202B: Functional Analysis

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1
PRODUCT MEASURES

1.1 January 17

Let's just get started.

1.1.1 Course Notes

Here are some course notes.

- The professor for this course is Michael Christ.
- There is a bCourses, which I don't have access to.
- There will be an exam in the evening in February.
- Problem sets will be due on Fridays.
- We will assume analysis on the level of Math 202A; see something like [Elb22].
- The text for the course is [Fol99].

1.1.2 Measures

Our first topic is to integrate on product spaces. Roughly speaking, we might have some measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ with some way to measure on them, and then we will want to measure $X \times Y$. Let's quickly recall what a measure is; we won't bother to recall the definition of a σ-algebra, but we will refer to [Elb22, Definition 5.25]. This requires the definition of a σ-algebra.

Definition 1.1 (σ-algebra)**.** Fix a set X. Then a collection $\mathcal{M} \subseteq \mathcal{P}(X)$ is a *σ-algebra* if and only if the following conditions are satisfied.

- $\varnothing \in \mathcal{M}$.
- $\mathcal{M}$ is closed under countable unions.
- $\mathcal{M}$ is closed under complements.

In the sequel, we will also want to produce σ-algebras.

Definition 1.2. Fix a set X. Given a collection $\mathcal{S} \subseteq \mathcal{P}(X)$, we will say that the smallest σ-algebra generated by $\mathcal{S}$ is the σ-algebra *generated by* $\mathcal{S}$.

It is lemma that a smallest (i.e., contained in all other such σ-algebras) such σ-algebra exists and is unique. Let's see this.

Lemma 1.3. Fix a set X and collection $\mathcal{S} \subseteq \mathcal{P}(X)$. Then there is a σ-algebra $\mathcal{M}$ containing $\mathcal{S}$ such that $\mathcal{M} \subseteq \mathcal{M}'$ for any σ-algebra $\mathcal{M}'$ containing $\mathcal{S}$. This $\mathcal{M}$ is also unique.

Proof. There is certainly some σ-algebra on X containing $\mathcal{S}$, namely $\mathcal{P}(X)$. So there is a nonempty collection $\underline{\mathcal{M}}$ of all σ-algebras containing $\mathcal{S}$, and then we define

$$\mathcal{M} := \bigcap_{\mathcal{M}' \in \underline{\mathcal{M}}} \mathcal{M}'.$$

Certainly $\mathcal{M}$ contains $\mathcal{S}$, and one can check directly that $\mathcal{M}$ is a σ-algebra. (See [Elb22, Lemma 5.28] for details.) And by construction, we see that $\mathcal{M} \subseteq \mathcal{M}'$ for any σ-algebra $\mathcal{M}'$ containing $\mathcal{S}$. Lastly, we note that $\mathcal{M}$ is unique because any two such σ-algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ will be contained in each other and hence equal. ∎

Anyway, here is our definition of a measure.

Definition 1.4 (measure)**.** Fix a σ-algebra $\mathcal{M}$ on a set X. Then a *measure* μ is a countably additive nonnegative function $\mu \colon \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, and we require that $\mu(\varnothing) = 0$. Here, being countably additive means that

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=0}^{\infty} \mu(E_i),$$

where the sum is allowed to be in ∞ (namely, diverge to infinity). We call the triple $(X, \mathcal{M}, \mu)$ a *measure space*.

Remark 1.5. If we have $\mu(\varnothing) > 0$, then the countably additive condition implies that $\mu(\varnothing) = \infty$ and then $\mu(E) = \infty$ for all $E \in \mathcal{M}$. This is in fact countably additive, but we would like to exclude it.

We will want to make our measures somewhat small.

Definition 1.6 (σ-finite)**.** Fix a measure space $(X, \mathcal{M}, \mu)$. Then μ is σ*-finite* if and only if X is a countable union of sets in $\mathcal{M}$ of finite measure.

This smallness condition is quite tame, and in practice all measures are σ-finite.

1.1.3 The Extension Theorem

We would like to discuss how to build measures from objects easier to construct. The following generalization of Definition 1.1 will be useful.

Definition 1.7 (algebra)**.** Fix a set X. Then a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* if and only if the following conditions are satisfied.

- $\varnothing \in \mathcal{A}$.
- $\mathcal{A}$ is closed under finite unions.
- $\mathcal{A}$ is closed under complements.

Example 1.8. Fix an uncountable set X, and let $\mathcal{A}$ denote the collection of finite and cofinite sets. Then $\mathcal{A}$ is an algebra (the finite union of finite sets is finite, and the finite union of cofinite sets is cofinite), but it need not be a σ-algebra because the countable union of finite sets need not be finite nor cofinite.

Example 1.9. Fix $X := \mathbb{R}$, and let $\mathcal{A}$ denote the collection of finite unions of open or closed intervals. Then $\mathcal{A}$ is an algebra but not a σ-algebra.

Additionally, the following generalization of Definition 1.4 will be useful.

Definition 1.10 (premeasure)**.** Fix an algebra $\mathcal{A}$ on a set X. Then a *premeasure* is a function $\rho\colon \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is satisfies the following.

- $\rho(\varnothing) = 0$.
- Finitely additive: we have $\rho(A \sqcup B) = \rho(A) + \rho(B)$ for $A, B \in \mathcal{A}$.
- Countably additive: suppose $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ is pairwise disjoint, and $\bigsqcup_{i=1}^\infty A_i \in \mathcal{A}$. Then

$$\rho\left(\bigsqcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \rho(A_i).$$

And now here is our theorem.

Theorem 1.11 (Extension)**.** Fix a set X and a premeasure ρ on an algebra $\mathcal{A}$ over X. Then there exists a measure μ on the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$ such that $\mu|_{\mathcal{A}} = \rho$. Additionally, if ρ is σ-finite, then μ is unique on $\mathcal{M}$.

Here, σ-finiteness for ρ takes the same definition as Definition 1.6.

Proof of Theorem 1.11. For existence, combine [Elb22, Lemma 6.16 and Theorems 6.21, 6.24]. Further, uniqueness is [Elb22, Theorem 6.35]. It will be helpful to say a few words about the construction. Essentially, one builds an "outer measure" ρ^* on $\mathcal{P}(X)$ by

$$\rho^*(E) := \inf\left\{\sum_{n=0}^\infty \rho(A_n) : \{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \text{ and } E \subseteq \bigcup_{n=0}^\infty A_n\right\}.$$

Then one restricts ρ^* to a smaller σ-algebra over which it becomes a bona fide measure. ■

1.1.4 Towards Product Measures

For our product measures, we take the following outline. Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$.

1. We will construct a special σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$. Then we will construct a measure $\mu \times \nu$ on $\mathcal{M} \otimes \mathcal{N}$.
2. Once the construction is in place, we will find a way to compare "double integrals" with "single integrals." Morally, one wants equalities comparing

$$\iint_{X\times Y} f\, d(\mu \times \nu) \qquad \text{and} \qquad \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x).$$

   The moral of the story is that we will be able to compare our product measure with the measures on X and Y which we already understand.
3. Lastly, we will specialize to Euclidean space $\mathbb{R}^d$.

Let's go ahead and begin.

Definition 1.12 (measurable rectangle)**.** Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. A *measurable rectangle* $E \subseteq X \times Y$ is a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

Example 1.13. The product of the circles $S^1 \subseteq \mathbb{R}^2$ and $S^1 \subseteq \mathbb{R}^2$ is the torus $S^1 \times S^1$ in $\mathbb{R}^4$ (identified with $\mathbb{R}^2 \times \mathbb{R}^2$).

Definition 1.14 (product algebra)**.** Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Then we define the *product algebra* $\mathcal{A}(X, Y)$ as the collection of all finite disjoint unions of measure rectangles.

Remark 1.15. The reason that we have taken finite disjoint unions of rectangles is because we know how to measure measurable rectangles, and we know how to sum their measures as disjoint unions.

It's not totally clear that we have actually defined an algebra. We'll show this next class.

1.2 January 19

Here we go.

1.2.1 The Product Algebra

We quickly pick up the following lemma.

Lemma 1.16. Fix finitely many subsets $A_1, \ldots, A_n \subseteq X$, and suppose that these subsets live in an algebra $\mathcal{A}$ on X. Then there exists a finite partition $\{C_\alpha\}_{\alpha \in \kappa}$ of X of sets in the algebra such that

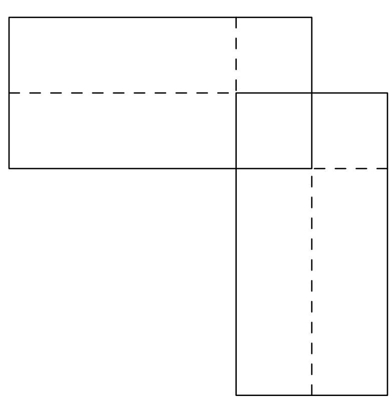
$$A_i = \bigsqcup_{\substack{\alpha \in \kappa \\ C_\alpha \subseteq A_i}} A_i.$$

Proof. We basically build a Venn diagram. Choose index set I to be $\{0,1\}^n$, and define C_α for $\alpha \in I$ to be the set of $x \in X$ such that $x \in A_i$ if and only if $\alpha_i = 1$. Note that we can write C_α as

$$C_\alpha := \bigcup_{\alpha_i = 1} A_i \Bigg\backslash \bigcup_{\alpha_i = 0} A_i.$$

Now, these $C_\bullet$s of course provide a partition satisfying the needed condition by its construction. ■

Anyway, let's return to showing that we have a product algebra. For example, it turns out that the union of two measure rectangles is again a measurable rectangle. Here's the image.



And here is our statement.

Lemma 1.17. Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Then $\mathcal{A}(X, Y)$ is actually an algebra.

Proof. Here are our checks.

- Note $\varnothing \times \varnothing = \varnothing$, so $\varnothing \in \mathcal{A}(X, Y)$.
- Finite union of rectangles: suppose that we have measurable rectangles $\{A_i \times B_i\}_{i=1}^n$. Then we show that the union is in $\mathcal{A}(X, Y)$. Now, the $A_\bullet$s produce some partition $\{C_\alpha\}_{\alpha \in I} \subseteq \mathcal{M}$ of X via Lemma 1.16, and the $B_\bullet$s produce some partition $\{D_\beta\}_{\beta \in J} \subseteq \mathcal{N}$ of Y via Lemma 1.16 again. Now
$$A_i \times B_i = \bigsqcup_{\substack{C_\alpha \subseteq A_i \\ D_\beta \subseteq B_i}} C_\alpha \times D_\beta,$$
so
$$\bigcup_{i=1}^n A_i \times B_i = \bigcup_{i=1}^n \bigsqcup_{\substack{C_\alpha \subseteq A_i \\ D_\beta \subseteq B_i}} C_\alpha \times D_\beta,$$
so our union is a union of measurable rectangles of the form $C_\alpha \times D_\beta$. But these measurable rectangles are all pairwise disjoint because the $C_\bullet$s and $D_\bullet$s are all pairwise disjoint, so the above union is in $\mathcal{A}$.
- Finite union: given $E_1, \ldots, E_n \in \mathcal{A}$, we need to show the union is in $\mathcal{A}$. Well, write
$$E_i = \bigsqcup_{j=1}^{n_i} A_{ij} \times B_{ij}$$
for some $A_\bullet \in \mathcal{M}$ and $B_\bullet \in \mathcal{N}$. Then
$$\bigcup_{i=1}^n E_i = \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} A_{ij} \times B_{ij}$$
is a union of measurable rectangles and hence lives in $\mathcal{A}$ by the above check.
- Complement: given $E \in \mathcal{A}$, write
$$E = \bigcup_{i=1}^n A_i \times B_i$$
for measurable rectangles $A_\bullet \times B_\bullet$. As before, the $A_\bullet$s produce some partition $\{C_\alpha\}_{\alpha \in I} \subseteq \mathcal{M}$ of X via Lemma 1.16, and the $B_\bullet$s produce some partition $\{D_\beta\}_{\beta \in J} \subseteq \mathcal{N}$ of Y via Lemma 1.16 again. This allows us to write
$$E = \bigsqcup_{i=1}^n \bigsqcup_{\substack{C_\alpha \subseteq A_i \\ D_\beta \subseteq B_i}} C_\alpha \times D_\beta,$$
and then the complement $(X \times Y) \setminus E$ will be the union of the measurable rectangles $C_\alpha \times D_\beta$ not in the above union. But these are still disjoint measurable rectangles, so the union remains in $\mathcal{A}$. ■

1.2.2 The Product Measure

Let's now define our product premeasure. Given the measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, we would like to define
$$\rho\left(\bigsqcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i)\nu(B_i),$$
but it is not obvious that this is well-defined. Instead of doing this, we will choose the following definition.

Definition 1.18 (product premeasure)**.** Fix measure spaces $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$. Given $E\in\mathcal{A}(X,Y)$, we define the *product premeasure* $\rho(E)$ as

$$\rho(E):=\int_X \nu(E_x)\,d\mu(x),$$

where $E_x:=\{y\in Y:(x,y)\in E\}$.

Remark 1.19. One should perhaps check that E_x is always in $\mathcal{N}$ and hence measurable. But for this we simply write $E=\bigsqcup_{i=1}^n(A_i\times B_i)$ for measurable rectangles $A_i\times B_i$ and note that

$$E_x=\{y\in Y:(x,y)\in A_i\times B_i\text{ for some }i\}=\bigcup_{\substack{i=1\\x\in A_i}}^n B_i,$$

which is a finite union of measurable sets and hence in $\mathcal{N}$. In fact,

Remark 1.20. One should perhaps check that $x\mapsto\nu(E_x)$ is integrable. Continuing from the above, we can see that these B_i must be disjoint if $x\in A_i$ for each of these i, so actually

$$\nu(E_x)=\sum_{\substack{i=1\\x\in A_i}}^n\nu(B_i)=\sum_{i=1}^n 1_{A_i}(x)\nu(B_i),$$

which is a linear combination of indicators of μ-measurable sets, so this is a μ-integrable function. Notably, we see that the measure of a measurable rectangle $A\times B$ is in fact $\mu(A)\nu(B)$.

Remark 1.21. It is notable that we can write

$$\rho(E)=\int_X\nu(E_x)\,d\mu(x)=\int_X\int_Y 1_E(x,y)\,d\nu(y)\,d\mu(x),$$

where the equality follows because the measure $\nu(E_x)$ is simply integrating Y over the indicator of $1_E(x,y)$.

We now check that we have a premeasure.

Proposition 1.22. Fix measure spaces $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$. Then the defined product premeasure ρ on $\mathcal{A}(X,Y)$ is in fact a premeasure.

Proof. Here are our checks.

- Note $\rho(\varnothing)=0$ because $\varnothing_x=\varnothing$ always.
- Finitely additive: fix disjoint $E_1,E_2\in\mathcal{A}(X,Y)$, and we want to compute $\rho(E_1\sqcup E_2)$. Well, we use Remark 1.21 to note

$$\begin{aligned}\rho(E_1\sqcup E_2)&=\int_X\nu((E_1\sqcup E_2)_x)\,d\mu(x)\\&=\int_X\int_Y 1_{E_1\sqcup E_2}(x,y)\,d\nu(y)\,d\mu(x)\\&=\int_X\int_Y(1_{E_1}(x,y)+1_{E_2}(x,y))\,d\nu(y)\,d\mu(x)\end{aligned}$$

Now, by linearity of integration, this is

$$\begin{aligned}\rho(E_1 \sqcup E_2) &= \int_X \int_Y 1_{E_1}(x,y)\,d\nu(y)\,d\,\mu(x) + \int_X \int_Y 1_{E_2}(x,y)\,d\nu(y)\,d\mu(x) \\ &= \rho(E_1) + \rho(E_2),\end{aligned}$$

as desired.

- Countably additive: we use the Monotone convergence theorem. Fix some disjoint subsets $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}(X,Y)$ such that $E := \bigcup_{i=1}^\infty E_i$ is in $\mathcal{A}(X,Y)$. Proceeding as in the previous check, we see that

$$\begin{aligned}\rho(E) &= \int_X \int_Y 1_E(x,y)\,d\nu(y)\,d\mu(x) \\ &= \int_X \int_Y 1_E(x,y)\,d\nu(y)\,d\mu(x) \\ &= \int_X \int_Y \left(\sum_{i=1}^\infty 1_{E_i}(x,y)\right) d\nu(y)\,d\mu(x).\end{aligned}$$

Now, the functions 1_{E_i} and 1_E are all integrable (for suitably fixed coordinates), so applying the Monotone convergence theorem [Elb22, Theorem 9.18] tells us that

$$\rho(E) = \sum_{i=1}^\infty \int_X \int_Y 1_{E_i}(x,y)\,d\nu(y)\,d\mu(x) = \sum_{i=1}^\infty \rho(E_i),$$

as desired. ■

We can now produce our product measure.

Definition 1.23 (product measure)**.** Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Define the *product σ-algebra* $\mathcal{M} \otimes \mathcal{N}$ to be the σ-algebra generated by $\mathcal{A}(X,Y) \subseteq \mathcal{P}(X \times Y)$. Then the product premeasure ρ on $\mathcal{A}(X,Y)$ extends by Theorem 1.11 to a measure $\mu \times \nu$ on $\mathcal{M} \otimes \mathcal{N}$.

Remark 1.24. By Theorem 1.11, if μ and ν are both σ-finite, then one can see that ρ is still σ-finite by some covering with measurable rectangles, so $\mu \times \nu$ becomes the unique measure on $\mathcal{M} \otimes \mathcal{N}$ extending ρ.

1.2.3 Tonelli's Theorem

The construction of our product premeasure in Definition 1.18 has a "handedness" in that we integrate with respect to Y and then with respect to X. This is somewhat upsetting, so we work to remedy this.

Theorem 1.25 (Tonelli)**.** Fix σ-finite measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Fix a measurable function $f\colon X \times Y \to [0, \infty]$. Then the following hold.

(a) The function $y \mapsto f(x,y)$ is $\mathcal{N}$-measurable.

(b) The function $x \mapsto \int_Y f(x,y)\,d\nu(y)$ is $\mathcal{M}$-measurable.

(c) We have

$$\int_{X \times Y} f\,d(\mu \times \nu) = \int_X \int_Y f(x,y)\,d\nu(y)\,d\mu(x).$$

Remark 1.26. Note that, once measurable, we can integrate a nonnegative function if we allow for infinite values. For example, see something like [Elb22, Proposition 9.22]

Reductions of Theorem 1.25. We begin with two reductions.

- We reduce to the case where f is the indicator of a function 1_E. Indeed, having the result for indicators shows the conclusions for any linear combination of these, so we get the result for simple measurable functions, and then we can get the general case by taking monotone limits via the Monotone convergence theorem [Elb22, Theorem 9.18].

  (Namely, (a) is direct by taking limits, (b) follows by the Monotone convergence theorem to move out the limit out of the integral and then taking limits to get measurable, and (c) is achieved directly by the Monotone convergence theorem repeatedly.)

- We reduce to the case where X and Y are spaces of finite measure. Indeed, by the σ-finiteness of X and Y, we can partition each into countable disjoint union of sets of finite measure, and then by taking rectangles, we see that $X \times Y$ is a countable union of disjoint sets of finite measure. So achieving the result on these disjoint sets of finite measure, we can check the conclusions by summing over all the disjoint spaces, again concluding via the Monotone convergence theorem [Elb22, Theorem 9.18]. Namely, one can do an identical argument to the parenthetical remark of the previous reduction.

Before doing anything, we note that the σ-algebra $\mathcal{M} \otimes \mathcal{N}$ is not obviously generated at finite steps from $\mathcal{A}(X, Y)$; in fact, there is no countable constructive procedure to do this. So we are not going to proceed by trying to build up to $\mathcal{M} \otimes \mathcal{N}$; instead we will have to do something difficult. ■

1.3 January 22

Here we go.

1.3.1 Proof of Tonelli's Theorem

Last class we reduced the proof of Theorem 1.25 to having $f = 1_E$ for some measurable set E and having X and Y be finite measure spaces. Today we will complete the proof. We proceed by a sequence of lemmas.

Lemma 1.27. Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. If $E \in \mathcal{M} \otimes \mathcal{N}$ and $x \in X$, then the slice

$$E_x := \{y \in Y : (x, y) \in E\}$$

is in $\mathcal{N}$.

Proof. The problem is that we know very little about $\mathcal{M} \otimes \mathcal{N}$, so we will have to do something indirect. Continue with x fixed, but we let E vary to define

$$\mathcal{D}_x := \{E \subseteq X \times Y : E_x \in \mathcal{N}\}.$$

Note $\mathcal{D}_x$ is a σ-algebra, as we now check.

- Note $\varnothing \subseteq X \times Y$ has $\varnothing = \varnothing_x$ in $\mathcal{N}$. So $\varnothing \in \mathcal{D}_x$.

- Complement: if $E \in \mathcal{D}_x$, then $((X \times Y) \setminus E)_x = Y \setminus E_x$ as this set contains exactly the $y \in Y$ such that $(x, y) \notin E$. Thus, $((X \times Y) \setminus E)_x \in \mathcal{N}$, so $(X \times Y) \setminus E \in \mathcal{N}$.

- Countable unions: fix a countable collection $\{E_i\}_{i=1}^\infty \subseteq \mathcal{D}_x$. Then

$$\left(\bigcup_{i=1}^\infty E_i\right)_x = \bigcup_{i=1}^\infty (E_i)_x$$

  because some y lives in this set if and only if (x,y) belongs to one of the E_i. The right-hand side lives in $\mathcal{N}$ by assumption, so we see $\bigcup_{i=1}^\infty E_i \in \mathcal{D}_x$.

Furthermore, we note that $\mathcal{D}_x$ contains $\mathcal{A}(X,Y)$. Indeed, it suffices to check that $\mathcal{D}_x$ contains measurable rectangles because $\mathcal{A}(X,Y)$ contains disjoint unions of these. Well, for a measurable rectangle $A \times B$, we see

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A, \\ \varnothing & \text{if } x \notin A, \end{cases} \tag{1.1}$$

always lives in $\mathcal{N}$, so $A \times B \in \mathcal{D}_x$. In total, it follows that $\mathcal{D}_x$ must contain the smallest σ-algebra containing $\mathcal{A}(X,Y)$, so $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{D}_x$. This is what we wanted. ■

> **Remark 1.28.** The above proof exemplifies how we will access $\mathcal{M} \otimes \mathcal{N}$: we will construct some σ-algebra characterizing the desirable properties, and then we will show that it contains $\mathcal{A}(X,Y)$ in order to contain $\mathcal{M} \otimes \mathcal{N}$.

We now prove (a) and (b) of Theorem 1.25.

> **Lemma 1.29.** Fix measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, with Y finite. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the function $f_E \colon x \mapsto \nu(E_x)$ is $\mathcal{M}$-measurable.

Proof. We consider

$$\mathcal{D} := \{E \subseteq X \times Y : f_E \text{ is } \mathcal{M}\text{-measurable and } E_x \text{ is always measurable}\}.$$

We would like to show that $\mathcal{D}$ contains $\mathcal{M} \otimes \mathcal{N}$. Let's show some properties of $\mathcal{D}$. We won't succeed at showing that $\mathcal{D}$ is actually a σ-algebra, but we will get close enough. Of course, $\mathcal{D}$ contains $\varnothing$ because $f_\varnothing$ is just the zero function. Additionally, taking complements uses finiteness of the measure spaces: if $\nu(Y) < \infty$, we can write

$$f_{(X \times Y) \setminus E}(x) = \nu(Y) - \nu(E_x) = \nu(Y) - f_E(x),$$

so we are done because the right-hand side is a measurable function in x. (Indeed, constant functions and sums of measurable functions are all measurable.)

> **Remark 1.30.** There is an issue with taking unions: given $E, F \in \mathcal{D}$, we want to look at $f_{E \cup F}$, but there is no obvious way to access this only in terms of f_E and f_F because there may be some intersection.

In light of Remark 1.30, we need a trick. Do note that we can show that $\mathcal{D}$ is closed under disjoint unions because then $f_{E \sqcup F} = f_E + f_F$, so $f_{E \sqcup F}$ being measurable is recovered from summing f_E and f_F. Thus, because $\mathcal{D}$ contains measurable rectangles (note that the measure of the output of the function (1.1) is measurable as it's basically an indicator), so $\mathcal{A}(X,Y) \subseteq \mathcal{D}$.

For our trick, we proceed in steps.

1. We begin by showing that $\mathcal{D}$ is closed under countable ascending unions: given an ascending sequence of sets $\{E_i\}_{i=1}^\infty \subseteq \mathcal{D}$, then we set $E := \bigcup_{i=1}^\infty E_i$ and see

$$\lim_{n \to \infty} \nu((E_n)_x) = \nu(E_x)$$

   because $\left(\bigcup_{i=1}^\infty E_i\right)_x = \bigcup_{i=1}^\infty (E_i)_x$ tells us that the $(E_n)_x$ are measurable sets ascending to E_x, so we get the above limit via [Elb22, Proposition 6.36]. Thus, f_E is the pointwise limit of the f_{E_n}s, so f_E is $\mathcal{M}$-measurable.

2. Additionally, $\mathcal{D}$ is closed under countable descending intersections: the same argument of the previous point works, exchanging the word "ascending" with "descending," exchanging unions with intersections, and exchanging the citation with [Elb22, Corollary 6.37]. Note that our sets are of finite measure because Y is finite!

To proceed with the proof, we pick up the following definition.

Definition 1.31 (monotone class)**.** Fix a set Ω. Then a *monotone class* is a collection $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ which contains $\varnothing$ and is closed under countable ascending unions and countable descending intersections.

In particular, we have shown that $\mathcal{D}$ is a monotone class. We will want the following fact about monotone classes.

Lemma 1.32. Fix a set Ω, and let $\mathcal{A}$ be an algebra on Ω. Then the smallest monotone class $\mathcal{C}$ containing $\mathcal{A}$ is a σ-algebra.

Proof. Note that the notion of a "smallest monotone class" makes sense because the intersection of monotone classes is another monotone class, so we can take $\mathcal{C}$ to be the intersection of all monotone classes containing $\mathcal{A}$. Anyway, here are our checks.

1. Fix $\mathcal{D}$ to be the collection of subsets of Ω whose complement is in $\mathcal{C}$. We claim that $\mathcal{D}$ is a monotone class; this will imply that $\mathcal{D}$ contains $\mathcal{C}$ (because $\mathcal{D}$ of course contains $\mathcal{A}$, which is closed under complements), meaning that $\mathcal{C}$ is closed under complements. For countable ascending unions of $E_1 \subseteq E_2 \subseteq \cdots$, we note that the union E has
$$\Omega \setminus E = \bigcap_{i=1}^{\infty} \Omega \setminus E_i,$$
which is in $\mathcal{C}$, so $E \in \mathcal{D}$. Replacing unions with intersections shows that $\mathcal{D}$ is closed under

2. If $A \in \mathcal{A}$ and $B \in \mathcal{C}$, then we claim $A \cup B \in \mathcal{C}$. Well, fix A, and we set
$$\mathcal{D}_A \coloneqq \{E \subseteq \Omega : A \cup E \in \mathcal{C}\}.$$
We claim that $\mathcal{D}_A$ is a monotone class, and it contains $\mathcal{A}$ (which is closed under unions), so $\mathcal{D}_A$ will contain $\mathcal{C}$, proving the claim. For ascending unions $E_1 \subseteq E_2 \subseteq \cdots$, we note
$$\left(\bigcup_{i=1}^{\infty} E_i\right) \cap A = \bigcup_{i=1}^{\infty} (E_i \cap A),$$
so the union is still in $\mathcal{D}_A$. Replacing the big $\bigcup$ with a big $\bigcap$ and working with a descending intersection shows that $\mathcal{D}_A$ is a monotone class, as needed.

3. If $A \in \mathcal{C}$ and $B \in \mathcal{C}$, then we claim $A \cup B \in \mathcal{C}$. Once again, we fix A and set
$$\mathcal{D}_A\{E \subseteq \Omega : A \cup E \in \mathcal{C}\}.$$
The previous check tells us that $\mathcal{D}_A$ contains $\mathcal{A}$. The same proof as the previous check tells us that $\mathcal{D}_A$ is a monotone class, so we once again are allowed to conclude that $\mathcal{D}_A$ contains $\mathcal{C}$, so the claim follows.

4. Lastly, we show $\mathcal{C}$ is closed under countable unions. Well, given a countable collection $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}$, we set
$$F_j \coloneqq \bigcup_{i \le j} E_i,$$
which is in $\mathcal{C}$ by the previous check. Then the union of the $E_\bullet$s is the union of the $F_\bullet$s, but $\mathcal{C}$ is a monotone class, so it contains the union of the $F_\bullet$s (which are ascending), so we are done. ■

Now, we see that Lemma 1.32 finishes the proof: $\mathcal{D}$ must contain the smallest monotone class containing $\mathcal{A}(X,Y)$, which is a σ-algebra by Lemma 1.32, so $\mathcal{D}$ contains the smallest σ-algebra containing $\mathcal{A}(X,Y)$, so $\mathcal{D}$ contains $\mathcal{M}\otimes\mathcal{N}$, as needed. ∎

We now complete the proof of Theorem 1.25; the following is the statement of (c) for one of the equalities where $f = 1_E$.

Lemma 1.33. Fix σ-finite measure spaces $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$. Then

$$(\mu\times\nu)(E) = \int_X \nu(E_x)\,d\mu(x).$$

Proof. Proceed as in Lemma 1.29. Explicitly, set

$$\mathcal{D} := \left\{E\subseteq X\times Y : x\mapsto \nu(E_x) \text{ is measurable and } (\mu\times\nu)(E) = \int_X \nu(E_x)\,d\mu(x)\right\}.$$

The construction of the measure $\mu\times\nu$ implies this equality when E is a measurable rectangle or even when E is a disjoint union of measurable rectangles, so $\mathcal{D}$ contains $\mathcal{A}(X,Y)$. A direct computation shows that $\mathcal{D}$ is closed under complements, and the Dominated convergence theorem [Elb22, Theorem 9.14] shows that $\mathcal{D}$ is closed under ascending unions and descending intersections. So $\mathcal{D}$ is a monotone class containing the algebra $\mathcal{A}(X,Y)$, which implies that $\mathcal{D}$ contains the smallest monotone class containing $\mathcal{A}(X,Y)$, which is a σ-algebra by Lemma 1.32, so $\mathcal{D}$ contains the smallest σ-algebra containing $\mathcal{A}$, so $\mathcal{D}$ contains $\mathcal{M}\otimes\mathcal{N}$. ∎

1.4 January 24

Let's begin.

1.4.1 Addenda to Tonelli's Theorem

Last class we completed the proof of Theorem 1.25. We take a moment to note that there is a "mirror" of Tonelli's theorem as follows.

Theorem 1.34 (Tonelli)**.** Fix σ-finite measure spaces $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$. Fix a measurable function $f\colon X\times Y\to[0,\infty]$. Then the following hold.

(a) The function $x\mapsto f(x,y)$ is $\mathcal{M}$-measurable.

(b) The function $y\mapsto \int_Y f(x,y)\,d\nu(y)$ is $\mathcal{N}$-measurable.

(c) We have

$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_Y\int_X f(x,y)\,d\mu(x)\,d\nu(y).$$

We will not write this proof because one can simply interchange X and Y in the provided proof of Theorem 1.25. Perhaps one will complain that the definition of the product premeasure Definition 1.18 appears asymmetric, but in fact it does not. Indeed, Remark 1.20 explains that the measure of a measurable rectangle is symmetric, which then explains how to measure anything in $\mathcal{A}(X,Y)=\mathcal{A}(Y,X)$ symmetrically, and then the Extension Theorem 1.11 tells us that this uniquely measures anything in $\mathcal{M}\otimes\mathcal{N}$ symmetrically.

Corollary 1.35. Fix σ-finite measure spaces $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$. If $f\colon X\times Y\to[0,\infty]$ is $(\mathcal{M}\otimes\mathcal{N})$-measurable, then

$$\int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x) = \int_Y\int_X f(x,y)\,d\mu(x)\,d\nu(y).$$

Proof. Combine Theorems 1.25 and 1.34. ■

Perhaps one might worry about spaces which are not σ-finite. Here are some examples.

Example 1.36. Fix an uncountable set X. Then one can define a measure μ on $\mathcal{M} := \mathcal{P}(X)$ by $\mu(E) := \#E$. This is not σ-finite because subsets of X has finite measure if and only if it is finite, and X cannot be covered by countably many finite sets.

Example 1.37. Fix an uncountable set X, and let $\mathcal{M}$ be the collection of countable and cocountable subsets. Then the function $\mu\colon \mathcal{M} \to [0,\infty]$ defined by

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{if } E \text{ is cocountable} \end{cases}$$

is a measure. Now, X fails to be σ-finite because the sets of finite measure are exactly the countable ones, and X cannot be covered by countably many countable subsets.

1.4.2 Fubini's Theorem

We are now ready to state Fubini's theorem. This requires the following definition.

Definition 1.38. Fix a measure space $(X, \mathcal{M}, \mu)$. Then we define $L^1(\mu)$ consists of the measurable functions $f\colon X \to \mathbb{C}$ (defined almost everywhere) such that

$$\int_X |f| \, d\mu < \infty.$$

Remark 1.39. If $f \in L^1(\mu)$, then one sees that $\int_X f\,d\mu$ makes sense. Namely, one has

$$\int_X f\,d\mu = \int_X \operatorname{Re} f\,d\mu + i\int_X \operatorname{Im} f\,d\mu,$$

and the integrals $\int_X \operatorname{Re} f\,d\mu$ and $\int_X \operatorname{Im} f\,d\mu$ are both bounded by $f \in L^1(\mu)$. Something like [Elb22, Proposition 9.22] assures us that this makes sense (upon taking differences).

Theorem 1.40 (Fubini)**.** Fix σ-finite measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Fix a measurable function $f\colon X \times Y \to \mathbb{C}$ such that $\int_{X\times Y} |f| \, d(\mu\times\nu) < \infty$. Then the following hold.

(a) For μ-almost every $x \in X$, the function $f_x\colon y \mapsto f(x,y)$ is defined and $\mathcal{N}$-measurable and in $L^1(\nu)$.

(b) The function $x \mapsto \int_Y f(x,y)\,d\nu(y)$ is defined almost everywhere and $\mathcal{M}$-measurable and in $L^1(\mu)$.

(c) We have
$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x).$$

Proof. Note that we have the result for nonnegative functions by Theorem 1.25. The idea is to reduce to this case. Here we go.

- By writing $f = u + iv$ for $u := \operatorname{Re} f$ and $v := \operatorname{Im} f$, we may assume that f is real-valued. Explicitly, note the set of functions f satisfying the conclusions of (a)–(c) is a $\mathbb{C}$-vector space by some addition and scalar multiplication. Notably, we still have the hypotheses that $\int_{X\times Y} |u| \, d(\mu\times\nu) < \infty$ and $\int_{X\times Y} |v| \, d(\mu\times\nu) < \infty$.

- By writing $f = f^+ - f^-$ for $f^+, f^- \geq 0$, we will reduce to the case that f is nonnegative. Namely, achieving the result for the two functions

$$f^+(x) := \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) \leq 0, \end{cases} \qquad \text{and} \qquad f^-(x) := \begin{cases} -f(x) & \text{if } f(x) \leq 0, \\ 0 & \text{if } f(x) \geq 0, \end{cases}$$

  will achieve the result for f by summing.

Note that there is a technicality hidden in the above reasoning with linear combinations: for example, for the second reduction, even though we have the conclusion for f_x^+ and f_x^- are $\mathcal{N}$-measurable for all x, their difference might not be in $L^1(\nu)$ always. Well, we note that we can compute

$$\int_X \left(\int_Y f^+(x,y)\,d\nu(y) \right) d\mu(x) = \int_{X\times Y} f^+\,d(\mu\times\nu) < \infty,$$

so the inner function $x \mapsto \int_Y f_x^+\,d\nu(y)$ must be finite almost everywhere, or else this integral would be infinite! So we do indeed achieve that f_x^+ and f_x^- are in $L^1(\nu)$ almost everywhere, so their difference is in $L^1(\nu)$ almost everywhere. The argument for taking linear combinations in (b) is similar. ■

Let's see an example of why we want the hypothesis in Theorem 1.40.

Example 1.41. Set $X := \mathbb{N}$, and let μ and ν denote the counting measures on $\mathcal{M} = \mathcal{N} := \mathcal{P}(X)$. Note that $\mathcal{A}(X, X) = \mathcal{P}(X^2)$, so the product measure $\mu\times\nu$ is defined on all subsets; furthermore, we can see that the measure of a singleton is 1, so $\mu\times\nu$ is the counting measure. Then we define the function

$$f(x,y) := \begin{cases} +1 & \text{if } x = y, \\ -1 & \text{if } y = x+1, \\ 0 & \text{otherwise.} \end{cases}$$

For each x, we compute $\int_Y f(x,y)\,d\nu(y) = 0$ because each value of x has two values of y where $f(x,y)$ is nonzero. On the other hand, for each y, we compute $\int_X f(x,y)\,d\mu(x) = 0$ if $y > 0$ but is 0 if $y = 0$. The problem here is that $\int_{X\times Y} f(x,y)\,d(\mu\times\nu) = 0$.

Remark 1.42. According to Professor Christ, the above example is a "catastrophic failure" of a theorem rather than a "technical" one.

Remark 1.43. By induction, we are able to take products of any finite product of σ-finite measure spaces. Alternatively, one can redo the entire theory to do measurable rectangular prisms and so on. There are some extra checks here (e.g., does forming products associate meaningfully?), but it will work out in the end, essentially by the uniqueness of the construction provided by Theorem 1.11. Namely, up to the identification of products, we get the identification of the product σ-algebras and product measures because they should all agree on measurable rectangles, from which everything is generated.

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List of Definitions