174: Category Theory

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Contents

THEME 1: BASIC DEFINITIONS

Category theory is much easier once you realize that it is designed to formalize and abstract things you already know.

—Ravi Vakil

1.1 January 19

Reportedly there is a lot of material that Bryce would like to cover today.

1.1.1 Our Definition

We're doing category theory, so let's define what a category is.

Definition 1.1 (Category)**.** A category $\mathcal{C}$ is a pair of objects and morphisms $(\operatorname{Ob}\mathcal{C}, \operatorname{Mor}\mathcal{C})$ satisfying the following.

- $\operatorname{Ob}\mathcal{C}$ is a collection of *objects*. By abuse of notation, when we write $c \in \mathcal{C}$
- $\operatorname{Mor}\mathcal{C}$ is a collection of *morphisms*. Morphisms might also be called arrows or maps or functions or continuous functions or similar.

  A morphism is written $f : x \to y$ where $x, y \in \operatorname{Ob}\mathcal{C}$. Here, x is the *domain*, and y is the *codomain*.

These morphisms have a little extra structure.

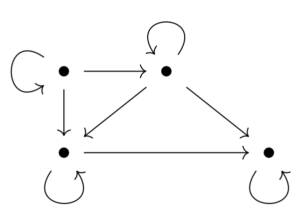
- For each $x \in \mathcal{C}$, there is a morphism $\operatorname{id}_x : x \to x$.
- Given any pair of morphisms $f : x \to y$ and $g : y \to z$, there exists a *composition* $gf : x \to z$. Importantly, the codomain of f is the domain of g.

Additionally, morphisms satisfy the following coherence conditions.

- Associativity: for any morphisms $f : a \to b$ and $g : b \to c$ and $h : c \to d$, we have that $h(gf) = (hg)f$.
- Identity: given any morphism $f : a \to b$, we have $\operatorname{id}_b f = f$ and $f \operatorname{id}_a = f$.

Yes, this is a long definition. For reference, it is on page 3 of Riehl.

The intuition to have here is that we have objects to be thought of as points a bunch of morphisms which are to be thought of arrows between them. Here is an example of some morphisms in a category.



The loops are identity morphisms. As an aside, it is reasonable to think that definition of a category is overly abstract. Most of the time we will be thinking about some concrete category.

Before continuing, we bring in the following definition.

Definition 1.2 (Hom-sets)**.** Fix a category $\mathcal{C}$. Then, given objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ or $\operatorname{Hom}_{\mathcal{C}}(x, y)$ or $\operatorname{Hom}(x, y)$ or $\operatorname{Mor}(x, y)$ for the set of morphisms $f : x \to y$. I personally prefer $\operatorname{Mor}(x, y)$.

Note that two objects need not have a morphism between them. For example, the following is a category even though the two objects have a morphism between them.

$$\begin{tikzcd} \bullet \arrow[loop left] & \bullet \arrow[loop right] \end{tikzcd}$$

As a less contrived example, there is no morphism between $\mathbb{F}_2$ and $\mathbb{F}_3$ in the category of fields.

1.1.2 Examples

Let's talk about examples.

Example 1.3. The category Set has objects which are all sets and its morphisms are the functions between sets.

Example 1.4. The category Grp has objects which are all groups and its morphisms are group homomorphisms. Similarly, Ab has abelian groups.

Example 1.5. The category Ring has objects which are all rings (with identity) and its morphisms are group homomorphisms.

Example 1.6. The category Field has objects which are all fields and its morphisms are field/ring homomorphisms.

Example 1.7. The category Vec_k has objects which are all k-vector spaces and its morphisms are k-linear transformations.

Those are the good examples. We like them because they are with familiar objects.

Here are some weirder examples.

Example 1.8 (Walking arrow)**.** The diagram

$$\bullet \longrightarrow \bullet$$

induces a category with a single non-identity morphism.

Note that we will stop writing down all the identity morphisms and all induced morphisms because they're annoying to write out.

Example 1.9 (Walking isomorphism)**.** The diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

induces a category with two non-identity morphisms. We declare that any composition of the two non-identity morphisms is the identity.

There are also such things as a poset category, but for this we should define a poset first.

Definition 1.10 (Poset)**.** A *poset* $(\mathcal{P}, \le)$ is a set $\mathcal{P}$ and a relation $\le$ on $\mathcal{P}$ which satisfies the following; let $a, b, c \in \mathcal{P}$.

- Reflexive: $a \le a$.
- Antisymmetric: $a \le b$ and $b \le a$ implies $a = b$.
- Transitive: $a \le b$ and $b \le c$ implies $a \le c$.

Now, it turns out that all posets induce a category.

Example 1.11 (Poset category)**.** Given any poset $(\mathcal{P}, \le)$, we can define the poset category as follows.

- The objects are elements of $\mathcal{P}$.
- For $x, y \in \mathcal{P}$, there is a morphism $x \to y$ if and only if $x \le y$, and there is only one morphism.

Checking that the poset category is in fact a category is not very interesting. The identity law comes from reflexivity, where id_a witnesses $a \le a$.

Additionally, transitivity defines our composition: if $a \le b$ and $b \le c$, then $a \le c$, and the morphism representing $a \le c$ is unambiguous because there is at most one morphism $a \to c$. This uniqueness is in fact crucial for our composition: if $f : a \to b$ and $g : b \to c$ and $h : c \to d$ are morphisms, then $h(gf) = (hg)f$ because they are both morphisms $a \to d$, of which there is at most one.

We continue with our examples. We will not check that these are actually categories formally; perhaps the reader can do the checks on their own time.

Example 1.12 (Groups)**.** Given a group G, we can define the category bG to have one object $*$ and morphisms $g : * \to *$ given by group elements $g \in G$. Composition in the category is group multiplication; the identity morphism id_* needed is the identity element of G; and the associativity check comes from associativity in G.

Example 1.13 (Pointer sets)**.** We define the category of pointed sets Set_* to consist of objects which are ordered pairs (X, x) where X is a set and $x \in X$ is an element. Then morphism are "based maps" $f : (X, x) \to (Y, y)$ to consist of the data of a function $f : X \to Y$ such that $f(x) = y$.

Example 1.14. Given any set S, we can define a category consisting of objects which are elements of S and morphisms which are only the required identity morphisms.

This last example generalizes.

Definition 1.15 (Discrete, indiscrete)**.** Fix a category $\mathcal{C}$. Then $\mathcal{C}$ is *discrete* if and only if the only morphisms are identity morphisms. Additionally, $\mathcal{C}$ is *indiscrete* if and only if $\mathrm{Mor}(x, y)$ has exactly one element for each pair of objects (x, y).

Warning 1.16. A total order with more than one element is not a category. Namely, if we have distinct objects x and y, then we cannot have both $x \le y$ and $y \le x$, so not both $\mathrm{Mor}(x, y)$ and $\mathrm{Mor}(y, x)$ inhabited.

1.1.3 Size Issues

Let's briefly talk about why we are calling $\operatorname{Ob}\mathcal{C}$ and $\operatorname{Mor}\mathcal{C}$ "collections." In short, we cannot have a set that contains all sets, but we would still like a category which contains all categories. There are a few ways around this; here are two.

- Grothendieck inaccessible categories: we essentially upper-bound the size of our sets and then let Set contain all of our sets.
- Proper classes: we add in things called "classes" to foundational mathematics we are allowed to be bigger than sets.

We will avoid doing anything like this in this course, so here is a definition making our avoidance concrete.

Definition 1.17 (Small, locally small)**.** Fix $\mathcal{C}$ a category. Then $\mathcal{C}$ is *small* if and only if $\operatorname{Mor}\mathcal{C}$ is a set. Alternatively, $\mathcal{C}$ is *locally small* if and only if $\operatorname{Mpr}(x, y)$ is a set.

Example 1.18. The category Set is locally small, but it is not small. To see that it is not small, note that $S \mapsto \operatorname{Mor}(\{*\}, S)$ is an injective map, so $\operatorname{Mor}\mathrm{Set}$ must be at least as big as Set.

It turns out that most of our categories will be locally small. It is a very nice property to have.

1.1.4 Isomorphism

In algebra (e.g., group theory), we are interested in when two objects are the same. In category theory, we focus on the morphisms between objects, so we need to be careful how we define this. Here is our definition.

Definition 1.19 (Isomorphism)**.** Fix a category $\mathcal{C}$. Then a morphism $f : x \to y$ is an *isomorphism* if and only if there is a morphism $g : y \to x$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. We call g the *inverse* of f and often notate it f^{-1}.

This is fairly intuitive: isomorphisms are those morphisms with a way to reverse them.

Observe that we called g "the" inverse of f, and we may do so because inverses are unique.

Proposition 1.20. Fix a category $\mathcal{C}$. Inverses of morphisms, if they exist, are unique.

Proof. Fix $f : x \to y$ some isomorphism, and suppose that we have found two inverse morphisms $g, h : y \to x$. Then

$$g = g\,\mathrm{id}_y = g(fh) = (gf)h = \mathrm{id}_x\,h = h,$$

so indeed the inverse morphisms that we found are the same. ■

Anyways, here are some examples.

Example 1.21. In Set, the isomorphisms are the bijective maps. For this we would have to show that bijective maps have inverse maps, which is not too hard to show.

Example 1.22. In Grp, the isomorphisms are group isomorphisms. Similarly, isomorphisms in Ring are ring isomorphisms.

As a warning, we will say now that lots of categories do not have a good categorical notion of injectivity or surjectivity, so we will not be able to say that isomorphisms are merely "bijective" morphisms.

1.2 January 21

By the way, this course is being run by Bryce (interested in category theory, homological algebra, and algebraic topology) and Chris (interested in representation theory and category theory).

1.2.1 Small Correction

Last class we discussed trying to a total order $(\mathcal{P}, \leq)$ into an indiscrete category. One way to do this is to say to give a morphism between two objects $a, b \in \mathcal{P}$ if and only if one of $a < b$ or $b < a$ or $a = a$ is true. Observe that the order does not actually matter here because any two objects have exactly one morphism anyways.

1.2.2 Groupoids

Reportedly, there will usually not be a lecture to begin out our discussion sections, but here is a lecture to begin out our first discussion section.

Last time we left off talking about indiscrete categories. Here is a nice fact.

Proposition 1.23. Fix $\mathcal{C}$ an indiscrete category. Then all maps are isomorphisms.

Proof. Fix any morphism $f : x \to y$. There is also a morphism $g : y \to x$, and we see that $gf \in \operatorname{Mor}(x, x)$. But $\operatorname{id}_x \in \operatorname{Mor}(x, x)$ as well, so we are forced to have $gf = \operatorname{id}_x$ by uniqueness of morphisms. Similar shows that $fg = \operatorname{id}_y$, finishing the proof. ■

Remark 1.24. This statement is also true for discrete categories but only because all identity morphisms are isomorphisms immediately.

The property of the proposition is nice enough to deserve a definition.

Definition 1.25 (Groupoid)**.** A category in which all morphisms are isomorphisms is called a *groupoid*.

Example 1.26. Viewing groups as one-element categories, we see that groups are groupoids because all elements (i.e., morphisms of the one-object set) have inverses and hence are isomorphisms.

Intuitively, a groupoid is a group but more "spread out."

1.2.3 Arrow Words

We close out with some miscellaneous definitions for our morphisms.

Definition 1.27 (Endo-, automorphism)**.** Fix a category $\mathcal{C}$. A morphism $f : x \to y$ is an *endomorphism* if and only if $x = y$. A morphism $f : x \to y$ is an *automorphism* if and only if it is an isomorphism and an endomorphism.

Example 1.28. In the category of abelian groups, the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 is an endomorphism but not an automorphism.

Definition 1.29 (Monic, epic)**.** Fix a category $\mathcal{C}$ and a morphism $f : x \to y$.

- We say f is a *monomorphism* (or is *monic*) if and only if $fg = fh$ implies $g = h$ for any morphisms $g, h : c \to x$. In other words, the map

$$\operatorname{Mor}(c, x) \stackrel{f\circ-}{\to} \operatorname{Mor}(c, y)$$

is injective. (This map is called "post-composition.") We might write $f : x \hookrightarrow y$ for emphasis.

- We say f is an *epimorphism* (or is *epic*) if and only if $gf = hf$ implies $g = h$ for any morphisms $g, h : y \to c$. In other words, the map

$$\operatorname{Mor}(y, c) \stackrel{-\circ f}{\to} \operatorname{Mor}(x, c)$$

is injective. (This map is called "pre-composition.") We might write $f : x \twoheadrightarrow y$ for emphasis.

Intuitively, the monomorphism condition looks like the injectivity condition (namely, $f(x) = f(y)$ implies $x = y$), so monic is supposed to be a generalization for injective.

Example 1.30. In the category of sets, monic is equivalent to injective, and epic is equivalent to surjective. Then it happens that being monic and epic implies being isomorphic. We will not fill in the details here.

Warning 1.31. It is not always true that being monic and epic implies being isomorphic. It is true in $\mathrm{Set}, \mathrm{Ab}, \mathrm{Grp}$ but not in, say, Ring as the below example shows.

Example 1.32. The inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring is both epic and monic but not an isomorphism. We run some checks.

- We show monic. Suppose $g, h : R \to \mathbb{Z}$ are morphisms with $fg = fh$. We claim $g = h$. Well, for any $r \in R$, we see $g(r) = f(g(r))$ and $h(r) = f(h(r))$ because f is merely an inclusion, so $g(r) = h(r)$ follows.
- We show epic. Suppose $g, h : \mathbb{Q} \to R$ are morphisms with $gf = hf$. We claim $g = h$. We start by noting any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}$ will have

$$g\left(n/m\right) \cdot g(m) = g(n)$$

and similar for h. However, $g(m) = g(f(m)) = h(f(m)) = h(m)$ and $g(n) = h(n)$ for the same reason, so $g\left(\frac{n}{m}\right) = g(n)/g(m) = h(n)/h(m) = h\left(\frac{n}{m}\right)$, and we are done because any rational can be expressed as some $\frac{n}{m}$.
- Lastly, f is not an isomorphism because $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. For example, $2x - 1$ has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

And now discussion begins.

1.3 January 24

Chris is giving the lecture today. Reportedly, it might be rough around the edges, but I have full faith in its coherence.

1.3.1 Review

Let's quickly talk about two fun types of categories.

Definition 1.33 (Slice categories)**.** Fix a category $\mathcal{C}$ and an object $c \in \mathcal{C}$.

- We define the *slice category* $\rfloor/\mathcal{C}$ to have objects which are morphisms $f : c \to x$ for objects $x \in \mathcal{C}$. The morphisms from $f : c \to x$ to $g : c \to y$ is a morphism $h : x \to y$ such that $f = gh$. Namely, we require the following triangle to commute.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

- Dual to this is the *slice category* $\mathcal{C}/c$ where we reverse all the arrows. For example, our objects are morphisms $f : x \to c$, and morphisms from $f : x \to c$ to $g : y \to c$ are morphisms $f : x \to y$ such that $g = hf$.

There are also groupoids, which we have defined previously.

1.3.2 Subcategories

We have the following definition.

Definition 1.34 (Subcategory)**.** A *subcategory* of a category $\mathcal{C}$ is a category $\mathcal{D}$ whose objects and morphisms come from $\mathcal{C}$ and that the composition law is inherited. Explicitly, we require $\mathcal{D}$ to have the identity morphisms and be closed under composition of $\mathcal{C}$ (i.e., if $f : x \to y$ and $g : y \to z$ are morphisms in $\mathcal{D}$, then gf is also a morphism in $\mathcal{D}$.)

We are going to want ways to generate subcategories. Here is one way.

Definition 1.35 (Full subcategory)**.** Fix a category $\mathcal{C}$. Then we define the *full subcategory* $\mathcal{D}$ of $\mathcal{C}$ to be defined by choosing some objects $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{C}$ and then choosing morphisms by taking all of them. Explicitly, for $x, y \in \operatorname{Ob}\mathcal{D}$, we have

$$\operatorname{Mor}_{\mathcal{D}}(x, y) = \operatorname{Mor}_{\mathcal{C}}(x, y).$$

Example 1.36. The category of abelian groups is a full subcategory of the category of groups. Namely, the category of abelian groups is made of the objects which are abelian groups and all arrows are simply all group homomorphisms, so no morphisms have been lost in this restriction.

Example 1.37. The category of finite sets is a full subcategory in the category of sets.

Example 1.38. Given a category $\mathcal{C}$, one can take the *maximal groupoid* of $\mathcal{C}$ to be the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the isomorphisms of $\mathcal{C}$. So as long as $\mathcal{C}$ has morphisms which are not isomorphisms, then the maximal groupoid will not be full.

Example 1.39. The category Rng is a subcategory of Ring, but it is not full. For example, in Ring, the map $\mathbb{Z} \overset{\times 2}{\mathbb{Z}}$ is not a morphism even though it is a morphism in Rng.

One has to be a bit careful with this, however.

Non-Example 1.40. The category Grp is not a subcategory of Set because one can endow the same set with different group structures.

1.3.3 Duality

Here is our main character.

Definition 1.41 (Opposite category)**.** Given a category $\mathcal{C}$, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to have objects which are objects of $\mathcal{C}$ and morphisms $f^{\mathrm{op}} : y \to x$ of $\mathcal{C}^{\mathrm{op}}$ are in one-to-one correspondence with morphisms $f : x \to y$ of $\mathcal{C}$. Lastly, composition is defined by, for $f^{\mathrm{op}} : y \to x$ and $g^{\mathrm{op}} : z \to y$, we have

$$f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}.$$

In pictures, the composition law reversed the diagram $x \overset{f}{\to} y \overset{g}{\to} z$ to

$$x \overset{f^{\mathrm{op}}}{\leftarrow} y \overset{g^{\mathrm{op}}}{\leftarrow} z.$$

Let's see some examples.

Example 1.42. Given a partial order $(\mathcal{P}, \leq)$, the opposite category is by (partial) ordering $\mathcal{P}$ simply by flipping the partial order: $b \leq_{\mathrm{op}} a$ if and only if $a \leq b$. Namely, the opposite category of a partial order remains a partial order.

Example 1.43. Fix a group G and form its category $\mathrm{B}G$. Now, when we reverse the arrows $(\mathrm{B}G)^{\mathrm{op}}$, we get a category corresponding to the group law G^{op} with group law defined by

$$h^{\mathrm{op}} g^{\mathrm{op}} = gh.$$

Namely, the opposite category of a group is still a group.

In fact, we have that $\mathrm{B}G \cong (\mathrm{B}G)^{\mathrm{op}}$ (for whatever $\cong$ means) by taking making our morphisms perform inversion by $\varphi : g \mapsto (g^{\mathrm{op}})^{-1}$. This map is bijective, and we can check the composition by writing

$$\varphi(gh) = \left((gh)^{\mathrm{op}}\right)^{-1} = \left(h^{\mathrm{op}} g^{\mathrm{op}}\right)^{-1} = \left(g^{\mathrm{op}}\right)^{-1}\left(h^{\mathrm{op}}\right)^{-1} = \varphi(g)\varphi(h),$$

so everything works.

Example 1.44. Algebraic geometry says that $\mathrm{CRing}^{\mathrm{op}}$ is equivalent to the category of affine schemes AffSch. The point here is that the opposite category is potentially very different from the original category. (Mnemonically, the opposite of algebra is geometry.)

Now, here is the idea of duality.

Idea 1.45. Theorem statements that hold for categories will need to be true for their opposite category as well.

As an example, let's work with monomorphisms and epimorphisms. For example, $f : y \to z$ is monic if and only if the commutativity of the diagram

$$\begin{tikzcd} x \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & y \arrow[r, "f"'] & z \end{tikzcd}$$

forces $g = h$. Similarly, $f : x \to y$ is epic if and only if the commutativity of the diagram

$$x \xrightarrow[f]{} y \overset{g}{\underset{h}{\rightrightarrows}} z$$

forces $g = h$. But notice that flipping the epic diagram notes that epic condition is equivalent to the commutativity of the diagram

$$x \overset{g^{\mathrm{op}}}{\underset{h^{\mathrm{op}}}{\rightrightarrows}} y \xrightarrow[f^{\mathrm{op}}]{} x$$

forces $g = h$, which is the same thing as $g^{\mathrm{op}} = h^{\mathrm{op}}$. Thus, we have the following lemma.

Lemma 1.46. Fix a category $\mathcal{C}$. Then a morphism f is monic if and only if f^{op} is epic in $\mathcal{C}$.

Proof. This comes from the discussion above. ■

The point is that we can prove theorems about monic and epic maps simultaneously by working with (say) monomorphisms general categories and then dualizing to get the statement about epimorphisms.

Let's see this strategy in action. We have the following definition.

Definition 1.47 (Section, retraction)**.** Suppose that $s : x \to y$ and $r : y \to x$ are morphisms such that $rs = \mathrm{id}_x$; i.e., the composition

$$x \xrightarrow{s} y \xrightarrow{r} x$$

is id_x. Then we say that s is a *section* of r, and r is a *retraction* of s.

Think about these as having a one-sided inverse. We have the following lemma.

Lemma 1.48. A morphism s in $\mathcal{C}$ is a section of some morphism if and only if s^{op} is a retraction in $\mathcal{C}$.

Proof. Fix $s : x \to y$. The condition that there exists r so that $rs = \mathrm{id}_x$ is equivalent to there exists r^{op} such that $s^{\mathrm{op}} r^{\mathrm{op}} = \mathrm{id}_x^{\mathrm{op}}$, which translates into the lemma. ■

And now let's actually see a proof.

Proposition 1.49. A morphism s in $\mathcal{C}$ is a section of some morphism implies that s is a monomorphism.

Proof. Suppose that $s : x \to y$ is a section for the morphism $r : y \to x$ so that $rs = \mathrm{id}_x$. Now, suppose that $sg = sh$ so that we want to show $g = h$. But we see that

$$g = \mathrm{id}_x g = (rs)g = r(sg) = r(sh) = (rs)h = \mathrm{id}_x h = h,$$

so we are done. ■

So here is our dual statement, which we get for free.

Proposition 1.50. A morphism r in $\mathcal{C}$ is a retraction of some morphism implies that r is an epimorphism.

Proof. We note that r is a retraction in $\mathcal{C}$ implies that r^{op} is a section in $\mathcal{C}^{\mathrm{op}}$, so by the above, r^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$. Thus, it follows that r is an epimorphism in $\mathcal{C}$. ■

We've been saying "section of" and "retraction of" a lot, so we optimize out these words in the following definition.

Definition 1.51 (Split mono-, split epi-morphism)**.** We say that a morphism f of $\mathcal{C}$ is a *split monomorphism* if and only if it is a section of some morphism. Similarly, we say that f is a *split epimorphism* if and only if it is the retraction of some morphism.

So the above statements show that split monomorphisms are in fact monomorphisms, and split epimorphisms are in fact epimorphisms.

1.3.4 Yoneda Lite

So far we have said that monic is similar to injective and epic is similar to surjective. We would like to make these sorts of correspondences a little more concrete, so we add more abstraction.

Definition 1.52 (Post- and pre-composition)**.** Fix a morphism $f : x \to y$ of $\mathcal{C}$. Then, given an object $c \in \mathcal{C}$, we define the maps $f_* : \operatorname{Mor}(c, x) \to \operatorname{Mor}(c, y)$ and $f^*(y, c) \to \operatorname{Mor}(x, c)$ by

$$f_*(g) := fg \qquad \text{and} f^*(g) := gf.$$

The map f_* is called *post-composition* because we apply f after; the map f^* is called *pre-composition* because we apply it after.

Note that f_* and f^* are nice because they are all real functions of sets (for locally small categories) with which we can use to understand f. Here are some equivalent conditions.

Proposition 1.53. Fix f a morphism of the category $\mathcal{C}$. Then the following are true.

(a) f is an isomorphism if and only if f_* is bijective if and only if f^* is bijective.

(b) f is monic if and only if f_* is injective.

(c) f is epic if and only if f^* is injective (!).

(d) f is split monic if and only if f^* is surjective.

(e) f is split epic if and only if f_* is surjective.

Proof. We omit most of these; let's show (b). We have two directions. Suppose that f is monic. Then fix an object c, and we show that the map

$$f_* : \operatorname{Mor}(c, x) \to \operatorname{Mor}(c, y)$$

by $f_*(g) := fg$ is injective. But indeed, $f_*(g) = f_*(h)$ implies $fg = fh$ implies $g = h$ by monic, so injectivity follows.

Conversely, suppose f_* is monic. Then suppose that $fg = fh$ for some morphisms $g, h : c \to x$, and we show that $g = h$. But f_* is injective! So

$$f_*(g) = fg = fh = f_*(h)$$

forces $g = h$, and we are done. ■

Theme 2: Functors and Natural Transformations

Mathematics is the art of giving the same names to different things

—Henri Poincaré

2.1 January 26

We will start on new things.

2.1.1 Functors

In this class, we will repeatedly talk about the following idea.

Idea 2.1. Everything is a special case of everything else.

In other words, we will want to abstract old ideas from new ones, and this will happen a lot.

The first time we are going to see this is by trying to consider categories of

Remark 2.2. Yes, Russel's paradox prevents a category of all categories. Nevertheless, we will try. One way to get around this is to do size declarations: for example, we can consider the category of all small categories, as we are about to do.

Anyways, we would like to give some categorical structure to (say, small) categories. Well, what will be our morphisms between categories? They will be "functors."

Before defining functors, we should describe what a functor $F : \mathcal{C} \to \mathcal{D}$ should do.

- Viewing $\mathcal{C}$ as consisting of the data of objects and morphisms, an initial requirement might be that F takes objects to objects and morphisms to morphisms.

- We would also like F to preserve the "structure" of our categories, which essentially means we want to preserve composition in our categories. So we will require a "functoriality" condition to preserve this structure.

Let's try to get an intuitive feeling for how functoriality should behave.

Example 2.3. Fix an abelian group A. Then there is a map $\mathrm{Hom}(A, -)$ sending abelian groups Ab to sets Set. In fact, we get a map of morphisms as well, for a morphism $f : X \to Y$ provides a post-composition mapping

$$f_* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$$

by $\varphi \mapsto f\varphi$. This association has some nice properties. For example, we have the following.

- We see $(\mathrm{id}_X)_* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, X)$ sends $\varphi \mapsto \varphi$, so $(\mathrm{id}_X)_* = \mathrm{id}_{\mathrm{Hom}(A,X)}$.
- Given $f : X \to Y$ and $g : Y \to Z$, we have $gf : X \to Z$, and we can see that

  $$(gf)_*(\varphi) = gf\varphi = g_*(f_*(\varphi)) = (g_*f_*)(\varphi),$$

  so we are "preserving composition" in some sense because we composed before and after.

Example 2.4. Given a topological space X, we can create the fundamental group $\pi_1(X)$. This mapping is nice because a continuous map $f : X \to Y$ will induce a map $\pi(f) : \pi_1(X) \to \pi_1(Y)$, and in fact we can check that $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X)}$ as well as preserving composition ($f : X \to Y$ and $g : Y \to Z$ gives $\pi_1(gf) = \pi_1(g)\pi_1(f)$).

With the above motivation, we are now ready to give the definition of a functor.

Definition 2.5 (Functor)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of "assignments" $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ satisfying the following coherence laws.

- Morphisms make sense: if $f : x \to y$ a morphism in $\mathcal{C}$, then Ff is a morphism with domain Fx and codomain Fy.
- Identity: given an object $c \in \mathcal{C}$, we require $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Composition: given morphisms $f : x \to y$ and $g : y \to z$ in $\mathcal{C}$, we require that $F(gf) = F(g)F(f)$.

2.1.2 More Examples

Let's do more examples.

Example 2.6 (Forgetful)**.** There is a functor $U : \mathrm{Grp} \to \mathrm{Set}$ which sends a group G to its underlying set G and a group homomorphism to the underlying function. In other words, we are simply forgetting the algebraic structure of the group. Because the composition law in groups is composition of functions, and identities in Grp do nothing like in Set.

Example 2.7 (Forgetful)**.** Here are more forgetful functors.

- $\mathrm{Ring} \to \mathrm{Grp}$ (by $R \mapsto R^\times$)
- $\mathrm{Field} \to \mathrm{Ring}$
- $\mathrm{Ring} \to \mathrm{Ab}$
- $\mathrm{Grp} \to \mathrm{Set}_*$ by sending $G \mapsto (G, e_G)$; namely, we point the set of G by its identity, which must be fixed by group homomorphisms anyways.

With all of our forgetful functors lying around, we have the following definition.

Definition 2.8 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if it has a forgetful functor to Set.

This is not terribly formal because we haven't defined what a forgetful functor means, but hopefully this is sufficiently intuitive: $\mathcal{C}$ should be sets with some extra structure.

Before our next example, we pick up the following example.

Definition 2.9 (Endofunctor)**.** A functor F is an *endofunctor* of its "domain" and "codomain" categories are the same category.

Example 2.10. There is an endofunctor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ sending a set X to its power set $\mathcal{P}(X)$. We send morphisms $f : X \to Y$ to $\mathcal{P}(f)$ by sending subsets $S_X \subseteq X$ in $\mathcal{P}(X)$ to the image $f(S_X) \in \mathcal{P}(Y)$. We will not check the functoriality conditions, but it can be done without too much effort.

And now for more examples.

Example 2.11. There is a functor $\mathrm{Top} \to \mathrm{Htpy}$ by sending a topological space X to the same space up to homotopy. Then we send continuous maps to continuous maps, up to homotopy.

Example 2.12. There is a "free" functor $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$ sending a set S to the abelian group

$$\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z}s.$$

Essentially, this is the free $\mathbb{Z}$-module generated by S; formally, $\mathbb{Z}[S]$ is made of finite $\mathbb{Z}$-linear combinations of elements of S.

Then we can take a function $f : S \to T$ to a group homomorphism $\mathbb{Z}[S] \to \mathbb{Z}[T]$ because we have described where to send the "basis elements" of S, and hence this f will uniquely determine the full map.

Example 2.13. Fix $\mathcal{C}$ a locally small category, and fix some $x \in \mathcal{C}$. Then there is a functor $\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ by sending

$$y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y) \qquad \text{and} \qquad (f : y \to z) \mapsto f_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z),$$

where $f_* : \varphi \mapsto f\varphi$ is again post-composition.

Example 2.14. There is an endofunctor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ by sending objects and morphisms to themselves.

2.1.3 Categories of Categories

While we're here, we note that we can create new functors from old ones by "composition."

Proposition 2.15. Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ functors. Then the naturally defined map $GF : \mathcal{C} \to \mathcal{E}$ is also a functor.

Proof. We do indeed send objects to objects, and a morphism $f : x \to y$ in $\mathcal{C}$ will be sent to $F(f) : Fx \to Fy$ and then

$$GF(f) : GFx \to GFy.$$

Further, we can check that $GF(\mathrm{id}_x) = G(\mathrm{id}_{Fx}) = \mathrm{id}_{GFx}$, so GF preserves identities. And then, given $f : x \to y$ and $g : y \to z$, we see that

$$GF(gf) = G(F(g)F(f)) = GF(g)GF(f),$$

which finishes the composition check. ■

The point of the above composition law, is that it lets us form a "category."

Definition 2.16. We define Cat to be the category of small categories where morphisms are functors. We define CAT to be the category of locally small categories where morphisms again are functors.

Remark 2.17. Fixing two small categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ can be identified with a function on merely the morphism sets $\operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{D}$, which is itself a set. Thus, Cat is a locally small category: $\mathrm{Cat} \in \mathrm{CAT}$.

2.1.4 Subcategories.

To finish out class, we have the following warning.

Warning 2.18. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We check that the naturally defined "image" $F(\mathcal{C})$ need not be a subcategory of $\mathcal{D}$.

Here is an example. Let $\mathcal{C}$ be the following category.

$$a \xrightarrow{f} b$$

$$a' \xrightarrow{f'} b'$$

Then let $\mathcal{D}$ be the following category.

$$0 \xrightarrow{x} 1 \xrightarrow{y} 2$$

Now we define $F : \mathcal{C} \to \mathcal{D}$ by $Ff = x$ and $Ff' = y$, which will make a perfectly fine functor. However, the composition $yx : 0 \to 2$ in $\mathcal{D}$ does not live in the image of F, so this image is not a subcategory.

To fix this problem, one often says something like "given a functor $F : \mathcal{C} \to \mathcal{D}$, consider the full subcategory of $F(\mathcal{C})$" to mean closing up $F(\mathcal{C})$'s potentially unclosed composition.

2.2 January 31

So class is in-person today.

2.2.1 Small Remark

A question was asked in the Discord server about dualizing. In theory, dualizing theorems should be very easy: simply state the theorem in the opposite category, provided we have shown the necessary machinery to make the theorem dualize as necessary.

2.2.2 Contravariance

Today we are talking about contravariance. A functor $F : \mathcal{C} \to \mathcal{D}$ is defined so far as what are called "covariant" functors. We would like to define contravariant functors. There are lots of equivalent ways to do this.

Definition 2.19 (Contravariance, I)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a mapping of objects and morphisms with the following coherence laws.

- If $f : a \to b$ in $\mathcal{C}$, then $Ff : Fb \to Fa$. (Note the reversal of direction!)
- Identity: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for each $c \in \mathcal{C}$.
- Contravariant (!) composition: if $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, then $F(gf) = F(f)F(g)$.

This in fact comes from dualizing.

Definition 2.20 (Contravariance, II)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

To be explicit, if we are given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, then a morphism $f : a \to b$ in $\mathcal{C}$ is first taken to a morphism $f^{\mathrm{op}} : b^{\mathrm{op}} \to a^{\mathrm{op}}$. And if we have another morphism $g : b \to c$ in $\mathcal{C}$, then we see the diagram

$$a \xrightarrow{f} b \xrightarrow{g} c$$

becomes

$$a^{\mathrm{op}} \xleftarrow{f^{\mathrm{op}}} b^{\mathrm{op}} \xleftarrow{g^{\mathrm{op}}} c^{\mathrm{op}}$$

becomes

$$Fa^{\mathrm{op}} \xleftarrow{Ff^{\mathrm{op}}} Fb^{\mathrm{op}} \xleftarrow{Fg^{\mathrm{op}}} Fc^{\mathrm{op}},$$

which gives our composition law.

We can also dualize in the opposite direction.

Definition 2.21 (Contravariance, III)**.** A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Warning 2.22. We will use Definition 2.20 as our definition of contravariance.

Example 2.23. We work with Vec_k the category whose objects are k-vector spaces and morphisms which are linear maps. Then we have a functor

$$-^* : \mathrm{Vec}_k^{\mathrm{op}} \to \mathrm{Vec}_k$$

by taking $V \mapsto V^*$. (Here, $V^* := \mathrm{Hom}_k(V, k)$.) As for morphisms, we need to take $f : V \to W$ to some map $f^* : W^* \to V^*$, which is

$$f^* : \varphi \mapsto \varphi f.$$

Example 2.24. We work with Poset the category whose objects are posets and morphisms which are order-preserving maps. I.e., a map $f : P \to Q$ is order-preserving if and only if $a \le b$ in P implies $f(a) \le f(b)$ in Q. Now we define the contravariant functor $\mathcal{O} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Poset}$ by taking

$$X \mapsto \{U : \text{open } U \subseteq X\},$$

where the order on the right is by inclusion. Then a continuous map $f : X \to Y$ becomes the order-preserving (!) map $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ by

$$\mathcal{O}(f)(U_Y) := f^{-1}(U_Y).$$

Explicitly, open subsets $U_1 \subseteq U_2$ of Y have $f^{-1}(U_1) \subseteq f^{-1}(U_2)$ back in X.

Remark 2.25. We can use the above example to define a presheaf. "Presheaf" can have lots of meanings.

- A "presheaf" can be any contravariant functor.
- A "presheaf" can be any contravariant functor with codomain Set.
- A "presheaf" can be any contravariant functor from $\mathcal{O}(X)^{\mathrm{op}}$. It is Set-valued (respectively, $\mathcal{C}$-valued) if its codomain is Set (respectively, $\mathcal{C}$).

2.2.3 A Lemma

It's a math class, so we should probably prove something today.

Theorem 2.26. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ preserves isomorphisms.

Remark 2.27. By convention, all functors will be covariant, and if we want a contravariant functor, we will write $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. In other words, I will now stop writing "(covariant)."

Proof. Let $f : a \to b$ be an isomorphism in $\mathcal{C}$ with inverse g. We want to show that $F(f)$ is an isomorphism; we claim that $F(g)$ is its inverse. Indeed,

$$F(f)F(g) = F(fg) = F(\mathrm{id}_b) = \mathrm{id}_{F(b)} \qquad \text{and} \qquad F(g)F(f) = F(gf) = F(\mathrm{id}_a) = \mathrm{id}_{F(a)},$$

so indeed, $F(g)$ is an inverse of $F(f)$. So $F(f)$ is an isomorphism, and we are done. ■

This example can do things.

Example 2.28. Fix groups G, H and their one-object categories $\mathrm{B}G, \mathrm{B}H$. We claim that functors $F : \mathrm{B}G \to \mathrm{B}G$ contain exactly the data of a group homomorphism $G \to H$. To see that F induces a group homomorphism, suppose $\sigma, \tau \in G$, we have by funtoriality

$$F(\sigma\tau) = F(\sigma)F(\tau),$$

which is exactly what we need to be a group homomorphism. Conversely, if $f : G \to H$ is a group homomorphism, then f induces a functor: $f(\sigma\tau) = f(\sigma)f(\tau)$ by definition, and $f(\mathrm{id}_G) = \mathrm{id}_H$ is a result of group theory.

Example 2.29. A functor $F : \mathrm{B}G \to \mathcal{C}$ is precisely the data of a G-action of an object $c \in \mathcal{C}$. We send the one object $* \in \mathrm{B}G$ somewhere, say to an object $c \in \mathcal{C}$. Then each $\sigma \in G$ goes to some morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ (which is in fact an isomorphism because σ is an isomorphism $\mathrm{B}G$). So in total we get a map

$$G \to \mathrm{Aut}\, c,$$

which is exactly the data of a group action. This unifies group actions on all sorts of structures.

The above definition is special enough to have a name.

Definition 2.30 (Functorial group action)**.** A *functorial group action* of G on a category $\mathcal{C}$ is a functor $\mathrm{B}G \to \mathcal{C}$.

Remark 2.31. Technically we will be viewing these functors as providing left actions. To get a right action, we want a functor $(\mathrm{B}G)^{\mathrm{op}} \to \mathcal{C}$.

Note, as in the example, the functor contains the same data as a group homomorphism $G \to \mathrm{Aut}\, c$ for some $c \in \mathcal{C}$.

Remark 2.32. Bryce would like to make us aware that writing down $G \to \mathrm{Aut}\, c$ as a group homomorphism is only legal when $\mathcal{C}$ is locally small.

Example 2.33. Given a group G, a G-representation V of G is a functor $\mathrm{B}G \to \mathrm{Vec}_k$ where $* \in \mathrm{B}G$ goes to $V \in \mathrm{Vec}_k$.

2.2.4 The Hom Bifunctor

We have a little time left, so let's do something fun. Given a (locally small) $\mathcal{C}$ and an object $x \in \mathcal{C}$, we get two functors

$$\mathrm{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set} \qquad \text{and} \qquad \mathrm{Mor}_{\mathcal{C}}(-, x) : \mathcal{C} \to \mathrm{Set}.$$

The former functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $\varphi : y \to z$ to $\varphi_* : \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$ to $\varphi_* : f \mapsto \varphi f$. We can check this functor is covariant because

$$\varphi_* \psi_*(f) = \varphi \psi f = (\varphi \psi)_*(f).$$

Now, the latter functor sends $y \mapsto \mathrm{Mor}_{\mathcal{C}}(y, x)$ and $\varphi : y \to z$ to $\varphi^* : \mathrm{Mor}_{\mathcal{C}}(z, x) \to \mathrm{Mor}_{\mathcal{C}}(y, z)$ by $\varphi^* : f \mapsto \varphi f$. We can check this functor is contravariant because

$$\psi^* \varphi^* f = f \varphi \psi = (\varphi \psi) * f.$$

2.3 February 2

Today we are talking about product categories and the Hom bifunctor.

2.3.1 Hom Bifunctor

Here is our definition.

Definition 2.34 (Product category)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then we define the *product category* $\mathcal{C} \times \mathcal{D}$ as follows.

- We define $\operatorname{Ob} \mathcal{C} \times \mathcal{D}$ to be the collection of ordered pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
- We define $\operatorname{Mor}((c, d), (c', d'))$ to be the collection of ordered pairs (f, g) with $f : c \to c'$ a morphism in $\mathcal{C}$ and $g : d \to d'$ a morphism in $\mathcal{D}$.

Lastly, we define identity to be the identity on each object and composition by composition component-wise.

From yesterday, we have the following functors.

Definition 2.35 (Functors represented by objects)**.** Fix $\mathcal{C}$ a locally small category and $x \in \mathcal{C}$ an object. Then we have the functors

$$\operatorname{Mor}_{\mathcal{C}}(x, -) : \mathcal{C} \to \operatorname{Set} \qquad \text{and} \qquad \operatorname{Mor}_{\mathcal{C}}(-, x) : \mathcal{C}^{\mathrm{op}} \to \operatorname{Set}.$$

The former functor is the *covariant functor represented by* x, and the latter is the *contravariant functor represented by* x.

We would like to codify the structure that having two functors gives us, so we have the following definition.

Definition 2.36 (Bifunctor)**.** A *bifunctor* is a functor whose domain is a product of categories.

In particular, here is our standard example.

Definition 2.37 (Hom bifunntor)**.** Fix $\mathcal{C}$ a locally small category. Then *Hom bifunctor* is the functor given by the functors representing a particular object $x \in \mathcal{C}$. Namely, we have

$$\operatorname{Mor}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \operatorname{Set}$$

by taking $(x, y) \mapsto \operatorname{Mor}_{\mathcal{C}}(x, y)$.

We will not check that this is actually a functor, but it is.

2.3.2 Category Isomorphism

We would like a notion of two categories being the same, but this is somewhat subtle. Here is a first approximation.

Definition 2.38 (Isomorphism)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if and only if there is an inverse functor $G : \mathcal{D} \to \mathcal{C}$ so that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. In this case we say that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*.

Remark 2.39. As usual, isomorphisms are unique and whatnot.

Let's make this definition a little more concrete.

Proposition 2.40. An isomorphism $F : \mathcal{C} \to \mathcal{D}$ descends to a bijective (i.e., injective and surjective) map $\operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D}$.

Remark 2.41. We are attempting to care about set-theoretic issues in our phrasing because Bryce cares about set-theoretic issues.

Proof of Proposition 2.40. Let G be the inverse morphism for F. Then we claim that the induced map $G : \operatorname{Ob}\mathcal{D} \to \operatorname{Ob}\mathcal{C}$ will be the inverse for the induced map for F. This is clear because $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$. ■

It turns out that isomorphisms are a little too strong: there are categories we want to be the same but are not actually isomorphic.

Example 2.42. The category

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

is not isomorphic to

$$\bullet$$

because there are a different number of objects, so there is no bijection.

2.3.3 Natural Transformation

To salvage our notion of categorical isomorphism, we need a notion of naturality. Naturality is more of something that we can feel as mathematicians rather than something we like to formalize.

Example 2.43. Any two trivial groups have a canonical isomorphism between them. In fact, there is only one homomorphism at all.

Non-Example 2.44. There is no "natural" or "canonical" isomorphism $\mathbb{Z}/3\mathbb{Z} \to A_3$, though the groups are isomorphic.

Non-Example 2.45. Given a two-dimensional $\mathbb{R}$-vector space named V, there is no canonical isomorphism $\mathbb{R}^2 \to V$.

We would like maps to preserve all the structure we could want. So here is our notion of naturality for functors.

Definition 2.46 (Natural transformation)**.** Fix functors $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\eta : F \Rightarrow G$ consists of the data of a morphism $\eta_X : Fc \to Gc$ for each $c \in \mathcal{C}$ such that the following diagram always commutes for any morphism $f : c \to c'$ in $\mathcal{C}$.

$$\begin{tikzcd} Fc \arrow[r, "\eta_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\eta_{c'}"'] & Gc' \end{tikzcd}$$

The maps φ_c are called the *components* of φ.

Quote 2.47. Burn this square into your minds. It is the most important square in this class.

As usual, we start with examples.

Exercise 2.48. We work in Vec_k. Then we consider the functor $-^{**} : \mathrm{Vec}_k \to \mathrm{Vec}_k$ by $V \mapsto V^{**}$. Then we claim that there is a natural transformation from $-^{**}$ to id, using the natural transformation

$$\mathrm{ev}_V : V \to V^{**}$$

by $\mathrm{ev}_V(x) := (\lambda \in V^* \mapsto \lambda x)$.

Proof. We need to check that the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\mathrm{ev}_W"'] & W^{**} \end{tikzcd}$$

Very quickly, we recall that $f^{**} : V^{**} \to W^{**}$ is by

$$f(\varphi) = (\lambda \in W^* \mapsto \varphi(\lambda f)).$$

Namely, $\lambda : W \to k$, so $\lambda f : V \to k$ lives in V^*, so $\varphi(\lambda f) \in k$.

Now we check the commutativity of the square. Fix some $x \in V$ and a linear functional $\lambda : W \to k$. Then we can carefully compute, after many tears and groans, that

$$f^{**}(\mathrm{ev}_V(x))(\lambda) = \mathrm{ev}_V(x)(\lambda f) = \lambda f(x) = \mathrm{ev}_W(f(x))(\lambda).$$

Because λ was arbitrary, we see that $f^{**}\,\mathrm{ev}_V(\lambda) = \mathrm{ev}_W\, f(x)$, which then gives us $f^{**}\,\mathrm{ev}_V = \mathrm{ev}_W\, f$. ■

We have the following definition.

Definition 2.49 (Natural isomorphism)**.** A natural transformation $\eta : F \to C$ is a *natural isomorphism* if and only if its component morphisms are isomorphisms.

Example 2.50. In finVec_k, the above ev is a natural isomorphism because $\mathrm{ev}_V : V \Rightarrow V^{**}$ is an isomorphism when V is finite-dimensional.

Here is a quick proposition.

Proposition 2.51. Let $\varphi : F \Rightarrow G$ be a natural isomorphism. Then the inverse morphisms $\psi_c := \varphi_c^{-1}$ assemble to make a natural transformation $\psi : G \Rightarrow F$.

Proof. We will be brief. Given a morphism $f : x \to y$, we need to check that the following diagram commutes.

$$\begin{tikzcd} Gx \arrow[r, "\psi_x"] \arrow[d, "Gf"'] & Fx \arrow[d, "Ff"] \\ Gy \arrow[r, "\psi_y"'] & Fy \end{tikzcd}$$

In other words, we need to know that $\psi_y Ff = Gf\psi_x$. Well, we already know that

$$\varphi_y Ff = Gf\varphi_x$$

by naturality, so

$$Ff\psi_x = \psi_y\varphi_y Ff\psi_x = \psi_y Gf\varphi_x\psi_x = \psi_y Gf$$

after checking through. ■

2.4 February 7

2.4.1 Examples of Natural Transformations

We're talking about more natural transformations today. For our first example, consider the covariant power set functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ by $S \mapsto \mathcal{P}(S)$ and $f : S \to T$ to $\mathcal{P}(f)(U) := f(U)$ for $U \subseteq S$.

Exercise 2.52. We define a natural transformation $\eta_\bullet : \mathrm{id}_{\mathrm{Set}} \Rightarrow \mathcal{P}$ a function $\eta_S : S \to \mathcal{P}(S)$ by

$$\eta_S(x) := \{x\}$$

Proof. Fix $f : S \to T$ a morphism in Set. After plugging everything in, we need the following diagram to commute.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\eta_S"'] & T \arrow[d, "\eta_T"] \\ \mathcal{P}(S) \arrow[r, "\mathcal{P}(f)"'] & \mathcal{P}(T) \end{tikzcd}$$

To see this commutes, fix some $x \in S$, and we run it through the diagram as follows.

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\eta_S"'] & f(x) \arrow[d, mapsto, "\eta_T"] \\ \{x\} \arrow[r, mapsto, "\mathcal{P}(f)"'] & \{f(x)\} \end{tikzcd}$$

So indeed, the diagram does commute. ■

Remark 2.53. We may call the second diagram an "internal" diagram because it is looking internally at our objects.

For our next example, recall we defined a functorial G-action on some object $c \in \mathcal{C}$ by a functor $F : \mathrm{B}G \to \mathcal{C}$. Our goal is to define a G-equivariant map between objects.

Exercise 2.54. We track the data between two G-representations $F, G : \mathrm{B}G \to \mathrm{Vec}_k$ by a natural transformation $\eta_\bullet : \mathrm{Vec}_k \Rightarrow \mathrm{Vec}_k$.

Proof. Because $\mathrm{B}G$ has only one object $*$, we set $V := F(*)$ and $W := G(*)$ and need to check the commutativity of the following diagram, for some $g : * \to *$ in G.

$$\begin{tikzcd} V \arrow[r, "Fg"] \arrow[d, "\eta_*"'] & V \arrow[d, "\eta_*"] \\ W \arrow[r, "Fw"'] & W \end{tikzcd}$$

Note that the natural transformation $\eta_\bullet$ really only consists of the map η_*, which is a linear map $V \to W$ which respects the group action: $\eta_*(gv) = g\eta_*(v)$. ■

These G-equivariant maps can be turned into a category.

Definition 2.55 (Rep_G)**.** We define the category of G-*representations* to be the category consisting of objects which are functors $F : \mathrm{B}G \to \mathrm{Vec}_k$ and morphisms which are natural transformations between the functors.

Exercise 2.56. We check that there is a category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose

Proof. To define our morphisms, suppose $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\eta_\bullet : F \Rightarrow G$ and $\nu_\bullet : G \Rightarrow H$. Lastly, we define our composition by

$$(\nu\eta)_X := \eta_X \nu_X.$$

To check that $(\eta\nu)_\bullet : F \Rightarrow H$ is in fact a natural transformation, we have the following ladder.

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"'] \arrow[dd, bend right, "(\nu\eta)_x"'] & Fy \arrow[d, "\eta_y"] \arrow[dd, bend left, "(\nu\eta)_y"] \\
Gx \arrow[r, "Gf"] \arrow[d, "\nu_x"'] & Gy \arrow[d, "\nu_y"] \\
Hx \arrow[r, "Hf"'] & Hy
\end{tikzcd}$$

Each square commutes, so the 2×1 rectangle will also commute. We check associativity by drawing a 3×1 rectangle and seeing that it commutes.

To define our identity maps for our category, we take $(\mathrm{id}_F)_X := \mathrm{id}_{F(x)} : Fx \to Fx$. We can check that this works with our composition without too many tears. ■

Definition 2.57 (Functor category)**.** The category of the above exercise is the *functor category*, notated $\mathcal{D}^{\mathcal{C}}$.

Example 2.58. We have that $\mathrm{Rep}_G = \mathrm{Vec}_k^{BG}$.

2.4.2 Yoneda, Contravariant It Is

For the discussion that follows, we fix $\mathcal{C}$ locally small and $f : w \to x$ and $h : y \to z$ some morphisms in $\mathcal{C}$. From this we get the following square.

$$\begin{tikzcd}
\mathrm{Mor}(x,y) \arrow[r, "h-"] \arrow[d, "-f"'] & \mathrm{Mor}(x,z) \arrow[d, "-f"] \\
\mathrm{Mor}(w,y) \arrow[r, "h-"'] & \mathrm{Mor}(w,z)
\end{tikzcd}$$

We can check that this square commutes. Here is the internal square.

$$\begin{tikzcd}
g \arrow[r, mapsto, "h-"] \arrow[d, mapsto, "-f"'] & hg \arrow[d, mapsto, "-f"] \\
gf \arrow[r, mapsto, "h-"'] & hgf
\end{tikzcd}$$

Hooray, it commutes. The point is that $h-$ and $-f$ are going to induce natural transformations of our Mor functors.

- The functors $\mathrm{Mor}(x,-), \mathrm{Mor}(w,-) : \mathcal{C} \to \mathrm{Set}$. Then any morphism $f : w \to x$ induces a natural transformation $-f : \mathrm{Mor}(x,-) \Rightarrow \mathrm{Mor}(w,-)$. The naturality check is the commutativity of the above square.

- Similarly, the functors $\text{Mor}(-, y), \text{Mor}(-, z) : \mathcal{C} \to \text{Set}$. Then any morphism $h : x \to y$ induces a natural transformation $h- : \text{Mor}(-, y) \Rightarrow \text{Mor}(-, w)$. The naturality check is again the commutativity of the above square.

We won't be more explicit about our squares because my head hurts.

Remark 2.59. Later in life we will talk about the Yoneda embedding, which is essentially about the embedding $\mathcal{C}^{\text{op}} \to \text{Set}^{\mathcal{C}}$, which takes $x \mapsto \text{Mor}(x, -)$ and $f : x \to y$ to the natural transformation $-f : \text{Mor}(x, -) \Rightarrow \text{Mor}(y, -)$. This will turn out to be a functor and very good. We will not say more for now.

2.4.3 Categorification

The category Set has some nice operations: we can talk about products $A \times B$, disjoint unions $A \sqcup B$, and functions $A^C = \{f : C \to A\}$. Note that these notations are suggestive of multiplication, addition (depending on whom you talk to), and exponentiation. For example,

$$\#(A \times B) = \#A \times \#B, \quad \#(A \sqcup B) = \#A + \#B, \quad \#\left(A^C\right) = \#A^{\#C}.$$

This gives us some notion of a "cardinality functor" $\# : \text{FinSet} \to \mathbb{N}$, which we can check does some things.

This lets us define "categorification." We will not give a formal definition of this, but here are some instructive examples.

Example 2.60. The functor $\# : \text{FinSet} \to \mathbb{N}$ is a decategorification functor. For example, we can categorify $a \times (b + c) = a \times b + a \times c$ in $\mathbb{N}$ to some natural isomorphism

$$A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C).$$

Example 2.61. There is a decategorification functor $\dim : \text{fdRep}_G \to \mathbb{N}$.

2.4.4 Equivalence: Advertisement

Let's close class by defining an equivalence of categories. Recall that we called a functor $F : \mathcal{C} \to \mathcal{D}$ an isomorphism if and only if it has an inverse functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \text{id}_{\mathcal{D}}$ and $GF = \text{id}_{\mathcal{C}}$.

This is a bad notion of saying two categories are the same.

Example 2.62. The categories of k-matrices and k-vector spaces are not isomorphic (they don't have the same), even though we often think about vector spaces as merely being some dimensional space.

Here is the fix

Definition 2.63 (Equivalence)**.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if and only if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \text{id}_{\mathcal{D}}$ and $GF \simeq \text{id}_{\mathcal{C}}$.

2.5 February 9

2.5.1 Equivalence

We can define a category Mat_k to have objects which are the natural numbers and morphisms which are $\text{Mat}_k(n, m)$ equal to the $m \times n$ matrices with coefficients in k. In linear algebra, we want to think about each natural number n as a k-vector space of dimension n, and we want to think about each matrix $n \to m$ as a linear map. In other words, Mat_k should be "the same" as fdVec_k.

However, fdVec_k and Mat_k do not even have the same number of objects, so they cannot be isomorphic. We still want them to be the same, so we weaken our notion of isomorphism.

Definition 2.64 (Equivalence)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ if and only if $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$. If an equivalence between $\mathcal{C}$ and $\mathcal{D}$ exists, then $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, denoted $\mathcal{C} \simeq \mathcal{D}$.

We should probably start by showing that our notion of equivalence forms what we think of as an equivalence relation.

Remark 2.65 (Bryce)**.** Equivalence does not form an equivalence relation for size reasons.

Lemma 2.66. Fix categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. Then the following hold.

- Reflexive: $\mathcal{C} \simeq \mathcal{C}$.
- Symmetric: $\mathcal{C} \simeq \mathcal{D}$ implies $\mathcal{D} \simeq \mathcal{C}$.
- Transitive: $\mathcal{C} \simeq \mathcal{D}$ and $\mathcal{D} \simeq \mathcal{E}$ implies $\mathcal{C} \simeq \mathcal{E}$.

Proof. We will be brief.

- We have that $\mathrm{id}_{\mathcal{C}}$ provides the needed equivalence.
- If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence with $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$, then G witnesses $\mathcal{D} \simeq \mathcal{C}$.
- Fix $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ witness $C \simeq D$, and fix $F' : \mathcal{D} \to \mathcal{E}$ and $G' : \mathcal{E} \to \mathcal{D}$ witness $D \simeq E$.

  In particular, we are promised natural isomorphisms $\varphi : G \simeq \mathrm{id}_{\mathcal{C}}$ and $\psi : FG \simeq \mathrm{id}_{\mathcal{D}}$ and $\varphi' : G'F' \simeq \mathrm{id}_{\mathcal{D}}$ and $\psi' : F'G' \simeq \mathrm{id}_{\mathcal{E}}$. We would like $GG'F'F \simeq \mathrm{id}_{\mathcal{C}}$, and then $F'FGG' \simeq \mathrm{id}_{\mathcal{E}}$ will follow in a very similar way.

  Well, for an object $c \in \mathcal{C}$, we define our natural transformation $\eta_\bullet$ as having component

  $$\eta_c := \varphi_c \circ G\varphi'_{Fc},$$

  which takes $GG'F'Fc$ to GFc to c. We show naturality directly. Fix some morphism $f : x \to y$ in $\mathcal{C}$. We need the following diagram to commute.

  $$\begin{tikzcd} GG'F'Fx \arrow[r, "\eta_x"] \arrow[d, "GG'F'Ff"'] & x \arrow[d, "f"] \\ GG'F'Fy \arrow[r, "\eta_y"'] & y \end{tikzcd}$$

  To see that this commutes, here is an expanded diagram.

  $$\begin{tikzcd} GG'F'Fx \arrow[r, "G\varphi'_{Fx}"] \arrow[d, "GG'F'Ff"'] \arrow[rr, bend left, "\eta_x"] & GFx \arrow[r, "\varphi_x"] \arrow[d, "GFf"] & x \arrow[d, "f"] \\ GG'F'Fx \arrow[r, "G\varphi'_{Fy}"'] \arrow[rr, bend right, "\eta_y"'] & GFy \arrow[r, "\varphi_y"'] & y \end{tikzcd}$$

By definition of $\eta_\bullet$, it now suffices to show that the left and right squares commute. The right square commutes by naturality of φ_x. To see that the left square commutes, we note that it is what we get after applying G to the naturality square for φ' on the morphism $GFf : GFx \to GFy$.

Lastly, to see that η is a natural isomorphism, we note that each component $\eta_c = \varphi_c \circ G\varphi'_{Fc}$ is the composite of isomorphisms, where we are using that φ and φ' are natural isomorphisms and that functors preserve isomorphisms. ■

This is nice because oftentimes showing that two categories are equivalent is easier by showing a chain of equivalences instead of doing it directly. For example, in our proof that $\mathrm{Mat}_k \simeq \mathrm{fdVec}_k$, we will instead show that both of these categories are equivalent to $\mathrm{fdVec}_k^{\mathrm{basis}}$ of vector spaces with given basis.

2.5.2 Lazy Equivalence

We want to provide a tool for constructing equivalences without having to actually write down a natural transformation. By way of analogy, when showing an "isomorphism of sets" we often show that a given map is both injective and surjective. We will do something similar.

Definition 2.67 (Adjectives for Functors)**.** Fix categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $F : \mathcal{C} \to \mathcal{D}$. We consider the map $F^\circ : F : \mathrm{Mor}_\mathcal{C}(x, y) \to \mathrm{Mor}_\mathcal{C}(Fx, Fy)$. Then

- F is *full* if and only if F° is surjective.
- F is *faithful* if and only if F° is injective.
- F is *fully faithful* if and only if F is full and faithful.
- F is *essentially surjective* on objects if and only if each $d \in \mathcal{D}$ has some $c \in \mathcal{C}$ such that $Fc \cong d$ in $\mathcal{D}$.
- F is an *embedding* if and only if F is faithful and injective on objects.
- F is a *full embedding* if and only if F is an embedding and full.

Remark 2.68. Technically we might want to require that $\mathcal{C}$ and $\mathcal{D}$ be locally small, but there are ways of stating "surjective" and "injective" to note require the underlying domain and codomain to be sets.

Remark 2.69. Being "essentially surjective" will give problems with the axiom of choice later in life because we are not requiring any notion of uniqueness.

We note that a functor being "full" or "faithful" are both local conditions on particular sets of morphisms. For example, if a functor doesn't even hit an object which is outside the image of F, then we can't touch those morphism sets.

Example 2.70. Full and faithful does not imply injective on objects. For example, consider the natural functor F from the left category to the right category, which causes full-on collisions but not locally on the morphism sets.

$$\begin{tikzcd} a_1 \arrow[d] & a_2 \arrow[d] \arrow[r, Rightarrow, "F"] & a \arrow[d] \\ b_1 & b_2 & b \end{tikzcd}$$

Namely, the maps $\mathrm{Mor}_\mathcal{C}(a_\bullet, b_\bullet) \to \mathrm{Mor}_\mathcal{C}(a, b)$.

Let's finish class by proving something.

Proposition 2.71. The following are closed under composition.

- Full functors.
- Faithful functors.
- Essentially surjective functors.

Proof. We will be very brief.

- Read the proof of the below and replace all instances of the word "surjective" with "injective."
- Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are faithful functors. Then fix $x, y \in \mathcal{C}$, and we know that the induced maps

$$F^\circ : \operatorname{Mor}_{\mathcal{C}}(x, y) \to \operatorname{Mor}_{\mathcal{D}}(Fx, Fy) \qquad \text{and} \qquad G^\circ : \operatorname{Mor}_{\mathcal{D}}(Fx, Fy) \to \operatorname{Mor}_{\mathcal{D}}(GFx, GFy)$$

  are both injective, so their composite is injective. To be explicit, if f and g have $(GF)f = (GF)g$, then $G(Ff) = G(Fg)$, so $Ff = Fg$ by injectivity of G°, so $f = g$ by
- Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are essentially surjective functors. Well, fix any $e \in \mathcal{E}$, and we are promised an object $d \in \mathcal{D}$ such that $Gd \cong e$. But now we are promised an object $c \in \mathcal{C}$ such that $Fc \cong d$, so $GFc \cong Fd \cong e$, which shows that GF is essentially surjective. ■

2.6 February 11

2.6.1 A Better Equivalence

Today we will be talking about the following theorem for our discussion.

Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are true.

(a) If F is an equivalence, then F is fully faithful and essentially surjective.

(b) Assuming a strong form of the axiom of choice, the converse holds.

Remark 2.73. The strong form of the Axiom of choice is for, not sets, but classes/categories depending on how we choose to construct our categories.

Proof of (a) in Theorem 2.72. We will want some lemmas.

Lemma 2.74. Fix a category $\mathcal{C}$. Further, fix a morphism $f : c \to d$ and isomorphisms $\varphi : c \cong c'$ and $\psi : d \cong d'$. Then there is a unique morphism $f' : c' \to d'$ such that one (or equivalently, all) of the following four squares commute.

$$\begin{tikzcd}
c' \arrow[r, shift left] \arrow[d, dashed, "f'"'] & c \arrow[l, shift left, "\varphi"] \arrow[d, "f"] \\
d' \arrow[r, shift left] & d \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

Here, the four squares are achieved by changing the direction of φ and ψ.

Proof. This is on the homework. ■

We now return to the proof of the theorem. In the easier direction, suppose that F is an equivalence with its inverse equivalence $G : \mathcal{D} \to \mathcal{C}$, witnessed by natural isomorphisms $\eta_\bullet : \mathrm{id}_\mathcal{C} \Rightarrow GF$ and $\varepsilon : GF \Rightarrow \mathrm{id}_\mathcal{D}$. We have the following checks.

- We show that F is essentially surjective. Indeed, for any object $d \in \mathcal{D}$, we set $c := Gd$. Then we see $FGd \cong d$ is witnessed by the component isomorphism ε_d.

- We show that F is faithful, for which we have to use Lemma 2.74. Indeed, suppose that we have morphisms $f, g : c \to d$ such that $Ff = Fg$. Then in fact $GFf = Gfg$, so the following diagrams will commute.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "GFf=GFg"'] & GFd \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "g"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "GFf=GFg"'] & GFd \end{tikzcd}$$

  It follows from Lemma 2.74 that there f and g are uniquely determined, so $f = g$.

We quickly remark that, by symmetry, G is also faithful.

- We show that F is full, which will use the lemma as well as the fact that G is faithful (!). Well, suppose that we have some morphism $g : Fc \to Fd$. Passing through to G, we get a morphism $Gg : GFg \to GFg$, so by Lemma 2.74, there is a unique morphism $f : c \to d$ so that the following diagram commutes.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, dashed, "f"'] & GFc \arrow[d, "Gg"] \\ d \arrow[r, "\eta_d"'] & GFd \end{tikzcd}$$

  Now, both GFf and Gg make the following diagram commute.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, dashed, "{Gg,GFf}"] \\ d \arrow[r, "\eta_d"'] & GFd \end{tikzcd}$$

  Thus, by Lemma 2.74, we see $GFf = Gg$, so $Ff = g$ by the faithfulness of G. This finishes. ■

Proof of (b) in Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a fully faithful and essentially surjective functor. We need to construct a $G : \mathcal{D} \to \mathcal{C}$ with some natural isomorphisms. We do this by hand.

- For each $d \in \mathcal{D}$, we callously choose[1] Gd to be any $c \in \mathcal{C}$ together with an isomorphism $\varepsilon_d : GFd \to d$. Indeed, such a d with isomorphism ε_d exists because F is essentially surjective.

- For each $f : d \to d'$ in $\mathcal{D}$, we use Lemma 2.74 to choose h to be the unique morphism making the following diagram commute.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[d, "f"'] & FGd \arrow[d, dashed, "h"] \\ d' \arrow[r, "\varepsilon_{d'}"'] & FGd' \end{tikzcd}$$

  But because F is fully faithful, there will be a unique morphism which we call Gf such that $F(Gf) = h$.

We would like to check that G is in fact our inverse equivalence. However, we don't even know if G is a functor yet.

[1] Note we are using some fuzzy form of the axiom of choice here. We will not say more about this.

- Fix $d \in \mathcal{D}$ and we compute $G(\mathrm{id}_d)$. We run through the definition. Well, we note that id_{FGd} makes the following diagram commute, so it will be the morphism generated by Lemma 2.74.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[d, "\mathrm{id}_d"'] & FGd \arrow[d, dashed, "\mathrm{id}_{FGd}"] \\ d \arrow[r, "\varepsilon_{d'}"'] & FGd' \end{tikzcd}$$

  But now we see that $F(\mathrm{id}_{Gd}) = \mathrm{id}_{FGd}$, so id_{Gd} must be the corresponding morphism promised by the fullness and faithfulness of F. In particular, by definition, $G(\mathrm{id}_d) = \mathrm{id}_{Gd}$.

- Suppose we have $f : d \to d'$ and $g : d' \to d''$. We want to show that $G(gf) = Gg \circ Gf$. For this, we have the following very big diagram.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"'] \arrow[d, "FGf"] \arrow[dd, bend right=60, "F(Gf \circ Gf)"'] & d \arrow[d, "g"] \arrow[dd, bend left=60, "gf"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] \arrow[d, "FGg"] & d' \arrow[d, "f"] \\ FGd'' \arrow[r, "\varepsilon_{d''}"'] & d'' \end{tikzcd}$$

  This diagram does commute, from which we see that the left arrow can be either $F(Gg \circ Gf)$ (by funtoriality of F) or $F(G(gf))$. So by Lemma 2.74, we have $F(Gg \circ Gf) = F(G(gf))$, so faithfulness of F implies $Gg \circ Gf = G(gf)$.

Now we construct our natural isomorphisms.

- By construction of the εs, the following diagram commutes.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FGf"'] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$

- For the other direction, we note that if $Fx \cong Fy$ in $\mathcal{D}$, then $x \cong y$, which we will prove on the homework.[2] In particular, to create an isomorphism $\eta_c : c \to GFc$, it suffices to create an isomorphism $Fc \to FGFc$, for which we use $F\eta_c := \varepsilon_{Fc}^{-1}$. For naturality, we suppose we have a morphism $f : c \to c'$, and we note that the following diagram commutes.

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[d, "Ff"'] & FGFc \arrow[r, "\varepsilon_{Fc}"] \arrow[d, "FGFf"] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "F\eta_{c'}"'] & FGFc' \arrow[r, "\varepsilon_{Fc'}"'] & Fc' \end{tikzcd}$$

  Indeed, the outer rectangle commutes by definition of the $\eta_\bullet$s, and the right square commutes by naturality of the $\varepsilon_\bullet$s. Then this forces the left square to commute by an argument by noting

$$\varepsilon_{Fc'} \circ F\eta_{c'} \circ Ff = \varepsilon_{Fc'} \circ FGFf \circ F\eta_c$$

  by the commutativity of the outer diagram, so we get the commutativity by inverting along $\varepsilon_{Fc'}$. ■

2.7 February 14

Here we go.

[2] Yes, I know.

2.7.1 Using Our Equivalence

Last time we proved the following theorem.

Theorem 2.72. Fix $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are true.

(a) If F is an equivalence, then F is fully faithful and essentially surjective.

(b) Assuming a strong form of the axiom of choice, the converse holds.

Let's use this for fun and profit.

Corollary 2.75 (Math 110)**.** The categories Mat_k and fdVec_k are equivalent.

Proof. Fix $\mathcal{C} := \mathrm{fdVec}_k^{\mathrm{basis}}$ to be the category consisting of objects which are ordered pairs $(V, \mathcal{B})$ of vector space equipped with a given ordered basis and morphisms which are linear transformations. I will call these based vector spaces because I can.

Observe that we have a functor $\mathcal{C} \to \mathrm{Mat}_k$ by sending the based vector space $(V, \mathcal{B})$ to $\dim V$ and the linear transformation $T : (V, \mathcal{B}) \to (V', \mathcal{B}')$ to the corresponding matrix representation. We run the following checks.

- The functor F is fully faithful because (based) linear transformations $(V, \mathcal{B}) \to (V', \mathcal{B}')$ are in bijective correspondence with matrices in $k^{\dim V' \times \dim V}$, which is exactly $\mathrm{Mor}_{\mathrm{Mat}_k}$
- This is essentially surjective because it is surjective: the vector space k^n goes to $n \in \mathrm{Mat}_k$.

Thus, F is an equivalence.

To continue, we use the forgetful functor $U : \mathcal{C} \to \mathrm{fdVec}_k$ by simply forgetting the basis. This is fully faithful because look at it, and it is essentially surjective because it is actually surjective. Thus, U witnesses $\mathcal{C} \simeq \mathrm{fdVec}_k$. Applying transitivity, we see

$$\mathrm{Mat}_k \simeq \mathcal{C} \simeq \mathrm{fdVec}_k,$$

which finishes. ■

We have the following definition.

Definition 2.76 (Essential image)**.** The *essential image* of a functor $F : \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ consisting of objects $d \in \mathcal{D}$ such that $d \cong Fc$ for some $c \in \mathcal{C}$.

We are saying "full subcategory" to just throw in all the morphisms, so we don't have to worry about potential composition problems in $\mathcal{D}$.

Corollary 2.77. A fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of $\mathcal{C}$ onto the essential image of F.

Proof. Apply Theorem 2.72, where being essentially surjective follows from the definition of the essential image. ■

2.7.2 Motivating Diagram Chasing

We're going to be talking about diagram-chasing for a little while. This is the technique by which we extract large amounts of information from a commutative diagram. Namely, we will get to formally define what a commutative diagram is and so on. For this, we will want to do a little graph theory.

Definition 2.78 (Path)**.** Fix a category $\mathcal{C}$. Then a *path* in $\mathcal{C}$ is finite sequence of the form

$$(A_1, f_1, A_2, f_2, \ldots, A_n, f_n, A_{n+1}),$$

where $A_1, \ldots, A_{n+1} \in \operatorname{Ob} \mathcal{C}$ and $f_k \in \operatorname{Mor}(A_k, A_{k+1})$ for each k.

Remark 2.79. Equivalently, we could encode this path by the sequence of morphism $f_1, \ldots, f_n$ such that $\operatorname{cod} f_k = \operatorname{dom} f_{k+1}$.

Let's see an example of the power of abstracting diagrams.

Definition 2.80 (Monoid)**.** A *monoid in the category* Set is a set M with morphisms $\mu : M \times M \to M$ and $\eta : \{*\} \to M$ such that the following diagrams commute.

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mu \times \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times \mu"'] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mathrm{id}_M \times \eta"] \arrow[dr, "\mathrm{id}_M"'] & M \times M \arrow[d, "\mu"] & M \arrow[l, "\eta \times \mathrm{id}_M"'] \arrow[dl, "\mathrm{id}_M"] \\ & M & \end{tikzcd}$$

Remark 2.81. Our monoid is made by the binary operation $\cdot_\mu : (a, b) \mapsto \mu(a, b)$ and an identity element $e := \eta(*)$. The left-hand diagram gives associativity in our "monoid" where μ is our binary operation: if $a, b, c \in M$, then we have

$$(a \cdot_\mu b) \cdot_\mu c = a \cdot_\mu (b \cdot_\mu c).$$

The right-hand diagram promises us an identity element $e := \eta(*)$: if $m \in M$, then

$$m \cdot_\mu e = m = e \cdot_\mu m.$$

Remark 2.82. It is not technically necessary for us to use sets M, but if we don't, then we need a good notion of product and one-element set. For example, Top can work instead of Set if we want to keep track of topologies.

Example 2.83. A unital ring R is a monoid in the category of Ab (where our products are tensor products and one-element set is $\mathbb{Z}$). Namely, we have morphisms $\mu : R \otimes R \to R$ and $\eta : \mathbb{Z} \to R$ with the following commutative diagrams.

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu \times \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \times \mu"'] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\mathrm{id}_R \times \eta"] \arrow[dr, "\mathrm{id}_R"'] & R \otimes R \arrow[d, "\mu"] & R \arrow[l, "\eta \times \mathrm{id}_R"'] \arrow[dl, "\mathrm{id}_R"] \\ & R & \end{tikzcd}$$

The left-hand diagram shows that multiplication is an associative bilinear map, and the right-hand diagram promises an identity. We will not be more explicit.

2.7.3 Commutative Diagrams

We should probably define a diagram now.

Definition 2.84 (Diagram)**.** Fix $\mathcal{J}$ and $\mathcal{C}$ categories. A *diagram* in $\mathcal{C}$ indexed by $\mathcal{J}$ is a functor $F : \mathcal{J} \to \mathcal{C}$.

Notably, we are not requiring this functor to be an embedding.

Example 2.85. A diagram of the shape $(0 \to 1)^2$ is a commutative square. To be explicit, our index category is as follows.

$$\begin{tikzcd} (0,0) \arrow[r, "\mathrm{id}\times f"] \arrow[d, "f\times\mathrm{id}"'] \arrow[dr, "f\times f" description] & (0,1) \arrow[d, "f\times\mathrm{id}"] \\ (1,0) \arrow[r, "\mathrm{id}\times f"'] & (1,1) \end{tikzcd}$$

Namely, if we send this to $\mathcal{C}$, we some diagram as follows.

$$\begin{tikzcd} c \arrow[r] \arrow[d] & c' \arrow[d] \\ d \arrow[r] & d' \end{tikzcd}$$

Because we embedded by a functor, we know that $c \to c' \to d'$ is the same as $c \to d \to d'$.

Example 2.86. We can think about triangles as images of squares which collapse a bit, as follows.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, equal] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Alternatively, we could just set the index category to be $\bullet \to \bullet \to \bullet$.

Definition 2.87 (Commutes)**.** A diagram $F : \mathcal{J} \to \mathcal{C}$ *commutes* if and only if, given $k, k' : i \to j$ in $\mathcal{J}$ has $Fk = Fk'$.

The point of this definition is that we don't want composition to matter too much in our index category. For example, if we have morphisms $0 \to 1$ and $1 \to 2$ in $\mathcal{J}$ which go to $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$, we want to be sure we have $0 \to 2$ goes to fg without having to look too hard at $\mathcal{J}$.

Example 2.88. Any diagram over a preorder will commute for free because any two i, j has at most one element in $\operatorname{Mor}(i, j)$.

It's a math class, so we should probably prove something today.

Proposition 2.89. Functors preserve commutative diagrams.

Proof. Fix $\mathcal{J}, \mathcal{C}, \mathcal{D}$ all diagrams with a commutative diagram $K : \mathcal{J} \to \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$. Indeed, if $k, k' : i \to j$ in $\mathcal{J}$, then $Kk = Kk'$, so $JKk = JKk'$, so $JK : \mathcal{J} \to \mathcal{D}$ is indeed a commutative diagram. ■

And here is a nice result on commutative diagrams.

Lemma 2.90. Fix $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are paths in $\mathcal{C}$. Then if we have an equality of composites

$$f_k f_{k-1} \cdots f_{i+1} f_i = g_n g_{n-1} \cdots g_2 g_1,$$

then

$$f_m \cdots f_1 = f_m \cdots f_k g_n \cdots g_1 f_{i-1} \cdots f_1.$$

Here is the image for the above lemma: we are allowed to take either path from A to B, given that the f-parts and g-parts are commuting.

$$\begin{tikzcd}
 & & & \bullet \arrow[drr, "g_3"] & & \\
A \arrow[r, "f_1"] & \bullet \arrow[r, "f_2"] & \bullet \arrow[d, "f_3"] & \bullet \arrow[u, "g_2"] & \bullet \arrow[dr, "g_5"] & \bullet \arrow[l, "g_4"] \\
 & & \bullet \arrow[ur, "g_1"] \arrow[dl, "f_4"'] & & & \bullet \arrow[dl, "g_6"'] \\
 & \bullet \arrow[r, "f_5"] & \bullet \arrow[r, "f_6"] & \bullet \arrow[dr, "f_7"] & \bullet \arrow[r, "f_8"] & B \\
 & & & & \bullet \arrow[u] &
\end{tikzcd}$$

Proof. Look at it. Namely, we have composition is well-defined, so take the given equality and add the required compositions on either end. ■

2.8 February 16

Here we go.

2.8.1 House-Keeping

Let's start with the attendance question from last class because it was a little tricky.

Exercise 2.91. All nonempty indiscrete categories are equivalent.

Proof. The first part of this problem is remembering that indiscrete categories are ones that have all morphism sets are singletons. The second part of the problem is recognizing the following lemma.

Lemma 2.92. Fix $\mathcal{C}$ be a nonempty indiscrete category. Then $\mathcal{C}$ is equivalent to $\mathrm{B}e$, where e is the single-element group.

Proof. We use the functor $F : \mathcal{C} \to \mathrm{B}e$ sending all objects to $*$ and all morphisms to id_*. It is surjective on objects because there is only one object to hit, and $\mathcal{C}$ is nonempty. Further, F is fully faithful because, for any $c, c' \in \mathcal{C}$, the induced map

$$F : \mathrm{Mor}(c, c') \to \mathrm{Mor}(*, *)$$

is a bijection because both of these are singletons. It follows from Theorem 2.72 that F is an equivalence. ■

So transitivity promises that all indiscrete categories are equivalent, finishing the proof. ■

Remark 2.93. In fact, one can use essentially the same proof to show that any functor between indiscrete categories is an equivalence. In particular, the (weak) inverse to the equivalence generated by Lemma 2.92 is not canonical.

2.8.2 Diagram-Chasing Philosophy

We recall that we proved Lemma 2.90 last time, which philosophically means that we should not try to show equalities of morphisms where there is some overlap between the morphisms. For example, to compare all paths in the rectangle

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

above, we merely have to check the commutativity of the squares.

We would like to have some tools to prove that diagrams commute.

Remark 2.94. We remark that the following force commutative diagrams immediately.

- Any diagram indexed by a preorder commutes.
- Any diagram in a preorder commutes because any two morphisms between objects must be equal, so we get the commuting in the image of the index category.

2.8.3 Initial and Final Objects

Let's keep building up our theory.

Definition 2.95 (Initial, final)**.** Fix a category $\mathcal{C}$.

- An object $i \in \mathcal{C}$ is *initial* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(i, c)$.
- The dual notion is that an object $t \in \mathcal{C}$ is *final* or *terminal* if and only if, for every $c \in \mathcal{C}$, there is a unique morphism in $\operatorname{Mor}(c, t)$.

Remark 2.96. It is true that initial and final objects are unique up to unique isomorphism. We will not show this here because it might appear on the homework.

And here are many, many examples.

Example 2.97. We work in Set.

- We have $\varnothing$ is initial. Namely, there is only one function $\varnothing \to S$ for any set S by taking all elements of $\varnothing$ to whatever one's heart desires in S, and there is only one way to do this because any two such functions always have the same outputs.
- The singleton set $\{*\}$ is final. Indeed, any set S has a unique function $S \to \{*\}$ by sending all elements of S to $*$.

Example 2.98. In Top, the initial object is $\varnothing$ and the final object is $\{*\}$.

Example 2.99. We work in Set_*, which are ordered pairs (S, s) where $s \in S$. Morphisms $(S, s) \to (T, t)$ are functions $f : S \to T$ such that $f(s) = t$. Singleton sets $\{*\}$ is both initial and final. It's final for the same reason as in Set, and it is initial because any pointed set (S, s) has the unique morphism $* \mapsto s$.

Example 2.100. We work in Ab or Grp. Then the trivial group 0 is the initial and final object by sending identities to identities.

Non-Example 2.101. The object $\mathbb{Z}/2\mathbb{Z}$ is not initial in Ring: there is no morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. Funnily enough, there is at most one morphism from $\mathbb{Z}/2\mathbb{Z}$ to anywhere.

Non-Example 2.102. We work in Ring.

- The object $\mathbb{Z}$ is initial by sending $1 \mapsto 1_R$ (which is forced) for any ring R, and this uniquely determines the rest of the morphism.
- The zero ring 0 is final in Ring because there is only one function $R \to 0$ for any ring R, and it is in fact a ring homomorphism.

Example 2.103. The category Field has no initial or final object. There is no final object because all morphisms are injections, and we cannot embed all fields into one large field. There is no initial object because there are no morphisms between fields of different characteristic. (One can fix this problem by considering the fields of characteristic p, where $\mathbb{F}_p$ is the initial object.)

Quote 2.104. I hate this category, and you should too.

Example 2.105. Let $\mathcal{P}$ be a preorder category.

- We claim that global minimums are equivalent to initial objects. To be explicit, there is surely at most one morphism between any two elements, so the object $m \in \mathcal{P}$ is an initial object if and only if there is a morphism $m \to x$ for each $x \in \mathcal{P}$ if and only if $m \le x$ for each x if and only if m is a global minimum.
- Dually, global maximums are equivalent to final objects.

These new definitions give us a quick criterion for diagram-chasing.

Lemma 2.106. Fix $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ be "parallel" paths in $\mathcal{C}$; i.e., $s := \operatorname{dom} f_1 = \dim g_1$ and $t := \operatorname{cod} f_n = \operatorname{cod} g_m$. If s is initial or t is final, then

$$f_n \cdots f_1 = g_m \cdots g_1.$$

Proof. We have two cases.

- Take s initial. Then $f_n \cdots f_1$ and $g_m \cdots g_1$ are both maps $s \to t$, of which there is a unique map by s being initial, so these are equal.
- Take t final. Then repeat the above sentence using the fact t is final instead of s being initial. ■

2.8.4 Concrete Categories

We have the following definition.

Definition 2.107 (Concrete)**.** A category $\mathcal{C}$ is *concrete* if and only if there is a fully faithful functor $U : \mathcal{C} \to \mathrm{Set}$. We call U the *forgetful functor*.

For example, this asserts that two morphisms $f, g : x \to y$ in $\mathcal{C}$ are equal if and only if their "restrictions" down in Set are equal, for which we can do an element-wise check on elements of sets.

Lemma 2.108. Fix $U : \mathcal{C} \to \mathcal{D}$ be faithful functors. A diagram in $\mathcal{C}$ commutes if and only if its image through U commutes.

Proof. Fix J our index category with the diagram $K : J \to \mathcal{C}$. We already know that K commuting implies that UK commutes by Proposition 2.89.

In the other direction, suppose that UK commutes. Then pick up $k, k' : i \to j$ in J so that $UKk = UKk'$, but then U being faithful forces

$$Kk = Kk',$$

which is exactly what we need to commute. ■

And here is why we care

Corollary 2.109. Commutativity of a diagram in a concrete category can be checked on "elements."

Proof. Essentially we use the forgetful functor in Lemma 2.108. To be explicit, checking on "elements" is doing the diagram-chase in Set, which we can then pull back to the original concrete category through the forgetful functor via Lemma 2.108. ■

In other words, we can diagram-chase by working everything in set.

2.8.5 Commutative Rectangles

We have the following warning.

Warning 2.110. Consider the following rectangle.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

We know that the squares commuting implies that the rectangle commutes. The converse is not true.

Example 2.111. We work in Ab. The outer rectangle of the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

will commute, but the inner squares do not. (The zero map is not the identity map.)

We can salvage Warning 2.110 as follows.

Lemma 2.112. Fix a rectangle as follows.

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[d, "f"'] & b \arrow[r, "k"] \arrow[d, "g"] & c \arrow[d, "h"] \\ a' \arrow[r, "j"'] & b' \arrow[r, "m"'] & c' \end{tikzcd}$$

Suppose the outer rectangle commutes. Then the diagram commutes if

- the right square commutes and m is monic, or
- the left square commutes and e is epic.

Proof. We have separate cases.

- Suppose the right square commutes and m is monic. The right square commutes, so $hk = mg$. Similarly, the outer rectangle commutes, so $hke = mjf$. But then

  $$mge = hke = mjf,$$

  so $ge = jf$ because m is monic. This shows the left square commutes, so we are done.
- This holds by running the proof of the above in the opposite category, where the main point is that the left and right squares flip, and m being monic turns into e being epic. ■

2.9 February 18

Apparently I have to take notes today.

2.9.1 Motivating Horizontal Composition

A while ago we discussed vertical composition of natural transformations: if $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, then we can define a natural transformation $(\beta\alpha) : F \Rightarrow H$ by $(\beta\alpha)_c := \beta_c\alpha_c$. To quickly review, the naturality condition can be checked by drawing the following commutative diagram.

$$\begin{tikzcd} Fc \arrow[r, "Ff"] \arrow[d, "\alpha_c"'] \arrow[dd, bend right=60, "\beta_c\alpha_c"'] & Fd \arrow[d, "\alpha_d"] \arrow[dd, bend left=60, "\beta_d\alpha_d"] \\ Gc \arrow[r, "Gf"] \arrow[d, "\beta_c"'] & Gd \arrow[d, "\beta_d"] \\ Hc \arrow[r, "Hf"'] & Hd \end{tikzcd}$$

We are going to discuss horizontal composition because Eckmann–Hamilton would like to know your location. The set-up is as follows: suppose that we have functors $F, G : \mathcal{C} \to \mathcal{D}$ with $\alpha : F \Rightarrow G$ and $F', G' : \mathcal{D} \to \mathcal{E}$ with $\beta : F' \Rightarrow G'$. Here is the diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

Our goal is to define $(\beta * \alpha) : F'F \Rightarrow G'G$.

2.9.2 Whiskering

To define this horizontal composition, we define "whiskering." There are two kinds of whiskering.

- Here is the diagram for left whiskering.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathcal{E} \end{tikzcd}$$

  We would like to define $\alpha H : FH \Rightarrow GH$. Well, we simply define $(\alpha H)_c := \alpha_{Hc}$, which defines a natural transformation by noting the following diagram commutes for a morphism $f : c \to d$ in $\mathcal{C}$ by the naturality of α on $Hf : Hc \to Hd$. This gives the following commutative naturality square.

$$\begin{tikzcd} FHc \arrow[r, "FHf"] \arrow[d, "\alpha_{Hc}"'] & FHd \arrow[d, "\alpha_{Hd}"] \\ GHc \arrow[r, "GHf"'] & GHd \end{tikzcd}$$

- There is also a notion of right whiskering. Here is the diagram.

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{E} \arrow[r, "H'"] & \mathcal{X} \end{tikzcd}$$

  We define $H'\alpha : H'F \Rightarrow H'G$ by $(H'\alpha)_d := H'\alpha_d$. This is a natural transformation because we can pick up some morphism $f : c \to d$ in $\mathcal{D}$ and apply H' to the naturality diagram for α, giving the following commutative naturality square.

$$\begin{tikzcd} H'Fc \arrow[r, "H'Ff"] \arrow[d, "H'\alpha_c"'] & H'Fd \arrow[d, "H'\alpha_d"] \\ H'Gc \arrow[r, "H'Gf"'] & H'Gd \end{tikzcd}$$

2.9.3 Horizontal Composition

From whiskering, there are two ways to define horizontal composition. To review, here is our diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

- We start by whiskering on the left and then whisker on the right. So we start by noting we have $\beta F : F'F \Rightarrow G'F$ induced by whiskering the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

  Then we have $G'\alpha : G'F \Rightarrow G'G$ by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend right, "G'"'] & \mathcal{E} \end{tikzcd}$$

  In total, we see that $(G'\alpha)(\beta F) : F'F \Rightarrow G'G$. Note this is a natural transformation by vertical composition!

- We start by whiskering on the right and then whisker on the left. So we start by noting we have $F'\alpha : F'F \Rightarrow F'G$ by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] & \mathcal{E} \end{tikzcd}$$

  Then we have $\beta G : F'G \Rightarrow G'G$ induced by whiskering along the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "G"'] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd}$$

  In total, we see that $(\beta G)(F'\alpha) : F'F \Rightarrow G'G$, which is a natural transformation by vertical composition.

We now claim that the two horizontal compositions that we just defined are the same. We could just track an element through, or we could simply note that this is the naturality of β applied to the morphism $\alpha_c : Fc \to Gc$. Indeed, we are showing that the following diagram commutes.

$$\begin{tikzcd} F'F \arrow[r, Rightarrow, "F'\alpha"] \arrow[d, Rightarrow, "\beta F"'] & F'G \arrow[d, Rightarrow, "\beta G"] \\ G'F \arrow[r, Rightarrow, "G'\alpha"'] & G'G \end{tikzcd}$$

Now, applying naturality of β to $\alpha_c : Fc \to Gc$, we see that the following diagram commutes.

$$\begin{tikzcd} F'Fc \arrow[r, "F'\alpha_c"] \arrow[d, "\beta_{Fc}"'] & F'Gc \arrow[d, "\beta_{Gc}"] \\ G'Fc \arrow[r, "G'\alpha_c"'] & G'Gc \end{tikzcd}$$

But this diagram is exactly what we wanted, so we are done.

2.9.4 Horizontal and Vertical Composition

For our last note, we show that horizontal composition of vertical compositions is the same as vertical composition of horizontal compositions. Here is our diagram.

$$\begin{tikzcd}[row sep=large] \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \arrow[r, bend left=60, "F'"] \arrow[r, "G'"] \arrow[r, bend right=60, "H'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{E} \end{tikzcd}$$

We claim that

$$(\beta'\alpha') * (\beta\alpha) \stackrel{?}{=} (\beta' * \beta)(\alpha'\alpha).$$

The point is to draw the following giant commuting square. The "morphisms" are induced by various kinds of whiskering in the diagram, and they all commute by uniqueness of horizontal composition.

$$\begin{tikzcd} F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\ G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\ H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H \end{tikzcd}$$

We now follow two paths. Consider the red path below.

$$\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\
G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\
H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H
\end{tikzcd}$$

By definition of horizontal composition, this is $(\beta' * \beta)(\alpha' * \alpha)$. Now consider the different red path below.

$$\begin{tikzcd}
F'F \arrow[r, "F'\alpha"] \arrow[d, "\alpha' F"'] & F'G \arrow[r, "F'\beta"] \arrow[d, "\alpha' G"'] & F'H \arrow[d, "\alpha' H"] \\
G'F \arrow[r, "G'\alpha"] \arrow[d, "\beta' F"'] & G'G \arrow[r, "G'\beta"] \arrow[d, "\beta' G"'] & G'H \arrow[d, "\beta' H"] \\
H'F \arrow[r, "H'\alpha"'] & H'G \arrow[r, "H'\beta"'] & H'H
\end{tikzcd}$$

The top leg is $\alpha' * \alpha$, and the right leg is $\beta' * \beta$, so this total red path comes out to $(\beta' * \beta)(\alpha' * \alpha)$.

So comparing our two red paths, we see that

$$(\beta'\alpha') * (\beta\alpha) = (\beta' * \beta)(\alpha'\alpha),$$

which is what we wanted.