Topology for the Impatient

Nir Elber

Fall 2022

Abstract

This document collects a variety of definitions and results from point-set topology.

Contents

1 Definitions

1.1 Metric Spaces

Definition 1 (Metric)**.** A *metric* d on a set X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following rules for any $x, y, z \in X$.

(a) Zero: $d(x, x) = 0$.

(b) Zero: $d(x, y) = 0$ implies $x = y$.

(c) Symmetry: $d(x, y) = d(y, x)$.

(d) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

We call (X, d) a *metric space*.

Definition 2 (Norm)**.** Fix a vector space V over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$ is a function satisfying the following, for any $r \in \mathbb{R}$ and $v, w \in V$.

(a) Zero: $\|v\| = 0$ if and only if $v = 0$.

(b) Scaling: $\|rv\| = |r| \cdot \|v\|$.

(c) Triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.

Definition 3 (Converge)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n > N \implies d(x_n, x) < \varepsilon.$$

We might write this as "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$." In this event, we may say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges*, and its limit is x.

Definition 4 (Cauchy)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy sequence* if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n, m > N \implies d(x_n, x_m) < \varepsilon.$$

Definition 5 (Complete)**.** A metric space (X, d) is *complete* if and only if every Cauchy sequence in X converges to a point in X.

Definition 6 (Bounded)**.** Fix a metric space (X, d) and a nonempty set A. A subset $A \subseteq X$ is *bounded* if and only if there is an open ball $B(x, r)$ containing A. More generally, a function $f\colon A \to X$ is *bounded* if and only if $\operatorname{im} f \subseteq X$ is bounded, and we let $B(A, X)$ denote the set of all bounded functions $f\colon A \to X$.

Definition 7 (Totally bounded)**.** Fix a metric space (X, d). A subset $A \subseteq X$ is *totally bounded* if and only if any $\varepsilon > 0$ has a finite set $\{x_i\}_{i=1}^n \subseteq A$ for which

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

If X is totally bounded, we say that (X, d) is totally bounded.

Definition 8 (Pointwise totally bounded)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and a metric space (M, d), and let $\mathcal{F}$ be a family of continuous functions $f : X \to M$. Then $\mathcal{F}$ is *pointwise totally bounded* if and only if any $x \in \mathcal{F}$ makes the set

$$\{f(x) : f \in \mathcal{F}\}$$

totally bounded.

Definition 9 (Equicontinuous)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and a metric space (M, d), and let $\mathcal{F}$ be a family of continuous functions $f : X \to M$. We say that the family $\mathcal{F}$ is *equicontinuous* at some $x \in X$ if and only if any $\varepsilon > 0$ has some open subset $U \subseteq X$ such that $y \in U$ has

$$d(f(y), f(x)) < \varepsilon$$

for all $f \in \mathcal{F}$. The entire family $\mathcal{F}$ is *equicontinuous* if any only if it is equicontinuous at all $x \in X$.

1.2 Basic Topology

Definition 10 (Topology)**.** Fix a set X. Then a *topology* $\mathcal{T}$ on X is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following.

(a) We have $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union $\bigcup_{U \in \mathcal{U}} U$ lives in $\mathcal{T}$.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $\bigcap_{i=1}^n U_i$ lives in $\mathcal{T}$.

We will say that the ordered pair $(X, \mathcal{T})$ is a *topological space*. We say that the sets in $\mathcal{T}$ are *open*.

Definition 11 (Continuous)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Then a function $f : X \to Y$ is *continuous* if and only if, for any $U_Y \in \mathcal{T}_Y$, we have $f^{-1}(U_Y) \in \mathcal{T}_X$.

Definition 12 (Sub-base)**.** Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{S} \subseteq \mathcal{T}$ is a *sub-base* for $\mathcal{T}$ if and only if the following hold.

(a) $\mathcal{S}$ covers X, in that $X = \bigcup_{U \in \mathcal{S}} U$.

(b) $\mathcal{T}$ is generated by $\mathcal{S}$.

Definition 13 (Base)**.** Fix a set X. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *base* (for a topology on X) if and only if the collection of arbitrary unions of $\mathcal{B}$ form a topology on X.

Proposition 14. Fix a set X and a collection $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a base if and only if

(a) $X = \bigcup_{B \in \mathcal{B}} B$, and

(b) any $B_1, B_2 \in \mathcal{B}$ has some collection $\mathcal{U} \subseteq \mathcal{B}$ such that

$$B_1 \cap B_2 = \bigcup_{B \in \mathcal{U}} B.$$

Definition 15 (Closed)**.** Fix a topological space $(X, \mathcal{T})$. A subset $V \subseteq X$ is *closed* if and only if $(X \setminus V) \in \mathcal{T}$.

Definition 16 (Closure)**.** Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we define the *closure* as

$$\overline{S} := \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V.$$

This is a closed set. In other words, the closure $\overline{S}$ is the unique smallest closed set containing S.

Definition 17 (Dense)**.** Fix a topological space $(X, \mathcal{T})$. Given subsets $A \subseteq B$, we say A is *dense* in B if and only if $B \subseteq \overline{A}$.

Definition 18 (Homeomorphism)**.** A function $f \colon X \to Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a *homeomorphism* if and only if f is continuous and has a continuous inverse. Formally, we require a continuous map $g \colon Y \to X$ such that

$$f \circ g = \mathrm{id}_Y \qquad \text{and} \qquad g \circ f = \mathrm{id}_X.$$

Definition 19 (Cluster point)**.** Fix a topological space $(X, \mathcal{T})$ and a net $\{x_\alpha\}_{\alpha \in \Lambda}$. Then $x \in X$ is a *cluster point* if and only if, for any open subset U containing x and $\alpha \in \Lambda$, there is some $\alpha' > \alpha$ for which $x_{\alpha'} \in U$.

1.3 Some Topologies

Definition 20 (Initial topology)**.** Fix a set X and a collection of topologies $\{(Y_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ with some functions $f_\alpha \colon X \to Y_\alpha$ for each $\alpha \in \lambda$. Then

$$\bigcup_{\alpha \in \lambda} \{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha\}$$

is a sub-base for an *initial topology*.

Definition 21 (Relative topology)**.** Fix $(Y, \mathcal{T})$ a topological space. Then the *relative topology* for a subset $X \subseteq Y$ is the topology initial for the natural embedding $\iota \colon X \hookrightarrow Y$.

Lemma 22. Fix $(Y, \mathcal{T}_Y)$ a topological space. Then the relative topology for a subset $X \subseteq Y$ consists of the subsets

$$\{X \cap U : U \in \mathcal{T}_Y\}.$$

Definition 23 (Product topology)**.** Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Then the *product topology* on $X := \prod_{\alpha\in\lambda} X_\alpha$ is initial topoloy for the canonical projection maps

$$\pi_\alpha : X \to X_\alpha.$$

Lemma 24. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Then the product topology on $X := \prod_{\alpha\in\lambda} X_\alpha$ has a base

$$\mathcal{B} := \left\{ \prod_{\alpha\in\lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha, U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \right\}.$$

Definition 25 (Final topology)**.** Fix a set Y and some topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Given functions $f_\alpha : X_\alpha \to Y$, we define the *final topology* on Y to be the "strongest" (i.e., with the most open sets) making the f_α continuous.

Definition 26 (Quotient topology)**.** Fix an equivalence relation $\sim$ on a set X with a topology $\mathcal{T}$. Then the *quotient topology* on $X/\sim$ is the final topology for the natural projection $X \twoheadrightarrow X/\sim$.

1.4 Adjectives for Spaces

Definition 27 (Hausdroff)**.** Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is *Hausdorff* if and only if, for any two distinct points $x, x' \in X$, there are disjoint open sets U and U' such that $x \in U$ and $x' \in U'$.

Definition 28 (Regular)**.** A topological space $(X, \mathcal{T})$ is *regular* if and only if each closed subset $A \subseteq X$ and $x \notin A$ have disjoint open subsets U and V with $A \subseteq U$ and $x \in V$.

Definition 29 (Normal)**.** Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is *Hausdorff* if and only if, for any two disjoint closed sets $V, V' \subseteq X$, there are disjoint open sets U and U' such that $V \in U$ and $V' \in U'$.

Definition 30 (Open cover)**.** Fix a topological space $(X, \mathcal{T})$. An *open cover* of X is a collection $\mathcal{U} \subseteq \mathcal{T}$ of open sets such that

$$X = \bigcup_{U\in\mathcal{U}} U.$$

Definition 31 (Compact)**.** Fix a topological space $(X, \mathcal{T})$. We say that $(X, \mathcal{T})$ is *compact* if and only if every open cover of X has a finite subcover.

Definition 32 (Locally compact)**.** A topological space $(X, \mathcal{T})$ is *locally compact* if and only if each point $x \in X$ has some open subset $U \in \mathcal{T}$ containing x such that $\overline{U}$ is compact.

Proposition 33. Fix a locally compact Hausdorff space $(X, \mathcal{T})$ and some compact subset $C \subseteq X$. Then any open subset U containing C has some open subset U_C containing C such that $\overline{U_C}$ is compact and $\overline{U_C} \subseteq U$.

2 Lemmas and Results

2.1 Metric Spaces

Lemma 34. Fix a metric space (X, d) and $V \subseteq X$. The following are equivalent.

(a) V is closed.

(b) Any sequence $\{x_n\}_{n\in\mathbb{N}}$ in V which converges to a point $x \in X$ actually converges to $x \in V$.

Corollary 35. Fix a complete metric space (X, d). Then a closed subset $V \subseteq X$ given the restricted metric is also complete.

Proposition 36. Fix a topological space $(X, \mathcal{T})$ and a metric space (Y, d). Let $B_c(X, Y) \subseteq B(X, Y)$ denote the metric subspace of bounded continuous functions $f: X \to Y$. Then $B_c(X, Y)$ is a closed subspace of $B(X, Y)$. In particular, if (Y, d) is complete, then $B_c(X, Y)$ is also complete.

Theorem 37. Fix a metric space (X, d). If X is complete and totally bounded, then X is compact.

Corollary 38. Fix a complete metric space (X, d). Then a subset $A \subseteq X$ is compact if and only if A is closed and totally bounded.

Theorem 39 (Arzelá–Ascoli). Fix a compact topological space $(X, \mathcal{T})$ and a metric space (M, d) so that we can give the space of bounded continuous functions $B_c(X, M)$ the uniform metric d_u. Then any family $\mathcal{F} \subseteq B_c(X, M)$ is totally bounded if and only if it is equicontinuous and pointwise totally bounded family.

2.2 Building Functions

Proposition 40. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Give the product $X := \prod_{\alpha\in\lambda} X_\alpha$ the projections $\pi_\alpha: X \to X_\alpha$ and the product topology $\mathcal{T}$. Given a topological space $(Y, \mathcal{T}_Y)$, a function $f: Y \to X$ is continuous if and only if the compositions $\pi_\alpha \circ f$ are continuous.

Proposition 41. Fix an equivalence relation $\sim$ on a set X with a topology $\mathcal{T}$; let $\pi: X \twoheadrightarrow (X/{\sim})$ be the natural projection. Then, for any continuous map $f: X \to Z$ such that any $x \sim x'$ has $f(x) = f(x')$, there is a unique continuous map $\overline{f}: (X/{\sim}) \to Z$ such that

$$f = \overline{f} \circ \pi.$$

Theorem 42 (Urysohn's lemma). Fix a topological space $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is normal, then for any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f: X \to [0, 1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

Theorem 43 (Tietze extension). Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f: A \to \mathbb{R}$, there exists a continuous function $\widetilde{f}: X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$. In fact, if $\operatorname{im} f \subseteq [a, b]$, then we may enforce $\operatorname{im} \widetilde{f} \subseteq [a, b]$ as well.

2.3 Running Checks

Lemma 44. Fix a topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$. Then $x \in \overline{A}$ if and only if every open subset $U \subseteq X$ containing x has $U \cap A \neq \varnothing$.

Lemma 45. Fix a compact topological space $(X, \mathcal{T})$. Then any closed subset $A \subseteq X$ is compact.

Lemma 46. Fix a Hausdorff topological space $(X, \mathcal{T})$, and let $A \subseteq X$ be compact. Then A is closed.

Proposition 47. Fix a compact Hausdorff space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is normal.

Lemma 48. Fix a continuous map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$. If $(X, \mathcal{T}_X)$ is compact, then $\operatorname{im} f \subseteq Y$ is also compact.

Proposition 49. Fix a compact topological space $(X, \mathcal{T}_X)$ and a Hausdorff topological space $(Y, \mathcal{T}_Y)$. Then any continuous bijection $f\colon X \to Y$ is a homeomorphism.

Proposition 50. Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is compact if and only if any collection $\mathcal{V}$ of closed subsets with the finite intersection property has

$$\bigcap_{V \in \mathcal{V}} V \neq \varnothing.$$

Theorem 51 (Tychonoff)**.** Fix a collection $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$ of compact topological spaces, and give the product space $X \coloneqq \prod_{\alpha \in \lambda} X_\alpha$ the product topology. Then X is compact.