256B: Algebraic Geometry

Nir Elber

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Contents 2

Theme 1
Curves

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

1.1 January 18

Here we go.

1.1.1 House-Keeping

Here are some notes on the course.

- We will continue to use [Har77]. Note that [Vak17] is also popular, as is [SP].
- Office hours will probably be after class on Wednesday and Friday.
- There is a bCourses.
- In the course, we plan to cover curves, some coherent cohomology (and maybe on Zariski sheaves), and some surfaces if we have time.
- Grading will be homework and a term paper. Homework will be challenging, so collaboration is encouraged.

In this course, we will discuss coherent cohomology, but we will begin by talking about curves.

1.1.2 Serre Duality Primer

For the next few weeks, we will focus on non-singular curves over an algebraically closed field. Here is our definition.

Definition 1.1 (curve)**.** Fix a field k. A k-*curve* is an integral, proper, normal scheme of dimension 1. Note that being normal is equivalent to being smooth, so we are requiring our curves to be smooth!

We will want to talk about genus a little. Here is a working definition.

Definition 1.2 (arithmetic genus)**.** Fix a projective k-variety X. Then the *arithmetic genus* $p_a(X)$ is defined

Definition 1.3 (geometric genus)**.** Fix an irreducible k-variety X. Then the *geometric genus* is $p_g(X) := \dim_k \Gamma(X, \omega_X)$, where ω_X is the canonical sheaf. Explicitly, ω_X is the top exterior power of the sheaf of differential forms on X.

In general, the above notions are not the same, but they will be for curves.

Proposition 1.4. Fix a k-curve X. Then $p_g(X) = p_a(X)$. We denote this genus by $g(X)$ or g when the curve is clear.

We would like to actually compute some genera, but this is a bit difficult. One goal of the class is to build a cohomology theory $H^i(X, \mathcal{F})$ for coherent sheaves $\mathcal{F}$ on X, and it turns out we can use these cohomology groups to compute the genus of X. Roughly speaking, we will derive (on the right) the left-exact functor $\Gamma(X, \cdot)$, so the cohomology will in some sense measure the difference between global sections and local sections. For example, flasque sheaves will have trivial cohomology.

For now, we will black-box various things. Here is an example of something we will prove.

Proposition 1.5. Fix a projective k-variety X, and let $\mathcal{F}$ be a coherent sheaf. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$, and $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces for all $i \geq 0$.

To show the Riemann–Roch theorem, we will black-box Serre duality, which we will prove much later. In the case of curves, it says the following.

Theorem 1.6 (Serre duality)**.** Fix a k-curve X. Then, for any vector bundle $\mathcal{L}$ on X, there is a duality

$$H^i(X, \mathcal{L}^\vee \otimes \omega_X) \otimes_k H^{1-i}(X, \mathcal{L}) \to k,$$

where $i \in \{0, 1\}$.

Remark 1.7. Notably, we see $p_g(X) = \dim_k \Gamma(X, \omega_X) = \dim_k H^0(X, \omega_X) = \dim_k H^1(X, \mathcal{O}_X)$.

We will also want the following fact.

Proposition 1.8. Fix a closed embedding $i \colon X \to Y$ of schemes. Given a sheaf $\mathcal{F}$ of abelian groups on Y, then

$$H^i(X, i_*\mathcal{F}) = H^i(Y, \mathcal{F}).$$

1.1.3 Divisors Refresher

We also want to recall a few facts about divisors. We begin with Weil divisors.

Definition 1.9 (Weil divisor)**.** Fix an irreducible k-variety X. A *Weil divisor* $\operatorname{Div}(X)$ are $\mathbb{Z}$-linear combinations of codimension-1 irreducible closed subschemes. Then the *principal divisors* are the image of the map $\operatorname{div} \colon K(X) \to \operatorname{Div}(X)$, where div takes rational functions to poles. The *class group* $\operatorname{Cl} X$ is the quotient.

More generally, we have Cartier divisors.

Definition 1.10 (Cartier divisor)**.** Fix a scheme X. A *Cartier divisor* in $\operatorname{CaDiv} X$ is a global section of $\Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$, where $K^\times$ is the sheafification of the presheaf $U \mapsto \operatorname{Frac} \mathcal{O}_X(U)$. The *principal divisors* are the image of $\Gamma(X, \mathcal{K}^\times)$, and the *class group* $\operatorname{CaCl} X$ is the quotient.

Notably, if X is an integral sheaf, then $\mathcal{K}$ is the constant sheaf $K(X)$. Then a global section is given by the pair $(\{U_i\}, \{f_i\})$ where the U_i cover X, and $f_i \in K(X)^\times$ so that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^\times$. (The coherence condition allows the Cartier divisors to glue.) Notably, each $f \in K(X)$ grants a principal divisor $(\{X\}, \{f\})$, which are exactly the principal divisors.

Here is the main result on these divisors.

Proposition 1.11. If X is an integral, separated, Noetherian, and locally factorial (notably, regular in codimension 1), then Weil divisors are in canonical isomorphism with Cartier divisors. Further, the principal divisors are in correspondence, and so the class groups are also isomorphic.

Example 1.12. Non-singular k-curves have all the required adjectives. Namely, codimension-1 means we are looking at points, and being smooth implies being regular, so all the local rings are dimension-1 regular local rings, which are discrete valuation rings. Notably, discrete valuation rings are

Yet another connection to divisors comes from invertible sheaves. Namely, for integral schemes X, the group of invertible sheaves $\operatorname{Pic} X$ is isomorphic to $\operatorname{CaCl} X$. The point here is that invertible shaves can be embedded into $\mathcal{K}^\times$ when X is integral.

We will be interested in some special divisors.

Definition 1.13 (effective)**.** Fix a k-curve X. Then an *effective Weil divisor* is a $\mathbb{Z}_{\geq 0}$ linear combination of closed points of X. Note that the collection of effective Weil divisors forms a submonoid of $\operatorname{Div} X$. We might be interested in knowing how many effective divisors are equivalent to some given divisor; the set of these is denoted $|D|$.

When our schemes X have enough adjectives, we note that the above correspondences tell us that there is a way to send a Cartier divisor $(\{U_i\}, \{f_i\})$ to a line bundle $\mathcal{L}$ embedded in $\mathcal{K}^\times$. Explicitly, we build $\mathcal{L}(D)$ by $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_X|_{U_i} \subseteq \mathcal{K}$, where the last isomorphism is by sending $1 \mapsto f_i^{-1}$. Notably, if D is effective, then the global section 1 of $\mathcal{K}^\times$ can be pulled back along to a nonzero global section on $\mathcal{L}(D)$ which is f_i on each U_i.

1.2 January 20

We continue moving towards Riemann–Roch.

1.2.1 Linear Systems

Let's discuss linear systems. Let X be a non-singular projective irreducible variety over a field k, and let D be a divisor of X.

Recall that a Cartier divisor $D = \{(U_i, f_i)\}$ on X is associated to the line bundle $\mathcal{L}(D)$ which is locally trivial on each U_i, given as $f_i^{-1}\mathcal{O}_X|_{U_i}$. Conversely, suppose that $\mathcal{L}$ is a line bundle on X. Then we pick up some nonzero global section $\Gamma(X, \mathcal{L})$. Give $\mathcal{L}$ a trivializing open cover $\{U_i\}$, where we are given isomorphisms $\varphi_i \colon \mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$. Setting $f_i := \varphi_i(s)$ recovers an (effective) Cartier divisor $\{(U_i, f_i)\}$ on X. We call this line bundle $\operatorname{div}(\mathcal{L}, s)$.

This thinking gives the following result.

Proposition 1.14. Let X be a non-singular projective integral variety over a field k. Given a Cartier divisor D_0, and let $\mathcal{L} := \mathcal{L}(D_0)$ be the corresponding line bundle.

(a) For each nonzero section $s \in \Gamma(X, \mathcal{L})$, the divisor $\operatorname{div}(\mathcal{L}, s)$ is an effective divisor linearly equivalent to D_0.

(b) Every effective divisor linearly equivalent tot D_0 is obtained in this way.

(c) If k is algebraically closed, we have $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s')$ if and only if s and s' differ by a scalar in $k^\times$.

The above result essentially says that we can study $\Gamma(X, \mathcal{L})$ as a k-vector space instead of trying to understand linear equivalence of divisors. For example, if $\Gamma(X, \mathcal{L}) = 0$, then D is not equivalent to any effective divisor!

Proof. We go one at a time.

(a) Embed $\mathcal{L} \subseteq \mathcal{K}_X$ as usual. Then $s \in \Gamma(X, \mathcal{L})$ becomes a rational function in $K(X)$. By the construction of $\mathcal{L}$, we have an open cover $\{U_i\}$ and some f_i so that $\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_X|_U$. Because we have a global section, we may write $\varphi_i(s) = f_i f$ for some fixed f, and then tracking through our Cartier divisor, we get
$$\operatorname{div}(\mathcal{L}, s) = D_0 + \operatorname{div}(f),$$
as needed.

(b) Suppose D is an effective divisor with $D = D_0 + \operatorname{div}(f)$. Then we see $(f) \ge -D_0$, so f determines a nonzero global section of $\mathcal{L}\mathcal{L}(D_0)$ by tracking through the above constructions: namely, set $s|_{U_i} = f_i^{-1} f$ and glue. (In particular, $(f) \ge -D_0$ means $f/f_i \in \mathcal{O}_X(U_i)$ for each i.) So we see $D = \operatorname{div}(\mathcal{L}, s)$.

(c) One can see directly that $s = cs'$ for $c \in k^\times$ will have $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s)$. Conversely, if $\operatorname{div}(\mathcal{L}, s) = \operatorname{div}(\mathcal{L}, s')$, then under the embedding $\mathcal{L} \subseteq \mathcal{K}_X$, we may correspond s and s' to $f, f' \in K(X)^\times$. Thus, $f/f' \in \Gamma(X, \mathcal{O}_X^\times)$. But because k is algebraically closed and X is proper over k, we have $\Gamma(X, \mathcal{O}_X) = k$, so we are done. ■

Remark 1.15. More generally, we have the following: let k be a field, and let X be a proper, geometrically reduced scheme over k. Then $\Gamma(X, \mathcal{O}_X) = k$ if and only if X is geometrically reduced.

So we have the following.

Corollary 1.16. Let X be a non-singular projective integral variety over a field k. The set $|D_0|$ of effective divisors linearly equivalent to a given divisor D_0 is in natural bijection with $(\Gamma(X, \mathcal{L}(D)) \setminus \{0\})/k^\times$.

With this in mind, we set the following notation.

Notation 1.17. Let X be a non-singular projective integral variety over a field k. Given a divisor D_0 of X, we define $\ell(D_0) := \dim_k \Gamma(X, \mathcal{L}(D))$ and $\dim D_0 := \ell(D_0) - 1$.

The Riemann–Roch theorem is interested in the values of $\ell(D_0)$. Here is a quick lemma.

Lemma 1.18. Let X be a non-singular projective integral variety over a field k. Fix a divisor D of X.

(a) If $\ell(D) \ne 0$, then $\deg D \ge 0$.

(b) If $\ell(D) \ne 0$ and $\deg D = 0$, then D is linearly equivalent to 0.

Proof. Note $\ell(D) \ne 0$ enforces $D \sim D_0$ for some effective divisor D, so $\deg D = \deg D_0 \ge 0$, which shows (a). Then for (b), we note $\deg D_0 = 0$ forces $D_0 = 0$. ■

1.2.2 Riemann–Roch for Curves

We now force $\dim X = 1$, meaning that X is a curve. Let $\Omega_{X/k}$ denote the sheaf of differentials, which is equal to the canonical sheaf $\omega_X = \bigwedge^{\dim X} \Omega_{X/k}$. Any divisor linearly equivalent to $\Omega_{X/k}$ will be denoted K and is called the "canonical divisor." Note that the canonical divisor is really a canonical divisor class.

Theorem 1.19 (Riemann–Roch)**.** Let D be a divisor on a k-curve X, and let g be the genus of X. Further, suppose k is algebraically closed. Then

$$\ell(D) - \ell(K - D) = \deg D + 1 - g.$$

Proof. Set $\mathcal{L} := \mathcal{L}(D)$ for brevity. Note $\mathcal{L}(K - D) \cong \omega_X \otimes \mathcal{L}^\vee$, so Serre duality implies

$$\ell(K - D) = \dim_k \Gamma(\omega_X \otimes \mathcal{L}^\vee) = \dim_k H^1(X, \mathcal{L}).$$

Thus, our left-hand side is $\chi(\mathcal{L}) := \dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L})$.[1] Quickly, note $D = 0$ can be seen directly by

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = \dim k - g = 1 - g,$$

which is what we wanted.

We now perturb D by a point. We show the formula holds for D if and only if the formula holds for $D+p$, where $p \in X$ is some closed point. Note we have a short exact sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{O}_X \to k(p) \to 0,$$

where $k(p)$ refers to the skyscraper sheaf which is the structure sheaf about p. Tensoring with $\mathcal{L}(D + p)$, we get

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + p) \to k(p) \to 0.$$

Now, χ is additive in short exact sequences by using the long exact sequence in cohomology, so

$$\chi(\mathcal{L}(D)) = \chi(\mathcal{L}(D + p)) + \chi(k(P)),$$

but $\chi(k(p)) = \dim_k \Gamma(X, k(p)) = \dim_k k = 1$ because k is algebraically closed. The conclusion now follows because $\deg(D + p) = \deg D + 1$. ■

1.3 January 23

Today we apply the Riemann–Roch theorem.

Remark 1.20. Here is a quick hint for the homework: fix a Weil divisor $D = \sum_P n_P P$ on a k-curve X, where k is algebraically closed. Then $\Gamma(X, \mathcal{O}_X(D))$ can be described as space of rational functions f on X such that $D + \operatorname{div}(f)$ is effective. In other words, for each point $P \in X$, we see f has a pole of order at worse n_P at P.

1.3.1 Applications of Riemann–Roch

Let's give a few applications of Theorem 1.19.

Example 1.21. Fix a k-curve X, where k is algebraically closed. Further, let g be the genus of X and K the canonical divisor. We can compute $\deg K$ as follows: plugging into Theorem 1.19, we see

$$g - 1 = \ell(K) - \ell(0) = \deg K - 1 + g,$$

so $\deg K = 2g - 2$.

[1] This is the Euler characteristic of $\mathcal{L}(D)$ because our higher cohomology groups vanish.

Remark 1.22. More generally, we can see that plugging in $K - D$ into Theorem 1.19 is only able to deduce $\deg K = 2g - 2$.

Example 1.23. Let D be a divisor on a k-curve X, where k is algebraically closed. Further, let g be the genus of X and K the canonical divisor. We would like to study $\dim |nD| = \ell(nD) - 1$ for $n \in \mathbb{Z}^+$. We have the following cases.

- If $\deg D < 0$, then $\deg(nD) < 0$ still, so $\ell(nD) = 0$, so $\dim |nD| = -1$ always.
- If $\deg D = 0$, then there are two possibilities. Namely, if nD is linearly equivalent to 0, then $\ell(nD) = 1$, so $\dim |nD| = 0$; otherwise, D will not be linearly equivalent to any effective divisor (the only effective divisor with degree 0 is 0), so $\dim |nD| = -1$.
- If $\deg D > 0$, then for n large enough, we see $\deg(K - nD) < 0$, so $\ell(K - nD) = 0$, so Theorem 1.19 implies $\ell(nD) = n \deg D + 1 - g$, so $\dim |nD| = n \deg D - g$. Here, "n large enough" is just $n > \deg K / \deg D$.

Here is a more interesting corollary.

Lemma 1.24. Let X be a k-curve, where k is algebraically closed. Suppose that two distinct closed points P and Q produce linearly equivalent Weil divisors. Then $X \cong \mathbb{P}^1_k$.

Proof. We are given that $\operatorname{div}(f) = P - Q$ for some $f \in K(X)$. Thus, we induce a map $k(t) \to K(X)$ given by $t \mapsto f$, where we view $k(t)$ as the fraction field of $\mathbb{P}^1_k$. Notably, t has a zero at 0 and a pole at ∞, and f has a zero at Q and a pole at P. This will induce a finite map $g \colon X \to \mathbb{P}^1$, which we can compute to have degree 1 by the following discussion (notably, the pull-back of the divisor $[0]$ is $[P]$), so g is a birational map and hence an isomorphism.

Now, for any finite map of curves $g \colon X \to Y$, recall there is a map on divisors $g^* \colon \operatorname{Cl}(Y) \to \operatorname{Cl}(X)$ as follows: given point $P \in Y$ inside an affine open subscheme $V \subseteq Y$, we can take the pre-image to X to produce a Weil divisor.[2] More formally, we send P to

$$g^*(P) := \sum_{Q \in g^{-1}(\{P\})} v_Q(t) Q,$$

What? where t is a uniformizer parameter for $Q_{X,P}$, and $v_Q(t)$ is its valuation at the local ring $\mathcal{O}_{X,Q}$.

In fact, we showed the following last semester, which we used in the proof above.

Proposition 1.25. Let $g \colon X \to Y$ be a finite map of k-curves. For any divisor D on Y, we have $\deg g^* D = (\deg g)(\deg D)$.

Proof. Let's recall the proof: it suffices to show this in the case where $D = P$ is a point. Plugging into the definition of g^*, we are showing that

$$\sum_{Q \in g^{-1}(\{P\})} v_Q(t) = \deg g^* P \stackrel{?}{=} \deg g.$$

This statement is local at P, so we may assume that $Y = \operatorname{Spec} B$, whereupon taking the pre-image along g enforces $X = \operatorname{Spec} A$ for some A. For dimension-theory reasons, we see that g is dominant, so the induced map $B \to A$ is injective.

Localizing, we set $A' := A \otimes_B \mathcal{O}_{Y,P}$, so we are really interested in the map $\mathcal{O}_{Y,P} \to A'$, which is still injective. It follows that A' is a finite (by g) torsion-free (by this injectivity argument) module over $\mathcal{O}_{Y,P}$. But

[2] Alternatively, one can view this operation as the pullback $g^* \colon \operatorname{Pic} Y \to \operatorname{Pic} X$ and then recall that each element of the class group corresponds to an isomorphism class in Pic.

$\mathcal{O}_{Y,P}$ is a principal ideal domain, so we may appeal to the structure theorem. Namely, we want to compute the rank of A' over $\mathcal{O}_{Y,P}$, for which it suffices to take fraction fields everywhere and instead compute

$$\operatorname{rank}_{\mathcal{O}_{Y,P}} A' = [\operatorname{Frac} A' : \operatorname{Frac} \mathcal{O}_{Y,P}] = [\operatorname{Frac} A : \operatorname{Frac} B].$$

On the other hand, given uniformizer t of $\mathcal{O}_{X,P}$, we can compute the corresponding rank of A'/tA' over $k = k(P)$ is $\deg g$. However, $\operatorname{Spec} A'/tA'$ is the pre-image of P, so we go ahead and note A'/tA' is a product of local Artinian rings which are quotients corresponding to points in $g^{-1}(\{P\})$. In particular, for each $Q \in g^{-1}(P)$, we see $v_Q(t) = m$ means that $\mathcal{O}_{X,Q}$ appears in A'/tA' as $\mathcal{O}_Q/(\varpi_Q)^m$, where ϖ_Q is a uniformizer at Q. So we can write

$$A'/tA' = \prod_{Q \in g^{-1}(\{P\})} \mathcal{O}_{X,Q}/(\varpi_Q)^{v_Q(t)}.$$

But the k-rank of this is $\deg g^*(\{P\})$ by definition of g^*, which must equal the $\mathcal{O}_{Y,P}$-rank of A', so we are done. ■

This proposition finishes justifying the first paragraph of the proof. ■

Corollary 1.26. Let X be a k-curve of genus 0, where k is algebraically closed. Then X is isomorphic to $\mathbb{P}^1_k$.

Proof. As an aside, we note that $\mathbb{P}^1_k$ is certainly a k-curve of genus 0.

Quickly, choose any two points P and Q on X. As such, we take $D := P - Q$ so that $\deg(K - D) = -2 < 0$, so $\ell(K - D) = 0$. Thus, Theorem 1.19 implies $\ell(D) = 1$. Thus, D is linearly equivalent to an effective divisor, but the only effective divisor with degree 0 is 0 itself, so we see that $P - Q$ is linearly equivalent to 0. This is enough to conclude that $X \cong \mathbb{P}^1_k$ by Lemma 1.24; note that X being a curve requires X to have infinitely many points and thus distinct points. ■

Lastly, let's give a corollary for elliptic curves.

Definition 1.27 (elliptic)**.** A (proper) k-curve X is *elliptic* if and only if X has genus 1.

Corollary 1.28. Let X be an elliptic k-curve, where k is algebraically closed. We give $X(k)$ a group law arising from $\operatorname{Pic} X$.

Proof. Let K be a canonical divisor for X, and we see $\deg K = 0$ by Example 1.21. However, $\ell(K) = 1$ is the genus, so K is linearly equivalent to some effective divisor, so as usual we note that K is linearly equivalent to 0.

Quickly, we note that the group structure on the Picard group $\operatorname{Pic} X$ of isomorphism classes of line bundles on $X(k)$ induces a group law on X. Indeed, fix some k-point $P_0 \in X$. We now claim that the map $X(k) \to \operatorname{Pic}^0 X$ given by

$$P \mapsto \mathcal{O}_X(P - P_0)$$

is a bijection. (Here, $\operatorname{Pic}^0 X$ is the subgroup of degree-0 line bundles.) This will give $X(k)$ a group law by stealing it from $\operatorname{Pic} X$.

Because we already know that $\operatorname{Pic} X$ is in bijection with divisors more generally, it's enough to show that any divisor D of degree 0 is linearly equivalent to a divisor of the form $P - P_0$ for $P \in X(k)$. Well, we use Theorem 1.19 with $D + P_0$, which yields

$$\ell(D + P_0) - \ell(K - D - P_0) = 1 + 1 - g = 1,$$

but $K - D - P_0$ has degree -1 and so $\ell(K - D - P_0) = 0$. Thus, $\dim |D + P_0| = 0$, so there is a unique effective divisor of degree 1 linearly equivalent to $D + P_0$. However, an effective divisor of degree 1 is just a point P, so we are done. ■

1.4 January 25

Today we discuss ramification.

1.4.1 Ramification

Given a finite morphism $f\colon X \to Y$ of k-curves, there are some numbers we can attach to f. For example, we have the degree $\deg f := [K(X) : K(Y)]$. Additionally, for each $y \in Y$, we have a map

$$\mathcal{O}_{Y,y} \to \prod_{x \in f^{-1}(\{y\})} \mathcal{O}_{X,x}.$$

Each of these discrete valuation rings have residue field k, so when k is algebraically closed, this is pretty simple to understand. Indeed, this gives rise to "ramification" information.

Definition 1.29 (ramification)**.** Let $f\colon X \to Y$ be a morphism of k-curves. Then for $x \in X$, we define the *ramification index* as $e_x := v_x(t_y)$, where $t_y \in \mathcal{O}_{Y,y}$ is a uniformizer for $\mathcal{O}_{Y,y}$, and $v_x(t_y)$ refers to the valuation of t_y embedded in $\mathcal{O}_{X,x}$.

Note $e_x > 0$ because $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a map of local rings.

When k is not algebraically closed, things get a little more complicated, and we will also want to keep track of the degree of the corresponding residue field extension.

Definition 1.30 (ramification)**.** Let $\varphi\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a map of discrete valuation rings. Fix a uniformizer $\varpi \in \mathfrak{m}$, and define $e := v_B(\varphi(\varpi))$ and $f := [B/\mathfrak{n} : A/\mathfrak{m}]$ and $p := \operatorname{char}(A/\mathfrak{m})$.

- φ is *unramified* if and only if $e = 1$ and $B/\mathfrak{n}$ is separable over $A/\mathfrak{m}$.
- φ is *tamely ramified* if and only if $p \nmid e$ and $B/\mathfrak{n}$ is separable over $A/\mathfrak{m}$.
- Otherwise, φ is *wildly ramified*.

Remark 1.31. Algebraic number theory has a lot to say about how the above process works for local fields (or even just number rings).

When k is algebraically closed, we see that the extension $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k$, so this extension is of course separable, so Definition 1.30 simplifies somewhat in our situation.

Definition 1.32. Let $f\colon X \to Y$ be a finite morphism of k-curves. Then we say that $x \in X$ is *unramified/tamely ramified/wildly ramified* if and only if the corresponding map $f^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is unramified/tamely ramified/wildly ramified.

Notably, last class we recalled that

$$f^*[y] = \sum_{x \in f^{-1}(\{y\})} e_x[x].$$

Now, even in our algebraically closed situation, we will want to care about separable extensions.

Definition 1.33 (separable)**.** Let $f\colon X \to Y$ be a finite morphism of k-curves. Then f is *separable* if and only if the extension $K(Y) \subseteq K(X)$ is separable.

Now, we would like to keep track of our ramification information all at once.

Lemma 1.34. Let $f\colon X \to Y$ be a finite separable morphism of k-curves. Then

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

is an exact sequence of line bundles on X.

Proof. We know from last semester that this map is exact on the right, so we need the map $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. Well, we may check exactness of quasicoherent sheaves on affine open subschemes, so we may assume that $X = \operatorname{Spec} A$, where the map looks like $A \to A$. Letting I denote the kernel of this map, we get an embedding $A/I \subseteq A$, but if I is nontrivial, then this means that the map $A \to A$ is zero at the generic point.

Thus, we want to check that the map $A \to A$ is nonzero at the generic point. Everything is compatible with localization, so we are now looking at

$$f^*\Omega_{K(Y)/k} \to \Omega_{K(X)/k} \to \Omega_{K(X)/K(Y)} \to 0.$$

Thus, to show that the map on the left is nonzero, it suffices to show that $\Omega_{K(X)/K(Y)} = 0$. We now use the fact that $K(X)/K(Y)$ is separable, for which the statement is true. ■

The point here is that $\Omega_{X/Y}$ precisely measures the "difference" between $\Omega_{Y/k}$ and $\Omega_{X/k}$.

In what follows, we fix the following notation. Let $f\colon X \to Y$ be a finite morphism of k-curves. Given $x \in X$ and $y := f(x)$, let ϖ_x be a uniformizer for $\mathcal{O}_{X,x}$ and ϖ_y be a uniformizer for $\Omega_{Y,y}$. Then we note $d\varpi_x$ generates $(\Omega_{X/k})_x$, and $d\varpi_y$ generates $(\Omega_{Y/k})_y$, and we have a map

$$(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x \simeq \mathcal{O}_{X,x}$$

by sending $f^*\colon d\varpi_y \mapsto d\varpi_y/d\varpi_x \cdot d\varpi_x$, where $d\varpi_y/d\varpi_x$ is an element of $\mathcal{O}_{X,x}$. (This is the definition of $d\varpi_y/d\varpi_x$.)

Proposition 1.35. Let $f\colon X \to Y$ be a finite separable morphism of k-curves.

(a) $\Omega_{X/Y}$ is supported on exactly the set of ramification points of f, so the set of ramified points is finite.

(b) For each $x \in X$, we have $(\Omega_{Y/X})_x$ is a principal $\mathcal{O}_{X,x}$-module of length $v_x(d\varpi_y/d\varpi_x)$.

(c) If f is tamely ramified at x, then the length of $(\Omega_{Y/X})_x$ is $e_x - 1$; if it's wildly ramified, then the length is larger.

Proof. We show these one at a time. As a warning, all uniformizers might be swapped here.

(a) Recall that $\Omega_{Y/X}$ is generically zero. Now, $(\Omega_{Y/X})_p = 0$ if and only if the map

$$(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x$$

is an isomorphism, which means that a uniformizer for $\mathcal{O}_{Y,f(x)}$ is going to a uniformizer of $\mathcal{O}_{X,x}$, which
What?
is equivalent to f being unramified at x. The point here is that the set of ramified points correspond to some dimension-zero subset and is therefore finite.

(b) The length of $(\Omega_{Y/X})_p$ is its k-dimension, which we can compute as

$$\operatorname{length}(\Omega_{X/k})_x - \operatorname{length}(f^*\Omega_{Y/K})_x,$$

which we can compute is e_x by hand. Notably, this has to do with how we identify $(\Omega_{X/k})_x$ with $\mathcal{O}_{X,x}$.

(c) Letting e denote our ramification index, we may set $\varpi_y = a\varpi_x^e$ where $a \in \mathcal{O}_{X,x}^\times$, which upon taking differentials reveals
$$d\varpi_y = ea\varpi_x^{e-1}d\varpi_x + \varpi_x^e da.$$
Now, if f is tamely ramified at x, we see $\operatorname{char} k \nmid e$, so we see that the valuation here is in fact $e_x - 1$. The statement for wild ramification follows similarly. ■

Remark 1.36. The length of the modules here coincides with the dimension as a k-vector space. This is because $\mathcal{O}_{X,x}$ is a discrete valuation ring with residue field k.

1.5 January 27

There is homework due tonight. Today we keep talking towards the Riemann–Hurwitz formula.

1.5.1 The Riemann–Hurwitz Formula

Throughout, $f\colon X \to Y$ is a finite separable morphism of k-curves, where k is algebraically closed.

Definition 1.37 (ramification divisor)**.** Let $f\colon X \to Y$ be a finite separable morphism of k-curves, where k is algebraically closed. Then the *ramification divisor* is the divisor
$$R(f) := \sum_{x\in X} \operatorname{length}(\Omega_{Y/X})_x x.$$
Note that there are only finitely many ramified points, so this is indeed a divisor.

In particular, in tame ramification, this length is in fact exactly our ramification.

Lemma 1.38. Let $f\colon X \to Y$ be a finite separable morphism of k-curves, where k is algebraically closed. Further, let K_X and K_Y denote the canonical divisors. Then K_X is linearly equivalent to $f^*K_Y + R$.

Proof. We can see by hand that the structure sheaf $\mathcal{O}_R$ of R as a closed subscheme is exactly $\Omega_{Y/X}$, so the exact sequence
$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{Y/X} \to 0$$
What? can be tensored with $\Omega_{X/k}^{-1}$ to give
$$0 \to f^*\Omega_{Y/k} \otimes \Omega_{X/k}^{-1} \to \mathcal{O}_X \to \mathcal{O}_R \to 0.$$
Now, we see $\mathcal{O}_R = \mathcal{O}_X(-R)$ by unwinding definitions, so $f^*\Omega_{Y/k} \otimes \Omega_{X/k}^{-1} = \mathcal{O}_X(-R)$, which is what we wanted. ■

Theorem 1.39 (Hurwitz)**.** Let $f\colon X \to Y$ be a finite separable map of k-curves, where k is algebraically closed. Letting $n := \deg f$, we have
$$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R(f).$$

Proof. Take degrees of Lemma 1.38. ■

Let's derive some corollaries.

Definition 1.40 (étale)**.** A morphism locally of finite presentation $f\colon X \to Y$ of schemes is *étale* at $x \in X$ if and only if the following conditions hold.

- f is flat at x. In other words, $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module.
- $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is a field and separable over $k(f(x))$. Equivalently, we are requiring $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and for the residue field extension to be separable.

Then f is étale if and only if f is étale at all points.

Remark 1.41. The locus of points for which a morphism is étale is open essentially because this is true for both conditions individually.

Remark 1.42. If the morphism of k-curves $f\colon X \to Y$ is étale, then it is unramified because we are essentially saying $\operatorname{length}\Omega_{X/Y} = 0$. In fact, unramified morphisms are étale, which should roughly be our intuition. Flatness is a bit mysterious, but such is life.

Remark 1.43. Equivalently, a morphism $f\colon X \to Y$ is étale if and only if f is a smooth morphism of relative dimension 0. For example, if both X and Y are varieties, then we are asking for the varieties to have the same sheaf of differentials.

Geometrically, we imagine finite étale morphisms as covering space maps. This motivates the following definition.

Definition 1.44 (simply connected)**.** A finite étale morphism $f\colon X \to Y$ is *trivial* if and only if X is a disjoint union of copies of Y, and f is a disjoint union of automorphisms. Then a curve Y is *simply connected* if and only if all finite étale morphisms $X \to Y$ are trivial.

Roughly speaking, we are trying to say that the fundamental group is trivial.

Proposition 1.45. Let k be an algebraically closed field. Then $\mathbb{P}^1_k$ is simply connected.

Example 1.46. We can see that the sphere $\mathbb{P}^1_{\mathbb{C}}$ is simply connected.

Proof. Fix some finite étale morphism $f\colon X \to Y$. By breaking this morphism into connected components, we may assume that X is connected. We show that $X = \mathbb{P}^1_k$. Because f is smooth, we see that the structure map $X \to Y \to k$ is smooth, so X is a (smooth) curve. (In particular, we can also see that X is irreducible.) Now, because f is étale, it is étale at the generic point, so f is also separable.

We are now ready to apply Theorem 1.39. Here, $R(f) = 0$ and $g(Y) = 0$, so we are left with

$$2g(X) - 2 = (\deg f)(0-2) = -2\deg f.$$

However, $g(X) \geq 0$ and $\deg f \geq 1$, so we must have $g(X) = 0$ and $\deg f = 1$, so f is in fact an isomorphism $X \cong \mathbb{P}^1_k$. ∎

Remark 1.47. In characteristic 0, we will have $\mathbb{A}^1_k$ is simply connected. However, in positive characteristic, this is no longer true; indeed, $\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{F}_p})$ is infinite.

1.6 January 30

Homework was assigned and still due on Friday, sadly.

Remark 1.48. Let k be an algebraically closed field of positive characteristic p. It turns out $G := \pi_1^{\text{ét}}(\mathbb{A}_k^1)$ is profinite but not topologically finitely generated—it's very large. In fact, one can show that any finite p-group arises as a quotient of $\pi_1^{\text{ét}}(\mathbb{A}_k^1)$. More generally, any finite quasi-p-group is a quotient, where a quasi-p-group is a finite group generated by its Sylow p-subgroups cover G.

1.6.1 Everything Is Frobenius

Thus far we roughly understand finite separable morphisms of curves. We now investigate the purely inseparable case. In particular, today k will be an algebraically closed field of positive characteristic p. Note there is a canonical embedding $\mathbb{F}_p \hookrightarrow k$, which gives rise to the Frobenius automorphism as follows.

Definition 1.49 (Frobenius)**.** Let k be a field of characteristic $p > 0$. Given a k-scheme X, we define the *Frobenius automorphism* $F\colon X \to X$ as being the identity on topological spaces and being the pth-power map $F_U^\sharp\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ for all open $U \subseteq X$.

We can see that this map takes units of $\mathcal{O}_{X,x}$ to units of $\mathcal{O}_{X,x}$ for any $x \in X$, so we have defined a morphism of locally ringed spaces.

Example 1.50. Let $X = \operatorname{Spec} A$ be an $\mathbb{F}_p$-scheme. Then the ring homomorphism $F\colon A \to A$ given by the pth power map is the Frobenius $F\colon X \to X$. To show the map is the identity on the topological space, we note $a^p \in \mathfrak{p}$ is equivalent to $a \in \mathfrak{p}$ for $a \in A$, where we are using the primality of $\mathfrak{p}$. Thus, $F^{-1}(\mathfrak{p}) = \mathfrak{p}$ for any prime $\mathfrak{p} \in \operatorname{Spec} A$.

The Frobenius map defined above is not k-linear because it is the pth power map on k too. To make this k-linear, we essentially cheat.

Definition 1.51. Fix a scheme X over a field k of characteristic p. Then we define the k-scheme X_p which is equal to X as a scheme but whose structure morphism to $\operatorname{Spec} k$ is given by

$$X_p \to \operatorname{Spec} k \xrightarrow{F} \operatorname{Spec} k.$$

The point is that the diagram

$$\begin{tikzcd} X_p \arrow[r, "F"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, "F"] & \operatorname{Spec} k \end{tikzcd} \tag{1.1}$$

commutes, so we do genuinely have a Frobenius morphism $F\colon X_p \to X$.

Remark 1.52. Explicitly, for each affine open $\operatorname{Spec} A \subseteq X$, we get an identical affine open $\operatorname{Spec} A_p \subseteq X_p$, but if the k-algebra structure on A is given by $g\colon k \to A$, then the k-algebra structure on A is given by $g_p(x) \cdot \alpha := g\left(x^p\right) \cdot \alpha$.

Namely, with everything being contravariant on functions, we see that

$$\begin{tikzcd} A & A \arrow[l, "F"'] \\ k \arrow[u] & k \arrow[l, "F"'] \arrow[u] \end{tikzcd}$$

commutes.

Why?

Remark 1.53. Certainly $X_p \cong X$ as schemes because they are literally the same data. If k is perfect, then $X_p \cong X$ is an isomorphism of k-schemes because the pth-power map on k is an isomorphism.

Example 1.54. Let k be a perfect field of characteristic $p > 0$. Set $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$. Then $F\colon X_p \to X$ is given by the morphism $k[t_p] \to k[t]$ by $f \mapsto f^p$. Thus, to witness our isomorphism of k-schemes, we note that we can post-compose $k[t_p] \to k[t]$ with the morphism which extends $k \cong k$ by $a^p \mapsto a$ by $t_p \mapsto t_p$, so we have made a k-linear isomorphism. One can essentially extend this construction to work in general when k is perfect.

Example 1.55. Let k be a field of characteristic $p > 0$, and let X be an integral k-scheme. Then $F\colon X_p \to X$ is given by a morphism $K(X) \to K(X_p)$ by $\alpha \mapsto \alpha^p$. (Yes, this is k-linear because the k-action on $K(X_p)$ is by pth powers.) As such, we have defined an embedding of $K(X)$ into an algebraic extension $K(X_p)$: namely, every $\alpha \in K(X_p)$ is a root of the polynomial $t^p - \alpha^p = 0$ in $K(X)[t]$.

Conversely, $t^p - \beta \in K(X)[t]$ always has a root in $K(X_p)$ because $\beta \in K(X)$ embeds into $K(X_p)$ as β^p, so this polynomial "looks like" $t^p - \beta^p$ in $K(X_p)$, where β is our root. Thus,

$$K(X_p) = K(X)^{1/p}.$$

The point is that a k-curve X will have the Frobenius morphism $X_p \to X$ induced by the embedding $K(X) \to K(X)^{1/p}$.

Now, here is our main result.

Theorem 1.56. Let k be an algebraically closed field of characteristic $p > 0$. Further, let $f\colon X \to Y$ be a finite map of k-curves which induces a purely inseparable extension $f^\sharp\colon K(Y) \to K(X)$. Then f is some iterate of the Frobenius morphism. In particular, X and Y are isomorphic as schemes and thus have the same genus.

Proof. Note $\deg f = [K(X) : K(Y)]$ is a power of p, which we call p^ν. Being purely inseparable then enforces $K(X) \subseteq K(Y)^{1/p^\nu}$; namely, the minimal polynomial if all elements $\alpha \in K(X)$ must be the minimal polynomial of the form $x^{p^r} - \beta = 0$, for otherwise there is a separable subextension, violating our pure inseparability.

Now, consider iterated Frobenius morphisms

$$Y_{p^\nu} \to Y_{p^{\nu-1}} \to \cdots \to Y_p \to Y,$$

which corresponds to the inclusion of fields

$$K(Y) \subseteq K(Y)^{1/p} \subseteq \cdots \subseteq K(Y)^{1/p^{\nu-1}} \subseteq K(Y)^{1/p^\nu},$$

where the inclusions have reversed.

Thus, to conclude, we would like to show $K(X) = K(Y)^{1/p^\nu}$. By degree arguments, it's enough to conclude $\left[K(Y)^{1/p^\nu} : K(Y)\right] = p^\nu$. By induction, it's enough to show $\left[K(Y)^{1/p} : K(Y)\right] = p$. We now must use the fact that Y is a smooth k-curve, so we will push its proof into the following lemma.

Lemma 1.57. Let k be an algebraically closed field of characteristic $p > 0$. If Y is a smooth k-curve, then $\left[K(Y)^{1/p} : K(Y)\right] = p$.

Proof. Equivalently, we would like to show that $[K(Y) : K(Y)^p] = p$. Note that $\Omega_{K(Y)/k}$ is a $K(Y)$-vector space of dimension 1; we refer to Theorem II.8.6.A, where the point is that k being perfect tells us $K(Y)/k$ is separably generated, so $\dim_k \Omega_{K(Y)/k}$ is the transcendence degree of $K(Y)$ over k, which is 1.

We now note that dx generates $\Omega_{K(Y)/k}$ if and only if $x \in K(Y)$ yields a power basis $\{1, x, \ldots, x^{p-1}\}$ of $K(Y)$ over $K(Y)^p$, which completes the proof. ■

Okay.

It remains to show that the genus does not change. Well, $g = \dim_k H^1(X, \mathcal{O}_X)$, and we see that the only difference between X and Y is the structure morphism, and this dimension does not change if we change the structure morphism. ■

Remark 1.58. The above proof basically shows that the Frobenius morphism $F\colon X_p \to X$ in our setting is a finite morphism of degree p. In particular, it cannot be an isomorphism.

1.7 February 1

We continue discussing the Frobenius morphism.

1.7.1 Relative Frobenius

We begin class with a few remarks on the Frobenius automorphism when k is not perfect. Roughly speaking, the issue is that a Frobenius morphism $F\colon X_p \to X$ is not p. In general, one sees $[K(X) : kK(X)^p] = p$ (note we have taken the composite with p!), so the extension $[K(X) : K(X)^p] > p$.

The idea is to generalize (1.1). Namely, we construct our Frobenius $X^{(p)} \colon X$ by pullback in the following square.

$$\begin{tikzcd}
X \arrow[dr, "F_{X/k}"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "F"] & \operatorname{Spec} k
\end{tikzcd}$$

Here $F_{X/k}$ is the relative Frobenius. Notably, when k is perfect, we see that the canonical projection $X^{(p)} \to X$ is an isomorphism of k-schemes because the Frobenius on the bottom is an isomorphism. One can even make this isomorphism explicit by arguing as in Example 1.54.

We quickly check that this roughly generalizes our earlier construction.

Example 1.59. Set $X = \operatorname{Spec} A$ for a k-algebra A. We track the diagram in this case.

Proof. Then $X^{(p)} = \operatorname{Spec}(A \otimes_k k)$, where the k-action on k is given by the Frobenius $F\colon k \to k$. Thus, when k is perfect, we are indeed looking k acting on A by $f \otimes \alpha^p = \alpha f \otimes 1$, so we have an isomorphism $A \otimes_k k \cong A$, where the k-action is given by

$$g^{(p)}(\alpha)a = g(\alpha^{1/p})a,$$

where $g\colon k \to A$ is the structure morphism.

To understand the relative Frobenius $X^{(p)} \to X$, we note that it sends $f \otimes \alpha \mapsto \alpha f^p$ by tracking the diagram. As such, when k is perfect, we may think of our morphism as

$$\alpha^{1/p} f \mapsto \alpha f^p,$$

so the map is in fact k-linear. ■

Remark 1.60. Thus, when k is not perfect, we see that $X^{(p)}$ need not be isomorphic to X, even absolutely. For example, take $A = k[x]/\left(x^2 - \alpha\right)$ for some $\alpha \in k$. In $A \otimes_k k$, we see that

$$(x \otimes 1)^2 = \alpha \otimes 1 = 1 \otimes \alpha^p.$$

Thus, $A \otimes_k k$, even though it is a two-dimensional k-algebra, is isomorphic to $k[x_p]/\left(x_p^2 - \alpha^p\right)$, which is not the same as A when k is not perfect!

Anyway, let's see the analogue of Theorem 1.56 in our setting.

Theorem 1.61. Fix a finite morphism $f\colon X \to Y$ of (smooth, proper, integral) k-curves, where k is a field of characteristic $p > 0$. Further, suppose that $f^\sharp\colon K(Y) \to K(X)$ is purely inseparable of degree p. Then $Y \cong X^{(p)}$ for some r, and f is the relative Frobenius under this isomorphism.

1.7.2 Inseparability for Fun and Profit

We will not prove this, but it's fun to know.

Corollary 1.62. Fix a finite morphism $f\colon X \to Y$ of k-curves, where k is algebraically closed. Then $g(X) \geq g(Y)$.

Proof. Factor the field extension $K(Y) \subseteq K(X)$ into a purely inseparable extension followed by a separable extension. Genus does not change when we are dealing with an inseparable extension by Theorem 1.56, so it suffices to show that the genus does not fall with separated morphisms. Well, by Theorem 1.39, we see

$$2g(X) - 2 = (\deg f)(2g(Y) - 2) + \deg R \geq 2g(Y) - 2,$$

so $g(X) \geq g(Y)$ follows. ■

Remark 1.63. For our equality cases, we see that one either has an isomorphism or an unramified finite morphism of elliptic curves. Such unramified maps of elliptic curves (which are not isomorphisms) do exist.

Classify them?

1.7.3 Embeddings of Curves

We now return to the case where k is algebraically closed. Our next goal is to show that every k-curve can be embedded into $\mathbb{P}^3_k$. As such, we are roughly speaking interested in showing that certain linear systems separate points (which means that we are base-point-free) and tangent vectors.

Let's begin by discussing being base-point-free.

Lemma 1.64. Fix a divisor D on a k-curve X. Then $\mathcal{O}_X(D)$ is base-point-free if and only if

$$\dim |D| = \dim |D - P| + 1.$$

Proof. We are essentially showing $\ell(D) = \ell(D - P) + 1$. Notably, we do always have $\ell(D) \leq \ell(D - P) + 1$ by staring at the short exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0$$

and then tensoring by $\mathcal{O}_X(D)$ to give

$$0 \to \mathcal{L}(D - P) \to \mathcal{L}(D) \to k(P) \to 0.$$

We now discuss the equality. Note $\ell(D) = \ell(D - P) + 1$ is just telling us that there is a global section in $\Gamma(X, \mathcal{O}_X(D)) \setminus \Gamma(X, \mathcal{O}_X(D - P))$. In other words, we have some $f \in K(X)$ such that $D + \operatorname{div}(f)$ is effective, but $D - P + \operatorname{div}(f)$ is not. Thus, we see that we are saying P is not in the support of f, so P is not a base-point for D. ■

Next time we will make a similar dimension condition to be ample and very ample.

1.8 February 3

We continue discussing the theory of things which go bump in the night.

1.8.1 Line Bundle Review

We begin by recalling a couple of facts.

Proposition 1.65. Fix an A-scheme X, where A is an affine scheme. We recall that a morphism $X \to \mathbb{P}^n_A$ has equivalent data to giving a line bundle $\mathcal{L}$ on X together with global sections $(s_0, \ldots, s_n)$ which generate $\mathcal{L}$.

Recall that the global sections generate $\mathcal{L}$ if and only if the induced map $\mathcal{O}_X^{n+1} \to \mathcal{L}$ is surjective, which means that the global sections generate all the talks of $\mathcal{L}$.

Remark 1.66. If $A = k$ is a field, then defining a morphism $X \to \mathbb{P}^n_k$ can be defined (up to automorphism on $\mathbb{P}^n_k$), then it suffices to just provide a globally generated line bundle $\mathcal{L}$, and the precise choice of spanning set $(s_0, \ldots, s_n)$ merely adds an automorphism.

Remark 1.67. The chosen global sections technically need not fully span $\Gamma(X, \mathcal{L})$.

Remark 1.68. We remark that the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^n_A}(1)$ under $X \to \mathbb{P}^1_A$ is (canonically) $\mathcal{L}$, and the pullback of the global section x_i is s_i. Explicitly, there is morphism $x_i \colon \mathcal{O}_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(1)$ which will pull back to $s_i \colon \mathcal{O}_X \to \mathcal{L}$ upon applying φ^*.

We might want to upgrade Proposition 1.65 to give a closed embedding into projective space. Here are the corresponding conditions.

Proposition 1.69. Fix an algebraically closed field k and a k-variety X. Fix a morphism $\varphi \colon X \to \mathbb{P}^n_k$ corresponding to the line bundle $\mathcal{L}$ equipped with global sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Further, set $V = \operatorname{span}(s_0, \ldots, s_n)$. Then φ is a closed immersion if and only if V satisfies the following.

- Separates points: for any distinct $x, x' \in X$, there is a section $s \in V$ such that $x \in \operatorname{supp}\operatorname{div}(\mathcal{L}, s)$ but $x' \notin \operatorname{supp}\operatorname{div}(\mathcal{L}, s)$ (i.e., $s \in \mathfrak{m}_x\mathcal{L}_x \setminus \mathfrak{m}_{x'}\mathcal{L}_{x'}$).
- Separates tangent vectors: for every $x \in X$, the set
$$\{s \in V : s \in \mathfrak{m}_x\mathcal{L}_x\}$$
spans the Zariski tangent place $\mathfrak{m}_x\mathcal{L}_x/(\mathfrak{m}_x\mathcal{L}_x)^2$.

Notably, when X is a curve, our Zariski tangent space has dimension 1, so we just want some section to show up in there.

We will also want the following definitions.

Definition 1.70 (ver ample)**.** A line bundle $\mathcal{L}$ on a scheme X is *very ample* relative to a scheme Y if and only if there is a locally closed embedding $\iota \colon X \to \mathbb{P}^n_Y$ for some $n > 0$ such that $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}^n_Y}(1)$.

Remark 1.71. In the case where X is a k-curve, because X is proper, any locally closed embedding $\iota \colon X \to \mathbb{P}^n_k$ is automatically closed. As such, in this situation we may as well talk about closed embeddings.

Definition 1.72 (ample)**.** A line bundle $\mathcal{L}$ on a scheme X is *ample* if each coherent sheaf $\mathcal{F}$ on X makes $\mathcal{F} \otimes \mathcal{L}^n$ globally generated for n sufficiently large.

Here is how these notions relate.

Proposition 1.73. Fix a scheme X of finite type over a Noetherian ring A. Then a line bundle $\mathcal{L}$ on X is ample if and only if $\mathcal{L}^n$ is very ample (relative to A) for some $n > 0$.

1.8.2 Projective Embeddings for Curves

We now return to talk about curves. We quickly extend our definition of ample.

Definition 1.74 (ample, very ample)**.** Fix a k-curve X. Then a divisor D on X is *ample* or *very ample* if and only if $\mathcal{O}_X(D)$ is as well.

So let's translate what we know about projective embeddings into our language of divisors.

Proposition 1.75. Fix a divisor D on a k-curve X.

(a) The complete linear system $|D|$ is base-point-free if and only if

$$\dim|D| = \dim|D-P|+1$$

for all closed points $P \in X$.

(b) The divisor D is very ample if and only if

$$\dim|D| = \dim|D-P-Q|+2$$

for all $P, Q \in X$.

Roughly speaking, (a) asks for us to separate points somehow, and (b) asks if we can separate points with the right multiplicity. Namely, we're asking for something to be in $\mathfrak{m}_P$ but not $\mathfrak{m}_P^2$.

Proof. We have two parts to show.

(a) Note $\dim|D| = \dim|D-P|+1$ is equivalent to

$$\ell(D) = \ell(D-P)+1,$$

which is equivalent to $\mathcal{O}_X(D)(X) \backslash \mathcal{O}_X(D-P)(X)$ being nonempty. (Recall that $\dim \mathcal{O}_X(D) \leq \mathcal{O}_X(D-P)+1$ at the very least.) Thus, we are saying there is $f \in K(X)$ such that $P \in \operatorname{supp} D$ but $P \in \operatorname{supp}(D+(f))$, so we have a global section f which does not vanish at P. Repeating this for all points P shows that $|D|$ is base-point-free, and running this argument in reverse gives us the converse implication.

(b) Certainly if D is very ample, then $\mathcal{L}(D)$ is base-point-free. Additionally, note that the conclusion of (b) implies the conclusion of (a) because

$$\dim|D-P-Q|+2 \leq \dim|D-P|+1 \leq \dim|D|,$$

so equalities must hold everywhere and in particular on the right inequality. In particular, we can assume that $|D|$ is base-point-free in either direction.

As such, in either direction, we already know that D determines a morphism to projective space, so we need to check that we have defined a closed embedding.

- Separate points: for every distinct $P, Q \in X$, some $s \in \Gamma(X, \mathcal{O}_X(D))$ has $P \in \operatorname{supp}\operatorname{div}(\mathcal{L}, s)$ but $Q \in \operatorname{supp}\operatorname{div}(\mathcal{L}, s)$ (namely, we vanish at P) is equivalent to $\operatorname{div}(\mathcal{L}, s) \in \Gamma(X, \mathcal{O}_X(D-P))$ but Q is not a base-point of $\mathcal{O}_X(D-P)$ at s (namely, we do not vanish at Q). Hitting this with (a) again, we are asking for
$$\dim|D-P| = \dim|D-P-Q| + 1.$$
We already have (a), so we conclude $\dim|D| = \dim|D-P-Q| + 2$. Running this argument in reverse gets the other implication.
- Separates tangent vectors: for every $P \in X$, we are asking for $s \in \Gamma(X, \mathcal{O}_X(D))$ which vanishes at order 1 at P. (Indeed, this is saying $s \in \mathfrak{m}_P\mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P$ is nonzero.) Arguing as above, we are asking for P to not be a base-point for $D-P$. So we can again hit this condition with (a) to say that we are asking for
$$\dim|D-P| = \dim|D-P-P| + 1$$
and use the fact that D is base-point-free already to finish.

The above discussion completes the proof. ■

Corollary 1.76. Fix a divisor D on a k-curve X of genus g.

(a) If $\deg D \geq 2g$, then D is base-point-free.

(b) If $\deg D \geq 2g+1$, then D is very ample.

Proof. We use Theorem 1.19. Recall $\deg K = 2g-2$, where K is the canonical divisor for X.

(a) Note $\deg(K-D) \leq 0$ and $\deg(K-(D-P)) \leq 0$, so $\ell(K-D) = \ell(K-D-P) = 0$, so we conclude that
$$\ell(D) = \deg D + 1 - G = 1 + \deg(D-P) + 1 - g = 1 + \ell(D-P),$$
so we are done by Proposition 1.75.

(b) Here, we also get $\deg(K-(D-P-Q)) \leq 0$, so $\ell(K-(D-P-Q)) = 0$ again, so arguing as above completes the proof by Proposition 1.75. ■

In particular, we see that every k-curve X has a closed embedding into projective space with a morphism of degree at most $2g+1$.

1.9 February 6

As is to be expected, we sleep with one eye open.

Remark 1.77. The next homework will be due Sunday night. The hope is that it is fun.

1.9.1 Small Projective Embeddings

We are still trying to embed curves into $\mathbb{P}^3_k$. Our main tool continues to be Proposition 1.75.

Corollary 1.78. Fix a divisor D on a k-curve X is ample if and only if $\deg D > 0$.

Proof. Certainly if $\deg D \leq 0$, then $\deg(nD) = 0$ and thus $\ell(nD) = 0$ for all positive integers n, so nD is never very ample, so D is not ample. (Alternatively, if nD is very ample, then nD is the pull-back of $\mathcal{O}_{\mathbb{P}^1_k}(1)$ some closed embedding $f\colon X \to \mathbb{P}^1_k$ and hence has positive degree.) Conversely, if $\deg D > 0$, then $\deg(2g+1)D \geq 2g+1$, so $(2g+1)D$ is very ample by Corollary 1.76, so D is ample. ■

Example 1.79. With $X = \mathbb{P}^1_k$, we note that $\mathcal{O}_{\mathbb{P}^1_k}(1)$ is very ample by the (identity) embedding $X \cong \mathbb{P}^1_k$. It follows that $\mathcal{O}_{\mathbb{P}^1_k}(n)$ are all very ample (and hence ample) for all $n > 0$. By Corollary 1.78, these are all the ample divisors of X.

Remark 1.80. It turns out that a very ample divisor D on X yielding a closed embedding $f\colon X \to \mathbb{P}^n_k$ has $\deg D = \deg f(X)$. Here, $\deg f(X)$ is defined using some intersection theory; for example, if $n = 2$, then this degree is the degree of the polynomial cutting out X. For example, if $g(X) = 1$ and D is a divisor of degree 3 (and hence very ample by Corollary 1.78). Further,

$$\ell(D) = \ell(K - D) + \deg D + (1 - g) = 0 + 3 + 0 = 3,$$

so we define a closed embedding to $\mathbb{P}^2_k$. We conclude that X has an embedding as a cubic curve in $\mathbb{P}^2_k$. Conversely, Exercise I.7.2(b) in [Har77] tells us that any cubic plane curve has genus $\frac{1}{2}(3-1)(3-2) = 1$. Note that adjusting the divisor's linear equivalence class can give us different embeddings to $\mathbb{P}^2_k$ (which are not the same up to an automorphism of $\mathbb{P}^2_k$).

Anyway, we are now almost ready to prove our main result.

K algebraically closed?

Theorem 1.81. Fix a k-curve X. Then X has an embedding to $\mathbb{P}^3_k$.

The outline here is as follows. To begin, fix some closed embedding $X \to \mathbb{P}^n_k$ for some $n > 0$. Then if $n > 3$, we will show that we can project down from $\mathbb{P}^n_k$ to $\mathbb{P}^{n-1}_k$ in a way which preserves us having a closed embedding. Inducting n downwards like this will complete the proof.

We know how to do this first step by Corollary 1.76, so we focus on the second step.

Proposition 1.82. Fix a k-curve X embedded into $\mathbb{P}^n_k$ for some positive integer n. Given $O \in \mathbb{P}^n_k \setminus X$, then the projection $\varphi\colon X \to \mathbb{P}^{n-1}_k$ from O is a closed embedding if and only if the following both hold.

- O does not belong on any secant line.
- O does not belong on any tangent line.

Note that the line we are projecting from O onto does not matter so much because it merely adjusts the map by an automorphism of the ambient space $\mathbb{P}^n_k$, which turns into an automorphism of $\mathbb{P}^{n-1}_k$ on the embedding.

Example 1.83. One can compute that the projection from $O = [0 : \cdots : 0 : 1]$ in $\mathbb{P}^n_k$ to $V(x_n)$ is given by the projection

$$[a_0 : \cdots : a_n] \mapsto [a_0 : \cdots : a_{n-1}].$$

Namely, the line connecting O and $[a_0 : \cdots : a_{n-1}]$ is parameterized by $[t_0 : t_1] \mapsto [t_0a_0 : \cdots : t_0a_{n-1} : t_1a_n]$, which tells us what the intersection with $V(x_n)$ should be.

In our theory of linear systems, we note that the global sections $x_0, \ldots, x_{n-1}$ span some subspace $V \subseteq \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$, and we can see that the only base-point here is the point O because this is the only point of simultaneous vanishing. Thus, our theory grants us a morphism $(\mathbb{P}^n_k \setminus \{0\}) \to \mathbb{P}^{n-1}_k$ given by the above formula! Explicitly, on the affine chart $U_i \subseteq \mathbb{P}^n_k$ where $x_i \neq x_0$ doesn't vanish, we are looking at the ring map

$$k\left[\frac{y_0}{y_i}, \ldots, \frac{y_{n-1}}{y_i}\right] \to k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$$

given by $y_j/y_i \mapsto x_j/x_i$.

Proof. Roughly speaking, we don't want O to be on any secant line so that the projection doesn't send two points to the same point. Additionally, we don't want O to be on any tangent line so that the closed embedding separates tangent vectors. We omit the remainder of the proof aside from this general intuition. ■

We now turn to the proof of Theorem 1.81.

Proof of Theorem 1.81. By Corollary 1.76, we have some closed embedding $X \to \mathbb{P}_k^n$ for some n large enough. Now, if $n > 3$, we use Proposition 1.82 to project down to $\mathbb{P}_k^{n-1}$. Thus, it suffices to find a point O not on any secant line or tangent line. Well, the "secant variety" $\operatorname{Sec} X$ defined by the image of the obvious map

$$(X \times X \setminus \Delta_X) \times \mathbb{P}_k^1 \to \mathbb{P}_k^n$$

we can see has dimension of the image at most 3. Explicitly, on points, this map is given by

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n], [t_0 : t_1]) \mapsto [a_0t_0 + b_0t_1 : \cdots : a_nt_0 + b_nt_1].$$

Similarly, the "tangent variety" $\operatorname{Tan} X$ defined by the image of the obvious map

$$X \times \mathbb{P}_k^1 \to \mathbb{P}_k^n$$

we can see has dimension of the image at most 2. Thus, with $n > 3$, these closed subschemes are proper and so have dense complement, meaning that we can find can point O in the complement of these varieties. This completes the proof. ■

1.10 February 8

Last class we showed that every curve can be embedded into $\mathbb{P}_k^3$.

Remark 1.84. We will not show this, but one can show with a little more work that any curve is birational to a curve in the plane with at worst nodes as singularities.

1.10.1 The Canonical Embedding

Throughout, X is a k-curve, where k is algebraically closed. We will set $g := g(X)$ and let K denote the canonical divisor.

Example 1.85. Notably, $g(X) = 0$ forces $\dim |K| = -1$, so $|K|$ is empty. (After all, $X \cong \mathbb{P}_k^1$.)

Example 1.86. If $g(X) = 1$, then we are looking at $\dim |K| = 0$, and here $\deg K = 0$, so we are just looking at the mapping from X to a point.

In higher genus, things get more interesting.

Lemma 1.87. Fix a k-curve X with $g(X) \geq 2$. Then the canonical divisor K is base-point-free.

Proof. By Proposition 1.75, it suffices to show that $\ell(K - P) = \ell(K) - 1 = g - 1$ for any $P \in X$. Now, because $g(X) \geq 2 > 0$, we know that $X \ncong \mathbb{P}_k^1$, so $\dim |P| = 0$ is forced. Now, using Theorem 1.39, we solve

$$\ell(P) = \ell(K - P) + \deg(P) + (1 - g),$$

What? so $\ell(K - P) = g - 1$, which is what we wanted. ■

In our study of curves, the following curves will make a somewhat large class.

Definition 1.88 (hyperelliptic)**.** A k-curve X is *hyperelliptic* if and only if it admits a degree-2 map to $\mathbb{P}^1_k$.

Remark 1.89. Given any divisor D on X with $\ell(D) = 2$ and $\deg D = 2$, one gets a rational map $X \dashrightarrow \mathbb{P}^1_k$ determined by D. (One can show that all such divisors are base-point-free using Proposition 1.75.) Notably, D is linearly equivalent to an effective divisor.

Remark 1.90. If $g(X) = 2$, then X is hyperelliptic. This was shown on the homework. In brief, K is base-point-free and has degree 2, so it determines a degree-2 map $X \to \mathbb{P}^1_k$.

One can improve Lemma 1.87 as follows.

Proposition 1.91. Fix a k-curve X with $g(X) \geq 2$. Then K is very ample if and only if X is not hyperelliptic.

Proof. By Proposition 1.75, we are interested in the condition

$$\ell(K - P - Q) = \ell(K) - 2 \stackrel{?}{=} g - 2$$

for any $P, Q \in X$. As such, by Theorem 1.39, we compute

$$\ell(P + Q) = \ell(K - P - Q) + \deg(P + Q) + (1 - g) = \ell(K - P - Q) + 3 - g,$$

so $\ell(K - P - Q) = g - 2$ is equivalent to

$$\ell(P + Q) = 1.$$

Now, if X is hyperelliptic, then one can find a divisor D with $\dim |D| = 1$ and $\deg D = 2$, so D linearly equivalent to an effective divisor which looks like $P + Q$. But then $\ell(P + Q) = 2 > 1$, which is a problem. In the other direction, if X is not hyperelliptic, then each effective divisor D with degree 2 must have $D = P + Q$, which forces $\ell(P + Q) < 2$ to retain being not hyperelliptic. ∎

Thus, we are interested in the embedding induced by this canonical divisor.

Definition 1.92 (canonical morphism)**.** Fix a k-curve X of genus $g(X) \geq 2$. Then the canonical divisor K determines the *canonical morphism* $X \to \mathbb{P}^{g-1}_k$ by Lemma 1.87. If X is not hyperelliptic curves, then we in fact get a *canonical embedding* by Proposition 1.91.

Remark 1.93. If $\deg \mathcal{L} = d$, and $\mathcal{L}$ is very ample, then $\mathcal{L}$ grants a closed embedding of X to $\mathbb{P}^{r-1}_k$ where $r := \dim_k \Gamma(X, \mathcal{L})$ (using the canonical embedding), and this embedding retains X as a curve of degree d.

1.11 February 10

Last class, we were in the middle of describing the canonical morphism of hyperelliptic curves. I have moved the entire proof of today because we did more work on it today.

1.11.1 Hyperelliptic Curves

We will spend the rest of class understanding hyperelliptic curves. The following is our statement.

Theorem 1.94. Fix a hyperelliptic curve X of genus $g(X) \geq 2$.

(a) Then X has a unique divisor class g_2^1 yielding the double-cover $\pi\colon X \to \mathbb{P}_k^1$.

(b) The canonical morphism $f\colon X \to \mathbb{P}^{g-1}$ can be written as
$$X \to \mathbb{P}_k^1 \to \mathbb{P}_k^{g-1},$$
where the last map is the $(g-1)$-uple embedding.

(c) Every effective canonical divisor can be written as the sum of $(g-1)$ effective divisors linearly equivalent to g_2^1.

Proof. For brevity, let X' be the image $f(X)$, and we fix some g_2^1 on X yielding a double-cover $X \to \mathbb{P}_k^1$.

Now, for any effective divisor $P+Q$ linearly equivalent to g_2^1, then we claim Q is a base-point of $K-P$: indeed, we know $\ell(K-P) = g-1$ because K is base-point-free by Lemma 1.87, but K is not very ample witnessed by P and Q by the proof of Proposition 1.91, so $\ell(K-P-Q) = \ell(K-P)$ is forced. Thus, Q is in fact a base-point.

We now claim that f is not birational. There are two cases.

- In the case where $P \neq Q$, we see that K does not separate the points P and Q. Explicitly, we see that $s \in \Gamma(X, \mathcal{O}_X(K-P))$ is equivalent to $s \in \Gamma(X, \mathcal{O}_X(K-P-Q))$ is equivalent to $s \in \Gamma(X, \mathcal{O}_X(K-Q))$ (by symmetry). In other words, we are saying that any divisor $K + \operatorname{div}(s)$ which retains P in its support will also retain Q in its support.

  It follows that $f(P) = f(Q)$: otherwise $f(P) \neq f(Q)$ lets us separate the two points in $\mathbb{P}_k^{g-1}$ and hence find a basis of $\Gamma(\mathbb{P}_k^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ separating them. But then we can pull back this basis to $\Gamma(X, \mathcal{O}_X)$ to find sections separating P and Q.

- Otherwise, we in the case where $P = Q$, we are now given that P is a base-point of $K-P$. Explicitly, we know that $s \in \Gamma(X, \mathcal{O}_X(K-P))$ is equivalent to $s \in \Gamma(X, \mathcal{O}_X(K-2P))$, which now means that f does not separate tangent vectors at P—namely, the image of $\Gamma(X, \mathcal{O}_X(K))$ does not generate $\mathfrak{m}_P\mathcal{O}_X(K)_P/\mathfrak{m}_P/\mathcal{O}_X(K)_P$. However, $\mathcal{O}_{\mathbb{P}_k^{g-1}}(1)$ does separate these tangent spaces by hyperplanes, so f is again not a closed embedding at the local ring at P.

By adjusting $P+Q$ appropriately in its linear equivalence class, we see that f fails to be a closed embedding at infinitely many points, so f is not a birational morphism.

As such, $\mu := \deg f$ is at least 2.[3] However, letting d denote the degree of $X' \subseteq \mathbb{P}_k^{g-1}$, we claim
$$d\mu \stackrel{?}{=} 2g-2.$$

For this, we must understand $d = \deg X'$, which we note is the number of intersections (counted with multiplicity) of a hyperplane section of $\mathbb{P}_k^{g-1}$ intersecting $Y = X' \setminus \operatorname{Sing} X'$, where $\operatorname{Sing} X'$ is the singular locus. Letting $H \cap Y$ denote such an intersection (with d points and hence degree d as a divisor), we restrict $f\colon X \to X'$ to $f\colon X \to \widetilde{X'}$, granting
$$\deg f^*(H \cap \widetilde{X'}) = (\deg f)d = \mu d$$
by explicitly writing down what does f^* does on points on the finite morphism of smooth curves $f|_{\widetilde{X'}}$. In total, we conclude that $d \leq g-1$.

Now, the composite $\widetilde{X'} \to X' \subseteq \mathbb{P}_k^{g-1}$ is a projective morphism and so grants a divisor $\mathfrak{d}$ on $\widetilde{X'}$, of dimension $g-1$. To compute the degree, we intersect $\widetilde{X'}$ with a generic hyperplane (for example, avoiding

[3] Alternatively, note that $\dim |g_2^1| = 1$ forces there to be infinitely many representatives of the form $P+Q$ with $P \neq Q$, for otherwise there are infinitely many representatives of the form $2P$, which would mean that our double-cover $X \to \mathbb{P}_k^1$ is ramified at infinitely many points, which is impossible.

the singular locus of X'), which still has d intersections after passing to the birational curve X'. As such, we see $\deg \mathfrak{d} = d$. But from the homework, we have the inequality

$$\dim \mathfrak{d} \leq \deg D \leq g-1,$$

so we conclude that equalities must hold everywhere, which in turn requires $\mathfrak{d}$ to be a complete linear system. Further, the homework also tells us that the equality case forces $g(X') = 0$, so we must have $X \cong \mathbb{P}^1_k$ (note the degree of $\mathfrak{d}$ is positive!), so comparing degrees enforces $\mathfrak{d} = \mathcal{O}_{\mathbb{P}^1_k}(g-1)$. Notably, we also get $\deg \mu = 2$ here.

Now, the map $\widetilde{X'} \to X' \subseteq \mathbb{P}^{g-1}_k$ is given by the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(g-1)$ and g spanning sections, but there is a basis of the g monomials of degree $g-1$ in the variables x_0 and x_1, so our morphism we can write out explicitly as

$$[x_0 : x_1] \mapsto \left[x_0^{g-1} : x_0^{g-2}x_1 : \cdots : x_0x_1^{g-2} : x_1^{g-1}\right]$$

by expanding out our sections. (One can also show that this is a closed embedding.) This is exactly the $(g-1)$-uple embedding (or "Veronese" embedding). We conclude that the canonical morphism $X \to \mathbb{P}^{g-1}_k$ has been factored as

$$X \to \widetilde{X'} \cong \mathbb{P}^1_k \to \mathbb{P}^{g-1}_k,$$

where the map $\mathbb{P}^1_k \to \mathbb{P}^{g-1}_k$. In particular, the image of the $(g-1)$-uple embedding can be checked to be nonsingular, so X' is successfully nonsingular, and we have $\widetilde{X'} \cong X'$.

Roughly speaking, it remains to discuss the uniqueness of our double-cover $\pi\colon X \to \mathbb{P}^1_k$. Notably, for any $P+Q$ linearly equivalent to g^1_2, pulling back the divisor $f(P) \in X' \cong \mathbb{P}^1_k$ up to $P+Q$ enforces $f(P) = f(Q)$. Running the same argument with π, which comes from a particular representative of g^1_2, we conclude that f and π coincide for infinitely many points of X, so they must coincide because $\operatorname{eq}(f,\pi)$ has positive codimension in X and therefore must vanish. It follows that π and f are uniquely determined by representatives of g^1_2. And quickly, we note that g^1_2 is also unique: it factors through the canonical morphism $f\colon X \to \mathbb{P}^{g-1}_k$ (followed by an embedding), which we already know to be unique, so the precise morphism π is unique (of course, up to automorphism of $\mathbb{P}^1_k$).

As such, we have shown (a) and (b). To show (c), we note that all effective canonical divisors are realizes as pullbacks of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^{g-1}_k}(1)$ on $X' \subseteq \mathbb{P}^{g-1}_k$. But we can pull this back along f factored as

$$X \xrightarrow{\pi} \mathbb{P}^1_k \cong X' \subseteq \mathbb{P}^{g-1}_k.$$

Well, this hyperplane intersects X' at $g-1$ points, and then each of those points in $\mathbb{P}^1_k$ will pull back to an effective divisor which represents g^1_2. ■

1.11.2 Special Divisors

We pick up the following definition.

Definition 1.95 (special)**.** Fix a divisor D on a k-curve X. Then D is *special* if and only if $\ell(K-D) > 0$.

Remark 1.96. If $\ell(K-D) = 0$, then Theorem 1.39 tells us

$$\ell(D) = \deg D + 1 - g,$$

so we understand the divisor pretty well.

We will show the following result.

Proposition 1.97 (Clifford)**.** Fix an effective special divisor D on the k-curve X. Then

$$\dim |D| \leq \frac{1}{2}\deg D.$$

To show this, we have the following lemma.

Lemma 1.98. Fix effective divisors D and E on a K-curve X. Then

$$\dim |D| + \dim |E| \leq \dim |D + E|.$$

Proof. Note that there is a map $\varphi\colon |D| \times |E| \to |D + E|$ by sending a pair (D', E') of effective divisors in the correct linear equivalence classes to $D' + E'$. Note that any effective divisor in $|D + E|$ can be written as a sum of effective divisors in only finitely many ways, so this map has finite fibers.

Going further, we can upgrade φ to a k-bilinear map of k-vector spaces given by

$$\Gamma(X, \mathcal{L}(D)) \times \Gamma(X, \mathcal{L}(E)) \to \Gamma(X, \mathcal{L}(D + E))$$

by sending $(f, g) \mapsto fg$. In particular, this grants us a map on projective spaces

$$\mathbb{P}\Gamma(X, \mathcal{L}(D)) \times \mathbb{P}\Gamma(X, \mathcal{L}(E)) \to \mathbb{P}\Gamma(X, \mathcal{L}(D + E))$$

agreeing with our earlier definition of φ by tracking through how $\mathbb{P}\Gamma(X, \mathcal{L}(D))$ agrees with $|D|$. Notably, this map still has finite fibers, so it is quasifinite, so taking dimensions (this morphism is basically finite) lets us conclude $\dim |D| + \dim |E| \leq \dim |D + E|$. ■

We are now ready to show Proposition 1.97.

Proof of inequality in Proposition 1.97. Because D is effective and special, we know $\ell(K - D) \neq 0$, so we can find an effective divisor E linearly equivalent to $K - D$. As such, Lemma 1.98 grants

$$\dim |D| + \dim |K - D| = \dim |D| + \dim |K - D| \leq \dim |K + E| = \dim |K| = g - 1,$$

but Theorem 1.39 tells us

$$\dim |D| - \dim |K - D| \leq \deg D + 1 - g,$$

so rearranging grants the inequality. ■

Remark 1.99. We can also check some equality cases, quickly. If $D = 0$, there is nothing to say. When $D = K$, we simply note $\dim |K| = g - 1$, but $\deg K = 2g - 2$, so we are done. Lastly, suppose X is hyperelliptic and $D \sim kg_2^1$ for some k. We only care about linear equivalence class, so we might as well assume $D = kg_2^1$ and proceed by induction on k. At $k = 1$, we note $\dim g_2^1 = 1$ and $\deg g_2^1 = 2$, so equality holds. Then for our induction, we note Lemma 1.98 gives

$$\dim \left|(k + 1)g_2^1\right| \geq \dim \left|kg_2^1\right| + \dim \left|kg_2^1\right| = k + 1,$$

but we just showed that $\dim \left|(k + 1)g_2^1\right| \leq \frac{1}{2} \deg \left((g + 1)g_2^1\right) = k + 1$ above, so we must have equalities.

1.12 February 13

Today we finish discussing curves. The fourth assignment will be released later today.

1.12.1 Finishing Clifford's Theorem

We are interested in the equality cases of Proposition 1.97. Here is the result.

Theorem 1.100 (Clifford)**.** Fix an effective special divisor D on the k-curve X. Then $\dim|D| = \frac{1}{2}\deg D$ if and only if $D = 0$, or $D = K$, or X is hyperelliptic where D is linearly equivalent to a multiple of X's g_2^1.

Proof. We showed each of these equality cases work in Remark 1.99. It remains to show that the equality implies one of these conditions.

Well, suppose that $\dim|D| = \frac{1}{2}\deg D$ and $D \neq 0$ and $D \neq K$; we will show that X is hyperelliptic, and D is a multiple of g_2^1. Note Theorem 1.39 tells us that

$$\dim|D| - \dim|K - D| \leq \deg D + 1 - g$$

must achieve equality, by the proof of Proposition 1.97 above. Now, we see $\deg D$ must be even. We will induct on $\deg D$. For example, if $\deg D = 2$ and $\ell(D) = 1$ forces D to induce a double-cover $X \to \mathbb{P}_k^1$ (meaning X is hyperelliptic), so $D \sim g_2^1$ follows from this uniqueness of this divisor from Theorem 1.94.

To finish the proof, suppose $\deg D \geq 4$ so that $\dim|D| \geq 2$. Because $\dim|K - D| \geq 0$ (recall D is special), we get an effective divisor $E \in |K - D|$. Without loss of generality, we may assume that $\deg E > 0$ (otherwise $D = K$), so we can find a point P in the support of E and a point Q not in the support of D. Without loss of generality, we can find an effective divisor linearly equivalent to D with support containing P and Q, which is doable because $\dim|D| \geq 2$: indeed, $|D|$ determines a projective morphism $f\colon X \to \mathbb{P}_k^n$ for some $n \geq 2$, and then one can pull back a hyperplane H of $\mathbb{P}_k^n$ containing $f(P)$ and $f(Q)$ so that $D \sim f^*H$ contains P and Q in its support.

We now set $D' := D \cap E$, which by definition is the maximal effective divisor such that $D' \leq D$ and $D' \leq E$. Note that $D' < D$ in fact because Q is in the support of D but not E, but $D' > 0$ because P is in the support of both D and E. We are going to want to use the inductive hypothesis on D'. Notably, we claim we have an exact sequence

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \oplus \mathcal{O}_X(E) \to \mathcal{O}_X(D + E - D') \to 0.$$

Here, $\mathcal{O}_X(D') \to \mathcal{O}_X(D) \oplus \mathcal{O}_X(E)$ is the diagonal map, and the second map is $(f, g) \mapsto (f - g)$. Notably, for any valuation v on $K(X)$, we see $v(a + b) \geq \min\{v(a), v(b)\}$ with equality provided $v(a) \neq v(b)$.

To see our exactness, let's work an example.

Example 1.101. Fix an open subscheme $U \subseteq X$, and take $D = P + 2Q$ and $E = 2P + Q$ on restriction to U so that $D' = P + Q$ here. Given some f with a pole of order 1 at P and order 2 at Q, and some g with a pole of order 2 at P and of order 1 at Q. Indeed, we see that $f - g$ has poles of order 2 at both P and Q, which tells us $f - g$ is in fact in $\mathcal{O}_X(D + E - D')$.

This example explains our exactness: by checking surjectivity at the stalks of some point R, we may assume that $D = aR$ and $E = bR$ where $a \geq b$ without loss of generality. Then the exact sequence looks roughly like

$$0 \to \mathcal{O}_X(bR)_R \to \mathcal{O}_X(aR) \oplus \mathcal{O}_X(E)_{bR} \to \mathcal{O}_X(aR) \to 0,$$

which is more clearly exact: we are injective on the left by definition, we are surjective on the right by taking elements of the form $(0, -g)$, and we are exact in the middle because the only elements which vanish in the right map are diagonal ones.

Now, taking global sections, we get the left-exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X(D')) \to \Gamma(X, \mathcal{O}_X(D)) \oplus \Gamma(X, \mathcal{O}_X(E)) \to \Gamma(X, \mathcal{O}_X(D + E - D')).$$

Taking dimensions everywhere, we see

$$\ell(D) + \ell(E) \leq \ell(D') + \ell(D + E + D'),$$

which implies

$$g - 1 = \dim|K| = \dim|D| + \dim|D - K| \leq \dim|D'| + \dim|K - D'| \leq \dim|K| = g - 1,$$

where we have used Lemma 1.98 repeatedly. Thus, equalities hold everywhere, and we are forced to have $\dim|D'| + \dim|K - D'| = g - 1$ as well, which means that D' also satisfies $\dim|D'| = \frac{1}{2}\deg D'$ by the argument of Proposition 1.97. However, $\deg D' < \deg D$, and $D' \neq 0, K$, so the inductive hypothesis now applies to D', telling us that X is hyperelliptic. It remains to show that D is a multiple of g_2^1.

The trick is to relate D to K and use Theorem 1.94 because we already know that K is a multiple of g_2^1. Well, set $r := \dim|D|$. Note that D being special implies that $\ell(K - D) > 0$, so $\deg D \leq 2g - 2$, so we go ahead and study like to show $S := D + (g - 1 - r)g_2^1$ is linearly equivalent to K, where the point is that $\deg S = 2g - 2$. We would like to compute $\dim|S|$. On one hand, we see

$$\dim|S| \geq \dim|D| \geq \dim|D| + (g - 1 - r) = g - 1.$$

On the other hand, Theorem 1.39 implies

$$|S| - |K - S| = (2g - 2) + (1 - g) = g - 1,$$

so $|K - S| \geq 0$. Thus, $K - S$ is linearly equivalent to an effective divisor of degree 0, which actually forces $K - S \sim 0$, so $S \sim K$ follows. However, this then implies that

$$D \sim K - (g - 1 - r)g_2^1,$$

which finishes because K is a multiple of g_2^1 by Theorem 1.94. This completes the proof. ■

Remark 1.102. This concludes our discussion of curves. We hope it motivates our discussion of cohomology, which will follow.

THEME 2
COHOMOLOGY

2.1 February 13

We now switch gears and talk about cohomology. Notably, this is not the usual algebraic topology version of cohomology from the topology on our scheme because the Zariski topology on a scheme is so terrible.

2.1.1 Some Starting Remarks

Fix a scheme X and an abelian sheaf $\mathcal{F}$ on X. In other words, $\mathcal{F}$ is a sheaf valued in an abelian category (though we personally will only request that $\mathcal{F}$ have sections which are modules). Later, we might ask for $\mathcal{F}$ to be an $\mathcal{O}_X$-module.

Roughly speaking, computing cohomology is about computing derived functors. Notably, it will turn out that all relevant categories have enough injectives, so we will be able to build an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^1 \to \mathcal{I}^2 \to \mathcal{I}^2 \to \cdots .$$

Now, $H^\bullet(X, \mathcal{F})$ is intended to be the right-derived functor of $\Gamma(X, -)$, so we will want to take global sections to get a complex

$$0 \overset{d^{-1}}{\to} \Gamma(X, \mathcal{I}^1) \overset{d^0}{\to} \Gamma(X, \mathcal{I}^2) \overset{d^1}{\to} \cdots ,$$

which is no longer exact. It then turns out by derived functor magic that we can compute cohomology from these groups. Namely, by definition, we will take

$$H^i(X, \mathcal{F}) \coloneqq \frac{\ker d^i}{\operatorname{im} d^{i-1}},$$

which turns out to be exactly the cohomology groups we desire.

Remark 2.1. The above recipe will work fine for any "Grothendieck topology," such as the étale cohomology. We won't see that in this class, but it is a useful general machine.

Remark 2.2. Classically, cohomology was computed using more general "acyclic" resolutions (most notably for Čech cohomology), but it turns out that injective resolutions are better-behaved for our abstract arguments. Nonetheless, we will see these acyclic resolutions in this class because they are useful; in particular, we will be building resolutions using flasque sheaves in this class.

2.2 February 15

Let's begin our discussion of cohomology.

2.2.1 Abelian Categories

Let's be somewhat formal with our definition of an abelian category.

Definition 2.3 (pre-additive)**.** A category $\mathcal{C}$ is *pre-additive* if and only if it is enriched over the category of abelian groups. In particular, for objects $X, Y.Z \in \mathcal{C}$, the set $\operatorname{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group, and for any $f: X \to Y$, the maps

$$(- \circ f)\colon \operatorname{Mor}_{\mathcal{C}}(Y, Z) \to \operatorname{Mor}_{\mathcal{C}}(X, Z) \qquad \text{and} \qquad (f \circ -)\colon \operatorname{Mor}_{\mathcal{C}}(Z, X) \to \operatorname{Mor}_{\mathcal{C}}(Z, Y)$$

are group homomorphisms.

Definition 2.4 (additive)**.** A pre-additive category $\mathcal{C}$ is *additive* if and only if it has finite products.

Remark 2.5. In a pre-additive category, finite products and coproducts are canonically isomorphic. In other words, given objects $X, Y \in \mathcal{C}$, if $X \oplus Y$ exists, then so does $X \times Y$ (and conversely), and there is a canonical isomorphism between them.

Remark 2.6. By Remark 2.5, the empty product and coproduct coincide, so we have an object which is both initial and final, which we see is our zero object. Thus, additive categories have a zero object.

Let's explain Remark 2.5 quickly. Roughly speaking, it turns out that both universal properties are equivalent to having a given object Z fit into the following commutative diagram.

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\iota_X"'] & & X \\
& Z \arrow[ur, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
Y \arrow[ur, "\iota_Y"] \arrow[rr, equal] & & Y
\end{tikzcd}$$

Here, we also require that $\pi_Y \iota_X = 0$ and $\pi_X \iota_Y = 0$ and $\mathrm{id}_Z = \iota_X \pi_X + \iota_Y \pi_Y$. It is not too hard to see that any object Z with the above data is both a product (Z, π_X, π_Y) and a coproduct (Z, ι_X, ι_Y). For example, given an object A with maps $\varphi_X : X \to A$ and $\varphi_Y : Y \to A$, we claim that having $\varphi\colon A \to Z$ with $\pi_X \varphi = \varphi_X$ and $\pi_Y \varphi = \varphi_Y$ is equivalent to

$$\varphi = \iota_X \varphi_X + \iota_Y \varphi_Y,$$

so our $\varphi\colon A \to Z$ and is unique. Indeed, this φ works, and it is forced because

$$\varphi = \mathrm{id}_Z \circ \varphi = (\iota_X \pi_X + \iota_Y \pi_Y) \circ \varphi = \iota_X \varphi_X + \iota_Y \varphi_Y.$$

Showing that Z is a coproduct follows similarly.

Conversely, if we (for example) have a coproduct (Z, ι_X, ι_Y), we can build the projection map $\pi_X : Z \to X$ by applying to the universal property in the following diagram.

$$\begin{tikzcd}
& X \arrow[d, "\iota_X"'] \arrow[ddr, equal, bend left] & \\
Y \arrow[r, "\iota_Y"] \arrow[drr, bend right, "0"'] & Z \arrow[dr, dashed, "\pi_X"] & \\
& & X
\end{tikzcd}$$

Defining π_Y similarly, we see that $\pi_Y \iota_X = 0$ and $\pi_X \iota_Y = 0$ for free, and then one can show that $\mathrm{id}_Z = \iota_X \pi_X + \iota_Y \pi_Y$.

We now continue our march towards abelian categories.

Definition 2.7 (pre-abelian)**.** An additive category $\mathcal{C}$ is *pre-abelian* if and only if it admits kernels and cokernels.

Remark 2.8. Given $f, g\colon X \to Y$, we see that $\mathrm{eq}(f,g) = \ker(f-g)$, so in fact we see we admit equalizers. Because we also admit coequalizers for similar reasons, we see that pre-abelian categories have finite limits and colimits.

Definition 2.9 (abelian)**.** A pre-abelian category $\mathcal{C}$ is *abelian* if and only if every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

Here, a monomorphism $f\colon X \to Y$ merely means that the induced maps $\mathrm{Hom}_{\mathcal{C}}(Z,X) \to \mathrm{Hom}_{\mathcal{C}}(Z,Y)$ is injective for any object Z, and epimorphisms are defined dually.

We can combine the above definitions into the following cleaner one.

Proposition 2.10. Fix a pre-abelian category $\mathcal{C}$. Then $\mathcal{C}$ is abelian if and only if the following hold.

(a) Every monomorphism is the kernel of its cokernel.

(b) Every epimorphism is the cokernel of its kernel.

(c) Further, any morphism can be factored as an epimorphism followed by a monomorphism.

Proof. Omitted. This more or less follows from expanding out our definitions. ■

Remark 2.11. Freyd's embedding theorem says that any locally small abelian category has a faithful embedding to Mod_R for some ring R. As such, we will usually just reason in Mod_R instead of general abelian categories.

2.2.2 Complexes

Throughout, we fix an abelian category $\mathcal{C}$. We now discuss complexes.

Definition 2.12 (complex)**.** Fix an abelian category $\mathcal{C}$. Then a *complex* is the data $(K^\bullet, d^\bullet)$ of objects $\{K^i : i \in \mathbb{Z}\}$ and maps $d^i\colon K^i \to K^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for each i. We will often notate this as

$$\cdots \to K^{-2} \xrightarrow{d^{-2}} K^{-1} \xrightarrow{d^{-1}} K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2 \to \cdots.$$

Definition 2.13 (cohomology)**.** Fix a complex $(K^\bullet, d^\bullet)$ in an abelian category $\mathcal{C}$, the ith *cohomology* for $i \in \mathbb{Z}$ is the object

$$H^i(K^\bullet) := \frac{\ker d^i}{\operatorname{im} d^{i-1}},$$

where we have embedded in K^i. Note that there is a map $\operatorname{im} d^{i-1} \to \ker d^i$ because $d^i \circ d^{i-1} = 0$.

We also have morphisms.

Definition 2.14 (complex morphism)**.** Fix an abelian category $\mathcal{C}$. Then a *morphism of complexes* $(K^\bullet, d_K^\bullet) \to (L^\bullet, d_L^\bullet)$ is a collection of maps $f^\bullet \colon K^\bullet \to L^\bullet$ making the following diagram commute.

$$\begin{tikzcd} \cdots \arrow[r] & K^{-2} \arrow[r] \arrow[d, "f^{-2}"] & K^{-1} \arrow[r] \arrow[d, "f^{-1}"] & K^0 \arrow[r] \arrow[d, "f^0"] & K^1 \arrow[r] \arrow[d, "f^1"] & K^2 \arrow[r] \arrow[d, "f^2"] & \cdots \\ \cdots \arrow[r] & L^{-2} \arrow[r] & L^{-1} \arrow[r] & L^0 \arrow[r] & L^1 \arrow[r] & L^2 \arrow[r] & \cdots \end{tikzcd}$$

Remark 2.15. A morphism $(f^\bullet) \colon (K^\bullet, d_K^\bullet) \to (L^\bullet, d_L^\bullet)$ gives rise (functorially) to maps

$$H^i(f^\bullet) \colon H^i(K^\bullet) \to H^i(L^\bullet)$$

for any $i \in \mathbb{Z}$. This is some technical argument dealing with the commutativity of the following diagram.

$$\begin{tikzcd} K^{i-1} \arrow[r, "d_K^{i-1}"] \arrow[d, "f^{i-1}"] & K^i \arrow[r, "d_K^i"] \arrow[d, "f^i"] & K^{i+1} \arrow[d, "f^{i+1}"] \\ L^{i-1} \arrow[r, "d_L^{i-1}"'] & L^i \arrow[r, "d_L^i"'] & L^{i+1} \end{tikzcd}$$

Quickly, we see that an element of the image of d_K^{i-1} does get mapped to the image of d_L^i because $f^i \circ d_K^{i-1} = d_K^i \circ f^{i-1}$; similarly, the kernel gets mapped to the kernel. As such, our map on cohomology is well-defined, and we get to define our morphism. Functoriality checks are routine.

And we even have a notion of exactness.

Definition 2.16 (exact)**.** Fix an abelian category $\mathcal{C}$. Given a sequence

$$(A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet) \to (C^\bullet, d_C^\bullet),$$

this is *exact* at $B^\bullet$ if and only if the corresponding maps on each degree i are exact at B^i.

Here is our central result for cohomology here.

Proposition 2.17. Fix an abelian category $\mathcal{C}$ and an exact sequence

$$0 \to (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet) \to (C^\bullet, d_C^\bullet) \to 0$$

of complexes. Then there is a long exact sequence in cohomology as follows.

$$\begin{tikzcd} \cdots \arrow[r] & H^{i-1}(A^\bullet) \arrow[r] & H^{i-1}(B^\bullet) \arrow[r] & H^{i-1}(C^\bullet) \arrow[dll] \\ & H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[r] & \cdots \end{tikzcd}$$

Proof. The argument of Remark 2.15 grants us a morphism of exact sequences as follows.

$$\begin{tikzcd} & A^i/\operatorname{im} d_A^{i-1} \arrow[r] \arrow[d, "d_A^i"] & B^i/\operatorname{im} d_B^{i-1} \arrow[r] \arrow[d, "d_B^i"] & C^i/\operatorname{im} d_C^{i-1} \arrow[r] \arrow[d, "d_C^i"] & 0 \\ 0 \arrow[r] & \ker d_A^{i+1} \arrow[r] & \ker d_B^{i+1} \arrow[r] & \ker d_C^{i+1} \end{tikzcd}$$

Hitting this with the snake lemma gives us the desired exact sequence. Zippering our exact sequences together creates the full long exact sequence. ■

We might be interested when two complexes give the same cohomology. This is typically done using a chain homotopy.

Definition 2.18 (chain homotopy)**.** Fix an abelian category $\mathcal{C}$ and complex morphisms $(f^\bullet),(g^\bullet)\colon (K^\bullet,d_K^\bullet)\to(L^\bullet,d_L^\bullet)$. Then a *chain homotopy* is a collection of maps $h^i\colon K^i\to L^{i+1}$ such that $f-g=d_Lh+hd_K$. Here, this arithmetic is taking place in the following squares.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{-2} \arrow[r] \arrow[d] & K^{-1} \arrow[r] \arrow[d] \arrow[dl, "h^{-1}"'] & K^0 \arrow[r] \arrow[d] \arrow[dl, "h^0"'] & K^1 \arrow[r] \arrow[d] \arrow[dl, "h^1"'] & K^2 \arrow[r] \arrow[d] \arrow[dl, "h^2"'] & \cdots \\
\cdots \arrow[r] & L^{-2} \arrow[r] & L^{-1} \arrow[r] & L^0 \arrow[r] & L^1 \arrow[r] & L^2 \arrow[r] & \cdots
\end{tikzcd}$$

Note that this diagram does not commute.

Here is the main result.

Proposition 2.19. Fix an abelian category $\mathcal{C}$ and complex morphisms $(f^\bullet),(g^\bullet)\colon (K^\bullet,d_K^\bullet)\to(L^\bullet,d_L^\bullet)$ such that we have a chain homotopy $h^\bullet$ between these morphisms. Then $f^\bullet$ and $g^\bullet$ induce the same morphism on cohomology.

Proof. We will show that $dh+hd$ induces the zero map on cohomology. This is now some diagram-chase: fixing some index i, we pick an element $a\in\ker d^i$ which we want to show has $(dh+hd)(a)\in\operatorname{im} d^{i-1}$. Well, we have $dha\in\operatorname{im} d^{i-1}$ automatically and $hda=0$ by construction of a, so we are done. ■

We will start talking about resolutions next class.

Definition 2.20 (injective)**.** Fix an abelian category $\mathcal{C}$. Then an object $I\in\mathcal{C}$ is *injective* if and only if any monomorphism $A\hookrightarrow B$ can extend any morphism $A\to I$ to a morphism $B\to I$.

Remark 2.21. We will show on the homework that Mod_R and $\operatorname{Mod}_{\mathcal{O}_X}$ both have enough injectives.

2.3 February 17

We continue talk about cohomology.

2.3.1 Functors for Abelian Categories

We might want to discuss functors between our categories, so we want them to respect some of our internal data.

Definition 2.22 (additive)**.** A functor $F\colon\mathcal{A}\to\mathcal{B}$ of abelian categories is *additive* if and only if $A,A'\in\mathcal{A}$ implies
$$F\colon \operatorname{Hom}_{\mathcal{A}}(A,A')\to\operatorname{Hom}_{\mathcal{B}}(FA,FA')$$
is a group homomorphism.

Remark 2.23. Note that additive functors preserve direct sums/products. Indeed, we saw last class that a direct sum/product of X and Y is an object Z with maps $\iota_X : X \to Z$ and $\iota_Y : Y \to Z$ and $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ satisfying the equations

$$\pi_X \iota_X = \mathrm{id}_X, \qquad \pi_Y \iota_Y = \mathrm{id}_Y, \qquad \mathrm{id}_Z = \iota_X \pi_X + \iota_Y \pi_Y.$$

These equations are preserved by an additive functor, so FZ is a direct sum of FX and FY.

Definition 2.24 (exact)**.** An additive functor $F : \mathcal{A} \to \mathcal{B}$ is *left-exact* if and only if it sends left-exact sequences

$$0 \to A' \to A \to A''$$

to left-exact sequences

$$0 \to FA' \to FA \to FA''.$$

The definition of *right-exact* is entirely dual, and the F is *exact* if and only if it is both left-exact and right-exact.

Remark 2.25. In fact, left-exact functors preserve all finite limits. Directly from the definition, left-exact functors preserve all kernels, so they also preserve equalizers because equalizers are just kernels in abelian categories. Further, because the functor is additive, it preserves finite products, so because all finite limits are an equalizer of some morphism of finite products, our functor does indeed preserve all finite limits.

Warning 2.26. Our definition of left-exact requires the functor to be additive.

2.3.2 Injective Resolutions

To make cohomology work, we will want the following definition.

Definition 2.27 (enough injectives)**.** An abelian category $\mathcal{A}$ has *enough injectives* if and only any object $A \in \mathcal{A}$ has a monomorphism $A \hookrightarrow I$ where I is an injective object.

Having enough injectives grant us injective resolutions.

Definition 2.28. Fix an abelian category $\mathcal{A}$. Given an object $A \in \mathcal{A}$, an *injective resolution* for A is a complex $0 \to I^\bullet$ of injective objects such that $H^0(I^\bullet) = A$ but $H^i(I^\bullet) = 0$ for $i > 0$.

Proposition 2.29. Fix an abelian category $\mathcal{A}$ with enough injectives. For any $A \in \mathcal{A}$, there are injective modules $I^0, I^1, I^2, \ldots$ fitting into an (augmented) injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots.$$

Proof. Because $\mathcal{A}$ has enough injectives, we can embed $A \hookrightarrow I^0$ for some injective module I^0. This gives us a short exact sequence

$$0 \to A \to I^0 \to I^0/A \to 0.$$

Now, because $\mathcal{A}$ has enough injectives, we can embed I^0/A into some injective module I^1, so we get to extend our injective resolution $0 \to A \to I^0 \to I^1$. Running this argument inductively, we get the exact augmented injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots,$$

as desired. ■

As an aside, Given an augmented injective resolution $A \to I^\bullet$, we note that there is a morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

which is an isomorphism on cohomology. This motivates the following definition.

Definition 2.30. Fix an abelian category $\mathcal{A}$. Then a morphism $A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if and only if it induces an isomorphism on cohomology.

It turns out that any two injective resolutions for A are quasi-isomorphic, but this takes some care.

Proposition 2.31. Fix an abelian category $\mathcal{A}$. Given injective resolutions $0 \to I^\bullet$ and $0 \to J^\bullet$ of an object $A \in \mathcal{A}$, they are chain homotopic.

Proof. The main difficulty is constructing some morphism $I^\bullet \to J^\bullet$. This is done using the injectivity. Consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \\ 0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots \end{tikzcd}$$

Because J^0 is injective, the inclusion $A \subseteq I^0$ allows us to extend $A \to J^0$ to a map $I^0 \to J^0$ commuting, so we get to extend this as follows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \\ 0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots \end{tikzcd}$$

We now continue this process. Namely, we have an inclusion $I^0/A \subseteq I^1$, so we can extend the map $I^0/A \to J^1$ to a map $I^1 \to J^1$ to make the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] & \cdots \\ 0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots \end{tikzcd}$$

We can now continue this process inductively.

Repeating this process, we also get a map $J^\bullet \to I^\bullet$ again induced by $\mathrm{id}_A \colon A \to A$. Composing our two morphisms of complexes and subtracting out the identity complex morphism, we want to show that any map of complexes looking like

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "d^{-1}"] \arrow[d, "0"'] & I^0 \arrow[r, "d^0"] \arrow[d, "f^0"] & I^1 \arrow[r, "d^1"] \arrow[d, "f^1"] & I^2 \arrow[r, "d^2"] \arrow[d, "f^2"] & \cdots \\ 0 \arrow[r] & A \arrow[r, "d^{-1}"] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r, "d^1"] & I^2 \arrow[r, "d^2"] & \cdots \end{tikzcd}$$

will have a chain homotopy with 0. We now go construct our chain homotopy. Well, we simply define $h^0 = 0$, and then we define $h^1 \colon I^1 \to I^0$ by extending the map $I^0/A \to J^0$ by the inclusion $I^0/A \to I^1$.

We now continue the process inductively. For example, having h^1, we extend the map $I^1/I^0 \to I^1$ given by $f^1 - dh^1$ by the inclusion $I^1/I^0 \to I^2$ to give a map $h^2 \colon I^2 \to I^1$ by our injectivity. This process will now work in general to complete the definition of our chain homotopy, which shows that we are in fact inducing the zero morphism on cohomology. ■

2.3.3 Right-Derived Functors

We are now reedy to define right-derived functors.

Definition 2.32 (right-derived functor)**.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. Then we define the *right-derived functor* $R^iF\colon \mathcal{A} \to \mathcal{B}$ by taking any object $A \in \mathcal{A}$, forming an injective resolution $A \to I^\bullet$ and computing $R^iF := H^i(FI^\bullet)$.

One ought to check that this construction is functorial and so on, but we won't bother with these checks.

Here are some facts about right-derived functors, which we mostly won't prove.

Proposition 2.33. Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. Fix $A \in \mathcal{A}$.

(a) For $i < 0$, we have $R^iF = 0$.

(b) The functors R^iF are additive, and they are independent of the choice of injective resolution up to some sufficiently canonical isomorphism.

(c) We have $R^0F = F$.

(d) If I is injective, then $R^iF = 0$ for $i > 0$.

Proof. Here we go.

(a) This is by definition of the complex we are taking cohomology of.

(b) The fact that the injective resolutions do not change R^iF is by Proposition 2.31. Additivity follows from an argument about extending the addition of two morphisms, as discussed in Proposition 2.31.

(c) Applying F to
$$0 \to A \to I^0 \to I^1$$
grants
$$0 \to FA \to FI^0 \to FI^1,$$
so $R^0F(A) = H^0(FI^\bullet) = A$.

(d) Use the injective resolution
$$0 \to I \to I \to 0 \to 0 \to \cdots$$
of I, so we are computing the cohomology of
$$0 \to FI \to 0 \to 0 \to \cdots,$$
so we do indeed vanish in higher degree. ■

Lastly, here is our main result.

Theorem 2.34. Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. Given an exact sequence
$$0 \to A' \to A \to A'' \to 0,$$
there is a long exact sequence in cohomology given as follows.

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A') \arrow[r] & R^0F(A) \arrow[r] & R^0F(A'') \arrow[dll] & \\
 & R^1F(A') \arrow[r] & R^1F(A) \arrow[r] & R^1F(A'') \arrow[r] & \cdots
\end{tikzcd}$$

In fact, this construction is functorial in the short exact sequence.

Proof. It turns out that one can turn our short exact sequence into a short exact sequence of injective resolutions by some technical lemma and then take cohomology, using Proposition 2.17. Lastly, one can check that a morphism of short exact sequences induces a morphism of the long exact sequences by using the functoriality of the Snake lemma, using Proposition 2.17 once more. ■

Injective objects are somewhat hard to handle, so we introduce acyclic resolutions.

Definition 2.35 (acyclic)**.** Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. An object $I \in \mathcal{A}$ is *acyclic* if and only if $R^iF(I) = 0$ for all $i > 0$.

Proposition 2.36. Fix a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, where $\mathcal{A}$ has enough injectives. For $A \in \mathcal{A}$, we can compute cohomology using an acyclic resolution of A instead of an injective one.

Remark 2.37. Theorem 2.34 essentially says that $\{R^iF\}_{i\geq 0}$ is a δ-functor. In fact, our right-derived functors are "universal," which means that any other δ-functor (T^i, δ^i) with a natural transformation $R^0F \Rightarrow T^0$ can extend this natural transformation uniquely to a full morphism of the δ-functors.

2.4 February 22

Another homework will be assigned probably later today.

2.4.1 Universal δ-Functors

Quickly, we state a useful abstract nonsense result.

Definition 2.38 (effaceable)**.** Fix an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories. Then F is *effaceable* if and only if each $A \in \mathcal{A}$ has some monomorphism $u\colon A \hookrightarrow M$ such that $F(u) = 0$. We define *coeffaceable* dually.

Of course, no left-exact functor will be effaceable, but we do have the following.

Theorem 2.39 (Grothendeick)**.** Fix a δ-functor $\{T^i, \delta^i\}_{i\geq 0} : \mathcal{A} \to \mathcal{B}$ between abelian categories. If each T^i is effaceable, then $\{T^i, \delta^i\}_{i\geq 0}$ is universal.

The point here is that we can show two δ-functors are the same by showing that one is universal and that the other is effaceable.

2.4.2 Sheaf Cohomology

On the homework, we established the following results.

Theorem 2.40. Fix a ring A. Then Mod_A has enough injectives.

Theorem 2.41. Fix a ringed space $(X, \mathcal{O}_X)$. Then $\operatorname{Mod}_{\mathcal{O}_X}$ has enough injectives.

Corollary 2.42. Fix a topological space X. Then the category Ab_X of abelian sheaves on X has enough injectives.

Proof. Turn X into a ringed space $(X, \mathcal{O}_X)$ by $\mathcal{O}_X = \underline{\mathbb{Z}}$. Then Theorem 2.41 completes the proof. ∎

In particular, we can use our abstract nonsense on the left-exact functor $\Gamma(X, -) \colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$. So we may define sheaf cohomology.

Definition 2.43. Fix a topological space. The right-derived functors of the functor $\Gamma(X, -) \colon \mathrm{Ab}_X \to \mathrm{Ab}$ exist and are denoted by $H^i(X, -)$.

Remark 2.44. If $\mathcal{F}$ is also a sheaf of $\mathcal{O}_X$-modules, then notably our abstract nonsense is forgetting about this extra structure. Nonetheless, we will shortly see that these cohomology groups are the same.

We are going to want to compute some cohomology groups. It will be helpful to be able to more easily build some acyclic sheaves.

Definition 2.45 (flasque)**.** Fix a topological space X. A sheaf $\mathcal{F}$ on X is *flasque* if and only if the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are always surjective.

Roughly speaking, it will turn out that these flasque sheaves are acyclic for our cohomology. Here is a quick result which suggests something along these lines.

Proposition 2.46. Fix a ringed space $(X, \mathcal{O}_X)$. Then every injective $\mathcal{O}_X$-module $\mathcal{F}$ is flasque.

Proof. Fix an open subset $i \colon U \subseteq X$. Then we note $\mathcal{O}_U$ is $i_! i^* \mathcal{O}_X = i_! \mathcal{O}_U$. The point is that $\mathrm{Hom}_X(\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_U(\mathcal{O}_U, \mathcal{F}|_U)$, which is $\mathcal{F}(U)$ because $\mathcal{F}$ is an $\mathcal{O}_U$-module. However, for any other open subset $j \colon V \subseteq U$, then we note our embedding $\mathcal{O}_V = j_! j^* \mathcal{O}_U \hookrightarrow \mathcal{O}_U$ allows us to push forward a section $s \in \mathcal{F}(V)$ inducing a morphism $\mathcal{O}_V \to \mathcal{F}$ to a section $s' \in \mathcal{F}(U)$ inducing a morphism $\mathcal{O}_U \to \mathcal{F}$ by the injectivity of $\mathcal{F}$. Unwinding,
What? we see that our restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. ∎

We quickly note the following properties of flasque sheaves.

Lemma 2.47. Fix an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves on a topological space X. If $\mathcal{F}'$ and $\mathcal{F}$ is flasque, then so is $\mathcal{F}''$ is flasque. Additionally, if $\mathcal{F}'$ is flasque, then

$$0 \to \Gamma(U, \mathcal{F}'0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'') \to 0$$

is exact for any open $U \subseteq X$.

Proof. We showed these last class. Notably, the first statement follows from the second by hitting ourselves with the Five lemma on the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{F}', U) \arrow[r] \arrow[d, two heads] & \Gamma(\mathcal{F}, U) \arrow[r] \arrow[d, two heads] & \Gamma(\mathcal{F}'', U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{F}', V) \arrow[r] & \Gamma(\mathcal{F}, V) \arrow[r] & \Gamma(\mathcal{F}'', V) \arrow[r] & 0
\end{tikzcd}$$

Indeed, the left two morphisms being surjective forces the right morphism to also be surjective. ∎

Proposition 2.48. Fix a flasque sheaf $\mathcal{F}$ of abelian groups on a topological space X. Then $\mathcal{F}$ is acyclic for the functor $\Gamma(X, -)$.

Proof. Embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$. Taking quotients, we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0,$$

so the previous lemma does tell us that $\mathcal{G}$ must be flasque. We now finish by dimension-shifting: by the long exact sequence, we see that $H^i(X, \mathcal{G}) = H^{i+1}(X, \mathcal{F})$. Furthermore, our long exact sequence has

$$\Gamma(X, \mathcal{I}) \twoheadrightarrow \Gamma(X, \mathcal{G}) \to H^1(X, \mathcal{F}) \to \underbrace{H^1(X, \mathcal{I})}_{0},$$

so we see $H^1(X, \mathcal{F}) = 0$ follows. But then $H^1(X, \mathcal{G}) = 0$ because $\mathcal{G}$ is flasque, so $H^2(X, \mathcal{F}) = 0$ also, and we can continue our induction upwards. ∎

Corollary 2.49. The right-derived functors of $\Gamma(X, -): \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ coincide with $H^\bullet(X, -)$.

Proof. Take an injective resolution of some $\mathcal{O}_X$-module $\mathcal{F}$. But this is an acyclic resolution for $\Gamma(X, -): \mathrm{Ab}_X \to \mathrm{Ab}$, so we are done by Proposition 2.36. ∎

Remark 2.50. Notably, if X actually has the structure of a ringed space $(X, \mathcal{O}_X)$, then deriving with $\Gamma(X, -): \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ grants $H^\bullet(X, -)$ a module structure over $\Gamma(X, \mathcal{O}_X)$. For example, if X is a k-variety, this tells us that the $H^\bullet(X, -)$ should be k-vector spaces.

2.4.3 Dimension-Bounding Cohomology

While we're here, we pick up the following vanishing result.

Theorem 2.51. Fix a Noetherian topological space X of dimension n. For any abelian sheaf $\mathcal{F}$, we have $H^i(X, \mathcal{F}) = 0$ for $i > n$.

We are going to want to pick up a few starting facts.

Lemma 2.52. Fix a continuous map $f: X \to Y$ of topological spaces. Given a flasque abelian sheaf $\mathcal{F}$ on X, then $f_*\mathcal{F}$ is flasque on Y.

Proof. For any open subsets $V \subseteq U \subseteq Y$. Then the restriction map $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ is simply the restriction map $\mathcal{F}\left(f^{-1}(U)\right) \to \mathcal{F}\left(f^{-1}(V)\right)$, which is surjective because $\mathcal{F}$ is flasque. ∎

Corollary 2.53. Fix a closed embedding $j: Z \to X$ of schemes. Given an abelian sheaf $\mathcal{F}$ on Z, we have $H^\bullet(X, j_*\mathcal{F}) = H^\bullet(Z, \mathcal{F})$.

Proof. Note that $j: Z \to X$ being a closed embedding means that j_* is just extension by zero and is thus exact by checking everything at stalks—in particular, $(j_*\mathcal{F})_x = 0$ for $x \notin j(Z)$. However, j_* also preserves flasque sheaves by Lemma 2.52, so we can push forward a flasque resolution of $\mathcal{F}$ on Z to a flasque resolution of $j_*\mathcal{F}$ on X.

Computing cohomology with these two resolutions completes the proof. Let's be more explicit: give $\mathcal{F}$ an injective resolution as follows
$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots .$$
Hitting this with j_* gives us an exact resolution of flasque sheaves
$$0 \to j_*\mathcal{F} \to j_*\mathcal{I}^0 \to j_*\mathcal{I}^1 \to j_*\mathcal{I}^2 \to \cdots .$$
Taking global sections of these two resolutions will give the same complex
$$0 \to \mathcal{I}^0(Z) \to \mathcal{I}^1(Z) \to \mathcal{I}^2(Z) \to \cdots ,$$
so they yield the same cohomology groups. ■

We also want to know how cohomology compares with directed colimits so that we can take about stalks. This is the content of the next lemma.

Definition 2.54 (directed set)**.** A partially ordered set $(\lambda, \le)$ is *directed* if and only if any two $a, b \in \lambda$ have some $c \in \lambda$ such that $a, c \le c$.

Lemma 2.55. Fix a Noetherian topological space X. If $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ is a collection of sheaves indexed by a directed set, then
$$H^\bullet(X, \operatorname{colim} \mathcal{F}) = \operatorname{colim} H^\bullet(X, \mathcal{F}_i).$$

This result is useful because it more or less bounds the size of $H^\bullet(X, \operatorname{colim} \mathcal{F})$ by the various $H^\bullet(X, \mathcal{F}_i)$s. Roughly speaking, we are going to use this result to prove Theorem 2.51 by replacing $\mathcal{F}$ with some kind of directed colimit of finitely generated subsheaves and work there instead.

2.5 February 24

Today we finish the proof of Theorem 2.51.

2.5.1 Finishing Dimension-Bounding Cohomology

We begin by proving the following result.

Lemma 2.55. Fix a Noetherian topological space X. If $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ is a collection of sheaves indexed by a directed set, then
$$H^\bullet(X, \operatorname{colim} \mathcal{F}) = \operatorname{colim} H^\bullet(X, \mathcal{F}_i).$$

For this we will want the following result.

Lemma 2.56. There are functorial injective resolutions $\operatorname{Inj}\colon \operatorname{Ab}_X \to \operatorname{Kom}(\operatorname{Ab}_X)$.

Proof. We have shown that every object produces some injective resolution. The difficulty here lies in the functoriality of the morphisms between our injective resolutions. Roughly speaking, if one goes back through the proof of having enough injectives, then everything is functorial at every step, so we are okay building these resolutions in a functorial manner. ■

Lemma 2.57. The categories Ab_X and $\operatorname{Mod}_{\mathcal{O}_X}$ are Grothendieck abelian categories. In other words, directed colimits exist and are exact.

Proof. Roughly speaking, we must show that a directed system of exact sequences of sheaves

$$0 \to \mathcal{F}'_\alpha \to \mathcal{F}_\alpha \to \mathcal{F}''_\alpha \to 0$$

produces an exact sequence

$$0 \to \operatorname{colim} \mathcal{F}'_\alpha \to \operatorname{colim} \mathcal{F}_\alpha \to \operatorname{colim} \mathcal{F}''_\alpha \to 0.$$

I think one can check this stalk-locally because colimits commute with taking stalks. ■

Lemma 2.58. Fix a Noetherian topological space X. If $\{\mathcal{F}_\alpha\}$ is a directed system of flasque sheaves, then the colimit is also flasque.

Proof. One can just check this directly. One can check that the presheaf

$$\mathcal{L}\colon U \mapsto \operatorname{colim} \mathcal{F}_\alpha(U)$$

is in fact a sheaf because our colimit is directed, and our space is directed. Thus, because colimits commute with cokernels, the surjectivity of the restriction maps commutes with the colimit.

Let's show that our presheaf $\mathcal{L}$ is in a sheaf.

- Identity: fix sections $a, b \in \mathcal{L}(U)$ and an open cover $\{U_\beta\}$ of U such that $a|_{U_\beta} = b|_{U_\beta}$ vanishes for each U_β. We must show that a vanishes. Well, X is Noetherian, so every open subset is quasicompact, so we may replace our open cover with a finite one $\{U_i\}_{i=1}^n$.

  Now, $a|_{U_i} = b|_{U_i}$ for each i. In fact, because we are in the colimit, we can find some α_i such that $a|_{U_i} = 0$ in merely $\mathcal{F}_{\alpha_i}(U)$. Taking some α exceeding all the α_i, we see that $a|_{U_i} = 0$ for each i in $\mathcal{F}_\alpha(U)$ (again, because are just in a colimit), so $a = 0$ in $\mathcal{F}_\alpha(U)$ because $\mathcal{F}_\alpha$ is a sheaf.
- The gluability condition now proceeds as usual. Namely, given some local conditions, we can glue up in some fixed $\mathcal{F}_\alpha$ and then use the above identity axiom to check that this works on all local conditions. We omit the details. ■

Remark 2.59. One can use the above result to show that the infinite direct sum of flasque sheaves is a flasque sheaf by taking a direct colimit over the finite direct sums.

We are now ready to show Lemma 2.55.

Proof of Lemma 2.55. Let the directed set by λ so that $\mathcal{F}_\bullet$ is really a functor $\lambda \to \mathrm{Ab}_X$. Then we get injective resolutions

$$\lambda \xrightarrow{\mathcal{F}_\bullet} \mathrm{Ab}_X \xrightarrow{\mathrm{Inj}} \mathrm{Kom}(\mathrm{Ab}_X).$$

Taking the colimit of the produced injective resolution term-wise produces still an exact flasque resolution: it's exact because taking directed colimits is exact, and it's flasque because the directed colimit of flasque sheaves remains flasque. Computing cohomology with this injective resolution will complete the proof because taking global sections commutes with taking directed colimits.

Let's see this more explicitly. For each $\alpha \in \lambda$, we get some injective resolution

$$0 \to \mathcal{F}_\alpha \to \mathcal{I}^0_\alpha \to \mathcal{I}^1_\alpha \to \mathcal{I}^2_\alpha \to \cdots,$$

and then we produced an injective resolution

$$0 \to \operatorname{colim} \mathcal{F}_\alpha \to \operatorname{colim} \mathcal{I}^0_\alpha \to \operatorname{colim} \mathcal{I}^1_\alpha \to \operatorname{colim} \mathcal{I}^2_\alpha \to \cdots.$$

Now, taking the global sections of these sheaves after colimits is okay because we showed that directed colimits commutes with taking sections above. Thus, we see global sections has

$$0 \to \operatorname{colim} \mathcal{F}_\alpha(U) \to \operatorname{colim} \mathcal{I}^0_\alpha(U) \to \operatorname{colim} \mathcal{I}^1_\alpha(U) \to \operatorname{colim} \mathcal{I}^2_\alpha(U) \to \cdots,$$

and we can now compute cohomology here to finish. ■

We now show Theorem 2.51.

Theorem 2.51. Fix a Noetherian topological space X of dimension n. For any abelian sheaf $\mathcal{F}$, we have $H^i(X,\mathcal{F})=0$ for $i>n$.

Proof. We show this in steps.

1. To begin, we reduce to the case where X is irreducible, by induction. As such, suppose that $X=Z\cup Y$ where Z is irreducible and closed, where we know the statement for Y. The point is that we can set $U:=X\setminus Z$ and use extension by zero. Let $j\colon U\subseteq X$ and $i\colon Z\subseteq X$ be the embeddings, and then any abelian sheaf $\mathcal{F}$ produces an exact sequence
$$0\to j_!j^*\mathcal{F}\to\mathcal{F}\to i_*i^*\mathcal{F}\to 0,$$
which we can see is exact by checking stalk-locally. (The important point is that each $x\in X$ lies in exactly one of Z or U.) We now want to hit this with the long exact sequence and use Lemma 2.55. The main claim is that
$$H^\bullet(X,j_!j^*\mathcal{F})\stackrel{?}{=}H^\bullet(\overline{U},j_!j^*\mathcal{F}),$$
which will complete the proof because $\overline{U}$ has fewer irreducible components than X, and we can use the long exact sequence above to finish the needed vanishing. Well, to see the claim, let $k\colon\overline{U}\subseteq X$ be the inclusion, and it suffices to show that k_*k^* sends injective resolutions to flasque resolutions so that we can compute cohomology in the usual way via our injectives.

   Well, taking k_* is certainly sending flasque sheaves to flasque sheaves, so it suffices to check that k^* sends flasque sheaves to flasque sheaves. For this, we see that
$$k^*\mathcal{I}(V)=\operatorname*{colim}_{V'\supseteq V}\mathcal{I}(V')$$
because we know that taking colimits here is safe because this is a directed colimit. Thus, we can check that this is flasque more locally on our global sections, which is fine because taking colimits preserves surjectivity.

2. We now induct on dimension, assuming that X is Noetherian. If $\dim X=0$, then being irreducible forces X to be a single point, so taking global sections induces an equivalence of categories $\mathrm{Ab}_X\to\mathrm{Ab}$. This is exact, so all higher cohomology vanishes (namely, strictly after degree 0), as needed.

   Now for our induction, take $\dim X>0$, and fix $\mathcal{F}\in\mathrm{Ab}_X$. Set
$$B:=\bigcup_{U\subseteq X}\mathcal{F}(U),$$
and we let $\mathcal{A}$ be the set of finite subsets of B; note $\mathcal{A}$ is directed by inclusion. Now, for each $\alpha\in\lambda$, we let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the finitely many sections in α. More formally, for each $a\in B$, we see a belongs to some $U_a\subseteq X$ and so induces a map
$$\underline{\mathbb{Z}}_{U_a}\to\mathcal{F}$$
by sending 1 to a; the image here is the subsheaf generated by a. We can generalize this construction to arbitrary $\alpha\in\mathcal{A}$. Now, it suffices to show that each $\mathcal{F}_\alpha$ has vanishing cohomology by taking colimits over $\mathcal{A}$ to get up to $\mathcal{F}$.

   Notably, for any α,α' with $\alpha'\subseteq\alpha$, we have an exact sequence
$$0\to\mathcal{F}_{\alpha'}\to\mathcal{F}_\alpha\to\overline{\mathcal{F}}_{\alpha\setminus\alpha'}\to 0,$$
where $\overline{\mathcal{F}}_{\alpha\setminus\alpha'}$ is some quotient of $\mathcal{F}_{\alpha\setminus\alpha'}$. Thus, by inducting on $|\alpha|$ (note that the quotient of $\mathcal{F}_{\alpha\setminus\alpha'}$ still has fewer generators), we may assume that $|\alpha|=1$.

   Thus, we see that we may assume $\mathcal{F}$ is generated by a single section $a\in\mathcal{F}(U)$, which means that we have an exact sequence
$$0\to\mathcal{K}\to\mathbb{Z}_U\to\mathcal{F}\to 0.$$

Notably, for each $x \in U$, we have an embedding $\mathcal{K}_x \subseteq \mathbb{Z}$ by taking stalks. Now, if $\mathcal{K}_x$ vanishes for all $x \in U$, then $\mathcal{K}$ itself will have to vanish because $\mathcal{K}$ is a sheaf. Otherwise, we can find the least positive integer d such that $\mathcal{R}_x$ embeds into $d\mathbb{Z} \subseteq \mathbb{Z} = (\mathbb{Z}_U)_x$ for all x.

We claim that this d is actually achieved. Well, we can find some small V where $d \in \underline{\mathbb{Z}}$ will extend to a section on $\mathcal{R}$, so we claim that $\mathcal{R}|_V = d\underline{\mathbb{Z}}$. However, all the other stalks over U are either vanishing or a subgroup of $d\underline{\mathbb{Z}}$, so we are forced to have equality by the minimality of d.

In total, our exact sequence now reads

$$0 \to \mathcal{R}_v \to \mathcal{R} \to \mathcal{R}_{U\setminus V} \to 0,$$

but irreducibility tells us that $\mathcal{R}_{U\setminus V}$ will have vanishing cohomology because it has smaller dimension. Thus, we are reduced to showing that $\underline{\mathbb{Z}}_U$ are going to have vanishing cohomology. But then we have the exact sequence

$$0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_{X\setminus U} \to 0.$$

The inductive hypothesis deals with $\underline{\mathbb{Z}}_{X\setminus U}$ as having smaller dimension, and $\underline{\mathbb{Z}}$ is flasque because it is a constant sheaf on an irreducible space! (Namely, all locally constant functions $V \to \mathbb{Z}$ must be constant because V is irreducible.) Thus, $\underline{\mathbb{Z}}$ is going to have vanishing cohomology, so $\underline{\mathbb{Z}}_U$ has vanishing cohomology in large enough dimension, so we are done. ■

2.6 February 27

We continue.

2.6.1 Cohomology of Quasicoherent Sheaves

We are going to show that quasicoherent sheaves over an affine scheme have vanishing cohomology. This is analogous to the result that Stein spaces have vanishing cohomology.

Theorem 2.60. Fix a Noetherian ring A and affine scheme $X = \operatorname{Spec} A$. If $\mathcal{F}$ is a quasicoherent sheaf, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

The main input to this result is the following result.

Proposition 2.61. Fix a Noetherian ring A and an injective A-module I. Then the quasicoherent sheaf $\widetilde{I}$ on $\operatorname{Spec} A$ is flasque.

Let's see how Proposition 2.61 implies Theorem 2.60.

Proof of Theorem 2.60. We may write $\mathcal{F} := \widetilde{M}$ for some A-module M, and we give M an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots .$$

Now, the functor $\widetilde{\cdot}$ is exact (this can be checked on stalks because localization is flat), so we get the resolution

$$0 \to \widetilde{M} \to \widetilde{I^0} \to \widetilde{I^1} \to \widetilde{I^2} \to \cdots ,$$

which is a flasque resolution by Proposition 2.61. Now, computing cohomology with this flasque resolution, we recover the original exact injective resolution for A. In particular, we see that we remain exact and therefore have vanishing higher cohomology. ■

Thus, we turn to the proof of Proposition 2.61. We require the following algebraic ingredient.

Theorem 2.62 (Krull)**.** Fix a Noetherian ring A. Given an ideal $\mathfrak{a} \subseteq A$ and finite A-modules $M \subseteq N$, then the $\mathfrak{a}$-adic topology on M is induced by that on N. In other words, for any $n > 0$, there exists $n' > 0$ such that $\mathfrak{a}^n M \supseteq (\mathfrak{a}^{n'} N) \cap M$.

Note that $(\mathfrak{a}^{n'} N) \cap M \supseteq \mathfrak{a}^{n'} M$ automatically, so our system of fundamental neighborhoods yielding topology on A to in fact produce the same topology on M.

To show Theorem 2.62, we will need a lemma.

Definition 2.63 (sheaf with support)**.** Given a topological space X and a closed subset $Z \subseteq X$, we define for any sheaf $\mathcal{F}$ of abelian groups the set

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) : s|_p = 0 \text{ for } p \notin Z\}.$$

This is the *sheaf of sections with support contained in* Z.

Definition 2.64. Fix a ring A and an ideal $\mathfrak{a}$. Then we define

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M : a^n m = 0 \text{ for some } n > 0 \text{ for all } a \in \mathfrak{a}\}.$$

Notably, $U \mapsto \Gamma_{Z \cap U}(U, \mathcal{F})$ defines a sheaf on $\mathcal{F}$, which is left exact and produces the cohomology groups $H^i_Z(X, \mathcal{F})$.

In our affine case, we have the following.

Lemma 2.65. Fix a Noetherian ring A. Then for $Z = V(\mathfrak{a})$,

$$\Gamma_Z(X, \widetilde{M}) := \{m \in M : a^n m = 0 \text{ for some } n > 0 \text{ for all } a \in \mathfrak{a}\}.$$

Proof. Quickly, let the right-hand set be $\Gamma_{\mathfrak{a}}(M)$.

In one direction, for $m \in \Gamma_{\mathfrak{a}}(M)$, we see $\mathfrak{a}^n m = 0$ for some $n > 0$ because A is Noetherian (so that $\mathfrak{a}$ is finitely generated, allowing us to find n large enough so that all generators to some sufficiently higher power n lie in $\operatorname{Ann} m$). Thus, if $\mathfrak{p} \notin Z$, then $\mathfrak{a} \not\subseteq \mathfrak{p}$, so we see there is $f \in \mathfrak{a} \setminus \mathfrak{p}$ which is invertible in $M_{\mathfrak{p}}$. In particular, $f^n m = 0$, so $m|_{\mathfrak{p}} = 0$, so we see that $\mathfrak{p}$ is not in the support of m. Thus, the support of m is contained in Z.

In the other direction, suppose $m \in M$ has support in Z. Now, for $\mathfrak{p} \notin Z$, we have $m|_{\mathfrak{p}} = 0$ by hypothesis, so there is $f \in A \setminus \mathfrak{p}$ such that $f^{n_f} m = 0$ for some $n_f > 0$. In particular, it follows that $\operatorname{Ann} m \not\subseteq \mathfrak{p}$ as witnessed by f. In other words, we see that $\operatorname{Ann} m \subseteq \mathfrak{p}$ implies that $\mathfrak{p} \in Z$ and so $\mathfrak{a} \subseteq \mathfrak{p}$, so we conclude that

$$\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \supseteq \operatorname{Ann} m} \mathfrak{p} = \operatorname{rad} \operatorname{Ann} m.$$

Because $\mathfrak{a}$ is finitely generated (as A is Noetherian), we conclude $\mathfrak{a}^n \subseteq \operatorname{Ann} m$ for some $n > 0$ by looking at each generator. It follows that $m \in \Gamma_{\mathfrak{a}}(M)$. ■

Remark 2.66. Thus, we would like to show that $H^0_Z(X,\mathcal{F})$ is quasicoherent when $\mathcal{F}$ is quasicoherent; notably, from the above, we will see that $H^0_Z(X,\mathcal{F}) = \widetilde{\Gamma_{\mathfrak{a}}(M)}$ by taking global sections.

Well, we note that we have an exact sequence

$$0 \to H^0_Z(X,\mathcal{F}) \to \mathcal{F} \to j_*\mathcal{F}|_{X\setminus Z},$$

where $j\colon (X\setminus Z) \subseteq X$ is the inclusion. Indeed, exactness here can be checked at stalks. But now this is a scheme-theoretic result: the kernel of a morphism of quasicoherent sheaves is in fact quasicoherent, so it will be enough to see that $j_*\mathcal{F}|_{X\setminus Z}$ is quasicoherent. But in our setting above, the direct image sheaf of a quasicoherent sheaf is still quasicoherent when the relevant scheme morphism is separated and quasicompact (when the source is Noetherian), but of course all these hypotheses are satisfied.

We now investigate this functor $\Gamma_{\mathfrak{a}}$ further.

Lemma 2.67. Fix a Noetherian ring A and an ideal $\mathfrak{a} \subseteq M$. Then the functor $\Gamma_{\mathfrak{a}}$ preserves injectives.

Proof. Fix an injective I, and set $J := \Gamma_{\mathfrak{a}}(I)$. By the homework, it is enough to show that any ideal $\mathfrak{b} \subseteq A$ with a morphism $\varphi\colon \mathfrak{b} \to J$ can be extended to a morphism $A \to J$.

Now, for each $b \in \mathfrak{b}$, we note that $\mathfrak{a}^n\varphi(b) = 0$ for some $n > 0$. In particular, $\mathfrak{a}^n b \subseteq \ker\varphi$. By choosing generators for $\mathfrak{b}$, we may actually assume that $\mathfrak{a}^n\mathfrak{b} \subseteq \ker\varphi$. However, by Theorem 2.62, we may thus find n' such that $\mathfrak{a}^{n'} \cap \mathfrak{b} \subseteq \mathfrak{a}^n\mathfrak{b}$. In total, we have the following diagram.

$$\begin{tikzcd}
A \arrow[r, two heads] & \frac{A}{\mathfrak{a}^{n'}} \arrow[drr, dashed] & & \\
\mathfrak{b} \arrow[u, hook] \arrow[r, two heads] & \frac{\mathfrak{b}}{\mathfrak{a}^{n'} \cap \mathfrak{b}} \arrow[u, hook] \arrow[r] & J \arrow[r, hook] & I
\end{tikzcd}$$

Well, we would like to show that the induced map $A/\mathfrak{a}^{n'} \to I$ factors through J, but this can just be checked directly from the construction. Indeed, the element $x \in A/\mathfrak{a}^{n'}$ in I must be killed by $\mathfrak{a}^{n'}$, so the image of 1 does in fact live in $\Gamma_{\mathfrak{a}}(I) = J$. ■

We now move towards showing Proposition 2.61.

Lemma 2.68. Fix a Noetherian ring A. Given an injective A-module I, any $f \in A$ has the localization map $I \to I_f$ surjective.

Proof. For each $n > 0$, we set $\mathfrak{b}_n := \operatorname{Ann} f^n$. Notably, this grants the ascending chain of ideals

$$\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \mathfrak{b}_3 \subseteq \cdots,$$

so this must stabilize at some $\mathfrak{b}_r$. Setting $\theta\colon I \to I_f$ to be the localization map, pick any $y/f^n \in I_f$ for $y \in I$, which we would like to be in the image of I.

Now, note there is a map $(f^{n+r}) \to I$ by $f^{n+r} \mapsto f^r y$. This is well-defined by construction of r. In particular, if $af^{n+r} = a'f^{n'+r}$, we can check that $af^r y = a'f^r y$ by computing some annihilators. Thus, the injectivity of I lifts this map to some

$$\psi\colon A \to I$$

such that $\psi(f^{n+r}) = f^r y$. Notably, setting $z := \psi(1)$, we see $f^{n+r}z = \psi(f^{n+r}) = f^r y$. Rearranging, we see that z gets sent to y/f^n in I_f, which is what we wanted. ■

We will continue the proof next class.

2.7 March 1

We continue talking about quasicoherent sheaves.

2.7.1 Finishing Cohomology of Quasicoherent Sheaves

Here is our result.

Proposition 2.61. Fix a Noetherian ring A and an injective A-module I. Then the quasicoherent sheaf $\widetilde{I}$ on $\operatorname{Spec} A$ is flasque.

Proof. We use Noetherian induction on the closed set $Y(I) := \overline{\operatorname{Supp} \widetilde{I}}$. As our base case, if $Y(I)$ is a singleton $\{x\}$, then the closed embedding $i\colon \{x\} \to X$ tells us that

$$\widetilde{I} = i_* i^* \widetilde{I}$$

by checking the obvious map at stalks. And now one can see that any skyscraper sheaf will be flasque by simply looking at it.

In general, we would like to show that $\Gamma(X, \mathcal{I}) \to \Gamma(U, \mathcal{I})$ is surjective by simply always lifting to a global section before restricting. Now, if $Y \cap U = \varnothing$, then $\Gamma(U, \mathcal{I}) = 0$, so there is nothing to say; as such, we may assume that $Y \cap U \neq \varnothing$. Using the standard base on the affine scheme, we may find $f \in A$ such that $D(f) \subseteq U$ and $D(f) \cap Y \neq \varnothing$.

Continuing, we set $Z := V(f)$ so that we have the following commutative diagram.

$$\begin{tikzcd}
\Gamma_Z(X, \widetilde{I}) \arrow[r] \arrow[d, hook] & \Gamma_{Z \cap U}(U, \widetilde{I}) \arrow[d, hook] & \\
\Gamma(X, \widetilde{I}) \arrow[r] & \Gamma(U, \widetilde{I}) \arrow[r] & \Gamma(D(f)), \widetilde{I})
\end{tikzcd}$$

Now, the bottom composite is surjective by Lemma 2.68.

We now proceed directly. Given $t \in \Gamma(U, \widetilde{I})$, we see that we can lift $t|_{D(f)}$ to some $s \in \Gamma(X, \mathcal{I})$ by this surjectivity. But then $s|_U - t$ will have support in $X \setminus D(f) = Z$. As such, if we could just show that the map

$$\Gamma_Z(X, \widetilde{I}) \to \Gamma_{Z \cap U}(U, \widetilde{I})$$

is surjective, then we could lift $s|_U - t$ to some $r \in \Gamma_Z(X, \widetilde{I})$ so that $(s-r)|_{D(f)} = s|_{D(f)} = t$, but $(s-r)|_U - t = (s|_U - t) - r|_U = 0$. Thus, we want to show that the above map is surjective, which amounts to showing that the sheaf $\Gamma_Z(-, \mathcal{I}) = \widetilde{\Gamma_{(f)}(I)}$ is flasque, where this equality is by Remark 2.66.

However, the support of $\widetilde{\Gamma_{(f)}(I)}$ is contained in $Z \cap Y$, where it is contained in Y by construction of I and contained in Z by construction of (f). But now we are done by Noetherian induction because $Z \cap Y$ is a smaller closed subset. Explicitly, we may suppose the result for all strictly smaller closed subsets than Y, for otherwise we would be able to build some infinite descending chain of closed sets in $\operatorname{Spec} A$, which is not possible because A is Noetherian. ■

Corollary 2.69. Fix a Noetherian scheme X. Then any quasicoherent sheaf $\mathcal{F}$ can be embedded into a flasque quasicoherent sheaf.

Proof. Over affine opens, we can embed into an injective module, which is flasque by the above. Taking the direct image sheaf to X and then taking the product produces the desired flasque sheaf and embedding. ■

Remark 2.70. Professor Barrett is unsure if injectives in the category of quasicoherent sheaf are actually acyclic for $\Gamma(X, -)$.

2.7.2 Serre's Criterion for Affines

Let's give a cohomological criterion for being affine.

Theorem 2.71 (Serre)**.** Fix a Noetherian scheme X. Then the following are equivalent.

(a) X is affine.

(b) $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and quasicoherent sheaves $\mathcal{F}$ on X.

(c) $H^1(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ of ideals.

For this, we want the following lemma.

Lemma 2.72. A scheme X is affine if and only if there exist global sections $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ which generate $\Gamma(X, \mathcal{O}_X)$ such that all the subschemes $D(f_i)$ are affine.

Proof. This is by Exercise II.2.17 of [Har77]. We did this for homework last semester. ■

Now let's show the theorem.

Proof of Theorem 2.71. We show our implications in sequence. Note (a) implies (b) by Theorem 2.60, and (b) implies (c) with no work, so the hard part is showing (c) implies (a). We would like to use Lemma 2.72. Well, for each $p \in X$, we let $p \in U$ be some affine open neighborhood and set $Y := X \setminus U$. Then we get some exact sequence

$$0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(p) \to 0,$$

where we can check exactness at stalks. Now, by the long exact sequence we see

$$\Gamma(X, \mathcal{I}_Y) \to k(p) \to H^1(X, \mathcal{I}_{Y \cup \{p\}}),$$

where the last cohomology group vanishes by hypothesis. In total, our surjectivity produces $f \in \Gamma(X, \mathcal{I}_Y) \subseteq \Gamma(X, \mathcal{O}_X)$ such that $f|_p = 1$ in $k(p)$. Further, f has support contained in Y, so $X_f \subseteq U$, so $X_f = X_f \cap U = U_f$ is an affine open neighborhood of U.

Thus, we have produced an affine open cover $\{X_f\}$ of X. Because X is Noetherian, we see that X is quasicompact, so we can reduce this to a finite affine open cover $\{X_{f_i}\}_{i=1}^r$. By Lemma 2.72, it suffices to show that the f_i generate $\Gamma(X, \mathcal{O}_X)$. Well, each f_i produces some map $\mathcal{O}_X \to \mathcal{O}_X$, so we get a map

$$\pi \colon \mathcal{O}_X^r \to \mathcal{O}_X$$

given by $(a_1, \ldots, a_r) \mapsto a_1 f_1 + \cdots + a_r f_r$. Notably, because the X_{f_i} cover X, we see that this map is surjective by checking at stalks because we do not have $f_i|_p \in \mathfrak{m}_p$ for all i at any fixed point p. Letting $\mathcal{F} := \ker \pi$, we get the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^r \xrightarrow{\pi} \mathcal{O}_X \to 0,$$

so it now suffices to show that $H^1(X, \ker \pi)$ because this will give us surjectivity on the right upon taking global sections. Well, we consider the filtration

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}_X^r \supseteq \mathcal{F} \cap \mathcal{O}_X^{r-1} \supseteq \cdots \supseteq \mathcal{F} \cap \mathcal{O}_X \supseteq 0.$$

Note that coherent sheaves are closed under taking kernels and finite products, so each of these intersections remains a coherent sheaf. Well, we do this by induction. Note $\mathcal{F} \cap \mathcal{O}_X$ is a coherent sheaf of ideals, so its H^1 vanishes by hypothesis; this is our base case. To go up, we look at the following short exact sequence.

$$0 \to \mathcal{F} \cap \mathcal{O}_X^n \to \mathcal{F} \cap \mathcal{O}_X^{n+1} \to \mathcal{C}_n \to 0.$$

Thus, it now suffices to show that $\mathcal{C}_n$ has vanishing H^1 by the long exact sequence. Well, we note that we have a morphism of exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \cap \mathcal{O}_X^n \arrow[r] \arrow[d, hook] & \mathcal{F} \cap \mathcal{O}_X^{n+1} \arrow[r] \arrow[d, hook] & \mathcal{C}_n \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{O}_X^n \arrow[r] & \mathcal{O}_X^{n+1} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

In particular, we see that $\mathcal{C}_n$ is coherent by the top, and it embeds into $\mathcal{O}_X$ by the Snake lemma, so it is a coherent sheaf of ideals and therefore has vanishing H^1 by hypothesis. ■

2.8 March 3

Here we go.

2.8.1 Introducing Čech Cohomology

One point of Čech cohomology is for computations, but we will see a little more. Fix a topological space X and open cover $\mathcal{U} := \{U_\alpha\}_{\alpha\in\lambda}$, and we fix a total order $\leq$ on λ. Now, given a sheaf of abelian groups $\mathcal{F}$ on X, we set

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0,\ldots,\alpha_p}),$$

where

$$U_{\alpha_0,\ldots,\alpha_p} := \bigcap_{i=0}^p U_{\alpha_i}.$$

In other words, an element of $C^p(\mathfrak{U},\mathcal{F})$ is an element $s_{\alpha_0,\ldots,\alpha_p} \in \mathcal{F}(U_{\alpha_0,\ldots,\alpha_p})$ for each $\alpha_0 < \cdots < \alpha_p$. For our differential, we set

$$(d^p s)_{\alpha_0<\cdots<\alpha_{p+1}} := \sum_{q=0}^{p+1} (-1)^k s_{\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_{q+1}},$$

where the hat denotes omission. Here, we have implicitly restricted all sections to $U_{\alpha_0,\ldots,\alpha_{p+1}}$.

Example 2.73. Give X an open covering by $\{U_0, U_1, U_2\}$. Then $C^1(\mathfrak{U},\mathcal{F})$ is a choice of sections $s_{01} \in \mathcal{F}(U_0 \cap U_1)$ and $s_{02} \in \mathcal{F}(U_0 \cap U_2)$ and $s_{12} \in \mathcal{F}(U_1 \cap U_2)$. Here, we have $\left(d^1 s\right)_{012} = s_{12} - s_{02} + s_{01}$. Note that all these sections make sense over U_{012}.

Remark 2.74. Notably, if $p > \#\lambda$, then our chain complex term vanishes.

One can check that $d^2 = 0$ by hand. Roughly speaking, this amounts to the fact that there are two ways to remove two fixed indices i and j from the tuple $(\alpha_0, \ldots, \alpha_{p+2})$, which produce different signs. This grants the following definition.

Definition 2.75 (Čech cohomology)**.** Fix everything as above. The *Čech cohomology* $\check{H}(\mathfrak{U},\mathcal{F})$ groups are the cohomology groups of the complex $C(\mathfrak{U},\mathcal{F})$.

Remark 2.76. These cohomology functors do not give a δ-functor. For example, if our open cover is given by $\mathfrak{U} = \{X\}$, then the complex is given by

$$\Gamma(X,\mathcal{F}) \to 0 \to 0 \to \cdots,$$

whose cohomology does not give a long exact sequence.

Exercise 2.77. We compute the cohomology of $X = \mathbb{P}^1_k$ where the sheaf is $\mathcal{F} = \Omega^1_X$.

Proof. Let our open cover be given by $\mathfrak{U} = \{U_0, U_\infty\}$ be the standard affine charts around 0 and ∞.

- Note $C^0(\mathfrak{U}, \mathcal{F})$ consists of a choice of a section in $\mathcal{F}(U_0) = k[x]\,dx$ and a section in $\mathcal{F}(U_\infty) = k[y]\,dy$.
- Note $C^1(\mathfrak{U}, \mathcal{F})$ consists of a choice of a section in $\mathcal{F}(U_0 \cap U_1) = k[x, 1/x]\,dx$.

In total, we want to compute the cohomology of the sequence

$$0 \to k[x]\,dx \oplus k[y]\,dy \to k[x, 1/x]\,dx \to 0,$$

where the middle map is given by $(f(x)\,dx, g(y)\,dy) \mapsto (-g(1/x)/x^2\,dx - f(x)\,dx)$.

- Notably, $-g(1/x)/x^2$ is a polynomial of negative degree, so there is no way for cancellation to occur, so $\check{H}^0(\mathfrak{U}, \mathcal{F}) = 0$.
- By letting f and g vary, we note that everything in $k[x, 1/x]\,dx$ is hit except for the terms of the form $c/x\,dx$ for $c \in k^\times$, so $\check{H}^1(\mathfrak{U}, \mathcal{F}) = k$. ■

Exercise 2.78. We compute the cohomology of S^1 where the sheaf is $\mathcal{F} = \underline{\mathbb{Z}}$.

Proof. Use the open cover $\mathfrak{U} = \{U_0, U_1\}$ of two large arcs of S^1 fully covering S^1.

- We see that $C^0(\mathfrak{U}, \underline{\mathbb{Z}})$ is a choice of a section on U_0 and a section on U_1, both of which are connected, so we are looking at $\mathbb{Z} \oplus \mathbb{Z}$.
- Similarly, $C^0(\mathfrak{U}, \underline{\mathbb{Z}})$ is a choice of a section of $U_0 \cap U_1$, which consists of two disconnected arcs, so this is also $\mathbb{Z} \oplus \mathbb{Z}$.

In total, we want the cohomology of the complex given by

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0,$$

where the middle map is given by $(a, b) \mapsto (b - a, b - a)$. As such, we can see $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \mathbb{Z}$ and $\check{H}^1(\mathfrak{U}, \mathcal{F}) = \mathbb{Z}$ by direct computation. ■

2.8.2 Abstract Remarks

To understand what Čech cohomology is doing, we want the following result.

Theorem 2.79. Fix a topological space X and open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \lambda}$ where λ is total-ordered. Further, suppose $\mathcal{F}$ is a sheaf on X which is acyclic on the restrictions U_α for $\Gamma(U_\alpha, -)$. Then

$$\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) = H^\bullet(X, \mathcal{F}).$$

Example 2.80. If X is a Noetherian scheme and $\mathcal{F}$ is a quasicoherent sheaf, then we can let $\mathfrak{U}$ be any affine open cover by Theorem 2.60.

Let's give some more general remarks. Roughly speaking, there is a spectral sequence

$$E_2^{p,q} := \check{H}^p(\mathfrak{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

where $\underline{H}^p(\mathcal{F})(V) = H^q(V, \mathcal{F})$ is a presheaf. Now, Čech cohomology is a tool for computing some cosimplicial limits: set Δ to be the "simplex" category whose objects are $[n] := \{0, \ldots, n\}$, and we consider the order preserving maps in our category of simplexes. For example, there are two order-preserving maps $[0] \to [1]$, and more generally there are some canonical maps $[n] \to [n+1]$ depending on where we choose to pause.

Remark 2.81. We have seen this in some context already. Namely, we are more or less considering ways to go back and forth along $U^{n+1} \to U^n$.

There is some "Dold–Kan" correspondence, which gives an equivalence of between cosimplicial objects $\operatorname{CoSimp}(\mathcal{A})$ of an abelian category $\mathcal{A}$ and the cochains of nonnegative degree in $\operatorname{CoCh}^{\geq 0}(\mathcal{A})$.

Let's be more concrete about this discussion to see where our cohomology comes from. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of X. Then we get a cosimplicial object in the category of abelian groups given as follows.

$$\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U}\times\mathcal{U}) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathcal{F}(\mathcal{U}\times\mathcal{U}\times\mathcal{U}) \longrightarrow \cdots$$

Here, $\mathcal{U}$ is the disjoint union of the open sets in our open cover $\{U_\alpha\}_{\alpha\in\lambda}$. In particular, under the Dold–Kan correspondence, the above "cosimplicial abelian group" goes to the Čech complex, and the limit of these is precisely $\mathcal{F}(X)$ because $\mathcal{F}$ is a sheaf. However, $\mathcal{F}$ is a sheaf, so we can actually identify $\mathcal{F}(X)$ with the equalizer of the maps

$$\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U}\times\mathcal{U}).$$

One can now upgrade the above ideas to give the desired equality in Theorem 2.79, for example replacing our limit with a homotopy limit.

2.9 March 3

Today we prove Theorem 2.79. As an aside, our goal is to compute the cohomology of projective space.

2.9.1 Čech Cohomology is Correct

Throughout, X will be a topological space, $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\lambda}$ will be an open cover where λ is total-ordered, and $\mathcal{F}$ is a sheaf on X. Recall our main result, which we will not prove in full generality but instead just in our special case.

Theorem 2.79. Fix a topological space X and open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\lambda}$ where λ is total-ordered. Further, suppose $\mathcal{F}$ is a sheaf on X which is acyclic on the restrictions U_α for $\Gamma(U_\alpha, -)$. Then

$$\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) = H^\bullet(X, \mathcal{F}).$$

We begin with the following lemma.

Lemma 2.82. Fix a topological space X and open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\lambda}$ where λ is total-ordered. Then $\check{H}^0(\mathfrak{U}, \mathcal{F}) = H^0(X, \mathcal{F})$.

Proof. Let $C^\bullet(\mathfrak{U}, \mathcal{F})$ denote the Čech complex. Then $\check{H}^0(\mathfrak{U}, \mathcal{F})$ is the kernel of the map $d^0\colon C^0(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{F})$. Well, $s \in C^0(\mathfrak{U}, \mathcal{F})$ lives in the kernel of this map if and only if each $\alpha < \beta$ in λ has

$$0 = (ds)_{\alpha\beta} = s_\alpha|_{U_\alpha\cap U_\beta} - s_\beta|_{U_\alpha\cap U_\beta},$$

so we are looking at exactly the coherence condition for our $\{s_\alpha\}_{\alpha\in\lambda}$ to assemble into a global section in $\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$. ■

Next we want to show that our Čech complex is a resolution we can use to compute cohomology, but we have to sheafify our complex first. Fix X, $\mathcal{F}$, and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\lambda}$ as usual. We then define

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{\alpha_0<\cdots<\alpha_p} j_*(\mathcal{F}|_{U_{\alpha_0,\ldots,\alpha_p}}),$$

where throughout j will denote the open embedding of the obvious open set. (Yes, the notation is ambiguous, but we already have too many indices floating around.) Note that this is a sheaf because restrictions, direct images, and products of sheaves produce sheaves. Notably, for any open $V \subseteq U$, we have

$$\Gamma(V, \mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = C^p(\{U_\alpha \cap V\}_\alpha, \mathcal{F}),$$

by construction, so we can define our differential as the one coming from $C^p(\{U_\alpha \cap V\}_\alpha, \mathcal{F})$. This makes us a complex by checking locally on open sets.

Now here is our resolution result.

Lemma 2.83. Fix everything as above. Then

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$

is an exact resolution of $\mathcal{F}$ with identity map chain homotopic to 0 on stalks.

The game plan is to show that the identity map on the complex is chain homotopic to 0, implying that we have an acyclic complex, which lets us compute cohomology from this. Anyway, let's prove the lemma.

Proof. This exact in degree 0 by the previous lemma.

Now, we may check exactness for positive degrees on stalks; it is enough to show that we have a chain homotopy between the identity and 0 because we already have a complex, so this establishes that every element of our kernel will arise from image. Thus, for our $x \in X$, we find β such that $x \in U_\beta$. Then for each index p, we define $h\colon \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x \to \mathcal{C}^{p-1}(\mathfrak{U}, \mathcal{F})_x$ as follows: with $s_x \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$, we may say $s_x = s|_x$ for some s defined in an open neighborhood V of x contained in U_β. As such, we define

$$h(s_x)_{\alpha_0,\ldots,\alpha_{p-1}} \coloneqq s_{\beta,\alpha_0,\ldots,\alpha_{p-1}},$$

where we are defining $s_{\alpha_0,\ldots,\alpha_p}$ to vanish if any of the indices fail to be distinct and to equal $(\operatorname{sgn}\sigma)s_{\sigma\alpha_0,\ldots,\sigma\alpha_p}$ if the indices are distinct, where $\sigma \in S_{p+1}$ is the permutation placing $(\alpha_0, \ldots, \alpha_p)$ so that $\sigma\alpha_0 < \cdots < \sigma\alpha_{p+1}$. Note that the above definition is compatible with restriction of s, so h is in fact defined on stalks.

It remains to show that $dh + hd = \mathrm{id}$, where d is our differential. As such, given $s_x \in \mathcal{C}^p(\mathfrak{U}, \mathcal{F})_x$, we want to show that

$$(dh + hd)(s)_{\alpha_0,\ldots,\alpha_p} \stackrel{?}{=} 0.$$

There are two cases.

- Suppose β is one of the indices. On one hand, $hd(s)_{\alpha_0,\ldots,\alpha_p}$ will vanish automatically because we are adding in β to our index list, so we have a repeat index, so this vanishes. On the other hand, we want to see that $dh(s)_{\alpha_0,\ldots,\alpha_p}$ vanishes. Well, we compute

$$dh(s)_{\alpha_0,\ldots,\alpha_p} = \sum_{k=0}^{p}(-1)^k h(s)_{\alpha_0,\ldots,\widehat{\alpha_k},\ldots,\alpha_p} = \sum_{k=0}^{p}(-1)^k s_{\beta,\alpha_0,\ldots,\widehat{\alpha_k},\ldots,\alpha_p}.$$

  However, all these terms vanish unless with removed α_k index was the one equal to p (of which there is exactly one), so transposing appropriately, we see that this last sum compresses down to just $s_{\alpha_0,\ldots,\alpha_p}$. Thus, $dh + hd = \mathrm{id}$ here.

- Suppose β is none of the indices. Then we must do some computation. On one hand, we see

$$hd(s)_{\alpha_0,\ldots,\alpha_p} = d(s)_{\beta,\alpha_0,\ldots,\alpha_p} = \sum_{k=0}^{p+1}(-1)^k s_{\beta,\alpha_0,\ldots,\widehat{\alpha_\bullet},\ldots,\alpha_p},$$

  where the removed term in the last sum depends on where β is placed among the remaining $\alpha_\bullet$s. On the other hand, we see

$$dh(s)_{\alpha_0,\ldots,\alpha_p} = \sum_{k=0}^{p}(-1)^k h(s)_{\alpha_0,\ldots,\widehat{\alpha_k},\ldots,\alpha_p} = \sum_{k=0}^{p-1} s_{\beta,\alpha_0,\ldots,\widehat{\alpha_k},\ldots,\alpha_p}.$$

Summing the above two expressions, the signs perfectly cancel except there is an extra term of the previous sum, which namely is where β is removed, totaling to $s_{\alpha_0,\ldots,\alpha_p}$, as needed.

Thus, we have $dh + hd$ here, completing the proof of exactness at our stalk. Explicitly, for any element $s \in \ker d$, we see $s = dh(s) + hd(s) = dh(s)$, so $s \in \operatorname{im} d$. ■

Our next step is to build some commonly acyclic sheaves.

Lemma 2.84. Fix everything as above. If $\mathcal{F}$ is flasque, then $\mathcal{F}$ is acyclic for $\check{H}^\bullet(\mathfrak{U}, -)$.

Proof. Note that restriction sends flasque sheaves to flasque sheaves, as do j_* and products, so we see that each $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ is flasque by definition. As such,

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$

is a flasque resolution of $\mathcal{F}$. So taking global sections and computing cohomology with our usual sheaf cohomology, we see

$$0 = H^i(X, \mathcal{F}) = H^p\Gamma(X, \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})) = H^p C^\bullet(\mathfrak{U}, \mathcal{F}) = \check{H}^p(X, \mathcal{F}),$$

which is what we wanted. ■

To continue, we recall the following fact.

Remark 2.85. Given objects A and B in an abelian category equipped with injective resolutions $A \to I^\bullet$ and $B \to J^\bullet$, then a morphism $\varphi\colon A \to B$ extends to a map on the injective resolutions unique up to homotopy. However, this proof really only requires $J^\bullet$ to be injective!

From Remark 2.85, we note that the identity on $\mathcal{F}$ induces maps on our resolutions as follows, where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{I}^0 \arrow[r] \arrow[d, dashed] & \mathcal{I}^1 \arrow[r] \arrow[d, dashed] & \mathcal{I}^2 \arrow[r] \arrow[d, dashed] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \arrow[r] & \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \arrow[r] & \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

As such, we induce a unique map on cohomology $\check{H}^\bullet(\mathfrak{U}, \mathcal{F}) \to H^\bullet(X, \mathcal{F})$.

Theorem 2.86. Fix everything as above, and suppose we have a separated Noetherian scheme X and a quasicoherent sheaf $\mathcal{F}$ on X. Then the above maps are isomorphisms.

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List of Definitions