# 250B: Commutative Algebra

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# Theme 1: Associated Primes

*And in a brilliant moment of word association...*

—John Mulaney

## 1.1 February 3

Today we are talking associated primes.

### 1.1.1 Associated Primes

Fix $M$ an $R$-module. Then given $m \in M$, recall that we can look at

$$\operatorname{Ann} m = \{r \in R : rm = 0\}.$$

Observe that $m \neq 0$ promises $\operatorname{Ann} m \neq R$ because $1_R m \neq 0$. In particular, these ideals are proper most of the time.

These annihilators give rise to the following definition.

Associated primes

**Definition 1.1** (Associated primes). Fix $M$ an $R$-module. Then a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ is *associated to* $M$ if and only if $\mathfrak{p} = \operatorname{Ann} m$ for some $m \in M$. We denote $\operatorname{Ass} M \subseteq \operatorname{Spec} R$ to be the set of all associated primes to $M$.

As usual, let's see some examples.

**Example 1.2.** We have that $\operatorname{Ass} 0 = \varnothing$ because the annihilator of any element in $0$ is $R$, which is not prime.

**Exercise 1.3.** Fix $M := \mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}$-module. Then we compute $\operatorname{Ass} M$.

*Proof.* Let $(p)$ be a prime. Then $(p)$ is associated to $M$ if and only if there exists some $m \in M$ such that

$$\operatorname{Ann} m = (p).$$

If $p \mid n$, then we note that $\operatorname{Ann} \frac{n}{p} = (p)$. Conversely, if $\operatorname{Ann} m$ is prime, then it follows $p \mid n$ because any annihilator will have to contain $(n)$. ■

**Example 1.4.** Fix $\mathfrak{p}$ a prime ideal and fix $M := R/\mathfrak{p}$. Certainly $\mathfrak{p} \in \operatorname{Ass} M/\mathfrak{p}$ because of $[1]_\mathfrak{p}$. Conversely, fix some $b \in R \setminus \mathfrak{p}$, and we want to know what primes can arise as

$$\operatorname{Ann}([b]_\mathfrak{p}) = \{a \in R : ab \in \mathfrak{p}\}.$$

But with $b \notin \mathfrak{p}$, the primality of $\mathfrak{p}$ forces $\operatorname{Ann}([b]_\mathfrak{p}) = \mathfrak{p}$. So $\operatorname{Ass} M = \{\mathfrak{p}\}$.

In the spirit of the above example, we have the following proposition.

**Proposition 1.5.** Fix $M$ an $R$-module. Suppose $I \subseteq R$ is an ideal maximal in

$$\mathcal{P} := \{\operatorname{Ann} m : m \in M \setminus \{0\}\}.$$

Then we claim $I$ is prime.

*Proof.* Because $I \in \mathcal{P}$, we can say $I = \operatorname{Ann} m$, so we have a natural map

$$\varphi : R/I \hookrightarrow M$$

by $\varphi : [r]_I \mapsto rm$.

Now take $a \notin I$, and we show that $ab \notin I$ implies $b \in I$. Well, $ab \in I$ would imply that $ab \in \operatorname{Ann} m$, or alternately, $a \in \operatorname{Ann} bm$. But with $a \notin I$, the maximality of $I$ forces $bm = 0$, so $b \in I$. ■

We have the following corollary.

**Corollary 1.6.** Fix $R$ a Noetherian ring and $M$ a nonzero ring. Then $\operatorname{Ass} M$ is nonempty.

*Proof.* Use the Noetherian condition to generate an ideal maximal in the sense of Proposition 1.5, which will finish. ■

### 1.1.2 Associateed Primes for Fun and Profit

For the sake of comparison, let's talk about $\operatorname{Supp} M$ now.

**Example 1.7.** Fix $R$ a domain. Then $\operatorname{Ass}_R R = \{(0)\}$ by the integral domain condition. However, the support $\operatorname{Supp} R = \operatorname{Spec} R$, so the associated primes appear smaller.

Indeed, we will find that the associated primes will be smaller.

Localization was able to tell us about maps by building upwards: an element was $0$ if and only if zero on all localizations.

**Proposition 1.8.** Fix $R$ Noetherian and $M, N$ as $R$-modules. The following are true; fix $m \in M$.

(a) We have $m = 0$ if and only if $\frac{m}{1} = 0$ in $M_\mathfrak{p}$ for each associated prime $\mathfrak{p} \in \operatorname{Ass} M$.

(b) In fact, a submodule $N \subseteq M$ has $N = 0$ if and only if $N_\mathfrak{p}$ if and only if $N_\mathfrak{p} = 0$ for each $\mathfrak{p} \in \operatorname{Ass} M$.

(c) A map $\varphi : N \to M$ is injective/surjective/isomorphic if and only if $\varphi : N_\mathfrak{p} \to M_\mathfrak{p}$ is injective/surjective if and only if $\mathfrak{p} \in \operatorname{Ass} M$.

*Proof.* It suffices to only prove (a), and the other two follow. For (a), the forwards direction is easy because we know that $m = 0$ if and only if $\frac{m}{1} = 0$ in $M_\mathfrak{p}$ for each prime $\mathfrak{p}$.

In the other direction, suppose $m \neq 0$, and we need to find an associated prime $\mathfrak{p}$ for which $\frac{m}{1} \neq 0$ in $M_\mathfrak{p}$. But we simply take the annihilator maximal among the annihilators containing $\operatorname{Ann} m$ by Proposition 1.5 to finish. ■

Associated primes also behave in short exact sequences, somewhat.

**Lemma 1.9.** Suppose

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of $R$-modules. Then $\operatorname{Ass} A \subseteq \operatorname{Ass} B \subseteq \operatorname{Ass} A \cup \operatorname{Ass} C$.

*Proof.* We have that $\operatorname{Ass} A \subseteq \operatorname{Ass} B$ because any annihilator in $A$ will end up being any annihilator in $B$ as well.

It remains to show $\operatorname{Ass} B \subseteq \operatorname{Ass} A \cup \operatorname{Ass} C$. Well, suppose $\mathfrak{p} \in \operatorname{Ass} B \setminus \operatorname{Ass} A$, and we show that $\mathfrak{p} \in \operatorname{Ass} C$. Namely, we can find $b \in B$ such that

$$\operatorname{Ann} b = \mathfrak{p}.$$

Then we note that $Rb \cong R/\mathfrak{p}$ by $[r]_\mathfrak{p} \mapsto rb$. However, $Rb \cap A = 0$ because otherwise $\mathfrak{p}$ will be appear in $\operatorname{Ass} A$ from the intersected submodule. It follows that

$$Rb \subseteq C$$

as an honest submodule (upon modding out by $A$), which gives us the result. ■

**Corollary 1.10.** Suppose $B = A \oplus C$ as $R$-modules. Then $\subseteq \operatorname{Ass} B = \operatorname{Ass} A \cup \operatorname{Ass} C$.

*Proof.* Use Lemma 1.9 on the split short exact sequences

$$0 \to A \to B \to C \to 0$$

and

$$0 \to C \to B \to A \to 0$$

to finish. ■

### 1.1.3 Finitely Many Associated Primes

Let's build towards the statement that the associated primes are finite. Before this, let's do an example.

**Example 1.11.** We work with $\mathbb{Z}$-modules. Then $\operatorname{Ass} \mathbb{Z} = \{(0)\}$ and $\operatorname{Ass} \mathbb{Z}/p^\bullet\mathbb{Z} = \{(p)\}$. Then from here we can build all finitely generated abelian groups as a direct sum of these and get our associated primes by Corollary 1.10.

Now here is our main lemma.

**Lemma 1.12.** Fix $M$ a finitely generated module over a Noetherian ring $R$. Then $M$ has a finite filtration

$$0 =: M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each quotient $M_{k+1}/M_k \cong R/\mathfrak{p}_k$ for some prime ideals $\{\mathfrak{p}_k\}_{k=0}^{n-1}$.

*Proof.* Because $R$ is Noetherian, $\operatorname{Ass} M$ is nonempty. So find some $\mathfrak{p}_0 = \operatorname{Ann} m_0$ and set $M_1 := Rm_1 \cong R/\mathfrak{p}_1$. Then we can continue this process to $M/M_1$ and pull back along the projection to get $M_2$. This gives us an ascending chain of $R$-submodules

$$M_0 \subseteq M_1 \subseteq \cdots,$$

which must eventually terminate because $M$ is Noetherian (as it is finitely generated over $R$). ■

**Theorem 1.13.** Fix $M$ a finitely generated module over a Noetherian ring $R$. The following are true.

(a) If $M$ is nonzero, then $\operatorname{Ass} M$ is finite.

(b) Fix $U \subseteq R$ a multiplicatively closed subset. Then we have that

$$\operatorname{Ass}_{R[U^{-1}]} M\left[U^{-1}\right] = \{\mathfrak{p}\left[U^{-1}\right] : \mathfrak{p} \in \operatorname{Ass} M, \mathfrak{p} \cap U = \varnothing\}.$$

(Recall that $\operatorname{Spec} R\left[U^{-1}\right]$ consists of localizations of primes of $R$.)

(c) We have that

$$\bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p} = \bigcup_{m \in M \setminus \{0\}} \operatorname{Ann} m.$$

(d) If $\mathfrak{p}$ is a minimal prime ideal containing $\operatorname{Ann} M$, then $\mathfrak{p} \in \operatorname{Ass} M$.

*Proof.* Here we go.

(a) Decompose the filtration for $M$ from Lemma 1.12 and use Lemma 1.9 to upper-bound $\operatorname{Ass} M$. Namely, if the factors of the filtration as $R/\mathfrak{p}_k$ for primes $\{\mathfrak{p}_k\}_{k=0}^{n-1}$, then

$$\operatorname{Ass} M \subseteq \{\mathfrak{p}_0, \ldots, \mathfrak{p}_{n-1}\}.$$

(b) In one direction, suppose that $\mathfrak{p} \in \operatorname{Ass} M$ and $\mathfrak{p} \cap U = \varnothing$. We already know that $\mathfrak{p}\left[U^{-1}\right]$ is prime. Because $\mathfrak{p} \in \operatorname{Ass} M$, we are promised some $m \in M$ such that

$$R/\mathfrak{p} \cong Rm \subseteq M,$$

so localization provides an injection $R\left[U^{-1}\right]/\mathfrak{p}\left[U^{-1}\right] \hookrightarrow M\left[U^{-1}\right]$, which can again be read off to be an associated prime.

In the other direction, suppose that $\mathfrak{p}\left[U^{-1}\right] \in \operatorname{Ass}_{R[U^{-1}]} M\left[U^{-1}\right]$, so we are promised some injection

$$\varphi : R\left[U^{-1}\right]/\mathfrak{p}\left[U^{-1}\right] \hookrightarrow M\left[U^{-1}\right].$$

Here is the key trick: because everything here is finitely generated and in particular finitely presented, we get an isomorphism

$$\operatorname{Hom}_{R[U^{-1}]}\left(R\left[U^{-1}\right]/\mathfrak{p}\left[U^{-1}\right], M\left[U^{-1}\right]\right) \cong \operatorname{Hom}_R(R/\mathfrak{p})\left[U^{-1}\right].$$

So we are promised some $\widetilde{\varphi} = \frac{\psi}{u}$ where $\psi : R/\mathfrak{p} \to M$ and for some $u \in U$. But because $\varphi$ was an injection, $\widetilde{\varphi}$ will be an injection, so $\psi$ will be an injection, and we do get that $\mathfrak{p}$ is an associated prime.

(c) In oe direction, suppose that $a \in \operatorname{Ann} m$. Then we can choose $\mathfrak{p}$ to be the ideal maximal among all annihilators connecting $\operatorname{Ann} m$, which makes $\mathfrak{p}$ an associated prime by Proposition 1.5. So $a \in \mathfrak{p}$ here.

(d) We leave this as an exercise. The main idea is to localize at $\mathfrak{p}$ so that $\operatorname{Ass} M_{\mathfrak{p}}$ is $\{\mathfrak{p}_{\mathfrak{p}}\}$. ■

**Quote 1.14.** I hope you see how powerful this idea is, of localization.

**Corollary 1.15.** Fix $M$ a finitely generated module over a Noetherian ring $R$, and fix any ideal $J \subseteq R$ and $m \in M$. Then $J \subseteq \operatorname{Ann} m$ (for some $m \notin M$) or there exists $a \in J$ such that $am = 0$ implies $m = 0$ for each $m \in M$.

*Proof.* The idea is to use (c) of the theorem. If every $a \in J$ annihilates some nonzero element of $M$, then

$$J \subseteq \bigcup_{m \in M \setminus \{0\}} \operatorname{Ann} m = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}.$$

We claim that $J \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$; otherwise, we can find $x_i \in \mathfrak{p}_j \setminus J$ for each $\mathfrak{p}_i \neq \mathfrak{p}_i$ while $x_i \in \mathfrak{p}_i$, for which $x_1 + x_2 + \cdots + x_n$ will not lie in any prime ideal. ■

### 1.1.4 Motivating Primay Decomposition

Let's give some motivational remarks for the primary decomposition. As an example, we consider $\mathbb{Z}$, where it happens that

$$(m) \cap (n) = (mn) = (m)(n).$$

This is a very nice property to have, with respect to proving unique prime factorization and such. Namely, to state unique prime factorization, we call an ideal "primary" if it is the power of some prime ideal, and then we see that any ideal $I$ is the intersection of finitely many "primary" ideals.

We will try to generalize this. Here is our definition of "primary."

$\mathfrak{p}$-primary

**Definition 1.16** ($\mathfrak{p}$-primary)**.** Fix $\mathfrak{p} \in \operatorname{Spec} R$ a prime ideal and $R$-modules $N \subseteq M$. Then $N$ is a $\mathfrak{p}$-*primary* submodule of $M$ if and only if

$$\operatorname{Ass} M/N = \{\mathfrak{p}\}.$$

**Example 1.17.** The ideals $(p^\bullet)$ are $p$-primary in $\mathbb{Z}$.

**Example 1.18.** Any prime ideal $\mathfrak{p}$ is $\mathfrak{p}$-primary in $R$ because $\operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

Let's prove one nice lemma to finish off today.

**Lemma 1.19.** Fix $N_1, \ldots, N_m$ a finite collection of $\mathfrak{p}$-primary submodules of an $R$-module $M$. Then

$$\bigcap_{k=1}^{n} N_k$$

is also $\mathfrak{p}$-primary.

*Proof.* By induction, it suffices to show the result for $m = 2$ so that we want to show $N_1 \cap N_2$ is $\mathfrak{p}$-primary. Now, we have the right-exact sequence

$$0 \to N_1 \cap N_2 \to M \to \frac{M}{N_1} \oplus \frac{M}{N_2},$$

which tells us that $\frac{M}{N_1 \cap N_2}$ is a submodule of $\frac{M}{N_1} \oplus \frac{M}{N_2}$. But then

$$\operatorname{Ass} \frac{M}{N_1 \cap N_2} \subseteq \operatorname{Ass} M/N_1 \cup \operatorname{Ass} M/N_2 = \{\mathfrak{p}\},$$

so we are done. ■

And we close by stating the theorem.

**Theorem 1.20.** Any finitely generated module $M$ over a Noetherian ring $R$ is an intersection of finitely many primary submodules.