250B: Commutative Algebra

Nir Elber

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Contents

THEME 1: PRIMARY DECOMPOSITION

And in a brilliant moment of word association...

—John Mulaney

1.1 February 8

Here we go.

1.1.1 Minimal Primes

Let's talk a little bit about minimal prime ideals. In particular, suppose that we have a striclty descending chain of prime ideals

$$\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \cdots .$$

Because prime ideals are closed under intersection in the chain, Zorn's lemma now promises us a minimal prime ideal. In fact, for any ideal I, there will be a minimal prime contained in I; indeed, there is a maximal ideal over I which will show that some prime exists, and Zorn's lemma lets us find a minimal ideal. We might call such a prime ideal a minimal prime ideal over I.

In the Noetherian case, our minimal primes are somewhat controlled.

Lemma 1.1. Fix R a Noetherian ring. Then there are only finitely many minimal prime ideals over a fixed ideal $I \subseteq R$.

Proof. Suppose for the sake of contradiction we have an ideal for which there are infinitely many prime ideals. Then by the Noetherian condition, we may find a maximal such ideal, and we name it I. But then we can look at R/I, so it suffices to consider $I = (0)$.

Now, I cannot itself be prime, so there are $a, b \in R$ such that $a, b \notin I$ while $ab \in I$. But now, for any minimal prime $\mathfrak{p}$ over I, we have $ab \in \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, so in fact $\mathfrak{p}$ is minimal over either $I + (a)$ or $I + (b)$. But $I + (a)$ and $I + (b)$ are strictly larger than I, so they have only finitely many minimal prime ideals, so it follows I will only have finitely minimal prime ideals. ■

Remark 1.2. Last time we showed that a finitely generated module M over a Noetherian ring R has $\operatorname{Ass} M$ is finite. But in fact **??** shows that any minimal prime over $\operatorname{Ann} M$ will be associated, so there will be finitely many of these. This provides an alternate proof of the result upon taking $M := R/I$ so that $\operatorname{Ann} M = I$.

To set up an application, we have the following definition.

Definition 1.3. Fix $X \subseteq \mathbb{A}^n(k)$ an affine algebraic set. Then X is *irreducible* if and only if $I(X) \subseteq k[x_1, \ldots, x_n]$ is prime; i.e., $A(X)$ is an integral domain.

As an application, consider any algebraic set $X \subseteq \mathbb{A}^n(k)$. Then a minimal prime $\mathfrak{p}$ over $I(X)$, of which we know there are finitely many, so we call them

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

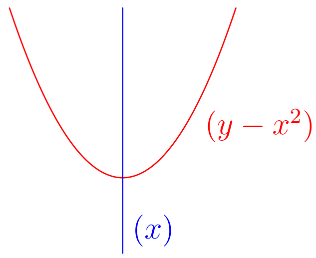
However, the intersection of all these primes will be the intersection of all the primes containing $I(X)$, which is the radical of $I(X)$, which is $I(X)$. Speaking geometrically, this means that any algebraic set is a finite union of maximal irreducible algebraic set.

Proposition 1.4. Fix $I := (yx - x^3)$, and we decompose $Z(I)$.

Proof. Well, $yx - x^3 = 0$ if and only if $x = 0$ or $y - x^2 = 0$, so

$$I = (x) \cap (y - x^2).$$

So here is the image of $Z(I)$.



Now we note (x) and $(y - x^2)$. ■

1.1.2 Primary Grab-Bag

We saw that, in the case where I was a radical ideal, we could write it as a finite intersection of prime ideals. Primary decomposition provides us with a general theory for non-radical ideals; let's start building towards that.

We recall the following definition.

$\mathfrak{p}$-primary

Definition 1.5 ($\mathfrak{p}$-primary)**.** Fix M a finitely generated module over R a Noetherian ring. Then a submodule $N \subseteq M$ is $\mathfrak{p}$-*primary* if and only if $\operatorname{Ass} M/N = \{\mathfrak{p}\}$.

$\mathfrak{p}$-coprimary

Definition 1.6 ($\mathfrak{p}$-coprimary)**.** Fix M a finitely generated module over R a Noetherian ring. Then M is $\mathfrak{p}$-*coprimary* if and only if $\operatorname{Ass} M = \{\mathfrak{p}\}$.

In other words, $N \subseteq M$ is $\mathfrak{p}$-primary if and only if M/N is $\mathfrak{p}$-coprimary.

We would like some more concrete conditions for being $\mathfrak{p}$-primary.

Proposition 1.7. Fix M a finitely generated module over R a Noetherian ring. Then the following are equivalent.

(a) M is $\mathfrak{p}$-coprimary.

(b) $\mathfrak{p}$ is the unique minimal prime over $\operatorname{Ann} M$, and $\mathfrak{p}$ contains $\operatorname{Ann} m$ for each $m \in M$.

(c) $\mathfrak{p}^n \subseteq \operatorname{Ann} M$ for some positive integer n, and $\mathfrak{p}$ contains $\operatorname{Ann} m$ for each $m \in M$.

Proof. We take our implications one at a time.

- We show that (a) implies (b). We are given that $\operatorname{Ass} M = \{\mathfrak{p}\}$. But we know that any prime minimal containing $\operatorname{Ann} M$ will be associated (by **??**), so $\mathfrak{p}$ must be a minimal prime over $\operatorname{Ann} M$.

  Additionally, we recall from **??** that
  $$\bigcup_{m\in M\setminus\{0\}} \operatorname{Ann} m = \bigcup_{\mathfrak{q}\in \operatorname{Ass} M} \mathfrak{q},$$
  but $\operatorname{Ass} M = \{\mathfrak{p}\}$, so the result follows.

- Recall that $\operatorname{rad}\operatorname{Ann} M$ is the intersection of all primes containing $\operatorname{Ann} M$, so $\mathfrak{p} = \operatorname{rad}\operatorname{Ann} M$, so $\mathfrak{p}^n \subseteq \operatorname{Ann} M$ for some positive integer n.

- We show that (c) implies (a). If $\mathfrak{p}^n \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ for some prime $\mathfrak{q}$ containing $\operatorname{Ann} M$, then $\mathfrak{q} = \mathfrak{p}$. So $\mathfrak{p}$ is the unique minimal prime over $\operatorname{Ann} M$, and the hypothesis implies that there are no other minimal primes. ■

Corollary 1.8. An ideal $I \subseteq R$ is $\mathfrak{p}$-primary if and only if $\mathfrak{p}^n \subseteq I$ and, for all $a \notin \mathfrak{p}$, we have $ab \in I$ implies $b \in \mathfrak{p}$.

Proof. This follows directly from the proposition. ■

1.1.3 Primary Decompositon

And here is our main result.

Theorem 1.9 (Primary decomposition, I)**.** Fix M a finitely generated module over a Noetherian ring R. Then every submodule $N \subseteq M$ is the intersection of finitely many primary submodules.

Proof. The key to this result is to instead talk about irreducible decomposition.

Irreducible

Definition 1.10 (Irreducible)**.** Fix M a module over a ring R. Then a submodule $N \subseteq M$ is *irreducible* if and only if $N = N_1 \cap N_2$ implies $N = N_1$ or $N = N_2$.

It will turn out that irreducible implies primary, but we do not know this.

Here is the key claim.

Lemma 1.11. Fix M a finitely generated module over a Noetherian ring R. Then every submodule $N \subseteq M$ is the intersection of finitely many irreducible submodules.

Proof. Suppose for the sake of contradiction that the statement is false. Because M is a Noetherian module, we can find a maximal such submodule N. Note that $N \neq M$ because M is irreducible.

However, N is not irreducible, so we can write $N = N_1 \cap N_2$ for $N \subsetneq N_1, N_2$. But by hypothesis on N, we can write N_1 and N_2 as intersections of irreducible modules, so N is also the intersection of irreducible modules, so we are done. ■

Remark 1.12 (Nir)**.** If you squint really hard, this is essentially the proof of existence of prime factorizatins.

And now we can finish up.

Lemma 1.13. Fix M a finitely generated module over a Noetherian ring R. Then any irreducible submodule $N \subsetneq M$ is primary.

Proof. We know that $\operatorname{Ass} M/N$ is nonempty because $M \neq N$, so we need to show that there is exactly one prime; we proceed by contrapositin. So suppose $\mathfrak{p}, \mathfrak{q} \in \operatorname{Ass} M/N$ which are distinct. But then by **??**, we get embeddings

$$\iota_{\mathfrak{p}} : R/\mathfrak{p} \hookrightarrow M/N \qquad \text{and} \qquad \iota_{\mathfrak{q}} : R/\mathfrak{p} \hookrightarrow M/N.$$

However, the annihilator of any nonzero element in $\operatorname{im} \iota_{\mathfrak{p}}$ and $\operatorname{im} \iota_{\mathfrak{q}}$ will be $\mathfrak{p}$ and $\mathfrak{q}$ respectively, so $\operatorname{im} \iota_{\mathfrak{p}} \cap \operatorname{im} \iota_{\mathfrak{q}} = 0$. So we can write

$$N = (\operatorname{im} \iota_{\mathfrak{p}} + N) \cap (\operatorname{im} \iota_{\mathfrak{q}} + N)$$

to break that N is irreducible. (These are strictly larger because of where $\operatorname{im} \iota_{\mathfrak{p}}$ and $\operatorname{im} \iota_{\mathfrak{q}}$ are supposed to map to.) ■

The above two lemmas finish the theorem. ■

Let's try to make primary decomposition a little more canonical.

Theorem 1.14 (Primary decomposition, II)**.** Fix M a finitely generated module over a Noetherian ring R. Then write some submodule $N \subseteq M$ as

$$N = \bigcap_{i=1}^{n} N_i$$

such that N_i is $\mathfrak{p}_i$-primary. Then the following are true.

(a) We have $\operatorname{Ass} M/N \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

(b) If we cannot remove any N_i from the decomposition, then equality in (a) holds.

(c) If n is as small as possible, then each $\mathfrak{p}_i$ is unique.

Proof. We show these one at a time. For psychological reasons, we note that we may swap M and N with M/N and (0) to assume that $N = (0)$ (without loss of generality).

(a) Note that we have the sequence

$$M \hookrightarrow \bigoplus_{i=1}^{n} M/N_i \to 0$$

with kernel $N = (0)$. It follows that $\operatorname{Ass} M$ is contained in $\bigcup_{i=1}^{n} \operatorname{Ass} M/N_i = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

(b) We are given that, for each j, we have

$$K_j := \bigcap_{i \neq j} N_i \neq 0.$$

Now, $N_j \cap K = 0$, so we can embed K_j into M/N_j, so $\operatorname{Ass} M/K_j = \{\mathfrak{p}_j\}$, making K_j into a $\mathfrak{p}_j$-primary module. It follows that $\mathfrak{p}_j \in \operatorname{Ass} M$.

(c) If there are any redundancies, then we intersect all of the redundancies for some prime $\mathfrak{p}$ to remove the redundancies, so there will be at most one $\mathfrak{p}$-primary ideal for any prime $\mathfrak{p}$ when N is minimal. ■

Primary decomposition also behaves with localization.

Theorem 1.15 (Primary decomposition, III)**.** Fix M a finitely generated module over a Noetherian ring R. Then write some submodule $N \subseteq M$ as

$$N = \bigcap_{i=1}^{n} N_i$$

such that N_i is $\mathfrak{p}_i$-primary. If $U \subseteq R$ is some multiplicatively closed subset, then

$$N\left[U^{-1}\right] = \bigcap_{\substack{i=1 \\ \mathfrak{p}_i \cap U = \varnothing}}^{n} \mathfrak{p}_i\left[U^{-1}\right].$$

Proof. We omit this proof. The main point is that all of the primes we avoided are actually fully $M\left[U^{-1}\right]$, and localization preserves intersections. ■

1.1.4 Examples of Primary Decomposition

Let's do some examples.

Example 1.16. For any nonzero integer $n \in \mathbb{Z} \setminus \{0\}$, we can write

$$(n) = \prod_{p \text{ prime}} \left(p^{\alpha_p}\right),$$

for some exponents α_p. This is our primary decomposition, and it is unique.

Example 1.17. In general, if R is a unique factorization domain, then any $f \in R$ will have

$$f = up_1^{d_1} \cdots p_n^{d_n}$$

for some unit u, primes $p_\bullet$, and exponents $d_\bullet$. Then

$$(f) = \bigcap_{i=1}^{n} \left(p_i^{d_i}\right).$$

In fact, we have the following statement.

Proposition 1.18. Fix R a Noetherian domain. Then R is a unique factorization domain if and only if every prime ideal over a principal ideal is principal.

Proof. We omit this proof. The main idea to the backwards direction is to take a minimal prime ideal (p) to sufficiently high power to read off the exponent. ■

We remark that primary decomposition is not unique, in general, however.

Example 1.19. Consider $R := k[x,y]/\left(x^2, xy\right)$. As a k-module, this has basis $\{1, x, y, y^2, y^3 \ldots\}$. Now, we see that we can write

$$(x) \cap (y^n) = (xy^n) = (0)$$

for any $n \in \mathbb{N}$. This is a primary decomposition: in particular, $R/\left(y^n\right)$ is a local ring (notably, all the basis elements other than 1 nilpotent, so R minus their k-span is the unique maximal ideal), and our associated prime is $(x,y)R/\left(y^n\right)$. Additionally, $R/(x)$ will have unique maximal ideal $(x,y)/(x)$, for similar reasons.

The point is that the above example provides "lots" of different minimal primary decompositions.

1.1.5 Graded Rings

We will want to consider primary decomposition for graded rings because later in life we will want to talk about projective space.

Proposition 1.20. Fix $R = R_0 \oplus R_1 \oplus \cdots$ a graded ring and M a graded module over R. Suppose that we have $m \in M$ and $\mathfrak{p} := \operatorname{Ann} m$ a prime ideal. Then $\mathfrak{p}$ is a graded ideal of R.

In other words, associated primes of a graded module are graded.

Proof. We have to show that

$$\mathfrak{p} = \bigoplus_{i=1}^{\infty} (R_i \cap \mathfrak{p}).$$

So we write, for some $f \in \mathfrak{p}$, that

$$f = \sum_{i=1}^{p} f_i,$$

where $f_i \in R_{d_i}$, and $d_1 < \cdots < d_n$. By induction, it will be enough to show that $f_1 \in \mathfrak{p}$ and then consider $f - f_1$ to continue.

Well, we may write

$$m = \sum_{j \in \mathbb{Z}}^{q} m_j,$$

where $m_j \in R_{e_j}$. Because $fm = 0$, the term of lowest degree, which is $f_1 m_1$, must itself vanish, so

$$f_1 m = \sum_{j=2}^{p} f_1 m_j.$$

Thus $\mathfrak{p} \subseteq \operatorname{Ann} f_1 m$, so we set $I := \operatorname{Ann} f_1 m$. If $\mathfrak{p} = \operatorname{Ann} f_1 m$, then we are done because we have effectively removed our decomposition. Otherwise, we can find $g \in I \setminus \mathfrak{p}$. But then gf_1 annihilates m, so $gf_1 \in \mathfrak{p}$, so $f_1 \in \mathfrak{p}$, and we are done. ■