214: Differential Topology

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1
INTRODUCTION

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.

—Charles Hermite

1.1 January 16

Let's just get started.

1.1.1 Course Structure

Here are some quick notes.

- There is a bCourses page: https://bcourses.berkeley.edu/courses/1533116. For example, it has the syllabus.
- The textbook is Lee's *Introduction to Smooth Manifolds* [Lee13]. We will read most of it.
- Our instructor is Professor Eric Chen, whose email can be reached at ecc@berkeley.edu. Office hours are after class in Evans 707.
- There is a GSI, who is Tahsia Saffat, whose email is tahsin_saffat@math.berkeley.edu. He will have some office hours and grade some homeworks.
- Homework will in general be due at 11:59PM on Thursdays via Gradescope.
- There will be an in-class midterm and a final.
- Grading is 30% homework, 30% midterm, and 40% final.
- This is a math class, not so geared towards applied subjects.
- In particular, we will assume a fair amount of topology, for which we use [Elb22] as a reference.

Let's also give a couple of notes on the course content. This course is on differential topology. The topology of interest will come from manifolds, and the differential part comes from some smoothness properties.

In some sense, our goal is to "do calculus" (e.g., differentiation, integration, vector fields, etc.) on spaces which look locally like some Euclidean space, such as a sphere. We also want to understand (smooth) manifolds on their own terms, such as understanding the maps between them and understanding some classical examples and constructions such as Lie groups or quotient manifolds.

1.1.2 Topology Review

Anyway let's get started. This is a class on manifolds, so perhaps we should begin by defining a manifold. These are going to form a special kind of topological space, so let's review topologies. We will freely use topological facts which we are too lazy to prove from [Elb22].

Definition 1.1 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X satisfying the following.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
- Finite intersection: given $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Union: for any subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have the union $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We say that the collection $\mathcal{T}$ is the collection of *open sets* of X. We will also suppress the collection $\mathcal{T}$ from the notation as much as possible.

Here is some helpful language.

Definition 1.2 (open, closed, neighborhood)**.** Fix a topological space $(X, \mathcal{T})$.

- An *open subset* $U \subseteq X$ is a subset in $\mathcal{T}$.
- A *closed subset* $V \subseteq X$ is one with $X \setminus V \in \mathcal{T}$.
- A *neighborhood* of a point $p \in X$ is an open subset $U \subseteq X$ containing p.

Example 1.3. Fix a metric space (X, d). Then there is a topology given by the metric. To be explicit, a set $U \subseteq X$ is open if and only if each $p \in U$ has some $\varepsilon > 0$ such that

$$\{x \in X : d(x, p) < \varepsilon\} \subseteq U.$$

See [Elb22, Example 2.13] for the details.

Sometimes it is easier to generate a topology from some subcollection.

Definition 1.4 (base)**.** Fix a topological space $(X, \mathcal{T})$. A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is a *base* for $\mathcal{T}$ if and only if the following holds: for each open $U \subseteq X$ and point $p \in U$, there is some $B \in \mathcal{B}$ such that $p \in B$ and $B \subseteq U$.

Example 1.5. Fix a metric space (X, d). Then the collection $\mathcal{B}$ of open balls

$$B(p, \varepsilon) :=,$$

over all $p \in X$ and $\varepsilon > 0$, forms a base of the topology. This is immediate from the construction of the topology in Example 1.3. In fact, one can merely take $\varepsilon \in \mathbb{Q}^+$ because $\mathbb{Q}$ is dense in $\mathbb{R}$.

With our objects of topological spaces in hand, we should discuss the maps between them.

Definition 1.6 (continuous)**.** Fix topological spaces X and X'. A function $\varphi\colon X \to X'$ is *continuous* if and only if $\varphi^{-1}(U')$ is open for each open $U' \subseteq X'$.

Definition 1.7 (homeomorphism)**.** Fix topological spaces X and X'. A function $\varphi\colon X \to X'$ is a *homeomorphism* if and only if φ is a bijection and both φ and φ^{-1} are continuous. We may write $X \cong X'$.

Remark 1.8. There is a continuous bijection $[0, 2\pi) \to S^1$ by $\theta \mapsto (\cos\theta, \sin\theta)$, but it is not a homeomorphism. (Here, both sets have the metric topology.) In particular, the inverse map is not continuous at 1 because the pre-image of $[0, \pi)$ is the subset $\left\{(x, y) \in S^1 : y > 0\right\} \cup \{(0, 0)\}$, which is not open in S^1 (because no $\varepsilon > 0$ has $B((0, 0), \varepsilon)$ lying in $\left\{(x, y) \in S^1 : y \geq 0\right\}$).

Exercise 1.9. Fix a nonnegative integer $n \geq 0$. Then $B(0, 1) \cong \mathbb{R}^n$.

Proof. We proceed as in [use14]. Define the functions $f\colon B(0, 1) \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to B(0, 1)$ by

$$f(x) \coloneqq \frac{x}{1 - |x|} \qquad \text{and} \qquad g(y) \coloneqq \frac{y}{1 + |y|}.$$

Notably, $|g(y)| < 1$ always, so g does indeed always output to $B(0, 1)$. These functions are both continuous, which can be checked on coordinates because they are rational functions in the coordinates, and the denominators never vanish on the domains. So we will be done once we show that f and g are inverse. In one direction, we note

$$f(g(y)) = \frac{g(y)}{1 - |g(y)|} = \frac{\dfrac{y}{1 + |y|}}{1 - \left|\dfrac{y}{1 + |y|}\right|} = \frac{y}{1 + |y| - |y|} = y.$$

In the other direction, we note

$$g(f(x)) = \frac{f(x)}{1 + |f(x)|} = \frac{\dfrac{x}{1 - |x|}}{1 + \left|\dfrac{x}{1 - |x|}\right|} = \frac{x}{1 - |x| + |x|} = x,$$

as desired. ∎

We would also like to be able to build new topologies from old ones.

Definition 1.10 (subspace)**.** Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we form a *subspace topology* by declaring the open subsets to be

$$\{U \cap S : U \in \mathcal{T}\}.$$

Example 1.11. The metric topology on $\mathbb{R}$ and the subspace topology on $X \coloneqq \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ are homeomorphic. Namely, the homeomorphism sends $x \mapsto (x, 0)$, and the inverse map is $(x, 0) \mapsto x$. Here are our continuity checks.

- The map $x \mapsto (x, 0)$ is continuous: the pre-image V of an open subset $U \subseteq X$ is open. Namely, for any $x \in V$, we see $(x, 0) \in V$, so there is $\varepsilon > 0$ such that $B((x, 0), \varepsilon) \cap X \subseteq U$, so $B(x, \varepsilon) \subseteq V$.
- The map $(x, 0) \mapsto x$ is continuous: the pre-image V of an open subset $U \subseteq \mathbb{R}$ is open. Namely, for each $(x, 0) \in V$, we see $x \in U$, so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V$, so $B((x, 0), \varepsilon) \cap X \subseteq U$.

Lastly, we will want some adjectives for our topologies.

Definition 1.12 (compact)**.** Fix a topological space X. A subset $K \subseteq X$ is *compact* if and only if any open cover can be reduced to a finite subcover. Explicitly, any collection $\mathcal{U}$ of open sets of X such that $K \subseteq \bigcup_{U\in\mathcal{U}} U$ (this is called an *open cover*) has some finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{U\in\mathcal{U}'} U$.

Example 1.13. The interval $[0,1] \subseteq \mathbb{R}$ is compact. See [Elb22, Example 4.4].

Definition 1.14 (Hausdorff)**.** Fix a topological space X. Then X is *Hausdorff* if and only if any two distinct points $p_1, p_2 \in X$ have disjoint open subsets $U_1, U_2 \subseteq X$ such that $p_1 \in U_1$ and $p_2 \in U_2$.

Example 1.15. Any metric space (X,d) is Hausdorff. Namely, for distinct points $p, q \in X$, we see $d(p,q) > 0$, so set $\varepsilon := d(p,q)/2$, and we see that $p \in B(p,\varepsilon)$ and $q \in B(q,\varepsilon)$, but $B(p,\varepsilon) \cap B(q,\varepsilon) = \varnothing$. For this last claim, we note r living in the intersection would imply

$$d(p,q) \le d(p,r) + d(r,q) < 2\varepsilon,$$

which is a contradiction to the construction of ε.

1.1.3 Topological Manifolds

For intuition, we state but not prove the following result.

Theorem 1.16 (Topological invariance of dimension)**.** Fix open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. If there is a homeomorphism $U \cong V$, then $m = n$.

Proof. The usual proofs go through (co)homology, which we may cover later in the class. For the interested, see [Elb23, Proposition 3.50]. ■

We will soon define topological manifolds. The main adjective we want is being "locally Euclidean."

Definition 1.17 (locally Euclidean)**.** Fix a topological space X. Then X is *locally Euclidean of dimension n at p* if and only if there is an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$. We say that X is *locally Euclidean of dimension n* if and only if it is locally Euclidean of dimension n at each point.

Remark 1.18. One can always take $\widetilde{U}$ to be either $B(0,1) \subseteq \mathbb{R}^n$ or even all of $\mathbb{R}^n$. Indeed, for $x \in X$, we are given an open neighborhood U of x and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. We produce open neighborhoods of x homeomorphic to $B(0,1)$ and $\mathbb{R}^n$.

- $B(0,1)$: there is $\varepsilon > 0$ such that $B(\varphi(x),\varepsilon) \subseteq \widehat{U}$. Then we let $U' := \varphi^{-1}(B(\varphi(x),\varepsilon))$ so that we have a chain of homeomorphisms

$$U' \overset{\varphi}{\cong} B(\varphi(x),\varepsilon) \cong B(0,\varepsilon) \cong B(0,1),$$

  where the second homeomorphism is a translation, and the last homeomorphism is a dilation.

- $\mathbb{R}^n$: in the light of the previous point, it suffices to note that Exercise 1.9 provides a homeomorphism $B(0,1) \cong \mathbb{R}^n$ and then post-compose with this homeomorphism.

Let's explain why we want Theorem 1.16.

Lemma 1.19. Fix a locally Euclidean space X. For each $p \in X$, there is a unique nonnegative integer n such that there exists an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$.

Proof. Suppose there are two such nonnegative integers m and n, so we get open neighborhoods $U, V \subseteq X$ and $\widetilde{U} \subseteq \mathbb{R}^m$ and $\widetilde{V} \subseteq \mathbb{R}^n$. Let $\varphi \colon U \cong \widetilde{U}$ and $\psi \colon V \cong \widetilde{V}$ be the needed homeomorphisms. Then the point is to use the intersection $U \cap V$: there is a composite isomorphism

$$\varphi(U \cap V) \cong U \cap V \cong \psi(U \cap V)$$

from an open subset in $\mathbb{R}^m$ to an open subset in $\mathbb{R}^n$. So Theorem 1.16 completes the proof. ■

Anyway, here is our definition of a topological manifold.

Definition 1.20 (topological manifold). An n-*dimensional topological manifold* is a topological space M with the following properties.

- M is Hausdorff.
- M is locally Euclidean of dimension n at each point.
- M is second countable (i.e., has a countable base).

We may abbreviate "n-dimensional topological manifold" to "topological n-manifold."

Let's give a few quick constructions.

Lemma 1.21. For each $n \geq 0$, the space $\mathbb{R}^n$ is an n-dimensional topological manifold.

Proof. Let's be quick. Being a metric space yields Hausdorff, locally Euclidean is immediate because it's $\mathbb{R}^n$, and second-countability follows by using the base

$$\left\{B(q, \varepsilon) : q \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}^+\right\}.$$

This is indeed a base because $\mathbb{Q}$ is dense in $\mathbb{R}$. Explicitly, for each $p \in \mathbb{R}^n$ living in some open subset $U \subseteq \mathbb{R}^n$, begin by replacing U with a smaller open subset of the form $B(p, \varepsilon)$ where $\varepsilon > 0$; by perhaps making ε smaller, we may assume that $\varepsilon > 0$ is rational. Now, choosing coordinates $p = (x_1, \ldots, x_n)$, choose rational numbers $q_1, \ldots, q_n$ so that $|x_i - q_i| < \varepsilon/(2\sqrt{n})$ for each i. Then $q := (q_1, \ldots, q_n)$ has $d(p, q) < \varepsilon/2$ and so

$$p \in B(q, \varepsilon/2) \subseteq B(p, \varepsilon) \subseteq U,$$

so $B(q, \varepsilon/2)$ is the needed open subset in our base. ■

The following lemma will be helpful in the sequel.

Lemma 1.22. Fix a topological space M and nonnegative integer $n \geq 0$. Suppose that there is a countable open cover $\{U_i\}_{i \in \mathbb{N}}$ of M such that each i has a homeomorphism $U_i \cong \widetilde{U}_i$ where $\widetilde{U}_i \subseteq \mathbb{R}^n$ is open. Then M is locally Euclidean of dimension n at each point, and M is second countable.

Proof. For locally Euclidean, we note that each $p \in M$ lives in some U_i, so we are done. As for second countability, we note that each $\widetilde{U}_i$ is second countable as a subspace of a second countable space (see Lemma 1.21), so each U_i is second countable by moving back through the homeomorphism, and so M is second countable by taking the union of the bases of the U_i.

To make this last step more explicitly, we note that each U_i has a countable base $\mathcal{B}_i$, so we claim that $\mathcal{B} := \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ becomes a countable base of M. Certainly $\mathcal{B}$ is countable, and every set in $\mathcal{B}$ is in one of the $\mathcal{B}_i$ and hence open in M. Lastly, to check that we have a base, we note that any open $U \subseteq M$ and $p \in M$ will have $p \in U_i$ for some i, so there is some $B \in \mathcal{B}_i \subseteq \mathcal{B}$ such that $p \in B \subseteq U \cap U_i$. ■

1.1.4 Examples and Non-Examples

Here are some non-examples to explain why we want all of these hypotheses.

> **Exercise 1.23.** Consider the space X defined as $\mathbb{R}\times\{0,1\}$ where we identify $(x,0)\sim(x,1)$ whenever $x\neq 0$. (The topology on X is the quotient topology [Elb22, Definition 2.81].) This space is not Hausdorff, but it is locally Euclidean and second countable.

Proof. We run our checks.

- This space is not Hausdorff because the points $(0,0)$ and $(0,1)$ are "infinitely close together." Explicitly, any open neighborhoods U and V of $(0,0)$ and $(0,1)$, respectively, the induced topology yields some $\varepsilon>0$ such that $B((0,0),\varepsilon)\subseteq U$ and $B((0,1),\varepsilon)\subseteq V$, but then $(-\varepsilon/2,0)=(-\varepsilon/2,1)$ is in both U and V.
- This space is locally Euclidean and second countable by Lemma 1.22. Explicitly, we note that $\mathbb{R}\cong\mathbb{R}\times\{0\}\subseteq X$ and $\mathbb{R}\cong\mathbb{R}\times\{1\}\subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ■

> **Exercise 1.24.** Consider the space X defined as $\mathbb{R}\times\{0,1\}$ where we identify $(x,0)\sim(x,1)$ whenever $x\leq 0$, again where we are using the quotient topology. Then X is Hausdorff and second countable, but it is not Euclidean of dimension 1 at $0\in X$.

Proof. We run our checks.

- This space is Hausdorff. We check this directly by casework.
  - Suppose we have distinct points $p=(x,a)$ and $q=(y,b)$ with $x\neq y$; for example, this includes the case where we may take $a=b$ and hence includes the case when $x,y\leq 0$. Then we may set $\varepsilon:=\frac{1}{2}\left|x-y\right|$ so that $B(p,\varepsilon)$ and $B(q,\varepsilon)$ are disjoint.
  - We now may assume that $x=y$; then $a\neq b$. Thus, we must have $x>0$ or $y>0$. As such, we may as well take $\varepsilon:=\min\{|x|,|y|\}$ so that $B(p,\varepsilon)$ and $B(q,\varepsilon)$ are disjoint.
- This space is not locally Euclidean at 0. Indeed, suppose that there is open subset $U\subseteq X$ around 0 which is homeomorphic to an open subset of $\mathbb{R}$. By shifting, we may as well assume that the homeomorphism sends 0 to 0. Additionally, the same statement will be true by any open subset of U, so we may as well as assume that U is of the form $(-\varepsilon,\varepsilon)\times\{0,1\}$ (in X). In particular, U is connected.

  But then the image $\widehat{U}$ of U in $\mathbb{R}$ is a connected open subset of $\mathbb{R}$, which must be an interval. Now, intervals have the property that deleting any point of an interval makes produces a topological space with two connected components. However, deleting 0 from U will produce three connected components: $(-\varepsilon,0)\times\{0,1\}$ and $(0,\varepsilon)\times\{0\}$ and $(0,\varepsilon)\times\{1\}$. So $\widehat{U}$ and U cannot actually be homeomorphic!
- This space is second countable by Lemma 1.22. Again, we note that $\mathbb{R}\cong\mathbb{R}\times\{0\}\subseteq X$ and $\mathbb{R}\cong\mathbb{R}\times\{1\}\subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ■

> **Remark 1.25.** Essentially the same argument implies that the above space fails to be locally Euclidean of any dimension at $0\in X$. Namely, a connected open subset of $\mathbb{R}^n$ for $n\geq 2$ will remain connected after removing any point, so it cannot be homeomorphic to $(-\varepsilon,\varepsilon)\times\{0,1\}$ in X.

Morally, the second countability is being required as a smallness condition; let's see some pathological examples without second countability. The following lemma approximately explains the problem.

Lemma 1.26. Fix a topological space X. Suppose that there is an uncountable subset $Y \subseteq X$ such that each $y \in Y$ has an open neighborhood $U_y \subseteq X$ where the U_y are pairwise disjoint. Then X fails to be second countable.

Proof. Suppose we have a base $\mathcal{B}$; we show $\mathcal{B}$ is uncountable. Each $y \in U_y$ has some $B_y \in \mathcal{B}$ with $B_y \subseteq U_y$. However, $y \neq y'$ implies that $B_y \neq B_{y'}$ because $y \in B_y$ while $p_y \notin U_{y'}$ implies $p_y \notin B_{y'}$. So $\{B_y\}_{y \in Y}$ is an uncountable subcollection of $\mathcal{B}$. ■

Exercise 1.27. Consider an uncountable set S with the discrete topology (namely, every subset is open), and then we form the product $X := \mathbb{R} \times S$. Then X is Hausdorff, locally Euclidean of dimension 1, but it is not second countable.

Proof. Here are our checks.

- Note that X is a product of Hausdorff spaces and hence is Hausdorff.
- This space is locally Euclidean of dimension 1: for each $(x, s) \in X$, we note that $\mathbb{R} \times \{s\}$ is an open subset of X (because S is discrete) where $\mathbb{R} \times \{s\} \cong \mathbb{R}$ by an argument similar to Example 1.11.
- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (0, s)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \mathbb{R} \times \{s\}$. ■

Exercise 1.28. Consider the first uncountable ordinal ω_1. Then define $X := (S \times [0,1)) \setminus \{(0,0)\}$, and we give X the order topology where the ordering is lexicographic. (Namely, the base consists of the "intervals" $\{x : x < b\}$ or $\{x : a < x\}$ or $\{x : a < x < b\}$.) This space is Hausdorff, locally Euclidean 1, but it is not second countable.

Proof. Here are our checks.

- This space is Hausdorff because it is a dense linear order. Explicitly, for $(s, a), (t, b) \in X$, we have the following cases.
  - Suppose $s = t$. In this case, $a \neq b$; suppose $a < b$ without loss of generality. Then $\{x : x < (s, (a + b)/2)\}$ and $\{x : x > (s, (a + b)/2)\}$ are the needed open sets.
  - Suppose $s \neq t$; take $s < t$ without loss of generality. If $a > 0$, then $\{s\} \times (0, (a + 1)/2)$ and $\{s\} \times ((a + 1)/2, 1) \cup \{t\} \times [0, 1)$ provide the needed open sets. Otherwise, if $a = 0$, then $\{x : x < (s, 1/2)\}$ and $\{x : x > (s, 1/2)\}$ provide the needed open sets.
- This space is locally Euclidean of dimension 1: fix any $(s, r) \in X$. Note that $s \in \omega_1$ is countable, so we claim that
  $$(s + 1) \times [0, 1) \cong [0, 1),$$
  sending $(0, 0)$ to 0, from which the claim follows by deleting $(0, 0)$. Because the relevant orders produce the needed topologies, we are really asking for an order-preserving bijection from $(s + 1) \times [0, 1)$ to $[0, 1)$.

  Well, for any $t \in \omega_1$, we claim that there is an increasing sequence $\{p_\alpha\}_{\alpha < t} \subseteq [0, 1)$ of order type t with $p_0 = 0$, from which the claim will follow by taking $s = t$ and sending $\alpha \times [0, 1) \subseteq (s + 1) \times [0, 1)$ to $[p_\alpha, p_{\alpha+1})$ (where we define $p_s := 1$). To see this claim, we argue by induction on s. For $s = 0$, take $p_0 := 0$. If s is a successor ordinal, divide all the existing p_α by 2 and then set $p_{s+1} := 1/2$.

  Lastly, if s is a limit ordinal, it is still only a countable limit ordinal, so we can find an increasing sequence of countable ordinals $\{s_i\}_{i \in \omega}$ approaching s. The sequence corresponding to s_0 will fit into $[0, 1/2)$ after scaling; then the sequence corresponding to s_1 but after s_0 will fit into $[1/2, 2/3)$ after scaling. We can continue this process inductively to complete the claim for s. I won't bother to write out the details.

- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (s, 1/2)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \{s\} \times (0,1)$. ■

Remark 1.29. What makes the locally Euclidean check above annoying is that we must show $(\omega, 0) \in X$ has a neighborhood isomorphic to an open subset of $\mathbb{R}$, which is not totally obvious.

Let's return to examples.

Example 1.30. Consider the unit circle S^1. We check that S^1 is a 1-dimensional topological manifold.

- S^1 is a metric space, so it is Hausdorff.
- S^1 is second countable: it is a subspace of $\mathbb{R}^2$, and $\mathbb{R}^2$ is second countable by Lemma 1.21 again.
- S^1 is locally Euclidean: we proceed explicitly. Define $U_1^\pm := \{(x,y) \in S^1 : \pm x > 0\}$; then $U_1^\pm \cong (-1,1)$ by $(x,y) \mapsto y$. Similarly, define $U_2^\pm := \{(x,y) \in S^1 : \pm y > 0\}$; then $U_2^\pm \cong (-1,1)$ by $(x,y) \mapsto x$.

1.2 January 18

The first homework has been posted. It is mostly a review of point-set topology things. It is due on the 25th of January.

Remark 1.31. Please read the section on fundamental groups of manifolds on your own. We will not discuss it in class.

To review, our current goal is to define smooth manifolds. Thus far we have defined a topological space and provided enough adjectives to turn it into a topological manifold. To proceed, we need to add smoothness to our structure. We will do this later.

1.2.1 Connectivity

For now, we will content ourselves with some extra adjectives for our topological manifolds which will later be helpful. Here are two notions of connectivity.

Definition 1.32 (connected)**.** Fix a topological space X. Then X is *disconnected* if and only if there exist disjoint nonempty open subsets $U, V \subseteq X$ such that $X = U \sqcup V$. If X is not disconnected, we say that X is connected.

Example 1.33. The interval $[0,1]$ is connected. See [Elb22, Lemma A.6].

Remark 1.34. Equivalently, we can say that X is connected if and only if X and $\varnothing$ are the only subsets of X which are both open and closed.

Definition 1.35 (path-connected)**.** Fix a topological space X. Then X is *path-connected* if and only if any two points $p, q \in X$ has some continuous map $\gamma \colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Example 1.36. The space $B(0,1) \subseteq \mathbb{R}^n$ is path-connected. Indeed, we show that the path-connected component of 0 is all of $B(0,1)$; see [Elb22, Definition A.19]. In other words, we must exhibit a path from 0 to v for any $v \in B(0,1)$. Well, define $\gamma\colon [0,1] \to B(0,1)$ by $\gamma(t) := tv$. This is continuous because it is linear, and it has $\gamma(0) = 0$ and $\gamma(1) = v$ as desired.

In general, these two notions do not coincide.

Example 1.37. Consider the topological space

$$X := \{(x, \sin(1/x)) : x \in (0,1)\} \cup \{(0,y) : y \in \mathbb{R}\}.$$

Then X is connected, but it is not path-connected. See [Elb22, Exercise A.20].

But one does in general apply the other.

Lemma 1.38. Fix a topological space X. If X is path-connected, then X is connected.

Proof. See [Elb22, Lemma A.16], though we will sketch the proof. We proceed by contraposition. Suppose that X is disconnected, so we may write $X = U \sqcup V$ where $U, V \subseteq X$ are disjoint nonempty open subsets. Now choose some $p \in U$ and $q \in V$, and we claim that there is no path $\gamma\colon [0,1] \to X$. Indeed, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would then be nonempty disjoint open subsets of $[0,1]$ covering $[0,1]$, which is a contradiction. ■

However, for topological manifolds, these notions do coincide.

Proposition 1.39. Fix a topological space M which is locally Euclidean of dimension n. Then M is path-connected if and only if it is connected.

Proof. The forward direction is by Lemma 1.38. Thus, we focus on showing the converse. Fix some $p \in M$, and we define the subset

$$U_p := \{q \in M : \text{there exists a path from } p \text{ to } q\}.$$

This is the path-connected component of p in M; see [Elb22, Definition A.19]. The main claim is that U_p is open.

Suppose $q \in M$, and we need to find an open neighborhood $B_q \subseteq M$ of q living inside U_p. Noting then that $U_p = \bigcup_{q \in U_p} B_q$ will complete the proof of this claim. Well, q has some open neighborhood $B \subseteq M$ equipped with a homeomorphism $\varphi\colon B \cong B(0,1)$ by Remark 1.18. Then $B(0,1)$ is path-connected by Example 1.36, so B is path-connected by going back through the homeomorphism. Thus, because U_p is an equivalence class, it is also the path-connected equivalence class of q, so U_p must contain B.

Now, let $\mathcal{U}$ denote the collection of path-connected components of M. This is a collection of disjoint open subsets covering M. Certainly it is nonempty, so select $U \in \mathcal{U}$. Then we write

$$M = U \cup \bigcup_{U' \in \mathcal{U} \setminus \{U\}} U'.$$

This is a decomposition of M into disjoint open subsets, so because M is connected, one of these must be empty. But U is empty, so instead the union of the U' must be nonempty. However, everything in $\mathcal{U}$ is nonempty, so instead we see that $\mathcal{U} \setminus \{U\}$ is empty, so $M = U$ is path-connected. ■

1.2.2 Local compactness

Here is our definition.

Definition 1.40 (local compactness)**.** Fix a topological space X. Then X is *locally compact* if and only if any $x \in X$ has some open neighborhood $U \subseteq X$ such that there exists a compact subset $K \subseteq X$ containing U.

Remark 1.41. If X is Hausdorff, then compact subsets are closed [Elb22, Corollary 4.13], and closed subsets of a compact space are still compact [Elb22, Lemma 4.10], so we may as well take $K = \overline{U}$ in the above definition.

The above remark motivates the following definition.

Definition 1.42 (precompact)**.** Fix a topological space X. An open subset $U \subseteq X$ is *precompact* if and only if $\overline{U}$ is compact.

Remark 1.43. Here is a quick check which will prove to be useful: if X is Hausdorff and $U \subseteq X$ is precompact, and $V \subseteq U$, then V is still precompact. Indeed, $\overline{U}$ is compact, and $\overline{V} \subseteq \overline{U}$ is a closed subset and hence compact [Elb22, Lemma 4.10].

Example 1.44. The topological space $\mathbb{R}$ is locally compact; see [Elb22, Example 4.71].

Non-Example 1.45. Infinite-dimensional normed vector spaces fail to be locally compact. Namely, open balls fail to be precompact, so local compactness fails.

Non-Example 1.46. The space $\mathbb{Q}$ is not locally compact. Indeed, suppose for the sake of contradiction that we have a precompact nonempty open neighborhood $U \subseteq \mathbb{Q}$ of $0 \in \mathbb{Q}$. Now, $\mathbb{Q}$ is Hausdorff (it's a metric space), so we can find some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U$ while $\varepsilon \notin \mathbb{Q}$, so Remark 1.43 tells us that $(\varepsilon/2, \varepsilon)$ is precompact so that $[\varepsilon/2, \varepsilon]$ is actually compact.

However, this is false. Let $\{\alpha_i\}_{i \geq 1}$ be an increasing sequence of irrationals in $[\varepsilon/2, \varepsilon]$ with $\alpha_i \to \varepsilon$. Explicitly, we can take $\alpha_i := \frac{i}{i+1} \cdot \varepsilon$. Then we define

$$U_i := [\alpha_i, \alpha_{i+1}]$$

for each $i \geq 1$. Note $[\alpha_i, \alpha_{i+1}] = (\alpha_i, \alpha_{i+1})$, so the $U_\bullet$s provide a countable sequence of disjoint open subsets covering $[\varepsilon/2, \varepsilon]$. Thus, $[\varepsilon/2, \varepsilon]$ cannot be compact.

One can check that manifolds are locally compact.

Proposition 1.47. Fix a topological n-manifold M. Then M is locally compact.

Proof. This follows from being locally Euclidean. Fix $p \in M$, and then we are promised some open subset $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. Then there is an open ball $B(\varphi(p), \varepsilon) \subseteq \widehat{U}$. Then $\overline{B(\varphi(p), \varepsilon/2)} \subseteq \widehat{U}$ is closed and bounded in $\mathbb{R}^n$ and hence compact, so $\varphi^{-1}(B(\varphi(p), \varepsilon/2))$ is a subset of the compact subset $\varphi^{-1}(\overline{B(\varphi(p), \varepsilon/2)})$. ■

Being locally compact approximately speaking allows one to understand a space by building it up from compact ones. Here is one way to do this.

Definition 1.48 (exhaustion)**.** Fix a topological space X. Then an *exhaustion* of X is a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of X satisfying the following.

- Ascending: $K_0 \subseteq K_1 \subseteq \cdots$.
- Covers: $X = \bigcup_{i\in\mathbb{N}} K_i$.
- Not too close: $K_i \subseteq K_{i+1}^\circ$.

Example 1.49. The space $\mathbb{R}^n$ has an exhaustion by $K_i := B(0,i)$.

Here is a way to build an exhaustion.

Proposition 1.50. Fix a topological space X. If X is second-countable, locally compact, and Hausdorff. Then X has an exhaustion. In particular, topological n-manifolds have an exhaustion.

Proof. The second claim follows from the first by Proposition 1.47 and the definition of a manifold. So we will focus on showing the first claim.

Fix a countable base $\mathcal{B}$ of X, and let $\mathcal{B}'$ be the subcollection of precompact open base elements. Quickly, we note that $\mathcal{B}'$ is still a base: certainly everything in $\mathcal{B}'$ is open, and then for any $p \in X$ and open neighborhood $U \subseteq X$, we need some $B' \in \mathcal{B}'$ such that B' is precompact.

Well, because X is locally compact, there is a precompact open neighborhood U' of p by Remark 1.41. Then $U \cap U$ is an open neighborhood of p, so we can find a base element $B \in \mathcal{B}$ containing p and inside $U' \cap U$. Then $B \subseteq U'$ is precompact by Remark 1.43.

We now construct our exhaustion. Enumerate $\mathcal{B} = \{B_0, B_1, \ldots\}$, and we proceed as follows.

1. Set $K_0 := \overline{B_0}$, which is compact by construction of B_0.
2. Now suppose we have a compact subset $K_i \subseteq X$, and we construct K_{i+1}. Note that $\mathcal{B}$ is an open cover of K_i, which can be reduced to a finite subcover, so there is some M_{i+1} such that K_i is covered by $\{B_i : i \le M_{i+1}\}$. We may as well suppose that $M_{i+1} \ge i+1$. Then we define
$$K_{i+1} := \bigcup_{i=1}^{M} \overline{B_i}.$$
Note that the finite union of compact sets remains compact.

The above construction produces an exhaustion. Here are our checks, which will complete the proof.

- Ascending: by construction, we see that
$$K_{i+1}^\circ \supseteq \bigcup_{i=1}^{M} B_i \supseteq K_i.$$
- Covers: any $x \in X$ lives in some B_i, and by construction, we have $B_i \subseteq K_i$, so $x \in K_i$. ■

1.2.3 Paracompactness

We will want to talk about covers in some more detail.

Definition 1.51 (cover)**.** Fix a topological space X. A *cover* is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ such that
$$X = \bigcup_{U\in\mathcal{U}} U.$$

Definition 1.52 (locally finite)**.** Fix a topological space X. A cover $\mathcal{U}$ of X is *locally finite* if and only if any $p \in X$ has some open neighborhood $U \subseteq X$ intersecting at most finitely many elements of $\mathcal{U}$.

Definition 1.53 (refinement)**.** Fix a cover $\mathcal{U}$ of a topological space X. Then a *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ such that any $V \in \mathcal{U}$ is contained in some $U \in \mathcal{U}$.

And here is our definition.

Definition 1.54 (paracompact)**.** Fix a topological space X. Then X is *paracompact* if and only if every open cover has a locally finite open refinement.

Approximately speaking, the point of desiring paracompactness is that it allows "reducing to Euclidean" arguments in the future will not have to deal with intersections which are infinitely bad. Anyway, here is our result.

Proposition 1.55. Fix a topological n-manifold M. Then M is paracompact.

Proof. In fact, we are only going to use the fact that M has an exhaustion, proven in Proposition 1.50.

Fix an open cover $\mathcal{U}$, and we want to produce a locally finite open refinement. To set us up, fix an exhaustion $\{K_i\}_{i\in\mathbb{N}}$, which exists by Proposition 1.50, and define the following sets for each $i \in \mathbb{N}$.

- For $i \geq -1$, define $V_i := K_{i+1} \setminus K_i^\circ$, which is a closed subset of the compact set K_{i+1} and hence compact [Elb22, Lemma 4.10]; take $K_{-1} = \varnothing$ without concern.
- For $i \geq 0$, define $W_i := K_{i+2}^\circ \setminus K_{i-1}$, which is open; here, take $K_{-1} = \varnothing$ without concern.

For intuition, we should think about the $W_\bullet$s as being a locally finite cover from which we will build the locally finite cover refinement of $\mathcal{U}$.

For the construction, we fix some $j \geq 0$ for the time being. For each $x \in V_j$, find some $U_x \in \mathcal{U}$ containing x. Note that $\{U_x\}_{x\in V_j}$ is an open cover of V_j, and because $V_j \subseteq W_j$, in fact $\{U_x \cap W_j\}_{x\in V_j}$ is an open cover. Because V_j is compact, we can thus reduce this open cover to a finite subcover $\mathcal{A}_j$.

Now letting j vary, we define

$$\mathcal{V} := \bigcup_{j\geq 0} \mathcal{A}_j.$$

Here are our checks.

- Open cover: each $x \in X$ lives in some K_{i+1} because we have an exhaustion, so lives in some V_i, so it lives in some open subset in $\mathcal{A}_j$, so it lives in some open subset in $\mathcal{V}$.
- Refinement: by construction, each open set in $\mathcal{A}_j$ is a subset in $\mathcal{U}$.
- Locally finite: this is essentially by construction. The main point is that any $x \in X$ lives in some K_i, so by choosing the least such K_i places x in some $V_i \subseteq W_i$. We now show that only finitely many open subsets in $\mathcal{V}$ intersect W_i. Note $W_i \subseteq K_{i+2}$, so $W_i \cap W_j = \varnothing$ for $j \geq i+2$. Thus, if $V \cap W_i \neq \varnothing$, we must have $V \in \mathcal{A}_j$ for $j < i+2$. But this is only finitely many indices, and each $\mathcal{A}_j$ is finite, so this is only finitely many candidates. ■

1.2.4 Products

We now discuss an in-depth example.

Proposition 1.56. Fix finitely many topological manifolds $M_1, \ldots, M_k$. Then the product

$$M_1 \times \cdots \times M_k$$

is also a topological manifold of dimension $\dim M_1 + \cdots + \dim M_k$.

We will do this via a sequence of lemmas.

Lemma 1.57. Fix a collection of Hausdorff topological spaces $\{X_\alpha\}_{\alpha \in \Lambda}$. Then the product

$$\prod_{\alpha \in \Lambda} X_\alpha$$

is also Hausdorff.

Proof. Fix distinct points $(x_\alpha)_{\alpha \in \Lambda}$ and $(y_\alpha)_{\alpha \in \Lambda}$ in the product. Then there is an index $\beta \in \Lambda$ such that $x_\beta \neq y_\beta$, so because X_β is Hausdorff, there are disjoint open neighborhoods $U_\beta, V_\beta \subseteq X_\beta$ of x_β and y_β, respectively. Then we define $U_\alpha = V_\alpha := X_\alpha$ for $\alpha \neq \beta$, and we note that the open subsets

$$\prod_{\alpha \in \Lambda} U_\alpha \qquad \text{and} \qquad \prod_{\alpha \in \Lambda} V_\alpha$$

are disjoint open neighborhoods of $(x_\alpha)_{\alpha \in \Lambda}$ and $(y_\alpha)_{\alpha \in \Lambda}$, respectively, so we are done. (These are disjoint because any point in the intersection will have the β coordinate in $U_\beta \cap V_\beta = \varnothing$.) ■

Lemma 1.58. Fix finitely many second countable topological spaces $\{X_i\}_{i=1}^n$. Then the product

$$\prod_{i=1}^n X_i$$

is also second countable.

Proof. Let the product be X. For each i, let $\mathcal{B}_i$ be a countable base for X_i. Then define

$$\mathcal{B} := \left\{ \prod_{i=1}^n B_i : B_i \in \mathcal{B}_i \text{ for each } i \right\}.$$

We claim that $\mathcal{B}$ is a base for the topology on the X. Indeed, suppose $(x_1, \ldots, x_n) \in X$ lives in some open subset $U \subseteq X$. From the standard base on X, we know that there are open subsets $U_i \subseteq X_i$ for each i such that $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$. Now, for each U_i, we note that $x_i \in U_i$ must have some $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$ and $B_i \subseteq U_i$. But then

$$(x_1, \ldots, x_n) \in B_1 \times \cdots \times B_n \subseteq U,$$

so $B_1 \times \cdots \times B_n \in \mathcal{B}$ is the desired base element. ■

We now prove Proposition 1.56.

Proof of Proposition 1.56. We get Hausdorff from Lemma 1.57 and second countable from Lemma 1.58. So it remains to check that we are locally Euclidean. For brevity, let M be the product, and set $n_i := \dim M_i$ for each i, and let $n := n_1 + \cdots + n_k$.

Now, fix some point $(x_1, \ldots, x_k) \in M$. For each i, we get some open neighborhood $U_i \subseteq M_i$ of x_i and some open $\widehat{U}_i \subseteq \mathbb{R}^{n_i}$ with a homeomorphism $\varphi_i \colon U_i \cong \widehat{U}_i$. Now, we see that the product map

$$(\varphi_1 \times \cdots \times \varphi_k) \colon U_1 \times \cdots \times U_k \to \widehat{U}_1 \times \cdots \times \widehat{U}_k$$

is still a homeomorphism, and the target is an open subset of

$$\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \cong \mathbb{R}^n,$$

where this last homeomorphism is obtained by simply concatenating the coordinates. So we have constructed a composite homeomorphism from an open neighborhood of $(x_1, \ldots, x_k)$ to an open subset of $\mathbb{R}^n$, as desired. ■

Example 1.59. Example 1.30 established S^1 as a topological 1-manifold, so the n-torus

$$T^n := \underbrace{S^1 \times \cdots \times S^1}_{n}$$

is a topological n-manifold. Note that the covering space $p\colon \mathbb{R} \to S^1$ will induce the covering space $p^n\colon \mathbb{R}^n \to T^n$, so we can also view T^n as $\mathbb{R}^n/\mathbb{Z}^n$; in other words, we have the unsurprising homeomorphism $\mathbb{R}^n/\mathbb{Z}^n \to (\mathbb{R}/\mathbb{Z})^n$.

1.2.5 Charts

The construction of our smooth structure will arise from more carefully understanding how a manifold is locally Euclidean. This arises from charts.

Definition 1.60 (chart)**.** Fix a topological n-manifold M. Then a *coordinate chart* or just *chart* is a pair (U, φ) where $U \subseteq M$ is open and $\varphi\colon U \cong \widehat{U}$ is a homeomorphism where $\widehat{U} \subseteq \mathbb{R}^n$ is open.

Essentially, the content of M being locally Euclidean is that it has an open cover by open subsets belonging to some chart. The reason we call it a chart is that we are (approximately speaking) providing "local coordinates" to an open subset of M.

Definition 1.61 (coordinate function)**.** Fix a chart (U, φ) if a topological n-manifold M. Then we may write

$$\varphi(p) := \left(x^1(p), \ldots, x^n(p)\right) \in \mathbb{R}^n$$

for each $p \in M$. We call these functions $x^\bullet\colon U \to \mathbb{R}$ the *coordinate functions.*

Note that these coordinate functions are continuous because they are simply the continuous function φ composed with the projection $\mathbb{R}^n \to \mathbb{R}$.

Example 1.62. Fix an open subset $V \subseteq \mathbb{R}^m$, and let $F\colon V \to \mathbb{R}^n$ be a continuous function. Then the graph

$$\Gamma := \{(x, F(x)) : x \in V\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

is a topological n-manifold. Because we are already a subspace of $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$, we see that Γ is also Hausdorff and second countable. (Subspaces inherit being Hausdorff directly, and we inherit being second countable by using the intersection of the given countable base.)

The main content comes from being locally Euclidean. Namely, there is a projection map $\pi\colon \Gamma \to V$ by $(x, y) \mapsto x$ which in fact is a homeomorphism (it's continuous inverse is $(\mathrm{id} \times F)\colon x \mapsto (x, F(x))$). So we have the single chart (V, π), which establishes being a topological n-manifold.

1.3 January 23

The first homework is due on Thursday. Today we discuss smooth structures.

1.3.1 Examples of Topological Manifolds

Let's provide a few more examples of topological manifolds.

Exercise 1.63 (sphere)**.** We show that the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold.

Proof. Explicitly, for each $i \in \{1, \ldots, n+1\}$, we define

$$U_i^{\pm} \coloneqq \{(x_1, \ldots, x_{n+1}) \in S^n : \pm x_i > 0\},$$

which has a projection $\pi_i^{\pm} \colon U_i^{\pm} \to B(0,1)$ (for $B(0,1) \subseteq \mathbb{R}^n$) given by erasing the x_i coordinate. One can show that the $\pi_i^{\pm}$ are all homeomorphisms—certainly, it is continuous, and the inverse map is given by

$$(x_1, \ldots, x_n) \coloneqq \left(x_1, \ldots, x_{i-1}, \pm\sqrt{1 - (x_1^2 + \cdots + x_n^2)}, x_i, \ldots, x_n\right),$$

which is also continuous. (We won't bother checking that the maps are mutually inverse.) Lastly, we note that the $U_i^{\pm}$ is an open cover of S^n because any point in S^n has some nonzero coordinate, and this nonzero coordinate will have a sign. ■

Exercise 1.64 (projective space)**.** Define the space $\mathbb{RP}^n$ as "lines in $\mathbb{R}^{n+1}$": it consists of equivalence classes of nonzero points in $\mathbb{R}^{n+1} \setminus \{0\}$, where $x \sim y$ if and only if there is some $\lambda \in \mathbb{R}^{\times}$ such that $x = \lambda y$. We show that $\mathbb{RP}^n$ is a topological n-manifold.

Proof. For notation, we let $[x_0 : \cdots : x_n]$ denote the equivalence class of $(x_1, \ldots, x_n)$ in $\mathbb{RP}^n$. Note there is a projection $p \colon \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to \mathbb{RP}^n$, and we give $\mathbb{RP}^n$ the induced (quotient) topology from $\mathbb{R}^{n+1} \setminus \{0\}$.

By Lemma 1.22, to achieve second countable, it suffices to provide a finite open cover by open subsets homeomorphic to open subsets of $\mathbb{R}^n$; this will also achieve locally Euclidean. Well, define

$$U_i \coloneqq \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\}.$$

Note that the pre-image in $\mathbb{R}^{n+1} \setminus \{0\}$ consists of the $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ with $x_i \neq 0$, so $U_i \subseteq \mathbb{RP}^n$ is open. Now, by scaling, we can write elements of U_i uniquely as $[y_0 : \cdots : y_n]$ with $y_i = 1$, which provides the required element in $\mathbb{R}^n$. Explicitly, we define $\varphi_i \colon U_i \to \mathbb{R}^n$ by

$$\varphi_i \colon [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right).$$

One sees that φ_i is continuous: by the quotient topology, we are trying to show that $\varphi_i \circ \pi \colon \pi^{-1}U_i \to \mathbb{R}^n$ is just $(x_0, \ldots, x_n) \mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i)$, which is continuous, so φ_i is continuous because $\mathbb{RP}^n$ has the quotient topology. Lastly, one notes that the inverse of φ_i is given by $(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$, which is continuous because it is the composite of the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$ and the projection $p \colon \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to \mathbb{RP}^n$.

Lastly, we show that $\mathbb{RP}^n$ is Hausdorff. Doing this in a slick way is surprisingly obnoxious. We claim that there is a 2-to-1 covering space map

$$p \colon S^n \to \mathbb{RP}^n.$$

To see why this implies that $\mathbb{RP}^n$ is Hausdorff, fix two distinct points $x, y \in \mathbb{RP}^n$. Then there are lifts $x_1, x_2 \in S^n$ of x and $y_1, y_2 \in S^n$. Because S^n is already Hausdorff (it's a subspace of $\mathbb{RP}^n$), we can find disjoint open subsets $U_1, U_2, V_1, V_2 \subseteq S^n$ around $x_1, x_2, y_1, y_2 \in S^n$ respectively, and we can make them all small enough so that p is a local homeomorphism. Then $p(U_1) \cap p(U_2)$ and $p(V_1) \cap p(V_2)$ are the desired open subsets.

So we are left showing that we have a double cover p. The map is given by the composite

$$S^n \subseteq \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \twoheadrightarrow \mathbb{RP}^n,$$

which we see is continuous automatically. To see that this is a 2-to-1 local homeomorphism, we note that the pre-image of the standard open subset $U_i \subseteq \mathbb{RP}^n$ is

$$\left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \neq 0\right\},$$

whose pre-image in S^n splits into the two open subsets $U_i^{\pm}$. So we have our continuous map $U_i^+ \sqcup U_i^- \to U_i$; it remains to show that $U_i^{\pm} \to U_i$ is a homeomorphism. We may as well assume $i = 0$; then the inverse map is given by sending $[1 : x_1 : \cdots : x_n]$ to the point on the hemisphere of S^n on this line, which is

$$\pm \frac{x}{|x|},$$

where the sign depends on $U_i^{\pm}$. This is continuous, so we are done. ■

Remark 1.65. Note S^n is continuous, so the surjectivity of the covering space map $S^n \twoheadrightarrow \mathbb{RP}^n$ implies that $\mathbb{RP}^n$ is compact.

1.3.2 Transition Functions

Defining smooth structures will come out of transition maps between coordinate charts.

Definition 1.66 (transition map)**.** Fix charts (U, φ) and (V, ψ) on a topological n-manifold M. Then the *transition map* is the map

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V).$$

Here, we are abusing notation a little: in order to make sense of $\psi \circ \varphi^{-1}$, we really want to work with the restrictions as $\psi|_{U\cap V} \circ (\varphi|_{U\cap V})^{-1}$.

Remark 1.67. Note $\varphi(U \cap V), \psi(U \cap V) \subseteq \mathbb{R}^n$, so this is a homeomorphism from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$. Namely, $\varphi|_{U\cap V}$ and $\psi|_{U\cap V}$ are both homeomorphisms, so the above composition is still a homeomorphism.

Example 1.68 (polar coordinates)**.** Consider the topological 2-manifold $M := \mathbb{R}^2$. There is the identity chart $\mathrm{id}_M \colon M \to \mathbb{R}^2$, and there is also "polar coordinates" on $U := \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$ with chart $\varphi \colon U \to \mathbb{R}_+ \times (0, \pi)$ defined by

$$\varphi((x, y)) := \left(\sqrt{x^2 + y^2}, \arg(x, y)\right),$$

where $\arg(x, y)$ is the angle of (x, y) with the positive x-axis. Note the inverse map of φ is given by $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, so φ is in fact a homeomorphism.

Now, the transition map $\psi \circ \varphi^{-1}$ sends

$$(r, \theta) \overset{\varphi^{-1}}{\mapsto} (r\cos\theta, r\sin\theta) \overset{\psi}{\mapsto} (r\cos\theta, r\sin\theta).$$

Example 1.69. Consider the topological 2-manifold $M := S^2$ from Exercise 1.63. We compute the transition maps between φ_1^+ and φ_3^+, which overlap on the open set consisting of $(x_1, x_2, x_3) \in S^2$ such that $x_1, x_3 > 0$. Well, we can directly compute that $\varphi_3^+ \circ \left(\varphi_1^+\right)^{-1}$ is given by

$$(x_2, x_3) \overset{(\varphi_1^+)^{-1}}{\to} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right) \overset{\varphi_3^+}{\to} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2\right).$$

In the above examples, we can note that the maps between the Euclidean smooths are smooth on their domains. This becomes our notion of smoothness.

Definition 1.70 (smoothly compatible)**.** Two charts (U, φ) and (V, ψ) of a topological manifold M are *smoothly compatible* if and only if both transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth (i.e., infinitely differentiable). Notably, this condition is vacuously satisfied if $U \cap V = \varnothing$.

1.3.3 Smooth Structures

We would like to cover M with smoothly compatible charts, so it will be helpful to have a language for such covers.

Definition 1.71 (atlas)**.** Fix a topological manifold M. An *atlas* $\mathcal{A}$ is a collection of charts "covering M" in the sense that

$$M = \bigcup_{(U,\varphi)} U.$$

An atlas is *smooth* if and only if its charts are pairwise smoothly compatible. A smooth atlas is *maximal* if and only if it is maximal in the sense of inclusion by smooth atlases.

The point of using a maximal atlas is that we would like a way to say when two atlases provide the same smooth structure for a topological manifold, but it will turn out to be easier to provide a "unique" atlas to look at, which will be the maximal smooth atlas. Quickly, we note that maximal smooth atlases exist. One could argue this by Zorn's lemma, but we don't have to.

Proposition 1.72. Fix a topological n-manifold M. Any smooth atlas $\mathcal{A}$ is contained in a unique maximal smooth atlas, denoted $\overline{\mathcal{A}}$.

Proof. We have to show existence and uniqueness. We will construct this directly: define $\overline{\mathcal{A}}$ to be the collection of charts (U, φ) which is smoothly compatible with each chart in $\mathcal{A}$. We show that $\overline{\mathcal{A}}$ is a maximal smooth atlas.

- Atlas: certainly $\overline{\mathcal{A}} \supseteq \mathcal{A}$, so $\overline{\mathcal{A}}$ covers M, so $\overline{\mathcal{A}}$ is an atlas.
- Smooth: fix any charts $(U_1, \varphi_1), (U_2, \varphi_2) \in \overline{\mathcal{A}}$, and we would like to show that they are smoothly compatible. If $U_1 \cap U_2 = \varnothing$, there is nothing to do, so we may assume that the intersection is nonempty. By symmetry, it will be enough to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth.

  The point is that differentiability is a local notion: explicitly, fix some $q \in \varphi_1(U_1 \cap U_2)$, and we want to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth at q. This can be checked on a small open neighborhood of q; in particular, find the $p \in U_1 \cap U_2$ such that $\varphi_1(p) = q$, and we can find some chart $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Then we note that

  $$\varphi_2|_{U_1 \cap U_2 \cap V} \circ (\varphi_1|_{U_1 \cap U_2 \cap V})^{-1} = \left(\varphi_2|_{U_1 \cap U_2 \cap V} \circ (\psi|_{U_1 \cap U_2 \cap V})^{-1}\right) \circ \left(\psi|_{U_1 \cap U_2 \cap V} \circ (\varphi_1|_{U_1 \cap U_2 \cap V})^{-1}\right)$$

  is smooth on $\varphi_1(U_1 \cap U_2 \cap V)$ as it is the composition of smooth maps. So our left-hand side is smooth on $U_1 \cap U_2 \cap V$ and in particular at $q \in \varphi_1(U_1 \cap U_2 \cap V)$.
- Maximal: suppose $\mathcal{A}'$ is a smooth atlas containing $\mathcal{A}$. We must show that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$; by supposing further that $\mathcal{A}'$ contains $\overline{\mathcal{A}}$, we achieve the maximality of $\overline{\mathcal{A}}$. Well, for each $(U, \varphi) \in \mathcal{A}'$, we see that (U, φ) is smoothly compatible with each chart in $\mathcal{A}$, so $(U, \varphi) \in \overline{\mathcal{A}}$. Thus, $(U, \varphi) \in \overline{\mathcal{A}}$, so $\mathcal{A}' \subseteq \overline{\mathcal{A}}$.
- Unique: suppose $\mathcal{A}'$ is a maximal smooth atlas containing $\mathcal{A}$. Then the previous point establishes that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$, but then we must have equality because $\mathcal{A}'$ is a maximal smooth atlas. ■

So we may make the following definition.

Definition 1.73 (maximal smooth atlas)**.** Fix a topological n-manifold M. Given a smooth atlas $\mathcal{A}$ on M, we let $\overline{\mathcal{A}}$ denote the unique maximal smooth atlas containing $\mathcal{A}$, which we know exists and is unique by Proposition 1.72.

Corollary 1.74. Fix a topological n-manifold M. Given smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is still a smooth atlas, then

$$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}.$$

Proof. Define $\mathcal{A} \coloneqq \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\overline{\mathcal{A}}$ is a maximal smooth atlas containing $\mathcal{A}$ and hence both $\mathcal{A}_1$ and $\mathcal{A}_2$, so we see that $\overline{\mathcal{A}_1} = \overline{\mathcal{A}}$ and $\overline{\mathcal{A}_2} = \overline{\mathcal{A}}$. Notably, we are using the uniqueness of Proposition 1.72. ■

At long last, here is our definition.

Definition 1.75 (smooth manifold)**.** Fix a topological n-manifold M. A *smooth structure* on M is a maximal smooth atlas on M. A *smooth n-manifold* is a pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a smooth structure on M.

Remark 1.76. Adjusting the "smoothness" on the manifold M produces different notions of manifold. For example, we can have twice differentiable manifolds, real analytic manifolds, complex manifolds, etc.

1.4 January 25

The first homework is due later today.

1.4.1 A Couple Lemmas on Atlases

Here are some basic properties of smooth manifolds which one can check.

Lemma 1.77. Fix a smooth n-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$, then for any open subset $U' \subseteq U$, we have $(U', \varphi|_{U'}) \in \mathcal{A}$.

Proof. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U', \varphi|_{U'})\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas of charts. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that $(U', \varphi|_{U'})$ and (V, ψ) are smoothly compatible. (The charts in $\mathcal{A}$ are already smoothly compatible with each other.) In other words, we must show that the transition functions are diffeomorphism: the transition maps are

$$\varphi|_{U' \cap V} \circ \psi|_{U' \cap V}^{-1} = \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}\right)|_{\psi(U' \cap V)}$$

and

$$\psi|_{U' \cap V} \circ \varphi|_{U' \cap V}^{-1} = \left(\psi|_{U \cap V} \circ \varphi|_{U \cap V}^{-1}\right)|_{\varphi(U' \cap V)},$$

and these are both smooth as the restrictions of smooth maps. (Namely, we are using the fact that (U, φ) and (V, ψ) are smoothly compatible already.) ■

Lemma 1.78. Fix a smooth n-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$ and diffeomorphism $\chi\colon \varphi(U) \to V$ for some open subset $V \subseteq \mathbb{R}^n$, we have $(U, \chi \circ \varphi) \in \mathcal{A}$.

Proof. The argument is similar to that of the above lemma. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U, \chi \circ \varphi)\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that (V, ψ) and $(U, \chi \circ \varphi)$ are smoothly compatible. (Indeed, the charts in $\mathcal{A}$ are already smoothly compatible with each other.) Well, the transition maps are

$$(\chi \circ \varphi)|_{U\cap V} \circ \psi|_{U\cap V}^{-1} = \chi|_{\varphi(U\cap V)} \circ \left(\varphi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}\right)$$

and

$$\psi|_{U\cap V} \circ (\chi \circ \varphi)|_{U\cap V}^{-1} = \psi|_{U\cap V} \circ \varphi|_{U\cap V}^{-1} \circ \chi|_{\varphi(U\cap V)}^{-1},$$

which are smooth maps because (U, φ) and (V, ψ) are already smoothly compatible, and χ is a diffeomorphism. ■

Lemma 1.79. Fix a smooth n-manifold $(M, \mathcal{A})$. If $\varphi\colon U \to \mathbb{R}^n$ is an injective map with $U \subseteq M$ is such that each $p \in U$ has some open neighborhood $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, then actually $(U, \varphi) \in \mathcal{A}$.

Proof. Proceed as above via gluing. ■

1.4.2 Examples of Smooth Manifolds

We go through some examples of smooth manifolds.

Example 1.80. Recall from Lemma 1.21 that $\mathbb{R}^n$ is a topological n-manifold. Then $\operatorname{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ provides an atlas on $\mathbb{R}^n$ consisting of a single chart, which is vacuously smooth; note Proposition 1.72 then gives us a smooth structure.

More generally, we have the following.

Proposition 1.81. Fix a smooth n-manifold $(M, \mathcal{A})$. For any open subset $M' \subseteq M$, we have that M' is a topological n-manifold, and

$$\mathcal{A}' := \{(U, \varphi) \in \mathcal{A} : U \subseteq M'\}$$

is a smooth structure on M.

Proof. Omitted. ■

Example 1.82. Any open subset of $\mathbb{R}^n$ is a smooth n-manifold by combining Example 1.80 and Proposition 1.81.

Example 1.83. The charts on S^n provided in Exercise 1.63 provide a smooth atlas on S^n and hence a smooth structure by Proposition 1.72. On the homework, we will see how to use stereographic projection to provide a smooth structure (in fact, the same smooth structure) on S^n.

Example 1.84. Fix an n-dimensional $\mathbb{R}$-vector space V. Then

$$\mathcal{A} := \{(V, \varphi) : \varphi \text{ is an isomorphism to } \mathbb{R}^n\}$$

is a smooth atlas on V and hence provides a smooth structure. These are smoothly compatible because the transition maps turn into linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$, which are automatically smooth.

Example 1.85. Fix the topological 1-manifold $\mathbb{R}$ of Lemma 1.21. Example 1.80 tells us $\mathcal{A} := \{(\mathbb{R}, \mathrm{id}_\mathbb{R})\}$ provides a smooth atlas, and $\mathcal{A}' := \{(\mathbb{R}, \varphi)\}$ given by $\varphi\colon x \mapsto x^3$ is also a smooth atlas (again, smoothness is vacuous). However, $\mathcal{A}$ and $\mathcal{A}'$ provide smooth structures: otherwise, they would be contained in the same maximal smooth atlas, so $(\mathbb{R}, \mathrm{id}_\mathbb{R})$ and $(\mathbb{R}, \varphi)$ would be smoothly compatible, but then the composite $\left(\mathrm{id}_\mathbb{R} \circ \varphi^{-1}\right) : x \mapsto \sqrt[3]{x}$ is not a smooth function $\mathbb{R} \to \mathbb{R}$.

Remark 1.86. In fact, if a topological n-manifold has some smooth structure, there are uncountably many distinct smooth structures on M. On the other hand, for n-manifolds of small dimensions (e.g., $n \le 3$), it turns out that these are diffeomorphic.

Remark 1.87. However, there do exist topological n-manifolds with no smooth structure, in dimensions $n \ge 4$. Even worse, there are topological n-manifolds with distinct smooth structures up to diffeomorphism, again in dimensions $n \ge 4$. Even for S^n, the story is complicated: there is only one smooth structure for $n \le 3$, we don't understand $n = 4$, and the story is complicated but somewhat understood for $n \ge 5$.

Remark 1.88. Please read the examples on $\mathrm{GL}_n(\mathbb{R})$, product manifolds, and some others.

1.4.3 Grassmannians

The construction of smooth manifolds is rather long: we build a topological space, define some charts, and then check that the charts are smoothly compatible. Here's a lemma to do all of this at once.

Lemma 1.89. Fix a set M with a nonnegative integer $n \ge 0$ and a collection of functions $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ where $U_\alpha \subseteq M$ and $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$ is open. Further, suppose the following.

(i) $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ is open for all $\alpha, \beta \in \kappa$.

(ii) The composite $\varphi_\alpha|_{U_\alpha\cap U_\beta} \circ \varphi_\beta|_{U_\alpha\cap U_\beta}^{-1}$ is smooth for all $\alpha, \beta \in \kappa$.

(iii) M is covered by a countable subcollection of $\{U_\alpha\}_{\alpha\in\kappa}$.

(iv) For distinct $p, q \in M$, either there is $\alpha \in \kappa$ such that $p, q \in U_\alpha$, or there are

Then M is a smooth n-manifold with smooth atlas given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$.

Proof. We sketch the steps.

1. We provide M with a topology. We would like for Well, we say that $A \subseteq M$ is open if and only if $\varphi_\alpha(A \cap U_\alpha)$ is open for all $\alpha \in \kappa$.
2. Then condition (i) makes the φ_α into homeomorphisms onto their images. Thus, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ is an atlas.
3. Condition (ii) implies that $\{(U_\alpha, \varphi_\alpha)\}$
4. Condition (iii) implies that M becomes second countable.
5. Lastly, condition (iv) implies that M is Hausdorff.

We leave the checks to the reader. ■

Let's see an example of this.

Exercise 1.90. Fix nonnegative integers $k \leq n$. Then let $M \coloneqq \mathrm{Gr}_k(\mathbb{R}^n)$ denote the set of k-dimensional linear subspaces V of $\mathbb{R}^n$. We show that M is a smooth $k(n-k)$-manifold.

Proof. We use Lemma 1.89. For concreteness, let we choose our index set I to consist of pairs (P, Q) of subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n = P \oplus Q$ and $\dim P = k$ and $\dim Q = n - k$. The point is that we are choosing a complement for our k-dimensional subspaces in order to help count them. In particular, we may define the subset

$$U_\alpha \coloneqq \{V \in \mathrm{Gr}_k(\mathbb{R}^n) : V \cap Q = \{0\}\}.$$

Notably, for any $V \in U_\alpha$, there is a unique linear map $M_{P,Q,V} \colon P \to Q$ such that

$$V = \{x + M_{P,Q,V}x \in P \oplus Q : x \in P\}.$$

Approximately speaking, we are viewing V as a graph. Anyway, this construction provides a map $\varphi_\alpha \colon U_\alpha \to \mathrm{Hom}_{\mathbb{R}}(P, Q)$ given by $V \mapsto M_{P,Q,V}$, where we identify $\mathrm{Hom}_{\mathbb{R}}(P, Q) \cong \mathbb{R}^{k(n-k)}$. We now conclude by noting that we can check the properties from Lemma 1.89. For example, to see that the transition maps are smooth, suppose we have two pairs $(P, Q), (P', Q') \in I$, and the vector space V decomposes into the two separate ways, and these matrices have rational functions in their coordinates, so smoothness follows. As another example, one can actually cover M by finitely many charts, and the last check follows because any k-dimensional subspaces $V, V' \subseteq \mathbb{R}^n$ has some $(n-k)$-dimensional subspace $Q \subseteq \mathbb{R}^n$ such that $V \cap Q = V' \cap Q = \{0\}$. ■

1.4.4 Manifolds with Boundary

Before moving on from our discussion of a single manifold, we discuss manifolds with boundary.

Definition 1.91 (topological manifold with boundary)**.** Fix a nonnegative integer n. A *topological n-manifold with boundary* is a Hausdorff, second countable topological space M with the following variant of being locally Euclidean: for any $p \in M$, there are open subsets $U \subseteq M$ and

$$\widehat{U} \subseteq \mathbb{H} \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

such that $p \in U$ and $U \cong \widehat{U}$. We continue to call (U, φ) a chart.

Example 1.92. Any topological n-manifold is a topological n-manifold with boundary: one can simply make the charts output to $\mathbb{H}^\circ$.

Example 1.93. The space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is a topological n-manifold with boundary.

The point is that we can pick up some "boundary" like the one in $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. This notion of boundary is in fact fairly intrinsic.

Definition 1.94 (boundary, interior)**.** Fix a topological n-manifold with boundary M and a point $p \in M$.

- Then p is a *boundary point* if and only if there is a chart (U, φ) such that $\varphi(p) \in \partial\mathbb{H}^n$.
- Then p is an *interior point* if and only if there is a chart (U, φ) such that $\varphi(p)$ is in the interior of $\mathbb{H}^n$.

We will show later that any point in M is exactly one of a boundary point or an interior point.

Anyway, let's discuss smoothness. This requires understanding smoothness on $\partial\mathbb{H}^n$.

Definition 1.95. Fix a subset $A \subseteq \mathbb{R}^n$. A function $f \colon A \to \mathbb{R}^m$ is *smooth* if and only if there is an open subset $V \subseteq \mathbb{R}^n$ containing A and a smooth extension $\widetilde{f} \colon V \to \mathbb{R}^n$ of f.

Remark 1.96. It turns out that (by Seeley's theorem) if $V \subseteq \mathbb{H}^n$ is open, it is enough to check that the partial derivatives of some function $f\colon V \to \mathbb{R}^m$ extend continuously to the boundary.

Bibliography

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[use14] user149792. *Is an open n-ball homeomorphic to $\mathbb{R}^n$?* Mathematics Stack Exchange. 2014. eprint: https://math.stackexchange.com/q/1072818. URL: https://math.stackexchange.com/q/1072818.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Elb22] Nir Elber. *Introduction to Topology and Analysis*. 2022. URL: https://dfoiler.github.io/notes/202A/notes.pdf.

[Elb23] Nir Elber. *Algebraic Topology*. 2023. URL: https://dfoiler.github.io/notes/215A/notes.pdf.

List of Definitions