250B: Commutative Algebra

Or, Eisenbud With Details

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CONTENTS

THEME 1

WORKING IN CHAINS

But this is like trying to scale a glacier. It's hard to get your footing, and your fingertips get all red and frozen and torn up.

—Anne Lamott

1.1 March 1

Welcome back everyone. The average and median for the exam was 32/50.

1.1.1 Krull's Intersection Theorem

Last time we showed the following.

Theorem 1.1 (Krull intersection)**.** Fix R a Noetherian ring with an ideal I and finitely generated module M. Then

$$N := \bigcap_{s \geq 0} I^s M$$

satisfies that there is some $x \in I$ such that $(1 - r)N = 0$.

The Noetherian condition is necessary here; consider the following example.

Exercise 1.2. Let R be the germ of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ at 0. Namely, two functions $f, g : \mathbb{R} \to \mathbb{R}$ are equivalent in R if and only if they coincide on an open neighborhood around 0. Then

$$\bigcap_{s>0} (x)^s$$

is nonzero.

Proof. The point is that

$$I := \bigcap_{s>0} (x)^s$$

is the set of germs represented by a function with all derivatives vanishing. However, it is a counterexample from real analysis that e^{-1/x^2} also has all derivatives vanish but is a nonzero function. ■

1.1.2 Flat Modules

Today we are talking about flatness and Tor. Let's start with flatness; we recall the definition.

Definition 1.3 (Flat)**.** Fix R a ring. Then an R-module M is *flat* if and only if the functor $M \otimes_R -$ is exact.

Remark 1.4. Because $M \otimes_R -$ is already left-exact, we merely have to check that $N \hookrightarrow N'$ induces an injection $M \otimes_R N \hookrightarrow M \otimes_R N'$.

We also had the following examples.

Example 1.5. We showed long ago that R and therefore free modules R^n are flat.

For our next example, we pick up the following definition.

Definition 1.6 (Projective)**.** An R-module P is *projective* if and only if one of the following four equivalent conditions are satisfied.

(a) The functor $\operatorname{Hom}_R(P, -)$ is exact.

(b) There exists an R-module K such that $P \oplus K$ is a free R-module.

(c) If we have a surjection $M \twoheadrightarrow M'$ and a map $P \to M'$, there is a map $P \to M$ making the following diagram commute.

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ M \arrow[r, two heads] & M'' \end{tikzcd}$$

(d) Any short exact sequence

$$0 \to A \to B \to Q \to 0$$

splits.

It is not obvious that these definitions are equivalent, but they are. For example, (a) and (c) are equivalent by writing out what the commutative diagram is asking for in terms of Hom sets. Further, (c) implies (d) by lifting from the following diagram.

$$\begin{tikzcd} & & & P \arrow[dl, dashed] \arrow[d, equal] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, two heads] & P \arrow[r] & 0 \end{tikzcd}$$

To show that (d) implies (b), we make the short exact sequence

$$0 \to \ker \pi \to \bigoplus_{m \in M} Rm \xrightarrow{\pi} M \to 0,$$

where π is defined in the natural way. Lastly, (b) implies (a) because it gives

$$\operatorname{Hom}_R(M \oplus K, -) \cong \operatorname{Hom}_R(M, -) \oplus \operatorname{Hom}_R(K, -).$$

This more or less completes the equivalences.

Example 1.7. Projective modules are flat, which we can see from the fact that $P \oplus K$ is free and then using the fact that free modules are flat already.

Example 1.8. For any multiplicative set $U \subseteq R$, the module $R\left[U^{-1}\right]$ is flat. We showed this a long time ago. As a small aside, we note that $R\left[U^{-1}\right] \otimes -$ is a priori only exact for $R\left[U^{-1}\right]$-modules, but this restricts to R-modules just fine (even when $R \to R\left[U^{-1}\right]$ is not injective).

And let's see a non-example.

Non-Example 1.9. The $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not exact. For example, we take

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

apply $- \otimes \mathbb{Z}/n\mathbb{Z}$ to get

$$0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

but this is no longer exact at $\mathbb{Z}/n\mathbb{Z}$ term because $\xrightarrow{\times n}$ is the zero map.

1.1.3 Flatness via Algebraic Geometry

In algebraic geometry, we are interested in families of affine varieties, which consists of a base B for a family and a map $\varphi : X \to B$. As usual, the algebraic story will reverse, so the family in the algebraic world becomes a function

$$\varphi^{-1} : A(B) \to A(X).$$

In particular, this is exactly the data of $A(X)$ being an $A(B)$-algebra. To make our notions more general, we set $S := A(X)$ an $R := A(B)$-algebra by $\varphi : R \to S$. As such, we have the following definition.

Definition 1.10 (Flat)**.** An R-algebra S is *flat* if and only if S is flat as an R-module.

To access flatness, we talk about fibers. In the algebraic world, the fiber of a "point" $\mathfrak{m}$ should be the ring of functions in S on the point $\mathfrak{m}$, which means we want to look at

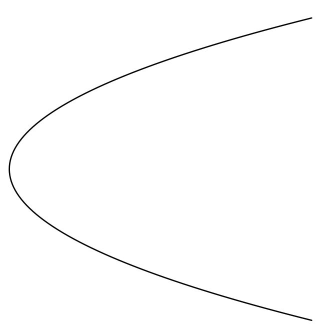
$$S/\mathfrak{m}S.$$

Flatness, roughly speaking, means that $S/\mathfrak{m}S$ varies continuously as the point $\mathfrak{m}$ moves.

Let's see some examples. We will take our base to be $B := \mathbb{A}^1(k)$ the affine line over an algebraically closed field k, which gives that $R := k[x]$.

Exercise 1.11. We consider the flatness of $S := R[x]/\left(x^2 - t\right)$ geometrically and algebraically.

Proof. This looks like the following.



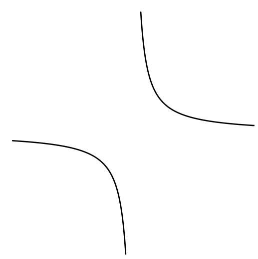
The fiber at $t = a$ as $a \in k$ varies is

$$\frac{k[x]}{(x^2 - a)} \cong \begin{cases} k^2 & a \neq 0, \\ k[x]/\left(x^2\right) & a = 0. \end{cases}$$

Visually, we can see that $a \neq 0$ has two points above it, and at $x = 0$, we are vertical. Because the dimension is constant as the point moves, we suspect S to be a flat R-algebra. And indeed, viewing $x^2 - t$ as a monic polynomial with coefficients in $R[t]$, we see that S is a free module over $R[t]$ of rank 2, so S is flat. ■

Exercise 1.12. We consider the flatness of $S := R[x]/\left(xt - 1\right)$ geometrically and algebraically.

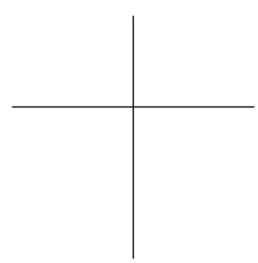
Proof. This looks like the following.



Visually, we can see that the fiber over any $t = a$ as $a \in k$ is one point, except when $a = 0$, where the fiber is empty. So we expect S to be flat, and indeed it is: $S = R\left[t^{-1}\right]$ is a localization and therefore flat. ■

Exercise 1.13. We consider the non-flatness of $S := R[x]/(tx - t)$ geometrically and algebraically.

Proof. This looks like the following.



The problem here is that the fiber is jumping at $t = 0$, so we expect S to not be flat as an R-module. For this, we have the following result.

Lemma 1.14. Fix R a ring $a \in R$ a non-zero-divisor. Further, if M is a flat R-module, then $am = 0$ implies $m = 0$ for $m \in M$.

Proof. The point is to look at the short exact sequence

$$0 \to (a) \to R \to R/(a) \to 0.$$

Upon tensoring with M, we see that $(a) \otimes_R M \hookrightarrow R \otimes_R M$, so $(a)M \hookrightarrow M$. In particular, multiplication by a is injective on M, so $am = 0 = a \cdot 0$ implies $m = 0$. ■

From the above lemma, we note that $t(x-1) = 0$ in S while t is not a zero-divisor, so S is not flat. ■

1.1.4 Homological Algebra

We will want to talk about Tor for our discussion, so we will want to talk about homological algebra.

> **Quote 1.15.** The difference between homology and cohomology is that homology indexes like H_i, and cohomology indexes like H^i.

We will want to talk about chains in homological algebra, so we will start with complexes.

> **Definition 1.16** (Complex)**.** Fix $C := \oplus_{i\geq 0} C_i$ a $\mathbb{N}$-graded R-module. Then C is a *chain* if and only if it is equipped with a (graded) morphism $\partial \in \operatorname{End}_R(C)$ such that $\partial^2 = 0$. If $\deg \partial = -1$, this is homology, and if $\deg \partial = +1$, this is cohomology.

In the homology case, we can view this like

$$\cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0.$$

If we wanted, we could index the arrows as $\partial_i : C_i \to C_{i-1}$, but it makes things a little harder to keep track of.

> **Definition 1.17** (Homology)**.** Given a chain (C, ∂), we define the *homology groups* as
>
> $$H_i(C) := \ker \partial_i / \operatorname{im} \partial_{i+1}$$

Note this is well-defined because $\partial^2 = 0$.

As usual in algebra, we will want morphisms between our objects.

> **Definition 1.18** (Chain morphism)**.** Fix chain complexes (C, ∂) and (C', ∂'), we define a morphism φ as a degree-0 morphism $\varphi : C \to C'$ preserving ∂ as in the following diagram.
>
> $$\begin{tikzcd} C_i \arrow[r, "\partial"] \arrow[d, "\varphi"'] & C_{i-1} \arrow[d, "\varphi"] \\ C'_i \arrow[r, "\partial'"'] & C'_{i-1} \end{tikzcd}$$

We can check that φ maps kernels of ∂ to kernels of ∂' and images of ∂ to images of ∂', so we get an induced map $H(\varphi) : H_i(C) \to H_i(C')$.

And because abstraction is all the rage, there is also a notion of morphisms being the same.

> **Definition 1.19** (Homotopically equivalent)**.** Two chain morphisms $\varphi, \psi : (C, \partial) \to (C', \partial')$ are *homotopically equivalent* if and only if there exists an R-module homomorphism $h : C \to C$ of degree 1 (i.e., $h : C_i \to C'_{i+1}$) such that $\varphi - \psi = h\partial + \partial' h$.

The image is as follows. As a warning, this diagram does not commute.

$$\begin{tikzcd} \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] \arrow[dl, "h"'] & C_0 \arrow[r] \arrow[d] \arrow[dl, "h"'] & 0 \\ \cdots \arrow[r] & C'_2 \arrow[r] & C'_1 \arrow[r] & C'_0 \arrow[r] & 0 \end{tikzcd}$$

The main point of this definition is the following.

> **Proposition 1.20.** Suppose $\varphi, \psi : (C, \partial) \to (C', \partial')$ are homotopically equivalent. Then $H(\varphi) = H(\psi)$.

Proof. It suffices (by taking $\gamma := \varphi - \psi$) to show that if γ is homotopically equivalent to 0, then $H(\gamma)$ vanishes. Now, suppose we have any $c \in \ker \partial$, and we want to show that $\gamma(c) \in \operatorname{im} \partial'$. Well, we compute

$$\gamma(c) = (h\partial + \partial' h)(c) = \partial'(h) \in \operatorname{im} \partial',$$

so we are done. ■

To close out class, we discuss the long exact sequence.

Theorem 1.21. Fix

$$0 \to C' \overset{\alpha}{\to} C \overset{\beta}{\to} C'' \to 0$$

a short exact sequence of complexes. Then there is a long exact sequence of homology

$$\cdots \to H_i(C') \overset{H(\alpha)}{\to} H_i(C) \overset{H(\beta)}{\to} H_i(C'') \overset{\delta}{\to} H_{i-1}(C') \to \cdots .$$

Proof. We will be very brief. The main point is the construction of δ. Fix some element $c \in \ker \partial_i''$ from $H_i(C'')$. Then we can pull it back to $\beta^{-1}(c)$ in $H_i(c)$, then push it forwards through ∂' to live in $H_{i-1}(C)$, which we can then lastly check lives in the image of α, so we finish by pulling backwards along α to get back to $H_{i-1}(C')$. ■