214: Differential Topology

Nir Elber

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Introduction

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.

—Charles Hermite

1.1 January 16

Let's just get started.

1.1.1 Course Structure

Here are some quick notes.

- There is a bCourses page: `https://bcourses.berkeley.edu/courses/1533116`. For example, it has the syllabus.
- The textbook is Lee's *Introduction to Smooth Manifolds* [Lee13]. We will read most of it.
- Our instructor is Professor Eric Chen, whose email can be reached at `ecc@berkeley.edu`. Office hours are after class in Evans 707.
- There is a GSI, who is Tahsia Saffat, whose email is `tahsin_saffat@math.berkeley.edu`. He will have some office hours and grade some homeworks.
- Homework will in general be due at 11:59PM on Thursdays via Gradescope.
- There will be an in-class midterm and a final.
- Grading is 30% homework, 30% midterm, and 40% final.
- This is a math class, not so geared towards applied subjects.
- In particular, we will assume a fair amount of topology, for which we use [Elb22] as a reference.

Let's also give a couple of notes on the course content. This course is on differential topology. The topology of interest will come from manifolds, and the differential part comes from some smoothness properties.

In some sense, our goal is to "do calculus" (e.g., differentiation, integration, vector fields, etc.) on spaces which look locally like some Euclidean space, such as a sphere. We also want to understand (smooth) manifolds on their own terms, such as understanding the maps between them and understanding some classical examples and constructions such as Lie groups or quotient manifolds.

1.1.2 Topology Review

Anyway let's get started. This is a class on manifolds, so perhaps we should begin by defining a manifold. These are going to form a special kind of topological space, so let's review topologies. We will freely use topological facts which we are too lazy to prove from [Elb22].

Definition 1.1 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X satisfying the following.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
- Finite intersection: given $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Union: for any subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have the union $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We say that the collection $\mathcal{T}$ is the collection of *open sets* of X. We will also suppress the collection $\mathcal{T}$ from the notation as much as possible.

Here is some helpful language.

Definition 1.2 (open, closed, neighborhood)**.** Fix a topological space $(X, \mathcal{T})$.

- An *open subset* $U \subseteq X$ is a subset in $\mathcal{T}$.
- A *closed subset* $V \subseteq X$ is one with $X \setminus V \in \mathcal{T}$.
- A *neighborhood* of a point $p \in X$ is an open subset $U \subseteq X$ containing p.

Example 1.3. Fix a metric space (X, d). Then there is a topology given by the metric. To be explicit, a set $U \subseteq X$ is open if and only if each $p \in U$ has some $\varepsilon > 0$ such that

$$\{x \in X : d(x, p) < \varepsilon\} \subseteq U.$$

See [Elb22, Example 2.13] for the details.

Sometimes it is easier to generate a topology from some subcollection.

Definition 1.4 (base)**.** Fix a topological space $(X, \mathcal{T})$. A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is a *base* for $\mathcal{T}$ if and only if the following holds: for each open $U \subseteq X$ and point $p \in U$, there is some $B \in \mathcal{B}$ such that $p \in B$ and $B \subseteq U$.

Example 1.5. Fix a metric space (X, d). Then the collection $\mathcal{B}$ of open balls

$$B(p, \varepsilon) :=,$$

over all $p \in X$ and $\varepsilon > 0$, forms a base of the topology. This is immediate from the construction of the topology in Example 1.3. In fact, one can merely take $\varepsilon \in \mathbb{Q}^+$ because $\mathbb{Q}$ is dense in $\mathbb{R}$.

With our objects of topological spaces in hand, we should discuss the maps between them.

Definition 1.6 (continuous)**.** Fix topological spaces X and X'. A function $\varphi\colon X \to X'$ is *continuous* if and only if $\varphi^{-1}(U')$ is open for each open $U' \subseteq X'$.

Definition 1.7 (homeomorphism)**.** Fix topological spaces X and X'. A function $\varphi\colon X \to X'$ is a *homeomorphism* if and only if φ is a bijection and both φ and φ^{-1} are continuous. We may write $X \cong X'$.

Remark 1.8. There is a continuous bijection $[0, 2\pi) \to S^1$ by $\theta \mapsto (\cos\theta, \sin\theta)$, but it is not a homeomorphism. (Here, both sets have the metric topology.) In particular, the inverse map is not continuous at 1 because the pre-image of $[0, \pi)$ is the subset $\left\{(x, y) \in S^1 : y > 0\right\} \cup \{(0, 0)\}$, which is not open in S^1 (because no $\varepsilon > 0$ has $B((0, 0), \varepsilon)$ lying in $\left\{(x, y) \in S^1 : y \geq 0\right\}$).

We would also like to be able to build new topologies from old ones.

Definition 1.9 (subspace)**.** Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we form a *subspace topology* by declaring the open subsets to be

$$\{U \cap S : U \in \mathcal{T}\}.$$

Example 1.10. The metric topology on $\mathbb{R}$ and the subspace topology on $X := \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ are homeomorphic. Namely, the homeomorphism sends $x \mapsto (x, 0)$, and the inverse map is $(x, 0) \mapsto x$. Here are our continuity checks.

- The map $x \mapsto (x, 0)$ is continuous: the pre-image V of an open subset $U \subseteq X$ is open. Namely, for any $x \in V$, we see $(x, 0) \in V$, so there is $\varepsilon > 0$ such that $B((x, 0), \varepsilon) \cap X \subseteq U$, so $B(x, \varepsilon) \subseteq V$.
- The map $(x, 0) \mapsto x$ is continuous: the pre-image V of an open subset $U \subseteq \mathbb{R}$ is open. Namely, for each $(x, 0) \in V$, we see $x \in U$, so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V$, so $B((x, 0), \varepsilon) \cap X \subseteq U$.

Lastly, we will want some adjectives for our topologies.

Definition 1.11 (compact)**.** Fix a topological space X. A subset $K \subseteq X$ is *compact* if and only if any open cover can be reduced to a finite subcover. Explicitly, any collection $\mathcal{U}$ of open sets of X such that $K \subseteq \bigcup_{U \in \mathcal{U}} U$ (this is called an *open cover*) has some finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{U \in \mathcal{U}'} U$.

Example 1.12. The interval $[0, 1] \subseteq \mathbb{R}$ is compact. See [Elb22, Example 4.4].

Definition 1.13 (Hausdorff)**.** Fix a topological space X. Then X is *Hausdorff* if and only if any two distinct points $p_1, p_2 \in X$ have disjoint open subsets $U_1, U_2 \subseteq X$ such that $p_1 \in U_1$ and $p_2 \in U_2$.

Example 1.14. Any metric space (X, d) is Hausdorff. Namely, for distinct points $p, q \in X$, we see $d(p, q) > 0$, so set $\varepsilon := d(p, q)/2$, and we see that $p \in B(p, \varepsilon)$ and $q \in B(q, \varepsilon)$, but $B(p, \varepsilon) \cap B(q, \varepsilon) = \varnothing$. For this last claim, we note r living in the intersection would imply

$$d(p, q) \leq d(p, r) + d(r, q) < 2\varepsilon,$$

which is a contradiction to the construction of ε.

1.1.3 Topological Manifolds

For intuition, we state but not prove the following result.

Theorem 1.15 (Topological invariance of dimension)**.** Fix open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. If there is a homeomorphism $U \cong V$, then $m = n$.

Proof. The usual proofs go through (co)homology, which we may cover later in the class. For the interested, see [Elb23, Proposition 3.50]. ∎

We will soon define topological manifolds. The main adjective we want is being "locally Euclidean."

Definition 1.16 (locally Euclidean)**.** Fix a topological space X. Then X is *locally Euclidean of dimension* n *at* p if and only if there is an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$. We say that X is *locally Euclidean of dimension* n if and only if it is locally Euclidean of dimension n at each point.

Remark 1.17. One can always take $\widetilde{U}$ to be either $B(0,1) \subseteq \mathbb{R}^n$ or even all of $\mathbb{R}^n$.

Let's explain why we want Theorem 1.15.

Lemma 1.18. Fix a locally Euclidean space X. For each $p \in X$, there is a unique nonnegative integer n such that there exists an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$.

Proof. Suppose there are two such nonnegative integers m and n, so we get open neighborhoods $U, V \subseteq X$ and $\widetilde{U} \subseteq \mathbb{R}^m$ and $\widetilde{V} \subseteq \mathbb{R}^n$. Let $\varphi\colon U \cong \widetilde{U}$ and $\psi\colon V \cong \widetilde{V}$ be the needed homeomorphisms. Then the point is to use the intersection $U \cap V$: there is a composite isomorphism

$$\varphi(U \cap V) \cong U \cap V \cong \psi(U \cap V)$$

from an open subset in $\mathbb{R}^m$ to an open subset in $\mathbb{R}^n$. So Theorem 1.15 completes the proof. ∎

Anyway, here is our definition of a topological manifold.

Definition 1.19 (topological manifold)**.** An n*-dimensional topological manifold* is a topological space M with the following properties.

- M is Hausdorff.
- M is locally Euclidean of dimension n at each point.
- M is second countable (i.e., has a countable base).

We may abbreviate "n-dimensional topological manifold" to "topological n-manifold."

Let's give a few quick constructions.

Lemma 1.20. For each $n \geq 0$, the space $\mathbb{R}^n$ is an n-dimensional topological manifold.

Proof. Let's be quick. Being a metric space yields Hausdorff, locally Euclidean is immediate because it's $\mathbb{R}^n$, and second-countability follows by using the base

$$\left\{B(q,\varepsilon) : q \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}^+\right\}.$$

This is indeed a base because $\mathbb{Q}$ is dense in $\mathbb{R}$. Explicitly, for each $p \in \mathbb{R}^n$ living in some open subset $U \subseteq \mathbb{R}^n$, begin by replacing U with a smaller open subset of the form $B(p, \varepsilon)$ where $\varepsilon > 0$; by perhaps making ε smaller, we may assume that $\varepsilon > 0$ is rational. Now, choosing coordinates $p = (x_1, \ldots, x_n)$, choose rational numbers $q_1, \ldots, q_n$ so that $|x_i - q_i| < \varepsilon/(2\sqrt{n})$ for each i. Then $q := (q_1, \ldots, q_n)$ has $d(p, q) < \varepsilon/2$ and so

$$p \in B(q, \varepsilon/2) \subseteq B(p, \varepsilon) \subseteq U,$$

so $B(q, \varepsilon/2)$ is the needed open subset in our base. ■

The following lemma will be helpful in the sequel.

Lemma 1.21. Fix a topological space M and nonnegative integer $n \geq 0$. Suppose that there is a countable open cover $\{U_i\}_{i \in \mathbb{N}}$ of M such that each i has a homeomorphism $U_i \cong \widetilde{U}_i$ where $\widetilde{U}_i \subseteq \mathbb{R}^n$ is open. Then M is locally Euclidean of dimension n at each point, and M is second countable.

Proof. For locally Euclidean, we note that each $p \in M$ lives in some U_i, so we are done. As for second countability, we note that each $\widetilde{U}_i$ is second countable as a subspace of a second countable space (see Lemma 1.20), so each U_i is second countable by moving back through the homeomorphism, and so M is second countable by taking the union of the bases of the U_i.

To make this last step more explicitly, we note that each U_i has a countable base $\mathcal{B}_i$, so we claim that $\mathcal{B} := \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ becomes a countable base of M. Certainly $\mathcal{B}$ is countable, and every set in $\mathcal{B}$ is in one of the $\mathcal{B}_i$ and hence open in M. Lastly, to check that we have a base, we note that any open $U \subseteq M$ and $p \in M$ will have $p \in U_i$ for some i, so there is some $B \in \mathcal{B}_i \subseteq \mathcal{B}$ such that $p \in B \subseteq U \cap U_i$. ■

1.1.4 Examples and Non-Examples

Here are some non-examples to explain why we want all of these hypotheses.

Exercise 1.22. Consider the space X defined as $\mathbb{R} \times \{0, 1\}$ where we identify $(x, 0) \sim (x, 1)$ whenever $x \neq 0$. (The topology on X is the quotient topology [Elb22, Definition 2.81].) This space is not Hausdorff, but it is locally Euclidean and second countable.

Proof. We run our checks.

- This space is not Hausdorff because the points $(0, 0)$ and $(0, 1)$ are "infinitely close together." Explicitly, any open neighborhoods U and V of $(0, 0)$ and $(0, 1)$, respectively, the induced topology yields some $\varepsilon > 0$ such that $B((0, 0), \varepsilon) \subseteq U$ and $B((0, 1), \varepsilon) \subseteq V$, but then $(-\varepsilon/2, 0) = (-\varepsilon/2, 1)$ is in both U and V.
- This space is locally Euclidean and second countable by Lemma 1.21. Explicitly, we note that $\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq X$ and $\mathbb{R} \cong \mathbb{R} \times \{1\} \subseteq X$ by an argument similar to Example 1.10. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.21. ■

Exercise 1.23. Consider the space X defined as $\mathbb{R} \times \{0, 1\}$ where we identify $(x, 0) \sim (x, 1)$ whenever $x \leq 0$, again where we are using the quotient topology. Then X is Hausdorff and second countable, but it is not Euclidean of dimension 1 at $0 \in X$.

Proof. We run our checks.

- This space is Hausdorff. We check this directly by casework.
  - Suppose we have distinct points $p = (x, a)$ and $q = (y, b)$ with $x \neq y$; for example, this includes the case where we may take $a = b$ and hence includes the case when $x, y \leq 0$. Then we may set $\varepsilon := \frac{1}{2}|x - y|$ so that $B(p, \varepsilon)$ and $B(q, \varepsilon)$ are disjoint.

  - We now may assume that $x = y$; then $a \neq b$. Thus, we must have $x > 0$ or $y > 0$. As such, we may as well take $\varepsilon := \min\{|x|, |y|\}$ so that $B(p, \varepsilon)$ and $B(q, \varepsilon)$ are disjoint.

- This space is not locally Euclidean at 0. Indeed, suppose that there is open subset $U \subseteq X$ around 0 which is homeomorphic to an open subset of $\mathbb{R}$. By shifting, we may as well assume that the homeomorphism sends 0 to 0. Additionally, the same statement will be true by any open subset of U, so we may as well as assume that U is of the form $(-\varepsilon, \varepsilon) \times \{0, 1\}$ (in X). In particular, U is connected.

  But then the image $\widehat{U}$ of U in $\mathbb{R}$ is a connected open subset of $\mathbb{R}$, which must be an interval. Now, intervals have the property that deleting any point of an interval makes produces a topological space with two connected components. However, deleting 0 from U will produce three connected components: $(-\varepsilon, 0) \times \{0, 1\}$ and $(0, \varepsilon) \times \{0\}$ and $(0, \varepsilon) \times \{1\}$. So $\widehat{U}$ and U cannot actually be homeomorphic!

- This space is second countable by Lemma 1.21. Again, we note that $\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq X$ and $\mathbb{R} \cong \mathbb{R} \times \{1\} \subseteq X$ by an argument similar to Example 1.10. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.21. ■

Remark 1.24. Essentially the same argument implies that the above space fails to be locally Euclidean of any dimension at $0 \in X$. Namely, a connected open subset of $\mathbb{R}^n$ for $n \geq 2$ will remain connected after removing any point, so it cannot be homeomorphic to $(-\varepsilon, \varepsilon) \times \{0, 1\}$ in X.

Morally, the second countability is being required as a smallness condition; let's see some pathological examples without second countability. The following lemma approximately explains the problem.

Lemma 1.25. Fix a topological space X. Suppose that there is an uncountable subset $Y \subseteq X$ such that each $y \in Y$ has an open neighborhood $U_y \subseteq X$ where the U_y are pairwise disjoint. Then X fails to be second countable.

Proof. Suppose we have a base $\mathcal{B}$; we show $\mathcal{B}$ is uncountable. Each $y \in U_y$ has some $B_y \in \mathcal{B}$ with $B_y \subseteq U_y$. However, $y \neq y'$ implies that $B_y \neq B_{y'}$ because $y \in B_y$ while $p_y \notin U_{y'}$ implies $p_y \notin B_{y'}$. So $\{B_y\}_{y \in Y}$ is an uncountable subcollection of $\mathcal{B}$. ■

Exercise 1.26. Consider an uncountable set S with the discrete topology (namely, every subset is open), and then we form the product $X := \mathbb{R} \times S$. Then X is Hausdorff, locally Euclidean of dimension 1, but it is not second countable.

Proof. Here are our checks.

- Note that X is a product of Hausdorff spaces and hence is Hausdorff.
- This space is locally Euclidean of dimension 1: for each $(x, s) \in X$, we note that $\mathbb{R} \times \{s\}$ is an open subset of X (because S is discrete) where $\mathbb{R} \times \{s\} \cong \mathbb{R}$ by an argument similar to Example 1.10.
- This space is not second countable by Lemma 1.25. Namely, we have the uncountably many points $p_s := (0, s)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \mathbb{R} \times \{s\}$. ■

Exercise 1.27. Consider the first uncountable ordinal ω_1. Then define $X := (S \times [0, 1)) \setminus \{(0, 0)\}$, and we give X the order topology where the ordering is lexicographic. (Namely, the base consists of the "intervals" $\{x : x < b\}$ or $\{x : a < x\}$ or $\{x : a < x < b\}$.) This space is Hausdorff, locally Euclidean 1, but it is not second countable.

Proof. Here are our checks.

- This space is Hausdorff because it is a dense linear order. Explicitly, for $(s,a),(t,b) \in X$, we have the following cases.
  - Suppose $s = t$. In this case, $a \neq b$; suppose $a < b$ without loss of generality. Then $\{x : x < (s,(a+b)/2)\}$ and $\{x : x > (s,(a+b)/2)\}$ are the needed open sets.
  - Suppose $s \neq t$; take $s < t$ without loss of generality. If $a > 0$, then $\{s\} \times (0,(a+1)/2)$ and $\{s\} \times ((a+1)/2, 1) \cup \{t\} \times [0,1)$ provide the needed open sets. Otherwise, if $a = 0$, then $\{x : x < (s,1/2)\}$ and $\{x : x > (s,1/2)\}$ provide the needed open sets.
- This space is locally Euclidean of dimension 1: fix any $(s,r) \in X$. Note that $s \in \omega_1$ is countable, so we claim that
$$(s+1) \times [0,1) \cong [0,1),$$
sending $(0,0)$ to 0, from which the claim follows by deleting $(0,0)$. Because the relevant orders produce the needed topologies, we are really asking for an order-preserving bijection from $(s+1) \times [0,1)$ to $[0,1)$.

  Well, for any $t \in \omega_1$, we claim that there is an increasing sequence $\{p_\alpha\}_{\alpha<t} \subseteq [0,1)$ of order type t with $p_0 = 0$, from which the claim will follow by taking $s = t$ and sending $\alpha \times [0,1) \subseteq (s+1) \times [0,1)$ to $[p_\alpha, p_{\alpha+1})$ (where we define $p_s := 1$). To see this claim, we argue by induction on s. For $s = 0$, take $p_0 := 0$. If s is a successor ordinal, divide all the existing p_α by 2 and then set $p_{s+1} := 1/2$.

  Lastly, if s is a limit ordinal, it is still only a countable limit ordinal, so we can find an increasing sequence of countable ordinals $\{s_i\}_{i\in\omega}$ approaching s. The sequence corresponding to s_0 will fit into $[0,1/2)$ after scaling; then the sequence corresponding to s_1 but after s_0 will fit into $[1/2,2/3)$ after scaling. We can continue this process inductively to complete the claim for s. I won't bother to write out the details.
- This space is not second countable by Lemma 1.25. Namely, we have the uncountably many points $p_s := (s,1/2)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \{s\} \times (0,1)$. ■

Remark 1.28. What makes the locally Euclidean check above annoying is that we must show $(\omega,0) \in X$ has a neighborhood isomorphic to an open subset of $\mathbb{R}$, which is not totally obvious.

Let's return to examples.

Example 1.29. Consider the unit circle S^1. We check that S^1 is a 1-dimensional topological manifold.

- S^1 is a metric space, so it is Hausdorff.
- S^1 is second countable: it is a subspace of $\mathbb{R}^2$, and $\mathbb{R}^2$ is second countable by Lemma 1.20 again.
- S^1 is locally Euclidean: we proceed explicitly. Define $U_1^\pm := \{(x,y) \in S^1 : \pm x > 0\}$; then $U_1^\pm \cong (-1,1)$ by $(x,y) \mapsto y$. Similarly, define $U_2^\pm := \{(x,y) \in S^1 : \pm y > 0\}$; then $U_2^\pm \cong (-1,1)$ by $(x,y) \mapsto x$.

1.2 January 18

The first homework has been posted. It is mostly a review of point-set topology things. It is due on the 25th of January.

Remark 1.30. Please read the section on fundamental groups of manifolds on your own. We will not discuss it in class.

To review, our current goal is to define smooth manifolds. Thus far we have defined a topological space and provided enough adjectives to turn it into a topological manifold. To proceed, we need to add smoothness to our structure. We will do this later.

1.2.1 Connectivity

For now, we will content ourselves with some extra adjectives for our topological manifolds which will later be helpful. Here are two notions of connectivity.

Definition 1.31 (connected)**.** Fix a topological space X. Then X is *disconnected* if and only if there exist disjoint nonempty open subsets $U, V \subseteq X$ such that $X = U \sqcup V$. If X is not disconnected, we say that X is connected.

Example 1.32. The interval $[0, 1]$ is connected. See [Elb22, Lemma A.6].

Remark 1.33. Equivalently, we can say that X is connected if and only if X and $\varnothing$ are the only subsets of X which are both open and closed.

Definition 1.34 (path-connected)**.** Fix a topological space X. Then X is *path-connected* if and only if any two points $p, q \in X$ has some continuous map $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Example 1.35. The space $B(0, 1) \subseteq \mathbb{R}^n$ is path-connected. Indeed, we show that the path-connected component of 0 is all of $B(0, 1)$; see [Elb22, Definition A.19]. In other words, we must exhibit a path from 0 to v for any $v \in B(0, 1)$. Well, define $\gamma\colon [0, 1] \to B(0, 1)$ by $\gamma(t) := tv$. This is continuous because it is linear, and it has $\gamma(0) = 0$ and $\gamma(1) = v$ as desired.

In general, these two notions do not coincide.

Example 1.36. Consider the topological space

$$X := \{(x, \sin(1/x)) : x \in (0, 1)\} \cup \{(0, y) : y \in \mathbb{R}\}.$$

Then X is connected, but it is not path-connected. See [Elb22, Exercise A.20].

But one does in general apply the other.

Lemma 1.37. Fix a topological space X. If X is path-connected, then X is connected.

Proof. See [Elb22, Lemma A.16], though we will sketch the proof. We proceed by contraposition. Suppose that X is disconnected, so we may write $X = U \sqcup V$ where $U, V \subseteq X$ are disjoint nonempty open subsets. Now choose some $p \in U$ and $q \in V$, and we claim that there is no path $\gamma\colon [0, 1] \to X$. Indeed, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would then be nonempty disjoint open subsets of $[0, 1]$ covering $[0, 1]$, which is a contradiction. ■

However, for topological manifolds, these notions do coincide.

Proposition 1.38. Fix a topological n-manifold M. Then M is path-connected if and only if it is connected.

Proof. The forward direction is by Lemma 1.37. Thus, we focus on showing the converse. Fix some $p \in M$, and we define the subset

$$U_p := \{q \in M : \text{there exists a path from } p \text{ to } q\}.$$

This is the path-connected component of p in M; see [Elb22, Definition A.19]. The main claim is that U_p is open.

Suppose $q \in M$, and we need to find an open neighborhood $B_q \subseteq M$ of q living inside U_p. Noting then that $U_p = \bigcup_{q \in U_p} B_q$ will complete the proof of this claim. Well, q has some open neighborhood $B \subseteq M$ equipped with a homeomorphism $\varphi \colon B \cong B(0,1)$ by Remark 1.17. Then $B(0,1)$ is path-connected by Example 1.35, so B is path-connected by going back through the homeomorphism. Thus, because U_p is an equivalence class, it is also the path-connected equivalence class of q, so U_p must contain B.

Now, let $\mathcal{U}$ denote the collection of path-connected components of M. This is a collection of disjoint open subsets covering M. Certainly it is nonempty, so select $U \in \mathcal{U}$. Then we write

$$M = U \cup \bigcup_{U' \in \mathcal{U} \setminus \{U\}} U'.$$

This is a decomposition of M into disjoint open subsets, so because M is connected, one of these must be empty. But U is empty, so instead the union of the U' must be nonempty. However, everything in $\mathcal{U}$ is nonempty, so instead we see that $\mathcal{U} \setminus \{U\}$ is empty, so $M = U$ is path-connected. ■

1.2.2 Local compactness

Here is our definition.

Definition 1.39 (local compactness)**.** Fix a topological space X. Then X is *locally compact* if and only if any $x \in X$ has some open neighborhood $U \subseteq X$ such that there exists a compact subset $K \subseteq X$ containing U.

Remark 1.40. If X is Hausdorff, then compact subsets are closed [Elb22, Corollary 4.13], and closed subsets of a compact space are still compact [Elb22, Lemma 4.10], so we may as well take $K = \overline{U}$ in the above definition.

The above remark motivates the following definition.

Definition 1.41 (precompact)**.** Fix a topological space X. An open subset $U \subseteq X$ is *precompact* if and only if $\overline{U}$ is compact.

Remark 1.42. Here is a quick check which will prove to be useful: if X is Hausdorff and $U \subseteq X$ is precompact, and $V \subseteq U$, then V is still precompact. Indeed, $\overline{U}$ is compact, and $\overline{V} \subseteq \overline{U}$ is a closed subset and hence compact [Elb22, Lemma 4.10].

Example 1.43. The topological space $\mathbb{R}$ is locally compact; see [Elb22, Example 4.71].

Non-Example 1.44. Infinite-dimensional normed vector spaces fail to be locally compact. Namely, open balls fail to be precompact, so local compactness fails.

Non-Example 1.45. The space $\mathbb{Q}$ is not locally compact. Indeed, suppose for the sake of contradiction that we have a precompact nonempty open neighborhood $U \subseteq \mathbb{Q}$ of $0 \in \mathbb{Q}$. Now, $\mathbb{Q}$ is Hausdorff (it's a metric space), so we can find some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U$ while $\varepsilon \notin \mathbb{Q}$, so Remark 1.42 tells us that $(\varepsilon/2, \varepsilon)$ is precompact so that $[\varepsilon/2, \varepsilon]$ is actually compact.

However, this is false. Let $\{\alpha_i\}_{i \geq 1}$ be an increasing sequence of irrationals in $[\varepsilon/2, \varepsilon]$ with $\alpha_i \to \varepsilon$. Explicitly, we can take $\alpha_i := \frac{i}{i+1} \cdot \varepsilon$. Then we define

$$U_i := [\alpha_i, \alpha_{i+1}]$$

for each $i \geq 1$. Note $[\alpha_i, \alpha_{i+1}] = (\alpha_i, \alpha_{i+1})$, so the $U_\bullet$s provide a countable sequence of disjoint open subsets covering $[\varepsilon/2, \varepsilon]$. Thus, $[\varepsilon/2, \varepsilon]$ cannot be compact.

One can check that manifolds are locally compact.

Proposition 1.46. Fix a topological n-manifold M. Then M is locally compact.

Proof. This follows from being locally Euclidean. Fix $p \in M$, and then we are promised some open subset $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi\colon U \cong \widehat{U}$. Then there is an open ball $B(\varphi(p), \varepsilon) \subseteq \widehat{U}$. Then $\overline{B(\varphi(p), \varepsilon/2)} \subseteq \widehat{U}$ is closed and bounded in $\mathbb{R}^n$ and hence compact, so $\varphi^{-1}(B(\varphi(p), \varepsilon/2))$ is a subset of the compact subset $\varphi^{-1}(\overline{B(\varphi(p), \varepsilon/2)})$. ■

Being locally compact approximately speaking allows one to understand a space by building it up from compact ones. Here is one way to do this.

Definition 1.47 (exhaustion)**.** Fix a topological space X. Then an *exhaustion* of X is a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of X satisfying the following.

- Ascending: $K_0 \subseteq K_1 \subseteq \cdots$.
- Covers: $X = \bigcup_{i\in\mathbb{N}} K_i$.
- Not too close: $K_i \subseteq K_{i+1}^\circ$.

Example 1.48. The space $\mathbb{R}^n$ has an exhaustion by $K_i := B(0, i)$.

Here is a way to build an exhaustion.

Proposition 1.49. Fix a topological space X. If X is second-countable, locally compact, and Hausdorff. Then X has an exhaustion. In particular, topological n-manifolds have an exhaustion.

Proof. The second claim follows from the first by Proposition 1.46 and the definition of a manifold. So we will focus on showing the first claim.

Fix a countable base $\mathcal{B}$ of X, and let $\mathcal{B}'$ be the subcollection of precompact open base elements. Quickly, we note that $\mathcal{B}'$ is still a base: certainly everything in $\mathcal{B}'$ is open, and then for any $p \in X$ and open neighborhood $U \subseteq X$, we need some $B' \in \mathcal{B}'$ such that B' is precompact.

Well, because X is locally compact, there is a precompact open neighborhood U' of p by Remark 1.40. Then $U \cap U$ is an open neighborhood of p, so we can find a base element $B \in \mathcal{B}$ containing p and inside $U' \cap U$. Then $B \subseteq U'$ is precompact by Remark 1.42.

We now construct our exhaustion. Enumerate $\mathcal{B} = \{B_0, B_1, \ldots\}$, and we proceed as follows.

1. Set $K_0 := \overline{B_0}$, which is compact by construction of B_0.
2. Now suppose we have a compact subset $K_i \subseteq X$, and we construct K_{i+1}. Note that $\mathcal{B}$ is an open cover of K_i, which can be reduced to a finite subcover, so there is some M_{i+1} such that K_i is covered by $\{B_i : i \le M_{i+1}\}$. We may as well suppose that $M_{i+1} \ge i+1$. Then we define
$$K_{i+1} := \bigcup_{i=1}^{M} \overline{B_i}.$$
Note that the finite union of compact sets remains compact.

The above construction produces an exhaustion. Here are our checks, which will complete the proof.

- Ascending: by construction, we see that
$$K_{i+1}^\circ \supseteq \bigcup_{i=1}^{M} B_i \supseteq K_i.$$
- Covers: any $x \in X$ lives in some B_i, and by construction, we have $B_i \subseteq K_i$, so $x \in K_i$. ■

1.2.3 Paracompactness

We will want to talk about covers in some more detail.

Definition 1.50 (cover)**.** Fix a topological space X. A *cover* is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ such that

$$X = \bigcup_{U \in \mathcal{U}} U.$$

Definition 1.51 (locally finite)**.** Fix a topological space X. A cover $\mathcal{U}$ of X is *locally finite* if and only if any $p \in X$ has some open neighborhood $U \subseteq X$ intersecting at most finitely many elements of $\mathcal{U}$.

Definition 1.52 (refinement)**.** Fix a cover $\mathcal{U}$ of a topological space X. Then a *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ such that any $V \in \mathcal{U}$ is contained in some $U \in \mathcal{U}$.

And here is our definition.

Definition 1.53 (paracompact)**.** Fix a topological space X. Then X is *paracompact* if and only if every open cover has a locally finite open refinement.

Approximately speaking, the point of desiring paracompactness is that it allows "reducing to Euclidean" arguments in the future will not have to deal with intersections which are infinitely bad. Anyway, here is our result.

Proposition 1.54. Fix a topological n-manifold M. Then M is paracompact.

Proof. In fact, we are only going to use the fact that M has an exhaustion, proven in Proposition 1.49.

Fix an open cover $\mathcal{U}$, and we want to produce a locally finite open refinement. To set us up, fix an exhaustion $\{K_i\}_{i \in \mathbb{N}}$, which exists by Proposition 1.49, and define the following sets for each $i \in \mathbb{N}$.

- For $i \geq -1$, define $V_i := K_{i+1} \backslash K_i^\circ$, which is a closed subset of the compact set K_{i+1} and hence compact [Elb22, Lemma 4.10]; take $K_{-1} = \varnothing$ without concern.
- For $i \geq 0$, define $W_i := K_{i+2}^\circ \setminus K_{i-1}$, which is open; here, take $K_{-1} = \varnothing$ without concern.

For intuition, we should think about the $W_\bullet$s as being a locally finite cover from which we will build the locally finite cover refinement of $\mathcal{U}$.

For the construction, we fix some $j \geq 0$ for the time being. For each $x \in V_j$, find some $U_x \in \mathcal{U}$ containing x. Note that $\{U_x\}_{x \in V_j}$ is an open cover of V_j, and because $V_j \subseteq W_j$, in fact $\{U_x \cap W_j\}_{x \in V_j}$ is an open cover. Because V_j is compact, we can thus reduce this open cover to a finite subcover $\mathcal{A}_j$.

Now letting j vary, we define

$$\mathcal{V} := \bigcup_{j \geq 0} \mathcal{A}_j.$$

Here are our checks.

- Open cover: each $x \in X$ lives in some K_{i+1} because we have an exhaustion, so lives in some V_i, so it lives in some open subset in $\mathcal{A}_j$, so it lives in some open subset in $\mathcal{V}$.
- Refinement: by construction, each open set in $\mathcal{A}_j$ is a subset in $\mathcal{U}$.
- Locally finite: this is essentially by construction. The main point is that any $x \in X$ lives in some K_i, so by choosing the least such K_i places x in some $V_i \subseteq W_i$. We now show that only finitely many open subsets in $\mathcal{V}$ intersect W_i. Note $W_i \subseteq K_{i+2}$, so $W_i \cap W_j = \varnothing$ for $j \geq i+2$. Thus, if $V \cap W_i \neq \varnothing$, we must have $V \in \mathcal{A}_j$ for $j < i+2$. But this is only finitely many indices, and each $\mathcal{A}_j$ is finite, so this is only finitely many candidates. ■

1.2.4 Products

We now discuss an in-depth example.

Proposition 1.55. Fix finitely many topological manifolds $M_1, \ldots, M_k$. Then the product

$$M_1 \times \cdots \times M_k$$

is also a topological manifold of dimension $\dim M_1 + \cdots + \dim M_k$.

We will do this via a sequence of lemmas.

Lemma 1.56. Fix a collection of Hausdorff topological spaces $\{X_\alpha\}_{\alpha\in\Lambda}$. Then the product

$$\prod_{\alpha\in\Lambda} X_\alpha$$

is also Hausdorff.

Proof. Fix distinct points $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$ in the product. Then there is an index $\beta\in\Lambda$ such that $x_\beta \neq y_\beta$, so because X_β is Hausdorff, there are disjoint open neighborhoods $U_\beta, V_\beta \subseteq X_\beta$ of x_β and y_β, respectively. Then we define $U_\alpha = V_\alpha := X_\alpha$ for $\alpha \neq \beta$, and we note that the open subsets

$$\prod_{\alpha\in\Lambda} U_\alpha \qquad \text{and} \qquad \prod_{\alpha\in\Lambda} V_\alpha$$

are disjoint open neighborhoods of $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$, respectively, so we are done. (These are disjoint because any point in the intersection will have the β coordinate in $U_\beta \cap V_\beta = \varnothing$.) ■

Lemma 1.57. Fix finitely many second countable topological spaces $\{X_i\}_{i=1}^n$. Then the product

$$\prod_{i=1}^n X_i$$

is also second countable.

Proof. Let the product be X. For each i, let $\mathcal{B}_i$ be a countable base for X_i. Then define

$$\mathcal{B} := \left\{ \prod_{i=1}^n B_i : B_i \in \mathcal{B}_i \text{ for each } i \right\}.$$

We claim that $\mathcal{B}$ is a base for the topology on the X. Indeed, suppose $(x_1, \ldots, x_n) \in X$ lives in some open subset $U \subseteq X$. From the standard base on X, we know that there are open subsets $U_i \subseteq X_i$ for each i such that $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$. Now, for each U_i, we note that $x_i \in U_i$ must have some $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$ and $B_i \subseteq U_i$. But then

$$(x_1, \ldots, x_n) \in B_1 \times \cdots \times B_n \subseteq U,$$

so $B_1 \times \cdots \times B_n \in \mathcal{B}$ is the desired base element. ■

We now prove Proposition 1.55.

Proof of Proposition 1.55. We get Hausdorff from Lemma 1.56 and second countable from Lemma 1.57. So it remains to check that we are locally Euclidean. For brevity, let M be the product, and set $n_i := \dim M_i$ for each i, and let $n := n_1 + \cdots + n_k$.

Now, fix some point $(x_1, \ldots, x_k) \in M$. For each i, we get some open neighborhood $U_i \subseteq M_i$ of x_i and some open $\widehat{U}_i \subseteq \mathbb{R}^{n_i}$ with a homeomorphism $\varphi_i \colon U_i \cong \widehat{U}_i$. Now, we see that the product map

$$(\varphi_1 \times \cdots \times \varphi_k)\colon U_1 \times \cdots \times U_k \to \widehat{U}_1 \times \cdots \times \widehat{U}_k$$

is still a homeomorphism, and the target is an open subset of

$$\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \cong \mathbb{R}^n,$$

where this last homeomorphism is obtained by simply concatenating the coordinates. So we have constructed a composite homeomorphism from an open neighborhood of $(x_1, \ldots, x_k)$ to an open subset of $\mathbb{R}^n$, as desired. ■

1.2.5 Charts

The construction of our smooth structure will arise from more carefully understanding how a manifold is locally Euclidean. This arises from charts.

Definition 1.58 (chart)**.** Fix a topological n-manifold M. Then a *coordinate chart* or just *chart* is a pair (U, φ) where $U \subseteq M$ is open and $\varphi \colon U \cong \widehat{U}$ is a homeomorphism where $\widehat{U} \subseteq \mathbb{R}^n$ is open.

Essentially, the content of M being locally Euclidean is that it has an open cover by open subsets belonging to some chart. The reason we call it a chart is that we are (approximately speaking) providing "local coordinates" to an open subset of M.

Definition 1.59 (coordinate function)**.** Fix a chart (U, φ) if a topological n-manifold M. Then we may write

$$\varphi(p) \coloneqq \left(x^1(p), \ldots, x^n(p)\right) \in \mathbb{R}^n$$

for each $p \in M$. We call these functions $x^\bullet \colon U \to \mathbb{R}$ the *coordinate functions*.

Note that these coordinate functions are continuous because they are simply the continuous function φ composed with the projection $\mathbb{R}^n \to \mathbb{R}$.

Example 1.60. Fix an open subset $V \subseteq \mathbb{R}^m$, and let $F \colon V \to \mathbb{R}^n$ be a continuous function. Then the graph

$$\Gamma \coloneqq \{(x, F(x)) : x \in V\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

is a topological n-manifold. Because we are already a subspace of $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$, we see that Γ is also Hausdorff and second countable. (Subspaces inherit being Hausdorff directly, and we inherit being second countable by using the intersection of the given countable base.)

The main content comes from being locally Euclidean. Namely, there is a projection map $\pi \colon \Gamma \to V$ by $(x, y) \mapsto x$ which in fact is a homeomorphism (it's continuous inverse is $(\mathrm{id} \times F) \colon x \mapsto (x, F(x))$). So we have the single chart (V, π), which establishes being a topological n-manifold.

Remark 1.61. Please read product manifolds and tori.

BIBLIOGRAPHY

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Elb22] Nir Elber. *Introduction to Topology and Analysis*. 2022. URL: https://dfoiler.github.io/notes/202A/notes.pdf.

[Elb23] Nir Elber. *Algebraic Topology*. 2023. URL: https://dfoiler.github.io/notes/215A/notes.pdf.

List of Definitions