256B: Algebraic Geometry

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How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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THEME 1

INTRODUCTION

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

1.1 January 17

Let's just get started.

1.1.1 Course Notes

Here are some notes about the course.

- The professor is Paul Vojta, whose email is `vojta@math.berkeley.edu`.
- The course webpage is `https://math.berkeley.edu/˜vojta/256b.html`.
- The textbook is [Har77].
- We will assume algebraic geometry on the level of Math 256A, which is a prerequisite for this course.
- This course focuses on (Zariski) cohomology of schemes, so we will spend most of our time going through [Har77, Chapter III]. We will also discuss smoothness, which lives in [Har77, Chapter III] as well. Along our way, we will want to discuss some topics in [Har77, Chapter II] in more detail, such as on divisors.
- Grading will be based on homework. Homework will be weekly or biweekly, due on Wednesdays (in general).

1.1.2 Abelian Categories

We'll assume some basic category theory (monomorphisms, epimorphisms, equalizers, coequalizers, etc.). Abelian categories are somewhat complex, so we provide their definition. Roughly speaking, our end goal is to do cohomology, which arises from homological algebra, and homological algebra lives in abelian categories.

Definition 1.1 (preadditive)**.** A *preadditive category* is a category $\mathcal{C}$ where the morphism set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ forms an abelian group for any $A, B \in \mathcal{C}$, and composition distributes over addition. Explicitly, the composition map

$$\circ\colon \operatorname{Hom}_{\mathcal{C}}(B, C) \times \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{C}}(A, C)$$

is bilinear.

It follows directly from having the preadditive structure that finite products and finite coproducts are canonically isomorphic. However, these (bi)products need not exist.

Definition 1.2 (additive)**.** An *additive category* is a preadditive category admitting all finite products/coproducts.

Definition 1.3 (abelian)**.** An *abelian category* is an additive category $\mathcal{C}$ in which the following hold.

- Every morphism admits a kernel and a cokernel; here, a (co)kernel is a (co)equalizer with the zero map.
- Every monomorphism is the kernel of some morphism.
- Every epimorphism is the cokernel of some morphism.

Let's give some examples.

Example 1.4. The following are abelian categories; we omit the checks.

- The category Ab of abelian groups is abelian.
- For a ring A, the category $\mathrm{Mod}(A)$ of A-modules is abelian. In particular, for a field k, the category $\mathrm{Vec}(k)$ of k-vector spaces is abelian.

Example 1.5. Here are more abelian categories, related to sheaves. All of their "abelian" hypotheses are done by passing to stalks or a similar local argument.

- For a topological space X, the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on X is abelian.
- Similarly, for a ringed space $(X, \mathcal{O}_X)$, the category $\mathrm{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules is abelian.
- For a scheme X, the category $\mathrm{QCoh}(X)$ of quasicoherent sheaves on X is abelian.
- Similarly, for a scheme X, the category $\mathrm{Coh}(X)$ of coherent sheaves on X is also abelian. Notably, we do not have infinite products here, but that's okay.

Example 1.6. For any abelian category $\mathcal{A}$, its opposite category $\mathcal{A}^{\mathrm{op}}$ is also abelian. One can see this by going through the conditions, all of which dualize.

1.1.3 Exact Functors

We will want to discuss exact functors in order to homological algebra in our abelian categories. Let's have at it.

Definition 1.7 (additive)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if and only if the map

$$F\colon \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{D}}(FA, FB)$$

(of F acting on morphisms $A \to B$) is a group homomorphism, for any $A, B \in \mathcal{C}$. Flipping arrows and using Example 1.6 produces the same definition for contravariant functors.

Example 1.8. Fix a topological space X. Then the functor $\Gamma(X, -)\colon \operatorname{Ab}(X) \to \operatorname{Ab}$ of global sections $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ is additive.

Remark 1.9. Being additive implies that the functor preserves biproducts. Roughly speaking, this holds because being a biproduct can be written as a set of equations for the object (and its inclusion/projection morphisms) to satisfy.

To define (left) exact for a functor, we need to define what it means to be exact.

Definition 1.10 (exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. Then a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* B if and only if $\ker g = \ker(\operatorname{coker} f)$ (up to some identification). Here, $\ker(\operatorname{coker} f)$ is intended to basically be the image.

Definition 1.11 (left exact)**.** Fix abelian categories $\mathcal{C}$ and $\mathcal{D}$. A (covariant) additive functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left-exact* if and only if a left exact sequence

$$0 \to A' \to A \to A''$$

produces a left exact sequence

$$0 \to FA' \to FA \to FA''.$$

Reversing the arrows produces the dual notion of right exactness.

Remark 1.12. Being left exact equivalently means that F preserves kernels, so by Remark 1.9 and a little category theory, F actually preserves all finite limits.

Example 1.13. The functor of global sections from Example 1.8 is left exact by [Har77, Exercise II.1.8].

To get us set up, let's approximately describe what we are trying to do. Basically, fix an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of abelian groups on a topological space X. Then there is a sequence of "cohomology" functors $\{H^i(X, -)\}_{i \in \mathbb{N}}$ with $H^0(X, -) = \Gamma(X, -)$ and a "long" exact sequence as follows

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{F}') \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{F}'') \arrow[dll] & \\ & H^1(X, \mathcal{F}') \arrow[r] & H^1(X, \mathcal{F}) \arrow[r] & H^1(X, \mathcal{F}'') \arrow[r] & \cdots \end{tikzcd}$$

where the maps $H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}')$ take some work to describe.

Remark 1.14. These functors will have a number of magical properties, which will amount to the main theorems of this course. Let's give an example. Fix a projective scheme X over a field k, where $i\colon X \to \mathbb{P}^n_k$ is the promised closed embedding; let $\mathcal{I}$ be the corresponding ideal sheaf of this closed embedding. Then we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n_k} \to i_*\mathcal{O}_X \to 0,$$

which one can do cohomology to. In fact, one can take the tensor product of this exact sequence with the twisting sheaves $\mathcal{O}_{\mathbb{P}^n_k}(m)$; for example, we will prove that $H^1(\mathbb{P}^n_k, \mathcal{I}(m)) = 0$ for sufficiently large m, which eventually implies that the map

$$\Gamma\left(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)\right) \to \Gamma(X, \mathcal{O}_X(m))$$

is surjective for sufficiently large m. In other words, global sections of $\mathcal{O}_X(m)$ are all restrictions of global sections of $\mathcal{O}_{\mathbb{P}^n_k}(m)$!

1.2 January 19

We'll do some homological algebra today.

1.2.1 Homological Algebra on Complexes

Homological algebra is something that comes out of understanding complexes, which we will now define.

Definition 1.15 (complex)**.** Fix an abelian category $\mathcal{A}$. A *complex* $(A^\bullet, d^\bullet)$ is a collection $\{A^i\}_{i\in\mathbb{Z}}$ together with some morphisms $d^i\colon A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$. We may abbreviate the differential $d^\bullet$ from the notation.

Remark 1.16. The above definition is usually a "cocomplex." We will have no need for the dual notion of a complex in this course.

Remark 1.17. By convention, if we state that we have a complex but only define A^i for a subset of $\mathbb{Z}$, then the full bona fide complex simply sets the undefined terms to zero.

Now that we have a complex, we should define a morphism.

Definition 1.18 (complex morphism)**.** Fix an abelian category $\mathcal{A}$. Given complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$, a morphism of complexes $\varphi^\bullet\colon A^\bullet \to B^\bullet$ is a collection of morphisms $\varphi^i\colon A^i \to B^i$ making the following diagram commute for each i.

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "\varphi^i"'] & A^{i+1} \arrow[d, "\varphi^{i+1}"] \\ B^i \arrow[r, "d^{i+1}"] & B^{i+1} \end{tikzcd}$$

Unsurprisingly, our definition of morphism provides us with a category of complexes, and in fact the category of complexes is an abelian category, where the point is that biproducts, kernels, and cokernels can all be computed pointwise at each term of the complex.

We are now ready to define cohomology.

Definition 1.19 (cohomology)**.** Fix a complex $(A^\bullet, d^\bullet)$ valued in an abelian category $\mathcal{A}$. Then we define the ith *cohomology* as

$$h^i(A^\bullet) := \frac{\ker d^i}{\operatorname{im} d^{i-1}}.$$

Here, $\operatorname{im} d^{i-1}$ has an induced map to $\ker d^i$ because $d^i \circ d^{i-1} = 0$.

Remark 1.20. Quickly, recall that the image $\operatorname{im} d^{i-1}$ is in fact $\ker(\operatorname{coker} d^{i-1})$.

Remark 1.21. In fact, cohomology is functorial: a morphism $f^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of complexes induces a morphism $h^i(f^\bullet)\colon h^i(A^\bullet) \to h^i(B^\bullet)$ on the ith cohomology, and one can check that this makes h^i into a functor. To be explicit, this morphism is induced by the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} d_A^{i-1} \arrow[r] \arrow[d, "f^i"] & \ker d_A^i \arrow[r] \arrow[d, "f^i"] & h^i(A^\bullet) \arrow[r] \arrow[d, dashed, "f^i"] & 0 \\
0 \arrow[r] & \operatorname{im} d_B^{i-1} \arrow[r] & \ker d_B^i \arrow[r] & h^i(B^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Namely, the morphisms on the left are well-defined because $f^\bullet$ is in fact a morphism.

The main result on these cohomology groups is the following.

Proposition 1.22. Fix an abelian category $\mathcal{A}$. Given a short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of complexes in $\mathcal{A}$, there are natural maps $\delta^i \colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ producing a long exact sequence as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & h^i(A^\bullet) \arrow[r] & h^i(B^\bullet) \arrow[r] & h^i(C^\bullet) \arrow[dll, "\delta^i"] \\
& h^{i+1}(A^\bullet) \arrow[r] & h^{i+1}(B^\bullet) \arrow[r] & h^{i+1}(C^\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. To produce the long exact sequence, use the Snake lemma. The proof is somewhat technical, so I will refer directly to [Elb22, Theorem 4.82], though the proof there is for the dual notion of homology instead of cohomology. (Note that we can replace $\mathcal{A}$ with $\mathcal{A}^{\mathrm{op}}$ to recover the result.) The naturality of the $\delta^\bullet$ can be checked directly from its construction. ■

We would like to measure a morphism of complexes based on what it does to cohomology: namely, two morphisms of complexes may induce the same map on cohomology despite being technically distinct. One way this might happen is by being "chain" homotopic.

Definition 1.23 (chain homotopy)**.** Fix morphisms $f^\bullet, g^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of the chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. A *chain homotopy* is a sequence of maps $k^i \colon A^i \to B^{i-1}$ such that

$$f^i - g^i = k^{i+1} \circ d_A^i + d_B^{i-1} \circ k^i.$$

In this case, we say that $f^\bullet$ and $g^\bullet$ are chain homotopic.

Remark 1.24. One can check directly that being chain homotopic is an equivalence relation on chain morphisms.

And here is our result.

Proposition 1.25. Fix morphisms $f^\bullet, g^\bullet\colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ of chain complexes $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ valued in an abelian category $\mathcal{A}$. If $f^\bullet \sim g^\bullet$, then $h^i(f^\bullet) = h^i(g^\bullet)$ for all i.

Proof. By some embedding theorem, we may as well work in $\operatorname{Mod}(R)$ for some ring R. Now, fix some $\alpha \in \ker d_A^i$, and we want to show that

$$\left[f^i(\alpha) - g^i(\alpha)\right] = 0$$

in $h^i(B^\bullet)$. But now let $k^j\colon A^j \to B^{j-1}$ for $j \in \mathbb{Z}$ provide our chain homotopy, so we see

$$f^i(\alpha) - g^i(\alpha) = k^{i+1}\big(\underbrace{d_A^i(\alpha)}_{0}\big) + d_B^{i-1}\left(k^i(\alpha)\right)$$

vanishes in $h^i(B^\bullet)$, as desired. ■

1.2.2 Injective Resolutions

We would now like to use our homological algebra to say something concrete about functors, which requires building injective resolutions. Injective resolutions are built out of injectives, so here is that definition.

Definition 1.26 (injective)**.** Fix an object I in an abelian category $\mathcal{A}$. Then I is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is right exact.

Remark 1.27. The functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is already left-exact (and contravariant), so it is equivalent to ask for this functor to be fully exact. Unwinding the definition, we may equivalently ask for short exact sequences

$$0 \to A' \to A \to A'' \to 0$$

to produce short exact sequences

$$0 \to \operatorname{Hom}_{\mathcal{A}}(A'', I) \to \operatorname{Hom}_{\mathcal{A}}(A, I) \to \operatorname{Hom}_{\mathcal{A}}(A', I) \to 0,$$

but this is already left-exact, so we are really only concerned about surjectivity on the right. So we may equivalently ask for injections $A' \hookrightarrow A$ to produce surjections $\operatorname{Hom}_{\mathcal{A}}(A', I) \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(A, I)$; i.e., any map $A' \to I$ can be extended to a full map $A \to I$.

We also have the following dual notion.

Definition 1.28 (projective)**.** Fix an object P in an abelian category $\mathcal{A}$. Then P is *injective* if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is right exact.

Remark 1.29. Exactly the dual arguments to Remark 1.27 say that being projective is equivalent to $\operatorname{Hom}_{\mathcal{A}}(P, -)$ being fully exact, or equivalently that any map $P \to A''$ can be pulled back to a map $P \to A$ whenever we have a surjection $A \twoheadrightarrow A''$.

And we now define our resolutions.

Definition 1.30 (resolution)**.** Fix an object A in an abelian category $\mathcal{A}$. A *coresolution* is an exact sequence

$$0 \to A \xrightarrow{\varepsilon} E^0 \to E^1 \to \cdots$$

in $\mathcal{A}$; we may write this as $0 \to A \to E^\bullet$. A *resolution* is an exact sequence

$$\cdots \to E_1 \to E_0 \xrightarrow{\varepsilon} A \to 0$$

in $\mathcal{A}$; again, we may write this as $E^\bullet \to A \to 0$. For any property $\mathcal{P}$ of objects in $\mathcal{A}$, we say that the resolution is $\mathcal{P}$ if and only if the Es are all $\mathcal{P}$.

Of interest to us right now are injective and projective resolutions, but we will find use for other kinds of resolutions.

We want to be able to build injective resolutions. The following provides the required adjective.

Definition 1.31 (enough injectives)**.** An abelian category $\mathcal{A}$ has *enough injective* if and only if any object $A \in \mathcal{A}$ has a monomorphism to an injective object.

And here is the relevant result.

Proposition 1.32. Fix an abelian category $\mathcal{A}$ with enough injectives. Then every object $A \in \mathcal{A}$ has an injective resolution.

Proof. By induction, it is enough to show that, for any map $f\colon A \to E$, there exists a map $g\colon E \to I$ where I is injective and the sequence $A \to E \to I$ is exact. Indeed, this will be enough because we can start with the sequence $0 \to A$, then extend to $0 \to A \to E^0$, then extend to $0 \to A \to E^0 \to E^1$, and so on.

Now, to show the claim of the previous paragraph, we note that we may find an injective object I and a monomorphism $\overline{g}\colon \operatorname{coker} f \to I$ because $\mathcal{A}$ has enough injectives. Then we note that the composite

$$A \to E \to \operatorname{coker} f \hookrightarrow I$$

produces the exact sequence $A \to E \to I$, as desired. ■

A nice property of injective resolutions is that they are, in some sense, functorial in their object.

Proposition 1.33. Fix a morphism $f\colon A \to B$ of objects in $\mathcal{A}$. Given injective resolutions $0 \to A \to E^\bullet$ and $0 \to B \to F^\bullet$, one can find maps $g^i\colon E^i \to F^i$ for each i inducing a chain morphism of the injective resolutions.

Proof. This is an exercise is induction and using the injective. ■

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