261A: Lie Groups

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

Theme 1
Introduction

1.1 August 28

Today we review differential topology. Here are some logistical notes.

- There will be weekly homeworks, of about 5 problems.
- There will be a final take-home exam.
- This course has a bCourses page.
- We will mostly follow Kirillov's book [Kir08].

1.1.1 Group Objects

The goal of this class is to study symmetries of geometric objects. As such, we are interested in studying (infinite) groups with some extra geometric structure, such as a real manifold or a complex manifold or a scheme structure. Speaking generally, we will have some category $\mathcal{C}$ of geometric objects, equipped with finite products (such as a final object), which allows us to have group objects in $\mathcal{C}$.

Definition 1.1 (group object)**.** Fix a category $\mathcal{C}$ with finite products, such as a final object $*$. A *group object* is the data (G, m, e, i) where $G \in \mathcal{C}$ is an object and $m\colon G \times G \to G$ and $e\colon * \to G$ and $i\colon G \to G$ are morphisms. We require this data to satisfy some associativity, identity, and inverse coherence laws.

For concreteness, we go ahead and write out the coherence diagrams, but they are not so interesting.

- Associative: the following diagram commutes.

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times m"] \arrow[d, "m \times \mathrm{id}_G"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

- Identity: the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}_G \times e"] \arrow[dr, equal] & G \times G \arrow[d, "m"] & G \arrow[l, "e \times \mathrm{id}_G"'] \arrow[dl, equal] \\ & G & \end{tikzcd}$$

- Inverses: the following diagram commutes.

$$\begin{tikzcd}
 & G \times G \arrow[dr, "m"] & \\
G \arrow[ur, "i \times \mathrm{id}_G"] \arrow[r] \arrow[dr, "\mathrm{id}_G \times i"'] & * \arrow[r, "e"] & G \\
 & G \times G \arrow[ur, "m"'] &
\end{tikzcd}$$

Example 1.2. In the case where $\mathcal{C} = \mathrm{Set}$, we recover the notion of a group, where G is the set, m is the multiplication law, e is the identity, and i is the inverse.

Example 1.3. Group objects in the category of manifolds will be Lie groups.

1.1.2 Review of Topology

This course requires some topology as a prerequisite, but let's review these notions for concreteness. We refer to [Elb22] for most of these notions.

Definition 1.4 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ of a set X and collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of open subsets of X, which we require to satisfy the following axioms.

- $\varnothing, X \in \mathcal{T}$.
- Finite intersection: for $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Arbitrary unions: for a subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We will suppress the notation $\mathcal{T}$ from our topological space as much as possible.

Example 1.5. The set $\mathbb{R}$ equipped with its usual (metric) topology is a topological space.

Example 1.6. Given a topological space X and a subset $Z \subseteq X$, we can make Z into a topological space with open subsets given by $U \cap Z$ whenever $U \subseteq Z$ is open.

Definition 1.7 (closed)**.** A subset Z of a topological space X is *closed* if and only if $X \setminus Z$ is open.

One way to describe topologies is via a base.

Definition 1.8 (base)**.** Given a topological space X, a *base* $\mathcal{B} \subseteq \mathcal{P}(X)$ for the topology such that any open subset $U \subseteq X$ is the union of a subcollection of $\mathcal{B}$. Equivalently, for any open subset $U \subseteq X$ and $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example 1.9. The collection of open intervals $(a, b) \subseteq \mathbb{R}$ generates the usual topology. In fact, one can even restrict ourselves to open intervals (a, b) where $a, b \in \mathbb{Q}$, so $\mathbb{R}$ has a countable base.

Our morphisms are continuous maps.

Definition 1.10 (continuous)**.** A function $f\colon X \to Y$ between topological spaces is *continuous* if and only if $f^{-1}(V) \subseteq X$ is open for each open subset $V \subseteq Y$.

Thus, we can define Top as the category of topological spaces equipped with continuous maps as its morphisms. Thinking categorically allows us to make the following definition.

Definition 1.11 (homeomorphism)**.** A *homeomorphism* is an isomorphism in Top. Namely, a function $f\colon X \to Y$ between topological spaces which is continuous and has a continuous inverse.

Remark 1.12. There are continuous bijections which are not homeomorphisms! For example, one can map $[0, 2\pi) \to S^1$ by sending $x \mapsto e^{ix}$, which is a continuous bijection, but the inverse is discontinuous at $1 \in S^1$.

Earlier, we wanted to have finite products in our category. Here is how we take products of pairs.

Definition 1.13 (product topology)**.** Given topological spaces X and Y, we define the topological space $X \times Y$ as having $X \times Y$ as its set and open subsets given by arbitrary unions of sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open.

Remark 1.14. Alternatively, we can say that the topology $X \times Y$ has a base given by the "rectangles" $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open. In fact, if $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, respectively, then we can check that the open subsets

$$\{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a base for $X \times Y$.

Remark 1.15. The final object in Top is the singleton space.

Now, group objects in Top are called topological groups, which are interesting in their own right. For example, locally compact topological groups have a good Fourier analysis theory.

Example 1.16. The group $\mathbb{R}$ under addition is a topological group. In fact, $\mathbb{Q}$ under addition is also a topological group, though admittedly a more unpleasant one.

Example 1.17. The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a topological group.

1.1.3 Review of Differential Topology

However, in this course, we will be more interested in manifolds, so let's define these notions. We refer to [Elb24] for (a little) more detail, and we refer to [Lee13] for (much) more detail. To begin, we note arbitrary topological spaces are pretty rough to handle; here are some niceness requirements. The following is a smallness assumption.

Definition 1.18 (separable)**.** A topological space X is *separable* if and only if it has a countable base.

The following says that points can be separated.

Definition 1.19 (Hausdorff)**.** A topological space X is *Hausdorff* if and only if any pair of distinct points $p, q \in X$ have disjoint open neighborhoods.

The following is another smallness assumption, which we will use frequently but not always.

Definition 1.20 (compact)**.** A topological space X is *compact* if and only if any open cover $\mathcal{U}$ (i.e., each $U \in \mathcal{U}$ is open, and $X = \bigcup_{U \in \mathcal{U}} U$) has a finite subcollection which is still an open cover.

We are now ready for our definition.

Definition 1.21 (topological manifold)**.** A *topological manifold of dimension* n is a topological space X satisfying the following.

- X is Hausdorff.
- X is separable.
- Locally Euclidean: X has an open cover $\{U_\alpha\}_{\alpha \in \kappa}$ such that there are open subsets $V_\alpha \subseteq \mathbb{R}^n$ and homeomorphisms $\varphi_\alpha \colon U_\alpha \to V_\alpha$.

Remark 1.22. By passing to open balls, one can require that all the V_α are open balls. By doing a little more yoga with such open balls (noting $B(0,1) \cong \mathbb{R}^n$), one can require that $V_\alpha = \mathbb{R}^n$ always.

Remark 1.23. It turns out that open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ can only be homeomorphic if and only if $n = m$. This implies that the dimension of a connected component of X is well-defined without saying what n is in advance. However, we should say what n is in advance in order to get rid of pathologies like $\mathbb{R} \sqcup \mathbb{R}^2$.

To continue, we must be careful about our choice of U_αs and φ_αs.

Definition 1.24 (chart, atlas, transition function)**.** Fix a topological manifold X of dimension n.

- A *chart* is a pair (U, φ) of an open subset $U \subseteq X$ and homeomorphism φ of U onto an open subset of $\mathbb{R}^n$.
- An *atlas* is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$ such that $\{U_\alpha\}_{\alpha \in \kappa}$ is an open cover of X.
- The *transition function* between two charts (U, φ) and (V, ψ) is the composite homeomorphism

$$\varphi(U \cap V) \xleftarrow{\varphi} (U \cap V) \xrightarrow{\psi} \psi(U \cap V).$$

  Note that there is also an inverse transition map going in the opposite direction.

Let's see some examples.

Example 1.25. The space $\mathbb{R}^n$ is a topological manifold of dimension n. It has an atlas with the single chart $\mathrm{id} \colon \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.26. The singleton $\{*\}$ is a topological manifold of dimension 0. In fact, $\{*\} = \mathbb{R}^0$.

Example 1.27. The hypersurface $S^n \subseteq \mathbb{R}^{n+1}$ cut out by the equation

$$x_0^2 + \cdots + x_n^2 = 1$$

is a topological manifold of dimension n. It has charts given by stereographic projection out of some choice of north and south poles. Alternatively, it has charts given by the projection maps $\mathrm{pr}_i \colon S^n \to \mathbb{R}^n$ given by deleting the ith coordinate, defined on the open subsets

$$U_i^\pm := \{(x_0, \ldots, x_n) \in \mathbb{R}^n : \pm x_i > 0\}$$

for choice of index i and sign in $\{\pm\}$.

Calculus on our manifolds will come from our transition maps.

Definition 1.28. An atlas $\mathcal{A}$ on a topological manifold X is C^k, real analytic, or complex analytic (if $\dim X$ is even) if and only if the transition maps have the corresponding condition.

1.2 August 30

Today we finish our review of smooth manifolds. Once again, we refer to [Elb24] for a few more details and [Lee13] for many more details.

Notation 1.29. We will use the word *regular* to refer to one of the regularity conditions C^k, smooth, real analytic, or complex analytic. We may abbreviate complex analytic to "complex" when no confusion is possible. We use the field $\mathbb{F}$ to denote the "ground field," which is $\mathbb{C}$ when considering the complex analytic case and $\mathbb{R}$ otherwise.

1.2.1 Smooth Manifolds

We now define a regular manifold.

Definition 1.30 (regular manifold)**.** A *regular manifold* of dimension n is a pair $(M, \mathcal{A})$ of a topological manifold M and a maximal regular atlas $\mathcal{A}$; a chart is called regular if and only if it is in $\mathcal{A}$. We will eventually suppress the $\mathcal{A}$ from our notation as much as possible.

The reason for using a maximal atlas is to ensure that it is more or less unique.

Remark 1.31. Here is perhaps a more "canonical" way to deal with atlas confusion. One can say that two regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if and only if the transition maps between them are also regular; this is the same as saying that $\mathcal{A}_1 \cup \mathcal{A}_2$ is regular. Compatibility forms an equivalence relation, and each equivalence class $[\mathcal{A}]$ has a unique maximal element, which one can explicitly define as

$$\mathcal{A}_{\max} := \{(U, \varphi) : \mathcal{A} \text{ and } (U, \varphi) \text{ are compatible}\}.$$

This explains why it is okay to just work with maximal atlases.

Example 1.32. One can give the topological manifold $\mathbb{R}^2$ many non-equivalent complex structures. For example, one has the usual choice of $\mathbb{R}^2 \cong \mathbb{C}$, but one can also make $\mathbb{R}^2$ homeomorphic to $B(0,1) \subseteq \mathbb{C}$.

Example 1.33. There are "exotic" smooth structures on S^7.

Example 1.34. Given regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, one can form the product manifold $X \times Y$. It should have maximal atlas compatible with the atlas

$$\{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B}\}.$$

1.2.2 Regular Functions

With any class of objects, we should have morphisms.

Definition 1.35. A function $f\colon X \to Y$ of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is regular if and only if any $p \in X$ has a choice of charts $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $f(U) \subseteq V$ and the composite

$$\varphi(U) \xleftarrow{\varphi} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is a regular function between open subsets of Euclidean space.

Remark 1.36. One can replace the single choice of charts above with any choice of charts satisfying $p \in U$ and $f(U) \subseteq V$.

Remark 1.37. Here is another way to state this: for any open $V \subseteq Y$ and smooth function $h\colon V \to \mathbb{F}$, the composite

$$f^{-1}(U) \xrightarrow{f} V \xrightarrow{h} \mathbb{F}$$

succeeds in being smooth (in any local coordinates).

Definition 1.38 (diffeomorphism)**.** A *diffeomorphism* of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is a regular map $f\colon X \to Y$ with regular inverse.

Remark 1.39. Alternatively, one can say that the charts in $\mathcal{A}$ and the charts in $\mathcal{B}$ are in natural bijection via f. Checking that these notions align is not too hard.

The above definition of regular map is a little rough to handle, so let's break it down into pieces.

Definition 1.40 (local coordinates)**.** Fix a regular manifold $(X, \mathcal{A})$ of dimension n. Then a system of *local coordinates* around some point $p \in X$ is a choice of regular chart $(U, \varphi) \in \mathcal{A}$ for which $\varphi(p) = 0$. From here, our local coordinates $(x_1, \ldots, x_n)$ are the composite of φ with a coordinate projection to the ground field. (In the complex analytic case, we want the ground field to be $\mathbb{C}$; otherwise, the ground field is $\mathbb{R}$.)

Now, we are able to see that a function $f\colon X \to \mathbb{R}$ is regular if and only if it becomes regular in local coordinates. One can even define regularity with respect to a subset of X.

Regularity allows us to produce lots of manifolds, as follows.

Theorem 1.41. Given regular maps $f_1, \ldots, f_m\colon X \to \mathbb{F}$, the subset

$$\{p \in \mathbb{F}^n : f_1(p) = \cdots = f_m(p) = 0 \text{ and } \{df_1(p), \ldots, df_n(p)\} \text{ are linearly independent}\}$$

is a manifold of dimension $n - m$.

Sketch. This is more or less by the implicit function theorem; for the $\mathbb{F} = \mathbb{R}$ cases, one can essentially follow [Lee13, Corollary 5.14]. ■

Example 1.42. The function $\mathbb{R}^{n+1} \to \mathbb{R}$ given by $(x_0, \ldots, x_n) \mapsto x_0^2 + \cdots + x_n^2$ is real analytic and sufficiently regular at the value 1, which establishes that S^n defined in Example 1.27 succeeds at being a real analytic manifold.

Functions to $\mathbb{F}$ have a special place in our hearts, so we take the following notation.

Notation 1.43. Give a regular manifold X and any open subset $U \subseteq X$, we let $\mathcal{O}_X(U)$ denote the set of regular functions $U \to \mathbb{F}$

Remark 1.44. One can check that the data $\mathcal{O}_X$ assembles into a sheaf. Namely, an inclusion $U \subseteq V$ produces restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Remark 1.45. Once we have all of our regular functions out of X, we note that some Yoneda-like philosophy explains that the sheaf of X determines its full regular structure. Here is an explicit statement: given a manifold X and two maximal regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ determining sheaves of regular functions $\mathcal{O}_1$ and $\mathcal{O}_2$, having $\mathcal{O}_1 = \mathcal{O}_2$ forces $\mathcal{A}_1 = \mathcal{A}_2$. Indeed, it is enough to show the inclusion $\mathcal{A}_1 \subseteq \mathcal{A}_2$, so suppose (U, φ) is a regular chart in $\mathcal{A}_1$. Then the corresponding local coordinates $(x_1, \ldots, x_n)$ all succeed at being regular for $\mathcal{A}_1$, so they are smooth functions in $\mathcal{O}_1$, so they live in $\mathcal{O}_2$ also, so (U, φ) will succeed at being a regular local diffeomorphism for $\mathcal{A}_2$ and hence be a regular chart.

Sheaf-theoretic notions tell us that we should be interested in germs.

Definition 1.46 (germ)**.** Fix a point p on a regular manifold X. A *germ* of a regular function $f \in \mathcal{O}_X(U)$ (where $p \in U$) is the equivalence class of functions $g \in \mathcal{O}_X(V)$ (for a possibly different open subset V containing p) such that $f|_{U \cap V} = g|_{U \cap V}$. The collection of equivalence classes is denoted $\mathcal{O}_{X,p}$ and is called the stalk at p.

1.3 September 4

Today we hope to finish our review of differential topology.

Convention 1.47. For the remainder of class, our manifolds will be smooth, real analytic, or complex analytic.

1.3.1 Tangent Spaces

Now that we are thinking locally about our functions via germs, we can think locally about our tangent spaces.

Definition 1.48 (derivation)**.** Fix a point p on a regular manifold X. A *derivation* at p is an $\mathbb{F}$-linear map $D \colon \mathcal{O}_{X,p} \to \mathbb{F}$ satisfying the Leibniz rule

$$D(fg) = g(p)D(f) + f(p)D(g).$$

Definition 1.49 (tangent space)**.** Fix a point p on a regular manifold X. Then the *tangent space* T_pX is the $\mathbb{F}$-vector space of all derivations on $\mathcal{O}_{X,p}$.

As with everything in this subject, one desires a local description of the tangent space.

Lemma 1.50. Fix an n-dimensional regular manifold X and a point $p \in X$. Equip p with a chart (U, φ) giving local coordinates $(x_1, \ldots, x_n)$. Then the maps $D_i\colon \mathcal{O}_{X,p} \to \mathbb{F}$ given by

$$D_i\colon [(V, f)] \mapsto \left.\frac{\partial f|_{U\cap V}}{\partial x_i}\right|_p$$

provide a basis for T_pX.

Proof. Checking that this is a derivation follows from the Leibniz rule on the chart. Linear independence of the $D_\bullet$s can also be checked locally by plugging in the germs $[(U, x_i)]$ into any linear dependence.

It remains to check that our derivations span. Well, fix any other derivation D which we want to be in the span of the $D_\bullet$s. By replacing D with $D - \sum_i D(x_i)D_i$, we may assume that $D(x_i) = 0$ for all i. We now want to show that $D = 0$. This amounts to some multivariable calculus. Fix a germ $[(V, f)]$, and shrink U and V enough so that f is defined on U; we want to show $D(f) = 0$. The fundamental theorem of calculus implies

$$f(x_1, \ldots, x_n) = f(0) + \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt.$$

However, one can expand out the derivative on the right by the chain rule to see that

$$f(x_1, \ldots, x_n) = f(0) + \sum_{i=1}^n x_i h_i(x_1, \ldots, x_n)$$

for some regular functions $h_1, \ldots, h_n\colon X \to \mathbb{F}$. Applying D, we see that

$$D(f) = \sum_{i=1}^n \underbrace{D(x_i)}_{0} h_i(p) + \underbrace{x_i(p)}_{0} D(h_i) = 0,$$

as required. ■

Tangent spaces have a notion of functoriality.

Definition 1.51. Fix a regular map $F\colon X \to Y$ of regular manifolds. Given $p \in X$, the *differential map* is the linear map $dF_p\colon T_pX \to T_{F(p)}Y$ defined by

$$dF_p(v)(g) := v(g \circ F)$$

for any $v \in T_pX$ and germ $g \in \mathcal{O}_{X,p}$. We may also denote $dF_p(v)$ by F_*v.

One has to check that dF_p is linear (which does not have much to check) and satisfies the Leibniz rule (which is a matter of expansion); we will omit these checks.

Remark 1.52. One also has a chain rule: for regular maps $F\colon X \to Y$ and $G\colon Y \to Z$, one has $d(G\circ F)_p = dG_{F(p)} \circ dF_p$.

1.3.2 Immersions and Submersions

This map at the tangent space is important enough to give us other definitions.

Definition 1.53 (submersion, immersion, embedding). Fix a regular function $F\colon X \to Y$.

- The map F is a *submersion* if and only if dF_p is surjective for all $p \in X$.
- The map F is an *immersion* if and only if dF_p is injective for all $p \in X$.
- The map F is an *embedding* if and only if F is an immersion and a homeomorphism onto its image.

Remark 1.54. One can check that submersions $F\colon X \to Y$ have local sections $Y \to X$. Explicitly, for $Q \in Y$, the fiber $F^{-1}(\{Q\}) \subseteq X$ is a manifold, and if $Q \in \operatorname{im} F$, the fiber has codimension $\dim Y$.

Remark 1.55. If $F\colon X \to Y$ is an embedding, then the image $F(X) \subseteq Y$ inherits a unique manifold structure so that the inclusion $F(X) \subseteq Y$ is smooth.

Example 1.56. The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x,y) := x$ is a submersion.

Example 1.57 (lemniscate). The function $F\colon S^1 \to \mathbb{R}^2$ given by

$$F(\theta) := \left(\frac{\cos\theta}{1+\sin^2\theta}, \frac{\cos\theta\sin\theta}{1+\sin^2\theta}\right)$$

can be checked to be an immersion (namely, $F'(\theta) \neq 0$ always), but it fails to be injective because $F(\pi/4) = F(3\pi/4) = (0,0)$, so it is not an embedding.

Example 1.58. The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$ is a smooth homeomorphism onto its image, but it is not an immersion.

Example 1.59. For any open subset $U \subseteq X$ of a manifold, the inclusion map $U \to X$ is an embedding. (In fact, it is also a submersion.)

We will want to distinguish between embeddings, notably to get rid of open embeddings.

Definition 1.60 (closed). An embedding $F\colon X \to Y$ of regular manifolds is *closed* if and only if $F(X) \subseteq Y$ is closed.

Example 1.61. Fix a submersion $F\colon X \to Y$. A point $Q \in Y$ gives rise to a fiber $Z := F^{-1}(\{Q\})$, which Remark 1.54 explains is a closed submanifold of X of codimension $\dim Y$. One can check that T_pZ is exactly the kernel of $dF_p\colon T_pX \to T_pY$; see [Lee13, Proposition 5.37].

1.3.3 Lie Groups

We now may stop doing topology.

Definition 1.62 (Lie group)**.** A regular *Lie group* is a group object in the category of regular manifolds. For brevity, we may call (real) smooth Lie groups simply "Lie groups" or "real Lie groups," and we may call complex analytic Lie groups simply "complex Lie groups."

Remark 1.63. If X is already a regular manifold, and we are equipped with continuous multiplication and inverse maps, to check that X becomes a regular Lie group, it is enough to check that merely the multiplication map is regular. See [Lee13, Exercise 7-3].

Remark 1.64. Hilbert's 5th problem asks when C^0 Lie groups can give rise to real Lie groups, and there is a lot of work in this direction. As such, we will content ourselves to focus on real Lie groups instead of any weaker regularity.

Remark 1.65. Any complex Lie group is also a real Lie group.

Here is a basic check which allows one to translate checks to the identity.

Lemma 1.66. Fix a regular Lie group G. For any $g \in G$, the maps $L_g \colon G \to G$ and $R_g \colon G \to G$ defined by $L_g(x) := gx$ and $R_g(x) := xg$ are regular diffeomorphisms.

Proof. Regulrity follows from regularity of multiplication. Our inverses of L_g and R_g are given by $L_{g^{-1}}$ and $R_{g^{-1}}$, which verifies that we have defined regular diffeomorphisms. ∎

BIBLIOGRAPHY

[Kir08] Alexander Kirillov Jr. *An Introduction to Lie Groups and Lie Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008.

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Elb22] Nir Elber. *Introduction to Topology and Analysis*. 2022. URL: https://dfoiler.github.io/notes/202A/notes.pdf.

[Elb24] Nir Elber. *Differential Topology*. 2024. URL: https://dfoiler.github.io/notes/214/notes.pdf.

List of Definitions