in V^*$.

Here is the main result we will need.

Theorem 1.106 (Harish-Chandra analyticity)**.** Fix a semisimple Lie group G with compact subgroup K. For each admissible Fréchet representation V of a Lie group G, every $v \in V_{\text{fin}}$ is weakly analytic.

To prove this, one uses elliptic regularity, so we spend a moment establishing some theory.

Definition 1.107 (differential operator)**.** Fix a smooth manifold X of dimension n. Then a *differential operator* d is a section of the vector bundle which is locally on a chart (x_i) is given by finite sums

$$\sum_{i \colon \{1, \ldots, n\} \to \mathbb{N}} \varphi_i \partial x_1^{i_1} \cdots \partial x_n^{i_n}.$$

The *order* of the operator is the maximum of $\sum_j i_j$ as $i \colon \{1, \ldots, n\} \to \mathbb{N}$ varies such that $\varphi_i \neq 0$.

Remark 1.108. The differential operators form a graded vector bundle. We define the principal symbol $\sigma(d)$ to be the entry in the associated graded ring.

Remark 1.109. We find that $\sigma(d)$ is a function on T^*X, and it is a polynomial of degree N.

Definition 1.110 (elliptic)**.** The operator d is *elliptic* if and only if $\sigma(d)(x, \xi) \neq 0$ for all x and ξ nonzero.

Appendix A
Functional Analysis

In this appendix, we introduce the small amount of functional analysis we will need in order to get going with infinite-dimensional vector spaces. In other words, we need to set up the theory of Fréchet spaces. Throughout this appendix, $\mathbb{F}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$. Our exposition is largely stolen from [Con90].

A.1 Locally Convex Spaces

We begin with the following definition.

Definition A.1 (topological vector space)**.** Fix a topological field k. Then a *topological vector space* is a vector space V over k equipped with a topology so that addition map $+\colon V \times V \to V$ and scalar multiplication map $\cdot\colon k \times V \to V$ are both continuous. In these notes, all topological vector spaces will be assumed to be Hausdorff.

A Fréchet space will be a complete topological vector space admitting two notable definitions: having its topology is defined by a countable family of seminorms, or being locally convex and metrizable. As such, let's quickly recall the definition of a seminorm.

Definition A.2 (seminorm)**.** Fix a vector space V over $\mathbb{F}$. Then a *seminorm* is a function $p\colon V \to \mathbb{R}$ satisfying the following.

- Subadditive: we have $p(x+y) \le p(x) + p(y)$ for any $x, y \in V$.
- Homogeneous: we have $p(\lambda x) = |\lambda|\, p(x)$ for any $x \in V$ and $\lambda \in \mathbb{F}$.

Remark A.3. The homogeneity implies that $p(0) = 0$, which is sometimes included in the definition. Similarly, the subadditivity now implies that $2p(x) = p(x) + p(-x)$ is at least $p(0) = 0$, so p is automatically nonnegative; this is also sometimes included in the definition.

It is worthwhile to have a few ways to check continuity.

Lemma A.4. Fix a topological vector space V over $\mathbb{F}$, and let $p\colon V \to \mathbb{R}$ be a seminorm. Then the following are equivalent.

(i) p is continuous.

(ii) $\{v : p(v) < 1\}$ is open.

(iii) p is continuous at 0.

Proof. This is [Con90, Proposition 1.3]. Of course (i) implies (ii). We show the remaining implications independently.

- We show (ii) implies (iii). For any net $\{x_i\}$ converging to 0, we want to show that $\{p(x_i)\} \to 0$, which is true because any x_i in the open neighborhood εU of 0 has $p(x_i) < \varepsilon$.
- We show (iii) implies (i). Note that any net $\{x_i\}$ converging to some x has

$$|p(x_i) - p(x)| \le p(x_i - x),$$

and $p(x_i - x) \to 0$ because $x_i - x \to 0$. ■

We now start talking about convex sets, but we will relate our definitions back to seminorms.

Definition A.5 (convex)**.** Fix a vector space V over $\mathbb{F}$. A subset $A \subseteq V$ is *convex* if and only if any two $a, b \in A$ has

$$ta + (1-t)b \in A$$

for any $t \in [0,1]$.

Example A.6. Let $p\colon V \to \mathbb{R}$ be a seminorm. Then we claim that $A := \{v \in V : p(v) < 1\}$ is convex. Indeed, for $a, b \in A$ and $t \in [0,1]$, we see that

$$p(ta + (1-t)b) = tp(a) + (1-t)p(b),$$

which is still less than 1, so $ta + (1-t)b \in A$.

Example A.7 (convex hull)**.** For any subset $A \subseteq V$, we may define the convex hull

$$\operatorname{conv}(A) := \left\{ \sum_{i=1}^n t_i a_i : \{a_i\}_i \subseteq A, \{t_i\}_i \in [0,1], t_1 + \cdots + t_n = 1 \right\}.$$

Note that $\operatorname{conv}(A)$ is convex: for two points $\sum_i t_i a_i$ and $\sum_j s_j b_j$ and $t \in [0,1]$, the sum $\sum_i t t_i a_i + \sum_j (1-t) s_j b_j$ still has $\sum_i t t_i + \sum_j (1-t) s_j = t + (1-t) = 1$. In fact, if B is convex and contains A, then $\operatorname{conv}(A) \subseteq B$ because the sums $\sum_i t_i a_i$ can be checked to be in B by induction.

Convex sets on their own turn out to not be good enough for our purposes, so we will need extra adjectives.

Definition A.8 (balanced)**.** Fix a vector space V over $\mathbb{F}$. A subset $A \subseteq V$ is *balanced* if and only if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{F}$ such that $|\lambda| \le 1$.

Example A.9. Let $p\colon V \to \mathbb{R}$ be a seminorm. Then we claim that $A := \{v \in V : p(v) < 1\}$ is balanced. Indeed, for $a \in A$ and λ with $|\lambda| \le 1$, we see that $p(\lambda a) = |\lambda|\, p(a)$, which is still less than 1, so $\lambda a \in A$.

Example A.10. For any subset $A \subseteq V$, the subset

$$\operatorname{bal}(A) := \bigcup_{|\lambda| \le 1} \lambda A$$

is balanced. Indeed, for any μ with $|\mu| \le 1$, we see that $\mu \operatorname{bal}(A) = \bigcup_\lambda \mu\lambda A$ is contained in $\operatorname{bal}(A)$ because $|\lambda\mu| \le 1$ whenever $|\lambda| \le 1$. Of course, we always have $A \subseteq \operatorname{bal}(A)$, and one can see that any balanced set containing A must contain each λA and hence contain $\operatorname{bal}(A)$.

It turns out that passing to balanced convex sets is not too big of a burden.

Lemma A.11. Fix a topological vector space V over $\mathbb{F}$. Any convex open neighborhood of 0 contains a balanced convex open neighborhood of 0.

Proof. Let U be a convex open neighborhood of 0. The point is to use the continuity of scalar multiplication: the continuity of

$$\cdot \colon \mathbb{F} \times V \to V$$

provides a basic open neighborhood $B(0, \varepsilon) \times U'$ of $(0,0)$ of $\mathbb{F} \times V$ such that $B(0,\varepsilon)U' \subseteq U$. We claim that $\operatorname{conv}(B(0,\varepsilon)U')$ is the desired open neighborhood. Here are our checks; set $U'' := B(0,\varepsilon)U'$ for brevity.

- Because U is already convex, Example A.7 explains that $U'' \subseteq U$ implies that $\operatorname{conv}(U'') \subseteq U$.
- Convex: note $\operatorname{conv}(U'')$ is convex by Example A.7.
- Open: because scalar multiplication by a nonzero number is a homeomorphism, we see that $U'' := B(0,\varepsilon)U'$, which is

  $$\bigcup_{0 < |\lambda| < \varepsilon} \lambda U',$$

  is open. Then once U' is open, we see that $\operatorname{conv}(U')$ can be written as a union

  $$\bigcup_{n \ge 1} \left(\bigcup_{t_1 + \cdots + t_n = 1} t_1 U'' + \cdots + t_n U'' \right),$$

  which is open because the sum of two open subsets is open (indeed, the sum of open sets is a union of translates of just one of the open sets).
- Balanced: the previous step realized U'' as a union $\bigcup_{0<|\lambda|<\varepsilon} \lambda U'$, which can be shown to be balanced exactly as in Example A.10. ■

Remark A.12. Here is a sample application: suppose that 0 has a neighborhood basis of convex sets. Then Lemma A.11 implies that 0 also admits a neighborhood basis of balanced convex sets: for each U in the neighborhood basis, the lemma produces a smaller open subset which is still convex but now also balanced. Similarly, admitting a countable neighborhood basis of convex sets can be upgraded to admitting a countable neighborhood basis of balanced convex sets.

The previous remark allows us to make the following definition.

Definition A.13 (locally convex)**.** A topological vector space V is *locally convex* if and only if 0 admits a neighborhood basis of convex sets.

Remark A.14. By translation, it is equivalent to require V to have a basis of convex sets. (Namely, if $\mathcal{U}$ is the neighborhood basis of 0, then $\bigcup_{v\in V} v+\mathcal{U}$ is the basis of V.) By Remark A.12, we can also upgrade these notions to having bases of balanced convex sets.

It is notable that the previous two definitions avoid the mention of any topology. In order to continue not doing any topology, we pick up the following definition, which provides a linear algebraic stand-in for "contains an open neighborhood of the origin."

Definition A.15 (absorbing). Fix a vector space V over $\mathbb{F}$. A subset $A \subseteq V$ is *absorbing* if and only if any $v \in V$ admits some $\varepsilon > 0$ such that $tv \in A$ for all $t \in [0, \varepsilon)$.

For example, we see that 0 is contained in any absorbing subset.

Remark A.16. Of course, if A is absorbing, and $A \subseteq B$, then B is absorbing: for each v, the ε which worked for A continues to work for B.

Example A.17. Let's explain the remark given before the definition. If V is a topological vector space over $\mathbb{F}$, then we claim that any open neighborhood U of 0 is absorbing. This will follow by the continuity of scalar multiplication: for any $v \in V$, the map $\mathbb{R} \to V$ given by $t \mapsto tv$ is continuous. Thus, because $0 \in U$, there must be $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon)v \subseteq U$.

Example A.18. Let $p\colon V \to \mathbb{R}$ be a seminorm, and set $A := \{v \in V : p(v) < 1\}$. Then we claim that $(-a) + A$ is absorbing for all $a \in A$; for example, setting $a = 0$ will imply that A is absorbing. Now, for any $v \in V$, we need to show that $a + tv \in A$ for small t. Well, $p(a + tv) \le p(a) + |t|\, p(v)$, so taking any t with $p(v)\,|t| < (1 - p(a))$ will do. (In particular, any t will work if $p(v) = 0$.)

We are now ready to construct some seminorms.

Notation A.19. Fix a vector space V over $\mathbb{F}$. For any absorbing subset $A \subseteq V$, we define $\|\cdot\|_A : V \to \mathbb{R}_{\ge 0}$ by

$$\|v\|_A = \inf\{t \ge 0 : v \in tA\}.$$

Here are some basic facts about this construction.

Remark A.20. Because A is absorbing, we see that any $v \in V$ does in fact have some $t > 0$ for which $(1/t)v \in A$ and hence $v \in tA$, so the infimum is a real number.

Remark A.21. Suppose further that A is convex. Then we claim that $v \in tA$ whenever $t > \|v\|_A$; note if $v = 0$, then $\|0\|_A = 0$, so there is nothing to do. The main point is to note that $sv \in A$ implies that $s'v \in A$ for any $s' \in [0, s]$ by convexity. Thus, if $t > \|v\|_A$, then we know there is $s < t$ such that $v \in sA$, so $(1/s)v \in A$ while $1/t < 1/s$, so $(1/t)v \in A$, so $v \in tA$.

Let's put all our adjectives together, finally explaining the relationship between seminorms and convex sets.

Proposition A.22. Fix a vector space V over $\mathbb{F}$ and a subset $A \subseteq V$.

(a) There is a seminorm $p\colon V \to \mathbb{R}$ such that $A = \{v \in V : p(v) < 1\}$ if and only if A is nonempty, convex, balanced, and $(-a) + A$ is absorbing for all $a \in A$.

(b) If A is merely convex, balanced, and absorbing, then there is a seminorm $p\colon V \to \mathbb{R}$ such that $\{v : p(v) < 1\} \subseteq A$.

Proof. This is [Con90, Proposition 1.14]. The forward direction of (a) follows from combining Examples A.6, A.9 and A.18. It remains to show the reverse direction of (a) and (b). For both of these, we will take $p := \|\cdot\|_A$, which is a well-defined function by Remark A.20 (once we know that A in (a) is absorbing). We will run our checks in a few pieces.

- In (a), we show that A is absorbing. It is enough to check that $0 \in A$. Well, being nonempty, there is some $a \in A$. Because A is balanced, we see $-a \in A$, and because A is convex, it follows that $0 \in A$.
- If A is absorbing and balanced, we check that $p(\lambda v) = |\lambda|\, p(v)$ for $v \in V$ and $\lambda \in \mathbb{F}$. Well, $\lambda v \in tA$ if and only if $v \in \frac{t}{\lambda}A$, which is equivalent to $v \in \frac{t}{|\lambda|}A$ because A is balanced! It follows that
$$\{t \geq 0 : \lambda v \in tA\} = |\lambda| \left\{ \frac{t}{|\lambda|} : v \in \frac{t}{|\lambda|}A \right\},$$
so the check follows.
- If A is convex and absorbing, then we check that $p(v+w) \leq p(v) + p(w)$ for $v, w \in V$. The geometric input is that $tA + sA \subseteq (t+s)A$ for any $t, s > 0$; this follows by convexity because
$$tA + sA = (t+s)\left(\frac{t}{t+s}A + \frac{s}{t+s}A\right)$$
is contained in $(t+s)A$ by convexity. Now, for the check, we note that having $p(v) < t$ and $p(w) < s$ implies that $v \in tA$ and $w \in sA$ by Remark A.21, so $v + w \in (t+s)A$, so $p(v+w) \leq t + s$. Sending $t \to p(v)$ and $s \to p(w)$ completes the check.
- We complete the proof of (b). The above checks show that p is a seminorm, so it remains to check that $\{v : p(v) < 1\} \subseteq A$. This follows from A being balanced: if $p(v) < 1$, then there is $t < 1$ such that $v \in tA$, and $tA \subseteq A$ because A is balanced.
- We complete the proof of (a). The previous check shows that $\{v : p(v) < 1\} \subseteq A$, so it remains to check the other inclusion. Well, for any $a \in A$, we see that $(-a) + A$ is absorbing, so $a + ta \in A$ for small $t > 0$. It follows that $a \in (1+t)^{-1}A$, so $p(a) < (1+t)^{-1} < 1$ follows. ■

Corollary A.23. Fix a topological vector space V over $\mathbb{F}$. The following are equivalent.

(i) V is locally convex.

(ii) The topology on V is induced by a family of seminorms.

Proof. We show the implications separately.

- Suppose that V is locally convex, so 0 admits a neighborhood basis $\mathcal{U}$ of balanced convex sets by Remark A.14. By Example A.17, we see that each $U \in \mathcal{U}$ has $(-a) + U$ absorbing for all $a \in U$, so Proposition A.22 provides a seminorm $p_U : V \to \mathbb{R}$ such that $U = \{v : p_U(v) < 1\}$. Note that p_U is continuous by Lemma A.4.

  Lastly, we should check that the topology given by the seminorms $\{p_U\}$ is the correct one. Well, this topology has basis given by finite intersections of sets of the form
$$\{v \in V : p_U(v) \in (a,b)\},$$
  where $(a,b) \subseteq \mathbb{R}$. The continuity of the p_Us implies that any such subset is open in V. Conversely, any open neighborhood of 0 in V contains some $U \in \mathcal{U}$ and therefore contains $\{v \in V : p(v) \in (-1,1)\}$, so a comparison of the neighborhood bases (via translation) implies that the open neighborhood of 0 is still open.

- Suppose that V has its topology generated by a family of seminorms $\{p_i\}$. Well, because $p_i(0) = 0$ for each i, an open neighborhood basis of 0 can be given by finite intersections of sets of the form $p_i^{-1}((-\varepsilon, \varepsilon))$. Of course, this is just
$$\varepsilon\{v \in V : p_i(v) < 1\},$$
which we note is convex by Example A.6. Thus, 0 admits a neighborhood basis of convex sets. ■

Now that we have an understanding of locally convex spaces, we may define Fréchet spaces.

Definition A.24 (complete)**.** Fix a topological vector space V. Then a *Cauchy net* in V is a net $\{v_i\}$ such that each neighborhood U of 0 has some N for which $v_i - v_j \in U$ for all $i, j > N$. We say that V is *complete* if every Cauchy net converges.

Remark A.25. If V is metrizable, then we may as well work with Cauchy sequences and being sequentially complete.

Definition A.26 (Fréchet)**.** A topological vector space X is *Fréchet* if and only if it is locally convex, metrizable, and complete.

The bizarre addition here is metrizable. This condition fits in with the other ones as follows.

Proposition A.27. Fix a locally convex topological vector space V over $\mathbb{F}$. Then the following are equivalent.

(i) V has its topology induced by a translation-invariant metric.

(ii) V is metrizable.

(iii) V has a countable neighborhood basis of 0.

(iv) The topology on V is induced by a countable family of seminorms.

Proof. The implication (i) to (ii) has no content, and (ii) to (iii) follows by taking the neighborhood basis of open subsets given by $\{v : d(v, 0) < 1/n\}$ for positive integers n. Next, (iii) implies (iv) by the proof of the forward direction of Corollary A.23, which built one seminorm for each balanced convex subset in the neighborhood basis of 0.

Lastly, we have to show that (iv) implies (i). Well, given the countable family of seminorms $\{p_i\}_{i\geq 1}$, we define the function $d\colon V \times V \to \mathbb{R}$ by

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x-y)}{1 + p_i(x-y)}.$$

Here are our checks on d.

- We check that d is a metric. The summation always converges because $\frac{p_i(x-y)}{1+p_i(x-y)} \leq 1$ always. Continuing, $d(x, x) = 0$ follows because $p_i(0) = 0$ for all i, and the triangle inequality follows from the subadditivity of each of the p_is.

  It remains to check the positivity of d. Well, if $x \neq y$, then because V is Hausdorff, we see that $p_i(x - y) > 0$ for some p_i. (Otherwise, the constant net $\{x - y\}$ would converge to both 0 and $x - y$.) Thus, $d(x, y) > 0$.

- We check that d is translation-invariant. Well, for any $a \in V$, we see that $d(x + a, y + a)$ is a function of $(x + a) - (y + a) = (x - y)$ and will equal $d(x, y)$.

- Lastly, we check that d induces the topology on V. It is enough to check that these two topologies have the same convergent nets. Well, a net $\{x_i\}$ converges to some $x \in V$ if and only if $p_\bullet(x_i - x) \to 0$ for all seminorms $p_\bullet$. This surely implies that $d(x_i, x) \to 0$, and conversely, $d(x_i, x) \to 0$ will require that $p_\bullet(x_i - x) \to 0$ for each $p_\bullet$. ■

A.2 Cauchy Sequences

Because we will have occasion to work with more general topological spaces than metric spaces, we go ahead and pick up some facts about Cauchy sequences and completeness.

Lemma A.28. Fix a continuous linear map $f\colon V \to V'$ of topological vector spaces. Then f sends Cauchy nets to Cauchy nets.

Proof. Suppose that $\{a_i\}_{i\in I}$ is a Cauchy net of V, and we would like to show that $\{f(a_i)\}_{i\in I}$ is a Cauchy net of V'. Well, for any open neighborhood $U' \subseteq V'$ of 0, we are granted an open neighborhood $U \subseteq V$ of 0 for which $x \in U$ implies that $f(x) \in U$. In particular, there is $N \in I$ for which $i, j > N$ implies that $a_i - a_j \in U$ and thus $f(a_i) - f(a_j) \in U'$, as required. ■

Definition A.29 (Cauchy sequence)**.** Fix a topological vector space V. Then a *Cauchy sequence* is a Cauchy net where the index set is a $\mathbb{N}$.

Proposition A.30. Fix a sequentially dense linear subspace $W \subseteq V$, and choose a sequentially continuous linear map $f\colon W \to V'$ of topological vector spaces.

(a) There is at most one sequentially continuous linear map $g\colon V \to V'$ such that $g|_W = f$.

(b) If V' is sequentially complete, then there is a sequentially continuous linear map $g\colon V \to V'$ such that $g|_W = f$.

Proof. We start with (a). By linearity of restriction, it is enough to show that any continuous linear map $g\colon V \to V'$ for which $g|_W = 0$ has $g = 0$. Well, for any $v \in V$, the density of $W \subseteq V$ grants a Cauchy sequence $\{w_i\}$ in W for which $\{w_i\} \to v$. Thus, the continuity of g requires that $\{g(w_i)\} \to g(v)$, so $g(v) = 0$.

It remains to show (b), which is trickier. We begin by defining g, and then we will run many checks on it. Because $W \subseteq V$ is sequentially complete, each $v \in V$ admits a Cauchy sequence $\{w_i\}$ of W such that $\{w_i\} \to v$. Then Lemma A.28 implies that $\{f(w_i)\}$ is still a Cauchy sequence in V', so we define

$$g(v) \coloneqq \lim_{i\to\infty} f(w_i).$$

(The limit exists because V' is sequentially complete!) It remains to run many checks on g.

- Well-defined: if $\{w_i\}$ and $\{w_i'\}$ are two Cauchy sequences approaching a given $v \in V$, then we need to check that $\lim f(w_i) = \lim f(w_i')$. Well, we see that $\{w_i - w_i'\} \to 0$, so $\{f(w_i) - f(w_i')\} \to f(0)$ by continuity of f, so the claim follows.

- Linear: given scalars a^1 and a^2 and $v^1, v^2 \in V$, we should check that $g\left(a^1v^1 + a^2v^2\right) = a^1$. Well, we simply pick up Cauchy sequences $\{w_i^1\}$ approaching v^1 and $\{w_i^2\}$ approaching v^2. Then $\{a^1w_i^1 + a^2w_i^2\}$ approaches $a^1v^1 + a^2v^2$, so we can pass everything through the definition of g.

- Continuous: given a convergent sequence $\{v_i\} \to v$ in V, we would like to check that $\{g(v_i)\} \to g(v)$. By linearity of g, we may assume that $v = 0$ (by subtracting out v everywhere). Now, for any open neighborhood $U' \subseteq V'$ of 0, we would like to show that $g(v_i) \in U'$ for large i. For technical reasons, we choose an open neighborhood $U_0' \subseteq U'$ of 0 such that $U_0' + U_0' \subseteq U'$, which is possible by the continuity of addition. The point of doing this is that the closure of U_0' is contained in U': for any Cauchy net $\{u_i'\}$ in U_0' converging to some u', we see that $u' - u_i' \in U_0'$ for any i large enough, so $u' \in U_0' + U_0'$.

Now, by continuity of f, we are granted an open neighborhood $U \subseteq V$ of 0 such that $x \in U \cap W$ implies $f(x) \in U_0'$. For any i large enough, we see that $v_i \in U_0$. Then for each i, we fix a Cauchy sequence $\{w_{ij}\}$ of W approaching v_i; for any j large enough, we see $w_{ij} \in U_0$ as well. Thus, $\{f(w_{ij})\} \to g(v_i)$ by definition of g, so $g(v_i)$ is in the closure of U_0', so $g(v_i) \in U'$, as required. ■

Remark A.31. By replacing all sequences with nets in the proof, we can upgrade the statement to show that continuous linear maps extend uniquely from dense linear subspaces when the target is complete.

A.3 The Open Mapping Theorem

In this section, we review the proof of the Open mapping theorem in order to extend the usual proof (for Banach spaces) to the setting of Fréchet spaces.

As usual, our proof will have to rely on the Baire category theorem. Before introducing any strange terminology, let's start with a statement on just metric spaces.

Lemma A.32. Let X be a nonempty complete metric space. Then a countable intersection of dense open subsets is dense.

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be our collection of dense open subsets. We would like to show that their intersection $\bigcap_{i\in\mathbb{N}} U_i$ intersects any open subset V of X. The idea is to recursively choose nearby elements in $U_i \cap V$ for each i, and then use completeness of X to finish the proof. We proceed in steps.

1. We build a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ recursively, as follows. To start us off, we note $V \cap U_0$ is nonempty and open (by density of U_0), so we are granted a point $x_0 \in U_0$ and ε_0 such that $B(x_0, \varepsilon_0) \subseteq V \cap U_0$. For the recursion, we suppose that we are given such an open neighborhood $B(x_n, \varepsilon_n)$, and then because U_{n+1} is open and dense, we are provided a point x_{n+1} in the intersection and some $\varepsilon_{n+1} < \varepsilon_n/3$ such that
$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \cap U_{n+1}.$$
2. We claim that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is (rapidly) Cauchy. Indeed, note $\varepsilon_{n+1} < \varepsilon_n/2$ for each n, so $\varepsilon_n < 2^{-n}$ follows by an induction. Thus, $d(x_n, x_{n+1}) < 2^{-n}$ for each n, so our sequence is rapidly Cauchy. To finish checking that it is Cauchy, we note that whenever $i < j$, we have
$$d(x_i, x_j) \le \sum_{k=i}^{j-1} \underbrace{d(x_k, x_{k+1})}_{<2^{-k}},$$
which is upper-bounded by 2^{-i+1}.
3. Now, we let x be a limit point of $\{x_n\}_{n\in\mathbb{N}}$. (This is where we used completeness!) Because our sequence is eventually in $B(x_n, \varepsilon_n)$ for any given n, we see that $x \in B(x_n, \varepsilon_n)$ for each n. Thus, $x \in V \cap U_0$ by the first step of the construction, and $x \in U_n$ for each $n \ge 1$ by the recursive step of the construction. ■

The previous lemma now upgrades to the Baire category theorem.

Theorem A.33 (Baire category)**.** Let X be a nonempty complete metric space. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable collection of dense open subsets. Then the intersection $\bigcap_{i\in\mathbb{N}} U_i$ is not contained in a countable union of nowhere dense subsets.

Proof. Suppose for the sake of contradiction that we have

$$\bigcap_{i\in\mathbb{N}} U_i \subseteq \bigcup_{j\in\mathbb{N}} A_j,$$

where each A_j is nowhere dense. It thus follows that

$$\bigcap_{i\in\mathbb{N}} U_i \cap \bigcap_{j\in\mathbb{N}} X \setminus \overline{A_j}$$

is empty. We claim that $X \setminus \overline{A_j}$ is open and dense, which yields the desired contradiction by Lemma A.32. Certainly $X\setminus\overline{A_j}$ is open; for density, note that $\overline{A_j}$ contains no open subset, which means that the complement intersects any open subset. ■

Corollary A.34. Let X be a nonempty complete metric space. Then X is not the countable union of nowhere dense subsets.

Proof. This follows from taking $U_i = X$ for each i in Theorem A.33. ■

We now proceed with the Open mapping theorem. We will isolate the application of the Baire category theorem to the following lemma: Corollary A.34 shows that the hypothesis is satisfied.

Lemma A.35. Let $f\colon X \to Y$ be a linear map of locally convex topological vector spaces. Suppose that $\operatorname{im} f$ is not the union of nowhere dense subsets. Then $\overline{f(U)}$ contains an open neighborhood of 0 for each open neighborhood U of 0.

Proof. This more or less follows from unwinding the hypothesis. Because V is locally convex, we may shrink U to make it convex and balanced (by Lemma A.11); we will only use this at the end of the proof. The hypothesis is applied as follows: because U is open, it is absorbing (by Example A.17), so $V = \bigcup_{i\in\mathbb{N}} iU$, so

$$\operatorname{im} f = \bigcup_{i>0} iU.$$

Thus, the hypothesis implies that one of the iU fails to be nowhere dense; because multiplication by $i > 0$ is a homeomorphism, we see that U is also fails to be nowhere dense, so it contains an open subset V.

We now upgrade V into an open neighborhood of 0. Well, simply set $V' := \frac{1}{2}(V - V)$. Then $f(V')$ is contained in $\frac{1}{2}(U - U)$ by linearity, which is $\frac{1}{2}(U + U)$ because U is balanced, which is contained in U because U is convex. ■

It remains to use the hypothesis that X is complete, which is done in the following lemma.

Lemma A.36. Let $f\colon X \to Y$ be a continuous linear map of metrizable locally convex topological spaces. Suppose that X is complete and that $\overline{f(U)}$ contains an open neighborhood of 0 for each open neighborhood U of 0. Then f is open.

Proof. The idea is to use the completeness of X to construct points of U which go to a required open neighborhood. We proceed in steps.

1. We are going to show that $f(U)$ contains an open neighborhood of 0 for each open neighborhood U of 0, so let's spend a moment to explain why this is enough. For each open subset $U' \subseteq X$ and $x \in U'$, we note that $f(U' - x)$ contains an open neighborhood V_x of the origin. Thus, $f(U')$ contains the open neighborhood $f(x) + V_x$, so $f(U')$ equals

$$\bigcup_{x\in U'} f(x) + V_x,$$

which is open because it is a union of open subsets.

2. We unwind the hypothesis on f. By shrinking our open neighborhood U of 0, we may assume that U is convex and balanced (by Lemma A.11), so there is a seminorm p on X for which U is $B(0,1)$ for some translation-invariant metric d on X, chosen via Proposition A.27. Similarly, by hypothesis on f, we know that $\overline{f(U)}$ contains some open neighborhood V of 0, which we may again shrink until it is $B(0,2)$ for some translation-invariant metric d on Y. It will be worthwhile to remove the closure from this statement. Well, for any $y \in Y$, we see that $y \in d(y,0)V$ and so $y \in \overline{q(y) \cdot f(U)}$, so any $\varepsilon > 0$ has some $x \in X$ for which $d(x,0) < d(y,0)$ and $d(y, f(x)) < \varepsilon$.

3. We will actually show that $V \subseteq f(U)$, so choose some $y \in V$. The completeness of X will be used via a limiting process to produce an element of U mapping to y. To start us off, fix some $\varepsilon > 0$ (to be fixed at the end of the proof), and we take $x_0 \coloneqq 0$. Now, if we are given $x_0 + \cdots + x_n$, we may select x_{n+1} so that $d(x_{n+1}, 0) < d(y - f(x_0 + \cdots + x_n), 0)$ and
$$d(y - f(x_0 + \cdots + x_n), f(x_{n+1})) < \varepsilon/2^{n+1}$$
by the above paragraph.

4. We complete the proof. Now, by construction, $d(x_n, 0) < \varepsilon/2^{n-1}$ for all $n \ge 2$, so the sequence of partial sums is rapidly Cauchy. As in the proof of Lemma A.32, it follows that these partial sums converge to some $x \in X$.

   We claim that this x is the desired element. To start, we see that $f(x_0 + \cdots + x_n) \to y$ as $n \to \infty$ by construction, so $f(x) = y$ by continuity of f!

   It remains to check that $x \in U$. Well, $d(x,0)$ is bounded by
$$\sum_{n=0}^{\infty} d(x_n, 0) < d(x_0, 0) + d(x_1, 0) + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}},$$
   and $d(x_1, 0) < d(y, 0) < 2$ and $\sum_{n=2}^{\infty} \frac{\varepsilon}{2^{n-1}} = \varepsilon$, so $x \in U$ for ε small enough. ■

Theorem A.37 (Open mapping)**.** Let $f \colon X \to Y$ be a continuous linear map of Fréchet spaces. If f is surjective, then f is open.

Proof. By Corollary A.34, we see that X is not a countable union of nowhere dense subsets. The result now follows from combining Lemmas A.35 and A.36. ■

Corollary A.38. Let $f \colon X \to Y$ be a bijective continuous linear map of Fréchet spaces. Then f has continuous inverse.

Proof. Let g be the inverse map. Checking that g is continuous is equivalent to checking that f is open, which follows from Theorem A.37. ■

A.4 The Hahn–Banach Theorems

In this section, we review the proof of the Hahn–Banach theorem. This section will be filled with plenty of nonsense. Ultimately, we are interested in extending continuous linear functionals on Fréchet spaces, but along the way, we will show that linear functionals separate convex sets.

For one of our applications, we will require a weaker notion than a seminorm.

Definition A.39. Fix a vector space V over $\mathbb{F}$. A function $p\colon V \to \mathbb{R}_{\geq 0}$ is a half-seminorm if and only if it is sublinear and satisfies $p(tx) = tp(x)$ for $t > 0$ and $x \in V$.

The lack of input for $p(-x)$ prevents p from being a seminorm.

As with Banach spaces, we check if a linear functional is continuous by checking if it is bounded, but the definition of bounded needs to be adjusted.

Definition A.40 (bounded). Fix a topological vector space V over $\mathbb{F}$. A linear functional $\ell\colon V \to \mathbb{F}$ is *bounded* if and only if there is an open neighborhood U of 0 and a constant $c > 0$ such that $|\ell(x)| \leq c$ for all $x \in U$.

Lemma A.41. Fix a topological vector space V over $\mathbb{F}$ and a linear functional ℓ on V. Then the following are equivalent.

(i) ℓ is continuous.

(ii) ℓ is bounded.

(iii) ℓ is continuous at 0.

Proof. We use Lemma A.4. The implication from (i) to (ii) is direct; the proof that (iii) implies (i) is identical to the proof in Lemma A.4. To show (ii) implies (iii), we note $|\ell|$ is a seminorm, and by considering nets, we see that it is enough to check that $|\ell|$ is continuous at 0, which follows from Lemma A.4(ii) and the fact that ℓ is bounded. ■

Corollary A.42. Fix a topological vector space V over $\mathbb{F}$, and let $p\colon V \to \mathbb{R}_{\geq 0}$ be a continuous half-seminorm. Given a linear functional $\ell\colon V \to \mathbb{F}$, if $-p(-v) \leq \ell(v) \leq p(v)$ for all v, then ℓ is continuous.

Proof. Because $-p(-v) \leq \ell(v) \leq p(v)$ for all v, we see that the open subset $U := \{v \in V : p(v) < 1\}$ bounds ℓ (with the constant 1), so we are done by Lemma A.41. ■

Thus, it will be important to be able to extend linear functionals along with an upper bound against a seminorm. By considerations with Zorn's lemma, we will find that the hard part is extending the linear functional one step, which is the content of the next lemma.

Lemma A.43. Fix a vector space V over $\mathbb{R}$, a half-seminorm p on V, and a linear functional ℓ on a subspace $W \subseteq V$ such that $-p(-w) \leq \ell(w) \leq p(w)$ for all $w \in W$. Given any $v' \in V$, there is an extension ℓ' to a linear functional on a subspace W' containing v' such that $-p(-v) \leq \ell'(v) \leq p(v)$ for all v.

Proof. If $v' \in W$ already, then there is nothing to do. Otherwise, for any real number c, we see that we may extend ℓ to a linear functional ℓ' on $W' := W + \mathbb{R}v'$ by setting $\ell'(v') := c$. Namely, we have

$$\ell'(w + tv') = \ell(w) + tc$$

for any $w \in W$ and $t \in \mathbb{R}$.

We would like to show that we can choose c so that $-p(-v) \leq \ell'(v) \leq p(v)$ for all v of the form $w + tv'$. This requires a little trickery. By scaling, it is enough to only check with $t \in \{\pm 1\}$ (because $t = 0$ follows by hypothesis). Thus, we need both $\ell(w) + c \leq p(w + v')$ and $\ell(w) - c \leq p(w - v')$ for all $w \in W$. Now, such a c exists if and only if

$$\sup_{w \in W} (\ell(w) - p(w - v')) \stackrel{?}{\leq} \inf_{w \in W} (p(w + v') - \ell(w)).$$

For this, we should check that $\ell(w) - p(w - v') \le p(w' + v') - \ell(w')$ for any $w, w' \in W$, which is equivalent to $\ell(w + w') \le p(w - v') + p(w' + v')$. This last inequality follows because $\ell(w) \le p(w)$ and the subadditivity of p. ■

Proposition A.44. Fix a vector space V over $\mathbb{R}$, a half-seminorm p on V, and a linear functional ℓ on a subspace $W \subseteq V$ such that $-p(-v) \le \ell(v) \le p(v)$ for all $v \in W$. Then ℓ extends to a linear functional ℓ' on V such that $-p(-v) \le \ell'(v) \le p(v)$ for all $v \in V$.

Proof. After Lemma A.43, the rest of this proof is largely formal nonsense. We use Zorn's lemma on the partially ordered set $\mathcal{P}$ of pairs (V', ℓ'), where V' is an intermediate subspace, and ℓ' is a functional on V' bounded above by p; the ordering is given by $(V', \ell') \le (V'', \ell'')$ if and only if $V' \subseteq V''$ and $\ell''|_{V'} = \ell'$. Our application of Zorn's lemma is in two steps.

- We claim that $\mathcal{P}$ has a maximal element, for which we use Zorn's lemma. First, note $\mathcal{P}$ is nonempty because it has (W, ℓ). Secondly, any ascending chain $\{(W_i, \ell_i)\}_i$ in $\mathcal{P}$ has upper bound given by setting $V' := \bigcup_i W_i$ and defining ℓ' as the union of the ℓ_is. We can see that V' is still a vector space, and the nature of the partial ordering verifies that ℓ' is a well-defined functional extending ℓ. Thus, so (V', ℓ') is indeed an upper bound for our chain.
- Let (V', ℓ') be a maximal element of $\mathcal{P}$. We claim that $V' = V$, which will complete the proof. We already have $V' \subseteq V$, so it remains to show the other inclusion. Well, for any $v \in V$, we see that (V', ℓ') can be extended up to $V' + \mathbb{R}v$ by Lemma A.43, so the maximality of (V', ℓ') requires $V' + \mathbb{R}v = V'$. Thus, $v \in V'$, so $V \subseteq V'$ follows. ■

In order to work with the base field $\mathbb{C}$, we use a trick.

Notation A.45. Fix a vector space V over $\mathbb{C}$. Given a real linear functional $\ell\colon V \to \mathbb{R}$, we define $C\ell\colon V \to \mathbb{C}$ by

$$C\ell(x) := \ell(x) - i\ell(ix).$$

On the other hand, for any complex linear functional ℓ on V, we define its real part as $\operatorname{Re}\ell := \frac{1}{2}(\ell + \overline{\ell})$.

Here are a few basic facts about this construction.

Lemma A.46. Fix a vector space V over $\mathbb{C}$.

(a) If ℓ is a real linear functional on V, then $C\ell\colon V \to \mathbb{C}$ is a complex linear functional. Furthermore, $\operatorname{Re} C\ell = \ell$.

(b) If ℓ is a complex linear functional on V, then $C \operatorname{Re}\ell = \ell$.

Proof. We show these separately.

(a) First, we check that $C\ell\colon V \to \mathbb{C}$ is a (complex) linear functional. Indeed, $C\ell$ is already real linear (it is a sum of real linear functionals), so it is enough to check that $C\ell(ix) = iC\ell(x)$ for any $x \in V$, which is true because quantities equal $\ell(ix) + \ell(x)$.

Secondly, for any $x \in V$, we see $\operatorname{Re} C\ell(x) = \operatorname{Re}(C\ell(x)) = \ell(x)$.

(b) This is a direct calculation. For any $x \in V$, we see $C \operatorname{Re}\ell(x) = \operatorname{Re}\ell(x) - i\operatorname{Re}\ell(ix)$, which further expands to

$$\frac{\ell(x) + \overline{\ell(x)} - i\ell(ix) - i\overline{\ell(ix)}}{2},$$

which is $\ell(x)$ after the dust settles. ■

Theorem A.47 (Hahn–Banach existence)**.** Fix a vector space V over $\mathbb{F}$, a half-seminorm p on V, and a linear functional ℓ on a subspace $W \subseteq V$ such that $\ell \leq p$. Then ℓ extends to a linear functional ℓ' on V such that $|\ell'| \leq p$.

Proof. The case of $\mathbb{F} = \mathbb{R}$ follows from Proposition A.44, so we may take $\mathbb{F} = \mathbb{C}$. For this, we freely use Lemma A.46. Even in this case, Proposition A.44 provides some real functional ℓ' on V extending $\operatorname{Re} \ell$ such that $|\ell'| \leq p$. Then $C\ell'$ extends $C \operatorname{Re} \ell = \ell$, so it remains to check that $|C\ell'| \leq p$. Well, for any $x \in V$, there is a unit complex number μ such that $C\ell'(\mu x) = |C\ell'(x)|$. But this means $C\ell'(\mu x)$ equals $\operatorname{Re} C\ell'(\mu x)$, which is just $\ell'(\mu x)$, which we now know to be upper-bounded by $p(\mu x)$ by construction. However, $p(\mu x) = p(x)$ because p is a half-seminorm! ■

We are now ready to give some applications. Let's start with showing that linear functionals separate convex sets.

Theorem A.48 (Hahn–Banach separation)**.** Fix a Fréchet space V over $\mathbb{F}$. Given disjoint convex subsets X and Y of V where $X \subseteq V$ is open, there is a continuous linear functional ℓ on V and a constant c so that

$$\operatorname{Re} \ell(x) < c \leq \operatorname{Re} \ell(y)$$

for all $x \in X$ and $y \in Y$.

Proof. We proceed in steps.

1. We quickly reduce to the case of $V = \mathbb{R}$: given the statement over $\mathbb{R}$, whatever real functional ℓ the statement would produce even with $\mathbb{F} = \mathbb{C}$ can be upgraded to a complex linear functional by working with $C\ell$, which then still satisfies the required inequality because $\operatorname{Re} C\ell = \ell$ by Lemma A.46.

2. Thus, we may assume that $\mathbb{F} = \mathbb{R}$. As another quick reduction, we remark that it is enough to find a continuous linear functional which merely achieves

   $$\ell(x) \stackrel{?}{<} \ell(y)$$

   for all $x \in X$ and $y \in Y$. Then we note that ℓ is surjective, so Theorem A.37 shows that $\ell(X)$ is open! Thus, $\ell(x)$ has a supremum c which it does not achieve, and this constant c suffices for the statement.

3. It remains to construct such a continuous linear functional ℓ; equivalently, we want to have $\ell(x-y) < 0$ for all $x \in X$ and $y \in Y$. We will construct ℓ via Proposition A.44. By translation, we may assume that $0 \in X$, which means that we want Y to take on large values. To this end, we define our starting functional on $\mathbb{R}y_0$ for some (nonzero!) basepoint $y_0 \in Y$ to send $y_0 \mapsto 1$.

4. To apply Proposition A.44, we must still build a half-seminorm, which we do with Proposition A.22. Because we want to upper-bound $\ell(X - Y)$, it would make sense to $X - Y$ as our convex set, but we need to have an open neighborhood of 0. Instead, we take

   $$U := X - (-y_0 + Y).$$

   This is a sum of convex sets and hence convex; similarly, this is a union of open sets and hence an open neighborhood of 0. Thus, Proposition A.22 provides us with a half-seminorm p on V such that $p\colon V \to \mathbb{R}$ has $U = \{v \in V : p(v) < 1\}$, and we remark that p is continuous by Lemma A.4. (Technically, we do not know if U is balanced, but convexity of U along with $0 \in U$ shows that $tU \subseteq U$ for all $t \in [0, 1)$, which is good enough to show that p is a half-seminorm.)

5. We are now ready to apply Proposition A.44. Quickly, we check that $-p(-y_0) \leq 1 \leq p(y_0)$, which holds because $y_0 \notin U$ (because this would imply $X \cap Y \neq \varnothing$). Thus, proposition A.44 provides us with a linear functional ℓ on V such that $\ell(x_0 - y_0) = 1$ and $-p(-v) \leq \ell(v) \leq p(v)$. In particular, ℓ is continuous by Lemma A.41.

6. We complete the proof. It remains to check that $\ell(x) < \ell(y)$ for all $x \in X$ and $y \in Y$. Well, for any $x \in X$ and $y \in Y$, we see that $x - (-y_0 + y) \in U$, so $p(y_0 + x - y) < 1$, but $\ell(y_0) = 1$, so $\ell(y_0 + x - y) < \ell(y_0)$, so
$$\ell(x) < \ell(y),$$
as required. ■

Here are some applications. If one has a compact convex set, then one can improve the separation result.

Corollary A.49. Fix a Fréchet space V over $\mathbb{F}$. Given disjoint convex subsets K and A where $K \subseteq V$ is compact and $A \subseteq V$ is closed, there is a continuous linear functional ℓ on V and constants c and d so that
$$\operatorname{Re} \ell(k) \le c < d \le \operatorname{Re} \ell(a)$$
for $k \in K$ and $a \in A$.

Proof. We apply Theorem A.48. For this, we need to produce an open convex set. Well, because K is compact, the main claim that there is an open neighborhood U of 0 such that $(K + U) \cap A = \varnothing$. Let's explain how this completes the proof. To start, note that we may shrink U to be convex so that $K + U$ is open and convex. Then Theorem A.48 provides us with a continuous linear functional ℓ and constant d for which
$$\operatorname{Re} \ell(k + u) < d \le \operatorname{Re} \ell(a)$$
for all $k + u \in K + U$ and $a \in A$. However, K is compact, so the continuous function $k \mapsto \operatorname{Re} \ell(k)$ must achieve its supremum, which we take to be c, and we are done after noting that we must have $c < d$.

It remains to show the main claim, which is purely topological in nature. Suppose that every open neighborhood U of 0 has $(K + U) \cap A \ne \varnothing$; we will show that $K \cap A \ne \varnothing$. Consider the collection $\{K \cap (A - U)\}$ of nonempty open subsets of K, which we note is closed under intersections, so we can extend it to a filter and then an ultrafilter $\mathcal{F}$. The compactness of K then shows that $\mathcal{F} \to k$ for some $k \in K$; by translation, we may assume $k = 0$. Unwinding, this means that $\mathcal{F}$ contains every open neighborhood U of k, so $K \cap (A - U)$ has nontrivial intersection with U, so A has nontrivial intersection with $U + U$, so we see $0 \in \overline{A}$ (by shrinking U). But because A is closed, we conclude $0 \in A \cap K$! ■

In the special case where K is a point, we are able to say something about the existence of continuous functionals.

Corollary A.50. Fix a Fréchet space V over $\mathbb{F}$. Given a vector $v \in V$ and subspace $W \subseteq V$, there is a continuous linear functional ℓ on V with $\ell|_W = 0$ and $\ell(v) = 1$ if and only if $v \notin \overline{W}$.

Proof. Of course, if such an ℓ exists, then $\overline{W} \subseteq \ker \ell$, so $v \notin \overline{W}$.

In the other direction, we use Corollary A.49 with $K = \{v\}$ and $A = \overline{W}$ to get a linear functional ℓ such that $\operatorname{Re} \ell(v) < \operatorname{Re} \ell(w)$ for all $w \in \overline{W}$. However, $\operatorname{Re} \ell$ is a (real) linear functional, so it must vanish on W if it is not surjective, so $\operatorname{Re} \ell|_W = 0$, so $\ell|_W = 0$ by Lemma A.46. Thus, we see $\operatorname{Re} \ell(v) < 0$, so we can complete the proof by dividing ℓ by $\ell(v)$. ■

At long last, we are ready to extend some functionals.

Corollary A.51. Fix a Fréchet space V over $\mathbb{F}$. Any continuous linear functional ℓ on a subspace $W \subseteq V$ extends to a continuous linear functional on V.

Proof. If $\ell = 0$, then there is nothing to do. Otherwise, we use Corollary A.50. Choose a vector $w_0 \in W$ such that $\ell(w_0) = 1$. Thus, we see that $w_0 \notin \overline{\ker \ell}$, so Corollary A.50 gives us a continuous linear functional ℓ' on the full space V for which $\ell'|_{\ker \ell} = 0$ and $\ell'(w_0) = \ell(w_0) = 1$. Because $W = \mathbb{R}w_0 + \ker \ell$, we conclude that ℓ' does in fact extend ℓ, so we are done. ■

Remark A.52. It may appear that we can use Lemma A.41 to immediately extend continuity from W up to V. However, there is something tricky going on: a priori, ℓ on W is only bounded by a continuous seminorm from W, and it is not obvious that such a seminorm should also extend to a suitable continuous seminorm on V!

A.5 Examples of Fréchet Spaces

To gain some experience with our new definition, we provide some examples of Fréchet spaces.

Definition A.53 (Banach)**.** A topological vector space V is *Banach* if and only if its topology is given by a norm, and V is complete.

Example A.54. Any Banach space is Fréchet. For example, one can take the countable family of seminorms to just be the single (actual) norm and conclude by Corollary A.23.

Of course, we are going to be interested in examples of Fréchet spaces which are not necessarily Banach spaces.

Example A.55. Fix a Hausdorff topological space X.

(a) Then the space $C(X)$ of continuous functions $X \to \mathbb{C}$ has a topology where $\{f_i\} \to f$ if and only if $f_i \to f$ uniformly on compact sets. Furthermore, $C(X)$ is locally convex.

(b) If X is compact, then $C(X)$ is a Banach space.

(c) If X is locally compact and second countable, then $C(X)$ is Fréchet.

Proof. Let's begin by explaining how to give $C(X)$ a topology so that it is always locally convex. Consider the collection $\mathcal{K}$ of compact subsets of X, which we give a partial ordering by inclusion. For each compact set K, we define a seminorm $\nu_K \colon C(X) \to \mathbb{R}$ by

$$\nu_K(f) := \max_{x\in K} |f(x)| .$$

To see that ν_K is a seminorm, we note that ν_K is surely homogeneous, and ν_K is subadditive because $|f(x) + g(x)| \le |f(x) + g(x)|$.

We may thus give $C(X)$ the topology given by these seminorms. Quickly, note that $C(X)$ is Hausdorff because $f \ne g$ has some $x \in X$ for which $f(x) \ne g(x)$, and then we have $\nu_{\{x\}}(f-g) > 0$. Thus, we see that this topology is locally convex for free! Lastly, observe that $\{f_i\} \to f$ if and only if $\{\nu_K(f_i - f)\} \to 0$ for all compact subsets $K \subseteq X$; by translation, we may assume that $f = 0$. But of course, having $\{\nu_K(f_i - f)\} \to 0$ is equivalent to having uniform convergence on compact sets.

Next up, let's suppose that X is compact, and we will show that $C(X)$ i Banach. Well, we claim that the topology is actually induced by the single seminorm ν_X, which is actually a norm. To see that the topology is induced by ν_X, note that $\nu_K \le \nu_X$ for all K, so having $\{\nu_K(f_i - f)\} \to 0$ for all $K \subseteq X$ is equivalent to having $\{\nu_X(f_i - f)\} \to 0$. To see that ν_X is a norm, it remains to check positivity, for which we note that any nonzero continuous function f is nonzero at some $x \in X$, so $\nu_X(f) \ge |f(x)|$ is positive.

Lastly, we assume that X is locally compact and second countable. This means that X will admit a countable cover by pre-compact open subsets, so taking the closure shows that X receives a countable cover $\{K_j\}_{j\in\mathbb{N}}$ by compact sets. By replacing each K_j with $\bigcup_{j'\le j} K_{j'}$, we may assume that

$$K_0 \subseteq K_1 \subseteq \cdots .$$

Accordingly, we claim that the topology is generated by the seminorms $\{\nu_{K_j}\}_{j\in\mathbb{N}}$. Well, for any compact set K, we see $\{\nu_K(f_i - f)\} \to 0$ is implied by having this convergence for any larger compact set, so it will be implied by $\{\nu_{K_n}(f_i - f)\} \to 0$ for some n sufficiently large. ■

Remark A.56. Suppose X is compact. Then it turns out that $C(X)$ is separable if and only if X is a metric space. Let's sketch the backwards direction: being metrizable implies that X is second countable, so there is a countable dense subset $\{x_i\}_{i\in\mathbb{N}}$ of X. Consider the algebra $\mathcal{A}$ over $\mathbb{C}$ of functions generated by 1 and $x \mapsto d(x_i, x)$. Note $x \mapsto d(x_i, x)$ is continuous, so $\mathcal{A} \subseteq C(X)$; additionally, $\mathcal{A}$ separates points by construction. It follows that $\mathcal{A}$ is dense in $C(X)$; considering corresponding algebra over $\mathbb{Q}$ provides us with a countable dense subset of $C(X)$.

Here is an example of the sort of check we can do.

Example A.57. Let $f\colon X \to Y$ be a continuous map. Then the map $(- \circ f)\colon C(Y) \to C(X)$ is well-defined and continuous. We may write this operation as $f^*\colon C(Y) \to C(X)$.

Proof. The map is well-defined because the composite of continuous maps is continuous. It remains to show continuity. Well, suppose that we have a converging net $\{\ell_i\} \to \ell$ in $C(Y)$. We would like to show that $\{\ell_i \circ f\} \to (\ell \circ f)$ in $C(X)$. Convergence is equivalent to converging uniformly on compacts, so we choose a compact set $K \subseteq X$, and we would like to check that

$$\|(\ell_i \circ f - \ell \circ f)|_K\|_\infty \stackrel{?}{\to} 0.$$

Well, this norm is bounded by $\left\|(\ell_i - \ell)|_{f(K)}\right\|_\infty$, which goes to 0 because $f(K)$ is compact and $\{\ell_i\} \to \ell$ uniformly on compact sets. ■

A.6 Categorical Properties

For my own personal use, we review some categorical properties of certain full subcategories of the category of topological vector spaces, which is equipped with continuous linear maps for its morphisms. We recall that all topological vector spaces are assumed to be Hausdorff.

Because they are easier, let's begin with some limits.

Definition A.58 (kernel)**.** Fix a continuous map $f\colon V \to W$ of topological vector spaces. Then we define the *kernel* $\ker f$ as the vector space kernel equipped with the subspace topology.

Remark A.59. Because $\ker f$ is a subspace, we see that it continues to be a Hausdorff topological space. In fact, it is the categorical kernel: it is already the kernel in the category of vector space, so any map $V' \to V$ vanishing under W factors via a linear map into $\ker f$. But $\ker f$ is just a subspace, so the induced map $W \to V'$ is continuous.

Remark A.60. Note that $\ker f \subseteq V$ is closed because f is continuous, so if V is complete, we see that $\ker f$ is also complete.

Remark A.61. If V is locally convex, then the subspace topology inherits the exact set of seminorms, so $\ker f$ is also locally convex. Once we combine this argument with Remark A.60, we see that V being Fréchet implies that $\ker f$ is Fréchet.

Definition A.62 (product)**.** Fix a family $\{V_i\}_{i\in I}$ of topological vector spaces. Then we define the *product* $\prod_{i\in I} V_i$ as the vector space product equipped with the product topology.

Remark A.63. When checking that $\prod_{i\in I} V_i$ is a categorical product, we note that the maps produced by being the vector space product and being the topological product are the same, so the result follows.

Remark A.64. If each V_i is complete, then we claim that the product $\prod_{i\in I} V_i$ is still complete. Indeed, given any Cauchy net $\{(v_{ij})_i\}_{j\in I}$, we see that each projection $\{v_{ij}\}_{j\in I}$ continues to be a Cauchy net and therefore has a limit $\{v_{ij}\} \to v_j$. But now having each projection converge means that $\{(v_{ij})\}_{j\in I} \to (v_j)_j$.

Remark A.65. If each V_i is locally convex with its topology determined by some family of seminorms $\{p_{i\alpha}\}_\alpha$, then the definition of the product means that the topology on the product are determined by the seminorms

$$\prod_{i\in I} V_i \overset{\mathrm{pr}_j}{\twoheadrightarrow} V_j \overset{p_{j\alpha}}{\to} \mathbb{R}.$$

This argument, when combined with Remark A.64, shows that a countable product of Fréchet spaces continues to be Fréchet.

Proposition A.66. Fix a diagram $\{V_i\}_{i\in I}$ of topological vector spaces. Then the limit

$$\lim_{i\in I} V_i$$

exists in the category of topological vector spaces. If each V_i is locally convex, so is the limit. If I is countable and each V_i is Fréchet, then so is the limit.

Proof. Existence follows from Remarks A.59 and A.63. The remaining claims follow from Remarks A.61 and A.65. ■

Example A.67. We recover Example A.55. Fix a locally compact topological space X. Then we claim that

$$C(X) \overset{?}{=} \lim_{K\subseteq X} C(K),$$

where the limit is taken over all compact subsets $K \subseteq X$. Indeed, there is a canonical map $f \mapsto (f|_K)_K$ from the left to the right; conversely, any tuple $(f_K)_K$ on the right extends to a unique function $f \in C(X)$ by construction of the limit. Lastly, we note that the map that we just described is actually a homeomorphism because a convergent net on the left $\{f_i\} \to f$ converges if and only if $\{f_i|_K\} \to f|_K$ uniformly for all compacts $K \subseteq X$, which is exactly the topology on the right-hand side.

We conclude that $C(X)$ is locally convex; if X is second countable, then we can replace the limit over all compact subsets with a countable ascending chain, thereby showing that the limit is also Fréchet.

We now turn to some colimits, which are harder.

Definition A.68 (quotient)**.** Fix a closed subspace W of a topological vector space V. Then we form the *quotient* V/W to be the vector space V/W to be equipped with the quotient topology induced by the projection $V \twoheadrightarrow V/W$.

Remark A.69. We claim that this quotient is the categorical quotient. Indeed, for any topological vector space V' and some map $f\colon V \to V'$ for which $W \subseteq \ker f$, we get an induced map $\overline{f}\colon V/W \to V'$, which we need to be continuous. Well, a map $\overline{f}\colon V/W \to V'$ is continuous if and only if the composite $V \twoheadrightarrow V/W \to V'$ is continuous, which is true.

Lemma A.70. Fix a closed subspace W of a topological vector space V.

(a) If V is locally convex, then V/W is locally convex.

(b) If V is Fréchet, then V/W is Fréchet.

Proof. We show the parts separately.

(a) For each seminorm p on V, we produce a continuous seminorm $\overline{p}\colon V/W \to \mathbb{R}$ by

$$\overline{p}(v+W) \coloneqq \inf_{w\in W} p(v+w).$$

Because p is a seminorm, we see that $\overline{p}$ is also a seminorm. Additionally, we note $\overline{p}$ is continuous: using Lemma A.4, it is enough to see that $\{v+W : \overline{p}(v+W) < 1\}$ is open, which is true because this subset equals

$$\{v+W : \overline{p}(v+W) < 1\} \stackrel{?}{=} \{v : p(v) < 1\} + W.$$

Certainly any vector v for which $p(v) < 1$ has $\overline{p}(v) < 1$. Conversely, if $\overline{p}(v+W) < 1$, then there is $w \in W$ for which $p(v+w) < 1$.

Lastly, we have to check that these seminorms actually produce the correct topology on V/W. Well, the open sets of V/W are exactly projections of the open sets of V, so we are done because the previous paragraph explains that the open sets produced by our seminorms $\overline{p}$ are exactly the projections of the open sets produced by the seminorms p of V.

(b) The argument in (a) shows that V/W has its topology determined by countable many seminorms, so it remains to show that V/W is complete. Because V/W is metrizable (Proposition A.27), it is enough to check that Cauchy sequences in V/W converge. As such, choose a Cauchy sequence $\{v_i+W\}_{i\in\mathbb{N}}$, which means that $\overline{p}(v_i - v_j + W) \to 0$ as $i,j\to\infty$. It is enough to show that any subsequence of this Cauchy sequence converges, so we may as well assume that $\overline{p}(v_{i+1} - v_i + W) < 2^{-i}$ for each i. By definition of $\overline{p}$, we may thus, find vectors $\{w_i\}$ for which

$$\overline{p}(v_{i+1} + w_{i+1} - (v_i + w_i)) < 2^{-i+1}.$$

Thus, the sequence $\{v_i+w_i\}_{i\in\mathbb{N}}$ is rapidly Cauchy in V, so it has a limit to some vector $v\in V$. Projecting back down, we find that $\{v_i+w_i\}\to v$ implies that $\{v_i+W\}\to v+W$, so we are done. ■

For our coproduct, we will retreat to locally convex vector spaces.

Definition A.71 (coproduct)**.** Fix a family $\{V_i\}_{i\in I}$ of locally convex vector spaces. Then we define the *coproduct* as given by the coproduct $\bigoplus_{i\in I} V_i$ of vector spaces, and we equip it with a topology whereby an open neighborhood basis of the origin takes the form

$$\operatorname{conv}\left(\sum_{i\in I} j_i(U_i)\right),$$

where $j_i\colon V_i \to \bigoplus_{i\in I} V_i$ is the inclusion, $U_i \subseteq V_i$ is an open neighborhood of 0 for each i. (This sum denotes the collection of any finite sum of vectors.)

Remark A.72. Let's verify that we have produced a categorical coproduct: given continuous inclusions $V_i \to W$ for each $i\in I$ into some locally convex vector space W, we need to check that the algebraically induced linear map $\bigoplus_{i\in I} V_i \to W$ is continuous. Well, it's enough to check continuity on an open neighborhood basis of the identity, so we note that the pre-image of some convex open neighborhood U of W consists of the finitely supported sequences $(v_i)_{i\in I}$ where $v_i \in j_i^{-1}(U)$ for each i. The given open neighborhood basis does in fact cover such an open subset!

For our application, we want to know something about the dual.

Definition A.73 (dual)**.** Fix a topological vector space V over $\mathbb{F}$. Then we define the *topological dual* V^* as the vector space of all continuous linear functionals $V \to \mathbb{F}$. We make V^* into a topological vector space by equipping it with the weak topology such that the evaluation maps $\operatorname{ev}_v \colon V^* \to \mathbb{R}$ are continuous for every $v \in V$.

Remark A.74. Given a continuous linear map $f \colon V \to W$, we see that composition provides a map $(- \circ f) \colon W^* \to V^*$. We may also call this map f^*, and it makes $(-)^*$ into a functor.

Lemma A.75. Given an exact sequence

$$0 \to A \to B \to C \to 0$$

of topological vector spaces, the sequence

$$0 \to C^* \to B^* \to A^* \to 0$$

is exact if B is Fréchet.

Proof. We show exactness by hand.

- Exact at C^*: by construction of the quotient $C = B/A$, we see that the map $B \to C$ is surjective. Thus, a functional $\ell \in C^*$ vanishes if and only if the composite $B \to C \to \mathbb{F}$ vanishes.
- Exact at A^*: we are being asked to show that any continuous linear functional on a closed subspace $A \subseteq B$ extends continuously to B, which is exactly Corollary A.51.
- Exact at B^*: we are given a continuous linear functional ℓ on B which vanishes on A. Thus, ℓ factors through as a continuous linear functional on $C = B/A$ because we have a categorical quotient, so we are done. ■

Lemma A.76. Fix a family $\{V_i\}_{i \in I}$ of locally convex topological vector spaces. Then the canonical map

$$\bigoplus_{i \in I} V_i^* \to \left(\prod_{i \in I} V_i \right)^*$$

is a continuous injection. If the family is countable, then it is an isomorphism.

Proof. The canonical map is induced by dualizing the canonical map $\prod_{i \in I} V_i \to \bigoplus_{i \in I} V_i$, which in turn is induced by the projections $\operatorname{pr}_j \colon \prod_{i \in I} V_i \to V_j$. Thus, we see that the map sends a finitely supported family $(\ell_i)_{i \in I}$ of linear functionals to the linear functional on $\prod_{i \in I} V_i$ defined by

$$(v_i)_i \mapsto \sum_{i \in I} \ell_i(v_i).$$

Here are the remaining checks on this map, which we call α.

- Injective: if this target linear functional vanishes, then the inclusions $V_j \to \prod_{i \in I} V_i$ allows us to check that each individual linear functional in $(\ell_i)_{i \in I}$ vanishes.
- Surjective if I is countable: fix a continuous linear functional ℓ on $\prod_{i \in I} V_i$, and we define ℓ_i by composing ℓ with the inclusion $V_i \to \prod_{i \in I} V_i$. It remains to show that $\ell = \alpha((\ell_i)_{i \in I})$. For this, we identify I with $\mathbb{N}$. Then for any vector $(v_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} V_i$, we note that the sequence of vectors

$$w_n := \sum_{i=0}^{n} v_i$$

approaches v as $n \to \infty$, which we can see by definition of the product topology. It follows that $\ell(v)$ is the limit of the $\ell(w_n)$, which equals

$$\ell(w_n) = \sum_{i=0}^{n} \ell_i(v_i).$$

Thus, it remains to show that ℓ_i vanishes for all but finitely many $i \in \mathbb{N}$. Well, otherwise, we can select v_i so that $\ell_i(v_i) = 1$ for infinitely many $i \in \mathbb{N}$, from which we find that the sequence $\{\ell(w_n)\}_{n\in\mathbb{N}}$ fails to converge.

- Open if I is countable: it is enough to check that α is open at 0. As such, we choose a family of open neighborhoods $U_i \subseteq V_i^*$ of 0 for each $i \in I$, and we need to show that

$$\operatorname{conv}\left(\sum_{i\in I} j_i(U_i)\right)$$

contains an open neighborhood of $0 \in \left(\prod_{i\in I} V_i\right)^*$. Fix our finite subset $S \subseteq I$ so that U_i is nonempty exactly when $i \in S$. Now, $U_i \subseteq V_i^*$ is open in the weak topology, so there is a finite family of vectors $\{v_{ij}\}_{j\in S_i}$ so that U_i contains an open neighborhood of the form

$$\bigcap_{j\in S_i} \{\ell : |\ell(v_{ij})| < \varepsilon_i\}$$

for some fixed very small $\varepsilon_i > 0$. This means that the image of our open set in $\left(\prod_{i\in I} V_i\right)^*$ is the convex hull of some finite sum of such vectors. But if we select any finite number of these conditions, we see that we cut out an open set in the weak topology of $\left(\prod_{i\in I} V_i\right)^*$, so we are done! ■

Proposition A.77. Fix a countable diagram $\{V_i\}_{i\in I}$ of Fréchet spaces. Then the canonical map

$$\operatorname*{colim}_{i\in I} V_i^* \to \left(\lim_{i\in I} V_i\right)^*$$

is an isomorphism.

Proof. This follows from combining Lemmas A.75 and A.76. ■

Example A.78. Continuing Example A.67, we see that

$$C(X)^* = \operatorname*{colim}_{K\subseteq X} C(K)^*,$$

where the induced maps $C(K)^* \to C(X)^*$ are given by sending $\mu \in C(K)^*$ to the map $f \mapsto \mu(f|_K)$.

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List of Definitions