

# 202B: Functional Analysis

Nir Elber

Spring 2024

# CONTENTS

---

*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

<b>Contents</b>	<b>2</b>
<b>1 Product Measures</b>	<b>4</b>
1.1 January 17	4
1.1.1 Course Notes	4
1.1.2 Measures	4
1.1.3 The Extension Theorem	5
1.1.4 Towards Product Measures	6
1.2 January 19	7
1.2.1 The Product Algebra	7
1.2.2 The Product Measure	8
1.2.3 Tonelli's Theorem	10
1.3 January 22	11
1.3.1 Proof of Tonelli's Theorem	11
1.4 January 24	14
1.4.1 Addenda to Tonelli's Theorem	14
1.4.2 Fubini's Theorem	15
1.5 January 26	16
1.5.1 Complete Measures	16
1.5.2 Measuring Euclidean Spaces	18
1.6 January 29	18
1.6.1 More on Measuring Euclidean Spaces	19
1.7 January 31	20
1.7.1 Measuring Affine Maps	21
1.8 February 2	22
1.8.1 Measuring Lipschitz Functions	22
1.9 February 5	24
1.9.1 Change of Variables	24
1.10 February 7	26
1.10.1 More on Change of Variables	26

<b>2 Banach Spaces</b>	<b>29</b>
2.1 February 9	29
2.1.1 Banach Spaces	29
2.1.2 A Little Linear Algebra	30
2.2 February 12	31
2.2.1 Linear Maps	31
2.2.2 Equivalence of Norms	32
2.2.3 Topology on Normed Vector Spaces	33
2.3 February 14	34
2.3.1 Closed Unit Balls	34
2.3.2 Functionals	35
2.4 February 16	36
2.4.1 The Hahn–Banach Theorem	36
2.5 February 21	38
2.5.1 Baire’s Theorem	38
2.5.2 Nowhere Dense Sets	39
2.5.3 The Open Mapping Theorem	40
2.6 February 23	40
2.6.1 Proof of the Open Mapping Theorem	41
2.6.2 The Closed Graph Theorem	42
2.7 February 28	43
2.7.1 The Uniform Boundedness Principle	43
2.7.2 A Little on Bases	45
<b>3 Weak Topologies</b>	<b>46</b>
3.1 March 1	46
3.1.1 Weak Topologies	46
3.2 March 4	47
3.2.1 An Example Dual Space	47
3.2.2 The Weak-* Topology	49
3.3 March 6	49
3.3.1 Reflexive Banach Spaces	49
3.3.2 Weak Compactness Results	50
3.4 March 8	51
3.4.1 Alaoglu’s Theorem	52
3.4.2 Hilbert Spaces	53
3.5 March 11	53
3.5.1 Bounding with Inner Products	53
3.5.2 Geometry in Hilbert Spaces	55
<b>Bibliography</b>	<b>57</b>
<b>List of Definitions</b>	<b>58</b>

# THEME 1

## PRODUCT MEASURES

---

### 1.1 January 17

Let's just get started.

#### 1.1.1 Course Notes

Here are some course notes.

- The professor for this course is Michael Christ.
- There is a bCourses, which I don't have access to.
- There will be an exam in the evening in February.
- Problem sets will be due on Fridays.
- We will assume analysis on the level of Math 202A; see something like [Elb22].
- The text for the course is [Fol99].

#### 1.1.2 Measures

Our first topic is to integrate on product spaces. Roughly speaking, we might have some measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  with some way to measure on them, and then we will want to measure  $X \times Y$ . Let's quickly recall what a measure is; we won't bother to recall the definition of a  $\sigma$ -algebra, but we will refer to [Elb22, Definition 5.25]. This requires the definition of a  $\sigma$ -algebra.

**Definition 1.1 ( $\sigma$ -algebra).** Fix a set  $X$ . Then a collection  $\mathcal{M} \subseteq \mathcal{P}(X)$  is a  $\sigma$ -algebra if and only if the following conditions are satisfied.

- $\emptyset \in \mathcal{M}$ .
- $\mathcal{M}$  is closed under countable unions.
- $\mathcal{M}$  is closed under complements.

In the sequel, we will also want to produce  $\sigma$ -algebras.

**Definition 1.2.** Fix a set  $X$ . Given a collection  $\mathcal{S} \subseteq \mathcal{P}(X)$ , we will say that the smallest  $\sigma$ -algebra generated by  $\mathcal{S}$  is the  $\sigma$ -algebra *generated by  $\mathcal{S}$* .

It is lemma that a smallest (i.e., contained in all other such  $\sigma$ -algebras) such  $\sigma$ -algebra exists and is unique. Let's see this.

**Lemma 1.3.** Fix a set  $X$  and collection  $\mathcal{S} \subseteq \mathcal{P}(X)$ . Then there is a  $\sigma$ -algebra  $\mathcal{M}$  containing  $\mathcal{S}$  such that  $\mathcal{M} \subseteq \mathcal{M}'$  for any  $\sigma$ -algebra  $\mathcal{M}'$  containing  $\mathcal{S}$ . This  $\mathcal{M}$  is also unique.

*Proof.* There is certainly some  $\sigma$ -algebra on  $X$  containing  $\mathcal{S}$ , namely  $\mathcal{P}(X)$ . So there is a nonempty collection  $\underline{\mathcal{M}}$  of all  $\sigma$ -algebras containing  $\mathcal{S}$ , and then we define

$$\mathcal{M} := \bigcap_{\mathcal{M}' \in \underline{\mathcal{M}}} \mathcal{M}'.$$

Certainly  $\mathcal{M}$  contains  $\mathcal{S}$ , and one can check directly that  $\mathcal{M}$  is a  $\sigma$ -algebra. (See [Elb22, Lemma 5.28] for details.) And by construction, we see that  $\mathcal{M} \subseteq \mathcal{M}'$  for any  $\sigma$ -algebra  $\mathcal{M}'$  containing  $\mathcal{S}$ . Lastly, we note that  $\mathcal{M}$  is unique because any two such  $\sigma$ -algebras  $\mathcal{M}_1$  and  $\mathcal{M}_2$  will be contained in each other and hence equal. ■

Anyway, here is our definition of a measure.

**Definition 1.4 (measure).** Fix a  $\sigma$ -algebra  $\mathcal{M}$  on a set  $X$ . Then a *measure*  $\mu$  is a countably additive non-negative function  $\mu: \mathcal{M} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ , and we require that  $\mu(\emptyset) = 0$ . Here, being countably additive means that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i),$$

where the sum is allowed to be in  $\infty$  (namely, diverge to infinity). We call the triple  $(X, \mathcal{M}, \mu)$  a *measure space*.

**Remark 1.5.** If we have  $\mu(\emptyset) > 0$ , then the countably additive condition implies that  $\mu(\emptyset) = \infty$  and then  $\mu(E) = \infty$  for all  $E \in \mathcal{M}$ . This is in fact countably additive, but we would like to exclude it.

We will want to make our measures somewhat small.

**Definition 1.6 ( $\sigma$ -finite).** Fix a measure space  $(X, \mathcal{M}, \mu)$ . Then  $\mu$  is  $\sigma$ -finite if and only if  $X$  is a countable union of sets in  $\mathcal{M}$  of finite measure.

This smallness condition is quite tame, and in practice all measures are  $\sigma$ -finite.

### 1.1.3 The Extension Theorem

We would like to discuss how to build measures from objects easier to construct. The following generalization of Definition 1.1 will be useful.

**Definition 1.7 (algebra).** Fix a set  $X$ . Then a collection  $\mathcal{A} \subseteq \mathcal{P}(X)$  is an *algebra* if and only if the following conditions are satisfied.

- $\emptyset \in \mathcal{A}$ .
- $\mathcal{A}$  is closed under finite unions.
- $\mathcal{A}$  is closed under complements.

**Example 1.8.** Fix an uncountable set  $X$ , and let  $\mathcal{A}$  denote the collection of finite and cofinite sets. Then  $\mathcal{A}$  is an algebra (the finite union of finite sets is finite, and the finite union of cofinite sets is cofinite), but it need not be a  $\sigma$ -algebra because the countable union of finite sets need not be finite nor cofinite.

**Example 1.9.** Fix  $X := \mathbb{R}$ , and let  $\mathcal{A}$  denote the collection of finite unions of open or closed intervals. Then  $\mathcal{A}$  is an algebra but not a  $\sigma$ -algebra.

Additionally, the following generalization of Definition 1.4 will be useful.

**Definition 1.10 (premeasure).** Fix an algebra  $\mathcal{A}$  on a set  $X$ . Then a *premeasure* is a function  $\rho: \mathcal{A} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$  which satisfies the following.

- $\rho(\emptyset) = 0$ .
- Finitely additive: we have  $\rho(A \sqcup B) = \rho(A) + \rho(B)$  for  $A, B \in \mathcal{A}$ .
- Countably additive: suppose  $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$  is pairwise disjoint, and  $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$ . Then

$$\rho\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \rho(A_i).$$

And now here is our theorem.

**Theorem 1.11 (Extension).** Fix a set  $X$  and a premeasure  $\rho$  on an algebra  $\mathcal{A}$  over  $X$ . Then there exists a measure  $\mu$  on the  $\sigma$ -algebra  $\mathcal{M}$  generated by  $\mathcal{A}$  such that  $\mu|_{\mathcal{A}} = \rho$ . Additionally, if  $\rho$  is  $\sigma$ -finite, then  $\mu$  is unique on  $\mathcal{M}$ .

Here,  $\sigma$ -finiteness for  $\rho$  takes the same definition as Definition 1.6.

*Proof of Theorem 1.11.* For existence, combine [Elb22, Lemma 6.16 and Theorems 6.21, 6.24]. Further, uniqueness is [Elb22, Theorem 6.35]. It will be helpful to say a few words about the construction. Essentially, one builds an “outer measure”  $\rho^*$  on  $\mathcal{P}(X)$  by

$$\rho^*(E) := \inf \left\{ \sum_{n=0}^{\infty} \rho(A_n) : \{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A} \text{ and } E \subseteq \bigcup_{n=0}^{\infty} A_n \right\}.$$

Then one restricts  $\rho^*$  to a smaller  $\sigma$ -algebra over which it becomes a bona fide measure. ■

### 1.1.4 Towards Product Measures

For our product measures, we take the following outline. Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ .

1. We will construct a special  $\sigma$ -algebra  $\mathcal{M} \otimes \mathcal{N}$  on  $X \times Y$ . Then we will construct a measure  $\mu \times \nu$  on  $\mathcal{M} \otimes \mathcal{N}$ .
2. Once the construction is in place, we will find a way to compare “double integrals” with “single integrals.” Morally, one wants equalities comparing

$$\iint_{X \times Y} f d(\mu \times \nu) \quad \text{and} \quad \int_X \left( \int_Y f(x, y) d\nu(y) \right) d\mu(x).$$

The moral of the story is that we will be able to compare our product measure with the measures on  $X$  and  $Y$  which we already understand.

3. Lastly, we will specialize to Euclidean space  $\mathbb{R}^d$ .

Let’s go ahead and begin.

**Definition 1.12** (measurable rectangle). Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . A measurable rectangle  $E \subseteq X \times Y$  is a set of the form  $A \times B$  where  $A \in \mathcal{M}$  and  $B \in \mathcal{N}$ .

**Example 1.13.** The product of the circles  $S^1 \subseteq \mathbb{R}^2$  and  $S^1 \subseteq \mathbb{R}^2$  is the torus  $S^1 \times S^1$  in  $\mathbb{R}^4$  (identified with  $\mathbb{R}^2 \times \mathbb{R}^2$ ).

**Definition 1.14** (product algebra). Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Then we define the product algebra  $\mathcal{A}(X, Y)$  as the collection of all finite disjoint unions of measure rectangles.

**Remark 1.15.** The reason that we have taken finite disjoint unions of rectangles is because we know how to measure measurable rectangles, and we know how to sum their measures as disjoint unions.

It's not totally clear that we have actually defined an algebra. We'll show this next class.

## 1.2 January 19

Here we go.

### 1.2.1 The Product Algebra

We quickly pick up the following lemma.

**Lemma 1.16.** Fix finitely many subsets  $A_1, \dots, A_n \subseteq X$ , and suppose that these subsets live in an algebra  $\mathcal{A}$  on  $X$ . Then there exists a finite partition  $\{C_\alpha\}_{\alpha \in \kappa}$  of  $X$  of sets in the algebra such that

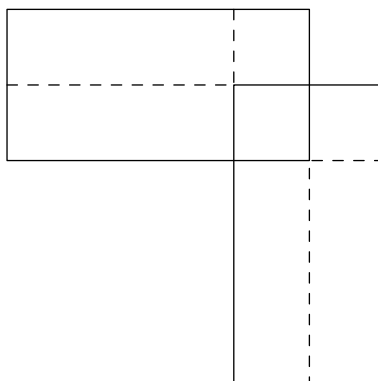
$$A_i = \bigsqcup_{\substack{\alpha \in \kappa \\ C_\alpha \subseteq A_i}} C_\alpha.$$

*Proof.* We basically build a Venn diagram. Choose index set  $I$  to be  $\{0, 1\}^n$ , and define  $C_\alpha$  for  $\alpha \in I$  to be the set of  $x \in X$  such that  $x \in A_i$  if and only if  $\alpha_i = 1$ . Note that we can write  $C_\alpha$  as

$$C_\alpha := \bigcup_{\alpha_i=1} A_i \setminus \bigcup_{\alpha_i=0} A_i.$$

Now, these  $C_\alpha$ 's of course provide a partition satisfying the needed condition by its construction. ■

Anyway, let's return to showing that we have a product algebra. For example, it turns out that the union of two measure rectangles is again a measurable rectangle. Here's the image.



And here is our statement.

**Lemma 1.17.** Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Then  $\mathcal{A}(X, Y)$  is actually an algebra.

*Proof.* Here are our checks.

- Note  $\emptyset \times \emptyset = \emptyset$ , so  $\emptyset \in \mathcal{A}(X, Y)$ .
- Finite union of rectangles: suppose that we have measurable rectangles  $\{A_i \times B_i\}_{i=1}^n$ . Then we show that the union is in  $\mathcal{A}(X, Y)$ . Now, the  $A_i$ 's produce some partition  $\{C_\alpha\}_{\alpha \in I} \subseteq \mathcal{M}$  of  $X$  via Lemma 1.16, and the  $B_i$ 's produce some partition  $\{D_\beta\}_{\beta \in J} \subseteq \mathcal{N}$  of  $Y$  via Lemma 1.16 again. Now

$$A_i \times B_i = \bigsqcup_{\substack{C_\alpha \subseteq A_i \\ D_\beta \subseteq B_i}} C_\alpha \times D_\beta,$$

so

$$\bigcup_{i=1}^n A_i \times B_i = \bigcup_{i=1}^n \bigsqcup_{\substack{C_\alpha \subseteq A_i \\ D_\beta \subseteq B_i}} C_\alpha \times D_\beta,$$

so our union is a union of measurable rectangles of the form  $C_\alpha \times D_\beta$ . But these measurable rectangles are all pairwise disjoint because the  $C_\alpha$ 's and  $D_\beta$ 's are all pairwise disjoint, so the above union is in  $\mathcal{A}$ .

- Finite union: given  $E_1, \dots, E_n \in \mathcal{A}$ , we need to show the union is in  $\mathcal{A}$ . Well, write

$$E_i = \bigsqcup_{j=1}^{n_i} A_{ij} \times B_{ij}$$

for some  $A_{ij} \in \mathcal{M}$  and  $B_{ij} \in \mathcal{N}$ . Then

$$\bigcup_{i=1}^n E_i = \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} A_{ij} \times B_{ij}$$

is a union of measurable rectangles and hence lives in  $\mathcal{A}$  by the above check.

- Complement: given  $E \in \mathcal{A}$ , write

$$E = \bigcup_{i=1}^n A_i \times B_i$$

for measurable rectangles  $A_i \times B_i$ . As before, the  $A_i$ 's produce some partition  $\{C_\alpha\}_{\alpha \in I} \subseteq \mathcal{M}$  of  $X$  via Lemma 1.16, and the  $B_i$ 's produce some partition  $\{D_\beta\}_{\beta \in J} \subseteq \mathcal{N}$  of  $Y$  via Lemma 1.16 again. This allows us to write

$$E = \bigsqcup_{i=1}^n \bigsqcup_{\substack{C_\alpha \subseteq A_i \\ D_\beta \subseteq B_i}} C_\alpha \times D_\beta,$$

and then the complement  $(X \times Y) \setminus E$  will be the union of the measurable rectangles  $C_\alpha \times D_\beta$  not in the above union. But these are still disjoint measurable rectangles, so the union remains in  $\mathcal{A}$ . ■

### 1.2.2 The Product Measure

Let's now define our product premeasure. Given the measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ , we would like to define

$$\rho\left(\bigsqcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i) \nu(B_i),$$

but it is not obvious that this is well-defined. Instead of doing this, we will choose the following definition.



**Definition 1.18** (product premeasure). Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Given  $E \in \mathcal{A}(X, Y)$ , we define the *product premeasure*  $\rho(E)$  as

$$\rho(E) := \int_X \nu(E_x) d\mu(x),$$

where  $E_x := \{y \in Y : (x, y) \in E\}$ .

**Remark 1.19.** One should perhaps check that  $E_x$  is always in  $\mathcal{N}$  and hence measurable. But for this we simply write  $E = \bigsqcup_{i=1}^n (A_i \times B_i)$  for measurable rectangles  $A_i \times B_i$  and note that

$$E_x = \{y \in Y : (x, y) \in A_i \times B_i \text{ for some } i\} = \bigcup_{\substack{i=1 \\ x \in A_i}}^n B_i,$$

which is a finite union of measurable sets and hence in  $\mathcal{N}$ . In fact,

**Remark 1.20.** One should perhaps check that  $x \mapsto \nu(E_x)$  is integrable. Continuing from the above, we can see that these  $B_i$  must be disjoint if  $x \in A_i$  for each of these  $i$ , so actually

$$\nu(E_x) = \sum_{\substack{i=1 \\ x \in A_i}}^n \nu(B_i) = \sum_{i=1}^n 1_{A_i}(x) \nu(B_i),$$

which is a linear combination of indicators of  $\mu$ -measurable sets, so this is a  $\mu$ -integrable function. Notably, we see that the measure of a measurable rectangle  $A \times B$  is in fact  $\mu(A)\nu(B)$ .

**Remark 1.21.** It is notable that we can write

$$\rho(E) = \int_X \nu(E_x) d\mu(x) = \int_X \int_Y 1_E(x, y) d\nu(y) d\mu(x),$$

where the equality follows because the measure  $\nu(E_x)$  is simply integrating  $Y$  over the indicator of  $1_E(x, y)$ .

We now check that we have a premeasure.

**Proposition 1.22.** Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Then the defined product premeasure  $\rho$  on  $\mathcal{A}(X, Y)$  is in fact a premeasure.

*Proof.* Here are our checks.

- Note  $\rho(\emptyset) = 0$  because  $\emptyset_x = \emptyset$  always.
- Finitely additive: fix disjoint  $E_1, E_2 \in \mathcal{A}(X, Y)$ , and we want to compute  $\rho(E_1 \sqcup E_2)$ . Well, we use Remark 1.21 to note

$$\begin{aligned} \rho(E_1 \sqcup E_2) &= \int_X \nu((E_1 \sqcup E_2)_x) d\mu(x) \\ &= \int_X \int_Y 1_{E_1 \sqcup E_2}(x, y) d\nu(y) d\mu(x) \\ &= \int_X \int_Y (1_{E_1}(x, y) + 1_{E_2}(x, y)) d\nu(y) d\mu(x) \end{aligned}$$

Now, by linearity of integration, this is

$$\begin{aligned}\rho(E_1 \sqcup E_2) &= \int_X \int_Y 1_{E_1}(x, y) d\nu(y) d\mu(x) + \int_X \int_Y 1_{E_2}(x, y) d\nu(y) d\mu(x) \\ &= \rho(E_1) + \rho(E_2),\end{aligned}$$

as desired.

- **Countably additive:** we use the Monotone convergence theorem. Fix some disjoint subsets  $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}(X, Y)$  such that  $E := \bigcup_{i=1}^\infty E_i$  is in  $\mathcal{A}(X, Y)$ . Proceeding as in the previous check, we see that

$$\begin{aligned}\rho(E) &= \int_X \int_Y 1_E(x, y) d\nu(y) d\mu(x) \\ &= \int_X \int_Y 1_E(x, y) d\nu(y) d\mu(x) \\ &= \int_X \int_Y \left( \sum_{i=1}^\infty 1_{E_i}(x, y) \right) d\nu(y) d\mu(x).\end{aligned}$$

Now, the functions  $1_{E_i}$  and  $1_E$  are all integrable (for suitably fixed coordinates), so applying the Monotone convergence theorem [Elb22, Theorem 9.18] tells us that

$$\rho(E) = \sum_{i=1}^\infty \int_X \int_Y 1_{E_i}(x, y) d\nu(y) d\mu(x) = \sum_{i=1}^\infty \rho(E_i),$$

as desired. ■

We can now produce our product measure.

**Definition 1.23 (product measure).** Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Define the *product  $\sigma$ -algebra*  $\mathcal{M} \otimes \mathcal{N}$  to be the  $\sigma$ -algebra generated by  $\mathcal{A}(X, Y) \subseteq \mathcal{P}(X \times Y)$ . Then the product premeasure  $\rho$  on  $\mathcal{A}(X, Y)$  extends by Theorem 1.11 to a measure  $\mu \times \nu$  on  $\mathcal{M} \otimes \mathcal{N}$ .

**Remark 1.24.** By Theorem 1.11, if  $\mu$  and  $\nu$  are both  $\sigma$ -finite, then one can see that  $\rho$  is still  $\sigma$ -finite by some covering with measurable rectangles, so  $\mu \times \nu$  becomes the unique measure on  $\mathcal{M} \otimes \mathcal{N}$  extending  $\rho$ .

### 1.2.3 Tonelli's Theorem

The construction of our product premeasure in Definition 1.18 has a “handedness” in that we integrate with respect to  $Y$  and then with respect to  $X$ . This is somewhat upsetting, so we work to remedy this.

**Theorem 1.25 (Tonelli).** Fix  $\sigma$ -finite measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Fix a measurable function  $f: X \times Y \rightarrow [0, \infty]$ . Then the following hold.

- The function  $y \mapsto f(x, y)$  is  $\mathcal{N}$ -measurable.
- The function  $x \mapsto \int_Y f(x, y) d\nu(y)$  is  $\mathcal{M}$ -measurable.
- We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

**Remark 1.26.** Note that, once measurable, we can integrate a nonnegative function if we allow for infinite values. For example, see something like [Elb22, Proposition 9.22]

*Reductions of Theorem 1.25.* We begin with two reductions.

- We reduce to the case where  $f$  is the indicator of a function  $1_E$ . Indeed, having the result for indicators shows the conclusions for any linear combination of these, so we get the result for simple measurable functions, and then we can get the general case by taking monotone limits via the Monotone convergence theorem [Elb22, Theorem 9.18].

(Namely, (a) is direct by taking limits, (b) follows by the Monotone convergence theorem to move out the limit out of the integral and then taking limits to get measurable, and (c) is achieved directly by the Monotone convergence theorem repeatedly.)

- We reduce to the case where  $X$  and  $Y$  are spaces of finite measure. Indeed, by the  $\sigma$ -finiteness of  $X$  and  $Y$ , we can partition each into countable disjoint union of sets of finite measure, and then by taking rectangles, we see that  $X \times Y$  is a countable union of disjoint sets of finite measure. So achieving the result on these disjoint sets of finite measure, we can check the conclusions by summing over all the disjoint spaces, again concluding via the Monotone convergence theorem [Elb22, Theorem 9.18]. Namely, one can do an identical argument to the parenthetical remark of the previous reduction.

Before doing anything, we note that the  $\sigma$ -algebra  $\mathcal{M} \otimes \mathcal{N}$  is not obviously generated at finite steps from  $\mathcal{A}(X, Y)$ ; in fact, there is no countable constructive procedure to do this. So we are not going to proceed by trying to build up to  $\mathcal{M} \otimes \mathcal{N}$ ; instead we will have to do something difficult. ■

## 1.3 January 22

Here we go.

### 1.3.1 Proof of Tonelli's Theorem

Last class we reduced the proof of Theorem 1.25 to having  $f = 1_E$  for some measurable set  $E$  and having  $X$  and  $Y$  be finite measure spaces. Today we will complete the proof. We proceed by a sequence of lemmas.

**Lemma 1.27.** Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . If  $E \in \mathcal{M} \otimes \mathcal{N}$  and  $x \in X$ , then the slice

$$E_x := \{y \in Y : (x, y) \in E\}$$

is in  $\mathcal{N}$ .

*Proof.* The problem is that we know very little about  $\mathcal{M} \otimes \mathcal{N}$ , so we will have to do something indirect. Continue with  $x$  fixed, but we let  $E$  vary to define

$$\mathcal{D}_x := \{E \subseteq X \times Y : E_x \in \mathcal{N}\}.$$

Note  $\mathcal{D}_x$  is a  $\sigma$ -algebra, as we now check.

- Note  $\emptyset \subseteq X \times Y$  has  $\emptyset = \emptyset_x$  in  $\mathcal{N}$ . So  $\emptyset \in \mathcal{D}_x$ .
- Complement: if  $E \in \mathcal{D}_x$ , then  $((X \times Y) \setminus E)_x = Y \setminus E_x$  as this set contains exactly the  $y \in Y$  such that  $(x, y) \notin E$ . Thus,  $((X \times Y) \setminus E)_x \in \mathcal{N}$ , so  $(X \times Y) \setminus E \in \mathcal{D}_x$ .

- Countable unions: fix a countable collection  $\{E_i\}_{i=1}^\infty \subseteq \mathcal{D}_x$ . Then

$$\left( \bigcup_{i=1}^\infty E_i \right)_x = \bigcup_{i=1}^\infty (E_i)_x$$

because some  $y$  lives in this set if and only if  $(x, y)$  belongs to one of the  $E_i$ . The right-hand side lives in  $\mathcal{N}$  by assumption, so we see  $\bigcup_{i=1}^\infty E_i \in \mathcal{D}_x$ .

Furthermore, we note that  $\mathcal{D}_x$  contains  $\mathcal{A}(X, Y)$ . Indeed, it suffices to check that  $\mathcal{D}_x$  contains measurable rectangles because  $\mathcal{A}(X, Y)$  contains disjoint unions of these. Well, for a measurable rectangle  $A \times B$ , we see

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A, \end{cases} \quad (1.1)$$

always lives in  $\mathcal{N}$ , so  $A \times B \in \mathcal{D}_x$ . In total, it follows that  $\mathcal{D}_x$  must contain the smallest  $\sigma$ -algebra containing  $\mathcal{A}(X, Y)$ , so  $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{D}_x$ . This is what we wanted. ■

**Remark 1.28.** The above proof exemplifies how we will access  $\mathcal{M} \otimes \mathcal{N}$ : we will construct some  $\sigma$ -algebra characterizing the desirable properties, and then we will show that it contains  $\mathcal{A}(X, Y)$  in order to contain  $\mathcal{M} \otimes \mathcal{N}$ .

We now prove (a) and (b) of Theorem 1.25.

**Lemma 1.29.** Fix measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ , with  $Y$  finite. If  $E \in \mathcal{M} \otimes \mathcal{N}$ , then the function  $f_E: x \mapsto \nu(E_x)$  is  $\mathcal{M}$ -measurable.

*Proof.* We consider

$$\mathcal{D} := \{E \subseteq X \times Y : f_E \text{ is } \mathcal{M}\text{-measurable and } E_x \text{ is always measurable}\}.$$

We would like to show that  $\mathcal{D}$  contains  $\mathcal{M} \otimes \mathcal{N}$ . Let's show some properties of  $\mathcal{D}$ . We won't succeed at showing that  $\mathcal{D}$  is actually a  $\sigma$ -algebra, but we will get close enough. Of course,  $\mathcal{D}$  contains  $\emptyset$  because  $f_\emptyset$  is just the zero function. Additionally, taking complements uses finiteness of the measure spaces: if  $\nu(Y) < \infty$ , we can write

$$f_{(X \times Y) \setminus E}(x) = \nu(Y) - \nu(E_x) = \nu(Y) - f_E(x),$$

so we are done because the right-hand side is a measurable function in  $x$ . (Indeed, constant functions and sums of measurable functions are all measurable.)

**Remark 1.30.** There is an issue with taking unions: given  $E, F \in \mathcal{D}$ , we want to look at  $f_{E \cup F}$ , but there is no obvious way to access this only in terms of  $f_E$  and  $f_F$  because there may be some intersection.

In light of Remark 1.30, we need a trick. Do note that we can show that  $\mathcal{D}$  is closed under disjoint unions because then  $f_{E \sqcup F} = f_E + f_F$ , so  $f_{E \sqcup F}$  being measurable is recovered from summing  $f_E$  and  $f_F$ . Thus, because  $\mathcal{D}$  contains measurable rectangles (note that the measure of the output of the function (1.1) is measurable as it's basically an indicator), so  $\mathcal{A}(X, Y) \subseteq \mathcal{D}$ .

For our trick, we proceed in steps.

1. We begin by showing that  $\mathcal{D}$  is closed under countable ascending unions: given an ascending sequence of sets  $\{E_i\}_{i=1}^\infty \subseteq \mathcal{D}$ , then we set  $E := \bigcup_{i=1}^\infty E_i$  and see

$$\lim_{n \rightarrow \infty} \nu((E_n)_x) = \nu(E_x)$$

because  $(\bigcup_{i=1}^\infty E_i)_x = \bigcup_{i=1}^\infty (E_i)_x$  tells us that the  $(E_n)_x$  are measurable sets ascending to  $E_x$ , so we get the above limit via [Elb22, Proposition 6.36]. Thus,  $f_E$  is the pointwise limit of the  $f_{E_n}$ s, so  $f_E$  is  $\mathcal{M}$ -measurable.

2. Additionally,  $\mathcal{D}$  is closed under countable descending intersections: the same argument of the previous point works, exchanging the word “ascending” with “descending,” exchanging unions with intersections, and exchanging the citation with [Elb22, Corollary 6.37]. Note that our sets are of finite measure because  $Y$  is finite!

To proceed with the proof, we pick up the following definition.

**Definition 1.31 (monotone class).** Fix a set  $\Omega$ . Then a *monotone class* is a collection  $\mathcal{C} \subseteq \mathcal{P}(\Omega)$  which contains  $\emptyset$  and is closed under countable ascending unions and countable descending intersections.

In particular, we have shown that  $\mathcal{D}$  is a monotone class. We will want the following fact about monotone classes.

**Lemma 1.32.** Fix a set  $\Omega$ , and let  $\mathcal{A}$  be an algebra on  $\Omega$ . Then the smallest monotone class  $\mathcal{C}$  containing  $\mathcal{A}$  is a  $\sigma$ -algebra.

*Proof.* Note that the notion of a “smallest monotone class” makes sense because the intersection of monotone classes is another monotone class, so we can take  $\mathcal{C}$  to be the intersection of all monotone classes containing  $\mathcal{A}$ . Anyway, here are our checks.

1. Fix  $\mathcal{D}$  to be the collection of subsets of  $\Omega$  whose complement is in  $\mathcal{C}$ . We claim that  $\mathcal{D}$  is a monotone class; this will imply that  $\mathcal{D}$  contains  $\mathcal{C}$  (because  $\mathcal{D}$  of course contains  $\mathcal{A}$ , which is closed under complements), meaning that  $\mathcal{C}$  is closed under complements. For countable ascending unions of  $E_1 \subseteq E_2 \subseteq \dots$ , we note that the union  $E$  has

$$\Omega \setminus E = \bigcap_{i=1}^{\infty} \Omega \setminus E_i,$$

which is in  $\mathcal{C}$ , so  $E \in \mathcal{D}$ . Replacing unions with intersections shows that  $\mathcal{D}$  is closed under

2. If  $A \in \mathcal{A}$  and  $B \in \mathcal{C}$ , then we claim  $A \cup B \in \mathcal{C}$ . Well, fix  $A$ , and we set

$$\mathcal{D}_A := \{E \subseteq \Omega : A \cup E \in \mathcal{C}\}.$$

We claim that  $\mathcal{D}_A$  is a monotone class, and it contains  $\mathcal{A}$  (which is closed under unions), so  $\mathcal{D}_A$  will contain  $\mathcal{C}$ , proving the claim. For ascending unions  $E_1 \subseteq E_2 \subseteq \dots$ , we note

$$\left( \bigcup_{i=1}^{\infty} E_i \right) \cap A = \bigcup_{i=1}^{\infty} (E_i \cap A),$$

so the union is still in  $\mathcal{D}_A$ . Replacing the big  $\bigcup$  with a big  $\bigcap$  and working with a descending intersection shows that  $\mathcal{D}_A$  is a monotone class, as needed.

3. If  $A \in \mathcal{C}$  and  $B \in \mathcal{C}$ , then we claim  $A \cup B \in \mathcal{C}$ . Once again, we fix  $A$  and set

$$\mathcal{D}_A \{E \subseteq \Omega : A \cup E \in \mathcal{C}\}.$$

The previous check tells us that  $\mathcal{D}_A$  contains  $\mathcal{A}$ . The same proof as the previous check tells us that  $\mathcal{D}_A$  is a monotone class, so we once again are allowed to conclude that  $\mathcal{D}_A$  contains  $\mathcal{C}$ , so the claim follows.

4. Lastly, we show  $\mathcal{C}$  is closed under countable unions. Well, given a countable collection  $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}$ , we set

$$F_j := \bigcup_{i \leq j} E_i,$$

which is in  $\mathcal{C}$  by the previous check. Then the union of the  $E_i$ s is the union of the  $F_j$ s, but  $\mathcal{C}$  is a monotone class, so it contains the union of the  $F_j$ s (which are ascending), so we are done. ■

Now, we see that Lemma 1.32 finishes the proof:  $\mathcal{D}$  must contain the smallest monotone class containing  $\mathcal{A}(X, Y)$ , which is a  $\sigma$ -algebra by Lemma 1.32, so  $\mathcal{D}$  contains the smallest  $\sigma$ -algebra containing  $\mathcal{A}(X, Y)$ , so  $\mathcal{D}$  contains  $\mathcal{M} \otimes \mathcal{N}$ , as needed. ■

We now complete the proof of Theorem 1.25; the following is the statement of (c) for one of the equalities where  $f = 1_E$ .

**Lemma 1.33.** Fix  $\sigma$ -finite measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Then

$$(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x).$$

*Proof.* Proceed as in Lemma 1.29. Explicitly, set

$$\mathcal{D} := \left\{ E \subseteq X \times Y : x \mapsto \nu(E_x) \text{ is measurable and } (\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) \right\}.$$

The construction of the measure  $\mu \times \nu$  implies this equality when  $E$  is a measurable rectangle or even when  $E$  is a disjoint union of measurable rectangles, so  $\mathcal{D}$  contains  $\mathcal{A}(X, Y)$ . A direct computation shows that  $\mathcal{D}$  is closed under complements, and the Dominated convergence theorem [Elb22, Theorem 9.14] shows that  $\mathcal{D}$  is closed under ascending unions and descending intersections. So  $\mathcal{D}$  is a monotone class containing the algebra  $\mathcal{A}(X, Y)$ , which implies that  $\mathcal{D}$  contains the smallest monotone class containing  $\mathcal{A}(X, Y)$ , which is a  $\sigma$ -algebra by Lemma 1.32, so  $\mathcal{D}$  contains the smallest  $\sigma$ -algebra containing  $\mathcal{A}$ , so  $\mathcal{D}$  contains  $\mathcal{M} \otimes \mathcal{N}$ . ■

## 1.4 January 24

Let's begin.

### 1.4.1 Addenda to Tonelli's Theorem

Last class we completed the proof of Theorem 1.25. We take a moment to note that there is a “mirror” of Tonelli's theorem as follows.

**Theorem 1.34 (Tonelli).** Fix  $\sigma$ -finite measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Fix a measurable function  $f: X \times Y \rightarrow [0, \infty]$ . Then the following hold.

- (a) The function  $x \mapsto f(x, y)$  is  $\mathcal{M}$ -measurable.
- (b) The function  $y \mapsto \int_Y f(x, y) d\nu(y)$  is  $\mathcal{N}$ -measurable.
- (c) We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y).$$

We will not write this proof because one can simply interchange  $X$  and  $Y$  in the provided proof of Theorem 1.25. Perhaps one will complain that the definition of the product premeasure Definition 1.18 appears asymmetric, but in fact it does not. Indeed, Remark 1.20 explains that the measure of a measurable rectangle is symmetric, which then explains how to measure anything in  $\mathcal{A}(X, Y) = \mathcal{A}(Y, X)$  symmetrically, and then the Extension Theorem 1.11 tells us that this uniquely measures anything in  $\mathcal{M} \otimes \mathcal{N}$  symmetrically.

**Corollary 1.35.** Fix  $\sigma$ -finite measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . If  $f: X \times Y \rightarrow [0, \infty]$  is  $(\mathcal{M} \otimes \mathcal{N})$ -measurable, then

$$\int_X \int_Y f(x, y) d\nu(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y).$$

*Proof.* Combine Theorems 1.25 and 1.34. ■

Perhaps one might worry about spaces which are not  $\sigma$ -finite. Here are some examples.

**Example 1.36.** Fix an uncountable set  $X$ . Then one can define a measure  $\mu$  on  $\mathcal{M} := \mathcal{P}(X)$  by  $\mu(E) := \#E$ . This is not  $\sigma$ -finite because subsets of  $X$  has finite measure if and only if it is finite, and  $X$  cannot be covered by countably many finite sets.

**Example 1.37.** Fix an uncountable set  $X$ , and let  $\mathcal{M}$  be the collection of countable and cocountable subsets. Then the function  $\mu: \mathcal{M} \rightarrow [0, \infty]$  defined by

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{if } E \text{ is cocountable} \end{cases}$$

is a measure. Now,  $X$  fails to be  $\sigma$ -finite because the sets of finite measure are exactly the countable ones, and  $X$  cannot be covered by countably many countable subsets.

## 1.4.2 Fubini's Theorem

We are now ready to state Fubini's theorem. This requires the following definition.

**Definition 1.38.** Fix a measure space  $(X, \mathcal{M}, \mu)$ . Then we define  $L^1(\mu)$  consists of the measurable functions  $f: X \rightarrow \mathbb{C}$  (defined almost everywhere) such that

$$\int_X |f| d\mu < \infty.$$

**Remark 1.39.** If  $f \in L^1(\mu)$ , then one sees that  $\int_X f d\mu$  makes sense. Namely, one has

$$\int_X f d\mu = \int_X \operatorname{Re} f d\mu + i \int_X \operatorname{Im} f d\mu,$$

and the integrals  $\int_X \operatorname{Re} f d\mu$  and  $\int_X \operatorname{Im} f d\mu$  are both bounded by  $f \in L^1(\mu)$ . Something like [Elb22, Proposition 9.22] assures us that this makes sense (upon taking differences).

**Theorem 1.40 (Fubini).** Fix  $\sigma$ -finite measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ . Fix a measurable function  $f: X \times Y \rightarrow \mathbb{C}$  such that  $\int_{X \times Y} |f| d(\mu \times \nu) < \infty$ . Then the following hold.

- (a) For  $\mu$ -almost every  $x \in X$ , the function  $f_x: y \mapsto f(x, y)$  is defined and  $\mathcal{N}$ -measurable and in  $L^1(\nu)$ .
- (b) The function  $x \mapsto \int_Y f(x, y) d\nu(y)$  is defined almost everywhere and  $\mathcal{M}$ -measurable and in  $L^1(\mu)$ .
- (c) We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

*Proof.* Note that we have the result for nonnegative functions by Theorem 1.25. The idea is to reduce to this case. Here we go.

- By writing  $f = u + iv$  for  $u := \operatorname{Re} f$  and  $v := \operatorname{Im} f$ , we may assume that  $f$  is real-valued. Explicitly, note the set of functions  $f$  satisfying the conclusions of (a)–(c) is a  $\mathbb{C}$ -vector space by some addition and scalar multiplication. Notably, we still have the hypotheses that  $\int_{X \times Y} |u| d(\mu \times \nu) < \infty$  and  $\int_{X \times Y} |v| d(\mu \times \nu) < \infty$ .

- By writing  $f = f^+ - f^-$  for  $f^+, f^- \geq 0$ , we will reduce to the case that  $f$  is nonnegative. Namely, achieving the result for the two functions

$$f^+(x) := \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) \leq 0, \end{cases} \quad \text{and} \quad f^-(x) := \begin{cases} -f(x) & \text{if } f(x) \leq 0, \\ 0 & \text{if } f(x) \geq 0, \end{cases}$$

will achieve the result for  $f$  by summing.

Note that there is a technicality hidden in the above reasoning with linear combinations: for example, for the second reduction, even though we have the conclusion for  $f_x^+$  and  $f_x^-$  are  $\mathcal{N}$ -measurable for all  $x$ , their difference might not be in  $L^1(\nu)$  always. Well, we note that we can compute

$$\int_X \left( \int_Y f^+(x, y) d\nu(y) \right) d\mu(x) = \int_{X \times Y} f^+ d(\mu \times \nu) < \infty,$$

so the inner function  $x \mapsto \int_Y f_x^+ d\nu(y)$  must be finite almost everywhere, or else this integral would be infinite! So we do indeed achieve that  $f_x^+$  and  $f_x^-$  are in  $L^1(\nu)$  almost everywhere, so their difference is in  $L^1(\nu)$  almost everywhere. The argument for taking linear combinations in (b) is similar. ■

Let's see an example of why we want the hypothesis in Theorem 1.40.

**Example 1.41.** Set  $X := \mathbb{N}$ , and let  $\mu$  and  $\nu$  denote the counting measures on  $\mathcal{M} = \mathcal{N} := \mathcal{P}(X)$ . Note that  $\mathcal{A}(X, X) = \mathcal{P}(X^2)$ , so the product measure  $\mu \times \nu$  is defined on all subsets; furthermore, we can see that the measure of a singleton is 1, so  $\mu \times \nu$  is the counting measure. Then we define the function

$$f(x, y) := \begin{cases} +1 & \text{if } x = y, \\ -1 & \text{if } y = x + 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each  $x$ , we compute  $\int_Y f(x, y) d\nu(y) = 0$  because each value of  $x$  has two values of  $y$  where  $f(x, y)$  is nonzero. On the other hand, for each  $y$ , we compute  $\int_X f(x, y) d\mu(x) = 0$  if  $y > 0$  but is 0 if  $y = 0$ . The problem here is that  $\int_{X \times Y} f(x, y) d(\mu \times \nu) = 0$ .

**Remark 1.42.** According to Professor Christ, the above example is a “catastrophic failure” of a theorem rather than a “technical” one.

**Remark 1.43.** By induction, we are able to take products of any finite product of  $\sigma$ -finite measure spaces. Alternatively, one can redo the entire theory to do measurable rectangular prisms and so on. There are some extra checks here (e.g., does forming products associate meaningfully?), but it will work out in the end, essentially by the uniqueness of the construction provided by Theorem 1.11. Namely, up to the identification of products, we get the identification of the product  $\sigma$ -algebras and product measures because they should all agree on measurable rectangles, from which everything is generated.

## 1.5 January 26

Here we go.

### 1.5.1 Complete Measures

Recall the following definition.



**Definition 1.44 (complete).** Fix a measure space  $(X, \mathcal{M}, \mu)$ . Then  $\mu$  is *complete* if and only if it has all null sets: if  $E \in \mathcal{M}$  has  $\mu(E) = 0$ , then any subset  $F \subseteq E$  has  $F \in \mathcal{M}$ .

**Non-Example 1.45.** Let  $\mu$  be the Lebesgue measure on the Borel algebra  $\mathcal{M}$  of  $\mathbb{R}$ . Then  $\mu$  is not complete: there are subsets of null sets which are not Borel. In fact, there are only  $|\mathbb{R}|$  many Borel sets by a counting construction, but there are more null sets.

We now recall the following construction.

**Proposition 1.46.** Fix a measure space  $(X, \mathcal{M}, \mu)$ . Then there is a measure space  $(X, \widetilde{\mathcal{M}}, \widetilde{\mu})$  such that  $\widetilde{\mathcal{M}} \supseteq \mathcal{M}$ ,  $\widetilde{\mu}$  extends  $\mu$ , and  $\widetilde{\mu}$  is complete.

*Proof.* We showed this in Math 202A. To sketch the idea, simply set  $\widetilde{\mathcal{M}}$  as the union of elements in  $\mathcal{M}$  with subsets of null sets, and define

$$\widetilde{\mu}(A \cup E) = \mu(A)$$

where  $A \in \mathcal{M}$  and  $E$  is a subset of a null set. Checking that this is actually a measure space is annoying and hence omitted; checking that  $\widetilde{\mu}$  is complete simply follows because  $\widetilde{\mu}(A \cup E) = 0$  and  $F \subseteq A \cup E$  implies that  $\mu(A) = 0$  actually, so  $A \cup E$  is a subset of a null set for  $\mu$  still, so  $F$  is a subset of a null set for  $\mu$ , so  $F \in \mathcal{M}$ . ■

**Remark 1.47.** One can see that the above constructions the “minimal” completion in the sense that any other completion  $(X, \widetilde{\mathcal{M}}', \widetilde{\mu}')$  has  $\widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}'$  and  $\widetilde{\mu}'|_{\widetilde{\mathcal{M}}} = \widetilde{\mu}$ .

We would like to examine completeness for our product measures. Sadly, in most cases, having complete metric spaces does not make the product measure complete.

**Example 1.48.** Let  $(\mathbb{R}, \mathcal{L}, \mu)$  be the completion of the Borel Lebesgue measure. Then the product measure  $(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, \mu \times \mu)$  fails to be complete. To see this, let  $A \subseteq \mathbb{R}$  be a subset not in  $\mathcal{L}$ , and let  $B \in \mathcal{L}$  be a nonempty null set. Then  $E := A \times B$  is the desired set.

- Note  $E \subseteq \mathbb{R} \times B$ , and  $\mathbb{R} \times B$  is a null set: note  $\mu([-r, r] \times B) = 2r\mu(B) = 0$ , so sending  $r \rightarrow \infty$  shows  $\mu(\mathbb{R} \times B) = 0$ . Alternatively, we simply recall that it is convention that  $\infty \times 0 = 0$  here.
- On the other hand,  $E \notin \mathcal{L} \otimes \mathcal{L}$ . Indeed, Theorem 1.25 would tell us that  $E \in \mathcal{L} \otimes \mathcal{L}$  implies that  $A = E_y$  is measurable for all  $y \in B$ , which is false.

To remedy our situation, we have the following result to recover Theorem 1.25.

**Theorem 1.49.** Fix  $\sigma$ -finite complete measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ , and let  $(X \times Y, \mathcal{L}, \lambda)$  denote the completion of  $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ . Let  $f: X \times Y \rightarrow [0, \infty]$  be  $\mathcal{L}$ -measurable.

- For  $\mu$ -almost every  $x$ , the function  $y \mapsto f(x, y)$  is  $\mathcal{N}$ -measurable.
- The function  $x \mapsto \int_Y f(x, y) d\nu(y)$  is defined  $\mu$ -almost everywhere, and it is  $\mathcal{M}$ -measurable.
- We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

One can recover Theorem 1.40 in the analogous way.

We will not prove Theorem 1.49 in detail. The main point is to show the following.

**Lemma 1.50.** Fix everything as in Theorem 1.49. For  $E \in \mathcal{L}$  with  $\lambda(E) = 0$ , we have  $E_x \in \mathcal{N}$  for  $\mu$ -almost every  $x \in X$ .

Indeed, as before, we can restrict to the case where  $f = 1_E$ , so this recovers (a), and then the arguments of Theorem 1.25 port over to prove (b) and (c) cleanly.

*Proof.* The main point is that we can find  $A \supseteq E$  such that  $A \in \mathcal{M} \otimes \mathcal{N}$ . But then Theorem 1.25 reassures us that

$$(\mu \times \nu)(A) = \int_X \nu(A_x) d\mu(x) = 0,$$

so  $\nu(A_x) = 0$  for  $\mu$ -almost every  $x \in X$ . ■

## 1.5.2 Measuring Euclidean Spaces

For our discussion, we will want two pieces of notation.

**Notation 1.51.** Fix a nonnegative integer  $d \geq 0$ .

- For a topological space  $X$ , we define  $\text{Borel}(X)$  to be the  $\sigma$ -algebra of Borel subsets in  $X$ .
- We define  $\mathcal{B}^d := \text{Borel}(\mathbb{R})^d$ .

There is some care one must take in our notation here, but not much care.

**Proposition 1.52.** For all nonnegative integers  $d \geq 0$ , we have  $\mathcal{B}^d = \text{Borel}(\mathbb{R}^d)$ .

We will require the following lemma.

**Lemma 1.53.** Fix a positive integer  $d \geq 1$ , and let  $U \subseteq \mathbb{R}^d$  be open. Then  $U$  is a disjoint union of countably many half-open cubes. Here, a “cube” is a product of intervals of the form

$$\prod_{i=1}^d [a_i, b_i).$$

*Proof.* For  $n \geq 0$ , let  $\mathcal{D}_n$  denote the collection of “dyadic cubes” of the form

$$\prod_{i=1}^d [a_i, b_i)$$

where  $b_i - a_i = 2^{-n}$  and  $a_i \in 2^{-n}\mathbb{Z}$  for each  $i$ . We now let  $\mathcal{C}$  denote the collection of cubes in some  $\mathcal{D}_n$  contained in  $U$ . Because any point  $x \in U$  is contained in some open ball  $B(x, r)$  such that  $B(x, 2r) \subseteq U$ , we can find a cube in  $\mathcal{D}_n$  for  $n$  large enough living inside  $B(x, 2r)$  containing  $x$ .

The issue is now to make the cubes  $\mathcal{C}$  disjoint. Well, define  $\mathcal{C}'$  to be the subcollection of “maximal” cubes in the sense that  $\prod_{i=1}^d [a_i, b_i)$  will be in  $\mathcal{C}'$  if and only if  $\prod_{i=1}^d [a_i, b_i + (b_i - a_i))$  is not in  $\mathcal{C}$ . Certainly  $\mathcal{C}'$  still covers  $U$ , and its cubes are disjoint: certainly no two cubes in  $\mathcal{C}'$  contain each other by construction, and dyadic cubes either contain each other or are disjoint. ■

## 1.6 January 29

Here we go.

### 1.6.1 More on Measuring Euclidean Spaces

We now prove the following statement from last class.

**Proposition 1.52.** For all nonnegative integers  $d \geq 0$ , we have  $\mathcal{B}^d = \text{Borel}(\mathbb{R}^d)$ .

*Proof.* The case of  $d = 0$  and  $d = 1$  have no content. Now, in one direction, we see that  $\text{Borel}(\mathbb{R}^d) \subseteq \mathcal{B}^d$ : note  $\mathcal{B}^d$  is a  $\sigma$ -algebra containing the cubes of the form in Lemma 1.53, so it also contains open subsets of  $\mathbb{R}^d$  by Lemma 1.53, so we conclude.

We now show that  $\mathcal{B}^d \subseteq \text{Borel}(\mathbb{R}^d)$ . By the definition of  $\mathcal{B}^d$ , it is enough to show that  $\text{Borel}(\mathbb{R}^d)$  contains all measurable rectangles  $A_1 \times \cdots \times A_d$  where  $A_1 \in \text{Borel}(\mathbb{R})$ . We proceed inductively, claiming that if  $A_i, A_{i+1}, \dots, A_d$  are all open, then the entire product is Borel. For  $i = 1$ , there is nothing to do. Now, for the induction, suppose we have the claim for  $i$ , and we want the claim for  $i + 1$ . Well, fix everything except  $A_i$ , and we define

$$\mathcal{D} := \{B \subseteq \mathbb{R} : A_1 \times \cdots \times A_{i-1} \times B \times A_{i+1} \times \cdots \times A_d \text{ is Borel}\}.$$

Certainly if  $B$  is open, then  $B \in \mathcal{D}$  by the induction. Additionally, arbitrary unions and intersections distribute over  $\times$ , so  $\mathcal{D}$  is closed arbitrary unions and intersections. Lastly, if  $B \in \mathcal{D}$ , we see that

$$(A_1 \times \cdots \times A_{i-1} \times (\mathbb{R} \setminus B) \times A_{i+1} \times \cdots \times A_d) = (A_1 \times \cdots \times A_{i-1} \times \mathbb{R} \times A_{i+1} \times \cdots \times A_d) \setminus (A_1 \times \cdots \times A_{i-1} \times B \times A_{i+1} \times \cdots \times A_d),$$

and the right-hand side is the subtraction of two Borel sets, so  $(\mathbb{R} \setminus B) \in \mathcal{D}$ . Thus,  $\mathcal{D}$  is a  $\sigma$ -algebra containing opens, so  $\mathcal{D}$  contains  $\text{Borel}(\mathbb{R})$ . ■

We now move towards some regularity conditions on our measures.

**Definition 1.54 (regular).** Fix a topological space  $X$  and a measure  $\mu$  on a  $\sigma$ -algebra  $\mathcal{M}$  containing the Borel sets.

- $\mu$  is *outer regular* if and only if any  $E \in \mathcal{M}$  has

$$\mu(E) = \inf_{\text{open } U \supseteq E} \mu(U).$$

- $\mu$  is *inner regular* if and only if any  $E \in \mathcal{M}$  has

$$\mu(E) = \sup_{\text{compact } K \subseteq E} \mu(K).$$

Here is our result.

**Theorem 1.55.** Fix a nonnegative integer  $d \geq 0$ . Then the measures  $\mu$  and  $\tilde{\mu}$  on  $\mathbb{R}^d$  is outer regular.

*Proof.* The statement for  $d = 0$  has no content. The outer regularity at  $d = 1$  follows by its construction [Elb22, Lemma 6.15]; we will prove inner regularity from outer regularity momentarily. We proceed in steps.

1. We now show outer regularity for  $d \geq 2$ , using the  $d = 1$  case. Take  $E \in \mathcal{B}^d$ , permissible by Proposition 1.52. If  $E$  has infinite measure, we can take  $\mathbb{R}^d$  as the needed open set. Otherwise, we take  $\varepsilon > 0$ . By construction of  $\mu$ , we get  $\{A_i^j\}_{1 \leq i \leq d, 1 \leq j \leq n} \in \mathcal{B}^1$  of positive measure such that

$$\sum_{j=1}^n \mu(A_1^j) \cdots \mu(A_d^j) < \mu(E) + \varepsilon.$$

We now use outer regularity in dimension 1. For each  $A_i^j$ , we get some  $U_i^j \supseteq A_i^j$  whose measure is within  $\varepsilon \min\{\mu(A_i^j)\} > 0$  of  $\mu(A_i^j)$ . Then we set  $U$  as the union of the  $U_1^j \times \cdots \times U_d^j$  and find

$$\mu(U) \leq \sum_{j=1}^n \left( \mu(A_1^j) + \min\{\mu(A_k^j) : k, \ell\} \varepsilon \right) \cdots \left( \mu(A_d^j) + \min\{\mu(A_k^j) : k, \ell\} \varepsilon \right),$$

which we see is upper-bounded by

$$\underbrace{\sum_{j=1}^n \mu(A_1^j) \cdots \mu(A_d^j)}_{< \mu(E) + \varepsilon} + 2^n n \varepsilon \sum_{j=1}^n \mu(A_1^j) \cdots \mu(A_d^j)$$

when the  $\varepsilon_{ij}$  are chosen to be sufficiently small, and we are flagrantly collecting terms without explanation. Regardless, sending  $\varepsilon \rightarrow 0^+$  recovers the result.

2. As an intermediate step, we note that we have the following form of outer regularity: for any  $E \in \mathcal{B}^d$  of finite measure and  $\varepsilon > 0$ , we can find an open  $U \supseteq E$  such that  $\mu(U \setminus E) < \varepsilon$ . In fact, we can even allow  $E$  to be of infinite measure: our measure is  $\sigma$ -finite, so we can write  $E = \bigcup_{n=1}^{\infty} E_n$  where  $E_n \in \mathcal{B}^d$  for each  $n$  and has finite measure. Then for any  $\varepsilon > 0$ , we find  $U_n \supseteq E_n$  with  $\mu(U_n) < \mu(E_n) + 2^{-n}\varepsilon$ , and we see

$$\bigcup_{n=1}^{\infty} (U_n \setminus E_n) \supseteq U \setminus E,$$

where  $U := \bigcup_{n=1}^{\infty} U_n$ . Now, the left-hand side has measure bounded by  $\sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon$ , as desired.

3. We now show inner regularity from outer regularity, for any  $d \geq 1$ . Fix  $E \in \mathcal{B}^d$  and some  $\varepsilon > 0$ . We use  $(-)^c$  to denote complement. Now, we are given some  $U \supseteq E^c$  such that  $\mu(U \setminus E^c) < \varepsilon$ , but  $U \setminus E^c = U \cap E = E \setminus U^c$ , so  $\mu(E \setminus U^c) < \varepsilon$ . Now, we note that the closed set  $U^c$  satisfies

$$\mu(U^c) = \sup_{\text{compact } K \subseteq U^c} \mu(K)$$

because we can set  $K_n := U^c \cap \overline{B(0, n)}$  to be compact subsets of  $U^c$  ascending to  $U^c$ , meaning  $\mu(K_n) \rightarrow \mu(U^c)$  as  $n \rightarrow \infty$ . In particular, we can find  $n$  large enough so that  $K_n \subseteq U^c$  has  $\mu(E \setminus K_n) < 2\varepsilon$ . Sending  $\varepsilon \rightarrow 0^+$  completes the proof. ■

## 1.7 January 31

Ok let's begin.

**Remark 1.56.** To produce the Lebesgue measure on  $\mathbb{R}^d$ , one can imagine completing  $(\mathbb{R}^d, \mathcal{L}^d, m^d)$  or completing  $(\mathbb{R}^d, \mathcal{B}^d, m^d)$ . Of course, one may just focus on showing that the  $\sigma$ -algebras are the same because then everything is the disjoint union of a null set and a measurable set.

For example, we note that  $\mathcal{B}^1 \subseteq \mathcal{L}^1$ , so completing makes  $\widehat{\mathcal{B}}^d \subseteq \widehat{\mathcal{L}}^d$  by construction of our completion. For the reverse inclusion, by construction of our completion, it suffices to show that  $\mathcal{L}^d \subseteq \widehat{\mathcal{B}}^d$ . Looking at these as  $\sigma$ -algebras, it suffices to show that  $\widehat{\mathcal{B}}^d$  contains measurable rectangles of  $\mathcal{L}^1$ -sets. Well, each set in  $\mathcal{L}^1$  can be written as the union of a Borel set and a null set, so we can write the needed measurable rectangle as

$$(B_1 \cup N_1) \times (B_2 \cup N_2) \times \cdots \times (B_d \cup N_d).$$

Expanding out the product, the "leading term"  $B_1 \times \cdots \times B_d$  is Borel, and then the remaining terms have null sets in them, so they are null sets. So the entire thing lives in  $\mathcal{B}^d$ .

### 1.7.1 Measuring Affine Maps

We will be interested in affine automorphisms of  $\mathbb{R}^d$ .

**Definition 1.57** (affine). An affine map  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is one which can be written as

$$f(v) := Tv + a$$

where  $T \in \text{GL}_d(\mathbb{R})$  and  $a \in \mathbb{R}^d$ .

**Remark 1.58.** Note that  $f$  is linear in its coordinates, so it is continuous. The inverse map is  $v \mapsto T^{-1}(v - a)$ , which is also affine, so  $f$  is in fact a homeomorphism and so sends open sets to open sets.

Let's quickly check that affine maps preserve Lebesgue sets.

**Lemma 1.59.** Fix a homeomorphism  $h: X \rightarrow X$ , where  $(X, \mathcal{L}, \hat{\mu})$  is a complete Borel measure space.

- (a)  $h$  sends Borel sets to Borel sets.
- (b) Suppose  $h$  sends Borel null sets to null sets. Then  $h$  preserves Lebesgue sets.

*Proof.* Here we go.

- (a) Let  $\mathcal{D}$  denote the collection of  $E \subseteq X$  such that  $h(E)$  is Borel. Well,  $h$  is an open map, so  $\mathcal{D}$  contains open sets. Further,  $\mathcal{D}$  is a  $\sigma$ -algebra because taking images preserves unions and complements because  $h$  is a bijection. Thus,  $\mathcal{D}$  contains all Borel sets.
- (b) Fix a Lebesgue set  $B \cup N$  where  $N$  is a null set. Then  $h(B \cup N) = h(B) \cup h(N)$  is the union of a Borel set  $h(B)$  (by (a)) and a null set  $h(N)$  by hypothesis. ■

**Remark 1.60.** Note that an affine map  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a homeomorphism, so (a) above tells us that Borel sets get sent to Borel sets.

We can actually measure our images pretty well.

**Proposition 1.61.** Fix an affine map  $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$  of the form  $f(v) := Tv + a$ . For a Lebesgue set  $E$ , we have

$$\mu(f(E)) = |\det T| \mu(E).$$

*Proof.* By definition, we can decompose  $f$  into a translation map  $\tau: v \mapsto v + a$  and a linear map  $T: v \mapsto Tv$ . It then suffices to check the result on translations and linear maps.

Well, for translations, we need  $\mu(E + a) = \mu(E)$  for any Lebesgue set  $E$ . It suffices to do this for Borel sets  $E$ . Letting  $\mathcal{D}$  denote the collection of Borel sets  $E$  with  $\mu(E + a) = \mu(E)$ , we note that  $\mathcal{D}$  contains all cubes (compute the measure as a product of the side lengths via, say, Theorem 1.25), so  $\mathcal{D}$  contains all open sets by Lemma 1.53. Further, we can see that  $\mathcal{D}$  is a  $\sigma$ -algebra because it is closed under unions and complements because translation is a bijection, and  $\mu$  preserves unions and complements (approximately speaking). (The complement argument needs to know that  $\mu$  is  $\sigma$ -finite.)

For linear maps, we break down our maps even more. We can write any linear map  $T$  as a composition of maps of the following kinds.

- Permutations of coordinates: for  $\sigma \in \text{Sym}(\{1, \dots, d\})$ , we have the linear map  $P_\sigma: (x_1, \dots, x_d) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(d)})$ .

- Dilation: for  $t \in \mathbb{R}^\times$ , we have the linear map  $D_t: (x_1, \dots, x_d) \mapsto (x_1, \dots, x_{d-1}, tx_d)$ .
- Skew shifts: for  $v \in \mathbb{R}^{d-1}$ , we have the linear map  $S_v: (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{d-1}, x_d + (x_1, \dots, x_{d-1}) \cdot v)$ .

Gaussian elimination shows any linear map  $T$  is the composite of maps of the above form, so it suffices to take of matrices of the above form.

Well, for permutations, one expands out the integral and exchanges integrals via Theorem 1.25. For skew shifts, a trick is required. We use Theorem 1.25 (at the start and end) in order to write

$$\begin{aligned}
 \mu(T(E)) &= \int_{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}} \mu(T(E)_{(x_1, \dots, x_{d-1})}) d\mu(x_1, \dots, x_{d-1}) \\
 &= \int_{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}} \mu(E_{(x_1, \dots, x_{d-1}) + v \cdot (x_1, \dots, x_{d-1})}) d\mu(x_1, \dots, x_{d-1}) \\
 &\stackrel{*}{=} \int_{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}} \mu(E_{(x_1, \dots, x_{d-1})}) d\mu(x_1, \dots, x_{d-1}) \\
 &= \mu(E),
 \end{aligned}$$

as needed. Notably,  $*$  used the fact that we already understand translations.

For dilations, we want to show that  $\mu(T(E)) = |t| \mu(E)$ . At  $d = 1$ , we note that the conclusion holds on open intervals by construction of the measure, so by taking finite unions, it holds on all open sets; then we can achieve the full conclusion at  $d = 1$  by using Theorem 1.55. Now, for higher dimensions, we argue as above via Theorem 1.25 to note

$$\begin{aligned}
 \mu(T(E)) &= \int_{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}} \mu(T(E)_{(x_1, \dots, x_{d-1})}) d\mu(x_1, \dots, x_{d-1}) \\
 &= \int_{(x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}} |t| \mu(E_{(x_1, \dots, x_{d-1})}) d\mu(x_1, \dots, x_{d-1}) \\
 &= |t| \mu(E),
 \end{aligned}$$

as needed. ■

**Remark 1.62.** In fact, the above proof shows that measure-zero sets go to measure-sets under affine maps. In particular, affine maps send null sets to null sets.

## 1.8 February 2

We began class by finishing a proof from last class, into which I have edited directly.

### 1.8.1 Measuring Lipschitz Functions

We would like to understand functions with more curves than affine maps.

**Definition 1.63 (Lipschitz).** Fix an open subset  $U \subseteq \mathbb{R}^d$ , and let  $f: U \rightarrow \mathbb{R}^k$  be a map. Then  $f$  is *Lipschitz* if and only if there is a finite positive real number  $A$  such that

$$|f(x) - f(y)| \leq A |x - y|$$

for all  $x, y \in U$ . We define the *Lipschitz constant*  $\|f\|_{\text{Lip}}$  to be the infimum of all such possible  $A$ .

We need to know that Lipschitz functions send Borel sets to Borel sets.

**Lemma 1.64.** Fix a Lipschitz function  $f: U \rightarrow \mathbb{R}^d$  where  $U \subseteq \mathbb{R}^d$ . Then for any Lebesgue measurable  $E \subseteq U$ , we have that  $f(E)$  is Lebesgue measurable. In fact, null sets map to null sets.

*Proof.* Certainly if  $K \subseteq U$  is compact, then  $f(K)$  is compact (and in particular closed) and hence Borel and hence Lebesgue measurable.

Lastly, let  $E$  be an arbitrary Lebesgue set. By Theorem 1.55, we get a sequence of ascending compact sets  $\{K_n\}_{n=1}^\infty$  such that  $S \cup \bigcup_{n=1}^\infty K_n = E$  where  $S$  is a null set (and  $S = \emptyset$  if  $E$  is actually Borel). Now, hitting this with  $f$ , we see that  $f(\bigcup_{n=1}^\infty K_n) = \bigcup_{n=1}^\infty f(K_n)$  is the countable union of Borel sets, which is Borel and hence Lebesgue. Note that we have so far shown that  $E$  being Borel makes  $f(E)$  measurable.

Lastly, we must show that  $f(S)$  is a null set. As an intermediate step, we claim that there is a constant  $C_f > 0$  such that

$$\mu(f(V)) \stackrel{?}{\leq} C_f \mu(V)$$

for any open  $V \subseteq U$ . To see why this is enough, note by Theorem 1.55, for any  $\varepsilon > 0$ , we can find an open  $V \supseteq S$  such that  $\mu(V) < \varepsilon$ , so  $\mu(f(S)) \leq \mu(f(V)) \leq C_f \varepsilon$ . Sending  $\varepsilon \rightarrow 0^+$  then shows  $\mu(f(S)) = 0$ .

It remains to show the claim. Well, use Lemma 1.53 to write  $V$  as the disjoint union of a countable collection of cubes  $\{Q_n\}_{n \in \mathbb{N}}$ , where  $Q_n$  has side length  $\ell_n$ . Then

$$\mu(f(V)) \leq \sum_{n \in \mathbb{N}} \mu(f(Q_n)).$$

Now, the diameter of  $Q_n$  is  $\ell_n \sqrt{d}$ , so it is contained in a ball of radius equal to half that, so  $f$  will send this ball to a ball of radius  $\frac{1}{2} \|f\|_{\text{Lip}} \ell_n \sqrt{d}$ , which is contained in a cube of side length  $\|f\|_{\text{Lip}} \ell_n \sqrt{d}$ , which has measure  $(\|f\|_{\text{Lip}} \ell_n \sqrt{d})^d$ . In total, we see

$$\mu(f(Q_n)) \leq \underbrace{\left( \|f\|_{\text{Lip}} \sqrt{d} \right)^d}_{C_f :=} \ell_n^d,$$

so by summing we have found our needed constant  $C_f$ . Importantly,  $C_f$  does not depend on  $V$  at all, so we are okay. ■

Anyway, here is our main result to change variables.

**Definition 1.65** ( $C^1$ -diffeomorphism). Fix open subsets  $U, V \subseteq \mathbb{R}^d$ . A map  $f: U \rightarrow V$  is a  $C^1$ -diffeomorphism if and only if  $f$  is bijective, and  $f$  and  $f^{-1}$  are both  $C^1$ ; i.e.,  $f$  and  $f^{-1}$  are both continuously differentiable everywhere.

**Remark 1.66.** If  $f$  is a  $C^1$ -diffeomorphism, then  $Df \circ Df^{-1} = \text{id}$ , so  $Df$  is actually invertible. In particular, the Jacobian is nonzero.

**Remark 1.67.** For example, the Inverse function theorem approximately says that a function  $f$  having invertible derivative at a point  $x$  implies that  $f$  becomes a  $C^1$ -diffeomorphism in a neighborhood of  $x$ .

**Theorem 1.68.** Fix open subsets  $U, V \subseteq \mathbb{R}^d$ , and fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$ . Then the following hold.

- (a) For each Lebesgue measurable  $E \subseteq U$ , the set  $\Phi(E)$  is still Lebesgue measurable.
- (b) For each Lebesgue measurable  $E \subseteq U$ , we have

$$\mu(\Phi(E)) = \int_U |J_\Phi(x)| \, d\mu(x),$$

where  $J_\Phi$  is the Jacobian  $\det D_\Phi$ .

- (c) For a Lebesgue measurable function  $f: V \rightarrow \mathbb{C}$ , we have

$$\int_V f \, d\mu = \int_U (f \circ \Phi) |J_\Phi| \, d\mu,$$

under hypotheses establishing the integrals make sense: either  $f$  is nonnegative, or  $f$  is in integrable, or  $(f \circ \Phi) |J_\Phi|$  is integrable.

We will prove this result next class.

## 1.9 February 5

We began class by finishing a proof from last class, into which I have edited directly.

### 1.9.1 Change of Variables

Before doing anything, here are some lemmas.

**Lemma 1.69.** Fix a compact convex subset  $K \subseteq \mathbb{R}^d$ . Suppose  $U \subseteq \mathbb{R}^d$  is an open subset containing  $K$ , and suppose  $f: U \rightarrow \mathbb{R}^n$  is continuously differentiable. Then  $f|_K$  is Lipschitz.

*Proof.* Fix  $x, y \in K$ . We would like to approximate  $f(y) - f(x)$  by  $Df(x)(y - x)$ , for which we use the Mean value theorem.

To bring our target down to one dimension, we note

$$|f(y) - f(x)| = w \cdot (f(y) - f(x)),$$

where  $w$  is the unit vector in the direction of  $f(y) - f(x)$ . In order to bring our source down to one dimension, we define

$$g(t) := w \cdot f(x + t(y - x)),$$

which is well-defined because  $K$  is convex. Note  $g$  is continuously differentiable because it is the composition of continuously differentiable functions; in particular, we find  $g'(t) = w \cdot Df(x + t(y - x))(y - x)$ . Everything is now in one dimension, so the Mean value theorem provides  $c \in (0, 1)$  such that

$$\frac{g(1) - g(0)}{1 - 0} = g'(c) = w \cdot Df(x + c(y - x))(y - x).$$

Thus, we may have the sequence of bounds

$$\begin{aligned} |f(y) - f(x)| &= |g(1) - g(0)| \\ &= |w \cdot Df(x + c(y - x))(y - x)| \\ &\leq |w| \cdot |Df(x + c(y - x))(y - x)| \\ &\leq |y - x| \cdot \max_{z \in K} \|Df(z)\|. \end{aligned}$$



Now,  $f$  is continuously differentiable, and  $K$  is compact, so  $Df$  is bounded on  $K$ . So we say that  $f$  is Lipschitz with Lipschitz constant bounded above by  $\max_{z \in K} \|Df(z)\|$ . ■

Now, the main content of Theorem 1.68 is in the following result.

**Lemma 1.70.** Fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$  where  $U, V \subseteq \mathbb{R}^d$  is open. Further, let  $U'$  be an open subset with compact closure  $\overline{U'} \subseteq U$  such that there is  $\delta_0 > 0$  such that any  $x \in U'$  has  $d(x, U^c) \geq 2\delta_0$ . Then we have

$$\mu(\varphi(U')) \leq \int_{U'} |J_\Phi| \, d\mu.$$

**Remark 1.71.** Here,  $d(x, U^c)$  denotes the (infimum of the) distance from  $x$  to the set  $U^c$ . Because  $\overline{U'}$  is compact, we note that we also get  $d(x, U^c) \geq 2\delta_0$  for each  $x \in \overline{U'}$  because  $x \mapsto d(x, U^c)$  is a continuous function.

To prove the above lemma, we will want the following lemma.

**Lemma 1.72.** Fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$  where  $U, V \subseteq \mathbb{R}^d$  is open. Further, let  $U' \subseteq U$  be an open subset with compact closure such that there is  $\delta > 0$  such that any  $x \in U'$  has  $d(x, U^c) \geq 2\delta$ . Then there is a decreasing “remainder” function  $R$  such that  $R(t) \rightarrow 0$  as  $t \rightarrow 0^+$  while the “Taylor remainder”

$$\mathcal{R}(x, u) := \Phi(x + u) - \Phi(x) - D\Phi(x)(u)$$

satisfies  $|\mathcal{R}(x, u)| \leq |u| R(|u|)$  for all  $x \in U$  and  $|u| \leq \delta$ .

The content here is that the remainder function  $\mathcal{R}$  (which ought to go to zero as  $u \rightarrow 0$ ) is bounded in a way that does not depend on  $x$ .

*Proof of Lemma 1.72.* For  $u$  such that  $|u| \leq \delta$ , the construction of  $\delta$  implies that any  $x \in U'$  has  $x + u \in U$ ; in fact, we have  $d(y, U^c) \geq \delta$  for all  $y$  of the form  $x + u$ . Notably,

$$\{y \in U : d(y, U^c) \geq \delta\}$$

is compact: it is closed because  $y \mapsto d(y, U^c)$  is continuous, and it is bounded because it is a fixed distance away from the compact set  $\overline{U'}$ .

We now use the Mean value theorem to conclude. The proof is similar to Lemma 1.69. Fix  $x$  and  $u$  as in the statement. Then there is a unit vector  $w$  such that

$$w \cdot \mathcal{R}(x, u) = |\mathcal{R}(x, u)|.$$

This allows us to define

$$g(t) := w \cdot \mathcal{R}(x, tu),$$

which is now a function  $g: [0, 1] \rightarrow \mathbb{R}$  which is continuously differentiable because everything in sight is continuously differentiable. As such, the Mean value theorem provides  $c \in (0, 1)$  such that

$$g(1) - g(0) = g'(c) = w \cdot (D\Phi(x + cu)(u) - D\Phi(x)(u)).$$

Thus,

$$\begin{aligned} |\mathcal{R}(x, u)| &= |g(1) - g(0)| \\ &\leq |w| \cdot |D\Phi(x + cu)(u) - D\Phi(x)(u)| \\ &\leq \max_{c \in [0, 1]} |D\Phi(x + cu)(u) - D\Phi(x)(u)| \\ &\leq |u| \cdot \max_{\substack{x' \in \overline{U'} \\ |u'| \leq |u|}} \|D\Phi(x' + u') - D\Phi(x')\|. \end{aligned}$$

The right-hand factor is our function  $R$  in terms of  $|u|$ , which is finite because we are taking the maximum of a continuous function on a compact set. Technically, we have not shown that  $R$  is decreasing, but we can make it decreasing by replacing  $R$  with  $t \mapsto \sup\{R(s) : s \geq t\}$ . ■

## 1.10 February 7

I have moved a proof to today's notes for continuity reasons.

### 1.10.1 More on Change of Variables

We are now ready to show Lemma 1.70.

**Lemma 1.70.** Fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$  where  $U, V \subseteq \mathbb{R}^d$  is open. Further, let  $U'$  be an open subset with compact closure  $\overline{U'} \subseteq U$  such that there is  $\delta_0 > 0$  such that any  $x \in U'$  has  $d(x, U^c) \geq 2\delta_0$ . Then we have

$$\mu(\varphi(U')) \leq \int_{U'} |J_\Phi| \, d\mu.$$

*Proof.* The point is to make everything as linear as possible. Fix  $\delta > 0$  less than  $\delta_0$ , which we will eventually send to  $0^+$ . Now, by Lemma 1.53, we may divide  $U$  into countably many dyadic cubes  $\{Q_j\}_{j \in \mathbb{N}}$ , where  $Q_j$  has side-length  $\ell_j$ ; by possibly decomposing cubes finitely, we may assume that  $\ell_j \leq \delta$  for each  $j \in \mathbb{N}$ . For our combinatorics, we let  $c_j \in Q_j$  be the center of the cube so that any point in  $Q_j$  can be written as  $c_j + u$  where  $u = (u_1, \dots, u_d)$  has  $|u_i| \leq \frac{1}{2}\ell_j$  for each  $i$ .

Now, by Lemma 1.72, we may upper-bound

$$\Phi(c_j + u) = \Phi(c_j) + D\Phi(c_j)(u) + |u| R(|u|)$$

where  $R$  is conjured from Lemma 1.72. For our bounding, it will help to recognize that

$$|u| R(|u|) \leq \ell_j \sqrt{d} R(\ell_j \sqrt{d}) \leq \sqrt{d} R(\delta \sqrt{d}) \ell_j.$$

To chart our progress, we note

$$\mu(\Phi(U')) \leq \sum_{j \in \mathbb{N}} \mu(\Phi(Q_j)),$$

and now  $\Phi$  on  $Q_j$  is basically linear.

Indeed, set  $T_j := D\Phi(c_j)$ , and for  $x = c_j + u$  in  $Q_j$ , we see

$$\underbrace{|\Phi(x) - \Phi(c_j) - T_j(u)|}_{\mathcal{R}(c_j, u) :=} \leq |u| R(|u|)$$

for some remainder function  $R$  provided by Lemma 1.72. Now, to bound  $\mu(\Phi(Q_j))$ , we see that  $\Phi(Q_j) \approx \Phi(c_j) + T_j(Q_j(0, \ell_j))$ , where  $Q$  denotes the cube. Notably, we have some identity like  $\Phi(c_j + u) = \Phi(c_j) + T_j(u + T_j^{-1}\mathcal{R}(c_j, u))$ , so we would like to bound  $T_j^{-1}\mathcal{R}(c_j, u)$ , which we do as

$$|T_j^{-1}\mathcal{R}(c_j, u)| \leq \|T_j\|^{-1} \cdot |u| \cdot R(|u|) \leq \|D\Phi(c_j)\|^{-1} \cdot \ell_j \sqrt{d} \cdot R(\delta \sqrt{d}).$$

Quickly, note that the function  $x \mapsto D\Phi(x)^{-1}$  is continuous (on  $U$ ) because  $\Phi$  is continuously differentiable, so with  $\overline{U'} \subseteq U$  is compact, we see that the function  $x \mapsto D\Phi(x)^{-1}$  is upper-bounded on  $U$  by some  $C$  (which does not depend on  $\delta$ !). So our bound becomes

$$|T_j^{-1}\mathcal{R}(c_j, u)| \leq C \ell_j \sqrt{d} R(\delta \sqrt{d}).$$

For brevity, set  $\varepsilon_\delta := C \sqrt{d} R(\delta \sqrt{d})$ . The point is that

$$u + T_j^{-1}\mathcal{R}(c_j, u) \in Q(0, \ell_j(1 + \varepsilon_\delta)),$$

so

$$\Phi(Q_j) \subseteq \Phi(c_j) + D\Phi(c_j)(Q(0, \ell_j(1 + \varepsilon_\delta))),$$

so

$$\mu(\Phi(Q_j)) \leq |J_\Phi(c_j)| \underbrace{\ell_j^d}_{\mu(Q_j)} (1 + \varepsilon_\delta)^d$$

by taking the measure of a cube under a linear transformation, so by summing over  $j$ , we achieve

$$\mu(\Phi(U')) \leq (1 + \varepsilon_\delta)^d \sum_{j \in \mathbb{N}} |J_\Phi(c_j)| \mu(Q_j).$$

It remains to relate the summation to the integral. Well, write

$$|J_\Phi(c_j)| \mu(Q_j) \leq \int_{Q_j} |J_\Phi(x)| d\mu(x) + \int_{Q_j} ||J_\Phi(x)| - |J_\Phi(c_j)|| d\mu(x),$$

and we see that we would like for  $|J_\Phi(x) - J_\Phi(c_j)|$  to be small. It will turn out to be small, so we can upper-bound it by the maximum over any pairs  $(x, y)$  with distance  $\delta\sqrt{d}$  apart, so we sum over all cubes to achieve

$$\mu(\Phi(U')) \leq (1 + \varepsilon_\delta)^d \left( \int_{U'} |J_\Phi| d\mu + \sup_{|x-y| \leq \delta\sqrt{d}} ||J_\Phi(x)| - |J_\Phi(c_j)|| \sum_{j \in \mathbb{N}} \mu(Q_j) \right).$$

Sending  $\delta \rightarrow 0^+$  will send  $\varepsilon_\delta \rightarrow 0^+$  and the supremum to 1 because that continuous function is uniformly continuous on the compact set  $\overline{U'} \subseteq U$ . ■

We now upgrade the lemma in various ways.

**Lemma 1.73.** Fix everything as in Lemma 1.70. For any Lebesgue measurable subset  $E \subseteq U'$ , we have

$$\mu(\Phi(E)) \leq \int_E |J_\Phi| d\mu.$$

*Proof.* Fix  $\varepsilon > 0$ , and regularity in Theorem 1.55 promises some open  $U'' \supseteq E$  contained in  $U'$  with  $\mu(U'' \setminus E) < \varepsilon$ . Then we rudely replace  $E$  with  $U''$  and apply Lemma 1.70: note

$$\mu(\Phi(E)) \leq \mu(\Phi(U'')) \leq \int_{U''} |J_\Phi| d\mu = \int_E |J_\Phi| d\mu + \int_{U'' \setminus E} |J_\Phi| d\mu.$$

So it remains to bound

$$\int_{U'' \setminus E} |J_\Phi| d\mu \leq \sup_{x \in \overline{U'}} |J_\Phi| \underbrace{\mu(U'' \setminus E)}_{< \varepsilon},$$

which we see goes to 0 as  $\varepsilon \rightarrow 0^+$ . Note that the supremum exists because  $\overline{U'}$  is compact. ■

We now upgrade away the  $U'$  entirely, which is mostly point-set topology.

**Lemma 1.74.** Fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$  where  $U, V \subseteq \mathbb{R}^d$  is open. For any Lebesgue measurable  $E \subseteq U$ , we have

$$\mu(\Phi(E)) \leq \int_E |J_\Phi| d\mu.$$

*Proof.* Fix a nonempty open subset  $U \subseteq \mathbb{R}^d$ . For each  $n \in \mathbb{N}$ , define  $U_n \subseteq U$  to consist of the  $x \in U$  such that  $|x| < n$  and  $\text{dist}(x, U^c) < 1/n$ . Then  $\overline{U_n} \subseteq U$  and has a distance at most  $1/n$  living inside  $U$ . So we may apply Lemma 1.73: set  $E_n := E \cap U_n$ , and we see

$$\mu(\Phi(E_n)) \leq \int_{E_n} |J_\Phi| d\mu \leq \int_E |J_\Phi| d\mu$$

for each  $n$ , and then sending  $n \rightarrow \infty$  has  $E = \bigcup_{n \in \mathbb{N}} E_n$  and so  $\mu(\Phi(E_n)) \rightarrow \mu(\Phi(E))$ , so we achieve the result. ■

We now upgrade from sets to functions.

**Lemma 1.75.** Fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$  where  $U, V \subseteq \mathbb{R}^d$  is open. Given a measurable function  $f: V \rightarrow [0, \infty]$ , we have

$$\int_V f d\mu \leq \int_U (f \circ \Phi) |J_\Phi| d\mu.$$

*Proof.* In the case where  $f = 1_E$  for a Lebesgue measurable  $E$ , this result is just Lemma 1.74. Taking linear combinations and approximating below achieves the result for general measurable functions  $f: V \rightarrow [0, \infty]$  via the Monotone convergence theorem [Elb22, Theorem 9.18]. ■

At long last, we produce equality.

**Lemma 1.76.** Fix a  $C^1$ -diffeomorphism  $\Phi: U \rightarrow V$  where  $U, V \subseteq \mathbb{R}^d$  is open. Given a measurable function  $f: V \rightarrow [0, \infty]$ , we have

$$\int_V f d\mu = \int_U (f \circ \Phi) |J_\Phi| d\mu.$$

*Proof.* Note  $\leq$  follows immediately from Lemma 1.75. For the other inequality, we set  $\Psi := \Phi^{-1}$  and  $g := (f \circ \Phi) |J_\Phi|$  so that  $f := |J_\Phi \circ \Phi^{-1}| (g \circ \Phi^{-1})$  and reapply Lemma 1.75 to  $g$ . ■

## THEME 2

# BANACH SPACES

---

### 2.1 February 9

Today we move on to talk about Banach spaces. Throughout,  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  will be a field.

#### 2.1.1 Banach Spaces

We begin with the definition of a normed vector space.

**Definition 2.1 (norm).** Fix an  $\mathbb{F}$ -vector space  $V$ . Then a *norm* on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following conditions.

- Positive-definite:  $\|x\| = 0$  if and only if  $x = 0$ .
- Homogeneous: for  $\lambda \in \mathbb{F}$ , one had  $\|\lambda x\| = |\lambda| \cdot \|x\|$ .
- Triangle inequality:  $\|x + y\| \leq \|x\| + \|y\|$ .

We might say that the pair  $(V, \|\cdot\|)$  is a *normed vector space*.

**Notation 2.2.** In the sequel, we may just say that  $V$  is a normed vector space, where we implicitly equip  $V$  with the norm  $\|\cdot\|_V$ .

**Remark 2.3.** Given a normed vector space  $(V, \|\cdot\|)$ . Then

$$d(x, y) := \|x - y\|$$

makes  $V$  into a metric space. Namely,  $d$  is a metric.

**Definition 2.4 (Banach space).** A normed vector space  $(V, \|\cdot\|)$  is a *Banach space* if and only if  $V$  is complete as a metric space.

**Remark 2.5.** Fix a normed vector space  $(V, \|\cdot\|)$ . Then we claim  $|\|x\| - \|y\|| \leq \|x - y\|$ . Indeed, the triangle inequality implies  $\|x\| \leq \|x - y\| + \|y\|$ , so

$$\|x\| - \|y\| \leq \|x - y\|.$$

Reversing  $x$  and  $y$  shows that  $\|y\| - \|x\| \leq \|x - y\|$  as well, so the claim follows.

We will also want a notion of convergence.

**Definition 2.6 (absolute convergence).** Fix a normed vector space  $(V, \|\cdot\|)$ . Then a sum  $\sum_{i=1}^{\infty} v_i$  for vectors  $\{v_i\}_{i=1}^{\infty} \subseteq V$  is absolutely convergent if and only if

$$\sum_{i=1}^{\infty} \|v_i\| < \infty.$$

This produces the following test of convergence.

**Lemma 2.7.** Fix a normed vector space  $(V, \|\cdot\|)$ . Then  $X$  is complete if and only if any absolutely convergent series converges (in  $V$ ).

*Proof.* In one direction, any absolutely convergent series has partial sums which are Cauchy (by the absolute convergence), so it will converge in  $V$ .

In the other direction, suppose absolutely convergent series converge. Then suppose  $\{v_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $V$ , which we would like to converge.

**Remark 2.8.** Here is an attempt which doesn't work: we might want to, define  $w_n := v_n - v_{n-1}$  (take  $w_0 = 0$ ), and we see that  $\sum_{k=1}^n w_k = v_n$ . So by hypothesis, it is enough to check that this series is absolutely convergent

$$\sum_{k=1}^{\infty} \|w_k\| < \infty,$$

but there is no reason to be true.

So we need to control how fast our series converges. Construct a strictly increasing sequence  $\{N_k\}_{k=1}^{\infty}$  such that  $m, n \geq N_k$  implies that  $\|v_m - v_n\| < 2^{-k}$ ; for this sequence to be strictly increasing, it must be defined recursively. Thus, we define  $w_k := v_{N_k}$  to be a subsequence of the  $v_n$ , and it continues to be Cauchy. In fact,  $\|w_k - w_{k-1}\| < 2^{-k-1}$  for each  $k$ , so the series

$$v_{N_n} := \sum_{k=1}^n w_k - w_{k-1}$$

is absolutely convergent. So a subsequence of  $\{v_n\}_{n \in \mathbb{N}}$  converges, so our actual sequence converges by using the Cauchy condition. ■

**Remark 2.9.** Of course, there are convergent series which are not absolutely convergent: work in  $\mathbb{R}$  with the usual norm, and take  $v_n := (-1)^n/n$  for each  $n \geq 1$ .

## 2.1.2 A Little Linear Algebra

We will want the notion of a basis, which we now build.

**Definition 2.10** (linearly independent). Fix a subset  $S$  of a vector space  $V$  if and only if, for any  $n \geq 1$  and distinct elements  $\{v_1, \dots, v_n\} \subseteq S$  has

$$a_1 v_1 + \dots + a_n v_n = 0$$

implies  $a_1 = \dots = a_n = 0$ .

**Definition 2.11** (finite dimensional). A vector space  $V$  is *finite-dimensional* if and only if there is a finite subset  $S \subseteq V$  such that any element of  $V$  can be written as a linear combination of elements of  $S$ .

**Example 2.12.** Let  $\ell^\infty$  denote the set of bounded infinite sequences in  $\mathbb{F}$ , and we give it the norm

$$\|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n|.$$

Then indicators  $e_n \in \ell^\infty$  given by  $(e_n)_i := 1_{n=i}$  are linearly independent.

It will turn out that subspaces of finite-dimensional normed vector spaces are closed, but this is not the case in general.

**Example 2.13.** Continue with  $\ell^\infty$  with the norm  $\|\cdot\|_\infty$ . Consider  $V$  to be the set of finitely supported sequences, which we can see is a subspace. However,  $\overline{V}$  contains the sequence  $v$  defined by  $v_n := 1/n$ . Indeed, for any  $\varepsilon > 0$ , we can find  $N > 1/\varepsilon$  and then note that we can define  $v' \in V$  by

$$v'_n := \begin{cases} 1/n & \text{if } n \leq N, \\ 0 & \text{if } n > N. \end{cases}$$

Then  $\|v_n - v'_n\| < \varepsilon$ . So we see  $v \in \overline{V} \setminus V$ .

But some parts of topology still make sense.

**Remark 2.14.** Fix a normed vector space  $(V, \|\cdot\|)$ . Then the open ball  $B(0, 1)$  is still open in  $V$  because it is just an open ball in the usual metric topology.

## 2.2 February 12

Let's talk about Banach spaces a little more.

### 2.2.1 Linear Maps

With our newfound topology, we want to control our linear maps.

**Definition 2.15** (bounded). Fix normed vector spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$ . Then a linear map  $T: X \rightarrow Y$  is *bounded* if and only if there is a finite constant  $C$  such that

$$\|Tx\|_Y \leq C \|x\|_X.$$

We let  $\mathcal{L}(X, Y)$  denote the space of bounded linear maps, and we define

$$\|T\|_{\mathcal{L}(X, Y)} := \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}.$$

**Remark 2.16.** We note  $\|\cdot\|_{\mathcal{L}(X,Y)}$  is in fact a norm. Homogeneity has little content, and positive-definiteness holds because we are looking at a sup: if  $\|T\|_{\mathcal{L}(X,Y)} = 0$ , then we must have  $\|Tx\|_Y = 0$  always, so  $Tx = 0$  always. Lastly, for the triangle inequality, we note

$$\|T + S\|_{\mathcal{L}(X,Y)} = \sup_{x \in X \setminus \{0\}} \frac{\|Tx + Sx\|_Y}{\|x\|_X} \leq \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} + \sup_{x \in X \setminus \{0\}} \frac{\|Sx\|_Y}{\|x\|_X} = \|T\|_{\mathcal{L}(X,Y)} + \|S\|_{\mathcal{L}(X,Y)}.$$

Here is the topological reason that we care.

**Lemma 2.17.** Fix normed vector spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$ . Then  $T: X \rightarrow Y$  is bounded if and only if continuous.

*Proof.* For the forward direction, suppose  $T$  is bounded with constant  $C > 0$ . Then for any  $\varepsilon > 0$ , choose  $\delta := \varepsilon/C$ : for any  $x, y \in X$ , we find that  $\|x - y\|_X < \delta$  implies that

$$\|Tx - Ty\|_Y \leq C \|x - y\|_X < C\delta = \varepsilon,$$

so in fact  $T$  is uniformly continuous.

For the reverse direction, we will only use continuity of  $T$  at 0. Then there exists  $\delta > 0$  such that  $\|x\|_X < \delta$  implies that  $\|Tx\|_Y < 1$ , so any nonzero vector  $x' \in X$  has

$$\frac{\|Tx'\|_Y}{\|x'\|_X} = \frac{\left\|T\left(\frac{\delta}{2\|x'\|_X}x'\right)\right\|_Y}{\left\|\frac{\delta}{2\|x'\|_X}x'\right\|_X} < \frac{1}{\delta/2} = \frac{2}{\delta}.$$

So we have a bound  $\|T\|_{\mathcal{L}(X,Y)} < 2/\delta$ , so  $T$  is bounded. ■

## 2.2.2 Equivalence of Norms

Let's try to classify our norms. Multiplying a norm by a scalar ought to be considered the same norm; generalizing this slightly produces the following definition.

**Definition 2.18 (equivalent).** Fix a vector space  $X$ . Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are *equivalent* if and only if there is a constant  $C > 1$  such that

$$C^{-1} \leq \frac{\|x\|_1}{\|x\|_2} \leq C$$

for any nonzero  $x \in X$ .

**Remark 2.19.** Equivalence as above is in fact an equivalence relation. Reflexivity has no content (take  $C = 1$ ), symmetry follows by taking the reciprocal of the given equation, and transitivity follows by multiplying the two constants together.

**Example 2.20.** The two norms  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are equivalent on any finite-dimensional space  $\mathbb{R}^d$ . Namely, we see that

$$\|x\|_\infty \leq \|x\|_2 = \sqrt{\sum_{k=1}^d x_k^2} \leq \sqrt{d} \|x\|_\infty,$$

so equivalence follows.

Here is our main result on equivalence.



**Proposition 2.21.** Fix a finite-dimensional vector space  $X$ . Then any two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $X$  are equivalent.

*Proof.* We will show that any norm  $\|\cdot\|_X$  on  $X$  is equivalent to a given one. Fix a basis  $\{v_1, \dots, v_n\}$  for our finite-dimensional vector space  $X$ . This basis defines an isomorphism  $\varphi: \mathbb{F}^n \rightarrow X$  given by  $(a_1, \dots, a_n) \mapsto (a_1 v_1 + \dots + a_n v_n)$ . We now define  $\|\cdot\|_\varphi$  to be a norm on  $\mathbb{F}^n$  defined by

$$\|x\|_\varphi := \left\| \sum_{i=1}^n x_i v_i \right\|.$$

We won't bother to check that  $\|x\|_\varphi$  is a norm, which hold by passing the norm checks through the isomorphism  $\varphi$  of vector spaces.

We claim that  $\|\cdot\|_\infty$  is continuous. Well, we compute

$$\|x\|_\varphi = \left\| \sum_{i=1}^n x_i v_i \right\| \leq \sum_{i=1}^n |x_i| \cdot \|v_i\| \leq \|x\|_\infty \underbrace{\sum_{i=1}^n \|v_i\|}_{A:=}$$

Thus, we see that  $\|\cdot\|_\varphi: \mathbb{F}^n \rightarrow \mathbb{R}_{\geq 0}$  is continuous: in fact,  $\|\cdot\|_\varphi$  is Lipschitz continuous, where we compute

$$\left| \|x\|_\varphi - \|x'\|_\varphi \right| \leq \|\varphi(x) - \varphi(x')\| = F(x - x') \leq A \|x - x'\|_\infty,$$

so  $\|\cdot\|_\varphi$  is in fact continuous.

Thus, by continuity, upon restricting  $\|\cdot\|_\varphi$  to the unit sphere  $S^{n-1}$ , we see that  $\|\cdot\|_\varphi$  will achieve its minimum. But 0 is never achieved on  $S^{n-1}$ , so let  $\delta$  be the minimum and find that

$$\delta \leq \frac{\|x\|_\varphi}{\|x\|_\infty} \leq A$$

for any  $x \in S^{n-1}$ , an inequality which extends to any nonzero  $x \in X$  by scaling. Setting  $C := \max\{\delta^{-1}, A\}$  to deduce that  $\|\cdot\|$  is equivalent to  $\|\cdot\|_\varphi$ . Pushing through the isomorphism  $\varphi$ , we see that

$$C^{-1} \leq \frac{\|x\|}{\|\varphi^{-1}(x)\|_\infty} \leq C$$

for any nonzero  $x \in X$ . Thus,  $\|\cdot\|$  is equivalent to the norm  $x \mapsto \|\varphi^{-1}(x)\|_\infty$ , but this latter norm is entirely independent of  $\|\cdot\|$ , so we have indeed shown that any norm on  $X$  is equivalent to a fixed one. ■

### 2.2.3 Topology on Normed Vector Spaces

Let's see some corollaries of Proposition 2.21.

**Corollary 2.22.** Fix a finite-dimensional normed vector space  $(X, \|\cdot\|)$ . Then  $X$  is complete.

*Proof.* Choose a basis, so we may let  $\varphi: \mathbb{F}^n \rightarrow X$  be some isomorphism of vector spaces. Then  $\|\cdot\|$  is equivalent to the norm  $x \mapsto \|\varphi^{-1}(x)\|_\infty$  by Proposition 2.21, so  $\varphi$  upgrades to a homeomorphism. But then completeness of  $X$  follows from completeness of  $\mathbb{F}^n$ . ■

**Corollary 2.23.** Fix a normed vector space  $(X, \|\cdot\|)$ . Then any finite-dimensional subspace  $V \subseteq X$  is closed.

*Proof.* We know that  $V$  is complete by Corollary 2.22. This is enough to see that  $V$  is closed: any convergent sequence  $x_n \rightarrow x$  where  $\{x_n\}_{n \in \mathbb{N}} \subseteq V$  will be Cauchy, but completeness of  $V$  implies that this Cauchy sequence has a limit  $x' \in V$ , but we must have  $x = x'$  by the uniqueness of limits. ■

## 2.3 February 14

Let's talk about Banach spaces a little more.

### 2.3.1 Closed Unit Balls

We may want our closed balls to be compact, but we are out of luck.

**Proposition 2.24.** Fix a normed vector space  $(X, \|\cdot\|)$ . Then the closed unit ball

$$B := \{x \in X : \|x\| \leq 1\}$$

is compact if and only if  $X$  is finite-dimensional.

To prove this, we will want a lemma.

**Definition 2.25.** Fix a normed vector space  $(X, \|\cdot\|)$ . For  $x \in X$  and subspace  $V \subseteq X$ , we define

$$d(x, V) := \inf_{v \in V} \|x - v\|.$$

Note that this exists and is finite because  $V$  is nonempty (for example,  $0 \in V$ ).

One of course has the bound  $d(x, V) \leq \|x\|$  because  $0 \in V$ . One cannot hope to do much better than this without explosion.

**Lemma 2.26.** Fix a normed vector space  $(X, \|\cdot\|)$ . Suppose that each  $x \in X$  has  $d(x, V) \leq \frac{1}{2} \|x\|$ . Then  $\bar{V} = X$ .

Let's see how this lemma implies Proposition 2.24.

*Proof of Proposition 2.24 from Lemma 2.26.* We will show that either  $B$  fails to be compact or  $X$  is not finite-dimensional. If  $X = 0$ , there is nothing to do; otherwise, for example, there is a nonzero vector, so scaling provides us with a unit vector.

We proceed with the following inductive process. Begin with some unit vector  $x_1 \in X$ . We then look for unit vectors  $x_2$  such that  $d(x_2, \text{span}\{x_1\}) \geq \frac{1}{2}$ ; if no such vector exists, we terminate our process. We then continue this process inductively to produce a set  $B$ : given the finite set  $\{x_1, \dots, x_n\}$ , we look for a unit vector  $x_{n+1}$  such that

$$d(x_{n+1}, \text{span}\{x_1, \dots, x_n\}) \geq \frac{1}{2}.$$

If no such vector exists, then the process terminates; otherwise, we add  $x_{n+1}$  to our finite set and continue. We have two cases.

- If we ever terminate, then we trigger Lemma 2.26: we know there is a finite set  $\{x_1, \dots, x_n\}$  of unit vectors such that every unit vector  $x$  has  $d(x, V) < 1/2$  where  $V := \text{span}\{x_1, \dots, x_n\}$ . Scaling, we actually know that

$$d(x, V) < \frac{1}{2} \|x\|$$

for any  $x \in X$ : for nonzero  $x$ , we simply divide both sides of the above inequality by  $\|x\|$ . So we trigger Lemma 2.26, so  $\bar{V} = X$ . But  $V$  is finite-dimensional by construction, so  $V \subseteq X$  is closed by Corollary 2.23, so  $X = \bar{V} = V$  is finite-dimensional.

- Otherwise, we have an infinite set  $\{x_1, x_2, \dots\}$  of unit vectors such that  $\|x_i - x_j\| \geq \frac{1}{2}$  for any indices  $i < j$ . In particular,  $B$  has a sequence with no convergent subsequence, so  $B$  fails to be compact. ■

We now prove Lemma 2.26.

*Proof of Lemma 2.26.* Fix  $x \in X$ . We want to show that  $d(x, V) = 0$ . In other words, for any  $\varepsilon > 0$ , we want  $y \in V$  such that  $\|x - y\| < \varepsilon$ .

Well, suppose for the sake of contradiction that we have  $\varepsilon > 0$  such that no  $y \in V$  achieves  $\|x - y\| < \varepsilon$ , and we may assume that  $\varepsilon := \inf\{\|x - y\| : y \in V\}$  is as small as possible. In particular, surely there will be some  $y_1 \in V$  such that  $\|x - y_1\| < \frac{3}{2}\varepsilon$ , but then we can find  $y_2 \in V$  such that

$$\|x - y_1 - y_2\| \leq \frac{1}{2} \|x - y_1\|,$$

so

$$\|x - (y_1 + y_2)\| \leq \frac{1}{2} \|x - y_1\| \leq \frac{3}{4}\varepsilon,$$

which is a contradiction to the construction of  $\varepsilon$ . ■

### 2.3.2 Functionals

It will be helpful to study linear maps to the ground field.

**Definition 2.27 (functional).** Fix a normed vector space  $(X, \|\cdot\|)$ . Then a *bounded linear functional* is an element of  $X^* := \mathcal{L}(X, \mathbb{F})$ .

**Example 2.28.** Certainly  $X^*$  is nonempty because it has the zero element.

However, it is rather hard to get anything else in  $X^*$ .

**Example 2.29.** If  $X$  is finite-dimensional, then choose a basis  $\{v_1, \dots, v_n\}$ . Then there are functionals

$$\sum_{i=1}^n a_i v_i \mapsto \sum_{i=1}^n c_i a_i$$

for any  $(c_1, \dots, c_n) \in \mathbb{F}^n$ .

It will turn out that we can build lots of functionals, but it is not so obvious how to do this. We will have the following results.

**Corollary 2.30.** Fix a normed vector space  $(X, \|\cdot\|)$ . For any  $x \in X$ , there is a bounded linear functional  $\ell: X \rightarrow \mathbb{F}$  such that  $\ell(x) = \|x\|$  and  $\|\ell\|_{X^*} = 1$ .

**Remark 2.31.** Certainly  $\ell(x) = \|x\|$  forces  $\|\ell\|_{X^*} \geq 1$ , so we are saying that  $\ell$  is small away from  $x$ .

**Corollary 2.32.** Fix a normed vector space  $(X, \|\cdot\|)$ . Given distinct  $x, y \in X$ , there is a bounded linear functional  $\ell: X \rightarrow \mathbb{F}$  such that  $\ell(x) \neq \ell(y)$ .

These results will arise as corollaries of the following result.

**Theorem 2.33 (Hahn–Banach).** Fix a normed vector space  $(X, \|\cdot\|)$ , and let  $V \subseteq X$  be a subspace. Given a bounded linear functional  $\ell: V \rightarrow \mathbb{F}$ , there is a bounded linear functional  $L: X \rightarrow \mathbb{F}$  such that  $L|_V = \ell$  and  $\|L\| = \|\ell\|_{V^*}$ .

One even has the following extension for  $\mathbb{R}$ .

**Definition 2.34.** Fix an  $\mathbb{R}$ -vector space  $X$ . A function  $p: X \rightarrow \mathbb{R}$  is a *sublinear functional* if and only if  $p(x + y) \leq p(x) + p(y)$  and  $p(tx) = tp(x)$  for  $t > 0$ .

**Example 2.35.** Fix a normed vector space  $(X, \|\cdot\|)$ , and let  $K \subseteq X$  be a closed convex subset, and suppose that  $K$  contains  $B(0, \varepsilon)$  for some  $\varepsilon > 0$ . Then we define

$$p(x) := \inf \left\{ t > 0 : \frac{1}{t}x \in K \right\}.$$

It turns out that  $p$  is a sublinear functional. Notably, if  $K$  is not symmetric, then  $p(x)$  need not equal  $p(-x)$ , so  $p$  need not be a norm.

**Theorem 2.36 (Hahn–Banach).** Fix an  $\mathbb{R}$ -vector space  $X$  equipped with sublinear functional  $p$ , and let  $V \subseteq X$  be a subspace. Given a linear functional  $\ell: V \rightarrow \mathbb{R}$  such that  $\ell \leq p$  pointwise, there is a bounded linear functional  $L: X \rightarrow \mathbb{R}$  such that  $L|_V = \ell$  and  $L \leq p$  pointwise.

## 2.4 February 16

Here we go.

### 2.4.1 The Hahn–Banach Theorem

Our proofs of Theorems 2.33 and 2.36 will be by transfinite induction on  $V$ . Let's state the successor step.

**Lemma 2.37.** Fix an  $\mathbb{R}$ -vector space  $X$  equipped with sublinear functional  $p$ , and let  $V \subseteq X$  be a subspace. Given a linear functional  $\ell: V \rightarrow \mathbb{R}$  such that  $\ell \leq p$  pointwise and some  $x \notin V$ , there exists some linear  $\tilde{\ell}: (V + \mathbb{R}x) \rightarrow \mathbb{R}$  such that  $\tilde{\ell}|_V = \ell$  and  $\tilde{\ell} \leq p$  pointwise.

*Proof.* Set  $\tilde{V} := V + \mathbb{R}x$ . Note that decomposition of an element of  $\tilde{V}$  into  $v + rx$  where  $v \in V$  and  $r \in \mathbb{R}$  is unique: if  $v + rx = v' + r'x$ , then  $(r - r')x = v' - v \in V$ , so  $r - r' = 0$ , so  $v' - v = 0$ . So we simply define

$$\tilde{\ell}(v + rx) := \ell(v) + r\alpha$$

for some  $\alpha$  to be determined later. For example,  $\tilde{\ell}$  is certainly linear by construction, and it restricts down to  $\ell$  on  $V$ . So we are looking for  $\alpha$  such that

$$t\alpha + \ell(v) \leq p(tx + v) \quad \text{and} \quad -t\alpha + \ell(v) \leq p(-tx + v)$$

for any  $t \geq 0$  and  $v \in V$ . Certainly we may assume that  $t > 0$  because  $t = 0$  reduces down to  $\ell$ , but then we can divide everything in sight by  $t$  so that it suffices for

$$\alpha + \ell(v) \leq p(x + v) \quad \text{and} \quad -\alpha + \ell(v) \leq p(-x + v)$$

for any  $v \in V$ . This rearranges to

$$\sup_{v \in V} (\ell(v) - p(-x + v)) \leq \alpha \leq \inf_{v \in V} (p(x + v) - \ell(v)).$$

So we can find the needed  $\alpha$  if and only if

$$\sup_{v \in V} (\ell(v) - p(-x + v)) \leq \inf_{v \in V} (p(x + v) - \ell(v)).$$

Fixing some vectors explicitly, it is enough to check that

$$\ell(u + v) = \ell(v) + \ell(u) \leq p(x + v) + p(-x + u)$$

for any  $v, u \in V$ . In particular, by hypothesis of  $\ell$ , it is enough for  $p(u + v) \leq p(x + v) + p(-x + u)$ , but this follows from sublinearity of  $p$  by writing  $u + v = (x + v) + (-x + u)$ . ■

**Remark 2.38.** Replacing  $p$  with an actual norm  $\|\cdot\|$  on  $X$  allows us to allow  $X$  to even be a  $\mathbb{C}$ -vector space, simply by repeating the proof verbatim.

We would now like to use Zorn's lemma to upgrade Lemma 2.37 to Theorem 2.36. We begin by stating Zorn's lemma in the form that we will use. This requires the notion of a partial order.

**Definition 2.39 (partial order).** A *partial order* on a set  $S$  is a binary relation  $\leq$  satisfying the following.

- Reflexive:  $x \leq x$  for all  $x \in S$ .
- Transitive: if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .
- Antisymmetric: if  $x \leq y$  and  $y \leq x$ , then  $x = y$ .

We say that  $\leq$  is *linearly ordered* (or is a *chain*) if and only if we satisfy the additional axiom of totality:  $x \leq y$  or  $y \leq x$  for all  $x, y \in S$ .

**Example 2.40.** Consider a set  $X$ . Then  $\mathcal{P}(X)$  is partially ordered by  $\subseteq$ . It is not linearly ordered by  $\subseteq$ .

And now here is Zorn's lemma.

**Theorem 2.41.** Suppose that  $(S, \leq)$  is a nonempty partially ordered set such that any subchain  $C \subseteq S$  is "bounded above" in the sense that there is some  $y \in S$  such that  $x \leq y$  for any  $x \in C$ . Then  $S$  has a maximal element; i.e., there is some  $y \in S$  such that  $x \leq y$  for any  $x \in S$ .

*Proof.* This is equivalent to the Axiom of choice. ■

To apply Zorn's lemma, here is the proof of Lemma 2.37.

**Theorem 2.36 (Hahn–Banach).** Fix an  $\mathbb{R}$ -vector space  $X$  equipped with sublinear functional  $p$ , and let  $V \subseteq X$  be a subspace. Given a linear functional  $\ell: V \rightarrow \mathbb{R}$  such that  $\ell \leq p$  pointwise, there is a bounded linear functional  $L: X \rightarrow \mathbb{R}$  such that  $L|_V = \ell$  and  $L \leq p$  pointwise.

*Proof.* We will use Zorn's lemma. For this, we let our partially ordered set  $S$  consisting of all pairs  $(W, \ell_W)$  where  $W \subseteq X$  is a subspace and  $\ell_W$  is a functional on  $W$  restricting to  $\ell$  and has  $\ell_W \leq p$  pointwise on  $W$ . Our partial order is given by  $(W_1, \ell_1) \leq (W_2, \ell_2)$  if and only if  $(W_2, \ell_2)$  "extends"  $(W_1, \ell_1)$  in the sense that  $W_1 \subseteq W_2$  and  $\ell_1 = \ell_2|_{W_1}$ .

Note  $S$  is nonempty because it has  $(V, \ell)$  by construction. Now, to apply Theorem 2.41, we need to check that any chain  $C \subseteq S$  has an upper bound. Well, define

$$W' := \bigcup_{(W, \ell_W) \in C} W,$$

and then we define  $\ell': W' \rightarrow \mathbb{R}$  by defining  $\ell'(v)$  to be  $\ell_W(v)$  where  $v \in W$  for some  $(W, \ell_W) \in C$ . Notably, the choice of  $\ell_W$  does not matter: if  $v \in W_1 \cap W_2$  where  $(W_1, \ell_1), (W_2, \ell_2) \in C$ , then without loss of generality we can take  $(W_1, \ell_1) \leq (W_2, \ell_2)$ , implying  $\ell_2(v) = \ell_1(v)$ . Thus,  $\ell'$  is well-defined, and one can check directly

that it is linear on  $W'$ ; similarly one can check that  $\ell'$  extends  $\ell$  and that  $\ell' \leq p$  pointwise. In conclusion,  $(W', \ell') \in S$  and bounds  $C$  above.

Thus, Theorem 2.41 provides a maximal element  $(W, \ell)$  of  $S$ . We claim that  $W = X$ , which will complete the proof. Well, if  $W \subsetneq X$ , then we can find  $x \in X \setminus W$ . But then Lemma 2.37 tells us that we can extend  $(W, \ell)$  to some  $(W + \mathbb{R}x, \tilde{\ell})$  in  $S$ , and  $W \subsetneq W + \mathbb{R}x$  contradicts the fact that  $(W, \ell)$  is maximal. This completes the proof. ■

**Remark 2.42.** It turns out that Theorem 2.33 is not actually equivalent to the Axiom of choice.

**Remark 2.43.** The proof of Theorem 2.33 is similar, so we will omit it. Alternatively, one can apply Theorem 2.36 to  $\operatorname{Re} \ell$ , viewing  $X$  as an  $\mathbb{R}$ -vector space. Then one can recover  $\ell$  from  $\operatorname{Re} \ell$  via the identity

$$\ell(x) = \operatorname{Re} \ell(x) - i \operatorname{Re} \ell(ix),$$

so similarly one can upgrade the lifted functional  $\operatorname{Re} \tilde{\ell}$  provided by Theorem 2.36 to  $\tilde{\ell}$  via the above formula.

## 2.5 February 21

Here we go.

### 2.5.1 Baire's Theorem

Compactness is not going to be so helpful for Banach spaces, so we will need a different notion, which we will slowly build up to. To start off our story, we consider the following example.

**Example 2.44.** One can build an open dense subset covering  $\mathbb{Q}$  but with arbitrarily small measure. Indeed, enumerate  $\mathbb{Q}$  as  $\{q_n\}_{n \in \mathbb{N}}$  and then set

$$U_\varepsilon := (q_n - \varepsilon 2^{-n}, q_n + \varepsilon 2^{-n}),$$

which we can compute has measure upper-bounded by  $\sum_{n \in \mathbb{N}} 2\varepsilon 2^{-n} = 4\varepsilon$ . Thus, for example,

$$A := \bigcap_{k \in \mathbb{N}} U_{1/k}$$

has measure zero but is dense and is the countable intersection of open sets.

One might be interested in what  $A$  looks like. It will turn out that  $A$  is uncountable, but of course it is rather hard to show there is a single element in  $A \setminus \mathbb{Q}$ .

Here is our theorem.

**Theorem 2.45 (Baire).** Fix a nonempty complete metric space  $X$ , and let  $\{U_n\}_{n \in \mathbb{N}}$  be a collection of open dense subsets of  $X$ . Then

$$\bigcap_{n \in \mathbb{N}} U_n$$

is nonempty and in fact dense.

*Proof.* We will show density directly, so let  $V$  be some open subset of  $X$  which we would like  $\bigcap_{n \in \mathbb{N}} U_n$  to intersect  $V$  nontrivially. Well, we proceed inductively. To begin, choose  $x_0 \in U_0 \cap V$ , and select  $r_0 < 1$  such

that  $B(x_0, 3r_0) \subseteq U_1 \cap V$ . Then given  $x_n$  and positive  $r_n < r_{n-1}/2$  (where  $r_{-1} = 2$ ), we select

$$x_{n+1} \in U_n \cap B(x_n, r_n)$$

and some positive  $r_{n+1} < r_n/2$  such that  $B(x_{n+1}, 3r_{n+1}) \subseteq U_n \cap B(x_n, r_n)$ .

We now claim that  $\{x_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence, and its limit will be the desired point. Well, by an induction, we see  $r_n < 2^{-n}$ , so

$$d_X(x_n, x_{n+1}) < r_n < 2^{-n},$$

so for  $m < n$ , we see

$$d_X(x_m, x_n) \leq \sum_{k=m}^{n-1} d_X(x_k, x_{k+1}) < \sum_{k=m}^{n-1} 2^{-k} < 2^{-m+1},$$

which does indeed vanish as  $m \rightarrow \infty$ .

Thus, by completeness, we get some  $x \in X$  with  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . We claim that  $x \in V \cap \bigcap_{n \in \mathbb{N}} U_n$ , which will complete the proof. This will require some understanding of  $d(x, x_m)$ . Well, for each  $m < n$ , we know

$$d_X(x_m, x_n) < \sum_{k=m}^{n-1} d_X(x_k, x_{k+1}) < \sum_{k=m}^{\infty} r_k < 2r_m$$

by an argument similar to the above one, so we must have  $d_X(x_m, x) \leq 2r_m < 3r_m$ , so  $x \in B(x_m, 3r_m) \subseteq U_{m-1} \cap B(x_{m-1}, r_{m-1})$ , where  $B(x_{-1}, r_{-1})$  means  $V$ . So indeed  $x$  lives in every  $U_m$  and also in  $V$ , so we are done. ■

**Remark 2.46.** Completeness of  $X$  is necessary. For example, take  $X = \mathbb{Q}$  with the standard metric and topology. But then we can enumerate  $\mathbb{Q}$  as  $\{q_n\}_{n \in \mathbb{N}}$  and define the open dense subset  $U_n := X \setminus \{q_n\}$ , but

$$\bigcap_{n \in \mathbb{N}} U_n = \emptyset.$$

At this point, we are able to say that  $A$  of Example 2.44 is at least dense.

## 2.5.2 Nowhere Dense Sets

To see that  $A$  of Example 2.44 is fairly large, we want the following definitions.

**Definition 2.47 (nowhere dense).** Fix a topological space  $X$ . Then a subset  $E \subseteq X$  is *nowhere dense* if and only if  $\overline{E}$  contains no nonempty subset.

**Example 2.48.** The Cantor set in  $\mathbb{R}$  is nowhere dense.

**Definition 2.49 (category).** Fix a topological space  $X$ . Then a subset  $E \subseteq X$  is *of the first category* if and only if  $E$  is a countable union of nowhere dense sets.

**Example 2.50.** If  $X$  is countable, then singletons are certainly nowhere dense, so  $X$  is of the first category.

We are now allowed to extend Theorem 2.45 as follows.

**Theorem 2.51 (Baire).** Fix a nonempty complete metric space  $X$ , and let  $\{U_n\}_{n \in \mathbb{N}}$  be a collection of open dense subsets of  $X$ . Then

$$\bigcap_{n \in \mathbb{N}} U_n$$

is not of the first category.

*Proof.* Suppose for contradiction that we can find countably many nowhere dense subsets  $\{E_k\}_{k \in \mathbb{N}}$  such that

$$\bigcap_{n \in \mathbb{N}} U_n \subseteq \bigcup_{k \in \mathbb{N}} E_k.$$

Rearranging, we see that

$$\left( \bigcap_{n \in \mathbb{N}} U_n \right) \cap \left( \bigcap_{k \in \mathbb{N}} X \setminus \overline{E_k} \right)$$

being empty. However,  $E_k$  being nowhere dense means that  $X \setminus \overline{E_k}$  is open and dense, so we have exhibited a countable union of open dense subsets of  $X$  with empty intersection. So we achieve contradiction from Theorem 2.45. ■

**Example 2.52.** If  $X$  is a complete metric space, one can take  $U_n := X$  for all  $n \in \mathbb{N}$ , so  $X$  itself is not of the first category. Thus, for example, a complete metric space cannot be countable by comparing with Example 2.50.

The point of Theorem 2.51 is that  $A$  of Example 2.44 cannot be of the first category, so for example it cannot be countable.

### 2.5.3 The Open Mapping Theorem

Let's now apply our results for fun and profit. Here is our next statement.

**Theorem 2.53 (Open mapping).** Fix a surjective bounded linear map  $T: X \rightarrow Y$  of Banach spaces.

- (a) Then  $T$  is open.
- (b) There is  $A > 0$  such that any  $y \in Y$  has some  $x \in X$  such that  $T(x) = y$  and  $\|x\|_X \leq A \|y\|_Y$ .

**Remark 2.54.** Theorem 2.53 is intended to recover some aspects of compactness. To see this, suppose  $f: X \rightarrow Y$  is a continuous bijection of compact Hausdorff topological spaces. Then it turns out that  $f$  is open: it's enough to see that  $f$  is closed by taking complements, but for closed  $K \subseteq X$ , we see  $K$  is compact, so  $f(K)$  is compact, so  $f(K) \subseteq Y$  is closed.

Here's an application, explaining how to recover Remark 2.54.

**Corollary 2.55.** Fix a bijective bounded linear map  $T: X \rightarrow Y$  of Banach spaces. Then  $T^{-1}$  is bounded.

*Proof.* Part (b) of Theorem 2.53 tells us that  $\|T^{-1}(y)\|_X \leq A \|y\|_Y$  (because  $x = T^{-1}(y)$  is unique!), which is what we wanted. ■

## 2.6 February 23

Today we prove the Open mapping theorem.



### 2.6.1 Proof of the Open Mapping Theorem

Recall the statement.

**Theorem 2.53 (Open mapping).** Fix a surjective bounded linear map  $T: X \rightarrow Y$  of Banach spaces.

- (a) Then  $T$  is open.
- (b) There is  $A > 0$  such that any  $y \in Y$  has some  $x \in X$  such that  $T(x) = y$  and  $\|x\|_X \leq A \|y\|_Y$ .

*Proof.* Let's focus on (b) for now. We proceed in steps; note that we may assume  $y \neq 0$ .

1. Consider  $Y_n := \overline{T(B_X(0, n))}$  to be a subset of  $Y$ , and we note that

$$\bigcup_{n \in \mathbb{N}} Y_n = Y.$$

Thus, Theorem 2.51 implies there is some  $N$  such that  $Y_N$  contains some  $B(\bar{y}, \delta)$  for  $\delta > 0$ . Namely, the complements have empty intersection, so at least one of the  $Y \setminus Y_\bullet$  must fail to be open and dense, so one of the  $Y_\bullet$  will contain an open ball. So we get

$$\overline{T(B_X(0, N))} \supseteq B(\bar{y}, \delta).$$

We now spend a couple steps upgrading this.

2. Now, choose some  $y \in B_Y(0, \delta)$  and  $\varepsilon > 0$ . Then by construction, we may choose  $x_1 \in X$  with  $\|Tx_1 - \bar{y}\|_Y < \varepsilon$  and  $\|x_1\|_X < N$  and  $x_2 \in X$  such that  $\|Tx_2 - (y + \bar{y})\|_Y < \varepsilon$  and  $\|x_2\|_X < N$ . So setting  $x := x_1 - x_2$ , we find

$$\|Tx - y\|_Y \leq \|Tx_1 - \bar{y}\|_Y + \|Tx_2 - (y + \bar{y})\|_Y < 2\varepsilon,$$

and  $\|x\|_X < 2N$ .

3. Continuing, set  $C := 2N/\delta$ . Then each nonzero  $y \in Y$  and  $\varepsilon > 0$  has some  $x \in X$  such that  $\|Tx - y\|_Y < \varepsilon$  and  $\|x\|_X < 2N/\delta$ . Indeed, this is direct from the prior step: replace  $y$  with  $y_0 := \frac{\delta}{2\|y\|}y$ , conjuring  $x_0$  with  $\|x_0\|_X < 2N$  and  $\|Tx_0 - y_0\|_Y < \frac{\delta}{2\|y\|}\varepsilon$  via the prior step, and then scale  $x_0$  back up to  $x := \frac{2\|y\|}{\delta}x_0$  to conclude.
4. We now complete the proof by a limiting procedure. To begin, we choose  $x_1 \in X$  such that  $\|x_1\|_X \leq C\|y\|_Y$  and  $\|Tx_1 - y\|_Y < 2^{-1}\|y\|_Y$ . Then given  $x_n$ , we choose  $x_{n+1} \in X$  such that

$$\|Tx_{n+1} - (y - Tx_1 - \cdots - Tx_n)\|_Y < 2^{-n-1}\|y\|_Y$$

such that  $\|x_{n+1}\|_X \leq C\|y - Tx_1 - \cdots - Tx_n\|_Y$ , and this upper bound is just  $C2^{-n}\|y\|_Y$  by construction. So we may define

$$x := \sum_{k=1}^{\infty} x_k,$$

which converges because  $X$  is complete (and this series is absolutely convergent). Additionally, we see  $\sum_{k=1}^N Tx_k \rightarrow y$  by construction, so continuity of  $T$  tells us  $Tx = y$ . Lastly, we bound  $\|x\|_X$  by the above absolutely convergent series, which is bounded by  $C\|y\|_Y \sum_{k=0}^{\infty} 2^{-k} = 2C\|y\|_Y$ , as required.

It remains to show (a). Well, (b) provides a constant  $A$  such that  $T(B_X(0, A))$  contains  $B_Y(0, 1)$ . So we see  $T(B_X(0, Ar)) \supseteq B_Y(0, r)$  for any  $r > 0$ . By translating, we see that any ball  $T(B_X(x, \varepsilon))$  contains  $B_Y(Tx, \varepsilon/A)$ . Thus, for any open subset  $U \subseteq X$ , we are able to conclude that  $T(U) \subseteq Y$  is open: for any  $y \in T(U)$ , write  $y = Tx$  for  $x \in U$ , but then there is  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq U$ , so

$$B(y, \varepsilon/A) \subseteq T(U)$$

is an open neighborhood of  $y$  contained in  $T(U)$ . So  $T(U)$  is in fact open. ■

**Remark 2.56.** The power of Theorem 2.53 is that it is able to provide a concrete bound “ $A$ ” on solving the “system” of equations  $Tx = y$ .

## 2.6.2 The Closed Graph Theorem

Here is our next result.

**Theorem 2.57 (Closed graph).** Fix a Banach spaces  $X$  and  $Y$ . Fix a linear map  $T: X \rightarrow Y$ . If  $T$  has a graph

$$\Gamma := \{(x, Tx) \in X \times Y : x \in X\}$$

which is closed in  $X \times Y$ , then  $T$  is bounded.

**Remark 2.58.** Again, the power of Theorem 2.57 is that we are able to get a concrete bound on the norm of  $T$  from a topological condition.

Wait, how do we give the product a topology? One could use the product topology, but we can see this on the level of norms as well.

**Definition 2.59 (product).** Given normed vector spaces  $X$  and  $Y$ , then the *product*  $X \times Y$  is a vector space with norm given by

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y.$$

**Remark 2.60.** If  $X$  and  $Y$  are both complete, then  $X \times Y$  is also complete, which we can see by taking limits coordinate-wise.

Let’s see an example of an unbounded operator.

**Example 2.61.** Let  $X$  be the collection of continuously differentiable functions  $f: [0, 1] \rightarrow \mathbb{C}$ , where we take  $\|f\|_X := \sup\{f(x) : x \in [0, 1]\}$ , which we can see is a norm. (Notably,  $\|f\|_X = 0$  means that  $f(x) = 0$  everywhere, so  $f = 0$ .) Then we let  $Y$  be the collection of continuous functions  $f: [0, 1] \rightarrow \mathbb{C}$  with the same norm  $\|f\|_Y := \sup\{f(x) : x \in [0, 1]\}$ , and we define  $D: X \rightarrow Y$  by  $Df := f'$ . Note  $D$  is well-defined and linear, and we see  $D$  is bounded is equivalent to requiring

$$\sup_{\substack{f \in X \\ f \neq 0}} \frac{\|Df\|_Y}{\|f\|_X} = \sup_{\substack{f \in X \\ f \neq 0}} \frac{\sup\{f'(x) : x \in [0, 1]\}}{\sup\{f(x) : x \in [0, 1]\}} < \infty,$$

but this is false; for example, we can consider the family  $f_n(x) = \sin nx$  which has  $\|f_n\|_X = 1$  but  $\|Df_n\|_Y \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Example 2.62.** We continue Example 2.61. Let  $\Gamma$  be the graph of  $D$ , but we note that  $\Gamma$  is closed! Unwinding definitions, we are being given a sequence of functions  $\{f_n\}_{n \in \mathbb{N}} \subseteq X$  and  $g$  and  $h$  such that  $f_n \rightarrow g$  uniformly and  $f'_n \rightarrow h$  uniformly, and we must show that  $g' = h$ . Well, write

$$f_n(x) = f_n(0) + \int_0^x f'_n(t) dt$$

and take limits everywhere to see  $g(x) = g(0) + \int_0^x h(t) dt$ , which implies  $g' = h$ .

It might look like the previous example violates Theorem 2.57, but in fact  $X$  is not complete and hence not Banach.

Here is another example.

**Example 2.63.** Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) := 1/x$  for nonzero  $x$  and  $f(0) = 0$ . Then the graph of  $f(x)$  is  $\{(0, 0)\} \sqcup \{(x, y) : xy = 1\}$ , which is a closed subset of  $\mathbb{R}^2$ . But  $f$  is certainly not continuous, which is not a violation of Theorem 2.57 because  $f$  is not linear!

Anyway, let's prove Theorem 2.57.

*Proof of Theorem 2.57.* Quickly, note that linearity of  $T$  means that the graph  $\Gamma$  is linear. Because  $\Gamma$  is closed, we actually see that  $\Gamma$  is a complete normed vector space, where the norm is given by  $\|\cdot\|_{X \times Y}$ . So we now define  $S: \Gamma \rightarrow X$  by  $S(x, y) := x$ , which is linear (because look at it) and bounded because

$$\|S(x, y)\|_X = \|x\|_X \leq \|x\|_X + \|y\|_Y = \|(x, y)\|_{X \times Y}.$$

Furthermore,  $S$  is a bijection, so in fact  $S^{-1}$  is bounded by Corollary 2.55, meaning that

$$\|(x, Tx)\|_{X \times Y} \leq C \|x\|_X$$

for some absolute constant  $C > 0$ . This unwinds into exactly showing that  $T$  is bounded, as required. ■

## 2.7 February 28

Today we discuss the uniform boundedness principle.

### 2.7.1 The Uniform Boundedness Principle

Here is our statement.

**Theorem 2.64 (Uniform boundedness principle).** Fix a Banach space  $X$ . Let  $\{T_\alpha\}_{\alpha \in \kappa} \subseteq X^*$  be a non-empty collection of linear functionals. Suppose that the family  $\{T_\alpha\}_{\alpha \in \kappa}$  is pointwise bounded; i.e., for each  $x \in X$ , we have

$$\sup_{\alpha \in \kappa} |T_\alpha(x)| < \infty.$$

Then in fact we are uniformly bounded:  $\sup \{\|T_\alpha\|_{X^*} : \alpha \in \kappa\} < \infty$ .

Here are some non-examples.

**Example 2.65.** Define a family  $f_n: [0, 1] \rightarrow \mathbb{R}$  (where  $n \in \mathbb{Z}^+$ ) by

$$f_n(x) := \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n, \\ n^2(x - 2/n) & \text{if } 1/n \leq x \leq 2/n, \\ 0 & \text{if } 2/n \leq x \leq 1. \end{cases}$$

Then  $\{f_n\}_{n \in \mathbb{Z}^+}$  the family is pointwise bounded because  $f_n(x) = 0$  for  $n > 3/x$  (say), so  $\{f_n(x)\}_{n \in \mathbb{N}}$ . Simply put, we do not violate Theorem 2.64 because  $[0, 1]$  is not a Banach space, and our functions  $f_n$  are not linear.

**Example 2.66.** Define  $X := \mathbb{C}^{\oplus \mathbb{N}}$  be an infinite direct sum of  $\mathbb{C}$ s. Explicitly,  $X$  consists of sequences which are eventually 0. We give  $X$  the norm

$$\|x\|_X := \max_{k \in \mathbb{Z}^+} \{|x_k|\}.$$

Now, define  $T_n: X \rightarrow \mathbb{C}$  by  $T_n(x) := nx_n$ . Then the family  $\{T_n\}_{n \in \mathbb{N}}$  is pointwise bounded: for any  $x \in X$ , there is some  $N > 0$  such that  $x_n = 0$  for  $n > N$ , so

$$\sup_{n \in \mathbb{N}} |T_n(x)| = \sup_{0 \leq n \leq N} |T_n(x)| = \max_{0 \leq n \leq N} |T_n(x)| < \infty.$$

However, we see that  $\|T_n\|_{X^*} \geq n$  for each  $n$  by taking the sequence  $x \in X$  given by  $x_k := 1_{n=k}$ , so this family is not uniformly bounded. (In fact, we see that  $\|T_n\|_{X^*} \leq n$  as well by bounding  $|T_n(x)| / \|x\|_X = n|x_n| / \sup\{|x_k|\} \leq n$ .) The reason we do not violate Theorem 2.64 is that  $X$  is not complete and hence not Banach.

Anyway, let's prove Theorem 2.64.

*Proof of Theorem 2.64.* We proceed in steps. Let  $\|\cdot\|$  be the norm on  $X$ .

1. For  $M > 0$ , define the subset  $E_M \subseteq X$  by

$$E_M := \left\{ x \in X : \sup_{\alpha \in \kappa} |T_\alpha(x)| \leq M \right\}.$$

Then we note that  $X$  is covered by the collection  $\{E_M\}_{M \in \mathbb{Z}^+}$  because the family  $\{T_\alpha\}_{\alpha \in \kappa}$  is pointwise bounded.

Further, we note that  $E_M$  is closed for each  $M > 0$ . For this, we should show that  $E_M$  contains its limit points. Well, suppose that  $\{x_n\}_{n \in \mathbb{N}} \subseteq X$  is a sequence in  $E_M$  converging to some  $x \in X$ , and we want to show that  $x \in E_M$ . To check this, fix any  $\alpha \in \kappa$ , and we must show that  $|T_\alpha(x)| \leq M$ . For this, we bound

$$|T_\alpha(x)| = \left| \lim_{n \rightarrow \infty} T_\alpha(x_n) \right| = \lim_{n \rightarrow \infty} |T_\alpha(x_n)|,$$

where the last equality is by continuity.

2. Thus,  $X$  is the union of the countably many closed sets  $\{E_M\}_{M \in \mathbb{Z}^+}$ , so Theorem 2.51 explains that there must be some  $M \in \mathbb{Z}^+$  and  $\delta > 0$  and  $x_0 \in X$  such that  $B(x, \delta) \subseteq E_M$ .

In fact, we claim that  $B(0, \delta) \subseteq E_{2M}$ . Indeed, if  $\|x\| < \delta$ , then define  $x := (x_0 + x) - x_0$  so that any  $\alpha \in \kappa$  has

$$|T_\alpha(x)| = |T_\alpha(x_0 + x)| - |T_\alpha(x_0)| \leq 2M,$$

so  $x \in E_{2M}$ .

3. We now do our bounding. Thus far we have shown that  $\|x\| < \delta$  implies that  $|T_\alpha(x)| \leq 2M$  for any  $\alpha \in \kappa$ . But then we see we can write

$$\|T_\alpha\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| = \delta/2}} \frac{|T_\alpha(x)|}{\|x\|} \leq \frac{2M}{\delta/2},$$

so we have achieved our uniform bound. ■

**Remark 2.67.** Later in life, we will use Theorem 2.64 do some Fourier analysis.

### 2.7.2 A Little on Bases

Let's talk a little about bases for vector spaces. Notably, we have never defined a "basis" for a Banach space, and we will avoid doing so ever. Instead, we will define bases with adjectives.

**Definition 2.68 (Hamel basis).** A *Hamel basis* for a vector space  $X$  is a collection  $\{v_\alpha\}_{\alpha \in \kappa}$  of vectors in  $X$  such that any  $x \in X$  can be written uniquely as a linear combination

$$x = \sum_{\alpha \in \kappa} c_\alpha v_\alpha,$$

where the scalars  $\{c_\alpha\}_{\alpha \in \kappa}$  are zero for all but finitely many  $\alpha$ . (This sum certainly converges because it is actually a finite sum.)

**Remark 2.69.** Equivalently, we can ask for the linear combination to merely exist and for the subset to be linearly independent in the sense that having any linear combination

$$\sum_{\alpha \in \kappa} c_\alpha v_\alpha = 0$$

(where  $c_\alpha$  is zero for all but finitely many  $\alpha$ ) will have  $c_\alpha = 0$  for all  $\alpha$ .

**Remark 2.70.** Any vector space has a Hamel basis by some argument with the Axiom of choice (e.g., via Zorn's lemma). In fact, any linearly independent collection of vectors can be extended to a Hamel basis. Rigorously, one builds a partially ordered set of linearly independent sets (ordered by inclusion), use Zorn's lemma to extract a maximal such set, and then show that a maximal linearly independent set is spanning.

**Remark 2.71.** Fix an infinite-dimensional normed vector space  $X$  over  $\mathbb{F}$  and a nonzero normed vector space  $Y$  over  $\mathbb{F}$ . Then we can show that there is an unbounded linear functional  $T: X \rightarrow Y$ . Indeed, let  $\{v_\alpha\}_{\alpha \in \kappa}$  be a Hamel basis, which is infinite because  $X$  is infinite-dimensional. Now, choose a nonzero vector  $y \in Y$  and some unbounded sequence  $\{c_\alpha\}_{\alpha \in \kappa} \subseteq \mathbb{F}$ , and define

$$Tv_\alpha := c_\alpha \|v_\alpha\|_X y$$

extended linearly to be a functional on  $X$ . (Explicitly,  $T$  sends  $\sum_\alpha c_\alpha v_\alpha = c_\alpha \|v_\alpha\|_X y$ , which is well-defined because we are working with a Hamel basis.) However,  $T$  is not bounded because

$$\|T\|_{X^*} \geq \sup_{\alpha \in \kappa} \frac{\|Tv_\alpha\|_Y}{\|v_\alpha\|_X} = \sup_{\alpha \in \kappa} |c_\alpha| \|y\|_Y = \infty.$$

Here are some more cautionary tales.

**Remark 2.72.** In general, closed subspaces of a Banach space do not have to have complements.

**Remark 2.73.** "Useful" bases for normed vector spaces  $X$  do not exist in general. Later in life, we will have Hilbert bases for some subclass of normed vector spaces.

**Remark 2.74.** Fix a normed vector space  $X$ . If  $V \subseteq X$  is a closed subspace and a point  $x \notin V$ , there is no well-defined notion of "closest point." In finite dimensions, it need not be unique, and in general, it need not even exist.

## THEME 3

# WEAK TOPOLOGIES

---

### 3.1 March 1

We began class with a review of topological spaces. We just refer to [Elb22].

#### 3.1.1 Weak Topologies

It will be helpful to have other topologies to work with on our normed vector spaces. We recall the following notion.

**Definition 3.1** (weak topology). Fix a nonempty set  $X$ , and let  $\{f_\alpha\}_{\alpha \in \kappa}$  be a nonempty collection of maps  $f_\alpha: X \rightarrow Y_\alpha$ , where  $\{Y_\alpha\}_{\alpha \in \kappa}$  is a collection of topological spaces. Then we define the *weak topology* on  $X$  to be the topology generated by the collection

$$\bigcup_{\alpha \in \kappa} \{f_\alpha^{-1}(U_\alpha) : \text{open } U_\alpha \subseteq Y_\alpha\}.$$

**Remark 3.2.** Notably, the given collection above is a subbasis because it contains  $\emptyset = f_\alpha^{-1}(\emptyset)$  and  $X = f_\alpha^{-1}(Y_\alpha)$ . As such, we can give ourselves a basis by taking finite intersections of the sets in our given collection.

**Remark 3.3.** Observe that the weak topology on  $X$  promises that each map  $f_\alpha: X \rightarrow Y_\alpha$  is continuous. Conversely, if  $\mathcal{T}$  is a topology on  $X$  such that the maps  $f_\alpha: X \rightarrow Y_\alpha$  are continuous, then  $\mathcal{T}$  must contain the weak topology.

**Remark 3.4.** We have the following universal property: a map  $g: Z \rightarrow X$  is continuous if and only if the composites  $(f_\alpha \circ g): Z \rightarrow Y_\alpha$  are all continuous. The forward direction occurs by taking composition. For the backward direction, it suffices to check that any  $g^{-1}(f_\alpha^{-1}(U_\alpha)) \subseteq Z$  is open for any  $\alpha \in \kappa$  and open  $U_\alpha \subseteq Y_\alpha$ , which holds by the continuity of  $f_\alpha \circ g$ .

**Example 3.5.** One gives the infinite product  $X := \prod_{\alpha \in \kappa} Y_\alpha$  of topological spaces  $\{Y_\alpha\}_{\alpha \in \kappa}$  the weak topology with respect to the projections  $\text{pr}_\alpha: X \rightarrow Y_\alpha$ .

**Remark 3.6.** Let's explain closures quickly. Let  $E \subseteq X$  be a subset as above. Then  $x \in \overline{E}$  if and only if  $B \cap E \neq \emptyset$  for any basic open neighborhood  $B \subseteq X$  of  $x$ . In other words, for any finite collection  $S \subseteq \kappa$  of indices and open subsets  $U_\alpha \subseteq Y_\alpha$  for  $\alpha \in S$ , we have

$$B \cap \bigcap_{\alpha \in S} f_\alpha^{-1} U_\alpha \neq \emptyset.$$

**Example 3.7.** Consider  $X := \{0, 1\}^{\mathbb{R}}$  endowed with the product topology. Set  $x := 0 \in X$  and  $E$  to be the collection of  $y \in X$  such that  $y_t = 1$  or all but countably many  $t \in \mathbb{R}$ . Then actually  $x \in \overline{E}$ : indeed, a basic open subset of  $X$  containing  $x$  merely requires that some finite subset of indices  $t \in \mathbb{R}$  vanish, which  $E$  does intersect.

However, there is no countable sequence  $\{x_n\}_{n \in \mathbb{Z}^+}$  in  $E$  which converge to  $x$ . Indeed, by taking a union over all  $x_n$ , we see that there is a cocountable subset  $T \subseteq \mathbb{R}$  such that  $(x_n)_t = 1$  for all  $n \in \mathbb{Z}^+$  and  $t \in T$ . So our sequence cannot approach  $x$  (for example) by using a basic open subset of  $X$  requiring  $y_t = 0$  for some  $t \in T$ .

Here is our chief example of interest.

**Definition 3.8.** Fix a normed vector space  $(X, \|\cdot\|)$ . Then the *weak topology* on  $X$  is the weak topology obtained by requiring that all bounded linear functionals  $f \in X^*$  are continuous.

**Remark 3.9.** The weak topology on  $X$  is Hausdorff: for any distinct  $x, y \in X$ , it is enough to show that there is a linear functional  $f \in X^*$  such that  $f(x) \neq f(y)$ . If  $x$  and  $y$  are linearly independent, then define  $f$  by  $f(x) = 1$  and extend  $f$  from  $\text{span}\{x\}$  to all  $X$  by Theorem 2.33. If they are not linearly dependent, then define  $f$  by  $f(x) = 1$  and  $f(y) = 0$  and extend  $f$  again via Theorem 2.33.

## 3.2 March 4

Here we go.

### 3.2.1 An Example Dual Space

We begin class with an example of convergence being strange. Let  $C_0$  denote the collection of sequences  $\{(x_n)_{n \in \mathbb{Z}^+}\}$  converging to 0, and we give  $C_0$  the norm  $\|\cdot\|_\infty$  defined by taking the maximum. One can see that  $C_0$  is now a normed vector space (inherited from  $L^\infty(\mathbb{N})$ ), and it is complete (one needs to check that  $C_0 \subseteq L^\infty(\mathbb{N})$  is closed, which can be done via limit points).

Let's attempt to understand  $C_0^*$ . For this, we consider  $L^1(\mathbb{N})$ , which consists of sequences  $(y_m)_m$  such that  $\sum_{m \geq 1} |y_m| < \infty$ . Then  $L^1(\mathbb{N})$  is a normed vector space with norm given by

$$\|y\|_1 := \sum_{m=1}^{\infty} |y_m|.$$

Now here is our duality check.

**Lemma 3.10.** The map  $L_\bullet : L^1(\mathbb{N}) \rightarrow C_0^*$  defined by

$$L_y(x) := \sum_{m=1}^{\infty} x_m y_m$$

is an isomorphism.

*Proof.* Here are our many checks.

- Note  $L_y(x) \in \mathbb{R}$ : the series (absolutely) converges because

$$\sum_{m=1}^{\infty} |x_m y_m| \leq \|x\|_{\infty} \sum_{m=1}^{\infty} |y_m| = \|x\|_{\infty} \|y\|_1.$$

- Note  $L_y$  is linear for each  $y \in L^1(\mathbb{N})$  by a direct check.
- In fact, note  $L_y$  is a bounded linear functional and hence in  $C_0^*$ ; this finishes showing that  $L_{\bullet}$  is well-defined. Indeed, we simply bound

$$\left| \sum_{m=1}^{\infty} x_m y_m \right| \leq \|x\|_{\infty} \|y\|_1$$

as above, meaning that  $|L_y(x)| / \|x\|_{\infty} \leq \|y\|_1$  for any nonzero  $x \in C_0$ .

- Note  $L_{\bullet}$  is linear by a direct check.
- In fact, we claim that  $\|L_y\|_{C_0^*} = \|y\|_{L^1}$  for any  $y \in L^1(\mathbb{N})$ ; note that this shows that  $L_{\bullet}$  is injective. Above we showed that  $|L_y(x)| / \|x\|_{\infty} \leq \|y\|_1$ , so we get  $\leq$  immediately. For the other inequality, fix some  $N > 0$  very large, and we define  $(x_n)$  to be

$$x_n := \begin{cases} +1 & \text{if } y_n \geq 0 \text{ and } n \leq N, \\ -1 & \text{if } y_n \leq 0 \text{ and } n \leq N, \\ 0 & \text{if } n > N. \end{cases}$$

Then we see that  $\|x\|_{\infty} = 1$

$$|L_y(x)| = \sum_{m=1}^N |y_m|$$

by construction, so  $|L_y(x)| \rightarrow \|y\|_1$  as  $N \rightarrow \infty$ , enforcing the equality.

- Lastly, we must check surjectivity. This requires some work. Fix a linear functional  $f \in C_0^*$ , and we need to show that  $f = L_y$  for some  $y \in L^1(\mathbb{N})$ . Well, simply define  $y_n := f(e_n)$  where  $e_n$  is the  $n$ -indicator sequence  $(e_n)_m := 1_{m=n}$ . For example, this implies that

$$L_y(e_n) = f(e_n)$$

for each  $n$ . Then  $L_y$  and  $f$  agree on the subspace  $C_{00} \subseteq C_0$  of sequences which are eventually 0. Note that we have not technically shown that  $y \in L^1(\mathbb{N})$  yet from its construction. Well, for large  $N$ , define the sequence  $(x_n)_n$  as in the previous point, and we see that

$$\sum_{m=1}^N |y_m| = |L_y(x)| = |f(x)| \leq \|f\| \cdot \|x\| = \|f\|,$$

so sending  $N \rightarrow \infty$  reveals that  $y \in L^1(\mathbb{N})$ . Thus, continuity of  $L_y$  (it is now known to be bounded) and density of  $C_{00}$  in  $C_0$  (by definition of converging to 0) forces  $L_y = f$  is forced. ■

**Remark 3.11.** Let's do a convergence check. Define the sequence  $e_n \in C_0$  by  $(e_n)_m := 1_{m=n}$ , and it turns out that  $e_n \rightarrow 0$  as  $n \rightarrow \infty$  in the weak topology, even though this is notably false in  $C_0$  because  $\|e_n\|_{\infty} = 1$  for each  $m$ ! In other words, we have  $f(e_n) \rightarrow 0$  as  $n \rightarrow \infty$  for any bounded linear functional  $f \in C_0^*$ . To see this, write  $f = L_y$  for  $y \in L^1(\mathbb{N})$  via Lemma 3.10, but now  $f(e_n) = y_n$ , which goes 0 as  $n \rightarrow \infty$  because  $y \in L^1(\mathbb{N})$ .



**Remark 3.12.** In general, if  $x_n \rightarrow x$  in the weak topology for  $C_0$ , then it turns out that  $(x_n)_m \rightarrow x_m$  for any fixed  $m$ . Indeed, let  $y = e_m$  be the  $m$ -indicator as above, and we must have  $L_y(x_n) \rightarrow L_y(x)$  as  $n \rightarrow \infty$ , from which the claim follows.

However, the converse is not true: let  $x_n := 2^n e_n$  be  $2^n$  times the  $n$ -indicator. For this to converge to any  $x$ , the previous paragraph enforces  $x = 0$ , but this sequence does not converge to 0 by using the bounded linear functional  $L_y$  where  $y_m := 1/2^m$  for each  $m$ . The problem here is that the norms are getting too large, and it will turn out that the converse holds as long as the norms are bounded (e.g., in the unit ball).

### 3.2.2 The Weak-\* Topology

Here is our definition.

**Definition 3.13** (weak-\* topology). Fix a normed vector space  $(X, \|\cdot\|)$ . Then the weak-\* topology on  $X^*$  is the weak topology obtained by requiring that the linear functionals  $\text{ev}_x: X^* \rightarrow \mathbb{F}$  defined by  $\text{ev}_x(f) := f(x)$  are continuous.

**Remark 3.14.** Note that  $\text{ev}_x$  is actually a bounded linear functional on  $X^*$ . Indeed, one computes that

$$\|\text{ev}_x\|_{X^{**}} = \sup_{\substack{f \in X^* \\ f \neq 0}} \frac{|\text{ev}_x(f)|}{\|f\|_{X^*}} = \sup_{\substack{f \in X^* \\ f \neq 0}} \frac{|f(x)|}{\|f\|_{X^*}} \leq \|x\|_X,$$

where the last inequality uses the definition of  $\|f\|_{X^*}$ . Thus,  $\text{ev}_x: X^* \rightarrow \mathbb{F}$  is already continuous, so the weak-\* topology is in fact coarser (has fewer open sets) than the usual topology.

**Example 3.15.** Given  $g \in X^*$  and  $\varepsilon > 0$  and  $x_0 \in X$ , the weak-\* topology has subbasic open subset given by

$$\{f \in X^* : |f(x_0) - g(x_0)| < \varepsilon\}.$$

Indeed, the point is that continuity of  $\text{ev}_{x_0}$  may as well be checked on basic open subsets of  $\mathbb{F}$ , and the above set is the pre-image of the open ball around  $g(x_0)$  with radius  $\varepsilon > 0$ .

**Example 3.16.** We continue the discussion of section 3.2.1 to explicate the weak-\* topology on  $C_0^* \cong L^1(\mathbb{N})$ . This amounts to choosing some  $z \in L^1(\mathbb{N})$  and  $\varepsilon > 0$  and  $x \in C_0$ , for which our subbasic open subset is

$$\left\{ y \in L^1(\mathbb{N}) : \left| \sum_{m=1}^{\infty} (y_m - z_m)x_m \right| < \varepsilon \right\}.$$

## 3.3 March 6

Here we go.

### 3.3.1 Reflexive Banach Spaces

We begin with a quick digression, upgrading Remark 3.14.

**Remark 3.17.** We note that there is a canonical linear map  $\text{ev}_\bullet: X \rightarrow X^{**}$  which preserves norms. The map is defined by defining  $\text{ev}_x \in X^{**}$  as  $\text{ev}_x(f) := f(x)$  for  $f \in X^*$ . Note Remark 3.14 shows that  $\text{ev}_x \in X^{**}$ , and we can see that  $\text{ev}_\bullet$  is linear. It remains to compute the norm of  $\text{ev}_x$ . For one inequality, we use the argument of Remark 3.14 to note

$$\|\text{ev}_x\|_{X^*} = \sup_{\substack{f \in X^* \\ f \neq 0}} \frac{|f(x)|}{\|f\|_{X^*}}.$$

To show the equality, we need to exhibit a good  $f$ . Well, by Theorem 2.33, one gets  $f \in X^*$  so that  $\|f\|_{X^*} = 1$  and  $|f(x)| = \|x\|_X$ ; explicitly,  $f$  is defined by extending  $f: \mathbb{R}x \rightarrow \mathbb{R}$  defined by  $f(x) := \|x\|_X$ . So the above computation tells us that  $\|\text{ev}_x\|$  is at least  $|f(x)| / \|f\|_{X^*} = \|x\|_X$ , as required.

**Remark 3.18.** If  $X$  is finite-dimensional, then  $\dim X^{**} = \dim X^* = \dim X$ , so the injective linear map  $\text{ev}_\bullet: X \rightarrow X^{**}$  must actually be an isomorphism. For infinite-dimensional  $X$ , then  $\text{ev}_\bullet$  need not be an isomorphism.

In light of the above remark, we pick up the following definition.

**Definition 3.19 (reflexive).** A Banach space  $X$  is *reflexive* if and only if the evaluation map  $\text{ev}_\bullet: X \rightarrow X^{**}$  is an isomorphism.

By Remark 3.17, being reflexive merely requires surjectivity.

**Remark 3.20.** On the homework, we will show that  $\text{im } \text{ev}_\bullet$  is dense in  $X^{**}$  in the weak-\* topology.

**Remark 3.21.** One can show that the closed unit ball of a Banach space  $X$  is compact in the weak topology if and only if  $X$  is reflexive.

**Non-Example 3.22.** Section 3.2.1 tells us that  $C_0$  is dual to  $L^1(\mathbb{N})$ . However, one can show that  $L^1(\mathbb{N})$  is dual to  $L^\infty(\mathbb{N})$ , which is the space of bounded sequences, but the evaluation map  $C_0 \rightarrow L^\infty(\mathbb{N})$  fails to be surjective (meaning  $C_0$  is not reflexive!). For example, let  $\mathcal{L} \subseteq L^1(\mathbb{N})$  of sequences with a limit; notably, the constant sequence of  $(1, 1, \dots)$  is in  $\mathcal{L} \setminus C_0$ , and one can use the Hahn–Banach theorem to bring this up to a linear functional on  $L^1(\mathbb{N})$  not coming from  $C_0$ .

**Non-Example 3.23.** One can also see that  $L^1(\mathbb{N})$  fails to be reflexive. One can again use the Hahn–Banach theorem or appeal to the following remark.

**Remark 3.24.** By functoriality, if  $X$  is reflexive, then  $X^*$  is also reflexive. We claim that the converse also holds! Suppose for the sake of contradiction that  $X^*$  is reflexive but  $X$  is not. Then  $\text{im } \text{ev}_\bullet^X \subsetneq X^{**}$ , so Theorem 2.33 permits us to find some bounded linear functional  $\lambda \in X^{***}$  such that  $\lambda \neq 0$  but  $\lambda|_{\text{im } \text{ev}_\bullet^X} = 0$ . However, we will show that  $\lambda|_{\text{im } \text{ev}_\bullet^X} = 0$  implies  $\lambda = 0$ . Indeed,  $X^*$  is reflexive, so we are granted  $\lambda_0 \in X^*$  such that  $\lambda = \text{ev}_{\lambda_0}^{X^*}$ ; it is enough now to show that  $\lambda_0$  vanishes. Well, for each  $x \in X$ , we see that

$$\lambda_0(x) = \text{ev}_{\lambda_0}^{X^*}(\text{ev}_x) = \lambda(\text{ev}_x) = 0.$$

### 3.3.2 Weak Compactness Results

Here is today's main result.

**Theorem 3.25 (Helly).** Fix a separable Banach space  $(X, \|\cdot\|)$ . Then the closed unit ball of  $X^*$  is sequentially compact in the weak-\* topology. In other words, given a sequence of bounded linear functionals  $\{f_n\}_{n \geq 1} \subseteq X^*$  such that  $\|f_n\|_{X^*} \leq 1$  for each  $n$ , there is a subsequence  $\{f_{n_k}\}_{k \geq 1}$  and  $f \in X^*$  such that each  $x \in X$  makes  $f_{n_k}(x) \rightarrow f(x)$  as  $k \rightarrow \infty$ .

*Proof.* This is a Cantor diagonal argument. Choose a countable dense subset  $\{x_n\}_{n \in \mathbb{N}}$ . We now proceed with an inductive construction.

1. Because  $|f_n(x_1)| \leq \|x_1\|_X$  for each  $n$ , we see that the sequence  $\{f_n(x_1)\}_{n \in \mathbb{N}}$  is bounded, so we can find a subsequence  $\{f_{n(i_1)}\}_{i_1 \in \mathbb{N}}$  with a subsequence where the limit  $\lim_{i_1 \rightarrow \infty} f_{n(i_1)}(x_1)$  exists.
2. Next up, we again pass to a subsequence where  $f_{n(i_1)(i_2)}$  so that  $\lim_{i_2 \rightarrow \infty} f_{n(i_1)(i_2)}(x_2)$  exists.

We have forced ourselves into the notation as above so that we can work with a subsequence  $\{g_i\}_{i \geq 1}$  where  $g_i := f_{n(i_1) \dots (i_k)}$  repeated  $i$  times. As such,  $g_i(x_n)$  converges as  $i \rightarrow \infty$  for any  $n$ .

It remains to show that  $\{g_i\}_{i \geq 1}$  actually converges to a function  $f$  in the closed unit ball. It suffices to show that this subsequence is Cauchy pointwise; then completeness of  $X^*$  tells us that our sequence converges to an actual bounded linear functional, and continuity of  $\|\cdot\|_{X^*}$  means that we will land in the closed unit ball. Now, to show Cauchy, for each  $x \in X$ , choose  $\varepsilon > 0$  and some  $x_j$  such that  $\|x_j - x\| < \varepsilon$ , and we find

$$\limsup_{m, n \rightarrow \infty} |g_m(x) - g_n(x)| \leq \limsup_{m, n \rightarrow \infty} |g_m(x_j) - g_n(x_j)| + 2 \sup_{n \in \mathbb{N}} |g_n(x) - g_n(x_j)|$$

by some rearrangement. But  $g_n$  is in the closed unit ball, so the right-hand term is bounded above by  $2\varepsilon$ . Additionally, the left term vanishes by construction of the  $g_\bullet$ , so are done. ■

**Remark 3.26.** Theorem 3.25 is useful because oftentimes one will construct a sequence of some functions (e.g., approximate solutions to a differential equation), and then we want to actually produce a limiting function. Then we see that it will merely remain to check some separability (which is the case when working in any Euclidean space) and a uniform boundedness (which is often not so bad).

Theorem 3.25 has the following analogous result without separability.

**Theorem 3.27 (Alaoglu).** Let  $X$  be a Banach space. Then the closed unit ball of  $X^*$  is compact in the weak-\* topology.

Note that we are working with the dual space here the result is not true for the closed unit ball in the weak topology on  $X$ ; see Remark 3.21.

It will be helpful to have the following extension result.

**Lemma 3.28.** Fix a normed vector space  $X$ , and let  $B \subseteq X$  be the closed unit ball. Suppose  $f: B \rightarrow \mathbb{F}$  is linear in the sense that  $f(x + y) = f(x) + f(y)$  and  $f(tz) = tf(z)$  whenever all inputs are in  $B$ . Then  $f$  has a unique linear extension  $F: X \rightarrow \mathbb{F}$ .

*Proof.* Define  $F(x) := f(x/\|x\|)$ . In fact, one finds that  $F(x) = tf(\frac{1}{t}x)$  whenever  $\|\frac{1}{t}x\| \leq 1$  by the given linearity; one can now check that  $F$  is linear. One can see that this construction of  $F$  is unique. ■

## 3.4 March 8

Today we complete our discussion of Banach spaces.

### 3.4.1 Alaoglu's Theorem

We are in the middle of proving the following result.

**Theorem 3.27 (Alaoglu).** Let  $X$  be a Banach space. Then the closed unit ball of  $X^*$  is compact in the weak-\* topology.

*Proof.* Let  $D := \{z \in \mathbb{F} : |z| \leq 1\}$  be the closed ball around 1, which we note is compact. Now, we set  $Y := D^B$  to be the infinite product, which we note is compact by Tychonoff's theorem.

We will turn compactness of  $Y$  to compactness of our closed unit ball  $B^* \subseteq X^*$ . As such, we define  $\varphi: B^* \rightarrow Y$  by

$$\varphi(f) := f|_B.$$

To see that  $f|_B$  is in fact a map  $B \rightarrow D$  note that  $f \in B^*$  must have  $|f(x)| \leq \|x\|$  for all  $x \in X$ , so in particular  $|f(x)| \leq 1$  for  $x \in B$ .

We claim that  $\varphi$  is a homeomorphism of  $B^*$  (with the weak topology) onto a closed subset of  $Y$ . This will finish the proof because the image  $\Lambda := \text{im } \varphi$  will be compact in  $Y$ , which then pulls back to compactness for  $B^*$  via our homeomorphism. So it remains to show our claim.

- We claim that  $\varphi$  is injective. Indeed, suppose that  $\varphi(f) = \varphi(g)$ . Then  $f|_B = g|_B$ , so  $f = g$  by Lemma 3.28.

The point is that  $\varphi: B^* \rightarrow \Lambda$  is now a bijection.

- We claim that  $\varphi: B^* \rightarrow \Lambda$  is a homeomorphism. This comes down to understanding the open subsets of  $B^*$  and of  $\Lambda$ .

For example, subbasic open subsets of  $\Lambda$  take the following form: choose some  $x_0 \in B$  and  $c \in \mathbb{F}$  and  $\varepsilon > 0$ , and we have a subbasic open subset of the form

$$\{g \in \Lambda : |g(x_0) - c| < \varepsilon\}.$$

Unraveling  $\Lambda$ , we see that we are looking at

$$\{f|_B \in \Lambda : |f(x_0) - c| < \varepsilon\}.$$

On the other hand, subbasic open subsets of  $B^*$  take the following form: choose some  $x_0 \in X$  and  $c \in \mathbb{F}$  and  $\varepsilon > 0$ , and we have a subbasic open subset of the form

$$\{f \in B^* : |f(x_0) - c| < \varepsilon\}.$$

(Namely, we are finding the subbasic open subset forced by continuity of  $\text{ev}_{x_0}: X^* \rightarrow \mathbb{F}$ .) It may appear that we are gaining subbasic open subsets by allowing  $x_0 \in X$  instead of restricting to  $x_0 \in B$  as previously, but in fact we gain nothing: if  $x_0 \notin B$ , simply replace  $x_0$  with  $x_0/\|x_0\|$  and  $c$  with  $c/\|x_0\|$  and  $\varepsilon$  with  $\varepsilon/\|x_0\|$ . So we see that our topologies are really the same, so we are done.

- We claim that  $\Lambda \subseteq Y$  is closed. Well, choose some  $g \in \overline{\Lambda}$ , and we claim that  $g(x+y) = g(x) + g(y)$  and  $g(tz) = tg(z)$  whenever the relevant inputs live in  $B$ . This will imply that  $g$  comes from some linear functional on  $X$  by Lemma 3.28, and in fact  $|g(x)| \leq \|x\|$  is forced because  $|g(x/\|x\|)| \leq 1$ , meaning  $g$  comes from some linear functional in  $B^*$ , as needed.

So it remains to show the claim of the previous paragraph. We will show that  $g(ax+by) = ag(x)+bg(y)$  whenever all inputs are in  $B$ . Well, note that

$$V := \{f \in Y : f(ax+by) = af(x) + bf(y)\}$$

is a closed subset of  $Y$ : the map  $f \mapsto (f(x+y) - (f(x) + f(y)))$  is continuous as the linear combination of projections from  $Y$ , and  $V$  is the pre-image of this map from  $\{0\}$ , so  $V$  is closed. Additionally,  $\Lambda \subseteq V$  because any element of  $\Lambda$  comes from a linear functional, so  $\overline{\Lambda} \subseteq V$  follows. Thus,  $g \in V$ . ■

**Remark 3.29.** One can use nets to prove Theorem 3.27, but Professor Christ is quite proud of never using nets, so we will not.

### 3.4.2 Hilbert Spaces

We will want the notion of inner product spaces.

**Definition 3.30 (inner product).** Fix an  $\mathbb{F}$ -vector space  $X$ . Then an *inner product*  $\langle \cdot, \cdot \rangle$  on  $X$  is a mapping  $X \times X \rightarrow \mathbb{C}$  satisfying the following.

- Additive:  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ .
- Conjugate symmetry:  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ .
- Positive definite:  $\langle x, x \rangle \geq 0$  for each  $x$  and only zero when  $x = 0$ .
- Scalar multiplication:  $\langle tx, y \rangle = t \langle x, y \rangle$ .

Inner products produce norms.

**Definition 3.31.** Fix an inner product space  $X$ . Then the *norm* on  $X$  is given by

$$\|x\| := \langle x, x \rangle^{1/2}.$$

We will not bother to check that  $\|\cdot\|$  is a norm, but it is. The most nontrivial check is the triangle inequality, which will be checked shortly in Proposition 3.36.

**Remark 3.32.** Note that

$$\|x - y\|^2 + \|x + y\|^2 = \langle x - y, x - y \rangle + \langle x + y, x + y \rangle \stackrel{*}{=} 2\langle x, x \rangle + 2\langle y, y \rangle = 2\|x\|^2 + 2\|y\|^2,$$

where  $\stackrel{*}{=}$  is by direct expansion. This is called the “parallelogram law.” It turns out that a norm satisfying the parallelogram law actually comes from an inner product. Explicitly, one can set

$$\langle x, y \rangle := \frac{\|x + y\|^2 - \|x - y\|^2}{4} + i \frac{\|ix - y\|^2 - \|ix + y\|^2}{4}$$

and check that the various identities hold by creative applications of the parallelogram law.

At long last, here is our definition.

**Definition 3.33 (Hilbert space).** A normed vector space  $X$  is a *Hilbert space* if and only if it is a complete inner product space (where the norm is induced by the inner product).

## 3.5 March 11

Here we go.

### 3.5.1 Bounding with Inner Products

Here are some fundamental bounding results.

**Proposition 3.34 (Cauchy–Schwarz).** Fix an inner product space  $X$ . Then each  $x, y \in X$  has

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$

*Proof.* Quickly, we take care of some special cases. If  $\langle x, y \rangle = 0$ , there is nothing to do. Let's also handle the case where  $x = 0$ . Then  $\langle 0, y \rangle = 0$  because  $\langle 0, y \rangle = \langle 0, y \rangle + \langle 0, y \rangle$  by bilinearity. So we conclude that  $\langle 0, y \rangle \leq \|0\| \cdot \|y\|$ .

Otherwise, by scaling everything via bilinearity, we may assume that  $\|x\| = \|y\| = 1$ . Then we find that

$$0 \leq \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y \rangle$$

by a direct expansion using the bilinearity. We now twist to get rid of  $\operatorname{Re}$ . Namely, choose  $\theta \in \mathbb{R}$  so that  $\langle e^{i\theta}x, y \rangle = -|\langle x, y \rangle|$ , doable because  $\langle e^{i\theta}x, y \rangle = e^{i\theta}\langle x, y \rangle$  allows us to adjust the angle of  $\langle x, y \rangle$  until it is a negative real number. As such, rerunning the argument at the start of the paragraph, we are able to conclude that

$$0 \leq \|x\|^2 + \|y\|^2 - 2|\langle x, y \rangle|,$$

so

$$2|\langle x, y \rangle| \leq \|x\|^2 + \|y\|^2 = 2 = 2\|x\| \cdot \|y\|,$$

where we are using  $\|x\| = \|y\| = 1$ . ■

**Remark 3.35.** Proposition 3.34 tells us that the map  $X \times X \rightarrow \mathbb{F}$  given by  $(x, y) \mapsto \langle x, y \rangle$  is continuous. Indeed, it is enough to check that this linear functional is bounded, so we note that  $|\langle x, y \rangle| \leq 1$  means  $\|x\|, \|y\| \leq 1$ , so

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \leq 1$$

by Proposition 3.34.

**Proposition 3.36 (Triangle inequality).** Fix an inner product space  $X$ . Then  $\|x + y\| \leq \|x\| + \|y\|$ .

*Proof.* It suffices to check that the squares have the inequality, but

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y \rangle.$$

Thus, Proposition 3.34 tells us that this is bounded above by  $\|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$ , which is precisely  $(\|x\| + \|y\|)^2$ . ■

We now take a moment for an example.

**Example 3.37.** Let  $L^2(\mathbb{N})$  denote the set of sequences  $\{x_n\}_{n=1}^\infty$  such that  $\sum_{n=1}^\infty |x_n|^2 < \infty$ . To see that this is a vector space, note that

$$\sum_{n=1}^\infty |ax_n + by_n|^2 = |a|^2 \sum_{n=1}^\infty |x_n|^2 + |b|^2 \sum_{n=1}^\infty |y_n|^2 + |ab|^2 \sum_{n=1}^\infty 2|x_n y_n|^2$$

Now,  $|x_n y_n| \leq |x_n|^2 + |y_n|^2$ , so the entire summation above absolutely converges. In fact, this same inequality tells us that

$$\langle x, y \rangle := \sum_{n=1}^\infty x_n \overline{y_n}$$

absolutely converges, and so this formula provides us with an inner product on  $L^2(\mathbb{N})$ . For example,  $\|x\|^2 = \langle x, x \rangle = \sum_{n=1}^\infty |x_n|^2$ .

**Remark 3.38.** One can check that  $L^2(\mathbb{N})$  is complete for its given norm, so in fact  $L^2(\mathbb{N})$  is a Hilbert space. The point is that a Cauchy sequence  $\{x_m\}_{m \in \mathbb{N}} \subseteq L^2(\mathbb{N})$  will converge component-wise because  $\|x_m - x_{m'}\| \geq |(x_m)_n - (x_{m'})_n|$  for each  $n$ , so we can define the desired limit sequence  $x$  so that  $(x_m)_n \rightarrow x_n$  as  $m \rightarrow \infty$ . Now the absolute convergence everywhere will tell us that  $x \in L^2(\mathbb{N})$  and that  $x_m \rightarrow x$  as  $m \rightarrow \infty$ .

### 3.5.2 Geometry in Hilbert Spaces

We now show the following nice geometric consequence: closest points to subspaces make sense.

**Lemma 3.39.** Fix a Hilbert space  $X$ . Given a closed subspace  $V \subseteq X$  and  $x \in X$  there is a unique  $v \in V$  such that

$$\|x - v\| = \inf_{u \in V} \|x - u\|.$$

*Proof.* We begin by showing existence of  $v$ . Let  $D$  be the minimal distance  $\inf\{\|x - v\| : v \in V\}$ . Note that  $D$  is finite because  $V$  is nonempty (e.g.,  $0 \in V$ ). Using this infimum, we are provided a sequence  $\{u_m\}_{m \in \mathbb{N}}$  such that  $\|x - u_m\| \rightarrow D$  as  $m \rightarrow \infty$ .

By continuity of  $\|\cdot\|$  (this is true even for normed vector spaces, or one can use Remark 3.35), it suffices to show that the sequence  $\{u_m\}_{m \in \mathbb{N}}$  converges in  $V$ , so we want to show that this sequence is Cauchy (and the limit will land in  $V$  because  $V$  is closed). Well, for  $m, n \in \mathbb{N}$ , we use the Parallelogram law Remark 3.32 to write

$$\|u_m - u_n\|^2 = \|(u_m - x) - (u_n - x)\|^2 = 2\|u_m - x\|^2 + 2\|u_n - x\|^2 - \|u_m + u_n - 2x\|^2.$$

The first two terms will be small, and the last term is  $4\|\frac{1}{2}(u_m + u_n) - x\|^2 \geq 4D^2$  because  $\frac{1}{2}(u_m + u_n) \in V$ . Thus,

$$\limsup_{m, n \rightarrow \infty} \|u_m - u_n\|^2 = 2 \limsup_{m, n \rightarrow \infty} \|u_m - x\|^2 + 2 \limsup_{n \rightarrow \infty} \|u_n - x\|^2 - 4D^2 = 0$$

because  $\|u_m - x\| \rightarrow D$  as  $m \rightarrow \infty$  (and similar for  $n$ ).

We now show that  $v$  is unique. Suppose that  $v_1$  and  $v_2$  both satisfy  $\|x - v_1\| = \|x - v_2\| = D$ . Then

$$\|v_1 - v_2\|^2 \leq 2\|v_1 - x\|^2 + 2\|v_2 - x\|^2 - 4D^2 = 0$$

by the above computations, so  $v_1 = v_2$  is forced. ■

With a notion of closest, we now talk about orthogonality.

**Definition 3.40** (orthogonal complement). Fix a subset  $S$  of an inner product space  $X$ . Then we define the *orthogonal complement*

$$S^\perp := \{v \in X : \langle x, v \rangle = 0 \text{ for all } x \in S\}.$$

**Remark 3.41.** One can check that  $S^\perp$  is always a subspace (by the bilinearity of the inner product). In fact,  $S^\perp$  is closed: it is the pre-image of 0 along the continuous map  $X \rightarrow \prod_{x \in S} \mathbb{F}$  defined by  $v \mapsto \{\langle x, v \rangle\}_{x \in S}$ . (This map is continuous because its coordinates are continuous: indeed, we want to check that the map  $v \mapsto \langle x, v \rangle$  is continuous, which is true.)

As a sanity check, orthogonal complements produce unique decomposition.

**Proposition 3.42.** Fix a Hilbert space  $X$  and a closed subspace  $V$ . Then  $V + V^\perp = X$ . In fact, for any  $x \in X$ , there are unique  $v \in V$  and  $u \in V^\perp$  such that  $x = v + u$ .

*Proof.* We begin with existence. Let  $v \in V$  be the closest vector to  $x$ , which exists by Lemma 3.39. We claim that  $\langle x - v, w \rangle = 0$  for all  $w \in V$ , which will complete the existence proof because then we may set  $u := x - v$ . The point is to consider the function  $F_w : \mathbb{R} \rightarrow \mathbb{C}$  defined by

$$F_w(t) := \|x - (v + tw)\|^2$$

for some fixed vector  $w \in V$ ; this is a continuous function in  $t$  because  $\|\cdot\|$  is continuous. Now,  $F_w$  is minimized at  $t = 0$  by assumption on  $v$ , so  $\frac{d}{dt}F_w|_{t=0} = 0$  provided we can show that this derivative makes sense. Well, we see that

$$F_w(t) = \|x - v\|^2 - 2t \operatorname{Re}\langle x - v, tw \rangle + t^2 \|w\|^2,$$

which is a polynomial in  $t$ , so this is indeed differentiable at 0, whereupon we can compute its derivative to find that  $\operatorname{Re}\langle x - v, w \rangle = 0$ . If  $\mathbb{F} = \mathbb{C}$ , this is enough to conclude that  $\langle x - v, w \rangle = 0$ , but in this case, we rerun the argument with  $w$  replaced by  $iw$  to see that  $\operatorname{Re}\langle x - v, iw \rangle = -i \operatorname{Im}\langle x - v, w \rangle$  also vanishes, so  $\langle x - v, w \rangle = 0$ .

It remains to check uniqueness. Suppose that  $v, v' \in V$  and  $u, u' \in V^\perp$  satisfies  $v + u = v' + u'$ . Then we want to check that  $v = v'$  and  $u = u'$ . Well, we see that

$$w := v - v' = u' - u$$

lives in  $V \cap V^\perp$ , but then  $\langle w, w \rangle = 0$  by definition of  $V^\perp$ , so  $\|w\|^2 = 0$ , so  $w = 0$ . ■



## BIBLIOGRAPHY

---

- [Fol99] Gerald B. Folland. *Real analysis*. Second. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xvi+386. ISBN: 0-471-31716-0.
- [Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.
- [Elb22] Nir Elber. *Introduction to Topology and Analysis*. 2022. URL: <https://dfoiler.github.io/notes/202A/notes.pdf>.

# LIST OF DEFINITIONS

---

absolute convergence, [30](#)  
affine, [21](#)  
algebra, [5](#)  
  
Banach space, [29](#)  
bounded, [31](#)  
  
category, [39](#)  
complete, [17](#)  
  
equivalent, [32](#)  
  
finite dimensional, [31](#)  
functional, [35](#)  
  
Hamel basis, [45](#)  
Hilbert space, [53](#)  
  
inner product, [53](#)  
  
linearly independent, [31](#)  
Lipschitz, [22](#)  
  
measurable rectangle, [7](#)

measure, [5](#)  
monotone class, [13](#)  
  
norm, [29](#)  
nowhere dense, [39](#)  
  
orthogonal complement, [55](#)  
  
partial order, [37](#)  
premeasure, [6](#)  
product, [42](#)  
product algebra, [7](#)  
product measure, [10](#)  
product premeasure, [9](#)  
  
reflexive, [50](#)  
regular, [19](#)  
  
 $\sigma$ -algebra, [4](#)  
 $\sigma$ -finite, [5](#)  
  
weak topology, [46](#)  
weak-\* topology, [49](#)