616: Homotopy Theory

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Homotopy Theory

What we didn't do is make the construction at all usable in practice!
This time we will remedy this.

—Kiran S. Kedlaya, [Ked21]

1.1 January 22

In this course, we would like to do computations.

1.1.1 CW Complexes

We would like to compute the homology and cohomology of interesting spaces. CW complexes are a good mix between being simple enough to work with while being flexible enough to have many interesting examples.

Definition 1.1 (CW complex). A *CW complex* is a topological space X equipped with an ascending chain of subspaces

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq$$

satisfying the following.

(i) $X = \bigcup_{k \geq -1} X_k$.

(ii) X_{k+1} is obtained inductively be adjoining $(k+1)$-disks $\{D_\alpha^{k+1}\}_\alpha$ to X_k as follows: some $x \in S_\alpha^k$ is identified with its image in X_k via some continuous map $f\colon \bigsqcup_\alpha S_\alpha^k \to X_k$.

(iii) The topology is on X is given by asserting $U \subseteq X$ is open if and only if $U \cap X_k$ is open for all k.

We call the interior $e_\alpha^n := (D_\alpha^n)^\circ$ an n-*cell*, and we call X_n the n-*skeleton*. If the total number of cells if finite, then X is *finite*; if merely each X_n is finite, then X is *finite type*.

Example 1.2. Compact manifolds have the structure of a CW-complex with only finitely many disks.

Remark 1.3. Note that we only ever identify points on the new disks for X_{k+1} along boundaries. In particular, we see that X can be seen as a union of the spaces e_α^n as n and α vary; thus, $e_\alpha^n \cap e_{\alpha'}^{n'} = \varnothing$ whenever $(n, \alpha) \neq (n', \alpha')$.

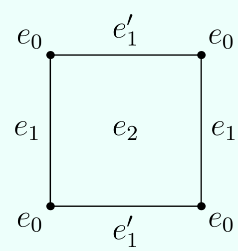
The point is that a CW complex allows us to work combinatorially with many topological spaces.

Example 1.4. We realize S^n as a CW complex as follows: simply attach D^n to a point $\{*\}$ by attaching the entire boundary to the point. Note that this is certainly continuous by definition of the quotient topology, and one can check that this is a bijection by doing casework at and away from $\{*\}$. Thus, we have a continuous bijection between compact Hausdorff spaces, so we have produced a homeomorphism.

Example 1.5. Here is another way to realize S^n. We do this inductively: note $n = 0$ has no content (simply take a point). Then supposing that we have already produced S^{n-1}, we simply attach a top and bottom hemisphere of n-cells to produce S^n.

Example 1.6. We can realize $\mathbb{RP}^n$ as a CW complex. Simply realize $\mathbb{RP}^n$ as S^n modulo the antipodal map, so we see that we can identify $\mathbb{RP}^n$ as $\mathbb{RP}^{n-1}$ union with a single n-cell (as in the previous example) but identified "twice-over" on the boundary.

Example 1.7. We realize $T^2 = S^1 \times S^1$ as a CW complex. Think of T^2 as a rectangle with the bottom/top and left/right edge pairs identified. Then we have a single point e_0 as our 0-cell, two 1-cells corresponding to the two edges (after identification), and then there is the 2-cell e^2 embedded. Here is the picture.



Example 1.8. We realize $\mathbb{CP}^n$ as a CW complex. The idea is that we can think about complex lines in $\mathbb{C}^{n+1}$ as intersecting

$$S^{2n+1} = \left\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots + |z_n|^2 = 1\right\},$$

and $p, q \in S^{n+1}$ both live on the same line if and only if $p/q \in S^1$. This explains that $\mathbb{CP}^{n+1}$ should be topologically a quotient S^{2n+1}/S^1.

We are now ready to give our CW structure, which we do inductively, beginning by noting that $\mathbb{CP}^0$ is a point. We would now like to attach D^{2n} to $\mathbb{CP}^{n-1}$ to produce $\mathbb{CP}^n$. To begin, note we can realize S^{2n} inside S^{2n+1} as above by adding the requirement $z_n \in \mathbb{R}$, and then we can realize D^{2n} inside this S^{2n} by further requiring $z_n \geq 0$. Then note that D^{2n} surjects onto S^{2n+1}/S^1 (by using the S^1-action), so we induce a homeomorphism of a quotient of D^{2n} onto S^{2n+1}/S^1. We take a moment to note that this quotient behaves as follows: distinct points $(z_0, \ldots, z_n)$ and $(w_0, \ldots, w_n)$ in D^{2n} are identified if and only if $z_n = w_n = 0$ (because we are only multiplying by scalars in S^1) and further $(z_0, \ldots, z_{n-1})$ and $(w_0, \ldots, w_{n-1})$ define the same line in $\mathbb{CP}^{n-1}$. In this way, we see that $\mathbb{CP}^{2n}$ is realized as $\mathbb{CP}^{2n-1}$ together with a D^{2n} attached as described above.

Of course, whenever one introduces combinatorics to study complicated objects, one should check that the combinatorics is able to keep track of the complications. For example, the relevant combinatorics suggests the following definition.

Definition 1.9 (cellular)**.** A continuous map $f\colon X \to Y$ of CW complexes is *cellular* if and only if $f(X_n) \subseteq Y_n$ for each $n \geq 0$.

Theorem 1.10 (cellular approximation)**.** Any continuous map $f\colon X \to Y$ of CW complexes is homotopic to a cellular one.

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LIST OF DEFINITIONS