250B: Commutative Algebra

For the Morbidly Curious

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Spring 2022

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Theme 1
Introduction to Dimension

In this sense the algebraic geometers have never left paradise: There is no snake (that is, Peano curve) in the garden.

—David Eisenbud

1.1 March 17

We're back, y'all.

1.1.1 A Quick Exercise

Let's start with an exercise, to review for the midterm. Recall the following result.

Lemma 1.1 (Eisenbud 6.4)**.** Fix R a ring and $S := R[x_1, \ldots, x_n]/(f)$, where f is some polynomial. Then S is a flat R-algebra if and only if $\operatorname{cont} f = R$.

Proof. This was on the homework. ■

And here is our exercise.

Exercise 1.2. Fix $R := k[x, y]$ with maximal ideal $\mathfrak{m} := (x, y)$, and we consider the blow-up ring

$$S := \operatorname{B}_{\mathfrak{m}} R := R \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \cdots .$$

Now, we ask if S is a flat R-module.

Proof. Heuristically, the fiber at any point except for $(0, 0)$ is a point, but the fiber over $(0, 0)$ is the full projective line. So these fibers are pretty poorly behaved, so we expect this to not be a flat module.

Well, by staring hard at our grading, we see that

$$S \cong k[x, y, tx, ty] \cong \frac{k[x, y, z, w]}{(yz - xw)},$$

where the point is that we induce the right-hand isomorphism by $tx \mapsto z$ and $ty \mapsto w$. As such, we see that this module is not flat because the polynomial $f(z,w) = yz - xw$ has coefficients which generate $\operatorname{cont}(f) = (x,y) \neq R$. In particular, we have "detected" the fiber over the origin. ■

1.1.2 The Krull Dimension

Warning 1.3. For effectively the rest of the course, all of our rings will be Noetherian.

Recall the following definition.

Definition 1.4 (Krull dimension)**.** The *Krull dimension* of a ring R, denoted $\dim R$, is the supremum of the length r of a chain of distinct primes

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

This gives rise to the following definitions.

Definition 1.5 (Krull dimension, ideals)**.** Fix a ring R and an ideal $I \subseteq R$. Then we define the *dimension* of an ideal I to be $\dim I := \dim R/I$.

Definition 1.6 (Codimension)**.** Fix I a prime ideal of a ring R. Then we define the *codimension* of I to be $\operatorname{codim} R_I$. For an arbitrary ideal I, we define

$$\operatorname{codim} I := \min_{\mathfrak{p} \supseteq I} \operatorname{codim} \mathfrak{p}.$$

Note that the above definition for codimension is well-defined because there are only finitely many minimal primes over an ideal.

Remark 1.7. Intuitively, the codimension of $\mathfrak{p}$, which is the dimension of $R_{\mathfrak{p}}$, can be computed as the length of the largest chain which goes up to $\mathfrak{p}$, because after that we can localize our chain. More explicitly, we are asking for the longest chain of the form

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \subsetneq \mathfrak{p}.$$

Example 1.8. Fix $R := \mathbb{Z}$, which has $\dim \mathbb{Z} = 1$. Then $\operatorname{codim}(0) = \dim \mathbb{Q} = 0$; in fact, $\dim(0) = 1$. Similarly, $\operatorname{codim}(p) = \dim \mathbb{Z}_p = 1$ (it helps to use the above remark), and $\dim(p) = \dim \mathbb{Z}/p\mathbb{Z} = 0$.

In all these examples, we see that

$$\dim \mathfrak{p} + \operatorname{codim} \mathfrak{p} = \dim R.$$

As such, we have the following statement.

Proposition 1.9. Fix $\mathfrak{p}$ a prime ideal of a ring R. Then

$$\dim \mathfrak{p} + \operatorname{codim} \mathfrak{p} \leq \dim R.$$

Proof. Use Remark 1.7 so that the left-hand side is the maximal length of a chain containing $\mathfrak{p}$. ■

Remark 1.10. Equality in Proposition 1.9 holds for affine domains (i.e., the ring of functions over a reduced variety).

Example 1.11. Consider $R := k[x] \times k[y, z]$, which is the coordinate. Here, $\dim R = 2$, but $\operatorname{codim}(x) = 1$ and $\dim(x) = 0$.

1.1.3 Dimension in Families

Here is a basic example of an affine variety.

Proposition 1.12. Fix R a ring.

(a) We have $\dim R = 0$ if and only if R is Artinian. In this case, R is the product of finitely many Artinian local rings.

(b) If X is an algebraic set, then $\dim A(X) = 0$ if and only if X is finite.

In algebraic geometry, we are interested in families of varieties, which in our algebraic context means morphisms of algebras. A helpful case to consider will be when we take an integral extension; this corresponds to the notion of a finite morphism of algebraic sets.

Proposition 1.13. Fix a ring homomorphism $\varphi : R \to S$ which makes S into an integral R-algebra. Then, for any $\mathfrak{p} \in \operatorname{Spec} R$ such that $\ker \varphi \subseteq \mathfrak{p}$, there exists $\mathfrak{q} \in \operatorname{Spec} S$ such that

$$\mathfrak{p} = \varphi^{-1}(\mathfrak{q}).$$

In fact, for any ideal $I \subseteq S$, we have $\dim S/I = \dim R/\varphi^{-1}(I)$.

Proof. By replacing R with $R/\ker\varphi$, we may assume that φ is an embedding. Now the point is to lift our prime $\mathfrak{p}$ upwards, which we know will give us our prime $\mathfrak{q}$ such that $\mathfrak{p} = \mathfrak{q} \cap R = \varphi^{-1}(\mathfrak{q})$.

For the latter statement, we first mod out by I to not have to worry about quotients, and we note that any chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq R$$

can be lifted to a chain

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n.$$

In fact, we know that the lifts $\mathfrak{q}_\bullet$ of a particular prime $\mathfrak{p}$ are incomparable, so we cannot make a chain like the one above longer, lest we be able to pull it back to R for a longer chain. This finishes proving the dimension statement. ■

Corollary 1.14. Fix $\varphi : X \to Y$ a morphism of algebraic varieties giving rise to a map $\varphi^* : A(Y) \to A(X)$. Further, suppose $A(X)$ is a finitely generated $A(Y)$-module. Then the following are true.

(a) The fibers of φ are finite.

(b) If $Z \subseteq X$ is Zariski closed, then $\varphi(Z) \subseteq Y$ is also Zariski closed.

(c) If φ^* is an injection, then φ is surjective.

Proof. We go one at a time.

(a) Fix a maximal ideal $\mathfrak{m} \subseteq R$. We want to compute the coordinate ring $S/\mathfrak{m}S$; in particular, we note

$$\dim S/\mathfrak{m}S = \dim R/\varphi^{-1}(\mathfrak{m}S) = \dim R/\mathfrak{m} = 0,$$

so the corresponding algebraic set is finite.

(b) We will show (c) first.

(c) Again, pick up a maximal ideal $\mathfrak{m} \subseteq R$, which is prime. Lifting to S, we can find some prime ideal $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{m}$, so because of the aforementioned incomparability, we conclude that $\mathfrak{q}$ must be maximal now. This gives our surjectivity.

(d) The point is to look at $R/\ker\varphi$ so that $Z(\ker\varphi) = \overline{\varphi(X)}$. At this point, we can apply (c) to see that φ is surjecting onto a Zariski closed set. ■

Example 1.15. Fix $S := k[x,y]/\left(x-y^2\right)$ and $R := k[x]$ so that we have a mapping $R \hookrightarrow S$. The mapping between the algebraic curves is in fact surjective, though this is not apparent from the image in $\mathbb{R}$.

We close this discussion with the following lemma.

Lemma 1.16. Fix a multiplicatively closed subset $U \subseteq R$, and set $S := R\left[U^{-1}\right]$, which gives the natural map $\varphi : R \to S$. Then, for any prime $\mathfrak{p} \subseteq R\left[U^{-1}\right]$, we have

$$\operatorname{codim} \varphi^{-1}(\mathfrak{p}) = \operatorname{codim} \mathfrak{p}.$$

Proof. Proceed directly from the definition and how our primes behave in localization. ■

1.1.4 The Principal Ideal Theorem

Here is our statement.

Theorem 1.17 (Principal ideal)**.** Fix a Noetherian ring R. Given $x \in R$, set $\mathfrak{p}$ to be a minimal prime over (x). Then

$$\operatorname{codim} \mathfrak{p} \le 1.$$

Proof. By moving from R to $R_\mathfrak{p}$, we may assume that R is local with maximal ideal $\mathfrak{p}$. We will show that, if we can find a prime $\mathfrak{q} \subsetneq \mathfrak{p}$ is strictly smaller than $\mathfrak{p}$, then $\operatorname{codim} \mathfrak{q} = 0$, which will be enough. As such, we look at $R_\mathfrak{q}$ and show that the ideal $\mathfrak{q}_\mathfrak{q}$ is nilpotent so that it has codimension 0. With this in mind, we set

$$\mathfrak{q}^{(n)} := \mathfrak{q}_\mathfrak{q}^n \cap R = \{r \in R : rs \in \mathfrak{q}^n \text{ for some } s \notin \mathfrak{q}\}.$$

We now return to our hypotheses. The fact that $\mathfrak{p}$ is minimal over (x) implies that $\mathfrak{p}/(x)$ is a maximal (by being local) and minimal ideal of $R/(x)$, so $R/(x)$ is an Artinian ring! As such, the descending chain

$$\mathfrak{q}^{(1)} + (x) \supseteq \mathfrak{q}^{(n)} + (x) \subseteq \cdots$$

must stabilize eventually. So we find our n for which $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$. In particular, $\mathfrak{q}^{(n)} \subseteq \mathfrak{q}^{(n+1)} + (x)$, so

$$\mathfrak{q}^{(n)} = x\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

Thus, Nakayama's lemma (note that x lives in the Jacobson radical) tells us that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. But now, looking in $R_\mathfrak{q}$, which is again a local ring, we see that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$ forces $\mathfrak{q}^{(n)} = 0$, which is what we wanted. ■

Remark 1.18. The analogous statement in linear algebra is that the codimension of a line in a space is 1 if the equation has a nonzero solution and 0 otherwise. More rigorously, by the implicit function theorem in differential geometry, having one equation in a tangent space will have a solution set with either the same dimension or one fewer dimension.

We can extend this result to any finitely generated ideal by an induction.

Theorem 1.19. Fix a Noetherian ring R and a minimal prime $\mathfrak{p}$ over the (finitely generated) ideal $I = (x_1, \ldots, x_s) \subseteq R$. Then $\operatorname{codim} \mathfrak{p} \leq s$.

Proof. We proceed by induction. We have already done the case of $s = 1$. For the inductive step, we would like some prime $\mathfrak{p}_1$ containing $\mathfrak{p}$ which is minimal over an ideal generated by $s - 1$ elements.

Well, if $x_s \in \mathfrak{p}_1$, then $\mathfrak{p} = \mathfrak{p}_1$ will do, so we assume henceforth that $x_s \notin \mathfrak{p}_1$. As before, we may pass to $R_\mathfrak{p}$ to assume that R is local with maximal ideal $\mathfrak{p}$. The idea, now, is to note that

$$\mathfrak{p}/(x_1, \ldots, x_s)$$

is nilpotent, using the same argument as in the previous theorem. Thus, there exists m such that

$$x_i^n \equiv 0 \pmod{\mathfrak{p}_1, x_s}$$

for any i. As such, we can write

$$x_i^n = a_i x_s + y_i,$$

where $y_i \in \mathfrak{p}_1$. So now we claim that $\mathfrak{p}_1$ is minimal over $(y_1, \ldots, y_{s-1})$, which holds by more or less looking at it, I guess. So we are done by induction. ■

Remark 1.20. The analogous statement in linear algebra is that we now have s equations, which will give rise to codimension s.

We close with some applications.

Example 1.21. Fix $R := k[x_1, \ldots, x_n]$. Then the codimension of $\mathfrak{p} := (x_1, \ldots, x_r)$ is upper-bounded by r by the above theorem, but we also have a chain

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots (x_1, \ldots, x_r) = \mathfrak{p},$$

so $\operatorname{codim} \mathfrak{p} = r$ follows.

Corollary 1.22. Fix $\mathfrak{p}$ a prime ideal of a ring R with codimension r. Then there are elements $x_1, \ldots, x_r$ such that $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_r)$.

Proof. The point is to do an induction. Starting with $r = 1$, we choose a minimal prime. Then we can choose an element x_2 which does not live in any of these finitely many minimal primes and finish by induction. ■

List of Definitions