18.917: The Chromatic Splitting Conjecture

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION

1.1 February 2

Here we go.

1.1.1 Idempotent Algebras

The goal of this class is to understand some topics related to the chromatic splitting conjecture. Thus, the first half of the class will try to understand the statement, and the second half of the class will explain how it relates to other problems in algebra.

Warning 1.1. All categories in this course are ∞-categories.

Example 1.2. Given a ring R, we have a stable, symmetric monoidal ∞-category $D(R)$ of chain complexes of R-modules, considered up to quasi-isomorphism. Notably, the symmetric monoidal structure is given by the derived tensor product.

We begin our story with idempotent algebras.

Definition 1.3 (idempotent algebra)**.** Fix a ring R. An *idempotent algebra* is an object $E \in D(R)$ equipped with a unit map $R \to E$ such that the composite

$$E = E \otimes_R R \to E \otimes_R E$$

is an equivalence.

Remark 1.4. Such an object E grants E a multiplication structure $E \otimes_R E \to E$, and E gains the structure of a differentially graded algebra.

Example 1.5. Consider $R = \mathbb{Z}$. Then for each prime p, the algebra $\mathbb{Z}_{(p)}$ is idempotent: localizing $\mathbb{Z}_{(p)}$ further at (p) does nothing!

Non-Example 1.6. The $\mathbb{Z}$-algebra $\mathbb{F}_p$ is not idempotent because the tensor product we are considering is derived. Indeed, we computed $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p$ last semester.

Here is a quick reason why one might care about idempotent algebras.

Theorem 1.7 (Neeman)**.** Fix a Noetherian ring R. Then the lattice of idempotent algebras is equivalent to the data of $\operatorname{Spec} R$ as a topological space.

Example 1.8. For $R = \mathbb{Z}$, it turns out that the idempotent algebras are either $\mathbb{Z}_{(p)}$ or $\mathbb{Q}$, and the maps between them look like the specializations of $\operatorname{Spec} R$.

Of course, we are homotopy theorists, so we have less reason to care about $\mathbb{Z}$. Recall that $\mathbb{Z}$ is obtained from $\mathbb{N}$ by formally adding inverses. But $\mathbb{N}$ is basically isomorphism classes of FinSet; if we had instead formally added inverses directly to FinSet (instead of taking isomorphism classes first), we would have found the sphere spectrum $\mathbb{S}$. In particular, we will be interested in the category $D(\mathbb{S})$ of $\mathbb{S}$-modules, also called spectra.

We now no longer have access to algebraic geometry directly on $\mathbb{S}$. Instead, Theorem 1.7 motivates us to look for the idempotent algebras for $\mathbb{S}$.

Remark 1.9. For any $x \in \pi_*\mathbb{S}$, there is an idempotent algebra $\mathbb{S}\left[x^{-1}\right]$. For example, $\pi_0\mathbb{S} = \mathbb{Z}$, so there is an idempotent algebra $\mathbb{S}_{(p)}$.

Here is our first main theorem.

Theorem 1.10 (Nishida)**.** Fix some $x \in \pi_*\mathbb{S}$ of positive degree. Then x is nilpotent.

Thus, the idempotent algebras $\mathbb{S}\left[x^{-1}\right]$ do not look genuinely "new." To get other idempotent algebras, we need more tools.

1.1.2 The Adams–Novikov Spectral Sequence

Recall the $\mathbb{S}$-algebra MU defined as the colimit of the embedding $\mathrm{BU} \to \mathrm{BLG}(\mathbb{S}) \subseteq \mathrm{Mod}(\mathbb{S})$. Let's compute its homotopy groups.

Definition 1.11 (formal group law)**.** Fix a commutative ring R. Then a *commutative formal group law* over R is a power series $f(x,y) \in R[[x,y]]$ satisfying

(a) $f(x,0) = x$ and $f(0,y) = y$,

(b) $f(x,y) = f(y,x)$, and

(c) $f(x, f(y,z)) = f(x, f(y,z))$.

Definition 1.12 (Lazard ring)**.** The *Lazard ring* is the ring L which is exactly the quotient of $\mathbb{Z}[\{a_{ij}\}_{ij}]$ by the relations dictating that

$$f(x,y) = \sum_{i,j \geq 0} a_{ij} x^i y^j$$

is a commutative formal group law.

Remark 1.13. In other words, L represents the collection formal group laws, in the sense that the data of a formal group law for a ring R amounts to the data of a ring homomorphism $L \to R$.

Remark 1.14. By definition, there is a "universal" formal group law f_L in L given exactly by

$$f_L(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j.$$

Theorem 1.15 (Quillen)**.** The ring $\pi_* \mathrm{MU}$ is exactly the Lazard ring.

Remark 1.16. Quillen also computed $\pi_*(\mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU})$ as well as the two natural maps $\pi_* \mathrm{MU} \to \pi_*(\mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU})$. It turns out that this is more or less related to some notion of isomorphism of the formal group laws.

The use of MU is that it produces a spectral sequence with which we can understand $\pi_* \mathbb{S}$. By Čech descent along the map $\mathbb{S} \to \mathrm{MU}$, we see that $\mathbb{S}$ is the limit of the diagram

$$\begin{tikzcd} \mathrm{MU} \arrow[r, shift left] \arrow[r, shift right] & \mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU} \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU} \otimes_{\mathbb{S}} \mathrm{MU} \arrow[l, shift left] \arrow[l, shift right] & \cdots \end{tikzcd}$$

which we can then truncate as $\operatorname{fil}^n \mathbb{S}$ in order to get a descending filtration to $\operatorname{fil}^0 \mathbb{S}$. Computing homotopy along this filtration produces the desired spectral sequence, as soon as we compute homotopy groups of the various tensor powers of the MUs and so on.

Theorem 1.17 (Adams–Novikov)**.** Let $\mathcal{M}_{\mathrm{fg}}$ be the moduli space of formal groups. Then there is a spectral sequence

$$E_2 = \mathrm{H}^s(\mathcal{M}_{\mathrm{fg}}; \omega^{\otimes t}) \Rightarrow \pi_{2t-s} \mathbb{S}.$$

Remark 1.18. It turns out that the spectral sequence is concentrated in the region $s \leq 2t - s$.

Example 1.19. Along the line $s = 2t - s$, there is some h_1 at $(s, 2t - s) = (1, 1)$, and then we can take powers of it to go up the line. It turns out that h_1 survives the spectral sequence, and it goes to the "Hopf map" $\eta \in \pi_1 \mathbb{S}$; however, $\eta^4 = 0$, though the Adams–Novikov spectral sequence cannot see it!

Thus, we see that the Adams–Novikov spectral sequence is not an amazing approximation: the E_2 page sees many classes which we know abstractly must vanish! Life is better if we pass to E_∞ instead; the following is our first main theorem.

Theorem 1.20 (Devinatz–Hopkins–Smith)**.** The E_∞ page of the Adams–Novikov spectral sequence lies under a curve which grows more slowly than any line.

Note that this immediately implies Theorem 1.10. On the other hand, we will see that the topological input of Theorem 1.10 plus some algebraic facts about formal group laws will prove the above big theorem.

Remark 1.21. The curve is known to be faster than logarithmic, but not much else is known. Our proof will not help us much because our proof of Theorem 1.10 will be ineffective.

1.1.3 Back to Idempotent Algebras

Let's return to trying to find some idempotent algebras.

Notation 1.22. Define the power series $[n] \in L[[x]]$ to be adding with f a total of n times.

Example 1.23. We see that $[2](x) = f(x,x)$ and $[5](x) = f(f(f(f(x,x),x),x),x)$.

Notation 1.24. Fix a prime p. Then we define the class $v_n \in \pi_*\mathrm{MU}$ to be the coefficient of x^{p^n} in the power series $[p](x)$.

Now, because localization is exact, we see that $\mathbb{S}_{(p)}$ is the limit of the nerve

$$\mathrm{MU}_{(p)} \rightrightarrows \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \qquad \cdots$$

so it is not unreasonable to consider the following limit.

Notation 1.25. Fix a prime p and some $n \geq 0$. Then we define $L_n\mathbb{S}_{(p)}$ as the limit of the following diagram.

$$\mathrm{MU}_{(p)}\left[v_n^{-1}\right] \rightrightarrows \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right]$$

We may abbreviate $L_n\mathbb{S}_{(p)}$ to $L_n\mathbb{S}$ if there is no possibility of confusion.

Remark 1.26. It turns out that there are natural maps $L_{n+1}\mathbb{S} \to L_n\mathbb{S}$.

These spectra $L_n\mathbb{S}$ give us new idempotent algebras, more or less granting us further understanding of the "spectrum" of $\mathbb{S}$.

Theorem 1.27 (Hopkins–Ravenel)**.** Fix a prime p and some $n \geq 0$. Then $L_n\mathbb{S}_{(p)}$ is an idempotent algebra.

Remark 1.28. Ravenel has conjectured that if E is a nonzero idempotent algebra under $\mathbb{S}_{(p)}$, then E is either $\mathbb{Q}$ or one of the $L_n\mathbb{S}$s. This was recently disproved. It is current work to attempt a classification.

Nonetheless, $\mathbb{S}_{(p)}$ can be understood well from the $L_n\mathbb{S}$s.

Theorem 1.29 (Hopkins–Ravenel)**.** Fix a prime p. Then $\mathbb{S}_{(p)}$ is the limit of the diagram

$$\cdots \to L_3\mathbb{S}_{(p)} \to L_2\mathbb{S}_{(p)} \to L_1\mathbb{S}_{(p)}.$$

1.1.4 Completion

Continue with our fixed prime p. For motivation, we return to abelian groups.

Remark 1.30. For any $M \in D(\mathbb{Z})$, the p-localization sits in a pullback square

$$\begin{tikzcd} M_{(p)} \arrow[r] \arrow[d] & M_p^\wedge \arrow[d] \\ M \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] & M_p^\wedge \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}$$

more or less corresponding to finding the "lattice" $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$.

Analogously, there is a completion of $L_n\mathbb{S}$ which fits into a diagram

$$\begin{tikzcd} L_nE \arrow[r] \arrow[d] & L_{K(n)}E \arrow[d] \\ L_{n-1}E \arrow[r] & L_{n-1}L_{K(n)}E \end{tikzcd} \tag{1.1}$$

where $L_nE := L_n\mathbb{S} \otimes_{\mathbb{S}} E$. We are now ready to state the chromatic splitting conjecture.

Conjecture 1.31 (Chromatic splitting)**.** For any $n \geq 2$, the inclusion

$$L_{K(n)}\mathbb{S} \to L_{n-1}L_{K(n)}\mathbb{S}$$

is an inclusion of a direct summand.

Remark 1.32. This implies that the natural map

$$\mathbb{S}_p^\wedge \to \prod_{n\geq 1} L_{K(n)}\mathbb{S}$$

is the inclusion of a direct summand. The point is that the squares (1.1) are rather degenerate, which would let us compute the homotopy groups of $L_n\mathbb{S}$ from the completions.

Remark 1.33. Conjecture 1.31 is known at $n = 2$ and all primes, by work of many people.

The goal of the present class is to review the homotopy theory required to understand the statement of Conjecture 1.31 formally, and then we will discuss why perfectoid geometry may be useful to prove it.

Let's see why passing to $L_{K(n)}\mathbb{S}$ is genuinely easier.

Example 1.34. For $p > 2$, we can define $L_{K(1)}\mathbb{S}$ as the homotopy fiber of the endomorphism $\psi^g - 1$ of $\mathrm{KU}_p^\wedge$, where g is a choice of topological generator of $\mathbb{Z}_p^\times$, and ψ is some action of $\mathbb{Z}_p^\times$ on $\mathrm{KU}_p^\wedge$.

Theorem 1.35 (Goerss–Hopkins–Miller, Rogres)**.** Fix a prime p. For each $n \geq 1$, there is an $\mathbb{S}$-algebra E_n and a profinite group $\mathbb{G}_n$ for which

$$L_{K(n)}\mathbb{S} = (E_n)^{\mathbb{G}_n}.$$

In fact, E_n is a Galois extension of $L_{K(n)}\mathbb{S}$.

Remark 1.36. We will only be able to keep track of this sort of "infinite Galois theory" with condensed mathematics.

Remark 1.37. The profinite group $\mathbb{G}_n$ is some subgroup of automorphisms of formal group laws.

Remark 1.38. For any spectrum X, there is some "Galois descent"

$$L_{K(n)}X = \left(L_{K(n)}(E_n \otimes X)\right)^{\mathbb{G}_n}.$$

This generalizes to a spectral sequence

$$\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n; \pi_*(L_{K(n)}(E_n \otimes X))) \Rightarrow \pi_* L_{K(n)}X.$$

The previous remark produces a spectral sequence

$$\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n; \pi_* E_n) \Rightarrow \pi_* L_{K(n)}\mathbb{S}.$$

If p is large compared to n, then it turns out that the spectral sequence collapses for degree reasons, so we are reduced to a pure algebra problem.

The end of the course will be interested $L_{K(n-1)}L_{K(n)}\mathbb{S}_{(p)}$ for general n but p very large. Conjecture 1.31 tells us that this should be fairly easy to understand, so we can view the end of the course as trying to provide some evidence for the conjecture. For example, work in progress by many people has recently culminated in the following strategy.

Notation 1.39. Fix $\mathbb{B} \coloneqq E_{n-1} \otimes_{\mathbb{S}} L_{K(n-1)}E_n$.

Remark 1.40. It turns out that $\mathbb{B}$ is Galois over $L_{K(n-1)}L_{K(n)}\mathbb{S}$ with Galois group $\mathbb{G}_{n-1} \times \mathbb{G}_n$. Thus, we can hope to be able to use some Galois descent spectral sequence to understand $L_{K(n-1)}L_{K(n)}\mathbb{S}$, as in Remark 1.38.

Now, $\pi_*\mathbb{B}$ is a local ring, so one becomes motivated to consider a perfection $\widehat{\mathbb{B}}$. In particular, it turns out that there is a $\mathbb{G}_n \times \mathbb{G}_{n-1}$-equivariant map $\mathbb{B} \to \widehat{\mathbb{B}}$, so taking fixed points produces a map out of $L_{K(n-1)}L_{K(n)}\mathbb{S}$. This is the sort of thing that Conjecture 1.31 asks us to do! Of course, the target is related to the perfection $\widehat{\mathbb{B}}$, which we now want to understand.

Theorem 1.41. The groups $\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n \times \mathbb{G}_{n-1}; \pi_*\widehat{\mathbb{B}})$ is the same as the cohomology of the structure sheaf of some diamond related to the Fargues–Fontaine curve.

Let's explain the application to Conjecture 1.31: this calculation tells us that $(\widehat{\mathbb{B}})^{\mathbb{G}_n \times \mathbb{G}_{n-1}}$ is $L_{K(n-1)}\mathbb{S} \oplus \Sigma L_{K(n-1)}\mathbb{S}$, from which our small piece of Conjecture 1.31 follows!

BIBLIOGRAPHY

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List of Definitions