214: Differential Topology

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CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

BUILDING MANIFOLDS

So the man gave him the bricks, and he built his house with them.

—Joseph Jacobs, "The Story of the Three Little Pigs" [Jac90]

1.1 January 16

Let's just get started.

1.1.1 Course Structure

Here are some quick notes.

- There is a bCourses page: https://bcourses.berkeley.edu/courses/1533116. For example, it has the syllabus.

- The textbook is Lee's *Introduction to Smooth Manifolds* [Lee13]. We will read most of it.

- Our instructor is Professor Eric Chen, whose email can be reached at ecc@berkeley.edu. Office hours are after class in Evans 707.

- There is a GSI, who is Tahsia Saffat, whose email is tahsin_saffat@math.berkeley.edu. He will have some office hours and grade some homeworks.

- Homework will in general be due at 11:59PM on Thursdays via Gradescope.

- There will be an in-class midterm and a final.

- Grading is 30% homework, 30% midterm, and 40% final.

- This is a math class, not so geared towards applied subjects.

- In particular, we will assume a fair amount of topology, for which we use [Elb22] as a reference.

Let's also give a couple of notes on the course content. This course is on differential topology. The topology of interest will come from manifolds, and the differential part comes from some smoothness properties.

In some sense, our goal is to "do calculus" (e.g., differentiation, integration, vector fields, etc.) on spaces which look locally like some Euclidean space, such as a sphere. We also want to understand (smooth) manifolds on their own terms, such as understanding the maps between them and understanding some classical examples and constructions such as Lie groups or quotient manifolds.

1.1.2 Topology Review

Anyway let's get started. This is a class on manifolds, so perhaps we should begin by defining a manifold. These are going to form a special kind of topological space, so let's review topologies. We will freely use topological facts which we are too lazy to prove from [Elb22].

> **Definition 1.1** (topological space). A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X satisfying the following.
>
> - $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
>
> - Finite intersection: given $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
>
> - Union: for any subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have the union $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.
>
> We say that the collection $\mathcal{T}$ is the collection of *open sets* of X. We will also suppress the collection $\mathcal{T}$ from the notation as much as possible.

Here is some helpful language.

> **Definition 1.2** (open, closed, neighborhood). Fix a topological space $(X, \mathcal{T})$.
>
> - An *open subset* $U \subseteq X$ is a subset in $\mathcal{T}$.
>
> - A *closed subset* $V \subseteq X$ is one with $X \setminus V \in \mathcal{T}$.
>
> - A *neighborhood* of a point $p \in X$ is an open subset $U \subseteq X$ containing p.

> **Example 1.3.** Fix a metric space (X, d). Then there is a topology given by the metric. To be explicit, a set $U \subseteq X$ is open if and only if each $p \in U$ has some $\varepsilon > 0$ such that
>
> $$\{x \in X : d(x, p) < \varepsilon\} \subseteq U.$$
>
> See [Elb22, Example 2.13] for the details.

Sometimes it is easier to generate a topology from some subcollection.

> **Definition 1.4** (base). Fix a topological space $(X, \mathcal{T})$. A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is a *base* for $\mathcal{T}$ if and only if the following holds: for each open $U \subseteq X$ and point $p \in U$, there is some $B \in \mathcal{B}$ such that $p \in B$ and $B \subseteq U$.

> **Example 1.5.** Fix a metric space (X, d). Then the collection $\mathcal{B}$ of open balls
>
> $$B(p, \varepsilon) \coloneqq,$$
>
> over all $p \in X$ and $\varepsilon > 0$, forms a base of the topology. This is immediate from the construction of the topology in Example 1.3. In fact, one can merely take $\varepsilon \in \mathbb{Q}^+$ because $\mathbb{Q}$ is dense in $\mathbb{R}$.

With our objects of topological spaces in hand, we should discuss the maps between them.

> **Definition 1.6** (continuous). Fix topological spaces X and X'. A function $\varphi \colon X \to X'$ is *continuous* if and only if $\varphi^{-1}(U')$ is open for each open $U' \subseteq X'$.

> **Definition 1.7** (homeomorphism). Fix topological spaces X and X'. A function $\varphi \colon X \to X'$ is a *homeomorphism* if and only if φ is a bijection and both φ and φ^{-1} are continuous. We may write $X \cong X'$.

Remark 1.8. There is a continuous bijection $[0, 2\pi) \to S^1$ by $\theta \mapsto (\cos\theta, \sin\theta)$, but it is not a homeomorphism. (Here, both sets have the metric topology.) In particular, the inverse map is not continuous at 1 because the pre-image of $[0, \pi)$ is the subset $\{(x, y) \in S^1 : y > 0\} \cup \{(0, 0)\}$, which is not open in S^1 (because no $\varepsilon > 0$ has $B((0, 0), \varepsilon)$ lying in $\{(x, y) \in S^1 : y \geq 0\}$).

Exercise 1.9. Fix a nonnegative integer $n \geq 0$. Then $B(0, 1) \cong \mathbb{R}^n$.

Proof. We proceed as in [use14]. Define the functions $f \colon B(0, 1) \to \mathbb{R}^n$ and $g \colon \mathbb{R}^n \to B(0, 1)$ by

$$f(x) := \frac{x}{1 - |x|} \qquad \text{and} \qquad g(y) := \frac{y}{1 + |y|}.$$

Notably, $|g(y)| < 1$ always, so g does indeed always output to $B(0, 1)$. These functions are both continuous, which can be checked on coordinates because they are rational functions in the coordinates, and the denominators never vanish on the domains. So we will be done once we show that f and g are inverse. In one direction, we note

$$f(g(y)) = \frac{g(y)}{1 - |g(y)|} = \frac{\dfrac{y}{1 + |y|}}{1 - \left|\dfrac{y}{1 + |y|}\right|} = \frac{y}{1 + |y| - |y|} = y.$$

In the other direction, we note

$$g(f(x)) = \frac{f(x)}{1 + |f(x)|} = \frac{\dfrac{x}{1 - |x|}}{1 + \left|\dfrac{x}{1 - |x|}\right|} = \frac{x}{1 - |x| + |x|} = x,$$

as desired. $\blacksquare$

We would also like to be able to build new topologies from old ones.

Definition 1.10 (subspace). Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we form a *subspace topology* by declaring the open subsets to be

$$\{U \cap S : U \in \mathcal{T}\}.$$

Example 1.11. The metric topology on $\mathbb{R}$ and the subspace topology on $X := \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ are homeomorphic. Namely, the homeomorphism sends $x \mapsto (x, 0)$, and the inverse map is $(x, 0) \mapsto x$. Here are our continuity checks.

- The map $x \mapsto (x, 0)$ is continuous: the pre-image V of an open subset $U \subseteq X$ is open. Namely, for any $x \in V$, we see $(x, 0) \in V$, so there is $\varepsilon > 0$ such that $B((x, 0), \varepsilon) \cap X \subseteq U$, so $B(x, \varepsilon) \subseteq V$.

- The map $(x, 0) \mapsto x$ is continuous: the pre-image V of an open subset $U \subseteq \mathbb{R}$ is open. Namely, for each $(x, 0) \in V$, we see $x \in U$, so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V$, so $B((x, 0), \varepsilon) \cap X \subseteq U$.

Lastly, we will want some adjectives for our topologies.

Definition 1.12 (compact). Fix a topological space X. A subset $K \subseteq X$ is *compact* if and only if any open cover can be reduced to a finite subcover. Explicitly, any collection $\mathcal{U}$ of open sets of X such that $K \subseteq \bigcup_{U \in \mathcal{U}} U$ (this is called an *open cover*) has some finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{U \in \mathcal{U}'} U$.

Example 1.13. The interval $[0, 1] \subseteq \mathbb{R}$ is compact. See [Elb22, Example 4.4].

Definition 1.14 (Hausdorff). Fix a topological space X. Then X is *Hausdorff* if and only if any two distinct points $p_1, p_2 \in X$ have disjoint open subsets $U_1, U_2 \subseteq X$ such that $p_1 \in U_1$ and $p_2 \in U_2$.

Example 1.15. Any metric space (X, d) is Hausdorff. Namely, for distinct points $p, q \in X$, we see $d(p, q) > 0$, so set $\varepsilon := d(p, q)/2$, and we see that $p \in B(p, \varepsilon)$ and $q \in B(q, \varepsilon)$, but $B(p, \varepsilon) \cap B(q, \varepsilon) = \varnothing$. For this last claim, we note r living in the intersection would imply

$$d(p, q) \leq d(p, r) + d(r, q) < 2\varepsilon,$$

which is a contradiction to the construction of ε.

1.1.3 Topological Manifolds

For intuition, we state but not prove the following result.

Theorem 1.16 (Topological invariance of dimension). Fix open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. If there is a homeomorphism $U \cong V$, then $m = n$.

Proof. The usual proofs go through (co)homology, which we may cover later in the class. For the interested, see [Elb23, Proposition 3.50]. ∎

We will soon define topological manifolds. The main adjective we want is being "locally Euclidean."

Definition 1.17 (locally Euclidean). Fix a topological space X. Then X is *locally Euclidean of dimension n at p* if and only if there is an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$. We say that X is *locally Euclidean of dimension n* if and only if it is locally Euclidean of dimension n at each point.

Remark 1.18. One can always take $\widetilde{U}$ to be either $B(0, 1) \subseteq \mathbb{R}^n$ or even all of $\mathbb{R}^n$. Indeed, for $x \in X$, we are given an open neighborhood U of x and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. We produce open neighborhoods of x homeomorphic to $B(0, 1)$ and $\mathbb{R}^n$.

- $B(0, 1)$: there is $\varepsilon > 0$ such that $B(\varphi(x), \varepsilon) \subseteq \widehat{U}$. Then we let $U' := \varphi^{-1}(B(\varphi(x), \varepsilon))$ so that we have a chain of homeomorphisms

$$U' \overset{\varphi}{\cong} B(\varphi(x), \varepsilon) \cong B(0, \varepsilon) \cong B(0, 1),$$

  where the second homeomorphism is a translation, and the last homeomorphism is a dilation.

- $\mathbb{R}^n$: in the light of the previous point, it suffices to note that Exercise 1.9 provides a homeomorphism $B(0, 1) \cong \mathbb{R}^n$ and then post-compose with this homeomorphism.

Let's explain why we want Theorem 1.16.

Lemma 1.19. Fix a locally Euclidean space X. For each $p \in X$, there is a unique nonnegative integer n such that there exists an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$.

Proof. Suppose there are two such nonnegative integers m and n, so we get open neighborhoods $U, V \subseteq X$ and $\widetilde{U} \subseteq \mathbb{R}^m$ and $\widetilde{V} \subseteq \mathbb{R}^n$. Let $\varphi \colon U \cong \widetilde{U}$ and $\psi \colon V \cong \widetilde{V}$ be the needed homeomorphisms. Then the point is to use the intersection $U \cap V$: there is a composite isomorphism

$$\varphi(U \cap V) \cong U \cap V \cong \psi(U \cap V)$$

from an open subset in $\mathbb{R}^m$ to an open subset in $\mathbb{R}^n$. So Theorem 1.16 completes the proof. ∎

Anyway, here is our definition of a topological manifold.

> **Definition 1.20** (topological manifold)**.** An n-*dimensional topological manifold* is a topological space M with the following properties.
>
> - M is Hausdorff.
>
> - M is locally Euclidean of dimension n at each point.
>
> - M is second countable (i.e., has a countable base).
>
> We may abbreviate "n-dimensional topological manifold" to "topological n-manifold."

Let's give a few quick constructions.

> **Lemma 1.21.** For each $n \geq 0$, the space $\mathbb{R}^n$ is an n-dimensional topological manifold.

Proof. Let's be quick. Being a metric space yields Hausdorff, locally Euclidean is immediate because it's $\mathbb{R}^n$, and second-countability follows by using the base

$$\left\{ B(q, \varepsilon) : q \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}^+ \right\}.$$

This is indeed a base because $\mathbb{Q}$ is dense in $\mathbb{R}$. Explicitly, for each $p \in \mathbb{R}^n$ living in some open subset $U \subseteq \mathbb{R}^n$, begin by replacing U with a smaller open subset of the form $B(p, \varepsilon)$ where $\varepsilon > 0$; by perhaps making ε smaller, we may assume that $\varepsilon > 0$ is rational. Now, choosing coordinates $p = (x_1, \ldots, x_n)$, choose rational numbers $q_1, \ldots, q_n$ so that $|x_i - q_i| < \varepsilon/(2\sqrt{n})$ for each i. Then $q := (q_1, \ldots, q_n)$ has $d(p, q) < \varepsilon/2$ and so

$$p \in B(q, \varepsilon/2) \subseteq B(p, \varepsilon) \subseteq U,$$

so $B(q, \varepsilon/2)$ is the needed open subset in our base. ∎

The following lemma will be helpful in the sequel.

> **Lemma 1.22.** Fix a topological space M and nonnegative integer $n \geq 0$. Suppose that there is a countable open cover $\{U_i\}_{i \in \mathbb{N}}$ of M such that each i has a homeomorphism $U_i \cong \widetilde{U}_i$ where $\widetilde{U}_i \subseteq \mathbb{R}^n$ is open. Then M is locally Euclidean of dimension n at each point, and M is second countable.

Proof. For locally Euclidean, we note that each $p \in M$ lives in some U_i, so we are done. As for second countability, we note that each $\widetilde{U}_i$ is second countable as a subspace of a second countable space (see Lemma 1.21), so each U_i is second countable by moving back through the homeomorphism, and so M is second countable by taking the union of the bases of the U_i.

To make this last step more explicitly, we note that each U_i has a countable base $\mathcal{B}_i$, so we claim that $\mathcal{B} := \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ becomes a countable base of M. Certainly $\mathcal{B}$ is countable, and every set in $\mathcal{B}$ is in one of the $\mathcal{B}_i$ and hence open in M. Lastly, to check that we have a base, we note that any open $U \subseteq M$ and $p \in M$ will have $p \in U_i$ for some i, so there is some $B \in \mathcal{B}_i \subseteq \mathcal{B}$ such that $p \in B \subseteq U \cap U_i$. ∎

1.1.4 Examples and Non-Examples

Here are some non-examples to explain why we want all of these hypotheses.

> **Exercise 1.23.** Consider the space X defined as $\mathbb{R} \times \{0, 1\}$ where we identify $(x, 0) \sim (x, 1)$ whenever $x \neq 0$. (The topology on X is the quotient topology [Elb22, Definition 2.81].) This space is not Hausdorff, but it is locally Euclidean and second countable.

Proof. We run our checks.

- This space is not Hausdorff because the points $(0, 0)$ and $(0, 1)$ are "infinitely close together." Explicitly, any open neighborhoods U and V of $(0, 0)$ and $(0, 1)$, respectively, the induced topology yields some $\varepsilon > 0$ such that $B((0, 0), \varepsilon) \subseteq U$ and $B((0, 1), \varepsilon) \subseteq V$, but then $(-\varepsilon/2, 0) = (-\varepsilon/2, 1)$ is in both U and V.

- This space is locally Euclidean and second countable by Lemma 1.22. Explicitly, we note that $\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq X$ and $\mathbb{R} \cong \mathbb{R} \times \{1\} \subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ∎

> **Exercise 1.24.** Consider the space X defined as $\mathbb{R} \times \{0, 1\}$ where we identify $(x, 0) \sim (x, 1)$ whenever $x \leq 0$, again where we are using the quotient topology. Then X is Hausdorff and second countable, but it is not Euclidean of dimension 1 at $0 \in X$.

Proof. We run our checks.

- This space is Hausdorff. We check this directly by casework.

  - Suppose we have distinct points $p = (x, a)$ and $q = (y, b)$ with $x \neq y$; for example, this includes the case where we may take $a = b$ and hence includes the case when $x, y \leq 0$. Then we may set $\varepsilon := \frac{1}{2} |x - y|$ so that $B(p, \varepsilon)$ and $B(q, \varepsilon)$ are disjoint.

  - We now may assume that $x = y$; then $a \neq b$. Thus, we must have $x > 0$ or $y > 0$. As such, we may as well take $\varepsilon := \min\{|x|, |y|\}$ so that $B(p, \varepsilon)$ and $B(q, \varepsilon)$ are disjoint.

- This space is not locally Euclidean at 0. Indeed, suppose that there is open subset $U \subseteq X$ around 0 which is homeomorphic to an open subset of $\mathbb{R}$. By shifting, we may as well assume that the homeomorphism sends 0 to 0. Additionally, the same statement will be true by any open subset of U, so we may as well as assume that U is of the form $(-\varepsilon, \varepsilon) \times \{0, 1\}$ (in X). In particular, U is connected.

  But then the image $\widehat{U}$ of U in $\mathbb{R}$ is a connected open subset of $\mathbb{R}$, which must be an interval. Now, intervals have the property that deleting any point of an interval makes produces a topological space with two connected components. However, deleting 0 from U will produce three connected components: $(-\varepsilon, 0) \times \{0, 1\}$ and $(0, \varepsilon) \times \{0\}$ and $(0, \varepsilon) \times \{1\}$. So $\widehat{U}$ and U cannot actually be homeomorphic!

- This space is second countable by Lemma 1.22. Again, we note that $\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq X$ and $\mathbb{R} \cong \mathbb{R} \times \{1\} \subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ∎

> **Remark 1.25.** Essentially the same argument implies that the above space fails to be locally Euclidean of any dimension at $0 \in X$. Namely, a connected open subset of $\mathbb{R}^n$ for $n \geq 2$ will remain connected after removing any point, so it cannot be homeomorphic to $(-\varepsilon, \varepsilon) \times \{0, 1\}$ in X.

Morally, the second countability is being required as a smallness condition; let's see some pathological examples without second countability. The following lemma approximately explains the problem.

> **Lemma 1.26.** Fix a topological space X. Suppose that there is an uncountable subset $Y \subseteq X$ such that each $y \in Y$ has an open neighborhood $U_y \subseteq X$ where the U_y are pairwise disjoint. Then X fails to be second countable.

Proof. Suppose we have a base $\mathcal{B}$; we show $\mathcal{B}$ is uncountable. Each $y \in U_y$ has some $B_y \in \mathcal{B}$ with $B_y \subseteq U_y$. However, $y \neq y'$ implies that $B_y \neq B_{y'}$ because $y \in B_y$ while $p_y \notin U_{y'}$ implies $p_y \notin B_{y'}$. So $\{B_y\}_{y \in Y}$ is an uncountable subcollection of $\mathcal{B}$. ∎

> **Exercise 1.27.** Consider an uncountable set S with the discrete topology (namely, every subset is open), and then we form the product $X := \mathbb{R} \times S$. Then X is Hausdorff, locally Euclidean of dimension 1, but it is not second countable.

Proof. Here are our checks.

- Note that X is a product of Hausdorff spaces and hence is Hausdorff.

- This space is locally Euclidean of dimension 1: for each $(x, s) \in X$, we note that $\mathbb{R} \times \{s\}$ is an open subset of X (because S is discrete) where $\mathbb{R} \times \{s\} \cong \mathbb{R}$ by an argument similar to Example 1.11.

- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (0, s)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \mathbb{R} \times \{s\}$. ∎

> **Exercise 1.28.** Consider the first uncountable ordinal ω_1. Then define $X := (S \times [0, 1)) \setminus \{(0, 0)\}$, and we give X the order topology where the ordering is lexicographic. (Namely, the base consists of the "intervals" $\{x : x < b\}$ or $\{x : a < x\}$ or $\{x : a < x < b\}$.) This space is Hausdorff, locally Euclidean 1, but it is not second countable.

Proof. Here are our checks.

- This space is Hausdorff because it is a dense linear order. Explicitly, for $(s, a), (t, b) \in X$, we have the following cases.

  - Suppose $s = t$. In this case, $a \neq b$; suppose $a < b$ without loss of generality. Then $\{x : x < (s, (a+b)/2)\}$ and $\{x : x > (s, (a+b)/2)\}$ are the needed open sets.
  - Suppose $s \neq t$; take $s < t$ without loss of generality. If $a > 0$, then $\{s\} \times (0, (a+1)/2)$ and $\{s\} \times ((a+1)/2, 1) \cup \{t\} \times [0, 1)$ provide the needed open sets. Otherwise, if $a = 0$, then $\{x : x < (s, 1/2)\}$ and $\{x : x > (s, 1/2)\}$ provide the needed open sets.

- This space is locally Euclidean of dimension 1: fix any $(s, r) \in X$. Note that $s \in \omega_1$ is countable, so we claim that
$$(s+1) \times [0, 1) \cong [0, 1),$$
sending $(0, 0)$ to 0, from which the claim follows by deleting $(0, 0)$. Because the relevant orders produce the needed topologies, we are really asking for an order-preserving bijection from $(s+1) \times [0, 1)$ to $[0, 1)$.

  Well, for any $t \in \omega_1$, we claim that there is an increasing sequence $\{p_\alpha\}_{\alpha < t} \subseteq [0, 1)$ of order type t with $p_0 = 0$, from which the claim will follow by taking $s = t$ and sending $\alpha \times [0, 1) \subseteq (s+1) \times [0, 1)$ to $[p_\alpha, p_{\alpha+1})$ (where we define $p_s := 1$). To see this claim, we argue by induction on s. For $s = 0$, take $p_0 := 0$. If s is a successor ordinal, divide all the existing p_α by 2 and then set $p_{s+1} := 1/2$.

  Lastly, if s is a limit ordinal, it is still only a countable limit ordinal, so we can find an increasing sequence of countable ordinals $\{s_i\}_{i \in \omega}$ approaching s. The sequence corresponding to s_0 will fit into $[0, 1/2)$ after scaling; then the sequence corresponding to s_1 but after s_0 will fit into $[1/2, 2/3)$ after scaling. We can continue this process inductively to complete the claim for s. I won't bother to write out the details.

- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (s, 1/2)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \{s\} \times (0, 1)$. ■

> **Remark 1.29.** What makes the locally Euclidean check above annoying is that we must show $(\omega, 0) \in X$ has a neighborhood isomorphic to an open subset of $\mathbb{R}$, which is not totally obvious.

Let's return to examples.

> **Example 1.30.** Consider the unit circle S^1. We check that S^1 is a 1-dimensional topological manifold.
>
> - S^1 is a metric space, so it is Hausdorff.
>
> - S^1 is second countable: it is a subspace of $\mathbb{R}^2$, and $\mathbb{R}^2$ is second countable by Lemma 1.21 again.
>
> - S^1 is locally Euclidean: we proceed explicitly. Define $U_1^\pm := \{(x, y) \in S^1 : \pm x > 0\}$; then $U_1^\pm \cong (-1, 1)$ by $(x, y) \mapsto y$. Similarly, define $U_2^\pm := \{(x, y) \in S^1 : \pm y > 0\}$; then $U_2^\pm \cong (-1, 1)$ by $(x, y) \mapsto x$.

1.2   January 18

The first homework has been posted. It is mostly a review of point-set topology things. It is due on the 25th of January.

> **Remark 1.31.** Please read the section on fundamental groups of manifolds on your own. We will not discuss it in class.

To review, our current goal is to define smooth manifolds. Thus far we have defined a topological space and provided enough adjectives to turn it into a topological manifold. To proceed, we need to add smoothness to our structure. We will do this later.

1.2.1   Connectivity

For now, we will content ourselves with some extra adjectives for our topological manifolds which will later be helpful. Here are two notions of connectivity.

> **Definition 1.32** (connected)**.** Fix a topological space X. Then X is *disconnected* if and only if there exist disjoint nonempty open subsets $U, V \subseteq X$ such that $X = U \sqcup V$. If X is not disconnected, we say that X is connected.

> **Example 1.33.** The interval $[0, 1]$ is connected. See [Elb22, Lemma A.6].

> **Remark 1.34.** Equivalently, we can say that X is connected if and only if X and $\varnothing$ are the only subsets of X which are both open and closed.

> **Definition 1.35** (path-connected)**.** Fix a topological space X. Then X is *path-connected* if and only if any two points $p, q \in X$ has some continuous map $\gamma \colon [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

> **Example 1.36.** The space $B(0,1) \subseteq \mathbb{R}^n$ is path-connected. Indeed, we show that the path-connected component of 0 is all of $B(0,1)$; see [Elb22, Definition A.19]. In other words, we must exhibit a path from 0 to v for any $v \in B(0,1)$. Well, define $\gamma \colon [0,1] \to B(0,1)$ by $\gamma(t) := tv$. This is continuous because it is linear, and it has $\gamma(0) = 0$ and $\gamma(1) = v$ as desired.

In general, these two notions do not coincide.

> **Example 1.37.** Consider the topological space
> $$X := \{(x, \sin(1/x)) : x \in (0,1)\} \cup \{(0,y) : y \in \mathbb{R}\}.$$
> Then X is connected, but it is not path-connected. See [Elb22, Exercise A.20].

But one does in general apply the other.

> **Lemma 1.38.** Fix a topological space X. If X is path-connected, then X is connected.

Proof. See [Elb22, Lemma A.16], though we will sketch the proof. We proceed by contraposition. Suppose that X is disconnected, so we may write $X = U \sqcup V$ where $U, V \subseteq X$ are disjoint nonempty open subsets. Now choose some $p \in U$ and $q \in V$, and we claim that there is no path $\gamma \colon [0,1] \to X$. Indeed, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would then be nonempty disjoint open subsets of $[0,1]$ covering $[0,1]$, which is a contradiction. ∎

However, for topological manifolds, these notions do coincide.

> **Proposition 1.39.** Fix a topological space M which is locally Euclidean of dimension n. Then M is path-connected if and only if it is connected.

Proof. The forward direction is by Lemma 1.38. Thus, we focus on showing the converse. Fix some $p \in M$, and we define the subset
$$U_p := \{q \in M : \text{there exists a path from } p \text{ to } q\}.$$
This is the path-connected component of p in M; see [Elb22, Definition A.19]. The main claim is that U_p is open.

Suppose $q \in M$, and we need to find an open neighborhood $B_q \subseteq M$ of q living inside U_p. Noting then that $U_p = \bigcup_{q \in U_p} B_q$ will complete the proof of this claim. Well, q has some open neighborhood $B \subseteq M$ equipped with a homeomorphism $\varphi \colon B \cong B(0,1)$ by Remark 1.18. Then $B(0,1)$ is path-connected by Example 1.36, so B is path-connected by going back through the homeomorphism. Thus, because U_p is an equivalence class, it is also the path-connected equivalence class of q, so U_p must contain B.

Now, let $\mathcal{U}$ denote the collection of path-connected components of M. This is a collection of disjoint open subsets covering M. Certainly it is nonempty, so select $U \in \mathcal{U}$. Then we write
$$M = U \cup \bigcup_{U' \in \mathcal{U} \setminus \{U\}} U'.$$

This is a decomposition of M into disjoint open subsets, so because M is connected, one of these must be empty. But U is empty, so instead the union of the U' must be nonempty. However, everything in $\mathcal{U}$ is nonempty, so instead we see that $\mathcal{U} \setminus \{U\}$ is empty, so $M = U$ is path-connected. ∎

1.2.2 Local compactness

Here is our definition.

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> **Definition 1.40** (local compactness). Fix a topological space X. Then X is *locally compact* if and only if any $x \in X$ has some open neighborhood $U \subseteq X$ such that there exists a compact subset $K \subseteq X$ containing U.

> **Remark 1.41.** If X is Hausdorff, then compact subsets are closed [Elb22, Corollary 4.13], and closed subsets of a compact space are still compact [Elb22, Lemma 4.10], so we may as well take $K = \overline{U}$ in the above definition.

The above remark motivates the following definition.

> **Definition 1.42** (precompact). Fix a topological space X. An open subset $U \subseteq X$ is *precompact* if and only if $\overline{U}$ is compact.

> **Remark 1.43.** Here is a quick check which will prove to be useful: if X is Hausdorff and $U \subseteq X$ is precompact, and $V \subseteq U$, then V is still precompact. Indeed, $\overline{U}$ is compact, and $\overline{V} \subseteq \overline{U}$ is a closed subset and hence compact [Elb22, Lemma 4.10].

> **Example 1.44.** The topological space $\mathbb{R}$ is locally compact; see [Elb22, Example 4.71].

> **Non-Example 1.45.** Infinite-dimensional normed vector spaces fail to be locally compact. Namely, open balls fail to be precompact, so local compactness fails.

> **Non-Example 1.46.** The space $\mathbb{Q}$ is not locally compact. Indeed, suppose for the sake of contradiction that we have a precompact nonempty open neighborhood $U \subseteq \mathbb{Q}$ of $0 \in \mathbb{Q}$. Now, $\mathbb{Q}$ is Hausdorff (it's a metric space), so we can find some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U$ while $\varepsilon \notin \mathbb{Q}$, so Remark 1.43 tells us that $(\varepsilon/2, \varepsilon)$ is precompact so that $[\varepsilon/2, \varepsilon]$ is actually compact.
>
> However, this is false. Let $\{\alpha_i\}_{i \geq 1}$ be an increasing sequence of irrationals in $[\varepsilon/2, \varepsilon]$ with $\alpha_i \to \varepsilon$. Explicitly, we can take $\alpha_i := \frac{i}{i+1} \cdot \varepsilon$. Then we define
>
> $$U_i := [\alpha_i, \alpha_{i+1}]$$
>
> for each $i \geq 1$. Note $[\alpha_i, \alpha_{i+1}] = (\alpha_i, \alpha_{i+1})$, so the $U_\bullet$s provide a countable sequence of disjoint open subsets covering $[\varepsilon/2, \varepsilon]$. Thus, $[\varepsilon/2, \varepsilon]$ cannot be compact.

One can check that manifolds are locally compact.

> **Proposition 1.47.** Fix a topological n-manifold M. Then M is locally compact.

Proof. This follows from being locally Euclidean. Fix $p \in M$, and then we are promised some open subset $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. Then there is an open ball $B(\varphi(p), \varepsilon) \subseteq \widehat{U}$. Then $\overline{B(\varphi(p), \varepsilon/2)} \subseteq \widehat{U}$ is closed and bounded in $\mathbb{R}^n$ and hence compact, so $\varphi^{-1}(B(\varphi(p), \varepsilon/2))$ is a subset of the compact subset $\varphi^{-1}(\overline{B(\varphi(p), \varepsilon/2)})$. ∎

Being locally compact approximately speaking allows one to understand a space by building it up from compact ones. Here is one way to do this.

> **Definition 1.48** (exhaustion)**.** Fix a topological space X. Then an *exhaustion* of X is a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of X satisfying the following.
>
> - Ascending: $K_0 \subseteq K_1 \subseteq \cdots$.
>
> - Covers: $X = \bigcup_{i\in\mathbb{N}} K_i$.
>
> - Not too close: $K_i \subseteq K_{i+1}^\circ$.

> **Example 1.49.** The space $\mathbb{R}^n$ has an exhaustion by $K_i := B(0, i)$.

Here is a way to build an exhaustion.

> **Proposition 1.50.** Fix a topological space X. If X is second-countable, locally compact, and Hausdorff. Then X has an exhaustion. In particular, topological n-manifolds have an exhaustion.

Proof. The second claim follows from the first by Proposition 1.47 and the definition of a manifold. So we will focus on showing the first claim.

Fix a countable base $\mathcal{B}$ of X, and let $\mathcal{B}'$ be the subcollection of precompact open base elements. Quickly, we note that $\mathcal{B}'$ is still a base: certainly everything in $\mathcal{B}'$ is open, and then for any $p \in X$ and open neighborhood $U \subseteq X$, we need some $B' \in \mathcal{B}'$ such that B' is precompact.

Well, because X is locally compact, there is a precompact open neighborhood U' of p by Remark 1.41. Then $U \cap U$ is an open neighborhood of p, so we can find a base element $B \in \mathcal{B}$ containing p and inside $U' \cap U$. Then $B \subseteq U'$ is precompact by Remark 1.43.

We now construct our exhaustion. Enumerate $\mathcal{B} = \{B_0, B_1, \ldots\}$, and we proceed as follows.

1. Set $K_0 := \overline{B_0}$, which is compact by construction of B_0.

2. Now suppose we have a compact subset $K_i \subseteq X$, and we construct K_{i+1}. Note that $\mathcal{B}$ is an open cover of K_i, which can be reduced to a finite subcover, so there is some M_{i+1} such that K_i is covered by $\{B_i : i \leq M_{i+1}\}$. We may as well suppose that $M_{i+1} \geq i + 1$. Then we define

$$K_{i+1} := \bigcup_{i=1}^{M} \overline{B_i}.$$

Note that the finite union of compact sets remains compact.

The above construction produces an exhaustion. Here are our checks, which will complete the proof.

- Ascending: by construction, we see that

$$K_{i+1}^\circ \supseteq \bigcup_{i=1}^{M} B_i \supseteq K_i.$$

- Covers: any $x \in X$ lives in some B_i, and by construction, we have $B_i \subseteq K_i$, so $x \in K_i$. ∎

1.2.3 Paracompactness

We will want to talk about covers in some more detail.

> **Definition 1.51** (cover)**.** Fix a topological space X. A *cover* is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ such that
>
> $$X = \bigcup_{U \in \mathcal{U}} U.$$

> **Definition 1.52** (locally finite). Fix a topological space X. A cover $\mathcal{U}$ of X is *locally finite* if and only if any $p \in X$ has some open neighborhood $U \subseteq X$ intersecting at most finitely many elements of $\mathcal{U}$.

> **Definition 1.53** (refinement). Fix a cover $\mathcal{U}$ of a topological space X. Then a *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ such that any $V \in \mathcal{U}$ is contained in some $U \in \mathcal{U}$.

And here is our definition.

> **Definition 1.54** (paracompact). Fix a topological space X. Then X is *paracompact* if and only if every open cover has a locally finite open refinement.

Approximately speaking, the point of desiring paracompactness is that it allows "reducing to Euclidean" arguments in the future will not have to deal with intersections which are infinitely bad. Anyway, here is our result.

> **Proposition 1.55.** Fix a topological n-manifold M. Then M is paracompact.

Proof. In fact, we are only going to use the fact that M has an exhaustion, proven in Proposition 1.50.

Fix an open cover $\mathcal{U}$, and we want to produce a locally finite open refinement. To set us up, fix an exhaustion $\{K_i\}_{i \in \mathbb{N}}$, which exists by Proposition 1.50, and define the following sets for each $i \in \mathbb{N}$.

- For $i \geq -1$, define $V_i := K_{i+1} \setminus K_i^\circ$, which is a closed subset of the compact set K_{i+1} and hence compact [Elb22, Lemma 4.10]; take $K_{-1} = \varnothing$ without concern.

- For $i \geq 0$, define $W_i := K_{i+2}^\circ \setminus K_{i-1}$, which is open; here, take $K_{-1} = \varnothing$ without concern.

For intuition, we should think about the $W_\bullet$s as being a locally finite cover from which we will build the locally finite cover refinement of $\mathcal{U}$.

For the construction, we fix some $j \geq 0$ for the time being. For each $x \in V_j$, find some $U_x \in \mathcal{U}$ containing x. Note that $\{U_x\}_{x \in V_j}$ is an open cover of V_j, and because $V_j \subseteq W_j$, in fact $\{U_x \cap W_j\}_{x \in V_j}$ is an open cover. Because V_j is compact, we can thus reduce this open cover to a finite subcover $\mathcal{A}_j$.

Now letting j vary, we define

$$\mathcal{V} := \bigcup_{j \geq 0} \mathcal{A}_j.$$

Here are our checks.

- Open cover: each $x \in X$ lives in some K_{i+1} because we have an exhaustion, so lives in some V_i, so it lives in some open subset in $\mathcal{A}_j$, so it lives in some open subset in $\mathcal{V}$.

- Refinement: by construction, each open set in $\mathcal{A}_j$ is a subset in $\mathcal{U}$.

- Locally finite: this is essentially by construction. The main point is that any $x \in X$ lives in some K_i, so by choosing the least such K_i places x in some $V_i \subseteq W_i$. We now show that only finitely many open subsets in $\mathcal{V}$ intersect W_i. Note $W_i \subseteq K_{i+2}$, so $W_i \cap W_j = \varnothing$ for $j \geq i+2$. Thus, if $V \cap W_i \neq \varnothing$, we must have $V \in \mathcal{A}_j$ for $j < i+2$. But this is only finitely many indices, and each $\mathcal{A}_j$ is finite, so this is only finitely many candidates. ∎

1.2.4 Products

We now discuss an in-depth example.

> **Proposition 1.56.** Fix finitely many topological manifolds $M_1, \ldots, M_k$. Then the product
> $$M_1 \times \cdots \times M_k$$
> is also a topological manifold of dimension $\dim M_1 + \cdots + \dim M_k$.

We will do this via a sequence of lemmas.

> **Lemma 1.57.** Fix a collection of Hausdorff topological spaces $\{X_\alpha\}_{\alpha \in \Lambda}$. Then the product
> $$\prod_{\alpha \in \Lambda} X_\alpha$$
> is also Hausdorff.

Proof. Fix distinct points $(x_\alpha)_{\alpha \in \Lambda}$ and $(y_\alpha)_{\alpha \in \Lambda}$ in the product. Then there is an index $\beta \in \Lambda$ such that $x_\beta \neq y_\beta$, so because X_β is Hausdorff, there are disjoint open neighborhoods $U_\beta, V_\beta \subseteq X_\beta$ of x_β and y_β, respectively. Then we define $U_\alpha = V_\alpha := X_\alpha$ for $\alpha \neq \beta$, and we note that the open subsets
$$\prod_{\alpha \in \Lambda} U_\alpha \qquad \text{and} \qquad \prod_{\alpha \in \Lambda} V_\alpha$$
are disjoint open neighborhoods of $(x_\alpha)_{\alpha \in \Lambda}$ and $(y_\alpha)_{\alpha \in \Lambda}$, respectively, so we are done. (These are disjoint because any point in the intersection will have the β coordinate in $U_\beta \cap V_\beta = \varnothing$.) ∎

> **Lemma 1.58.** Fix finitely many second countable topological spaces $\{X_i\}_{i=1}^n$. Then the product
> $$\prod_{i=1}^n X_i$$
> is also second countable.

Proof. Let the product be X. For each i, let $\mathcal{B}_i$ be a countable base for X_i. Then define
$$\mathcal{B} := \left\{ \prod_{i=1}^n B_i : B_i \in \mathcal{B}_i \text{ for each } i \right\}.$$

We claim that $\mathcal{B}$ is a base for the topology on the X. Indeed, suppose $(x_1, \ldots, x_n) \in X$ lives in some open subset $U \subseteq X$. From the standard base on X, we know that there are open subsets $U_i \subseteq X_i$ for each i such that $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$. Now, for each U_i, we note that $x_i \in U_i$ must have some $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$ and $B_i \subseteq U_i$. But then
$$(x_1, \ldots, x_n) \in B_1 \times \cdots \times B_n \subseteq U,$$
so $B_1 \times \cdots \times B_n \in \mathcal{B}$ is the desired base element. ∎

We now prove Proposition 1.56.

Proof of Proposition 1.56. We get Hausdorff from Lemma 1.57 and second countable from Lemma 1.58. So it remains to check that we are locally Euclidean. For brevity, let M be the product, and set $n_i := \dim M_i$ for each i, and let $n := n_1 + \cdots + n_k$.

Now, fix some point $(x_1, \ldots, x_k) \in M$. For each i, we get some open neighborhood $U_i \subseteq M_i$ of x_i and some open $\widehat{U}_i \subseteq \mathbb{R}^{n_i}$ with a homeomorphism $\varphi_i : U_i \cong \widehat{U}_i$. Now, we see that the product map
$$(\varphi_1 \times \cdots \times \varphi_k) : U_1 \times \cdots \times U_k \to \widehat{U}_1 \times \cdots \times \widehat{U}_k$$

is still a homeomorphism, and the target is an open subset of

$$\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \cong \mathbb{R}^n,$$

where this last homeomorphism is obtained by simply concatenating the coordinates. So we have constructed a composite homeomorphism from an open neighborhood of $(x_1, \ldots, x_k)$ to an open subset of $\mathbb{R}^n$, as desired. ∎

> **Example 1.59.** Example 1.30 established S^1 as a topological 1-manifold, so the n-torus
>
> $$T^n := \underbrace{S^1 \times \cdots \times S^1}_{n}$$
>
> is a topological n-manifold. Note that the covering space $p \colon \mathbb{R} \to S^1$ will induce the covering space $p^n \colon \mathbb{R}^n \to T^n$, so we can also view T^n as $\mathbb{R}^n/\mathbb{Z}^n$; in other words, we have the unsurprising homeomorphism $\mathbb{R}^n/\mathbb{Z}^n \to (\mathbb{R}/\mathbb{Z})^n$.

1.2.5 Open Submanifolds

We proceed with a sequence of lemmas.

> **Lemma 1.60.** Suppose X is a Hausdorff topological space. If $X' \subseteq X$ is a subspace, then X' is still Hausdorff.

Proof. Fix distinct points $p, q \in X'$. Then X is Hausdorff, so there exist disjoint open neighborhoods $U, V \subseteq X$ of p and q, respectively, so $U \cap X'$ and $V \cap X'$ are the needed disjoint open subsets of X', respectively. ∎

> **Lemma 1.61.** Suppose that X is a second countable topological space. Then for any subset $X' \subseteq X$, the topological (sub)space X' is still second countable.

Proof. Well, let $\mathcal{B}$ be a countable base for X, and we claim that the collection

$$\mathcal{B}' := \{B \cap X' : B \in \mathcal{B}\}$$

makes a countable base for X'. Note that $\mathcal{B}'$ is certainly countable because there is a surjective map $\mathcal{B} \to \mathcal{B}'$ by $B \mapsto (B \cap X')$, and $\mathcal{B}$ is countable. (This map is surjective by construction.)

So it remains to show that $\mathcal{B}'$ is a base. Quickly, we claim that every $B' \in \mathcal{B}'$ is open in X'. Indeed, for any $B' \in \mathcal{B}'$, we can find some $B \in \mathcal{B}$ such that $B' = B \cap X'$. Now, $\mathcal{B}$ is a base, so $B \subseteq X$ is open, so $B' = B \cap X'$ is open in the subspace topology of X'.

To finish checking that we have a base, fix some $x' \in X'$ and open $U' \subseteq X'$ containing x'. Then we need some $B' \in \mathcal{B}'$ such that $x' \in B'$ and $B' \subseteq U'$. Well, by the subspace topology, we can write $U' = U \cap X'$ for some open $U \subseteq X$, but then $x' \in U$, so there is some $B \in \mathcal{B}'$ such that $x' \in B$ and $B \subseteq U$. To finish, we set

$$B' := B \cap X',$$

which is in $\mathcal{B}'$ by construction, and we have $x' \in B \cap X' = B'$ and $B' = B \cap X' \subseteq U \cap X' = U'$, so B' is indeed the required basic open set. ∎

> **Lemma 1.62.** Suppose that X is locally Euclidean of dimension n. Then for any open subset $X' \subseteq X$, the topological (sub)space X' is locally Euclidean of dimension n.

Proof. For any $x' \in X'$, we must find open subsets $U' \subseteq X'$ and $\widehat{U}' \subseteq \mathbb{R}^n$ such that $x' \in U'$ and there is a homeomorphism $U' \cong \widehat{U}'$.

Well, $x' \in X$, so there are open subsets $U \subseteq X$ and $\widehat{U} \subseteq \mathbb{R}^n$ such that $x' \in U$ and there is a homeomorphism $\varphi\colon U \cong \widehat{U}$. Now, set

$$U' := U \cap X'.$$

Then φ is a homeomorphism, so $\varphi' := \varphi|_{U'}$ continues to be a homeomorphism onto its image $\widehat{U}' := \varphi(U')$. Indeed, the inverse of the bijection $\varphi|_{U'}\colon U' \to \widehat{U}'$ is $\varphi'|_{\widehat{U}'}$. Both of these maps are continuous by, so $\varphi|_{U'}$ is in fact a homeomorphism.

Now, $U' \subseteq U$ is open, so because φ is a homeomorphism, we see that $\varphi(U') \subseteq \widehat{U}$ is open: $\varphi(U')$ is the pre-image of the open subset $U' \subseteq U$ under the continuous map $\varphi^{-1}\colon \widehat{U} \to U$, so $\varphi(U')$ being open follows. Continuing, because $\widehat{U} \subseteq \mathbb{R}^n$ is open, we conclude that $\widehat{U}' \subseteq \mathbb{R}^n$ is open.[1] So $U' \subseteq X'$ is open (by the subspace topology), contains x', and it is homeomorphic to an open subset $\widehat{U}'$ of $\mathbb{R}^n$. ∎

> **Proposition 1.63.** Fix a topological n-manifold M. For any nonempty open subset $U \subseteq M$, we have that U is a topological n-manifold.

Proof. Combine Lemmas 1.60 to 1.62. ∎

1.2.6 Charts

The construction of our smooth structure will arise from more carefully understanding how a manifold is locally Euclidean. This arises from charts.

> **Definition 1.64** (chart)**.** Fix a topological n-manifold M. Then a *coordinate chart* or just *chart* is a pair (U, φ) where $U \subseteq M$ is open and $\varphi\colon U \cong \widehat{U}$ is a homeomorphism where $\widehat{U} \subseteq \mathbb{R}^n$ is open.

Essentially, the content of M being locally Euclidean is that it has an open cover by open subsets belonging to some chart. The reason we call it a chart is that we are (approximately speaking) providing "local coordinates" to an open subset of M.

> **Definition 1.65** (coordinate function)**.** Fix a chart (U, φ) if a topological n-manifold M. Then we may write
> $$\varphi(p) := \big(x^1(p), \ldots, x^n(p)\big) \in \mathbb{R}^n$$
> for each $p \in M$. We call these functions $x^\bullet\colon U \to \mathbb{R}$ the *coordinate functions*.

Note that these coordinate functions are continuous because they are simply the continuous function φ composed with the projection $\mathbb{R}^n \to \mathbb{R}$.

---

[1] Namely, an open subset of an open subset U is still an open subset. This sentence has some content because the larger open subset uses the subspace topology; the proof simply notes that being open in U is equivalent to being the intersection of an open subset and U, which is open because finite intersections of open subsets continues to be open.

> **Example 1.66.** Fix an open subset $V \subseteq \mathbb{R}^m$, and let $F \colon V \to \mathbb{R}^n$ be a continuous function. Then the graph
> $$\Gamma := \{(x, F(x)) : x \in V\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$
> is a topological n-manifold. Because we are already a subspace of $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$, we see that Γ is also Hausdorff and second countable. (Subspaces inherit being Hausdorff directly, and we inherit being second countable by using the intersection of the given countable base.)
>   The main content comes from being locally Euclidean. Namely, there is a projection map $\pi \colon \Gamma \to V$ by $(x, y) \mapsto x$ which in fact is a homeomorphism (it's continuous inverse is $(\mathrm{id} \times F) \colon x \mapsto (x, F(x))$). So we have the single chart (V, π), which establishes being a topological n-manifold.

1.3 January 23

The first homework is due on Thursday. Today we discuss smooth structures.

1.3.1 Examples of Topological Manifolds

Let's provide a few more examples of topological manifolds.

> **Exercise 1.67** (sphere). We show that the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold.

Proof. Explicitly, for each $i \in \{1, \dots, n+1\}$, we define
$$U_i^{\pm} := \{(x_1, \dots, x_{n+1}) \in S^n : \pm x_i > 0\},$$
which has a projection $\pi_i^{\pm} \colon U_i^{\pm} \to B(0, 1)$ (for $B(0, 1) \subseteq \mathbb{R}^n$) given by erasing the x_i coordinate. One can show that the $\pi_i^{\pm}$ are all homeomorphisms—certainly, it is continuous, and the inverse map is given by
$$(x_1, \dots, x_n) := \left(x_1, \dots, x_{i-1}, \pm\sqrt{1 - (x_1^2 + \cdots + x_n^2)}, x_i, \dots, x_n \right),$$
which is also continuous. (We won't bother checking that the maps are mutually inverse.) Lastly, we note that the $U_i^{\pm}$ is an open cover of S^n because any point in S^n has some nonzero coordinate, and this nonzero coordinate will have a sign. $\blacksquare$

> **Exercise 1.68** (projective space). Define the space $\mathbb{RP}^n$ as "lines in $\mathbb{R}^{n+1}$": it consists of equivalence classes of nonzero points in $\mathbb{R}^{n+1} \setminus \{0\}$, where $x \sim y$ if and only if there is some $\lambda \in \mathbb{R}^{\times}$ such that $x = \lambda y$. We show that $\mathbb{RP}^n$ is a topological n-manifold.

Proof. For notation, we let $[x_0 : \cdots : x_n]$ denote the equivalence class of $(x_1, \dots, x_n)$ in $\mathbb{RP}^n$. Note there is a projection $p \colon (\mathbb{R}^{n+1} \setminus \{0\}) \to \mathbb{RP}^n$, and we give $\mathbb{RP}^n$ the induced (quotient) topology from $\mathbb{R}^{n+1} \setminus \{0\}$.
    By Lemma 1.22, to achieve second countable, it suffices to provide a finite open cover by open subsets homeomorphic to open subsets of $\mathbb{R}^n$; this will also achieve locally Euclidean. Well, define
$$U_i := \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\}.$$
Note that the pre-image in $\mathbb{R}^{n+1} \setminus \{0\}$ consists of the $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ with $x_i \neq 0$, so $U_i \subseteq \mathbb{RP}^n$ is open. Now, by scaling, we can write elements of U_i uniquely as $[y_0 : \cdots : y_n]$ with $y_i = 1$, which provides the required element in $\mathbb{R}^n$. Explicitly, we define $\varphi_i \colon U_i \to \mathbb{R}^n$ by
$$\varphi_i \colon [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i} \right).$$

One sees that φ_i is continuous: by the quotient topology, we are trying to show that $\varphi_i \circ \pi \colon \pi^{-1}U_i \to \mathbb{R}^n$ is just $(x_0, \ldots, x_n) \mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i)$, which is continuous, so φ_i is continuous because $\mathbb{RP}^n$ has the quotient topology. Lastly, one notes that the inverse of φ_i is given by $(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$, which is continuous because it is the composite of the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$ and the projection $p \colon (\mathbb{R}^{n+1} \setminus \{0\}) \to \mathbb{RP}^n$.

Lastly, we show that $\mathbb{RP}^n$ is Hausdorff. Doing this in a slick way is surprisingly obnoxious. We claim that there is a 2-to-1 covering space map

$$p \colon S^n \to \mathbb{RP}^n.$$

To see why this implies that $\mathbb{RP}^n$ is Hausdorff, fix two distinct points $x, y \in \mathbb{RP}^n$. Then there are lifts $x_1, x_2 \in S^n$ of x and $y_1, y_2 \in S^n$. Because S^n is already Hausdorff (it's a subspace of $\mathbb{RP}^n$), we can find disjoint open subsets $U_1, U_2, V_1, V_2 \subseteq S^n$ around $x_1, x_2, y_1, y_2 \in S^n$ respectively, and we can make them all small enough so that p is a local homeomorphism. Then $p(U_1) \cap p(U_2)$ and $p(V_1) \cap p(V_2)$ are the desired open subsets.

So we are left showing that we have a double cover p. The map is given by the composite

$$S^n \subseteq (\mathbb{R}^{n+1} \setminus \{0\}) \twoheadrightarrow \mathbb{RP}^n,$$

which we see is continuous automatically. To see that this is a 2-to-1 local homeomorphism, we note that the pre-image of the standard open subset $U_i \subseteq \mathbb{RP}^n$ is

$$\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \neq 0 \right\},$$

whose pre-image in S^n splits into the two open subsets $U_i^{\pm}$. So we have our continuous map $U_i^+ \sqcup U_i^- \to U_i$; it remains to show that $U_i^{\pm} \to U_i$ is a homeomorphism. We may as well assume $i = 0$; then the inverse map is given by sending $[1 : x_1 : \cdots : x_n]$ to the point on the hemisphere of S^n on this line, which is

$$\pm \frac{x}{|x|},$$

where the sign depends on $U_i^{\pm}$. This is continuous, so we are done. ∎

> **Remark 1.69.** Note S^n is continuous, so the surjectivity of the covering space map $S^n \twoheadrightarrow \mathbb{RP}^n$ implies that $\mathbb{RP}^n$ is compact.

1.3.2 Transition Functions

Defining smooth structures will come out of transition maps between coordinate charts.

> **Definition 1.70** (transition map)**.** Fix charts (U, φ) and (V, ψ) on a topological n-manifold M. Then the *transition map* is the map
> $$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V).$$
> Here, we are abusing notation a little: in order to make sense of $\psi \circ \varphi^{-1}$, we really want to work with the restrictions as $\psi|_{U \cap V} \circ (\varphi|_{U \cap V})^{-1}$.

> **Remark 1.71.** Note $\varphi(U \cap V), \psi(U \cap V) \subseteq \mathbb{R}^n$, so this is a homeomorphism from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$. Namely, $\varphi|_{U \cap V}$ and $\psi|_{U \cap V}$ are both homeomorphisms, so the above composition is still a homeomorphism.

Example 1.72 (polar coordinates)**.** Consider the topological 2-manifold $M := \mathbb{R}^2$. There is the identity chart $\mathrm{id}_M \colon M \to \mathbb{R}^2$, and there is also "polar coordinates" on $U := \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$ with chart $\varphi \colon U \to \mathbb{R}_+ \times (0, \pi)$ defined by

$$\varphi((x,y)) := \left(\sqrt{x^2 + y^2}, \arg(x,y) \right),$$

where $\arg(x, y)$ is the angle of (x, y) with the positive x-axis. Note the inverse map of φ is given by $(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$, so φ is in fact a homeomorphism.

Now, the transition map $\psi \circ \varphi^{-1}$ sends

$$(r, \theta) \overset{\varphi^{-1}}{\mapsto} (r \cos \theta, r \sin \theta) \overset{\psi}{\mapsto} (r \cos \theta, r \sin \theta).$$

Example 1.73. Consider the topological 2-manifold $M := S^2$ from Exercise 1.67. We compute the transition maps between φ_1^+ and φ_3^+, which overlap on the open set consisting of $(x_1, x_2, x_3) \in S^2$ such that $x_1, x_3 > 0$. Well, we can directly compute that $\varphi_3^+ \circ \left(\varphi_1^+ \right)^{-1}$ is given by

$$(x_2, x_3) \overset{(\varphi_1^+)^{-1}}{\mapsto} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3 \right) \overset{\varphi_3^+}{\mapsto} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2 \right).$$

In the above examples, we can note that the maps between the Euclidean smooths are smooth on their domains. This becomes our notion of smoothness.

Definition 1.74 (smoothly compatible)**.** Two charts (U, φ) and (V, ψ) of a topological manifold M are *smoothly compatible* if and only if both transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth (i.e., infinitely differentiable). Notably, this condition is vacuously satisfied if $U \cap V = \varnothing$.

1.3.3 Smooth Structures

We would like to cover M with smoothly compatible charts, so it will be helpful to have a language for such covers.

Definition 1.75 (atlas)**.** Fix a topological manifold M. An *atlas* $\mathcal{A}$ is a collection of charts "covering M" in the sense that

$$M = \bigcup_{(U, \varphi)} U.$$

An atlas is *smooth* if and only if its charts are pairwise smoothly compatible. A smooth atlas is *maximal* if and only if it is maximal in the sense of inclusion by smooth atlases.

The point of using a maximal atlas is that we would like a way to say when two atlases provide the same smooth structure for a topological manifold, but it will turn out to be easier to provide a "unique" atlas to look at, which will be the maximal smooth atlas. Quickly, we note that maximal smooth atlases exist. One could argue this by Zorn's lemma, but we don't have to.

Proposition 1.76. Fix a topological n-manifold M. Any smooth atlas $\mathcal{A}$ is contained in a unique maximal smooth atlas, denoted $\overline{\mathcal{A}}$.

Proof. We have to show existence and uniqueness. We will construct this directly: define $\overline{\mathcal{A}}$ to be the collection of charts (U, φ) which is smoothly compatible with each chart in $\mathcal{A}$. We show that $\overline{\mathcal{A}}$ is a maximal smooth atlas.

- Atlas: certainly $\overline{\mathcal{A}} \supseteq \mathcal{A}$, so $\overline{\mathcal{A}}$ covers M, so $\overline{\mathcal{A}}$ is an atlas.

- Smooth: fix any charts $(U_1, \varphi_1), (U_2, \varphi_2) \in \overline{\mathcal{A}}$, and we would like to show that they are smoothly compatible. If $U_1 \cap U_2 = \varnothing$, there is nothing to do, so we may assume that the intersection is nonempty. By symmetry, it will be enough to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth.

  The point is that differentiability is a local notion: explicitly, fix some $q \in \varphi_1(U_1 \cap U_2)$, and we want to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth at q. This can be checked on a small open neighborhood of q; in particular, find the $p \in U_1 \cap U_2$ such that $\varphi_1(p) = q$, and we can find some chart $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Then we note that

  $$\varphi_2|_{U_1 \cap U_2 \cap V} \circ (\varphi_1|_{U_1 \cap U_2 \cap V})^{-1} = \left(\varphi_2|_{U_1 \cap U_2 \cap V} \circ (\psi|_{U_1 \cap U_2 \cap V})^{-1}\right) \circ \left(\psi|_{U_1 \cap U_2 \cap V} \circ (\varphi_1|_{U_1 \cap U_2 \cap V})^{-1}\right)$$

  is smooth on $\varphi_1(U_1 \cap U_2 \cap V)$ as it is the composition of smooth maps. So our left-hand side is smooth on $U_1 \cap U_2 \cap V$ and in particular at $q \in \varphi_1(U_1 \cap U_2 \cap V)$.

- Maximal: suppose $\mathcal{A}'$ is a smooth atlas containing $\mathcal{A}$. We must show that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$; by supposing further that $\mathcal{A}'$ contains $\overline{\mathcal{A}}$, we achieve the maximality of $\overline{\mathcal{A}}$. Well, for each $(U, \varphi) \in \mathcal{A}'$, we see that (U, φ) is smoothly compatible with each chart in $\mathcal{A}$, so $(U, \varphi) \in \overline{\mathcal{A}}$. Thus, $(U, \varphi) \in \overline{\mathcal{A}}$, so $\mathcal{A}' \subseteq \overline{\mathcal{A}}$.

- Unique: suppose $\mathcal{A}'$ is a maximal smooth atlas containing $\mathcal{A}$. Then the previous point establishes that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$, but then we must have equality because $\mathcal{A}'$ is a maximal smooth atlas. ∎

So we may make the following definition.

> **Definition 1.77** (maximal smooth atlas)**.** Fix a topological n-manifold M. Given a smooth atlas $\mathcal{A}$ on M, we let $\overline{\mathcal{A}}$ denote the unique maximal smooth atlas containing $\mathcal{A}$, which we know exists and is unique by Proposition 1.76.

> **Corollary 1.78.** Fix a topological n-manifold M. Given smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is still a smooth atlas, then
> $$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}.$$

Proof. Define $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\overline{\mathcal{A}}$ is a maximal smooth atlas containing $\mathcal{A}$ and hence both $\mathcal{A}_1$ and $\mathcal{A}_2$, so we see that $\overline{\mathcal{A}_1} = \overline{\mathcal{A}}$ and $\overline{\mathcal{A}_2} = \overline{\mathcal{A}}$. Notably, we are using the uniqueness of Proposition 1.76. ∎

At long last, here is our definition.

> **Definition 1.79** (smooth manifold)**.** Fix a topological n-manifold M. A *smooth structure* on M is a maximal smooth atlas on M. A *smooth n-manifold* is a pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a smooth structure on M.

> **Remark 1.80.** Adjusting the "smoothness" on the manifold M produces different notions of manifold. For example, we can have twice differentiable manifolds, real analytic manifolds, complex manifolds, etc.

1.4   January 25

The first homework is due later today.

1.4.1   A Couple Lemmas on Atlases

Here are some basic properties of smooth manifolds which one can check.

> **Lemma 1.81.** Fix a smooth n-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$, then for any open subset $U' \subseteq U$, we have $(U', \varphi|_{U'}) \in \mathcal{A}$.

Proof. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U', \varphi|_{U'})\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas of charts. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that $(U', \varphi|_{U'})$ and (V, ψ) are smoothly compatible. (The charts in $\mathcal{A}$ are already smoothly compatible with each other.) In other words, we must show that the transition functions are diffeomorphism: the transition maps are

$$\varphi|_{U' \cap V} \circ \psi|_{U' \cap V}^{-1} = \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1} \right)|_{\psi(U' \cap V)}$$

and

$$\psi|_{U' \cap V} \circ \varphi|_{U' \cap V}^{-1} = \left(\psi|_{U \cap V} \circ \varphi|_{U \cap V}^{-1} \right)|_{\varphi(U' \cap V)},$$

and these are both smooth as the restrictions of smooth maps. (Namely, we are using the fact that (U, φ) and (V, ψ) are smoothly compatible already.) ∎

> **Lemma 1.82.** Fix a smooth n-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$ and diffeomorphism $\chi \colon \varphi(U) \to V$ for some open subset $V \subseteq \mathbb{R}^n$, we have $(U, \chi \circ \varphi) \in \mathcal{A}$.

Proof. The argument is similar to that of the above lemma. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U, \chi \circ \varphi)\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that (V, ψ) and $(U, \chi \circ \varphi)$ are smoothly compatible. (Indeed, the charts in $\mathcal{A}$ are already smoothly compatible with each other.) Well, the transition maps are

$$(\chi \circ \varphi)|_{U \cap V} \circ \psi|_{U \cap V}^{-1} = \chi|_{\varphi(U \cap V)} \circ \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1} \right)$$

and

$$\psi|_{U \cap V} \circ (\chi \circ \varphi)|_{U \cap V}^{-1} = \psi|_{U \cap V} \circ \varphi|_{U \cap V}^{-1} \circ \chi|_{\varphi(U \cap V)}^{-1},$$

which are smooth maps because (U, φ) and (V, ψ) are already smoothly compatible, and χ is a diffeomorphism. ∎

> **Lemma 1.83.** Fix a smooth n-manifold $(M, \mathcal{A})$. If $\varphi \colon U \to \mathbb{R}^n$ is an injective map with $U \subseteq M$ is such that each $p \in U$ has some open neighborhood $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, then actually $(U, \varphi) \in \mathcal{A}$.

Proof. By the definition of being a maximal smooth atlas, it suffices to show that (U, φ) is smoothly compatible with all charts in $\mathcal{A}$. Well, pick up some chart (V, ψ), and we would like to show that the transition map

$$\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}$$

is a diffeomorphism. Well, we can being a diffeomorphism locally by checking it at all point $\psi(p) \in \psi(U \cap V)$ where $p \in U \cap V$. But for some fixed p, we are promised some open subset $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, so the map

$$\varphi|_{U_p \cap V} \circ \psi|_{U_p \cap V}^{-1} = \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1} \right)|_{\psi(U_p \cap V)}$$

is a diffeomorphism. So we produce smoothness at the images of p of the function and its inverse. ∎

1.4.2   Examples of Smooth Manifolds

We go through some examples of smooth manifolds.

> **Example 1.84.** Recall from Lemma 1.21 that $\mathbb{R}^n$ is a topological n-manifold. Then $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ provides an atlas on $\mathbb{R}^n$ consisting of a single chart, which is vacuously smooth; note Proposition 1.76 then gives us a smooth structure.

More generally, we have the following.

> **Proposition 1.85.** Fix a smooth n-manifold $(M, \mathcal{A})$. For any nonempty open subset $M' \subseteq M$, we have that M' is a topological n-manifold, and
>
> $$\mathcal{A}' := \{(U, \varphi) \in \mathcal{A} : U \subseteq M'\}$$
>
> is a smooth structure on M.

Proof. By Proposition 1.63, we see that M' is a topological n-manifold. It remains to show that $\mathcal{A}'$ is a smooth structure. Here are our checks.

- Chart: for any $x \in M'$, we know $\mathcal{A}$ is a chart on M, so there is a chart $(U, \varphi) \in \mathcal{A}$ with $x \in U$. Now, $U \subseteq M$ is open, so Lemma 1.81 tells us that $(U \cap M', \varphi|_{U \cap M'})$ is a chart in $\mathcal{A}$. But now $U \cap M' \subseteq M'$, so $(U \cap M', \varphi|_{U \cap M'}) \in \mathcal{A}'$ by construction, so we conclude because $x \in U \cap M'$.

- Smooth: for any two charts $(U, \varphi), (V, \psi) \in \mathcal{A}'$, we note that these charts belong to the smooth atlas $\mathcal{A}$ already, so they are already smoothly compatible.

- Maximal: by definition of being a maximal smooth atlas, it suffices to show that if (U, φ) is a chart of M' smoothly compatible with $\mathcal{A}'$, then it must be in $\mathcal{A}'$. Well, $U \subseteq M'$ already, so it suffices to show that $(U, \varphi) \in \mathcal{A}$. Because $\mathcal{A}$ is already a maximal smooth atlas, it suffices to show that (U, φ) is compatible with all the charts in $\mathcal{A}$. Well, for any chart $(V, \psi) \in \mathcal{A}$, we need the composite

$$\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}$$

  to be a diffeomorphism. But we simply note that $(U \cap V, \psi|_{U \cap V}) \in \mathcal{A}$ by Lemma 1.81 will live in $\mathcal{A}'$, so the above is a diffeomorphism because the hypothesis on (U, φ) implies that it would be smoothly compatible with $(U \cap V, \psi|_{U \cap V}) \in \mathcal{A}'$. ∎

> **Example 1.86.** Any nonempty open subset of $\mathbb{R}^n$ is a smooth n-manifold by combining Example 1.84 and Proposition 1.85. For example,
>
> $$\mathrm{GL}_n(\mathbb{R}) := \left\{M \in \mathbb{R}^{n \times n} : \det M \neq 0\right\}$$
>
> is an open subset of $\mathbb{R}^{n \times n}$, so $\mathrm{GL}_n(\mathbb{R})$ is a smooth manifold. (Notably, $\det\colon \mathbb{R}^{n \times n} \to \mathbb{R}$ is a polynomial and hence continuous, so the pre-image of $\mathbb{R} \setminus \{0\}$ is open.)

> **Example 1.87.** From Example 1.66, we know that the graph Γ of a smooth function $f\colon V \to \mathbb{R}^n$, where $V \subseteq \mathbb{R}^m$ is open, is a topological n-manifold, where we have a chart given by the projection $\pi\colon \Gamma \to V$. Using this chart alone produces a smooth atlas and makes Γ into a smooth n-manifold as well.

Example 1.88. We claim that the charts on S^n provided in Exercise 1.67 provide a smooth atlas on S^n and hence a smooth structure by Proposition 1.76. Indeed, we must show that the transition maps

$$\varphi_i^\pm|_{U_i^\pm \cap U_j^\pm} \circ \varphi_j^\pm|_{U_i^\pm \cap U_j^\pm}^{-1} (x_1, \ldots, x_n) = \left(x_1, \ldots, \widehat{x}_j, \ldots, x_{i-1}, \pm\sqrt{1 - (x_1^2 + \cdots + x_n^2)}, x_i, \ldots, x_n \right)$$

is a diffeomorphism (for any choice of signs). The above equation shows that our map is smooth for $i > j$, and the computation for $i < j$ simply switches the ith and jth coordinates. On the homework, we will see how to use stereographic projection to provide a smooth structure (in fact, the same smooth structure) on S^n.

Example 1.89. Fix an n-dimensional $\mathbb{R}$-vector space V. Then we claim

$$\mathcal{A} := \{(V, \varphi) : \varphi \text{ is an isomorphism to } \mathbb{R}^n\}$$

is a smooth atlas on V and hence provides a smooth structure. Indeed, certainly this is an atlas: there is some isomorphism $\varphi \colon V \to \mathbb{R}^n$, and this chart will cover V. Further, these are smoothly compatible because the transition map between the two arbitrary charts (V, φ) and (V, ψ) is the linear isomorphism $\left(\varphi \circ \psi^{-1}\right) \colon \mathbb{R}^n \to \mathbb{R}^n$, which is linear and hence smooth.

Example 1.90. Fix the topological 1-manifold $\mathbb{R}$ of Lemma 1.21. Example 1.84 tells us $\mathcal{A} := \{(\mathbb{R}, \mathrm{id}_\mathbb{R})\}$ provides a smooth atlas, and $\mathcal{A}' := \{(\mathbb{R}, \varphi)\}$ given by $\varphi \colon x \mapsto x^3$ is also a smooth atlas (again, smoothness is vacuous). However, $\mathcal{A}$ and $\mathcal{A}'$ provide smooth structures: otherwise, they would be contained in the same maximal smooth atlas, so $(\mathbb{R}, \mathrm{id}_\mathbb{R})$ and $(\mathbb{R}, \varphi)$ would be smoothly compatible, but then the composite $\left(\mathrm{id}_\mathbb{R} \circ \varphi^{-1}\right) \colon x \mapsto \sqrt[3]{x}$ is not a smooth function $\mathbb{R} \to \mathbb{R}$.

Example 1.91. Recall that $\mathbb{RP}^n$ is a topological n-manifold by Exercise 1.68. We claim that the charts (U_i, φ_i) actually form a smooth atlas on $\mathbb{RP}^n$, thus making $\mathbb{RP}^n$ into a smooth atlas. We already checked that these charts cover $\mathbb{RP}^n$, and they are smoothly compatible because we can compute the transition between (U_i, φ_i) and (U_j, φ_j) is

$$\varphi_i|_{U_i \cap U_j} \circ \varphi_j|_{U_i \cap U_j}^{-1} (x_0, \ldots, \widehat{x}_j, \ldots, x_n) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right),$$

which we can see is a rational and hence smooth function.

Example 1.92. Fix smooth manifolds $(M_1, \mathcal{A}_1), \ldots, (M_k, \mathcal{A}_k)$, where M_i is a smooth n_i-manifold. The product $M := M_1 \times \cdots \times M_k$ is a smooth manifold by Proposition 1.56, and the proof implies that

$$\mathcal{A} := \{(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k) : (U_i, \varphi_i) \in \mathcal{A}_i \text{ for each } i\}$$

is an atlas on M. In fact, this is a smooth atlas, thus providing M with a smooth structure by Proposition 1.76. Well, the transition map between the charts $(U, \varphi) := (U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$ and $(V, \psi) := (V_1 \times \cdots \times V_k, \psi_1 \times \cdots \times \psi_k)$ is

$$\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1} = \left(\varphi_1|_{U_1 \cap V_1} \circ \psi_1|_{U_1 \cap V_1}^{-1}\right)^{-1} \times \cdots \times \left(\varphi_k|_{U_k \cap V_k} \circ \psi_k|_{U_k \cap V_k}^{-1}\right),$$

which we can see is smooth as it is the product of smooth functions.

Remark 1.93. In fact, if a topological n-manifold has some smooth structure, there are uncountably many distinct smooth structures on M. On the other hand, for n-manifolds of small dimensions (e.g., $n \leq 3$), it turns out that these are diffeomorphic.

Remark 1.94. However, there do exist topological n-manifolds with no smooth structure, in dimensions $n \geq 4$. Even worse, there are topological n-manifolds with distinct smooth structures up to diffeomorphism, again in dimensions $n \geq 4$. Even for S^n, the story is complicated: there is only one smooth structure for $n \leq 3$, we don't understand $n = 4$, and the story is complicated but somewhat understood for $n \geq 5$.

1.4.3 Grassmannians

The construction of smooth manifolds is rather long: we build a topological space, define some charts, and then check that the charts are smoothly compatible. Here's a lemma to do all of this at once.

Lemma 1.95. Fix a set M with a nonnegative integer $n \geq 0$ and a collection of functions $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$ where $U_\alpha \subseteq M$ and $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ is open. Further, suppose the following.

(i) $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ is open for all $\alpha, \beta \in \kappa$.

(ii) The composite $\varphi_\alpha|_{U_\alpha \cap U_\beta} \circ \varphi_\beta|_{U_\alpha \cap U_\beta}^{-1}$ is smooth for all $\alpha, \beta \in \kappa$.

(iii) M is covered by a countable subcollection of $\{U_\alpha\}_{\alpha \in \kappa}$.

(iv) For distinct $p, q \in M$, either there is $\alpha \in \kappa$ such that $p, q \in U_\alpha$, or there are disjoint U_α and U_β containing p and q, respectively.

Then M is a smooth n-manifold with smooth atlas given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$.

Proof. We sketch the steps.

1. We provide M with a topology. We would like for Well, we say that $A \subseteq M$ is open if and only if $\varphi_\alpha(A \cap U_\alpha)$ is open for all $\alpha \in \kappa$.

2. Then condition (i) makes the φ_α into homeomorphisms onto their images. Thus, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$ is an atlas.

3. Condition (ii) implies that $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas.

4. Condition (iii) implies that M becomes second countable.

5. Lastly, condition (iv) implies that M is Hausdorff.

We leave the checks to the reader. ∎

Let's see an example of this.

Exercise 1.96. Fix nonnegative integers $k \leq n$. Then let $M := \mathrm{Gr}_k(\mathbb{R}^n)$ denote the set of k-dimensional linear subspaces V of $\mathbb{R}^n$. We show that M is a smooth $k(n-k)$-manifold.

Sketch. We use Lemma 1.95. For concreteness, let we choose our index set I to consist of pairs (P, Q) of subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n = P \oplus Q$ and $\dim P = k$ and $\dim Q = n - k$. The point is that we are choosing a complement for our k-dimensional subspaces in order to help count them. In particular, we may define the subset
$$U_\alpha := \{V \in \mathrm{Gr}_k(\mathbb{R}^n) : V \cap Q = \{0\}\}.$$

Notably, for any $V \in U_\alpha$, there is a unique linear map $M_{P,Q,V} \colon P \to Q$ such that

$$V = \{x + M_{P,Q,V}x \in P \oplus Q : x \in P\}.$$

Approximately speaking, we are viewing V as a graph. Anyway, this construction provides a map $\varphi_\alpha \colon U_\alpha \to \mathrm{Hom}_\mathbb{R}(P, Q)$ given by $V \mapsto M_{P,Q,V}$, where we identify $\mathrm{Hom}_\mathbb{R}(P, Q) \cong \mathbb{R}^{k(n-k)}$. We now conclude by noting that we can check the properties from Lemma 1.95. For example, to see that the transition maps are smooth, suppose we have two pairs $(P, Q), (P', Q') \in I$, and the vector space V decomposes into the two separate ways, and these matrices have rational functions in their coordinates, so smoothness follows. As another example, one can actually cover M by finitely many charts, and the last check follows because any k-dimensional subspaces $V, V' \subseteq \mathbb{R}^n$ has some $(n-k)$-dimensional subspace $Q \subseteq \mathbb{R}^n$ such that $V \cap Q = V' \cap Q = \{0\}$.   ∎

1.4.4 Manifolds with Boundary

Before moving on from our discussion of a single manifold, we discuss manifolds with boundary.

> **Definition 1.97** (topological manifold with boundary)**.** Fix a nonnegative integer n. A *topological n-manifold with boundary* is a Hausdorff, second countable topological space M with the following variant of being locally Euclidean: for any $p \in M$, there are open subsets $U \subseteq M$ and
>
> $$\widehat{U} \subseteq \mathbb{H} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$
>
> such that $p \in U$ and $U \cong \widehat{U}$. We continue to call (U, φ) a chart.

> **Example 1.98.** Any topological n-manifold is a topological n-manifold with boundary: one can simply make the charts output to $\mathbb{H}^\circ$.

> **Example 1.99.** The space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is a topological n-manifold with boundary.

The point is that we can pick up some "boundary" like the one in $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Anyway, let's discuss smoothness. This requires understanding smoothness on $\partial \mathbb{H}^n$.

> **Definition 1.100.** Fix a subset $A \subseteq \mathbb{R}^n$. A function $f \colon A \to \mathbb{R}^m$ is *smooth* if and only if there is an open subset $V \subseteq \mathbb{R}^n$ containing A and a smooth extension $\widetilde{f} \colon V \to \mathbb{R}^n$ of f.

> **Remark 1.101.** It turns out that (by Seeley's theorem) if $V \subseteq \mathbb{H}^n$ is open, it is enough to check that the partial derivatives of some function $f \colon V \to \mathbb{R}^m$ extend continuously to the boundary.

> **Definition 1.102** (smooth manifold with boundary)**.** Fix a nonnegative integer n. A *smooth n-manifold with boundary* is a pair $(M, \mathcal{A})$ where M is a topological n-manifold with boundary, and $\mathcal{A}$ is a maximal smooth atlas, where we are taking atlas in the sense

We will not bother to redo the proof of Proposition 1.76 to explain that the notion of a maximal smooth atlas makes sense with subsets of $\mathbb{H}^n$ in addition to subsets of $\mathbb{R}^n$; all the proofs are the same.

Note that boundary is in fact an intrinsic notion.

> **Definition 1.103** (boundary, interior)**.** Fix a smooth n-manifold with boundary M and a point $p \in M$.
>
> - Then p is a *boundary point* if and only if there is a smooth chart (U, φ) such that $\varphi(p) \in \partial \mathbb{H}^n$.
>
> - Then p is an *interior point* if and only if there is a smooth chart (U, φ) such that $\varphi(p)$ is in the interior of $\mathbb{H}^n$.

We will show in Theorem 1.104 that any point in M is exactly one of a boundary point or an interior point.

1.5 January 30

Here we go.

1.5.1 Smooth Manifolds with Boundary

We would like for the boundary of a smooth manifold with boundary to make sense.

> **Theorem 1.104.** Fix a smooth n-manifold with boundary M, and fix some $p \in M$. Given two charts (U, φ) and (V, ψ) with $p \in U \cap V$), then $\varphi(p) \in \partial \mathbb{H}^n$ if and only if $\psi(p) \in \partial \mathbb{H}^n$.

Proof. Suppose this is not the case. Then, up to rearranging, we get $\varphi(p) \in (\mathbb{H}^n)^\circ$ and $\psi(p) \in \partial \mathbb{H}^n$. Our transition maps are smooth, so we have produced a diffeomorphism from the open subsets $U' \subseteq \mathbb{H}^n$ and $V' \subseteq \mathbb{H}^n$ such that $U' \cap \partial \mathbb{H}^n = \varnothing$ but $V' \cap \partial \mathbb{H}^n \neq \varnothing$. Now, for smoothness, the transition map $\tau \colon V' \to U'$ must have an extension $\widetilde{\tau} \colon \widetilde{V}' \to \widetilde{U}'$. But then $\widetilde{\tau}$ is an invertible map, so the Inverse function theorem implies that τ is locally invertible and in particular must be an open map. But V' goes to U', which is not open in $\mathbb{R}^n$, so we have our contradiction. ∎

> **Remark 1.105.** In fact,
> $$\psi \circ \varphi^{-1}|_{\partial \mathbb{H}^n \cap \varphi(U \cap V)} \colon (\partial \mathbb{H}^n \cap \varphi(U \cap V)) \to (\partial \mathbb{H}^n \cap \psi(U \cap V))$$
> is a smooth transition map, though we will not check this here.

> **Remark 1.106.** People in the modern day might allow ∂M to be a manifold with boundary itself, which is a "manifold with corners."

> **Remark 1.107.** One can remove the smoothness assumption here as well, but it will require some cohomology or similar.

The boundary/interior for a smooth manifold may not actually be its boundary/interior when embedded into a space.

> **Example 1.108.** Consider $M := \{x \in \mathbb{R}^n : x_n > 0\}$. Then M is a smooth manifold with boundary, but $\partial M = \{x \in \mathbb{R}^n : x_n = 0\}$ when viewed as a subset of $\mathbb{R}^n$.

> **Example 1.109.** Consider $M = S^n \subseteq \mathbb{R}^{n+1}$. Then M is a smooth manifold (without boundary), but as a subspace of $\mathbb{R}^{n+1}$, we have $\partial M = M$.

> **Example 1.110.** Consider $M := \mathbb{H}^n \cap B(0, 1)$. Then M is a smooth manifold whose boundary (as a manifold) is $\partial \mathbb{H}^n \cap B(0, 1)$, but the topological boundary is $\partial \mathbb{H}^n \cup (\partial B(0, 1) \cap \mathbb{H}^n)$.

MAPS BETWEEN MANIFOLDS

*I can assure you, at any rate, that my intentions are honourable and my
results invariant, probably canonical, perhaps even functorial.*

—Andre Weil, [Wei59]

2.1   January 30-map

We continue.

2.1.1   Smooth Maps to $\mathbb{R}^n$

We will define smooth maps in steps. To begin, we say what it means to have a smooth map $M \to \mathbb{R}^n$.
Basically, we look locally at the points on our manifold and check smoothness on charts.

> **Definition 2.1** (smooth)**.** Fix a smooth manifold M, possibly with boundary. Then a function $f \colon M \to \mathbb{R}^m$
> is *smooth* if and only if each $p \in M$ has some smooth chart (U, φ) with $p \in U$ and
>
> $$f|_U \circ \varphi|_U^{-1}$$
>
> is a smooth map $\varphi(U) \to \mathbb{R}^m$.

> **Example 2.2.** Any smooth map $f \colon U \to \mathbb{H}^m$, where $U \subseteq \mathbb{H}^n$ is open, is smooth in the above sense.
> Indeed, U as an n-manifold has a smooth atlas given by $\{(U, \mathrm{id}_U)\}$, and this witnesses the smoothness
> of f for any $p \in U$.

Here is a quick sanity check: the charts don't matter.

> **Lemma 2.3.** Fix a smooth map $f \colon M \to \mathbb{H}^m$, where M is a smooth manifold, possibly with boundary.
> For any smooth chart (V, ψ), the composition $f|_U \circ \varphi|_U^{-1}$ is smooth.

Proof. This is a matter of tracking through all the definitions. Fix some $p \in V$, and we would to test smooth-
ness around p. Well, p has some smooth chart (U, φ) such that $p \in U$ and $f|_U \circ \varphi|_U^{-1}$ is smooth. But now we

write
$$f|_{U \cap V} \circ \psi|_{U \cap V}^{-1} = \left(f|_{U \cap V} \circ \varphi|_{U \cap V}^{-1}\right) \circ \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}\right),$$
which is the composition of smooth maps: the left map is smooth by construction of (U, φ), and the right map is smooth by compatibility of smooth charts. ∎

We are now ready to define smooth maps between manifolds. Approximately speaking, we simply add in a check locally on the target.

> **Definition 2.4** (smooth). Fix smooth manifolds M and N, possibly with boundary. A map $F \colon M \to N$ is *smooth* if and only if each $p \in M$ has smooth charts (U, φ) and (V, ψ) such that $p \in U$ and $F(U) \subseteq V$ and the composite
> $$\psi \circ F|_U \circ \varphi|_U^{-1}$$
> is a smooth map $\mathbb{H}^m \to \mathbb{H}^n$. We may call the above composite a *coordinate representation*.

> **Example 2.5.** Any smooth map $F \colon U \to V$, where $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open, is smooth in the above sense. Indeed, U and V have smooth atlases given by $\{(U, \mathrm{id}_U)\}$ and $\{(V, \mathrm{id}_V)\}$ (respectively), and these charts witness that F is smooth at each $p \in U$ because the composite
> $$\mathrm{id}_V \circ F \circ \mathrm{id}_U{}^{-1} = F$$
> is smooth by hypothesis.

Here's the same sanity check: the charts don't matter.

> **Lemma 2.6.** Fix a smooth map $F \colon M \to N$ of manifolds, possibly with boundary. If (U, φ) and (V, ψ) are smooth charts on M and N, respectively, and $F(U) \subseteq V$, then the composite $\psi \circ F|_U \circ \varphi|_U^{-1}$ is smooth.

Proof. Again, we track through locally, tracking through all the definitions. To check that $\psi \circ F|_U \circ \varphi|_U^{-1}$ is smooth, it suffices to check it an open cover of $\varphi(U)$. Pick $\varphi(p) \in \varphi(U)$ where $p \in U$, and we know that we have smooth charts (U_p, φ_p) and (V_p, ψ_p) in M and N, respectively, such that $F(U_p) \subseteq V_p$ and the composite $\psi_p|_{F(U_p)} \circ F|_{U_p} \circ \varphi_p|_{U_p}^{-1}$ is smooth. Then we see that

$$\psi \circ F|_U \circ \varphi|_U^{-1} = \left(\psi|_{V \cap V_p} \circ \psi_p|_{V \cap V_p}^{-1}\right) \circ \left(\psi_p \circ F|_U \circ \varphi_p|_{U_p}^{-1}\right)|_{\varphi_p(U \cap U_p)} \circ \left(\varphi_p|_{U \cap U_p} \circ \varphi|_{U \cap U_p}^{-1}\right)$$

is smooth, where the left and right maps are smooth by smooth compatibility, and the middle map is smooth by construction. ∎

> **Remark 2.7.** One can write out the above proof diagrammatically by noting that having smooth charts (U, φ) and (U', φ') of M and smooth charts (V, ψ) and (V', ψ') of N such that $F(U) \subseteq V$ and $F(U') \subseteq V'$ will have the following diagram.
> $$\begin{array}{ccc} \varphi(U) & \xrightarrow{\psi \circ F \circ \varphi^{-1}} & \psi(V) \\ \updownarrow & & \updownarrow \\ \varphi'(U') & \xrightarrow[\psi' \circ F \circ (\varphi')^{-1}]{} & \psi'(V') \end{array}$$
> Here, the vertical maps are only defined on the corresponding intersections, but it is smooth when defined by the smooth compatibility.

> **Remark 2.8.** Please read more of chapter 2 to get helpful properties of smooth maps.

2.1.2 Partition of Unity

By way of motivation, suppose we have two smooth functions $f, g \colon \mathbb{R} \to \mathbb{R}$, and we want to build a smooth function $h \colon \mathbb{R} \to \mathbb{R}$ such that $f|_{(-\infty, -1)} = h|_{(-\infty, -1)}$ and $g|_{(1, \infty)} = h|_{(1, \infty)}$. One way to do this is to find smooth functions $\varphi, \psi \colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \varphi|_{(-\infty, -1)} = 1, \\ \varphi|_{(1, \infty)} = 0. \end{cases}$$

Then $h := \varphi f + (1 - \varphi) g$ is smooth by construction, and it satisfies the restriction conditions also by construction. This idea of "splitting up the 1 function" is known as partition of unity.

> **Definition 2.9** (partition of unity). Fix a topological space X, and let $\{U_\alpha\}_{\alpha \in \kappa}$ be an open cover on M. Then a *partition of unity subordinate to* $\{U_\alpha\}_{\alpha \in \kappa}$ is a collection of continuous functions $\{\varphi_\alpha\}_{\alpha \in \kappa}$ on X satisfying the following.
>
> - $\operatorname{im} \varphi_\alpha \subseteq [0, 1]$ always.
>
> - $\operatorname{supp} \varphi_\alpha \subseteq U_\alpha$ for each α.
>
> - The collection $\{\operatorname{supp} \varphi_\alpha\}_{\alpha \in \kappa}$ is locally finite.
>
> - For each $x \in X$, we have
> $$\sum_{\alpha \in \kappa} \varphi_\alpha(x) = 1.$$

Of course, we must show that these exist.

2.2 February 1

The second homework is due later today. We began class by completing a proof, so I edited directly into those notes.

2.2.1 Partition of Unity for Manifolds

We will show that partitions of unity exist for manifolds.

> **Theorem 2.10.** Fix a smooth manifold M. For any open cover $\{U_\alpha\}_{\alpha \in \kappa}$, there is a partition of unity $\{\varphi_\alpha\}_{\alpha \in \kappa}$ (of smooth functions) subordinate to $\{U_\alpha\}_{\alpha \in \kappa}$.

Proof. We begin by constructing smooth functions $\{\widetilde{\varphi}\}_{\alpha \in \kappa}$ satisfying the following constraints.

- $\operatorname{im} \widetilde{\varphi}_\alpha \subseteq [0, \infty)$.

- $\operatorname{supp} \widetilde{\varphi}_\alpha \subseteq U_\alpha$.

- The collection $\{\operatorname{supp} \widetilde{\varphi}_\alpha\}_{\alpha \in \kappa}$ is a locally finite open cover of M.

Dividing out by the summation of the $\widetilde{\varphi}_\bullet$s completes the proof. Notably, for each $x \in M$, the sum

$$\widetilde{\varphi}(x) := \sum_{\alpha \in \kappa} \widetilde{\varphi}_\alpha(x)$$

is finite (x can only belong to finitely many of the supports); in fact, there is an open neighborhood U of x such that U only intersects finitely many of the supports, so

$$\widetilde{\varphi}|_U = \sum_{\substack{\alpha \in \kappa \\ \operatorname{supp} \widetilde{\varphi}_\alpha \cap U \neq \varnothing}} \widetilde{\varphi}_\alpha|_U$$

is just a finite sum of smooth functions, so $\widetilde{\varphi}$ is smooth on U. Thus, by gluing, $\widetilde{\varphi}$ is smooth on M, and we note that it is nonzero because each $x \in M$ is in some support, so we can define $\varphi_\alpha := \widetilde{\varphi}_\alpha / \varphi$ to satisfy all the needed conditions, most notable being that these functions are smooth, have support contained in U_α, and $\sum_{\alpha \in \kappa} \varphi_\alpha = 1$.

It remains to construct the $\widetilde{\varphi}_\alpha$s. We proceed in steps.

1. We construct a nice open cover. For each $x \in M$, we can find some open neighborhood U such that we have a homeomorphism $\varphi \colon U \to B(\varphi(x), 2)$. Then $\left\{\varphi^{-1}(B(\varphi(x), 1))\right\}_{x \in M}$ is an open cover of M, so we can refine this to a locally finite open cover $\mathcal{U}$ of precompact open sets. By looking down on compact, we may as well assume that $\mathcal{U}$ is made of coordinate balls $B(\varphi(x), r)$ contained in larger coordinate balls $B(\varphi(x), r')$ for $r' > r$.

2. Now, for each coordinate ball $\varphi \colon U \cong B(0, r)$ for $U \in \mathcal{U}$ extending to $\varphi' \colon U' \cong B(0, r')$. Then we construct f_U which is nonzero on $B(0, r)$ but vanishes on $B(0, r')$.

    Now, for each $U \in \mathcal{U}$, select $\alpha_U \in \kappa$ such that $\overline{U} \subseteq U_{\alpha_U}$. From here, we may set

    $$\widetilde{\varphi}_\alpha := \sum_{\alpha_U = \alpha} f_U,$$

    which satisfies all the needed conditions. For example, one finds that the support of $\widetilde{\varphi}_\alpha$ is

    $$\overline{\bigcup_{U \subseteq U_\alpha} U} \subseteq \bigcup_{U \subseteq U_\alpha} \overline{U} \subseteq U.$$

    One needs local finiteness in order to verify the first inclusion; the point is that one can reduce this large union to a finite one around any given point, so the closures must agree. ∎

Let's give some applications.

> **Corollary 2.11.** Fix a smooth manifold M. For any closed set $A \subseteq M$ contained in an open set $U \subseteq M$, there exists a smooth function $\psi \colon M \to \mathbb{R}$ such that $\psi|_A = 1$ and $\psi|_{M \setminus U} = 0$.

Proof. Consider the open cover $\{U, M \setminus A\}$; this is an open cover because $U \cup (M \setminus A) = M$ is equivalent to $A \subseteq U$. Then Theorem 2.10 produces two nonnegative smooth functions ψ_0 and ψ_1 such that $\operatorname{supp} \psi_0 \subseteq U$ and $\operatorname{supp} \psi_1 \subseteq M \setminus A$ and $\psi_0 + \psi_1 = 1$ everywhere. But now ψ_0 is the desired function: $\operatorname{supp} \psi_0 \subseteq U$ implies $\psi_0|_{M \setminus U}$, and $\psi_0|_A + \psi_1|_A = 1$, but $\psi_1|_A = 0$ because $\operatorname{supp} \psi_1 \subseteq M \setminus A$. ∎

> **Corollary 2.12** (Extension lemma). Fix a smooth manifold M. Further, fix a closed subset $A \subseteq M$ contained in an open set $U \subseteq M$. Given a smooth function $f \colon A \to \mathbb{R}^k$, there is a smooth function $\widetilde{f} \colon M \to \mathbb{R}^k$ extending f and with $\operatorname{supp} \widetilde{f} \subseteq U$.

Proof. Omitted. ∎

> **Corollary 2.13.** Fix a smooth manifold M. There is a nonnegative function $f \colon M \to \mathbb{R}$ such that all the sets
> $$f^{-1}([0, c])$$
> are compact for any $c \geq 0$.

Proof. Fix a countable cover $\{U_n\}_{n \in \mathbb{N}}$ of M by precompact open subsets, and let $\{\psi_n\}_{n \in \mathbb{N}}$ be the corresponding partition of unity. Then we set

$$f := \sum_{n=0}^{\infty} n\psi_n.$$

Notably, for each $c \in \mathbb{R}$, we see

$$f^{-1}([0, c]) \subseteq \bigcup_{n \leq c} \operatorname{supp} \psi_n,$$

so $f^{-1}([0, c])$ is a closed subset of a finite union of compact sets (which is compact), so we are done. ∎

> **Corollary 2.14.** Fix a closed subset K of a smooth manifold M. Then there is a nonnegative smooth function $f \colon M \to \mathbb{R}$ such that $f^{-1}(\{0\}) = K$.

Proof. One begins with $M = \mathbb{R}^n$ and then does the general case from there. ∎

2.2.2 Diffeomorphisms

Here is our definition.

> **Definition 2.15** (diffeomorphism). Fix a map $F \colon M \to N$ of smooth manifolds, possibly with boundary. Then F is a *diffeomorphism* if and only if F is bijective, smooth, and has smooth inverse.

> **Remark 2.16.** Invariance of the boundary under smooth charts implies F must send boundary points to boundary points.

> **Remark 2.17.** If F is a diffeomorphism, then $\dim M = \dim N$. Simply put, we can work locally on a chart, and then we are providing a diffeomorphism $\mathbb{R}^m \to \mathbb{R}^n$, but this can only happen when $m = n$. For example, it means that DF and DF^{-1} are invertible linear maps $\mathbb{R}^m \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively, which manifestly requires $m = n$.

> **Remark 2.18.** It turns out that topological n-manifolds with a smooth structure admit a unique smooth structure up to diffeomorphism, for $n \geq 3$. For $n \geq 4$, even $\mathbb{R}^4$ fails to have a unique smooth structure.

> **Remark 2.19.** The collection $\operatorname{Diff}(M)$ of diffeomorphisms $M \to M$ is a group, and one can give it a topology. For example, one can compute that $\operatorname{Diff}\left(S^2\right)$ is homotopy equivalent to $O(3)$, given approximately by rotations.

2.2.3 Tangent Spaces

Fix a smooth n-manifold M. One would like to provide each point $p \in M$ with an n-dimensional tangent vector space T_pM. If M is embedded into Euclidean space reasonably, we can imagine using the embedding to realize the tangent space; for example, if M is a (smooth) curve in $\mathbb{R}^2$, we can imagine that the tangent vectors tell us what direction we are moving in. We would also like to actually be able to compute these things in charts.

Anyway, here is our definition of tangent vectors. This definition is a bit awkward to handle because we want to be invariant.

> **Definition 2.20** (tangent space)**.** Fix a smooth n-manifold M and some point $p \in M$. A *derivation at p* is an $\mathbb{R}$-linear map $v \colon C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule
>
> $$v(fg) = f(p)v(g) + g(p)v(f)$$
>
> for any $f, g \in C^\infty(M)$. Then the *tangent space $T_p(M)$ at p* is the collection of derivations.

> **Remark 2.21.** Note that $T_p(M)$ is an $\mathbb{R}$-subspace of the collection of linear maps $C^\infty(M) \to \mathbb{R}$.

> **Example 2.22.** Fix $M \coloneqq \mathbb{R}^n$ and some $p \in M$. Then any $v \in \mathbb{R}^n$ has a "directional derivative" given by
>
> $$f \mapsto \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}\bigg|_p.$$
>
> This is simply by the product rule in multivariable calculus.

2.3 February 6

Today we continue talking about tangent vectors.

2.3.1 Derivations

Let's provide some basic properties of derivations.

> **Lemma 2.23.** Fix a smooth n-manifold M and a derivation $v \colon C^\infty(M) \to \mathbb{R}$ at a point $p \in M$. If $f \colon M \to \mathbb{R}$ is constant, then $v(p) = 0$.

Proof. By scaling, it suffices to do the case where $f \equiv 1$. Then we see that $f^2 = f$, so

$$v(f) = v\left(f^2\right) = 2f(p)v(f) = 2v(f),$$

so $v(f) = 1$ is forced. ∎

> **Lemma 2.24.** Fix a smooth n-manifold M and a derivation $v \colon C^\infty(M) \to \mathbb{R}$ at a point $p \in M$. Given $f, g \in C^\infty(M)$ such that $f|_U = g|_U$ for some open $U \subseteq M$ containing p, we have $v(f) = v(g)$.

Proof. Set $h \coloneqq f - g$ so that we want to show $v(h) = 0$ by linearity. The moral of the story is to extend being zero on U to all of M; in other words, we will want some bump functions. Because M is locally Euclidean, we can find a precompact open neighborhood V of p such that $\overline{V} \subseteq U$. Thus, Corollary 2.11 provides a smooth bump function $\psi \colon M \to \mathbb{R}$ such that $\psi|_{\overline{V}} \equiv 1$, and $\operatorname{supp} \psi \subseteq U$. Notably, $\psi \cdot h$ has support contained in U, but h vanishes on U, so $\psi \cdot h = 0$, so

$$0 = v(\psi \cdot h) = \psi(p)v(h) + h(p)v(\psi) = v(h),$$

as desired. ∎

Manifolds are understood by passing to local charts, and the above lemma somewhat allows us to do this. As such, we are now motivated to understand local charts.

> **Lemma 2.25.** Fix a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$. For each $v \in \mathbb{R}^n$, define $D_v|_a \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ by
>
> $$D_v|_a(f) := \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_a.$$
>
> Then $D_v|_a$ is a derivation at a. In fact, the map $D \colon \mathbb{R}^n \to T_a\mathbb{R}^n$ given by $v \mapsto D_v|_a$ is an isomorphism of vector spaces.

Proof. To check that $D_v|_a$ is a derivation, one proceeds via the product rule in multivariable calculus. We omit this check. It remains to check that we have an isomorphism.

- Linear: given $c, d \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, we compute

$$D_{cv+dw}|_a f = \sum_{i=1}^n (cv_i + dw_i) \frac{\partial f}{\partial x_i}\bigg|_a = c \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_a + d \sum_{i=1}^n w_i \frac{\partial f}{\partial x_i}\bigg|_a = (cD_v|_a + dD_v|_a)f,$$

as desired.

- Injective: by linearity, it is enough to show that having $D_v|_a = 0$ implies $v = 0$. Well, it is enough to check that $v_j = 0$ for each j. For this, we let $p_j \colon \mathbb{R}^n \to \mathbb{R}$ denote the jth projection so that

$$\frac{\partial p_j}{\partial x_i} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases}$$

so we see that

$$D_v|_a(p_j) = \sum_{i=1}^n v_i \frac{\partial p_j}{\partial x_i}\bigg|_a = v_j$$

must vanish for each j, as desired.

- Surjective: this is the heart of the matter. Fix a derivation $v \in T_a\mathbb{R}^n$. We need a candidate vector, so we define $u_i := v(p_i)$, where $p_i \colon \mathbb{R}^n \to \mathbb{R}$ is the ith projection. We claim that

$$v = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}\bigg|_a,$$

which will complete the proof. This requires a quick digression into a Taylor expansion. Given a smooth function $f \colon M \to \mathbb{R}$ and points $x, a \in \mathbb{R}^n$, we see

$$f(x) = f(a) + \int_0^1 \frac{d}{dt} f(x + t(x - a))\, dt,$$

$$= f(a) + \sum_{i=1}^n \left((x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x - a))\, dt}_{h_i(x):=} \right),$$

where in the last equality we have used the multivariable chain rule. Applying the derivation, we see

$$v(f) = \underbrace{v(f(a))}_{0} + \sum_{i=1}^n v(x_i - a_i) h_i(a) + \sum_{i=1}^n \underbrace{(a_i - a_i)}_{0} v(h_i),$$

where $v(f(a)) = 0$ by Lemma 2.23. Additionally, $v(x_i - a_i) = v(x_i) = u_i$ using Lemma 2.23 again. Notably, $h_i(a) = \frac{\partial f}{\partial x_i}|_a$, so

$$v(f) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}\bigg|_a,$$

as desired. ∎

2.3.2 Differentials of Smooth Maps

Derivations explain how to take derivatives of functions in $M \to \mathbb{R}$. We now upgrade to taking derivatives of functions between manifolds.

> **Definition 2.26** (differential). Fix smooth manifolds M and N. Given a smooth map $F\colon M \to N$, the *differential of F at $p \in M$* is the map $dF_p\colon T_pM \to T_{F(p)}N$ defined by
> $$dF_p(v)(f) := v(f \circ F)$$
> for any $f \in C^\infty(N)$.

> **Remark 2.27.** The composition of smooth functions is smooth, so $f \circ F$ is smooth, so the definition of dF_p at least makes sense. Notably, $f \mapsto (f \circ F)$ is a map $C^\infty(N) \to C^\infty(M)$ of $\mathbb{R}$-algebras, so $f \mapsto v(f \circ F)$ remains a derivation. Explicitly, it is surely $\mathbb{R}$-linear (as the composition of $\mathbb{R}$-linear maps), and we satisfy the Leibniz rule because
> $$\begin{aligned} dF_p v(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))dF_p v(g) + g(F(p))dF_p v(f). \end{aligned}$$

> **Remark 2.28.** The map $dF_p\colon T_pM \to T_{F(p)}N$ is linear, essentially by definition. Namely, for $a, b \in \mathbb{R}$ and $v, w \in T_pM$ and $f \in C^\infty(N)$, we compute
> $$dF_p(av + bw)(f) = (av + bw)(f \circ F) = av(f \circ F) + bw(f \circ G) = (adF_p(v) + bdF_p(w))(f).$$

> **Example 2.29.** Take $M := \mathbb{R}^m$ and $N := \mathbb{R}^n$, and let $F\colon M \to N$ be a smooth map, which we may as well write as $F = (F_1, \ldots, F_n)$. Now, fix some $p \in M$, and identify $\mathbb{R}^m \cong T_pM$ and $\mathbb{R}^n \cong T_{F(p)}N$ as in Lemma 2.25. Well, given some smooth $f\colon N \to \mathbb{R}$, we see
> $$dF_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(f) = \frac{\partial}{\partial x_i}(f \circ F) \overset{*}{=} \sum_{j=1}^n \left.\frac{\partial f}{\partial y_j}\right|_{F(p)} \left.\frac{\partial F_j}{\partial x_i}\right|_p = \left(\sum_{j=1}^n \left.\frac{\partial F_j}{\partial x_i}\right|_p \cdot \left.\frac{\partial}{\partial y_j}\right|_{F(p)}\right)(f),$$
> where the main point is the application of the Chain rule in $\overset{*}{=}$.

> **Remark 2.30.** Differentials behave under composition. Explicitly, let $F_1\colon M_1 \to M_2$ and $F_2\colon M_2 \to M_3$ be smooth maps. Given $p \in M_1$, we claim that
> $$d(F_2 \circ F_1)_p \overset{?}{=} (dF_2)_{F_1(p)} \circ (dF_1)_p.$$
> This can be checked directly.

> **Example 2.31.** Fix a point p on a smooth n-manifold M. Then we claim $d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}$. Indeed, we simply compute
> $$d(\mathrm{id}_M)_p(v)(f) = v(f \circ \mathrm{id}_M) = v(f).$$

2.3.3 Back to Tangent Spaces

Now that we understand how to take differentials of maps, we may realize the remark that derivations ought to be understood locally, as alluded to in Lemma 2.24.

> **Proposition 2.32.** Fix a smooth n-manifold M. Given an open neighborhood U of a point $p \in M$, the inclusion $i \colon U \hookrightarrow M$ is smooth, and $di_p \colon T_pU \to T_pM$ is an isomorphism of vector spaces.

Proof. Remark 2.28 tells us that this map is linear. It remains to check injectivity and surjectivity, which we do by hand.

- Injective: if $di_p(v) = 0$, then $v(f \circ i) = 0$ for all $f \in C^\infty(M)$, or equivalently, $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We would now like to show that v is actually zero. Well, pick up some $g \in C^\infty(U)$, and we want to show that $v(g) = 0$.

  Well, choose some open precompact open neighborhood B around p such that $\overline{B} \subseteq U$. Then Corollary 2.11 provides us with a smooth bump function $\psi \colon M \to \mathbb{R}$ which is 1 on $\overline{B}$ and vanishes outside U. Then $g\psi$ is actually smooth (it is smooth on U because g and ψ are both smooth there, and it is smooth outside U because the function is zero there), so $v(g\psi|_U) = 0$. But $g\psi$ and g agree on B, so $v(g) = v(g\psi|_U)$ by Lemma 2.24, as needed.

- Surjective: fix some derivation $\widetilde{v} \in T_pM$, and we want some $v \in T_pU$ such that $\widetilde{v}(f) = v(f|_U)$ for all $f \in C^\infty(M)$. The main point is the construction of U.

  Given a smooth function $f \in C^\infty(U)$, we want to define $\widetilde{v}(f)$. Well, as in the previous step, we may define $\widetilde{f} \colon M \to \mathbb{R}$ such that there is an open neighborhood $B \subseteq U$ of p with $f|_B = \widetilde{f}|_B$. Then we define $v(f) := \widetilde{v}(\widetilde{f})$. Note that $v(\widetilde{f})$ does not depend on the choice of $\widetilde{f}$ and B: well, given another pair of $\widetilde{f}'$ and B', we see that $\widetilde{f}|_{B \cap B'} = \widetilde{f}|_{B \cap B'}$, so they have the same value of $\widetilde{v}$ under Lemma 2.24.

  Additionally, we note that v is in fact a derivation: given $f, g \in C^\infty(U)$ and smooth extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)$ agreeing on $B_f, B_g \subseteq M$, respectively, we see

  $$\widetilde{v}(\widetilde{f}\widetilde{g}) = \widetilde{f}(p)\widetilde{v}(\widetilde{g}) + \widetilde{g}(p)\widetilde{v}(\widetilde{f})$$

  because $\widetilde{v}$ is a derivation, but then this immediately produces $v(fg) = f(p)v(g) + g(p)v(f)$ by checking the definitions. Similarly, we have

  $$\widetilde{v}(a\widetilde{f} + b\widetilde{g}) = a\widetilde{v}(\widetilde{g}) + b\widetilde{v}(\widetilde{g}),$$

  so $v(af + bg) = av(f) + bv(g)$, so v is linear.

  Lastly, we note that $\widetilde{v}(f) = v(f|_U)$ for any $f \in C^\infty(M)$ by construction. Namely, f is a perfectly fine extension of $f|_U$ agreeing on some open neighborhood of p contained in U (for example, taking U to be the needed open neighborhood itself will work), so we conclude. ∎

> **Corollary 2.33.** Fix a smooth n-manifold M. For any $p \in M$, we have $\dim_\mathbb{R} T_pM = n$.

Proof. Fix a smooth chart (U, φ) around $p \in M$. Then we have the sequence of isomorphisms

$$T_pM \cong T_pU \cong T_{\varphi(p)}\varphi(U) \cong T_p\mathbb{R}^n \cong \mathbb{R}^n.$$

The first and third isomorphisms are by Proposition 2.32. The second isomorphism is by functoriality of the tangent space from Remark 2.30 and Example 2.31; namely, the differential of a diffeomorphism must be an isomorphism by functoriality. And the last isomorphism is by Lemma 2.25. ∎

While we're here, we take a moment to understand how these derivations behave under coordinates.

> **Remark 2.34.** Please read about how to provide the differential of a smooth map on coordinates.

So here are some coordinate computations.

- Fix a smooth n-manifold M and a point $p \in M$. Given a smooth chart (U, φ) around p, we give φ its coordinates $\varphi := (x_1, \ldots, x_n)$. For example, given $f \in C^\infty(U)$, we are able to define

$$\left. \frac{\partial}{\partial x_i} \right|_p f := \left. \frac{\partial f}{\partial \widetilde{x}_i} \right|_{\varphi(p)} \left(f \circ \varphi^{-1} \right),$$

  where $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are the coordinates of M. By tracking the isomorphisms of Corollary 2.33 through, we can see that the above derivations form a basis for $T_p M$. Indeed, it suffices to show that they are a basis for the derivations on $T_p U$, and by passing through φ, it is enough to see that $\partial f / \partial \widetilde{x}_i|_{\varphi(p)}$ form a basis of derivations on $T_{\varphi(p)} U$. But it's now enough to see that we have a basis on $T_p \mathbb{R}^n$, which is simply Lemma 2.25.

- We examine change of coordinates. Fix a smooth n-manifold M and a point $p \in M$ covered by the charts (U, φ) and (V, ψ). As above, we give coordinates as $\varphi := (x_1, \ldots, x_n)$ and $\psi := (y_1, \ldots, y_n)$, and we give the target spaces the coordinates $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$, respectively.

  Well, on the restrictions, we will choose coordinate representations by

$$(\psi \circ \varphi^{-1})(\widetilde{x}) := (\overline{y}_1(\widetilde{x}), \ldots, \overline{y}_n(\widetilde{x})),$$

  and we in particular see that

$$\left. \frac{\partial}{\partial y_j} \right|_p y_k = \left((d\psi^{-1})_{\psi(p)} \left. \frac{\partial}{\partial \widetilde{y}_j} \right|_{\psi(p)} \right) y_k$$

$$= \left. \frac{\partial}{\partial \widetilde{y}_j} \right|_{\psi(p)} \left(y_k \circ \psi^{-1} \right)$$

$$= \left. \frac{\partial}{\partial \widetilde{y}^j} \right|_{\psi(p)} \widetilde{y}_k$$

$$= 1_{j=k}.$$

  The moral of the story is that some $v = \sum_{k=1}^m v_k \partial / \partial y_k|_p$ will have

$$\left. \frac{\partial}{\partial x_i} \right|_p = \sum_{k=1}^n \left. \frac{\partial \overline{y}_k}{\partial \widetilde{x}_i} \right|_{\varphi(p)} \left. \frac{\partial}{\partial y_j} \right|_p.$$

2.4   February 8

Here we go.

2.4.1   Velocity Vectors

Let's discuss a more geometric variant of tangent vectors.

> **Definition 2.35** (velocity vector)**.** Fix a smooth n-manifold M and a point $p \in M$. Define the space $\mathcal{J}_p M$ to be the set of smooth curves $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ (and $\varepsilon > 0$). We say that $\gamma_1, \gamma_2 \in \mathcal{J}_p$ are equivalent, written $\gamma_1 \sim \gamma_2$, if and only if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for any $f \in C^\infty(M)$.

> **Remark 2.36.** We won't bother checking that $\sim$ is an equivalence relation; it holds because we are basically checking equalities after passing to $\mathbb{R}^{C^\infty(M)}$ by sending $\gamma \mapsto ((f \circ \gamma)'(0))_f$.

And here is how this relates to tangent vectors.

> **Lemma 2.37.** Fix a smooth n-manifold M and a point $p \in M$. Then T_pM is in natural bijection with $\mathcal{J}_pM/\sim$.

Proof. In one direction, one can send some $[\gamma] \in (\mathcal{J}_pM/\sim)$ to the derivation $v_{[\gamma]} \colon f \mapsto (f \circ \gamma)'(0)$. Note that this only depends on the class $[\gamma]$ rather than the representative γ by definition of the equivalence relation $\sim$. This map is injective essentially by construction, and one can show by hand that it is surjective, for example by working locally on charts and then using lines as the needed curve to realize a differential in T_pM. ∎

2.4.2   The Tangent Bundle

Let's glue our tangent spaces together.

> **Remark 2.38.** Given $p, q \in \mathbb{R}^n$, there is a natural identification $T_p\mathbb{R}^n \to T_q\mathbb{R}^n$. One can see this on velocity vectors by moving the curves over by hand. Alternatively, let $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation sending $T \colon p \mapsto q$, which is a diffeomorphism, and then we know we have an isomorphism $dT_p \colon T_p\mathbb{R}^n \to T_q\mathbb{R}^n$. (Recall functoriality of T_p implies that diffeomorphisms produce isomorphisms.)

In general, it is somewhat difficult to identify these tangent spaces naturally.

> **Definition 2.39** (tangent bundle)**.** Fix a smooth n-manifold M. Then *tangent bundle* TM is
> $$TM := \bigsqcup_{p \in M} T_pM.$$
> Morally, TM consists of all the tangent spaces glued together.

> **Proposition 2.40.** Fix a smooth n-manifold M. Then TM is a smooth $2n$-manifold.

Proof. We will use Lemma 1.95. Quickly, note that we have a projection $\pi \colon TM \to M$ given by $\pi(p, v) := p$.

Now, for each smooth chart (U, φ) on M, we define the chart $(\pi^{-1}U, \widetilde{\varphi})$ on TM, where $\widetilde{\varphi} \colon \pi^{-1}U \to (\operatorname{im}\varphi) \times \mathbb{R}^n$ is defined by
$$\widetilde{\varphi} \colon \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\bigg|_p \mapsto (\varphi(p), (v_1, \ldots, v_n)).$$
Recall $(\partial/\partial x_i)|_p = d\varphi^{-1}_{\varphi(p)}(\partial/\partial \widetilde{x}_i)$, where $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are coordinates chosen on is We now have to check our various conditions. For example, $\widetilde{\varphi}$ is a bijection to an open subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{n+1}$ by construction.

(i) Given two (U, φ) and (V, ψ), we need $\widetilde{\varphi}\left(\pi^{-1}U \cap \pi^{-1}V\right)$ to be open in $\mathbb{R}^{2n}$. But this is $\widetilde{\varphi}\left(\pi^{-1}(U \cap V)\right)$, which is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ because $(U \cap V, \varphi|_{U \cap V})$ is a smooth chart on M, so the argument above applies.

(ii) Given two (U, φ) and (V, ψ), we need the composite $\widetilde{\varphi} \circ \widetilde{\psi}^{-1}$ to be smooth, when suitably restricted. Well, one simply commutes the change-of-coordinates for the part on the tangent spaces, and on points, we simply use that $\varphi \circ \psi^{-1}$ is smooth already. Explicitly, one finds that this is
$$(\widetilde{x}, v) \mapsto \left(\left(\varphi \circ \psi^{-1}\right)(\widetilde{x}), \sum_{i=1}^n v_i \frac{\partial \overline{y}_\bullet}{\partial \widetilde{x}_i} \frac{\partial}{\partial \widetilde{y}_\bullet} \right).$$

(iii) A countable cover of M by charts produces a countable cover of TM by charts upon pulling back by π.

(iv) Fix distinct $(p, v), (q, w) \in TM$. If $p \neq q$, then we can find disjoint smooth charts (U, φ) and (V, ψ) on M, so $\left(\pi^{-1}U, \widetilde{\varphi}\right)$ and $\left(\pi^{-1}V, \widetilde{\psi}\right)$ provided the needed disjoint charts. Otherwise, $p = q$, and then p and q are of course contained in the same chart (U, φ), so (p, v) and (q, w) are contained in the same chart $\left(\pi^{-1}U, \widetilde{\varphi}\right)$. ∎

Example 2.41. One has $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

Example 2.42. One has $TS^1 = S^1 \times \mathbb{R}$ and $TS^3 = S^3 \times \mathbb{R}^3$ and even $TS^7 = S^7 \times \mathbb{R}^7$.

Example 2.43. For even n, one has $TS^n \neq S^n \times \mathbb{R}^n$, which is essentially a consequence of the Hairy ball theorem: one would be able to produce n linearly independent elements of $S^n \times \mathbb{R}^n$ and then pull them back to n linearly independent vector fields TS^n, which do not exist for even n. The same inequality holds for odd $n \notin \{1, 3, 7\}$.

2.4.3 Maps of Constant Rank

We are going to want some inverse function theorems. Here is the most basic case. Morally, the statement is that invertible derivative should mean locally invertible.

Theorem 2.44 (Inverse function). Fix a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Given $x_0 \in \mathbb{R}^n$, if the map $(Tf)_{x_0}\colon T_{x_0}\mathbb{R}^n \to T_{f(x_0)}\mathbb{R}^n$ is invertible, then there is an open neighborhood $U \subseteq \mathbb{R}^n$ around x_0 such that $f|_U$ is a diffeomorphism.

By working on charts, the following result is basically immediate.

Theorem 2.45 (Inverse function). Fix a smooth function $f\colon M \to N$ of n-manifolds. Given $x_0 \in \mathbb{R}^n$, if the map $(Tf)_{x_0}\colon T_{x_0}M \to T_{f(x_0)}N$ is invertible, then there is an open neighborhood $U \subseteq M$ around x_0 such that $f|_U$ is a diffeomorphism.

This condition is good enough to make into a definition.

Definition 2.46. Fix a smooth function $F\colon M \to N$ of n-manifolds. Then F is a *local diffeomorphism* at p if and only if dF_p is invertible. Equivalently, by Theorem 2.45, there is an open neighborhood U of p such that $F|_U$ is a diffeomorphism onto its image.

Remark 2.47. Of course, the converse direction (local diffeomorphism implies invertible derivative) is just by functoriality of the tangent space construction.

Remark 2.48. By gluing, if F has invertible derivative at all points, and F is a bijection, then one can see that F^{-1} must be locally a diffeomorphism at all points, so in particular F^{-1} is smooth, so F is fully a diffeomorphism.

Example 2.49. The map $F\colon \mathbb{R} \to S^1$ given by $x \mapsto (\cos x, \sin x)$ is not injective, but it is a local diffeomorphism.

More generally, one could require something weaker than full invertibility.

> **Definition 2.50** (immersion, submersion, full rank, constant rank)**.** Fix a map $F \colon M \to N$ of smooth manifolds, where $m \coloneqq \dim M$ and $n \coloneqq \dim N$.
>
> - F is an *immersion* if and only if dF_p is injective for all $p \in M$.
>
> - F is a *submersion* if and only if dF_p is surjective for all $p \in M$.
>
> - F has *full rank* if and only if $\operatorname{rank} dF_p = \min\{m, n\}$ for all $p \in M$ (notably, this is as large as possible).
>
> - F has *constant rank* if and only if dF_p has the same rank for all $p \in M$ (notably, this is as large as possible).

We now state the following theorem.

> **Theorem 2.51.** Fix a map $F \colon M \to N$ of smooth manifolds. If dF_p has full rank for some $p \in M$, then there is an open neighborhood U of p such that $F|_U$ has full rank.

Proof. The condition that dF_p having full rank is equivalent to the determinant of some largest submatrix being nonzero. So one has a map $M \to \mathbb{R}^N$ for some large N taking $p \in M$ to the list of determinants of these submatrices of dF_p, and this map is continuous, so the set of points not going to zero is open and contains p. ∎

> **Example 2.52.** Fix two manifolds M and N, and fix some $y_0 \in N$.
>
> - The map $x \mapsto (x, y_0)$ is an immersion.
>
> - The projection map $M \times N \to M$ is a submersion.

> **Example 2.53.** Fix a smooth curve $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ with non-vanishing derivative everywhere. Then γ is an immersion.

2.5 February 13

Here we go.

2.5.1 The Rank Theorem

It will turn out that maps of constant rank basically look like projections.

> **Example 2.54.** The projection $F \colon \mathbb{R}^2 \to \mathbb{R}$ given by $F \colon (x, y) \mapsto x$ is a submersion. Namely, $dF = (1, 0)$ for each p, so $\operatorname{rank} dF_p = 1$ for all p.

Our result will arise from some change of basis.

> **Proposition 2.55.** Fix a linear map $L \colon V \to W$ of finite-dimensional $\mathbb{R}$-vector spaces of rank r. Then there is a basis of V and a basis of W such that L has matrix representation given by
>
> $$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$
>
> where I is an $r \times r$ identity matrix.

Proof. Put any given matrix L in row-reduced Echelon form and then move the columns around as needed. Row and column operations correspond to adjusting bases of V and W. ∎

So here is our result.

> **Theorem 2.56** (Constant rank). Fix a smooth m-manifold M and a smooth n-manifold N, and fix a smooth map $F\colon M \to N$ of constant rank r. For each p, there are smooth coordinate charts (U, φ) on M and (V, ψ) such that $p \in U$, $F(U) \subseteq V$, and F has a coordinate representation given by
> $$F(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

Proof. Smoothness allows us to choose some coordinate representation, so we may assume that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. In our choice of coordinate representation, we may also assume that $p = 0 \in \mathbb{R}^m$ and $F(p) = 0 \in \mathbb{R}^n$. We are basically trying to "straighten out" F around p.

The name of the game is to find a diffeomorphism φ on an open neighborhood $U \subseteq \mathbb{R}^m$ of 0 and a diffeomorphism ψ on an open neighborhood $V \subseteq \mathbb{R}^n$ of 0 such that
$$\psi \circ F \circ \varphi^{-1}$$
is going to look as in the statement. We proceed in steps.

1. Using change-of-basis isomorphisms $A\colon \mathbb{R}^m \to \mathbb{R}^m$ and $B\colon \mathbb{R}^n \to \mathbb{R}^n$ so that $d(B \circ F \circ A)_0 = dB_0 \circ dF_0 \circ dF_0$ now looks like
   $$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$
   (We are using Proposition 2.55 to find A and B.) The point is that F looks how we want locally at 0.

2. We apply the Inverse function theorem to straighten out the first r coordinates. While we're here, we establish our coordinate as follows: given the domain of F the coordinates $(x_1, \ldots, x_r, y_1, \ldots, y_{m-r})$, and give the codomain of F the coordinates $(x'_1, \ldots, x'_r, y'_1, \ldots, y'_{n-r})$. Under these coordinates, say F is $F(x, y) = (Q(x, y), R(x, y))$.

   To straighten out Q, we set $\varphi(x, y) := (Q(x, y), y)$. We would like for φ to be a diffeomorphism local at 0, which we can compute as $\mathrm{id}_{\mathbb{R}^m}$: on the first r coordinates, we are $Q(x, y)$, which is I_m locally, and on the last $n - r$ coordinates, we are y, which continues to be the identity. Thus, φ is in fact locally a diffeomorphism on some open neighborhood U of 0. So we may compute
   $$\left(F \circ \varphi^{-1}\right)(x, y) = \left(x, (R \circ \varphi^{-1})(x, y)\right).$$

3. We remove the dependence of $F \circ \varphi^{-1}$ on y. Computing our current differential, we get
   $$d(F \circ \varphi^{-1})_{(x,y)} = \begin{bmatrix} I_r & 0 \\ \frac{\partial(R \circ \varphi^{-1})}{\partial x} & \frac{\partial(R \circ \varphi^{-1})}{\partial y} \end{bmatrix}.$$

   However, for F to have constant rank r, we see that we must have $\frac{\partial(R \circ \varphi^{-1})}{\partial y} = 0$; in other words, this composite does not depend on y. (In other words, it is constant with respect to y.) So we set $S(x) := (R \circ \varphi^{-1})(x, y)$. So we now have
   $$\left(F \circ \varphi^{-1}\right)(x, y) = (x, S(x)).$$

4. We straighten out the remaining $n - r$ coefficients using the Inverse function theorem. Namely, define $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ by
   $$\psi(x', y') := (x', S(x') - y').$$
   Computing the differential at 0 shows that ψ is locally a diffeomorphism, so we may use it as a chart. We now conclude by computing $\left(\psi \circ F \circ \varphi^{-1}\right)(x, y) = (x, 0)$, as required. ∎

> **Remark 2.57.** Please read the Global rank theorem.

2.5.2 Embeddings

Here is our definition.

> **Definition 2.58** (embedding)**.** Fix smooth manifolds M and N. A smooth map $F\colon M \to N$ is an *embedding* if and only if F is an injective immersion and a homeomorphism onto its image.

> **Remark 2.59.** The image of a smooth map does not necessarily make sense as a smooth manifold, which is why we are only requiring a homeomorphism onto the image instead of a diffeomorphism onto its image.

Here is how one might check this.

> **Lemma 2.60.** Fix a smooth map $F\colon M \to N$. Then F is an embedding if and only if F is an injective immersion, and given any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M$ and $x \in M$ such that $Fx_n \to Fx$ as $n \to \infty$, we have $x_n \to x$ as $n \to \infty$.

Proof. The forward direction is clear because the inverse homeomorphism must take convergent sequences to convergent sequences. The reverse direction amounts to checking the continuity of F^{-1}, which is basically what the condition says on sequences. ∎

> **Example 2.61.** Fix smooth manifolds M and N. For $p \in N$, the inclusion map $M \times \{p\} \to M \times N$ is an embedding.

> **Non-Example 2.62.** Any curve $\gamma\colon [0,1] \to \mathbb{R}^n$ with self-intersection fails to be injective, so γ fails to be an embedding.

> **Non-Example 2.63.** Consider the map $\gamma\colon [0,2\pi) \to \mathbb{R}^2$ by $\gamma(x) := (\cos x, \sin x)$. Then as $x \to 2\pi$, we have $\gamma(x) \to \gamma(0)$, which contradicts Lemma 2.60.

> **Non-Example 2.64.** Consider the map $F\colon \mathbb{R}^+ \to \mathbb{R}^2$ by $F(t) := (t, \sin 1/t)$. One can see that F is in fact an embedding, but if we add in some $(-1,1) \to \mathbb{R}^2$ by $s \mapsto (0,s)$, then $F\colon (\mathbb{R}^+ \sqcup (-1,1)) \to \mathbb{R}^2$ is no longer an embedding. The point is that there are points in $\operatorname{im} F$ converging to $\{0\} \times (-1,1)$, but this is bad news because points in $\mathbb{R}^+$ are not going to converge to $(-1,1)$.

> **Non-Example 2.65.** Fix $T^2 := S^1 \times S^1$, and realize S^1 as $\mathbb{R}/\mathbb{Z}$. Then $F\colon \mathbb{R} \to T^2$ defined by $t \mapsto (\alpha t, \beta t)$ for $\alpha, \beta \in \mathbb{R}^\times$ is never an embedding.
>
> - If $\alpha/\beta \in \mathbb{Q}$, then one can see that F is periodic, so it fails to be injective. Namely, if $\beta = (r/s)\alpha$, then $F(st)$.
>
> - When $\alpha/\beta \notin \mathbb{Q}$, some Diophantine approximation implies that $\operatorname{im} F$ is dense in T^2, so it cannot be an embedding.

> **Non-Example 2.66.** Consider $F\colon \mathbb{R} \to \mathbb{R}$ by $F(t) := t^3$. Then F does not have constant rank, so F is not an embedding.

Compactness makes many of these pathologies disappear.

> **Proposition 2.67.** Fix an injective immersion $F\colon M \to N$ of smooth manifolds. Then F is an embedding.

Proof. We need to show that F is a homeomorphism onto its image. Because F is a continuous injection, it suffices to show that the map $F\colon M \to \operatorname{im} F$ is an open map, for which it suffices to show that it is actually a closed map. Well, any closed subset $V \subseteq M$ is compact because M is compact, so $F(V)$ is compact, so $F(V) \subseteq \operatorname{im} F$ is closed because $\operatorname{im} F \subseteq N$ is Hausdorff. ∎

Similarly, looking locally makes many of these pathologies disappear.

> **Proposition 2.68.** Fix an immersion $F\colon M \to N$. Given $p \in M$, there is an open neighborhood U of p such that $F|_U$ is an embedding.

Proof. This follows somewhat quickly from Theorem 2.56. ∎

> **Remark 2.69.** If $\dim M = \dim N$, then the above result follows rather quickly from the Inverse function theorem.

> **Remark 2.70.** Please read about submersions and smooth covering maps.

2.5.3 Submanifolds

Our naïve definition is simply that we are a subset with inherited smooth structure.

> **Definition 2.71** (embedded smooth submanifolds)**.** Fix a smooth manifold M. Then a subspace $S \subseteq M$ is an *embedded smooth submanifold* if and only if S is a manifold with smooth structure such that the inclusion $S \hookrightarrow M$ is a smooth embedding. In other words, we are asking that S is the image of a smooth embedding $F\colon N \to M$.

> **Example 2.72.** Fix an open subset $S \subseteq M$. Then the inclusion $S \hookrightarrow M$ is of course an embedding, so S is a submanifold.

> **Example 2.73.** Fix a countable discrete set of points $S \subseteq M$. Then the inclusion $S \hookrightarrow M$ is smooth of rank 0.

2.6 February 15

The midterm is in two weeks.

2.6.1 Proper Embeddings

The following notion will be useful.

> **Definition 2.74.** An embedded smooth submanifold $S \subseteq M$ is *properly embedded* if and only if the inclusion $S \hookrightarrow M$ is proper; i.e., the inverse image of a compact subset of M is still compact in S.

Non-Example 2.75. There is an embedding $\mathbb{R}^2 \to S^2$ by inverting the stereographic projection map $\left(S^2 \setminus \{(0,0,1)\}\right) \to \mathbb{R}^2$. However, this is not proper: all of S^2 is compact, but its pre-image in $\mathbb{R}^2$ is all of $\mathbb{R}^2$, which is not compact.

Here is a nice way to check properness.

Proposition 2.76. Fix an embedded smooth submanifold $S \subseteq M$. Then S is properly embedded if and only if $S \subseteq M$ is closed.

Proof. We have two directions to show.

- Suppose $S \subseteq M$ is closed. Well, for any compact subset $K \subseteq M$, we see that $S \cap K$ is closed in M (it is the intersection of two closed subsets of M), so $S \cap K$ is a closed subset of the compact set K, so $S \cap K$ continues to be compact.

- Suppose $S \subseteq M$ is properly embedded. Then we want to show that $S \subseteq M$ is closed. Well, it suffices to check that S contains all of its limit points, so suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of points in S which converges to some point $x \in M$; then we want to show that $x \in S$.

  Well, we note that the subset $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact (any open cover has an open neighborhood of x, and this open neighborhood has all but finitely many of the x_ns), so $(\{x_n : n \in \mathbb{N}\} \cup \{x\}) \cap S$ continues to be compact by the proper embedding. But if $x \notin S$, then $\{x_n : n \in \mathbb{N}\}$ fails to be compact, so instead we must have $x \in S$. ∎

2.6.2 Slice Charts

Here is our definition.

Definition 2.77 (slice). Fix a smooth n-manifold and a smooth chart (U, φ), where we give φ the coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then a k-*slice of* (U, φ) is the slice

$$S(c_{k+1}, \ldots, c_n) := \{p \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell > k\}.$$

Conversely, a chart (U, φ) is a k-*slice chart* for a given subset $S \subseteq U$ if and only if $S = S(c_{k+1}, \ldots, c_n)$ for some real numbers $(c_{k+1}, \ldots, c_n)$. Then a subset $S \subseteq M$ satisfies the *local k-slice condition* if and only if any $p \in S$ has a smooth chart (U, φ) around p such that (U, φ) is a k-slice chart for $S \cap U$.

Example 2.78. Fix a smooth function $f \colon \mathbb{R}^m \to \mathbb{R}^n$, and define the graph

$$\Gamma(f) := \left\{(x, f(x)) \in \mathbb{R}^{m+n} : x \in \mathbb{R}^m\right\}.$$

Then $\Gamma(f) \subseteq \mathbb{R}^{m+n}$ is a (global) m-slice chart for the chart $(\mathbb{R}^{m+n}, \varphi)$, where φ is the map $\varphi(x, y) := (x, y - f(x))$. (Note that φ is of course smooth, and it has smooth inverse given by $(x, y) \mapsto (x, y + f(x))$.) Namely,

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : \varphi(x, y) = (x, 0)\},$$

so we are indeed a slice chart.

Here is our theorem. Approximately, we are saying embedded submanifolds locally look like slices.

Theorem 2.79 (Slice). Fix a smooth n-manifold M. A subset $S \subseteq M$ is an embedded k-dimensional submanifold if and only if S satisfies the local k-slice condition.

Proof. We have two implications to show, which we do separately.

- Suppose that S is an embedded k-dimensional submanifold of M, and let $F \colon S \to M$ to be the embedding. We need to show that S satisfies the local k-slice condition. Well, fix some $p \in S$, and we need a k-slice chart (U, φ) around $p \in U$. For this, we use Theorem 2.56, which provides us with smooth charts (U, φ) and (V, ψ) around $p \in S$ and $F(p) \in M$, respectively, such F has a coordinate representation given by

$$\widehat{F}(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0),$$

where $\widehat{F} := \psi \circ F \circ \varphi^{-1}$.

We are almost done, except for a technicality that V might contain other parts of S. For brevity, let $\widehat{U} := \varphi(U)$ and $\widehat{V} := \psi(V)$ to be subsets of Euclidean space; notably, $\widehat{F}(\widehat{U}) = \widehat{U} \times \{0\}$. To begin our restriction, set $\widehat{V}' := \widehat{V} \cap \left(\widehat{U} \times \mathbb{R}^{n-k} \right)$ and $V' := \psi^{-1}(\widehat{V}')$, so we are excluding points of S not in U which are near p. To exclude points not near p, note we can write $U = U' \cap S$ where $U' \subseteq M$ is open, so we define

$$V'' := V' \cap U'.$$

We set $\psi'' := \psi|_{V''}$.

We now claim that (V'', ψ'') is the needed local k-slice chart of S around p. Indeed, we claim that

$$S \cap V'' \overset{?}{=} \{ q \in V'' : \psi''_\ell(q) = 0 \text{ for } \ell > k \}.$$

In one direction, $q \in V'' \cap S$ implies $q \in U$ by construction, but then $\psi''(q) = \psi(q) \in \mathbb{R}^{n-k} \times \{0\}$ by definition of ψ. In the other direction, if $q \in V''$ has $\psi''_\ell(q) = 0$ for $\ell > k$, then (for example) $\psi(q) \in \widehat{U} \times \mathbb{R}^{n-k}$ because that is where V' goes to, so actually $\psi(p) \in \widehat{U} \times \{0\} = \widehat{F}(\widehat{U})$, so $p \in \varphi^{-1}(\widehat{U})$ by undoing $\widehat{F}$, so $p \in S$ by definition.

- Suppose that S satisfies the local k-slice condition. Then we want to give a smooth structure to S so that the inclusion makes S into a smooth embedded submanifold. Well, give $S \subseteq M$ the subspace topology; then this makes S a homeomorphism onto its image automatically, so notably S is Hausdorff and second countable.

It remains to give S some smooth charts. Well, fix some $p \in S$, and satisfying the k-slice chart condition promises us a chart (U, φ) around p so that

$$S \cap U = \{ q \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell > k \}$$

for some given real numbers $c_{k+1}, \dots, c_n$. These last $(n - k)$ coordinates shouldn't matter, so we let $\pi \colon \mathbb{R}^n \to \mathbb{R}^k$ denote the projection onto the first k coordinates. As such, we set $V := U \cap S$ and $\widehat{V} := (\pi \circ \varphi)(V)$, which is an open subset of

$$\varphi(U) \cap \{ x \in \mathbb{R}^n : x_\ell = c_\ell \text{ for } \ell > k \}.$$

The above is open in the subspace defined by the plane at the right, so it is open when projected down to π, which can be checked because π is a quotient map.

So we will let $(V, \pi \circ \varphi)$ become the relevant chart. For example, we can check that $\pi \circ \varphi$ is a homeomorphism: indeed, its inverse map is given by $\varphi^{-1} \circ j$, where $j(x_1, \dots, x_k) := (x_1, \dots, x_k, c_{k+1}, \dots, c_n)$, and φ^{-1} and j are both smooth. This concludes the proof that S is a topological k-manifold.

We now check smooth compatibility of the given charts to show that we have actually given S a smooth structure. Well, choose two charts (V, ψ) and (V', ψ') of S which are constructed as above from charts (U, φ) and (U', φ') of M. Well, the transition map $\psi' \circ \psi^{-1}$ is given by

$$\pi' \circ \varphi' \circ \varphi^{-1} \circ j,$$

where j and π and j' and π' are given as above. This transition map is smooth because it is the composition of smooth maps.

Lastly, we must check that the embedding $S \to M$ is smooth. Well, for any $p \in S$, choose a smooth chart (V, ψ) arising from the smooth chart (U, φ) on M, as constructed above. Then the inclusion $F \colon S \subseteq M$ sends $V \subseteq U$, and the composite $\varphi \circ F \circ \psi^{-1}$ is just the identity, so it is smooth. $\blacksquare$

Here is a consequence of the above proof.

> **Corollary 2.80.** Fix a smooth n-manifold M, and let $S \subseteq M$ be a smooth embedded submanifold. Then for any k-slice chart (U, φ) of S, one finds that $(U \cap S, (\varphi_1, \ldots, \varphi_k))$ where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a coordinate expansion.

Proof. The second part of the proof of Theorem 2.79 establishes this.   ∎

2.6.3 Level Sets

A common way to build embedded submanifolds is via level sets. Let's begin with a couple examples.

> **Example 2.81.** Consider the smooth function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 + y^2$. For example, $f^{-1}(\{1\}) = S^1$ and $f^{-1}(\{0\}) = \{(0,0)\}$ and $f^{-1}(\{-1\}) = \varnothing$.

> **Example 2.82.** Consider the smooth function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 - y^2$. Then $f^{-1}(\{1\})$ is a hyperbola with two connected components, but $f^{-1}(\{0\})$ looks like two crossing lines.

> **Remark 2.83.** Given any closed set $A \subseteq M$, we remarked earlier that there is a smooth function $f \colon M \to \mathbb{R}$ such that $f(\{0\}) = A$. So it cannot be the case that level sets always give nice submanifolds.

We do expect that we should get a submanifold "generically." Here is one instance of this.

> **Theorem 2.84.** Fix a smooth map $F \colon M \to N$ of constant rank r between the m-manifold M and n-manifold N. Then any $q \in \operatorname{im} F$ makes the level set $F^{-1}(\{q\})$ is a proper embedded submanifold of M of dimension $(m - r)$.

Morally, the dimensions of M must go somewhere, and there are r dimensions going out into N.

> **Example 2.85.** Consider the smooth function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 - y^2$. Then
> $$df_{(x,y)} = \begin{bmatrix} 2x & -2y \end{bmatrix},$$
> so the function $f|_{\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})}$ is a smooth map of constant rank 1, so Theorem 2.84 tells us that all of its fibers will be proper submanifolds of M of dimension $2 - 1 = 1$.

2.7 February 20

The fun never ends.

2.7.1 More on Level Sets

Last class we were stated the following result.

> **Theorem 2.84.** Fix a smooth map $F \colon M \to N$ of constant rank r between the m-manifold M and n-manifold N. Then any $q \in \operatorname{im} F$ makes the level set $F^{-1}(\{q\})$ is a proper embedded submanifold of M of dimension $(m - r)$.

Proof. We apply Theorem 2.79 to $S := F^{-1}(\{q\})$, for which we will use Theorem 2.56. For each $p_0 \in S$, we receive smooth charts (U, φ) on M (around p_0) and (V, ψ) on N (with $F(U) \subseteq V$) such that

$$\left(\psi \circ F \circ \varphi^{-1}\right)(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

In particular, write $\psi(q) = (c_1, \ldots, c_r, 0, \ldots, 0)$, and we see that

$$S \cap U = \{p \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell \leq r\},$$

which is in fact an $(m-r)$-slice. Thus, Theorem 2.79 applies, and to finish up, we note that $S \subseteq M$ is certainly closed and hence proper by Proposition 2.76. ∎

> **Example 2.86.** If $F \colon M \to N$ is a submersion, then $F^{-1}(\{q\})$ is a proper embedded submanifold of dimension $(\dim M) - (\dim N)$ for any $q \in N$.

2.7.2 Regularity

We will want to understand the differential of a smooth map pointwise, for which we provide some language.

> **Definition 2.87** (regular, critical). Fix a smooth map $F \colon M \to N$.
>
> - A point $p \in M$ is *regular* if and only if dF_p is surjective; otherwise, $p \in M$ is *critical*.
>
> - A value $q \in N$ is *regular* if and only if all points in $F^{-1}(\{q\})$ are regular; otherwise, $q \in N$ is *critical*.

> **Example 2.88.** Given a function $f \colon \mathbb{R} \to \mathbb{R}$, we see that the point $x_0 \in \mathbb{R}$ is regular if and only if $f'(x_0) = 0$, based on some Jacobian computation reducing $T_{x_0} f$ to $\frac{d}{dx} f\big|_{x_0}$. Accordingly, the critical values are exactly when some point in the fiber is critical.

> **Example 2.89.** Continuing from Example 2.85, we see that the regular points of $\mathbb{R}^2$ are just $\mathbb{R}^2 \setminus \{0\}$, so the collection of regular values is $\mathbb{R} \setminus \{0\}$, which has pre-image $\mathbb{R}^2 \setminus \{(x, y) : xy = 0\}$.

It will turn out that the set of critical values will always be small (namely, measure zero).

> **Remark 2.90.** Note that the set of regular points M' of M is open: the map sending $p \in M$ to the ordered list of determinants of the largest square minors of M is continuous by checking on charts (where this function is a polynomial), and being regular means that we are interested in the pre-image where at least one coordinate is nonzero. Thus, so $F|_{M'} \colon M' \to N$ will be a submersion provided that there is some regular input to F.

Anyway, we get the following result.

> **Proposition 2.91.** Fix a smooth map $F \colon M \to N$ from the m-manifold M to the n-manifold N, and let $q \in N$ be a regular value. Then $F^{-1}(\{q\})$ is a proper embedded submanifold of dimension $m - n$.

Proof. Let $U \subseteq M$ be the set of regular points in M, which is nonempty because N has a regular value; in particular, $F^{-1}(\{q\}) \subseteq U$. Now, $F|_U \colon U \to N$ is a submersion by the regularity of each $p \in U$, so Example 2.86 tells us that $F^{-1}(\{q\}) \subseteq M$ is an embedded submanifold of dimension $m - n$. Lastly, $F^{-1}(\{q\})$ is still proper by Proposition 2.76 because it is closed. ∎

Example 2.92. Define $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ by $F(p) := |x|^2$. Then $S^n = F^{-1}(\{1\})$ will be a proper embedded submanifold of dimension n by Proposition 2.91. Indeed, it is enough to check that $1 \in \mathbb{R}$ is a regular value of F. Well, for $p = (x_0, \ldots, x_n) \in F$, we can compute dF_p as the Jacobian matrix

$$\begin{bmatrix} 2x_0 & \cdots & 2x_n \end{bmatrix}.$$

Notably, this has full rank 1 unless $p = 0$, and $F^{-1}(\{1\}) \cap \{0\} = \varnothing$, so we are safe.

Example 2.93. Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) := \left(x^2 + y^2 - 1\right)^2$. Then for $p = (x_0, y_0)$, we can compute dF_p as the Jacobian matrix
$$\begin{bmatrix} 4x_0 \left(x_0^2 + x_1^2 - 1\right) & 4x_0 \left(x_0^2 + x_1^2 - 1\right) \end{bmatrix},$$
so $S^1 \subseteq \mathbb{R}^2$ now contains entirely critical points even though $S^1 = F^{-1}(\{1\})$ is a perfectly fine smooth embedded submanifold of dimension 1.

Example 2.94. Consider the torus $T^2 := S^1 \times S^1$, and define $F\colon T^2 \to \mathbb{R}$ by some kind of height function, achieved by embedding $T^2 \subseteq \mathbb{R}^3$. Then the pre-image of the critical values of this height make figure-8s, which are not smooth embedded submanifolds.

These regular values also allow us to sensibly discuss defining functions.

Definition 2.95 (defining function)**.** Fix a smooth embedded submanifold $S \subseteq M$, where $k := \dim S$ and $m := \dim M$. Then a smooth function $F\colon M \to N$ is a *defining function for S* if and only if $S = F^{-1}(\{q\})$ for some regular value $q \in N$. Locally over some open subset $U \subseteq M$, we say that a smooth map $F\colon M \to N$ is a *local defining function for F at some $p \in S$* if and only if $S \cap U = F^{-1}(\{q\})$ for some regular value $q \in N$.

The local notion is useful because it is universal.

Proposition 2.96. Fix a subset S of a smooth m-manifold M. Then S is a k-dimensional embedded submanifold of M if and only if any $p \in S$ has some open neighborhood $U \subseteq M$ of p such that there is a local defining function $F\colon U \to \mathbb{R}^{m-k}$ for any $p \in S$.

Sketch. Use Theorem 2.79 to realize F as a projection onto the relevant coordinates. ∎

2.7.3 Tangent Vectors

Embedded submanifolds will produce a natural embedding on tangent spaces, which we now use.

Definition 2.97 (tangent vector)**.** Fix an embedded k-submanifold S of the smooth m-manifold M. For any $p \in S$, we define $T_p^{\mathrm{extrinsic}} S := \operatorname{im} d\iota_p$, where $\iota\colon S \to M$ is the inclusion. Namely, we are viewing $T_p S$ as a k-dimensional subspace of $T_p M$.

Example 2.98. Let (U, φ) be a local k-slice chart for S so that

$$S \cap U = \{p \in U : \varphi_\ell(p) = c_\ell \text{ for } \ell > k\}.$$

Then we see $T_p^{\mathrm{extrinsic}}$ is just the span of $\left\{ \frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_k}\big|_p \right\}$.

Example 2.99. Let $F\colon U \to N$ be a local defining function for S so that $U \cap S = F^{-1}(\{q\})$ for some regular value $q \in N$. Then $T_p^{\mathrm{extrinsic}} S = \ker dF_p$ by tracking through what being a defining function means.

Example 2.100. Consider the subset

$$\mathrm{O}(n) := \left\{ A \in \mathbb{R}^{n \times n} : A^{\mathsf{T}} A = I_m \right\}.$$

We have a natural defining map $F\colon \mathbb{R}^{n \times n} \to \mathrm{Sym}(n)$ by $A \mapsto A^{\mathsf{T}} A$, and F is certainly smooth because it is a polynomial in the coordinates. We claim that $I_m \in \mathrm{Sym}(n)$ is a regular value for F, which implies $\mathrm{O}(n) \subseteq \mathbb{R}^{n \times n}$ is a smooth embedded submanifold of codimension 1 by Proposition 2.91.

Well, we compute dF_A for $A \in \mathbb{R}^{n \times n}$ via curves. A curve producing the differential $B \in T_A \mathbb{R}^{n \times n}$ is simply given by $t \mapsto A + tB$, so

$$dF_A(B) = \frac{d}{dt} F(A + tB)\bigg|_{t=0} = \frac{d}{dt}(A^{\mathsf{T}} + tB^{\mathsf{T}})(A + tB)\bigg|_{t=0} = \frac{d}{dt}\left(A^{\mathsf{T}}A + t(B^{\mathsf{T}}A + A^{\mathsf{T}}B) + t^2 B^{\mathsf{T}}B\right)\bigg|_{t=0},$$

which is $B^{\mathsf{T}}A + A^{\mathsf{T}}B$. So we need the map $B \mapsto B^{\mathsf{T}}A + A^{\mathsf{T}}B$ to be surjective, so we will just check that it has kernel of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. Well, B lives in the kernel if and only if $B^{\mathsf{T}}A = -A^{\mathsf{T}}B$, or equivalently $A^{\mathsf{T}}B$ is alternating. Taking A to be invertible, we are looking at A times the space of alternating matrices, which is in fact of dimension $\frac{1}{2}n(n-1)$.

Remark 2.101. While we're here, we note that we have already computed $T_{I_m} \mathrm{O}(n)$ extrinsically as

$$\ker dF_A = \left\{ B \in \mathbb{R}^{n \times n} : B^{\mathsf{T}} + B = 0 \right\},$$

which we will later understand as the Lie alegbra.

2.8 February 22

The midterm is next week. It will be about four questions. More information will be sent out soon.

2.8.1 Null Sets

Sard's theorem will tell us that most values are regular values. In particular, we will show that critical values have measure zero. The notion of measure zero will be glued together from charts.

Definition 2.102 (null set)**.** A subset $A \subseteq \mathbb{R}^n$ has *measure zero* or is a *null set* if and only if any $\varepsilon > 0$ has some countable list of balls $\{B(x_i, r_i)\}_{i \geq 1}$ such that

$$A \subseteq \bigcup_{i \geq 1} B(x_i, r_i) \qquad \text{and} \qquad \sum_{i=1}^{\infty} r_i^n < \varepsilon.$$

Example 2.103. According to the above definition, any countable subset is a null set, even if we are in $\mathbb{R}^0$.

The point of the r_i^n is that it is the volume of $B(x_i, r_i)$, up to a constant, so we are saying that A can be covered by sets of arbitrarily small measure.

Here are some quick checks.

Lemma 2.104. Fix a positive integer n.

(a) If $A \subseteq \mathbb{R}^n$ is a null set and $B \subseteq A$, then B is a null set.

(b) If $\{A_j\}_{j \geq 1}$ is a countable collection of null sets, then $\bigcup_{i=1}^{\infty} A_i$ is a null set.

(c) If $A \subseteq \mathbb{R}^n$ makes $A \cap (\{c\} \times \mathbb{R}^{n-1}) \subseteq \{c\} \times \mathbb{R}^{n-1}$ into a null set for each $c \in \mathbb{R}$, then A is a null set.

(d) If $f \colon U \to \mathbb{R}$ is a continuous function with $U \subseteq \mathbb{R}^{n-1}$, then the graph

$$\Gamma(f) := \{(x, f(x)) : x \in U\}$$

is a null set.

(e) Every nontrivial affine subspace of $\mathbb{R}^n$ (not equal to $\mathbb{R}^n$) is a null set.

(f) If $A \subseteq \mathbb{R}^n$ is a null set, then $\mathbb{R}^n \setminus A$ is dense.

(g) A subset $A \subseteq \mathbb{R}^n$ is a null set if and only if each $p \in A$ has some open neighborhood $U_p \subseteq \mathbb{R}^n$ such that $A \cap U_p$ is a null set.

(h) If a subset $A \subseteq \mathbb{R}^n$ is a null set, and a function $F \colon A \to \mathbb{R}^n$ is Lipschitz, then $F(A)$ is a null set.

(i) Let $S \subseteq \mathbb{R}^n$ be a submanifold of positive codimension. Then S has measure zero.

Proof. Here we go.

(a) For any $\varepsilon > 0$, use the countable list $\{B(x_i, r_i)\}_{i \geq 1}$ such that

$$A \subseteq \bigcup_{i \geq 1} B(x_i, r_i) \qquad \text{and} \qquad \sum_{i=1}^{\infty} r_i^n < \varepsilon.$$

Indeed, the same conclusion will hold for $B \subseteq A$.

(b) Fix $\varepsilon > 0$. For each j, build a countable list $\{B(x_{ij}, r_{ij})\}_{i \geq 1}$ such that

$$A_j \subseteq \bigcup_{i \geq 1} B(x_{ij}, r_{ij}) \qquad \text{and} \qquad \sum_{i=1}^{\infty} r_{ij}^n < \frac{\varepsilon}{2^j}.$$

Then zipper the countable lists together to cover the union, and the total measure is bounded above by $\sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon$.

(c) Use Fubini's theorem, integrating over $c \in \mathbb{R}$. Explicitly, now using some heavier measure theory,

$$\mu(A) = \int_{\mathbb{R}^n} 1_A(x) \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} 1_A(c, x) \, dx \, dc = \int_{\mathbb{R}} 0 \, dc = 0.$$

(d) Apply the previous part, slicing by the last coordinate inductively.

(e) Apply the previous part because an affine subspace is the image of a linear map composed with a translation.

(f) If $\mathbb{R}^n \setminus A$ fails to be dense, then there is an open subset in the complement of $\mathbb{R}^n \setminus A$, so A contains an open ball, so A cannot be a null set.

(g) The forward direction is immediate by taking subsets. Looping over all $p \in \mathbb{R}^n$ produces an open cover of $\mathbb{R}^n$, which can be reduced to a countable cover by a compactness argument of closed balls. Then we see that A is a countable union of null sets and hence a null set.

(h) For $\varepsilon > 0$, cover A with open balls of measure smaller than ε. Passing the open balls through F (with Lipschitz constant K) will have image contained in an open ball with K times the radius. So we have bounded the measure of A by $K^n \varepsilon$, up to some constants, which vanishes as $\varepsilon \to 0^+$.

(i) Use k-slice charts to realize S^k locally as a slice chart, which have measure zero. Notably, if $A \subseteq U$ is a null set where $U \subseteq \mathbb{R}^n$ is open, and $\varphi \colon U \to \widehat{U}$ is a diffeomorphism to some other $\widehat{U} \subseteq \mathbb{R}^n$, then $\varphi(A)$ continues to be a null set by using (g) to allow us to check locally and then note that diffeomorphisms are locally Lipschitz by taking the Lipschitz constant to be the norm of the Jacobian matrix. ∎

And now let's glue.

> **Definition 2.105** (null set)**.** Let M be a smooth n-manifold. Then a subset $A \subseteq M$ is a *null set* if and only if any smooth chart (U, φ) of M makes $\varphi(A \cap U) \subseteq \mathbb{R}^n$ into a null set.

> **Remark 2.106.** If $A \subseteq \mathbb{R}^n$ is a null set, then actually A has measure zero where we view $\mathbb{R}^n$ as an n-manifold. The backward direction is clear because $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is a smooth chart; the forward direction follows because having measure zero is a diffeomorphism invariant as argued in (i) of Lemma 2.104.

> **Remark 2.107.** A quick argument shows that (a), (b), (f), (g), (h), and (i).

2.8.2 Sard's Theorem

Recall from our examples that there simply were not many critical values; for example, see Examples 2.85 and 2.92. This is in general true.

> **Theorem 2.108** (Sard)**.** Fix a smooth map $F \colon M \to N$. Then the set of critical values of F has measure zero.

> **Remark 2.109.** Here's a heuristic argument when $\dim M = \dim N$. Let $C \subseteq M$ consist of the critical points. Then one has
> $$\mu(F(c)) = \int_{F(c)} 1 \, dy \leq \int_C |\det dF_p(x)| \, dx = 0,$$
> where the content is in justifying the inequality above via some change-of-variables argument.

Anyway, let's start the proof.

Proof of Theorem 2.108. Let $D \subseteq N$ be the set of critical points. By Lemma 2.104, we know that it suffices to show that each $q \in N$ has some open neighborhood U_q such that $D \cap U_q$ is a null set. As such, it suffices to replace N with $\mathbb{R}^n$ (using diffeomorphism invariance of null sets) where $n := \dim N$, and then restrictions of F by pullback mean that we may as well replace M also with an open subset $U \subseteq \mathbb{R}^m$ where $m := \dim M$.

We are going to induct on m. Starting with $m = 0$, it means that M is a 0-manifold, so M is countable, so $F(M)$ is countable, so its image has measure zero. We also note that if $n = 0$, then the image is always countable and hence a null set. So we are left with the case $m, n \geq 1$.

To set up, let $C \subseteq U$ denote the critical points of F, and we set

$$C_k := \left\{ p \in U : \left. \frac{\partial F_{\bullet}}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \right|_p = 0 \text{ for all } \ell \leq k \text{ and } i_1, \ldots, i_\ell \in \{1, \ldots, m\} \right\}.$$

Notably, we have a chain $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$; note all these sets are closed because taking these derivatives is continuous. The game for the proof is to show that the differences are small, and that these sets are small

for large k. Explicitly, we find

$$F(C) = \left(F(C \setminus C_1)\right) \cup \bigcup_{i=2}^{k} \left(F(C_i \setminus C_{i+1})\right) \cup F(C_{k+1})$$

where k is some large integer to be determined later. So we see that our sets divide up into three classes (as above), and we will show that each class is a null set.

1. We show that $F(C \setminus C_1)$ is a null set. Well, choose some $p \in C \setminus C_1$; we would like an open subset $U_p \subseteq U$ such that $F(C \cap U_p)$ is a null set, which will complete the argument by looping over all p and then reducing to a countable cover of C. Because C_1 is closed, we may as well replace U by $U \setminus C_1$, meaning that some partial derivative of F fails to vanish at each point in U. We can cover U by the open subsets where each partial derivative fails to vanish, of which there are finitely many, so we may as well assume that there's a fixed partial derivative that fails to vanish by passing to this open set. By rearranging, we may then assume that $\frac{\partial F_1}{\partial x_1} \neq 0$, and by scaling, we'll just go ahead and take $\frac{\partial F_1}{\partial x_1} = 1$.

   Set $y_1 \coloneqq F_1$ and $y_i \coloneqq x_i$ for each $2 \leq i \leq m$ so that the matrix of partial derivatives $\left[\frac{\partial y_j}{\partial x_i}\right]_{1 \leq i,j \leq m}$ is invertible at p. In particular, $\Phi \coloneqq (y_1, \ldots, y_m)$ is a local diffeomorphism around p, so passing to an open neighborhood of $p \in U$ allows us to make Φ into a genuine diffeomorphism $\Phi \colon U \to U'$. Because Φ is a diffeomorphism, we see that showing the critical values of F is a null set is then equivalent to show that the critical values of $\widetilde{F} \coloneqq F \circ \Phi^{-1}$ is a null set, so we will focus on $\widetilde{F}$.

   Now, the point of passing to $\widetilde{F}$ is that

   $$(F_1(x_1, \ldots, x_m), \ldots) = F(x_1, \ldots, x_m) = \left(\widetilde{F} \circ \Phi\right)(x_1, \ldots, x_m) = (y_1(x_1, \ldots, x_m), \ldots),$$

   so the moral of the story is that
   $$\widetilde{F}(x_1, \ldots, x_m) = (x_1, \ldots),$$

   where the "$\ldots$" simply means that we have some other functions that we haven't bothered to write out. The point is that we can compute the Jacobian of $\widetilde{F}$ as a block matrix

   $$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & \partial \widetilde{F}_2/\partial x_2 & \cdots & \partial \widetilde{F}_2/\partial x_m \\ \vdots & \vdots & \ddots & \vdots \\ * & \partial \widetilde{F}_n/\partial x_2 & \cdots & \partial \widetilde{F}_n/\partial x_m \end{bmatrix}.$$

   The moral of the story is that surjectivity of F is equivalent to surjectivity of $\widetilde{F}$. Now set

   $$\widetilde{C}_s \coloneqq C \cap \left(\{s\} \times \mathbb{R}^{n-1}\right)$$

   to be the critical points of F whose first coordinate is s. So we can integrate over s to get the desired null sets, using the inductive hypothesis because we moved down in coordinates.

2. We show that $F(C_k \setminus C_{k+1})$ is a null set. Note that $p \in C_k \setminus C_{k+1}$ must have some $(k+1)$-derivative which is nonzero, say
   $$\left. \frac{\partial^{k+1} F^j}{\partial x_{i_1} \cdots \partial x_{i_{k+1}}} \right|_p \neq 0,$$

   so we set $h \coloneqq \partial^k F_j / (\partial x_{i_1} \cdots \partial x_{i_k})$ to be a function $M \to \mathbb{R}$. Then $h(p) = 0$ but $\left. \frac{\partial}{\partial x_{k+1}} h \right|_p \neq 0$. Thus, p is a regular point (having nonzero derivative is enough for a map to $\mathbb{R}$), so we may as well take $U_p \subseteq M$ to be the regular locus of h.

   In particular, we see that $h^{-1}(\{0\}) \cap U_p$ is a lower-dimensional embedded submanifold $S \subseteq M$, and $C_k \cap U_p \subseteq h^{-1}(\{0\}) \cap U_p$, so $F(C_k \cap U_p)$ is contained in the critical values of $F|_S \colon S \to N$, which we see has measure zero by the induction. Looping over all $p \in M$ (and then reducing $\{U_p\}_{p \in M}$ to a countable subcover), we conclude.

3. We show that $F(C_k)$ is a null set for $k > \frac{m}{n} - 1$. This is rather technical. Recall we realized M as an open subset $U \subseteq \mathbb{R}^m$, so we may as well show that each $p \in M$ is contained in some cube $Q \subseteq \mathbb{R}^m$ such that $F(C_k \cap Q)$ is a null set. By shifting and scaling, we may as well assume that $Q = [0,1]^m$.

   Take some large N to be determined later. The point is that F has very slow polynomial growth on the scale of $1/N$ when living in C_k, made rigorous by Taylor's theorem, so we are able to bound the size of the image of F. Indeed, we go ahead and subdivide the cube Q into the N^m cubes $\{Q_v\}_{v \in (\mathbb{Z} \cap [0,N))^m}$ given by

   $$Q_v = \prod_{i=1}^m \left[\frac{v_i}{N}, \frac{v_i + 1}{N} \right].$$

   Now, for each $v \in (\mathbb{Z} \cap [0,N))^m$, we bound the size of $F(Q_v)$ under the assumption that $C_k \cap Q_v$ is nonempty. Say $a \in C_k \cap Q_v$.

   So we claim that

   $$|F(x) - F(a)| \stackrel{?}{\leq} C \, |x - a|^{k+1}$$

   for some constant $C > 0$ depending only on F. Let's quickly see why this is enough. Indeed, it follows that the value of F on Q_v is contained in a cube of radius $C(1/N)^{k+1}$. But there are only N^m total cubes, so the volume of our images is bounded above by

   $$N^m (1/N)^{n(k+1)},$$

   up to some unnamed constant depending only on F. Because $k > \frac{m}{n} - 1$, sending $N \to \infty$ will complete our bound.

   It remains to show the bound of the previous paragraph. This follows from an analogue of Taylor's theorem. It suffices to get this bound when F is valued in $\mathbb{R}$ by working on each coordinate function $f := F_\ell$ and then summing the bounds for each coordinate. (Note now that the derivatives for f all vanish to the order k.) So now we claim more generally that

   $$f(x) \stackrel{?}{=} f(a) + \sum_{i=1}^k \frac{1}{i!} \sum_{\substack{I \subseteq \{1,\dots,m\} \\ \#I = i}} \partial_I f(a)(x-a)^I + R_k(x), \tag{2.1}$$

   where our remainder is

   $$R_k(x) := \frac{1}{k!} \sum_{\substack{I \subseteq \{1,\dots,m\} \\ \#I = k+1}} (x-a)^I \int_0^1 (1-t)^k \partial_I f(a + (t-a)x) \, dt.$$

   This is enough for our inequality because all the terms vanish except for $f(a) + R_k(x)$, and we can upper-bound our remainder by hand because these derivatives are taking place over the compact set Q, the integral can be bounded. One now shows (2.1) by an induction on k: if $k = 0$, there is nothing to say (this is just the Fundamental theorem of calculus), and for the induction, one uses integration by parts to expand out $\partial_I f$ again. ∎

2.9 February 27

Today we completed the proof of Sard's theorem. I have edited there for completeness.

2.9.1 Applications of Sard's Theorem

Here are some applications.

> **Corollary 2.110.** Fix a smooth map $F\colon M \to N$ where $\dim M < \dim N$. Then $F(M)$ has measure zero.

Proof. Because $\dim M < \dim N$, it is required that every value of F is critical: $dF_p\colon T_pM \to T_{F(p)}N$ can never be surjective! So we conclude by Theorem 2.108. ∎

For the next application, we need the following notion.

> **Definition 2.111** (regular domain)**.** A regular domain D of a smooth manifold M is a properly embedded codimension-0 smooth submanifold (possibly with boundary).

> **Corollary 2.112.** Fix a closed subset K of a smooth manifold M. Then there are descending regular domains $\{Q_i\}_{i\in\mathbb{N}}$ such that
> $$M \supseteq Q_0 \supseteq Q_1 \supseteq \cdots$$
> and $K = \bigcap_{i\in\mathbb{N}} Q_i$.

Proof. To begin, we recall that we can find a nonnegative smooth function $f \in C^\infty(M)$ such that $f^{-1}(\{0\}) = K$. Now, Theorem 2.108 allows us to find a regular sequence of values $\{s_i\}_{i\in\mathbb{N}}$ such that $s_i \to 0^+$ monotonically. Then $Q_i \coloneqq f^{-1}([0, s_i])$ will work. (We will not show that $f^{-1}([0, s_i])$ is a regular domain; this is essentially on the homework.) ∎

2.9.2 The Whitney Embedding Theorem

As another application, we will show that any smooth manifold can be embedded into some Euclidean space. To begin, we discuss how to decrease the dimensionality of the target space.

> **Lemma 2.113.** Fix a smooth m-manifold M embedded in some $\mathbb{R}^N$. For each $v \in \mathbb{R}^N \setminus \mathbb{R}^{N-1}$, let $\pi_v\colon \mathbb{R}^N \to \mathbb{R}^{N-1}$ denote the projection map with kernel $\mathbb{R}v$. If $N > 2m + 1$, then there exists some v for which $\pi_v|_M$ is an injective immersion $M \to \mathbb{R}^{N-1}$.

Proof. Injectivity of $\pi_v|_M$ is equivalent to asking for $p - q$ to never be parallel to v for $p, q \in M$. Being a smooth immersion is equivalent to asking for $T_pM \cap \ker d(\pi_v)_p = 0$; note $(\pi_v)_p = \pi_v$ up to the identification $T_p\mathbb{R}^N = \mathbb{R}^N$, so we are asking for T_pM to not have any nonzero vectors parallel to v.

 We now build a smooth map to check these two facts. Set $\Delta_M \subseteq M \times M$ to be the diagonal subset $\{(p, p) : p \in M\}$; this allows us to define $\kappa\colon (M \times M) \setminus \Delta_M \to \mathbb{RP}^{N-1}$ by $\kappa(x, y) \coloneqq [x - y]$. Analogously, we define $M_0 \coloneqq \{(p, 0) \in TM : p \in M\}$ by $\tau\colon TM \setminus M_0 \to \mathbb{RP}^{N-1}$ by $\tau(p, w) \coloneqq [w]$. We are now choosing $v \in \mathbb{RP}^{N-1}$ to avoid the images of κ and τ, which are both null sets by Corollary 2.110, so we conclude. ∎

Next up, we show that we can embed compact manifolds.

> **Lemma 2.114.** Fix a smooth compact m-manifold M. Then M can be embedded in $\mathbb{R}^N$ for some $N > 0$.

Proof. Choose a finite smooth atlas $\{(U_i, \varphi_i)\}_{i=1}^d$. By adding in some more charts (and then using compactness to reduce), we may assume that $\operatorname{im}\varphi_i = B(0, 1) \subseteq \mathbb{R}^m$ by some shifting and that the open subsets $\varphi_i^{-1}(B(0, 1/2))$ actually fully cover M. By smoothly extending, we are able to find some $\eta\colon B(0, 1) \to [0, 1]$ which is 0 on $\partial B(0, 1)$ but 1 on $B(0, 1/2)$. We now define
$$F \coloneqq ((\eta \circ \varphi_1)\varphi_1, \ldots, (\eta \circ \varphi_m)\varphi_m).$$

A quick counting argument tells us that the target is $\mathbb{R}^{m(n+1)}$. Now one checks that F is injective and an immersion and hence a smooth embedding (by compactness of M). ∎

Bibliography

[Jac90]   Joseph Jacobs. *English Fairy Tales*. English Fairy Tales. David Nutt, 1890.

[Wei59]   Andre Weil. "Correspondence". In: *Annals of Mathematics* 69.1 (1959), pp. 247–251.

[Lee13]   John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[use14]   user149792. *Is an open n-ball homeomorphic to $\mathbb{R}^n$?* Mathematics Stack Exchange. 2014. eprint: https://math.stackexchange.com/q/1072818. URL: https://math.stackexchange.com/q/1072818.

[Shu16]   Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Elb22]   Nir Elber. *Introduction to Topology and Analysis*. 2022. URL: https://dfoiler.github.io/notes/202A/notes.pdf.

[Elb23]   Nir Elber. *Algebraic Topology*. 2023. URL: https://dfoiler.github.io/notes/215A/notes.pdf.

LIST OF DEFINITIONS