# 214: Differential Topology

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# Contents

*How strange to actually have to see the path of your journey in order to make it.*

—Neal Shusterman, [Shu16]

THEME 1
# BUILDING MANIFOLDS

*So the man gave him the bricks, and he built his house with them.*

—Joseph Jacobs, "The Story of the Three Little Pigs" [Jac90]

## 1.1 January 16

Let's just get started.

### 1.1.1 Course Structure

Here are some quick notes.

- There is a bCourses page: https://bcourses.berkeley.edu/courses/1533116. For example, it has the syllabus.
- The textbook is Lee's *Introduction to Smooth Manifolds* [Lee13]. We will read most of it.
- Our instructor is Professor Eric Chen, whose email can be reached at ecc@berkeley.edu. Office hours are after class in Evans 707.
- There is a GSI, who is Tahsia Saffat, whose email is tahsin_saffat@math.berkeley.edu. He will have some office hours and grade some homeworks.
- Homework will in general be due at 11:59PM on Thursdays via Gradescope.
- There will be an in-class midterm and a final.
- Grading is 30% homework, 30% midterm, and 40% final.
- This is a math class, not so geared towards applied subjects.
- In particular, we will assume a fair amount of topology, for which we use [Elb22] as a reference.

Let's also give a couple of notes on the course content. This course is on differential topology. The topology of interest will come from manifolds, and the differential part comes from some smoothness properties.

In some sense, our goal is to "do calculus" (e.g., differentiation, integration, vector fields, etc.) on spaces which look locally like some Euclidean space, such as a sphere. We also want to understand (smooth) manifolds on their own terms, such as understanding the maps between them and understanding some classical examples and constructions such as Lie groups or quotient manifolds.

### 1.1.2 Topology Review

Anyway let's get started. This is a class on manifolds, so perhaps we should begin by defining a manifold. These are going to form a special kind of topological space, so let's review topologies. We will freely use topological facts which we are too lazy to prove from [Elb22].

**Definition 1.1** (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where $X$ is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of $X$ satisfying the following.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
- Finite intersection: given $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Union: for any subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have the union $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We say that the collection $\mathcal{T}$ is the collection of *open sets* of $X$. We will also suppress the collection $\mathcal{T}$ from the notation as much as possible.

Here is some helpful language.

**Definition 1.2** (open, closed, neighborhood)**.** Fix a topological space $(X, \mathcal{T})$.

- An *open subset* $U \subseteq X$ is a subset in $\mathcal{T}$.
- A *closed subset* $V \subseteq X$ is one with $X \setminus V \in \mathcal{T}$.
- A *neighborhood* of a point $p \in X$ is an open subset $U \subseteq X$ containing $p$.

**Example 1.3.** Fix a metric space $(X, d)$. Then there is a topology given by the metric. To be explicit, a set $U \subseteq X$ is open if and only if each $p \in U$ has some $\varepsilon > 0$ such that

$$\{x \in X : d(x, p) < \varepsilon\} \subseteq U.$$

See [Elb22, Example 2.13] for the details.

Sometimes it is easier to generate a topology from some subcollection.

**Definition 1.4** (base)**.** Fix a topological space $(X, \mathcal{T})$. A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is a *base* for $\mathcal{T}$ if and only if the following holds: for each open $U \subseteq X$ and point $p \in U$, there is some $B \in \mathcal{B}$ such that $p \in B$ and $B \subseteq U$.

**Example 1.5.** Fix a metric space $(X, d)$. Then the collection $\mathcal{B}$ of open balls

$$B(p, \varepsilon) :=,$$

over all $p \in X$ and $\varepsilon > 0$, forms a base of the topology. This is immediate from the construction of the topology in Example 1.3. In fact, one can merely take $\varepsilon \in \mathbb{Q}^+$ because $\mathbb{Q}$ is dense in $\mathbb{R}$.

With our objects of topological spaces in hand, we should discuss the maps between them.

**Definition 1.6** (continuous)**.** Fix topological spaces $X$ and $X'$. A function $\varphi\colon X \to X'$ is *continuous* if and only if $\varphi^{-1}(U')$ is open for each open $U' \subseteq X'$.

**Definition 1.7** (homeomorphism)**.** Fix topological spaces $X$ and $X'$. A function $\varphi\colon X \to X'$ is a *homeomorphism* if and only if $\varphi$ is a bijection and both $\varphi$ and $\varphi^{-1}$ are continuous. We may write $X \cong X'$.

**Remark 1.8.** There is a continuous bijection $[0, 2\pi) \to S^1$ by $\theta \mapsto (\cos\theta, \sin\theta)$, but it is not a homeomorphism. (Here, both sets have the metric topology.) In particular, the inverse map is not continuous at $1$ because the pre-image of $[0, \pi)$ is the subset $\{(x, y) \in S^1 : y > 0\} \cup \{(0, 0)\}$, which is not open in $S^1$ (because no $\varepsilon > 0$ has $B((0, 0), \varepsilon)$ lying in $\{(x, y) \in S^1 : y \geq 0\}$).

**Exercise 1.9.** Fix a nonnegative integer $n \geq 0$. Then $B(0, 1) \cong \mathbb{R}^n$.

*Proof.* We proceed as in [use14]. Define the functions $f \colon B(0, 1) \to \mathbb{R}^n$ and $g \colon \mathbb{R}^n \to B(0, 1)$ by

$$f(x) := \frac{x}{1 - |x|} \qquad \text{and} \qquad g(y) := \frac{y}{1 + |y|}.$$

Notably, $|g(y)| < 1$ always, so $g$ does indeed always output to $B(0, 1)$. These functions are both continuous, which can be checked on coordinates because they are rational functions in the coordinates, and the denominators never vanish on the domains. So we will be done once we show that $f$ and $g$ are inverse. In one direction, we note

$$f(g(y)) = \frac{g(y)}{1 - |g(y)|} = \frac{\frac{y}{1 + |y|}}{1 - \left|\frac{y}{1 + |y|}\right|} = \frac{y}{1 + |y| - |y|} = y.$$

In the other direction, we note

$$g(f(x)) = \frac{f(x)}{1 + |f(x)|} = \frac{\frac{x}{1 - |x|}}{1 + \left|\frac{x}{1 - |x|}\right|} = \frac{x}{1 - |x| + |x|} = x,$$

as desired. ■

We would also like to be able to build new topologies from old ones.

**Definition 1.10** (subspace)**.** Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we form a *subspace topology* by declaring the open subsets to be

$$\{U \cap S : U \in \mathcal{T}\}.$$

**Example 1.11.** The metric topology on $\mathbb{R}$ and the subspace topology on $X := \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ are homeomorphic. Namely, the homeomorphism sends $x \mapsto (x, 0)$, and the inverse map is $(x, 0) \mapsto x$. Here are our continuity checks.

- The map $x \mapsto (x, 0)$ is continuous: the pre-image $V$ of an open subset $U \subseteq X$ is open. Namely, for any $x \in V$, we see $(x, 0) \in V$, so there is $\varepsilon > 0$ such that $B((x, 0), \varepsilon) \cap X \subseteq U$, so $B(x, \varepsilon) \subseteq V$.
- The map $(x, 0) \mapsto x$ is continuous: the pre-image $V$ of an open subset $U \subseteq \mathbb{R}$ is open. Namely, for each $(x, 0) \in V$, we see $x \in U$, so there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V$, so $B((x, 0), \varepsilon) \cap X \subseteq U$.

Lastly, we will want some adjectives for our topologies.

**Definition 1.12** (compact)**.** Fix a topological space $X$. A subset $K \subseteq X$ is *compact* if and only if any open cover can be reduced to a finite subcover. Explicitly, any collection $\mathcal{U}$ of open sets of $X$ such that $K \subseteq \bigcup_{U \in \mathcal{U}} U$ (this is called an *open cover*) has some finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{U \in \mathcal{U}'} U$.

**Example 1.13.** The interval $[0,1] \subseteq \mathbb{R}$ is compact. See [Elb22, Example 4.4].

**Definition 1.14** (Hausdorff)**.** Fix a topological space $X$. Then $X$ is *Hausdorff* if and only if any two distinct points $p_1, p_2 \in X$ have disjoint open subsets $U_1, U_2 \subseteq X$ such that $p_1 \in U_1$ and $p_2 \in U_2$.

**Example 1.15.** Any metric space $(X,d)$ is Hausdorff. Namely, for distinct points $p, q \in X$, we see $d(p,q) > 0$, so set $\varepsilon := d(p,q)/2$, and we see that $p \in B(p,\varepsilon)$ and $q \in B(q,\varepsilon)$, but $B(p,\varepsilon) \cap B(q,\varepsilon) = \varnothing$. For this last claim, we note $r$ living in the intersection would imply

$$d(p,q) \leq d(p,r) + d(r,q) < 2\varepsilon,$$

which is a contradiction to the construction of $\varepsilon$.

### 1.1.3 Topological Manifolds

For intuition, we state but not prove the following result.

**Theorem 1.16** (Topological invariance of dimension)**.** Fix open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. If there is a homeomorphism $U \cong V$, then $m = n$.

*Proof.* The usual proofs go through (co)homology, which we may cover later in the class. For the interested, see [Elb23, Proposition 3.50]. ■

We will soon define topological manifolds. The main adjective we want is being "locally Euclidean."

**Definition 1.17** (locally Euclidean)**.** Fix a topological space $X$. Then $X$ is *locally Euclidean of dimension $n$ at $p$* if and only if there is an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$. We say that $X$ is *locally Euclidean of dimension $n$* if and only if it is locally Euclidean of dimension $n$ at each point.

**Remark 1.18.** One can always take $\widetilde{U}$ to be either $B(0,1) \subseteq \mathbb{R}^n$ or even all of $\mathbb{R}^n$. Indeed, for $x \in X$, we are given an open neighborhood $U$ of $x$ and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. We produce open neighborhoods of $x$ homeomorphic to $B(0,1)$ and $\mathbb{R}^n$.

- $B(0,1)$: there is $\varepsilon > 0$ such that $B(\varphi(x),\varepsilon) \subseteq \widehat{U}$. Then we let $U' := \varphi^{-1}(B(\varphi(x),\varepsilon))$ so that we have a chain of homeomorphisms

  $$U' \overset{\varphi}{\cong} B(\varphi(x),\varepsilon) \cong B(0,\varepsilon) \cong B(0,1),$$

  where the second homeomorphism is a translation, and the last homeomorphism is a dilation.
- $\mathbb{R}^n$: in the light of the previous point, it suffices to note that Exercise 1.9 provides a homeomorphism $B(0,1) \cong \mathbb{R}^n$ and then post-compose with this homeomorphism.

Let's explain why we want Theorem 1.16.

**Lemma 1.19.** Fix a locally Euclidean space $X$. For each $p \in X$, there is a unique nonnegative integer $n$ such that there exists an open neighborhood $U \subseteq X$ and open subset $\widetilde{U} \subseteq \mathbb{R}^n$ such that $U \cong \widetilde{U}$.

*Proof.* Suppose there are two such nonnegative integers $m$ and $n$, so we get open neighborhoods $U, V \subseteq X$ and $\widetilde{U} \subseteq \mathbb{R}^m$ and $\widetilde{V} \subseteq \mathbb{R}^n$. Let $\varphi\colon U \cong \widetilde{U}$ and $\psi\colon V \cong \widetilde{V}$ be the needed homeomorphisms. Then the point is to use the intersection $U \cap V$: there is a composite isomorphism

$$\varphi(U \cap V) \cong U \cap V \cong \psi(U \cap V)$$

from an open subset in $\mathbb{R}^m$ to an open subset in $\mathbb{R}^n$. So Theorem 1.16 completes the proof. ■

Anyway, here is our definition of a topological manifold.

**Definition 1.20** (topological manifold)**.** An $n$-*dimensional topological manifold* is a topological space $M$ with the following properties.

- $M$ is Hausdorff.
- $M$ is locally Euclidean of dimension $n$ at each point.
- $M$ is second countable (i.e., has a countable base).

We may abbreviate "$n$-dimensional topological manifold" to "topological $n$-manifold."

Let's give a few quick constructions.

**Lemma 1.21.** For each $n \geq 0$, the space $\mathbb{R}^n$ is an $n$-dimensional topological manifold.

*Proof.* Let's be quick. Being a metric space yields Hausdorff, locally Euclidean is immediate because it's $\mathbb{R}^n$, and second-countability follows by using the base

$$\left\{B(q, \varepsilon) : q \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}^+\right\}.$$

This is indeed a base because $\mathbb{Q}$ is dense in $\mathbb{R}$. Explicitly, for each $p \in \mathbb{R}^n$ living in some open subset $U \subseteq \mathbb{R}^n$, begin by replacing $U$ with a smaller open subset of the form $B(p, \varepsilon)$ where $\varepsilon > 0$; by perhaps making $\varepsilon$ smaller, we may assume that $\varepsilon > 0$ is rational. Now, choosing coordinates $p = (x_1, \ldots, x_n)$, choose rational numbers $q_1, \ldots, q_n$ so that $|x_i - q_i| < \varepsilon/(2\sqrt{n})$ for each $i$. Then $q := (q_1, \ldots, q_n)$ has $d(p, q) < \varepsilon/2$ and so

$$p \in B(q, \varepsilon/2) \subseteq B(p, \varepsilon) \subseteq U,$$

so $B(q, \varepsilon/2)$ is the needed open subset in our base. ■

The following lemma will be helpful in the sequel.

**Lemma 1.22.** Fix a topological space $M$ and nonnegative integer $n \geq 0$. Suppose that there is a countable open cover $\{U_i\}_{i \in \mathbb{N}}$ of $M$ such that each $i$ has a homeomorphism $U_i \cong \widetilde{U}_i$ where $\widetilde{U}_i \subseteq \mathbb{R}^n$ is open. Then $M$ is locally Euclidean of dimension $n$ at each point, and $M$ is second countable.

*Proof.* For locally Euclidean, we note that each $p \in M$ lives in some $U_i$, so we are done. As for second countability, we note that each $\widetilde{U}_i$ is second countable as a subspace of a second countable space (see Lemma 1.21), so each $U_i$ is second countable by moving back through the homeomorphism, and so $M$ is second countable by taking the union of the bases of the $U_i$.

To make this last step more explicitly, we note that each $U_i$ has a countable base $\mathcal{B}_i$, so we claim that $\mathcal{B} := \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ becomes a countable base of $M$. Certainly $\mathcal{B}$ is countable, and every set in $\mathcal{B}$ is in one of the $\mathcal{B}_i$ and hence open in $M$. Lastly, to check that we have a base, we note that any open $U \subseteq M$ and $p \in M$ will have $p \in U_i$ for some $i$, so there is some $B \in \mathcal{B}_i \subseteq \mathcal{B}$ such that $p \in B \subseteq U \cap U_i$. ■

### 1.1.4 Examples and Non-Examples

Here are some non-examples to explain why we want all of these hypotheses.

**Exercise 1.23.** Consider the space $X$ defined as $\mathbb{R}\times\{0,1\}$ where we identify $(x,0)\sim(x,1)$ whenever $x\neq 0$. (The topology on $X$ is the quotient topology [Elb22, Definition 2.81].) This space is not Hausdorff, but it is locally Euclidean and second countable.

*Proof.* We run our checks.

- This space is not Hausdorff because the points $(0,0)$ and $(0,1)$ are "infinitely close together." Explicitly, any open neighborhoods $U$ and $V$ of $(0,0)$ and $(0,1)$, respectively, the induced topology yields some $\varepsilon>0$ such that $B((0,0),\varepsilon)\subseteq U$ and $B((0,1),\varepsilon)\subseteq V$, but then $(-\varepsilon/2,0)=(-\varepsilon/2,1)$ is in both $U$ and $V$.
- This space is locally Euclidean and second countable by Lemma 1.22. Explicitly, we note that $\mathbb{R}\cong\mathbb{R}\times\{0\}\subseteq X$ and $\mathbb{R}\cong\mathbb{R}\times\{1\}\subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ■

**Exercise 1.24.** Consider the space $X$ defined as $\mathbb{R}\times\{0,1\}$ where we identify $(x,0)\sim(x,1)$ whenever $x\leq 0$, again where we are using the quotient topology. Then $X$ is Hausdorff and second countable, but it is not Euclidean of dimension $1$ at $0\in X$.

*Proof.* We run our checks.

- This space is Hausdorff. We check this directly by casework.
  - Suppose we have distinct points $p=(x,a)$ and $q=(y,b)$ with $x\neq y$; for example, this includes the case where we may take $a=b$ and hence includes the case when $x,y\leq 0$. Then we may set $\varepsilon:=\frac{1}{2}|x-y|$ so that $B(p,\varepsilon)$ and $B(q,\varepsilon)$ are disjoint.
  - We now may assume that $x=y$; then $a\neq b$. Thus, we must have $x>0$ or $y>0$. As such, we may as well take $\varepsilon:=\min\{|x|,|y|\}$ so that $B(p,\varepsilon)$ and $B(q,\varepsilon)$ are disjoint.
- This space is not locally Euclidean at $0$. Indeed, suppose that there is open subset $U\subseteq X$ around $0$ which is homeomorphic to an open subset of $\mathbb{R}$. By shifting, we may as well assume that the homeomorphism sends $0$ to $0$. Additionally, the same statement will be true by any open subset of $U$, so we may as well as assume that $U$ is of the form $(-\varepsilon,\varepsilon)\times\{0,1\}$ (in $X$). In particular, $U$ is connected.

  But then the image $\widehat{U}$ of $U$ in $\mathbb{R}$ is a connected open subset of $\mathbb{R}$, which must be an interval. Now, intervals have the property that deleting any point of an interval makes produces a topological space with two connected components. However, deleting $0$ from $U$ will produce three connected components: $(-\varepsilon,0)\times\{0,1\}$ and $(0,\varepsilon)\times\{0\}$ and $(0,\varepsilon)\times\{1\}$. So $\widehat{U}$ and $U$ cannot actually be homeomorphic!
- This space is second countable by Lemma 1.22. Again, we note that $\mathbb{R}\cong\mathbb{R}\times\{0\}\subseteq X$ and $\mathbb{R}\cong\mathbb{R}\times\{1\}\subseteq X$ by an argument similar to Example 1.11. So we have a finite cover by open subsets of $\mathbb{R}^n$, completing the check in Lemma 1.22. ■

**Remark 1.25.** Essentially the same argument implies that the above space fails to be locally Euclidean of any dimension at $0\in X$. Namely, a connected open subset of $\mathbb{R}^n$ for $n\geq 2$ will remain connected after removing any point, so it cannot be homeomorphic to $(-\varepsilon,\varepsilon)\times\{0,1\}$ in $X$.

Morally, the second countability is being required as a smallness condition; let's see some pathological examples without second countability. The following lemma approximately explains the problem.

**Lemma 1.26.** Fix a topological space $X$. Suppose that there is an uncountable subset $Y \subseteq X$ such that each $y \in Y$ has an open neighborhood $U_y \subseteq X$ where the $U_y$ are pairwise disjoint. Then $X$ fails to be second countable.

*Proof.* Suppose we have a base $\mathcal{B}$; we show $\mathcal{B}$ is uncountable. Each $y \in U_y$ has some $B_y \in \mathcal{B}$ with $B_y \subseteq U_y$. However, $y \neq y'$ implies that $B_y \neq B_{y'}$ because $y \in B_y$ while $p_y \notin U_{y'}$ implies $p_y \notin B_{y'}$. So $\{B_y\}_{y \in Y}$ is an uncountable subcollection of $\mathcal{B}$. ■

**Exercise 1.27.** Consider an uncountable set $S$ with the discrete topology (namely, every subset is open), and then we form the product $X := \mathbb{R} \times S$. Then $X$ is Hausdorff, locally Euclidean of dimension $1$, but it is not second countable.

*Proof.* Here are our checks.

- Note that $X$ is a product of Hausdorff spaces and hence is Hausdorff.
- This space is locally Euclidean of dimension $1$: for each $(x, s) \in X$, we note that $\mathbb{R} \times \{s\}$ is an open subset of $X$ (because $S$ is discrete) where $\mathbb{R} \times \{s\} \cong \mathbb{R}$ by an argument similar to Example 1.11.
- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (0, s)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \mathbb{R} \times \{s\}$. ■

**Exercise 1.28.** Consider the first uncountable ordinal $\omega_1$. Then define $X := (S \times [0,1)) \setminus \{(0,0)\}$, and we give $X$ the order topology where the ordering is lexicographic. (Namely, the base consists of the "intervals" $\{x : x < b\}$ or $\{x : a < x\}$ or $\{x : a < x < b\}$.) This space is Hausdorff, locally Euclidean $1$, but it is not second countable.

*Proof.* Here are our checks.

- This space is Hausdorff because it is a dense linear order. Explicitly, for $(s, a), (t, b) \in X$, we have the following cases.
  - Suppose $s = t$. In this case, $a \neq b$; suppose $a < b$ without loss of generality. Then $\{x : x < (s, (a + b)/2)\}$ and $\{x : x > (s, (a + b)/2)\}$ are the needed open sets.
  - Suppose $s \neq t$; take $s < t$ without loss of generality. If $a > 0$, then $\{s\} \times (0, (a + 1)/2)$ and $\{s\} \times ((a+1)/2, 1) \cup \{t\} \times [0, 1)$ provide the needed open sets. Otherwise, if $a = 0$, then $\{x : x < (s, 1/2)\}$ and $\{x : x > (s, 1/2)\}$ provide the needed open sets.
- This space is locally Euclidean of dimension $1$: fix any $(s, r) \in X$. Note that $s \in \omega_1$ is countable, so we claim that
  $$(s + 1) \times [0, 1) \cong [0, 1),$$
  sending $(0, 0)$ to $0$, from which the claim follows by deleting $(0, 0)$. Because the relevant orders produce the needed topologies, we are really asking for an order-preserving bijection from $(s + 1) \times [0, 1)$ to $[0, 1)$.

  Well, for any $t \in \omega_1$, we claim that there is an increasing sequence $\{p_\alpha\}_{\alpha < t} \subseteq [0, 1)$ of order type $t$ with $p_0 = 0$, from which the claim will follow by taking $s = t$ and sending $\alpha \times [0, 1) \subseteq (s + 1) \times [0, 1)$ to $[p_\alpha, p_{\alpha+1})$ (where we define $p_s := 1$). To see this claim, we argue by induction on $s$. For $s = 0$, take $p_0 := 0$. If $s$ is a successor ordinal, divide all the existing $p_\alpha$ by $2$ and then set $p_{s+1} := 1/2$.

  Lastly, if $s$ is a limit ordinal, it is still only a countable limit ordinal, so we can find an increasing sequence of countable ordinals $\{s_i\}_{i \in \omega}$ approaching $s$. The sequence corresponding to $s_0$ will fit into $[0, 1/2)$ after scaling; then the sequence corresponding to $s_1$ but after $s_0$ will fit into $[1/2, 2/3)$ after scaling. We can continue this process inductively to complete the claim for $s$. I won't bother to write out the details.

- This space is not second countable by Lemma 1.26. Namely, we have the uncountably many points $p_s := (s, 1/2)$ (one for each $s \in S$) contained in the pairwise disjoint open neighborhoods $U_s := \{s\} \times (0,1)$. ■

**Remark 1.29.** What makes the locally Euclidean check above annoying is that we must show $(\omega, 0) \in X$ has a neighborhood isomorphic to an open subset of $\mathbb{R}$, which is not totally obvious.

Let's return to examples.

**Example 1.30.** Consider the unit circle $S^1$. We check that $S^1$ is a $1$-dimensional topological manifold.

- $S^1$ is a metric space, so it is Hausdorff.
- $S^1$ is second countable: it is a subspace of $\mathbb{R}^2$, and $\mathbb{R}^2$ is second countable by Lemma 1.21 again.
- $S^1$ is locally Euclidean: we proceed explicitly. Define $U_1^\pm := \{(x,y) \in S^1 : \pm x > 0\}$; then $U_1^\pm \cong (-1,1)$ by $(x,y) \mapsto y$. Similarly, define $U_2^\pm := \{(x,y) \in S^1 : \pm y > 0\}$; then $U_2^\pm \cong (-1,1)$ by $(x,y) \mapsto x$.

## 1.2 January 18

The first homework has been posted. It is mostly a review of point-set topology things. It is due on the 25th of January.

**Remark 1.31.** Please read the section on fundamental groups of manifolds on your own. We will not discuss it in class.

To review, our current goal is to define smooth manifolds. Thus far we have defined a topological space and provided enough adjectives to turn it into a topological manifold. To proceed, we need to add smoothness to our structure. We will do this later.

### 1.2.1 Connectivity

For now, we will content ourselves with some extra adjectives for our topological manifolds which will later be helpful. Here are two notions of connectivity.

**Definition 1.32** (connected)**.** Fix a topological space $X$. Then $X$ is *disconnected* if and only if there exist disjoint nonempty open subsets $U, V \subseteq X$ such that $X = U \sqcup V$. If $X$ is not disconnected, we say that $X$ is connected.

**Example 1.33.** The interval $[0,1]$ is connected. See [Elb22, Lemma A.6].

**Remark 1.34.** Equivalently, we can say that $X$ is connected if and only if $X$ and $\varnothing$ are the only subsets of $X$ which are both open and closed.

**Definition 1.35** (path-connected)**.** Fix a topological space $X$. Then $X$ is *path-connected* if and only if any two points $p, q \in X$ has some continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

**Example 1.36.** The space $B(0,1) \subseteq \mathbb{R}^n$ is path-connected. Indeed, we show that the path-connected component of $0$ is all of $B(0,1)$; see [Elb22, Definition A.19]. In other words, we must exhibit a path from $0$ to $v$ for any $v \in B(0,1)$. Well, define $\gamma\colon [0,1] \to B(0,1)$ by $\gamma(t) := tv$. This is continuous because it is linear, and it has $\gamma(0) = 0$ and $\gamma(1) = v$ as desired.

In general, these two notions do not coincide.

**Example 1.37.** Consider the topological space

$$X := \{(x, \sin(1/x)) : x \in (0,1)\} \cup \{(0,y) : y \in \mathbb{R}\}.$$

Then $X$ is connected, but it is not path-connected. See [Elb22, Exercise A.20].

But one does in general apply the other.

**Lemma 1.38.** Fix a topological space $X$. If $X$ is path-connected, then $X$ is connected.

*Proof.* See [Elb22, Lemma A.16], though we will sketch the proof. We proceed by contraposition. Suppose that $X$ is disconnected, so we may write $X = U \sqcup V$ where $U, V \subseteq X$ are disjoint nonempty open subsets. Now choose some $p \in U$ and $q \in V$, and we claim that there is no path $\gamma\colon [0,1] \to X$. Indeed, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would then be nonempty disjoint open subsets of $[0,1]$ covering $[0,1]$, which is a contradiction. ■

However, for topological manifolds, these notions do coincide.

**Proposition 1.39.** Fix a topological space $M$ which is locally Euclidean of dimension $n$. Then $M$ is path-connected if and only if it is connected.

*Proof.* The forward direction is by Lemma 1.38. Thus, we focus on showing the converse. Fix some $p \in M$, and we define the subset

$$U_p := \{q \in M : \text{there exists a path from } p \text{ to } q\}.$$

This is the path-connected component of $p$ in $M$; see [Elb22, Definition A.19]. The main claim is that $U_p$ is open.

Suppose $q \in M$, and we need to find an open neighborhood $B_q \subseteq M$ of $q$ living inside $U_p$. Noting then that $U_p = \bigcup_{q \in U_p} B_q$ will complete the proof of this claim. Well, $q$ has some open neighborhood $B \subseteq M$ equipped with a homeomorphism $\varphi\colon B \cong B(0,1)$ by Remark 1.18. Then $B(0,1)$ is path-connected by Example 1.36, so $B$ is path-connected by going back through the homeomorphism. Thus, because $U_p$ is an equivalence class, it is also the path-connected equivalence class of $q$, so $U_p$ must contain $B$.

Now, let $\mathcal{U}$ denote the collection of path-connected components of $M$. This is a collection of disjoint open subsets covering $M$. Certainly it is nonempty, so select $U \in \mathcal{U}$. Then we write

$$M = U \cup \bigcup_{U' \in \mathcal{U} \setminus \{U\}} U'.$$

This is a decomposition of $M$ into disjoint open subsets, so because $M$ is connected, one of these must be empty. But $U$ is empty, so instead the union of the $U'$ must be nonempty. However, everything in $\mathcal{U}$ is nonempty, so instead we see that $\mathcal{U} \setminus \{U\}$ is empty, so $M = U$ is path-connected. ■

### 1.2.2 Local compactness

Here is our definition.

**Definition 1.40** (local compactness)**.** Fix a topological space $X$. Then $X$ is *locally compact* if and only if any $x \in X$ has some open neighborhood $U \subseteq X$ such that there exists a compact subset $K \subseteq X$ containing $U$.

**Remark 1.41.** If $X$ is Hausdorff, then compact subsets are closed [Elb22, Corollary 4.13], and closed subsets of a compact space are still compact [Elb22, Lemma 4.10], so we may as well take $K = \overline{U}$ in the above definition.

The above remark motivates the following definition.

**Definition 1.42** (precompact)**.** Fix a topological space $X$. An open subset $U \subseteq X$ is *precompact* if and only if $\overline{U}$ is compact.

**Remark 1.43.** Here is a quick check which will prove to be useful: if $X$ is Hausdorff and $U \subseteq X$ is precompact, and $V \subseteq U$, then $V$ is still precompact. Indeed, $\overline{U}$ is compact, and $\overline{V} \subseteq \overline{U}$ is a closed subset and hence compact [Elb22, Lemma 4.10].

**Example 1.44.** The topological space $\mathbb{R}$ is locally compact; see [Elb22, Example 4.71].

**Non-Example 1.45.** Infinite-dimensional normed vector spaces fail to be locally compact. Namely, open balls fail to be precompact, so local compactness fails.

**Non-Example 1.46.** The space $\mathbb{Q}$ is not locally compact. Indeed, suppose for the sake of contradiction that we have a precompact nonempty open neighborhood $U \subseteq \mathbb{Q}$ of $0 \in \mathbb{Q}$. Now, $\mathbb{Q}$ is Hausdorff (it's a metric space), so we can find some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U$ while $\varepsilon \notin \mathbb{Q}$, so Remark 1.43 tells us that $(\varepsilon/2, \varepsilon)$ is precompact so that $[\varepsilon/2, \varepsilon]$ is actually compact.

However, this is false. Let $\{\alpha_i\}_{i \geq 1}$ be an increasing sequence of irrationals in $[\varepsilon/2, \varepsilon]$ with $\alpha_i \to \varepsilon$. Explicitly, we can take $\alpha_i := \frac{i}{i+1} \cdot \varepsilon$. Then we define

$$U_i := [\alpha_i, \alpha_{i+1}]$$

for each $i \geq 1$. Note $[\alpha_i, \alpha_{i+1}] = (\alpha_i, \alpha_{i+1})$, so the $U_\bullet$s provide a countable sequence of disjoint open subsets covering $[\varepsilon/2, \varepsilon]$. Thus, $[\varepsilon/2, \varepsilon]$ cannot be compact.

One can check that manifolds are locally compact.

**Proposition 1.47.** Fix a topological $n$-manifold $M$. Then $M$ is locally compact.

*Proof.* This follows from being locally Euclidean. Fix $p \in M$, and then we are promised some open subset $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ with a homeomorphism $\varphi \colon U \cong \widehat{U}$. Then there is an open ball $B(\varphi(p), \varepsilon) \subseteq \widehat{U}$. Then $\overline{B(\varphi(p), \varepsilon/2)} \subseteq \widehat{U}$ is closed and bounded in $\mathbb{R}^n$ and hence compact, so $\varphi^{-1}(B(\varphi(p), \varepsilon/2))$ is a subset of the compact subset $\varphi^{-1}(\overline{B(\varphi(p), \varepsilon/2)})$. ■

Being locally compact approximately speaking allows one to understand a space by building it up from compact ones. Here is one way to do this.

**Definition 1.48** (exhaustion)**.** Fix a topological space $X$. Then an *exhaustion* of $X$ is a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of $X$ satisfying the following.

- Ascending: $K_0 \subseteq K_1 \subseteq \cdots$.
- Covers: $X = \bigcup_{i\in\mathbb{N}} K_i$.
- Not too close: $K_i \subseteq K_{i+1}^\circ$.

**Example 1.49.** The space $\mathbb{R}^n$ has an exhaustion by $K_i := B(0,i)$.

Here is a way to build an exhaustion.

**Proposition 1.50.** Fix a topological space $X$. If $X$ is second-countable, locally compact, and Hausdorff. Then $X$ has an exhaustion. In particular, topological $n$-manifolds have an exhaustion.

*Proof.* The second claim follows from the first by Proposition 1.47 and the definition of a manifold. So we will focus on showing the first claim.

Fix a countable base $\mathcal{B}$ of $X$, and let $\mathcal{B}'$ be the subcollection of precompact open base elements. Quickly, we note that $\mathcal{B}'$ is still a base: certainly everything in $\mathcal{B}'$ is open, and then for any $p \in X$ and open neighborhood $U \subseteq X$, we need some $B' \in \mathcal{B}'$ such that $B'$ is precompact.

Well, because $X$ is locally compact, there is a precompact open neighborhood $U'$ of $p$ by Remark 1.41. Then $U \cap U$ is an open neighborhood of $p$, so we can find a base element $B \in \mathcal{B}$ containing $p$ and inside $U' \cap U$. Then $B \subseteq U'$ is precompact by Remark 1.43.

We now construct our exhaustion. Enumerate $\mathcal{B} = \{B_0, B_1, \ldots\}$, and we proceed as follows.

1. Set $K_0 := \overline{B_0}$, which is compact by construction of $B_0$.
2. Now suppose we have a compact subset $K_i \subseteq X$, and we construct $K_{i+1}$. Note that $\mathcal{B}$ is an open cover of $K_i$, which can be reduced to a finite subcover, so there is some $M_{i+1}$ such that $K_i$ is covered by $\{B_i : i \le M_{i+1}\}$. We may as well suppose that $M_{i+1} \ge i+1$. Then we define
$$K_{i+1} := \bigcup_{i=1}^{M} \overline{B_i}.$$
Note that the finite union of compact sets remains compact.

The above construction produces an exhaustion. Here are our checks, which will complete the proof.

- Ascending: by construction, we see that
$$K_{i+1}^\circ \supseteq \bigcup_{i=1}^{M} B_i \supseteq K_i.$$
- Covers: any $x \in X$ lives in some $B_i$, and by construction, we have $B_i \subseteq K_i$, so $x \in K_i$. ■

### 1.2.3 Paracompactness

We will want to talk about covers in some more detail.

**Definition 1.51** (cover)**.** Fix a topological space $X$. A *cover* is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ such that
$$X = \bigcup_{U\in\mathcal{U}} U.$$

**Definition 1.52** (locally finite)**.** Fix a topological space $X$. A cover $\mathcal{U}$ of $X$ is *locally finite* if and only if any $p \in X$ has some open neighborhood $U \subseteq X$ intersecting at most finitely many elements of $\mathcal{U}$.

**Definition 1.53** (refinement)**.** Fix a cover $\mathcal{U}$ of a topological space $X$. Then a *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ such that any $V \in \mathcal{U}$ is contained in some $U \in \mathcal{U}$.

And here is our definition.

**Definition 1.54** (paracompact)**.** Fix a topological space $X$. Then $X$ is *paracompact* if and only if every open cover has a locally finite open refinement.

Approximately speaking, the point of desiring paracompactness is that it allows "reducing to Euclidean" arguments in the future will not have to deal with intersections which are infinitely bad. Anyway, here is our result.

**Proposition 1.55.** Fix a topological $n$-manifold $M$. Then $M$ is paracompact.

*Proof.* In fact, we are only going to use the fact that $M$ has an exhaustion, proven in Proposition 1.50.

Fix an open cover $\mathcal{U}$, and we want to produce a locally finite open refinement. To set us up, fix an exhaustion $\{K_i\}_{i\in\mathbb{N}}$, which exists by Proposition 1.50, and define the following sets for each $i \in \mathbb{N}$.

- For $i \geq -1$, define $V_i := K_{i+1} \setminus K_i^\circ$, which is a closed subset of the compact set $K_{i+1}$ and hence compact [Elb22, Lemma 4.10]; take $K_{-1} = \varnothing$ without concern.
- For $i \geq 0$, define $W_i := K_{i+2}^\circ \setminus K_{i-1}$, which is open; here, take $K_{-1} = \varnothing$ without concern.

For intuition, we should think about the $W_\bullet$s as being a locally finite cover from which we will build the locally finite cover refinement of $\mathcal{U}$.

For the construction, we fix some $j \geq 0$ for the time being. For each $x \in V_j$, find some $U_x \in \mathcal{U}$ containing $x$. Note that $\{U_x\}_{x\in V_j}$ is an open cover of $V_j$, and because $V_j \subseteq W_j$, in fact $\{U_x \cap W_j\}_{x\in V_j}$ is an open cover. Because $V_j$ is compact, we can thus reduce this open cover to a finite subcover $\mathcal{A}_j$.

Now letting $j$ vary, we define

$$\mathcal{V} := \bigcup_{j\geq 0} \mathcal{A}_j.$$

Here are our checks.

- Open cover: each $x \in X$ lives in some $K_{i+1}$ because we have an exhaustion, so lives in some $V_i$, so it lives in some open subset in $\mathcal{A}_j$, so it lives in some open subset in $\mathcal{V}$.
- Refinement: by construction, each open set in $\mathcal{A}_j$ is a subset in $\mathcal{U}$.
- Locally finite: this is essentially by construction. The main point is that any $x \in X$ lives in some $K_i$, so by choosing the least such $K_i$ places $x$ in some $V_i \subseteq W_i$. We now show that only finitely many open subsets in $\mathcal{V}$ intersect $W_i$. Note $W_i \subseteq K_{i+2}$, so $W_i \cap W_j = \varnothing$ for $j \geq i+2$. Thus, if $V \cap W_i \neq \varnothing$, we must have $V \in \mathcal{A}_j$ for $j < i+2$. But this is only finitely many indices, and each $\mathcal{A}_j$ is finite, so this is only finitely many candidates. ■

### 1.2.4 Products

We now discuss an in-depth example.

**Proposition 1.56.** Fix finitely many topological manifolds $M_1, \ldots, M_k$. Then the product

$$M_1 \times \cdots \times M_k$$

is also a topological manifold of dimension $\dim M_1 + \cdots + \dim M_k$.

We will do this via a sequence of lemmas.

**Lemma 1.57.** Fix a collection of Hausdorff topological spaces $\{X_\alpha\}_{\alpha\in\Lambda}$. Then the product

$$\prod_{\alpha\in\Lambda} X_\alpha$$

is also Hausdorff.

*Proof.* Fix distinct points $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$ in the product. Then there is an index $\beta \in \Lambda$ such that $x_\beta \neq y_\beta$, so because $X_\beta$ is Hausdorff, there are disjoint open neighborhoods $U_\beta, V_\beta \subseteq X_\beta$ of $x_\beta$ and $y_\beta$, respectively. Then we define $U_\alpha = V_\alpha := X_\alpha$ for $\alpha \neq \beta$, and we note that the open subsets

$$\prod_{\alpha\in\Lambda} U_\alpha \qquad \text{and} \qquad \prod_{\alpha\in\Lambda} V_\alpha$$

are disjoint open neighborhoods of $(x_\alpha)_{\alpha\in\Lambda}$ and $(y_\alpha)_{\alpha\in\Lambda}$, respectively, so we are done. (These are disjoint because any point in the intersection will have the $\beta$ coordinate in $U_\beta \cap V_\beta = \varnothing$.) ∎

**Lemma 1.58.** Fix finitely many second countable topological spaces $\{X_i\}_{i=1}^n$. Then the product

$$\prod_{i=1}^n X_i$$

is also second countable.

*Proof.* Let the product be $X$. For each $i$, let $\mathcal{B}_i$ be a countable base for $X_i$. Then define

$$\mathcal{B} := \left\{\prod_{i=1}^n B_i : B_i \in \mathcal{B}_i \text{ for each } i\right\}.$$

We claim that $\mathcal{B}$ is a base for the topology on the $X$. Indeed, suppose $(x_1, \ldots, x_n) \in X$ lives in some open subset $U \subseteq X$. From the standard base on $X$, we know that there are open subsets $U_i \subseteq X_i$ for each $i$ such that $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$. Now, for each $U_i$, we note that $x_i \in U_i$ must have some $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$ and $B_i \subseteq U_i$. But then

$$(x_1, \ldots, x_n) \in B_1 \times \cdots \times B_n \subseteq U,$$

so $B_1 \times \cdots \times B_n \in \mathcal{B}$ is the desired base element. ∎

We now prove Proposition 1.56.

*Proof of Proposition 1.56.* We get Hausdorff from Lemma 1.57 and second countable from Lemma 1.58. So it remains to check that we are locally Euclidean. For brevity, let $M$ be the product, and set $n_i := \dim M_i$ for each $i$, and let $n := n_1 + \cdots + n_k$.

Now, fix some point $(x_1, \ldots, x_k) \in M$. For each $i$, we get some open neighborhood $U_i \subseteq M_i$ of $x_i$ and some open $\widehat{U}_i \subseteq \mathbb{R}^{n_i}$ with a homeomorphism $\varphi_i \colon U_i \cong \widehat{U}_i$. Now, we see that the product map

$$(\varphi_1 \times \cdots \times \varphi_k) \colon U_1 \times \cdots \times U_k \to \widehat{U}_1 \times \cdots \times \widehat{U}_k$$

is still a homeomorphism, and the target is an open subset of

$$\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \cong \mathbb{R}^n,$$

where this last homeomorphism is obtained by simply concatenating the coordinates. So we have constructed a composite homeomorphism from an open neighborhood of $(x_1, \ldots, x_k)$ to an open subset of $\mathbb{R}^n$, as desired. ■

**Example 1.59.** Example 1.30 established $S^1$ as a topological $1$-manifold, so the $n$-torus

$$T^n \coloneqq \underbrace{S^1 \times \cdots \times S^1}_{n}$$

is a topological $n$-manifold. Note that the covering space $p\colon \mathbb{R} \to S^1$ will induce the covering space $p^n\colon \mathbb{R}^n \to T^n$, so we can also view $T^n$ as $\mathbb{R}^n/\mathbb{Z}^n$; in other words, we have the unsurprising homeomorphism $\mathbb{R}^n/\mathbb{Z}^n \to (\mathbb{R}/\mathbb{Z})^n$.

### 1.2.5 Open Submanifolds

We proceed with a sequence of lemmas.

**Lemma 1.60.** Suppose $X$ is a Hausdorff topological space. If $X' \subseteq X$ is a subspace, then $X'$ is still Hausdorff.

*Proof.* Fix distinct points $p, q \in X'$. Then $X$ is Hausdorff, so there exist disjoint open neighborhoods $U, V \subseteq X$ of $p$ and $q$, respectively, so $U \cap X'$ and $V \cap X'$ are the needed disjoint open subsets of $X'$, respectively. ■

**Lemma 1.61.** Suppose that $X$ is a second countable topological space. Then for any subset $X' \subseteq X$, the topological (sub)space $X'$ is still second countable.

*Proof.* Well, let $\mathcal{B}$ be a countable base for $X$, and we claim that the collection

$$\mathcal{B}' \coloneqq \{B \cap X' : B \in \mathcal{B}\}$$

makes a countable base for $X'$. Note that $\mathcal{B}'$ is certainly countable because there is a surjective map $\mathcal{B} \to \mathcal{B}'$ by $B \mapsto (B \cap X')$, and $\mathcal{B}$ is countable. (This map is surjective by construction.)

So it remains to show that $\mathcal{B}'$ is a base. Quickly, we claim that every $B' \in \mathcal{B}'$ is open in $X'$. Indeed, for any $B' \in \mathcal{B}'$, we can find some $B \in \mathcal{B}$ such that $B' = B \cap X'$. Now, $\mathcal{B}$ is a base, so $B \subseteq X$ is open, so $B' = B \cap X'$ is open in the subspace topology of $X'$.

To finish checking that we have a base, fix some $x' \in X'$ and open $U' \subseteq X'$ containing $x'$. Then we need some $B' \in \mathcal{B}'$ such that $x' \in B'$ and $B' \subseteq U'$. Well, by the subspace topology, we can write $U' = U \cap X'$ for some open $U \subseteq X$, but then $x' \in U$, so there is some $B \in \mathcal{B}'$ such that $x' \in B$ and $B \subseteq U$. To finish, we set

$$B' \coloneqq B \cap X',$$

which is in $\mathcal{B}'$ by construction, and we have $x' \in B \cap X' = B'$ and $B' = B \cap X' \subseteq U \cap X' = U'$, so $B'$ is indeed the required basic open set. ■

**Lemma 1.62.** Suppose that $X$ is locally Euclidean of dimension $n$. Then for any open subset $X' \subseteq X$, the topological (sub)space $X'$ is locally Euclidean of dimension $n$.

*Proof.* For any $x' \in X'$, we must find open subsets $U' \subseteq X'$ and $\widehat{U}' \subseteq \mathbb{R}^n$ such that $x' \in U'$ and there is a homeomorphism $U' \cong \widehat{U}'$.

Well, $x' \in X$, so there are open subsets $U \subseteq X$ and $\widehat{U} \subseteq \mathbb{R}^n$ such that $x' \in U$ and there is a homeomorphism $\varphi\colon U \cong \widehat{U}$. Now, set

$$U' := U \cap X'.$$

Then $\varphi$ is a homeomorphism, so $\varphi' := \varphi|_{U'}$ continues to be a homeomorphism onto its image $\widehat{U}' := \varphi(U')$. Indeed, the inverse of the bijection $\varphi|_{U'}\colon U' \to \widehat{U}'$ is $\varphi'|_{\widehat{U}'}$. Both of these maps are continuous by, so $\varphi|_{U'}$ is in fact a homeomorphism.

Now, $U' \subseteq U$ is open, so because $\varphi$ is a homeomorphism, we see that $\varphi(U') \subseteq \widehat{U}$ is open: $\varphi(U')$ is the pre-image of the open subset $U' \subseteq U$ under the continuous map $\varphi^{-1}\colon \widehat{U} \to U$, so $\varphi(U')$ being open follows. Continuing, because $\widehat{U} \subseteq \mathbb{R}^n$ is open, we conclude that $\widehat{U}' \subseteq \mathbb{R}^n$ is open.[1] So $U' \subseteq X'$ is open (by the subspace topology), contains $x'$, and it is homeomorphic to an open subset $\widehat{U}'$ of $\mathbb{R}^n$. ■

**Proposition 1.63.** Fix a topological $n$-manifold $M$. For any nonempty open subset $U \subseteq M$, we have that $U$ is a topological $n$-manifold.

*Proof.* Combine Lemmas 1.60 to 1.62. ■

### 1.2.6 Charts

The construction of our smooth structure will arise from more carefully understanding how a manifold is locally Euclidean. This arises from charts.

**Definition 1.64** (chart)**.** Fix a topological $n$-manifold $M$. Then a *coordinate chart* or just *chart* is a pair $(U, \varphi)$ where $U \subseteq M$ is open and $\varphi\colon U \cong \widehat{U}$ is a homeomorphism where $\widehat{U} \subseteq \mathbb{R}^n$ is open.

Essentially, the content of $M$ being locally Euclidean is that it has an open cover by open subsets belonging to some chart. The reason we call it a chart is that we are (approximately speaking) providing "local coordinates" to an open subset of $M$.

**Definition 1.65** (coordinate function)**.** Fix a chart $(U, \varphi)$ if a topological $n$-manifold $M$. Then we may write

$$\varphi(p) := \big(x^1(p), \ldots, x^n(p)\big) \in \mathbb{R}^n$$

for each $p \in M$. We call these functions $x^\bullet\colon U \to \mathbb{R}$ the *coordinate functions*.

Note that these coordinate functions are continuous because they are simply the continuous function $\varphi$ composed with the projection $\mathbb{R}^n \to \mathbb{R}$.

[1] Namely, an open subset of an open subset $U$ is still an open subset. This sentence has some content because the larger open subset uses the subspace topology; the proof simply notes that being open in $U$ is equivalent to being the intersection of an open subset and $U$, which is open because finite intersections of open subsets continues to be open.

**Example 1.66.** Fix an open subset $V \subseteq \mathbb{R}^m$, and let $F\colon V \to \mathbb{R}^n$ be a continuous function. Then the graph

$$\Gamma \coloneqq \{(x, F(x)) : x \in V\} \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

is a topological $n$-manifold. Because we are already a subspace of $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$, we see that $\Gamma$ is also Hausdorff and second countable. (Subspaces inherit being Hausdorff directly, and we inherit being second countable by using the intersection of the given countable base.)

The main content comes from being locally Euclidean. Namely, there is a projection map $\pi\colon \Gamma \to V$ by $(x, y) \mapsto x$ which in fact is a homeomorphism (it's continuous inverse is $(\mathrm{id} \times F)\colon x \mapsto (x, F(x))$). So we have the single chart $(V, \pi)$, which establishes being a topological $n$-manifold.

## 1.3 January 23

The first homework is due on Thursday. Today we discuss smooth structures.

### 1.3.1 Examples of Topological Manifolds

Let's provide a few more examples of topological manifolds.

**Exercise 1.67** (sphere)**.** We show that the $n$-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a topological $n$-manifold.

*Proof.* Explicitly, for each $i \in \{1, \ldots, n+1\}$, we define

$$U_i^\pm \coloneqq \{(x_1, \ldots, x_{n+1}) \in S^n : \pm x_i > 0\},$$

which has a projection $\pi_i^\pm\colon U_i^\pm \to B(0,1)$ (for $B(0,1) \subseteq \mathbb{R}^n$) given by erasing the $x_i$ coordinate. One can show that the $\pi_i^\pm$ are all homeomorphisms—certainly, it is continuous, and the inverse map is given by

$$(x_1, \ldots, x_n) \coloneqq \left(x_1, \ldots, x_{i-1}, \pm\sqrt{1 - (x_1^2 + \cdots + x_n^2)}, x_i, \ldots, x_n\right),$$

which is also continuous. (We won't bother checking that the maps are mutually inverse.) Lastly, we note that the $U_i^\pm$ is an open cover of $S^n$ because any point in $S^n$ has some nonzero coordinate, and this nonzero coordinate will have a sign. ■

**Exercise 1.68** (projective space)**.** Define the space $\mathbb{RP}^n$ as "lines in $\mathbb{R}^{n+1}$": it consists of equivalence classes of nonzero points in $\mathbb{R}^{n+1} \setminus \{0\}$, where $x \sim y$ if and only if there is some $\lambda \in \mathbb{R}^\times$ such that $x = \lambda y$. We show that $\mathbb{RP}^n$ is a topological $n$-manifold.

*Proof.* For notation, we let $[x_0 : \cdots : x_n]$ denote the equivalence class of $(x_1, \ldots, x_n)$ in $\mathbb{RP}^n$. Note there is a projection $p\colon \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to \mathbb{RP}^n$, and we give $\mathbb{RP}^n$ the induced (quotient) topology from $\mathbb{R}^{n+1} \setminus \{0\}$.

By Lemma 1.22, to achieve second countable, it suffices to provide a finite open cover by open subsets homeomorphic to open subsets of $\mathbb{R}^n$; this will also achieve locally Euclidean. Well, define

$$U_i \coloneqq \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_i \neq 0\}.$$

Note that the pre-image in $\mathbb{R}^{n+1} \setminus \{0\}$ consists of the $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ with $x_i \neq 0$, so $U_i \subseteq \mathbb{RP}^n$ is open. Now, by scaling, we can write elements of $U_i$ uniquely as $[y_0 : \cdots : y_n]$ with $y_i = 1$, which provides the required element in $\mathbb{R}^n$. Explicitly, we define $\varphi_i\colon U_i \to \mathbb{R}^n$ by

$$\varphi_i\colon [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right).$$

One sees that $\varphi_i$ is continuous: by the quotient topology, we are trying to show that $\varphi_i \circ \pi \colon \pi^{-1}U_i \to \mathbb{R}^n$ is just $(x_0, \ldots, x_n) \mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i)$, which is continuous, so $\varphi_i$ is continuous because $\mathbb{RP}^n$ has the quotient topology. Lastly, one notes that the inverse of $\varphi_i$ is given by $(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$, which is continuous because it is the composite of the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $(x_0, \ldots, \widehat{x}_i, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$ and the projection $p\colon \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \to \mathbb{RP}^n$.

Lastly, we show that $\mathbb{RP}^n$ is Hausdorff. Doing this in a slick way is surprisingly obnoxious. We claim that there is a $2$-to-$1$ covering space map

$$p\colon S^n \to \mathbb{RP}^n.$$

To see why this implies that $\mathbb{RP}^n$ is Hausdorff, fix two distinct points $x, y \in \mathbb{RP}^n$. Then there are lifts $x_1, x_2 \in S^n$ of $x$ and $y_1, y_2 \in S^n$. Because $S^n$ is already Hausdorff (it's a subspace of $\mathbb{RP}^n$), we can find disjoint open subsets $U_1, U_2, V_1, V_2 \subseteq S^n$ around $x_1, x_2, y_1, y_2 \in S^n$ respectively, and we can make them all small enough so that $p$ is a local homeomorphism. Then $p(U_1) \cap p(U_2)$ and $p(V_1) \cap p(V_2)$ are the desired open subsets.

So we are left showing that we have a double cover $p$. The map is given by the composite

$$S^n \subseteq \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \twoheadrightarrow \mathbb{RP}^n,$$

which we see is continuous automatically. To see that this is a $2$-to-$1$ local homeomorphism, we note that the pre-image of the standard open subset $U_i \subseteq \mathbb{RP}^n$ is

$$\left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \neq 0\right\},$$

whose pre-image in $S^n$ splits into the two open subsets $U_i^{\pm}$. So we have our continuous map $U_i^+ \sqcup U_i^- \to U_i$; it remains to show that $U_i^{\pm} \to U_i$ is a homeomorphism. We may as well assume $i = 0$; then the inverse map is given by sending $[1 : x_1 : \cdots : x_n]$ to the point on the hemisphere of $S^n$ on this line, which is

$$\pm\frac{x}{|x|},$$

where the sign depends on $U_i^{\pm}$. This is continuous, so we are done. ■

**Remark 1.69.** Note $S^n$ is continuous, so the surjectivity of the covering space map $S^n \twoheadrightarrow \mathbb{RP}^n$ implies that $\mathbb{RP}^n$ is compact.

### 1.3.2 Transition Functions

Defining smooth structures will come out of transition maps between coordinate charts.

**Definition 1.70** (transition map)**.** Fix charts $(U, \varphi)$ and $(V, \psi)$ on a topological $n$-manifold $M$. Then the *transition map* is the map

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V).$$

Here, we are abusing notation a little: in order to make sense of $\psi \circ \varphi^{-1}$, we really want to work with the restrictions as $\psi|_{U\cap V} \circ (\varphi|_{U\cap V})^{-1}$.

**Remark 1.71.** Note $\varphi(U \cap V), \psi(U \cap V) \subseteq \mathbb{R}^n$, so this is a homeomorphism from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$. Namely, $\varphi|_{U\cap V}$ and $\psi|_{U\cap V}$ are both homeomorphisms, so the above composition is still a homeomorphism.

**Example 1.72** (polar coordinates)**.** Consider the topological 2-manifold $M := \mathbb{R}^2$. There is the identity chart $\mathrm{id}_M\colon M \to \mathbb{R}^2$, and there is also "polar coordinates" on $U := \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$ with chart $\varphi\colon U \to \mathbb{R}_+ \times (0, \pi)$ defined by

$$\varphi((x, y)) := \left(\sqrt{x^2 + y^2}, \arg(x, y)\right),$$

where $\arg(x, y)$ is the angle of $(x, y)$ with the positive $x$-axis. Note the inverse map of $\varphi$ is given by $(r, \theta) \mapsto (r \cos\theta, r \sin\theta)$, so $\varphi$ is in fact a homeomorphism.

Now, the transition map $\psi \circ \varphi^{-1}$ sends

$$(r, \theta) \overset{\varphi^{-1}}{\mapsto} (r\cos\theta, r\sin\theta) \overset{\psi}{\mapsto} (r\cos\theta, r\sin\theta).$$

**Example 1.73.** Consider the topological 2-manifold $M := S^2$ from Exercise 1.67. We compute the transition maps between $\varphi_1^+$ and $\varphi_3^+$, which overlap on the open set consisting of $(x_1, x_2, x_3) \in S^2$ such that $x_1, x_3 > 0$. Well, we can directly compute that $\varphi_3^+ \circ \left(\varphi_1^+\right)^{-1}$ is given by

$$(x_2, x_3) \overset{(\varphi_1^+)^{-1}}{\to} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3\right) \overset{\varphi_3^+}{\to} \left(\sqrt{1 - x_2^2 - x_3^2}, x_2\right).$$

In the above examples, we can note that the maps between the Euclidean smooths are smooth on their domains. This becomes our notion of smoothness.

**Definition 1.74** (smoothly compatible)**.** Two charts $(U, \varphi)$ and $(V, \psi)$ of a topological manifold $M$ are *smoothly compatible* if and only if both transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth (i.e., infinitely differentiable). Notably, this condition is vacuously satisfied if $U \cap V = \varnothing$.

### 1.3.3 Smooth Structures

We would like to cover $M$ with smoothly compatible charts, so it will be helpful to have a language for such covers.

**Definition 1.75** (atlas)**.** Fix a topological manifold $M$. An *atlas* $\mathcal{A}$ is a collection of charts "covering $M$" in the sense that

$$M = \bigcup_{(U,\varphi)} U.$$

An atlas is *smooth* if and only if its charts are pairwise smoothly compatible. A smooth atlas is *maximal* if and only if it is maximal in the sense of inclusion by smooth atlases.

The point of using a maximal atlas is that we would like a way to say when two atlases provide the same smooth structure for a topological manifold, but it will turn out to be easier to provide a "unique" atlas to look at, which will be the maximal smooth atlas. Quickly, we note that maximal smooth atlases exist. One could argue this by Zorn's lemma, but we don't have to.

**Proposition 1.76.** Fix a topological $n$-manifold $M$. Any smooth atlas $\mathcal{A}$ is contained in a unique maximal smooth atlas, denoted $\overline{\mathcal{A}}$.

*Proof.* We have to show existence and uniqueness. We will construct this directly: define $\overline{\mathcal{A}}$ to be the collection of charts $(U, \varphi)$ which is smoothly compatible with each chart in $\mathcal{A}$. We show that $\overline{\mathcal{A}}$ is a maximal smooth atlas.

- Atlas: certainly $\overline{\mathcal{A}} \supseteq \mathcal{A}$, so $\overline{\mathcal{A}}$ covers $M$, so $\overline{\mathcal{A}}$ is an atlas.

- Smooth: fix any charts $(U_1, \varphi_1), (U_2, \varphi_2) \in \overline{\mathcal{A}}$, and we would like to show that they are smoothly compatible. If $U_1 \cap U_2 = \varnothing$, there is nothing to do, so we may assume that the intersection is nonempty. By symmetry, it will be enough to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth.

  The point is that differentiability is a local notion: explicitly, fix some $q \in \varphi_1(U_1 \cap U_2)$, and we want to show that $\varphi_2 \circ \varphi_1^{-1}$ is smooth at $q$. This can be checked on a small open neighborhood of $q$; in particular, find the $p \in U_1 \cap U_2$ such that $\varphi_1(p) = q$, and we can find some chart $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Then we note that

  $$\varphi_2|_{U_1\cap U_2\cap V} \circ (\varphi_1|_{U_1\cap U_2\cap V})^{-1} = \left(\varphi_2|_{U_1\cap U_2\cap V} \circ (\psi|_{U_1\cap U_2\cap V})^{-1}\right) \circ \left(\psi|_{U_1\cap U_2\cap V} \circ (\varphi_1|_{U_1\cap U_2\cap V})^{-1}\right)$$

  is smooth on $\varphi_1(U_1 \cap U_2 \cap V)$ as it is the composition of smooth maps. So our left-hand side is smooth on $U_1 \cap U_2 \cap V$ and in particular at $q \in \varphi_1(U_1 \cap U_2 \cap V)$.

- Maximal: suppose $\mathcal{A}'$ is a smooth atlas containing $\mathcal{A}$. We must show that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$; by supposing further that $\mathcal{A}'$ contains $\overline{\mathcal{A}}$, we achieve the maximality of $\overline{\mathcal{A}}$. Well, for each $(U, \varphi) \in \mathcal{A}'$, we see that $(U, \varphi)$ is smoothly compatible with each chart in $\mathcal{A}$, so $(U, \varphi) \in \overline{\mathcal{A}}$. Thus, $(U, \varphi) \in \overline{\mathcal{A}}$, so $\mathcal{A}' \subseteq \overline{\mathcal{A}}$.

- Unique: suppose $\mathcal{A}'$ is a maximal smooth atlas containing $\mathcal{A}$. Then the previous point establishes that $\mathcal{A}' \subseteq \overline{\mathcal{A}}$, but then we must have equality because $\mathcal{A}'$ is a maximal smooth atlas. ■

So we may make the following definition.

**Definition 1.77** (maximal smooth atlas)**.** Fix a topological $n$-manifold $M$. Given a smooth atlas $\mathcal{A}$ on $M$, we let $\overline{\mathcal{A}}$ denote the unique maximal smooth atlas containing $\mathcal{A}$, which we know exists and is unique by Proposition 1.76.

**Corollary 1.78.** Fix a topological $n$-manifold $M$. Given smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is still a smooth atlas, then

$$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}.$$

*Proof.* Define $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\overline{\mathcal{A}}$ is a maximal smooth atlas containing $\mathcal{A}$ and hence both $\mathcal{A}_1$ and $\mathcal{A}_2$, so we see that $\overline{\mathcal{A}_1} = \overline{\mathcal{A}}$ and $\overline{\mathcal{A}_2} = \overline{\mathcal{A}}$. Notably, we are using the uniqueness of Proposition 1.76. ■

At long last, here is our definition.

**Definition 1.79** (smooth manifold)**.** Fix a topological $n$-manifold $M$. A *smooth structure* on $M$ is a maximal smooth atlas on $M$. A *smooth $n$-manifold* is a pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a smooth structure on $M$.

**Remark 1.80.** Adjusting the "smoothness" on the manifold $M$ produces different notions of manifold. For example, we can have twice differentiable manifolds, real analytic manifolds, complex manifolds, etc.

## 1.4 January 25

The first homework is due later today.

### 1.4.1 A Couple Lemmas on Atlases

Here are some basic properties of smooth manifolds which one can check.

**Lemma 1.81.** Fix a smooth $n$-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$, then for any open subset $U' \subseteq U$, we have $(U', \varphi|_{U'}) \in \mathcal{A}$.

*Proof.* By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U', \varphi|_{U'})\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas of charts. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that $(U', \varphi|_{U'})$ and $(V, \psi)$ are smoothly compatible. (The charts in $\mathcal{A}$ are already smoothly compatible with each other.) In other words, we must show that the transition functions are diffeomorphism: the transition maps are

$$\varphi|_{U' \cap V} \circ \psi|_{U' \cap V}^{-1} = \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}\right)|_{\psi(U' \cap V)}$$

and

$$\psi|_{U' \cap V} \circ \varphi|_{U' \cap V}^{-1} = \left(\psi|_{U \cap V} \circ \varphi|_{U \cap V}^{-1}\right)|_{\varphi(U' \cap V)},$$

and these are both smooth as the restrictions of smooth maps. (Namely, we are using the fact that $(U, \varphi)$ and $(V, \psi)$ are smoothly compatible already.) ■

**Lemma 1.82.** Fix a smooth $n$-manifold $(M, \mathcal{A})$. Given a chart $(U, \varphi) \in \mathcal{A}$ and diffeomorphism $\chi \colon \varphi(U) \to V$ for some open subset $V \subseteq \mathbb{R}^n$, we have $(U, \chi \circ \varphi) \in \mathcal{A}$.

*Proof.* The argument is similar to that of the above lemma. By maximality of $\mathcal{A}$, it suffices to show that $\mathcal{A} \cup \{(U, \chi \circ \varphi)\}$ is a smooth atlas. It contains $\mathcal{A}$, so this is at least an atlas. For smooth compatibility, we pick up some $(V, \psi) \in \mathcal{A}$, and we must show that $(V, \psi)$ and $(U, \chi \circ \varphi)$ are smoothly compatible. (Indeed, the charts in $\mathcal{A}$ are already smoothly compatible with each other.) Well, the transition maps are

$$(\chi \circ \varphi)|_{U \cap V} \circ \psi|_{U \cap V}^{-1} = \chi|_{\varphi(U \cap V)} \circ \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}\right)$$

and

$$\psi|_{U \cap V} \circ (\chi \circ \varphi)|_{U \cap V}^{-1} = \psi|_{U \cap V} \circ \varphi|_{U \cap V}^{-1} \circ \chi|_{\varphi(U \cap V)}^{-1},$$

which are smooth maps because $(U, \varphi)$ and $(V, \psi)$ are already smoothly compatible, and $\chi$ is a diffeomorphism. ■

**Lemma 1.83.** Fix a smooth $n$-manifold $(M, \mathcal{A})$. If $\varphi \colon U \to \mathbb{R}^n$ is an injective map with $U \subseteq M$ is such that each $p \in U$ has some open neighborhood $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, then actually $(U, \varphi) \in \mathcal{A}$.

*Proof.* By the definition of being a maximal smooth atlas, it suffices to show that $(U, \varphi)$ is smoothly compatible with all charts in $\mathcal{A}$. Well, pick up some chart $(V, \psi)$, and we would like to show that the transition map

$$\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}$$

is a diffeomorphism. Well, we can being a diffeomorphism locally by checking it at all point $\psi(p) \in \psi(U \cap V)$ where $p \in U \cap V$. But for some fixed $p$, we are promised some open subset $U_p \subseteq U$ such that $(U_p, \varphi|_{U_p}) \in \mathcal{A}$, so the map

$$\varphi|_{U_p \cap V} \circ \psi|_{U_p \cap V}^{-1} = \left(\varphi|_{U \cap V} \circ \psi|_{U \cap V}^{-1}\right)|_{\psi(U_p \cap V)}$$

is a diffeomorphism. So we produce smoothness at the images of $p$ of the function and its inverse. ■

### 1.4.2 Examples of Smooth Manifolds

We go through some examples of smooth manifolds.

**Example 1.84.** Recall from Lemma 1.21 that $\mathbb{R}^n$ is a topological $n$-manifold. Then $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ provides an atlas on $\mathbb{R}^n$ consisting of a single chart, which is vacuously smooth; note Proposition 1.76 then gives us a smooth structure.

More generally, we have the following.

**Proposition 1.85.** Fix a smooth $n$-manifold $(M, \mathcal{A})$. For any nonempty open subset $M' \subseteq M$, we have that $M'$ is a topological $n$-manifold, and

$$\mathcal{A}' := \{(U, \varphi) \in \mathcal{A} : U \subseteq M'\}$$

is a smooth structure on $M$.

*Proof.* By Proposition 1.63, we see that $M'$ is a topological $n$-manifold. It remains to show that $\mathcal{A}'$ is a smooth structure. Here are our checks.

- Chart: for any $x \in M'$, we know $\mathcal{A}$ is a chart on $M$, so there is a chart $(U, \varphi) \in \mathcal{A}$ with $x \in U$. Now, $U \subseteq M$ is open, so Lemma 1.81 tells us that $(U \cap M', \varphi|_{U\cap M'})$ is a chart in $\mathcal{A}$. But now $U \cap M' \subseteq M'$, so $(U \cap M', \varphi|_{U\cap M'}) \in \mathcal{A}'$ by construction, so we conclude because $x \in U \cap M'$.

- Smooth: for any two charts $(U, \varphi), (V, \psi) \in \mathcal{A}'$, we note that these charts belong to the smooth atlas $\mathcal{A}$ already, so they are already smoothly compatible.

- Maximal: by definition of being a maximal smooth atlas, it suffices to show that if $(U, \varphi)$ is a chart of $M'$ smoothly compatible with $\mathcal{A}'$, then it must be in $\mathcal{A}'$. Well, $U \subseteq M'$ already, so it suffices to show that $(U, \varphi) \in \mathcal{A}$. Because $\mathcal{A}$ is already a maximal smooth atlas, it suffices to show that $(U, \varphi)$ is compatible with all the charts in $\mathcal{A}$. Well, for any chart $(V, \psi) \in \mathcal{A}$, we need the composite

$$\varphi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}$$

to be a diffeomorphism. But we simply note that $(U \cap V, \psi|_{U\cap V}) \in \mathcal{A}$ by Lemma 1.81 will live in $\mathcal{A}'$, so the above is a diffeomorphism because the hypothesis on $(U, \varphi)$ implies that it would be smoothly compatible with $(U \cap V, \psi|_{U\cap V}) \in \mathcal{A}'$. ■

**Example 1.86.** Any nonempty open subset of $\mathbb{R}^n$ is a smooth $n$-manifold by combining Example 1.84 and Proposition 1.85. For example,

$$\mathrm{GL}_n(\mathbb{R}) := \left\{M \in \mathbb{R}^{n\times n} : \det M \neq 0\right\}$$

is an open subset of $\mathbb{R}^{n\times n}$, so $\mathrm{GL}_n(\mathbb{R})$ is a smooth manifold. (Notably, $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a polynomial and hence continuous, so the pre-image of $\mathbb{R} \setminus \{0\}$ is open.)

**Example 1.87.** From Example 1.66, we know that the graph $\Gamma$ of a smooth function $f\colon V \to \mathbb{R}^n$, where $V \subseteq \mathbb{R}^m$ is open, is a topological $n$-manifold, where we have a chart given by the projection $\pi\colon \Gamma \to V$. Using this chart alone produces a smooth atlas and makes $\Gamma$ into a smooth $n$-manifold as well.

**Example 1.88.** We claim that the charts on $S^n$ provided in Exercise 1.67 provide a smooth atlas on $S^n$ and hence a smooth structure by Proposition 1.76. Indeed, we must show that the transition maps

$$\varphi_i^\pm|_{U_i^\pm\cap U_j^\pm}\circ\varphi_j^\pm|_{U_i^\pm\cap U_j^\pm}^{-1}(x_1,\ldots,x_n)=\left(x_1,\ldots,\widehat{x}_j,\ldots,x_{i-1},\pm\sqrt{1-(x_1^2+\cdots+x_n^2)},x_i,\ldots,x_n\right)$$

is a diffeomorphism (for any choice of signs). The above equation shows that our map is smooth for $i>j$, and the computation for $i<j$ simply switches the $i$th and $j$th coordinates. On the homework, we will see how to use stereographic projection to provide a smooth structure (in fact, the same smooth structure) on $S^n$.

**Example 1.89.** Fix an $n$-dimensional $\mathbb{R}$-vector space $V$. Then we claim

$$\mathcal{A}:=\{(V,\varphi):\varphi\text{ is an isomorphism to }\mathbb{R}^n\}$$

is a smooth atlas on $V$ and hence provides a smooth structure. Indeed, certainly this is an atlas: there is some isomorphism $\varphi\colon V\to\mathbb{R}^n$, and this chart will cover $V$. Further, these are smoothly compatible because the transition map between the two arbitrary charts $(V,\varphi)$ and $(V,\psi)$ is the linear isomorphism $\left(\varphi\circ\psi^{-1}\right):\mathbb{R}^n\to\mathbb{R}^n$, which is linear and hence smooth.

**Example 1.90.** Fix the topological 1-manifold $\mathbb{R}$ of Lemma 1.21. Example 1.84 tells us $\mathcal{A}:=\{(\mathbb{R},\mathrm{id}_\mathbb{R})\}$ provides a smooth atlas, and $\mathcal{A}':=\{(\mathbb{R},\varphi)\}$ given by $\varphi\colon x\mapsto x^3$ is also a smooth atlas (again, smoothness is vacuous). However, $\mathcal{A}$ and $\mathcal{A}'$ provide smooth structures: otherwise, they would be contained in the same maximal smooth atlas, so $(\mathbb{R},\mathrm{id}_\mathbb{R})$ and $(\mathbb{R},\varphi)$ would be smoothly compatible, but then the composite $\left(\mathrm{id}_\mathbb{R}\circ\varphi^{-1}\right):x\mapsto\sqrt[3]{x}$ is not a smooth function $\mathbb{R}\to\mathbb{R}$.

**Example 1.91.** Recall that $\mathbb{RP}^n$ is a topological $n$-manifold by Exercise 1.68. We claim that the charts $(U_i,\varphi_i)$ actually form a smooth atlas on $\mathbb{RP}^n$, thus making $\mathbb{RP}^n$ into a smooth atlas. We already checked that these charts cover $\mathbb{RP}^n$, and they are smoothly compatible because we can compute the transition between $(U_i,\varphi_i)$ and $(U_j,\varphi_j)$ is

$$\varphi_i|_{U_i\cap U_j}\circ\varphi_j|_{U_i\cap U_j}^{-1}(x_0,\ldots,\widehat{x}_j,\ldots,x_n)=\left(\frac{x_0}{x_i},\ldots,\frac{x_{j-1}}{x_i},\frac{1}{x_i},\frac{x_{j+1}}{x_i},\ldots,\frac{x_n}{x_i}\right),$$

which we can see is a rational and hence smooth function.

**Example 1.92.** Fix smooth manifolds $(M_1,\mathcal{A}_1),\ldots,(M_k,\mathcal{A}_k)$, where $M_i$ is a smooth $n_i$-manifold. The product $M:=M_1\times\cdots\times M_k$ is a smooth manifold by Proposition 1.56, and the proof implies that

$$\mathcal{A}:=\{(U_1\times\cdots\times U_k,\varphi_1\times\cdots\times\varphi_k):(U_i,\varphi_i)\in\mathcal{A}_i\text{ for each }i\}$$

is an atlas on $M$. In fact, this is a smooth atlas, thus providing $M$ with a smooth structure by Proposition 1.76. Well, the transition map between the charts $(U,\varphi):=(U_1\times\cdots\times U_k,\varphi_1\times\cdots\times\varphi_k)$ and $(V,\psi):=(V_1\times\cdots\times V_k,\psi_1\times\cdots\times\psi_k)$ is

$$\varphi|_{U\cap V}\circ\psi|_{U\cap V}^{-1}=\left(\varphi_1|_{U_1\cap V_1}\circ\psi_1|_{U_1\cap V_1}^{-1}\right)^{-1}\times\cdots\times\left(\varphi_k|_{U_k\cap V_k}\circ\psi_k|_{U_k\cap V_k}^{-1}\right),$$

which we can see is smooth as it is the product of smooth functions.

**Remark 1.93.** In fact, if a topological $n$-manifold has some smooth structure, there are uncountably many distinct smooth structures on $M$. On the other hand, for $n$-manifolds of small dimensions (e.g., $n \leq 3$), it turns out that these are diffeomorphic.

**Remark 1.94.** However, there do exist topological $n$-manifolds with no smooth structure, in dimensions $n \geq 4$. Even worse, there are topological $n$-manifolds with distinct smooth structures up to diffeomorphism, again in dimensions $n \geq 4$. Even for $S^n$, the story is complicated: there is only one smooth structure for $n \leq 3$, we don't understand $n = 4$, and the story is complicated but somewhat understood for $n \geq 5$.

### 1.4.3 Grassmannians

The construction of smooth manifolds is rather long: we build a topological space, define some charts, and then check that the charts are smoothly compatible. Here's a lemma to do all of this at once.

**Lemma 1.95.** Fix a set $M$ with a nonnegative integer $n \geq 0$ and a collection of functions $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ where $U_\alpha \subseteq M$ and $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ is open. Further, suppose the following.

(i) $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ is open for all $\alpha, \beta \in \kappa$.

(ii) The composite $\varphi_\alpha|_{U_\alpha\cap U_\beta} \circ \varphi_\beta|_{U_\alpha\cap U_\beta}^{-1}$ is smooth for all $\alpha, \beta \in \kappa$.

(iii) $M$ is covered by a countable subcollection of $\{U_\alpha\}_{\alpha\in\kappa}$.

(iv) For distinct $p, q \in M$, either there is $\alpha \in \kappa$ such that $p, q \in U_\alpha$, or there are disjoint $U_\alpha$ and $U_\beta$ containing $p$ and $q$, respectively.

Then $M$ is a smooth $n$-manifold with smooth atlas given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$.

*Proof.* We sketch the steps.

1. We provide $M$ with a topology. We would like for Well, we say that $A \subseteq M$ is open if and only if $\varphi_\alpha(A \cap U_\alpha)$ is open for all $\alpha \in \kappa$.
2. Then condition (i) makes the $\varphi_\alpha$ into homeomorphisms onto their images. Thus, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\kappa}$ is an atlas.
3. Condition (ii) implies that $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas.
4. Condition (iii) implies that $M$ becomes second countable.
5. Lastly, condition (iv) implies that $M$ is Hausdorff.

We leave the checks to the reader. ■

Let's see an example of this.

**Exercise 1.96.** Fix nonnegative integers $k \leq n$. Then let $M := \operatorname{Gr}_k(\mathbb{R}^n)$ denote the set of $k$-dimensional linear subspaces $V$ of $\mathbb{R}^n$. We show that $M$ is a smooth $k(n-k)$-manifold.

*Sketch.* We use Lemma 1.95. For concreteness, let we choose our index set $I$ to consist of pairs $(P, Q)$ of subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n = P \oplus Q$ and $\dim P = k$ and $\dim Q = n - k$. The point is that we are choosing a complement for our $k$-dimensional subspaces in order to help count them. In particular, we may define the subset

$$U_\alpha := \{V \in \operatorname{Gr}_k(\mathbb{R}^n) : V \cap Q = \{0\}\}.$$

Notably, for any $V \in U_\alpha$, there is a unique linear map $M_{P,Q,V}\colon P \to Q$ such that

$$V = \{x + M_{P,Q,V}x \in P \oplus Q : x \in P\}.$$

Approximately speaking, we are viewing $V$ as a graph. Anyway, this construction provides a map $\varphi_\alpha\colon U_\alpha \to \operatorname{Hom}_{\mathbb{R}}(P,Q)$ given by $V \mapsto M_{P,Q,V}$, where we identify $\operatorname{Hom}_{\mathbb{R}}(P,Q) \cong \mathbb{R}^{k(n-k)}$. We now conclude by noting that we can check the properties from Lemma 1.95. For example, to see that the transition maps are smooth, suppose we have two pairs $(P,Q),(P',Q') \in I$, and the vector space $V$ decomposes into the two separate ways, and these matrices have rational functions in their coordinates, so smoothness follows. As another example, one can actually cover $M$ by finitely many charts, and the last check follows because any $k$-dimensional subspaces $V,V' \subseteq \mathbb{R}^n$ has some $(n-k)$-dimensional subspace $Q \subseteq \mathbb{R}^n$ such that $V \cap Q = V' \cap Q = \{0\}$. ■

### 1.4.4 Manifolds with Boundary

Before moving on from our discussion of a single manifold, we discuss manifolds with boundary.

**Definition 1.97** (topological manifold with boundary)**.** Fix a nonnegative integer $n$. A *topological $n$-manifold with boundary* is a Hausdorff, second countable topological space $M$ with the following variant of being locally Euclidean: for any $p \in M$, there are open subsets $U \subseteq M$ and

$$\widehat{U} \subseteq \mathbb{H} := \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

such that $p \in U$ and $U \cong \widehat{U}$. We continue to call $(U,\varphi)$ a chart.

**Example 1.98.** Any topological $n$-manifold is a topological $n$-manifold with boundary: one can simply make the charts output to $\mathbb{H}^\circ$.

**Example 1.99.** The space $\mathbb{H}^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is a topological $n$-manifold with boundary.

The point is that we can pick up some "boundary" like the one in $\mathbb{H}^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Anyway, let's discuss smoothness. This requires understanding smoothness on $\partial\mathbb{H}^n$.

**Definition 1.100.** Fix a subset $A \subseteq \mathbb{R}^n$. A function $f\colon A \to \mathbb{R}^m$ is *smooth* if and only if there is an open subset $V \subseteq \mathbb{R}^n$ containing $A$ and a smooth extension $\widetilde{f}\colon V \to \mathbb{R}^n$ of $f$.

**Remark 1.101.** It turns out that (by Seeley's theorem) if $V \subseteq \mathbb{H}^n$ is open, it is enough to check that the partial derivatives of some function $f\colon V \to \mathbb{R}^m$ extend continuously to the boundary.

**Definition 1.102** (smooth manifold with boundary)**.** Fix a nonnegative integer $n$. A *smooth $n$-manifold with boundary* is a pair $(M,\mathcal{A})$ where $M$ is a topological $n$-manifold with boundary, and $\mathcal{A}$ is a maximal smooth atlas, where we are taking atlas in the sense

We will not bother to redo the proof of Proposition 1.76 to explain that the notion of a maximal smooth atlas makes sense with subsets of $\mathbb{H}^n$ in addition to subsets of $\mathbb{R}^n$; all the proofs are the same.

Note that boundary is in fact an intrinsic notion.

**Definition 1.103** (boundary, interior)**.** Fix a smooth $n$-manifold with boundary $M$ and a point $p \in M$.

- Then $p$ is a *boundary point* if and only if there is a smooth chart $(U,\varphi)$ such that $\varphi(p) \in \partial\mathbb{H}^n$.
- Then $p$ is an *interior point* if and only if there is a smooth chart $(U,\varphi)$ such that $\varphi(p)$ is in the interior of $\mathbb{H}^n$.

We will show in Theorem 1.104 that any point in $M$ is exactly one of a boundary point or an interior point.

## 1.5 January 30

Here we go.

### 1.5.1 Smooth Manifolds with Boundary

We would like for the boundary of a smooth manifold with boundary to make sense.

**Theorem 1.104.** Fix a smooth $n$-manifold with boundary $M$, and fix some $p \in M$. Given two charts $(U, \varphi)$ and $(V, \psi)$ with $p \in U \cap V$), then $\varphi(p) \in \partial\mathbb{H}^n$ if and only if $\psi(p) \in \partial\mathbb{H}^n$.

*Proof.* Suppose this is not the case. Then, up to rearranging, we get $\varphi(p) \in (\mathbb{H}^n)^\circ$ and $\psi(p) \in \partial\mathbb{H}^n$. Our transition maps are smooth, so we have produced a diffeomorphism from the open subsets $U' \subseteq \mathbb{H}^n$ and $V' \subseteq \mathbb{H}^n$ such that $U' \cap \partial\mathbb{H}^n = \varnothing$ but $V' \cap \partial\mathbb{H}^n \neq \varnothing$. Now, for smoothness, the transition map $\tau \colon V' \to U'$ must have an extension $\widetilde{\tau} \colon \widetilde{V}' \to \widetilde{U}'$. But then $\widetilde{\tau}$ is an invertible map, so the Inverse function theorem implies that $\tau$ is locally invertible and in particular must be an open map. But $V'$ goes to $U'$, which is not open in $\mathbb{R}^n$, so we have our contradiction. ■

**Remark 1.105.** In fact,

$$\psi \circ \varphi^{-1}|_{\partial\mathbb{H}^n \cap \varphi(U\cap V)} \colon (\partial\mathbb{H}^n \cap \varphi(U \cap V)) \to (\partial\mathbb{H}^n \cap \psi(U \cap V))$$

is a smooth transition map, though we will not check this here.

**Remark 1.106.** People in the modern day might allow $\partial M$ to be a manifold with boundary itself, which is a "manifold with corners."

**Remark 1.107.** One can remove the smoothness assumption here as well, but it will require some cohomology or similar.

The boundary/interior for a smooth manifold may not actually be its boundary/interior when embedded into a space.

**Example 1.108.** Consider $M := \{x \in \mathbb{R}^n : x_n > 0\}$. Then $M$ is a smooth manifold with boundary, but $\partial M = \{x \in \mathbb{R}^n : x_n = 0\}$ when viewed as a subset of $\mathbb{R}^n$.

**Example 1.109.** Consider $M = S^n \subseteq \mathbb{R}^{n+1}$. Then $M$ is a smooth manifold (without boundary), but as a subspace of $\mathbb{R}^{n+1}$, we have $\partial M = M$.

**Example 1.110.** Consider $M := \mathbb{H}^n \cap B(0,1)$. Then $M$ is a smooth manifold whose boundary (as a manifold) is $\partial\mathbb{H}^n \cap B(0,1)$, but the topological boundary is $\partial\mathbb{H}^n \cup (\partial B(0,1) \cap \mathbb{H}^n)$.

THEME 2

# MAPS BETWEEN MANIFOLDS

*I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial.*

—Andre Weil, [Wei59]

## 2.1 January 30-map

We continue.

### 2.1.1 Smooth Maps to $\mathbb{R}^n$

We will define smooth maps in steps. To begin, we say what it means to have a smooth map $M \to \mathbb{R}^n$. Basically, we look locally at the points on our manifold and check smoothness on charts.

**Definition 2.1** (smooth)**.** Fix a smooth manifold $M$, possibly with boundary. Then a function $f\colon M \to \mathbb{R}^m$ is *smooth* if and only if each $p \in M$ has some smooth chart $(U, \varphi)$ with $p \in U$ and

$$f|_U \circ \varphi|_U^{-1}$$

is a smooth map $\varphi(U) \to \mathbb{R}^m$.

**Example 2.2.** Any smooth map $f\colon U \to \mathbb{H}^m$, where $U \subseteq \mathbb{H}^n$ is open, is smooth in the above sense. Indeed, $U$ as an $n$-manifold has a smooth atlas given by $\{(U, \mathrm{id}_U)\}$, and this witnesses the smoothness of $f$ for any $p \in U$.

Here is a quick sanity check: the charts don't matter.

**Lemma 2.3.** Fix a smooth map $f\colon M \to \mathbb{H}^m$, where $M$ is a smooth manifold, possibly with boundary. For any smooth chart $(V, \psi)$, the composition $f|_U \circ \varphi|_U^{-1}$ is smooth.

*Proof.* This is a matter of tracking through all the definitions. Fix some $p \in V$, and we would to test smoothness around $p$. Well, $p$ has some smooth chart $(U, \varphi)$ such that $p \in U$ and $f|_U \circ \varphi|_U^{-1}$ is smooth. But now we write

$$f|_{U\cap V}\circ\psi|_{U\cap V}^{-1}=\left(f|_{U\cap V}\circ\varphi|_{U\cap V}^{-1}\right)\circ\left(\varphi|_{U\cap V}\circ\psi|_{U\cap V}^{-1}\right),$$

which is the composition of smooth maps: the left map is smooth by construction of $(U,\varphi)$, and the right map is smooth by compatibility of smooth charts. ■

We are now ready to define smooth maps between manifolds. Approximately speaking, we simply add in a check locally on the target.

**Definition 2.4** (smooth)**.** Fix smooth manifolds $M$ and $N$, possibly with boundary. A map $F\colon M\to N$ is *smooth* if and only if each $p\in M$ has smooth charts $(U,\varphi)$ and $(V,\psi)$ such that $p\in U$ and $F(U)\subseteq V$ and the composite

$$\psi\circ F|_U\circ\varphi|_U^{-1}$$

is a smooth map $\mathbb{H}^m\to\mathbb{H}^n$. We may call the above composite a *coordinate representation*.

**Example 2.5.** Any smooth map $F\colon U\to V$, where $U\subseteq\mathbb{R}^m$ and $V\subseteq\mathbb{R}^n$ are open, is smooth in the above sense. Indeed, $U$ and $V$ have smooth atlases given by $\{(U,\mathrm{id}_U)\}$ and $\{(V,\mathrm{id}_V)\}$ (respectively), and these charts witness that $F$ is smooth at each $p\in U$ because the composite

$$\mathrm{id}_V\circ F\circ{\mathrm{id}_U}^{-1}=F$$

is smooth by hypothesis.

Here's the same sanity check: the charts don't matter.

**Lemma 2.6.** Fix a smooth map $F\colon M\to N$ of manifolds, possibly with boundary. If $(U,\varphi)$ and $(V,\psi)$ are smooth charts on $M$ and $N$, respectively, and $F(U)\subseteq V$, then the composite $\psi\circ F|_U\circ\varphi|_U^{-1}$ is smooth.

*Proof.* Again, we track through locally, tracking through all the definitions. To check that $\psi\circ F|_U\circ\varphi|_U^{-1}$ is smooth, it suffices to check it an open cover of $\varphi(U)$. Pick $\varphi(p)\in\varphi(U)$ where $p\in U$, and we know that we have smooth charts $(U_p,\varphi_p)$ and $(V_p,\psi_p)$ in $M$ and $N$, respectively, such that $F(U_p)\subseteq V_p$ and the composite $\psi_p|_{F(U_p)}\circ F|_{U_p}\circ\varphi_p|_{U_p}^{-1}$ is smooth. Then we see that

$$\psi\circ F|_U\circ\varphi|_U^{-1}=\left(\psi|_{V\cap V_p}\circ\psi_p|_{V\cap V_p}^{-1}\right)\circ\left(\psi_p\circ F|_U\circ\varphi_p|_{U_p}^{-1}\right)|_{\varphi_p(U\cap U_p)}\circ\left(\varphi_p|_{U\cap U_p}\circ\varphi|_{U\cap U_p}^{-1}\right)$$

is smooth, where the left and right maps are smooth by smooth compatibility, and the middle map is smooth by construction. ■

**Remark 2.7.** One can write out the above proof diagrammatically by noting that having smooth charts $(U,\varphi)$ and $(U',\varphi')$ of $M$ and smooth charts $(V,\psi)$ and $(V',\psi')$ of $N$ such that $F(U)\subseteq V$ and $F(U')\subseteq V'$ will have the following diagram.

$$\begin{tikzcd}
\varphi(U) \arrow[r, "\psi\circ F\circ\varphi^{-1}"] \arrow[d, leftrightarrow] & \psi(V) \arrow[d, leftrightarrow] \\
\varphi'(U') \arrow[r, "\psi'\circ F\circ(\varphi')^{-1}"'] & \psi'(V')
\end{tikzcd}$$

Here, the vertical maps are only defined on the corresponding intersections, but it is smooth when defined by the smooth compatibility.

**Remark 2.8.** Please read more of chapter 2 to get helpful properties of smooth maps.

### 2.1.2 Partition of Unity

By way of motivation, suppose we have two smooth functions $f, g\colon \mathbb{R} \to \mathbb{R}$, and we want to build a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ such that $f|_{(-\infty,-1)} = h|_{(-\infty,-1)}$ and $g|_{(1,\infty)} = h|_{(1,\infty)}$. One way to do this is to find smooth functions $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \varphi|_{(-\infty,-1)} = 1, \\ \varphi|_{(1,\infty)} = 0. \end{cases}$$

Then $h \coloneqq \varphi f + (1-\varphi)g$ is smooth by construction, and it satisfies the restriction conditions also by construction. This idea of "splitting up the $1$ function" is known as partition of unity.

**Definition 2.9** (partition of unity)**.** Fix a topological space $X$, and let $\{U_\alpha\}_{\alpha\in\kappa}$ be an open cover on $M$. Then a *partition of unity subordinate to* $\{U_\alpha\}_{\alpha\in\kappa}$ is a collection of continuous functions $\{\varphi_\alpha\}_{\alpha\in\kappa}$ on $X$ satisfying the following.

- $\operatorname{im}\varphi_\alpha \subseteq [0,1]$ always.
- $\operatorname{supp}\varphi_\alpha \subseteq U_\alpha$ for each $\alpha$.
- The collection $\{\operatorname{supp}\varphi_\alpha\}_{\alpha\in\kappa}$ is locally finite.
- For each $x \in X$, we have
$$\sum_{\alpha\in\kappa} \varphi_\alpha(x) = 1.$$

Of course, we must show that these exist.

## 2.2 February 1

The second homework is due later today. We began class by completing a proof, so I edited directly into those notes.

### 2.2.1 Partition of Unity for Manifolds

We will show that partitions of unity exist for manifolds.

**Theorem 2.10.** Fix a smooth manifold $M$. For any open cover $\{U_\alpha\}_{\alpha\in\kappa}$, there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in\kappa}$ (of smooth functions) subordinate to $\{U_\alpha\}_{\alpha\in\kappa}$.

*Proof.* We begin by constructing smooth functions $\{\widetilde{\varphi}\}_{\alpha\in\kappa}$ satisfying the following constraints.

- $\operatorname{im}\widetilde{\varphi}_\alpha \subseteq [0,\infty)$.
- $\operatorname{supp}\widetilde{\varphi}_\alpha \subseteq U_\alpha$.
- The collection $\{\operatorname{supp}\widetilde{\varphi}_\alpha\}_{\alpha\in\kappa}$ is a locally finite open cover of $M$.

Dividing out by the summation of the $\widetilde{\varphi}_\bullet$s completes the proof. Notably, for each $x \in M$, the sum

$$\widetilde{\varphi}(x) \coloneqq \sum_{\alpha\in\kappa} \widetilde{\varphi}_\alpha(x)$$

is finite ($x$ can only belong to finitely many of the supports); in fact, there is an open neighborhood $U$ of $x$ such that $U$ only intersects finitely many of the supports, so

$$\widetilde{\varphi}|_U = \sum_{\substack{\alpha\in\kappa \\ \operatorname{supp}\widetilde{\varphi}_\alpha\cap U\neq\varnothing}} \widetilde{\varphi}_\alpha|_U$$

is just a finite sum of smooth functions, so $\widetilde{\varphi}$ is smooth on $U$. Thus, by gluing, $\widetilde{\varphi}$ is smooth on $M$, and we note that it is nonzero because each $x \in M$ is in some support, so we can define $\varphi_\alpha := \widetilde{\varphi}_\alpha/\varphi$ to satisfy all the needed conditions, most notable being that these functions are smooth, have support contained in $U_\alpha$, and $\sum_{\alpha\in\kappa}\varphi_\alpha = 1$.

It remains to construct the $\widetilde{\varphi}_\alpha$s. We proceed in steps.

1. We construct a nice open cover. For each $x \in M$, we can find some open neighborhood $U$ such that we have a homeomorphism $\varphi\colon U \to B(\varphi(x), 2)$. Then $\left\{\varphi^{-1}(B(\varphi(x), 1))\right\}_{x\in M}$ is an open cover of $M$, so we can refine this to a locally finite open cover $\mathcal{U}$ of precompact open sets. By looking down on compact, we may as well assume that $\mathcal{U}$ is made of coordinate balls $B(\varphi(x), r)$ contained in larger coordinate balls $B(\varphi(x), r')$ for $r' > r$.

2. Now, for each coordinate ball $\varphi\colon U \cong B(0,r)$ for $U \in \mathcal{U}$ extending to $\varphi'\colon U' \cong B(0,r')$. Then we construct $f_U$ which is nonzero on $B(0,r)$ but vanishes on $B(0,r')$.

   Now, for each $U \in \mathcal{U}$, select $\alpha_U \in \kappa$ such that $\overline{U} \subseteq U_{\alpha_U}$. From here, we may set

$$\widetilde{\varphi}_\alpha := \sum_{\alpha_U=\alpha} f_U,$$

   which satisfies all the needed conditions. For example, one finds that the support of $\widetilde{\varphi}_\alpha$ is

$$\overline{\bigcup_{U\subseteq U_\alpha} U} \subseteq \bigcup_{U\subseteq U_\alpha} \overline{U} \subseteq U.$$

   One needs local finiteness in order to verify the first inclusion; the point is that one can reduce this large union to a finite one around any given point, so the closures must agree. ■

Let's give some applications.

**Corollary 2.11.** Fix a smooth manifold $M$. For any closed set $A \subseteq M$ contained in an open set $U \subseteq M$, there exists a smooth function $\psi\colon M \to \mathbb{R}$ such that $\psi|_A = 1$ and $\psi|_{M\setminus U} = 0$.

*Proof.* Consider the open cover $\{U, M \setminus A\}$; this is an open cover because $U \cup (M \setminus A) = M$ is equivalent to $A \subseteq U$. Then Theorem 2.10 produces two nonnegative smooth functions $\psi_0$ and $\psi_1$ such that $\operatorname{supp}\psi_0 \subseteq U$ and $\operatorname{supp}\psi_1 \subseteq M \setminus A$ and $\psi_0 + \psi_1 = 1$ everywhere. But now $\psi_0$ is the desired function: $\operatorname{supp}\psi_0 \subseteq U$ implies $\psi_0|_{M\setminus U}$, and $\psi_0|_A + \psi_1|_A = 1$, but $\psi_1|_A = 0$ because $\operatorname{supp}\psi_1 \subseteq M \setminus A$. ■

**Corollary 2.12** (Extension lemma)**.** Fix a smooth manifold $M$. Further, fix a closed subset $A \subseteq M$ contained in an open set $U \subseteq M$. Given a smooth function $f\colon A \to \mathbb{R}^k$, there is a smooth function $\widetilde{f}\colon M \to \mathbb{R}^k$ extending $f$ and with $\operatorname{supp}\widetilde{f} \subseteq U$.

*Proof.* Omitted. ■

**Corollary 2.13.** Fix a smooth manifold $M$. There is a nonnegative function $f\colon M \to \mathbb{R}$ such that all the sets

$$f^{-1}([0,c])$$

are compact for any $c \geq 0$.

*Proof.* Fix a countable cover $\{U_n\}_{n\in\mathbb{N}}$ of $M$ by precompact open subsets, and let $\{\psi_n\}_{n\in\mathbb{N}}$ be the corresponding partition of unity. Then we set

$$f := \sum_{n=0}^{\infty} n\psi_n.$$

Notably, for each $c \in \mathbb{R}$, we see

$$f^{-1}([0,c]) \subseteq \bigcup_{n\leq c} \operatorname{supp}\psi_n,$$

so $f^{-1}([0,c])$ is a closed subset of a finite union of compact sets (which is compact), so we are done. ■

**Corollary 2.14.** Fix a closed subset $K$ of a smooth manifold $M$. Then there is a nonnegative smooth function $f\colon M \to \mathbb{R}$ such that $f^{-1}(\{0\}) = K$.

*Proof.* One begins with $M = \mathbb{R}^n$ and then does the general case from there. ■

### 2.2.2 Diffeomorphisms

Here is our definition.

**Definition 2.15** (diffeomorphism)**.** Fix a map $F\colon M \to N$ of smooth manifolds, possibly with boundary. Then $F$ is a *diffeomorphism* if and only if $F$ is bijective, smooth, and has smooth inverse.

**Remark 2.16.** Invariance of the boundary under smooth charts implies $F$ must send boundary points to boundary points.

**Remark 2.17.** If $F$ is a diffeomorphism, then $\dim M = \dim N$. Simply put, we can work locally on a chart, and then we are providing a diffeomorphism $\mathbb{R}^m \to \mathbb{R}^n$, but this can only happen when $m = n$. For example, it means that $DF$ and $DF^{-1}$ are invertible linear maps $\mathbb{R}^m \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively, which manifestly requires $m = n$.

**Remark 2.18.** It turns out that topological $n$-manifolds with a smooth structure admit a unique smooth structure up to diffeomorphism, for $n \geq 3$. For $n \geq 4$, even $\mathbb{R}^4$ fails to have a unique smooth structure.

**Remark 2.19.** The collection $\operatorname{Diff}(M)$ of diffeomorphisms $M \to M$ is a group, and one can give it a topology. For example, one can compute that $\operatorname{Diff}\left(S^2\right)$ is homotopy equivalent to $O(3)$, given approximately by rotations.

### 2.2.3 Tangent Spaces

Fix a smooth $n$-manifold $M$. One would like to provide each point $p \in M$ with an $n$-dimensional tangent vector space $T_pM$. If $M$ is embedded into Euclidean space reasonably, we can imagine using the embedding to realize the tangent space; for example, if $M$ is a (smooth) curve in $\mathbb{R}^2$, we can imagine that the tangent vectors tell us what direction we are moving in. We would also like to actually be able to compute these things in charts.

Anyway, here is our definition of tangent vectors. This definition is a bit awkward to handle because we want to be invariant.

**Definition 2.20** (tangent space)**.** Fix a smooth $n$-manifold $M$ and some point $p \in M$. A *derivation at* $p$ is an $\mathbb{R}$-linear map $v\colon C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule

$$v(fg) = f(p)v(g) + g(p)v(f)$$

for any $f, g \in C^\infty(M)$. Then the *tangent space* $T_p(M)$ *at* $p$ is the collection of derivations.

**Remark 2.21.** Note that $T_p(M)$ is an $\mathbb{R}$-subspace of the collection of linear maps $C^\infty(M) \to \mathbb{R}$.

**Example 2.22.** Fix $M := \mathbb{R}^n$ and some $p \in M$. Then any $v \in \mathbb{R}^n$ has a "directional derivative" given by

$$f \mapsto \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_p.$$

This is simply by the product rule in multivariable calculus.

## 2.3 February 6

Today we continue talking about tangent vectors.

### 2.3.1 Derivations

Let's provide some basic properties of derivations.

**Lemma 2.23.** Fix a smooth $n$-manifold $M$ and a derivation $v\colon C^\infty(M) \to \mathbb{R}$ at a point $p \in M$. If $f\colon M \to \mathbb{R}$ is constant, then $v(p) = 0$.

*Proof.* By scaling, it suffices to do the case where $f \equiv 1$. Then we see that $f^2 = f$, so

$$v(f) = v\left(f^2\right) = 2f(p)v(f) = 2v(f),$$

so $v(f) = 1$ is forced. ■

**Lemma 2.24.** Fix a smooth $n$-manifold $M$ and a derivation $v\colon C^\infty(M) \to \mathbb{R}$ at a point $p \in M$. Given $f, g \in C^\infty(M)$ such that $f|_U = g|_U$ for some open $U \subseteq M$ containing $p$, we have $v(f) = v(g)$.

*Proof.* Set $h := f - g$ so that we want to show $v(h) = 0$ by linearity. The moral of the story is to extend being zero on $U$ to all of $M$; in other words, we will want some bump functions. Because $M$ is locally Euclidean, we can find a precompact open neighborhood $V$ of $p$ such that $\overline{V} \subseteq U$. Thus, Corollary 2.11 provides a smooth bump function $\psi\colon M \to \mathbb{R}$ such that $\psi|_{\overline{V}} \equiv 1$, and $\operatorname{supp}\psi \subseteq U$. Notably, $\psi \cdot h$ has support contained in $U$, but $h$ vanishes on $U$, so $\psi \cdot h = 0$, so

$$0 = v(\psi \cdot h) = \psi(p)v(h) + h(p)v(\psi) = v(h),$$

as desired. ■

Manifolds are understood by passing to local charts, and the above lemma somewhat allows us to do this. As such, we are now motivated to understand local charts.

**Lemma 2.25.** Fix a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$. For each $v \in \mathbb{R}^n$, define $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ by

$$D_v|_a(f) := \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_a.$$

Then $D_v|_a$ is a derivation at $a$. In fact, the map $D\colon \mathbb{R}^n \to T_a\mathbb{R}^n$ given by $v \mapsto D_v|_a$ is an isomorphism of vector spaces.

*Proof.* To check that $D_v|_a$ is a derivation, one proceeds via the product rule in multivariable calculus. We omit this check. It remains to check that we have an isomorphism.

- Linear: given $c, d \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, we compute

  $$D_{cv+dw}|_a f = \sum_{i=1}^n (cv_i + dw_i)\frac{\partial f}{\partial x_i}\bigg|_a = c\sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\bigg|_a + d\sum_{i=1}^n w_i \frac{\partial f}{\partial x_i}\bigg|_a = (cD_v|_a + dD_v|_a)f,$$

  as desired.

- Injective: by linearity, it is enough to show that having $D_v|_a = 0$ implies $v = 0$. Well, it is enough to check that $v_j = 0$ for each $j$. For this, we let $p_j : \mathbb{R}^n \to \mathbb{R}$ denote the $j$th projection so that

  $$\frac{\partial p_j}{\partial x_i} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases}$$

  so we see that

  $$D_v|_a(p_j) = \sum_{i=1}^n v_i \frac{\partial p_j}{\partial x_i}\bigg|_a = v_j$$

  must vanish for each $j$, as desired.

- Surjective: this is the heart of the matter. Fix a derivation $v \in T_a\mathbb{R}^n$. We need a candidate vector, so we define $u_i := v(p_i)$, where $p_i : \mathbb{R}^n \to \mathbb{R}$ is the $i$th projection. We claim that

  $$v = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}\bigg|_a,$$

  which will complete the proof. This requires a quick digression into a Taylor expansion. Given a smooth function $f\colon M \to \mathbb{R}$ and points $x, a \in \mathbb{R}^n$, we see

  $$\begin{aligned} f(x) &= f(a) + \int_0^1 \frac{d}{dt} f(x + t(x-a))\,dt, \\ &= f(a) + \sum_{i=1}^n \left( (x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\,dt}_{h_i(x):=} \right), \end{aligned}$$

  where in the last equality we have used the multivariable chain rule. Applying the derivation, we see

  $$v(f) = \underbrace{v(f(a))}_{0} + \sum_{i=1}^n v(x_i - a_i)h_i(a) + \sum_{i=1}^n \underbrace{(a_i - a_i)}_{0} v(h_i),$$

  where $v(f(a)) = 0$ by Lemma 2.23. Additionally, $v(x_i - a_i) = v(x_i) = u_i$ using Lemma 2.23 again. Notably, $h_i(a) = \frac{\partial f}{\partial x_i}|_a$, so

  $$v(f) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}\bigg|_a,$$

  as desired. ■

### 2.3.2 Differentials of Smooth Maps

Derivations explain how to take derivatives of functions in $M \to \mathbb{R}$. We now upgrade to taking derivatives of functions between manifolds.

**Definition 2.26** (differential)**.** Fix smooth manifolds $M$ and $N$. Given a smooth map $F\colon M \to N$, the *differential of $F$ at $p \in M$* is the map $dF_p\colon T_pM \to T_{F(p)}N$ defined by

$$dF_p(v)(f) := v(f \circ F)$$

for any $f \in C^\infty(N)$.

**Remark 2.27.** The composition of smooth functions is smooth, so $f \circ F$ is smooth, so the definition of $dF_p$ at least makes sense. Notably, $f \mapsto (f \circ F)$ is a map $C^\infty(N) \to C^\infty(M)$ of $\mathbb{R}$-algebras, so $f \mapsto v(f \circ F)$ remains a derivation. Explicitly, it is surely $\mathbb{R}$-linear (as the composition of $\mathbb{R}$-linear maps), and we satisfy the Leibniz rule because

$$\begin{aligned} dF_pv(fg) &= v((fg) \circ F) \\ &= v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))dF_pv(g) + g(F(p))dF_pv(f). \end{aligned}$$

**Remark 2.28.** The map $dF_p\colon T_pM \to T_{F(p)}N$ is linear, essentially by definition. Namely, for $a, b \in \mathbb{R}$ and $v, w \in T_pM$ and $f \in C^\infty(N)$, we compute

$$dF_p(av + bw)(f) = (av + bw)(f \circ F) = av(f \circ F) + bw(f \circ G) = (adF_p(v) + bdF_p(w))(f).$$

**Example 2.29.** Take $M := \mathbb{R}^m$ and $N := \mathbb{R}^n$, and let $F\colon M \to N$ be a smooth map, which we may as well write as $F = (F_1, \ldots, F_n)$. Now, fix some $p \in M$, and identify $\mathbb{R}^m \cong T_pM$ and $\mathbb{R}^n \cong T_{F(p)}N$ as in Lemma 2.25. Well, given some smooth $f\colon N \to \mathbb{R}$, we see

$$dF_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(f) = \frac{\partial}{\partial x_i}(f \circ F) \overset{*}{=} \sum_{j=1}^n \left.\frac{\partial f}{\partial y_j}\right|_{F(p)} \left.\frac{\partial F_j}{\partial x_i}\right|_p = \left(\sum_{j=1}^n \left.\frac{\partial F_j}{\partial x_i}\right|_p \cdot \left.\frac{\partial}{\partial y_j}\right|_{F(p)}\right)(f),$$

where the main point is the application of the Chain rule in $\overset{*}{=}$.

**Remark 2.30.** Differentials behave under composition. Explicitly, let $F_1\colon M_1 \to M_2$ and $F_2\colon M_2 \to M_3$ be smooth maps. Given $p \in M_1$, we claim that

$$d(F_2 \circ F_1)_p \overset{?}{=} (dF_2)_{F_1(p)} \circ (dF_1)_p.$$

This can be checked directly.

**Example 2.31.** Fix a point $p$ on a smooth $n$-manifold $M$. Then we claim $d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}$. Indeed, we simply compute

$$d(\mathrm{id}_M)_p(v)(f) = v(f \circ \mathrm{id}_M) = v(f).$$

### 2.3.3 Back to Tangent Spaces

Now that we understand how to take differentials of maps, we may realize the remark that derivations ought to be understood locally, as alluded to in Lemma 2.24.

**Proposition 2.32.** Fix a smooth $n$-manifold $M$. Given an open neighborhood $U$ of a point $p \in M$, the inclusion $i \colon U \hookrightarrow M$ is smooth, and $di_p \colon T_pU \to T_pM$ is an isomorphism of vector spaces.

*Proof.* Remark 2.28 tells us that this map is linear. It remains to check injectivity and surjectivity, which we do by hand.

- Injective: if $di_p(v) = 0$, then $v(f \circ i) = 0$ for all $f \in C^\infty(M)$, or equivalently, $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We would now like to show that $v$ is actually zero. Well, pick up some $g \in C^\infty(U)$, and we want to show that $v(g) = 0$.

  Well, choose some open precompact open neighborhood $B$ around $p$ such that $\overline{B} \subseteq U$. Then Corollary 2.11 provides us with a smooth bump function $\psi \colon M \to \mathbb{R}$ which is $1$ on $\overline{B}$ and vanishes outside $U$. Then $g\psi$ is actually smooth (it is smooth on $U$ because $g$ and $\psi$ are both smooth there, and it is smooth outside $U$ because the function is zero there), so $v(g\psi|_U) = 0$. But $g\psi$ and $g$ agree on $B$, so $v(g) = v(g\psi|_U)$ by Lemma 2.24, as needed.

- Surjective: fix some derivation $\widetilde{v} \in T_pM$, and we want some $v \in T_pU$ such that $\widetilde{v}(f) = v(f|_U)$ for all $f \in C^\infty(M)$. The main point is the construction of $U$.

  Given a smooth function $f \in C^\infty(U)$, we want to define $\widetilde{v}(f)$. Well, as in the previous step, we may define $\widetilde{f} \colon M \to \mathbb{R}$ such that there is an open neighborhood $B \subseteq U$ of $p$ with $f|_B = \widetilde{f}|_B$. Then we define $v(f) := \widetilde{v}(\widetilde{f})$. Note that $v(\widetilde{f})$ does not depend on the choice of $\widetilde{f}$ and $B$: well, given another pair of $\widetilde{f}'$ and $B'$, we see that $\widetilde{f}|_{B \cap B'} = \widetilde{f}|_{B \cap B'}$, so they have the same value of $\widetilde{v}$ under Lemma 2.24.

  Additionally, we note that $v$ is in fact a derivation: given $f, g \in C^\infty(U)$ and smooth extensions $\widetilde{f}, \widetilde{g} \in C^\infty(M)$ agreeing on $B_f, B_g \subseteq M$, respectively, we see

  $$\widetilde{v}(\widetilde{f}\widetilde{g}) = \widetilde{f}(p)\widetilde{v}(\widetilde{g}) + \widetilde{g}(p)\widetilde{v}(\widetilde{f})$$

  because $\widetilde{v}$ is a derivation, but then this immediately produces $v(fg) = f(p)v(g) + g(p)v(f)$ by checking the definitions. Similarly, we have

  $$\widetilde{v}(a\widetilde{f} + b\widetilde{g}) = a\widetilde{v}(\widetilde{g}) + b\widetilde{v}(\widetilde{g}),$$

  so $v(af + bg) = av(f) + bv(g)$, so $v$ is linear.

  Lastly, we note that $\widetilde{v}(f) = v(f|_U)$ for any $f \in C^\infty(M)$ by construction. Namely, $f$ is a perfectly fine extension of $f|_U$ agreeing on some open neighborhood of $p$ contained in $U$ (for example, taking $U$ to be the needed open neighborhood itself will work), so we conclude. ■

**Corollary 2.33.** Fix a smooth $n$-manifold $M$. For any $p \in M$, we have $\dim_{\mathbb{R}} T_pM = n$.

*Proof.* Fix a smooth chart $(U, \varphi)$ around $p \in M$. Then we have the sequence of isomorphisms

$$T_pM \cong T_pU \cong T_{\varphi(p)}\varphi(U) \cong T_p\mathbb{R}^n \cong \mathbb{R}^n.$$

The first and third isomorphisms are by Proposition 2.32. The second isomorphism is by functoriality of the tangent space from Remark 2.30 and Example 2.31; namely, the differential of a diffeomorphism must be an isomorphism by functoriality. And the last isomorphism is by Lemma 2.25. ■

While we're here, we take a moment to understand how these derivations behave under coordinates.

> **Remark 2.34.** Please read about how to provide the differential of a smooth map on coordinates.

So here are some coordinate computations.

- Fix a smooth $n$-manifold $M$ and a point $p \in M$. Given a smooth chart $(U, \varphi)$ around $p$, we give $\varphi$ its coordinates $\varphi := (x_1, \ldots, x_n)$. For example, given $f \in C^\infty(U)$, we are able to define

$$\frac{\partial}{\partial x_i}\bigg|_p f := \frac{\partial f}{\partial \widetilde{x}_i}\bigg|_{\varphi(p)} \left(f \circ \varphi^{-1}\right),$$

  where $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are the coordinates of $M$. By tracking the isomorphisms of Corollary 2.33 through, we can see that the above derivations form a basis for $T_pM$. Indeed, it suffices to show that they are a basis for the derivations on $T_pU$, and by passing through $\varphi$, it is enough to see that $\partial f/\partial\widetilde{x}_i|_{\varphi(p)}$ form a basis of derivations on $T_{\varphi(p)}U$. But it's now enough to see that we have a basis on $T_p\mathbb{R}^n$, which is simply Lemma 2.25.

- We examine change of coordinates. Fix a smooth $n$-manifold $M$ and a point $p \in M$ covered by the charts $(U, \varphi)$ and $(V, \psi)$. As above, we give coordinates as $\varphi := (x_1, \ldots, x_n)$ and $\psi := (y_1, \ldots, y_n)$, and we give the target spaces the coordinates $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$, respectively.

  Well, on the restrictions, we will choose coordinate representations by

$$(\psi \circ \varphi^{-1})(\widetilde{x}) := (\overline{y}_1(\widetilde{x}), \ldots, \overline{y}_n(\widetilde{x})),$$

  and we in particular see that

$$\begin{aligned}
\frac{\partial}{\partial y_j}\bigg|_p y_k &= \left(\left(d\psi^{-1}\right)_{\psi(p)} \frac{\partial}{\partial \widetilde{y}_j}\bigg|_{\psi(p)}\right) y_k \\
&= \frac{\partial}{\partial \widetilde{y}_j}\bigg|_{\psi(p)} \left(y_k \circ \psi^{-1}\right) \\
&= \frac{\partial}{\partial \widetilde{y}^j}\bigg|_{\psi(p)} \widetilde{y}_k \\
&= 1_{j=k}.
\end{aligned}$$

  The moral of the story is that some $v = \sum_{k=1}^m v_k \partial/\partial y_k|_p$ will have

$$\frac{\partial}{\partial x_i}\bigg|_p = \sum_{k=1}^n \frac{\partial \overline{y}_k}{\partial \widetilde{x}_i}\bigg|_{\varphi(p)} \frac{\partial}{\partial y_j}\bigg|_p.$$

## 2.4 February 8

Here we go.

### 2.4.1 Velocity Vectors

Let's discuss a more geometric variant of tangent vectors.

> **Definition 2.35** (velocity vector)**.** Fix a smooth $n$-manifold $M$ and a point $p \in M$. Define the space $\mathcal{J}_pM$ to be the set of smooth curves $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ (and $\varepsilon > 0$). We say that $\gamma_1, \gamma_2 \in \mathcal{J}_p$ are equivalent, written $\gamma_1 \sim \gamma_2$, if and only if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for any $f \in C^\infty(M)$.

**Remark 2.36.** We won't bother checking that $\sim$ is an equivalence relation; it holds because we are basically checking equalities after passing to $\mathbb{R}^{C^\infty(M)}$ by sending $\gamma \mapsto ((f \circ \gamma)'(0))_f$.

And here is how this relates to tangent vectors.

**Lemma 2.37.** Fix a smooth $n$-manifold $M$ and a point $p \in M$. Then $T_pM$ is in natural bijection with $\mathcal{J}_pM/\sim$.

*Proof.* In one direction, one can send some $[\gamma] \in (\mathcal{J}_pM/\sim)$ to the derivation $v_{[\gamma]}\colon f \mapsto (f \circ \gamma)'(0)$. Note that this only depends on the class $[\gamma]$ rather than the representative $\gamma$ by definition of the equivalence relation $\sim$. This map is injective essentially by construction, and one can show by hand that it is surjective, for example by working locally on charts and then using lines as the needed curve to realize a differential in $T_pM$. ■

### 2.4.2 The Tangent Bundle

Let's glue our tangent spaces together.

**Remark 2.38.** Given $p, q \in \mathbb{R}^n$, there is a natural identification $T_p\mathbb{R}^n \to T_q\mathbb{R}^n$. One can see this on velocity vectors by moving the curves over by hand. Alternatively, let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation sending $T\colon p \mapsto q$, which is a diffeomorphism, and then we know we have an isomorphism $dT_p\colon T_p\mathbb{R}^n \to T_q\mathbb{R}^n$. (Recall functoriality of $T_p$ implies that diffeomorphisms produce isomorphisms.)

In general, it is somewhat difficult to identify these tangent spaces naturally.

**Definition 2.39** (tangent bundle)**.** Fix a smooth $n$-manifold $M$. Then *tangent bundle* $TM$ is

$$TM := \bigsqcup_{p \in M} T_pM.$$

Morally, $TM$ consists of all the tangent spaces glued together.

**Proposition 2.40.** Fix a smooth $n$-manifold $M$. Then $TM$ is a smooth $2n$-manifold.

*Proof.* We will use Lemma 1.95. Quickly, note that we have a projection $\pi\colon TM \to M$ given by $\pi(p, v) := p$.

Now, for each smooth chart $(U, \varphi)$ on $M$, we define the chart $\left(\pi^{-1}U, \widetilde{\varphi}\right)$ on $TM$, where $\widetilde{\varphi}\colon \pi^{-1}U \to (\operatorname{im}\varphi) \times \mathbb{R}^n$ is defined by

$$\widetilde{\varphi}\colon \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\bigg|_p \mapsto (\varphi(p), (v_1, \ldots, v_n)).$$

Recall $(\partial/\partial x_i)|_p = d\varphi^{-1}_{\varphi(p)}(\partial/\partial \widetilde{x}_i)$, where $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are coordinates chosen on is We now have to check our various conditions. For example, $\widetilde{\varphi}$ is a bijection to an open subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{n+1}$ by construction.

(i) Given two $(U, \varphi)$ and $(V, \psi)$, we need $\widetilde{\varphi}\left(\pi^{-1}U \cap \pi^{-1}V\right)$ to be open in $\mathbb{R}^{2n}$. But this is $\widetilde{\varphi}\left(\pi^{-1}(U \cap V)\right)$, which is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ because $(U \cap V, \varphi|_{U\cap V})$ is a smooth chart on $M$, so the argument above applies.

(ii) Given two $(U, \varphi)$ and $(V, \psi)$, we need the composite $\widetilde{\varphi} \circ \widetilde{\psi}^{-1}$ to be smooth, when suitably restricted. Well, one simply commutes the change-of-coordinates for the part on the tangent spaces, and on points, we simply use that $\varphi \circ \psi^{-1}$ is smooth already. Explicitly, one finds that this is

$$(\widetilde{x}, v) \mapsto \left(\left(\varphi \circ \psi^{-1}\right)(\widetilde{x}), \sum_{i=1}^n v_i \frac{\partial \widetilde{y}_\bullet}{\partial \widetilde{x}_i} \frac{\partial}{\partial \widetilde{y}_\bullet}\right).$$

(iii) A countable cover of $M$ by charts produces a countable cover of $TM$ by charts upon pulling back by $\pi$.

(iv) Fix distinct $(p,v),(q,w)\in TM$. If $p\neq q$, then we can find disjoint smooth charts $(U,\varphi)$ and $(V,\psi)$ on $M$, so $\left(\pi^{-1}U,\widetilde{\varphi}\right)$ and $\left(\pi^{-1}V,\widetilde{\psi}\right)$ provided the needed disjoint charts. Otherwise, $p=q$, and then $p$ and $q$ are of course contained in the same chart $(U,\varphi)$, so $(p,v)$ and $(q,w)$ are contained in the same chart $\left(\pi^{-1}U,\widetilde{\varphi}\right)$. ■

**Example 2.41.** One has $T\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n$.

**Example 2.42.** One has $TS^1=S^1\times\mathbb{R}$ and $TS^3=S^3\times\mathbb{R}^3$ and even $TS^7=S^7\times\mathbb{R}^7$.

**Example 2.43.** For even $n$, one has $TS^n\neq S^n\times\mathbb{R}^n$, which is essentially a consequence of the Hairy ball theorem: one would be able to produce $n$ linearly independent elements of $S^n\times\mathbb{R}^n$ and then pull them back to $n$ linearly independent vector fields $TS^n$, which do not exist for even $n$. The same inequality holds for odd $n\notin\{1,3,7\}$.

### 2.4.3 Maps of Constant Rank

We are going to want some inverse function theorems. Here is the most basic case. Morally, the statement is that invertible derivative should mean locally invertible.

**Theorem 2.44 (Inverse function).** Fix a smooth function $f\colon\mathbb{R}^n\to\mathbb{R}^n$. Given $x_0\in\mathbb{R}^n$, if the map $(Tf)_{x_0}\colon T_{x_0}\mathbb{R}^n\to T_{f(x_0)}\mathbb{R}^n$ is invertible, then there is an open neighborhood $U\subseteq\mathbb{R}^n$ around $x_0$ such that $f|_U$ is a diffeomorphism.

By working on charts, the following result is basically immediate.

**Theorem 2.45 (Inverse function).** Fix a smooth function $f\colon M\to N$ of $n$-manifolds. Given $x_0\in\mathbb{R}^n$, if the map $(Tf)_{x_0}\colon T_{x_0}M\to T_{f(x_0)}N$ is invertible, then there is an open neighborhood $U\subseteq M$ around $x_0$ such that $f|_U$ is a diffeomorphism.

This condition is good enough to make into a definition.

**Definition 2.46.** Fix a smooth function $F\colon M\to N$ of $n$-manifolds. Then $F$ is a *local diffeomorphism* at $p$ if and only if $dF_p$ is invertible. Equivalently, by Theorem 2.45, there is an open neighborhood $U$ of $p$ such that $F|_U$ is a diffeomorphism onto its image.

**Remark 2.47.** Of course, the converse direction (local diffeomorphism implies invertible derivative) is just by functoriality of the tangent space construction.

**Remark 2.48.** By gluing, if $F$ has invertible derivative at all points, and $F$ is a bijection, then one can see that $F^{-1}$ must be locally a diffeomorphism at all points, so in particular $F^{-1}$ is smooth, so $F$ is fully a diffeomorphism.

**Example 2.49.** The map $F\colon\mathbb{R}\to S^1$ given by $x\mapsto(\cos x,\sin x)$ is not injective, but it is a local diffeomorphism.

More generally, one could require something weaker than full invertibility.

**Definition 2.50** (immersion, submersion, full rank, constant rank)**.** Fix a map $F\colon M\to N$ of smooth manifolds, where $m:=\dim M$ and $n:=\dim N$.

- $F$ is an *immersion* if and only if $dF_p$ is injective for all $p\in M$.
- $F$ is a *submersion* if and only if $dF_p$ is surjective for all $p\in M$.
- $F$ has *full rank* if and only if $\operatorname{rank} dF_p=\min\{m,n\}$ for all $p\in M$ (notably, this is as large as possible).
- $F$ has *constant rank* if and only if $dF_p$ has the same rank for all $p\in M$ (notably, this is as large as possible).

We now state the following theorem.

**Theorem 2.51.** Fix a map $F\colon M\to N$ of smooth manifolds. If $dF_p$ has full rank for some $p\in M$, then there is an open neighborhood $U$ of $p$ such that $F|_U$ has full rank.

*Proof.* The condition that $dF_p$ having full rank is equivalent to the determinant of some largest submatrix being nonzero. So one has a map $M\to\mathbb{R}^N$ for some large $N$ taking $p\in M$ to the list of determinants of these submatrices of $dF_p$, and this map is continuous, so the set of points not going to zero is open and contains $p$. ■

**Example 2.52.** Fix two manifolds $M$ and $N$, and fix some $y_0\in N$.

- The map $x\mapsto(x,y_0)$ is an immersion.
- The projection map $M\times N\to M$ is a submersion.

**Example 2.53.** Fix a smooth curve $\gamma\colon\mathbb{R}\to\mathbb{R}^2$ with non-vanishing derivative everywhere. Then $\gamma$ is an immersion.

# BIBLIOGRAPHY


[Jac90] Joseph Jacobs. *English Fairy Tales*. English Fairy Tales. David Nutt, 1890.

[Wei59] Andre Weil. “Correspondence”. In: *Annals of Mathematics* 69.1 (1959), pp. 247–251.

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[use14] user149792. *Is an open $n$-ball homeomorphic to $\mathbb{R}^n$?* Mathematics Stack Exchange. 2014. eprint: https://math.stackexchange.com/q/1072818. URL: https://math.stackexchange.com/q/1072818.

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.

[Elb22] Nir Elber. *Introduction to Topology and Analysis*. 2022. URL: https://dfoiler.github.io/notes/202A/notes.pdf.

[Elb23] Nir Elber. *Algebraic Topology*. 2023. URL: https://dfoiler.github.io/notes/215A/notes.pdf.

# List of Definitions