18.917: The Chromatic Splitting Conjecture

Nir Elber

Spring 2026

Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Contents 2

1 Introduction 3
- 1.1 February 2 . . . 3
  - 1.1.1 Idempotent Algebras . . . 3
  - 1.1.2 The Adams–Novikov Spectral Sequence . . . 4
  - 1.1.3 Back to Idempotent Algebras . . . 5
  - 1.1.4 Completion . . . 6
- 1.2 February 2 . . . 8
  - 1.2.1 Spectra . . . 8
  - 1.2.2 Module Categories . . . 9
  - 1.2.3 Some Free Constructions . . . 10
  - 1.2.4 The Transfer . . . 12

Bibliography 14

List of Definitions 15

THEME 1
INTRODUCTION

1.1 February 2

Here we go.

1.1.1 Idempotent Algebras

The goal of this class is to understand some topics related to the chromatic splitting conjecture. Thus, the first half of the class will try to understand the statement, and the second half of the class will explain how it relates to other problems in algebra.

Warning 1.1. All categories in this course are ∞-categories.

Example 1.2. Given a ring R, we have a stable, symmetric monoidal ∞-category $D(R)$ of chain complexes of R-modules, considered up to quasi-isomorphism. Notably, the symmetric monoidal structure is given by the derived tensor product.

We begin our story with idempotent algebras.

Definition 1.3 (idempotent algebra)**.** Fix a ring R. An *idempotent algebra* is an object $E \in D(R)$ equipped with a unit map $R \to E$ such that the composite

$$E = E \otimes_R R \to E \otimes_R E$$

is an equivalence.

Remark 1.4. Such an object E grants E a multiplication structure $E \otimes_R E \to E$, and E gains the structure of a differentially graded algebra.

Example 1.5. Consider $R = \mathbb{Z}$. Then for each prime p, the algebra $\mathbb{Z}_{(p)}$ is idempotent: localizing $\mathbb{Z}_{(p)}$ further at (p) does nothing!

Non-Example 1.6. The $\mathbb{Z}$-algebra $\mathbb{F}_p$ is not idempotent because the tensor product we are considering is derived. Indeed, we computed $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p$ last semester.

Here is a quick reason why one might care about idempotent algebras.

Theorem 1.7 (Neeman)**.** Fix a Noetherian ring R. Then the lattice of idempotent algebras is equivalent to the data of $\operatorname{Spec} R$ as a topological space.

Example 1.8. For $R = \mathbb{Z}$, it turns out that the idempotent algebras are either $\mathbb{Z}_{(p)}$ or $\mathbb{Q}$, and the maps between them look like the specializations of $\operatorname{Spec} R$.

Of course, we are homotopy theorists, so we have less reason to care about $\mathbb{Z}$. Recall that $\mathbb{Z}$ is obtained from $\mathbb{N}$ by formally adding inverses. But $\mathbb{N}$ is basically isomorphism classes of FinSet; if we had instead formally added inverses directly to FinSet (instead of taking isomorphism classes first), we would have found the sphere spectrum $\mathbb{S}$. In particular, we will be interested in the category $D(\mathbb{S})$ of $\mathbb{S}$-modules, also called spectra.

We now no longer have access to algebraic geometry directly on $\mathbb{S}$. Instead, Theorem 1.7 motivates us to look for the idempotent algebras for $\mathbb{S}$.

Remark 1.9. For any $x \in \pi_*\mathbb{S}$, there is an idempotent algebra $\mathbb{S}\left[x^{-1}\right]$. For example, $\pi_0\mathbb{S} = \mathbb{Z}$, so there is an idempotent algebra $\mathbb{S}_{(p)}$.

Here is our first main theorem.

Theorem 1.10 (Nishida)**.** Fix some $x \in \pi_*\mathbb{S}$ of positive degree. Then x is nilpotent.

Thus, the idempotent algebras $\mathbb{S}\left[x^{-1}\right]$ do not look genuinely "new." To get other idempotent algebras, we need more tools.

1.1.2 The Adams–Novikov Spectral Sequence

Recall the $\mathbb{S}$-algebra MU defined as the colimit of the embedding $\mathrm{BU} \to \mathrm{BLG}(\mathbb{S}) \subseteq \mathrm{Mod}(\mathbb{S})$. Let's compute its homotopy groups.

Definition 1.11 (formal group law)**.** Fix a commutative ring R. Then a *commutative formal group law* over R is a power series $f(x,y) \in R[[x,y]]$ satisfying

(a) $f(x,0) = x$ and $f(0,y) = y$,

(b) $f(x,y) = f(y,x)$, and

(c) $f(x,f(y,z)) = f(x,f(y,z))$.

Definition 1.12 (Lazard ring)**.** The *Lazard ring* is the ring L which is exactly the quotient of $\mathbb{Z}[\{a_{ij}\}_{ij}]$ by the relations dictating that
$$f(x,y) = \sum_{i,j\geq 0} a_{ij}x^iy^j$$
is a commutative formal group law.

Remark 1.13. In other words, L represents the collection formal group laws, in the sense that the data of a formal group law for a ring R amounts to the data of a ring homomorphism $L \to R$.

Remark 1.14. By definition, there is a "universal" formal group law f_L in L given exactly by

$$f_L(x,y) = \sum_{i,j\geq 0} a_{ij}x^iy^j.$$

Theorem 1.15 (Quillen)**.** The ring $\pi_*\mathrm{MU}$ is exactly the Lazard ring.

Remark 1.16. Quillen also computed $\pi_*(\mathrm{MU}\otimes_{\mathbb{S}}\mathrm{MU})$ as well as the two natural maps $\pi_*\mathrm{MU} \to \pi_*(\mathrm{MU}\otimes_{\mathbb{S}} \mathrm{MU})$. It turns out that this is more or less related to some notion of isomorphism of the formal group laws.

The use of MU is that it produces a spectral sequence with which we can understand $\pi_*\mathbb{S}$. By Čech descent along the map $\mathbb{S} \to \mathrm{MU}$, we see that $\mathbb{S}$ is the limit of the diagram

$$\mathrm{MU} \rightrightarrows \mathrm{MU}\otimes_{\mathbb{S}}\mathrm{MU} \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; \mathrm{MU}\otimes_{\mathbb{S}}\mathrm{MU}\otimes_{\mathbb{S}}\mathrm{MU} \qquad \cdots$$

which we can then truncate as $\operatorname{fil}^n\mathbb{S}$ in order to get a descending filtration to $\operatorname{fil}^0\mathbb{S}$. Computing homotopy along this filtration produces the desired spectral sequence, as soon as we compute homotopy groups of the various tensor powers of the MUs and so on.

Theorem 1.17 (Adams–Novikov)**.** Let $\mathcal{M}_{\mathrm{fg}}$ be the moduli space of formal groups. Then there is a spectral sequence

$$E_2 = \mathrm{H}^s(\mathcal{M}_{\mathrm{fg}};\omega^{\otimes t}) \Rightarrow \pi_{2t-s}\mathbb{S}.$$

Remark 1.18. It turns out that the spectral sequence is concentrated in the region $s \leq 2t-s$.

Example 1.19. Along the line $s = 2t-s$, there is some h_1 at $(s, 2t-s) = (1,1)$, and then we can take powers of it to go up the line. It turns out that h_1 survives the spectral sequence, and it goes to the "Hopf map" $\eta \in \pi_1\mathbb{S}$; however, $\eta^4 = 0$, though the Adams–Novikov spectral sequence cannot see it!

Thus, we see that the Adams–Novikov spectral sequence is not an amazing approximation: the E_2 page sees many classes which we know abstractly must vanish! Life is better if we pass to E_∞ instead; the following is our first main theorem.

Theorem 1.20 (Devinatz–Hopkins–Smith)**.** The E_∞ page of the Adams–Novikov spectral sequence lies under a curve which grows more slowly than any line.

Note that this immediately implies Theorem 1.10. On the other hand, we will see that the topological input of Theorem 1.10 plus some algebraic facts about formal group laws will prove the above big theorem.

Remark 1.21. The curve is known to be faster than logarithmic, but not much else is known. Our proof will not help us much because our proof of Theorem 1.10 will be ineffective.

1.1.3 Back to Idempotent Algebras

Let's return to trying to find some idempotent algebras.

Notation 1.22. Define the power series $[n] \in L[[x]]$ to be adding with f a total of n times.

Example 1.23. We see that $[2](x) = f(x,x)$ and $[5](x) = f(f(f(f(x,x),x),x),x)$.

Notation 1.24. Fix a prime p. Then we define the class $v_n \in \pi_* \mathrm{MU}$ to be the coefficient of x^{p^n} in the power series $[p](x)$.

Now, because localization is exact, we see that $\mathbb{S}_{(p)}$ is the limit of the nerve

$$\mathrm{MU}_{(p)} \rightrightarrows \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \otimes_{\mathbb{S}} \mathrm{MU}_{(p)} \qquad \cdots$$

so it is not unreasonable to consider the following limit.

Notation 1.25. Fix a prime p and some $n \geq 0$. Then we define $L_n \mathbb{S}_{(p)}$ as the limit of the following diagram.

$$\mathrm{MU}_{(p)}\left[v_n^{-1}\right] \rightrightarrows \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right] \otimes_{\mathbb{S}} \mathrm{MU}_{(p)}\left[v_n^{-1}\right]$$

We may abbreviate $L_n \mathbb{S}_{(p)}$ to $L_n \mathbb{S}$ if there is no possibility of confusion.

Remark 1.26. It turns out that there are natural maps $L_{n+1}\mathbb{S} \to L_n \mathbb{S}$.

These spectra $L_n \mathbb{S}$ give us new idempotent algebras, more or less granting us further understanding of the "spectrum" of $\mathbb{S}$.

Theorem 1.27 (Hopkins–Ravenel)**.** Fix a prime p and some $n \geq 0$. Then $L_n \mathbb{S}_{(p)}$ is an idempotent algebra.

Remark 1.28. Ravenel has conjectured that if E is a nonzero idempotent algebra under $\mathbb{S}_{(p)}$, then E is either $\mathbb{Q}$ or one of the $L_n \mathbb{S}$s. This was recently disproved. It is current work to attempt a classification.

Nonetheless, $\mathbb{S}_{(p)}$ can be understood well from the $L_n \mathbb{S}$s.

Theorem 1.29 (Hopkins–Ravenel)**.** Fix a prime p. Then $\mathbb{S}_{(p)}$ is the limit of the diagram

$$\cdots \to L_3 \mathbb{S}_{(p)} \to L_2 \mathbb{S}_{(p)} \to L_1 \mathbb{S}_{(p)}.$$

1.1.4 Completion

Continue with our fixed prime p. For motivation, we return to abelian groups.

Remark 1.30. For any $M \in D(\mathbb{Z})$, the p-localization sits in a pullback square

$$\begin{tikzcd} M_{(p)} \arrow[r] \arrow[d] & M_p^\wedge \arrow[d] \\ M \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] & M_p^\wedge \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}$$

more or less corresponding to finding the "lattice" $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$.

Analogously, there is a completion of $L_n\mathbb{S}$ which fits into a diagram

$$\begin{tikzcd} L_nE \arrow[r] \arrow[d] & L_{K(n)}E \arrow[d] \\ L_{n-1}E \arrow[r] & L_{n-1}L_{K(n)}E \end{tikzcd} \tag{1.1}$$

where $L_nE := L_n\mathbb{S} \otimes_{\mathbb{S}} E$. We are now ready to state the chromatic splitting conjecture.

Conjecture 1.31 (Chromatic splitting)**.** For any $n \geq 2$, the inclusion

$$L_{K(n)}\mathbb{S} \to L_{n-1}L_{K(n)}\mathbb{S}$$

is an inclusion of a direct summand.

Remark 1.32. This implies that the natural map

$$\mathbb{S}_p^\wedge \to \prod_{n\geq 1} L_{K(n)}\mathbb{S}$$

is the inclusion of a direct summand. The point is that the squares (1.1) are rather degenerate, which would let us compute the homotopy groups of $L_n\mathbb{S}$ from the completions.

Remark 1.33. Conjecture 1.31 is known at $n = 2$ and all primes, by work of many people.

The goal of the present class is to review the homotopy theory required to understand the statement of Conjecture 1.31 formally, and then we will discuss why perfectoid geometry may be useful to prove it.

Let's see why passing to $L_{K(n)}\mathbb{S}$ is genuinely easier.

Example 1.34. For $p > 2$, we can define $L_{K(1)}\mathbb{S}$ as the homotopy fiber of the endomorphism $\psi^g - 1$ of $\mathrm{KU}_p^\wedge$, where g is a choice of topological generator of $\mathbb{Z}_p^\times$, and ψ is some action of $\mathbb{Z}_p^\times$ on $\mathrm{KU}_p^\wedge$.

Theorem 1.35 (Goerss–Hopkins–Miller, Rogres)**.** Fix a prime p. For each $n \geq 1$, there is an $\mathbb{S}$-algebra E_n and a profinite group $\mathbb{G}_n$ for which

$$L_{K(n)}\mathbb{S} = (E_n)^{\mathbb{G}_n}.$$

In fact, E_n is a Galois extension of $L_{K(n)}\mathbb{S}$.

Remark 1.36. We will only be able to keep track of this sort of "infinite Galois theory" with condensed mathematics.

Remark 1.37. The profinite group $\mathbb{G}_n$ is some subgroup of automorphisms of formal group laws.

Remark 1.38. For any spectrum X, there is some "Galois descent"

$$L_{K(n)}X = \left(L_{K(n)}(E_n \otimes X)\right)^{\mathbb{G}_n}.$$

This generalizes to a spectral sequence

$$\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n; \pi_*(L_{K(n)}(E_n \otimes X))) \Rightarrow \pi_* L_{K(n)}X.$$

The previous remark produces a spectral sequence

$$\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n; \pi_* E_n) \Rightarrow \pi_* L_{K(n)}\mathbb{S}.$$

If p is large compared to n, then it turns out that the spectral sequence collapses for degree reasons, so we are reduced to a pure algebra problem.

The end of the course will be interested $L_{K(n-1)}L_{K(n)}\mathbb{S}_{(p)}$ for general n but p very large. Conjecture 1.31 tells us that this should be fairly easy to understand, so we can view the end of the course as trying to provide some evidence for the conjecture. For example, work in progress by many people has recently culminated in the following strategy.

Notation 1.39. Fix $\mathbb{B} \coloneqq E_{n-1} \otimes_{\mathbb{S}} L_{K(n-1)}E_n$.

Remark 1.40. It turns out that $\mathbb{B}$ is Galois over $L_{K(n-1)}L_{K(n)}\mathbb{S}$ with Galois group $\mathbb{G}_{n-1} \times \mathbb{G}_n$. Thus, we can hope to be able to use some Galois descent spectral sequence to understand $L_{K(n-1)}L_{K(n)}\mathbb{S}$, as in Remark 1.38.

Now, $\pi_*\mathbb{B}$ is a local ring, so one becomes motivated to consider a perfection $\widehat{\mathbb{B}}$. In particular, it turns out that there is a $\mathbb{G}_n \times \mathbb{G}_{n-1}$-equivariant map $\mathbb{B} \to \widehat{\mathbb{B}}$, so taking fixed points produces a map out of $L_{K(n-1)}L_{K(n)}\mathbb{S}$. This is the sort of thing that Conjecture 1.31 asks us to do! Of course, the target is related to the perfection $\widehat{\mathbb{B}}$, which we now want to understand.

Theorem 1.41. The groups $\mathrm{H}^*_{\mathrm{cts}}(\mathbb{G}_n \times \mathbb{G}_{n-1}; \pi_*\widehat{\mathbb{B}})$ is the same as the cohomology of the structure sheaf of some diamond related to the Fargues–Fontaine curve.

Let's explain the application to Conjecture 1.31: this calculation tells us that $(\widehat{\mathbb{B}})^{\mathbb{G}_n \times \mathbb{G}_{n-1}}$ is $L_{K(n-1)}\mathbb{S} \oplus \Sigma L_{K(n-1)}\mathbb{S}$, from which our small piece of Conjecture 1.31 follows!

1.2 February 2

Our next goal is to prove Theorem 1.10.

1.2.1 Spectra

Let's set some notation. We are interested in the category Spaces of spaces, also called anima.

Definition 1.42 (pointed space)**.** The category of pointed spaces is denoted Space_*.

Remark 1.43. There is a functor $(-)_+ \colon \mathrm{Spaces} \to \mathrm{Spaces}_*$ which simply adds a basepoint to any topological space.

Our next category will be spectra.

Definition 1.44 (spectra)**.** A *spectrum* is an infinite tuple $(X_0, X_1, \ldots)$ of spaces, equipped with isomorphisms $X_0 \cong \Omega X_1 \cong \Omega^2 X_1 \cong \cdots$. The category of spectra is, unsurprisingly, denoted $\mathrm{Spectra}$.

Remark 1.45. The functor $\Omega \colon \mathrm{Spectra} \to \mathrm{Spectra}$, given by shifting all the spaces down, is an autoequivalence. Its inverse functor is Σ. Thus, $\mathrm{Spectra}$ is a stable category.

Remark 1.46. The functor $\Omega^\infty\colon \mathrm{Spectra} \to \mathrm{Spaces}$ given by sending a spectrum X to X_0 admits an adjoint Σ^∞. We may denote the composite functor $X \mapsto \Sigma^\infty(X_+)$ by Σ^∞_+.

Remark 1.47. The category $\mathrm{Spectra}$ admits products and coproducts, which are the same and denoted $\oplus$.

Example 1.48. We let $\mathbb{S}$ denote the sphere spectrum $\Sigma^\infty S^0$. We can also think about this as $\Sigma^\infty_+\mathrm{pt}$.

Remark 1.49. The category $\mathrm{Spectra}$ admits internal Homs which has a left adjoint $\otimes_{\mathbb{S}}$. This gives a symmetric monoidal structure to $\mathrm{Spectra}$.

Remark 1.50. For any space X, the constant map $X \to \mathrm{pt}$ induces a map $\Sigma^\infty_+ X \to \mathbb{S}$. On the other hand, for any pointed space X, the canonical map $\mathrm{pt} \to X$ induces a map $\mathbb{S} \to \Sigma^\infty X$. Thus, when X is pointed, we find

$$\Sigma^\infty_+ X = \Sigma^\infty X \oplus \mathbb{S}.$$

1.2.2 Module Categories

We now pass to other rings.

Definition 1.51 ($\mathbb{E}_\infty$-ring)**.** An $\mathbb{E}_\infty$*-ring* R is a spectrum R equipped with a multiplication $R \otimes_{\mathbb{S}} R \to R$ and a unit $\mathbb{S} \to R$, as well as much other data (e.g., many other operations all required to cohere with each other).

One can define modules in an expected way.

Example 1.52. For any $\mathbb{E}_\infty$-ring R, the category $\mathrm{Mod}(R)$ of R-modules continues to be a stable, symmetric monoidal category, where the symmetric monoidal structure is given by some tensor product $\otimes_R$.

Remark 1.53. Conversely, given a symmetric monoidal stable category $\mathcal{C}$, we can recover $\mathcal{C}$ as a module category over the ring $R := \mathrm{End}_{\mathcal{C}}\,1$, where 1 is a tensor unit. Indeed, if $\mathcal{C} = \mathrm{Mod}_R$, we see that

$$\mathrm{Hom}_{\mathcal{C}}(R, R) = \mathrm{Hom}_{\mathbb{S}}(\mathcal{S}, R)$$

by the tensor–hom adjunction. This is then $\mathrm{Hom}_{\mathbb{S}}(\Sigma^\infty_+\mathrm{pt}, R)$, which is just $\Omega^\infty R$ because Σ^∞ is left adjoint to Ω^∞.

Here are some constructions with modules.

Definition 1.54. Fix an $\mathbb{E}_\infty$-ring R. For any $x \in \pi_i R$, we receive a spectrum map $x\colon \Sigma^i\mathbb{S} \to R$ and hence an R-module map $\Sigma^i R \to R$. We now define $R\left[x^{-1}\right]$ to be the R-module

$$\mathrm{colim}\left(R \overset{x}{\to} \Sigma^{-i}R \overset{x}{\to} \Sigma^{-2i}R \to \cdots\right).$$

Remark 1.55. By construction, there is a unit $R \to R\left[x^{-1}\right]$. Additionally, by understanding maps from $\mathbb{S}$ to the colimit, we see that $\pi_* R\left[x^{-1}\right] = (\pi_* R)\left[x^{-1}\right]$.

Note that we have only defined $R\left[x^{-1}\right]$ as a module. One may hope to upgrade it to a ring.

Remark 1.56. One can check that the canonical map

$$R \otimes_R R\left[x^{-1}\right] \to R\left[x^{-1}\right] \otimes_R R\left[x^{-1}\right]$$

is an equivalence: on homotopy groups, both sides are $(\pi_* R)\left[x^{-1}\right]$. It follows that $R\left[x^{-1}\right]$ is an idempotent algebra, defined by taking the inverse of the equivalence.

Remark 1.57. Here is another way to find that $R\left[x^{-1}\right]$ is a ring. Let $\mathcal{C}$ be the full subcategory of R-modules M such that the action of x on $\pi_* M$ is invertible. Then $\mathcal{C}$ is closed under the tensor product, so it is a stable symmetric monoidal category, and then Remark 1.53 allows us to find $R\left[x^{-1}\right]$ as the corresponding ring.

1.2.3 Some Free Constructions

Here is a way to construct an $\mathbb{E}_\infty$-ring.

Remark 1.58. Given a spectrum R, note that $X \otimes_{\mathbb{S}} X$ admits a natural action by the symmetric group Σ_2. This produces a functor $\mathrm{B}\Sigma_2 \to \mathrm{Spectra}$ which sends the one object to $X \otimes_{\mathbb{S}} X$ and the nontrivial morphism to the swapping action. We may let $(X \otimes_{\mathbb{S}} X)_{h\Sigma_2}$ denote the "quotient." In general, one can form the "quotient" $\left(R^{\otimes_{\mathbb{S}} n}\right)_{h\Sigma_n}$.

Remark 1.59. If R is an $\mathbb{E}_\infty$-ring, then there is a canonical map $(R \otimes_{\mathbb{S}} R)_{h\Sigma_2} \to R$ given by the commutativity. The associativity and commutativity produce a map

$$\left(R^{\otimes_{\mathbb{S}} n}\right)_{h\Sigma_n} \to R.$$

Note that the existence of this map requires us to remember the higher coherences present in the definition of the $\mathbb{E}_\infty$-ring R.

Definition 1.60 (free ring)**.** Fix a spectrum X. Then we define the *free $\mathbb{E}_\infty$-ring* $\operatorname{Free}_{\mathbb{E}_\infty} X$ to be

$$\operatorname{Free}_{\mathbb{E}_\infty} X := \mathbb{S} \oplus X \oplus (X \otimes_{\mathbb{S}} X)_{h\Sigma_2} \oplus \cdots .$$

The operation is induced by the construction.

Remark 1.61. The free rings have a grading, but the general ones do not.

Remark 1.62. There is an analogous notion of a free $\mathbb{E}_\infty$-space of a space X, which is

$$\operatorname{Free}_{\mathbb{E}_\infty} X := \mathrm{pt} \sqcup X \sqcup (X \times X)_{h\Sigma_2} \sqcup \cdots .$$

Notably, Σ^∞_+ preserves the symmetric monoidal structure and is a left adjoint, so one knows by pure nonsense that

$$\operatorname{Free}_{\mathbb{E}_\infty} \Sigma^\infty_+ X = \Sigma^\infty_+ \operatorname{Free}_{\mathbb{E}_\infty} X.$$

Recall that $\mathbb{E}_\infty$-spaces are interesting because they allow us to access many spectra.

Remark 1.63. If X is an $\mathbb{E}_\infty$-space, then $\pi_0 X$ is a commutative monoid.

Definition 1.64 (group-like)**.** An $\mathbb{E}_\infty$-space is *group-like* if and only if $\pi_0 X$ is a group.

Remark 1.65. The embedding from group-like $\mathbb{E}_\infty$-spaces to $\mathbb{E}_\infty$-spaces admits a right adjoint $\mathrm{B}\Omega$.

Theorem 1.66 (May)**.** Consider the category of group-like $\mathbb{E}_\infty$-spaces.

(a) There is a fully faithful embedding from the category of group-like $\mathbb{E}_\infty$-spaces to the category of spectra.

(b) Its essential image consists of those spectra with no homotopy groups in negative degrees.

(c) The inverse functor to the fully faithful embedding is Ω^∞.

Note that we currently have two ways to construct an $\mathbb{E}_\infty$-space: there is the free construction, but we could also take $\Omega^\infty \Sigma^\infty_+$ because Ω^∞ automatically outputs $\mathbb{E}_\infty$-spaces. These constructions are related, but a modification is required because Ω^∞ outputs group-like $\mathbb{E}_\infty$-spaces.

Theorem 1.67. For any space X, there is an isomorphism

$$\Omega\mathrm{B}\,\mathrm{Free}_{\mathbb{E}_\infty} X \cong \Omega^\infty \Sigma^\infty_+ X$$

of group-like $\mathbb{E}_\infty$-spaces.

Proof. We use the Yoneda lemma: for any group-like $\mathbb{E}_\infty$-space A, which we may naturally view as a spectrum, we see

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{E}_\infty}(\Omega^\infty \Sigma^\infty_+ X, A) &= \mathrm{Hom}_{\mathbb{S}}(\Sigma^\infty_+ X, A) \\
&= \mathrm{Hom}_{\mathrm{Spaces}}(X, A) \\
&= \mathrm{Hom}_{\mathbb{E}_\infty}(\mathrm{Free}_{\mathbb{E}_\infty} X, A) \\
&= \mathrm{Hom}_{\mathbb{E}_\infty}(\Omega\mathrm{B}\,\mathrm{Free}_{\mathbb{E}_\infty} X, A),
\end{aligned}$$

where the last equality holds because $\Omega^\infty A$ is group-like. ∎

Example 1.68. Note that $\Omega\mathrm{B}\,\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt} = \Omega^\infty \Sigma^\infty_+ \mathrm{pt}$, which is $\Omega^\infty \mathbb{S}$. On the other hand, $\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt}$ by definition(!) is

$$\mathrm{pt} \sqcup \mathrm{pt} \sqcup (\mathrm{pt} \times \mathrm{pt})_{h\Sigma_2} \sqcup \cdots,$$

which is $\mathrm{B}\Sigma_0 \sqcup \mathrm{B}\Sigma_1 \sqcup \mathrm{B}\Sigma_2 \sqcup \cdots$, also known as the category of finite sets up to isomorphism. Thus, we see that the "group completion" of the category of finite sets (up to isomorphism) is $\Omega^\infty \mathbb{S}$, indicating that the group completion of the category of finite sets ought to be $\mathbb{S}$.

We can also take some free $\mathbb{E}_\infty$-spaces of pointed spaces.

Definition 1.69. Fix a pointed space X. Then $\mathrm{Free}_{\mathbb{E}_\infty,*} X$ is the pushout of the following diagram.

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt} \arrow[r] \arrow[d] & \mathrm{Free}_{\mathbb{E}_\infty} X \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{Free}_{\mathbb{E}_\infty,*} X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Theorem 1.70 (May)**.** If X is a pointed space, then

$$\Omega \mathrm{B}\, \mathrm{Free}_{\mathbb{E}_\infty,*} X = \Omega^\infty \Sigma^\infty X.$$

Example 1.71. If X is a connected pointed space, then one can compute that $\mathrm{Free}_{\mathbb{E}_\infty,*} X$ is connected (intuitively, the connected components of $\mathrm{Free}_{\mathbb{E}_\infty} X$ have collapsed). Thus, $\mathrm{Free}_{\mathbb{E}_\infty,*} X$ is already group-like (it's π_0 is just a point!), so $\mathrm{Free}_{\mathbb{E}_\infty,*} X = \Omega^\infty \Sigma^\infty X$. Thus, we may view $\Omega^\infty \Sigma^\infty X$ as the "group completion" of X.

Here is an application of our "free" constructions which does not use the word "free."

Theorem 1.72 (Snaith, Jones)**.** Fix a connected pointed space X. Then

$$\Sigma^\infty_+ \Omega^\infty \Sigma^\infty X = \mathbb{S} \oplus \Sigma^\infty X \oplus (\Sigma^\infty X)^{\otimes 2}_{h\Sigma_2} \oplus \cdots .$$

Proof. Recall that we have a pushout

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{E}_\infty} \mathrm{pt} \arrow[r] \arrow[d] & \mathrm{Free}_{\mathbb{E}_\infty} X \arrow[d] \\
\mathrm{pt} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Free}_{\mathbb{E}_\infty,*} X
\end{tikzcd}$$

of $\mathbb{E}_\infty$-spaces. Note that the bottom-right is $\Omega^\infty \Sigma^\infty X$ by Theorem 1.70. Hitting this with the left adjoint Σ^∞_+ produces a pushout

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{E}_\infty} \mathbb{S} \arrow[r] \arrow[d] & \mathrm{Free}_{\mathbb{E}_\infty} \Sigma^\infty_+ X \arrow[d] \\
\mathbb{S} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma^\infty_+ \Omega^\infty \Sigma^\infty X
\end{tikzcd}$$

of $\mathbb{E}_\infty$-rings, where the top-right is has $\Sigma^\infty_+ X = \Sigma^\infty X \oplus \mathbb{S}$ and so

$$\begin{aligned}
\mathrm{Free}_{\mathbb{E}_\infty} \Sigma^\infty_+ X &= \mathrm{Free}_{\mathbb{E}_\infty} (\Sigma^\infty X \oplus \mathbb{S}) \\
&= \mathrm{Free}_{\mathbb{E}_\infty} \Sigma^\infty X \otimes_{\mathbb{S}} \mathrm{Free}_{\mathbb{E}_\infty} \mathbb{S} \\
&= \mathrm{Free}_{\mathbb{E}_\infty} \Sigma^\infty X.
\end{aligned}$$

The result now follows by computing our pushout. ■

Remark 1.73. This is a remarkable theorem! Basically, it tells us that taking Σ^∞_+ of some group-like $\mathbb{E}_\infty$-space should have homology which splits in a very natural way.

Remark 1.74. It turns out that $\Sigma^\infty_+ \Omega^\infty \mathbb{S} = \mathrm{Free}_{\mathbb{E}_\infty} \mathbb{S}\left[x^{-1}\right]$ for some class x arising from the class of a point (in degree one). Note that Theorem 1.72 does not apply here because the space S^0 is not connected! The proof of this statement amounts to "forcing" π_0 to be a group.

1.2.4 The Transfer

Here is a more complicated free example.

Example 1.75. Consider the category of finite Σ_2-sets, considered up to isomorphism. Such a set is a union of some points and some two-element sets with a swapping action. Enumerating all such finite sets, we find that this category is $\mathrm{Free}_{\mathbb{E}_\infty}(\mathrm{pt} \sqcup \mathrm{B}\Sigma_2)$. By Theorem 1.67, we thus see that

$$\Omega \mathrm{B}\,\mathrm{Free}_{\mathbb{E}_\infty}(\mathrm{pt} \sqcup \mathrm{B}\Sigma_2) = \Omega^\infty \left(\mathbb{S} \oplus \Sigma^\infty_+ \mathrm{B}\Sigma_2\right).$$

Continuing, the example, we note that there is a forgetful functor $\mathrm{FinSet}(\Sigma_2) \to \mathrm{FinSet}$ which descends to isomorphism classes. By taking group completion, one receives a map

$$\Sigma^\infty_+ \mathrm{B}\Sigma_2 \oplus \mathbb{S} \to \mathbb{S},$$

so there is a "transfer" map $\mathrm{tr}\colon \Sigma^\infty_+ \mathrm{B}\Sigma_2 \to \mathbb{S}$. (Note that we have silently removed our Ω^∞, which we can do because the forgetful functor preserves disjoint unions, so Ω^∞ can be removed by Theorem 1.66 because these things are group-like.)

Remark 1.76. The canonical map $\mathbb{RP}^\infty \to \mathrm{pt}$ produces a different map $\varepsilon\colon \Sigma^\infty_+ \mathrm{B}\Sigma_2 \to \mathbb{S}$.

Remark 1.77 (Siegel's conjecture)**.** It turns out that all maps $\Sigma^\infty_+ \mathrm{B}\Sigma_2 \to \mathbb{S}$ can be obtained from tr and ε.

Theorem 1.78 (Kahn–Priddy)**.** The map $\pi_* \mathrm{tr}\colon \pi_* \Sigma^\infty_+ \mathrm{B}\Sigma_2 \to \pi_* \mathbb{S}$ is surjective in positive degrees.

Bibliography

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

List of Definitions

$\mathbb{E}_\infty$-ring, 9

formal group law, 4
free ring, 10

group-like, 11

idempotent algebra, 3

Lazard ring, 4

pointed space, 8

spectra, 8