256A: Algebraic Geometry

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Contents

THEME 1
SHEAF THEORY

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 24

A feeling of impending doom overtakes your soul.

1.1.1 Administrative Notes

Here are housekeeping notes.

- Here are some housekeeping notes. There is a syllabus on `bCourses`.
- We hope to cover Chapter II of [Har77], mostly, supplemented with examples from curves.
- There are lots of books.
  - We use [Har77] because it is short.
  - There is also [Vak17], which has more words.
  - The book [Liu06] has notes on curves.
  - There are more books in the syllabus. Professor Tang has some opinions on these.
- Some proofs will be skipped in lecture. Not all of these will appear on homework.
- Some examples will say lots of words, some of which we won't have good definitions for until later. Do not be afraid of words.

Here are assignment notes.

- Homework is 70% of the class.
- Homework is due on noon on Fridays. There will be 6–8 problems, which means it is pretty heavy. The lowest homework score will be dropped.

- Office hours exist. Professor Tang also answers emails.
- The term paper covers the last 30% of the grade. They are intended to be extra but interesting topics we don't cover in this class.

1.1.2 Motivation

We're going to talk about schemes. Why should we care about schemes? The point is that schemes are "correct."

Example 1.1. In algebraic topology, there is a cup product map in homology, which is intended to algebraically measure intersections. However, intersections are hard to quantify when we aren't dealing with, say manifolds.

Here is an example of algebraic geometry helping us with this rigorization.

Theorem 1.2 (Bézout)**.** Let C_1 and C_2 be curves in $\mathbb{P}^2(k)$, for some algebraically closed k, where C_1 and C_2 are defined by homogeneous polynomials f_1 and f_2. Then the "intersection number" between the curves C_1 and C_2 is $(\deg f_1)(\deg f_2)$.

This is a nice result, for example because it automatically accounts for multiplicities, which would be difficult to deal with directly using (say) geometric techniques. Schemes will help us with this.

Example 1.3. Moduli spaces are intended to be geometric objects which represent a family of geometric objects of interest. For example, we might be interested in the moduli space of some class of elliptic curves.

It turns out that the correct way to define these objects is using schemes as a functor; we will see this in this class.

Remark 1.4. One might be interested in when a functor is a scheme. We will not cover this question in this class in full, but it is an interesting question, and we will talk about this in special cases.

1.1.3 Elliptic Curves

For the last piece of motivation, let's talk about elliptic curves, over a field k.

Definition 1.5 (Elliptic curve)**.** An *elliptic curve* over k is a smooth projective curve of genus 1, with a marked k-rational point.

Remember that we said that we not to be afraid of words. However, we should have some notion of what these words mean: being a curve means that we are one-dimensional, being smooth is intuitive, and having genus 1 roughly means that base-changing to a complex manifold has one hole. Lastly, the k-rational point requires defining a scheme as a functor.

Here's another (more concrete) definition of an elliptic curve.

Definition 1.6 (Elliptic curve)**.** An *elliptic curve* over k is an affine variety in $\mathbb{A}^2(k)$ cut out be a polynomial of the form

$$y^2 + a_1xy + a_3y^2 = x^3 + a_2x^2 + a_4x + a_6$$

with nonzero discriminant plus a point $\mathcal{O}$ at infinity.

Remark 1.7. Why are these equivalent? Well, the Riemann–Roch theorem approximately lets us take a smooth projective curve of genus 1 and then write it as an equation; the marked point goes to $\mathcal{O}$. In the reverse direction, one merely needs to embed our affine curve into projective space and verify its smoothness and genus.

Instead of working with affine varieties, we can also give a concrete description of an elliptic curve using projective varieties.

Definition 1.8 (Elliptic curve)**.** An *elliptic curve* over k is a projective variety in $\mathbb{P}^2(k)$ cut out be a polynomial of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with nonzero discriminant.

We get the equivalence of the previous two definitions via the embedding $\mathbb{A}_2(k) \hookrightarrow \mathbb{P}^2(k)$ by $(x, y) \mapsto [x : y : 1]$; the point at infinity $\mathcal{O}$ is $[0 : 1 : 0]$.

1.1.4 Crackpot Varieties

In order to motivate schemes, we should probably mention varieties, so we will spend some time in class discussing affine and projective varieties. For convenience, we work over an algebraically closed field k.

Definition 1.9 (Affine space)**.** Given a field k, we define *affine n-space over k*, denoted $\mathbb{A}^n(k)$. An *affine variety* is a subset $Y \subseteq \mathbb{A}^n(k)$ of the form

$$Y = V(S) := \{p \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } f \in S\},$$

where $S \subseteq k[x_1, \ldots, x_n]$.

Remark 1.10. The set $S \subseteq k[x_1, \ldots, x_n]$ in the above definition need not be finite or countable. In certain cases, we can enforce this condition; for example, if $n = 1$, then $k[x]$ is a principal ideal domain, so we may force $\#S = 1$.

Note that we have defined vanishing sets $V(S)$ from subsets $S \subseteq k[x_1, \ldots, x_n]$. We can also go from vanishing sets to subsets.

Definition 1.11. Fix a field k and subset $Y \subseteq \mathbb{A}^n(k)$. Then we define the ideal

$$I(Y) := \{f \in \mathbb{A}^n(k) : f(p) = 0 \text{ for all } p \in Y\}.$$

Remark 1.12. One should check that this is an ideal, but we won't bother.

So we've defined some geometry. But we're in an algebraic geometry class; where's the algebra?

Theorem 1.13 (Hilbert's Nullstellensatz)**.** Fix an algebraically closed field k and ideal $J \subseteq k[x_1, \ldots, x_n]$. Then

$$I(V(J)) = \operatorname{rad} I,$$

where $\operatorname{rad} I$ is the radical of I.

Remark 1.14. The Nullstellensatz has no particularly easy proof.

The point of this result is that it ends up giving us a contravariant equivalence of posets of radical ideals and affine varieties.

Why do we care? In some sense, we prefer to work with ideals because it "remembers" more information than merely the points on the variety. To see this, note that elements $f \in k[x_1, \ldots, x_n]$ we are viewing as giving functions on $\mathbb{A}^n(k)$. However, when we work on a variety $Y \subseteq \mathbb{A}^n(k)$, then sometimes two functions will end up being identical on Y. So the correct ring of functions on Y is

$$k[x_1, \ldots, x_n]/I(Y),$$

so indeed keeping track of the (algebraic) ideal $V(Y)$ gets us some extra (geometric) information.

We will use this discussion as a jumping-off point to discuss affine schemes and then schemes. Affine schemes will have the following data.

- A commutative ring A, which we should think of as the ring of functions on a variety.
- A topological space $\operatorname{Spec} A$, which has more information than merely points on the variety.
- A structure sheaf of functions on $\operatorname{Spec} A$.

Remark 1.15. Our topological space $\operatorname{Spec} A$ will contain more points than just the points on the variety. In some sense, these extra points make the topology more apparent.

Remark 1.16. Going forward, one might hope to remove requirements that the field k is algebraically closed (e.g., to work with a general ring) or talk about ideals which are not radical. This is the point of scheme theory.

1.2 August 26

Let's finish up talking varieties, and then we'll move on to affine schemes.

1.2.1 Projective Varieties

We're going to briefly talk about projective varieties. Let's start with projective space.

Definition 1.17 (Projective space)**.** Given a field k, we define *projective n-space over k*, denoted $\mathbb{P}^n(k)$ as

$$\frac{k^{n+1} \setminus \{(0, \ldots, 0\}}{\sim},$$

where $\sim$ assigns two points being equivalent if and only if they span the same 1-dimensional subspace of k^{n+1}. We will denote the equivalence class of a point $(a_0, \ldots, a_n)$ by $[a_0 : \ldots : a_n]$.

To work with varieties, we don't quite cut out by general polynomials but rather by homogeneous polynomials.

Definition 1.18 (Projective variety)**.** Given a field k and a set of some homogeneous polynomials $T \subseteq k[x_1, \ldots, x_n]$, we define the *projective variety* cut out by T as

$$V(T) := \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in T\}.$$

Example 1.19. The elliptic curve corresponding to the affine algebraic variety in $\mathbb{A}^2(k)$ cut out by $y^2 - x^3 - 1$ becomes the projective variety in $\mathbb{P}^2(k)$ cut out by

$$Y^2Z - X^3 - Z^3 = 0.$$

Remark 1.20. One can give projective varieties some Zariski topology as well, which we will define later in the class.

What to remember about projective varieties is that we can cover $\mathbb{P}^2(k)$ (say) by affine spaces as

$$\begin{aligned}\mathbb{P}^2(k) &= \{[X:Y:Z] : X,Y,Z\in k \text{ not all } 0\}\\ &= \{[X:Y:Z] : X,Y,Z\in k \text{ and } X\neq 0\}\\ &\quad\cup\{[X:Y:Z] : X,Y,Z\in k \text{ and } Y\neq 0\}\\ &= \{[1:y:z] : y,z\in k\}\\ &\quad\cup\{[x:1:z] : x,z\in k\}\\ &\simeq \mathbb{A}^2(k)\cup\mathbb{A}^2(k).\end{aligned}$$

The point is that we can decompose $\mathbb{P}^2(k)$ into an affine cover.

Example 1.21. Continuing from Example 1.19, we can decompose $Z\left(Y^2Z-X^3-Z^3\right)$ into having an affine open cover by

$$\underbrace{\left\{(x,y):y^2-x^3-1=0\right\}}_{z\neq 0}\cup\underbrace{\left\{(x,z):z-x^3-z^3=0\right\}}_{y\neq 0}\cup\underbrace{\left\{(y,z):y^2z-1-z^3=0\right\}}_{x\neq 0}.$$

Notably, we get almost everything from just one of the affine chunks, and we get the point at infinity by taking one of the other chunks.

Remark 1.22. It is a general fact that we only need two affine chunks to cover our projective curve.

1.2.2 The Spectrum

The definition of a(n affine) scheme requires a topological space and its ring of functions. We will postpone talking about the ring of functions until we discuss sheaves, so for now we will focus on the space.

Definition 1.23 (Spectrum)**.** Given a ring A, we define the *spectrum*

$$\operatorname{Spec} A := \{\mathfrak{p}\subseteq A : \mathfrak{p} \text{ is a prime ideal}\}.$$

Example 1.24. Fix a field k. Then $\operatorname{Spec} k = \{(0)\}$. Namely, non-isomorphic rings can have homeomorphic spectra.

Exercise 1.25. Fix a field k. We show that

$$\operatorname{Spec} k[x] = \{(0)\}\cup\{(\pi) : \pi \text{ is monic, irred.}, \deg\pi > 0\}.$$

Proof. To begin, note that (0) is prime, and (π) is prime for irreducible non-constant polynomials π because irreducible elements are prime in principal ideal domains. Additionally, we note that all the given primes are distinct: of course (0) is distinct from any prime of the form (π), but further, given monic non-constant irreducible polynomials α and β, having

$$(\alpha)=(\beta)$$

forces $\alpha=c\beta$ for some $c\in k[x]^\times$. But $k[x]^\times=k^\times$, so $c\in k^\times$, so $c=1$ is forced by comparing the leading coefficients of α and β.

It remains to show that all prime ideals $\mathfrak{p} \subseteq k[x]$ take the desired form. Well, $k[x]$ is a principal ideal domain, so we may write $\mathfrak{p} = (\pi)$ for some $\pi \in k[x]$. If $\pi = 0$, then we are done. Otherwise, $\deg \pi \geq 0$, but $\deg \pi > 0$ because $\deg \pi = 0$ implies $\pi \in k[x]^\times$. By adjusting by a unit, we may also assume that π is monic. And lastly, note that (π) is prime means that π is prime, so π is irreducible. ■

Example 1.26. If k is an algebraically closed field, then the only nonconstant irreducible polynomials are linear (because all nonconstant polynomials have a root and hence a linear factor), and of course any linear polynomial is irreducible. Thus,

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(x-\alpha) : \alpha \in k\}.$$

Set $\mathfrak{m}_\alpha := (x-\alpha)$ so that $\alpha \mapsto \mathfrak{m}_\alpha$ provides a natural map from $\mathbb{A}^1_k$ to $\operatorname{Spec} k[x]$. In this way we can think of $\operatorname{Spec} k[x]$ as $\mathbb{A}^1_k$ with an extra point (0).

Remark 1.27. Continuing from Example 1.26, observe that we can also recover function evaluation at a point $\alpha \in \mathbb{A}^1_k$: given $f \in k[x]$, the value of $f(\alpha)$ is the image of f under the canonical map

$$k[x] \twoheadrightarrow \frac{k[x]}{\mathfrak{m}_\alpha} \cong k,$$

where the last map is the forced $x \mapsto \alpha$. Observe running this construction at the point $(0) \in \operatorname{Spec} k[x]$ makes the "evaluation" map just the identity.

Example 1.28. Similar to $k[x]$, we can classify $\operatorname{Spec} \mathbb{Z}$: all ideals are principal, so our primes look like (p) where $p = 0$ or is a rational prime. Namely, essentially the same proof gives

$$\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) : p \text{ prime}, p > 0\}.$$

The condition $p > 0$ is to ensure that all the points on the right-hand side are distinct; certainly we can write all nonzero primes $(p) \subseteq \mathbb{Z}$ for some nonzero (p), and we can adjust p by a unit to ensure $p > 0$. Conversely, $(p) = (q)$ with $p, q > 0$ forces $p \mid q$ and $q \mid p$ and so $p = q$.

We might hope to have a way to view $\operatorname{Spec} k[x]$ as points even when k is not algebraically closed.

Example 1.29. Set $k = \mathbb{Q}$. There is a map sending a nonconstant monic irreducible polynomial $\pi \in \mathbb{Q}[x]$ to its roots in $\overline{\mathbb{Q}}$, and note that this map is injective because one can recover a polynomial from its roots. Further, all the roots of π are Galois conjugate because π is irreducible, and a Galois orbit S_α of a root α corresponds to the polynomial

$$\pi(x) = \prod_{\beta \in S_\alpha} (x - \beta),$$

where $\pi(x) \in \mathbb{Q}[x]$ because its coefficients are preserved the Galois action. Thus, there is a bijection between the nonconstant monic irreducible polynomials $\pi \in \mathbb{Q}[x]$ and Galois orbits of elements in $\overline{\mathbb{Q}}$.

So far, all of our examples have been "dimension 0" (namely, a field k) or "dimension 1" (namely, $\mathbb{Z}$ and $k[x]$). Here is an example in dimension 2.

Exercise 1.30. Let k be algebraically closed. Any $\mathfrak{p} \in \operatorname{Spec} k[x,y]$ is one of the following types of prime.

- Dimension 2: $\mathfrak{p} = (0)$.
- Dimension 1: $\mathfrak{p} = (f(x,y))$ where f is nonconstant and irreducible.
- Dimension 0: $\mathfrak{p} = (x-\alpha, y-\beta)$, where $\alpha, \beta \in k$.

Proof. We follow [Vak17, Exercise 3.2.E]. If $\mathfrak{p} = (0)$, then we are done. If $\mathfrak{p}$ is principal, then we can write $\mathfrak{p} = (f)$ where $f \in k[x, y]$ is a prime element and hence irreducible. Observe that if f is irreducible, then f is also a prime element because $k[x, y]$ is a unique factorization domain.

Lastly, we suppose that $\mathfrak{p}$ is not principal. We start by finding $f, g \in \mathfrak{p}$ with no nonconstant common factors. Because $\mathfrak{p} \neq 0$, we can find $f_0 \in \mathfrak{p} \setminus \{0\}$, and assume that (f_0) is maximal with respect to this (namely, $f_0 \notin (f_0')$ for any $f_0' \in \mathfrak{p}$). Because $\mathfrak{p}$ is not principal, we can find $g_0 \in \mathfrak{p} \setminus (f_0)$. Now, we can use unique prime factorization of f_0 and g_0 to find some $d \in k[x, y]$ such that

$$f_0 = fd \qquad \text{and} \qquad g_0 = gd$$

where f and g share no common factors. (Namely, $\nu_\pi(d) = \min\{\nu_\pi(f_0), \nu_\pi(g_0)\}$ for all irreducible factors $\pi \in k[x, y]$.) Note $d \notin \mathfrak{p}$ by the maximality of f_0, so $f, g \in \mathfrak{p}$ is forced.

Continuing, embedding f and g into $k(x)[y]$ and using the Euclidean algorithm there, we can write

$$af + bg = 1$$

where $a, b \in k(x)[y]$, because f and g have no common factors in $k(x)[y]$. (Any common factor would lift to a common factor in $k[x, y]$.[1]) Clearing denominators, we see that we can find $h(x) \in k[x] \cap \mathfrak{p}$, but by factoring $h(x)$ using the fact that k is algebraically closed, we see that we can actually enforce $(x - \alpha) \in \mathfrak{p}$ for some $\alpha \in k$.

By symmetry, we can force $(y - \beta) \in \mathfrak{p}$ for some $\beta \in \mathfrak{p}$ as well, so $(x - \alpha, y - \beta) \subseteq \mathfrak{p}$. However, we see that $(x - \alpha, y - \beta)$ is maximal because of the isomorphism

$$\frac{k[x, y]}{(x - \alpha, y - \beta)} \to k$$

by $x \mapsto \alpha$ and $y \mapsto \beta$. Thus, $\mathfrak{p} = (x - \alpha, y - \beta)$ follows. ■

Remark 1.31. The intuition behind Exercise 1.30 is that the prime ideal $(x - \alpha, y - \beta)$ "cuts out" the zero-dimensional point $(\alpha, \beta) \in \mathbb{A}^2_k$. Then the prime ideal (f) cuts out some one-dimensional curve in $\mathbb{A}^2_k$, and the prime ideal (0) cuts out the entire two-dimensional plane. We have not defined dimension rigorously, but hopefully the idea is clear.

Remark 1.32. It is remarkable that the number of equations we need to cut out a variety of dimension d is $2 - d$. This is not always true.

The point is that we seem to have recovered $\mathbb{A}^1_k$ by looking at $\operatorname{Spec} k[x]$ and $\mathbb{A}^2_k$ by looking at $\operatorname{Spec} k[x, y]$, so we can generalize this to arbitrary rings cleanly, realizing some part of Remark 1.16.

Definition 1.33 (Affine space)**.** Given a ring R, we define *affine n-space over R* as

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n].$$

So far all the rings we've looked at so far have been integral domains, but it's worth pointing out that working with general rings allows more interesting information.

Example 1.34. We classify $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$. Notably, all prime ideals here must correspond to prime ideals of $k[\varepsilon]$ containing $\left(\varepsilon^2\right)$ and hence contain $\operatorname{rad}\left(\varepsilon^2\right) = (\varepsilon)$, which allows only the prime (ε). (We will make this correspondence precise later.) So $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ has a single point.

[1] If $d(x, y)/e(x)$ divides both f and g in $k(x)[y]$, where d and e share no common factors, then $d \mid fe, ge$ in $k[x, y]$. Unique prime factorization now forces $d \mid f, g$ in $k[x, y]$.

Remark 1.35. In some sense, $\operatorname{Spec} k[\varepsilon]/\left(\varepsilon^2\right)$ will be able to let us talk about differential information algebraically: ε is some very small nonzero element such that $\varepsilon^2 = 0$. So we can study a "function" $f \in k[x]$ locally at a point p by studying $f(p+\varepsilon)$. Rigorously, $f(x) = \sum_{i=0}^d a_i x^i$ has

$$f(x+\varepsilon) = \sum_{i=0}^d a_i (x+\varepsilon)^i = \sum_{i=0}^d a_i x^i + \sum_{i=1}^d i a_i x^{i-1}\varepsilon = f(x) + f'(x)\varepsilon.$$

One can recover more differential information by looking at $k[\varepsilon]/\left(\varepsilon^n\right)$ for larger n.

1.2.3 The Zariski Topology

Thus far we've defined our space. Here's our topology.

Definition 1.36 (Zariski topology)**.** Fix a ring A. Then, for $S \subseteq A$, we define the *vanishing set*

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} A : S \subseteq \mathfrak{p}\}$$

Then the *Zariski topology* on $\operatorname{Spec} A$ is the topology whose closed sets are the $V(S)$.

Intuitively, we are declaring A as the (continuous) functions on $\operatorname{Spec} A$, and the evaluation of the function $f \in A$ at the point $\mathfrak{p} \in \operatorname{Spec} A$ is $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27). Then the vanishing sets of a continuous function must be closed, and without easy access to any other functions on $\operatorname{Spec} A$, we will simply declare that these are all of our closed sets.

In the affine case, we can be a little more rigorous.

Example 1.37. Set $A := k[x_1, \ldots, x_n]$, where k is algebraically closed. Then, given $f \in k[x_1, \ldots, x_n]$, we want to be convinced that $V(\{f\})$ matches up with the affine k-points $(a_1, \ldots, a_n)$ which vanish on f. Well, $(a_1, \ldots, a_n)$ corresponds to the prime ideal $(x_1 - a_1, \ldots, x_n - a_n) \in \operatorname{Spec} A$, and

$$\{f\} \subseteq (x_1 - a_1, \ldots, x_n - a_n)$$

is equivalent to f vanishing in the evaluation map

$$k[x_1, \ldots, x_n] \twoheadrightarrow \frac{k[x_1, \ldots, x_n]}{(x_1 - a_1, \ldots, x_n - a_n)} \to k,$$

which is equivalent to $f(a_1, \ldots, a_n) = 0$. So indeed, f vanishes on $(a_1, \ldots, a_n)$ if and only if the corresponding maximal ideal is in $V(\{f\})$.

With intuition out of the way, we should probably check that the sets $V(S)$ make a legitimate topology. To begin, here are some basic properties.

Lemma 1.38. Fix a ring A.

(a) If subsets $S, T \subseteq A$ have $S \subseteq T$, then $V(T) \subseteq V(S)$.

(b) A subset $S \subseteq A$ has $V(S) = V((S))$.

(c) An ideal $\mathfrak{a} \subseteq A$ has $V(\mathfrak{a}) = V(\operatorname{rad} \mathfrak{a})$.

Proof. We go in sequence.

(a) Note $\mathfrak{p} \in V(T)$ implies that $T \subseteq \mathfrak{p}$, which implies $S \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(S)$.

(b) Surely $S \subseteq (S)$, so $V((S)) \subseteq V(S)$. Conversely, if $\mathfrak{p} \in V(S)$, then $S \subseteq \mathfrak{p}$, but then the generated ideal (S) must also be contained in $\mathfrak{p}$, so $\mathfrak{p} \in V((S))$.

(c) Surely $\mathfrak{a} \subseteq \operatorname{rad} \mathfrak{a}$, so $V(\operatorname{rad} \mathfrak{a}) \subseteq V(I)$. Conversely, if $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{p} \subseteq \mathfrak{a}$, so

$$\mathfrak{p} \subseteq \bigcap_{\mathfrak{q} \supseteq \mathfrak{a}} \mathfrak{q} = \operatorname{rad} \mathfrak{a},$$

so $\mathfrak{p} \in V(\operatorname{rad} \mathfrak{a})$. ■

Remark 1.39. In light of (b) and (c) of Lemma 1.38, we can actually write all closed subsets of $\operatorname{Spec} A$ as $V(\mathfrak{a})$ for a radical ideal $\mathfrak{a}$. We will use this fact freely.

And here are our checks.

Lemma 1.40. Fix a ring A.

(a) $V(A) = \varnothing$ and $V((0)) = \operatorname{Spec} A$.

(b) Given ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$, then $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(c) Given a collection of ideals $\mathcal{I} \subseteq \mathcal{P}(A)$, we have

$$\bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a}) = V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right).$$

Proof. We go in sequence.

(a) All primes are proper, so no prime $\mathfrak{p}$ has $A \subseteq \mathfrak{p}$, so $V(A) = \varnothing$. Also, 0 is an element of all ideals, so all $\mathfrak{p} \in \operatorname{Spec} A$ have $(0) \subseteq \mathfrak{p}$, so $V((0)) = \operatorname{Spec} A$.

(b) Note $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$, so $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ follows. Conversely, take $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, and suppose $\mathfrak{p} \notin V(\mathfrak{a})$ so that we need $\mathfrak{p} \in V(\mathfrak{b})$. Well, $\mathfrak{p} \notin V(\mathfrak{a})$ implies $\mathfrak{a} \not\subseteq \mathfrak{p}$, so we can find $a \in \mathfrak{a} \setminus \mathfrak{p}$. Now, for any $b \in \mathfrak{b}$, we see

$$ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p},$$

so $a \notin \mathfrak{p}$ forces $b \in \mathfrak{p}$. Thus, $\mathfrak{b} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{b})$.

(c) Certainly any $\mathfrak{b} \in \mathcal{I}$ has $\mathfrak{b} \subseteq \sum_{\mathfrak{a} \in \mathcal{I}} I$, so $V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right) \subseteq \bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a})$ follows.

Conversely, suppose $\mathfrak{p} \in \bigcap_{\mathfrak{a} \in \mathcal{I}} V(\mathfrak{a})$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{a} \in \mathcal{I}$, so $\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a} \subseteq \mathfrak{p}$ follows. Thus, $\mathfrak{p} \in V\left(\sum_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}\right)$. ■

Remark 1.41. For ideals $I, J \subseteq A$, note that $IJ \subseteq I \cap J$. Additionally, $I \cap J \subseteq \operatorname{rad}(IJ)$: if $f \in I \cap J$, then $f^2 \in (I \cap J)^2 \subseteq IJ$. It follows from Lemma 1.38 that

$$V(IJ) \supseteq V(I \cap J) \supseteq V(\operatorname{rad}(IJ)) = V(IJ),$$

so $V(I) \cup V(J) = V(IJ) = V(I \cap J)$. So V does respect some poset structure.

It follows that the collection of vanishing sets is closed under finite union and arbitrary intersection, so they do indeed specify the closed sets of a topology.

1.2.4 Easy Nullstellensatz

While we're here, let's also generalize Definition 1.11 in the paradigm that $\operatorname{Spec} A$ is the analogue for affine space.

Definition 1.42. Fix a ring A. Then, given a subset $Y \subseteq \operatorname{Spec} A$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Remark 1.43. To see that this is the correct definition, note we want $f \in I(Y)$ if and only if f vanishes at all points $\mathfrak{p} \in Y$. We said earlier that the value of f at $\mathfrak{p}$ should be $f \pmod{\mathfrak{p}}$ (using the ideas of Remark 1.27), so f vanishes at $\mathfrak{p}$ if and only if $f \in \mathfrak{p}$. So we want

$$I(Y) = \{f \in A : f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in Y\} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

As before, we'll write in a few basic properties of I.

Lemma 1.44. Fix a ring A, and fix subsets $X, Y \subseteq \operatorname{Spec} A$.

(a) If $X \subseteq Y$, then $I(Y) \subseteq I(X)$.

(b) The ideal $I(X)$ is radical.

Proof. We go in sequence.

(a) Note

$$I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = I(X).$$

(b) Suppose that $f^n \in I(X)$ for some positive integer n, and we need to show $f \in I(X)$. Then $f^n \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, so $f \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, so $f \in I(X)$. ■

And here is our nice version of Theorem 1.13.

Proposition 1.45. Fix a ring A.

(a) Given an ideal $\mathfrak{a} \subseteq A$, we have $I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$.

(b) Given a subset $X \subseteq \operatorname{Spec} A$, we have $V(I(X)) = \overline{X}$.

(c) The functions V and I provide an inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. We go in sequence.

(a) Observe

$$I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} = \operatorname{rad} \mathfrak{a}.$$

(b) Using Lemma 1.40, we find

$$\overline{X} = \bigcap_{V(\mathfrak{a}) \supseteq X} V(\mathfrak{a}) = V\left(\sum_{V(\mathfrak{a}) \supseteq X} \mathfrak{a}\right).$$

Now, $X \subseteq V(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in X$, which is equivalent to $\mathfrak{a} \subseteq I(X)$. Thus,

$$\overline{X} = V\left(\sum_{\mathfrak{a} \subseteq I(X)} \mathfrak{a}\right) = V(I(X)).$$

(c) Note that V sends (radical) ideals to closed subsets of $\operatorname{Spec} A$ by the definition of the Zariski topology. Also, I sends (closed) subsets of $\operatorname{Spec} A$ to radical ideals by Lemma 1.44. Additionally, for a closed subset $X \subseteq \operatorname{Spec} A$, we have

$$V(I(X)) = \overline{X} = X,$$

and for a radical ideal $\mathfrak{a}$, we have

$$I(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a} = \mathfrak{a},$$

so I and V are in fact mutually inverse. ■

Remark 1.46. Given $X \subseteq \operatorname{Spec} A$, we claim $I(X) = I(\overline{X})$. Well, these are both radical ideals, so it suffices by Proposition 1.45 (c) to show $V(I(X)) = V(I(\overline{X}))$, which is clear because these are both $\overline{X}$.

Remark 1.47. Intuitively, what makes proving Proposition 1.45 so much easier than Theorem 1.13 is that we've added extra points to our space in order to track varieties better.

1.2.5 Some Continuous Maps

As a general rule, we will make continuous maps between our spectra by using ring homomorphisms. Here is the statement.

Lemma 1.48. Given a ring homomorphism $\varphi\colon A \to B$, the pre-image function $\varphi^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$ induces a continuous function $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$.

Proof. We begin by showing $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is well-defined: given a prime $\mathfrak{q} \subseteq \operatorname{Spec} B$, we claim that $\varphi^{-1}\mathfrak{q}$ is a prime in $\operatorname{Spec} A$. Well, if $ab \in \varphi^{-1}\mathfrak{q}$, then $\varphi(a)\varphi(b) \in \mathfrak{q}$, so $\varphi(a) \in \mathfrak{q}$ or $\varphi(b) \in \mathfrak{q}$. So indeed, $\varphi^{-1}\mathfrak{q}$ is prime.

We now show that $\varphi^{-1}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous. It suffices to show that the pre-image of a closed set $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ under φ^{-1} is a closed set. For concreteness, we will make $\operatorname{Spec} \varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ our pre-image map so that we want to show $(\operatorname{Spec} \varphi)^{-1}(V(\mathfrak{a}))$ is closed. Well,

$$\begin{aligned}(\operatorname{Spec} \varphi)^{-1}(V(\mathfrak{a})) &= \{\mathfrak{q} \in \operatorname{Spec} B : (\operatorname{Spec} \varphi)(\mathfrak{q}) \in V(\mathfrak{a})\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{a} \subseteq (\operatorname{Spec} \varphi)(\mathfrak{q})\} \\ &= \left\{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}\right\}.\end{aligned}$$

Now, if $\mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}$, then any $a \in \mathfrak{a}$ has $\varphi(a) \in \mathfrak{q}$, so $\varphi(\mathfrak{a}) \subseteq \mathfrak{q}$. Conversely, if $\varphi(\mathfrak{a}) \subseteq \mathfrak{q}$, then any $a \in \mathfrak{a}$ has $\varphi(a) \in \mathfrak{q}$ and hence $a \in \varphi^{-1}\mathfrak{q}$, so $\mathfrak{a} \subseteq \varphi^{-1}\mathfrak{q}$ follows. In total, we see

$$(\operatorname{Spec} \varphi)^{-1}(V(\mathfrak{a})) = \{\mathfrak{q} \in \operatorname{Spec} B : \varphi(\mathfrak{a}) \subseteq \mathfrak{q}\} = V(\varphi(\mathfrak{a})),$$

which is closed. ■

In fact, we have defined a (contravariant) functor.

Proposition 1.49. The mapping Spec sending rings A to topological spaces $\operatorname{Spec} A$ and ring homomorphisms $\varphi\colon A \to B$ to continuous maps $\operatorname{Spec}\varphi = \varphi^{-1}$ assembles into a functor $\operatorname{Spec}\colon \operatorname{Ring}^{\mathrm{op}} \to \operatorname{Top}$.

Proof. Thus far our data is sending objects to objects and morphisms to (flipped) morphisms, so we just need to run the functoriality checks.

- Identity: note that $\operatorname{Spec}\operatorname{id}_A$ sends a prime $\mathfrak{p} \in \operatorname{Spec} A$ to
$$(\operatorname{Spec}\operatorname{id}_A)(\mathfrak{p}) = \operatorname{id}_A^{-1}(\mathfrak{p}) = \{a \in A : \operatorname{id}_A a \in \mathfrak{p}\} = \mathfrak{p},$$
so indeed, $\operatorname{Spec}\operatorname{id}_A = \operatorname{id}_{\operatorname{Spec} A}$.
- Functoriality: given morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$, as well as a prime $\mathfrak{r} \in \operatorname{Spec} C$, we compute
$$\begin{aligned}(\operatorname{Spec}(\psi\circ\varphi))(\mathfrak{r}) &= (\psi\circ\varphi)^{-1}(\mathfrak{r}) \\ &= \{a \in A : \psi(\varphi(a)) \in \mathfrak{r}\} \\ &= \{a \in A : \varphi(a) \in (\operatorname{Spec}\psi)(\mathfrak{r})\} \\ &= \{a \in A : a \in (\operatorname{Spec}\varphi)((\operatorname{Spec}\psi)(\mathfrak{r}))\} \\ &= (\operatorname{Spec}\varphi \circ \operatorname{Spec}\psi)(\mathfrak{r}).\end{aligned}$$
So indeed, $\operatorname{Spec}(\psi\circ\varphi) = \operatorname{Spec}\varphi\circ\operatorname{Spec}\psi$. ■

Here is a quick example.

Definition 1.50 (k-points)**.** Given a ring A and field k, a *k-point* of $\operatorname{Spec} A$ is a ring homomorphism $\iota\colon A \to k$.

Remark 1.51. To see that Definition 1.50 does indeed cut out a single point, note $\iota\colon A \to k$ induces $\operatorname{Spec}\iota\colon \operatorname{Spec} k \to \operatorname{Spec} A$ and therefore picks out a single point of $\operatorname{Spec} A$ because $\operatorname{Spec} k = \{(0)\}$.

Remark 1.52. To see that Definition 1.50 is reasonable, let $A = k[x_1, \ldots, x_n]$ so that $\operatorname{Spec} A = \mathbb{A}_k^n$. Then a map $\iota\colon A \to k$ is determined by $a_i := \iota(x_i)$, so we expect this ι to correspond to the point $(a_1, \ldots, a_n)$. Indeed, Remark 1.51 says we should compute
$$(\operatorname{Spec}\iota)((0)) = \iota^{-1}((0)) = \ker\iota = (x_1 - a_1, \ldots, x_n - a_n),$$
which does indeed correspond to the point $(a_1, \ldots, a_n)$.

Here is a more elaborate example: closed subsets can be realized as spectra themselves!

Exercise 1.53. Fix a ring A and ideal $\mathfrak{a} \subseteq A$. Letting $\pi\colon A \twoheadrightarrow A/\mathfrak{a}$ be the natural projection, we have that
$$\operatorname{Spec}\pi\colon \operatorname{Spec} A/\mathfrak{a} \to V(\mathfrak{a})$$
is a homeomorphism.

Proof. To be more explicit, we claim that the maps
$$\begin{aligned}\operatorname{Spec} A/\mathfrak{a} &\cong V(\mathfrak{a}) \\ \mathfrak{q} &\mapsto \pi^{-1}\mathfrak{q} \\ \pi(\mathfrak{p}) &\leftarrow\!\shortmid \mathfrak{p}\end{aligned}$$
are continuous inverses. Here are our well-definedness and continuity checks.

- That $\mathfrak{q} \mapsto \pi^{-1}\mathfrak{q}$ is continuous follows from Lemma 1.48. Note $\pi^{-1}\mathfrak{q}$ contains $\mathfrak{a}$ because any $a \in \mathfrak{q}$ has $\pi(a) = [0]_\mathfrak{a} \in \mathfrak{q}$.
- For any $\mathfrak{p}$ containing $\mathfrak{a}$, we need to show that $\pi(\mathfrak{p})$ is prime. Of course, if $\mathfrak{p}$ is proper, then $\pi(\mathfrak{p})$ is proper as well. For the primality check, note $[a]_\mathfrak{a} \cdot [b]_\mathfrak{a} \in \pi(\mathfrak{p})$ implies $ab \in \mathfrak{p} + \mathfrak{a} = \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, so $[a]_\mathfrak{a} \in \mathfrak{p}$ or $[b]_\mathfrak{a} \in \mathfrak{p}$.
- To show that $\mathfrak{p} \mapsto \pi\mathfrak{p}$ is continuous, note that a closed set $V(\overline{S}) \subseteq \operatorname{Spec} A/\mathfrak{a}$ has pre-image
$$\pi^{-1}(V(\overline{S})) = \{\mathfrak{p} : \pi\mathfrak{p} \supseteq \overline{S}\}.$$
Now, set $S = \pi^{-1}(\overline{S})$. Now $\pi\mathfrak{p} \supseteq \overline{S}$ if and only if each $a \in S$ has $\pi(a) \in \pi\mathfrak{p}$, which is equivalent to $a \in \mathfrak{a} + \mathfrak{p} = \mathfrak{p}$. Thus,
$$\pi^{-1}(V(\overline{S})) = V(S),$$
which is closed.

Here are our inverse checks.

- Given $\mathfrak{p} \in V(\mathfrak{a})$, note
$$\pi^{-1}(\pi\mathfrak{p}) = \{a \in A : \pi(a) \in \pi\mathfrak{p}\} = \{a \in A : a \in \mathfrak{a} + \mathfrak{p}\} = \mathfrak{a} + \mathfrak{p} = \mathfrak{p}.$$
- Given $\mathfrak{q} \in \operatorname{Spec} A/\mathfrak{a}$, note
$$\pi\left(\pi^{-1}\mathfrak{q}\right) = \pi\left(\{a \in A : \pi(a) \in \mathfrak{q}\}\right).$$
Because $\pi \colon A \twoheadrightarrow A/\mathfrak{a}$ is surjective, the output here is just $\mathfrak{q}$. ■

A similar story exists for open sets, but we must be more careful. Here are our open sets.

Definition 1.54 (Distinguished open sets)**.** Given a ring A and element $f \in A$, we define the *distinguished open set*
$$D(f) := (\operatorname{Spec} A) \setminus V(\{f\}) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}.$$

Intuitively, these are the points on which f does not vanish.

Remark 1.55. In fact, the distinguished open sets form a base: any open set takes the form $(\operatorname{Spec} A) \setminus V(S)$ for some $S \subseteq A$, so we write
$$(\operatorname{Spec} A) \setminus V(S) = \{\mathfrak{p} : S \not\subseteq \mathfrak{p}\} = \bigcup_{f \in S} \{\mathfrak{p} : f \notin \mathfrak{p}\} = \bigcup_{f \in S} D(f).$$

Remark 1.56. The distinguished open base is good in that $D(f) \cap D(g) = \{\mathfrak{p} : f \notin \mathfrak{p}, g \notin \mathfrak{p}\} = \{\mathfrak{p} : fg \notin \mathfrak{p}\} = D(fg)$.

Here is our statement.

Exercise 1.57. Fix a ring A and element $f \in A$. Letting $\iota \colon A \to A_f$ be the localization map,
$$\operatorname{Spec} \iota \colon \operatorname{Spec} A_f \to D(f)$$
is a homeomorphism.

Proof. The arguments here are analogous to Exercise 1.53. To be explicit, we will say that our maps are
$$\begin{array}{rrcl} & \operatorname{Spec} A_f & \cong & D(f) \\ \varphi\colon & \mathfrak{P} & \mapsto & \iota^{-1}\mathfrak{P} \\ \psi\colon & \mathfrak{p}A_f & \leftarrow\!\shortmid & \mathfrak{p} \end{array}$$
for which it remains to run the various checks.

- We show φ is well-defined. Namely, we need to show that $\iota^{-1}\mathfrak{P}$ does not contain f for any $\mathfrak{P} \in \operatorname{Spec} A_f$. Well, if $f \in \iota^{-1}\mathfrak{P}$, then $f/1 \in \mathfrak{P}$, but $f/1 \in A_f^\times$, so this would violate $\mathfrak{P}$ being a proper ideal.
- We show ψ is well-defined. More formally, we have
$$\mathfrak{p}A_f = \{a/f^n : a \in \mathfrak{p}, n \in \mathbb{N}\}.$$
Quickly, if $a/f^k \cdot b/f^\ell \in \mathfrak{p}$, then $(ab)/f^{k+\ell} \in \mathfrak{p}$, so there exists $c \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that
$$\frac{ab}{f^{k+\ell}} = \frac{c}{f^n}.$$
Clearing denominators, there is some $N, M \in \mathbb{N}$ such that $f^N ab = f^M c \in \mathfrak{p}$, but $f \notin \mathfrak{p}$ forces $ab \in \mathfrak{p}$, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. It follows $a/f^k \in \mathfrak{p}A_f$ or $b/f^\ell \in \mathfrak{p}A_f$.
We should also check that $\mathfrak{p}A_f$ is proper. Indeed, if $1/1 \in \mathfrak{p}A_f$, there exists $a \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that $a/f^n = 1/1$, so there exists $N, M \in \mathbb{N}$ such that $f^N a = f^M$, which is a contradiction because $f^N a \in \mathfrak{p}$ while $f^N \notin \mathfrak{p}$.
- Note φ is continuous by Lemma 1.48.
- We show ψ is continuous. It suffices to check this on the distinguished base; for $a/f^m \in A_f$, we need to compute $\psi^{-1}(D(a/f^m))$. Well,
$$\psi^{-1}(D(a/f^m)) = \{\mathfrak{p} \in D(f) : a/f^m \notin \mathfrak{p}A_f\}.$$
Now, $a/f^m \in \mathfrak{p}A_f$ means there is $b \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that $a/f^m = b/f^n$, so clearing denominators promises $N, M \in \mathbb{N}$ such that
$$f^N a = f^M b \in \mathfrak{p},$$
so $a \in \mathfrak{p}$ follows. Conversely, $a \in \mathfrak{p}$ of course $a/f^m \in \mathfrak{p}A_F$, so we see that
$$\psi^{-1}(D(a/f^m)) = \{\mathfrak{p} \in D(f) : a/f^m \notin \mathfrak{p}A_f\} = \{\mathfrak{p} \in D(f) : a \notin \mathfrak{p}\} = D(f) \cap D(a)$$
is certainly open in $D(f) \subseteq \operatorname{Spec} A$.

And here our are inverse checks.

- We show $\psi \circ \varphi$ is the identity. Namely, given $\mathfrak{P} \in \operatorname{Spec} A_f$, we have to show that $(\iota^{-1}\mathfrak{P})A_f = \mathfrak{P}$.
In one direction, elements in $(\iota^{-1}\mathfrak{P})A_f$ take the form a/f^n where $a \in \iota^{-1}\mathfrak{P}$, which is equivalent to being in the form a/f^n where $a/1 \in \mathfrak{P}$, from which $a/f^n \in \mathfrak{P}$ certainly follows.
In the other direction, pick up some $a/f^n \in \mathfrak{P}$. Then $a/1 \in \mathfrak{P}$, so $a \in \iota^{-1}\mathfrak{P}$, so $a/f^n \in (\iota^{-1}\mathfrak{P})A_f$.
- We show $\varphi \circ \psi$ is the identity. Namely, given $\mathfrak{p} \in D(f)$, we have to show that $\iota^{-1}(\mathfrak{p}A_f) = \mathfrak{p}$. In one direction, if $a \in \mathfrak{p}$, then $a/1 \in \mathfrak{p}A_f$, so $a \in \iota^{-1}(\mathfrak{p}A_f)$.
In the other direction, if $a \in \iota^{-1}(\mathfrak{p}A_f)$, then $a/1 \in \mathfrak{p}A_f$. Then there exists $b \in \mathfrak{p}$ and $n \in \mathbb{N}$ such that $a/1 = b/f^n$, so clearing denominators promises $N, M \in \mathbb{N}$ such that
$$f^N a = f^M b \in \mathfrak{p},$$
so $f \notin \mathfrak{p}$ forces $a \in \mathfrak{p}$. ■

Remark 1.58. Not every open set is a distinguished open set. For example, taking k algebraically closed,
$$\mathbb{A}_k^2 \setminus \{(0,0)\} \subseteq \mathbb{A}_k^2$$
is an open set not in the form $D(f)$; equivalently, we need to show $V(\{f\}) \neq \{(x,y)\}$ for any $f \in k[x,y]$. Intuitively, this is impossible because a curve cuts out a one-dimensional variety of $\mathbb{A}_k^2$, not a zero-dimensional point.

Rigorously, we are requiring $f \in k[x,y]$ to have $f \in \mathfrak{p}$ if and only if $\mathfrak{p} = (x,y)$. However, f is certainly nonzero and nonconstant, so f has an irreducible factor π, which means that $f \in (\pi)$, where (π) is prime because $k[x,y]$ is a unique factorization domain.

1.3 August 29

Today we talk about the structure sheaf. To review, so far we have defined the spectrum $\operatorname{Spec} A$ of a ring A and given it a topology. The goal for today is to define its structure sheaf. Here is a motivating example.

Example 1.59. Set $A := \mathbb{C}[x_1, \ldots, x_n]$ so that $\operatorname{Spec} A = \mathbb{A}^n_k$. Recall that $\{D(f)\}_{f \in A}$ is a base for the Zariski topology, and we would like the functions on this ring to be A_f, the rational polynomials which allow some f in the denominator. In other words, these are rational functions on $\mathbb{C}^n$ whose poles are allowed on $V(\{f\})$ only.

1.3.1 Sheaves

Sheaves are largely a topological object, so we will forget that we are interested in the Zariski topology for now. Throughout, X will be a topological space.

Notation 1.60. Given a topological space X, we let $\operatorname{Op} X$ denote the poset (category) of its open sets.

Namely, the objects of $\operatorname{Ob} X$ are open sets, and

$$\operatorname{Mor}(V, U) = \begin{cases} \{*\} & V \subseteq U, \\ \varnothing & \text{else.} \end{cases}$$

Here is our definition.

Definition 1.61 (Presheaf)**.** A *presheaf* $\mathcal{F}$ on a topological space X valued in a category $\mathcal{C}$ is a contravariant functor $\mathcal{F}\colon (\operatorname{Ob} X)^{\text{op}} \to \mathcal{C}$. More concretely, $\mathcal{F}$ has the following data.

- Given an open set $U \subseteq X$, we have $\mathcal{F}(U) \in \mathcal{C}$.
- Given open sets $V \subseteq U \subseteq X$, we have a restriction map $\operatorname{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$.

This data satisfies the following coherence conditions.

- Identity: given an open set $U \subseteq X$, $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(U)}$.
- Functoriality: given open sets $W \subseteq V \subseteq U$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\operatorname{res}_{U,V}"] \arrow[dr, "\operatorname{res}_{U,W}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,W}"] \\ & \mathcal{F}(W) \end{tikzcd}$$

Notation 1.62. We might call an element $f \in \mathcal{F}(U)$ a *section over* U.

As suggested by our language and notation, we should think about (pre)sheaves as mostly being "sheaves of functions." We will see a few examples shortly.

Notation 1.63. Given $f \in \mathcal{F}(U)$, we might write $f|_V := \operatorname{res}_{U,V} f$.

Remark 1.64. In principle, one can have any target category $\mathcal{C}$ for our presheaf. However, we will only work Set, Ab, Ring, Mod_R in this class. In particular, we will readily assume that $\mathcal{C}$ is a concrete category.

Now that we've defined an algebraic object, we should discuss its morphisms.

Definition 1.65 (Presheaf morphism)**.** Fix a topological space X. A *presheaf morphism* between $\mathcal{F}$ and $\mathcal{G}$ is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$. In other words, for each open set $U \subseteq X$, we have a morphism $\eta_U\colon \mathcal{F}(U) \to \mathcal{F}(V)$; these morphisms make the following diagram commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"'] & \mathcal{G}(V) \end{tikzcd}$$

We've talked about presheaves a lot; where are sheaves?

Definition 1.66 (Sheaf)**.** Fix a topological space X. A presheaf $\mathcal{F}\colon (\operatorname{Ob} X)^{\text{op}} \to \mathcal{C}$ is a *sheaf* if and only if it satisfies the following for any open set $U \subseteq X$ with an open cover $\mathcal{U}$.

- Identity: if $f_1, f_2 \in \mathcal{F}(U)$ have $f_1|_V = f_2|_V$ for all $V \in \mathcal{U}$, then $f_1 = f_2$.
- Gluability: if we have $f_V \in \mathcal{F}(V)$ for all $V \in \mathcal{U}$ such that

$$f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$$

  for all $V_1, V_2 \in \mathcal{U}$, then there is $f \in \mathcal{F}(U)$ such that $f|_V = f_V$ for all $V \in \mathcal{U}$.

Ok, so we've defined the sheaf as an algebraic object, so here are its morphisms.

Definition 1.67 (Sheaf morphism)**.** A *sheaf morphism* is a morphism of the (underlying) presheaves.

Because there is an identity natural transformation and because the composition of natural transformations is a natural transformation, we see that we have the necessary data for a category PreSh_X of presheaves on X and a category Sh_X of sheaves on X.

As an aside, we note that we can succinctly write the sheaf conditions in an exact sequence.

Lemma 1.68. Fix a topological space X and presheaf $\mathcal{F}\colon (\operatorname{Ob} X)^{\text{op}} \to \mathcal{C}$, where $\mathcal{C}$ is an abelian category or Grp. Then $\mathcal{F}$ is a sheaf if and only if the sequence

$$\begin{array}{rcl} 0 \to \mathcal{F}(U) \to & \displaystyle\prod_{V\in\mathcal{U}} \mathcal{F}(V) \to & \displaystyle\prod_{V_1,V_2\in\mathcal{U}} \mathcal{F}(V_1\cap V_2) \\ f \mapsto & (f|_V)_{V\in\mathcal{U}} & \\ & (f_V)_{V\in\mathcal{U}} \mapsto & (f_{V_1}|_{V_1\cap V_2} - f_{V_2}|_{V_1\cap V_2})_{V_1,V_2} \end{array} \tag{1.1}$$

is exact for any open cover $\mathcal{U}$ of an open subset U.

Proof. In one direction, suppose that $\mathcal{F}$ is a sheaf, and we will show that (1.1) is exact for any open cover $\mathcal{U}$ of an open set U.

- Exact at $\mathcal{F}(U)$: suppose $f_1, f_2 \in \mathcal{F}(U)$ have the same image in $\prod_{V\in\mathcal{U}} \mathcal{F}(V)$. This means that

$$f_1|_V = f_2|_V$$

  for all $V \in \mathcal{U}$, so the identity axiom tells us that $f_1 = f_2$.
- Exact at $\prod_{V\in\mathcal{U}} \mathcal{F}(V)$: of course any $f \in \mathcal{F}(U)$ goes to $(f|_V)_{V\in\mathcal{U}}$, which goes to

$$f|_{V_1}|_{V_1\cap V_2} - f|_{V_2}|_{V_1\cap V_2} = f|_{V_1\cap V_2} - f|_{V_1\cap V_2} = 0 \in \prod_{V_1,V_2\in\mathcal{U}} \mathcal{F}(V_1\cap V_2)$$

and therefore lives in the kernel. Conversely, suppose $(f_V)_{V\in\mathcal{U}}$ vanishes in $\prod_{V_1,V_2}\mathcal{F}(V_1\cap V_2)$. Rearranging, this means that

$$f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2},$$

so the gluability axiom tells us that we can find $f\in\mathcal{F}(U)$ such that $f|_V = f_V$. This finishes.

Conversely, suppose that $\mathcal{F}$ makes (1.1) always exact, and we will show that $\mathcal{F}$ is a sheaf. Fix an open cover $\mathcal{U}$ of an open set U.

- Identity: suppose that $f_1, f_2 \in \mathcal{F}(U)$ have $f_1|_V = f_2|_V$ for any $V\in\mathcal{U}$. This means that f_1 and f_2 have the same image in $\prod_{V\in\mathcal{U}}\mathcal{F}(V)$, so the exactness of (1.1) at $\mathcal{F}(U)$ enforces $f_1 = f_2$.
- Gluability: suppose that we have $f_V\in\mathcal{F}(V)$ for each $V\in\mathcal{U}$ in such a way that $f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$ for all $V_1, V_2\in\mathcal{U}$. Then the image of $(f_V)_{V\in\mathcal{U}}$ in $\prod_{V_1,V_2\in\mathcal{U}}\mathcal{F}(V_1\cap V_2)$ is

$$(f_{V_1}|_{V_1\cap V_2} - f_{V_2}|_{V_1\cap V_2})_{V_1,V_2} = (0)_{V_1,V_2},$$

so exactness of (1.1) forces there to be $f\in\mathcal{F}(U)$ such that $f|_V = f_V$ for each $V\in\mathcal{U}$. This finishes. ■

Remark 1.69. One might want to continue this left-exact sequence. To see this, we will have to talk about cohomology, which is a task for later in life.

1.3.2 Examples of Sheaves

Sheaves of functions will be our key example here. Intuitively, any type of function which can be determined "locally" will form a sheaf; for example, here are continuous functions.

Remark 1.70. For most of our examples, the identity axiom is easily satisfied: intuitively, the identity axiom says that two sections are equal if and only if they agree locally. However, gluability is usually the tricky one: it requires us to build a function from local behavior.

Exercise 1.71. Fix topological spaces X and Y. For each $U\subseteq X$, let $\mathcal{F}(U)$ denote the set of continuous functions $f\colon U\to Y$, and equip these sets with the natural restriction maps. Then $\mathcal{F}$ is a sheaf.

Proof. To begin, here are the functoriality checks.

- Identity: for any $f\in\mathcal{F}(U)$, we have $f|_U = f$.
- Functoriality: if $W\subseteq V\subseteq U$, any $f\in\mathcal{F}(U)$ will have $(f|_V|_W)(w) = f(w) = (f|_W)(w)$ for any $w\in W$, so $f|_V|_W = f|_W$ follows.

Here are sheaf checks. Fix an open cover $\mathcal{U}$ of an open set $U\subseteq X$.

- Identity: suppose $f_1, f_2\in\mathcal{U}$ have $f_1|_V = f_2|_V$ for all $V\in\mathcal{U}$. Now, for all $x\in U$, we see $x\in U_x$ for some $U_x\in\mathcal{U}$, so

$$f_1(x) = (f_1|_{U_x})(x) = (f_2|_{U_x})(x) = f_2(x),$$

so $f_1 = f_2$ follows.
- Gluability: suppose we have $f_V\in\mathcal{F}(V)$ for each $V\in\mathcal{U}$ such that $f_{V_1}|_{V_1\cap V_2} = f_{V_2}|_{V_1\cap V_2}$ for each $V_1, V_2\in\mathcal{U}$. Now, for each $x\in\mathcal{U}$, find $U_x\in\mathcal{U}$ with $x\in U_x$ and set

$$f(x) := f_{U_x}(x).$$

Note this is well-defined: if $x \in U_x$ and $x \in U_{x'}$, then $f_{U_x}(x) = f_{U_x}|_{U_x \cap U'_x}(x) = f_{U'_x}|_{U_x \cap U'_x}(x) = f_{U'_x}(x)$. Additionally, we see that, for each $V \in \mathcal{U}$ and $x \in V$, we have

$$f|_V(x) = f(x) = f_V(x)$$

by construction, so we are done.

Lastly, we need to check that f is continuous. Well, for any open set $V_0 \subseteq Y$, we can compute

$$f^{-1}(V_0) = \{x \in U : f(x) \in V_0\} = \bigcup_{V \in \mathcal{U}} \{x \in V : f(x) \in V_0\} = \bigcup_{V \in \mathcal{U}} \{x \in V : f_V(x) \in V_0\} = \bigcup_{V \in \mathcal{U}} f_V^{-1}(V_0),$$

which is open as the arbitrary union of open sets because $f_V : V \to Y$ is a continuous function. ■

Another key geometric example going forward will be the following.

Exercise 1.72. Set $X := \mathbb{C}$. For each open $U \subseteq X$, let $\mathcal{O}_X(U)$ denote the set of holomorphic functions $U \to \mathbb{C}$, and equip these sets with the natural restriction maps. Then $\mathcal{O}_X$ is a sheaf.

Proof. Again, the point here is that being differentiable can be checked locally. Anyway, we note that our presheaf checks are exactly the same as in Exercise 1.71, as is the check of the sheaf identity axiom.

The gluability axiom is also mostly the same. Given an open cover $\mathcal{U}$ of an open set $U \subseteq X$, pick up $f_V \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that

$$f_{V_1}|_{V_1 \cap V_2} = f_{V_2}|_{V_1 \cap V_2}.$$

As before, we note that each $x \in U$ has some $U_x \in \mathcal{U}$ containing x, so we may define $f : U \to \mathbb{C}$ by $f(x) := f_{U_x}(x)$. The arguments of Exercise 1.71 tell us that this function f is well-defined and has $f|_V = f_V$ for each $V \in \mathcal{U}$.

It remains to check that f is actually holomorphic. This requires that, for each $x \in X$, the limit

$$\lim_{x' \to x} \frac{f(x) - f(x')}{x - x'}.$$

However, this limit can be computed locally for $x' \in U_x$ because U_x contains an open neighborhood around x. As such, it suffices to show that the limit

$$\lim_{x' \to x} \frac{f|_{U_x}(x) - f|_{U_x}(x')}{x - x'} = \lim_{x' \to x} \frac{f_{U_x}(x) - f_{U_x}(x')}{x - x'}$$

exists, which is true because $f_{U_x} \in \mathcal{F}(U_x)$ is holomorphic. ■

In contrast, sheaves have trouble keeping track of "global" information.

Example 1.73. For each $U \subseteq \mathbb{R}$, let $\mathcal{F}(\mathbb{R})$ denote the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, and equip these sets with the natural restriction maps. Then $\mathcal{F}$ is not a sheaf: for each open set $(n-1, n+1)$ for $n \in \mathbb{N}$, the function $f_{(n-1,n+1)} := \mathrm{id}_{(n-1,n=1)}$ is bounded and continuous, but the glued function $f = \mathrm{id}_{\mathbb{R}}$ is not bounded on all of $\mathbb{R}$. (We glued using Exercise 1.71, which does force the definition of f.)

1.3.3 Sheaf on a Base

In light of our sheaf language, we are trying to define a "structure" sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$, and we wanted to have

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f.$$

We aren't going to be able to specify a presheaf with this data, but we can specify a sheaf. In some sense, the presheaf is unable to build up locally in the way that a sheaf can, so having the data on a base like $\{D(f)\}_{f \in A}$ need not be sufficient to define the full presheaf.

But as alluded to, we can do this for sheaves. We begin by defining a sheaf on a base.

Definition 1.74 (Sheaf on a base)**.** Fix a topological space X and a base $\mathcal{B}$ for its topology. Then a *sheaf on a base valued in* $\mathcal{C}$ is a contravariant functor $F\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ satisfying the following identity and gluability axioms: for any $B \in \mathcal{B}$ with a basic cover $\{B_i\}_{i\in I}$, we have the following.

- Identity: if we have $f_1, f_2 \in F(B)$ such that $f_1|_{B_i} = f_2|_{B_i}$ for all B_i, then $f_1 = f_2$.
- Gluability: if we have $f_i \in F(B_i)$ for each i such that $f_i|_B = f_j|_B$ for each $B \subseteq B_i \cap B_j$, then there is $f \in F(B)$ such that $f|_{B_i} = f_i$ for each i.

Example 1.75. Given a topological space X and a base $\mathcal{B}$, any sheaf $\mathcal{F}\colon (\operatorname{Op} X)^{\mathrm{op}} \to \mathcal{C}$ "restricts" to a sheaf on a base $\mathcal{F}_\mathcal{B}$ by setting $\mathcal{F}_\mathcal{B}(B) := \mathcal{F}(B)$ for all $B \in \mathcal{B}$ and reusing the same restriction maps. The identity and gluability axioms follow from their (stronger) sheaf counterparts; checking this amounts writing down the axioms.

Morphisms are constructed in the obvious way.

Definition 1.76 (Sheaf on a base morphisms)**.** Fix a topological space X and a base $\mathcal{B}$ for its topology. Then a *morphism* between two sheaves F and G on the base $\mathcal{B}$ is a natural transformation of the (underlying) contravariant functors.

Example 1.77. Given a topological space X and a base $\mathcal{B}$, any sheaf morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ restricts in the obvious way to a morphism $\eta_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$ (namely, $(\eta_\mathcal{B})_B = \eta_B$) on the corresponding sheaves on a base. Checking this amounts to saying out loud that the diagram on the left commutes for any $B' \subseteq B$ because it is the same as the diagram on the right.

$$\begin{tikzcd}
\mathcal{F}_\mathcal{B}(B) \arrow[r, "(\eta_\mathcal{B})_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{G}_\mathcal{B}(B) \arrow[d, "\mathrm{res}_{B,B'}"] \\
\mathcal{F}_\mathcal{B}(B') \arrow[r, "(\eta_\mathcal{B})_{B'}"'] & \mathcal{G}_\mathcal{B}(B')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(B) \arrow[r, "\eta_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{G}(B) \arrow[d, "\mathrm{res}_{B,B'}"] \\
\mathcal{F}(B') \arrow[r, "\eta_{B'}"'] & \mathcal{G}(B')
\end{tikzcd}$$

Remark 1.78. Example 1.75 and Example 1.77 combine into the data of a forgetful functor $(-)_\mathcal{B}$ from sheaves on X to sheaves on a base $\mathcal{B}$. Here are the last two checks.

- Identity: given a sheaf $\mathcal{F}$ on X, note $(\mathrm{id}_\mathcal{F})_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{F}_\mathcal{B}$ sends $s \in \mathcal{F}_\mathcal{B}(B) = \mathcal{F}(B)$ to itself, so $(\mathrm{id}_\mathcal{F})_\mathcal{B} = \mathrm{id}\,\mathcal{F}_\mathcal{B}$.
- Functoriality: given sheaf morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ and some $B \in \mathcal{B}$, we see
$$(\varphi\circ\psi)_\mathcal{B}(B) = (\varphi\circ\psi)_B = \varphi_B \circ \psi_B = (\varphi_\mathcal{B}\circ\psi_\mathcal{B})(B).$$

We are interested in showing that we can build a sheaf from a sheaf on a base uniquely, but it will turn out to be fruitful to spend a moment to discuss how this behaves on morphisms first for the uniqueness part of this statement.

Lemma 1.79. Fix a topological space X with a base $\mathcal{B}$ for its topology. Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X with values in $\mathcal{C}$ and a morphism of the (underlying) sheaves on a base $\eta_\mathcal{B}\colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$, there is a unique sheaf morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ such that $(\eta_\mathcal{B})_B = \eta_B$ for each $B \in \mathcal{B}$.

Proof. We show uniqueness before existence.

- Uniqueness: fix any open $U \subseteq X$, and we will try to solve for $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$. Well, fix a basic open cover $\mathcal{U}$ of U; then, for any $B \in \mathcal{U}$, we need the following diagram to commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,B}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,B}"] \\ \mathcal{F}(B) \arrow[r, "\eta_B = (\eta_{\mathcal{B}})_B"'] & \mathcal{G}(B) \end{tikzcd}$$

  In particular, for any $f \in \mathcal{F}(U)$, we need $\eta_U(f)|_B = (\eta_{\mathcal{B}})_B(f|_B)$. Thus, $\eta_U(f)|_B$ is fully specified by the data provided by $\eta_{\mathcal{B}}$, so the identity axiom for $\mathcal{G}$ forces $\eta_U(f)$ to be unique.

- Existence: to begin, fix any open $U \subseteq X$, and we will define $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$. As alluded to above, we let $\mathcal{U}$ be the set of basis elements which are contained in U so that $\mathcal{U}$ is a (large) basic cover of U.

  Then, picking up $f \in \mathcal{F}(U)$, we will try to use the gluability axiom by setting $g_B := (\eta_{\mathcal{B}})_B(f|_B)$ for each $B \in \mathcal{U}$. In particular, for any $B, B' \in \mathcal{U}$, any basic $B_0 \subseteq B \cap B'$ has

$$(g_B|_{B\cap B'})|_{B_0} = g_B|_{B_0} = (\eta_{\mathcal{B}})_B(f|_B)|_{B_0} = (\eta_{\mathcal{B}})_{B_0}(f|_B|_{B_0}) = \eta_{B_0}(f|_{B_0}) = g_{B_0},$$

  which is also $(g_{B'}|_{B\cap B'})_{B_0}$ by symmetry, so the identity axiom applied to $B \cap B'$ implies $g_B|_{B\cap B'} = g_{B'}|_{B\cap B'}$. Thus, the gluability axiom applied to U gives us a unique $g \in \mathcal{G}(U)$ such that

$$g|_B = (\eta_{\mathcal{B}})_B(f|_B)$$

  for each basic set $B \subseteq U$. We define $\eta_U(f) := g$.

  It remains to show that η does in fact assemble into a sheaf morphism. Fix open sets $V \subseteq U$, and we need the following diagram to commute.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\eta_V"'] & \mathcal{G}(V) \end{tikzcd}$$

  Well, pick up any $f \in \mathcal{F}(U)$. Then, for any basic $B \subseteq V \subseteq U$, we see that

$$\eta_U(f)|_B = (\eta_{\mathcal{B}})_B(f|_B) = (\eta_{\mathcal{B}})_B(f|_V|_B),$$

  so the uniqueness of $\eta_V(f|_V)$ forces $\eta_V(f|_V) = \eta_U(f)|_U$. This finishes. ■

1.3.4 Extending a Sheaf on a Base

We dedicate this subsection to the following result, describing how to extend a sheaf on a base to a full sheaf.

Proposition 1.80. Fix a topological space X with a base $\mathcal{B}$ for its topology. Given a sheaf on a base $F : \mathcal{B}^{\text{op}} \to \mathcal{C}$, there is a sheaf $\mathcal{F}$ and isomorphism (of sheaves on a base) $\iota : F \to \mathcal{F}_{\mathcal{B}}$ satisfying the following universal property: any sheaf $\mathcal{G}$ with a morphism (of sheaves on a base) $\varphi : F \to \mathcal{G}_{\mathcal{B}}$ has a unique sheaf morphism $\psi : \mathcal{F} \to \mathcal{G}$ making the following diagram commute.

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{F}_{\mathcal{B}} \arrow[d, "\psi_{\mathcal{B}}"] \\ & \mathcal{G}_{\mathcal{B}} \end{tikzcd} \tag{1.2}$$

Proof. We begin by providing a construction of $\mathcal{F}$. For each open set $U \subseteq X$, define

$$\mathcal{F}(U) := \varprojlim_{B\subseteq U} F(B) = \left\{(f_B)_{B\subseteq U} \in \prod_{B\subseteq U} F(B) : f_B|_{B'} = f_{B'} \text{ for each } B' \subseteq B \subseteq U\right\}.$$

(Namely, we are implicitly assuming that our target category has limits.) Observe that, when $V \subseteq U$, the natural surjection

$$\prod_{B\subseteq U} \mathcal{F}(B) \to \prod_{B\subseteq V} \mathcal{F}(B)$$

induces a map $\mathcal{F}(U) \to \mathcal{F}(V)$. Indeed, an element $(f_B)_{B\subseteq U} \in \mathcal{F}(U)$ gets sent to $(f_B)_{B\subseteq V}$, and it is still the case that $B' \subseteq B \subseteq V$ implies $f_B|_{B'} = f_{B'}$ because actually $B' \subseteq B \subseteq U$. Thus, $(f_B)_{B\subseteq V} \in \mathcal{F}(V)$, so we have a well-defined map

$$\begin{aligned} \operatorname{res}_{U,V} : \quad \mathcal{F}(U) &\to \mathcal{F}(V) \\ (f_B)_{B\subseteq U} &\mapsto (f_B)_{B\subseteq V} \end{aligned}$$

which will serve as our restrictions. We start by checking that these data assemble into a presheaf.

- When $U = V$, we are sending $(f_B)_{B\subseteq U} \in \mathcal{F}(U)$ to itself, so $\operatorname{res}_{U,U} = \operatorname{id}_{\mathcal{F}(I)}$.
- Given $W \subseteq V \subseteq U$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\operatorname{res}_{U,V}"] \arrow[dr, "\operatorname{res}_{U,W}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,W}"] \\ & \mathcal{F}(W) \end{tikzcd} \qquad \begin{tikzcd} (f_B)_{B\subseteq U} \arrow[r, mapsto] \arrow[dr, mapsto] & (f_B)_{B\subseteq V} \arrow[d, mapsto] \\ & (f_B)_{B\subseteq W} \end{tikzcd}$$

  commutes, which is our functoriality check.

We now show that these data make a sheaf. Fix an open set $U \subseteq X$ with an open cover $\mathcal{U}$. To help our constructions, given any open subset $V \subseteq X$, let $\mathcal{B}_V$ denote the collection of basis elements B contained in V; notably $\mathcal{B}_V$ is a basic cover for V. Then, for any open $U' \subseteq U$, we let

$$\mathcal{S}_{U'} := \bigcup_{V\subseteq U} \mathcal{S}_{U'\cap V}.$$

Notably, $\mathcal{S}_{U'}$ is a basic cover for U' such that any $B \in \mathcal{S}_{U'}$ is contained in some element of $\mathcal{U}$.

- Identity: suppose that $(f_B)_{B\subseteq U}, (g_B)_{B\subseteq U} \in \mathcal{F}(U)$ restrict to the same element on any $V \in \mathcal{U}$. Now, fix any $B_0 \subseteq U$, and we will show $f_{B_0} = g_{B_0}$.

  Now consider $\mathcal{S}_{B_0}$: for each $B' \in \mathcal{S}$, we can find $V \in \mathcal{U}$ so that $B' \subseteq V$, for which we know

$$(f_B)_{B\subseteq V} = (g_B)_{B\subseteq V}.$$

  In particular $f_{B_0}|_{B'} = f_{B'} = g_{B'} = g_{B_0}|_{B'}$ for any $B \in \mathcal{S}$, so the identity axiom for the sheaf on a base F forces $f_{B_0} = g_{B_0}$.
- Gluability: suppose we are given some $(f_{V,B})_{B\subseteq V} \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that

$$(f_{V,B})_{B\subseteq V\cap V'} = (f_{V,B})_{B\subseteq V}|_{V\cap V'} = (f_{V',B})_{B\subseteq V}|_{V\cap V'} = (f_{V',B})_{B\subseteq V\cap V'}$$

  for any $V, V' \in \mathcal{U}$. In other words, for any basic $B \subseteq V \cap V'$, we have $f_{V,B} = f_{V',B}$.

  Now, for any basic $B_0 \subseteq U$, we will solve for f_{B_0}. Using $\mathcal{S}_{B_0}$, note that any $B \in \mathcal{S}_{B_0}$ has some $V_B \in \mathcal{U}$ such that $B \subseteq V_B$, so we will use $f_{V_B,B}$ at this point. Note that if $B \subseteq V_B'$ as well, then $f_{V_B,B} = f_{V_B',B}$, so our $f_{V_B,B}$ is independent of V_B. Continuing, if we have $B \subseteq B_1 \cap B_2$, then

$$f_{V_{B_1},B_1}|_B = f_{V_{B_1},B} = f_{V_{B_2},B} = f_{V_{B_2},B_2}|_B,$$

so gluability applied to our sheaf F on a base promises us a unique f_{B_0} such that $f_{B_0}|_B = f_{V_B,B}$ for any $B \in \mathcal{S}_{B_0}$.

We now need to show that the $(f_B)_{B \subseteq U}$ assemble into an element of $\mathcal{F}(U)$. Namely, if we have $B_0' \subseteq B_0$, we need to show that $f_{B_0}|_{B_0'} = f_{B_0'}$. Well, for any $B \in \mathcal{S}_{B_0'}$, we compute

$$f_{B_0}|_{B_0'}|_B = f_{B_0}|_B = f_{V_B,B} = f_{B_0}|_B,$$

so the uniqueness of f_{B_0} gives the equality.

For our next step, we define $\iota_{B_0} \colon F(B) \to \mathcal{F}_\mathcal{B}(B_0)$ by

$$\iota_{B_0}(f) := (f|_B)_{B \subseteq B_0}.$$

Here are the checks on ι.

- Well-defined: note $\iota_{B_0}(f)$ is an element of $\mathcal{F}_\mathcal{B}(B_0)$ because $B' \subseteq B \subseteq B_0$ will have $f|_B|_{B'} = f|_{B'}$.
- Natural: if $B \subseteq B'$, then note that the diagrams

$$\begin{tikzcd} F(B_0) \arrow[r, "\iota_B"] \arrow[d, "\mathrm{res}_{B,B'}"'] & \mathcal{F}_\mathcal{B}(B_0) \arrow[d, "\mathrm{res}_{B,B'}"] \\ F(B_0') \arrow[r, "\iota_{B'}"'] & \mathcal{F}_\mathcal{B}(B_0') \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & (f|_B)_{B \subseteq B_0} \arrow[d, mapsto] \\ f|_{B_0'} \arrow[r, mapsto] & (f|_B)_{B \subseteq B_0'} \end{tikzcd}$$

  commute, finishing.
- Injective: suppose that $f, g \in F(B_0)$ have the same image in $\mathcal{F}_\mathcal{B}(B_0)$. This means that $(f|_B)_{B \subseteq B_0} = (g|_B)_{B \subseteq B_0}$, so $f = f|_{B_0} = g|_{B_0} = g$, so we are done.
- Surjective: fix some $(f_B)_{B \subseteq B_0} \in \mathcal{F}_\mathcal{B}(B_0)$. Notably, for any basic $B_1, B_2 \subseteq B_0$ with some basic $B \subseteq B_1 \cap B_2$, we have

$$f_{B_1}|_B = f_B = f_{B_2}|_B,$$

  so gluability applied to F promises $f \in F(B_0)$ such that $f|_B = f_B$ for all basic $B \subseteq B_0$. So $\iota_{B_0}(f) = (f_B)_{B \subseteq B_0}$.

We now begin showing that $\mathcal{F}$ satisfies the universal property. Fix some sheaf $\mathcal{G}$ on X with a morphism $\varphi \colon F \to \mathcal{G}_\mathcal{B}$.

In light of Lemma 1.79, it suffices to show the existence and uniqueness of a morphism $\psi_\mathcal{B} \colon \mathcal{F}_\mathcal{B} \to \mathcal{G}_\mathcal{B}$ on the base $\mathcal{B}$ making (1.2) commute. Namely, the existence of $\psi_\mathcal{B}$ promises a full sheaf morphism $\psi \colon \mathcal{F} \to \mathcal{G}$ extending via Lemma 1.79; for uniqueness, two possible $\psi, \psi' \colon \mathcal{F} \to \mathcal{G}$ with $\psi_\mathcal{B}$ and $\psi'_\mathcal{B}$ both commuting will enforce $\psi_\mathcal{B} = \psi'_\mathcal{B}$ and then $\psi = \psi'$ by the uniqueness of Lemma 1.79.

Continuing with the proof, we note that the fact that ι is an isomorphism means that the commutativity of (1.2) is equivalent to the diagram

$$\begin{tikzcd} F \arrow[dr, "\varphi"'] & \mathcal{F}_\mathcal{B} \arrow[l, "\iota^{-1}"'] \arrow[d, "\psi_\mathcal{B}"] \\ & \mathcal{G}_\mathcal{B} \end{tikzcd}$$

commuting. However, the commutativity of this diagram is equivalent to setting $\psi_\mathcal{B} := \varphi \circ \iota^{-1}$. Thus, uniqueness of $\psi_\mathcal{B}$ is immediate, and existence of $\psi_\mathcal{B}$ amounts to noting the composition of natural transformations remains a natural transformation. ■

Remark 1.81. One can also define $\mathcal{F}(U)$ as compatible systems of stalks, but we have not defined stalks yet.

Remark 1.82. The universal property implies that the pair $(\mathcal{F}, \iota)$ is unique up to unique isomorphism, for a suitable notion of unique isomorphism. Namely, the usual abstract nonsense arguments with universal properties is able to show that if we have another sheaf $\mathcal{F}'$ with isomorphism $\iota' \colon F \to \mathcal{F}'_{\mathcal{B}}$ satisfying the universal property, then $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic. (This isomorphism $\eta \colon \mathcal{F} \cong \mathcal{F}'$ is unique if we ask for the corresponding diagram

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \mathcal{F}_{\mathcal{B}} \arrow[d, "\eta_{\mathcal{B}}"] \\ & \mathcal{F}'_{\mathcal{B}} \end{tikzcd}$$

to commute.)

Remark 1.83. Here is another universal property: given a sheaf $\mathcal{G}$ with a morphism $\varphi \colon \mathcal{G}_{\mathcal{B}} \to F$ of sheaves on the base $\mathcal{B}$, there is a unique sheaf morphism $\psi \colon \mathcal{G} \to \mathcal{F}$ making the diagram

$$\begin{tikzcd} \mathcal{G}_{\mathcal{B}} \arrow[d, "\psi_{\mathcal{B}}"'] \arrow[dr, "\varphi"] & \\ \mathcal{F}_{\mathcal{B}} \arrow[r, "\iota^{-1}"'] & F \end{tikzcd}$$

commute. Indeed, reversing the arrow ι shows that we are asking for a unique sheaf morphism ψ such that $\psi_{\mathcal{B}} = \iota \circ \varphi$, which we get from Lemma 1.79.

The universal property actually gives a functor from sheaves F on a base $\mathcal{B}$ to sheaves $\mathcal{F}_B$ on X.

Lemma 1.84. Fix a topological space X with a base $\mathcal{B}$ for its topology. Then the map sending a sheaf F on a base to its sheaf $\mathcal{E}(F)$ describes the action of a functor on objects.

Proof. Given a sheaf on a base F, let $\iota_F \colon F \to \mathcal{E}(F)_{\mathcal{B}}$ be the inclusion. Now, given a morphism $\varphi \colon F \to G$ of sheaves on a base, note that there is a unique morphism $\mathcal{E}(\varphi)$ making the diagram

$$\begin{tikzcd} F \arrow[r, "\iota_F"] \arrow[d, "\varphi"'] & \mathcal{E}(F)_{\mathcal{B}} \arrow[d, dashed, "\mathcal{E}(\varphi)_{\mathcal{B}}"] \\ G \arrow[r, "\iota_G"] & \mathcal{E}(G)_{\mathcal{B}} \end{tikzcd}$$

commute by Lemma 1.79. We now need to show that this data assembles into a functor.

- Identity: given a sheaf on a base F, note that id_F induces the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\iota_F"] \arrow[d, "\mathrm{id}_F"'] & \mathcal{E}(F)_{\mathcal{B}} \arrow[d, dashed, "(\mathrm{id}_{\mathcal{E}(F)})_{\mathcal{B}}"] \\ F \arrow[r, "\iota_F"] & \mathcal{E}(F)_{\mathcal{B}} \end{tikzcd}$$

  which makes us conclude $\mathcal{E}(\mathrm{id}_F) = \mathrm{id}_{\mathcal{E}(F)}$.

- Functoriality: given morphisms $\varphi \colon F \to G$ and $\psi \colon G \to H$ of sheaves on a base, we note that the diagram

$$\begin{tikzcd} F \arrow[r, "\iota_F"] \arrow[d, "\varphi"'] & \mathcal{E}(F)_{\mathcal{B}} \arrow[d, "\mathcal{E}(\varphi)_{\mathcal{B}}"'] \arrow[dd, bend left, dashed, "\mathcal{E}(\psi\circ\varphi)_{\mathcal{B}}"] \\ G \arrow[r, "\iota_G"] \arrow[d, "\psi"'] & \mathcal{E}(G)_{\mathcal{B}} \arrow[d, "\mathcal{E}(\psi)_{\mathcal{B}}"'] \\ H \arrow[r, "\iota_H"] & \mathcal{E}(H)_{\mathcal{B}} \end{tikzcd}$$

commutes, so the uniqueness of the arrow $\mathcal{E}(\psi \circ \varphi)_{\mathcal{B}}$ forces $\mathcal{E}(\psi \circ \varphi) = \mathcal{E}(\psi) \circ \mathcal{E}(\varphi)$. ■

Remark 1.85. In fact, the functor $\mathcal{E}$ is the right adjoint to the forgetful functor $(-)_{\mathcal{B}}$ from sheaves on a base $\mathcal{B}$ to sheaves on X, which also essentially follows from the universal property. We will not bother showing this.

1.4 August 31

We finish defining the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ of an affine scheme today.

Remark 1.86. One complaint about sheaves on a base is that we have to choose a base. To be more canonical, we will discuss stalks today, which treats all points the same.

1.4.1 The Structure Sheaf

We are now ready to define the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ of a ring A, which we will define a sheaf on a base. Recall from Remark 1.55 that $\{D(f)\}_{f\in A}$ forms a base of the Zariski topology of $\operatorname{Spec} A$, so it will suffice to set

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) := A_{S(D(f))},$$

where

$$S(D(f)) := \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f)\}.$$

In other words, $S(D(f))$ consists of the set of functions in A which only vanish outside $D(f)$ so that we can invert them on $D(f)$. Namely, $g \in S(D(f))$ means that $g \in \mathfrak{p}$ implies $\mathfrak{p} \notin D(f)$, which is the same as $f \in \mathfrak{p}$.

Remark 1.87. In essence, $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ is supposed to be the functions on $D(f)$, which is why we want to be able to invert functions which only vanish on $(\operatorname{Spec} A) \setminus D(f)$.

Remark 1.88. The subset $S(D(f))$ only depends on $D(f)$, not f, so $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ is well-defined. With that said, we note that $f \in S(D(f))$ gives a natural localization map $A_f \to A_{S(D(f))}$ induced by id_A. Similarly, any $g \in S(D(f))$ has $V((g)) \subseteq V((f))$ and so Proposition 1.45 tells us that

$$\operatorname{rad}(f) = I(V((f))) \subseteq I(V((g))) = \operatorname{rad}(g),$$

so $f \in \operatorname{rad}(g)$, so $f^n = ag$ for some positive integer n and $a \in A$; this means $g \in A_f^\times$, so actually $S(D(f)) \subseteq A_f^\times$, allowing another natural localization map $A_{S(D(f))} \to A_f$ induced by id_A. These natural localization maps are inverse (their compositions are induced by id_A), so $\mathcal{O}_{\operatorname{Spec} A}(D(f)) \cong A_f$.

Remark 1.89 (Nir). In class, Professor Tang defined the structure sheaf on a base by $\mathcal{O}_{\operatorname{Spec} A}(D(f)) := A_f$. I have chosen to follow [Vak17] here because I don't like $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ to depend on $f \in A$ when it should only depend on $D(f)$.

To define our (pre)sheaf on a base, we also need to provide restriction maps. Well, for $f, f' \in A$ with $D(f') \subseteq D(f)$, we see that

$$S(D(f)) = \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f)\} \subseteq \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f')\} = S(D(f')),$$

so there is a natural localization map

$$\operatorname{res}_{D(f),D(f')} \colon A_{S(D(f))} \to A_{S(D(f'))}$$

induced by id_A. These data give all the data we need to define a sheaf on a base. We will throw the remaining checks into the following lemma.

Lemma 1.90. Fix a ring A. The above data define a sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on the base $\{D(f)\}_{f \in A}$.

Proof. We begin by showing that the data gives a presheaf.

- Identity: if $D(f) = D(f')$, then $S(D(f)) = S(D(f'))$, so the localization map

$$\operatorname{res}_{D(f),D(f)} \colon A_{S(D(f))} \to A_{S(D(f))}$$

  is simply $\operatorname{id}_{A_{S(D(f))}}$.
- Functoriality: suppose $D(f'') \subseteq D(f') \subseteq D(f)$. Then we note that the diagram

$$\begin{tikzcd} A_{S(D(f))} \arrow[r, "\text{res}"] \arrow[dr, "\text{res}"'] & A_{S(D(f'))} \arrow[d, "\text{res}"] \\ & A_{S(D(f''))} \end{tikzcd} \qquad \begin{tikzcd} a/g \arrow[r, mapsto] \arrow[dr, mapsto] & a/g \arrow[d, mapsto] \\ & a/g \end{tikzcd}$$

  commutes because everything is induced by id_A, so we are done.

It remains to check the identity and gluability axioms. For this, we will need a basis set $D(f)$ and a basic cover $\{D(f_\alpha)\}_{\alpha\in\lambda}$. To access this cover, we have the following lemma.

Lemma 1.91. Fix a ring A. Then, given $f \in A$ and $\{f_\alpha\}_{\alpha\in\lambda} \subseteq A$, the following are equivalent.

(a) $D(f) \subseteq \bigcup_{\alpha\in\lambda} D(f_\alpha)$.

(b) $f \in \operatorname{rad}(f_\alpha)_{\alpha\in\lambda}$.

Proof. Note that

$$\bigcup_{\alpha\in\lambda} D(f_\alpha) = \operatorname{Spec} A \setminus \bigcap_{\alpha\in\lambda} V((f_\alpha)) = \operatorname{Spec} A \setminus V((f_\alpha)_{\alpha\in\lambda}),$$

so (a) is equivalent to $V((f_\alpha)_{\alpha\in\lambda}) \subseteq V((f))$. Now, Proposition 1.45 tells us that (a) implies

$$\operatorname{rad}(f) = I(V((f))) \subseteq I(V((f_\alpha)_{\alpha\in\lambda})) = \operatorname{rad}(f_\alpha)_{\alpha\in\lambda},$$

from which (b) follows. Conversely, if (b) holds, then $\operatorname{rad}(f) \subseteq \operatorname{rad}(f_\alpha)_{\alpha\in\lambda}$ by taking radicals, so Proposition 1.45 again promises

$$V((f_\alpha)_{\alpha\in\lambda}) = V(\operatorname{rad}(f_\alpha)_{\alpha\in\lambda}) \subseteq V(\operatorname{rad}(f)) = V((f)),$$

which we showed is equivalent to (a). ■

Corollary 1.92. Fix a ring A. Then any cover $\{D(f_\alpha)\}_{\alpha\in\lambda}$ of $D(f)$ has a finite subcover.

Proof. Note Lemma 1.91 tells us that $f \in \operatorname{rad}(f_\alpha)_{\alpha\in\lambda}$, so there is a positive integer n and finite subset $\lambda \subseteq \lambda'$ so that

$$f^n = \sum_{\alpha\in\lambda'} a_\alpha f_\alpha,$$

but then $f \in \operatorname{rad}(f_\alpha)_{\alpha\in\lambda'}$, so $D(f)$ is covered by the (finite) cover $\{D(f_\alpha)\}_{\alpha\in\lambda'}$. ■

We now show the identity and gluability axioms separately.

- Identity: note Corollary 1.92 promises us some $\lambda' \subseteq \lambda$ such that the $\{D(f_\alpha)\}_{\alpha\in\lambda'}$ still covers $D(f)$. We will now forget about λ entirely and deal with the finite λ' instead.

  For identity, we suppose that we have $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ such that $s|_{D(f_\alpha)} = 0$ for all $\alpha \in \lambda'$, and we want to show that $s = 0$. Under the (canonical) isomorphism $\mathcal{O}_{\operatorname{Spec} A}(D(f_\alpha)) \simeq A_{f_\alpha}$, we see that we must have

  $$f_\alpha^{d_\alpha} s = 0$$

  for some d_α, for each α. Now, $D(f_\alpha) = D\left(f_\alpha^{d_\alpha}\right)$, so the $D\left(f_\alpha^{d_\alpha}\right)$ still cover $D(f)$; it follows from Lemma 1.91 that there is some d for which

  $$f^d = \sum_{\alpha\in\lambda'} c_\alpha f_\alpha^{d_\alpha}.$$

  Multiplying both sides by s (after embedding in $A_{S(D(f))}$) tells us that $f^d s = 0$ in $A_{S(D(f))}$, so $s = 0$ because $f \in A_{S(D(f))}^\times$.

- Finite gluability: fix sections $s_\alpha \in \mathcal{O}_{\operatorname{Spec} A}(D(f_\alpha))$ such that

  $$s_\alpha|_{D(f_\alpha)\cap D(f_\beta)} = s_\beta|_{D(f_\alpha)\cap D(f_\beta)}.$$

  For concreteness, use $\mathcal{O}_{\operatorname{Spec} A}(D(f)) \simeq A_f$ to write $s_\alpha := a_\alpha/f_\alpha^n$, where n is the maximum of all the possibly needed denominators.

  Noting that $D(f_\alpha) \cap D(f_\beta) = D(f_\alpha f_\beta)$, our coherence is equivalent to asking for

  $$(f_\alpha f_\beta)^m \left(f_\beta^n a_\alpha - f_\alpha^n a_\beta\right) = 0,$$

  where again m is chosen to be large enough among the finitely many possibilities for α and β. We now notice that

  $$s_\alpha = \frac{a_\alpha}{f_\alpha^n} = \frac{f_\alpha^m a_\alpha}{f_\alpha^{n+m}},$$

  so we set $b_\alpha := f_\alpha^m a_\alpha$ and $g_\alpha := f_\alpha^{n+m}$, which means

  $$g_\beta b_\alpha = g_\alpha b_\beta$$

  for all α, β. Notably, $\operatorname{rad}(f_\alpha) = \operatorname{rad}(g_\alpha)$, so $D(f_\alpha) = D(g_\alpha)$, so the $\{D(g_\alpha)\}_{\alpha\in\lambda}$ still cover $D(f)$, so Lemma 1.91 tells us that we can write

  $$f^n = \sum_{\alpha\in\lambda} c_\alpha g_\alpha$$

  for some positive integer n. In particular, we set $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f)) \simeq A_f$ by

  $$s := \frac{1}{f^n}\sum_{\alpha\in\lambda} c_\alpha b_\alpha.$$

  In particular, for any $\beta \in \lambda$, we see

  $$g_\beta s = \frac{1}{f^n}\sum_{\alpha\in\lambda} c_\alpha g_\beta b_\alpha = \frac{1}{f^n}\sum_{\alpha\in\lambda} c_\alpha g_\alpha b_\beta = b_\beta$$

  in A_f, so our restriction is $s|_{D(g_\beta)} = b_\beta/g_\beta = s_\beta$, which is what we wanted.

- Gluability: we show general gluability from finite gluability. Fix sections $s_\alpha \in \mathcal{O}_{\operatorname{Spec} A}(D(f_\alpha))$ such that

$$s_\alpha|_{D(f_\alpha)\cap D(f_\beta)} = s_\beta|_{D(f_\alpha)\cap D(f_\beta)} \tag{1.3}$$

  for each $\alpha, \beta \in \lambda$. Using Corollary 1.92, we can find a finite subcover using $\lambda' \subseteq \lambda$, and the sections $\{s_\alpha\}_{\alpha\in\lambda'}$ still satisfy (1.3), so finite gluability (and identity!) gives a unique $s \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ with

$$s|_{D(f_\alpha)} = s_\alpha.$$

  We claim that actually $s|_{D(f_\alpha)} = s_\alpha$ for all $\alpha \in \lambda$. Well, for any $\beta \in \lambda$, apply finite gluability to $\lambda' \cup \{\beta\}$ to find $s' \in \mathcal{O}_{\operatorname{Spec} A}(D(f))$ such that $s'|_{D(f_\alpha)} = s_\alpha$ for all $\alpha \in \lambda' \cup \{\beta\}$.
  It follows from the identity axiom that on the open cover $\{D(f_\alpha)\}_{\alpha\in\lambda'}$ that $s = s'$, so we conclude

$$s|_{D(f_\beta)} = s'|_{D(f_\beta)} = s_\beta$$

  for any $\beta \in \lambda$. ■

Having finished the last of our checks, we see that our data make a sheaf on a base, so Proposition 1.80 promises a unique sheaf extending this sheaf on a base. This is the (affine) structure sheaf, and it finishes our definition of an affine scheme.

Definition 1.93 (Affine scheme)**.** Fix a ring A. An *affine scheme* is the topological space $\operatorname{Spec} A$ (given the Zariski topology) together with the sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ such that

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_{S(D(f))}$$

for each $f \in A$; here $S(D(f)) = \{g \in A : D(f) \subseteq D(g)\}$.

Note that we are somewhat sloppily identifying the outputs of the structure sheaf with its outputs on the base.

1.4.2 Stalks

To define a morphism of schemes, we will want to discuss stalks.

Remark 1.94. We might expect a morphism of (affine) schemes to be merely a continuous map together with a natural transformation of the structure sheaves (perhaps with some coherence conditions). However, this will not be enough data. Namely, we want all of our morphisms of affine schemes to be induced by ring homomorphisms, and this will require exploiting a little more data.

The extra data in those morphisms will come from stalks.

Definition 1.95 (Stalk)**.** Fix a presheaf $\mathcal{F}$ on a topological space X. For a point $p \in X$, we define the *stalk of $\mathcal{F}$ at $p \in X$* to be the direct limit

$$\mathcal{F}_p := \varinjlim_{U \ni p} \mathcal{F}(U).$$

Concretely, elements of $\mathcal{F}_p$ are ordered pairs (U, s) where $s \in \mathcal{F}(U)$ with $p \in U$, modded out by an equivalence relation $\sim$; here, $(U, s) \sim (U', s')$ if and only if there is $W \subseteq U \cap U'$ such that $s|_W = s'|_W$.

Remark 1.96 (Nir)**.** In some sense, the stalk is intended to encode "local information" at the point $p \in X$ in a particularly violent way: whenever two functions $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ (where $p \in U_1 \cap U_2$) are equal locally on some open set U containing p, then we identify s_1 and s_2. As such, $\mathcal{F}_p$ can really study functions locally at p.

Remark 1.97. We go ahead and check that $\sim$ forms an equivalence relation. Fix (U_i, s_i) with $s_i \in \mathcal{F}(U_i)$ for $i \in \{1,2,3\}$.

- Reflexive: note $U_1 \subseteq U_1$ and $s_1|_{U_1} = s_1 = s_1|_{U_1}$, so $(U_1, s_1) \sim (U_1, s_1)$.
- Symmetry: if $(U_1, s_1) \sim (U_2, s_2)$, we can find an open $V \subseteq U_1 \cap U_2$ with $s_1|_V = s_2|_V$, which implies $s_2|_V = s_1|_V$, so $(U_2, s_2) \sim (U_1, s_1)$.
- Transitive: if $(U_1, s_1) \sim (U_2, s_2)$ and $(U_2, s_2) \sim (U_3, s_3)$, we can find open $V_1 \subseteq U_1 \cap U_2$ and $V_2 \subseteq U_2 \cap U_3$ such that $s_1|_{V_1} = s_2|_{V_1}$ and $s_2|_{V_2} = s_3|_{V_2}$. Then $V_1 \cap V_2 \subseteq U_1 \cap U_3$, and we can see
$$s_1|_{V_1\cap V_2} = s_1|_{V_1}|_{V_1\cap V_2} = s_2|_{V_1}|_{V_1\cap V_2} = s_2|_{V_1\cap V_2} = s_2|_{V_2}|_{V_1\cap V_2} = s_3|_{V_2}|_{V_1\cap V_2} = s_3|_{V_1\cap V_2}.$$

Definition 1.98 (Germ)**.** Fix a presheaf $\mathcal{F}$ on a topological space X. For a point $p \in X$ and section $s \in \mathcal{F}(U)$ with $p \in U$, the *germ of s at p* is the element
$$[(U, s)] \in \mathcal{F}_p.$$

Notation 1.99. I will write the germ of $f \in \mathcal{F}(U)$ at $p \in U$ as $f|_p$. This notation is not standard, but I like it because I think of taking the germ of a section at p as analogous to "restricting" to the point p.

As a warning, later on, we will want to consider tuples of sections $(f_p)_p$, and we will want to distinguish the notation for an element of this tuple as f_p with the notation for the corresponding germ $f|_p$.

Remark 1.100. As justification for my notation, if $f \in \mathcal{F}(U)$ while $p \in V \subseteq U$, then
$$f|_p = f|_V|_p$$
because $[(U, f)] = [(V, f|_V)]$ can be witnessed by $f|_V = f|_V|_V$.

Here are some examples of stalks.

Lemma 1.101. Fix a presheaf $\mathcal{F}$ on a topological space X, and give the topology on X a base $\mathcal{B}$. For a point $p \in X$, we have the isomorphism
$$\begin{aligned} \varphi\colon \varinjlim_{B \ni p} \mathcal{F}(B) &\simeq \mathcal{F}_p \\ [(B, s)] &\mapsto [(B, s)] \end{aligned}$$
where the colimit is taken over $B \in \mathcal{B}$ such that $p \in B$.

Proof. The main point to show that φ is well-defined is that the system of maps $\mathcal{F}(B) \to \mathcal{F}_p$ for each $B \in \mathcal{B}$ containing p induce the map φ by the universal property. Concretely, if $(B_1, s_1) \sim (B_2, s_2)$, then we can find $B \subseteq B_1 \cap B_2$ such that $s_1|_B = s_2|_B$, which means that $[(B_1, s_1)] = [(B_2, s_2)]$ in $\varinjlim_{B \ni p} \mathcal{F}(B)$ implies the equality in $\mathcal{F}_p$. Now, any structure that φ needs to preserve (e.g., being a homomorphism of some kind) will be immediately preserved.

We now exhibit the map in the reverse direction. Note that any $U \subseteq X$ containing p can find some basis element $B \in \mathcal{B}$ such that $p \in B \subseteq U$. As such, we define $\psi\colon \mathcal{F}_p \to \varinjlim_{B \ni p} \mathcal{F}(B)$ by
$$\psi\colon [(U, s)] \mapsto [(B, s|_B)].$$
To show that this map is well-defined, first note that ψ does not depend on B: if we have basis sets B_1 and B_2 inside U containing p, we can find basis sets $B \subseteq B_1 \cap B_2$ giving $s|_{B_1}|_B = s|_B = s|_{B_2}|_B$, so
$$[(B_1, s|_{B_1})] = [(B_2, s|_{B_2})].$$

Second, note that ψ does not depend on the representative of $[(U,s)]$. Indeed, if $(U_1,s_1)\sim(U_2,s_2)$, then we are promised $U\subseteq U_1\cap U_2$ such that $s_1|_U=s_2|_U$. Now, find B contained in U containing p, so we see $s_1|_B=s_2|_B$, so

$$[(B,s_1|_B)]=[(B,s_2|_B)].$$

So we have a well-defined map ψ.

We now show that ψ and φ are inverse. In one direction, given some $[(B,s)]$, we note we can write

$$\psi(\varphi([(B,s)]))=\psi([(B,s)])=[(B,s)],$$

where the last equality is legal because B is a basis set containing p which is contained in B. In the other direction, given some $[(U,s)]$, find a basis set $B\subseteq U$ containing p so that

$$\varphi(\psi([(U,s)]))=\varphi([(B,s|_B)])=[(B,s|_B)],$$

and we note that $[(B,s|_B)]=[(U,s)]$ because $B\subseteq U$ has $s|_B|_B=s|_B$. ■

Lemma 1.102. Fix a ring A. Then, for any prime $\mathfrak{p}$, $A_\mathfrak{p}\simeq\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$ induced by $a\mapsto a|_\mathfrak{p}$.

Proof. The point here is that $\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$ permits denominators from anyone in $A\setminus\mathfrak{p}$. In one direction, note that $A=\mathcal{O}_{\operatorname{Spec}A}(\operatorname{Spec}A)$, so there is a canonical map

$$\begin{array}{ccc}\mathcal{O}_{\operatorname{Spec}A}(\operatorname{Spec}A) & \to & \mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}\\ s & \mapsto & s|_\mathfrak{p}\end{array}$$

because $\mathfrak{p}\in\operatorname{Spec}A$. Call this map φ. Note, for any $f\in A\setminus\mathfrak{p}$, we see that $\mathfrak{p}\in D(f)$, so the canonical map

$$\mathcal{O}_{\operatorname{Spec}A}(D(f))\to\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$$

permits us to write

$$[(\operatorname{Spec}A,f)]\cdot[(D(f),1/f)]=[(D(f),f)]\cdot[(D(f),1/f)]=[(D(f),1)]$$

is the unit element of $\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$. Thus, $\varphi(f)\in\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}^\times$ for each $f\in A\setminus\mathfrak{p}$, so φ induces a natural map $\varphi\colon A_\mathfrak{p}\to\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$ sending a/f to $[(D(f),a/f)]$.

In the other direction, we can directly pick up any $[(D(f),a/f^n)]\in\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$, where we are thinking about the colimit as happening over the distinguished base according to Lemma 1.101. Now, $\mathfrak{p}\in D(f)$ is equivalent to $f\notin\mathfrak{p}$, so $f\in A_\mathfrak{p}^\times$, so we can define $\psi\colon\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}\to A_\mathfrak{p}$ by

$$\psi\colon\ [(D(f),a/f^n)]\mapsto a/f^n.$$

To see that ψ is well-defined, note $(D(f_1),a_1/f_1^{n_1})\sim(D(f_2),a_2/f_2^{n_2})$ means we can find $D(f)\subseteq D(f_1)\cap D(f_2)$ containing $\mathfrak{p}$ with

$$f^n\left(f_2^{n_2}a_1-f_1^{n_1}a_2\right)=0$$

in A. Rearranging, it follows that $a_1/f_1^{n_1}=a_2/f_2^{n_2}$ in $A_\mathfrak{p}$.

We won't bother checking that ψ is a ring map; just look at it. However, we will check that ψ and φ are inverses (which tells us that ψ is a ring map automatically). Well, given $a/f\in A_\mathfrak{p}$, we see

$$\psi(\varphi(a/f))=\psi([(D(f),a/f)])=a/f.$$

On the other hand, given $[(D(f),a/f^n)]\in\mathcal{O}_{\operatorname{Spec}A,\mathfrak{p}}$, write

$$\varphi(\psi([(D(f),a/f^n)]))=\varphi\left(a/f^n\right)=[(D(f^n),a/f^n)]=[(D(f),a/f)],$$

where the last equality holds because $D(f^n)=(\operatorname{Spec}A)\setminus V((f^n))=(\operatorname{Spec}A)\setminus V(\operatorname{rad}(f^n))=(\operatorname{Spec}A)\setminus V((f))=D(f)$. ■

Remark 1.103. Notably, $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ is always a local ring, and the maximal ideal $\mathfrak{p}A_\mathfrak{p}$ corresponds to germs $[(D(f), a/f)]$ such that $a/f \in \mathfrak{p}A_\mathfrak{p}$, or equivalently, such that $a \in \mathfrak{p}$. Namely, the maximal ideal consists of our germs which vanish at $\mathfrak{p}$.

Example 1.104. Continuing from Exercise 1.72, set $X := \mathbb{C}$ and $\mathcal{O}_X$ to be the sheaf of holomorphic functions. Then, for any $z_0 \in X$, we have

$$\mathcal{O}_{X,z_0} = \left\{ \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ with positive radius of convergence} \right\}.$$

Indeed, any germ $[(U, f)]$ with f holomorphic actually has f analytic, so f is equal to a (unique) power series of the given form in some small enough neighborhood. And of course, each power series with positive radius of convergence gives rise to a germ.

Remark 1.105 (Nir). As in Remark 1.103, we note that $\mathcal{O}_{X,z_0}$ is a local ring with maximal ideal

$$\mathfrak{m}_{X,z_0} = \left\{ \sum_{n=1}^{\infty} a_n(z-z_0)^n \text{ with positive radius of convergence} \right\}.$$

Of course $\mathfrak{m}_{X,z_0} \subseteq \mathcal{O}_{X,z_0}$ is an ideal. Conversely, one can see that any germ $[(f,U)]$ with $f(z_0) \neq 0$ is nonzero in some neighborhood around z_0 (by continuity) and therefore is invertible in $\mathcal{O}_{X,z_0}$, so $\mathcal{O}_{X,z_0} \setminus \mathfrak{m}_{X,z_0} = \mathcal{O}_{X,z_0}^\times$.

1.4.3 Stalk Memory

Here is why we care about stalks.

Idea 1.106. Stalks remember everything about a sheaf.

Again, the reason why we expect Idea 1.106 to be true is that the stalk is able to remember local information, so having all the local information should be able to recover the original sheaf. Here is a rigorization.

Proposition 1.107. Fix a sheaf $\mathcal{F}$ and a presheaf $\mathcal{G}$ on X. Also, fix an open subset $U \subseteq X$.

(a) The natural embedding

$$\begin{aligned} \iota\colon \mathcal{F}(U) &\to \prod_{p\in U} \mathcal{F}_p \\ f &\mapsto (f|_p)_{p\in U} \end{aligned}$$

is injective.

(b) A tuple $(f_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p$ is in $\operatorname{im}\iota$ if and only if, for each $p \in U$, there is an open set U_p containing p such that we can find $\widetilde{f_p} \in \mathcal{F}(U_p)$ such that all $q \in U_p$ have $f_q = \widetilde{f_p}|_q$.

Remark 1.108. Intuitively, part (b) is saying that all stalks in a small neighborhood come from a single section.

Proof. Here we go.

(a) We use the identity axiom on $\mathcal{F}$. Suppose that $f, g \in \mathcal{F}(U)$ have $f|_p = g|_p$ for all $p \in U$. Thus, for each $p \in U$, we can find $U_p \subseteq U$ containing p such that $f|_{U_p} = g|_{U_p}$.

Now, $U \subseteq \bigcup_{p \in U} U_p \subseteq U$, so $\{U_p\}_{p \in U}$ is an open cover for U, so the identity axiom on $\mathcal{F}$ forces $f = g$.

(b) We use the gluability axiom on $\mathcal{F}$. In one direction, suppose $\iota(f) = (f_p)_{p \in U}$ so that $f|_p = f_p$ for each $p \in U$. This means that, for each $p \in U$, we can set $U_p := U$ and $\widetilde{f}_p := f \in \mathcal{F}(U_p)$ so that any $q \in U_p$ have

$$f_q = f|_q = \widetilde{f}_p|_q.$$

In the other direction, suppose we have germs $(f_p)_{p \in U} \in \prod_{p \in U} \mathcal{F}_p$ such that any $p \in U$ has an open set U_p and a section $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $f_q = \widetilde{f}_p|_q$ for any $q \in U_p$. We claim that

$$\widetilde{f}_p|_{U_p \cap U_q} \stackrel{?}{=} \widetilde{f}_q|_{U_p \cap U_q}. \tag{1.4}$$

Well, for any $r \in U_p \cap U_q$, we know that $\widetilde{f}_p|_{U_p \cap U_q}|_r = f_r = \widetilde{f}_q|_{U_p \cap U_q}|_r$, so there is an open set $V_r \subseteq U_p \cap U_q$ containing r such that

$$\widetilde{f}_p|_{U_p \cap U_q}|_{V_r} = \widetilde{f}_q|_{U_p \cap U_q}|_{V_r}.$$

Now, applying the identity axiom of $\mathcal{F}$ on the open cover $\{V_r\}_{r \in U_p \cap U_q}$ forces (1.4). Thus, the gluability axioms grants $f \in \mathcal{F}(U)$ such that $f|_{U_p} = \widetilde{f}_p$ for each $p \in U$, so it follows that

$$f|_p = f|_{U_p}|_p = f_p$$

for each $p \in U$. ■

We are going to want a name for the condition in Proposition 1.107 (b).

Definition 1.109 (Compatible germ)**.** Fix a sheaf $\mathcal{F}$ on a topological space X. Then, given a subset $U \subseteq X$, a *system of compatible germs* is a tuple $(f_p)_{p \in U}$ such that, for each $p \in U$, there is an open set U_p containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that all $q \in U_p$ have $f_q = \widetilde{f}_p|_q$.

As a quick sanity check, we can see by hand that morphisms preserve compatibility.

Lemma 1.110. Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. If $(f_p)_{p \in U}$ is a system of compatible germs for $\mathcal{F}(U)$, then $(\varphi_p f_p)_{p \in U}$ is a system of compatible germs for $\mathcal{G}(U)$.

Proof. For each $p \in U$, we can find $U_p \subseteq U$ containing p and a lift $\widetilde{f}_p$ so that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. Thus, for each p, we set $\widetilde{g}_p := \varphi_{U_p}(\widetilde{f}_p)$ so that any $q \in U_p$ has

$$\widetilde{g}_p|_q = \varphi_{U_p}(\widetilde{f}_p)|_q = [(U_p, \varphi_{U_p}(\widetilde{f}_p))] = \varphi_q([(U_p, \widetilde{f}_p)]) = \varphi_q(\widetilde{f}_p|_q) = \varphi_q(f_q),$$

which finishes our check. ■

In addition to sections, stalks also remember morphisms.

Proposition 1.111. Fix presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X with a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$.

(a) For any $p \in X$, there is a natural map $\varphi_p \colon \mathcal{F}_p \to \mathcal{G}_p$.

(b) Suppose $\mathcal{G}$ is a sheaf. Given presheaf morphisms $\varphi, \varphi' \colon \mathcal{F} \to \mathcal{G}$ such that $\varphi_p = \varphi'_p$ for all $p \in X$, we have $\varphi = \varphi'$.

Proof. We go in sequence.

(a) It is possible to induce this map from abstract nonsense. Alternatively, we can write this explicitly as being induced by

$$\varphi_p\colon [(U,s)] \mapsto [(U,\varphi_U(s))].$$

To see that φ_p is well-defined, suppose $(U_1,s_1) \sim (U_2,s_2)$ so that we have some $U \subseteq U_1 \cap U_2$ with $s_1|_U = s_2|_U$. Then

$$\varphi_{U_1}(s_1)|_U = \varphi_U(s_1|_U) = \varphi_U(s_2|_U) = \varphi_{U_2}(s_2)|_U,$$

so $(U_1,\varphi_U(s_1)) \sim (U_2,\varphi_U(s_2))$. Now, φ_p will preserve whatever extra structure we need it to because it is essentially induced by the φ_U.

(b) Fix an open set $U \subseteq X$ so that we need $(\varphi_1)_U = (\varphi_2)_U$. Now, the point is that any $\psi\colon \mathcal{F} \to \mathcal{G}$ will make the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\psi_U"'] & \prod_{p\in U} \mathcal{F}_p \arrow[d, "\prod \psi_p"] \\ \mathcal{G}(U) \arrow[r, hook] & \prod_{p\in U} \mathcal{G}_p \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & (f|_p)_{p\in U} \arrow[d, mapsto] \\ \psi_U f \arrow[r, mapsto] & ((\psi_U f)|_p)_{p\in U} \end{tikzcd}$$

commute. In particular, if $\varphi_p = \varphi_p'$ for all $p \in X$, then we see that $\varphi_U(f)|_p = \varphi_p(f) = \varphi_p'(f) = \varphi_U(f)|_p$. Thus, the injectivity of the map $\mathcal{G}(U) \to \prod_{p\in U} \mathcal{G}_p$ of Proposition 1.107 forces $\varphi_U(f) = \varphi_U(f)$. ■

Remark 1.112. It is not hard to see that $(-)_p\colon \mathrm{PreSh}_X \to \mathcal{C}$ is a functor, where $\mathcal{C}$ is the target category for our sheaves. We can see this because we're just computing limits, but we can also see this concretely. We have already described the action on (pre)sheaves and morphisms, so it remains to check functoriality. Fix $p \in X$.

- Identity: note that $[(U,f))] \in \mathcal{F}_p$ has $(\mathrm{id}_{\mathcal{F}})_p\colon [(U,f)] \mapsto [(U,f)]$.
- Functoriality: given $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ as well as $[(U,f)] \in \mathcal{F}_p$, we have

$$\psi_p(\varphi_p([(U,f)])) = \psi_p([U,\varphi_U f]) = [(U,(\psi\circ\varphi)_U f)] = (\psi\circ\varphi)_p([(U,f)]).$$

1.4.4 The Category of Sheaves Is Additive

We are going to want to do category theory on sheaves, so let's begin. Our end goal is to show that the category of (pre)sheaves over a topological space X valued in an abelian category is itself abelian. Throughout, our target category for our sheaves will be abelian (and concrete). Explicitly, the target category will essentially be a subcategory of Mod_R always.

To begin, we need to show that we can give morphisms of sheaves an abelian group structure.

Lemma 1.113. Fix presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X. Then, given morphisms $\varphi,\psi\colon \mathcal{F} \to \mathcal{G}$, we can define

$$(\varphi+\psi)_U \coloneqq \varphi_U + \psi_U$$

for each $U \subseteq X$. Then $(\varphi+\psi)\colon \mathcal{F} \to \mathcal{G}$ is a presheaf morphism. This operation $+$ makes $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ an abelian group, and composition of morphisms distributes over addition.

Proof. To check that $\varphi+\psi$ is a presheaf morphism, pick up a containment of open sets $V \subseteq U$, and we need to check that the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "(\varphi+\psi)_U"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "(\varphi+\psi)_V"] & \mathcal{G}(B) \end{tikzcd}$$

commutes. Well, for any $s \in \mathcal{F}(U)$, we note

$$(\varphi + \psi)_U(s)|_V = (\varphi_U s + \psi_U s)|_V = \varphi_U(s)|_V + \psi_U(s)|_V \stackrel{*}{=} \varphi_V(s|_V) + \psi_V(s|_V) = (\varphi + \psi)_V(s|_V),$$

where we have used the fact that φ and ψ are presheaf morphisms in $\stackrel{*}{=}$.

To check that $\operatorname{Mor}(\mathcal{F}, \mathcal{G})$ is an abelian group under $+$, we note that

$$\operatorname{Mor}(\mathcal{F}, \mathcal{G}) \subseteq \prod_{U \subseteq X} \operatorname{Mor}(\mathcal{F}(U), \mathcal{G}(U)),$$

where the latter is a product group under the same addition operation. We have already established that $\operatorname{Mor}(\mathcal{F}, \mathcal{G})$ is closed under the addition operation. So we have two more checks to establish that we have a subgroup.

- Zero: the zero element $0 \in \operatorname{Mor}(\mathcal{F}, \mathcal{G})$ is then made of the zero morphisms $0_U : \mathcal{F}(U) \to \mathcal{G}(U)$ sending all elements to zero. The uniqueness of zero morphisms ensures that $0 : \mathcal{F} \to \mathcal{G}$ is a presheaf morphism. Namely, any $V \subseteq U$ and $s \in \mathcal{F}(U)$ gives $0_U(s)|_V = 0 = 0_V(s|_V)$.
- Inverses: given a sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$, we define $(-\varphi)_U := -\varphi_U$ for each $U \subseteq X$. The $(-\varphi)$ assembles into a presheaf morphism: for any $V \subseteq U$ and $s \in \mathcal{F}(U)$, we see that $(-\varphi)_U(s)|_V = -\varphi_U(s)|_V = -\varphi_V(s|_V) = (-\varphi)_V(s|_V)$.

It remains to check distributivity. Let $\varphi_1, \varphi_1 : \mathcal{F} \to \mathcal{G}$ and $\psi_1, \psi_2 : \mathcal{G} \to \mathcal{H}$ be presheaf morphisms. Then, for any $U \subseteq X$ and $s \in \mathcal{F}(U)$, we compute

$$\begin{aligned}
((\psi_1 + \psi_2) \circ (\varphi_1 + \varphi_2))_U(s) &= (\psi_1 + \psi_2)_U\big((\varphi_1 + \varphi_2)_U(s)\big) \\
&= ((\psi_1)_U + (\psi_2)_U)\big((\varphi_1)_U(s) + (\varphi_2)_U(s)\big) \\
&= ((\psi_1)_U + (\psi_2)_U)\big((\varphi_1)_U(s)\big) + ((\psi_1)_U + (\psi_2)_U)\big((\varphi_2)_U(s)\big) \\
&= (\psi_1 \circ \varphi_1)_U(s) + (\psi_2 \circ \varphi_1)_U(s) + (\psi_1 \circ \varphi_2)_U(s) + (\psi_2 \circ \varphi_2)_U(s) \\
&= (\psi_1 \circ \varphi_1 + \psi_2 \circ \varphi_1 + \psi_1 \circ \varphi_2 + \psi_2 \circ \varphi_2)_U(s),
\end{aligned}$$

so $(\psi_1 + \psi_2) \circ (\varphi_1 + \varphi_2) = \psi_1 \circ \varphi_1 + \psi_2 \circ \varphi_1 + \psi_1 \circ \varphi_2 + \psi_2 \circ \varphi_2$ follows. ■

Remark 1.114. Of course, replacing all presheaves with sheaves in Lemma 1.113 makes the statement still true because sheaf morphisms are just presheaf morphisms. This will be a recurring theme.

Continuing, we should define a zero presheaf.

Definition 1.115 (Zero presheaf)**.** Given a topological space X, the *zero presheaf* on X is the presheaf $\mathcal{Z}$ such that $\mathcal{Z}(U) = 0$ for all open $U \subseteq X$.

Lemma 1.116. The zero presheaf $\mathcal{Z}$ on X is the zero object in the category PreSh_X.

Proof. The restriction maps for $\mathcal{Z}$ are all zero maps; the functoriality checks are all immediate because zero maps are unique (namely, $\operatorname{id}_0 = 0$ and $0 \circ 0 = 0$). Now, given any presheaf $\mathcal{F}$, we need to exhibit unique presheaf morphisms to and from $\mathcal{Z}$.

- Initial: we show there is a unique sheaf morphism $\varphi : \mathcal{Z} \to \mathcal{F}$. For uniqueness, note that any $U \subseteq X$ needs a map

$$\varphi_U : \mathcal{Z}(U) \to \mathcal{F}(U),$$

so because $\mathcal{Z}(U) = 0$ is initial, there is a unique possible map. To check that this data actually assembles into a presheaf morphism, we need to check that any containment of open sets $V \subseteq U$ causes the diagram

$$\begin{tikzcd} \mathcal{Z}(U) \arrow[r, "0"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ \mathcal{Z}(V) \arrow[r, "0"] & \mathcal{F}(V) \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "0"] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ 0 \arrow[r, "0"] & \mathcal{F}(V) \end{tikzcd}$$

commutes, which is clear by the uniqueness of our zero maps. Namely, the map $0 \to 0 \to \mathcal{F}(V)$ and $0 \to \mathcal{F}(U) \to \mathcal{F}(V)$ must both just be the map $0 \to \mathcal{F}(V)$.

- Terminal: one merely has to reverse all the arrows in the previous argument. Notably, the zero object 0 in the target category of $\mathcal{Z}$ is terminal in addition to being initial. ■

As in Remark 1.114, we can quickly move the zero presheaf to being the zero sheaf.

Corollary 1.117. The zero presheaf $\mathcal{Z}$ on a topological space X is a sheaf and hence the zero object in the category Sh_X.

Proof. The main point here is that the zero presheaf $\mathcal{Z}$ is in fact a sheaf. This is easy to check: fix an open cover $\mathcal{U}$ of an open set $U \subseteq V$. If we are given sections $f, g \in \mathcal{Z}(U)$, then we don't even need any other conditions to know that

$$f = g \in \mathcal{Z}(U) = 0$$

because there is only one element in the zero object. Similarly, given sections $f_V \in \mathcal{Z}(V)$ for each $V \in \mathcal{U}$, we note that $f_V = 0$ everywhere, so we can set $f_U = 0 \in \mathcal{Z}(U)$ so that $f|_V = f_V$; this proves the gluability axiom.

We now check the universal property. Given any sheaf $\mathcal{F}$, we know from Lemma 1.116 that there are unique presheaf morphisms $\mathcal{F} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{F}$. Because sheaf morphisms are presheaf morphisms, it follows that there are unique sheaf morphisms as well. ■

To show that our category of (pre)sheaves is additive, it remains to exhibit (finite) products.

Definition 1.118 (Product presheaf)**.** Given presheaves $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ on a topological space X, the *product presheaf* $\mathcal{F} := \prod_{\alpha\in\lambda} \mathcal{F}_\alpha$ by

$$\mathcal{F}(U) := \prod_{\alpha\in\lambda} \mathcal{F}_\alpha(U)$$

with the restriction maps induced by the $\mathcal{F}_\alpha$.

Lemma 1.119. Given presheaves $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ on X, the product presheaf $\mathcal{F} := \prod_{\alpha\in\lambda} \mathcal{F}_\alpha$ is the categorical product in PreSh_X.

Proof. We begin by showing that $\mathcal{F}$ is in fact a presheaf. To be explicit, our restriction maps for opens $V \subseteq U \subseteq X$ are

$$\begin{aligned} \mathrm{res}_{U,V} \colon \mathcal{F}(U) &\to \mathcal{F}(V) \\ (f_\alpha)_\alpha &\mapsto (f_\alpha|_V)_\alpha. \end{aligned}$$

Here are our presheaf checks.

- Identity: with an open $U \subseteq X$ and $(f_\alpha) \in \mathcal{F}(U)$, we have $(f_\alpha)_\alpha|_U = (f_\alpha|_U)_\alpha = (f_\alpha)_\alpha$.
- Functoriality: with opens $W \subseteq V \subseteq U$ and $(f_\alpha)_\alpha \in \mathcal{F}(U)$, we have

$$(f_\alpha)_\alpha|_V|_W = (f_\alpha|_V|_W)_\alpha = (f_\alpha|_W)_\alpha = (f_\alpha)_\alpha|_W.$$

It remains to show our universal property for products. Given an open $U \subseteq X$, define $(\pi_\alpha)_U : \mathcal{F}(U) \to \mathcal{F}_1(U)$ by projection onto the α coordinate. To show that π_α assembles into a presheaf morphism, pick up opens $V \subseteq U \subseteq X$ and $(f_\alpha)_\alpha \in \mathcal{F}(U)$ and check

$$(\pi_\alpha)_U((f_\alpha)_\alpha)|_V = f_\alpha|_V = (\pi_\alpha)_V((f_\alpha|_V)_\alpha) = (\pi_\alpha)_V((f_\alpha)_\alpha|_V).$$

For our universal property, suppose that we have a presheaf $\mathcal{G}$ with maps $\varphi_\alpha : \mathcal{G} \to \mathcal{F}_\alpha$. We need a unique presheaf morphism $\varphi : \mathcal{G} \to \mathcal{F}$ making the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\varphi"] \arrow[dr, "\varphi_\alpha"'] & \mathcal{F} \arrow[d, "\pi_\alpha"] \\ & \mathcal{F}_\alpha \end{tikzcd} \tag{1.5}$$

commute for each α. We show uniqueness and existence separately.

- Uniqueness: if $\varphi : \mathcal{G} \to \mathcal{F}$ makes (1.5) commute for each α, at any given open $U \subseteq X$ and $g \in \mathcal{G}(U)$, we must have
$$(\pi_\alpha)_U(\varphi_U g) = (\varphi_\alpha)_U(g)$$
for each α, so $\varphi_U(g) \coloneqq ((\varphi_\alpha)_U g)_\alpha$ is forced.
- Existence: as above, given an open $U \subseteq X$ and $g \in \mathcal{G}(U)$, define
$$\varphi_U(g) \coloneqq ((\varphi_\alpha)_U g)_\alpha.$$
We can see, as above, that $(\pi_\alpha)_U \circ \varphi_U = (\varphi_\alpha)_U$, so (1.5) will commute as long as φ actually assembles into a presheaf morphism.
Well, given $V \subseteq U \subseteq X$ and $g \in \mathcal{G}(U)$, note
$$\varphi_U(g)|_V = ((\varphi_\alpha)_U g)_\alpha|_V = ((\varphi_\alpha)_U(g)|_V)_\alpha = ((\varphi_\alpha)_V(g|_V))_\alpha = \varphi_V(g|_V),$$
which finishes. ■

Corollary 1.120. Given sheaves $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ on X, the product presheaf $\mathcal{F} \coloneqq \prod_{\alpha\in\lambda} \mathcal{F}_\alpha$ is a sheaf and hence the categorical product in Sh_X.

Proof. As in Corollary 1.117, the main point is to show that $\mathcal{F}$ is in fact a sheaf. Fix an open cover $\mathcal{U}$ of U.

- Identity: given $(f_\alpha)_\alpha \in \mathcal{F}(U)$ with $(f_\alpha)_\alpha|_V = 0$ for all $V \in \mathcal{U}$, we see $f_\alpha|_V = 0$ for each α is forced for all $V \in \mathcal{U}$, so the identity axiom on $\mathcal{F}_\alpha$ forces $f_\alpha = 0$ for each α. Thus, $(f_\alpha)_\alpha = 0$.
- Gluability: pick up sections $(f_{\alpha,V})_\alpha \in \mathcal{F}(V)$ for each $V \in \mathcal{U}$ such that any $V, V' \in \mathcal{U}$ have
$$(f_{\alpha,V}|_{V\cap V'})_\alpha = (f_{\alpha,V})_\alpha|_{V\cap V'} = (f_{\alpha,V'})_\alpha|_{V\cap V'} = (f_{\alpha,V'}|_{V\cap V'})_\alpha.$$
Thus, for each α, the gluability axiom on $\mathcal{F}_\alpha$ promises $f_\alpha \in \mathcal{F}_\alpha(U)$ such that $f_\alpha|_V = f_{\alpha,V}$ for each $V \in \mathcal{U}$. Thus, $(f_\alpha)_\alpha|_V = (f_\alpha|_V)_\alpha = (f_{\alpha,V})_\alpha$ for each $V \in \mathcal{U}$, as needed.

We now discuss the universal property. This immediately follows from the corresponding statement in the category of presheaves, but for completeness, we will say out loud what's going on. Let $\pi_\alpha : \mathcal{F} \to \mathcal{F}_\alpha$ be the projection (pre)sheaf morphisms.

Suppose we have a sheaf $\mathcal{G}$ with sheaf morphisms $\varphi_\alpha : \mathcal{G} \to \mathcal{F}_\alpha$ for each α. Then we are promised a unique presheaf morphism $\varphi : \mathcal{G} \to \mathcal{F}$ such that $\varphi_\alpha = \pi_\alpha \circ \varphi$ for each α. Thus, there is also a unique sheaf morphism φ satisfying the same constraint because sheaf morphisms are just presheaf morphisms. ■

Remark 1.121. The above discussion immediately generalizes to arbitrary products, but we will not need these.

Corollary 1.122. The category Sh_X of sheaves on a topological space X valued in a (concrete) abelian category $\mathcal{C}$ is additive.

Proof. Combine Lemma 1.113, Corollary 1.117, and Corollary 1.120. ■

1.4.5 Sheaf Kernels

We continue working with (pre)sheaves valued in a concrete abelian category. The next step to show that the category is abelian is to exhibit kernels and cokernels. Cokernels will turn out to be difficult, so we begin with kernels.

Definition 1.123 (Presheaf kernel)**.** Given a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X, we define the *presheaf kernel* as

$$(\ker\varphi)(U) := \ker\varphi_U$$

for each $U \subseteq X$, where restriction maps are induced by $\mathcal{F}$. Then $\ker\varphi$ is our *presheaf kernel*.

Lemma 1.124. Given a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ on a topological space X, the presheaf kernel $\ker\varphi$ is a categorical kernel.

Proof. We haven't actually defined the restriction maps for the presheaf kernel, so we do this now: for each open $U \subseteq X$ with $V \subseteq U$, note $\ker\varphi_U \subseteq \mathcal{F}(U)$, so we can restrict the map $\mathrm{res}_{U,V}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ to a map

$$\ker\varphi_U \to \mathcal{F}(V).$$

Now, for any $s \in \ker\varphi_U$, we note that actually $\varphi_V(s|_V) = \varphi_U(s)|_V = 0$, so our restriction map restricts to $\mathrm{res}_{U,V}\colon \ker\varphi_U \to \ker\varphi_V$ as needed. The presheaf checks on $\ker\varphi$ of identity and functoriality checks are inherited from $\mathcal{F}$.

It remains to check the universal property: we need $\ker\varphi$ to be the limit of the following diagram.

$$\begin{tikzcd} & \mathcal{Z} \arrow[d] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

There is an inclusion $\ker\varphi_U \subseteq \mathcal{F}(U)$ for each open $U \subseteq X$, which induces maps $\iota_U\colon (\ker\varphi)(U) \to \mathcal{F}(U)$. To see that ι_U assembles into a presheaf morphism, pick up a containment $V \subseteq U$ and $s \in \mathcal{F}(U)$, and we check $\iota_U(s)|_V = s|_V = \iota_V(s|_V)$. Additionally, there is a canonical 0 map $0\colon (\ker\varphi) \to \mathcal{Z}$, so we claim that the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"] & \mathcal{Z} \arrow[d] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

commutes. Well, for any $U \subseteq X$ and $f \in (\ker\varphi)(U)$, note $\varphi_U(\iota_U(f)) = 0$, so the presheaf morphism $\varphi \circ \iota$ is just the zero morphism, as needed.

We are now ready to show the universal property. Fix a presheaf $\mathcal{H}$ with a map $\psi\colon \mathcal{H} \to \mathcal{F}$ such that $\varphi \circ \psi = 0$. Then we claim that there is a unique map $\overline{\psi}$ making the diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[dr, dashed, "\overline{\psi}"] \arrow[drr, bend left] \arrow[ddr, bend right, "\psi"'] & & \\
& \ker\varphi \arrow[r] \arrow[d, "\iota"] & \mathcal{Z} \arrow[d] \\
& \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G}
\end{tikzcd} \tag{1.6}$$

commute. We show uniqueness and existence separately.

- Uniqueness: for any subset $U \subseteq X$ and $h \in \mathcal{H}(U)$, (1.6) forces

$$\iota_U(\overline{\psi}_U(h)) = \psi_U(h).$$

  However, ι_U is just an inclusion (of sets, say), so we must have $\overline{\psi}_U(h) = \iota_U^{-1}(\psi_U(h))$. As such, $\overline{\psi}$ is uniquely determined.

- Existence: for any subset $U \subseteq X$ and $h \in \mathcal{H}(U)$, (1.6) forces $\varphi_U(\psi_U(h)) = 0$, so $\psi_U(h) \in \ker\varphi_U$. So we can restrict the image of ψ_U to define a map

$$\overline{\psi}_U(h) := \psi_U(h).$$

  Of course, $\iota_U(\overline{\psi}_U(h)) = \psi_U(h)$, so (1.6) will commute as long as $\overline{\psi}$ assembles into a presheaf morphism. Well, for a containment $V \subseteq U$ and $h \in \mathcal{H}(U)$, we see

$$\overline{\psi}_U(h)|_V = \psi_U(h)|_V = \psi_V(h|_V) = \overline{\psi}_V(h|_V),$$

  as needed. ■

What makes the presheaf kernel nice is that is actually the sheaf kernel.

Lemma 1.125. Fix a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$. Then $\ker\varphi$ is a sheaf and hence the categorical kernel.

Proof. As usual, the main point is to show that $\ker\varphi$ is a sheaf. For clarity, label the (canonical) inclusion $\iota\colon (\ker\varphi) \to \mathcal{F}$; note ι_U is injective at each open $U \subseteq X$. Now, fix an open cover $\mathcal{U}$ for an open set $U \subseteq X$.

- Identity: fix $f, g \in (\ker\varphi)(U)$ such that $f|_V = g|_V$ for all $V \in \mathcal{U}$. However, all of this is embedded in $\mathcal{F}$ by ι, so we really have $\iota f, \iota g \in \mathcal{F}(U)$ with $(\iota_U f)|_V = \iota_V(f|_V) = \iota_V(g|_V) = (\iota_U g)|_V$ for all $V \in \mathcal{U}$, so the identity axiom promises that $\iota_U f = \iota_U g$. Thus, $f = g$ follows.

- Gluability: fix sections $f_V \in (\ker\varphi)(V)$ for each $V \in \mathcal{F}(V)$ such that

$$f_V|_{V\cap V'} = f_{V'}|_{V\cap V'}$$

  for each $V, V' \in \mathcal{U}$. Embedding everything in $\mathcal{F}$, we see

$$(\iota_V f_V)|_{V\cap V'} = \iota_{V\cap V'}(f_V|_{V\cap V'}) = \iota_{V\cap V'}(f_{V'}|_{V\cap V'}) = (\iota_{V'} f_{V'})|_{V\cap V'},$$

  so the gluability axiom on $\mathcal{F}(U)$ tells us there is $f \in \mathcal{F}(U)$ with $f|_V = \iota_V(f_V)$ for each $V \in \mathcal{U}$.

  We now need to show $f \in (\ker\varphi)(U)$. Well, for each $V \in \mathcal{U}$, we see

$$\varphi_U(f)|_V = \varphi_V(f|_V) = \varphi_V(f_V) = 0,$$

  where the last equality is because $f_V \in (\ker\varphi)(V)$. Thus, the identity axiom on $\mathcal{G}$ tells us $f \in \ker\varphi_U$, so we can pull f back to an element $f \in (\ker\varphi)(U)$ such that $f|_V = f_V$ for each $V \in \mathcal{U}$.

Checking the universal property is a matter of stating it and noting that working in the category PreSh_X immediately forces the universal property to work in the subcategory Sh_X. We showed what this looks like in the last paragraph of Corollary 1.120. ■

Now, having a kernel gives us a definition.

Definition 1.126 (Injective morphism)**.** A morphism of (pre)sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if and only if the kernel (pre)sheaf $\ker\varphi$ is identically zero. Equivalently, we are asking for φ_U to be injective everywhere.

We briefly convince ourselves that this is the correct definition.

Lemma 1.127. Let $\mathcal{C}$ be a category with a zero object and kernels, and fix a morphism $\varphi\colon A \to B$. Then φ is monic if and only if $\ker\varphi$ vanishes.

Proof. This is purely categorical; let $\iota\colon (\ker\varphi) \to A$ be the kernel map. In one direction, suppose that $\ker\varphi$ vanishes. To show φ is monic, write down

$$\begin{tikzcd} C \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] & A \arrow[r, "\varphi"] & B \end{tikzcd}$$

with $\varphi\circ\psi_1 = \varphi\circ\psi_2$, we need to show $\psi_1 = \psi_2$. Well, $\psi := \psi_1 - \psi_2$ has $\varphi\circ\psi = 0$, so our kernel promises a unique map $\overline{\psi}\colon C \to (\ker\varphi)$ with $\psi = \iota\circ\overline{\psi}$. However, $\ker\varphi$ is the zero object, so we conclude $\psi = 0$.

In the other direction, suppose φ is monic, and we show that the zero object Z satisfies the universal property of the kernel. Well, fix an object C with a map $\psi\colon C \to A$ such that $\varphi\circ\psi = 0$. Then we need a unique map $\overline{\psi}$ making

$$\begin{tikzcd} C \arrow[r, dashed, "\overline{\psi}"] \arrow[d, "\psi"'] & Z \arrow[dl] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

commute. Well, the map $C \to Z$ is certainly unique because Z is terminal. Additionally, we note that the zero map $C \to Z$ does indeed make the diagram commute: $\varphi\circ\psi = 0 = \varphi\circ 0$ forces $\psi = 0$, so ψ is the zero map. ■

1.4.6 Injectivity at Stalks

In our stalk philosophy, we might hope we can detect injectivity at stalks. Indeed, we can.

Lemma 1.128. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then, for any $p \in X$, the inclusion $(\ker\varphi) \to \mathcal{F}$ induces an isomorphism

$$(\ker\varphi)_p \simeq \ker\varphi_p.$$

Proof. Let $\iota\colon (\ker\varphi) \to \mathcal{F}$ denote the inclusion. Then Proposition 1.111 grants us a map $\iota_p\colon (\ker\varphi)_p \to \mathcal{F}_p$. Now, for any $[(U,f)] \in (\ker\varphi)_p$, we have

$$\varphi_p(\iota_p([(U,f)])) = [(U,\varphi_U(\iota_U(f)))] = [(U,0)] = 0$$

by how these maps are defined in Proposition 1.111. Thus, we can restrict the image of ι_p to $\ker\varphi_p \subseteq \mathcal{F}_p$.

In the other direction, suppose that we have a germ $[(U,f)] \in \ker\varphi_p$ so that $[(U,\varphi_U(f))] = 0$, which means there is $V \subseteq U$ containing p such that $\varphi_V(f|_V) = \varphi_U(f)|_V = 0$. In particular, $f|_V \in \ker\varphi_V$, so we have $[(V,f|_V)] \in (\ker\varphi)_p$. Thus, we define the map $\pi\colon \ker\varphi_p \to (\ker\varphi)_p$ by

$$\pi\colon [(U,f)] \mapsto [(V,f|_V)].$$

Note that π does not depend on the choice of $V \subseteq U$: if $V' \subseteq U$ also have $\varphi_{V'}(f|_{V'}) = 0$, then we note $(V, f|_V) \sim (V', f|_{V'})$ because $f|_V|_{V\cap V'} = f|_{V'}|_{V\cap V'}$. Additionally, π does not depend on the choice of representative for $[(U, f)]$: if $(U, f) \sim (U', f')$ in $\ker \varphi_p$, then find $V \subseteq U \cap U'$ small enough so that $f|_V = f'|_V$ and $\varphi_V(f|_V) = \varphi_V(f'|_V) = 0$ so that $\pi([(U, f)]) = [(V, f|_V)] = [(V, f'|_V)] = \pi([(U, f')])$.

Lastly, we check ι_p and π are inverse. In one direction, given $[(U, f)] \in (\ker\varphi)_p$, we note $\varphi_U(f) = 0$, so

$$\pi(\iota_p([(U, f)])) = \pi([(U, f)]) = [(U, f)].$$

In the other direction, given $[(U, f)] \in \ker\varphi_p$, find $V \subseteq U$ small enough so that $\varphi_V(f|_V) = 0$. Then

$$\iota_p(\pi([(U, f)])) = \iota([(V, f|_V)]) = [(V, f|_V)] = [(U, f)],$$

finishing. ■

Lemma 1.129. Fix a sheaf $\mathcal{F}$ on a topological space X. The following are equivalent.

(a) $\mathcal{F}$ is the zero sheaf.

(b) $\mathcal{F}(U) \simeq 0$ for each open $U \subseteq X$.

(c) $\mathcal{F}_p \simeq 0$ for each $p \in X$.

Proof. Our construction of the zero presheaf tells us that (a) implies (c): any germ $[(U, f)] \in \mathcal{Z}_p$ has $f \in \mathcal{Z}(U) = 0$, so $[(U, f)] = 0$. Note we are using the fact that isomorphic sheaves have isomorphic stalks. To show that (c) implies (b), we note that $\mathcal{F}$ being a sheaf grants us the inclusion

$$\mathcal{F}(U) \hookrightarrow \prod_{p\in U} \mathcal{F}_p$$

by Proposition 1.107. However, the right-hand side is 0, so the left-hand side must also be 0.

Lastly, we show that (b) implies (a). Well, note that the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for an inclusion $V \subseteq U$ are forced because zero morphisms are unique. Similarly, letting $\mathcal{Z}$ denote the zero sheaf, we have isomorphisms $\varphi_U : \mathcal{F}(U) \simeq \mathcal{Z}(U)$ induced by these zero maps for all $U \subseteq X$, and we thus assemble into a natural isomorphism $\varphi: \mathcal{F} \to \mathcal{Z}$ because the uniqueness of zero maps makes the naturality square commute. ■

Proposition 1.130. Fix a morphism of sheaves $\varphi: \mathcal{F} \to \mathcal{G}$. The following are equivalent.

(a) φ is monic.

(b) φ_U is monic for each open $U \subseteq X$.

(c) φ_p is monic for each $p \in X$.

Proof. By Lemma 1.127, these are equivalent to the following.

(a') $\ker\varphi$ vanishes.

(b') $(\ker\varphi)(U)$ vanishes for each open $U \subseteq X$.

(c') $\ker\varphi_p$ vanishes for each $p \in X$. By Lemma 1.128, this is equivalent to $(\ker\varphi)_p$ vanishing for each $p \in X$.

These are equivalent by Lemma 1.129. ■

Remark 1.131. Technically, we only need to know that $\mathcal{F}$ is a sheaf for Proposition 1.130.

Being careful, one can extend Proposition 1.130 as follows.

Proposition 1.132. Fix a morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}$ is a sheaf. The following are equivalent.

(a) φ is an isomorphism.

(b) φ_U is an isomorphism for each open $U \subseteq X$.

(c) φ_p is an isomorphism for each $p \in X$.

Proof. To begin, (a) and (b) are equivalent by category theory: natural isomorphisms are just natural transformations whose component morphisms are isomorphisms. The main check here is that the inverse morphisms $\varphi^{-1}(U)\colon \mathcal{G}(U) \to \mathcal{F}(U)$ cohere into a bona fide natural transformation, which is true because, for any containment $V \subseteq U$, the commutativity of the left diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(U) \arrow[d, "\operatorname{res}_{U,V}"'] & \mathcal{G}(U) \arrow[l, "\varphi_U^{-1}"'] \arrow[d, "\operatorname{res}_{U,V}"] \\ \mathcal{F}(V) & \mathcal{G}(V) \arrow[l, "\varphi_V^{-1}"] \end{tikzcd}$$

is equivalent to the commutativity of the right diagram.

Additionally, it is also fairly easy that (a) implies (c); fix some $p \in X$. Give φ an inverse morphism ψ, and we claim that φ_p is the inverse of ψ_p. Indeed, for any $[(U, f)] \in \mathcal{F}_p$, we see

$$\psi_p(\varphi_p([(U, f)])) = \psi_p([(U, \varphi_U(f))]) = [(U, \psi_U\varphi_U(f))] = [(U, f)].$$

By symmetry, we see $\varphi_p \circ \psi_p = \operatorname{id}_{\mathcal{G}_p}$ as well, finishing.

Thus, the hard direction is showing that φ_p being an isomorphism for all $p \in X$ promises that φ_U is an isomorphism for each $U \subseteq X$. Injectivity is easier: by Proposition 1.107 and Proposition 1.111, we see the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\varphi_U"'] & \prod_{p\in U} \mathcal{F}_p \arrow[d, "\prod \varphi_p"] \\ \mathcal{G}(U) \arrow[r] & \prod_{p\in U} \mathcal{G}_p \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & (f|_p)_{p\in U} \arrow[d, mapsto] \\ \varphi_U f \arrow[r, mapsto] & ((\varphi_U f)|_p)_{p\in U} \end{tikzcd}$$

commutes. However, the top arrow is injective by Proposition 1.107, and the right arrow is injective because each of the φ_p is injective, so the product morphism is also injective. It follows that the left arrow must be injective, which is what we wanted.

We now focus on showing φ_U is surjective. Well, for any $g \in \mathcal{G}(U)$, we get a system of compatible germs $(g|_p)_{p\in U}$ (by Proposition 1.130), so because φ_p is an isomorphism, we may set

$$f_p := \varphi_p^{-1}(g|_p).$$

We claim that f_p is a set of compatible germs, which gives rise to a section $f \in \mathcal{F}(U)$ by Proposition 1.130. Well, giving f_p some representative $[(V_p, s_p)]$, we see $\varphi_p(f_p) = [(V_p, \varphi_{V_p}(s_p))]$. Thus, by appropriately restricting V_p, we see $\varphi_p(f_p) = g|_p$ means that we can find some open $U_p \subseteq U$ containing p and a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that

$$\varphi_{U_p}(\widetilde{f}_p) = g|_{U_p}.$$

In particular, for all $q \in U_p$, we see that

$$\varphi_q(\widetilde{f}_p|_q) = g|_q,$$

so we see that $\widetilde{f}_p|_q = \varphi_q^{-1}(g|_q) = f_q$. This finishes the compatibility check.

Thus, $(f_p)_{p\in U}$ is a system of compatible germs and therefore lifts to some $f \in \mathcal{F}(U)$ with $f|_p = f_p$ everywhere. So $\varphi_U(f)|_p = \varphi_p(f|_p) = \varphi_p(f_p) = g|_p$ for each $p \in X$, so Proposition 1.130 gives $\varphi_U(f) = g$. ■

Remark 1.133. We are avoiding surjectivity for the moment because it is a little trickier. In particular, a morphism φ will be able to be epic without being each φ_U being epic. However, surjectivity will still be equivalent to surjectivity on the stalks.

Remark 1.134. It is possible for sheaves to isomorphic stalks but to not be isomorphic. At a high level, any line bundle over S^1 has stalks isomorphic to $\mathbb{R}$, but not all line bundles are homeomorphic (e.g., the Möbius strip and the trivial line bundle are not homeomorphic). The issue here is that there need not even be a candidate isomorphism between line bundles over S^1 at all!

1.5 September 2

It is another day.

Remark 1.135. Facts used on the homework from Vakil which are in Vakil without proof should be proven on the homework.

We begin lecture by providing an example which we don't quite have the language to describe yet, but we will elaborate on it more later.

elaborate

Example 1.136. Fix $X = \mathbb{C}$ with the usual topology, and give it the sheaf $\mathcal{O}_X$ of holomorphic functions. There is a constant sheaf $\underline{\mathbb{Z}}$ returning $\mathbb{Z}$ at its stalks. Then there is an exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \overset{2\pi i}{\to} \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^\times \to 1 \tag{1.7}$$

even though the last map is not always surjective for any $U \subseteq \mathbb{C}$; for example, take $U = \mathbb{C} \setminus \{0\}$. (However, if U is simply connected, then the map will be surjective.)

Remark 1.137. Cohomology applied to (1.7) (with X some smooth projective curve) shows a special case of the Hodge conjecture.

The point here is that surjectivity cannot be checked on open sets the way that injectivity can. At some level, the issue here is that the cokernel presheaf is not a sheaf, so we have to apply a sheafification operation to fix this.

Remark 1.138. Setting

$$\mathcal{F}(U) := \operatorname{im}\exp(U)$$

makes $\mathcal{F}$ a presheaf but does not give a sheaf.

1.5.1 Sheafification

We introduce sheafification by its universal property.

Definition 1.139 (Sheafification)**.** Fix a presheaf $\mathcal{F}$ on X valued in a (concrete) category $\mathcal{C}$. The *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^{\mathrm{sh}}, \mathrm{sh})$ where $\mathrm{sh}\colon \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ satisfies the following universal property: any sheaf $\mathcal{G}$ with a presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ has a unique sheaf morphism $\overline{\varphi}\colon \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\varphi"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, "\overline{\varphi}"] \\ & \mathcal{G} \end{tikzcd}$$

Of course, there are some checks we should do before using this object.

Lemma 1.140. The sheafification of a presheaf $\mathcal{F}$ on X exists and is unique up to (a suitable notion of) unique isomorphism.

Proof. The idea of the construction is to set $\mathcal{F}^{\mathrm{sh}}(U)$ to be systems of compatible germs; precisely,

$$\mathcal{F}^{\mathrm{sh}}(U) := \left\{ (f_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p : (f_p)_{p\in U} \text{ is a compatible system of germs} \right\}.$$

Given open sets $V \subseteq U$, we define the restriction map

$$\begin{aligned} \mathrm{res}_{U,V}\colon \mathcal{F}^{\mathrm{sh}}(U) &\to \mathcal{F}^{\mathrm{sh}}(V) \\ (f_p)_{p\in U} &\mapsto (f_p)_{p\in V} \end{aligned}$$

though we do have to check this is well-defined: to show $(f_p)_{p\in V} \in \mathcal{F}^{\mathrm{sh}}(V)$, we note $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$ promises that each $p \in U$ has $U_p \subseteq U$ containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ so that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. As such, each $p \in V$ has $V_p := U_p \cap V$ containing p with a lift $\widetilde{f}_p|_{U_p\cap V} \in \mathcal{F}(U_p \cap V)$ so that $\widetilde{f}_p|_{U_p\cap V}|_q = \widetilde{f}_p|_q = f_q$ for each $q \in U_p \cap V$. Thus, $(f_p)_{p\in V}$ is indeed a system of compatible germs.

We now check that $\mathcal{F}^{\mathrm{sh}}$ is a presheaf.

- Identity: given $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, we see $(f_p)_{p\in U}|_U = (f_p)_{p\in U}$.
- Functoriality: given open sets $W \subseteq V \subseteq U$, we see $(f_p)_{p\in U}|_V|_W = (f_p)_{p\in V}|_W = (f_p)_{p\in W} = (f_p)_{p\in U}|_W$.

Next up, we check that $\mathcal{F}^{\mathrm{sh}}$ is a sheaf. Fix an open cover $\mathcal{U}$ of an open set $U \subseteq X$.

- Identity: suppose that $(f_p)_{p\in U}, (g_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$ have $(f_p)_{p\in U}|_V = (g_p)_{p\in U}|_V$ for each $V \in \mathcal{U}$. Now, for each $q \in U$, there is some $V \in \mathcal{U}$ containing q, so we note

  $$(f_p)_{p\in V} = (f_p)_{p\in U}|_V = (g_p)_{p\in U}|_V = (g_p)_{p\in V}$$

  forces $f_q = g_q$. Thus, $(f_p)_{p\in U} = (g_p)_{p\in U}$.
- Gluability: suppose we have $(f_{V,p})_{p\in V} \in \mathcal{F}^{\mathrm{sh}}(V)$ for each $V \in \mathcal{U}$ so that

  $$(f_{V,p})_{p\in V\cap V'} = (f_{V,p})_{p\in V}|_{V\cap V'} = (f_{V',p})_{p\in V'}|_{V\cap V'} = (f_{V',p})_{p\in V\cap V'}.$$

  Now, for each $q \in U$, find any $V \in \mathcal{U}$ containing q, and set $f_q := f_{V,q}$. Note that this is independent of the choice of V: if we have $q \in V \cap V'$ with $V, V' \in \mathcal{U}$, then $(f_{V,p})_{p\in V\cap V'} = (f_{V',p})_{p\in V\cap V'}$ tells us that $f_{V,q} = f_{V',q}$. Further, we note that $(f_p)_{p\in U}|_V = (f_p)_{p\in V} = (f_{V,p})_{p\in V}$ for any $V \in \mathcal{U}$.

  So it remains to show that $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$. Well, for each $p \in U$, find some $V \in \mathcal{U}$ containing p. Then $(f_{V,p})_{p\in V}$ is a system of compatible germs, so we can find $U_p \subseteq V$ containing p and a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that

  $$\widetilde{f}_p|_q = f_{V,q} = f_q$$

  for each $q \in U_p$. This finishes checking that $(f_p)_{p\in U}$ is a compatible system of germs.

We now begin showing the universal property. The sheafification map is defined as

$$\begin{array}{rl} \mathrm{sh}_U\colon \mathcal{F}(U) \to & \mathcal{F}^{\mathrm{sh}}(U) \\ f \quad \mapsto & (f|_p)_{p\in U} \end{array}$$

for any open set $U \subseteq X$. Note $f \in \mathcal{F}(U)$ does indeed give $(f|_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$ because each $p \in U$ can choose $U_p := U$ (which contains p) with lift $\widetilde{f}_p := f$ so that $\widetilde{f}_p|_q = f|_q$ for each $q \in U_p$.

Additionally, it is fairly quick to check that sh is actually a presheaf morphism: given open sets $V \subseteq U$ and $f \in \mathcal{F}(U)$, we compute

$$\mathrm{sh}_U(f)|_V = (f|_p)_{p\in U}|_V = (f|_p)_{p\in V} = (f|_V|_p)_{p\in V} = \mathrm{sh}_V(f|_V).$$

We are now ready to prove the universal property. Fix any sheaf $\mathcal{G}$ with a presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$. We need to show there is a unique sheaf morphism $\overline{\varphi}\colon \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ such that $\varphi = \overline{\varphi} \circ \mathrm{sh}$. We show these separately.

- Uniqueness: fix an open set $U \subseteq X$ and $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, and we will solve for $\overline{\varphi}_U((f_p)_{p\in U})$. Well, each $p \in U$ has some $U_p \subseteq U$ containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. As such, for each $q \in U$,

$$\overline{\varphi}_U((f_p)_{p\in U})|_{U_q} = \overline{\varphi}_{U_q}((f_p)_{p\in U}|_{U_q}) = \overline{\varphi}_{U_q}((f_p)_{p\in U_q}) = \overline{\varphi}_{U_q}\big((\widetilde{f}_q|_p)_{p\in U_q}\big) = \overline{\varphi}_{U_q}(\mathrm{sh}_{U_q}\widetilde{f}_q) = \varphi_{U_q}(\widetilde{f}_q).$$

  Thus, restrictions $\overline{\varphi}_U((f_p)_{p\in U})|_{U_q}$ are fixed by φ, so the identity axiom on $\mathcal{G}$ makes $\overline{\varphi}_U((f_p)_{p\in U})$ unique.

- Existence: fix an open set $U \subseteq X$ and $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, and we will define $\overline{\varphi}_U((f_p)_{p\in U})$. Well, $(\varphi_p f_p)_{p\in U}$ is a system of compatible germs in $\mathcal{G}(U)$ by Lemma 1.110, so there is a unique $g \in \mathcal{G}(U)$ such that $g|_p = \varphi_p(f_p)$ for each $p \in U$. (Uniqueness is by Proposition 1.107.) As such, we will set $\overline{\varphi}_U((f_p)_{p\in U}) := g$ so that $\overline{\varphi}_U((f_p)_{p\in U})$ is the unique section in $\mathcal{G}(U)$ such that

$$\overline{\varphi}_U((f_p)_{p\in U})|_q = \varphi_q(f_q)$$

  for each $q \in U$. Note any section $f \in \mathcal{F}(U)$ has

$$(\overline{\varphi} \circ \mathrm{sh})_U(f)|_q = \overline{\varphi}_U((f|_p)_{p\in U})|_q = \varphi_q(f|_q) = \varphi_U(f)|_q$$

  for any $q \in U$, so Proposition 1.107 applied to the sheaf $\mathcal{G}$ forces equality, implying $\overline{\varphi} \circ \mathrm{sh} = \varphi$.

  So we will be done as soon as we can show $\overline{\varphi}_U$ is a (pre)sheaf morphism. Well, given open sets $V \subseteq U$ and some $(f_p)_{p\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$, we note any $q \in V$ has

$$\overline{\varphi}_U((f_p)_{p\in U})|_V|_q = \overline{\varphi}_U((f_p)_{p\in U})|_q = \varphi_q(f_q),$$

  so the uniqueness of $\overline{\varphi}_V((f_p)_{p\in V})$ forces $\overline{\varphi}_U((f_p)_{p\in U})|_V = \overline{\varphi}_V((f_p)_{p\in U}|_V)$, as desired.

Lastly, we see that $\mathcal{F}^{\mathrm{sh}}$ is unique up to unique isomorphism from the usual universal property arguments: given two sheafifications $\psi\colon \mathcal{F} \to \mathcal{G}$ and $\psi'\colon \mathcal{F} \to \mathcal{G}'$, the universal property induces unique morphisms making the various triangles of the diagrams in

$$\begin{tikzcd} & \mathcal{G} \arrow[d, dashed, "\overline{\psi}"'] \arrow[dd, dashed, bend left, "\mathrm{id}_{\mathcal{G}}"] \\ \mathcal{F} \arrow[ur, "\psi"] \arrow[r, "\psi'"] \arrow[dr, "\psi"'] & \mathcal{G}' \arrow[d, dashed, "\overline{\psi'}"'] \\ & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{G}' \arrow[d, dashed, "\overline{\psi'}"'] \arrow[dd, dashed, bend left, "\mathrm{id}_{\mathcal{G}'}"] \\ \mathcal{F} \arrow[ur, "\psi'"] \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & \mathcal{G} \arrow[d, dashed, "\overline{\psi}"'] \\ & \mathcal{G}' \end{tikzcd}$$

commute (namely, $\mathrm{id}_{\mathcal{G}}$ and $\mathrm{id}_{\mathcal{G}'}$ are the unique morphisms making the outer triangles commute), so $\overline{\psi}$ and $\overline{\psi'}$ are the isomorphisms we want. ■

Here are some basic properties.

Proposition 1.141. Fix a presheaf $\mathcal{F}$ on X with a sheafification $\mathrm{sh}\colon \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$. For given $p \in X$, the induced map $\mathrm{sh}_p\colon \mathcal{F}_p \to (\mathcal{F}^{\mathrm{sh}})_p$ on stalks is an isomorphism.

Proof. We use the explicit description of the sheafification from Lemma 1.140. To be explicit, our map $\mathrm{sh}_p\colon \mathcal{F}_p \to (\mathcal{F}^{\mathrm{sh}})_p$ sends $[(U,f)]$ to $[(U,(f|_q)_{q\in U})]$.

For the inverse morphism $\pi_p\colon (\mathcal{F}^{\mathrm{sh}})_p \to \mathcal{F}_p$, we simply send

$$\pi_p\colon [(U,(f_q)_{q\in U})] \mapsto f_p.$$

Notably, this is well-defined: $[(U,(f_q)_{q\in U})] = [(U',(f'_q)_{q\in U})]$, then there is $V \subseteq U \cap U'$ such that $(f_q)_{q\in U}|_V = (f'_q)_{q\in U'}|_V$, which implies $f_p = f'_p$.

It remains to show that these are inverse. Well, for $[(U,f)] \in \mathcal{F}_p$, we see

$$\pi_p(\mathrm{sh}_p([(U,f)])) = \pi_p([(U,(f|_q)_{q\in U})]) = f|_p.$$

And for $[(U,(f_q)_{q\in U})] \in (\mathcal{F}^{\mathrm{sH}})_p$, we see

$$\mathrm{sh}_p(\pi_p([(U,(f_q)_{q\in U})])) = \mathrm{sh}_p(f_p).$$

Now, because $(f_q)_{q\in U}$ is a compatible system of germs, we may find $U_p \subseteq U$ containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $\widetilde{f}_p|_q = f_q$ for each $q \in U_p$. It follows

$$\mathrm{sh}_p(f_p) = \mathrm{sh}_p(\widetilde{f}_p|_p) = [(U_p,(\widetilde{f}_p|_q)_{q\in U_p})] = [(U_p,(f_q)_{q\in U})] = [(U,(f_q)_{q\in U})],$$

finishing this check. ■

Remark 1.142. If $\mathcal{F}$ is itself a sheaf, then we can see fairly directly that $\mathcal{F}$ satisfies the universal property for $\mathcal{F}^{\mathrm{sh}}$. Alternatively, the sheafification map $\mathrm{sh}\colon \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ is a sheaf morphism which is an isomorphism on stalks by Proposition 1.141 and thus an isomorphism of sheaves by Proposition 1.132.

Proposition 1.143. Sheafification $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{sh}}$ defines a functor $(-)^{\mathrm{sh}}\colon \mathrm{PreSh}_X \to \mathrm{Sh}_X$ which is left adjoint to the forgetful functor $U\colon \mathrm{Sh}_X \to \mathrm{PreSh}_X$.

Proof. We begin by describing the functor $(-)^{\mathrm{sh}}$. We know its behavior on objects, so we still need to know its behavior on morphisms $\eta\colon \mathcal{F} \to \mathcal{G}$. Well, note that we have a composite map $\mathcal{F} \to \mathcal{G} \to \mathcal{G}^{\mathrm{sh}}$, and $\mathcal{G}^{\mathrm{sh}}$ is a sheaf, so the universal property of $\mathcal{F}^{\mathrm{sh}}$ induces a unique map $\eta^{\mathrm{sh}}\colon \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\eta"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, dashed, "\eta^{\mathrm{sh}}"] \\ \mathcal{G} \arrow[r] & \mathcal{G}^{\mathrm{sh}} \end{tikzcd}$$

commute. We quickly check functoriality.

- Identity: note $\mathrm{id}_{\mathcal{F}^{\mathrm{sh}}}$ makes the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, dashed, "\mathrm{id}_{\mathcal{F}^{\mathrm{sh}}}"] \\ \mathcal{F} \arrow[r] & \mathcal{F}^{\mathrm{sh}} \end{tikzcd}$$

  commute, so by definition, we see $(\mathrm{id}_{\mathcal{F}})^{\mathrm{sh}} = \mathrm{id}_{\mathcal{F}^{\mathrm{sh}}}$.

- Functoriality: given presheaf morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$, we note that $\psi^{\text{sh}} \circ \varphi^{\text{sh}}$ makes the outer rectangle of

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\varphi"] \arrow[dd, bend right, "\psi\circ\varphi"'] & \mathcal{F}^{\text{sh}} \arrow[d, "\varphi^{\text{sh}}"'] \arrow[dd, bend left, dashed, "\psi^{\text{sh}}\circ\varphi^{\text{sh}}"] \\ \mathcal{G} \arrow[r] \arrow[d, "\psi"] & \mathcal{G}^{\text{sh}} \arrow[d, "\psi^{\text{sh}}"'] \\ \mathcal{H} \arrow[r] & \mathcal{H}^{\text{sh}} \end{tikzcd}$$

  commute, so by definition, we see $\psi^{\text{sh}} \circ \varphi^{\text{sh}} = (\psi \circ \varphi)^{\text{sh}}$.

We will not check that the forgetful functor U is a functor; the main point is that it does nothing to morphisms. Also, we will not formally check the adjoint pair, but we will say that it requires exhibit a natural isomorphism

$$\operatorname{Mor}_{\operatorname{Sh}_X}(F^{\text{sh}}, \mathcal{G}) \simeq \operatorname{Mor}_{\operatorname{PreSh}_X}(F, U\mathcal{G})$$

where $F \in \operatorname{PreSh}_X$ and $\mathcal{G} \in \operatorname{Sh}_X$. And we will describe this isomorphism: if $\operatorname{sh}\colon F \to F^{\text{sh}}$ is the sheafification map, the isomorphism is given by

$$\begin{array}{ccc} \operatorname{Mor}_{\operatorname{Sh}_X}(F^{\text{sh}}, \mathcal{G}) & \simeq & \operatorname{Mor}_{\operatorname{PreSh}_X}(F, U\mathcal{G}) \\ \varphi & \mapsto & \varphi \circ \operatorname{sh} \\ \overline{\psi} & \leftarrow\!\shortmid & \psi \end{array}$$

where $\overline{\psi}$ is the morphism induced by the universal property of sheafification applied to the presheaf morphism $\psi\colon F \to \mathcal{G}$. That this is an isomorphism follows from the universal property, and the naturality checks for the adjoint pair are a matter of writing down the squares and checking them. ■

Remark 1.144. Sheafification being a left adjoint means that it preserves limits. Kernels and limits, so we see that the sheafification of the presheaf kernel is just the presheaf kernel again. The point here is that we don't need to sheafify the kernel, which is why we could talk about them before sheafification, but we will not be so lucky with cokernels.

1.5.2 Sheaf Cokernels

Now that we have sheafification, we may continue showing that the category sheaves valued in an abelian category is abelian. For this, we need to understand cokernels.

Definition 1.145 (Sheaf cokernel)**.** Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then the *presheaf cokernel* $\operatorname{coker} \varphi$ is the sheaf found by setting

$$(\operatorname{coker}^{\text{pre}} \varphi)(U) := \operatorname{coker} \varphi_U = \mathcal{G}(U)/\operatorname{im} \varphi_U.$$

We define the *sheaf cokernel* as the sheafification of the presheaf $\operatorname{coker}^{\text{pre}} \varphi$.

We begin by running our checks on the presheaf cokernel.

Lemma 1.146. Fix a presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$. Then $\operatorname{coker}^{\text{pre}} \varphi$ is a presheaf, and it is the cokernel of φ in the category PreSh_X.

Proof. To begin, we must exhibit our restriction maps. Given open sets $V \subseteq U$ and $[g] \in (\operatorname{coker}^{\text{pre}} \varphi)(U) = \operatorname{coker} \varphi_U$, we define

$$\operatorname{res}_{U,V}([g]) := [g|_V].$$

Note this is well-defined: if $[g] = [g']$, then $g - g' \in \operatorname{im}\varphi_U$, so write $g - g' = \varphi_U(f)$ for $f \in \mathcal{F}(U)$. Thus, $g|_V - g'|_V = (g - g')|_V = \varphi_U(f)|_V = \varphi_V(f|_V)$ is in the image of φ_V, so $[g|_V] = [g'|_V]$.

We quickly check that this data assembles into a presheaf.

- Identity: given $g \in (\operatorname{coker}^{\mathrm{pre}}\varphi)(U)$, note $[g]|_U = [g|_U] = [g]$.
- Functoriality: given open sets $W \subseteq V \subseteq U$ and some $g \in (\operatorname{coker}^{\mathrm{pre}}\varphi)(U)$, we see $[g]|_V|_W = [g|_V]|_W = [g|_V|_W] = [g|_W] = [g]|_W$.

It remains to check the universal property: we need $\operatorname{coker}^{\mathrm{pre}}\varphi$ to be the colimit of the following diagram.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \mathcal{Z} & \end{tikzcd}$$

To begin, we define a morphism $\pi\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}}\varphi$. Well, for each open $U \subseteq X$, there is a natural projection $\pi_U\colon \mathcal{G}(U) \twoheadrightarrow \operatorname{coker}\varphi_U$ by $\pi_U\colon g \mapsto [g]$, which we need to assemble into a natural transformation. Indeed, given open sets $V \subseteq U$ and a section $g \in \mathcal{G}(U)$, we compute

$$\pi_U(g)|_V = [g]|_V = [g|_V] = \pi_V(g|_V).$$

This map $\pi\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}}\varphi$ induces the other needed maps $\mathcal{F} \to \operatorname{coker}^{\mathrm{pre}}\varphi$ (as $\pi\circ\varphi$) and $\mathcal{Z} \to \operatorname{coker}^{\mathrm{pre}}\varphi$ (which is the zero map). Further, note that any open $U \subseteq X$ has $(\pi\circ\varphi)_U = \pi_U \circ \varphi_U = 0$ because π_U returns 0 on $\operatorname{im}\varphi_U$; thus, $\pi\circ\varphi = 0$. Thus, the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \\ \mathcal{Z} \arrow[r] & \operatorname{coker}^{\mathrm{pre}}\varphi \end{tikzcd}$$

commutes.

We are now ready to show the universal property. Fix a presheaf $\mathcal{H}$ with a map $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $\psi\circ\varphi = 0$. Then we need a unique map $\overline{\psi}\colon (\operatorname{coker}^{\mathrm{pre}}\varphi) \to \mathcal{H}$ such that $\psi = \overline{\psi}\circ\pi$; i.e., such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \arrow[ddr, bend left, "\psi"] & \\ \mathcal{Z} \arrow[r] \arrow[drr, bend right] & \operatorname{coker}^{\mathrm{pre}}\varphi \arrow[dr, dashed, "\overline{\psi}"] & \\ & & \mathcal{H} \end{tikzcd}$$

commutes. We show uniqueness and existence of $\overline{\psi}$ separately.

- Uniqueness: given an open set $U \subseteq X$ and some $[g] \in (\operatorname{coker}^{\mathrm{pre}})(U)$, we must have

  $$\overline{\psi}_U([g]) = \overline{\psi}_U(\pi_U g) = \psi_U(g),$$

  so $\overline{\psi}_U$ is uniquely determined.
- Existence: given an open set $U \subseteq X$ and some $[g] \in (\operatorname{coker}^{\mathrm{pre}}\varphi)(U)$, we simply define

  $$\overline{\psi}_U([g]) \coloneqq \psi_U(g).$$

  Note this is well-defined: if $[g] = [g']$, then $g - g' \in \operatorname{im}\varphi_U$, so write $g - g' = \varphi_U(f)$. Then $\psi_U(g) - \psi_U(g') = \psi_U(\varphi_U f) = 0$, so $\psi_U(g) = \psi_U(g')$.

Additionally, we note that any $g \in \mathcal{G}(U)$ will have $\overline{\psi}_U(\pi_U g) = \overline{\psi}_U([g]) = \psi_U(g)$, so we conclude $\overline{\psi} \circ \pi = \psi$. It remains to show that $\overline{\psi}$ is actually a presheaf morphism. Well, any open sets $V \subseteq U$ and $[g] \in (\operatorname{coker}^{\text{pre}} \varphi)(U)$ has

$$\overline{\psi}_U([g])|_V = \psi_U(g)|_V = \psi_V(g|_V) = \overline{\psi}_V([g|_V]) = \overline{\psi}_V([g]|_V),$$

finishing. ■

And now we run the checks on the sheaf kernel.

Lemma 1.147. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X. Then $\operatorname{coker} \varphi$ is the cokernel in the category Sh_X.

Proof. Let $\pi^{\text{pre}}\colon \mathcal{G} \to \operatorname{coker}^{\text{pre}} \varphi$ be the projection map of Lemma 1.146 and $\operatorname{sh}\colon \operatorname{coker}^{\text{pre}} \varphi \to \operatorname{coker} \varphi$ be the sheafification map. Then we define $\pi := \operatorname{sh} \circ \pi^{\text{pre}}$, so we claim that this map makes $\operatorname{coker} \varphi$ the colimit of the following diagram.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\
\mathcal{Z}
\end{tikzcd}$$

Notably, we have $\pi \circ \varphi = \operatorname{sh} \circ \pi^{\text{pre}} \circ \varphi = \operatorname{sh} \circ 0 = 0$, so π at least works as a candidate morphism.

To show the universal property, fix a sheaf $\mathcal{H}$ with a map $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $\psi \circ \varphi = 0$. Then we need a unique map $\overline{\psi}\colon \operatorname{coker} \varphi \to \mathcal{H}$ such that $\psi = \overline{\psi} \circ \pi$, or equivalently, making

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \arrow[ddr, bend left, "\psi"] \\
\mathcal{Z} \arrow[r] \arrow[drr, bend right] & \operatorname{coker} \varphi \arrow[dr, dashed, "\overline{\psi}"] \\
& & \mathcal{H}
\end{tikzcd}$$

commute. We show existence and uniqueness separately.

- Existence: working in PreSh_X for a moment, the fact that $\psi \circ \varphi = 0$ promises a map $\overline{\psi}^{\text{pre}}\colon \operatorname{coker}^{\text{pre}} \varphi \to \mathcal{H}$ such that $\overline{\psi}^{\text{pre}} \circ \pi^{\text{sh}} = \psi$. Now, from the definition of sheafification, we get a map $\overline{\psi}\colon \operatorname{coker} \varphi \to \mathcal{H}$ such that
$$\overline{\psi} \circ \operatorname{sh} = \overline{\psi}^{\text{pre}}.$$
Thus, $\overline{\psi} \circ \pi = \overline{\psi} \circ \operatorname{sh} \circ \pi^{\text{pre}} = \overline{\psi}^{\text{pre}} \circ \pi^{\text{pre}} = \psi$, as needed.
- Uniqueness: suppose $\overline{\psi}_1, \overline{\psi}_2\colon \operatorname{coker} \varphi \to \mathcal{H}$ have $\psi = \overline{\psi}_1 \circ \pi = \overline{\psi}_2 \circ \pi$. Then we see that actually
$$\psi = (\overline{\psi}_1 \circ \operatorname{sh}) \circ \pi^{\text{pre}} = (\overline{\psi}_2 \circ \operatorname{sh}) \circ \pi^{\text{pre}},$$
but the universal property of $\operatorname{coker}^{\text{pre}} \varphi$ has a uniqueness forcing $\overline{\psi}_1 \circ \operatorname{sh} = \overline{\psi}_2 \circ \operatorname{sh}$. But then the universal property of sheafification says there is a unique map $\overline{\psi}\colon \operatorname{coker} \varphi \to \mathcal{H}$ such that
$$\overline{\psi} \circ \operatorname{sh} = \overline{\psi}_1 \circ \operatorname{sh} = \overline{\psi}_2 \circ \operatorname{sh},$$
so $\overline{\psi} = \overline{\psi}_1 = \overline{\psi}_2$ follows. ■

As before, we take a moment to verify that vanishing cokernel does indeed mean epic.

Lemma 1.148. Let $\mathcal{C}$ be a category with a zero object and cokernels. Then a morphism $\varphi\colon A \to B$ is epic if and only if $\operatorname{coker} \varphi$ vanishes.

Proof. Reverse all the arrows in Lemma 1.127. Notably, the dual of the kernel is the cokernel, the dual of a monic map is an epic map, and the dual of the zero object is still the zero object. ■

1.5.3 Surjectivity at Stalks

We are now ready to fix our surjectivity. Just like injectivity, we can check surjectivity at stalks.

Lemma 1.149. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then, for any p, the projection $\mathcal{G} \to \operatorname{coker}^{\text{pre}} \varphi$ induces an isomorphism

$$\operatorname{coker} \varphi_p \to (\operatorname{coker}^{\text{pre}} \varphi)_p.$$

Thus, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the projection $\mathcal{G} \to \operatorname{coker} \varphi$ induces an isomorphism $\operatorname{coker} \varphi_p \simeq (\operatorname{coker} \varphi)_p$.

Proof. Let $\pi^{\text{pre}}\colon \mathcal{G} \to \operatorname{coker}^{\text{pre}} \varphi$ be the natural projection witnessing that $\operatorname{coker}^{\text{pre}} \varphi$ is the presheaf cokernel.

To show the second sentence note π^{pre} induces a map $\mathcal{G}_p \to (\operatorname{coker}^{\text{pre}} \varphi)_p$ as

$$\pi_p^{\text{pre}}\colon [(U, g)] \mapsto [(U, \pi_U^{\text{pre}} g)].$$

Note that, if $[(U, g)] \in \operatorname{im} \varphi_p$, then we can write $[(U, g)] = [(V, \varphi_V f)]$ for some $f \in \mathcal{F}(V)$, so

$$\pi_p^{\text{pre}}(f|_p) = (\pi_V^{\text{pre}} f)|_p = 0|_p = 0,$$

so $\operatorname{im} \varphi_p \subseteq \ker \pi_p^{\text{pre}}$, so we have actually induced a map $\operatorname{coker} \varphi_p \to (\operatorname{coker}^{\text{pre}} \varphi)_p$.

In the other direction, we define $\varphi_p\colon (\operatorname{coker}^{\text{pre}} \varphi)_p \to \operatorname{coker} \varphi_p$ by

$$\varphi_p\colon [(U, [g])] \mapsto (g|_p + \operatorname{im} \varphi_p).$$

We do need to check that this is well-defined: if $(U, [g]) \sim (U', [g'])$, then we can find $V \subseteq U \cap U'$ such that $[(g - g')|_V] = [g]|_V - [g']|_V = 0$, so there is $f \in \mathcal{F}(V)$ such that $(g - g')|_V = \varphi_V(f)$. Thus, $g|_p - g'|_p = (g - g')|_p = (g - g')|_V|_p = \varphi_V(f)|_p$ is in $\operatorname{im} \varphi_p$.

Lastly, we need to check that π_p^{pre} and φ_p are inverse. Given $[(U, g)] + \operatorname{im} \varphi_p \in \operatorname{coker} \varphi_p$, we note

$$\varphi_p \left(\pi^{\text{pre}}([(U, g)] + \operatorname{im} \varphi_p)\right) = \varphi_p([(U, [g])]) = g|_p + \operatorname{im} \varphi_p.$$

Conversely, given $[(U, [g])] \in (\operatorname{coker}^{\text{pre}} \varphi)_p$, we note

$$\pi_p^{\text{pre}}(\varphi_p([(U, [g])])) = \pi^{\text{pre}}([(U, g)] + \operatorname{im} \varphi_p) = [(U, [g])],$$

finishing.

We now show the last sentence. Let $\operatorname{sh}\colon \operatorname{coker}^{\text{pre}} \varphi \to \operatorname{coker} \varphi$ be the sheafification map. Then $\pi_p = (\operatorname{sh} \circ \pi^{\text{pre}})_p$ we can check to be $\operatorname{sh}_p \circ \pi_p^{\text{pre}}$ (by, say, Remark 1.112). Stringing these isomorphisms together, we see

$$\operatorname{coker} \varphi_p \to (\operatorname{coker}^{\text{pre}} \varphi)_p \simeq (\operatorname{coker} \varphi)_p,$$

which is what we wanted. ■

And here is our result.

Proposition 1.150. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X. The following are equivalent.

(a) φ is epic.

(b) $(\operatorname{coker} \varphi)(U)$ vanishes for all open $U \subseteq X$.

(c) φ_p is epic for all $p \in X$.

Proof. By Lemma 1.148, these are equivalent to the following.

(a') $\operatorname{coker} \varphi$ vanishes.

(b') $(\operatorname{coker} \varphi)(U)$ vanishes for all open $U \subseteq X$.

(c') $\operatorname{coker} \varphi_p$ vanishes for all $p \in X$. By Lemma 1.149, this is equivalent to $(\operatorname{coker} \varphi)_p$ vanishing for all $p \in X$.

These are equivalent by Lemma 1.129. ■

1.5.4 The Category of Sheaves Is Abelian

Now that our category of sheaves (valued in an abelian category) has kernels and cokernels for our morphisms, we have two more conditions to check.

Lemma 1.151. Fix a monic morphism $\varphi\colon \mathcal{F}\to\mathcal{G}$ of sheaves on X. Then actually $\varphi\colon\mathcal{F}\to\mathcal{G}$ makes $\mathcal{F}$ the kernel of the cokernel $\pi\colon\mathcal{G}\to\operatorname{coker}\varphi$.

Proof. We need to show that $\mathcal{F}$ is the limit of the following diagram.

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

To begin, note that $\varphi\colon\mathcal{F}\to\mathcal{G}$ makes the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

commute by the construction of $\pi\colon\mathcal{G}\to\operatorname{coker}\varphi$. We now show that $\mathcal{F}$ satisfies the universal property. Fix a sheaf morphism $\psi\colon\mathcal{H}\to\mathcal{G}$ such that $\pi\circ\psi=0$. Then we need a unique map $\overline{\psi}\colon\mathcal{H}\to\mathcal{F}$ making the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[dr, dashed, "\overline{\psi}"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right] & & \\ & \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"] \\ & \mathcal{Z} \arrow[r] & \operatorname{coker}\varphi \end{tikzcd}$$

commute; i.e., we need $\psi=\varphi\circ\overline{\psi}$. We show uniqueness and existence separately.

- Uniqueness: this follows because φ is monic. Indeed, if $\overline{\psi}_1,\overline{\psi}_2$ have $\varphi\circ\overline{\psi}_1=\psi=\varphi\circ\overline{\psi}_2$, then $\overline{\psi}_1=\overline{\psi}_2$ because φ is monic.
- Existence: this is trickier. Let $\pi^{\mathrm{pre}}\colon\mathcal{G}\to\operatorname{coker}^{\mathrm{pre}}\varphi$ be the natural projection, and let $\mathrm{sh}\colon\operatorname{coker}^{\mathrm{pre}}\varphi\to\operatorname{coker}\varphi$ be the sheafification map.

  Now, given $U\subseteq X$ and $h\in\mathcal{H}(U)$, set $g:=\psi_U(h)$ for brevity. Notably, we have $\pi_U\circ\psi_U=0$, so $\pi_U(g)=0$. It follows $\pi_p(g|_p)=\pi_U(g)|_p=0$ for each $p\in U$, so $g|_p\in\ker\pi_p$ for each $p\in U$. Now, for each $p\in U$, by Lemma 1.149, $\ker\pi_p=\operatorname{im}\varphi_p$, and by Proposition 1.130, φ_p is monic, so is a unique $f_p\in\mathcal{F}_p$ such that
  $$\varphi_p(f_p)=g|_p.$$
  We claim that $(f_p)_{p\in U}$ is a system of compatible germs. To begin, choose some representative $f_p=[(U_p',\widetilde{f}_p')]$ and note that we have
  $$[(U,g)]=g|_p=\varphi_p(f_p)=[(U_p',\varphi_{U_p'}(\widetilde{f}_p'))],$$
  so we can find $U_p\subseteq U_p'$ containing p with $\widetilde{f}_p=\widetilde{f}_p'|_{U_p}$ small enough so that $g|_{U_p}=\varphi_{U_p}(\widetilde{f}_p)$. As such, any $q\in U_p$ has
  $$\varphi_q(\widetilde{f}_p|_q)=[(U_p,\varphi_{U_p}(\widetilde{f}_p))]=[(U_p,g|_{U_p})]=g|_p,$$
  so $\widetilde{f}_p|_q=f_q$ follows.

Thus, Proposition 1.107 promises a unique $f \in \mathcal{F}(U)$ such that $f|_p = f_p$ for each $p \in U$. So we define $\overline{\psi}_U(h) := f$ to be the unique element such that

$$\varphi_p(\overline{\psi}_U(h)|_p) = \psi_U(h)|_p$$

for all $p \in U$.

It remains to show that $\overline{\psi}$ assembles into a presheaf morphism. Well, for open sets $V \subseteq U$ and $h \in \mathcal{H}(U)$, we see that any $p \in V$ will have

$$\varphi_p(\overline{\psi}_U(h)|_V|_p) = \varphi_p(\overline{\psi}_U(h)|_p) = \psi_U(h)|_p = \psi_V(h|_V)|_p,$$

so the uniqueness of $\psi_V(h|_V)$ forces $\overline{\psi}_U(h)|_V = \psi_V(h|_V)$. ■

Lemma 1.152. Fix an epic morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X. Then actually $\varphi\colon \mathcal{F} \to \mathcal{G}$ makes $\mathcal{G}$ the cokernel of the kernel $\iota\colon \ker\varphi \to \mathcal{F}$.

Proof. We need to show that $\mathcal{G}$ is the colimit of the following diagram.

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"'] & \mathcal{Z} \\ \mathcal{F} & \end{tikzcd}$$

To begin, note that $\varphi\colon \mathcal{F} \to \mathcal{G}$ makes the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"'] & \mathcal{Z} \arrow[d] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

commute by the construction of $\iota\colon \ker\varphi \to \mathcal{G}$. We are now ready to show that $\mathcal{G}$ satisfies the universal property. Fix a sheaf $\mathcal{H}$ with a morphism $\psi\colon \mathcal{F} \to \mathcal{H}$ such that $\psi \circ \iota = 0$. We need a unique map $\overline{\psi}\colon \mathcal{G} \to \mathcal{H}$ such that $\psi = \overline{\psi} \circ \varphi$, or equivalently, making the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d, "\iota"'] & \mathcal{Z} \arrow[d] \arrow[ddr, bend left] & \\ \mathcal{F} \arrow[r, "\varphi"] \arrow[drr, bend right, "\psi"'] & \mathcal{G} \arrow[dr, dashed, "\overline{\psi}"] & \\ & & \mathcal{H} \end{tikzcd}$$

commute. We show uniqueness and existence separately.

- Uniqueness: this follows because φ is epic. Indeed, if $\overline{\psi}_1, \overline{\psi}_2\colon \mathcal{G} \to \mathcal{H}$ have $\overline{\psi}_1 \circ \varphi = \psi = \overline{\psi}_2 \circ \varphi$, then $\overline{\psi}_1 = \overline{\psi}_2$ because φ is epic.
- Existence: given $U \subseteq X$ and $g \in \mathcal{G}(U)$, we define $\overline{\psi}_U(g)$ by hand. By Proposition 1.150, we see that φ being epic means that φ_p is surjective for each $p \in U$, we can find $f_p \in \mathcal{F}_p$ with $\varphi_p(f_p) = g|_p$ for each p. We now set

$$h_p := \psi_p(f_p).$$

We claim that h_p is independent of the choice for f_p. Indeed, if we have $[(U, f)]$ and $[(U', f')]$ in $\mathcal{F}_p$ with $[(U, \varphi_U f)] = [(U', \varphi_{U'} f')] = g|_p$, then there is an open $V \subseteq U \cap U'$ such that $\varphi_V(f|_V - f'|_V) = 0$. Thus, $f - f' \in \ker\varphi_V = (\ker\varphi)(V)$, so it follows $\psi_V((f - f')|_V) = 0$. Thus, so

$$\psi_p([(U, f)]) - \psi_p([(U', f')]) = \psi_p([(V, \psi_V((f - f')|_V))]) = \psi_p([(V, 0)]) = 0.$$

Next, we claim that the $(h_p)_{p\in U}$ forms a compatible system of germs. Well, for each $p \in U$, we can find a sufficiently small open set U_p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $\varphi_{U_p}(\widetilde{f}_p) = g|_{U_p}$. Set $\widetilde{h}_p := \psi_{U_p}(\widetilde{f}_p)$ so that for each $q \in U_p$ has $\varphi_q(\widetilde{f}_p|_q) = \varphi_{U_p}(\widetilde{f}_p)|_q = g|_q$ and thus

$$h_q = \psi_q(\widetilde{f}_p|_q) = \psi_{U_p}(\widetilde{f}_p)|_q.$$

It follows Proposition 1.107 that we have a unique $h \in \mathcal{H}(U)$ such that $h|_p = h_p$ for each $p \in U$, so we define $\overline{\psi}_U(g) := h$. Explicitly, $\overline{\psi}_U(g)$ is the unique element of $\mathcal{H}(U)$ such that

$$\overline{\psi}_U(g)|_p = \psi_p\left(\varphi_p^{-1}(g|_p)\right)$$

for each $p \in U$.

We now run checks on $\overline{\psi}$. To see that we have a morphism $\mathcal{G} \to \mathcal{H}$, note that any opens $V \subseteq U$ and $g \in \mathcal{G}(U)$ will have, for each $p \in U$,

$$\overline{\psi}_U(g)|_V|_p = \overline{\psi}_U(g)|_p = \psi_p\left(\varphi_p^{-1}(g|_p)\right) = \psi_p\left(\varphi_p^{-1}(g|_V|_p)\right),$$

so the uniqueness of $\overline{\psi}_V(g|_V)$ forces $\overline{\psi}_U(g)|_V = \overline{\psi}_V(g|_V)$.

Lastly, we note that $\psi = \overline{\psi} \circ \varphi$: for any open $U \subseteq X$ and $f \in \mathcal{F}(U)$, all points $p \in U$ give

$$\overline{\psi}_U(\varphi_U(f))|_p = \psi_p\left(\varphi_p^{-1}(\varphi_U(f)|_p)\right) = \psi_p\left(\varphi_p^{-1}(\varphi_p(f|_p))\right) = \psi_p(f|_p) = \psi_U(f)|_p,$$

so the injectivity of Proposition 1.107 forces our equality. ■

And here is our result.

Theorem 1.153. The category Sh_X of sheaves on a topological space X valued in a (concrete) abelian category $\mathcal{C}$ is additive.

Proof. The category is additive by Corollary 1.122. Kernels exist by Lemma 1.125, and cokernels exist by Lemma 1.147. The last conditions to check are Lemma 1.151 and Lemma 1.152. ■

1.5.5 Exactness via Stalks

It is a general philosophy, well-exhibited by Theorem 1.153, that we can prove (categorical) facts about sheaves by passing to stalks. Here is an example.

Proposition 1.154. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X valued in an abelian category. Then $\operatorname{coker}\varphi \simeq \operatorname{coker}\varphi^{\mathrm{sh}}$.

Proof. We merely need to exhibit a candidate isomorphism and then check that it is an isomorphism on stalks. Here is our diagram.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\mathrm{sh}"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\mathrm{sh}"] & \operatorname{coker}^{\mathrm{pre}}\varphi \arrow[r, "\mathrm{sh}"] & \operatorname{coker}\varphi \\
\mathcal{F}^{\mathrm{sh}} \arrow[r, "\varphi^{\mathrm{sh}}"] & \mathcal{G}^{\mathrm{sh}} \arrow[r, "\pi'"] & \operatorname{coker}^{\mathrm{pre}}\varphi^{\mathrm{sh}} \arrow[r, "\mathrm{sh}"] & \operatorname{coker}\varphi^{\mathrm{sh}}
\end{tikzcd}$$

Note that the composite $\mathcal{F} \to \mathcal{G} \to \mathcal{G}^{\mathrm{sh}} \to \operatorname{coker}^{\mathrm{pre}}\varphi \to \operatorname{coker}\varphi^{\mathrm{sh}}$ is the zero map because it is the same as the same as

$$\mathcal{F} \to \underbrace{\mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}} \to \operatorname{coker}^{\mathrm{pre}}\varphi^{\mathrm{sh}}}_{0} \to \operatorname{coker}\varphi^{\mathrm{sh}}.$$

Thus, the universal property of $\operatorname{coker}\varphi$ induces a unique sheaf morphism $\psi\colon \operatorname{coker}^{\mathrm{pre}}\varphi \to \operatorname{coker}\varphi^{\mathrm{sh}}$ making

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\mathrm{sh}"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\mathrm{sh}"] & \operatorname{coker}^{\mathrm{pre}}\varphi \arrow[r, "\mathrm{sh}"] \arrow[d, dashed, "\psi"] & \operatorname{coker}\varphi \\ \mathcal{F}^{\mathrm{sh}} \arrow[r, "\varphi^{\mathrm{sh}}"] & \mathcal{G}^{\mathrm{sh}} \arrow[r, "\pi'"] & \operatorname{coker}^{\mathrm{pre}}\varphi^{\mathrm{sh}} \arrow[r, "\mathrm{sh}"] & \operatorname{coker}\varphi^{\mathrm{sh}} \end{tikzcd}$$

commute. Now, sheafification promises a unique map ψ^{sh} making

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\mathrm{sh}"'] & \mathcal{G} \arrow[r, "\pi"] \arrow[d, "\mathrm{sh}"] & \operatorname{coker}^{\mathrm{pre}}\varphi \arrow[r, "\mathrm{sh}"] \arrow[d, "\psi"] & \operatorname{coker}\varphi \arrow[d, dashed, "\psi^{\mathrm{sh}}"] \\ \mathcal{F}^{\mathrm{sh}} \arrow[r, "\varphi^{\mathrm{sh}}"] & \mathcal{G}^{\mathrm{sh}} \arrow[r, "\pi'"] & \operatorname{coker}^{\mathrm{pre}}\varphi^{\mathrm{sh}} \arrow[r, "\mathrm{sh}"] & \operatorname{coker}\varphi^{\mathrm{sh}} \end{tikzcd}$$

commute. We claim that ψ^{sh} is the desired isomorphism, for which it suffices by Proposition 1.132 to take stalks at $p \in X$ everywhere. This gives the following diagram.

$$\begin{tikzcd} \mathcal{F}_p \arrow[r, "\varphi_p"] \arrow[d, "\mathrm{sh}_p"'] & \mathcal{G}_p \arrow[r, "\pi_p"] \arrow[d, "\mathrm{sh}_p"] & (\operatorname{coker}^{\mathrm{pre}}\varphi)_p \arrow[r, "\mathrm{sh}_p"] \arrow[d, "\psi_p"] & (\operatorname{coker}\varphi)_p \arrow[d, "\psi_p^{\mathrm{sh}}"] \\ \mathcal{F}_p^{\mathrm{sh}} \arrow[r, "\varphi_p^{\mathrm{sh}}"] & \mathcal{G}_p^{\mathrm{sh}} \arrow[r, "\pi'_p"] & (\operatorname{coker}^{\mathrm{pre}}\varphi^{\mathrm{sh}})_p \arrow[r, "\mathrm{sh}_p"] & (\operatorname{coker}\varphi^{\mathrm{sh}})_p \end{tikzcd}$$

All the sh_p morphisms are isomorphisms by Proposition 1.141, so to show ψ_p^{sh} is an isomorphism, it suffices to show that ψ_p is an isomorphism. Now, by Lemma 1.149, we see that $\operatorname{im}\varphi_p$ lives in the kernel of $\mathcal{G}_p \to (\operatorname{coker}^{\mathrm{pre}}\varphi)_p$ and analogously for the bottom row. So the fact that the sh_ps are isomorphisms induces the diagram

$$\begin{tikzcd} \mathcal{G}_p/\operatorname{im}\varphi_p \arrow[r, "\overline{\pi}_p"] \arrow[d, "\mathrm{sh}_p"] & (\operatorname{coker}^{\mathrm{pre}}\varphi)_p \arrow[d, "\psi_p"] \\ \mathcal{G}_p^{\mathrm{sh}}/\operatorname{im}\varphi_p \arrow[r, "\overline{\pi}'_p"] & (\operatorname{coker}^{\mathrm{pre}}\varphi^{\mathrm{sh}})_p \end{tikzcd}$$

where sh_p is still an isomorphism because it was an isomorphism before. However, Lemma 1.149 actually tells us that this map π_p from $\mathcal{G}_p/\operatorname{im}\varphi_p = \operatorname{coker}\varphi_p$ to $(\operatorname{coker}^{\mathrm{pre}}\varphi)_p$ is an isomorphism, and analogous holds for the bottom row, so it follows that ψ_p is an isomorphism. This finishes. ■

Remark 1.155. Thinking about cokernels as quotients, Proposition 1.154 roughly says that $(\mathcal{F}/\mathcal{G})^{\mathrm{sh}} \simeq (\mathcal{F}^{\mathrm{sh}}/\mathcal{G}^{\mathrm{sh}})^{\mathrm{sh}}$, where the "embedding" $\varphi\colon \mathcal{F} \to \mathcal{G}$ has been made implicit.

As an example application, we define the sheaf image.

Definition 1.156 (Sheaf image)**.** Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then the *sheaf image* $\operatorname{im}\varphi$ of φ is the sheafification of the presheaf image

$$(\operatorname{im}^{\mathrm{pre}}\varphi)(U) = \operatorname{im}\varphi_U.$$

We go ahead and check that we have an image presheaf very quickly.

Lemma 1.157. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves on X. Then $\operatorname{im}^{\mathrm{pre}}\varphi$ is a presheaf on X.

Proof. We quickly define restriction maps in the obvious way. Given a containment $U \subseteq V$, we define $\mathrm{res}_{U,V}\colon \operatorname{im}\varphi_U \to \operatorname{im}\varphi_V$ by restricting $\mathrm{res}_{U,V}\colon \mathcal{G}(U) \to \mathcal{G}(V)$. This is well-defined: if $g \in (\operatorname{im}^{\mathrm{pre}}\varphi)(U) = \operatorname{im}\varphi_U$, then we write $g = \varphi_U(f)$ for some $f \in \mathcal{F}(U)$, so $g|_V = \varphi_U(f)|_V = \varphi_V(f|_V) \in \operatorname{im}\varphi_V$.

Now, here are our presheaf checks.

- Identity: note $g \in \operatorname{im} \varphi_U$ has $g|_U = g$.
- Functoriality: given open sets $W \subseteq V \subseteq U$ and $g \in \operatorname{im} \varphi_U$, we have $g|_V|_W = g|_W$. ■

Remark 1.158. Note there is an obvious inclusion $\iota_U^{\text{pre}} : (\operatorname{im}^{\text{pre}} \varphi)(U) \to \mathcal{G}(U)$ by $g \mapsto g$. This assembles into a presheaf morphism: given open sets $V \subseteq U$ and $g \in (\operatorname{im}^{\text{pre}} \varphi)(U)$, we have

$$\iota_U^{\text{pre}}(g)|_V = g|_V = \iota_V^{\text{pre}}(g|_V).$$

Thus, when $\mathcal{G}$ is a sheaf, sheafification induces a unique sheaf morphism $\iota \colon \operatorname{im} \varphi \to \mathcal{G}$.

We quickly check that our sheaf image is the categorical image.

Proposition 1.159. Fix a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of sheaves on X, and let $\pi \colon \mathcal{G} \to \operatorname{coker} \varphi$ be the canonical projection. Then

$$\operatorname{im} \varphi \simeq \ker \pi.$$

In other words, the canonical inclusion $\iota \colon \operatorname{im} \varphi \to \mathcal{G}$ is a kernel for π.

Proof. Pass to stalks. ■

Having defined an image sheaf, we may deal with exactness.

Definition 1.160 (Exact sequence)**.** Fix an abelian category $\mathcal{C}$. Then a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at* B if and only if $\operatorname{im} f \simeq \ker g$. More precisely, this is asking for the image of f, thought of as $\iota \colon \operatorname{im} f \to B$, to be a kernel of g.

And here is our main result.

Proposition 1.161. A sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

of sheaves on X is exact (at $\mathcal{G}$) if and only if it is exact at all stalks.

Proof. Unsurprisingly, pass to stalks. ■

Of course, because kernels don't require sheafification, we can just look at open subsets to get left-exactness.

Proposition 1.162. Fix a topological space X and an open subset $U \subseteq X$. Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

of sheaves on X (valued in an abelian category). Then

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is still exact.

Proof. We are given that

$$0 \to \mathcal{F}' \xrightarrow{\iota} \mathcal{F} \xrightarrow{\varphi} \mathcal{F}''$$

is exact, so Proposition 1.161 tells us that

$$0 \to \mathcal{F}' \xrightarrow{\iota_p} \mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{F}''_p \tag{1.8}$$

is exact at each $p \in X$. Now, as in Proposition 1.111, for any morphism of sheaves $\psi\colon \mathcal{F} \to \mathcal{G}$, we note that we have the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\psi_U"] \arrow[d, "i_{\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "i_{\mathcal{G}}"] \\
\prod_{p\in U} \mathcal{F}_p \arrow[r, "\prod \psi_p"'] & \prod_{p\in U} \mathcal{G}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi_U(f) \arrow[d, mapsto] \\
(f|_p)_{p\in U} \arrow[r, mapsto] & (\psi_U(f)|_p)_{p\in U}
\end{tikzcd}$$

so the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(U) \arrow[r, "\iota_U"] \arrow[d, "i'"] & \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "i"] & \mathcal{F}''(U) \arrow[d, "i''"] \\
0 \arrow[r] & \prod_{p\in U} \mathcal{F}'_p \arrow[r, "\prod \iota_p"] & \prod_{p\in U} \mathcal{F}_p \arrow[r, "\prod \varphi_p"] & \prod_{p\in U} \mathcal{F}''_p
\end{tikzcd} \tag{1.9}$$

commutes. We want to show that the top row is exact; note that the bottom row is exact as the product of (1.8), where exactness is preserved because commutes with kernels and hence preserve left-exact sequences. In particular, note that the vertical morphisms are injective by Proposition 1.107.

We now show our exactness of the top row of (1.9). There are two checks.

- Exact at $\mathcal{F}'(U)$: we need to show that ι_U is injective. This follows from a result showed in class which asserted that $\iota\colon \mathcal{F}' \to \mathcal{F}$ being injective implies that ι_U is injective for any open $U \subseteq X$.

- Exact at $\mathcal{F}(U)$: we need to show that the kernel of φ_U is the image of ι_U. In one direction, pick up some element $\iota_U(f')$ where $f' \in \mathcal{F}'(U)$. Then we can track this element through the diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & f' \arrow[r, "\iota_U"] \arrow[d, "i'"] & \iota_U f' \arrow[r, "\varphi_U"] \arrow[d, "i"] & \varphi_U \iota_U f' \arrow[d, "i''"] \\
0 \arrow[r] & (f'|_p)_{p\in U} \arrow[r, "\prod \iota_p"] & (\iota_p(f'|_p))_{p\in U} \arrow[r, "\prod \varphi_p"] & (\varphi_p(\iota_p(f'|_p))_{p\in U}
\end{tikzcd}$$

  Now, note that the exactness of (1.8) forces $\varphi_p \circ \iota_p = 0$, so actually $(\varphi_p(\iota_p(f'|_p)))_{p\in U}$ vanishes everywhere. Thus, the injectivity of i'' forces $\varphi_U(\iota_U(f')) = 0$, so we conclude that $\varphi_U \circ \iota_U = 0$.

  It remains to show that any element of $\ker \psi_U$ comes from the image $\operatorname{im} \iota_U$. Well, find some $f \in \mathcal{F}(U)$ such that $\varphi_U(f) = 0$. Using the commutativity of (1.9), we see that $i''(\varphi_U(f)) = 0$, so

$$\varphi_p(f|_p) = 0$$

  for all $p \in U$.

  Now, the exactness of (1.8), tells us that there is a unique germ $f''_p \in \mathcal{F}_p$ such that $\iota_p(f''_p) = f|_p$ for each $p \in U$. We claim that the tuple $(f''_p)_{p\in U}$ is a compatible system of germs. Well, for each $p \in U$, find a representative for f''_p named $\widetilde{f''_p} \in \mathcal{F}''(U_p)$. Because $\iota_p(f''_p) = f|_p$, we get the equality

$$[(U_p, \iota_{U_p}(\widetilde{f''_p}))] = [(U, f)]$$

of germs in $\mathcal{F}_p$. By making U_p small enough and restricting $\widetilde{f}_p$ appropriately, we may assume that actually $U_p \subseteq U$ and $\iota_{U_p}(\widetilde{f}_p'') = f|_{U_p}$. Now, for each $q \in U_p$, we see that

$$\iota_q(\widetilde{f}_p''|_q) = \iota_{U_p}(\widetilde{f}_p'')|_q = f|_{U_p}|_q = f|_q,$$

so the uniqueness of f_q (or equivalently, the injectivity of ι_q) forces $\widetilde{f}_p''|_q = f_q$.

Thus, we do in fact have a compatible system of germs $(f_p'')_{p\in U}$, so there is a unique element $f'' \in \mathcal{F}''(U)$ such that $f''|_p = f_p''$. This implies that

$$\left(\iota_p(f''|_p)\right)_{p\in U} = \left(\iota_p(f_p'')\right)_{p\in U} = (f|_p)_{p\in U}$$

for each $p \in U$, so the commutativity of (1.9) forces $\iota_U(f) = f'$. ∎

1.5.6 The Direct Image Sheaf

We now discuss how to build some new sheaves from old.

Definition 1.163 (Direct image sheaf)**.** Fix a continuous map $f\colon X \to Y$ of topological spaces. Given a (pre)sheaf $\mathcal{F}$ on X, we define the *direct image (pre)sheaf* on Y to be

$$f_*\mathcal{F}(U) := \mathcal{F}\left(f^{-1}(U)\right).$$

Here are our checks on the direct image sheaf.

Lemma 1.164. Fix a continuous map $f\colon X \to Y$.

(a) If $\mathcal{F}$ is a presheaf on X, then $f_*\mathcal{F}$ defines a presheaf on Y.

(b) If $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ defines a sheaf on Y.

Proof. We do these one at a time.

(a) We begin by defining our restriction maps. Well, if we have open sets $V \subseteq U \subseteq Y$, then $f^{-1}(V) \subseteq f^{-1}(U) \subseteq X$, so there is a restriction map

$$\operatorname{res}_{f^{-1}(U),f^{-1}(V)}\colon \mathcal{F}\left(f^{-1}(U)\right) \to \mathcal{F}\left(f^{-1}(V)\right).$$

Thus, we set our restriction map $\operatorname{res}_{U,V}\colon f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ as $\operatorname{res}_{U,V} := \operatorname{res}_{f^{-1}(U),f^{-1}(V)}$.

Here are our presheaf checks.

- Identity: given $s \in f_*\mathcal{F}(U) = \mathcal{F}\left(f^{-1}(U)\right)$, note $s|_U = s|_{f^{-1}(U)} = s$.
- Functoriality: given open sets $W \subseteq V \subseteq U$ and some $s \in f_*\mathcal{F}(U)$, we compute

$$f|_V|_W = f|_{f^{-1}(V)}|_{f^{-1}(W)} = f|_{f^{-1}(W)} = f|_W.$$

(b) Suppose $\mathcal{F}$ is a sheaf. We now run our sheaf checks. Fix an open cover $\mathcal{U}$ for an open set $U \subseteq Y$. Then define $V := f^{-1}(U)$ and $\mathcal{V} :- \left\{f^{-1}(U_0) : U_0 \in \mathcal{U}\right\}$; notably, $\mathcal{U}$ being an open cover of $U \subseteq Y$ promises that $\mathcal{V}$ is an open cover for V.

- Identity: suppose $s_1, s_2 \in f_*\mathcal{F}(U) = \mathcal{F}(V)$ has

$$s_1|_{U_0} = s_2|_{U_0}$$

for each $U_0 \in \mathcal{U}$. Then, moving back to X, we have $s_1|_{V_0} = s_2|_{V_0}$ for each $V_0 \in \mathcal{V}$, so it follows $s_1 = s_2$ as sections in $\mathcal{F}(V) = f_*\mathcal{F}(U)$ by the identity axiom of $\mathcal{F}$.

- Gluability: suppose we have sections $s_{U_0} \in f_*\mathcal{F}(U_0) = \mathcal{F}\left(f^{-1}(U_0)\right)$ for each $U_0 \in \mathcal{U}$ such that
$$s_{U_0}|_{U_0 \cap U_0'} = s_{U_0'}|_{U_0 \cap U_0'}.$$
Moving back to X, we have sections $t_{f^{-1}(U_0)} := s_{U_0}$ such that
$$t_{f^{-1}(U_0)}|_{f^{-1}(U_0)\cap f^{-1}(U_0')} = t_{f^{-1}(U_0')}|_{f^{-1}(U_0)\cap f^{-1}(U_0')}.$$
As such, the gluability axiom of $\mathcal{F}$ applied to the open cover $\mathcal{V}$ promises $s \in \mathcal{F}(V) = f_*\mathcal{F}(U)$ such that $s|_{U_0} = s|_{f^{-1}(U_0)} = t_{f^{-1}(U_0)} = s_{U_0}$ for each $U_0 \in \mathcal{U}$. This finishes. ■

In fact, we can build a functor out of this.

Lemma 1.165. Fix a continuous map $f\colon X \to Y$. Given a morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ of (pre)sheaves on X, there is an induced morphism $f_*\eta\colon f_*\mathcal{F} \to f_*\mathcal{G}$ of (pre)sheaves on Y. This makes $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$ into a functor.

Proof. For open $U \subseteq Y$, define $f_*\eta_U\colon f_*\mathcal{F}(U) \to f_*\mathcal{G}(U)$ by $f_*\eta_U := \eta_{f^{-1}(U)}$. Note this makes sense because
$$\eta_{f^{-1}(U)}\colon \mathcal{F}\left(f^{-1}(U)\right) \to \mathcal{G}\left(f^{-1}(U)\right).$$
Observe quickly that $f_*\eta$ is indeed a morphism of (pre)sheaves: given open sets $U' \subseteq U$ and some $s \in f_*\mathcal{F}(U)$, we have
$$f_*\eta_U(s)|_{U'} = \eta_{f^{-1}(U)}(s)|_{f^{-1}(U')} = \eta_{f^{-1}(U')}(s|_{f^{-1}(U')}) = f_*\eta_{U'}(s|_{U'}).$$
We now run functoriality checks on the functor $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$.

- Identity: given a (pre)sheaf $\mathcal{F}$ on X, an open set $U \subseteq Y$, and a section $s \in f_*\mathcal{F}(U)$, we compute
$$(f_*\mathrm{id}_{\mathcal{F}})_U(s) = (\mathrm{id}_{\mathcal{F}})_{f^{-1}(U)}(s) = s = (\mathrm{id}_{f_*\mathcal{F}})_U(s).$$
- Functoriality: given morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of (pre)sheaves on X, pick up an open set $U \subseteq Y$ and compute
$$f_*(\psi \circ \varphi)_U = (\psi \circ \varphi)_{f^{-1}(U)} = \varphi_{f^{-1}(U)} \circ \psi_{f^{-1}(U)} = f_*\psi \circ f_*\varphi,$$
which is what we wanted. ■

Remark 1.166. Given continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$, we have
$$(g \circ f)_* = g_* \circ f_*$$
as functors $\mathrm{Sh}_X \to \mathrm{Sh}_Z$. To see this, we have two checks. Fix any $U \subseteq Z$ and morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X.

- On objects, we see $(g\circ f)_*\mathcal{F}(U) = \mathcal{F}\left((g \circ f)^{-1}(U)\right) = \mathcal{F}\left(f^{-1}(g^{-1}(U))\right) = g_*(f_*\mathcal{F})(U)$. Additionally, given $U' \subseteq U$, the restriction map for $(g \circ f)_*\mathcal{F}$ is $\mathrm{res}_{(g\circ f)^{-1}(U),(g\circ f)^{-1}(U')}$ of $\mathcal{F}$. This matches the restriction map for $g_*f_*\mathcal{F}$.
- On morphisms, we see $(g \circ f)_*\varphi_U = \varphi_{((g\circ f)^{-1}(U))} = \varphi_{f^{-1}(g^{-1}(U))} = g_*(f_*\varphi)_U$.

Philosophically, we see that the point of the direct image sheaf is to use a continuous map $f\colon X \to Y$ to take a (pre)sheaf on X to a (pre)sheaf on Y. Under our stalk philosophy, we might want something like $(f_*\mathcal{F})_{f(x)} = \mathcal{F}_x$, but this need not be the case; essentially, $(f_*\mathcal{F})_{f(x)}$ is a colimit over all open sets containing $f(x)$, but we want to only consider the ones of the form $f^{-1}(U)$ where $x \in U$.

Nonetheless, there is a canonical map.

Lemma 1.167. Fix a continuous map $f\colon X \to Y$ and a (pre)sheaf $\mathcal{F}$ on X. Then, at any $x \in X$, there is a canonical map

$$(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x.$$

Proof. A germ in $(f_*\mathcal{F})_{f(x)}$ looks like $[(U,s)]$ where $f(x) \in U$ and $s \in f_*\mathcal{F}(U) = \mathcal{F}\left(f^{-1}(U)\right)$. As such, we will callously define

$$\begin{array}{rcc} \varphi\colon (f_*\mathcal{F})_{f(x)} & \to & \mathcal{F}_x \\ [(U,s)] & \mapsto & \left[(f^{-1}(U),s)\right] \end{array}$$

which we will only have to verify is well-defined. Well, suppose $[(s,U)] = [(s',U')]$ in $(f_*\mathcal{F})_{f(x)}$ so that we can find an open $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. Moving back to $\mathcal{F}$, this translates to

$$s|_{f^{-1}(V)} = s'|_{f^{-1}(V)},$$

so $\left[(f^{-1}(U),s)\right] = \left[(f^{-1}(U'),s')\right]$ follows. ■

Remark 1.168. If f is a homeomorphism (with inverse $g\colon Y \to X$), then this canonical map is an isomorphism. Indeed, we can see that the maps

$$\begin{array}{rcc} (f_*\mathcal{F})_{f(x)} & \to & \mathcal{F}_x \\ [(U,s)] & \mapsto & \left[(f^{-1}(U),s)\right] \\ \left[g^{-1}(V),s\right] & \leftarrow\!\shortmid & [(V,s)] \end{array}$$

are well-defined, essentially by the above argument, and they are inverse because $g^{-1}(f^{-1}(U)) = f(f^{-1}(U)) = U$ and similar on the other side.

Nonetheless, we can kind of feel that $f_*\mathcal{F}$ only has access to the open subsets coming from X. Let's see this.

Remark 1.169. Fix a continuous map $f\colon X \to Y$ and a sheaf $\mathcal{F}$ on X. If $y \notin \overline{\operatorname{im} f}$, then we claim that $(f_*\mathcal{F})_y = 0$. Indeed, we can restrict any germ $[(U,g)] \in (f_*\mathcal{O}_Z)_y$ to have $U \subseteq Z \setminus \overline{\operatorname{im} f}$, which means $g \in \mathcal{O}_Z(f^{-1}U) = \mathcal{O}_Z(\varnothing) = 0$, so $[(U,g)] = 0$. Thus, $(f_*\mathcal{O}_Z)_y = 0$.

1.5.7 The Inverse Image Sheaf

Given a continuous map $f\colon X \to Y$, the direct image sheaf tells us how to take a sheaf on X to a sheaf on Y. We can also define an inverse image sheaf.

Definition 1.170 (Inverse image sheaf)**.** Fix a continuous map $f\colon X \to Y$ of topological spaces. Given a (pre)sheaf $\mathcal{G}$ on Y, we define the *inverse image sheaf* $f^{-1}\mathcal{G}$ on X to be the sheafification of the presheaf

$$f^{-1,\mathrm{pre}}\mathcal{G}(U) \coloneqq \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \left\{(V,s) : s \in \mathcal{F}(V) \text{ and } V \supseteq f(U)\right\}/\sim,$$

where $(V,s) \sim (V',s')$ if and only if there is some $V'' \subseteq V \cap V'$ containing $f(U)$ such that $s|_{V''} = s'|_{V''}$.

As usual, here are the checks on the inverse image sheaf.

Lemma 1.171. Fix a continuous map $f\colon X \to Y$.

(a) If $\mathcal{G}$ is a presheaf on Y, then $f^{-1,\mathrm{pre}}\mathcal{G}$ defines a presheaf on X.

(b) If $\mathcal{G}$ is a sheaf on Y, then $f^{-1}\mathcal{G}$ defines a sheaf on X.

Proof. Note that (b) is immediate from (a) because $f^{-1}\mathcal{G}$ is the sheafification of $f^{-1,\mathrm{pre}}\mathcal{G}$. So we will focus on showing (a).

To begin, we define our restriction maps for open sets $U' \subseteq U$ as

$$\begin{aligned} \mathrm{res}_{U,U'} \colon f^{-1,\mathrm{pre}}\mathcal{G}(U) &\to f^{-1,\mathrm{pre}}\mathcal{G}(U') \\ [(V,s)] &\mapsto [(V,s)] \end{aligned}$$

which at least makes sense because $V \supseteq f(U) \supseteq f(U')$. To see that this is well-defined, suppose $[(V,s)] = [(V',s')]$ as elements of $f^{-1,\mathrm{pre}}\mathcal{G}(U)$. Then there is $V'' \subseteq V \cap V'$ with $V'' \supseteq f(U)$ such that $s|_{V''} = s'|_{V''}$. As such, $V'' \supseteq f(U')$ as well while $s|_{V''} = s'|_{V''}$, so $[(V,s)] = [(V',s')]$ as elements of $f^{-1,\mathrm{pre}}\mathcal{G}(U')$.

We now check our presheaf conditions.

- Identity: observe that $[(V,s)] \in f^{-1,\mathrm{pre}}\mathcal{G}(U)$ has $[(V,s)]|_U = [(V,s)]$.
- Functoriality: fix open sets $U'' \subseteq U' \subseteq U$ and some $[(V,s)] \in f^{-1,\mathrm{pre}}\mathcal{G}(U)$. Then

  $$[(V,s)]|_{U'}|_{U''} = [(V,s)]|_{U''} = [(V,s)] = [(V,s)]|_{U''},$$

  finishing. ■

As before, we actually have a functor.

Lemma 1.172. Fix a continuous map $f\colon X \to Y$. Given a morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ of sheaves on Y, there is an induced morphism $f^{-1}\eta\colon f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$ of sheaves on X. This makes $f^{-1}\colon \mathrm{Sh}_Y \to \mathrm{Sh}_X$ into a functor.

Proof. For open $U \subseteq X$, define

$$\begin{aligned} f^{-1,\mathrm{pre}}\eta_U \colon f^{-1,\mathrm{pre}}\mathcal{F}(U) &\to f^{-1,\mathrm{pre}}\mathcal{G}(U) \\ [(V,s)] &\mapsto [(V,\eta_V(s))] \end{aligned}$$

where $[(V,\eta_V(s))] \in f^{-1,\mathrm{pre}}\mathcal{G}(U)$ again at least makes sense because $V \supseteq f(U)$. To see that this is well-defined, note $[(V,s)] = [(V',s')]$ as elements of $f^{-1,\mathrm{pre}}\mathcal{F}(U)$ promises some $V'' \subseteq V \cap V'$ containing $f(U)$ such that $s|_{V''} = s'|_{V''}$. Then

$$\eta_V(s)|_{V''} = \eta_{V''}(s|_{V''}) = \eta_{V''}(s'|_{V''}) = \eta_{V'}(s')|_{V''},$$

so we conclude $[(V,\eta_V(s))] = [(V',\eta_{V'}(s'))]$ as elements of $f^{-1,\mathrm{pre}}\mathcal{G}(U)$.

Additionally, we see that $f^{-1,\mathrm{pre}}\eta$ assembles into a presheaf morphism: given open sets $U' \subseteq U$, note that the diagram

$$\begin{tikzcd} f^{-1,\mathrm{pre}}\mathcal{F}(U) \arrow[r, "f^{-1,\mathrm{pre}}\eta_U"] \arrow[d, "\mathrm{res}_{U,U'}"'] & f^{-1,\mathrm{pre}}\mathcal{G}(U) \arrow[d, "\mathrm{res}_{U,U'}"] \\ f^{-1,\mathrm{pre}}\mathcal{F}(U') \arrow[r, "f^{-1,\mathrm{pre}}\eta_{U'}"] & f^{-1,\mathrm{pre}}\mathcal{G}(U') \end{tikzcd} \qquad \begin{tikzcd} {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V,\eta_V(s))]} \arrow[d, mapsto] \\ {[(V,s)]} \arrow[r, mapsto] & {[(V,\eta_V(s))]} \end{tikzcd}$$

commutes. We now run the functoriality checks on $f^{-1,\mathrm{pre}}$.

- Identity: given a (pre)sheaf $\mathcal{F}$ on X, we see

  $$(f^{-1,\mathrm{pre}}\mathrm{id}_{\mathcal{F}})_U\big([(V,s)]\big) = [(V, \mathrm{id}_{\mathcal{F}(V)}\, s)] = [(V,s)].$$

- Functoriality: given morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of (pre)sheaves on X, pick up an open set $U \subseteq X$ and some $[(V,s)] \in f^{-1,\mathrm{pre}}\mathcal{F}(U)$. Then we see

  $$f^{-1,\mathrm{pre}}(\psi\circ\varphi)\big([(V,s)]\big) = [(V, \psi_V\varphi_V(s))] = (f^{-1,\mathrm{pre}}\psi \circ f^{-1,\mathrm{pre}}\varphi)_U\big([(V,s)]\big).$$

To finish, we define $f^{-1}\eta := \left(f^{-1,\text{pre}}\eta\right)^{\text{sh}}$ to be a map $f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$. Here are the functoriality checks.

- Identity: given a sheaf $\mathcal{F}$ on X, we see $f^{-1}\text{id}_\mathcal{F} = (f^{-1,\text{pre}}\,\text{id}_\mathcal{F})^{\text{sh}} = \text{id}^{\text{sh}}_{f^{-1,\text{pre}}\mathcal{F}} = \text{id}_{f^{-1}\mathcal{F}}$.
- Functoriality: given morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of sheaves on X, we see

$$f^{-1}(\varphi \circ \psi) = \left(f^{-1,\text{pre}}(\varphi \circ \psi)\right)^{\text{sh}} = \left(f^{-1,\text{pre}}\varphi \circ f^{-1,\text{pre}}\psi\right)^{\text{sh}} = f^{-1}\varphi \circ f^{-1}\psi,$$

finishing. ■

Remark 1.173. As in Remark 1.166, continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ give $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ as functors $\text{Sh}_Z \to \text{Sh}_X$. Dealing with the implicit intermediate sheafification is not something that fits into a remark, so we will omit showing this. I suspect that we will not use this fact.

Here is, approximately, the reason that we like the inverse image sheaf.

Lemma 1.174. Fix a continuous map $f\colon X \to Y$ and a sheaf $\mathcal{G}$ on Y. Then, for any $x \in X$, we have $\left(f^{-1}\mathcal{G}\right)_x \simeq \mathcal{G}_{f(x)}$.

Proof. By Proposition 1.141, it suffices to work with $f^{-1,\text{pre}}\mathcal{G}$, though this is somewhat annoying because $\left(f^{-1,\text{pre}}\mathcal{G}\right)_x$ involves equivalence classes of equivalence classes. In particular, a generic element looks like $[(U, [(V, s)])]$ where $[(V, s)] \in f^{-1,\text{pre}}\mathcal{G}(U)$, meaning $s \in \mathcal{F}(V)$ while $V \supseteq f(U)$. Thus, we see $x \in U$ gives $f(x) \in V$, so we define our map as

$$\begin{aligned} \varphi\colon \left(f^{-1,\text{pre}}\mathcal{G}\right)_x &\to [\mathcal{G}_{f(x)}] \\ [(U, [(V, s)])] &\mapsto [(V, s)] \end{aligned}$$

which again makes sense because $s \in \mathcal{F}(V)$ and $f(x) \in V$. We have the following checks on φ.

- Well-defined: if $[(U, [(V, s)])] = [(U', [(V', s')])]$, then there is an open set $U'' \subseteq U \cap U'$ containing $f(x)$ such that $[(V, s)] = [(V', s')]$ as elements of $f^{-1,\text{pre}}\mathcal{G}(U'')$. Thus, we are promised $V'' \subseteq V \cap V'$ containing $f(U'')$ and thus $f(x)$ such that $s|_{V''} = s'|_{V''}$. It follows $[(V, s)] = [(V', s')]$ as elements of $\mathcal{G}_{f(x)}$.
- Injective: suppose that $[(U, [(V, s)])]$ and $[(U', [(V', s')])]$ have $[(V, s)] = [(V', s')]$ as elements of $\mathcal{G}_{f(x)}$. This then promises some open $V'' \subseteq V \cap V'$ containing $f(x)$ such that $s|_{V''} = s'|_{V''}$. As such, set

$$U'' := f^{-1}(V'') \cap U \cap U'.$$

We automatically see $U'' \subseteq U \cap U'$ and $V'' \supseteq f(U'')$, so we note $[(V, s)] = [(V', s')]$ as elements of $f^{-1,\text{pre}}\mathcal{G}(U'')$. Thus,

$$[(U, [(V, s)])] = [(U'', [(V, s)]|_{U''})] = [(U'', [(V, s)])] = [(U'', [(V', s')])] = [(U', [(V', s')])].$$

- Surjective: pick up some $[(V, s)] \in \mathcal{G}_{f(x)}$ we would like to hit with φ. Well, set $U := f^{-1}(V)$ so that $V \supseteq f(U)$ and $x \in U$, meaning that $[(U, [(V, s)])]$ is a valid element of $\left(f^{-1,\text{pre}}\mathcal{G}\right)_x$, which we can fairly directly check goes to $[(V, s)]$ under φ.

Lastly, to check naturality, we see that a morphism $\varphi\colon \mathcal{G} \to \mathcal{G}'$ of sheaves on Y makes the diagram

$$\begin{tikzcd} \left(f^{-1,\text{pre}}\mathcal{G}\right)_x \arrow[r] \arrow[d] & \mathcal{G}_{f(x)} \arrow[d] \\ \left(f^{-1,\text{pre}}\mathcal{G}'\right)_x \arrow[r] & \mathcal{G}'_{f(x)} \end{tikzcd} \qquad \begin{tikzcd} {[(U, [(V, s)])]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V, s)]} \arrow[d, mapsto] \\ {[(U, \varphi_U[(V, s)])]} \arrow[r, mapsto] & {[(V, \varphi_V s)]} \end{tikzcd}$$

commute. ■

1.5.8 A Sheaf Adjunction

The two sheaves we just introduced are intertwined, as follows.

Proposition 1.175. There is a natural bijection

$$\operatorname{Mor}_{\mathrm{Sh}_X}\left(f^{-1}\mathcal{G},\mathcal{F}\right)\simeq\operatorname{Mor}_{\mathrm{Sh}_Y}\left(\mathcal{G},f_*\mathcal{F}\right).$$

In other words, we have a pair of adjoint functors.

Proof. We proceed with in steps. The main point is to define a unit and counit map.

1. We define the natural map $\varepsilon\colon f^{-1}f_*\mathcal{F}\to\mathcal{F}$ given a sheaf $\mathcal{F}$ on X. Well, for any open set $U\subseteq X$, we compute
$$\left(f^{-1,\mathrm{pre}}f_*\mathcal{F}\right)(U)=\varinjlim_{V\supseteq f(U)}f_*\mathcal{F}(V)=\varinjlim_{V\supseteq f(U)}\mathcal{F}\left(f^{-1}(V)\right).$$
Notably, $V\supseteq f(U)$ implies $U_X\subseteq f^{-1}(U)$, so we can take some $[(V,s)]$ with $s\in\mathcal{F}\left(f^{-1}(V)\right)$ to $s|_U\in\mathcal{F}(U)$. As such, we define
$$\begin{array}{rccc}\varepsilon_U^{\mathrm{pre}}\colon & \left(f^{-1,\mathrm{pre}}f_*\mathcal{F}\right)(U) & \to & \mathcal{F}(U_X)\\ & [(V,s)] & \mapsto & s|_U\end{array}$$
for which we have the following checks.
    - Well-defined: if $(V,s)\sim(V',s')$, then there is some open set $V''\subseteq V\cap V'$ with $V\supseteq f(U_X)$ such that $s|_{f^{-1}(V'')}=s'|_{f^{-1}(V'')}$. As such, we find
    $$s|_U=s|_{f^{-1}(V'')}|_U=s'|_{f^{-1}(V'')}|_U=s'|_U,$$
    so $\varepsilon_U^{\mathrm{pre}}([(V,s)])$ is in fact well-defined.
    - Natural: we verify that $\varepsilon^{\mathrm{pre}}$ is a (pre)sheaf morphism. Fix open sets $U'\subseteq U\subseteq X$. Then we see that the diagram
    $$\begin{tikzcd}\left(f^{-1,\mathrm{pre}}f_*\mathcal{F}\right)(U) \arrow[r, "\varepsilon_U^{\mathrm{pre}}"] \arrow[d, "\mathrm{res}_{U,U'}"'] & \mathcal{F}(U) \arrow[d, "\mathrm{res}_{U,U'}"] & {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & s|_U \arrow[d, mapsto] \\ \left(f^{-1,\mathrm{pre}}f_*\mathcal{F}\right)(U') \arrow[r, "\varepsilon_{U'}^{\mathrm{pre}}"] & \mathcal{F}(U') & {[(V,s)]} \arrow[r, mapsto] & s|_V\end{tikzcd}$$
    commutes.
    - Natural: we verify that $\varepsilon^{\mathrm{pre}}$ assembles into a natural transformation $f^{-1,\mathrm{pre}}f_*\Rightarrow\mathrm{id}_{\mathrm{PreSh}_X}$. Indeed, given a presheaf morphism $\varphi\colon\mathcal{F}\to\mathcal{F}'$, observe that the left diagram
    $$\begin{tikzcd}f^{-1,\mathrm{pre}}f_*\mathcal{F} \arrow[r, "f^{-1,\mathrm{pre}}f_*\varphi"] \arrow[d, "\varepsilon^{\mathrm{pre}}"'] & f^{-1,\mathrm{pre}}f_*\mathcal{F}' \arrow[d, "\varepsilon^{\mathrm{pre}}"] & {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V,\varphi_V(s))]} \arrow[d, mapsto] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{F}' & s|_U \arrow[r, mapsto] & \varphi_V(s)|_U\end{tikzcd}$$
    commutes at each open set $U\subseteq X$, as shown in the right diagram.

   The universal property of sheafification tells us that there is a unique sheaf morphism $\varepsilon\colon f^{-1}f_*\mathcal{F}\to\mathcal{F}$ making the diagram
$$\begin{tikzcd}f^{-1,\mathrm{pre}}f_*\mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\varepsilon^{\mathrm{pre}}"'] & f^{-1}f_*\mathcal{F} \arrow[d, dashed, "\varepsilon"] \\ & \mathcal{F}\end{tikzcd}$$

commute. We quickly check the naturality of $\varepsilon\colon f^{-1}f_* \Rightarrow \mathrm{id}_{\mathrm{Sh}_X}$: given a sheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$, the outer square of

$$\begin{tikzcd}
f^{-1,\mathrm{pre}}f_*\mathcal{F} \arrow[rrr, "f^{-1,\mathrm{pre}}f_*\varphi"] \arrow[dr, "\mathrm{sh}"] \arrow[dd, "\varepsilon^{\mathrm{pre}}_{\mathcal{F}}"'] & & & f^{-1,\mathrm{pre}}f_*\mathcal{F}' \arrow[dl, "\mathrm{sh}"'] \arrow[dd, "\varepsilon^{\mathrm{pre}}_{\mathcal{F}'}"] \\
& f^{-1}f_*\mathcal{F} \arrow[r, "f^{-1}f_*\varphi"] \arrow[dl, "\varepsilon_{\mathcal{F}}"] & f^{-1}f_*\mathcal{F}' \arrow[dr, "\varepsilon_{\mathcal{F}'}"] & \\
\mathcal{F} \arrow[rrr, "\varphi"] & & & \mathcal{F}'
\end{tikzcd}$$

commutes by our previous naturality check. Additionally, the triangles and top square commutes by sheafification. We want the bottom square to commute. Well, the path

$$f^{-1,\mathrm{pre}}f_*\mathcal{F} \to \mathcal{F} \to \mathcal{F}'$$

by sheafification induces a unique morphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}'$ making the diagram commute. Comparing our two candidates, we see that $\varphi \circ \varepsilon_{\mathcal{F}} = \varepsilon_{\mathcal{F}'} \circ f^{-1}f_*\varphi$. This finishes our check.

2. We define the natural map $\eta\colon \mathcal{G} \to f_*f^{-1}\mathcal{G}$ given a sheaf $\mathcal{G}$ on Y. Well, for any open set $U \subseteq Y$, we compute
$$\left(f_*f^{-1,\mathrm{pre}}\mathcal{G}\right)(U) = f^{-1,\mathrm{pre}}\mathcal{G}\left(f^{-1}(U)\right) = \varinjlim_{V \supseteq f(f^{-1}(U))} \mathcal{G}(V).$$

As such, there is a natural map

$$\begin{aligned} \eta^{\mathrm{pre}}_U\colon \mathcal{G}(U) &\to \left(f_*f^{-1,\mathrm{pre}}\mathcal{G}\right)(U) \\ s &\mapsto [(U,s)] \end{aligned}$$

which makes sense because $U \supseteq f\left(f^{-1}(U)\right)$. We have the following naturality checks on η^{pre}_U.

- Natural: we verify that $\varepsilon^{\mathrm{pre}}$ assembles into a (pre)sheaf morphism. Indeed, given open sets $U' \subseteq U \subseteq Y$, the diagram

$$\begin{tikzcd}
\mathcal{G}(U) \arrow[r, "\eta^{\mathrm{pre}}_U"] \arrow[d, "\mathrm{res}_{U,U'}"'] & f_*f^{-1,\mathrm{pre}}\mathcal{G}(U) \arrow[d, "\mathrm{res}_{U,U'}"] \\
\mathcal{G}(U') \arrow[r, "\eta^{\mathrm{pre}}_{U'}"] & f_*f^{-1,\mathrm{pre}}\mathcal{G}(U')
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[r, mapsto] \arrow[d, mapsto] & {[(U,s)]} \arrow[d, mapsto] \\
s|_{U'} \arrow[r, mapsto] & {[(U,s)]}
\end{tikzcd}$$

commutes because $s|_{U'}|_{U'} = s|_{U'}$ verifies $[(U', s|_{U'})] = [(U,s)]$.

- Natural: we verify that $\varepsilon^{\mathrm{pre}}$ assembles into a natural transformation $\mathrm{id}_{\mathrm{PreSh}_Y} \Rightarrow f_*f^{-1}$. Indeed, given a presheaf morphism $\varphi\colon \mathcal{G} \to \mathcal{G}'$, observe the left diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\varphi"] \arrow[d, "\eta^{\mathrm{pre}}_{\mathcal{G}}"'] & \mathcal{G}' \arrow[d, "\eta^{\mathrm{pre}}_{\mathcal{G}'}"] \\
f_*f^{-1,\mathrm{pre}}\mathcal{G} \arrow[r, "f_*f^{-1,\mathrm{pre}}\varphi"] & f_*f^{-1,\mathrm{pre}}\mathcal{G}'
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_U(s) \arrow[d, mapsto] \\
{[(U,s)]} \arrow[r, mapsto] & {[(U,\varphi_U(s))]}
\end{tikzcd}$$

commutes at each open set $U \subseteq X$ as shown in the right diagram.

Denoting our sheafification map by $\mathrm{sh}\colon f^{-1,\mathrm{pre}}\mathcal{G} \to \mathcal{G}$, we define $\eta := f_*\,\mathrm{sh} \circ \eta^{\mathrm{pre}}$. We automatically know that η is always a sheaf morphism, but to see that a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Sh}_Y} \Rightarrow f_*f^{-1}$, observe that a morphism $\varphi\colon \mathcal{G} \to \mathcal{G}'$ makes the diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\varphi"] \arrow[d, "\eta_{\mathcal{G}'}"'] & \mathcal{G}' \arrow[d, "\eta_{\mathcal{G}'}"] \\
f_*f^{-1,\mathrm{pre}}\mathcal{G} \arrow[r, "f_*f^{-1,\mathrm{pre}}\varphi"] \arrow[d, "f_*\,\mathrm{sh}"'] & f_*f^{-1,\mathrm{pre}}\mathcal{G}' \arrow[d, "f_*\,\mathrm{sh}"] \\
f_*f^{-1}\mathcal{G} \arrow[r, "f_*f^{-1}\varphi"] & f_*f^{-1}\mathcal{G}'
\end{tikzcd}$$

commute: the top square commutes as shown above, and the bottom square by applying f_* to the usual sheafification square. As such, the outer rectangle commutes, finishing.

3. We verify the triangle identities.
   - Given a sheaf $\mathcal{F}$ on X, we verify that the diagram

$$\begin{tikzcd}
f_*\mathcal{F} \arrow[r, "\eta_{f_*\mathcal{F}}"] \arrow[dr, equal] & f_*f^{-1}f_*\mathcal{F} \arrow[d, "f_*\varepsilon_{\mathcal{F}}"] \\
& f_*\mathcal{F}
\end{tikzcd}$$

commutes. Indeed, for any open set $U \subseteq Y$ and $s \in \mathcal{F}\left(f^{-1}(U)\right)$, we see

$$\begin{tikzcd}
f_*\mathcal{F}(U) \arrow[r, "\eta^{\mathrm{pre}}_{f_*\mathcal{F}}(U)"] \arrow[dr, equal] & f_*f^{-1,\mathrm{pre}}f_*\mathcal{F}(U) \arrow[d, "f_*\varepsilon^{\mathrm{pre}}_{\mathcal{F}}(U)"] \\
& f_*\mathcal{F}(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
s \arrow[r, mapsto] \arrow[dr, mapsto] & {[(U,s)]} \arrow[d, mapsto] \\
& s|_U = s|_{f^{-1}(U)} = s
\end{tikzcd}$$

commutes. As such, the outer triangle of

$$\begin{tikzcd}
f_*\mathcal{F} \arrow[dr, "\eta^{\mathrm{pre}}_{f_*\mathcal{F}}"'] \arrow[drr, "\eta_{f_*\mathcal{F}}"] \arrow[dd, equal] & & \\
& f_*f^{-1,\mathrm{pre}}f_*\mathcal{F} \arrow[r, dashed, "\mathrm{sh}"] \arrow[dl, "f_*\varepsilon^{\mathrm{pre}}_{\mathcal{F}}"'] & f_*f^{-1}f_*\mathcal{F} \arrow[dll, "f_*\varepsilon_{\mathcal{F}}"] \\
f_*\mathcal{F} & &
\end{tikzcd}$$

commutes, making the inner triangle commute by definition of those morphisms.
   - Give a sheaf $\mathcal{G}$ on Y, we verify that the diagram

$$\begin{tikzcd}
f^{-1}\mathcal{G} \arrow[r, "f^{-1}\eta_{\mathcal{G}}"] \arrow[dr, equal] & f^{-1}f_*f^{-1}\mathcal{G} \arrow[d, "\varepsilon_{f^{-1}\mathcal{G}}"] \\
& f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes. Indeed, for any open set $U \subseteq X$ and $[(V,s)] \in f^{-1}\mathcal{G}(U)$, we see

$$\begin{tikzcd}
f^{-1,\mathrm{pre}}\mathcal{G}(U) \arrow[r, "f^{-1,\mathrm{pre}}\eta^{\mathrm{pre}}_{f^{-1}\mathcal{G}}(U)"] \arrow[dr, equal] & f^{-1,\mathrm{pre}}f_*f^{-1,\mathrm{pre}}\mathcal{G}(U) \arrow[d, "\varepsilon^{\mathrm{pre}}_{f^{-1,\mathrm{pre}}\mathcal{G}}(U)"] \\
& f^{-1,\mathrm{pre}}\mathcal{G}(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(V,s)]} \arrow[r, mapsto] \arrow[dr, mapsto] & {[(V,\eta_{\mathcal{G}}(s))] = [(V,[(V,s)])]} \arrow[d, mapsto] \\
& {[(V,s)]|_U = [(V,s)]}
\end{tikzcd}$$

commutes; here we have extended η^{pre} and ε^{pre} in the natural way to all presheaves. Thus, we claim that the diagram

$$\begin{tikzcd}
f^{-1}\mathcal{G} \arrow[r, "f^{-1}\eta^{\text{pre}}_{\mathcal{G}}"] \arrow[rr, bend left, "f^{-1}\eta_{\mathcal{G}}"] \arrow[dr, phantom, "(1)"] & f^{-1}f_*f^{-1,\text{pre}}\mathcal{G} \arrow[r, "f^{-1}f_*\,\text{sh}"] \arrow[dr, phantom, "(2)"] & f^{-1}f_*f^{-1}\mathcal{G} \arrow[dd, bend left, "\varepsilon_{f^{-1}\mathcal{G}}"] \\
f^{-1,\text{pre}}\mathcal{G} \arrow[u, "\text{sh}"] \arrow[r, "f^{-1,\text{pre}}\eta^{\text{pre}}_{\mathcal{G}}"] \arrow[dr, equal] & f^{-1,\text{pre}}f_*f^{-1,\text{pre}}\mathcal{G} \arrow[u, "\text{sh}"] \arrow[r, "f^{-1,\text{pre}}f_*\,\text{sh}"] \arrow[d, "\varepsilon^{\text{pre}}_{f^{-1,\text{pre}}\mathcal{G}}"] \arrow[dr, phantom, "(3)"] & f^{-1,\text{pre}}f_*f^{-1}\mathcal{G} \arrow[u, "\text{sh}"] \arrow[d, "\varepsilon^{\text{pre}}_{f^{-1}\mathcal{G}}"] \\
& f^{-1,\text{pre}}\mathcal{G} \arrow[r, "\text{sh}"] & f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes. The triangle commutes as checked above; (1) and (2) both commute because f^{-1} is sheafification applied to the functor $f^{-1,\text{pre}}$. Lastly, (3) is a naturality square for ε^{pre} applied to $\text{sh}\colon f^{-1,\text{pre}}\mathcal{G} \to \mathcal{G}$. Collapsing the above diagram, we conclude that

$$\begin{tikzcd}
f^{-1,\text{pre}}\mathcal{G} \arrow[r, equal] \arrow[d, "\text{sh}"'] & f^{-1,\text{pre}}\mathcal{G} \arrow[d, "\text{sh}"] \\
f^{-1}\mathcal{G} \arrow[r, "\varepsilon_{f^{-1}\mathcal{G}}\circ f^{-1}\eta_{\mathcal{G}}"'] & f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes, but because sheafification is a functor, we are forced to have $\varepsilon_{f^{-1}\mathcal{G}} \circ f^{-1}\eta_{\mathcal{G}} = \text{id}_{f^{-1}\mathcal{G}}$, which finishes this check.

4. We now exhibit our natural bijection as follows; fix sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y.

$$\begin{array}{ccc}
\text{Mor}_{\text{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) & \simeq & \text{Mor}_{\text{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}) \\
\varphi & \mapsto & f_*\varphi \circ \eta_{\mathcal{G}} \\
\varepsilon_{\mathcal{F}} \circ f^{-1}\psi & \mapsfrom & \psi
\end{array}$$

We have the following checks.

- Bijective: starting with $\varphi\colon f^{-1}\mathcal{G} \to \mathcal{F}$, we get mapped to

$$\begin{aligned}
\varepsilon_{\mathcal{F}} \circ f^{-1}\left(f_*\varphi \circ \eta_{\mathcal{G}}\right) &= \varepsilon_{\mathcal{F}} \circ f^{-1}f_*\varphi \circ f^{-1}\eta_{\mathcal{G}} \\
&\overset{*}{=} \varphi \circ \varepsilon_{f^{-1}\mathcal{G}} \circ f^{-1}f_*\varphi \circ f^{-1}\eta_{\mathcal{G}} \\
&= \varphi,
\end{aligned}$$

  where in $\overset{*}{=}$ we used the naturality of ε, and the last equality used the triangle equalities. Similarly, starting with $\psi\colon \mathcal{G} \to f_*\mathcal{F}$, we get mapped to

$$\begin{aligned}
f_*\left(\varepsilon_{\mathcal{F}} \circ f^{-1}\psi\right) \circ \eta_{\mathcal{G}} &= f_*\varepsilon_{\mathcal{F}} \circ f_*f^{-1}\psi \circ \eta_{\mathcal{G}} \\
&\overset{*}{=} f_*\varepsilon_{\mathcal{F}} \circ \eta_{f_*\mathcal{F}} \circ \psi \\
&= \psi,
\end{aligned}$$

  where in $\overset{*}{=}$ we used the naturality of η, and the last equality used the triangle equalities.
- Natural: given a morphism $\alpha\colon \mathcal{F} \to \mathcal{F}'$ of sheaves on X, the square

$$\begin{tikzcd}
\text{Mor}_{\text{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \arrow[r, phantom, "\simeq"] \arrow[d, "\alpha\circ-"'] & \text{Mor}_{\text{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}) \arrow[d, "f_*\alpha\circ-"] & \varphi \arrow[r, mapsto] \arrow[d, mapsto] & f_*\varphi \circ \eta_{\mathcal{G}} \arrow[d, mapsto] \\
\text{Mor}_{\text{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}') \arrow[r, phantom, "\simeq"] & \text{Mor}_{\text{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}') & \alpha \circ \varphi \arrow[r, mapsto] & f_*(\alpha \circ \varphi) \circ \eta_{\mathcal{G}}
\end{tikzcd}$$

commutes. Similarly, given a morphism $\beta\colon \mathcal{G} \to \mathcal{G}'$ of sheaves on Y, the square

$$\begin{tikzcd}
\operatorname{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G}', \mathcal{F}) \arrow[r, phantom, "\simeq"] \arrow[d, "-\circ f^{-1}\beta"'] & \operatorname{Mor}_{\mathrm{Sh}_Y}(\mathcal{G}', f_*\mathcal{F}) \arrow[d, "-\circ\beta"] & \varepsilon_{\mathcal{F}} \circ f^{-1}\psi \arrow[d, mapsto] & \psi \arrow[l, mapsto] \arrow[d, mapsto] \\
\operatorname{Mor}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \arrow[r, phantom, "\simeq"] & \operatorname{Mor}_{\mathrm{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}) & \varepsilon_{\mathcal{F}} \circ f^{-1}(\psi\circ\beta) & \psi\circ\beta \arrow[l, mapsto]
\end{tikzcd}$$

commutes.

The above checks finish the proof. ■

1.5.9 The Restriction Sheaf

One particular example of the inverse image sheaf is for an embedding.

Definition 1.176 (Restriction sheaf)**.** Fix a topological space X and a subset $S \subseteq X$; let $\iota\colon S \to X$ be the embedding. Then a sheaf $\mathcal{F}$ on X restricts to a sheaf $\mathcal{F}|_S \coloneqq \iota^{-1}\mathcal{F}$ on S.

For example, our computation of stalks for the inverse image sheaf tells us that any $p \in S$ has

$$(\mathcal{F}|_S)_p = \left(\iota^{-1}\mathcal{F}\right)_{\iota(p)} \simeq \mathcal{F}_p$$

by Lemma 1.174.

A special case of this embedding will be of interest.

Lemma 1.177. Fix a topological space X and an open subset $U \subseteq X$; let $\iota\colon U \to X$ be the embedding. Then a sheaf $\mathcal{F}$ on X actually restricts to a sheaf

$$\iota^{-1,\mathrm{pre}}\mathcal{F}(V) \coloneqq \mathcal{F}(V)$$

on U.

Proof. We already know that $\iota^{-1,\mathrm{pre}}\mathcal{F}$ is a presheaf by Lemma 1.171, where the restriction map for $V' \subseteq V \subseteq U$ going $\iota^{-1,\mathrm{pre}}\mathcal{F}(V') \to \iota^{-1,\mathrm{pre}}\mathcal{F}(V)$ is just $\mathcal{F}(V') \to \mathcal{F}(V)$. It remains to show the sheaf axioms. Fix an open cover $\mathcal{V}$ of an open set $V \subseteq U$.

- Identity: if $f_1, f_2 \in \iota^{-1,\mathrm{pre}}\mathcal{F}(V)$ have $f_1|_W = f_2|_W$ for each $W \in \mathcal{V}$, then we are actually saying that $f_1, f_2 \in \mathcal{F}(V)$ have
$$f_1|_W = f_2|_W$$
for all $W \in \mathcal{V}$, so $f_1 = f_2$ follows from the identity axiom of $\mathcal{F}$.
- Gluability: suppose we have $f_W \in \iota^{-1,\mathrm{pre}}\mathcal{F}(W)$ for each $W \in \mathcal{V}$ such that $f_W|_{W\cap W'} = f_{W'}|_{W\cap W'}$ for each $W, W' \in \mathcal{V}$. This actually translates into $f_W \in \mathcal{F}(W)$ and
$$f_W|_{W\cap W'} = f_{W'}|_{W\cap W'}$$
for each $W, W' \in \mathcal{V}$, from which it follows we can find $f \in \mathcal{F}(V) = \iota^{-1,\mathrm{pre}}\mathcal{F}(V)$ such that $f|_W = f_W$ for each $W \in \mathcal{V}$. ■

Remark 1.178. Note that there is a natural isomorphism

$$\begin{array}{ccc}
\varinjlim_{W\supseteq V} \mathcal{F}(W) & \simeq & \mathcal{F}(V) \\
(W, s) & \mapsto & s|_W \\
(V, s) & \leftarrow\!\shortmid & s
\end{array}$$

which motivates makes our definition of $\iota^{-1,\mathrm{pre}}\mathcal{F}$ above make sense.

With the above in mind, in order to avoid a level of sheafification in this special case, we will sloppily set the following notation.

Notation 1.179. Fix a topological space X and an open subset $U \subseteq X$. Then, given a (pre)sheaf $\mathcal{F}$ we will set the restriction (pre)sheaf $\mathcal{F}|_U$ to actually be $\iota^{-1,\text{pre}}\mathcal{F}$, where $\iota\colon U \to X$ is the embedding.

Notably, because $\mathcal{F}|_U$ is already a sheaf when $\mathcal{F}$ is a sheaf, the isomorphism class remains well-defined among our notation.

Restricting to open sets is a good notion for a number of reasons. Let's first make Lemma 1.174 more concrete.

Lemma 1.180. Fix a topological space X and an open subset $U \subseteq X$. Then for any presheaf $\mathcal{F}$ on X and point $p \in X$, we exhibit a natural isomorphism $\mathcal{F}_p \simeq (\mathcal{F}|_U)_p$.

Proof. Our isomorphism will be as follows.

$$\begin{array}{rcl} \mathcal{F}_p & \simeq & (\mathcal{F}|_U)_p \\ \varphi\colon [(V,f)] & \mapsto & [(V\cap U, f|_{V\cap U})] \\ \psi\colon [(V,f)] & \leftarrow\!\shortmid & [(V,f)] \end{array}$$

Here are our checks.

- φ is well-defined: if $(V,f) \sim (V',f')$ in $\mathcal{F}_p$, then there is $V'' \subseteq V \cap V'$ containing p with $f|_{V''} = f'|_{V''}$. Thus,
$$f|_{V\cap U}|_{V''\cap U} = f|_{V'\cap U}|_{V''\cap U},$$
so $(V\cap U, f) \sim (V'\cap U, f')$ in $(\mathcal{F}|_U)_p$.
- ψ is well-defined: if $(V,f) \sim (V',f')$ in $(\mathcal{F}|_U)_p$, then there is $V'' \subseteq V \cap V'$ containing p with $f|_{V''} = f'|_{V''}$. It follows $(V,f) \sim (V',f')$ in $\mathcal{F}_p$.
- Inverse: starting with $[(V,f)] \in \mathcal{F}_p$, we go to $[(V\cap U, f|_{V\cap U})] \in (\mathcal{F}|_U)_p$ and then back to $[(V \cap U, f|_{V\cap U})] \in \mathcal{F}_p$. But
$$[(V\cap U, f|_{V\cap U})] = [(V,f)].$$
- Inverse: starting with $[(V,f)] \in (\mathcal{F}|_U)_p$, we go to $[(V,f)] \in \mathcal{F}_p$ and then back to $[(V,f)] \in (\mathcal{F}|_U)_p$.
- Naturality: fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves. Then we see that the diagram
$$\begin{tikzcd} (\mathcal{F}|_U)_p \arrow[d, "(\varphi|_U)_p"'] \arrow[r, phantom, "\simeq"] & \mathcal{F}_p \arrow[d, "\varphi_p"] & {[(V,f)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V,f)]} \arrow[d, mapsto] \\ (\mathcal{G}|_U)_p \arrow[r, phantom, "\simeq"] & \mathcal{G}_p & {[(V,\varphi_V f)]} \arrow[r, mapsto] & {[(V,\varphi_V f)]} \end{tikzcd}$$
commutes, which is what we wanted. ■

Unsurprisingly, understanding stalks gives us exactness.

Lemma 1.181. Fix a topological space X and an open subset $U \subseteq X$ and some abelian category $\mathcal{C}$. Then $\cdot|_U$ defines an exact functor $\mathrm{Sh}_X \to \mathrm{Sh}_U$.

Proof. Functoriality is by our functoriality of Lemma 1.172 applied to the (continuous) embedding $\iota\colon U \hookrightarrow X$. As such, we begin with a sequence
$$\mathcal{F} \stackrel{\alpha}{\to} \mathcal{G} \stackrel{\beta}{\to} \mathcal{H}$$

exact at $\mathcal{G}$. Now, for any $p \in U$, Lemma 1.180 grants us the commutative diagram

$$\begin{tikzcd} \mathcal{F}_p \arrow[r, "\alpha_p"] \arrow[d, "\simeq"'] & \mathcal{G}_p \arrow[r, "\beta_p"] \arrow[d, "\simeq"'] & \mathcal{H}_p \arrow[d, "\simeq"'] \\ (\mathcal{F}|_U)_p \arrow[r, "(\alpha|_U)_p"] & (\mathcal{G}|_U)_p \arrow[r, "(\beta|_U)_p"] & (\mathcal{H}|_U)_p \end{tikzcd}$$

where the vertical morphisms are isomorphisms. Now, Proposition 1.161 tells us that the top row is exact in $\mathcal{C}$, so the bottom row is also exact in $\mathcal{C}$ because the diagram commutes and the vertical morphisms are isomorphisms. Thus, applying Proposition 1.161 in the other direction, we see

$$\mathcal{F}|_U \stackrel{\alpha|_U}{\to} \mathcal{G}|_U \stackrel{\beta|_U}{\to} \mathcal{H}|_U$$

is also exact. This is what we wanted. ■

Having a good understanding of stalks gives us an understanding of sheafification here as well.

Lemma 1.182. Fix a topological space X and an open subset $U \subseteq X$. Then restriction commutes with sheafification: for a presheaf $\mathcal{F}$ on X, we have $(\mathcal{F}|_U)^{\mathrm{sh}} \simeq \mathcal{F}^{\mathrm{sh}}|_U$.

Proof. We begin with the sheafification map $\mathrm{sh}\colon \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$, we found that $\cdot|_U$ is functorial in Lemma 1.172, so we induce a presheaf morphism $\mathrm{sh}|_U\colon \mathcal{F}|_U \to \mathcal{F}^{\mathrm{sh}}|_U$. Now, $\mathcal{F}^{\mathrm{sh}}|_U$ is a sheaf, so sheafification induces a unique morphism $\varphi\colon (\mathcal{F}|_U)^{\mathrm{sh}} \to \mathcal{F}^{\mathrm{sh}}|_U$ making the diagram

$$\begin{tikzcd} \mathcal{F}|_U \arrow[r] \arrow[dr, "\mathrm{sh}|_U"'] & (\mathcal{F}|_U)^{\mathrm{sh}} \arrow[d, "\varphi"] \\ & \mathcal{F}^{\mathrm{sh}}|_U \end{tikzcd}$$

commute, where the top arrow is sheafification.

It remains to check that φ is an isomorphism, for which it suffices by Proposition 1.132 to check that φ is an isomorphism on stalks. Well, we note that the sheafification map $\mathcal{F}|_U \to (\mathcal{F}|_U)^{\mathrm{sh}}$ is an isomorphism on stalks by Proposition 1.141, so it suffices to show that $\mathrm{sh}|_U$ is an isomorphism on stalks. Well, fix some $p \in U$, and we use Lemma 1.180 to note that

$$\begin{tikzcd} (\mathcal{F}|_U)_p \arrow[r, phantom, "\simeq"] \arrow[d, "(\mathrm{sh}|_U)_p"'] & \mathcal{F}_p \arrow[d, "\mathrm{sh}_p"] \\ (\mathcal{F}^{\mathrm{sh}}|_U)_p \arrow[r, phantom, "\simeq"] & \mathcal{F}^{\mathrm{sh}}_p \end{tikzcd}$$

commutes, where the horizontal morphisms are isomorphisms. We note the right arrow is an isomorphism by Proposition 1.132, so it follows that the left arrow is an isomorphism as well, which is what we wanted. ■

1.5.10 More Sheaves

Let's see a few more examples, for fun.

Definition 1.183 (Constant sheaf)**.** Fix a set S and a topological space X. Then the *constant sheaf* is

$$\underline{S}(U) := \mathrm{Mor}_{\mathrm{Top}}(U, S),$$

where S has been turned into a topological space by giving it the discrete topology.

Remark 1.184. Intuitively, one should think of $\underline{S}(U)$ as $S^{\oplus \pi_0(U)}$ where $\pi_0(U)$ is the number of connected components in U. We have chosen not to do this because this definition is hard to work with for proofs.

Remark 1.185. All the stalks of $\underline{S}$ are S.

Definition 1.186 (Skyscraper sheaf)**.** Fix a topological space Y and a set S. For $y \in Y$, set $X := \{y\}$ so that there is a continuous map $\iota\colon X \hookrightarrow Y$. Then we define the *skyscraper sheaf* as

$$\iota_* S(U) := \begin{cases} S & y \in U, \\ \{*\} & y \notin U. \end{cases}$$

Remark 1.187. For $z \in Y$, we can compute the stalk of the skyscraper sheaf as

$$(\iota_* S)_z = \begin{cases} S & z \in \overline{\{y\}}, \\ \{*\} & z \notin \overline{\{y\}}. \end{cases}$$

For another remark, we pick up the following definition.

Definition 1.188 (Support)**.** Fix a sheaf $\mathcal{F}$ on a topological space x. Then we define the *support* of $\mathcal{F}$ to be

$$\operatorname{supp} \mathcal{F} := \{x \in X : \#\mathcal{F}_x \text{ is not terminal}\}.$$

Remark 1.189. The support of $\iota_* S$ is $\overline{\{y\}}$.

Here is another result, which explains why we care about the skyscraper sheaf.

Proposition 1.190. There is a natural bijection

$$\operatorname{Mor}_{\{y\}}(\mathcal{F}_y, \mathcal{G}) \simeq \operatorname{Mor}_Y(\mathcal{F}, \iota_* \mathcal{G}).$$

In other words, understanding maps from stalks is roughly the same as understanding maps to the corresponding skyscraper sheaf.

Theme 2
Building Schemes

when it is right, the things you reach for in life, the things you deeply hope for, they will reach back.

—Bianca Sparacino, [Spa18]

2.1 September 7

Today we define schemes.

2.1.1 Locally Ringed Spaces

Schemes will be a special kind of locally ringed space, so we take a moment to define these.

Definition 2.1 (Locally ringed space)**.** A *locally ringed space* is an ordered pair $(X, \mathcal{O}_X)$ of a topological space X and sheaf of rings $\mathcal{O}_X$ such that all stalks are local rings.

Example 2.2. Affine schemes are locally ringed spaces by Lemma 1.102.

Example 2.3. Fix a locally ringed space $(X, \mathcal{O}_X)$. For any open subset $U \subseteq X$, we see that $(U, \mathcal{O}_X|_U)$ is a locally ringed space as well. Namely, $\mathcal{O}_X|_U$ is certainly a sheaf of rings on U, and by Lemma 1.174 tells us that any $x \in U$ makes

$$(\mathcal{O}_X|_U)_x = \mathcal{O}_{X,x}$$

a local ring, so all stalks are indeed local rings.

Having been introduced to a new algebraic object, one should ask how to define a morphism. This is somewhat subtle. We begin by just giving the definition.

Definition 2.4 (Morphism of locally ringed spaces)**.** Given locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ locally ringed spaces, a *morphism* is a pair $(\varphi, \varphi^\sharp)$ of a continuous map $\varphi\colon X \to Y$ and a sheaf morphism $\varphi^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Further, we require that, at each $x \in X$, the induced map

$$\begin{array}{ccccc} \mathcal{O}_{Y,\varphi(x)} & \overset{\varphi^\sharp_{\varphi(x)}}{\to} & (\varphi_*\mathcal{O}_X)_{\varphi(x)} & \to & \mathcal{O}_{X,x} \\ [(V,s)] & \mapsto & \left[(V, \varphi^\sharp_V(s))\right] & \mapsto & \varphi^\sharp_V(s)|_x \end{array}$$

is a morphism of local rings; i.e., the image of $\mathfrak{m}_{Y,\varphi(x)}$ is contained in $\mathfrak{m}_{X,x}$, or equivalently the pre-image of $\mathfrak{m}_{X,x}$ is $\mathfrak{m}_{Y,\varphi(x)}$.

Example 2.5. Given an open subset $U \subseteq X$, the embedding $\iota\colon U \to X$ combined with the sheaf map $\iota^\sharp\colon \mathcal{O}_X \to \iota_*(\mathcal{O}_X|_U)$ by $\iota^\sharp(s) := s|_{U\cap V}$ assembles into a morphism of locally ringed spaces $(\iota, \iota^\sharp)$.

- To see that $\iota^\sharp$ is a sheaf morphism, we see that any open $V' \subseteq V \subseteq X$ and $s \in \mathcal{O}_X(V)$ has $\iota^\sharp_V(s)|_{V'} = s|_{U\cap V}|_{U\cap V'} = s|_{U\cap V'} = s|_{V'}|_{U\cap V'} = \iota^\sharp_{V'}(s|_{V'})$.
- To see that we have a morphism of locally ringed spaces, we fix $p \in U$ and compute that

  $$\begin{array}{ccccc} \mathcal{O}_{X,p} & \overset{\iota^\sharp_p}{\to} & (\iota_*\mathcal{O}_X)_p & \to & \mathcal{O}_{X,p} \\ [(V,s)] & \mapsto & \left[(V, \iota^\sharp_V(s))\right] & \mapsto & [V\cap U, s|_{U\cap V}] \end{array}$$

  which we can check directly is a map of local rings: if $[(V,s)] = 0$, then there is some open neighborhood V' containing p where s vanishes; but then $[V\cap U, s|_{U\cap V}]$ will also vanish upon restricting to $V' \cap U$.

Notably, the last map $(f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ above is the canonical map of Lemma 1.167.

Remark 2.6. Using the inverse image sheaf instead of the direct image sheaf, we can use Proposition 1.175 to think about $f^\sharp$ as

$$f^\flat\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

One might want to do this because the stalks of $f^{-1}\mathcal{O}_Y$ are nicely behaved by Lemma 1.174.

We take a moment to provide two ways to motivate Definition 2.4.

1. On the algebraic side, it will turn out that this definition makes the only morphisms of affine schemes (which are locally ringed spaces) come from ring homomorphisms, so we can "check" that this definition is the correct one.

   To help see why Definition 2.4 looks the way that it does a ring homomorphism $f\colon A \to B$ gives rise to a continuous map $\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$, but the function data still goes to $A \to B$. This explains why $\varphi^\sharp$ should go $\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$.

   Lastly, we can view the local ring condition as checking that we cohere with the "local" part of a locally ringed space.

2. On the geometric side, we should imagine that a morphism of locally ringed spaces is like a map $\varphi\colon X \to Y$ of manifolds, where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the sheaf of holomorphic functions on each. Then the sheaf morphism

   $$\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$$

   is saying that a holomorphic function $f\colon V \to \mathbb{C}$ (for some open $V \subseteq Y$) should pull back through φ to a differential function

   $$\varphi^{-1}(V) \overset{\varphi}{\to} V \overset{f}{\to} \mathbb{C} \tag{2.1}$$

which is simply true. Importantly, there isn't really a way to take a holomorphic function $X \to \mathbb{C}$ and "push it" through φ to a holomorphic function $Y \to \mathbb{C}$.

Lastly, the local ring condition is saying that a germ $f \in \mathcal{O}_{Y,\varphi(x)}$ will vanish at $\varphi(x)$ will pull back via (2.1) to a germ in $\mathcal{O}_{X,x}$ which vanishes at x. Again, this is simply true.

Here are some quick checks on locally ringed spaces.

Lemma 2.7. All locally ringed spaces equipped with the defined morphisms makes a category.

Proof. Here is the extra data we need to define.

- Identity: given a locally ringed space $(X, \mathcal{O}_X)$, we define $\mathrm{id}_{(X,\mathcal{O}_X)}$ as given by the continuous map $\mathrm{id}_X \colon X \to X$ and sheaf morphism $\mathrm{id}_{\mathcal{O}_X} \colon \mathcal{O}_X \to \mathcal{O}_X$. (Notably, $(\mathrm{id}_X)_*\mathcal{O}_X$ is the same as $\mathcal{O}_X$ by Lemma 1.165.) Checking stalks, we see that any $x \in X$ has

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & \stackrel{\mathrm{id}_{\mathcal{O}_X,x}}{\to} & ((\mathrm{id}_{\mathcal{O}_X})_*\mathcal{O}_X)_x & \to & \mathcal{O}_{X,x} \\ [(U,s)] & \mapsto & [(U,s)] & \mapsto & [(U,s)] \end{array}$$

  is the identity and hence a map of local rings.

- Composition: given two morphisms $(\varphi, \varphi^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(\psi, \psi^\sharp) \colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, we define the composition as having the continuous map $\psi \circ \varphi \colon X \to Y$ and sheaf morphism

$$\mathcal{O}_Z \stackrel{\psi^\sharp}{\to} \psi_*\mathcal{O}_Y \stackrel{\psi_*\varphi^\sharp}{\to} \psi_*\varphi_*\mathcal{O}_X.$$

  Notably, $\psi_*\varphi_*\mathcal{O}_X = (\psi \circ \varphi)_*\mathcal{O}_X$ by Remark 1.166, so at least all of our data look correct.

  Checking stalks, fix $x \in X$ and $[(U,s)] \in \mathfrak{m}_{Z,\psi(\varphi(z))}$. Because $(\varphi, \varphi^\sharp)$ and $(\psi, \psi^\sharp)$ are morphisms of locally ringed spaces, we see that $[(\psi^{-1}U, \psi^\sharp_U z)] \in \mathfrak{m}_{Y,\varphi(z)}$, so

$$\left[((\psi \circ \varphi)^{-1}U, (\psi_*\varphi^\sharp \circ \psi^\sharp)_U s)\right] = \left[(\varphi^{-1}\psi^{-1}U, \varphi^\sharp_{\psi^{-1}U}\psi^\sharp_U s)\right] \in \mathfrak{m}_{X,x},$$

  which finishes the check.

We have the following coherence checks.

- Identity: given a morphism $(\varphi, \varphi^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, we compute

$$(\varphi, \varphi^\sharp) \circ \mathrm{id}_{(X,\mathcal{O}_X)} = \left(\varphi \circ \mathrm{id}_X, \varphi_*\mathrm{id}_{\mathcal{O}_X} \circ \varphi^\sharp\right) = \left(\varphi \circ \mathrm{id}_X, \mathrm{id}_{\varphi_*\mathcal{O}_X} \circ \varphi^\sharp\right) = (\varphi, \varphi^\sharp),$$

  and

$$\mathrm{id}_{Y,\mathcal{O}_Y} \circ (\varphi, \varphi^\sharp) = \left(\mathrm{id}_Y \circ \varphi, (\mathrm{id}_Y)_*\varphi^\sharp \circ \mathrm{id}_{\mathcal{O}_Y}\right) = \left(\mathrm{id}_Y \circ \varphi, \varphi^\sharp \circ \mathrm{id}_{\mathcal{O}_Y}\right) = (\varphi, \varphi^\sharp).$$

- Associativity: given morphisms $(\alpha, \alpha^\sharp) \colon (A, \mathcal{O}_A) \to (B, \mathcal{O}_B)$ and $(\beta, \beta^\sharp) \colon (B, \mathcal{O}_B) \to (C, \mathcal{O}_C)$ and $(\gamma, \gamma^\sharp) \colon (C, \mathcal{O}_C) \to (D, \mathcal{O}_D)$, we compute

$$\begin{aligned} (\gamma, \gamma^\sharp) \circ \left((\beta, \beta^\sharp) \circ (\alpha, \alpha^\sharp)\right) &= (\gamma, \gamma^\sharp) \circ (\beta \circ \alpha, \beta_*\alpha^\sharp \circ \beta^\sharp) \\ &= (\gamma \circ \beta \circ \alpha, \gamma_*\beta_*\alpha^\sharp \circ \gamma_*\beta^\sharp \circ \gamma^\sharp) \\ &= (\gamma \circ \beta, \gamma_*\beta^\sharp \circ \gamma^\sharp) \circ (\alpha, \alpha^\sharp) \\ &= \left((\gamma, \gamma^\sharp) \circ (\beta, \beta^\sharp)\right) \circ (\alpha, \alpha^\sharp), \end{aligned}$$

  finishing. ∎

Thus, an isomorphism of locally ringed spaces is, of course, an isomorphism in the category. This carries a lot of data, so it will be helpful to have a shorter version of the data to carry around.

Lemma 2.8. A morphism of ringed spaces $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if and only if φ is a homeomorphism and $\varphi^\sharp$ is an isomorphism of sheaves.

Proof. Note that $(\varphi, \varphi^\sharp)$ is a morphism of locally ringed spaces already because any $x \in X$ has

$$\mathcal{O}_{Y,\varphi(x)} \stackrel{\varphi^\sharp_{\varphi(x)}}{\to} (\varphi_*\mathcal{O}_X)_{\varphi(x)} \to \mathcal{O}_{X,x}$$

is a string of isomorphisms: the former is an isomorphism because $\varphi^\sharp$ is, and the last is an isomorphism by Remark 1.168. Thus, this is a map of local rings for free.

We now construct the inverse for $(\varphi, \varphi^\sharp)$. Let $\psi\colon Y \to X$ be the inverse continuous map for φ. Also, for each $V \subseteq \mathcal{O}_Y$, define the morphism $\psi^\sharp_V\colon \mathcal{O}_Y(V) \to \psi_*\mathcal{O}_X(V)$ as the inverse of the morphism

$$\varphi^\sharp_{\varphi(V)}\colon \mathcal{O}_X(\varphi(V)) \to \varphi_*\mathcal{O}_Y(\varphi(V))$$

which makes sense because $\psi^{-1}(V) = \varphi(V)$ and $\varphi^{-1}(\varphi(V)) = V$. To check that $\psi^\sharp$ assembles into a sheaf morphism, we note that open subsets $V' \subseteq V \subseteq Y$ make the left diagram below

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\psi^\sharp_V} & \psi_*\mathcal{O}_X(V) \\ {\scriptstyle \mathrm{res}_{V,V'}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{V,V'}} \\ \mathcal{O}_Y(V') & \xrightarrow{\psi^\sharp_{V'}} & \psi_*\mathcal{O}_X(V') \end{array} \qquad \begin{array}{ccc} \varphi_*\mathcal{O}_Y(\varphi(V)) & \xleftarrow{\varphi^\sharp_{\varphi(V)}} & \mathcal{O}_X(\varphi(V)) \\ {\scriptstyle \mathrm{res}_{\varphi(V),\varphi(V')}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{\varphi(V),\varphi(V')}} \\ \varphi_*\mathcal{O}_Y(\varphi(V')) & \xleftarrow{\varphi^\sharp_{\varphi(V')}} & \mathcal{O}_X(\varphi(V')) \end{array}$$

commute because it is the same as the one on the right. Additionally, we can quickly check that we have a morphism of locally ringed spaces; by Remark 1.168, we are actually given that any $x \in X$ has

$$\mathcal{O}_{Y,\varphi(x)} \to (\varphi_*\mathcal{O}_X)_{\varphi(x)} \simeq \mathcal{O}_{X,x}$$

is a map of local rings. Inverting this map, we see that any $y \in Y$ has

$$\mathcal{O}_{X,\psi(y)} \to (\psi_*\mathcal{O}_Y)_{\psi(y)} \simeq \mathcal{O}_{Y,y}$$

is also a map of local rings.

It remains to see that $(\psi, \psi^\sharp)$ is actually the inverse of $(\varphi, \varphi^\sharp)$. On one side, we see that

$$(\varphi, \varphi^\sharp) \circ (\psi, \psi^\sharp) = (\varphi \circ \psi, \varphi_*\psi^\sharp \circ \varphi^\sharp).$$

Now, $\varphi \circ \psi = \mathrm{id}_Y$ by definition of ψ, and for any $U \subseteq X$, we note $\psi^\sharp_{\varphi^{-1}(U)} = (\varphi^\sharp_U)^{-1}$ by definition of $\psi^\sharp$. So the above is indeed $\mathrm{id}_{(Y,\mathcal{O}_Y)}$. The other side inverse check is entirely symmetric. ■

For schemes, we will be very interested in special (open) subsets of the underlying topological space. The following lemma will be of use.

Lemma 2.9. Fix a morphism $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces. Then, for any open subset $U \subseteq Y$, φ will restrict to a morphism of locally ringed spaces

$$(\varphi, \varphi^\sharp)|_U\colon \left(\varphi^{-1}(U), \mathcal{O}_X|_{\varphi^{-1}U}\right) \to (U, \mathcal{O}_Y|_U).$$

In particular, if $(\varphi, \varphi^\sharp)$ is an isomorphism, then $(\varphi, \varphi^\sharp)|_U$ is an isomorphism.

Proof. We will define $(\varphi, \varphi^\sharp)_U$ by hand. We set $\psi\colon \varphi^{-1}(U) \to U$ to just be $\varphi|_{\varphi^{-1}(U)}$, which is continuous by restriction. Additionally, for any open subset $V \subseteq U$, we define

$$\psi^\sharp_V\colon \underbrace{\mathcal{O}_Y|_U(V)}_{\mathcal{O}_Y(V)} \to \underbrace{\psi_*(\mathcal{O}_X|_{\varphi^{-1}U})(V)}_{\mathcal{O}_X(\psi^{-1}(V))}$$

as just $\varphi^\sharp_V$, which makes sense because $\mathcal{O}_X(\psi^{-1}(V)) = \mathcal{O}_X(\varphi^{-1}(V)) = \varphi_*\mathcal{O}_X(V)$. To see that $\psi^\sharp$ assembles into a morphism of sheaves, we see that any $V' \subseteq V \subseteq U$ makes the left diagram of

$$\begin{tikzcd} \mathcal{O}_Y|_U(V) \arrow[r, "\psi^\sharp_V"] \arrow[d, "\operatorname{res}_{V,V'}"'] & \psi_*(\mathcal{O}_X|_{\varphi^{-1}U})(V) \arrow[d, "\operatorname{res}_{V,V'}"] \\ \mathcal{O}_Y|_U(V') \arrow[r, "\psi^\sharp_{V'}"] & \psi_*(\mathcal{O}_X|_{\varphi^{-1}U})(V) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\varphi^\sharp_V"] \arrow[d, "\operatorname{res}_{V,V'}"'] & \varphi_*\mathcal{O}_X(V) \arrow[d, "\operatorname{res}_{V,V'}"] \\ \mathcal{O}_Y(V') \arrow[r, "\varphi^\sharp_{V'}"] & \varphi_*\mathcal{O}_X(V') \end{tikzcd}$$

commutes because it is the same as the right diagram. Continuing, $(\psi, \psi^\sharp)$ is a morphism of locally ringed spaces because any $x \in \varphi^{-1}(U)$ makes the diagram

$$\begin{tikzcd} (\mathcal{O}_Y|_U)_{\psi(x)} \arrow[r, "\psi^\sharp_{\psi(x)}"] \arrow[d] & \left(\psi_*(\mathcal{O}_X|_{\varphi^{-1}(U)})\right)_{\psi(x)} \arrow[r] & (\mathcal{O}_X|_{\varphi^{-1}(U)})_x \arrow[d] \\ \mathcal{O}_{Y,\varphi(x)} \arrow[r, "\varphi^\sharp_{\varphi(x)}"] & (\varphi_*\mathcal{O}_X)_{\varphi(x)} \arrow[r] & \mathcal{O}_{X,x} \end{tikzcd} \qquad \begin{tikzcd} {[(V,s)]} \arrow[r, mapsto] \arrow[d, equal] & \psi^\sharp_V(s)|_x \arrow[d, equal] \\ {[(V,s)]} \arrow[r, mapsto] & \varphi^\sharp_V(s)|_x \end{tikzcd}$$

commute, where the vertical morphisms are the isomorphisms of Lemma 1.174. In particular, the top composite is a map of local rings because the bottom one is.

It remains to show that $(\varphi, \varphi^\sharp)$ being an isomorphism forces $(\psi, \psi^\sharp)$ is an isomorphism. Well, to see that $\psi \colon \varphi^{-1}(U) \to U$ is a homeomorphism, note that ψ is an injective, continuous, open map as inherited from φ, and ψ is surjective onto U by construction. Additionally, $\psi^\sharp$ is an isomorphism because its components morphisms come from $\varphi^\sharp$, which are all isomorphisms. Thus, we are done by Lemma 2.8. ■

Remark 2.10. We can see that the diagram

$$\begin{tikzcd} (\varphi^{-1}U, \mathcal{O}_X|_{\varphi^{-1}U}) \arrow[r, hook, "(\iota,\iota^\sharp)"] \arrow[d, "(\varphi,\varphi^\sharp)|_U"'] & (X, \mathcal{O}_X) \arrow[d, "(\varphi,\varphi^\sharp)"] \\ (U, \mathcal{O}_Y|_U) \arrow[r, hook, "(\jmath,\jmath^\sharp)"] & (Y, \mathcal{O}_Y) \end{tikzcd}$$

commutes, where the horizontal embeddings are from Example 2.5. On topological spaces, this is clear because the left map is just the restriction of φ to $\varphi^{-1}U$. On sheaves, pick up some open $V \subseteq Y$, and we see that the following diagram commutes.

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\varphi^\sharp_V"] \arrow[d, "\jmath^\sharp_V"'] & \varphi_*\mathcal{O}_X(V) \arrow[d, "(\varphi_*\iota^\sharp)_V"] \\ \jmath_*(\mathcal{O}_Y|_U) \arrow[r, "(\varphi^\sharp|_U)_V"] & \varphi_*\iota_*(\mathcal{O}_X|_{\varphi^{-1}U}) \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & \varphi^\sharp_V(s) \arrow[d, mapsto] \\ s|_{U\cap V} \arrow[r, mapsto] & \varphi^\sharp_{U\cap V}(s|_{U\cap V}) \end{tikzcd}$$

Remark 2.11. As an example of when this might be useful, suppose a morphism $(\varphi, \varphi^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ actually outputs topologically into an open subset $V \subseteq Y$. Then $\varphi^{-1}V = X$, so our restriction will just end up being

$$(\varphi, \varphi^\sharp)|_V \colon (X, \mathcal{O}_X) \to (V, \mathcal{O}_Y|_V),$$

which by Remark 2.10 becomes $(\varphi, \varphi^\sharp)$ after embedding $(V, \mathcal{O}_Y|_V) \hookrightarrow (Y, \mathcal{O}_Y)$ again.

2.1.2 K-points

The morphism of a locally ringed space contains a lot of data, so it will be helpful to see all this data go to use. Here's an example.

Definition 2.12 (Residue field)**.** Fix a locally ringed space $(X, \mathcal{O}_X)$. Given a point $x \in X$, define $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ to be the unique maximal ideal of $\mathcal{O}_{X,x}$. Then the *residue field* of x is $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Exercise 2.13. Fix a locally ringed space $(X, \mathcal{O}_X)$ and a field K. Then the data of a morphism of locally ringed spaces $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (X, \mathcal{O}_X)$ can be equivalently presented as a point $p \in X$ equipped with an inclusion $\iota\colon k(p) \to K$.

Intuitively, we are saying that morphisms from the affine scheme over K correspond to "K-points of X," for a suitable definition of K-points.

Proof. Let M be the set of morphisms $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (X, \mathcal{O})$, and let P be the set of ordered pairs (p, ι) where $p \in X$ is a point and $\iota\colon k(p) \hookrightarrow K$ is an embedding. We exhibit a bijection between M and P. Here are the maps.

- We exhibit a map $\alpha\colon M \to P$. Well, given a morphism $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (X, \mathcal{O})$, we have an underlying continuous map $\varphi\colon \operatorname{Spec} K \to X$ and sheaf morphism $\varphi^\sharp\colon \mathcal{O} \to \varphi_*\mathcal{O}_{\operatorname{Spec} K}$.

  Now, $(0) \in \operatorname{Spec} K$, so we set $p := \varphi((0))$. Then $\varphi^\sharp$ will provide a map

  $$\mathcal{O}_p \stackrel{\varphi_p^\sharp}{\to} (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p \to \mathcal{O}_{\operatorname{Spec} K,(0)} \cong K_{(0)} = K.$$

  This is supposed to be a map of local rings, so the pre-image of the maximal ideal $(0) \subseteq K$ is supposed to equal $\mathfrak{m}_p$, so we actually induce an embedding $\iota\colon \mathcal{O}_p/\mathfrak{m}_p \hookrightarrow K$. Thus, we set $\alpha((\varphi, \varphi^\sharp)) = (p, \iota)$.

- We exhibit a map $\beta\colon P \to M$. We are provided with a point $p \in X$ and an inclusion $\mathcal{O}_p/\mathfrak{m}_p \to K$. Here is the defining data.

  - Define $\varphi\colon \operatorname{Spec} K \to X$ by $\varphi((0)) := p$. To see that this continuous, note any open subset $U \subseteq X$ containing p have $\varphi^{-1}(U) = \{(0)\} = \operatorname{Spec} K$, which is open. Otherwise, the open subset $U \subseteq X$ does not contain p, so $\varphi^{-1}(U) = \varnothing$, which is still open.
  - Given an open subset $U \subseteq X$, we define $\varphi_U^\sharp\colon \mathcal{O}(U) \to \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U)$. If U does not contain p, then

    $$\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) = \mathcal{O}_{\operatorname{Spec} K}\left(\varphi^{-1}(U)\right) = \mathcal{O}_{\operatorname{Spec} K}(\varnothing) = 0,$$

    so we set $\varphi_U^\sharp$ to be the zero map. Otherwise, when U contains p, we see

    $$\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) = \mathcal{O}_{\operatorname{Spec} K}\left(\varphi^{-1}(U)\right) = \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K) = K,$$

    so we need to exhibit a map $\varphi_U^\sharp\colon \mathcal{O}(U) \to K$. For this, we use the composite map

    $$\begin{array}{ccccccc} \mathcal{O}(U) & \to & \mathcal{O}_p & \to & \mathcal{O}_p/\mathfrak{m}_p & \stackrel{\iota}{\to} & K \\ s & \mapsto & s|_p & \mapsto & (s|_p + \mathfrak{m}_p) & \mapsto & \iota(s|_p + \mathfrak{m}_p) \end{array}$$

    as our $\varphi_U^\sharp$.

  We quickly check that $\varphi^\sharp$ assembles into a map of sheaves. Fix open sets $U' \subseteq U$, and we want the diagram

  $$\begin{tikzcd} \mathcal{O}(U) \arrow[r, "\varphi_U^\sharp"] \arrow[d, "\operatorname{res}_{U,U'}"'] & \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) \arrow[d, "\operatorname{res}_{U,U'}"] \\ \mathcal{O}(U') \arrow[r, "\varphi_{U'}^\sharp"] & \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U') \end{tikzcd} \tag{2.2}$$

  to commute. We have two cases.

  - If $p \notin U'$, then $\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U') = 0$, so (2.2) commutes for free.

- If $p \in U'$, then $p \in U$ as well, so (2.2) becomes the diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, "\varphi_U^\sharp"] \arrow[d, "\mathrm{res}_{U,U'}"'] & K \arrow[d, "\mathrm{res}_{U,U'}"] \\ \mathcal{O}(U') \arrow[r, "\varphi_{U'}^\sharp"] & K \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & \iota(s|_p + \mathfrak{m}_p) \arrow[d, mapsto] \\ s|_{U'} \arrow[r, mapsto] & \iota(s|_p + \mathfrak{m}_p) \end{tikzcd}$$

  which does indeed commute.

Next we check that $(\varphi, \varphi^\sharp)$ assembles into a morphism of locally ringed spaces. For this we have to check that, for any $\mathfrak{p} \in \operatorname{Spec} K$, the composite

$$\mathcal{O}_{\varphi(\mathfrak{p})} \overset{\varphi_{\mathfrak{p}}^\sharp}{\to} (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_{\varphi(\mathfrak{p})} \to (\mathcal{O}_{\operatorname{Spec} K})_{\mathfrak{p}}$$

is a map of local rings. Notably, the only point we have to check this is on $\mathfrak{p} = (0)$ because $\operatorname{Spec} K = \{(0)\}$, and $\varphi((0)) = p$, so we are checking that

$$\begin{array}{ccccc} \mathcal{O}_p & \overset{\varphi_p^\sharp}{\to} & (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p & \to & (\mathcal{O}_{\operatorname{Spec} K})_0 \\ [(U,s)] & \mapsto & \big[(U, \iota(s|_p + \mathfrak{m}_p))\big] & \mapsto & \big[(\varphi^{-1}(U), \iota(s|_p + \mathfrak{m}_p))\big] \end{array}$$

is a map of local rings. Notably, $\varphi^{-1}(U) = \{(0)\} = \operatorname{Spec} K = D(1)$, so we can chain the above composite with the isomorphism $(\mathcal{O}_{\operatorname{Spec} K})_0 \cong K_{(0)} = K$, which will send $[(D(1), \overline{s})]$ to $\overline{s}$. So we are showing that

$$\begin{array}{ccc} \mathcal{O}_p & \to & K \\ [(U,s)] & \mapsto & \iota(s|_p + \mathfrak{m}_p) \end{array}$$

is a map of local rings. (Namely, isomorphisms are maps of local rings, so we can "unchain" the above map with the previous isomorphisms to recover the needed map of local rings.) Well, the pre-image of the maximal ideal $(0) \subseteq K$ consists of sections $s_p \in \mathcal{O}_p$ such that $\iota(s_p + \mathfrak{m}_p) = 0$; because ι is injective, we see that this is equivalent to $s_p \in \mathfrak{m}_p$.

So indeed, the pre-image of the maximal ideal (0) is $\mathfrak{m}_p$, verifying that we have a map of local rings. As such, we may define $\beta((p, \iota)) := (\varphi, \varphi^\sharp)$.

We now have to show that α and β are inverses.

- Fix some $(p, \iota) \in P$. We show that $(\alpha \circ \beta)((p,\iota)) = (p, \iota)$. Set $(\varphi, \varphi^\sharp) := \beta((p,\iota))$, and we need to compute $\alpha((\varphi, \varphi^\sharp))$. To start, by construction, we see

$$\varphi((0)) = p,$$

  as it should be. To solve for ι, we note that above we tracked through the map

$$\begin{array}{ccccccc} \mathcal{O}_p & \overset{\varphi_p^\sharp}{\to} & (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p & \to & (\mathcal{O}_{\operatorname{Spec} K})_0 & \cong K_{(0)} = & K \\ [(U,s)] & \mapsto & \big[(U, \iota(s|_p + \mathfrak{m}_p))\big] & \mapsto & \big[(\varphi^{-1}(U), \iota(s|_p + \mathfrak{m}_p))\big] & \mapsto & \iota(s|_p + \mathfrak{m}_p) \end{array}$$

  as having kernel $\mathfrak{m}_p$, so the induced map $\mathcal{O}_p/\mathfrak{m}_p \to K$ is just $(s_p + \mathfrak{m}_p) \mapsto \iota(s_p + \mathfrak{m}_p)$. Thus, this map induced by $\alpha((\varphi, \varphi^\sharp))$ is exactly ι, as needed.

- Fix some $(\varphi, \varphi^\sharp) \in M$. We show that $(\beta \circ \alpha)((\varphi, \varphi^\sharp)) = (\varphi, \varphi^\sharp)$. Set $(p, \iota) := \alpha((\varphi, \varphi^\sharp))$ and $(\psi, \psi^\sharp) := \beta((p, \iota))$, and we will show $(\psi, \psi^\sharp) = (\varphi, \varphi^\sharp)$. By construction, we see $\psi((0)) = p$, so we get $\psi = \varphi$ immediately.

  To show $\psi^\sharp = \varphi^\sharp$, we need to show that $\psi_U^\sharp = \varphi_U^\sharp$ as functions $\mathcal{O}(U) \to \varphi_*\mathcal{O}_{\operatorname{Spec} K}(U)$ for each open $U \subseteq X$. We have two cases.

  - If $p \notin U$, then $\varphi_*\mathcal{O}_{\operatorname{Spec} K}(U) = \mathcal{O}_{\operatorname{Spec} K}\left(\varphi^{-1}(U)\right) = \mathcal{O}_{\operatorname{Spec} K}(\varnothing) = 0$, so $\psi_U^\sharp$ and $\varphi_U^\sharp$ must both be the zero map because 0 is terminal.

- Otherwise, we have $p \in U$; note that $\varphi_U^\sharp, \psi_U^\sharp \colon \mathcal{O}(U) \to K$ now. By definition, $\psi_U^\sharp$ sends a section $s \in \mathcal{O}(U)$ to $\iota(s|_p + \mathfrak{m}_p)$; by definition, ι sends some $s|_p + \mathfrak{m}_p$ down the composite

$$\begin{array}{ccccccc} \mathcal{O}_p & \overset{\varphi_p^\sharp}{\to} & (\varphi_*\mathcal{O}_{\operatorname{Spec} K})_p & \to & \mathcal{O}_{\operatorname{Spec} K,(0)} & \cong K_{(0)} = & K \\ [(U,s)] & \mapsto & \left[(U, \varphi_U^\sharp(s))\right] & \mapsto & \left[(\varphi^{-1}(U), \varphi_U^\sharp(s))\right] & \mapsto & \varphi_U^\sharp(s) \end{array}$$

which verifies that $\psi_U^\sharp$ is sending $s \in \mathcal{O}(U)$ all the way to $\varphi_U^\sharp(s)$.

From the above, it follows that $\psi_U^\sharp = \varphi_U^\sharp$, which finishes this last check. ■

Remark 2.14. There is a similar story one can tell for $K[\varepsilon]/(\varepsilon^2)$, where we can see that we will also want to keep track of some differential information from the ε.

2.1.3 Schemes

We finally arrive at the definition of a scheme.

Definition 2.15 (Scheme)**.** A *scheme* is a ringed space $(X, \mathcal{O}_X)$ such that, for each $x \in X$, there is an open set $U \subseteq X$ containing x such that the restriction

$$(U, \mathcal{O}_X|_U)$$

is isomorphic (as a locally ringed space) to an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. We call the $(U, \mathcal{O}_X|_U)$ an *affine open subscheme* of $(X, \mathcal{O}_X)$.

Here are some quick facts about this definition.

Lemma 2.16. Fix a ring A and a distinguished open set $D(f) \subseteq \operatorname{Spec} A$. Then

$$(A_f, \mathcal{O}_{\operatorname{Spec} A_f}) \cong (D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}).$$

Proof. The underlying homeomorphism is provided by Exercise 1.57; namely, set $\psi \coloneqq D(f) \to \operatorname{Spec} A_f$ by $\psi \colon \mathfrak{p} \mapsto \mathfrak{p}A_f$, which we showed is a homeomorphism. By Lemma 2.8, it suffices to exhibit a sheaf isomorphism

$$\psi^\sharp \colon \mathcal{O}_{\operatorname{Spec} A_f} \to \psi_*\mathcal{O}_{\operatorname{Spec} A}|_{D(f)}.$$

By Lemma 1.84, it suffices provide a sheaf isomorphism on the distinguished base. Well, given a distinguished basis set $D(a/f^m) \subseteq \operatorname{Spec} A_f$, we showed in Exercise 1.57 that $\psi^{-1}(D(a/f^m)) = D(a) \subseteq D(f)$, so we define

$$\psi^\sharp_{D(a/f^m)} \colon \mathcal{O}_{\operatorname{Spec} A_f}(D(a/f^m)) \to \mathcal{O}_{\operatorname{Spec} A}\left(\psi^{-1}(D(a))\right)$$

as the string of isomorphisms

$$\mathcal{O}_{\operatorname{Spec} A_f}(D(a/f^m)) = \mathcal{O}_{\operatorname{Spec} A_f}(D(a)) \simeq A_{f,a} \simeq A_a \simeq \mathcal{O}_{\operatorname{Spec} A}(D(a)),$$

where $A_{f,a} \simeq A_a$ because $D(a) \subseteq D(f)$. In particular, this assembles into a morphism of sheaves on a base because, for $D(a/f^m) \subseteq D(b/f^n)$, the diagram

$$\begin{array}{ccccccccc} \mathcal{O}_{\operatorname{Spec} A_f}(D(a/f^m)) & = & \mathcal{O}_{\operatorname{Spec} A_f}(D(a)) & \simeq & A_{f,a} & \simeq & A_a & \simeq & \mathcal{O}_{\operatorname{Spec} A}(D(a)) \\ \downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle\text{res}} & & & & & & \downarrow{\scriptstyle\text{res}} \\ \mathcal{O}_{\operatorname{Spec} A_f}(D(b/f^n)) & = & \mathcal{O}_{\operatorname{Spec} A_f}(D(b)) & \simeq & A_{f,b} & \simeq & A_b & \simeq & \mathcal{O}_{\operatorname{Spec} A}(D(b)) \end{array}$$

because all the maps are just localization maps in various orders. In particular, all elements in the top row can be written in the form s/a^m for some $s \in A$ and $m \in \mathbb{N}$, which makes all maps on the top the identity. Then transporting this element to the bottom row, all elements become exactly s/a^m, so the diagram is indeed commuting. ■

Corollary 2.17. Fix a scheme $(X, \mathcal{O}_X)$. Then any open subset $U \subseteq X$ induces an "open subscheme" $(U, \mathcal{O}_X|_U)$.

Proof. The affine case follows from Lemma 2.16. The general case follows by reducing to an affine open cover.

To see this explicitly, fix some $p \in U$. We need to find an open subset $U_p \subseteq U$ such that $(U_p, \mathcal{O}_X|_U|_{U_p})$ is an affine open subscheme of X; quickly, note we will have $\mathcal{O}_X|_U|_{U_p} = \mathcal{O}_X|_{U_p}$, which is clear on the level of open sets, and the restriction maps are just induced.

Because X is a scheme, we can find some affine open subset $V_p \subseteq X$ containing p, so find our isomorphism

$$(\varphi, \varphi^\sharp) : (V_p, \mathcal{O}_X|_{V_p}) \cong (\operatorname{Spec} B_p, \mathcal{O}_{\operatorname{Spec} B_p}).$$

Now, $U \cap V_p \subseteq V_p$ is an open subset, so $\varphi(U) \subseteq \operatorname{Spec} B_p$ is still an open subset. Thus, we can find an element $D(f) \subseteq \varphi(U)$ of the distinguished base containing $\varphi(p)$. Setting $U_p := \varphi^{-1}(D(f))$ (which is still open), we have the chain of isomorphisms

$$(U_p, \mathcal{O}_X|_{U_p}) \overset{(\varphi,\varphi^\sharp)|_{D(f)}}{\cong} (D(f), \mathcal{O}_{\operatorname{Spec} B_p}|_{D(f)}) \cong (\operatorname{Spec}(B_p)_f, \mathcal{O}_{\operatorname{Spec}(B_p)_f}).$$

Namely, the first isomorphism is from Lemma 2.9, and the second isomorphism is from Lemma 2.16. ■

2.1.4 Geometry Is Opposite Algebra

We are going to build towards the following result.

Theorem 2.18. The functors

$$\begin{array}{ccc} \mathrm{Rings}^{\mathrm{op}} & \simeq & \mathrm{AffSch} \\ A & \mapsto & (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\ \mathcal{O}_X(X) & \leftarrow & (X, \mathcal{O}_X) \end{array}$$

define an equivalence of categories.

It turns out that we can extend Theorem 2.18 to work with general locally ringed spaces. We will state this as an adjunction of two functors. Here are our two functors.

Lemma 2.19. The mapping $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$ defines the action of a functor

$$\Gamma : \mathrm{LocRingSpace} \to \mathrm{Ring}^{\mathrm{op}}$$

on objects.

Proof. Given a morphism $(\varphi, \varphi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, we induce a morphism

$$\mathcal{O}_Y(Y) \overset{\varphi^\sharp_Y}{\to} \varphi_* \mathcal{O}_X(Y) = \mathcal{O}_X(X),$$

so we define $\Gamma((\varphi, \varphi^\sharp)) := \varphi^\sharp_Y$. Here are our functoriality checks.

- Identity: note that $\mathrm{id}_{(X,\mathcal{O}_X)} = (\mathrm{id}_X, \mathrm{id}_{\mathcal{O}_X})$, so this will induce the morphism $\Gamma(\mathrm{id}_{(X,\mathcal{O}_X)}) = (\mathrm{id}_{\mathcal{O}_X})_X = \mathrm{id}_{\mathcal{O}_X(X)}$.

- Functoriality: given morphisms $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(\psi, \psi^\sharp)\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, we note the composite $(\psi, \psi^\sharp) \circ (\varphi, \varphi^\sharp)$ taken through Γ acts as

$$\mathcal{O}_Z(Z) \overset{\psi^\sharp_Z}{\to} \psi_*\mathcal{O}_Y(Z) \overset{(\psi_*\varphi^\sharp)_Z}{\to} \psi_*\varphi_*\mathcal{O}_X(Z) = \mathcal{O}_X(X)$$

on global sections. Now, we can see that this composite is $\varphi^\sharp_Y \circ \psi^\sharp_Z = \Gamma((\varphi, \varphi^\sharp)) \circ \Gamma((\psi, \psi^\sharp))$ on global sections. ■

Lemma 2.20. The mapping $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ defines the action of a functor

$$\operatorname{Spec}\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{LocRingSpace}$$

on objects.

Proof. Given a ring homomorphism $f\colon A \to B$, we need to induce a morphism $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. For brevity, given a ring R, we will define $\mathcal{O}_R \coloneqq \mathcal{O}_{\operatorname{Spec} R}$.

- On topological spaces, we define $\varphi \coloneqq f^{-1}$ to be our continuous map $\operatorname{Spec} B \to \operatorname{Spec} A$. This is continuous by Lemma 1.48 and functorial by Lemma 1.48.
- On sheaves, it suffices to induce the morphism $\varphi^\sharp\colon \mathcal{O}_A \to \varphi_*\mathcal{O}_B$ on the distinguished base $\{D(a)\}_{a \in A}$ by Lemma 1.48. Well, given $a \in A$, we can compute

$$\varphi^{-1}(D(a)) = \{\mathfrak{q} \in \operatorname{Spec} B : a \notin f^{-1}\mathfrak{q}\} = \{\mathfrak{p} \in \operatorname{Spec} A : f(a) \notin \mathfrak{q}\} = D(f(a)),$$

so $\varphi^\sharp_{D(a)}$ is a map $A_a \to B_{f(a)}$. However, this is induced directly from the localization map $A \to B \to B_{f(a)}$ upon noting that $a \in A$ goes to a unit $f(a) \in B_{f(a)}$.

To finish, we do need to check that this is a morphism of sheaves on a base. Suppose $D(a') \subseteq D(a)$, which means that there is a canonical localization map $A_a \simeq A_{S(D(a))} \to A_{S(D(a'))} \simeq A_{a'}$. (Namely, $a \in A_{a'}^\times$.) Then we see that the left diagram of

$$\begin{array}{ccc} \mathcal{O}_A(D(a)) & \xrightarrow{\varphi^\sharp_{D(a)}} & \varphi_*\mathcal{O}_B(D(a)) \\ {\scriptstyle \operatorname{res}_{D(a),D(a')}}\downarrow & & \downarrow{\scriptstyle \operatorname{res}_{D(a),D(a')}} \\ \mathcal{O}_A(D(a')) & \xrightarrow{\varphi^\sharp_{D(a')}} & \varphi_*\mathcal{O}_B(D(a')) \end{array} \qquad \begin{array}{ccc} A_a & \xrightarrow{f} & B_{f(a)} \\ \downarrow & & \downarrow \\ A_{a'} & \xrightarrow{f} & B_{f(a')} \end{array} \qquad \begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x/1 & \longmapsto & f(x)/1 \end{array}$$

commutes because it is the same as the middle diagram.

To finish our construction, we need to know that $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ assembles into a map of locally ringed spaces. Namely, we need to verify that, for each $\mathfrak{q} \in \operatorname{Spec} B$, the map

$$\begin{array}{ccccc} \mathcal{O}_{A,\varphi(\mathfrak{q})} & \overset{\varphi^\sharp_{\mathfrak{q}}}{\to} & (\varphi_*\mathcal{O}_B)_{\mathfrak{q}} & \to & \mathcal{O}_{B,\mathfrak{q}} \\ [(D(a), s/a^m)] & \mapsto & [(D(a), f(s)/f(a)^m)] & \mapsto & [(D(f(a)), f(s)/f(a)^m)] \end{array}$$

is a map of local rings; notably, we are using Lemma 1.101 to define the stalk on the distinguished base. Well, given $[(D(a), s/a^m)] \in \mathfrak{m}_{\varphi(\mathfrak{q})}$ implies that $s \in \varphi(\mathfrak{q}) = f^{-1}(\mathfrak{q})$, so $f(s) \in \mathfrak{q}$, so $[(D(f(a)), f(s)/f(a)^m)] \in \mathfrak{m}_{\mathfrak{q}}$.

Thus, we define $\operatorname{Spec} f \coloneqq (\varphi, \varphi^\sharp)$. It remains to run our functoriality checks.

- Identity: note that $f = \mathrm{id}_A$ makes $\varphi\colon \operatorname{Spec} A \to \operatorname{Spec} A$ the identity Lemma 1.48, and each $a \in A$ induces the localization map $A_a \to A_{f(a)}$, which we can see visually is just the identity map. Thus, $\varphi^\sharp_{D(a)}$ is the identity for each distinguished base element $D(a)$, so $\varphi^\sharp\colon \mathcal{O}_A \to \varphi_*\mathcal{O}_A$ is just the identity by Lemma 1.79.

  Thus, $(\varphi, \varphi^\sharp)$ is the identity morphism on the locally ringed space $(\operatorname{Spec} A, \mathcal{O}_A)$.

- Functoriality: fix ring morphisms $f\colon A \to B$ and $g\colon B \to C$ yielding morphisms of locally ringed spaces $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ and $(\psi, \psi^\sharp)\colon (\operatorname{Spec} C, \mathcal{O}_C) \to (\operatorname{Spec} B, \mathcal{O}_B)$. For brevity, we will also define $(\gamma, \gamma^\sharp) = \operatorname{Spec}(g \circ f)$ to be the morphism induced by the composite. We need to show $(\gamma, \gamma^\sharp) = (\varphi, \varphi^\sharp) \circ (\psi, \psi^\sharp)$.

  We already know we are functorial on the level of topological spaces (i.e., $\gamma = \varphi \circ \psi$) by Lemma 1.48. Then on sheaves, we need to check that $\gamma^\sharp = \varphi_*\psi^\sharp \circ \varphi^\sharp$. Well, for some distinguished open set $D(a) \subseteq \operatorname{Spec} A$, our composite is

  $$(\varphi_*\psi^\sharp \circ \varphi^\sharp)_{D(a)} = \psi^\sharp_{\varphi^{-1}(D(a))} \circ \varphi^\sharp_{D(a)} = \psi^\sharp_{D(f(a))} \circ \varphi^\sharp_{D(a)}$$

  using computations above. Unwrapping our construction, we see that this composite is the composite of the localized maps

  $$A_a \xrightarrow{f} B_{f(a)} \xrightarrow{g} C_{g(f(a))},$$

  which of course is just the single map $A_a \to C_{g(f(a))}$ induced by localizing $g \circ f$. So we do indeed match with $\gamma^\sharp$ on the base, so we have an equality of sheaves on the base by Lemma 1.79. ■

We now exhibit the first of our natural maps.

Lemma 2.21. We exhibit a map $\varepsilon_\bullet\colon\ \mathrm{id}_{\mathrm{Ring}} \Rightarrow \Gamma\operatorname{Spec}$.

Proof. Fix an object A. To begin, we compute

$$\Gamma((\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A,$$

so we set $\varepsilon_A \coloneqq \mathrm{id}_A$. ■

Going in the other direction is a little more subtle because we need to construct a morphism of locally ringed spaces from a morphism of just the global sections. For example, we will need to construct a continuous map, so we will need access to some open sets. Here are the ones we will need.

Lemma 2.22. Fix a locally ringed space $(X, \mathcal{O}_X)$. Then, for some $f \in \mathcal{O}_X(X)$, the subset

$$X_f \coloneqq \left\{p \in X : f|_p \in \mathcal{O}_{X,p}^\times\right\}$$

is open in X.

Proof. For each $p \in X_f$, we need to provide an open neighborhood $U_p \subseteq X_f$ containing p. Well, we are given $f|_p \in \mathcal{O}_{X,p}^\times$, so there is some germ g_p such that

$$g_p \cdot f|_p = 1.$$

In particular, giving g_p a sufficiently restricted representative, we can find an open subset U_p containing p and some $g \in \mathcal{O}_X(U_p)$ such that

$$g \cdot f|_{U_p} = 1.$$

In particular, any $q \in U_p$ will thus have $g|_q \cdot f|_q = 1$, so $f|_q \in \mathcal{O}_{X,q}^\times$, so $q \in X_f$. Thus, $U_p \subseteq X_f$ does the trick. ■

Here is another quick fact we will want.

Lemma 2.23. Fix a locally ringed space $(X, \mathcal{O}_X)$. For some $f \in \mathcal{O}_X(X)$, consider the open set X_f of Lemma 2.22. Then $f \in \mathcal{O}_X(X_f)^\times$.

Proof. For each $p \in X_f$, we know that $f|_p \in \mathcal{O}_{X,p}$ is a unit, so find $g_p \in \mathcal{O}_{X,p}$ with $f|_p \cdot g_p = 1$. We claim that $(g_p)_{p\in X_f}$ is a compatible system of germs. Well, for each $p \in U$, the equation

$$f|_p \cdot g_p = 1$$

promises an open set $U_p \subseteq X_f$ containing p and a lift $\widetilde{g}_p \in \mathcal{O}_X(U_p)$ such that $f|_{U_p} \cdot \widetilde{g}_p = 1$. Thus, for any $q \in U_p$, we see

$$f|_q \cdot \widetilde{g}_p|_q = 1,$$

so uniqueness of multiplicative inverses forces $\widetilde{g}_p|_q = g_q$. This finishes the compatibility check.

Thus, we are granted $g \in \mathcal{O}_X(X_f)$ such that $(fg)|_p = f|_p \cdot g|_p = 1$ for each $p \in X_f$. It follows that $fg = 1$ by Proposition 1.107, so we have witnessed $f \in \mathcal{O}_X(X_f)^\times$. ■

Remark 2.24. As an aside, this construction behaves through morphisms. Fix some open subscheme $U \subseteq Y$ and a morphism $\pi\colon X \to Y$. Then, for some $f \in \mathcal{O}_Y(U)$, we compute

$$\pi^{-1}(U_f) = \{x \in X : \pi(x) \in U_f\} = \{x \in X : f \notin \mathfrak{m}_{Y,\pi(x)}\}.$$

Now, recall that the composite $\mathcal{O}_{Y,\pi(x)} \overset{\pi^\sharp_{\pi(x)}}{\to} (\pi_*\mathcal{O}_X)_{\pi(x)} \to \mathcal{O}_{X,x}$ is a map of local rings, so $f \notin \mathfrak{m}_{Y,\pi(x)}$ is equivalent to $\pi^\sharp_U(f) \notin \mathfrak{m}_{X,x}$, so we see $\pi^{-1}(U_f) = \pi^{-1}(U)_{\pi^\sharp_U(f)}$.

And here is the result.

Lemma 2.25. We exhibit a map $\eta_\bullet\colon \mathrm{id}_{\mathrm{LocRingSpace}} \Rightarrow \operatorname{Spec}\Gamma$.

Proof. Fix a locally ringed space $(X, \mathcal{O}_X)$ so that we need to exhibit a map

$$\varepsilon_X\colon (X, \mathcal{O}_X) \to (\operatorname{Spec}\mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}).$$

We define this map in pieces.

- We need a continuous map $\varphi\colon X \to \operatorname{Spec}\mathcal{O}_X(X)$. Well, given $p \in X$, we define $\varphi(p) \in \operatorname{Spec}\mathcal{O}_X(X)$ as the kernel of the composite
$$\mathcal{O}_X(X) \to \mathcal{O}_{X,p} \twoheadrightarrow \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}.$$
This kernel makes a prime ideal because modding out by it induces a subring of $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$, which must be an integral domain.
To see that φ is continuous, it suffices to check on the distinguished open base. Well, for some $f \in \mathcal{O}_X(X)$, we see that
$$\varphi^{-1}(D(f)) = \{p \in X : \varphi(p) \in D(f)\} = \{p \in X : f \notin \varphi(p)\} = \{p \in X : f|_p \notin \mathfrak{m}_{X,p}\}.$$
However, this last condition is equivalent to $f|_p \in \mathcal{O}_{X,p}^\times$, so our pre-image is the open set X_f by Lemma 2.22.
- Next we need a sheaf morphism $\varphi^\sharp\colon \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)} \to \varphi_*\mathcal{O}_X$. By Lemma 1.79, it suffices to exhibit $\varphi^\sharp$ on the distinguished base of $\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}$. Well, for some $f \in \mathcal{O}_X(X)$, we need a map
$$\mathcal{O}_X(X)_f \simeq \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(f)) \to \mathcal{O}_X\left(\varphi^{-1}(D(f))\right) = \mathcal{O}_X(X_f).$$
Now, there is the obvious restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_f)$, and $f \in \mathcal{O}_X(X_f)^\times$ by Lemma 2.23 will exhibit the required map $\varphi^\sharp_{D(f)}\colon \mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$.

We now check that we have built a morphism of sheaves on the distinguished base. Well, given that $D(f') \subseteq D(f)$ for $f, f' \in \mathcal{O}_X(X)$, we see that the diagram

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(f)) \arrow[d, "\operatorname{res}_{X_f,X_{f'}}"'] & \simeq & \mathcal{O}_X(X)_f \arrow[r] & \mathcal{O}_X(X_f) \arrow[d, "\operatorname{res}_{X_f,X_{f'}}"'] & a\cdot f^{-n} \arrow[r, mapsto] \arrow[d, mapsto] & a|_{X_f}\cdot(f|_{X_f})^{-n} \arrow[d, mapsto] \\ \mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(f')) & \simeq & \mathcal{O}_X(X)_{f'} \arrow[r] & \mathcal{O}_X(X_{f'}) & a\cdot f^{-n} \arrow[r, mapsto] & a|_{X_{f'}}\cdot(f|_{X_{f'}})^{-n} \end{tikzcd}$$

commutes, so we have uniquely induced $\varphi^\sharp$ by Lemma 1.79.

We should also verify that $(\varphi, \varphi^\sharp)$ assembles into a morphism of locally ringed spaces. Fixing a point $p \in X$, we need to show

$$\begin{array}{ccccc} (\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)})_{\varphi(p)} & \overset{\varphi^\sharp_{\varphi(p)}}{\to} & (\varphi_*\mathcal{O}_X)_{\varphi(p)} & \to & \mathcal{O}_{X,p} \\ {}[(D(f), s/f^n)] & \mapsto & [(D(f), s|_{X_f}\cdot(f|_{X_f})^{-n})] & \mapsto & [(X_f, s|_{X_f}\cdot(f|_{X_f})^{-n})] \end{array}$$

is a map of local rings; notably, we are using Lemma 1.101 to define the stalk on the distinguished base. Well, $[(D(f), s/f^n)] \in \mathfrak{m}_{\varphi(p)}$ implies $s \in \varphi(p)$, so $s|_p \in \mathfrak{m}_p$, so $\left[(X_f, s|_{X_f}\cdot(f|_{X_f})^{-n})\right] \in \mathfrak{m}_p$. Thus, we have indeed defined a morphism $\eta_{X,\mathcal{O}_X} := (\varphi, \varphi^\sharp)$. ■

We are now ready to show the main result.

Theorem 2.26. Given a locally ringed space $(X, \mathcal{O}_X)$ and a ring A, there is a natural bijection

$$\begin{array}{ccc} \operatorname{Hom}((X,\mathcal{O}_X),(\operatorname{Spec}A,\mathcal{O}_{\operatorname{Spec}A})) & \simeq & \operatorname{Hom}(A,\mathcal{O}_X(X)) \\ (\varphi,\varphi^\sharp) & \mapsto & A = \mathcal{O}_{\operatorname{Spec}A}(A) \to \mathcal{O}_X(X) \\ (X,\mathcal{O}_X) \to (\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}) \overset{\operatorname{Spec}f}{\to} (\operatorname{Spec}A,\mathcal{O}_{\operatorname{Spec}A}) & \leftarrow\!\shortmid & f \end{array}$$

where we are using the natural maps constructed in Lemma 2.21 and Lemma 2.25.

Proof. Naturality will follow easily from what we've already done as soon as we show that these are inverses. We have two checks.

- Beginning with a ring homomorphism $f\colon A \to \mathcal{O}_X(X)$, for brevity set $(\varphi,\varphi^\sharp) := \operatorname{Spec} f$ and $\eta_X = (\psi,\psi^\sharp)$. Now, we are studying the composite

  $$(X,\mathcal{O}_X) \overset{(\psi,\psi^\sharp)}{\to} (\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}) \overset{(\varphi,\varphi^\sharp)}{\to} (\operatorname{Spec}A,\mathcal{O}_{\operatorname{Spec}A}).$$

  We are interested in what this composite looks like on global sections. On sheaves, we are looking at

  $$\mathcal{O}_{\operatorname{Spec}A} \overset{\varphi^\sharp}{\to} \varphi_*\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)} \overset{\varphi_*\psi^\sharp}{\to} \varphi_*\psi_*\mathcal{O}_X.$$

  However, on global sections, this simplifies to

  $$\underbrace{\mathcal{O}_{\operatorname{Spec}A}(\operatorname{Spec}A)}_{A} \overset{\varphi^\sharp_{D(1)}}{\to} \underbrace{\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(\operatorname{Spec}\mathcal{O}_X(X))}_{\mathcal{O}_X(X)} \overset{\psi^\sharp_{D(1)}}{\to} \mathcal{O}_X(X).$$

  Expanding out our definitions, $\varphi^\sharp_{D(1)}$ is supposed to be f localized at 1, but this is just f; also, $\psi^\sharp_{D(1)}$ is supposed to be some localized restriction map, but it is just the identity. So indeed, our composite is f.

- Begin with a morphism of locally ringed spaces $(\pi,\pi^\sharp)\colon(X,\mathcal{O}_X)\to(\operatorname{Spec}A,\mathcal{O}_{\operatorname{Spec}A})$. Under Γ, this gives rise to the ring homomorphism $\pi^\sharp_{\operatorname{Spec}A}\colon A\to\mathcal{O}_X(X)$; set $f:=\pi^\sharp_{\operatorname{Spec}A}$ and $(\varphi,\varphi^\sharp)=\operatorname{Spec}f$. Going backward, we label our natural map by $\eta_{(X,\mathcal{O}_X)}$ by $(\alpha,\alpha^\sharp)\colon(X,\mathcal{O}_X)\to(\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)})$, and we want to show that $(\pi,\pi^\sharp)$ agrees with the composite
$$(X,\mathcal{O}_X)\overset{(\alpha,\alpha^\sharp)}{\to}(\operatorname{Spec}\mathcal{O}_X(X),\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)})\overset{(\varphi,\varphi^\sharp)}{\to}(\operatorname{Spec}A,\mathcal{O}_{\operatorname{Spec}A}).$$
As before, there are two checks.
  - On topological spaces, we want the diagram
$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\pi"'] & \operatorname{Spec}\mathcal{O}_X(X) \arrow[d, "\varphi"] \\ & \operatorname{Spec}A \end{tikzcd}\qquad\begin{tikzcd} p \arrow[r, mapsto] \arrow[dr, mapsto] & \ker(\mathcal{O}_X(X)\to\mathfrak{m}_p) \\ & \pi(p) \end{tikzcd}$$
to commute. As such, recalling φ is given by f^{-1}, we compute
$$\begin{aligned} f^{-1}(\ker(\mathcal{O}_X(X)\to\mathfrak{m}_p)) &= f^{-1}(\{r\in\mathcal{O}_X(X):r|_p\in\mathfrak{m}_p\}) \\ &= \{a\in A: f(a)|_p\in\mathfrak{m}_p\} \\ &= \{a\in A:\pi^\sharp_{\operatorname{Spec}A}(a)|_p\in\mathfrak{m}_p\}. \end{aligned}$$
However, $(\pi,\pi^\sharp)$ being a morphism of locally ringed spaces says that $\pi^\sharp_{\operatorname{Spec}A}(a)|_p\in\mathfrak{m}_p$ is equivalent to $a|_{\pi(p)}\in\mathfrak{m}_{\pi(p)}$.
Now, for a prime $\mathfrak{p}\in\operatorname{Spec}A$, we have $a|_\mathfrak{p}\in\mathfrak{m}_\mathfrak{p}=\mathfrak{p}A_\mathfrak{p}$ if and only if $fa\in\mathfrak{p}$ for some $f\notin\mathfrak{p}$, which is equivalent to $a\in\mathfrak{p}$. Thus, we do indeed get out $\pi(p)$ from the above computation, as needed.
  - On sheaves, we want the diagram
$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec}A} \arrow[r, "\varphi^\sharp"] \arrow[dr, "\pi^\sharp"'] & \varphi_*\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)} \arrow[d, "\varphi_*\alpha^\sharp"] \\ & \pi_*\mathcal{O}_X \end{tikzcd}$$
to commute, which at least see makes sense from the above topological check. By Lemma 1.79, it suffices to check this on the distinguished base of $\operatorname{Spec}A$, so fix some $a\in A$. Plugging in $D(a)$, we want the diagram
$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec}A}(D(a)) \arrow[r, "\varphi^\sharp_{D(a)}"] \arrow[dr, "\pi^\sharp_{D(a)}"'] & \varphi_*\mathcal{O}_{\operatorname{Spec}\mathcal{O}_X(X)}(D(a)) \arrow[d, "(\varphi_*\alpha^\sharp)_{D(a)}"] \\ & \pi_*\mathcal{O}_X(D(a)) \end{tikzcd}$$
to commute. Recalling the computations from Lemma 2.20, we see that $\varphi^{-1}(D(a))=D(f(a))$, and the map $\varphi^\sharp_{D(a)}\colon A_a\to\mathcal{O}_X(X)_{f(a)}$ is induced by localizing f. Similarly, the definition of $\alpha^\sharp$ says that $(\varphi_*\alpha^\sharp)_{D(a)}=\alpha^\sharp_{D(f(a))}$ is the localization of the restriction map $\mathcal{O}_X(X)\to\mathcal{O}_X(X_{f(a)})$. So we are staring at the diagram
$$\begin{tikzcd} A_a \arrow[r, "f"] \arrow[dr, "\pi^\sharp_{D(a)}"'] & \mathcal{O}_X(X)_{f(a)} \arrow[d, "\operatorname{res}_{X,X_{f(a)}}"] \\ & \mathcal{O}_X(X_{f(a)}) \end{tikzcd}$$

which commutes because $f = \pi^\sharp_{\operatorname{Spec} A}$. Indeed, this triangle is the localization of the diagram

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \arrow[r, "\pi^\sharp_{\operatorname{Spec} A}"] \arrow[d, "\operatorname{res}_{\operatorname{Spec} A, D(a)}"'] & \mathcal{O}_X(X) \arrow[d, "\operatorname{res}_{X, X_{\pi^\sharp_{\operatorname{Spec} A}(a)}}"] \\
\mathcal{O}_{\operatorname{Spec} A}(D(a)) \arrow[r, "\pi^\sharp_{D(a)}"] & \mathcal{O}_X(X_{\pi^\sharp_{\operatorname{Spec} A}(a)})
\end{tikzcd}$$

at $a \in A$.

The above checks show that we have defined inverse morphisms. There are two naturality checks.

- Given a ring homomorphism $h\colon B \to A$, we see that the diagram

$$\begin{tikzcd}
\operatorname{Hom}((X, \mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \arrow[r, phantom, "\simeq"] \arrow[d, "\operatorname{Spec} h \circ -"] & \operatorname{Hom}(A, \mathcal{O}_X(X)) \arrow[d, "- \circ h"] & \operatorname{Spec} f \circ \eta_{(X, \mathcal{O}_X)} \arrow[d, mapsto] & f \arrow[l, mapsto] \arrow[d, mapsto] \\
\operatorname{Hom}((X, \mathcal{O}_X), (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})) \arrow[r, phantom, "\simeq"] & \operatorname{Hom}(B, \mathcal{O}_X(X)) & \operatorname{Spec}(f \circ h) \circ \eta_{(X, \mathcal{O}_X)} & f \circ h \arrow[l, mapsto]
\end{tikzcd}$$

  commutes.

- Given a scheme morphism $(\pi, \pi^\sharp)\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, we see that the diagram

$$\begin{tikzcd}
\operatorname{Hom}((X, \mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \arrow[r, phantom, "\simeq"] \arrow[d, "- \circ (\pi, \pi^\sharp)"] & \operatorname{Hom}(A, \mathcal{O}_X(X)) \arrow[d, "\Gamma((\pi, \pi^\sharp)) \circ -"] & (\varphi, \varphi^\sharp) \arrow[r, mapsto] \arrow[d, mapsto] & \Gamma((\varphi, \varphi^\sharp)) \arrow[d, mapsto] \\
\operatorname{Hom}((Y, \mathcal{O}_Y), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \arrow[r, phantom, "\simeq"] & \operatorname{Hom}(A, \mathcal{O}_X(Y)) & (\varphi, \varphi^\sharp) \circ (\pi, \pi^\sharp) \arrow[r, mapsto] & \Gamma((\varphi, \varphi^\sharp) \circ (\pi, \pi^\sharp))
\end{tikzcd}$$

  commutes.

The above checks complete the proof. ■

Remark 2.27. The asymmetry of Theorem 2.26 (namely, we can only map out of general schemes) is intentionally. Indeed, it will turn out that control over maps out of an affine scheme to a particular scheme $(X, \mathcal{O}_X)$ is enough information to fully recover the scheme $(X, \mathcal{O}_X)$; one can get a feel from this because maps from $(\operatorname{Spec}, \mathcal{O}_{\operatorname{Spec} K})$ reads off all K-points of $(X, \mathcal{O}_X)$.

As a nice consequence, we get a pretty nice check for a scheme to be affine.

Corollary 2.28. If $(X, \mathcal{O}_X)$ is an affine scheme, then the map $\eta_X\colon (X, \mathcal{O}_X) \to (\mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)})$ of Lemma 2.25 is an isomorphism.

Proof. Because $(X, \mathcal{O}_X)$ is affine, there is some ring A with an isomorphism $(\varphi, \varphi^\sharp)\colon (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong (X, \mathcal{O}_X)$. However, on global sections, we see are granted an isomorphism

$$\Gamma((\varphi, \varphi^\sharp))\colon \mathcal{O}_X(X) \to A.$$

Set $f := \Gamma((\varphi, \varphi^\sharp))$, so applying Spec gives us the composite

$$(X, \mathcal{O}_X)(\varphi, \varphi^\sharp)^{-1} \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \overset{\operatorname{Spec} f}{\cong} (\operatorname{Spec} \mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}),$$

which on global sections behaves as

$$\mathcal{O}_X(X) \overset{(\varphi^\sharp_{\operatorname{Spec} A})^{-1}}{\cong} \mathcal{O}_{\operatorname{Spec} A}(A) = A \overset{f}{\cong} \mathcal{O}_X(X),$$

where this composite is just the identity by definition.

Thus, we have induced an isomorphism $(X, \mathcal{O}_X) \cong (\operatorname{Spec} \mathcal{O}_X(X), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)})$ which is the identity on global sections. Applying Theorem 2.26, we see that Spec of the identity is still the identity, so this morphism is just given by η_X. ■

It's also fairly easily to see the final object in the category of schemes now.

Corollary 2.29. Fix a locally ringed space $(X, \mathcal{O}_X)$. Then there is a unique scheme map $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$.

Proof. Using the adjunction, we note that

$$\operatorname{Hom}((X, \mathcal{O}_X), (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})) \simeq \operatorname{Hom}(\mathbb{Z}, \mathcal{O}_X(X)).$$

However, $\mathbb{Z}$ is initial in the category of rings—there is only one ring map from $\mathbb{Z}$ to $\mathcal{O}_X(X)$ by extending $1 \mapsto 1$. As such, there is only one scheme map $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$. ■

And now here is our last result.

Theorem 2.18. The functors

$$\begin{array}{ccc} \mathrm{Rings}^{\mathrm{op}} & \simeq & \mathrm{AffSch} \\ A & \mapsto & (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\ \mathcal{O}_X(X) & \leftarrow\!\shortmid & (X, \mathcal{O}_X) \end{array}$$

define an equivalence of categories.

Proof. The leftward functor Spec is essentially surjective by definition of an affine scheme, so it remains to show that Spec is fully faithful. Namely, we must show

$$\operatorname{Mor}_{\mathrm{AffSch}}((\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \cong \operatorname{Hom}_{\mathrm{Ring}}(A, B).$$

Because $\mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) = B$, this follows directly from Theorem 2.26. ■

Remark 2.30. In some sense, Theorem 2.18 is intended to be fact-checking: at the end of the day, we really just want the categorical equivalence and don't care much for its proof.

We will quickly provide an example that says that we really do need to pay attention to morphisms of locally ringed spaces.

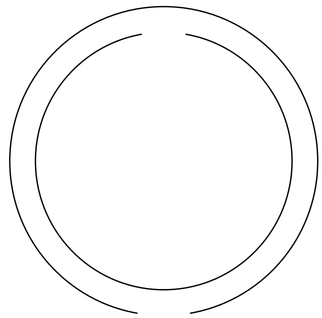
Non-Example 2.31. Consider ring homomorphisms $\mathbb{Z}_p \to \mathbb{Q}_p$. Notably, $\operatorname{Spec} \mathbb{Z}_p = \{(0), (p)\}$ while $\operatorname{Spec} \mathbb{Q}_p = \{(0)\}$. From the natural embedding $\iota \colon \mathbb{Z}_p \to \mathbb{Q}_p$, we get a map sending $(0) \mapsto (0)$, and it will not be possible to get a ring homomorphism to send (0) to (p) because this forces $\mathbb{Q}_p$ to have torsion. Nonetheless, one can upgrade sending $(0) \mapsto (p)$ to a full morphism of sheaves even though it will not be a morphism of locally ringed spaces.

2.1.5 Scheme Examples

Schemes have a lot of data. Let's try to make it more concrete; we'll be satisfied with just two examples today. We won't be very rigorous because we haven't defined gluing yet.

Remark 2.32. Today, we are only gluing two things together at a time because we don't want to worry about the "cocycle condition" for gluing.

Our first example is the projective line. Here is the image of our affine cover.

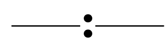


Here is the rigorization of our affine cover.

Example 2.33 (Projective line)**.** Let R be a ring. Then we can glue two copies of $\mathbb{A}^1_R$ (which is $\operatorname{Spec} R[x]$) as subrings of $\operatorname{Spec} R[x, x^{-1}]$. Then we can identify our copies $\operatorname{Spec} R[x, x^{-1}]$ and $\operatorname{Spec} R[y, y^{-1}]$ by sending $x \mapsto y^{-1}$.

To be rigorous, one should also define our full sheaf on this topological space; this comes from the homework problem explaining how to glue together sheaves.

Here is the image for our next example.



Here is the rigorization.

Example 2.34 (Doubled origin)**.** Let R be a ring. Then we can glue two copies of $\mathbb{A}^1_R$ (which is $\operatorname{Spec} R[x]$) as subrings of $\operatorname{Spec} R[x, x^{-1}]$. Then we can identify our copies $\operatorname{Spec} R[x, x^{-1}]$ and $\operatorname{Spec} R[y, y^{-1}]$ by sending $x \mapsto y$.

Remark 2.35. Later on, we will add certain adjectives (namely, "separated") which disallow the above scheme.

For our last example, we return to elliptic curves.

Example 2.36. We build the elliptic curve carved out by $Y^2Z = X^3 - Z^3$. Our two affine patches are

$$\operatorname{Spec} \frac{k[x,y]}{(y^2 - x^3 + 1)} \qquad \text{and} \qquad \operatorname{Spec} \frac{k[x,z]}{(z - x^3 + z^3)}.$$

To glue these together, we identify

$$\operatorname{Spec} \frac{k[x,y,y^{-1}]}{(y^2 - x^3 + 1)} \qquad \text{and} \qquad \operatorname{Spec} \frac{k[x,z,z^{-1}]}{(z - x^3 + z^3)}$$

by sending $x \mapsto x/z$ and $y \mapsto z^{-1}$.

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The fun continues.

2.2.1 Gluing Scheme Morphisms

We are going to want to build new schemes from old ones. Because all schemes are covered by affine schemes, our primary way of making schemes is going to be by gluing (especially affine) schemes together. Because affine schemes make open sets, we are going to want to glue "open subschemes."

Definition 2.37 (Open subscheme)**.** Fix a scheme $(X, \mathcal{O}_X)$ and an open subset $U \subseteq X$. Then we define the scheme $(U, \mathcal{O}_X|_U)$ to be an *open subscheme*.

In particular, $(U, \mathcal{O}_X|_U)$ is still a scheme, as we saw in Corollary 2.17.

Example 2.38. Given an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we note that taking $U := D(f)$ has

$$(U, \mathcal{O}_{\operatorname{Spec} A}|_U) = (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f}),$$

where we are using Exercise 1.57.

As a taste of how we glue schemes, we will start by gluing scheme morphisms. To be able to glue, we need to be able to restrict, so here is our restriction.

Lemma 2.39. Fix a scheme morphism $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and open subset $U \subseteq X$ and $V \subseteq Y$ such that $\varphi(U) \subseteq V$. Then $(\varphi|_U, \varphi^\sharp|_U)$ assembles into a scheme morphism $(U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$.

Proof. The restriction of a continuous function is still continuous, so $\varphi|_U\colon U \to Y$ is still continuous; restricting the image is okay because open subsets of V are also open subsets of Y because $V \subseteq Y$ is open.

We now define $\varphi^\sharp|_U\colon \mathcal{O}_Y|_V \to (\varphi|_U)_*(\mathcal{O}_X|_U)$: for each open $W \subseteq V$, we see W is open in Y, and $(\varphi|_U)^{-1}(W) = U \cap \varphi^{-1}(W)$, so we want a map $\mathcal{O}_Y(W) \to \mathcal{O}_X(U \cap \varphi^{-1}(W))$, so we define $(\varphi^\sharp|_U)_W$ as the composite

$$\mathcal{O}_Y|_V(W) \overset{\varphi^\sharp_W}{\to} \mathcal{O}_X\left(\varphi^{-1}(W)\right) \overset{\operatorname{res}_{\varphi^{-1}W, U\cap\varphi^{-1}W}}{\to} \mathcal{O}_X\left(U \cap \varphi^{-1}(W)\right).$$

To see that $\varphi^\sharp|_U$ assembles into a sheaf morphism, fix open sets $W' \subseteq W \subseteq V$, and we note the diagram

$$\begin{tikzcd}
\mathcal{O}_Y(W) \arrow[r, "(\varphi^\sharp|_U)_W"] \arrow[d, "\operatorname{res}_{W,W'}"'] & (\varphi|_U)_*(\mathcal{O}_X|_U)(W) \arrow[d, "\operatorname{res}_{W,W'}"] \\
\mathcal{O}_Y(W') \arrow[r, "(\varphi^\sharp|_U)_{W'}"] & (\varphi|_U)_*(\mathcal{O}_X|_U)(W')
\end{tikzcd}$$

commutes because it is the same as

$$\begin{tikzcd}
\mathcal{O}_Y(W) \arrow[r, "\varphi^\sharp_W"] \arrow[d, "\operatorname{res}_{W,W'}"'] & \mathcal{O}_X(\varphi^{-1}(W)) \arrow[r, "\operatorname{res}"] \arrow[d, "\operatorname{res}_{\varphi^{-1}W,\varphi^{-1}W'}"] & \mathcal{O}_X(U \cap \varphi^{-1}(W)) \arrow[d, "\operatorname{res}_{U\cap\varphi^{-1}W, U\cap\varphi^{-1}W'}"] \\
\mathcal{O}_Y(W') \arrow[r, "\varphi^\sharp_{W'}"] & \mathcal{O}_X(\varphi^{-1}(W')) \arrow[r, "\operatorname{res}"] & \mathcal{O}_X(U \cap \varphi^{-1}(W'))
\end{tikzcd}$$

where now the left square commutes because $\varphi^\sharp$ is a sheaf morphism, and the right diagram commutes because $\mathcal{O}_X$ is a sheaf.

Remark 2.40. If we only wanted a morphism of ringed spaces, then we could end the proof here. We remark here to point out that we can also restrict "morphisms of ringed spaces."

It remains to check that $(\varphi|_U, \varphi^\sharp|_U)$ assembles into a morphism of locally ringed spaces. Well, fixing some point $p \in U$, we want to show that the composite

$$\begin{array}{ccccc} (\mathcal{O}_Y|_V)_{\varphi(p)} & \overset{(\varphi^\sharp|_U)_{\varphi(p)}}{\to} & ((\varphi|_U)_*\mathcal{O}_X)_{\varphi(p)} & \to & (\mathcal{O}_X|_U)_p \\ [(W,s)] & \mapsto & [(W, \varphi^\sharp_W(s)|_{U\cap\varphi^{-1}W})] & \mapsto & [(U\cap\varphi^{-1}W, \varphi^\sharp_W(s)|_{U\cap\varphi^{-1}W})] \end{array}$$

is a map of local rings. It will suffice to show that the diagram

$$\begin{tikzcd} (\mathcal{O}_Y|_V)_{\varphi(p)} \arrow[r, "(\varphi^\sharp|_U)_{\varphi(p)}"] \arrow[d, equal] & (\varphi|_U)_*(\mathcal{O}_X|_U)_{\varphi(p)} \arrow[r] & (\mathcal{O}_X|_U)_p \arrow[d, equal] \\ (\mathcal{O}_Y|_V)_{\varphi(p)} \arrow[r, "\varphi^\sharp_{\varphi(p)}"] & (\varphi_*\mathcal{O}_X)_{\varphi(p)} \arrow[r] & \mathcal{O}_{X,p} \end{tikzcd}$$

commutes. Notably, $[(U\cap\varphi^{-1}W, \varphi^\sharp_W(s)|_{U\cap\varphi^{-1}W})] = [(\varphi^{-1}W, \varphi^\sharp_W(s)]$. Additionally, Lemma 1.174 (combined with the notation that $\mathcal{O}_X|_U$ refers to the inverse image presheaf) tells us that $(\mathcal{O}_X|_U)_p \simeq \mathcal{O}_{X,p}$ by $[(W,s)] \mapsto [(W,s)]$. Thus, we can track through the above diagram as

$$\begin{tikzcd} {[(W,s)]} \arrow[r, mapsto, "(\varphi^\sharp|_U)_{\varphi(p)}"] \arrow[d, equal] & {[(W, \varphi^\sharp_W(s)|_{U\cap\varphi^{-1}W})]} \arrow[r, mapsto] & {[(U\cap\varphi^{-1}W, \varphi^\sharp_W(s)|_{U\cap\varphi^{-1}W})]} \arrow[d, equal] \\ {[(W,s)]} \arrow[r, mapsto, "\varphi^\sharp_{\varphi(p)}"] & {[(W, \varphi^\sharp_W(s))]} \arrow[r, mapsto] & {[(\varphi^{-1}W, \varphi^\sharp_W(s))]} \end{tikzcd}$$

which does indeed commute. ■

Remark 2.41. From the above proof, it will be worth our time to keep the commutative diagram

$$\begin{tikzcd} \mathcal{O}_{Y,\varphi(p)} \arrow[r, "(\varphi^\sharp|_U)_{\varphi(p)}"] \arrow[d, equal] & (\varphi|_U)_*(\mathcal{O}_X|_U)_{\varphi(p)} \arrow[r] & (\mathcal{O}_X|_U)_p \arrow[d, equal] \\ \mathcal{O}_{Y,\varphi(p)} \arrow[r, "\varphi^\sharp_p"] & (\varphi_*\mathcal{O}_X)_{\varphi(p)} \arrow[r] & \mathcal{O}_{X,p} \end{tikzcd}$$

in a safe place. Notably, the proof of the commutativity of this diagram does not need to know that $(\varphi, \varphi^\sharp)$ is actually a morphism of local rings.

Remark 2.42. It will be worth pointing out that open subsets $\jmath\colon U \subseteq X$ and $\iota\colon V \subseteq Y$ with $\varphi(U) \subseteq V$ will make the diagram

$$\begin{tikzcd} (U, \mathcal{O}_X|_U) \arrow[r, "(\jmath,\jmath^\sharp)"] \arrow[d, "(\varphi,\varphi^\sharp)|_U"'] & (X, \mathcal{O}_X) \arrow[d, "(\varphi,\varphi^\sharp)"] \\ (V, \mathcal{O}_Y|_V) \arrow[r, "(\iota,\iota^\sharp)"] & (Y, \mathcal{O}_Y) \end{tikzcd}$$

commute. This commutes on topological spaces because the map $(\varphi,\varphi^\sharp)|_U$ on spaces is just $\varphi|_U$. This commutes on sheaves by noting any open $W \subseteq V$ makes the following diagram commute.

$$\begin{tikzcd} \varphi_*\jmath_*(\mathcal{O}_X|_U)(W) & \varphi_*\mathcal{O}_X(W) \arrow[l, "(\varphi_*\jmath^\sharp)_W"'] \\ \iota_*(\mathcal{O}_Y|_V)(W) \arrow[u, "\iota_*(\varphi|_U)^\sharp_W"] & \mathcal{O}_Y(W) \arrow[l, "\iota^\sharp_W"'] \arrow[u, "\varphi^\sharp_W"'] \end{tikzcd} \qquad \begin{tikzcd} \varphi^\sharp_W(s)|_{U\cap\varphi^{-1}W} & \varphi^\sharp_W(s) \arrow[l, mapsto] \\ s|_{V\cap W} \arrow[u, mapsto] & s \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

Remark 2.43. Additionally, restriction is functorial. Given scheme morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ with open subsets $U \subseteq X$ and $V \subseteq Y$ and $W \subseteq Z$ such that $\varphi(U) \subseteq V$ and $\psi(V) \subseteq W$, we see that

$$\begin{tikzcd}
(U, \mathcal{O}_X|_U) \arrow[r, "(\varphi,\varphi^\sharp)|_U"] \arrow[dr, "((\psi,\psi^\sharp)\circ(\varphi,\varphi^\sharp))|_U"'] & (V, \mathcal{O}_Y|_V) \arrow[d, "(\psi,\psi^\sharp)|_V"] \\
& (W, \mathcal{O}_Z|_W)
\end{tikzcd}$$

commutes. This commutes on the level of topological spaces because we're just restricting continuous maps. This commutes on the level of sheaves because any $W' \subseteq W$ makes the following diagram commute.

$$\begin{tikzcd}
\varphi_*\psi_*\mathcal{O}_X|_U & (\psi|_V)_*\mathcal{O}_Y|_V(W') \arrow[l, "(\varphi^\sharp|_U)_{V\cap\psi^{-1}W'}"'] \\
& \mathcal{O}_Z|_W(W') \arrow[u, "(\psi^\sharp|_V)_W"'] \arrow[ul, "((\varphi_*\psi^\sharp\circ\varphi^\sharp)|_U)_{W'}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi^\sharp_{\psi^{-1}W'}(\psi^\sharp_{W'}(s))|_{U\cap\varphi^{-1}\psi^{-1}W'} & \psi^\sharp_{W'}(s)|_{V\cap\psi^{-1}W'} \arrow[l, mapsto] \\
& s \arrow[u, mapsto] \arrow[ul, mapsto]
\end{tikzcd}$$

And now we can glue morphisms.

Proposition 2.44. Fix schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover for X; then, given scheme morphisms $(\varphi_\alpha, \varphi^\sharp_\alpha)\colon (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \to (Y, \mathcal{O}_Y)$ such that any $\alpha, \beta \in \lambda$ have

$$(\varphi_\alpha, \varphi^\sharp_\alpha)|_{U_\alpha\cap U_\beta} = (\varphi_\beta, \varphi^\sharp_\beta)|_{U_\alpha\cap U_\beta},$$

there is a unique scheme morphism $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $(\varphi, \varphi^\sharp)|_{U_\alpha} = (\varphi_\alpha, \varphi^\sharp_\alpha)$.

Proof. Notably, we are using Lemma 2.9 to discuss the restriction of our morphisms. We show uniqueness and existence separately.

- Uniqueness: suppose that we have two scheme morphisms $(\varphi, \varphi^\sharp), (\psi, \psi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ which restrict to $(\varphi_\alpha, \varphi^\sharp_\alpha)$ on each $U \in \mathcal{U}$. On the level of topological spaces, we see that any $p \in X$ has $p \in U_\alpha$ for some α, so
$$\varphi(p) = \varphi_\alpha(p) = \psi(p).$$
On the level of sheaves, pick up some open subset $V \subseteq Y$. Because $(\varphi, \varphi^\sharp)|_{U_\alpha} = (\varphi_\alpha, \varphi^\sharp_\alpha)$, we are promised that the diagram
$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "\varphi^\sharp_V"] \arrow[d, equal] & \varphi_*\mathcal{O}_X(V) \arrow[r, equal] & \mathcal{O}_X(\varphi^{-1}(V)) \arrow[d, "\operatorname{res}_{\varphi^{-1}V, U_\alpha\cap\varphi^{-1}V}"] \\
\mathcal{O}_Y(V) \arrow[r, "(\varphi^\sharp_\alpha)_V"] & (\varphi_\alpha)_*(\mathcal{O}_X|_{U_\alpha})(V) \arrow[r, equal] & \mathcal{O}_X(U_\alpha\cap\varphi^{-1}(V))
\end{tikzcd} \tag{2.3}$$
commutes for each $\alpha \in \lambda$. In particular, tracking some $s \in \mathcal{O}_X(V)$ through, we see that
$$\varphi^\sharp_V(s)|_{U_\alpha\cap\varphi^{-1}V} = \varphi^\sharp_\alpha(s)$$
for each $\alpha \in \lambda$, so the identity axiom on $\mathcal{O}_X$ uniquely determines $\varphi^\sharp(s)$. Thus, $\varphi^\sharp_V$ is uniquely determined.
- Existence: On the level of topological spaces, we pick up any $p \in X$ and find some U_α containing p, so we define $\varphi(p) := \varphi_\alpha(p)$. This is well-defined because $p \in U_\alpha \cap U_\beta$ implies
$$\varphi_\alpha(p) = \varphi_\alpha|_{U_\alpha\cap U_\beta}(p) = \varphi_\beta|_{U_\alpha\cap U_\beta}(p) = \varphi_\beta(p).$$

Additionally, φ is continuous because any open $V \subseteq Y$ will have

$$\varphi^{-1}(V) = \bigcup_{\alpha\in\lambda}\{p \in U_\alpha : \varphi(p) \in V\} = \bigcup_{\alpha\in\lambda}\{p \in U_\alpha : \varphi_\alpha(p) \in V\} = \bigcup_{\alpha\in\lambda}\varphi_\alpha^{-1}(V),$$

and $\varphi_\alpha^{-1}(V)$ is open in U_α and hence in X, so this arbitrary union remains open.

It remains to define our map on the level of sheaves. Fix some $s \in \mathcal{O}_Y(V)$. Using the diagram of (2.3) as our intuition, we define $s_\alpha := (\varphi_\alpha^\sharp)_V(s) \in \mathcal{O}_X(U_\alpha \cap \varphi^{-1}(V))$, which we would like to glue. Well, for any $\alpha, \beta \in \lambda$, we see that

$$\begin{aligned} s_\alpha|_{U_\alpha\cap U_\beta\cap\varphi^{-1}(V)} &= (\varphi_\alpha^\sharp)_V(s)|_{U_\alpha\cap U_\beta\cap\varphi^{-1}(V)} \\ &= (\varphi_\alpha^\sharp|_{U_\alpha\cap U_\beta})_V(s) \\ &= (\varphi_\beta^\sharp|_{U_\alpha\cap U_\beta})_V(s) \\ &= (\varphi_\beta^\sharp)_V(s)|_{U_\alpha\cap U_\beta\cap\varphi^{-1}(V)} \\ &= s_\alpha|_{U_\alpha\cap U_\beta\cap\varphi^{-1}(V)}, \end{aligned}$$

so the $\{s_\alpha\}_{\alpha\in\lambda}$ on the open cover $\{U_\alpha \cap \varphi^{-1}V\}_{\alpha\in\lambda}$ of $\varphi^{-1}(V)$ grants a unique $s \in \varphi_*\mathcal{O}_X(V)$ such that $s|_{U_\alpha\cap\varphi^{-1}V} = s_\alpha$. In other words, $\varphi_V^\sharp(s) \in \mathcal{O}_X(\varphi^{-1}(V))$ should be uniquely defined to have

$$\varphi_V^\sharp(s)|_{U_\alpha\cap\varphi^{-1}(V)} = (\varphi_\alpha^\sharp)_V(s). \tag{2.4}$$

We now run our checks.

- Sheaf morphism: given open sets $V' \subseteq V \subseteq Y$, we see that

$$\begin{aligned} (\varphi_V^\sharp(s)|_{V'})|_{U_\alpha\cap\varphi^{-1}(V')} &= \varphi_V^\sharp(s)|_{U_\alpha\cap\varphi^{-1}(V)}|_{U_\alpha\cap\varphi^{-1}(V')} \\ &= (\varphi_\alpha^\sharp)_V(s)|_{U_\alpha\cap\varphi^{-1}(V')} \\ &= (\varphi_\alpha^\sharp)_{V'}(s|_{V'})|_{U_\alpha\cap\varphi^{-1}(V')} \end{aligned}$$

  for each $\alpha \in \lambda$, so $\varphi_V^\sharp(s)|_{V'} = (\varphi_\alpha^\sharp)_{V'}(s|_{V'})$ follows.
- Ring morphism: we won't write this out, but one should just use the uniqueness of (2.4) to verify the various properties for a ring morphism. For example, $(\varphi_\alpha^\sharp)_V(1) = 1$ everywhere because we have ring maps, so it follows $\varphi_V^\sharp(s)$ by uniqueness. Similarly,

$$(\varphi_\alpha^\sharp)_V(ax+by) = (\varphi_\alpha^\sharp)_V(a)\cdot(\varphi_\alpha^\sharp)_V(x) + (\varphi_\alpha^\sharp)_V(b)\cdot(\varphi_\alpha^\sharp)_V(y)$$

  because $(\varphi_\alpha^\sharp)_V$ is a ring map, so gluing all $\alpha\in\lambda$ forces $\varphi_V^\sharp(ax+by) = \varphi_V^\sharp(a)\varphi_V^\sharp(x) + \varphi_V^\sharp(b)\varphi_V^\sharp(y)$.
- Restrictions: we show that $(\varphi, \varphi^\sharp)|_{U_\alpha} = (\varphi_\alpha, \varphi_\alpha^\sharp)$ as morphisms of ringed spaces. (Namely, the algorithm to restrict from Lemma 2.39 also works for ringed spaces, as noted in Remark 2.40.) On the level of topological spaces, by construction we have $\varphi|_{U_\alpha}(p) = \varphi_\alpha(p)$ for each $p \in U_\alpha$. On the level of sheaves, we note that the map $(\varphi^\sharp)|_{U_\alpha} : \mathcal{O}_Y \to (\varphi|_{U_\alpha})_*\mathcal{O}_X$ by definition sends $s \in \mathcal{O}_Y(V)$ to

$$((\varphi^\sharp)|_{U_\alpha})_V(s) = \varphi_V^\sharp(s)|_{U_\alpha\cap\varphi^{-1}V} = (\varphi_\alpha^\sharp)_V(s),$$

  where the last equality is by (2.4). This finishes.
- Locally ringed space: we show that $(\varphi, \varphi^\sharp)$ is a morphism of locally ringed spaces. Well, for some $p \in X$, place p in some $U := U_\alpha$, and we note that we want the bottom row of the diagram in Remark 2.41 to be a map of local rings. Thus, it suffices for the top row of the diagram in Remark 2.41 to be a map of local rings, but this is clear because $\varphi^\sharp|_{U_\alpha} = \varphi_\alpha^\sharp$ and $\varphi|_U = \varphi_\alpha$ already makes

$$\mathcal{O}_{Y,\varphi(p)} \stackrel{(\varphi_\alpha^\sharp)_{\varphi(p)}}{\to} ((\varphi_\alpha)_*(\mathcal{O}_X|_{U_\alpha})_{\varphi_\alpha(p)} \to (\mathcal{O}_X)_p$$

  a map of local rings because $(\varphi_\alpha, \varphi_\alpha^\sharp)$ is a morphism of locally ringed spaces.

The above checks complete the proof. ■

2.2.2 Gluing Sheaves

So far we've glued together morphisms. It remains to glue together schemes. Unsurprisingly, this will happen in two steps: we will first glue the topological spaces (which is comparatively easy) and then we will glue the sheaves. Gluing sheaves is hard enough on its own, so we will discuss how to do that now.

Lemma 2.45 (Vakil 2.5.D)**.** Fix a topological space X with an open cover $\{U_\alpha\}_{\alpha\in\lambda}$. Suppose we have sheaves $\mathcal{F}_\alpha$ on U_α, along with isomorphisms $\varphi_{\alpha\beta}\colon \mathcal{F}_\alpha|_{U_\alpha\cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha\cap U_\beta}$ (with $\varphi_{\alpha\alpha}$ the identity) that agree on triple overlaps such that

$$\varphi_{\beta\gamma}\circ\varphi_{\alpha\beta} = \varphi_{\alpha\gamma} \qquad \text{on} \qquad U_i\cap U_j\cap U_k.$$

Then these sheaves can be glued together into a sheaf $\mathcal{F}$ on X (unique up to unique isomorphism) equipped with isomorphisms $\pi_\alpha\colon \mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ making the following diagram commute.

$$\begin{tikzcd}
\mathcal{F}|_{U_\alpha\cap U_\beta} \arrow[r, "\pi_\alpha"] \arrow[d, equal] & \mathcal{F}_\alpha|_{U_\alpha\cap U_\beta} \arrow[d, "\varphi_{\alpha\beta}"] \\
\mathcal{F}|_{U_\alpha\cap U_\beta} \arrow[r, "\pi_\beta"] & \mathcal{F}_\beta|_{U_\alpha\cap U_\beta}
\end{tikzcd}$$

Proof. Because $X = \bigcup_{\alpha\in\lambda} U_\alpha$, we may set

$$\mathcal{B} := \bigcup_{\alpha\in\lambda}\{\text{open } U\subseteq X : U\subseteq U_\alpha\}$$

to be a basis for X. (Indeed, for any open $U\subseteq X$, we have $U=\bigcup_{\alpha\in\lambda}(U\cap U_\alpha)$ where $U\cap U_\alpha\in\mathcal{B}$ for each α.) The point is to build $\mathcal{F}$ as a sheaf on the base $\mathcal{B}$.

For each $B\in\mathcal{B}$, make some arbitrary choice and set $\alpha(B)\in\lambda$ to be such that $B\subseteq U_{\alpha(B)}$. We now define

$$F(B) := \mathcal{F}_{\alpha(B)}(B).$$

To define our restriction maps, we note that $B'\subseteq B$ implies that $B'\subseteq U_{\alpha(B)}\cap U_{\alpha(B')}$, so the sheaf isomorphism

$$\varphi_{\alpha(B),\alpha(B')}\colon \mathcal{F}_{\alpha(B)}|_{U_{\alpha(B)}\cap U_{\alpha(B')}} \to \mathcal{F}_{\alpha(B')}|_{U_{\alpha(B)}\cap U_{\alpha(B')}}$$

grants us an isomorphism

$$\varphi_{\alpha(B),\alpha(B')}(B')\colon \mathcal{F}_{\alpha(B)}(B') \to \mathcal{F}_{\alpha(B')}(B'),$$

so we may define our restriction map as the composite

$$\mathcal{F}_{\alpha(B)}(B) \stackrel{\operatorname{res}_{B,B'}}{\to} \mathcal{F}_{\alpha(B)}(B') \stackrel{\varphi_{\alpha(B),\alpha(B')}(B')}{\to} \mathcal{F}_{\alpha(B')}(B').$$

For concreteness, denote this composite by $r_{B,B'}$. We now check that F assembles to a presheaf on the base $\mathcal{B}$.

- Identity: given $B\in\mathcal{B}$, our restriction to B is the composite

$$\mathcal{F}_{\alpha(B)}(B) \stackrel{\operatorname{res}_{B,B}}{\to} \mathcal{F}_{\alpha(B)}(B) \stackrel{\varphi_{\alpha(B),\alpha(B)}(B')}{\to} \mathcal{F}_{\alpha(B)}(B),$$

  but both these maps are the identity (the first because $\mathcal{F}_{\alpha(B)}$ is a sheaf, and the second by hypothesis on the $\varphi_\bullet$).

- Functoriality: given basis elements $B'' \subseteq B' \subseteq B$, we are interested in showing that the following diagram

$$\begin{tikzcd}
\mathcal{F}_{\alpha(B)}(B) \arrow[r, "\operatorname{res}_{B,B'}"] \arrow[dr, "\operatorname{res}_{B,B''}"'] & \mathcal{F}_{\alpha(B)}(B') \arrow[r, "\varphi_{\alpha(B),\alpha(B')}(B')"] \arrow[d, "\operatorname{res}_{B',B''}"] & \mathcal{F}_{\alpha(B')}(B') \arrow[d, "\operatorname{res}_{B',B''}"] \\
& \mathcal{F}_{\alpha(B)}(B'') \arrow[r, "\varphi_{\alpha(B),\alpha(B')}(B'')"] \arrow[dr, "\varphi_{\alpha(B),\alpha(B'')}(B'')"'] & \mathcal{F}_{\alpha(B')}(B'') \arrow[d, "\varphi_{\alpha(B'),\alpha(B'')}(B'')"] \\
& & \mathcal{F}_{\alpha(B'')}(B'')
\end{tikzcd} \tag{2.5}$$

commutes. Namely, the top row is $r_{B,B'}$, the right column is $r_{B',B''}$, and we want to know that the composite of these is the diagonal $r_{B,B''}$.

Now, the top-left triangle of (2.5) commutes by the functoriality of the (pre)sheaf $\mathcal{F}_{\alpha(B)}$. The square of (2.5) commutes because $\varphi_{\alpha(B),\alpha(B')}$ is a (pre)sheaf morphism. Lastly, the bottom-right triangle of (2.5) commutes because $\varphi_{\alpha(B),\alpha(B'')} = \varphi_{\alpha(B'),\alpha(B'')} \circ \varphi_{\alpha(B),\alpha(B')}$ by the cocycle condition.

It remains to check that we have a sheaf on the base. Fix some $B \in \mathcal{B}$ with a basic cover $\{B_i\}_{i\in I} \subseteq \mathcal{B}$. For brevity, set $\beta_i := \alpha(B_i)$ and $\beta := \alpha(B)$.

- Identity: suppose $s, s' \in F(B)$ have $r_{B,B_i}(s) = r_{B,B_i}(s')$ for each $i \in I$. Expanding out what restriction really means here, we are saying that $s, s' \in \mathcal{F}_\beta(B)$ has

$$\varphi_{\beta,\beta_i}(B_i)(\operatorname{res}_{B,B_i}(s)) = \varphi_{\beta,\beta_i}(B_i)(\operatorname{res}_{B,B_i}(s'))$$

as elements of $\mathcal{F}_{\beta_i}(B_i)$. Undoing the isomorphism, we see that

$$\operatorname{res}_{B,B_i}(s) = \operatorname{res}_{B,B_i}(s')$$

as elements of $\mathcal{F}_\beta(B_i)$. Thus, the identity axiom of $\mathcal{F}_\beta$ forces $s = s'$.

- Gluability: suppose we have $s_i \in F(B_i)$ for each $i \in I$ such that

$$r_{B_i,B'}(s_i) = r_{B_j,B'}(s_j)$$

for each $i, j \in I$ and basis element $B' \subseteq B_i \cap B_j$. Notably, $B' \subseteq B_i \subseteq U_{\alpha(B_i)}$ implies that $B' \in \mathcal{B}$, so we might as well assume this for $B' = B_i \cap B_j$. Expanding out the restriction, we are asserting

$$\varphi_{\beta_i,\alpha(B')}(B')(\operatorname{res}_{B_i,B'}(s_i)) = \varphi_{\beta_j,\alpha(B')}(B')(\operatorname{res}_{B_j,B'}(s_j)).$$

Hitting both sides with $\varphi_{\alpha(B'),\beta}(B_i)$ (which is legal because $B' \subseteq B$ after all), the cocycle condition gives

$$\varphi_{\beta_i,\beta}(B')(\operatorname{res}_{B_i,B'}(s_i)) = \varphi_{\beta_j,\beta}(B')(\operatorname{res}_{B_j,B'}(s_j)).$$

Using the fact that the φ are sheaf morphisms, this is equivalent to

$$\operatorname{res}_{B_i,B'}(\varphi_{\beta_i,\beta}(B_i)(s_i)) = \operatorname{res}_{B_j,B'}(\varphi_{\beta_j,\beta}(B_j)(s_j)).$$

Setting $t_i := \varphi_{\beta_i,\beta}(B_i)(s_i) \in \mathcal{F}_\beta(B_i)$, we see that $t_i|_{B_i\cap B_j} = t_j|_{B_i\cap B_j}$ for each $i, j \in I$, so gluability on $\mathcal{F}_\beta$ gives some $s \in \mathcal{F}_\beta(B)$ such that

$$s|_{B_i} = \varphi_{\beta_i,\beta}(B_i)(s_i)$$

for each $i \in I$. Equivalently, we have

$$r_{B,B_i}(s) = \varphi_{\beta,\beta_i}(B_i)(\operatorname{res}_{B,B_i}(s)) = s_i,$$

which is what we wanted.

Thus, F does indeed define a sheaf on the base $\mathcal{B}$, which extends to a sheaf $\mathcal{F}$ (unique up to unique isomorphism) by Proposition 1.80. Here are our last checks on $\mathcal{F}$.

- We exhibit an isomorphism $\pi_\alpha\colon \mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ for each $\alpha \in \lambda$. Indeed, for any $B \subseteq U_\alpha$, note $B \in \mathcal{B}$, so we define $\pi_\alpha(B)$ as the composite

$$\mathcal{F}|_{U_\alpha}(B) = \mathcal{F}(B) = F(B) = \mathcal{F}_{\alpha(B)}(B) \overset{\varphi_{\alpha(B),\alpha}(B)}{\to} \mathcal{F}_\alpha(B),$$

where $\varphi_{\alpha(B),\alpha}(B)$ makes sense because $B \subseteq U_{\alpha(B)} \cap U_\alpha$.

Now, to see that π_α assembles into a natural transformation, we suppose $B' \subseteq B \subseteq U_\alpha$ and draw the diagram

$$\begin{tikzcd} \mathcal{F}|_{U_\alpha}(B) \arrow[r, equal] \arrow[d] & \mathcal{F}(B) \arrow[r, equal] & F(B) \arrow[r, equal] & \mathcal{F}_{\alpha(B)}(B) \arrow[r, "\varphi_{\alpha(B),\alpha}(B)"] \arrow[d, "r_{B,B'}"'] & \mathcal{F}_\alpha(B) \arrow[d, "\mathrm{res}_{B,B'}"] \\ \mathcal{F}|_{U_\alpha}(B') \arrow[r, equal] & \mathcal{F}(B') \arrow[r, equal] & F(B') \arrow[r, equal] & \mathcal{F}_{\alpha(B')}(B') \arrow[r, "\varphi_{\alpha(B'),\alpha}(B')"'] & \mathcal{F}_\alpha(B') \end{tikzcd}$$

which we would like to commute. Well, we compute directly that

$$\begin{aligned} \varphi_{\alpha(B'),\alpha}(B') \circ r_{B,B'} &= \varphi_{\alpha(B'),\alpha}(B') \circ \varphi_{\alpha(B),\alpha(B')}(B') \circ \mathrm{res}_{B,B'} \\ &= \varphi_{\alpha(B),\alpha}(B') \circ \mathrm{res}_{B,B'} \\ &= \mathrm{res}_{B,B'} \circ \varphi_{\alpha(B),\alpha}(B). \end{aligned}$$

Lastly, we note that π_α is component-wise an isomorphism, so it follows that π_α is an isomorphism on the level of sheaves.

- We show that the diagram

$$\begin{tikzcd} \mathcal{F}|_{U_\alpha \cap U_\beta} \arrow[r, "\pi_\alpha"] \arrow[d, equal] & \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \arrow[d, "\varphi_{\alpha\beta}"] \\ \mathcal{F}|_{U_\alpha \cap U_\beta} \arrow[r, "\pi_\beta"] & \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \end{tikzcd}$$

commutes. Indeed, for any $B \subseteq U_\alpha \cap U_\beta$, we have $B \in \mathcal{B}$, and as such we compute

$$\begin{aligned} \pi_\beta(B) \circ \pi_\alpha(B)^{-1} &= \varphi_{\alpha(B),\beta}(B) \circ \varphi_{\alpha(B),\alpha}(B)^{-1} \\ &= \varphi_{\alpha(B),\beta}(B) \circ \varphi_{\alpha,\alpha(B)}(B) \\ &= \varphi_{\alpha,\beta}(B), \end{aligned}$$

where the last two equalities hold by the cocycle condition. ■

This sheaf we just built satisfies the following universal property.

Proposition 2.46. Work in the context of Lemma 2.45. Given any sheaf $\mathcal{G}$ on X with maps $\gamma_\alpha\colon \mathcal{G}|_{U_\alpha} \to \mathcal{F}_\alpha$ making the diagram

$$\begin{tikzcd} \mathcal{G}|_{U_\alpha \cap U_\beta} \arrow[r, "\gamma_\alpha"] \arrow[d, equal] & \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \arrow[d, "\varphi_{\alpha\beta}"] \\ \mathcal{G}|_{U_\alpha \cap U_\beta} \arrow[r, "\gamma_\beta"] & \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \end{tikzcd}$$

commute, there is a unique map $\gamma\colon \mathcal{G} \to \mathcal{F}$ such that $\gamma_\alpha = \pi_\alpha \circ \gamma|_{U_\alpha}$.

Proof. We continue to work with the explicit description of $\mathcal{F}$ as coming from the sheaf F on the base $\mathcal{B}$. We show existence and uniqueness of γ separately.

- Uniqueness: on the base $\mathcal{B}$, for any $B \in \mathcal{B}$ with $B \subseteq U_{\alpha(B)}$, we see that γ_B must have

$$\gamma_B = (\pi_{\alpha(B)}^{-1})_B \circ (\gamma_{\alpha(B)})_B,$$

so γ is uniquely determined on the base $\mathcal{B}$. It follows from Lemma 1.79 that γ is unique.

- Existence: on the base $\mathcal{B}$, for any $B \in \mathcal{B}$ with $B \subseteq U_{\alpha(B)}$, we set

$$\gamma_B \coloneqq (\pi_{\alpha(B)}^{-1})_B \circ (\gamma_{\alpha(B)})_B.$$

We claim that γ will at least assemble into a morphism of sheaves $\mathcal{G}_{\mathcal{B}} \to \mathcal{F}_{\mathcal{B}}$ on sheaves on the base $\mathcal{B}$. Well, given $B' \subseteq B \subseteq U_\alpha$ for $\alpha = \alpha(B)$, we need the diagram

$$\begin{tikzcd}
\mathcal{G}(B) \arrow[r, "\gamma_{\alpha(B)}(B)"] \arrow[d, "\operatorname{res}_{B,B'}"'] & \mathcal{F}_{\alpha(B)}(B) & \mathcal{F}(B) \arrow[l, "\pi_{\alpha(B)}(B)"'] \arrow[d, "\operatorname{res}_{B,B'}"] \\
\mathcal{G}(B') \arrow[r, "\gamma_{\alpha(B')}(B')"] & \mathcal{F}_{\alpha(B')}(B') & \mathcal{F}(B') \arrow[l, "\pi_{\alpha(B)}(B')"']
\end{tikzcd}$$

to commute. Well, we expand this diagram into

$$\begin{tikzcd}
\mathcal{G}(B) \arrow[r, "\gamma_{\alpha(B)}(B)"] \arrow[d, "\operatorname{res}_{B,B'}"'] & \mathcal{F}_{\alpha(B)}(B) \arrow[d, "\operatorname{res}_{B,B'}"] & \mathcal{F}(B) \arrow[l, "\pi_{\alpha(B)}(B)"'] \arrow[d, "\operatorname{res}_{B,B'}"] \\
\mathcal{G}(B') \arrow[r, "\gamma_{\alpha(B)}(B')"] \arrow[d, equal] & \mathcal{F}_{\alpha(B)}(B') \arrow[d, "\varphi_{\alpha(B),\alpha(B')}(B')"] & \mathcal{F}(B') \arrow[l, "\pi_{\alpha(B)}(B')"'] \arrow[d, equal] \\
\mathcal{G}(B') \arrow[r, "\gamma_{\alpha(B')}(B')"'] & \mathcal{F}_{\alpha(B')}(B') & \mathcal{F}(B') \arrow[l, "\pi_{\alpha(B')}(B')"]
\end{tikzcd}$$

and check commutativity square-by-square: the top squares commute because $\gamma_{\alpha(B)}$ and $\pi_{\alpha(B)}$ are natural transformations, and the bottom squares commute by coherence of the γ and π.

In total, we are thus promised some sheaf morphism γ such that $\gamma_B = (\pi_{\alpha(B)}^{-1})_B \circ (\gamma_{\alpha(B)})_B$ for each $B \in \mathcal{B}$. We claim that $\gamma \circ \pi_\alpha = \gamma_\alpha$ for each α, for which it suffices to check on the base $\mathcal{B}$. Well, for any fixed $B \in \mathcal{B}$, we are asking for

$$(\pi_{\alpha(B)}^{-1})_B \circ (\gamma_{\alpha(B)})_B \circ \pi_\alpha = \gamma_\alpha,$$

which translates into the diagram

$$\begin{tikzcd}
\mathcal{G}(B) \arrow[r, "\gamma_{\alpha(B)}"] \arrow[d, equal] & \mathcal{F}_{\alpha(B)}(B) \arrow[r, "\pi_{\alpha(B)}(B)"] \arrow[d, "\varphi_{\alpha(B),\alpha}(B)"] & \mathcal{F}(B) \arrow[d, equal] \\
\mathcal{G}(B) \arrow[r, "\gamma_\alpha(B)"'] & \mathcal{F}_\alpha(B) \arrow[r, "\pi_\alpha(B)"'] & \mathcal{F}(B)
\end{tikzcd}$$

commuting, which holds by the coherence of π and γ. ■

Remark 2.47. Reversing the direction of the γs in Proposition 2.46 gives another universal property: given any sheaf $\mathcal{G}$ on X with maps $\gamma_\alpha \colon \mathcal{F}_\alpha \to \mathcal{G}|_{U_\alpha}$ making the diagram

$$\begin{tikzcd}
\mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \arrow[r, "\gamma_\alpha"] \arrow[d, "\varphi_{\alpha\beta}"'] & \mathcal{G}|_{U_\alpha \cap U_\beta} \arrow[d, equal] \\
\mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \arrow[r, "\gamma_\beta"] & \mathcal{G}|_{U_\alpha \cap U_\beta}
\end{tikzcd}$$

commute, there is a unique map $\gamma \colon \mathcal{F} \to \mathcal{G}$ such that $\gamma_\alpha \circ \pi_\alpha = \gamma|_{U_\alpha}$.

2.2.3 Gluing Schemes

We are finally ready to glue schemes.

Proposition 2.48. Fix a collection of schemes $(X_\alpha, \mathcal{O}_\alpha)$ for each $\alpha \in \lambda$, with an open subset $U_{\alpha\beta} \subseteq X_\alpha$ for each $\alpha, \beta \in \lambda$, where $X_{\alpha\alpha} = X_\alpha$; let $(X_{\alpha\beta}, \mathcal{O}_{\alpha\beta}) := (U_{\alpha\beta}, \mathcal{O}_{X_\alpha}|_{U_{\alpha\beta}})$ be the induced open subscheme. Further, pick up some isomorphisms $(\varphi_{\alpha\beta}, \varphi^\sharp_{\alpha\beta}) \colon (X_{\alpha\beta}, \mathcal{O}_{\alpha\beta}) \to (X_{\beta\alpha}, \mathcal{O}_{\beta\alpha})$ satisfying the "cocycle condition"

$$(\varphi_{\alpha\gamma}, \varphi^\sharp_{\alpha\gamma}) = (\varphi_{\beta\gamma}, \varphi^\sharp_{\beta\gamma}) \circ (\varphi_{\alpha\beta}, \varphi^\sharp_{\alpha\beta}),$$

on $(U_{\alpha\beta} \cap U_{\alpha\gamma}, \mathcal{O}_{X_\alpha}|_{U_{\alpha\beta} \cap U_{\alpha\gamma}}) \subseteq X_\alpha$, where we implicitly assume that $\varphi_{\alpha\beta}(U_{\alpha\beta} \cap U_{\alpha\gamma}) \subseteq U_{\beta\gamma}$. Then there is a scheme $(X, \mathcal{O})$ equipped with an open cover $\{U_\alpha\}_{\alpha\in\lambda}$ and isomorphisms $(\iota_\alpha, \iota^\sharp_\alpha) \colon (X_\alpha, \mathcal{O}_\alpha) \to (U_\alpha, \mathcal{O}|_{U_\alpha})$ covering X such that

- $(\iota_\alpha, \iota^\sharp_\alpha)|_{X_{\alpha\beta}} = (\iota_\beta, \iota^\sharp_\beta)|_{X_{\beta\alpha}} \circ (\varphi_{\alpha\beta}, \varphi^\sharp_{\alpha\beta})$, and
- $U_\alpha \cap U_\beta = \iota_\alpha(X_{\alpha\beta}) = \iota_\beta(X_{\beta\alpha})$.

Proof. As a quick aside, we note that setting $\alpha = \beta = \gamma$ in the cocycle condition tells us that $(\varphi_{\alpha\alpha}, \varphi^\sharp_{\alpha\alpha})$ is the identity. Then setting $\alpha = \gamma$ in the cocycle condition tells us that $(\varphi_{\alpha\beta}, \varphi^\sharp_{\alpha\beta})$ is the inverse of $(\varphi_{\beta\alpha}, \varphi^\sharp_{\beta\alpha})$.

We begin by gluing the topological space. Let $\widetilde{X}$ denote the disjoint union of the X_αs, equipped with inclusions $\widetilde{\jmath}_\alpha \colon X_\alpha \to \widetilde{X}$. Then we define the equivalence relation $\sim$ by taking $x_\alpha \in U_{\alpha\beta}$ and identifying $\widetilde{\jmath}_\alpha x_\alpha \sim \widetilde{\jmath}_\beta \varphi_{\alpha\beta} x_\alpha$. Quickly, we show $\sim$ forms an equivalence relation.

- Reflexive: note $x_\alpha \in U_{\alpha\alpha}$ has $\widetilde{\jmath}_\alpha x_\alpha \sim \widetilde{\jmath}_\alpha x_\alpha$.
- Symmetric: given $\widetilde{\jmath}_\alpha x_\alpha \sim \widetilde{\jmath}_\beta \varphi_{\alpha\beta} x_\alpha$, we set $x_\beta := \varphi_{\alpha\beta} x_\alpha$ so that $x_\alpha = \varphi_{\beta\alpha} x_\beta$, from which

  $$\jmath_\beta \varphi_{\alpha\beta} x_\alpha = \widetilde{\jmath}_\beta x_\beta \sim \widetilde{\jmath}_\alpha \varphi_{\beta\alpha} x_\beta = \widetilde{\jmath}_\alpha x_\alpha$$

  follows.
- Transitive: given $\widetilde{\jmath}_\alpha x_\alpha \sim \widetilde{\jmath}_\beta \varphi_{\alpha\beta} x_\alpha$ and $\widetilde{\jmath}_\beta \varphi_{\alpha\beta} x_\alpha \sim \widetilde{\jmath}_\gamma \varphi_{\beta\gamma} \varphi_{\alpha\beta} x_\alpha$, we see that $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ by the cocycle condition, so it follows

  $$\widetilde{\jmath}_\alpha x_\alpha \sim \widetilde{\jmath}_\gamma \varphi_{\alpha\gamma} x_\alpha = \widetilde{\jmath}_\gamma \varphi_{\beta\gamma} \varphi_{\alpha\beta} x_\alpha.$$

Thus, we give $X := \widetilde{X}/{\sim}$ the quotient topology, where $\pi \colon \widetilde{X} \twoheadrightarrow X$ is the canonical projection. Let $\jmath_\alpha \colon X_\alpha \to X$ be the composite $\pi \circ \widetilde{\jmath}_\alpha$. In particular, $U \subseteq X$ is open if and only if $\pi^{-1}(U) \subseteq \widetilde{X}$ is open, which is true if and only if $\jmath_\alpha^{-1}(U) \subseteq X_\alpha$ is open for each $\alpha \in \lambda$. We now have two topological checks.

- Note that $\jmath_\beta(x_\beta) \in \jmath_\alpha(X_\alpha)$ if and only if $\widetilde{\jmath}_\beta(x_\beta) \sim \widetilde{\jmath}_\alpha(x_\alpha)$ for some $x_\alpha \in X_\alpha$. But the only elements in $\widetilde{\jmath}_\beta(X_\beta)$ which can be identified with an element of $\widetilde{\jmath}_\beta(X_\beta)$ live in $U_{\beta\alpha}$, so $x_\beta \in U_{\beta\alpha}$. Conversely, if $x_\beta \in U_{\beta\alpha}$ give

  $$\widetilde{\jmath}_\beta x_\beta \sim \widetilde{\jmath}_\alpha \varphi_{\beta\alpha} x_\beta.$$

  It follows that $\jmath_\beta(X_\beta) \cap \jmath_\alpha(X_\alpha) = \jmath_\beta(U_{\beta\alpha})$. Analogously, we get $\jmath_\alpha(X_\alpha) \cap \jmath_\beta(X_\beta) = \jmath_\alpha(U_{\alpha\beta})$.
- We show $\jmath_\alpha \colon X_\alpha \to X$ is an open embedding (i.e., with open image and a homeomorphism onto its image). To begin, note that $\jmath_\alpha$ is injective: we have $\widetilde{\jmath}_\alpha x_\alpha \sim \widetilde{\jmath}_\alpha x'_\alpha$ if and only if $x_\alpha = \varphi_{\alpha\alpha} x'_\alpha$, from which $x_\alpha = x'_\alpha$ follows.

  Continuing, note $\operatorname{im} \jmath_\alpha$ is open because $\jmath_\beta^{-1}(\operatorname{im} \jmath_\alpha) = U_{\beta\alpha} \subseteq X_\beta$ is always open by construction. Lastly, $\jmath_\alpha$ is an open map: for any open subset $U \subseteq X_\alpha$, we see that

  $$\jmath_\beta^{-1}(\jmath_\alpha(U)) = \{x_\beta \in U_{\beta\alpha} : \varphi_{\alpha\beta} x_\beta \in U\} = \varphi_{\beta\alpha}^{-1}(U)$$

  is open for each β, so $\jmath_\alpha(U)$ is in fact open.

Thus, we now set $\iota_\alpha\colon X_\alpha \to \operatorname{im} \jmath_\alpha$ to be the map $\jmath_\alpha$ restricted to its image.

It remains to glue our structure sheaves together. By construction of X, the map π is surjective, and $\widetilde{X}$ is covered by the sets $\widetilde{\iota}_\alpha(X_\alpha)$, so X is covered by the sets $U_\alpha := \iota_\alpha(X_\alpha)$; note that this is an open cover because U_α is open by the above check. Importantly, the following diagram commutes.

$$\begin{tikzcd} U_{\alpha\beta} \arrow[r, "\iota_\alpha"] \arrow[d, "\varphi_{\alpha\beta}"'] & U_\alpha & \subseteq & X \arrow[d, equal] & x_\alpha \arrow[r, mapsto] \arrow[d, mapsto] & {[\widetilde{\jmath}_\alpha(x_\alpha)]} \arrow[d, equal] \\ U_{\beta\alpha} \arrow[r, "\iota_\beta"] & U_\beta & \subseteq & X & \varphi_{\alpha\beta}(x_\alpha) \arrow[r, mapsto] & {[\widetilde{\jmath}_\beta(\varphi_{\alpha\beta}(x_\alpha))]} \end{tikzcd} \tag{2.6}$$

Now, we can take each structure sheaf $\mathcal{O}_\alpha$ and push it to the sheaf $\mathcal{F}_\alpha := (\iota_\alpha)_*\mathcal{O}_\alpha$ on U_α. We will glue these together using Lemma 2.45.

- We exhibit isomorphisms $\psi^\sharp_{\beta\alpha}\colon \mathcal{F}_\alpha|_{U_\alpha\cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha\cap U_\beta}$.

  For any $\alpha, \beta \in \lambda$, we note that any $U \subseteq U_{\beta\alpha}$ gives the isomorphism

  $$(\varphi^\sharp_{\alpha\beta})_U\colon \mathcal{O}_\beta(U) \to \mathcal{O}_\alpha(\varphi^{-1}_{\alpha\beta}(U)).$$

  Hitting this isomorphism with $(\iota_\beta)_*$, we see that (2.6) tells us that $\varphi^{-1}_{\alpha\beta} \circ \iota^{-1}_\beta = \iota^{-1}_\alpha$, so we have the isomorphism

  $$(\varphi^\sharp_{\alpha\beta})_{\iota_\beta^{-1}U}\colon \underbrace{\mathcal{O}_\beta\left(\iota^{-1}_\beta U\right)}_{\mathcal{F}_\alpha(U)} \to \underbrace{\mathcal{O}_\alpha\left(\iota^{-1}_\alpha U\right)}_{\mathcal{F}_\beta(U)}$$

  for any $\iota^{-1}_\beta U \subseteq U_{\beta\alpha}$, which is equivalent to $U \subseteq U_\alpha \cap U_\beta$.

  Thus, we set $(\psi^\sharp_{\beta\alpha})_U := (\varphi^\sharp_{\alpha\beta})_{\iota_\beta^{-1}U}$ for $U \subseteq U_\alpha \cap U_\beta$. To see that $\psi^\sharp_{\beta\alpha}$ assembles into a sheaf isomorphism, we pick open sets $U' \subseteq U \subseteq U_\alpha \cap U_\beta$ and check that the left diagram of

  $$\begin{tikzcd} \mathcal{F}_\beta(U) \arrow[r, "\psi^\sharp_{\beta\alpha}(U)"] \arrow[d, "\operatorname{res}_{U,U'}"'] & \mathcal{F}_\alpha(U) \arrow[d, "\operatorname{res}_{U,U'}"] \\ \mathcal{F}_\beta(U') \arrow[r, "\psi^\sharp_{\beta\alpha}(U')"] & \mathcal{F}_\alpha(U') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_\beta(\iota_\beta^{-1}U) \arrow[r, "\varphi^\sharp_{\alpha\beta}(\iota_\beta^{-1}U)"] \arrow[d, "\operatorname{res}_{\iota_\beta^{-1}U,\iota_\beta^{-1}U'}"'] & (\varphi_{\alpha\beta})_*\mathcal{O}_\alpha(\iota_\beta^{-1}U) \arrow[d, "\operatorname{res}_{\iota_\beta^{-1}U,\iota_\beta^{-1}U'}"] \\ \mathcal{O}_\beta(\iota_\beta^{-1}U') \arrow[r, "\varphi^\sharp_{\alpha\beta}(\iota_\beta^{-1}U')"] & (\varphi_{\alpha\beta})_*\mathcal{O}_\alpha(\iota_\beta^{-1}U') \end{tikzcd}$$

  commutes, which holds because it is the same as the right diagram, which commutes by the naturality of $\varphi^\sharp_{\alpha\beta}$. Thus, we have induced a sheaf isomorphism $\psi^\sharp_{\beta\alpha}\colon \mathcal{F}_\alpha|_{U_\alpha\cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha\cap U_\beta}$.

- We now check the cocycle condition. Namely, for $U \subseteq U_\alpha \cap U_\beta \cap U_\gamma$, we need the diagram

  $$\begin{tikzcd} \mathcal{F}_\alpha(U) \arrow[r, "(\psi^\sharp_{\alpha\beta})_U"] \arrow[dr, "(\psi^\sharp_{\alpha\gamma})_U"'] & \mathcal{F}_\beta(U) \arrow[d, "(\psi^\sharp_{\beta\gamma})_U"] \\ & \mathcal{F}_\gamma(U) \end{tikzcd}$$

  to commute. Well, this diagram is the same as

  $$\begin{tikzcd} \mathcal{O}_\alpha(\iota_\alpha^{-1}U) \arrow[r, "(\varphi^\sharp_{\beta\alpha})_{\iota_\alpha^{-1}U}"] \arrow[dr, "(\varphi^\sharp_{\alpha\gamma})_{\iota_\alpha^{-1}U}"'] & \mathcal{O}_\beta(\varphi^{-1}_{\beta\alpha}\iota_\alpha^{-1}U) \arrow[d, "(\varphi^\sharp_{\gamma\beta})_{\varphi^{-1}_{\gamma\beta}\iota_\alpha^{-1}U}"] \\ & \mathcal{O}_\gamma(\varphi^{-1}_{\gamma\alpha}\iota_\alpha^{-1}U) \end{tikzcd}$$

where we have used the cocycle condition on the topological spaces to see that $\varphi_{\alpha\gamma}^{-1} = \varphi_{\gamma\beta}^{-1} \circ \varphi_{\beta\alpha}^{-1}$. As such, we set $V := \iota_\alpha^{-1}U$ so that we are asking for

$$\begin{tikzcd}
\mathcal{O}_\alpha(V) \arrow[r, "(\varphi^\sharp_{\beta\alpha})_V"] \arrow[dr, "(\varphi^\sharp_{\alpha\gamma})_V"'] & (\varphi_{\beta\alpha})_*\mathcal{O}_\beta(V) \arrow[d, "(\varphi_{\beta\alpha})_*(\varphi^\sharp_{\gamma\beta})_V"] \\
& (\varphi_{\gamma\alpha})_*\mathcal{O}_\gamma(V)
\end{tikzcd}$$

to commute, which is by the given cocycle condition on sheaves.

In total, we get promised a sheaf $\mathcal{O}$ on X which glues the $\mathcal{F}_\alpha$. Namely, we are equipped with isomorphisms $\pi_\alpha^\sharp \colon \mathcal{O}|_{U_\alpha} \to \mathcal{F}_\alpha$ which makes the diagram

$$\begin{tikzcd}
\mathcal{O}|_{U_\alpha \cap U_\beta} \arrow[r, "\pi_\alpha^\sharp"] \arrow[d, equal] & \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \arrow[d, "\varphi^\sharp_{\beta\alpha}"] \\
\mathcal{O}|_{U_\alpha \cap U_\beta} \arrow[r, "\pi_\beta^\sharp"] & \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}
\end{tikzcd} \tag{2.7}$$

commute. Here are our checks.

- We build our isomorphisms of ringed spaces $(\iota_\alpha, \iota_\alpha^\sharp) \colon (X_\alpha, \mathcal{O}_\alpha) \to (U_\alpha, \mathcal{O}|_{U_\alpha})$. Above we noted we already have homeomorphisms $\iota_\alpha \colon X_\alpha \to U_\alpha$. By Lemma 2.8, it suffices to exhibit an isomorphism $\iota_\alpha^\sharp \colon \mathcal{O}|_{U_\alpha} \to (\iota_\alpha)_*\mathcal{O}_\alpha$, but $\mathcal{F}_\alpha = (\iota_\alpha)_*\mathcal{O}_\alpha$, so we just set $\iota_\alpha^\sharp := \pi_\alpha^\sharp$.
- We show that $(X, \mathcal{O})$ is a scheme. Indeed, any $p \in X$ has some α such that $p \in U_\alpha$. Pulling back, find some affine open subset $U \subseteq X_\alpha$ containing $\iota_\alpha^{-1}(p)$ with $(\mu, \mu^\sharp) \colon (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong (U, \mathcal{O}_\alpha|_U)$.

  Finishing, we set $V := \iota_\alpha(U)$. Then we have a homeomorphism
  $$\operatorname{Spec} A \overset{\mu}{\cong} U \overset{\iota_\alpha}{\cong} V$$
  and isomorphisms
  $$\mathcal{O}(V') \overset{\mu_V^\sharp}{\cong} \mathcal{O}_\alpha(\iota_\alpha^{-1}V') \overset{\mu^\sharp_{\iota_\alpha^{-1}V'}}{\cong} \mathcal{O}_{\operatorname{Spec} A}(\mu^{-1}\iota_\alpha^{-1}V'),$$
  for any $V' \subseteq V$; these isomorphisms are natural in V because both $\mu^\sharp$ and $(\iota_\alpha)_*\mu^\sharp$ are natural transformations. (Checking this is a matter of writing down the appropriate square.) It follows that we have an isomorphism of ringed spaces
  $$(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong (V, \mathcal{O}|_V)$$
  where $V \subseteq X$ contains p. Thus, we have given p an affine open neighborhood, which is enough.
- We show $(\iota_\alpha, \iota_\alpha^\sharp)|_{X_{\alpha\beta}} = (\iota_\beta, \iota_\beta^\sharp)|_{X_{\beta\alpha}} \circ (\varphi_{\alpha\beta}, \varphi_{\alpha\beta}^\sharp)$. Some trickery is required to make this equality make sense. Namely, note that $\iota_\alpha(U_{\alpha\beta}) = U_\alpha \cap U_\beta$ as discussed earlier; additionally, for each open subset $U \subseteq U_\alpha \cap U_\beta$, we are granted an isomorphism
  $$\iota_\alpha^\sharp \colon \mathcal{O}(U) \cong \mathcal{O}_\alpha(\iota_\alpha^{-1}U)$$
  which is natural by the naturality of $\iota_\alpha^\sharp$, so this assembles into a sheaf isomorphism $\iota_\alpha^\sharp \colon \mathcal{O}|_{U_\alpha \cap U_\beta} \cong (\iota_\alpha)_*(\mathcal{O}_\alpha|_{U_{\alpha\beta}})$. In total, we see that we have assembled an isomorphism
  $$(\iota_\alpha, \iota_\alpha^\sharp) \colon (U_{\alpha\beta}, \mathcal{O}_\alpha|_{U_{\alpha\beta}}) \cong (U_\alpha \cap U_\beta, \mathcal{O}|_{U_\alpha \cap U_\beta}).$$
  Doing the same for $(\iota_\beta, \iota_\alpha^\beta)$, we are being asked for the diagram
  $$\begin{tikzcd}
  (U_{\alpha\beta}, \mathcal{O}_\alpha|_{U_{\alpha\beta}}) \arrow[r, "(\iota_\alpha, \iota_\alpha^\sharp)"] \arrow[d, "(\varphi_{\alpha\beta}, \varphi_{\alpha\beta}^\sharp)"'] & (U_\alpha \cap U_\beta, \mathcal{O}|_{U_\alpha \cap U_\beta}) \arrow[d, equal] \\
  (U_{\beta\alpha}, \mathcal{O}_\beta|_{U_{\beta\alpha}}) \arrow[r, "(\iota_\beta, \iota_\beta^\sharp)"] & (U_\alpha \cap U_\beta, \mathcal{O}|_{U_\alpha \cap U_\beta})
  \end{tikzcd}$$

to commute. Well, on topological spaces, we have $\iota_\alpha = \iota_\beta \circ \varphi_{\alpha\beta}$ by (2.6). On sheaves, we are asking for the diagram

$$\begin{array}{ccc} \mathcal{O}|_{U_\alpha\cap U_\beta} & \xrightarrow{\iota_\beta^\sharp} & (\iota_\beta)_*(\mathcal{O}_\beta|_{U_{\beta\alpha}}) \\ \| & & \downarrow{\scriptstyle \varphi_{\alpha\beta}^\sharp} \\ \mathcal{O}|_{U_\alpha\cap U_\beta} & \xrightarrow{\iota_\alpha^\sharp} & (\iota_\alpha)_*(\mathcal{O}_\alpha|_{U_{\alpha\beta}}) \end{array}$$

to commute. Well, we note that any open set $U \subseteq X_\beta$ will have the equality of sheaves

$$(\iota_\beta)_*(\mathcal{O}_\beta|_{U_{\beta\alpha}})(U) = \mathcal{O}_\beta(\iota_\beta^{-1}(U_{\beta\alpha}\cap U)) = \mathcal{O}_\beta(U_\alpha\cap U_\beta\cap \iota_\beta^{-1}(U)) = ((\iota_\beta)_*\mathcal{O}_\beta)|_{U_\alpha\cap U_\beta}(U),$$

and similar for α, so we are really staring at the commuting square (2.7).

The above checks complete the proof. ■

2.2.4 Projective Space by Gluing

Fix a ring R. Let's define $\mathbb{P}^n_R$ by gluing $n+1$ different affine sets $\mathbb{A}^n_R$. Intuitively, we want to define projective space to have the topological space of homogeneous coordinates

$$[X_0 : X_1 : \ldots : X_n],$$

and we would like the ith affine piece of this space to be given by

$$\left(\frac{X_0}{X_i}, \frac{X_1}{X_i}, \ldots, \frac{X_n}{X_i}\right).$$

Notably, this has killed a coordinate with $X_i/X_i = 1$.

As such, to glue properly, we define the ith affine piece to be

$$X_i := \operatorname{Spec} R\left[x_{0/i}, x_{1/i}, \ldots, x_{(i-1)/i}, x_{(i+1)/i}, \ldots, x_{n/i}\right].$$

To glue this X_i piece to the X_j piece, we need to force $x_{j/i}$ to be nonzero (namely, to invert it), so we look at the open subscheme

$$X_{ij} := \operatorname{Spec} R\left[x_{0/i}, x_{1/i}, \ldots, x_{(i-1)/i}, x_{(i+1)/i}, \ldots, x_{n/i}, x_{j/i}^{-1}\right].$$

To glue these open subschemes directly, we remember that $x_{i/j}$ is supposed to mean X_i/X_j as a quotient not always defined, so we define our isomorphism as

$$\begin{aligned} f_{ji} \colon X_{ij} &\to X_{ji} \\ x_{k/i} &\mapsto x_{k/j}/x_{i/j} \end{aligned}$$

from which we can pretty directly check the cocycle condition. (The f_{ji} is an isomorphism because we can see its inverse is f_{ij}.) This gives us our definition.

Definition 2.49 (Projective space)**.** Fix a ring R. Then we define *projective n-space over R*, denoted $\mathbb{P}^n_R$ to be the scheme obtained from the above gluing data.

Remark 2.50. One can see that

$$\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R) = R.$$

Indeed, any global section $s \in \mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R)$ must restrict to each affine open set X_i; however, looking at our gluing data X_i and X_j tells us that we cannot use a non-constant polynomial because having any positive degree (in, say $x_{i/j}$), would induce a denominator when pushing to X_i. Thus, $\mathbb{P}^n_R$ is not an affine scheme unless $n = 0$, for we would be asserting that $\mathbb{P}^n_R$ is the affine scheme $\operatorname{Spec} R$.

2.2.5 Graded Rings

Another way to look at projective schemes is to approach them from graded rings.

Definition 2.51 (Graded rings)**.** Fix a commutative monoid $(M,+)$. An M-*graded ring* S is a ring S equipped with a decomposition of abelian groups

$$S=\bigoplus_{d\in M}S_d$$

such that $S_k\cdot S_k\subseteq S_{k+\ell}$ for any $k,\ell\in M$. By convention, a *graded ring* will be an $\mathbb{N}$-graded ring.

Remark 2.52. If S is an M-graded ring, then $S_0\subseteq S$ is a subring. Here are our checks.

- Note $0\in S_0$ and that S_0 is closed under addition and subtraction because S_0 is an abelian group.
- We check $1\in S_0$. Well, suppose $1=\sum_{d\in M}^N s_d$. Observe that $s_0s_d\in S_0S_d\subseteq S_d$ for each $d\in M$, so by comparing degrees, we are forced to have $s_0s_d=s_d$. But then

$$s_0=s_0\cdot 1=s_0\sum_{d\in M}s_d=\sum_{d\in M}s_d=1,$$

  so $1=s_0\in S_0$ follows.
- For $s,s'\in S_0$, we see $ss'\in S_0S_0\subseteq S_0$.

Remark 2.53. Certainly, if S is an $\mathbb{N}$-graded ring, then S is a $\mathbb{Z}$-graded ring by just setting $S_d=0$ for $d<0$.

Example 2.54. Take $S:=R[x_0,\ldots,x_n]$ graded by degree; namely, S_k is the set of homogeneous polynomials of degree k with 0. Because $\deg(fg)=\deg f+\deg g$, we do indeed have $S_kS_\ell\subseteq S_{k+\ell}$.

Example 2.55. If S is a graded ring, and $f\in S_n$, then S_f is a $\mathbb{Z}$-graded ring, where we are allowing negative degrees coming from $1/f$.

We will want our ideals to keep track of the grading, so we have the following definition.

Definition 2.56 (Homogeneous element)**.** Fix an M-graded ring S. Then an element $f\in S$ is *homogeneous* if and only if $f\in S_d$ for some $d\in M$. If $s\in S_d\setminus\{0\}$ is nonzero and homogeneous, we set $\deg s:=S_d$.

Definition 2.57 (Homogeneous ideal)**.** Fix an M-graded ring S. An ideal $I\subseteq S$ is *homogeneous* if and only if I is generated by homogeneous elements.

Remark 2.58. Directly from the definition, we can see that the (arbitrary) sum of homogeneous ideals is homogeneous by just taking the union of the homogeneous generators. Also, if $I=(r_\alpha)_{\alpha\in\lambda}$ and $J=(s_\beta)_{\beta\in\kappa}$ are homogeneous ideals, we see

$$IJ=(r_\alpha s_\beta)_{(\alpha,\beta)\in\lambda\times\kappa},$$

so IJ is homogeneous as well; namely, $r_\alpha s_\beta\in S_{\deg r_\alpha}S_{\deg s_\beta}=S_{\deg r_\alpha+\deg s_\beta}$.

This definition of a homogeneous ideal is easy to think about, but it is not yet clear why it "respects the grading."

Lemma 2.59. Fix an M-graded ring S and ideal $I \subseteq S$. The following are equivalent.

(a) I is generated by homogeneous elements.

(b) If $s = \sum_{d \in M} s_d$ lives in I, then $s_d \in I$ for each $d \in M$.

Proof. To see that (b) implies (a), note that I is generated by

$$I = \left(\sum_{d \in M} s_d : \sum_{d \in M} s_d \in I \right) \subseteq \left(s_d : \sum_{d \in M} s_d \in I \right).$$

However, $\sum_{d \in M} s_d \in I$ implies $s_d \in I$ for each $d \in M$, so in fact

$$\left(s_d : \sum_{d \in M} s_d \in I \right) \subseteq I,$$

giving the needed equality. Thus, we have shown I to be generated by homogeneous elements.

We now show that (a) implies (b). Suppose I is generated by the homogeneous elements $\{s_\alpha\}_{\alpha \in \lambda}$, where the degree of s_α is d_α. Now, for any $s \in I$, write $s = \sum_{d \in M} s_d$ for $s_d \in S_d$. Of course, we can also write

$$\sum_{d \in M} s_d = s = \sum_{\alpha \in \lambda} r_\alpha s_\alpha$$

for some $r_\alpha \in S$. Writing $r_\alpha = \sum_{d \in M} r_{\alpha,d}$, we have

$$\sum_{d \in M} s_d = \sum_{\alpha \in \lambda} \sum_{d \in M} r_{\alpha,d} s_\alpha.$$

Comparing the dth degree on both sides, we see that

$$s_d = \sum_{\alpha \in \lambda} r_{\alpha, d_\alpha - d} s_d,$$

which is indeed an element of I. This finishes. ■

Corollary 2.60. Fix an M-graded ring S and homogeneous ideal $I \subseteq S$. Then, setting $I_d := I \cap S_d$, we see S/I is an M-graded ring by $(S/I)_d \simeq S_d/I_d$ for each $d \in M$.

Proof. Note we have the surjection

$$\begin{aligned} S \quad &\simeq \bigoplus_{d \in M} S_d \twoheadrightarrow \bigoplus_{d \in M} S_d/I_d \\ \sum_{d \in M} s_d &\mapsto (s_d)_{d \in M} \mapsto (s_d + I_d)_{d \in M} \end{aligned}$$

which is indeed a surjection because some $(s_d + I_d)_{d \in M} \in \bigoplus_{d \in M} S_d/I_d$ will just lift right back to $(s_d)_{d \in M} \in \bigoplus_{d \in M} S_d$, where $s_d = 0$ if $s_d + I_d = I_d$ (which occurs all but finitely often). Additionally, an element $\sum_{d \in M} s_d \in S$ lives in the kernel of this map if and only if $s_d \in I_d$ for each $d \in M$, which by Lemma 2.59 is equivalent to $\sum_{d \in M} s_d \in I$. So we actually have the isomorphism

$$\begin{aligned} S/I \quad &\simeq \bigoplus_{d \in M} S_d/I_d \\ \sum_{d \in M} s_d + I &\mapsto (s_d + I_d)_{d \in M} \end{aligned}$$

which becomes a grading upon noting that $k,\ell\in M$ with $s_k+I\in(S/I)_k\simeq S_k/I_k$ and $s_\ell+I\in(S/I)_\ell\simeq S_\ell/I_\ell$ will have $s_ks_\ell+I\in(S/I)_{k+\ell}\simeq S_{k+\ell}/I_{k+\ell}$. ■

Here are some other quick facts about homogeneous ideals.

> **Corollary 2.61.** Fix an M-graded ring S and homogeneous ideals $\{I_\alpha\}_{\alpha\in\lambda}$. Then $\bigcap_{\alpha\in\lambda}I_\alpha$ is also a homogeneous ideal.

Proof. Set $I:=\bigcap_{\alpha\in\lambda}I_\alpha$. We use Lemma 2.59. Indeed, if $s=\sum_{d\in M}s_d$ lives in I, then $s\in I_\alpha$ for each $\alpha\in\lambda$, so each $d\in M$ has $s_d\in I_\alpha$ for each $\alpha\in\lambda$. Thus, $s_d\in I$ for each $d\in M$. ■

> **Lemma 2.62.** Fix an M-graded ring S and homogeneous ideal I. Then I is prime if and only if, for any homogeneous elements $ab\in I$, we have $a,b\in I$.

Proof. Certainly if I is prime, then the conclusion holds. Conversely, we need to show that I is prime. Well, suppose $a=\sum_{d\in M}a_d$ and $b=\sum_{d\in M}b_d$ have $ab\notin I$. Expanding,

$$ab=\sum_{d\in M}\left(\sum_{k+\ell=d}a_kb_\ell\right)\notin I,$$

so there is some term $a_kb_\ell\notin I$. Using the hypothesis, we see $a_k\notin I$ and $b_\ell\notin I$, so because I is homogeneous, we conclude $a\notin I$ and $b\notin I$ by Lemma 2.59. ■

> **Lemma 2.63.** Fix an M-graded ring S and homogeneous ideal I. Then
>
> $$\operatorname{rad}I=\bigcap_{\substack{\mathfrak p\supseteq I\\ \mathfrak p\text{ homogeneous}}}\mathfrak p.$$
>
> In particular, $\operatorname{rad}I$ is homogeneous.

Proof. We follow [Kid12]. The main claim is the first one; that $\operatorname{rad}I$ is homogeneous will follow by Corollary 2.61. Now, for any prime ideal $\mathfrak p$ containing I, let $\mathfrak p'$ denote the ideal generated by the homogeneous elements of $\mathfrak p$. We collect the following facts.

- By definition, $\mathfrak p'$ is homogeneous, and $\mathfrak p'\subseteq\mathfrak p$.
- Note $\mathfrak p'$ is prime by Lemma 2.62: given homogeneous elements a,b with $ab\in\mathfrak p'$, we see $ab\in\mathfrak p$, so $a\in\mathfrak p$ or $b\in\mathfrak p$, so $a\in\mathfrak p'$ or $b\in\mathfrak p'$ by definition of $\mathfrak p'$.
- If $s=\sum_{d\in M}s_d$ lives in I, then $s_d\in I\subseteq\mathfrak p$ for each $d\in M$, so $s_d\in\mathfrak p'$ for each $d\in M$, so $s\in\mathfrak p'$. Thus, $I\subseteq\mathfrak p'$.

From the above, we see

$$\operatorname{rad}I=\bigcap_{\mathfrak p\supseteq I}\mathfrak p\supseteq\bigcap_{\mathfrak p\supseteq I}\mathfrak p'\supseteq\bigcap_{\substack{\mathfrak p\supseteq I\\ \mathfrak p\text{ homogeneous}}}\mathfrak p\supseteq\bigcap_{\mathfrak p\supseteq I}\mathfrak p,$$

which is what we wanted. ■

It turns out that some ideals do not carry geometric information.

Definition 2.64 (Irrelevant ideal)**.** Fix a graded ring S. Then the *irrelevant ideal* S_+ is the ideal of S generated by the homogeneous elements of positive degree.

We will see why this ideal is called the irrelevant ideal shortly. For now, note that S_+ is a homogeneous ideal, and because

$$(S_+)_d := S_+ \cap S_d = \begin{cases} 0 & d = 0, \\ S_d & d > 0, \end{cases}$$

we see that

$$S/S_+ \simeq \bigoplus_{d \in \mathbb{N}} (S_d/(S_+)_d) = S_0 \oplus \bigoplus_{d \in \mathbb{N}} 0 \simeq S_0.$$

2.2.6 The Topological Space Proj

Fix a graded ring S. We now construct $\operatorname{Proj} S$. Intuitively, we want to have $\operatorname{Proj} R[x_0, \ldots, x_n] = \mathbb{P}^n_R$ and $\operatorname{Proj} S[x_0, \ldots, x_n]/I = V(I)$ when I is a homogeneous ideal. Rigorously, we are going to retell the affine story but add the word homogeneous everywhere.

Let's speak a little non-rigorously for a moment. In some sense, the point $p = [\lambda_0 : \lambda_1 : \ldots : \lambda_n] \in \mathbb{P}^n_R$ should correspond to the ideal of $R[x_0, \ldots, x_n]$ which cuts out this line. Supposing $\lambda_0 \neq 0$ without loss of generality, we can see that the correct ideal is

$$\mathfrak{m}_p = (\lambda_0 x_1 - \lambda_1 x_0, \lambda_0 x_2 - \lambda_2 x_0, \ldots, \lambda_0 x_n - \lambda_n x_0).$$

In particular, $x_i \in \mathfrak{m}_p$ if and only if $\lambda_i = 0$, so we can encode the condition that $\lambda_i \neq 0$ for some i by requiring $\mathfrak{p} \not\supseteq R[x_0, \ldots, x_n]_+$—namely, our irrelevant ideal $R[x_0, \ldots, x_n]_+$ carves out no points.[1] This gives our definition.

Definition 2.65 (Proj)**.** Given a graded ring S, we define

$$\operatorname{Proj} S := \{\mathfrak{p} \in \operatorname{Spec} S : \mathfrak{p} \text{ homogeneous}, \mathfrak{p} \not\supseteq S_+\}.$$

Having defined a version of our spectrum, we should give it a Zariski topology.

Definition 2.66 (Zariski topology)**.** Fix a graded ring S. Given a homogeneous ideal $\mathfrak{a} \subseteq S$, define

$$V_+(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a}\}.$$

In other words, $V_+(\mathfrak{a}) = V(\mathfrak{a}) \cap \operatorname{Proj} S$.

Remark 2.67. As before, we see homogeneous ideals $\mathfrak{a} \subseteq \mathfrak{b}$ give

$$V_+(\mathfrak{b}) = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{b}\} \subseteq \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a}\} = V_+(\mathfrak{a}).$$

Remark 2.68. In light of Lemma 2.63, we may say

$$V_+(\mathfrak{a}) = V(\mathfrak{a}) \cap \operatorname{Proj} S = V(\operatorname{rad} \mathfrak{a}) \cap \operatorname{Proj} S = V_+(\mathfrak{a}).$$

Here is the check that we have defined a topology.

[1] This is why S_+ is called the irrelevant ideal.

Lemma 2.69. Fix a graded ring S. Then the subsets $\{V_+(\mathfrak{a})\}$ define a topology of closed sets on $\operatorname{Proj} S$. In particular, we have the following.

(a) $V_+(S_+) = \varnothing$ and $V_+((0)) = \operatorname{Proj} S$.

(b) Arbitrary intersection: homogeneous ideals $\{\mathfrak{a}_\alpha\}_{\alpha\in\lambda}$ give $\bigcap_{\alpha\in\lambda} V_+(\mathfrak{a}_\alpha) = V_+\left(\sum_{\alpha\in\lambda}\mathfrak{a}_\alpha\right)$.

(c) Finite union: homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ give $V_+(\mathfrak{ab}) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b})$.

Proof. This largely follows straight from Lemma 1.40.

(a) Note there is no $\mathfrak{p} \in \operatorname{Proj} S$ with $\mathfrak{p} \supseteq S_+$ by construction, so $V_+(S_+) = \varnothing$. Also, all ideals contain (0), so $V_+((0)) = \operatorname{Proj} S$. We also note that S_+ and (0) are both homogeneous ideals.

(b) Using Lemma 1.40, we see

$$V_+\left(\sum_{\alpha\in\lambda}\mathfrak{a}_\alpha\right) = V\left(\sum_{\alpha\in\lambda}\mathfrak{a}_\alpha\right)\cap\operatorname{Proj} S = \left(\bigcap_{\alpha\in\lambda} V(\mathfrak{a}_\alpha)\right)\cap\operatorname{Proj} S = \bigcap_{\alpha\in\lambda}\underbrace{(V(\mathfrak{a}_\alpha)\cap\operatorname{Proj} S)}_{V_+(\mathfrak{a}_\alpha)}.$$

We close by noting that $\sum_{\alpha\in\lambda}\mathfrak{a}_\alpha$ is a homogeneous ideal by Remark 2.58.

(c) Again using Lemma 1.40, we see

$$V_+(\mathfrak{ab}) = V(\mathfrak{ab})\cap\operatorname{Proj} S = (V(\mathfrak{a})\cup V(\mathfrak{b}))\cap\operatorname{Proj} S = \underbrace{(V(\mathfrak{a})\cap\operatorname{Proj} S)}_{V_+(\mathfrak{a})}\cup\underbrace{(V(\mathfrak{b})\cap\operatorname{Proj} S)}_{V_+(\mathfrak{b})}.$$

We close by noting that $\mathfrak{ab}$ is a homogeneous ideal by Remark 2.58. ■

As before, we will have a distinguished base, but we will be a little more careful.

Definition 2.70 (Distinguished open sets)**.** Fix a graded ring S. For a homogeneous element $f \in S_+$, we define

$$D_+(f) \coloneqq \{\mathfrak{p}\in\operatorname{Proj} S : f\notin\mathfrak{p}\}.$$

As before, we see $D_+(f) = D(f)\cap\operatorname{Proj} S$.

Here is the analogue of Remark 1.55.

Lemma 2.71. Fix a graded ring S. The open sets $\{D_+(f)\}_{f\in S_+}$ form a base of the Zariski topology on $\operatorname{Proj} S$.

Proof. Given any open subset $(\operatorname{Proj} S)\setminus V_+(\mathfrak{a})$ and point $\mathfrak{p}\in(\operatorname{Proj} S)\setminus V_+(\mathfrak{a})$, we need to find $f\in S_+$ such that $D_+(f)$ contains $\mathfrak{p}$ and $D_+(f)\subseteq(\operatorname{Proj} S)\setminus V_+(\mathfrak{a})$. In other words, we need $f\notin\mathfrak{p}$ while $V_+(\mathfrak{a})\subseteq V_+((f))$. As such, it will suffice to find $f\notin\mathfrak{p}$ with $f\in\mathfrak{a}$ by Remark 2.67.

Note that $\mathfrak{a}$ is generated by homogeneous elements, so there certainly must exist some homogeneous element in $\mathfrak{a}$ which is not in $\mathfrak{p}$. If this element has positive degree, we are done immediately. Otherwise, suppose for contradiction the only homogeneous elements $f\in\mathfrak{a}\setminus\mathfrak{p}$ have degree zero. Then any homogeneous $s\in S_+$ of positive degree will have

$$fs\in\mathfrak{a}$$

while fs has positive degree, but then we forced ourselves into having $s\in\mathfrak{p}$. Thus, $\mathfrak{p}$ contains all homogeneous elements of S_+, so $\mathfrak{p}\supseteq S_+$ because S_+ is homogeneous (!), which contradicts $\mathfrak{p}\in\operatorname{Proj} S$. ■

2.2.7 Easy Nullstellensatz for Proj

For fun, we take a moment to establish the analogue for Proposition 1.45.

Definition 2.72. Fix a graded ring S. Then, given a subset $Y \subseteq \operatorname{Proj} S$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Remark 2.73. Identically as in Lemma 1.44, we have $X \subseteq Y \subseteq \operatorname{Proj} S$ implies $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = I(X)$.

Remark 2.74. Because the intersection of homogeneous radical ideals is homogeneous (Corollary 2.61) and radical, we see that $I(Y)$ is a homogeneous radical ideal for any $Y \subseteq \operatorname{Proj} S$.

And here is our analogue.

Proposition 2.75. Fix a graded ring S.

(a) Given a homogeneous ideal $\mathfrak{a} \subseteq A$, we have $I(V_+(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$.

(b) Given a subset $X \subseteq \operatorname{Proj} S$, we have $V_+(I(X)) = \overline{X}$.

(c) The functions V_+ and I provide an inclusion-reversing bijection between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Proof. The proof are essentially analogous to Proposition 1.45; we record them for completeness.

(a) Note

$$I(V_+(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V_+(\mathfrak{a})} \mathfrak{a} = \bigcap_{\substack{\mathfrak{p} \supseteq \mathfrak{a} \\ \mathfrak{p} \text{ homogeneous}}} \mathfrak{p} = \operatorname{rad} \mathfrak{a},$$

where the last equality follows from Lemma 2.63.

(b) Using Lemma 2.69, we see

$$\overline{X} = \bigcap_{V_+(\mathfrak{a}) \supseteq X} V_+(\mathfrak{a}) = V_+\left(\sum_{V_+(\mathfrak{a}) \supseteq X} \mathfrak{a} \right).$$

In particular, $X \subseteq V_+(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in X$, which means $\mathfrak{a} \subseteq I(X)$. Thus, $\overline{X} = V\left(\sum_{\mathfrak{a} \subseteq I(X)} \mathfrak{a}\right) = V(I(X))$.

(c) As before, V_+ sends radical homogeneous ideals to closed subsets of $\operatorname{Proj} S$ (by definition of the topology), and I sends closed subsets of $\operatorname{Proj} S$ to radical homogeneous ideals by Remark 2.74. These mappings are inclusion-reversing by Remark 2.67 and Remark 2.73. Lastly, (a) and (b) show that these mappings compose to the identity. ■

2.2.8 The Structure Sheaf for Proj

One can again check that this makes a topology. In fact, given $f \in S$, we can define

$$D_+(f) := (\operatorname{Proj} S) \setminus V((f))$$

and then check that this makes a basis for our topology, essentially for the same reason.

Remark 2.76. One can check that the map

$$\begin{aligned} D_+(f) &\simeq \operatorname{Spec}(S_f)_0 \\ \mathfrak{p} &\mapsto (\mathfrak{p}S_f) \cap (S_f)_0 \end{aligned}$$

is a homeomorphism.

As such, we give the open set $D_+(f)$ the structure sheaf $\mathcal{O}_{\operatorname{Spec}((S_f)_0)}$. To glue these together, we choose the affine open subset

$$\operatorname{Spec}((S_f)_0)_{g^{\deg f}/f^{\deg g}} \subseteq \operatorname{Spec}(S_f)_0$$

and identify them with $\operatorname{Spec}(S_{fg})_0$.

2.3 September 12

The classroom is emptier than usual.

2.3.1 Projective Schemes from Proj

We quickly finish our definition of a projective scheme.

Definition 2.77 (Projective scheme)**.** Fix a ring R. A scheme $(X, \mathcal{O}_X)$ is a *projective scheme over* R if and only if $(X, \mathcal{O}_X)$ is isomorphic (as schemes) to some

$$\operatorname{Proj} R[x_0, \ldots, x_n]/I$$

for a homogeneous ideal $I \subseteq R[x_0, \ldots, x_n]$. Equivalently, $(X, \mathcal{O}_X)$ is isomorphic to some $\operatorname{Proj} S$, where S is a finitely generated graded R-algebra.

Intuitively, the ring map

$$R[x_0, \ldots, x_n] \twoheadrightarrow R[x_0, \ldots, x_n]/I$$

will induce an embedding from $(X, \mathcal{O}_X)$ into $\mathbb{P}^n_R$. So a projective scheme is really just one which has an embedding into projective space.

Remark 2.78. It is not totally trivial that we may allow S to be finitely generated from elements outside S_1. See [Vak17, Section 7.4.4].

Here is another equivalent definition.

Definition 2.79 (Projective scheme)**.** Fix a ring R. A scheme $(X, \mathcal{O}_X)$ is a *projective scheme over* R if and only if there is a "closed embedding" $X \hookrightarrow \mathbb{P}^n_R$ of schemes.

We haven't defined a closed embedding yet, but we will do this soon.

2.3.2 Topological Adjectives

We start by describing a scheme by focusing on its topological space.

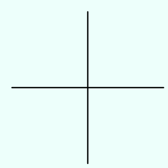
Definition 2.80 (Connected)**.** A scheme $(X, \mathcal{O}_X)$ is *connected* if and only if X is connected as a topological space. In other words, if $X = V_1 \sqcup V_2$ for closed subsets $V_1, V_2 \subseteq X$, then one of the $V_1 = X$ or $V_2 = X$.

Definition 2.81 (Irreducible)**.** A scheme $(X, \mathcal{O}_X)$ is *irreducible* if and only if X is irreducible as a topological space. In other words, we require X to be nonempty, and if $X = V_1 \cup V_2$ for closed subsets $V_1, V_2 \subseteq X$, then one of the $V_1 = X$ or $V_2 = X$.

Example 2.82. Let X be a topological space. Then, for any $x \in X$, the subset $\overline{\{x\}} \subseteq X$ is irreducible: if closed sets $V_1, V_2 \subseteq X$ have $\overline{\{x\}} \in V_1 \cup V_2$, then $x \in V_1$ or $x \in V_2$, so $\overline{\{x\}} \subseteq V_1$ or $\overline{\{x\}} \subseteq V_2$.

Example 2.83. Projective space $\mathbb{P}^n_k = \operatorname{Proj} k[x_1, \ldots, x_{n+1}]$ is irreducible. Indeed, the homogeneous prime ideal (0) certainly does not contain $k[x_1, \ldots, x_{n+1}]_+$, so $(0) \in \mathbb{P}^n_k$. However, we claim $\mathbb{P}^n_k = \overline{\{(0)\}}$, which will finish by Example 2.82. Indeed, setting $\overline{\{0\}} = V_+(\mathfrak{a})$ for an ideal $\mathfrak{a}$, we see $(0) \supseteq \mathfrak{a}$, so any homogeneous $\mathfrak{p}$ has $\mathfrak{p} \supseteq (0) \supseteq \mathfrak{a}$, so $\mathbb{P}^n_k \subseteq \overline{\{(0)\}}$.

Non-Example 2.84. The scheme $\operatorname{Spec} k[x,y]/(xy)$ is connected but not irreducible. The picture is as follows.



We will explain this example in more detail shortly in Remark 2.95.

We like compact topological spaces, so here is the scheme analogue.

Definition 2.85 (Quasicompact)**.** A scheme $(X, \mathcal{O}_X)$ is *quasicompact* if and only if any open cover of the topological space X has a finite subcover.

Example 2.86. The scheme $\operatorname{Spec} A$ is quasicompact.

Non-Example 2.87. The infinite disjoint union $X := \bigsqcup_{i \in \mathbb{N}} y \operatorname{Spec} \mathbb{Z}$ is not quasicompact. Explicitly, this scheme is constructed by setting $X_i := \operatorname{Spec} \mathbb{Z}$ for any natural $i \in \mathbb{N}$ and setting all gluing data for Proposition 2.48 to be empty; for example, the cocycle condition is satisfied because the intersections X_{ij} are all empty.

Then X has an open cover $\{U_i\}_{i \in \mathbb{N}}$ where $U_i \cap U_j = \varnothing$ for each i, j by construction of the gluing; however, any finite subcollection $\{U_{i_k}\}_{k=1}^n$ has a maximum index j and therefore does not cover U_j and thus does not cover X.

Non-Example 2.88. The scheme $\operatorname{Proj} k[x_1, x_2, \ldots]$ is not quasicompact.

2.3.3 Components

Having discussed the entire topological space, we might be interested in studying some controlled subspaces.

Definition 2.89 (Connected component)**.** Fix a topological space X. A *connected component* is a maximal connected subset of X.

Definition 2.90 (Irreducible component)**.** Fix a topological space X. An *irreducible component* is a maximal irreducible subset of X.

Here are some quick facts.

Lemma 2.91. Fix a topological space X.

(a) If a subset $V \subseteq X$ is irreducible, then V is connected.

(b) If a subset $V \subseteq X$ is irreducible (respectively, connected), then so is $\overline{V}$.

(c) All points $x \in X$ are contained in an irreducible component. Also, all points of $x \in X$ are contained in a connected component.

Proof. We go one at a time.

(a) Suppose that $V \subseteq V_1 \sqcup V_2$ where $V_1, V_2 \subseteq X$ are closed subsets. Being irreducible forces $V \subseteq V_1$ or $V \subseteq V_2$, so connectivity of V follows.

(b) We have two claims to show.

- Take V irreducible so that we want to show $\overline{V}$ is irreducible. Suppose $\overline{V} \subseteq V_1 \cup V_2$ where $V_1, V_2 \subseteq X$ are closed. Then $V \subseteq V_1 \cup V_2$, so $V \subseteq V_1$ or $V \subseteq V_2$, so properties of the closure promise $\overline{V} \subseteq V_1$ or $\overline{V} \subseteq V_2$.
- Take V connected so that we want to show $\overline{V}$ is connected. Well, replace the $\cup$ in the previous proof with a $\sqcup$, and the proof goes through verbatim.

(c) Observe that $\{x\}$ is irreducible: if $\{x\} \subseteq V_1 \cup V_2$ where $V_1, V_2 \subseteq X$ are closed, then $x \in V_1$ or $x \in V_2$, so $\{x\} \subseteq V_1$ or $\{x\} \subseteq V_2$.

We now apply Zorn's lemma twice.

- Let $\mathcal{I}_x$ denote the set of irreducible subsets of X containing x. We need to show that $\mathcal{I}_x$ has a maximal element, which will finish because any maximal element of $\mathcal{I}_x$ will be maximal among all irreducible subsets.

  Note $\{x\} \in \mathcal{I}_x$ means that $\mathcal{I}_x$ is nonempty. We now show that $\mathcal{I}_x$ satisfies the ascending chain condition: given a totally ordered set λ and nonempty ascending chain $\{V_\alpha\}_{\alpha\in\lambda} \subseteq \mathcal{I}_x$, we claim that
  $$V \coloneqq \bigcup_{\alpha\in\lambda} V_\alpha$$
  and contains x. That $x \in V$ is clear because x lives in any of the V_α. To see irreducibility, suppose that $V \subseteq V_1 \cup V_2$.

  If $V \subseteq V_1$, then we are done, so suppose that we can find $p \in V \setminus V_1$. This means that $p \in V_\beta \setminus V_1$ for some $\beta \in \lambda$, so $p \in V_\beta \setminus V_1$ for all $\alpha \geq \beta$. However, $V_\alpha \subseteq V_1 \cup V_2$ still even though $V_1 \not\subseteq V_1$, so we must instead have
  $$V_\alpha \subseteq V_2$$
  for all $\alpha \geq \beta$. It follows $V = \bigcup_{\alpha\geq\beta} V_\alpha \subseteq V_2$.
- Let $\mathcal{C}_x$ denote the set of connected subsets of X containing x. We actually claim that
  $$V \coloneqq \bigcup_{C\in\mathcal{C}_x} C$$
  is connected. This will finish because V is a connected component containing x: if $V \subseteq V'$ with V' connected, then $x \in V'$, so $V' \in \mathcal{C}_x$, so $V' \subseteq V$.

  We now check V is connected. Suppose $V \subseteq V_1 \sqcup V_2$ for closed subsets $V_1, V_2 \subseteq X$.

The main point is that $x \in V_1$ or $x \in V_2$. Without loss of generality, take $x \in V_1$ so that $x \notin V_2$. Now, any $C \in \mathcal{C}_x$ has $C \subseteq V_1 \sqcup V_2$, so $C \subseteq V_1$ or $C \subseteq V_2$. However, $x \in C \setminus V_2$, so we must have $C \subseteq V_1$ instead, meaning that actually $V \subseteq C_1$. ■

Remark 2.92. It follows from the above proof that any connected subset C of x is contained in the connected component of x.

Here is another nice result.

Proposition 2.93. If X is an irreducible topological space, then all nonempty open subsets $U \subseteq X$ have U irreducible and $\overline{U}$ dense in X.

Proof. We have two claims to show.

- We show U is irreducible. Suppose $U \subseteq V_1 \cup V_2$ for closed subsets $V_1, V_2 \subseteq X$. It follows that

$$X \subseteq \big((X \setminus U) \cup V_1\big) \cup V_2$$

  has covered X by closed subsets. It follows that either $V_2 = X$ (and hence covers U) or $(X \setminus U) \cup V_1 = X$ (and so $V_1 \supseteq U$).

- We show $\overline{U} = X$. Indeed, we can cover X by closed sets as

$$X = (X \setminus U) \cup \overline{U},$$

  so either $X \setminus U = X$, which is impossible because U is nonempty, or $\overline{U} = X$, which finishes. ■

Even though irreducible components are a little weird in typical point-set topology, they are of interest in scheme theory.

Lemma 2.94. Fix a ring A.

(a) Given an ideal $I \subseteq A$, the subset $V(I) \subseteq \operatorname{Spec} A$ is irreducible if and only if $\operatorname{rad} I$ is prime.

(b) The irreducible components of X are

$$\{V(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A \text{ is a minimal prime}\}.$$

Proof. As usual, we go in sequence.

(a) We have two claims to show.

- Suppose that $\mathfrak{p} := \operatorname{rad} I$ is prime. Then Proposition 1.45 tells us

$$V(I) = V(\operatorname{rad} I) = V(\mathfrak{p}) = V(I(\{\mathfrak{p}\})) = \overline{\{\mathfrak{p}\}},$$

  which is irreducible by Example 2.82.

- Suppose that $V(I)$ is irreducible; by replacing I with $\operatorname{rad} I$, we may assume that I is radical. We want to show that I is prime. Well, if $a, b \in A$ have $ab \in I$, we want to show $a \notin I$ and $b \notin I$. Well, $ab \in I$ means any $\mathfrak{p}$ containing $V(I)$ contains (ab), so

$$V(I) \subseteq V(ab) = V((a)) \cup V((b))$$

  using Lemma 1.40. Because $V(I)$ is irreducible, we conclude that $V(I) \subseteq V((a))$ without loss of generality. Thus, $\operatorname{rad}((a)) \subseteq \operatorname{rad} I = I$ by Proposition 1.45, so $a \in I$.

(b) The inclusion reversing-bijection of Proposition 1.45 takes prime ideals $\mathfrak{p} \subseteq A$ to $V(\mathfrak{p})$, which we have seen is an irreducible closed subset; and it takes closed subsets $X \subseteq \operatorname{Spec} A$, which we know must take the form $V(\mathfrak{p})$ for a prime $\mathfrak{p}$, to a prime ideal $I(X) = I(V(\mathfrak{p})) = \operatorname{rad} \mathfrak{p} = \mathfrak{p}$.

Thus, the inclusion-reversing bijection restricts to an inclusion-reversing bijection between prime ideals of A and irreducible closed subsets of $\operatorname{Spec} A$. Thus, the maximal irreducible closed subsets of $\operatorname{Spec} A$ correspond (under this bijection) to minimal prime ideals of A.

The claim follows, upon remarking that irreducible components are equal to their closure and hence closed (by Lemma 2.91 (b)), so maximal irreducible closed subsets are just maximal irreducible subsets. ■

Remark 2.95. We are now ready to explain Non-Example 2.84.

- Not irreducible: note that any prime $\mathfrak{p} \in \operatorname{Spec} k[x, y]/(xy)$ contains $xy = 0 \in \mathfrak{p}$, so $(x) \subseteq \mathfrak{p}$ or $(y) \subseteq \mathfrak{p}$. On the other hand, (x) and (y) are primes (with quotients isomorphic to $k[t]$), so (x) and (y) are our minimal primes. Thus, Lemma 2.94 tells us that $V((x))$ and $V((y))$ are our irreducible components; in particular, the entire space is not irreducible.
- Connected: note $(x, y) \in V((x))$ and $(x, y) \in V((y))$, so the connected component of (x, y) contains $V((x)) \cup V((y))$, which is the entire space because we have taken the union of our irreducible components (and all points live in some irreducible component by Lemma 2.91). So the full space is connected.

2.3.4 Closed and Generic Points

We like our topological spaces to be Hausdorff, but we have seen that this need not happen in our schemes. So let's keep track of the good points we try to be Hausdorff

Definition 2.96 (Closed point)**.** Fix a topological space X. Then a point $x \in X$ is a *closed point* if and only if $\overline{\{x\}} = \{x\}$.

In the variety setting, we are more interested in counting closed points, which correspond to the "actual" points on our variety. As such, we might hope that we have "lots" of closed points in our schemes, and under suitable smallness conditions, we do.

Lemma 2.97. Let $(X, \mathcal{O}_X)$ be a quasicompact scheme. Then any nonempty closed subset $V \subseteq X$ contains a closed point.

Proof. Note that this is essentially ring theory for affine schemes: for an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we see that a closed subset $V(I) \subseteq \operatorname{Spec} A$ being nonempty forces I to be proper, so I is contained in some maximal ideal $\mathfrak{m}$. So $\mathfrak{m} \in V(I)$ while Proposition 1.45 says

$$\overline{\{\mathfrak{m}\}} = V(I(\{\mathfrak{m}\})) = V(\mathfrak{m}) = \{\mathfrak{m}\}$$

because $\mathfrak{m}$ is maximal.

We now attack the general case. Because X has an affine open cover, the quasicompactness condition gives V (which is closed in a quasicompact space and hence quasicompact) a finite affine open cover $\{U_i\}_{i=1}^n$ so that we have rings A_i such that

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$$

for each i. We may assume that none of the $\{V\cap U_i\}_{i=1}^n$ is contained in the union of the other, for otherwise we could remove the offending U_i.[2] Now,

$$U_1\cap\left(V\cap\bigcap_{i=1}^n(X\setminus U_i)\right)$$

is a closed subset of U_1, so because $(U_1,\mathcal{O}_X|_{U_1})$ is an affine scheme, it will have a closed point $p\in U_1$.

Notably, $p\in V$ by construction, so it remains to show that $\{p\}\subseteq X$ is closed. Well, $\{p\}\subseteq U_1$ is closed, so $U_1\setminus\{p\}\subseteq U_1$ is open, so there is some open set $U'\subseteq X$ such that $U'\cap U_1=U_1\setminus\{p\}$. It follows that

$$X\setminus\{p\}=(U_1\setminus\{p\})\cup\bigcup_{i=2}^n U_i$$

because $p\notin U_i$ for any $i\neq 1$; in particular, $X\setminus\{p\}\subseteq X$ is open, finishing. ■

Remark 2.98. Sadly, there are examples of schemes with no closed points.

Having kept track of our closed points, we don't want to shame our "unclosed points," so we give them a name as well.

Definition 2.99 (Generic point)**.** Fix a topological space X. Then a point $x\in X$ is a *generic point* of an irreducible subset $V\subseteq X$ if and only if $V=\overline{\{x\}}$.

Example 2.100. Given a ring A, the point $\mathfrak{p}\in\operatorname{Spec}A$ is the (unique!) generic point of $V(\mathfrak{p})$. Indeed, certainly $V(\mathfrak{p})=V(I(\{\mathfrak{p}\}))=\overline{\{\mathfrak{p}\}}$. On the other hand, if some $\mathfrak{q}$ has

$$V(\mathfrak{p})=\overline{\{\mathfrak{q}\}}=V(\mathfrak{q}),$$

then $\mathfrak{p}\supseteq\mathfrak{q}$ and $\mathfrak{q}\supseteq\mathfrak{p}$, so $\mathfrak{p}=\mathfrak{q}$.

Example 2.101. Fix an integral domain A. Then (0) is the generic point for $\operatorname{Spec}A$. Notably, $\overline{\{(0)\}}=V((0))=\operatorname{Spec}A$.

The relationship between generic points will be important to keep track of.

Definition 2.102 (Specialization, generalization)**.** Fix a topological space X and two points $x,y\in X$. We say that x is a *specialization* of y (or equivalently, y is a *generalization* of x) if and only if $x\in\overline{\{y\}}$.

Example 2.103. Given a ring A, we see that $\mathfrak{q}\in\overline{\{\mathfrak{p}\}}=V(\mathfrak{p})$ if and only if $\mathfrak{q}\supseteq\mathfrak{p}$.

This provides a sort of ordering on our space. Closed points are the most "specific," so let's keep track of the most generic.

Definition 2.104 (Generic point)**.** Fix a topological space X. A point $p\in X$ is a *generic point* if and only if the only point specializing to $\overline{\{p\}}$ is p.

We saw in Example 2.100 that in fact every irreducible closed subset had a unique generic point. This can be extended so schemes.

[2] Here we implicitly use the fact that there are only finitely many U_i.

Lemma 2.105. Fix a scheme $(X, \mathcal{O}_X)$. Then any nonempty irreducible closed subset $Z \subseteq X$ has a unique generic point $p \in X$. In other words, we can write any nonempty irreducible closed subset $Z \subseteq X$ as $Z = \overline{\{p\}}$ for some $p \in X$.

Proof. Give X an affine open cover $\mathcal{U}$ so that each $U \in \mathcal{U}$ has $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A_U, \mathcal{O}_{\operatorname{Spec} A_U})$ for some ring A_U. Now, let $\mathcal{V}$ be the open sets $U \in \mathcal{U}$ such that $Z \cap U \neq \varnothing$. As such,

$$Z = \bigcup_{V \in \mathcal{V}} (Z \cap V).$$

Now, $(V, \mathcal{O}_X|_V)$ is an affine scheme for each $V \in \mathcal{V}$. Further, $Z \cap V \subseteq V$ is a closed subset by the induced topology, and it is irreducible because $Z \cap V \subseteq (V_1 \cap V) \cup (V_2 \cap V)$ for closed subsets $V_1, V_2 \subseteq V$ tells us that

$$Z \subseteq (X \setminus V) \cup V_1 \cup V_2,$$

so irreducibility of Z and $Z \cap V \neq \varnothing$ forces $Z \subseteq V_1$ or $Z \subseteq V_2$.

Thus, Example 2.100 promises a unique point $p_V \in Z \cap V$ such that $Z \cap V = \overline{\{p_V\}} \cap V$ for each $V \in \mathcal{V}$. Fixing some p_V, we claim that $Z \cap W = \overline{\{p_V\}} \cap W$ for each $W \in \mathcal{V}$. Indeed, note

$$Z \cap W \subseteq (Z \cap V \cap W) \cup (W \setminus V) \subseteq (\overline{\{p_V\}} \cap W) \cup (W \setminus V).$$

The right-hand side exhibits $Z \cap W$ as the union of two closed subsets of W, so the irreducibility of $Z \cap W$ tells us $Z \cap W \subseteq (\overline{\{p_V\}} \cap W)$ or $Z \cap W \subseteq (W \setminus V)$. The second case would imply $Z \subseteq (X \setminus V) \cup (X \setminus W)$, which by irreducibility of Z forces $Z \cap V = \varnothing$ or $Z \cap W = \varnothing$, which is assumed false. So instead, we have

$$Z \cap W \subseteq \overline{\{p_V\}} \cap W,$$

but of course $p_V \in Z$ forces the other inclusion.

It follows that

$$Z = \bigcup_{W \in \mathcal{V}} (Z \cap W) = \bigcup_{W \in \mathcal{V}} (\overline{\{p_V\}} \cap W) \subseteq \overline{\{p_V\}}.$$

But $p_V \in Z$ and the fact that Z is closed gives the other inclusion, so we conclude $Z = \overline{\{p_V\}}$. So we have indeed given Z some generic point.

It remains to show that this generic point is unique. Well, suppose $p, q \in X$ have $\overline{\{p\}} = \overline{\{q\}}$. The affine open cover $\mathcal{U}$ from earlier grants us some open set U containing q. Note that $p \notin U$ would imply that $\{p\} \subseteq X \setminus U$ and so $\overline{\{p\}} \subseteq X \setminus U$, meaning that $q \notin \overline{\{p\}}$, which is assumed false. So we must have $p \in U$ as well. But then

$$\overline{\{p\}} \cap U = \overline{\{q\}} \cap U$$

tells us that $p = q$ by the uniqueness of generic points of irreducible closed subsets in affine schemes, from Example 2.100. ■

Remark 2.106. It is not in general case that nonempty irreducible closed subsets of a topological space X can be uniquely written as $\overline{\{x\}}$ for some $x \in X$. Here are two examples.

- If $X = \mathbb{R}$ has the indiscrete topology, the closure of any point is the full space X.
- If $X = \mathbb{R}$ has the cofinite topology, the full space X is irreducible (because all proper closed subsets are finite, so the finite union of proper closed subsets cannot cover X), but X is not the closure of any point (because all points are closed).

2.3.5 Noetherian Conditions

Noetherian rings are good, so we will want to push this to our schemes as well.

Definition 2.107 (Locally Noetherian)**.** A scheme $(X, \mathcal{O}_X)$ is *locally Noetherian* if and only if X has an open cover $\mathcal{U}$ where each $U \in \mathcal{U}$ has $(U, \mathcal{O}_X|_U)$ isomorphic to the affine scheme of a Noetherian ring.

Noetherian is about making infinite things finite, so we want to add a quasicompact condition this.

Definition 2.108 (Noetherian)**.** A scheme $(X, \mathcal{O}_X)$ is *Noetherian* if and only if X is quasicompact and locally Noetherian.

Example 2.109. The scheme X from Non-Example 2.87 is locally Noetherian (as the infinite disjoint union of $\operatorname{Spec} \mathbb{Z}$s, and $\mathbb{Z}$ is Noetherian), but X is not quasicompact and hence not Noetherian.

There is also a separate notion of a Noetherian topological space.

Definition 2.110 (Noetherian)**.** A topological space X is *Noetherian* if and only if the closed subsets of X satisfy the descending chain condition.

Example 2.111. If A is a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space: given a totally ordered set λ, any descending chain of closed subsets $\{V_\alpha\}_{\alpha\in\lambda}$ gives an ascending chain of A-ideals $\{I(V_\alpha)\}_{\alpha\in\lambda}$. Because A is Noetherian, $\{I(V_\alpha)\}_{\alpha\in\lambda}$ must stabilize past some β, so for $\alpha \geq \beta$, we have $I(V_\alpha) = I(V_\beta)$, so

$$V_\alpha = \overline{V_\alpha} = V(I(V_\alpha)) = V(I(V_\beta)) = \overline{V_\beta} = V_\beta,$$

so the descending chain of closed subsets stabilizes past β.

Non-Example 2.112. Fix a field k and ring $A := k[x_1, x_2, x_3, \ldots]/\left(x_1, x_2^2, x_3^3, \ldots\right)$. Then $\operatorname{Spec} A$ a Noetherian topological space even though A is not a Noetherian ring.

- Observe that any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ must contain $x_k^k = 0$ for each $k \geq 1$, so $x_k \in \mathfrak{p}$. Thus, $\mathfrak{m} := (x_1, x_2, x_3, \ldots)$ is contained in $\mathfrak{p}$, but $\mathfrak{m}$ is maximal in A because $A/\mathfrak{m} \cong k$. So we conclude that $\mathfrak{p} = \mathfrak{m}$, meaning that $\operatorname{Spec} A = \{\mathfrak{m}\}$ has a single point and so is Noetherian as a topological space.
- The ascending chain
$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$$
shows that A is not a Noetherian ring.

Having seen that Noetherian rings give Noetherian spaces, we might hope that a similar result holds for schemes. As in Lemma 2.97, we must add a quasicompactness hypothesis.

Lemma 2.113. If $(X, \mathcal{O}_X)$ is a Noetherian scheme, then X is a Noetherian topological space.

Proof. As usual, give X an affine open cover $\mathcal{U}$ where the affine schemes are from Noetherian rings; we may make $\mathcal{U}$ finite because X is quasicompact. Now, let λ be a totally ordered set, and pick up some descending chain $\{V_\alpha\}_{\alpha\in\lambda}$ of closed subsets of X.

Now, for each $U \in \mathcal{U}$, so we see that $\{U \cap V_\alpha\}_{\alpha\in\lambda}$ is a descending chain of closed subsets of the affine open set U, so because $(U, \mathcal{O}_X|_U)$ is the affine scheme of a Noetherian ring, Example 2.111 tells us that $\{U \cap V_\alpha\}$ will stabilize after some β_U. Thus, we set

$$\beta := \max\{\beta_U : U \in \mathcal{U}\},$$

which exists because $\mathcal{U}$ is finite. Then any $\alpha \geq \beta$ and $U \in \mathcal{U}$ will have

$$V_\alpha \cap U = V_{\beta_U} \cap U = V_\beta \cap U,$$

so $V_\alpha = V_\beta$ after taking the union over $\mathcal{U}$. So indeed, our descending chain stabilized after β. ■

Remark 2.114. More generally, we have shown that a finite union of Noetherian topological spaces is a Noetherian topological space.

Remark 2.115. There are some nice philosophical remarks in [Vak17, Section 3.6.21] about when we might care about non-Noetherian things.

As a benefit to keeping things finite, we have the following.

Lemma 2.116. Fix a Noetherian topological space X. Then any open subset $U \subseteq X$ is quasicompact.

Proof. We proceed by contraposition. Suppose that we can find an open cover $\mathcal{V}$ of U with no finite subcover. In other words, any finite subset of $\mathcal{V}$ cannot cover U, so we can build some strictly ascending chain of open subsets

$$V_1 \subsetneq V_1 \cup V_2 \subsetneq V_1 \cup V_2 \cup V_3 \subsetneq \cdots$$

by choosing each $V_n \in \mathcal{V}$ inductively to not be contained in $\bigcup_{k<n} V_k$. (If no such V_n existed, then we would have $U = \bigcup_{V \in \mathcal{V}} V \subseteq \bigcup_{k<n} V_k$, granting a finite open subcover.) For brevity, define

$$V_n' := \bigcup_{k \leq n} V_k$$

so that $\{V_n'\}_{n\geq 1}$ is a strictly ascending chain of open subsets. Taking complements, we see that $\{X \setminus V_n'\}_{n\geq 1}$ is a strictly descending chain of closed subsets, which means that our space X is not Noetherian. ■

Here are some applications to affine open subschemes.

Lemma 2.117. Fix a scheme $(X, \mathcal{O}_X)$.

(a) If $(X, \mathcal{O}_X)$ is locally Noetherian, then any open subset $U \subseteq X$ makes a locally Noetherian subscheme $(U, \mathcal{O}_X|_U)$.

(b) If $(X, \mathcal{O}_X)$ is Noetherian, then any open subset $U \subseteq X$ makes a Noetherian subscheme $(U, \mathcal{O}_X|_U)$.

(c) All stalks $\mathcal{O}_{X,p}$ of a locally Noetherian scheme $(X, \mathcal{O}_X)$ are Noetherian rings.

Proof. We go in sequence.

(a) As usual, begin by giving X an affine open cover $\mathcal{U}$, and use the locally Noetherian condition to promise that each $V \in \mathcal{U}$ has $\varphi^V : (V, \mathcal{O}_X|_V) \cong (\operatorname{Spec} A_V, \mathcal{O}_{\operatorname{Spec} A_V})$ for a Noetherian ring A_U.

Now, pick up some $V \in \mathcal{U}$ and focus on $U \cap V$; it suffices to cover $U \cap V$ with affine open subschemes of Noetherian rings. Notably, φ^V makes $U \cap V$ an open subset of $(\operatorname{Spec} A_V, \mathcal{O}_{\operatorname{Spec} A_V})$. In particular, using the distinguished open base of $\operatorname{Spec} A_V$, we can write

$$\varphi^V(U \cap V) = \bigcup_{f \in F_V} D(f),$$

where $F_V \subseteq A_V$ is some subset. Now let $U_f \subseteq U \cap V$ be the pre-image of $D(f)$ under the homeomorphism φ^V, and we can use our isomorphism to give the scheme isomorphisms

$$(U \cap V, \mathcal{O}_X|_U|_{U\cap V}) \cong (D(f), \mathcal{O}_{\operatorname{Spec} A_U}|_{D(f)}) \cong (\operatorname{Spec} A_{U,f}, \mathcal{O}_{\operatorname{Spec} A_{U,f}}),$$

where we have used Lemma 2.9 for the first isomorphism and Lemma 2.16 for the second isomorphism. Now, $A_{U,f}$ is the localization of the Noetherian ring and hence Noetherian. This is what we wanted.

(b) This follows directly from (a) and Lemma 2.116: by (a), we see that $(U, \mathcal{O}_X|_U)$ is locally Noetherian, and because X is a Noetherian topological space by Lemma 2.113, Lemma 2.116 tells us that U is quasicompact as a topological space. It follows $(U, \mathcal{O}_X|_U)$ is a Noetherian scheme.

(c) For our point $p \in X$, the locally Noetherian condition promises an open set $U \subseteq X$ containing p and a Noetherian ring A such that

$$(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

Now, Lemma 1.174 tells us that

$$\mathcal{O}_{X,p} \simeq (\mathcal{O}_X|_U)_p = \mathcal{O}_{\operatorname{Spec} A,p},$$

which is $A_\mathfrak{p}$ by Lemma 1.102. This ring $A_\mathfrak{p}$ is Noetherian because A is Noetherian. ■

2.4 September 14

We continue to study things which go bump in the night.

2.4.1 The Affine Communication Lemma

One annoying thing about our locally Noetherian definition is that we are being forced to choose a very special affine open cover.

In fact, in the discussion that follows, we will want to check many properties as being affine-local, so we will go ahead and state the relevant lemma now.

Lemma 2.118 (Affine communication). Fix a scheme $(X, \mathcal{O}_X)$, and let P be a class of its affine open subschemes. Suppose the following conditions are met.

(i) If $U \in P$, then $U_f \in P$ for any $f \in \mathcal{O}_X(U)$.

(ii) Given an affine open subscheme $U \subseteq X$ with some elements $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ generating $\mathcal{O}_X(U)$, if $U_{f_i} \in P$ for each i, then $U \in P$.

(iii) There is an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ such that $U_\alpha \in P$ for each α.

Then P contains all affine open subsets of X.

Remark 2.119. To explain (ii), we note that $(f_1, \ldots, f_n) = \mathcal{O}_X(U)$ if and only if the U_{f_i} cover U. Set $A := \mathcal{O}_X(U)$ for brevity, and let $\varphi\colon (U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be the canonical isomorphism.

- Suppose the U_{f_i} cover U; set $A := \mathcal{O}_X(U)$ for brevity. Passing through φ, we are told that the sets $D(f_i)$ cover $\operatorname{Spec} A$, and we want to show that $(f_1, \ldots, f_n) = A$. Well, each prime $\mathfrak{p}$ does not contain at least of the f_i by our cover, so $(f_1, \ldots, f_n)$ is contained in no maximal ideal, so this ideal is not proper, so $(f_1, \ldots, f_n) = A$.
- Conversely, suppose $(f_1, \ldots, f_n) = A$. Then no prime $\mathfrak{p} \in \operatorname{Spec} A$ contains all the f_i, so the $D(f_i)$ cover $\operatorname{Spec} A$. Passing back through φ, it follows that the U_{f_i} cover U.

The proof of this result rests on the following result, interesting in its own right.

Lemma 2.120. Fix a scheme $(X, \mathcal{O}_X)$ and affine open subschemes $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong (U, \mathcal{O}_X|_U) \subseteq (X, \mathcal{O}_X)$ and $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \cong (V, \mathcal{O}_X|_V) \subseteq (X, \mathcal{O}_X)$. Then, for any $p \in U \cap V$, we can find an open subset $W \subseteq U \cap V$ containing p such that $W \subseteq U$ and $W \subseteq V$ both make W in to a distinguished open subscheme.

Proof. We follow [Vak17, Proposition 5.3.1]. Without loss of generality, we may use Corollary 2.28 to make $A = \mathcal{O}_X|_U(U) = \mathcal{O}_X(U)$ and $B = \mathcal{O}_X|_V(V) = \mathcal{O}_X(V)$, where the corresponding scheme isomorphisms are the canonical ones.

Namely, as we computed in Lemma 2.25, the underlying homeomorphism associates $D(f) \subseteq \operatorname{Spec} A$ with U_f where $f \in \mathcal{O}_X(U) = A$ (and similar for V), so we will elect to work with the open subsets which look like as our "distinguished open sets." In particular, because we still have a homeomorphism, the fact that distinguished opens $D(f)$ make a basis of the topology on $\operatorname{Spec} A$ means that the open subsets U_f will make a basis of the topology on U.

Now, $U \cap V \subseteq U$ is an open subset containing p, so we begin by giving it a distinguished open $U_f \subseteq U \cap V$ containing p, where $f \in A$. Continuing, $U_f \cap V \subseteq V$ is an open subset containing p, so we may next find some $g \in B$ such that $V_g \subseteq U_f \cap V$ containing p.

It might look like we're in an infinite loop, but we're not: $V_g \subseteq U_f$ means that $g \in \mathcal{O}_X(V)$ will restrict to some element g' in $\mathcal{O}_X(U_f) \simeq D(f) = \mathcal{O}_X(U)_f$, so we let $g' := g''/f^n$. Now, we compute

$$\begin{aligned} V_g &= U_f \cap \{x \in X : g|_x \notin \mathfrak{m}_{X,x}\} \\ &= \{x \in U_f : g''/f^n|_x \notin \mathfrak{m}_{X,x}\} \\ &\overset{*}{=} \{x \in U_f : g''|_x \notin \mathfrak{m}_{X,x}\} \\ &= \{x \in U : (fg'')|_x \notin \mathfrak{m}_{X,x}\} \\ &= U_{fg''}, \end{aligned}$$

which is what we wanted. Namely, $\overset{*}{=}$ holds by thinking as in an affine scheme: a prime $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_X(U)_f$ has $g''/f^n \notin \mathfrak{p}$ if and only if $g'' \notin \mathfrak{p}$, ■

We are now ready to prove Lemma 2.118

Proof of Lemma 2.118. We follow [Vak17, Lemma 5.3.2]. Use (iii) to pick up an affine open cover $\{U_\alpha\}_{\alpha \in \lambda}$ of X.

Now, fix any affine open subscheme $U \subseteq X$, and we need to show $U \in P$. For each α and $p \in U \cap U_\alpha$, we can use Lemma 2.120 to find some open subset $V_{\alpha,p} \subseteq U \cap U_\alpha$ containing which is simultaneously a distinguished open set of both U and U_α. Notably, each $V_{\alpha,p}$ is a distinguished open subset of U_α, so $V_{\alpha,p} \in P$ by (i).

Now, we recognize that we have an open cover

$$U = \bigcup_{\alpha \in \lambda} (U \cap U_\alpha) = \bigcup_{\alpha \in \lambda} \bigcup_{p \in U \cap U_\alpha} V_{\alpha,p}.$$

However, U is an affine open subscheme and therefore quasicompact, so we extract a finite subcover. Because these are all distinguished open sets of U, we may write our open subcover as $\{U_{f_i}\}_{i=1}^n \subseteq P$ where $\{f_i\}_{i=1}^n \subseteq \mathcal{O}_X(U)$.

Continuing, U is covered by the U_{f_i}, so we claim that $(f_1, \ldots, f_n) = \mathcal{O}_X(U)$; set $A := \mathcal{O}_X(U)$ for brevity. Letting $\varphi \colon (U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we are told that the sets $D(f_i)$ cover $\operatorname{Spec} A$, and we want to show that $(f_1, \ldots, f_n) = A$. Well, each prime $\mathfrak{p}$ does not contain at least of the f_i by our cover, so $(f_1, \ldots, f_n)$ is contained in no maximal ideal, so this ideal is not proper, so $(f_1, \ldots, f_n) = A$. It follows that $U \in P$ by (ii). ■

2.4.2 A Better Noetherian

As a first application of Lemma 2.118, we fix our definition of a locally Noetherian scheme.

Proposition 2.121. Fix a locally Noetherian scheme $(X, \mathcal{O}_X)$. Then any affine open subset $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for a ring A has A a Noetherian ring.

Proof. We use Lemma 2.118. Call an affine open subscheme $U \subseteq X$ "good" if and only if $\mathcal{O}_X(U)$ is Noetherian. Because $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ induces $\mathcal{O}_X(U) \cong A$ on global sections, we are interested in showing that all affine open subschemes of X are good. We now run the checks of Lemma 2.118.

(i) Suppose an affine open subscheme $U \subseteq X$ has $\mathcal{O}_X(U)$ Noetherian. Then, for any $f \in \mathcal{O}_X(U)$, the canonical isomorphism of Corollary 2.28 tells us that $\mathcal{O}_X(U_f) \simeq \mathcal{O}_X(U)_f$, so we see that U_f is good because the localization of a Noetherian ring is still Noetherian.

(ii) Fix an affine open subscheme $U \subseteq X$ with elements $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ generating $\mathcal{O}_X(U)$ such that U_{f_i} is good for each P. For brevity, set $A := \mathcal{O}_X(U)$, and we are given that $A_{f_i} = \mathcal{O}_X(U)_{f_i} \simeq \mathcal{O}_X(U_{f_i})$ (using the canonical isomorphism of Corollary 2.28) is Noetherian for each i.

We need to show that A is Noetherian, so pick up some ideal $I \subseteq A$, and we need to show that I is Noetherian. Well, for each i, we see $IA_{f_i} \subseteq A_{f_i}$ is an ideal and must be finitely generated, so find generators $\{x_{i,1}, \ldots, x_{i,n_i}\}$. Now, for each $x_{i,j}$, we may write

$$x_{i,j} = \frac{y_{i,j}}{f_i^{e_{i,j}}}$$

where $y_{i,j} \in I$, and we see that the $y_{i,j}$ will also generate IA_{f_i} as an A_{f_i}-module by just multiplying the necessary linear combinations by $1/f_i^{e_{i,j}}$ as is necessary.

Now, let J be the ideal generated by the $y_{i,j}$ over all i and j, which is finitely generated because there are only finitely many i and only finitely many j for each i. Thus, we claim $I = J$, which will finish the proof.

On one hand, we note $J \subseteq I$ because the generators of J are contained in I. In the other direction, we note that any $a \in I$ can be embedded as $a/1 \in IA_{f_i}$, for which we use our above generators $y_{i,j}$ to write

$$\frac{a}{1} = \sum_{j=1}^{n_i} \frac{a_{i,j} y_{i,j}}{f_i^{d_{i,j}}}$$

for some $a_{i,j} \in A$ and $d_{i,j} \in \mathbb{N}$. Collecting denominators on the right-hand side and using the equality in A_{f_i}, there is some N_i such that

$$f^{N_i} a = \sum_{j=1}^{n_i} b_{i,j} y_{i,j} \in J$$

for some $b_{i,j} \in A$.

We are now essentially done. Note that

$$\varnothing = V(A) = V((f_1, \ldots, f_n)) = V(\operatorname{rad}(f_1, \ldots, f_n)) = V\left((f_1^{N_1}, \ldots, f_n^{N_n})\right),$$

so it follows $(f_1^{N_1}, \ldots, f_n^{N_n}) = A$, so we may find $c_1, \ldots, c_n \in A$ such that $\sum_{i=1}^n c_i f_i^{N_i} = 1$, so

$$a = \sum_{i=1}^n c_i \left(f_i^{N_i} a\right) \in J.$$

This finishes showing that $I \subseteq J$.

(iii) Note that we are promised an open cover by good affine open subschemes by hypothesis on $(X, \mathcal{O}_X)$.

Thus, Lemma 2.118 kicks in and finishes the proof. ■

Corollary 2.122. Fix a ring A. Then $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is locally Noetherian if and only if A is Noetherian.

Proof. If A is Noetherian, then we use the affine open cover $\{(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})\}$ of $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ to show that $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is locally Noetherian by definition. Conversely, if $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is locally Noetherian, then its affine open subscheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ must give $A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$ Noetherian. ■

2.4.3 Reduced Schemes

Here is the definition.

Definition 2.123 (Reduced)**.** A scheme $(X, \mathcal{O}_X)$ is *reduced* if and only if each $p \in X$ give a reduced local ring $\mathcal{O}_{X,p}$; i.e., we are asking for $\operatorname{nilrad} \mathcal{O}_{X,p} = (0)$.

Example 2.124. Fix a reduced ring A; we claim that $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a reduced scheme. Indeed, by Lemma 1.102, it suffices to show that $A_\mathfrak{p}$ is reduced for any prime $\mathfrak{p} \in \operatorname{Spec} A$. Well, suppose $a/x \in A_\mathfrak{p}$ (where $x \notin \mathfrak{p}$) has $(a/x)^n = 0$ for some $n \in \mathbb{N}$; if $n = 0$, there is nothing to say. Otherwise, $a^n/x^n = 0$, so $ya^n = 0$ for some $y \notin \mathfrak{p}$. Multiplying both sides by y^{n-1}, we conclude $(ay)^n = 0$, so $ay = 0$, so $a/x = 0$.

We can also think about reduced schemes on the level of open sets.

Lemma 2.125. A scheme $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for each open subset $U \subseteq X$.

Proof. We show our implications separately.

- Suppose that $\mathcal{O}_X(U)$ is reduced for each open $U \subseteq X$. Given $p \in X$, we need to show that $\mathcal{O}_{X,p}$ is also reduced. Well, fix some germ $f_p \in \mathcal{O}_{X,p}$ for $p \in U$ such that $f_p^n = 0$ for some $n \in \mathbb{N}$; we need to show that $f_p = 0$.

  Well, write $f_p = [(U, f)]$ so that $f \in \mathcal{F}(U)$ where U contains p. Then we see

  $$[(U, 0)] = 0 = f_p^n = [(U, f)]^n = [(U, f^n)]$$

  implies that there is some open subset $V \subseteq U$ containing p such that $f^n|_V = 0$. Thus, $(f|_V)^n = 0$, so $f|_V = 0$ because $\mathcal{O}_X(V)$ is reduced. It follows

  $$f_p = [(U, f)] = [(V, f|_V)] = 0,$$

  which is what we wanted.

- Suppose that $\mathcal{O}_{X,p}$ is reduced for all $p \in X$. Then, for each open set $U \subseteq X$, we need to show that $\mathcal{O}_X(U)$ is also reduced. Well, suppose that $f \in \mathcal{O}_X(U)$ has $f^n = 0$ for some $n \in \mathbb{N}$. We need to show that $f = 0$.

  For this, we note that all $p \in U$ will have

  $$(f|_p)^n = (f^n)\,|_p = 0|_p = 0 \in \mathcal{O}_{X,p},$$

  so because $\mathcal{O}_{X,p}$ is reduced, we conclude that $f|_p = 0$. Thus, f lives in the kernel of the natural map

  $$\begin{array}{ccc} \mathcal{O}_X(U) & \to & \prod_{p \in U} \mathcal{O}_{X,p} \\ f & \mapsto & (f|_p)_{p \in U} \end{array}$$

  which is actually injective because $\mathcal{O}_X$ is a sheaf. It follows that $f = 0$. ■

Remark 2.126. For example, if a ring A makes $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ into a reduced scheme, then we see $A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$ is a reduced ring. This provides the converse to Example 2.124.

Non-Example 2.127. For example, the scheme $\operatorname{Spec} k[x,y]/(x^2)$ is not reduced because its ring of global sections is $k[x,y]/(x^2)$, which is not reduced.

We can also think of the reduced condition on the level of affine open subschemes.

Lemma 2.128. Fix a scheme $(X, \mathcal{O}_X)$. The following are equivalent.

(a) $(X, \mathcal{O}_X)$ is reduced.

(b) $\mathcal{O}_X(U)$ is a reduced ring for every affine open subscheme $U \subseteq X$.

(c) There is an affine open cover $\mathcal{U}$ of X such that $\mathcal{O}_X(U)$ is reduced for each $U \in \mathcal{U}$.

Proof. Of course, (a) implies (b) by Lemma 2.125, and (b) implies (c) by choosing any affine open cover X.

Lastly, we show that (c) implies (a). Fix our affine open cover $\mathcal{U}$, where $\mathcal{U}$ is made of affine rings. Well, for any $p \in X$, we need to show that $\mathcal{O}_{X,p}$ is reduced. For this, pick up some affine open subscheme $U \subseteq X$ containing p, where $\mathcal{O}_X(U)$ is a reduced ring. Then, using the canonical isomorphism of Corollary 2.28,

$$(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} \mathcal{O}_X(U), \mathcal{O}_X|_U)$$

is a reduced scheme by Example 2.124. Thus, the stalk $(\mathcal{O}_X|_U)_p$ is a reduced, but this is canonically isomorphic to $\mathcal{O}_{X,p}$ by Lemma 1.174, so we are done. ■

Example 2.129. Projective space $\mathbb{P}_k^n$ is reduced because its affine open patches are reduced.

Being reduced is a nice property, so we might want to force schemes to be reduced.

Definition 2.130 (Reduced scheme associted)**.** Given a scheme $(X, \mathcal{O}_X)$, the *reduced scheme associated* to $(X, \mathcal{O}_X)$ is the scheme $(X, \mathcal{O}_X/\mathcal{N})$, where $\mathcal{N}(U) := \{s \in \mathcal{O}_X(U) : s|_x \in \mathcal{O}_{X,x} \text{ is nilpotent}\}$ for each $U \subseteq X$.

This satisfies the following universal property.

Lemma 2.131. Fix a scheme $(X, \mathcal{O}_X)$, and let $(X^{\text{red}}, \mathcal{O}_X^{\text{red}})$ be the reduced scheme. Then, for any reduced scheme $(Y, \mathcal{O}_Y)$ and map $\varphi\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, there is a unique map $\overline{\varphi}\colon (Y, \mathcal{O}_Y) \to (X^{\text{red}}, \mathcal{O}_X^{\text{red}})$ making the following diagram commute.

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[r, "\varphi"] \arrow[dr, dashed, "\overline{\varphi}"'] & (X, \mathcal{O}_X) \\ & (X^{\text{red}}, \mathcal{O}_X^{\text{red}}) \arrow[u] \end{tikzcd}$$

Proof. On the homework. ■

Example 2.132. The reduced scheme associated to $\operatorname{Spec} k[x,y]/(x^2)$ just becomes $\operatorname{Spec} k[y]$. Intuitively, we are "deleting" all of our differential information.

2.4.4 Integral Schemes

Here is the definition.

Definition 2.133 (Integral)**.** A scheme $(X, \mathcal{O}_X)$ is *integral* if and only if all nonempty open subsets $U \subseteq X$ give an integral domain $\mathcal{O}_X(U)$.

Remark 2.134. Note that X being integral will imply that each $\mathcal{O}_{X,x}$ is an integral domain: if $f, g \in \mathcal{O}_{X,x}$ are germs with $fg = 0$ while $f \neq 0$, we need to show $g = 0$. Well, find a sufficiently small W containing x so that we can define $f = [(W, \widetilde{f})]$ and $g = [(W, \widetilde{g})]$. Now, we see $\widetilde{f} \cdot \widetilde{g}$ needs to restrict to the zero germ, so restrict W enough so that $\widetilde{f}\widetilde{g} = 0$. However, $\widetilde{f} \neq 0$ because $f \neq 0$, so we must instead have $\widetilde{g} = 0$, which finishes.

Proposition 2.135. A scheme $(X, \mathcal{O}_X)$ is integral if and only if $(X, \mathcal{O}_X)$ is reduced and irreducible.

Proof. We show the directions separately.

- In the forward direction, note that $(X, \mathcal{O}_X)$ is easily reduced: indeed, all stalks are reduced because they are all integral domains by Remark 2.134. Further, if $(X, \mathcal{O}_X)$ is not irreducible, then we have two proper closed subsets $V_1, V_2 \subseteq X$ covering X. Setting $U_\bullet := X \setminus V_\bullet$, we see that $U_1 \cap U_2 = \varnothing$ while $U_1, U_2 \neq \varnothing$.

  Now, we let $u_1 \in \mathcal{O}_X(U_1)$ and $u_2 \in \mathcal{O}_X(U_2)$ correspond to the units. However, $u_1|_{U_1 \cap U_2} = 0|_{U_1 \cap U_2}$, so we can glue u_1 and 0 together into some $e_1 \in \mathcal{O}_X(U_1 \cup U_2)$. Similarly, $0|_{U_1 \cap U_2} = u_2|_{U_1 \cap U_2}$ let us glue 0 and u_2 into some $e_2 \in \mathcal{O}_X(U_1 \cup U_2)$.

  Thus, we see that $\mathcal{O}_X(U_1 \cup U_2)$ is not an integral domain: note $e_1|_{U_1} \neq 0$ and $e_2|_{U_2} \neq 0$, so $e_1, e_2 \neq 0$, but
  $$(e_1 e_2)|_{U_1} = (u_1 \cdot 0) = 0 \qquad \text{and} \qquad (e_1 e_2)|_{U_2} = (0 \cdot u_2) = 0,$$
  so $e_1 e_2 = 0$ by gluing.

- In the other direction, suppose $(X, \mathcal{O}_X)$ is irreducible and reduced. Well, fix any open subset $U \subseteq X$ and $f, g \in \mathcal{O}_X(U)$ such that $fg = 0$. We now define
  $$V(a) := \{x \in U : g_x \in \mathfrak{m}_x\},$$
  and we see that $V(f)$ and $V(g)$ are going to be closed subsets of U because they are the complement of the open ones in Lemma 2.22.

  Continuing, because X is irreducible, U is as well by Lemma 2.22, so $U = V(g) \cup V(f)$ forces U to be contained in one of these closed subsets, so without loss of generality take $U = V(f)$. We are now ready to claim $f = 0$, which will finish.

  Because U is an open subscheme, it suffices to show that $f = 0$ on an affine open subcover $\{U_\alpha\}_{\alpha \in \lambda}$ of U. Well, for each $\alpha \in \lambda$, let $f_\alpha := f|_{U_\alpha}$, and we see that $U_\alpha \subseteq U = V(f)$ implies that $f|_p \in \mathfrak{m}_p$ for each $p \in U_\alpha$. Thus, using the canonical isomorphism $\varepsilon \colon (U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} \mathcal{O}_X(U), \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(U)})$ of Corollary 2.28, we see
  $$\mathcal{O}_X(U)_p \simeq \varinjlim_{U_h \ni p} \mathcal{O}_X(U_h) \stackrel{\varepsilon}{\simeq} \varinjlim_{D(h) \ni \varepsilon(p)} \mathcal{O}_X(U)_h \simeq \mathcal{O}_X(U)_{\varepsilon(p)}$$
  where we have also used Lemma 1.102. Thus, we see $f/1 \in \mathfrak{p}\mathcal{O}_X(U)_\mathfrak{p}$ for each prime $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_X(U)$, so
  $$f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \operatorname{nilrad} A.$$
  But $\mathcal{O}_X(U)$ is reduced as a ring because $(X, \mathcal{O}_X)$ is a reduced scheme by Lemma 2.125, so now forces $f = 0$. ■

We are now prepared to give some examples.

Example 2.136. A reduced scheme whose stalks are integral domains even though X is not irreducible will not make X in total integral. Somehow being integral is a more global property, which the irreducibility tracks.

Example 2.137. Projective space $\mathbb{P}^n_k$ is integral: this is reduced by Example 2.129 and irreducible by Example 2.83.

Example 2.138. Fix an integral domain A. Then $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is reduced because A is reduced (Example 2.124), and $\operatorname{Spec} A$ is irreducible with generic point (0) by using Example 2.82:

$$\overline{\{(0)\}} = V((0)) = \{\mathfrak{p} \in \operatorname{Spec} A : 0 \in \mathfrak{p}\} = \operatorname{Spec} A.$$

The above example inspires us into the following.

Lemma 2.139. An integral scheme $(X, \mathcal{O}_X)$ has a unique generic point ξ for X. Then the following are true.

- If $(X, \mathcal{O}_X)$ is affine, then $\mathcal{O}_{X,\xi} \simeq \operatorname{Frac} \mathcal{O}_X(X)$.
- For any affine open subscheme $U \subseteq X$, we have $\mathcal{O}_{X,\xi} \simeq \operatorname{Frac} \mathcal{O}_X(U)$. In particular, the stalk $\mathcal{O}_{X,\xi}$ is a field.

Proof. Because X is irreducible by Proposition 2.135, we see that the closed irreducible subset X of X has a unique generic point ξ by Lemma 2.105. We now show the remaining claims in sequence.

- Note that (0) is the generic point of $\operatorname{Spec} \mathcal{O}_X(X)$ as shown in Example 2.138, so we see that the canonical isomorphism of Corollary 2.28 gives

  $$\mathcal{O}_{X,\xi} \simeq \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X),(0)} \simeq \mathcal{O}_X(X)_{(0)}.$$

  Namely, the first isomorphism holds because ξ must be taken to a point in $\operatorname{Spec} \mathcal{O}_X(X)$ whose closure is the entire space, which is just (0) by uniqueness of the generic point in Lemma 2.105; the second isomorphism holds by Lemma 1.102. The claim now follows upon noticing $\operatorname{Frac} \mathcal{O}_X(X) = \operatorname{Frac} \mathcal{O}_X(X)$.

- Given an affine open subscheme $U \subseteq X$, we note that $U \cap \overline{\{\xi\}} \neq \varnothing$ forces $U \cap \{\xi\} \neq \varnothing$ by properties of the closure, so $\xi \in U$. Thus, we see

  $$\mathcal{O}_{X,\xi} \simeq (\mathcal{O}_X|_U)_\xi \simeq \operatorname{Frac} \mathcal{O}_X(U),$$

  where the first isomorphism is by Lemma 1.174, and the second isomorphism is by the above. ■

Remark 2.140. In general, it is not true that $\mathcal{O}_{X,\xi}$ is the fraction field of $\mathcal{O}_X(U)$ for any open subset $U \subseteq X$. For example, take $X := \mathbb{P}^1_k$. To begin, recall $\mathcal{O}_X(X) \simeq k$ from Remark 2.50, so $\operatorname{Frac} \mathcal{O}_X(X) = k$. However, recalling we can build $\mathbb{P}^1_k$ by gluing together $\operatorname{Spec} k[x]$s, we see that the function field should be $\operatorname{Frac} k[x] = k(x)$ by Lemma 2.139.

So we get the following nice definition.

Definition 2.141 (Function field)**.** An integral scheme $(X, \mathcal{O}_X)$ with generic point ξ has *function field* $\mathcal{O}_{X,\xi}$.

The point here is that we can retrieve the field out from some $\operatorname{Spec} k[x,y]$, say. This also allows us to define regular functions.

Definition 2.142 (Regular)**.** Fix an integral scheme $(X, \mathcal{O}_X)$ with generic point ξ. Then $f \in \mathcal{O}_{X,\xi}$ is *regular* at a point $x \in X$ if and only if f lifts to $\mathcal{O}_{X,x}$.

2.4.5 Closed Subschemes

Open subschemes had natural subscheme structure by just taking restriction. Closed subschemes are a little harder.

Example 2.143. Set $X := \operatorname{Spec} k[x,y]$. Then the closed subset $V(x)$ will have lots of natural homeomorphisms

$$V(x) \cong \operatorname{Spec} \frac{k[x,y]}{(x^n)},$$

for any $n \geq 1$, so there is no canonical way to set the structure sheaf.

The idea to define a closed subscheme is to instead keep track of the morphism which does the embedding.

Definition 2.144 (Closed embedding)**.** A scheme morphism $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a *closed embedding* if and only if the following two conditions hold.

- The map $f\colon Z \to X$ is a homeomorphism from Z onto a closed subset of X.
- The map $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Z$ is surjective on stalks.

If in fact $Z \subseteq X$ is a closed subset, then we will say Z is a closed subscheme.

Remark 2.145. The geometric intuition behind requiring $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Z$ to be surjective on stalks is that we want holomorphic functions on a closed subset $V \subseteq \mathbb{C}$ to pull back to meromorphic ones on $\mathbb{C}$. Locally, this means that a germ of a holomorphic function on V should come from a germ on a holomorphic function on X, which is exactly what this sheaf morphism being epic means.

Example 2.146. Of course, scheme isomorphisms are homeomorphisms topologically and give sheaf isomorphisms of the structure sheaves by Lemma 2.8, so isomorphisms are closed embeddings.

The main point here is that we would like our closed embeddings in the affine case to be induced by $A \twoheadrightarrow A/I$ as in Exercise 1.53.

Proposition 2.147. Fix an affine scheme $(X, \mathcal{O}_X) := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

(a) Each ideal $I \subseteq A$ induces a closed immersion

$$\operatorname{Spec} A/I \to \operatorname{Spec} A$$

from the projection map $A \twoheadrightarrow A/I$. In particular, this gives $V(I) \subseteq \operatorname{Spec} A$ the structure of a closed subscheme.

(b) The map of (a) provides a bijection between ideals of A and closed subschemes of $\operatorname{Spec} A$.

Proof. Here we go.

(a) From Exercise 1.53, we already have the natural homeomorphism $\operatorname{Spec} A/I \cong V(I)$. On the level of sheaves, we only need to check surjectivity at stalks, for which we look at the distinguished open base.

Namely, at some $D(f)$, we are studying the map

$$A_f = \mathcal{O}_X(D(f)) \to \mathcal{O}_Z(D(f+I)) = (A/I)_f \simeq A_f/I_f,$$

which we can see is surjective here. Taking the direct limit shows that we remain surjective on the level of stalks.

(b) This proof will be able to simplified later in life when we have talked about coherent sheaves. Fix a closed subscheme $\iota\colon Z \to X$ with $\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_Z$. Now, define

$$I_Z := \ker(\iota_X^\sharp).$$

Then one can show that I_Z is the ideal we want, providing the inverse to (a). In particular, one can show that Z is identified with $\operatorname{Spec} A/I_Z$ as schemes. Notably, there is an embedding $Z \hookrightarrow Y$ by first looking on the level of topological spaces and then carrying over to schemes. It remains to show that this is an isomorphism.

- We show that we have a homeomorphism of our topological spaces. Note that Z is quasicompact, so we can write it as a finite union of affine open subschemes. Now, for $Z \subseteq V(s)$, we will show that $V(s) = Y$. We see $s \in B := A/I_Z$, so note that $s \in \mathcal{O}_X(U_i)$ will be nilpotent because of its definition. Looping over entire affine open cover forces $s^n = 0$, so the injectivity of the open cover

$$A/I_Z \hookrightarrow \mathcal{O}_Z(Z)$$

forces $s^N = 0$ in A/I_Y, meaning $Y \subseteq V(s)$.
- We now need to show $\mathcal{O}_Y \to \iota_*\mathcal{O}_Z$. Surjectivity follows from the construction looking at the global sections. The injectivity follows by looking at stalks and making an argument similar to the above. ■

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We continue.

2.5.1 Schemes over a scheme

Here is our definition.

Definition 2.148 (Schmes over a scheme)**.** Fix a scheme $(S, \mathcal{O}_S)$. An S-scheme is a scheme $(X, \mathcal{O}_X)$ equipped with a morphism $(\pi, \pi^\sharp)\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ of schemes. We might write X/S.

Example 2.149. All schemes are naturally a scheme over $\operatorname{Spec} \mathbb{Z}$.

A morphism of two S-schemes $\pi\colon X \to S$ and $\pi'\colon X' \to S$ is a morphism $\varphi\colon X \to X'$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

commute.

Remark 2.150. Fix $S := \operatorname{Spec} A$. Then we can define a projective scheme as a scheme X over S with a closed embedding into $\mathbb{P}^n_S$. We will show the equivalence of this definition later; the point is that Proposition 2.147 should have an analogue to graded rings.

2.5.2 Reduced Schemes

Given a closed subscheme $Z \subseteq X$, there might be many natural scheme structures from the topology. However, if we ask for it to be reduced, then it is unique.

Proposition 2.151. Fix a scheme X and a closed subset $Z \subseteq X$. Then there is a unique reduced, closed subscheme $Z \subseteq X$ whose topological space agrees with Z.

Proof. We start by showing uniqueness. Note that we may replace X with its reduced scheme without headaches, so we assume that X is a reduced scheme. Fix an affine open subscheme $U \subseteq X$, and write $U = \operatorname{Spec} A$ for some ring A. Now, $U \cap Z$ is a reduced, closed subscheme of U, so by Proposition 2.147 tells us that

$$(U \cap Z, \mathcal{O}_Z|_U)$$

comes from a radical ideal I, and we see that this is unique.

This also tells us how to construct Z. Namely, each affine open subscheme $U \subseteq X$ with $U = \operatorname{Spec} A$ can take some $I = I(U \cap Z) \subseteq \operatorname{Spec} A$. As such, we can give $U \cap Z \subseteq U$ a reduced scheme structure from $\mathcal{O}_{\operatorname{Spec} A/I}$, and we finish the construction by gluing these subschemes together. Notably, the gluing is possible because the uniqueness forces the cocycle condition. ■

Remark 2.152. Given a scheme morphism $f\colon X \to Y$, we might want to think about the image of f. The correct way to think about this is to say that there is a unique closed subscheme $Z \subseteq Y$ such that f factors through Z, with the following universal property: for all closed subschemes $Z' \subseteq Y$ factoring through f, we have $Z \subseteq Z'$.

At a high level, in some nice cases one takes Z to be the kernel of $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Then we will look at $\mathcal{O}_Y \to \mathcal{O}_Y / \ker f^\sharp$, at least when X is reduced.

2.5.3 Quasiprojective Schemes

We might want to talk about affine and projective schemes at the same time. Here is how we do this.

Definition 2.153 (Quasiprojective)**.** Fix an affine scheme $S := \operatorname{Spec} A$. Then a scheme X/S is *quasiprojective* if and only if X/S is a quasicompact open S-subscheme of some projective S-scheme.

Example 2.154. Affine k-varieties are quasiprojective.

Here is a related definition.

Definition 2.155 (Locally closed embedding)**.** A scheme morphism $\pi\colon X \to Y$ is a *locally closed embedding* if and only if we can factor π into

$$X \hookrightarrow Z \hookrightarrow Y$$

where $X \hookrightarrow Z$ is a closed embedding, and $Y \hookrightarrow Z$ is an open embedding.

The reason that this is called a "locally closed embedding" is because it becomes a closed embedding on a sufficiently small open subset.

Remark 2.156. For a locally closed embedding $\pi\colon X \to Y$, then under suitable smallness conditions (i.e., for π to be quasicompact), we can find Z' so that π factors as

$$X \hookrightarrow Z' \hookrightarrow Y$$

where $X \hookrightarrow Z'$ is open, and $Z' \hookrightarrow Y$ is closed. The idea here is that we want to generalize the notion of "constructible subsets" which are intersections of open and closed subsets. This finiteness result is telling us that these are approximately the same notion.

2.5.4 Dimension

The last topological property we will talk about is dimension.

Definition 2.157 (Dimension)**.** Fix a topological space X. Then the *dimension* of X is the longest possible length n of a chain of closed irreducible subsets

$$Z_0 \subsetneq Z_1 \subsetneq Z_3 \subsetneq \cdots \subsetneq Z_n \subseteq X.$$

Example 2.158. If $X = \operatorname{Spec} A$, then $\dim X$ is the Krull dimension $\dim A$.

Example 2.159. We see $\dim \mathbb{A}_k^n = \dim \mathbb{P}_k^n = n$.

Sometimes a topological space might have components of larger dimension than others, which is undesirable.

Definition 2.160 (Pure dimension)**.** Fix a topological space X. Then X is of *pure dimension* n if and only if all irreducible components of X have dimension n.

Having defined a notion of dimension, we can now define codimension.

Definition 2.161 (Codimension)**.** Fix a topological space X and an irreducible closed subset $Z \subseteq X$. Then the *codimension* $\operatorname{codim}_Z X$ is the supremum of the length n of a chain of irreducible closed subsets

$$Z = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n \subseteq X.$$

Remark 2.162. Of course, $\dim Z + \operatorname{codim}_X Z \le \dim Z$, but equality is not true in general because we lack pure dimension.

2.5.5 The Functor of Points

Here is our definition.

Definition 2.163 (Functor of points)**.** Fix an S-scheme X. Then the functor of points of X is the functor defined as follows.

$$\begin{aligned} h_X\colon (\mathrm{Sch}_S)^{\mathrm{op}} &\to \mathrm{Set} \\ Y &\mapsto \mathrm{Mor}_{\mathrm{Sch}_S}(Y, X) \end{aligned}$$

This provides the correct intuitive definition, say when $S = k$ is some field. In particular, the A-points of X are made of the scheme morphisms

$$\operatorname{Spec} A \to X,$$

so we are taking a general scheme Y to its "Y-points." Notably, a morphism of schemes $X \to X'$ will induce a natural transformation $h_X \Rightarrow h_{X'}$.

A little category theory will be informative.

Theorem 2.164 (Yoneda's lemma)**.** Fix a category $\mathcal{C}$, and define the functor よ: $\mathcal{C}^{\text{op}} \to \text{Set}$ taking objects X to h_X.

(a) Natural transformations from h_X to a functor $\mathcal{F}\colon \mathcal{C}^{\text{op}} \to \text{Set}$ are canonically isomorphic to $\mathcal{F}(X)$.

(b) The functor $X \mapsto h_X$ is fully faithful.

We will be interested in the following special case.

Corollary 2.165. Fix $\mathcal{C}$ to be the category of schemes over $S := \operatorname{Spec} R$. Then the functor $h_\bullet$ taking X to h_X^{aff} (where h_X maps R-algbras A to $\operatorname{Mor}_{\text{Sch}_{\operatorname{Spec} R}}(\operatorname{Spec} A, X)$) is fully faithful.

Proof. For a given S-scheme Y, reduce to the case of affine open subsets, and then in the affine case we get to appeal directly to Yoneda's lemma.

Namely, cover X with some affine open cover $\mathcal{U}$. Then, given a natural transformation $\varphi\colon h_X^{\text{aff}} \to h_{X'}^{\text{aff}}$, we need to construct a (unique) morphism $X \to X'$. Well, we simply go down to each affine piece $U \in \mathcal{U}$, use the affine case which provides some

$$\varphi(A_U)\colon \operatorname{Mor}_R(U, X) \to \operatorname{Mor}_R(U, X')$$

for each $U \in \mathcal{U}$. Passing the inclusion $U \hookrightarrow X$ through this proof, we get a bunch of morphisms $U \hookrightarrow X'$, which we then glue to a morphism. ■

2.6 September 19

Bump, bump, bump.

Warning 2.166. Today we will begin more aggressively notating a scheme $(X, \mathcal{O}_X)$ by its topological space X, where the structure sheaf will always be $\mathcal{O}_X$. Similarly, a morphism $\varphi\colon X \to Y$ refers to its continuous map, and the map of structure sheaves is $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$.

2.6.1 Fiber Products

Here is our definition.

Definition 2.167 (Fiber product)**.** Fix a category $\mathcal{C}$. Then, given morphisms $\psi_X\colon X \to S$ and $\psi_Y\colon Y \to S$, the *fiber product* $X \times_S Y$ is the limit of the following diagram.

$$\begin{tikzcd} & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"'] & S \end{tikzcd}$$

Example 2.168. In the category Set, one can show that

$$X \times_S Y = \{(x, y) \in X \times Y : \psi_X(x) = \psi_Y(y)\},$$

where the projections are the canonical ones.

Notation 2.169. Given a fiber product $X \times_S Y$ in a category, we call the resulting square

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

a *pullback square*.

And here is our main result.

Theorem 2.170. Fix two S-schemes X and Y. Then the fiber product $X \times_S Y$ exists.

Remark 2.171. Even if X and Y are Noetherian, it does not follow that $X \times_S Y$ is Noetherian. For example, taking $\operatorname{Spec} \overline{\mathbb{Q}}$ and $\operatorname{Spec} \overline{\mathbb{Q}}$ are both Noetherian, but the fiber product turns out to be

$$\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

Namely, $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is zero-dimensional but has infinitely many points and is therefore not Noetherian.

We will provide one proof of Theorem 2.170 today and another next class involving the representability of functors. To get a taste for this functor business, we note that we can concretely describe the functor of points for the fiber product.

Lemma 2.172. Fix a category $\mathcal{C}$ and two maps $\psi_X : X \to S$ and $\psi_Y : Y \to S$ such that the fiber product $X \times_S Y$ exists. Then, for any $Z \in \mathcal{C}$ with a map $\psi_Z : Z \to S$,

$$\operatorname{Mor}_S(Z, X \times_S Y) \cong \operatorname{Mor}_S(Z, X) \times_{\operatorname{Mor}_S(Z,S)} \operatorname{Mor}_S(Z, Y).$$

Here, Mor_S refers to morphisms in the category over S.

Proof. This follows straight from the universal property. Namely, let $\pi_X : X \times_S Y \to X$ and $\pi_Y : X \times_S Y \to Y$ be the canonical projections. Using Example 2.168, we are really just asking for an isomorphism

$$\operatorname{Mor}_S(Z, X \times_S Y) \cong \{(\varphi_X, \varphi_Y) \in \operatorname{Mor}_S(Z, X) \times \operatorname{Mor}_S(Z, Y) : \psi_X \circ \varphi_X = \psi_Y \circ \varphi_Y\}$$

matching up with the projections. Namely, observe that we have a map from the left to the right simply taking a map $\varphi \mapsto (\pi_X \circ \varphi, \pi_Y \circ \varphi)$, which works because

$$\psi_X \circ \pi_X \circ \varphi = \psi_Y \circ \pi_Y \circ \varphi$$

should both be ψ_Z. In the reverse direction, we can take a pair of maps $\varphi_X : Z \to X$ and $\varphi_Y : Z \to Y$ such that $\psi_X \circ \varphi_X = \psi_Y \circ \varphi_Y$ and recover a unique map $\varphi : Z \to X \times_S Y$ (by the universal property) such that $\pi_X \circ \varphi = \varphi_X$ and $\pi_Y \circ \varphi_Y$. ■

The difficulty will be in actually finding a scheme which can represent the functor

$$\operatorname{Mor}_S(-, X) \times_{\operatorname{Mor}_S(-,S)} \operatorname{Mor}_S(-, Y).$$

Namely, even though we are sure that this object is unique up to unique isomorphism (by Theorem 2.164), it is not actually clear that it exists at all!

2.6.2 Stacking Squares

We are going to want a few basic facts about pullback squares in life, so we pick them up now.

Lemma 2.173. Suppose that the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\iota''"] \arrow[d, "\varphi'"'] & B \arrow[d, "\pi'"] \\ C \arrow[r, "\iota'"] \arrow[d, "\varphi"'] & D \arrow[d, "\pi"] \\ E \arrow[r, "\iota"] & F \end{tikzcd}$$

has the big rectangle and the bottom square both pullback squares. Then the top square is also a pullback.

Proof. We check the universal property. Fix some object Z with maps $\psi_B\colon Z \to B$ and $\psi_C\colon Z \to C$ such that $\iota' \circ \pi_C = \pi' \circ \pi_B$, and we want a unique morphism $\psi\colon Z \to A$ making the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\psi_B"] \arrow[ddr, bend right, "\psi_C"'] & & \\ & A \arrow[r, "\iota''"] \arrow[d, "\varphi'"'] & B \arrow[d, "\pi'"] \\ & C \arrow[r, "\iota'"] \arrow[d, "\varphi"'] & D \arrow[d, "\pi"] \\ & E \arrow[r, "\iota"] & F \end{tikzcd} \tag{2.8}$$

commute. We show uniqueness and existence separately.

- Uniqueness: set $\psi_E := \varphi \circ \psi_C$ so that π makes the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\psi_B"] \arrow[ddr, bend right, "\psi_E"'] & & \\ & A \arrow[r, "\iota''"] \arrow[d, "\varphi\circ\varphi'"'] & B \arrow[d, "\pi\circ\pi'"] \\ & E \arrow[r, "\iota"] & F \end{tikzcd} \tag{2.9}$$

  commute, but this morphism ψ is uniquely induced by the diagram because the square here is a pullback square.

- Existence: as suggested by the above proof, set $\psi_E := \varphi \circ \psi_C$, and use the commutativity of (2.9) to induce a morphism π. We need to show that the full diagram (2.8) commutes. We get $\iota'' \circ \psi = \psi_B$ for free, so the main concern is showing $\varphi' \circ \psi = \psi_C$. Well, note that the commutativity of (2.9) means that both $\varphi' \circ \psi$ and ψ_C can fit the dashed arrow in the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left, "\pi'\circ\pi_B"] \arrow[ddr, bend right, "\psi_E"'] & & \\ & C \arrow[r, "\iota'"] \arrow[d, "\varphi"'] & D \arrow[d, "\pi"] \\ & E \arrow[r, "\iota"] & F \end{tikzcd}$$

  even though the square here is a pullback square. We conclude that $\varphi' \circ \pi = \pi_C$ is forced. ■

Lemma 2.174. Suppose that the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\iota''"] \arrow[d, "\varphi'"'] & B \arrow[d, "\pi'"] \\ C \arrow[r, "\iota'"] \arrow[d, "\varphi"'] & D \arrow[d, "\pi"] \\ E \arrow[r, "\iota"] & F \end{tikzcd}$$

has the two square as both pullback squares. Then the big rectangle is a pullback square.

Proof. We proceed by force. Fix some object Z with morphisms $\psi_B\colon Z \to B$ and $\psi_E\colon Z \to E$ such that $\pi \circ \pi' \circ \psi_B = \iota \circ \psi_E$. We need a unique morphism $\psi\colon Z \to A$ making the diagram

$$\begin{tikzcd} Z \arrow[drr, bend left, "\psi_B"] \arrow[dr, dashed, "\psi"] \arrow[ddr, bend right, "\psi_E"'] & & \\ & A \arrow[r, "\iota''"'] \arrow[d, "\varphi\circ\varphi'"] & B \arrow[d, "\pi\circ\pi'"] \\ & E \arrow[r, "\iota"'] & F \end{tikzcd} \tag{2.10}$$

commute. We show uniqueness and existence separately.

- Uniqueness: set $\psi_D := \pi' \circ \psi_B$ so that $\varphi' \circ \psi$ makes the diagram

$$\begin{tikzcd} Z \arrow[drr, bend left, "\psi_D"] \arrow[dr, dashed, "\varphi'\circ\psi"] \arrow[ddr, bend right, "\psi_E"'] & & \\ & C \arrow[r, "\iota'"'] \arrow[d, "\varphi"] & D \arrow[d, "\pi"] \\ & E \arrow[r, "\iota"'] & F \end{tikzcd}$$

  commute, so this pullback square tells us that $\psi_C := \varphi' \circ \psi$ is unique. Continuing, we see that

$$\iota' \circ \psi_C = \iota' \circ \varphi' \circ \psi = \pi' \circ \iota'' \circ \psi = \pi' \circ \psi_B,$$

  so we see that the diagram

$$\begin{tikzcd} Z \arrow[drr, bend left, "\psi_B"] \arrow[dr, dashed, "\psi"] \arrow[ddr, bend right, "\psi_C"'] & & \\ & A \arrow[r, "\iota''"'] \arrow[d, "\varphi'"] & B \arrow[d, "\pi'"] \\ & C \arrow[r, "\iota'"'] & D \end{tikzcd}$$

  commutes. However, ψ_C and ψ_B are both uniquely determined, so we see that the morphism ψ is thus also uniquely determined.

- Existence: we unwind the above proof. Because (2.10) commutes, we set $\psi_D := \pi' \circ \psi_B$ to make the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left, "\psi_D"] \arrow[ddr, bend right, "\psi_E"'] & & \\
& C \arrow[r, "\iota'"'] \arrow[d, "\varphi"] & D \arrow[d, "\pi"] \\
& E \arrow[r, "\iota"'] & F
\end{tikzcd}$$

commute, so the fact that we have a pullback square induces a unique morphism $\psi_C \colon Z \to C$ making the above diagram commute. In particular, $\iota' \circ \psi_C = \psi_D = \pi' \circ \psi_B$ by construction of ψ_C, so the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left, "\psi_B"] \arrow[ddr, bend right, "\psi_C"'] & & \\
& A \arrow[r, "\iota''"'] \arrow[d, "\varphi'"] & B \arrow[d, "\pi'"] \\
& C \arrow[r, "\iota'"'] & D
\end{tikzcd}$$

commutes, thus inducing a unique morphism $\psi \colon Z \to C$ making the above diagram commute. In particular, we have $\iota'' \circ \psi = \psi_B$ and $\varphi \circ \varphi' \circ \psi = \varphi \circ \psi_C = \psi_E$, which is what we wanted. ■

Lemma 2.175. Fix a commuting square

$$\begin{tikzcd}
A \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & B \arrow[d, "\psi"] \\
C \arrow[r, "\varphi"] & D
\end{tikzcd}$$

and a monic morphism $\iota \colon D \to E$. Then the above square is a pullback square if and only if

$$\begin{tikzcd}
A \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & B \arrow[d, "\iota \circ \psi"] \\
C \arrow[r, "\iota \circ \varphi"] & E
\end{tikzcd}$$

is a pullback square.

Proof. We proceed by force. In the forward direction, fix an object Z with morphisms $\pi_B \colon Z \to B$ and $\pi_C \colon Z \to C$ such that $\iota \circ \psi \circ \pi_B = \iota \circ \varphi \circ \pi_C$, and we need a unique morphism $\pi \colon Z \to A$ making the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\pi"] \arrow[drr, bend left, "\pi_B"] \arrow[ddr, bend right, "\pi_C"'] & & \\
& A \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & B \arrow[d, "\iota \circ \psi"] \\
& C \arrow[r, "\iota \circ \varphi"] & E
\end{tikzcd} \tag{2.11}$$

commute. We show uniqueness and existence separately.

- Uniqueness: note that $\varphi' \circ \pi = \pi_B$ and $\psi' \circ \pi = \pi_C$, so we see that the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\pi_B"] \arrow[ddr, bend right, "\pi_C"'] \\ & A \arrow[r, "\varphi'"] \arrow[d, "\psi'"] & B \arrow[d, "\pi"] \\ & C \arrow[r, "\varphi"] & D \end{tikzcd} \tag{2.12}$$

commutes. However, the morphism ψ is uniquely determined by the above diagram, finishing.

- Existence: note that $\iota \circ \psi \circ \pi_B = \iota \circ \varphi \circ \pi_C$ implies $\psi \circ \pi_B = \varphi \circ \pi_C$ because ι is monic, so the outer square of (2.12) commutes and induces a morphism $\pi\colon Z \to A$ such that $\varphi \circ \psi' \circ \pi = \iota \circ \psi \circ \varphi' \circ \pi$. It follows $\iota \circ \varphi \circ \psi' \circ \pi = \iota \circ \varphi \circ \psi' \circ \pi$, which is what we wanted.

We now discuss the other direction. Fix an object Z with morphisms $\pi_B\colon Z \to B$ and $\pi_C\colon Z \to C$ such that $\psi \circ \pi_B = \varphi \circ \pi_C$, and we need a unique morphism $\pi\colon Z \to A$ such that $\varphi' \circ \pi = \pi_B$ and $\psi' \circ \pi = \pi_C$. We show uniqueness and existence separately.

- Uniqueness: we are given that (2.12) commutes, so (2.11) also commutes by appending an ι to the end. However, (2.11) has a pullback square uniquely inducing the arrow π, which finishes.

- Existence: note that $\iota \circ \varphi \circ \pi_C = \iota \circ \psi \circ \pi_B$, so the outer square of (2.11) commutes and induces $\pi\colon Z \to A$ making (2.11) commute, so $\psi' \circ \pi = \pi_C$ and $\varphi' \circ \pi = \pi_B$, which is what we wanted. ■

2.6.3 Fiber Products: Easy Cases

We will now start marching toward a proof of Theorem 2.170. Schemes are made of affine schemes, so we will begin with affine schemes, with the hope of patching these together later.

Lemma 2.176. Fix affine schemes X, Y, and S, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Then we may set

$$X \times_S Y = \operatorname{Spec} A \otimes_R B,$$

where the canonical projections $X \times_S Y \to X$ and $X \times_S Y \to Z$ are induced by the canonical inclusions $\iota_A\colon A \to A \otimes_R B$ and $\iota_B\colon B \to A \otimes_R B$.

Proof. Let $f_A\colon R \to A$ and $f_B\colon R \to B$ be the maps associated to the maps $\psi_A\colon X \to S$ and $\psi_B\colon Y \to S$. Now, for some scheme Z over S by ψ_Z, and we see that

$$\operatorname{Mor}_S(Z, \operatorname{Spec} A \otimes_R B) \simeq \operatorname{Hom}_R(A \otimes_R B, \mathcal{O}_Z(Z))$$

by Theorem 2.26. Now, it is a fact of commutative algebra[3] that R-algebras maps $\varphi\colon A \otimes_R S \to \mathcal{O}_Z(Z)$ are in bijection with pairs of R-algebra maps $\varphi_A\colon A \to \mathcal{O}_Z(Z)$ and $\varphi_B\colon B \to \mathcal{O}_Z(Z)$ such that $\varphi_A = \varphi \circ \iota_A$ and $\varphi_B = \varphi \circ \iota_B$. In other words, we have a natural isomorphism

$$\operatorname{Mor}_S(Z, \operatorname{Spec} A \otimes_R B) \simeq \operatorname{Hom}_R(A, \mathcal{O}_Z(Z)) \times_{\operatorname{Hom}_R(R, \mathcal{O}_Z(Z))} \operatorname{Hom}_R(B, \mathcal{O}_Z(Z)).$$

Applying the adjunction again, we see

$$\operatorname{Mor}_S(Z, \operatorname{Spec} A \otimes_R B) \simeq \operatorname{Mor}_S(Z, X) \times_{\operatorname{Mor}_S(Z,S)} \operatorname{Mor}_S(Z, Y),$$

which finishes by Lemma 2.172. ■

[3] Namely, the tensor product is the fiber coproduct.

Remark 2.177. One can view the above proof as basically preserving the fact that $A \otimes_R B$ is the fiber coproduct of A and B as R-algebras.

We are also going to want to take bigger fiber products and find small ones inside them to be able to be glue them properly. For this, we note that open subschemes induce pullbacks.

Lemma 2.178. Fix a scheme morphism $\varphi\colon X \to Y$. Then, for any open $U \subseteq Y$, the square

$$\begin{tikzcd} \varphi^{-1}U \arrow[r, hook] \arrow[d, "\varphi|_{\varphi^{-1}U}"'] & X \arrow[d, "\varphi"] \\ U \arrow[r, hook] & Y \end{tikzcd}$$

is a pullback square.

Proof. Label our maps as follows.

$$\begin{tikzcd} \varphi^{-1}U \arrow[r, hook, "\iota"] \arrow[d, "\varphi|_{\varphi^{-1}U}"'] & X \arrow[d, "\varphi"] \\ U \arrow[r, hook, "\jmath"] & Y \end{tikzcd}$$

Observe that the right arrow of the diagram is induced as $(\varphi, \varphi^\sharp)|_U$ by Lemma 2.9. The horizontal arrows are the open embeddings of Example 2.5, and the diagram commutes by Remark 2.42.

It remains to show the universal property. Suppose that Z is a scheme with morphisms $\psi_X : Z \to X$ and $\psi_U : Z \to U$ such that $\varphi \circ \psi_X = \jmath \circ \psi_U$. We need a unique scheme morphism $\psi\colon Z \to \varphi^{-1}U$ making the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\psi_X"] \arrow[ddr, bend right, "\psi_U"'] & & \\ & \varphi^{-1}U \arrow[r, hook, "\iota"'] \arrow[d, "\varphi|_{\varphi^{-1}U}"] & X \arrow[d, "\varphi"] \\ & U \arrow[r, hook, "\jmath"'] & Y \end{tikzcd} \tag{2.13}$$

commute. We show uniqueness and existence separately.

- Uniqueness: on topological spaces, we require any $z \in Z$ to $\psi(z) = \iota(\psi(z)) = \psi_X(z)$, so ψ is uniquely determined topologically. On sheaves, we note that we need the diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\iota^\sharp"] \arrow[dr, "\psi_X^\sharp"'] & \iota_*(\mathcal{O}_X|_{\varphi^{-1}U}) \arrow[d, "\iota_*\psi^\sharp"] \\ & \mathcal{O}_Z \end{tikzcd}$$

to commute. In particular, for open subset $V \subseteq \varphi^{-1}U$, we see that $\iota_V^\sharp = \operatorname{res}_{V,\varphi^{-1}UV}$ is just the identity, so we are asking for the diagram

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r, equal, "\iota_V^\sharp"] \arrow[dr, "(\psi_X^\sharp)_V"'] & \mathcal{O}_X(V) \arrow[d, "\psi_V^\sharp"] \\ & \mathcal{O}_Z \end{tikzcd}$$

to commute, which we can see forces ψ_V.

- Existence: we follow the above formula. Define $\psi(z) := \psi_X(z) \in X$. Notably, this makes sense because $\varphi(\psi_X(z)) = \psi_Y(z) \in U$, so $\psi_X(z) \in \varphi^{-1}U$. Now, (2.13) commutes on topological spaces by tracking everything through:

$$\iota(\psi(z)) = \psi(z) = \psi_X(z) \qquad \text{and} \qquad \varphi(\psi(z)) = \varphi(\psi_Z(z)) = \psi_Y(z).$$

  On sheaves, for any open $V \subseteq \varphi^{-1}U$, we need a map $\psi_V : \mathcal{O}_{\varphi^{-1}U}(V) \to \psi_*\mathcal{O}_Z(V)$, but $\mathcal{O}_{\varphi^{-1}U}(V) = \mathcal{O}_X(\varphi^{-1}U \cap V)$ and $\psi_*\mathcal{O}_Z(V) = \mathcal{O}_Z(\psi^{-1}V) = \mathcal{O}_Z(\psi_X^{-1}V)$. However, we note that $\operatorname{im}\psi_X \subseteq \varphi^{-1}U$ as discussed previously, so we define our map as the composite

$$\mathcal{O}_{\varphi^{-1}U}(V) = \mathcal{O}_X(\varphi^{-1}U \cap V) \overset{(\psi_X^\sharp)_{\varphi^{-1}U\cap V}}{\to} (\psi_X)_*\mathcal{O}_Z(\varphi^{-1}U \cap V) = \mathcal{O}_Z(\psi_X^{-1}V).$$

  We now run the necessary checks.

  – Sheaf morphism: given open subsets $V' \subseteq V$, we see that the diagram

$$\begin{tikzcd} \mathcal{O}_{\varphi^{-1}U}(V) \arrow[r, equal] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_X(\varphi^{-1}U \cap V) \arrow[r, "(\psi_X^\sharp)_{\varphi^{-1}U\cap V}"] \arrow[d, "\mathrm{res}"'] & (\psi_X)_*\mathcal{O}_Z(\varphi^{-1}U \cap V) \arrow[r, equal] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_Z(\psi_X^{-1}V) \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_{\varphi^{-1}U}(V') \arrow[r, equal] & \mathcal{O}_X(\varphi^{-1}U \cap V') \arrow[r, "(\psi_X^\sharp)_{\varphi^{-1}U\cap V'}"] & (\psi_X)_*\mathcal{O}_Z(\varphi^{-1}U \cap V') \arrow[r, equal] & \mathcal{O}_Z(\psi_X^{-1}V') \end{tikzcd}$$

  commutes, where the only square which isn't made of horizontal identities is a naturality square for $\psi_X^\sharp$.

  – Morphism of locally ringed spaces: this is essentially inherited directly from $\psi_X^\sharp$. Given $z \in Z$, we need to know that the composite

$$\begin{array}{ccccc} \mathcal{O}_{\varphi^{-1}U,\psi(z)} & \overset{\psi^\sharp_{\psi(z)}}{\to} & (\psi_*\mathcal{O}_Z)_{\psi(z)} & \to & \mathcal{O}_{Z,z} \\ [(V,s)] & \mapsto & [(V, (\psi_X^\sharp)_{\varphi^{-1}U\cap V}(s))] & \mapsto & (\psi_X^\sharp)_{\varphi^{-1}U\cap V}(s)|_z \end{array}$$

  is a map of local rings. Well, we note that any $[(V,s)] \in \mathcal{O}_{\varphi^{-1}U,\psi(z)}$ is canonically also a germ in $\mathcal{O}_{X,\psi(z)}$, and the fact that ψ_Z is a morphism of locally ringed spaces tells us $(\psi_X^\sharp)_{\varphi^{-1}U\cap V}(s)|_z \in \mathfrak{m}_{Z,z}$, which is what we wanted.

  – Commutes: we already checked that the needed diagram (2.13) commutes on the level of topological spaces, so we just need to check that it commutes on the level of sheaves. This has two checks.

    * We verify that

$$\begin{tikzcd} \mathcal{O}_U \arrow[r, "\varphi^\sharp"] \arrow[dr, "\psi_U^\sharp"'] & \varphi_*\mathcal{O}_{\varphi^{-1}U} \arrow[d, "\varphi_*\psi^\sharp"] \\ & (\psi_U)_*\mathcal{O}_Z \end{tikzcd}$$

    commutes. Indeed, for any open subset $V \subseteq U$, we could verify that

$$\begin{tikzcd} \mathcal{O}_U(V) \arrow[r, "\varphi_V^\sharp"] \arrow[dr, "(\psi_U^\sharp)_V"'] & \mathcal{O}_{\varphi^{-1}U}(\varphi^{-1}U \cap V) \arrow[d, "\psi^\sharp_{\varphi^{-1}U\cap V}"] \\ & \mathcal{O}_Z(\psi^{-1}V) \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[dr, mapsto] & \varphi_V^\sharp(s) \arrow[d, mapsto] \\ & (\psi_X^\sharp)_{\varphi^{-1}U\cap V}(\varphi_V^\sharp s) \end{tikzcd}$$

    commutes appropriately using the given commuting square.

* We verify that

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\iota^\sharp"] \arrow[dr, "(\psi_X^\sharp)"'] & \iota_*\mathcal{O}_{\varphi^{-1}U} \arrow[d, "\iota_*\psi^\sharp"] \\ & (\psi_X)_*\mathcal{O}_Z \end{tikzcd}$$

  commutes. Indeed, for any open subset $V \subseteq U$, we could verify that

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r, "\iota_V^\sharp"] \arrow[dr, "(\psi_X^\sharp)_V"'] & \mathcal{O}_{\varphi^{-1}U}(\varphi^{-1}U \cap V) \arrow[d, "\psi^\sharp_{\varphi^{-1}U\cap V}"] \\ & \mathcal{O}_Z(\psi_X^{-1}V) \end{tikzcd}$$

  commutes directly from the construction of $\psi^\sharp$.

The above checks complete the proof. ■

Corollary 2.179. Let X be a scheme and $\iota\colon U \to X$ be an open subscheme. Then ι is a monic morphism of schemes.

Proof. Lemma 2.178 tells us that the square

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, equal] & U \arrow[d, hook, "\iota"] \\ U \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

is a pullback square. Technically, one should check that $\iota|_U : U \to U$ is the identity, but our restriction is functorial enough for this to be okay. (Plugging in the identity makes all of our constructions trivialize.)

As such, suppose we have a scheme Z with two morphisms $\varphi, \varphi' : Z \to U$ such that $\iota \circ \varphi = \iota \circ \varphi'$. Then note that any morphism ψ filling the dashed arrow of

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left, "\varphi'"] \arrow[ddr, bend right, "\varphi"'] & & \\ & U \arrow[r, equal] \arrow[d, equal] & U \arrow[d, hook, "\iota"] \\ & U \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

must have $\varphi = \psi = \varphi'$ follows because we have a pullback square. ■

Corollary 2.180. Fix a scheme X and two open subschemes $U_1, U_2 \subseteq X$. Then $U_1 \times_X U_2 \simeq U_1 \cap U_2$.

Proof. Let $\iota_1 : U_1 \hookrightarrow X$ and $\iota_2 : U_2 \hookrightarrow X$ be our open embeddings. Then

$$\begin{tikzcd} \iota_1^{-1}U_2 \arrow[r, "\iota_2"] \arrow[d, "\iota_1"] & U_1 \arrow[d, "\iota_1"] \\ U_2 \arrow[r, "\iota_2"] & X \end{tikzcd}$$

is a pullback square by Lemma 2.178. However, on the level of topological spaces $\iota_1^{-1}U_2 = \{x \in U_1 : x \in U_2\} = U_1 \cap U_2$, which completes the proof. ■

And now we manifest what we mean by "small fiber products inside large ones."

Corollary 2.181. Fix schemes X and Y over a scheme S. Given an open subscheme $U \hookrightarrow Y$, if $X \times_S Y$ exists, then $X \times_S U$ also exists and is (canonically) isomorphic to $\pi_Y^{-1}(U)$, where $\pi_Y : X \times_S Y \to Y$ is the canonical projection.

Proof. To begin, label our relevant maps as follows.

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

Our fiber product is going to be $\pi_Y^{-1}(U)$, which induces the restricted ring map $\pi_U := \pi_Y|_{\pi_Y^{-1}U}$ by Lemma 2.9. This gives us the diagram

$$\begin{tikzcd} \pi_Y^{-1}(U) \arrow[r, hook, "\iota"] \arrow[d, "\pi_U"'] & X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\psi_X"] \\ U \arrow[r, hook, "\jmath"] & Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

which we can see commutes on the left by Remark 2.42. Now, the right square is a pullback square by construction of $X \times_S Y$, and the left square is a pullback square by Lemma 2.178, so the larger rectangle is a pullback square by Lemma 2.174. This completes the proof. ■

2.6.4 Fiber Products: Gluing the Factors

We now begin our gluing. Here is the key idea to keep track of.

Idea 2.182. After checking the affine case, the gluing follows by chasing universal properties around.

Indeed, we will have to do no actual algebra in the argument that follows. For this subsection, we will glue along the factors of the fiber product; here is the statement.

Lemma 2.183. Fix schemes X and Y over S. Given an open cover $\mathcal{U}$ of Y, if the fiber products $X \times_S U$ exists for each open subscheme $U \in \mathcal{U}$, then the fiber product $X \times_S Y$ also exists and has a cover by schemes isomorphic to $X \times_S U$. (The implicit scheme map $U \to S$ is induced by appending the open embedding $U \hookrightarrow X$ to get $U \hookrightarrow X \to S$.)

Here is why we care about Lemma 2.183.

Corollary 2.184. Fix schemes X and Y over an affine scheme S. Then the fiber product $X \times_S Y$ exists.

Proof. We have two steps.

1. Suppose that X and S are affine. Then we can give Y an affine open cover $\mathcal{U}$, and we know that the fiber product $X \times_S U$ exists because now everything is affine, so Lemma 2.176 is good enough. It follows that $X \times_S Y$ also exists by Lemma 2.183.

2. Suppose that S is affine. Then we can give X an affine open cover $\mathcal{U}$, and we know that the fiber product $U \times_S Y$ exists by the previous point. So again, $X \times_S Y$ exists by applying the form Lemma 2.183 achieved by swapping the Xs and Ys. ■

We now prove Lemma 2.183.

Proof of Lemma 2.183. Unsurprisingly, we begin by giving Y the promised open cover $\{Y_\alpha\}_{\alpha\in\lambda}$. To prepare for gluing, we write $Y_{\alpha\beta} := Y_\alpha \cap Y_\beta$ for any $\alpha,\beta \in \lambda$ and think of $Y_{\alpha\beta}$ as a subset of Y_α with embedding $\jmath_{\alpha\beta} \colon Y_{\alpha\beta} \to Y_\alpha$. We proceed in steps.

1. By hypothesis on the open cover $\{Y_\alpha\}_{\alpha\in\lambda}$, there are fiber products $W_\alpha := X \times_S Y_\alpha$; label our pullback square as

$$\begin{tikzcd} W_\alpha \arrow[r, "\pi_{Y,\alpha}"] \arrow[d, "\pi_{X,\alpha}"'] & Y_\alpha \arrow[d, "\psi_Y|_{Y_\alpha}"] \\ X \arrow[r, "\psi_X"'] & S \end{tikzcd}$$

where we explicitly recognize that the canonical map $Y_\alpha \to S$ is the embedding $\jmath_\alpha \colon Y_\alpha \to Y$ followed by ψ_α, which is the restriction $\psi_\alpha|_{Y_\alpha}$ (using Remark 2.42).

2. The construction in Corollary 2.181 grants us fiber products

$$W_{\alpha\beta} := (X \times_S Y_{\alpha\beta}) := \pi_{Y,\alpha}^{-1}(Y_{\alpha\beta}).$$

For brevity, we label the canonical maps as $\pi_{X,\alpha\beta} := \pi_{X,\alpha}|_{\pi_{Y,\alpha}^{-1}(Y_{\alpha\beta})}$ and $\pi_{Y,\alpha\beta} := \pi_{Y,\alpha}|_{\pi_{Y,\alpha}^{-1}(Y_{\alpha\beta})}$. Now, $Y_{\alpha\beta} = Y_{\beta\alpha}$ by construction, so we must have a canonical isomorphism $W_{\alpha\beta} \simeq W_{\beta\alpha}$. Namely, the universal property of the fiber product promises a unique isomorphism $\varphi_{\alpha\beta} \colon W_{\alpha\beta} \to W_{\beta\alpha}$ such that

$$\begin{tikzcd} W_{\alpha\beta} \arrow[dr, dashed, "\varphi_{\alpha\beta}"] \arrow[drr, bend left, "\pi_{Y,\alpha\beta}"] \arrow[ddr, bend right, "\pi_{X,\alpha\beta}"'] & & \\ & W_{\beta\alpha} \arrow[r, "\pi_{Y,\beta\alpha}"'] \arrow[d, "\pi_{X,\beta\alpha}"] & Y_{\alpha\beta} \arrow[d, "\psi_Y|_{Y_{\alpha\beta}}"] \\ & X \arrow[r, "\psi_X"'] & S \end{tikzcd}$$

commutes. For example, when $\alpha = \beta$, we see that $Y_{\alpha\beta} = Y_\alpha = Y_\beta$, and the identity morphism will work for $\varphi_{\alpha\beta}$, so we must have $\varphi_{\alpha\beta} = \mathrm{id}_{Y_\alpha}$.

3. We are going to glue the schemes W_α along the isomorphisms $\varphi_{\alpha\beta}$, but we must check the cocycle condition. This is most clearly seen by manually checking triple intersections: given $\alpha,\beta,\gamma \in \lambda$, set $Y_{\alpha\beta\gamma} := Y_\alpha \cap Y_\beta \cap Y_\gamma = Y_{\alpha\beta} \cap Y_{\alpha\gamma}$ so that Corollary 2.181 grants us a fiber product

$$W_{\alpha\beta\gamma} := X \times_S Y_{\alpha\beta\gamma} = \pi_{Y,\alpha}^{-1}(Y_{\alpha\beta\gamma}) = \underbrace{\pi_{Y,\alpha}^{-1}(Y_{\alpha\beta})}_{W_{\alpha\beta}} \cap \underbrace{\pi_{Y,\alpha}^{-1}(Y_{\alpha\gamma})}_{W_{\beta\alpha}}.$$

Notably, $W_{\alpha\beta\gamma} = W_{\alpha\gamma\beta}$. Again, we will label the canonical maps as $\pi_{X,\alpha\beta\gamma} := \pi_{X,\alpha}|_{W_{\alpha\beta\gamma}}$ and $\pi_{Y,\alpha\beta\gamma} := \pi_{Y,\alpha}|_{W_{\alpha\beta\gamma}}$.

4. Because $Y_{\alpha\beta\gamma} = Y_{\beta\gamma\alpha}$, we see that there is a unique morphism $W_{\beta\gamma\alpha} \to W_{\beta\gamma\alpha}$ making the diagram

$$\begin{tikzcd} W_{\beta\gamma\alpha} \arrow[dr, dashed] \arrow[drr, bend left, "\pi_{Y,\beta\gamma\alpha}"] \arrow[ddr, bend right, "\pi_{X,\beta\gamma\alpha}"'] & & \\ & W_{\alpha\beta\gamma} \arrow[r, "\pi_{Y,\alpha\beta\gamma}"'] \arrow[d, "\pi_{X,\alpha\beta\gamma}"] & Y_{\alpha\beta\gamma} \arrow[d, "\psi_Y|_{Y_{\alpha\beta\gamma}}"] \\ & X \arrow[r, "\psi_X"'] & S \end{tikzcd} \tag{2.14}$$

commute. However, we claim that we can put $\varphi_{\beta\alpha}|_{W_{\beta\gamma\alpha}}$ into the dashed arrow to make the diagram commute. At the very least, note we can induce a restricted morphism $\varphi_{\beta\alpha}|_{W_{\beta\gamma\alpha}} \colon W_{\beta\gamma\alpha} \to W_{\alpha\beta\gamma}$: note that

$$\begin{aligned}\varphi_{\beta\alpha}^{-1}(W_{\alpha\beta\gamma}) &= \{w \in W_{\beta\alpha} : \varphi_{\beta\alpha} w \in W_{\alpha\beta\gamma}\} \\ &= \{w \in W_{\beta\alpha} : \pi_{Y,\alpha}\varphi_{\beta\alpha} w \in Y_\alpha \cap Y_\beta \cap Y_\gamma\} \\ &= \{w \in W_{\beta\alpha} : \pi_{Y,\beta} w \in Y_\alpha \cap Y_\beta \cap Y_\gamma\} \\ &= W_{\beta\gamma\alpha},\end{aligned}$$

so we get the needed morphism by restricting as in Lemma 2.9.

There are now two checks; let $\iota_{\alpha\beta\gamma} := W_{\alpha\beta\gamma} \subseteq W_{\alpha\beta}$ be the canonical embeddings.

- We check that the top triangle of (2.14) commutes with $\varphi_{\beta\alpha}|_{W_{\beta\alpha\gamma}}$ in the dashed arrow. Unraveling everything, we are asking for the diagram

$$\begin{tikzcd} \pi_{Y,\beta}^{-1}(Y_\alpha \cap Y_\beta \cap Y_\gamma) \arrow[r, "\jmath_{\beta\alpha\gamma}"] \arrow[d, "\varphi_{\beta\alpha}|_{W_{\beta\alpha\gamma}}"] & \pi_{Y,\beta}^{-1}(Y_\alpha \cap Y_\beta) \arrow[r, "\pi_{Y,\beta\alpha}"] \arrow[d, "\varphi_{\beta\alpha}"] & Y_\alpha \cap Y_\beta \arrow[d, equal] \\ \pi_{Y,\alpha}^{-1}(Y_\alpha \cap Y_\beta \cap Y_\gamma) \arrow[r, "\jmath_{\alpha\beta\gamma}"] & \pi_{Y,\alpha}^{-1}(Y_\alpha \cap Y_\beta) \arrow[r, "\pi_{Y,\alpha\beta}"] & Y_\alpha \cap Y_\beta \end{tikzcd}$$

  to commute. The right square commutes by construction of $\varphi_{\alpha\beta}$, and the left square commutes by Remark 2.42.

- We check that the left triangle of (2.14) commutes with $\varphi_{\beta\alpha}|_{W_{\beta\alpha\gamma}}$ in the dashed arrow. Unraveling everything, we want

$$\begin{tikzcd} \pi_{Y,\beta}^{-1}(Y_\alpha \cap Y_\beta \cap Y_\gamma) \arrow[r, "\jmath_{\beta\alpha\gamma}"] \arrow[d, "\varphi_{\beta\alpha}|_{W_{\beta\alpha\gamma}}"] & \pi_{Y,\beta}^{-1}(Y_\alpha \cap Y_\beta) \arrow[r, "\pi_{X,\beta\alpha}"] \arrow[d, "\varphi_{\beta\alpha}"] & X \arrow[d, equal] \\ \pi_{Y,\alpha}^{-1}(Y_\alpha \cap Y_\beta \cap Y_\gamma) \arrow[r, "\jmath_{\alpha\beta\gamma}"] & \pi_{Y,\alpha}^{-1}(Y_\alpha \cap Y_\beta) \arrow[r, "\pi_{X,\alpha\beta}"] & X \end{tikzcd}$$

  to commute. Again, the right square commutes by construction of the $\varphi_{\beta\alpha}$, and the left square commutes by Remark 2.42.

5. We are now ready to verify the cocycle condition. Essentially, we are asking for the diagram

$$\begin{tikzcd} W_{\alpha\beta\gamma} \arrow[r, "\varphi_{\alpha\beta}|_{W_{\alpha\beta\gamma}}"] \arrow[dr, "\varphi_{\alpha\gamma}|_{W_{\alpha\beta\gamma}}"'] & W_{\beta\gamma\alpha} \arrow[d, "\varphi_{\beta\gamma}|_{W_{\beta\gamma\alpha}}"] \\ & W_{\gamma\alpha\beta} \end{tikzcd}$$

to commute. By construction of $\varphi_{\alpha\gamma}|_{W_{\alpha\beta\gamma}}$, it suffices to show we can place the composite $\varphi_{\beta\gamma}|_{W_{\beta\gamma\alpha}} \circ \varphi_{\alpha\beta}|_{W_{\alpha\beta\gamma}}$ into the dashed arrow of

$$\begin{tikzcd} W_{\alpha\beta\gamma} \arrow[dr, dashed] \arrow[drr, bend left, "\pi_{Y,\alpha\beta\gamma}"] \arrow[ddr, bend right, "\pi_{X,\alpha\beta\gamma}"'] & & \\ & W_{\gamma\alpha\beta} \arrow[r, "\pi_{Y,\gamma\alpha\beta}"'] \arrow[d, "\pi_{X,\gamma\alpha\beta}"] & Y_{\alpha\beta\gamma} \arrow[d, "\psi_Y|_{Y_{\alpha\beta\gamma}}"] \\ & X \arrow[r, "\psi_X"'] & S \end{tikzcd}$$

to make the diagram commute. For this, we note that checking the commutativity of the two needed triangles comes down to writing the diagrams

$$\begin{tikzcd} W_{\alpha\beta\gamma} \arrow[r, "\pi_{Y,\alpha\beta\gamma}"] \arrow[d, "\varphi_{\alpha\beta}|_{W_{\alpha\beta\gamma}}"'] & Y_{\alpha\beta\gamma} \arrow[d, equal] \\ W_{\beta\gamma\alpha} \arrow[r, "\pi_{Y,\beta\gamma\alpha}"] \arrow[d, "\varphi_{\beta\gamma}|_{W_{\beta\gamma\alpha}}"'] & Y_{\beta\alpha\gamma} \arrow[d, equal] \\ W_{\gamma\alpha\beta} \arrow[r, "\pi_{Y,\gamma\alpha\beta}"] & Y_{\gamma\alpha\beta} \end{tikzcd} \qquad \begin{tikzcd} W_{\alpha\beta\gamma} \arrow[r, "\pi_{X,\alpha\beta\gamma}"] \arrow[d, "\varphi_{\alpha\beta}|_{W_{\alpha\beta\gamma}}"'] & X \arrow[d, equal] \\ W_{\beta\gamma\alpha} \arrow[r, "\pi_{X,\beta\gamma\alpha}"] \arrow[d, "\varphi_{\beta\gamma}|_{W_{\beta\gamma\alpha}}"'] & X \arrow[d, equal] \\ W_{\gamma\alpha\beta} \arrow[r, "\pi_{X,\gamma\alpha\beta}"] & X \end{tikzcd}$$

(the left rectangle is the top triangle, and the right rectangle is the left triangle), and we note that everything commutes by construction of the morphisms on the left.

6. The above steps allow us to glue together the W_α along the isomorphisms $\varphi_{\alpha\beta}$ to get a single scheme W by Proposition 2.48. Explicitly, W has an open cover $\{W'_\alpha\}_{\alpha\in\lambda}$ and embeddings $\iota'_\alpha\colon W_\alpha \cong W_{\alpha'}$ such that $\iota'_\alpha(W_{\alpha\beta}) = W_\alpha \cap W_\beta$ and $\iota'_\beta \circ \varphi_{\alpha\beta} = \iota'_\alpha|_{W_{\alpha\beta}}$.

   However, the only property we needed for W_α is that it is the fiber product of $X \times_S Y_\alpha$, so we will (for psychological reasons) go ahead and identify W_α with its image in W so that W has an open subscheme $W_\alpha \subseteq W$ which is a fiber product of $X \times_S Y_\alpha$.

   Notably, the gluing process, along with the above identification, tells us that $W_\alpha \cap W_\beta$ was isomorphic to both $\iota'_\alpha(W_{\alpha\beta})$ and $\iota'_\beta(W_{\beta\alpha})$ and therefore serves as a fiber product $X \times_S Y_{\alpha\beta}$. However, earlier we had to use the isomorphism $\varphi_{\alpha\beta}$ to translate between these two, but $\iota'_\beta \circ \varphi_{\alpha\beta} = \iota'_\alpha$ tells us that $\varphi_{\alpha\beta}$ becomes literally the identity in W.

   Now, we had maps $\pi_{X,\alpha}\colon W_\alpha \to X$ for each $\alpha \in \lambda$ such that

$$\pi_{X,\alpha}|_{W_{\alpha\beta}} = \pi_{X,\beta}|_{W_{\beta\alpha}} \circ \varphi_{\alpha\beta},$$

   which again upon identifying everything into W will glue by Proposition 2.44 to grant us a unique morphism $\pi_X\colon W \to X$ such that $\pi_X|_{W_\alpha} = \pi_{X,\alpha}$.

   Continuing, we note that we had maps $\pi_{Y,\alpha}\colon W_\alpha \to Y_\alpha$ for each $\alpha \in \lambda$ such that

$$\pi_{Y,\alpha}|_{W_{\alpha\beta}} = \pi_{Y,\beta}|_{W_{\beta\alpha}} \circ \varphi_{\alpha\beta}. \tag{2.15}$$

   Once we've identified everything into W, (2.15) tells us that the morphisms $\pi_{Y,\alpha}$ glue together by Proposition 2.44 to a unique morphism $\pi_Y : W \to Y$ such that $\pi_Y|_{W_\alpha} = \pi_{Y,\alpha}$. (Technically, one must post-compose $\pi_{Y,\alpha}$ with the embedding $Y_\alpha \hookrightarrow Y$ first, and then note that we can also post-compose (2.15) with $Y_{\alpha\beta} \hookrightarrow Y$, but this causes no problems.)

   We take a moment to recognize that $\pi_Y^{-1}(Y_\alpha) = W_\alpha$. Indeed, if $w \in W$ has $\pi_Y(w) \in Y_\alpha$, then we can at least place w in some W_β so that $\pi_Y(w) = \pi_{Y,\beta}(w) \in Y_\beta$. But then $\pi_{Y,\beta}(w) \in Y_{\alpha\beta}$, so $w \in W_{\alpha\beta} \subseteq W_\beta$, so $w \in W_\alpha$.

7. We now verify the universal property. Fix morphisms $\varphi_X : Z \to X$ and $\varphi_Y : Y \to Z$ making the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\ & W \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "\psi_Y"] \\ & X \arrow[r, "\psi_X"'] & S \end{tikzcd} \tag{2.16}$$

   commute, and we want to induce a unique morphism $\varphi\colon Z \to W$ making the diagram commute. We show uniqueness and existence separately.

- Uniqueness: suppose we have some $\varphi\colon Z \to W$ making (2.16) commute. Restricting Y with Y_α, we see that φ_Y will restrict to $\varphi_{Y,\alpha} \coloneqq \varphi_Y|_{\varphi_Y^{-1}Y_\alpha}$ by Lemma 2.9. Similarly, we should replace W with $\pi_Y^{-1}(Y_\alpha) = W_\alpha$ so that π_Y becomes $\pi_{Y,\alpha}$ and π_X becomes $\pi_{X,\alpha}$, and we should replace Z with $Z_\alpha \coloneqq \varphi_Y^{-1}(Y_\alpha)$. Notably, $\varphi|_{Z_\alpha}\colon Z_\alpha \to W_\alpha$ makes sense as restricted by Lemma 2.9. Lastly, we should replace φ_X with $\varphi_{X,\alpha} \coloneqq \varphi_X|_{Z_\alpha}$.
  In total functoriality of restricting morphisms tells us that the diagram
$$\begin{tikzcd}
Z_\alpha \arrow[dr, dashed, "\varphi|_{Z_\alpha}"] \arrow[drr, bend left, "\varphi_{Y,\alpha}"] \arrow[ddr, bend right, "\varphi_{X,\alpha}"'] \\
& W_\alpha \arrow[r, "\pi_{Y,\alpha}"'] \arrow[d, "\pi_{X,\alpha}"] & Y_\alpha \arrow[d, "\psi_{Y,\alpha}"] \\
& X \arrow[r, "\psi_X"'] & S
\end{tikzcd}$$
  commutes; namely, the two triangles commute by applying Remark 2.43 everywhere—the top triangle has restricted Y to Y_α, and the left triangle has restricted W to W_α. However, we note that the above diagram features a pullback square, so the morphisms $\varphi|_{Z_\alpha}\colon Z_\alpha \to W_\alpha$ are uniquely determined. It follows that the morphisms $\varphi|_{Z_\alpha}\colon Z_\alpha \to W$ are uniquely determined, so Proposition 2.44 tells us that φ itself is uniquely determined.
- Existence: we unwrap the construction above. Setting variables as in the previous step (except φ of course), we see that the diagram
$$\begin{tikzcd}
Z_\alpha \arrow[rr, "\varphi_{Y,\alpha}"] \arrow[dr, hook] \arrow[ddr, bend right, "\varphi_{X,\alpha}"'] & & Y_\alpha \arrow[d, hook] \\
& Z \arrow[r, "\varphi_Y"] \arrow[d, "\varphi_X"] & Y \arrow[d, "\psi_Y"] \\
& X \arrow[r, "\psi_X"] & S
\end{tikzcd}$$
  commutes for any $\alpha \in \lambda$, by the functoriality of restriction (namely, apply Remark 2.43 everywhere). However, this will induce a unique morphism $\varphi_\alpha\colon Z \to W_\alpha$ making the diagram
$$\begin{tikzcd}
Z_\alpha \arrow[dr, dashed, "\varphi_\alpha"] \arrow[drr, bend left, "\varphi_{Y,\alpha}"] \arrow[ddr, bend right, "\varphi_{X,\alpha}"'] \\
& W_\alpha \arrow[r, "\pi_{Y,\alpha}"'] \arrow[d, "\pi_{X,\alpha}"] & Y_\alpha \arrow[d, "\psi_{Y,\alpha}"] \\
& X \arrow[r, "\psi_X"'] & S
\end{tikzcd}$$
  commute. To glue the morphisms φ_α together, we restrict our diagram to $Y_\alpha \cap Y_\beta$. Then we replace W_α with $W_\alpha \cap W_\beta$ and Z_α with $Z_\alpha \cap Z_\beta$, so functoriality of restriction (namely, Remark 2.43) tells us that
$$\begin{tikzcd}
Z_\alpha \cap Z_\beta \arrow[dr, dashed, "\varphi_\alpha|_{Z_\alpha \cap Z_\beta}"] \arrow[drr, bend left, "\varphi_Y|_{Z_\alpha \cap Z_\beta}"] \arrow[ddr, bend right, "\varphi_X|_{Z_\alpha \cap Z_\beta}"'] \\
& W_\alpha \cap W_\beta \arrow[r, "\pi_Y|_{W_\alpha \cap W_\beta}"'] \arrow[d, "\pi_X|_{W_\alpha \cap W_\beta}"] & Y_\alpha \cap Y_\beta \arrow[d, "\psi_Y|_{W_\alpha \cap W_\beta}"] \\
& X \arrow[r, "\psi_X"'] & S
\end{tikzcd}$$
  commutes. However, the morphism $Z_\alpha \cap Z_\beta \to W_\alpha \cap W_\beta$ is unique making this diagram commute because the square is a pullback square, under our identification of $W_\alpha \subseteq W$. (Namely, our

projections are $\pi_{Y,\alpha\beta} = \pi_{Y,\alpha}|_{W_{\alpha\beta}} = \pi_Y|_{W_\alpha \cap W_\beta}$, and similar for X.) Thus, swapping all αs and βs in the above diagram changes nothing except this morphism, so we conclude that

$$\varphi_\alpha|_{Z_\alpha \cap Z_\beta} = \varphi_\beta|_{Z_\alpha \cap Z_\beta}$$

for any $\alpha, \beta \in \lambda$. Because the Y_α cover Y, we see that the Z_α cover Z, so we do indeed glue into a morphism $\varphi\colon Z \to W$ such that $\varphi|_{Z_\alpha} = \varphi_\alpha$. (Technically, to glue, we have to post-compose with the embeddings into W everywhere, but this causes no problems.)

Now, for any Z_α, we see that

$$(\pi_Y \circ \varphi)|_{Z_\alpha} = \pi_Y|_{W_\alpha} \circ \varphi|_{Z_\alpha} = \pi_{Y,\alpha} \circ \varphi_\alpha = \varphi_{Y,\alpha} = \varphi_Y|_{Z_\alpha}$$

by repeatedly using functoriality of restriction (Remark 2.43). Because the $\{Z_\alpha\}_{\alpha\in\lambda}$ form an open cover of Z, Proposition 2.44 tells us that $\pi_Y \circ \varphi = \varphi_Y$. Replacing all Ys with Xs in the above argument tells us that $\pi_X \circ \varphi = \varphi_X$.

The above steps have been able to glue together a fiber product and show it satisfies the universal property. This finishes. ■

2.6.5 Fiber Products: Gluing the Base

In order to give the gluing data for gluing on the base, we should pick up the following lemma.

Lemma 2.185. Fix schemes X and Y over a scheme S such that $X \times_S Y$ exists. Then for any open subscheme $S' \subseteq S$, the scheme $X \times_S Y$ satisfies the universal property of $X \times_{S'} Y$.

Proof. We are given that

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

is a pullback square. However, $\jmath\colon S' \to S$ is an open embedding and hence monic by Corollary 2.179, so it follows that

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\iota\circ\psi_X"] \\ Y \arrow[r, "\iota\circ\psi_Y"] & S' \end{tikzcd}$$

is a pullback square by Lemma 2.175. This finishes. ■

Lastly, here is the general case.

Proof of Theorem 2.170. Give S an affine open cover $\{S_\alpha\}_{\alpha\in\lambda}$, with our natural maps $f_X\colon X \to S$ and $f_Y\colon Y \to S$. Then we set $X_\alpha := f_X^{-1}(S_\alpha)$ and similar for Y_α. By previous work, we have the fiber products $X_\alpha \times_{S_\alpha} Y_\alpha$, which we may glue together to finish. Indeed, the large rectangle of

$$\begin{tikzcd} X_\alpha \times_{S_\alpha} Y_\alpha \arrow[r, "\pi_{X,\alpha}"] \arrow[d, "\pi_{Y,\alpha}"'] & X_\alpha \arrow[r, hook] \arrow[d, "\psi_X|_{X_\alpha}"] & X \arrow[d, "\psi_X"] \\ Y_\alpha \arrow[r, "\psi_{Y,\alpha}"'] & S_\alpha \arrow[r, hook] & S \end{tikzcd}$$

is a pullback square by Lemma 2.174: the left square is a pullback by construction of $X_\alpha \times_{S_\alpha} Y_\alpha$, and the right square is a pullback by Lemma 2.178. Thus, $X_\alpha \times_{S_\alpha} Y_\alpha$ is canonically isomorphic to a fiber product of $X \times_S Y_\alpha$, and now we just have to glue along Y_α via Lemma 2.183. ■

It will be useful later to have a convenient open cover $X \times_S Y$ later on, especially when we want to reduce S to being affine. As such, we pick up the following lemma.

Lemma 2.186. Fix schemes X and Y with morphisms $\psi_X \colon X \to S$ and $\psi_Y \colon Y \to S$ inducing the fiber product $X \times_S Y$ with canonical projections $\pi_X \colon X \times_S Y \to X$ and $\pi_Y \colon X \times_S Y$. Set $\pi_S \coloneqq \psi_X \circ \pi_X = \psi_Y \circ \pi_Y$. Then, for any open subscheme $S' \subseteq S$, the following is a pullback square.

$$\begin{tikzcd}
\pi_S^{-1}(S') \arrow[r, "\pi_X|_{\pi_S^{-1}(S')}"] \arrow[d, "\pi_Y|_{\pi_S^{-1}(S')}"'] & \psi_X^{-1}(S') \arrow[d, "\psi_X|_{\psi_X^{-1}(S')}"] \\
\psi_Y^{-1}(S') \arrow[r, "\psi_Y|_{\psi_Y^{-1}(S')}"] & S'
\end{tikzcd}$$

Proof. The point is to chase around the following commutative cube.

$$\begin{tikzcd}
\pi_S^{-1}(S') \arrow[rr, "\pi_Y|_{\pi_S^{-1}(S')}"] \arrow[dr, hook] \arrow[dd, "\pi_X|_{\pi_S^{-1}(S')}"'] & & \psi_X^{-1}(S') \arrow[dr, hook] \arrow[dd, "\psi_X|_{\psi_X^{-1}(S')}" near start] & \\
& X \times_S Y \arrow[rr, "\pi_X" near start] \arrow[dd, "\pi_Y" near start] & & X \arrow[dd, "\psi_X"] \\
\psi_Y^{-1}(S') \arrow[rr, "\psi_Y|_{\psi_Y^{-1}(S')}" near end] \arrow[dr, hook] & & S' \arrow[dr, hook] & \\
& Y \arrow[rr, "\psi_Y"] & & S
\end{tikzcd}$$

Namely, the front face commutes by construction of the fiber product, each adjacent face commutes by Remark 2.10, and the back face commutes by Remark 2.43.

We know that the front face is a pullback square, and we know that the adjacent squares are also pullback squares. We want to show that back face is a pullback square. To begin, note that the right square of

$$\begin{tikzcd}
\pi_S^{-1}(S') \arrow[r, "\pi_X|_{\pi_S^{-1}(S')}"] \arrow[d, "\pi_Y|_{\pi_S^{-1}(S')}"'] & \psi_X^{-1}(S') \arrow[r, hook] \arrow[d, "\psi_X|_{\psi_X^{-1}(S')}"] & X \arrow[d, "\psi_X"] \\
\psi_Y^{-1}(S') \arrow[r, "\psi_Y|_{\psi_Y^{-1}(S')}"] & S' \arrow[r, hook] & S
\end{tikzcd}$$

is a pullback square, so to show that the left square is a pullback square, it suffices by Lemma 2.173 to show that the big rectangle is a pullback square. Now, using the commutative cube, this is the same as showing that the outer rectangle of

$$\begin{tikzcd}
\pi_S^{-1}(S') \arrow[r, hook] \arrow[d, "\pi_Y|_{\pi_S^{-1}(S')}"'] & X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "\psi_X"] \\
\psi_Y^{-1}(S') \arrow[r, hook] & Y \arrow[r, "\psi_Y"] & S
\end{tikzcd}$$

is a pullback square. However, the right square is a pullback square by hypothesis, and the left square is a pullback square by Lemma 2.178, so the full rectangle is a pullback square by Lemma 2.174. ■

Remark 2.187. We will provide a more categorical viewpoint of this construction next class. This categorical viewpoint will be helpful for when we want to define the Grassmannian.

2.7 September 21

Today we return to fiber products and discuss some applications.

2.7.1 Representability

We start with a few definitions.

Definition 2.188 (Zariski sheaf)**.** A functor $F\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ is a *Zariski sheaf* if and only if F is a sheaf when the scheme is viewed as merely a topological space. Namely, each scheme T has

$$0 \to F(T) \to \prod_i F(T_i) \to \prod_{i,j} F(T_i \cap T_j)$$

exact for any open cover $\{T_i\}$ of T.

Definition 2.189 (Open subfunctor)**.** Fix functors $F, F'\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$. Then $F' \subseteq F$ is an *open subfunctor* if and only if each scheme T, every natural transformation $\psi\colon h_T \Rightarrow F$ yielding a pullback square

$$\begin{tikzcd} F_{i,\psi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & h_T \arrow[d] \\ F_i \arrow[r] & F \end{tikzcd}$$

already has each $F_{i,\psi}$ represented by a scheme T_i with the natural transformation $F_{i,\psi} \hookrightarrow h_T$ given by an open embedding $T_i \hookrightarrow T$.

Here is an abstract lemma.

Lemma 2.190. A functor $F\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ is representable if and only if the following conditions are satisfied.

- F is a Zariski sheaf.
- Locally representable: there are representable subfunctors $F_i \subseteq F$ for each i, and $F_i \subseteq F$ is an open subfunctor such that $\{F_i\}$ covers F. Here, covering means that each field K has $F(\operatorname{Spec} K) = \bigcup_i F_i(\operatorname{Spec} K)$.

The point is that each F_i is represented by some X_i, and we just want to glue these X_i together. This is the idea of the proof.

Remark 2.191. One can replace $\mathrm{Sch}^{\mathrm{op}}$ with Ring or $\mathrm{Sch}_S^{\mathrm{op}}$.

Remark 2.192. One can show that Lemma 2.190 implies that the fiber product exists. Namely, the fiber product forms a Zariski sheaf, which we can see from the part where we glued to make W in the key case. Then the F_i come from the purely affine case, which was comparatively easier. Lastly, the F_i cover F roughly speaking comes from the rest of the proof.

We will not need Lemma 2.190 for the time being.

2.7.2 Fibers

As a first application, we discuss fibers. Given a scheme morphism $\varphi\colon Y \to S$, we might be interested in the fibers here to pull-back. Namely, pulling back to a "subscheme" X of S, we can imagine the fibers of Y over

X as the fiber product, as in the following diagram.

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

For example, in the case of $Y = \operatorname{Spec} k[x,y,z]/\left(y^2 - x(x-1)(x-s)\right)$ and $S = \operatorname{Spec} k[s]$, we see that we have the obvious map $Y \to S$ sending $x, y \mapsto 0$.

Now, if we want to understand the fiber at a given point $s_0 \in S$ with $s_0 \in k$ for concreteness, the corresponding scheme is a $\operatorname{Spec} k$ over $\operatorname{Spec} S$ induced by the map $k[s] \to k$ by $s \mapsto s_0$. Then we can track our fibers in this affine case as given by

$$\begin{tikzcd} Y \times_{\operatorname{Spec} k[s]} \operatorname{Spec} k \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[s] \end{tikzcd}$$

where we can compute directly from commutative algebra that

$$Y \times_{\operatorname{Spec} k[s]} \operatorname{Spec} k \simeq \operatorname{Spec} \frac{k[x,y]}{(y^2 - x(x-1)(x-s_0))}$$

here.

As such, we are convinced that the following is a good definition of a fiber.

Definition 2.193 (Fiber)**.** Fix a scheme Y over a scheme S. Given a point $s_0 \in S$, the fiber product $Y \times_S \{s_0\}$ is the *fiber* of $Y \to S$ over s_0.

Remark 2.194. If $X \hookrightarrow S$ is a closed embedding, then $Y \times_S X \hookrightarrow Y$ is a closed embedding as well. In particular, if $s_0 \in S$ is a closed point, then our fiber is in fact a closed embedding. More generally, if S is irreducible with generic point η, we call $Y \times_S \{\eta\}$ the generic fiber.

Notably, we can check purely topologically that the fiber defined by the fiber product is the correct fiber at a point. Here is the affine case.

Lemma 2.195. Fix a ring homomorphism $\pi \colon A \to B$ and a prime $\mathfrak{p} \in \operatorname{Spec} A$, and let $S := \pi^{-1}(A \setminus \mathfrak{p})$. Then

$$\operatorname{Spec} k(\mathfrak{p}) \times_{\operatorname{Spec} A} \operatorname{Spec} B \simeq \operatorname{Spec} S^{-1}B/\mathfrak{p}(S^{-1}B)$$

and is homeomorphic to $(\operatorname{Spec} \pi)^{-1}(\{\mathfrak{p}\})$.

Proof. Note $k(\mathfrak{p}) = (A/\mathfrak{p})_\mathfrak{p}$. Now, we see from the construction of the fiber product that

$$\operatorname{Spec} k(\mathfrak{p}) \times_{\operatorname{Spec} A} \operatorname{Spec} B \simeq \operatorname{Spec} B \otimes_A (A/\mathfrak{p})_\mathfrak{p}.$$

We now note the isomorphisms

$$\begin{aligned} B \otimes_A (A/\mathfrak{p})_\mathfrak{p} &\simeq B \otimes_A A_\mathfrak{p} \otimes_A A/\mathfrak{p} \\ &\simeq S^{-1}B \otimes_A A/\mathfrak{p} \\ &\simeq S^{-1}B/\mathfrak{p}(S^{-1}B), \end{aligned}$$

which finishes the first claim. For the homeomorphism, we know that the localization map $B \to S^{-1}B$ induces a homeomorphism

$$\operatorname{Spec} S^{-1}B \simeq \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \cap \pi(S) = \varnothing\} = \{\mathfrak{q} \in \operatorname{Spec} B : \mathfrak{q} \cap \pi(S) = \varnothing\} = \{\mathfrak{q} \in \operatorname{Spec} A : \pi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}\}.$$

Next up, the projection map $S^{-1}B \twoheadrightarrow \operatorname{Spec} S^{-1}B/\mathfrak{p}(S^{-1}B)$ induces a homeomorphism

$$\begin{aligned}\operatorname{Spec} S^{-1}B/\mathfrak{p}(S^{-1}B) &\cong \{\mathfrak{q} \in \operatorname{Spec} S^{-1}B : \mathfrak{q} \supseteq \mathfrak{p}(S^{-1}B)\} \\ &\cong \{\mathfrak{q} \in \operatorname{Spec} S^{-1}B : \mathfrak{q} \supseteq \pi\mathfrak{p}\} \\ &\cong \{\mathfrak{q} \in \operatorname{Spec} B : \pi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p} \text{and} \mathfrak{q} \supseteq \mathfrak{p}\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B : \pi^{-1}\mathfrak{q} = \mathfrak{p}\},\end{aligned}$$

which is the needed $(\operatorname{Spec}\pi)^{-1}(\{\mathfrak{p}\})$. Note that we have used our previous homeomorphism in the second $\cong$. ■

Remark 2.196. Tracking through the above proof shows that the canonical map $B \to S^{-1}B/\mathfrak{p}(S^{-1}B)$ is providing the needed homeomorphism.

We now proceed with the proof.

Lemma 2.197. Fix a scheme morphism $\varphi\colon X \to Y$ and a point $y \in Y$. Fixing $X_y := X \times_Y \{y\}$, the canonical projection $X_y \to X$ is a homeomorphism onto $\varphi^{-1}(\{y\})$.

Proof. We optimize the proof that the fiber product exists, essentially skipping over all the checks because they are covered in any of the books (as well as in class). To begin, place $y \in Y$ inside an affine open subset $U \subseteq Y$, which induces the commutative diagram

$$\begin{tikzcd} X_y \arrow[r, dashed, "\iota'"] \arrow[d, "\pi"'] & \varphi^{-1}U \arrow[r, "\jmath'"] \arrow[d] & X \arrow[d, "\varphi"] \\ \{y\} \arrow[r, hook, "\iota"] & U \arrow[r, hook, "\jmath"] & Y \end{tikzcd}$$

where in particular $\iota'\colon X \to \varphi^{-1}U$ is induced because the right square is a pullback square (it's made by open embeddings), so we can use the maps $X_y \to \{y\} \to U$ and $X_y \to X$ to induce ι by pullback.

Quickly, we note that the left square is a pullback square by Lemma 2.173. As such, we may replace Y with $U = \operatorname{Spec} A$ and X with $\varphi^{-1}U$ and X_y with $U_y = \varphi^{-1}U \times_Y \{y\}$ without actually changing the fiber product. Applying the suitable isomorphisms everywhere, we may directly assume that $Y = \operatorname{Spec} A$ for a ring A.

Now, give X an affine open cover $\widetilde{X}$. Letting $\iota'\colon X_y \to X$ and $\iota\colon \{y\} \to X$ be our embeddings, we note that the diagram

$$\begin{tikzcd} (\iota')^{-1}U_\alpha \arrow[r, "\iota'"] \arrow[d, hook] & U_\alpha \arrow[d, hook] \\ X_y \arrow[r, "\iota'"] \arrow[d, "\pi"'] & X \arrow[d, "\varphi"] \\ \{y\} \arrow[r, "\iota"] & Y \end{tikzcd}$$

has both squares as pullback squares (the top is an open embedding square, and the bottom is by definition of X_y), so we see that $(\iota')^{-1}(U_\alpha) = U_{\alpha,y}$. However, from the affine case, we know that the map $\iota'\colon (\iota')^{-1}U_\alpha \to U_\alpha$ provides a homeomorphism of $(\iota')^{-1}(U_\alpha)$ onto $U_\alpha \cap \varphi^{-1}(\{y\})$. Thus, we see that the full map $\iota'\colon X_y \to X$ provides a homeomorphism of X_y onto

$$\bigcup_{\alpha\in\lambda} \iota'\left((\iota')^{-1}U_\alpha\right) = \bigcup_{\alpha\in\lambda} (\varphi^{-1}(\{y\}) \cap U_\alpha) = \varphi^{-1}(\{y\}).$$

This finishes the proof. ■

2.7.3 Base Extension

We again begin with a special case. Take $S = \operatorname{Spec} K$ to be our base and $X = \operatorname{Spec} K'$ where K'/K is a field extension; the embedding $K \hookrightarrow K'$ induces a map $X \to S$.

Now, if we have a scheme Y over S, we might want to pull Y back to a scheme over X, where we are applying some base-change operation. To do this, we unsurprisingly want the fiber product, as in the following diagram.

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

As an example, if we have $Y = \operatorname{Spec} K[x,y]/\left(y^2 - x^3 + x\right)$, we can compute that

$$X \times_S Y = \operatorname{Spec} \frac{K'[x,y]}{(y^2 - x^3 + x)},$$

which agrees with our intuition of what base-change should do.

Exercise 2.198. Given a ring A and a nonnegative integer n, the canonical morphism $\mathbb{Z} \to A$ makes the diagram

$$\begin{tikzcd} \mathbb{P}^n_A \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \mathbb{P}^n_{\mathbb{Z}} \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

into a pullback square.

Proof. We have to track open covers; let $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ be the canonical map induced by $\mathbb{Z} \to A$ on global sections. Now, define

$$X_i := \operatorname{Spec} \frac{\mathbb{Z}[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)}$$

so that the X_i form an open cover of $\mathbb{P}^n_{\mathbb{Z}}$ by the usual isomorphisms. Explicitly, we set

$$X_{ij} := D(x_{j/i}) \simeq \operatorname{Spec} \frac{\mathbb{Z}[x_{0/i}, \ldots, x_{n/i}, x_{j/i}^{-1}]}{(x_{i/i} - 1)}$$

and use isomorphisms $\varphi_{ij}\colon X_{ij} \to X_{ji}$ by the ring maps $x_{k/i} \mapsto x_{k/j}/x_{i/j}$.

By the construction of Lemma 2.183, we can give $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$ an affine open cover by using

$$X_i \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A = \operatorname{Spec} \left(\frac{\mathbb{Z}[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)} \otimes_{\mathbb{Z}} A \right)$$

by Lemma 2.176, but this ring is canonically isomorphic to $A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i-1})$ by $f \otimes a \mapsto af$.[4] Thus, we see $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$ has an affine open cover by

$$Y_i := \operatorname{Spec} \frac{A[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)},$$

which are precisely the affine open subschemes covering $\mathbb{P}^n_A$.

[4] One can check that $\mathbb{Z}[x] \otimes_{\mathbb{Z}} A \simeq A[x]$ by this isomorphism, where the inverse morphism is by $\sum_{k=0}^n a_k x^k \mapsto \sum_{k=0}^n a_k \otimes x^k$. Then the quotient follows by the right-exactness of $- \otimes_{\mathbb{Z}} A$.

It remains to check the gluing data. Under the canonical isomorphism $X_i \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A \simeq Y_i$, we see that the affine open subscheme $X_{ij} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$ goes to

$$Y_{ij} := D(x_{j/i}) \simeq \operatorname{Spec} \frac{A[x_{0/i}, \ldots, x_{n/i}, x_{j/i}^{-1}]}{(x_{i/i} - 1)},$$

with the correct gluing data! To see this, we have two steps.

1. Note that the canonical projection $X_i \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A \to X_i$ corresponds to the ring map given by $x_{k/i} \mapsto x_{k/i} \otimes 1$, by Lemma 2.176, so computations of Remark 2.24 tell us that this pre-image is $D(x_{j/i} \otimes 1)$.

   While we're here, we recall that the isomorphisms $\varphi_{ij} \colon D(x_{j/i}) \to D(x_{i/j})$ are given by the ring maps $x_{k/j} \mapsto x_{k/i}/x_{j/i}$. So the corresponding isomorphisms $D(x_{j/i} \otimes 1) \to D(x_{i/j} \otimes 1)$ are given by $x_{k/j} \otimes 1 \mapsto x_{k/i}/x_{j/i} \otimes 1$.

2. Next, the canonical isomorphism $Y_i \cong X_i \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$ is given by the ring map $a \otimes x_i \mapsto ax_i$, so computations of Remark 2.24 tell us that the pre-image of $D(x_{j/i} \otimes 1)$ is just $D(x_{j/i})$, which is what we wanted.

   But further, the corresponding isomorphisms $D(x_{j/i}) \to D(x_{i/j})$ from the isomorphisms $D(x_{j/i} \otimes 1) \to D(x_{i/j} \otimes 1)$ above can be projected back down to see that they are

   $$x_{k/j} \mapsto x_{k/i}/x_{j/i},$$

   which is the correct one.

So indeed, we see that the affine open cover of $\mathbb{P}^n_{\mathbb{Z}}$ lists naturally to the affine open cover of $\mathbb{P}^n_A$. This finishes. ■

The above exercise motivates the following definition.

Definition 2.199 (Projective space)**.** Given a scheme S and a nonnegative integer n, we define *projective n-space over S* to be $\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$.

Note this definition agrees with our affine case by Exercise 2.198.

This notation is a bit cumbersome, so we will abbreviate it.

Notation 2.200. Fix a morphism $S' \to S$. If X is a scheme over S, we might denote the base-change of X to a scheme over S' as $X_{S'} = X \times_S S'$.

Remark 2.201. Given a field extension K'/K and a scheme X over $\operatorname{Spec} K$, we can check that $X_{K'}(K') = X(K')$. This is purely formal: a morphism $\operatorname{Spec} K' \to X$ induces a unique morphism $\operatorname{Spec} K' \to X_{K'}$ making the diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, equal, bend right] & & \\
& X_{K'} \arrow[r] \arrow[d] & X \arrow[d] \\
& \operatorname{Spec} K' \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

commute. Notably, we are using the identity map $\operatorname{Spec} K' \to \operatorname{Spec} K'$ because $X(K')$ consists of the K-morphisms $\operatorname{Spec} K' \to X$.

In applications, the base-change to the algebraic closure will be especially important because certain aspects become clear only once passing to an algebraic closure. This gives the following definition.

Definition 2.202 (Goemetrically irreducible, reduced, connected)**.** A scheme X over a field K is *geometrically irreducible/reduced/connected* if and only if $X_{\overline{K}}$ is irreducible/reduced/connected.

Example 2.203. Fix the scheme $X = \operatorname{Spec}\mathbb{Q}(\sqrt{2})$ over the scheme $\operatorname{Spec}\mathbb{Q}$. Even though X is irreducible, it is not geometrically irreducible because $X_{\overline{\mathbb{Q}}}$ becomes two copies of $\operatorname{Spec}\overline{\mathbb{Q}}$. Indeed,

$$X_{\overline{\mathbb{Q}}} = \operatorname{Spec}(\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) = \operatorname{Spec}(\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}) = \operatorname{Spec}\overline{\mathbb{Q}} \sqcup \operatorname{Spec}\overline{\mathbb{Q}}.$$

Namely, $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$ by decomposing some element $(a + b\sqrt{2}) \otimes \alpha$ as $1 \otimes a\alpha + \sqrt{2} \otimes b\alpha$.

Intuitively, being irreducible but not geometrically irreducible means that passing to the algebraic closure gives rise to Galois conjugate pieces. This allows us to separate geometric information from Galois information.

2.7.4 The Relative Frobenius

For concreteness, fix a scheme S over $\mathbb{F}_p$. Notably, a scheme X over S has a map $X \to S$, so because the sheaf $p\mathcal{O}_S$ vanishes, we have that $p\mathcal{O}_X$ will also vanish. The point is that the pth-power map $f \mapsto f^p$ is going to induce a scheme morphism $F_X \colon X \to X$.

To see morphism topologically, let's see an example. When $X = \operatorname{Spec} A$ is affine, we see that A is an $\mathbb{F}_p$-algebra, and the Frobenius mapping $\varphi \colon a \mapsto a^p$ we can see directly sends $\mathfrak{p}$ by φ^{-1} to itself. Thus, the Frobenius is indeed nothing at all topologically.

Remark 2.204. The above morphism is called the absolute Frobenius.

Now, we have another Frobenius morphism $F_S \colon S \to S$, and we can see that the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

commutes. However, the fiber product now promises us a morphism $X \to X \times_S S$, where the two Ss are different copies with a Frobenius morphism going between them.

Remark 2.205. The above morphism is called the relative Frobenius.

Example 2.206. With $S = \operatorname{Spec}\mathbb{F}_p[s]$ and $X = \operatorname{Spec}\mathbb{F}_p[s, x]$, the relative Frobenius keeps s fixed and sends $x \mapsto x^p$.

Theme 3
Morphisms of Schemes

I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial.

—Andre Weil, [Wei59]

3.1 September 23

Today we place some finiteness conditions on morphisms.

3.1.1 Quasicompact and Quasiseparated

For today, we will denote a morphism of schemes $\pi\colon X \to Y$.

Definition 3.1 (Quasicompact)**.** A scheme morphism $\pi\colon X \to Y$ is *quasicompact* if and only if all affine open subschemes $U \subseteq Y$ make $\pi^{-1}(U) \subseteq X$ a quasicompact topological space.

Example 3.2. Fix a morphism $\pi\colon X \to Y$, where X is a Noetherian scheme and hence a Noetherian space. Then π is quasicompact because any (affine) open subset $U \subseteq Y$ makes $\pi^{-1}U \subseteq X$ an open and hence quasicompact set, where we are using Lemma 2.116.

Example 3.3. Any closed embedding $\pi\colon X \to Y$ is quasicompact. Indeed, given an affine open subset $U \subseteq Y$, we note that $\pi^{-1}(U)$ is homeomorphic through π to $\pi(X) \cap U$. However, U is quasicompact (it's affine), so the closed subset $\pi(X) \cap U$ is a closed subset in a quasicompact space and therefore quasicompact.

Example 3.4. Of course, isomorphisms of schemes are homeomorphisms by Lemma 2.8, so isomorphisms are quasicompact.

Non-Example 3.5. An open embedding need not be quasicompact. For example, an affine scheme can have open subschemes which are not quasicompact.

To define quasiseparated, we will need to have the adjective on topological spaces.

Definition 3.6 (Quasiseperated)**.** A topological space X is *quasiseparated* if and only if the intersection of two quasicompact open subsets is still quasicompact.

Example 3.7. Any Noetherian space is quasiseparated because any open subset is quasicompact by Lemma 2.116.

Example 3.8. Locally Noetherian schemes X are quasiseparated: given quasicompact open subsets $U, V \subseteq X$, give U a finite affine open covers $\{U_i\}_{i=1}^n$. Then U_i is an affine scheme of a Noetherian ring, so U_i is a Noetherian space by Example 2.111. Thus, $U_i \cap V$ is quasicompact for each i by Lemma 2.116, so the finite union $U \cap V = \bigcup_{i=1}^n (U_i \cap V)$ is still quasicompact.

Example 3.9. If X is a quasiseparated space, and $U \subseteq X$ is an open subset, then U is still quasiseparated: any quasicompact open subsets $U_1, U_2 \subseteq U$ are also quasicompact open subsets of X, which means $U_1 \cap U_2$ is quasicompact because X is quasiseparated.

Example 3.10. Affine schemes $X \cong \operatorname{Spec} A$ are quasiseparated. Indeed, any quasicompact open subschemes $U, V \subseteq X$ can be given finite distinguished open covers $\{D(f_i)\}_{i=1}^m$ for U and $\{D(g_j)\}_{j=1}^n$ for V, which then by Remark 1.56 has the finite affine open cover given by

$$U \cap V = \left(\bigcup_{i=1}^m D(f_i)\right) \cap \left(\bigcup_{j=1}^n D(g_j)\right) = \bigcup_{i=1}^m \bigcup_{i=1}^n \underbrace{D(f_i) \cap D(g_j)}_{D(f_i g_j)}.$$

And here is our definition.

Definition 3.11 (Quasiseparated)**.** A scheme morphism $\pi \colon X \to Y$ is *quasiseparated* if and only if all affine open subschemes $U \subseteq Y$ makes $\pi^{-1}(U)$ a quasiseparated topological space.

Remark 3.12. Equivalently, $\pi \colon X \to Y$ is quasiseparated if and only if given any affine open subset $U \subseteq Y$ and more affine open subsets $V_1, V_2 \subseteq \pi^{-1}(U)$, we can give $V_1 \cap V_2$ a finite affine open cover.

- If π is quasiseparated, then $\pi^{-1}(U)$ is quasiseparated, so $V_1 \cap V_2$ is quasicompact and thus has a finite affine open cover (from any affine open cover).
- If π satisfies the condition, suppose we have quasicompact subsets $V_1, V_2 \subseteq \pi^{-1}(U)$. Then we can give V_1 and V_2 finite affine open covers $\mathcal{V}_1$ and $\mathcal{V}_2$, and we see

$$V_1 \cap V_2 = \left(\bigcup_{W_1 \in \mathcal{V}_1} W_1\right) \cap \left(\bigcup_{W_2 \in \mathcal{V}_2} W_2\right) = \bigcup_{W_1 \in \mathcal{V}_1, W_2 \in \mathcal{V}_2} (W_1 \cap W_2),$$

  where $W_1 \cap W_2$ is quasicompact and thus has a finite affine open cover. Synthesizing our finite cover by finite affine open covers, we see that we in total have given $V_1 \cap V_2$ a finite affine open cover.

Example 3.13. Fix a scheme morphism $\pi \colon X \to Y$. If X is quasiseparated (e.g., X is locally Noetherian, using Example 3.8), then π is quasiseparated. Indeed, for any (affine) open subset $V \subseteq Y$, we see that $\pi^{-1}(V) \subseteq X$ is open and therefore quasiseparated by Example 3.9.

It turns out that a scheme $(X, \mathcal{O}_X)$ is quasicompact/quasiseparated if and only if its morphism $(X, \mathcal{O}_X) \to (\operatorname{Spec}\mathbb{Z}, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}})$ is quasicompact/quasiseparated; we will show this later.

Remark 3.14. A scheme being quasiseparated is a very reasonable smallness condition, weaker than being locally Noetherian. We will later define what it means for a morphism/scheme to be "separated," which will be stronger than this and approximately mean Hausdorff.

3.1.2 Quasicompactness is Reasonable

Here are some equivalent definitions for being quasicompact.

Lemma 3.15. A morphism $\pi\colon X \to Y$ is quasicompact if and only if every quasicompact subset $U \subseteq Y$ has $\pi^{-1}U$ also quasicompact.

Proof. If the conclusion is true, then π is certainly quasicompact because affine open subsets are necessarily quasicompact.

On the other hand, suppose π is quasicompact, and pick up a quasicompact subset $U \subseteq Y$. Now, U as an open subscheme can be given an affine open cover $\mathcal{V}$, but because U is quasicompact, we may assume that $\mathcal{V}$ is finite. But then

$$\pi^{-1}(U) = \bigcup_{V \in \mathcal{V}} \pi^{-1}(V)$$

is the finite union of quasicompact sets, where the $\pi^{-1}(V)$ is quasicompact because the V are affine. Thus, $\pi^{-1}(U)$ is quasicompact. ■

Lemma 3.16. Fix a morphism $\pi\colon X \to Y$ of schemes. Then π is quasicompact if and only if there is an affine open cover $\mathcal{U}$ of Y such that each $\pi^{-1}(U)$ is quasicompact for each $U \in \mathcal{U}$.

Proof. In one direction, if π is quasicompact, then any affine open cover $\mathcal{U}$ has each $U \in \mathcal{U}$ affine, so we see $\pi^{-1}(U) \subseteq X$ is quasicompact by hypothesis on π.

The other direction is harder. Fix an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of Y with $\varphi_\alpha\colon \operatorname{Spec} A_\alpha \cong U_\alpha$, and we are given that $\pi^{-1}(U_\alpha)$ is quasicompact for each α. Now, for any quasicompact open subset $U \subseteq Y$, we need to show that $\pi^{-1}(U)$ is quasicompact.

Well, using the distinguished base of each $U_\alpha \cong \operatorname{Spec} A_\alpha$, we can write

$$U \cap U_\alpha = \bigcup_{\beta\in\lambda_\alpha} \varphi(D(f_{\alpha,\beta}))$$

for some elements $\alpha, \beta \in \lambda$ (Remark 1.55). It follows that

$$U = \bigcup_{\alpha\in\lambda} (U \cap U_\alpha) = \bigcup_{\alpha\in\lambda} \bigcup_{\beta\in\lambda_\alpha} \varphi(D(f_{\alpha,\beta})).$$

This provides an open cover of U, so the quasicompactness of U forces us to have a finite subcover; let λ' denote the finite set of (α, β) such that $\varphi(D(f_{\alpha,\beta}))$ cover U.

It follows that

$$\pi^{-1}(U) = \bigcup_{(\alpha,\beta)\in\lambda'} \pi^{-1}(\varphi(D(f_{\alpha,\beta}))).$$

However, each $\varphi(D(f_{\alpha,\beta}))$ is affine, so their preimages under π are quasicompact, so $\pi^{-1}(U)$ is the finite union of quasicompact sets and hence quasicompact. This finishes. ■

So here are some quick results.

Corollary 3.17. Fix a morphism $\pi\colon X \to Y$ of schemes. If Y is affine, then π is quasicompact if and only if X is quasicompact.

Proof. We apply Lemma 3.16. If π is quasicompact, then the affine open subset $Y \subseteq Y$ must have $X = \pi^{-1}Y$ quasicompact by definition. Conversely, if X is quasicompact, then we use the affine open cover $\{Y\}$ on Y to note that π is quasicompact because $\pi^{-1}(Y) = X$ is, by Lemma 3.16. ■

Example 3.18. We see from Corollary 3.17 that a scheme X is quasicompact if and only if its unique morphism $X \to \operatorname{Spec}\mathbb{Z}$ is quasicompact. (Recall this morphism is unique by Corollary 2.29.)

Remark 3.19. In fact, if a class of morphisms P is affine-local on the target, then it is actually local on the target. Indeed, fix a morphism $\pi\colon X \to Y$ and give Y an open cover $\{Y_\alpha\}_{\alpha\in\lambda}$, and give each Y_α an affine open cover $\{U_{\alpha,\beta}\}_{\beta\in\lambda_\alpha}$.

- Suppose $\pi \in P$; we want to show $\pi|_{\pi^{-1}Y'_\alpha}$ is in P for some fixed α'. Well, the $\{U_{\alpha,\beta}\}$ form affine open cover of Y, so $\pi|_{\pi^{-1}U_{\alpha',\beta}} \in P$ for each β, so $\pi|_{\pi^{-1}Y_{\alpha'}} \in P$ because P is affine-local on the target.
- Suppose $\pi|_{Y_\alpha} \in P$ for each α. Then, using the affine open covers of Y_α, we see that $\pi|_{\pi^{-1}U_{\alpha,\beta}} \in P$ for each α and β, so $\pi \in P$ follows.

In light of the above remark, we will make little distinction between being local on the target and affine-local on the target.

The above results are important enough that we will want to give it a name.

Definition 3.20 (Affine-local on the target)**.** Let P be a class of morphisms. We say that P is *affine-local on the target* if and only if a morphism $\pi\colon X \to Y$ is in P if and only if there is an affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$ such that all the restricted maps $\pi|_{\pi^{-1}Y_\alpha}\colon \pi^{-1}Y_\alpha \to Y_\alpha$ are also in P.

Example 3.21. Quasicompact morphisms are affine-local on the target, from Lemma 3.16. Certainly if π is quasicompact, then for any affine open subset $U \subseteq Y$, we see $\pi^{-1}U$ is quasicompact, so the restriction $\pi|_{\pi^{-1}U}\colon \pi^{-1}U \to U$ is quasicompact by Corollary 3.17. Conversely, if all the restrictions to $\pi|_{\pi^{-1}Y_\alpha}$ are quasicompact, then because $Y_\alpha \subseteq Y$ is quasicompact, $\pi^{-1}(Y_\alpha)$ is quasicompact for each α, so π is quasicompact by Lemma 3.16.

Here are a few niceness checks.

Corollary 3.22. Fix quasicompact scheme morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. Then $\psi\circ\varphi$ is quasicompact.

Proof. We use Lemma 3.15. Pick up any quasicompact subset $W \subseteq Z$. Then $\psi^{-1}(W) \subseteq Y$ is quasicompact by Lemma 3.15, so $(\psi\circ\varphi)^{-1}(W) = \varphi^{-1}(\psi^{-1}(W))$ is quasicompact again by Lemma 3.15. ■

Once more, it will be useful to have language to describe the above.

Definition 3.23 (Preserved by composition)**.** Let P be a class of morphisms. We say that P is *preserved by composition* if and only if, for any pair of morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ in P, we have $\psi\circ\varphi$ also in P.

Example 3.24. By Corollary 3.22, quasicompact morphisms are preserved by composition.

Lemma 3.25. Suppose we have a pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

of schemes. If ψ_Y is quasicompact, then π_X is quasicompact.

Proof. The main point is to reduce to the affine case, where everything is clear. Let $\pi_S := \psi_Y \circ \pi_Y = \psi_X \circ \pi_X$, for brevity. Give S an affine open cover $\{S_\alpha\}_{\alpha\in\lambda}$. For each $\alpha \in \lambda$, we give $\psi_X^{-1}(S_\alpha) \subseteq X$ an affine open cover $\{X_{\alpha,\beta}\}_{\alpha\in\lambda,\beta\in\kappa_\alpha}$. Then we build the tower

$$\begin{tikzcd} \pi_X^{-1}(X_{\alpha,\beta}) \arrow[r, "\pi_X|_{\pi_X^{-1}(X_{\alpha,\beta})}"] \arrow[d, hook] & X_{\alpha,\beta} \arrow[d, hook] \\ \pi_S^{-1}(S_\alpha) \arrow[r, "\pi_X|_{\pi_S^{-1}(S_\alpha)}"] \arrow[d, "\pi_Y|_{\pi_S^{-1}(S_\alpha)}"] & \psi_X^{-1}(S_\alpha) \arrow[d, "\psi_X"] \\ \psi_Y^{-1}(S_\alpha) \arrow[r, "\psi_Y|_{\psi_Y^{-1}(S_\alpha)}"'] & S_\alpha \end{tikzcd}$$

and note that the bottom square is a pullback square by Lemma 2.186, the top square is a pullback square by Lemma 2.178, so the total rectangle is a pullback square by Lemma 2.174. Now, because being quasicompact is affine-local on the target by Lemma 3.16, it suffices to show that each restricted map $\pi_X|_{\pi_X^{-1}(X_{\alpha,\beta})}$ is quasicompact. Notably, by Lemma 3.15, the restriction $\psi_Y|_{\psi_Y^{-1}(S_\alpha)}$ is quasicompact.

Thus, we fix some α and β. We now rename our variables, replacing S_α with S, $X_{\alpha,\beta}$ with X, and $\psi_Y^{-1}(S_\alpha)$ with Y, and we rename our morphisms to fit the pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

though we now have both X and S affine. We are given that ψ_Y is quasicompact, and we would like to show that π_X is also quasicompact. By Corollary 3.17, it suffices to show that $X \times_S Y$ is quasicompact.

However, we note that S is affine, so because ψ_Y is quasicompact, Corollary 3.17 tells us that Y is quasicompact. As such, we give Y a finite affine open cover $\{Y_i\}_{i=1}^n$. By the construction of the fiber product in Lemma 2.183, we note that $X \times_S Y$ is covered by the schemes $X \times_S Y_i$.

But now each scheme $X \times_S Y_i$ can have $X \cong \operatorname{Spec} A$ and $S \cong \operatorname{Spec} R$ and $Y_i \cong \operatorname{Spec} B_i$ so that

$$X \times_S Y_i \cong \operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B_i,$$

which is isomorphic to $\operatorname{Spec} A \otimes_R B_i$ by Lemma 2.176. Notably, $X \times_S Y_i$ is an affine scheme and hence quasicompact, so $X \times_S Y$ is a finite union of quasicompact subschemes and therefore quasicompact. ■

And here is the name.

Definition 3.26 (Preserved by base change)**.** Let P be a class of morphisms. We say that P is *preserved by base change* if and only if $\varphi \colon X \to S$ being in P implies that the canonical morphism $\pi_Y : X \times_S Y \to Y$ is still in P, for any scheme Y over S.

We are going to use the above argument for being preserved by base change many times, so we take a second to write it down.

Lemma 3.27. Let P be a class of morphisms which is affine-local on the target. Given affine schemes S and Y, suppose that $\varphi\colon X \to S$ being in P implies that the canonical morphism $\pi_Y\colon X \times_S Y \to Y$ is still in P. Then P is preserved by base change.

Proof. Well, suppose we have a pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

of schemes such that $\psi_Y \in P$; we need to show $\pi_X \in P$.

For this, we just reduce to the affine case. Let $\pi_S := \psi_Y \circ \pi_Y = \psi_X \circ \pi_X$, for brevity. Give S an affine open cover $\{S_\alpha\}_{\alpha\in\lambda}$. For each $\alpha \in \lambda$, we give $\psi_X^{-1}(S_\alpha) \subseteq X$ an affine open cover $\{X_{\alpha,\beta}\}_{\alpha\in\lambda,\beta\in\kappa_\alpha}$. Then we build the tower

$$\begin{tikzcd} \pi_X^{-1}(X_{\alpha,\beta}) \arrow[r, "\pi_X|_{\pi_X^{-1}(X_{\alpha,\beta})}"] \arrow[d, hook] & X_{\alpha,\beta} \arrow[d, hook] \\ \pi_S^{-1}(S_\alpha) \arrow[r, "\pi_X|_{\pi_S^{-1}(S_\alpha)}"] \arrow[d, "\pi_Y|_{\pi_S^{-1}(S_\alpha)}"] & \psi_X^{-1}(S_\alpha) \arrow[d, "\psi_X"] \\ \psi_Y^{-1}(S_\alpha) \arrow[r, "\psi_Y|_{\psi_Y^{-1}(S_\alpha)}"'] & S_\alpha \end{tikzcd}$$

and note that the bottom square is a pullback square by Lemma 2.186, the top square is a pullback square by Lemma 2.178, so the total rectangle is a pullback square by Lemma 2.174. Now, because P is affine-local on the target, it suffices to show that each restricted map $\pi_X|_{\pi_X^{-1}(X_{\alpha,\beta})}$ is in P. Notably, because P is affine-local on the target, the restriction $\psi_Y|_{\psi_Y^{-1}(S_\alpha)}$ is in P.

Thus, we fix some α and β. We now rename our variables, replacing S_α with S, $X_{\alpha,\beta}$ with X, and $\psi_Y^{-1}(S_\alpha)$ with Y, and we rename our morphisms to fit the pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

though we now have both Y and S affine. Now the hypothesis on P kicks in because we know that the restricted version of ψ_Y is in P, implying that π_X is in P. ■

Remark 3.28. Being preserved under composition and base-change shows that the product of two quasicompact morphisms is quasicompact. Namely, $\varphi\colon X \to Y$ and $\varphi'\colon X' \to Y'$ makes us build the product morphism $\varphi \times \varphi'\colon X \times_{\operatorname{Spec}\mathbb{Z}} X' \to Y \times_{\operatorname{Spec}\mathbb{Z}} Y'$ by applying base-change twice.

3.1.3 Isomorphisms Are Reasonable

For some more practice with the definitions, we work with the class of isomorphisms.

Lemma 3.29. The class of isomorphisms is preserved by composition.

Proof. This is purely category theory. Fix isomorphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$, and let $\varphi^{-1}\colon Y \to X$ and $\psi^{-1}Z \to Y$ be their inverses. Then we see that

$$\psi^{-1} \circ \varphi^{-1} \circ \varphi \circ \psi = \mathrm{id}_Z \qquad \text{and} \qquad \varphi \circ \psi \circ \psi^{-1} \circ \varphi^{-1} = \mathrm{id}_X,$$

so $\varphi \circ \psi$ is an isomorphism with inverse $\psi^{-1} \circ \varphi^{-1}$. ■

Lemma 3.30. The class of isomorphisms is preserved by base change.

Proof. This is also purely category theory. Suppose that

$$\begin{tikzcd} W \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

is a pullback square with ψ_Y an isomorphism. Then we claim that π_X is also isomorphism. For this, the main claim is that

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\psi_Y^{-1}\psi_X"] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"'] & S \end{tikzcd}$$

is a pullback square. Indeed, fixing an object Z and morphisms $\alpha_X\colon Z \to X$ and $\alpha_Y\colon Z \to Y$ making the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "\alpha_X"] \arrow[ddr, bend right, "\alpha_Y"'] & & \\ & X \arrow[r, equal] \arrow[d, "\psi_Y^{-1}\psi_X"] & X \arrow[d, "\psi_X"] \\ & Y \arrow[r, "\psi_Y"'] & S \end{tikzcd}$$

minus the dashed arrow, we need to show that there is a unique morphism $\alpha\colon Z \to X$ making the diagram commute. Well, α is unique because we see that we are requiring $\alpha = \alpha_X$ in the commutativity. And α exists because α_X does give $\psi_X \circ \alpha_X = \psi_Y \circ \alpha_X$ by hypothesis on α_X and α_Y.

Thus, we see that X and W are canonically isomorphic. To see that π_X is the needed canonical isomorphism, we note that the canonical isomorphism $W \to X$ is the one induced by the diagram

$$\begin{tikzcd} W \arrow[dr, dashed] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_Y"'] & & \\ & X \arrow[r, equal] \arrow[d, "\psi_Y^{-1}\psi_X"] & X \arrow[d, "\psi_X"] \\ & Y \arrow[r, "\psi_Y"'] & S \end{tikzcd}$$

which we can see must be π_X from the upper-right triangle. ■

Lemma 3.31. The class of isomorphisms is local on the target.

Proof. In one direction, if $\varphi\colon X \to Y$ is an isomorphism, then any open $V \subseteq Y$ makes $\varphi|_{\varphi^{-1}V}\colon \varphi^{-1}V \to V$ is an isomorphism by Lemma 2.9.

In the other direction, fix a morphism $\varphi\colon X \to Y$ of schemes and an open cover $\{V_\alpha\}_{\alpha\in\lambda}$ of Y such that each restricted morphism $\varphi|_{\varphi^{-1}V_\alpha}$ is an isomorphism for each $\alpha \in \lambda$. Then φ is an isomorphism.

We construct the inverse by hand. Recall from our construction of restriction (Lemma 2.9) that the topological map $\varphi_\alpha\colon \varphi^{-1}(V_\alpha) \to V_\alpha$ is the restricted map $\varphi_\alpha := \varphi|_{\varphi^{-1}(V_\alpha)}$, and the sheaf map

$$\varphi_\alpha^\sharp\colon \mathcal{O}_Y|_{V_\alpha} \to (\varphi_\alpha)_*(\mathcal{O}_X|_{\varphi^{-1}(V_\alpha)})$$

is made of the morphisms

$$\mathcal{O}_Y|_{V_\alpha}(V) = \mathcal{O}_Y(V) \stackrel{\varphi_V^\sharp}{\to} \varphi_*\mathcal{O}_X(V) = \mathcal{O}_X\left(\varphi^{-1}(V)\right) = (\varphi_\alpha)_*(\mathcal{O}_X|_{\varphi^{-1}(V_\alpha)})(V)$$

for each $V \subseteq V_\alpha$.

We now show that φ is an isomorphism; it suffices to show that φ is a homeomorphism and that $\varphi^\sharp$ is an isomorphism of sheaves (by Lemma 2.8). There are two checks.

- We show φ is a homeomorphism. Let $\varphi_\alpha\colon \varphi^{-1}(V_\alpha) \to V_\alpha$ be the restricted map. For each $\alpha \in \lambda$, there is a continuous map $\psi_\alpha\colon V_\alpha \to \varphi^{-1}(V_\alpha)$ which is the inverse for φ_α.

  Now, for each $y \in Y$, find α with $y \in V_\alpha$, and define $\psi(y) := \psi_\alpha(y)$. This is well-defined: if $y \in V_\alpha \cap V_\beta$, then we can find $x \in \varphi^{-1}(V_\alpha)$ with $\varphi(x) = y$, so it follows that actually $x \in \varphi^{-1}(V_\alpha) \cap \varphi^{-1}(V_\beta)$ as well, so $\psi_\alpha(y) = x = \psi_\beta(y)$.

  To see that ψ is continuous, observe that any $U \subseteq X$ has

  $$\psi^{-1}(U) = \psi^{-1}\left(U \cap \bigcup_{\alpha\in\lambda} \varphi_\alpha^{-1}(V_\alpha)\right) = \bigcup_{\alpha\in\lambda} \psi_\alpha^{-1}\left(U \cap \varphi^{-1}(V_\alpha)\right),$$

  which is open as the arbitrary union of open subsets of X.

  Lastly, for any $y \in Y$, find α with $y \in V_\alpha$, and we see $\varphi(\psi(y)) = \varphi_\alpha(\psi_\alpha(y)) = y$. On the other side, for any $x \in X$, find α with $x \in \varphi^{-1}(V_\alpha)$, so $\psi(\varphi(x)) = \psi_\alpha(\varphi_\alpha(x)) = x$.

- We show $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is an isomorphism of sheaves. Define the collection $\mathcal{B}$ as the collection of all open subsets of Y contained in a V_α for some $\alpha \in \lambda$. Because $\{V_\alpha\}_{\alpha\in\lambda}$ covers Y, we see that $\mathcal{B}$ is a base for the topology on X: for any $V \subseteq X$, we can write

  $$V = \bigcup_{\alpha\in\lambda} (V \cap V_\alpha).$$

  Now, we showed on previous homework that the construction of a morphism of sheaves on a base is unique, so it will roughly suffice to show $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ is an isomorphism on the base.

  Well, for any $B \in \mathcal{B}$, we want to show that $\varphi_V^\sharp$ is an isomorphism. Well, we can find $\alpha \in \lambda$ with $B \subseteq V_\alpha$, so we note that by hypothesis

  $$\mathcal{O}_Y|_{V_\alpha}(B) = \mathcal{O}_Y(B) \stackrel{\varphi_V^\sharp}{\to} \varphi_*\mathcal{O}_X(B) = \mathcal{O}_X\left(\varphi^{-1}(B)\right) = (\varphi_\alpha)_*(\mathcal{O}_X|_{\varphi^{-1}(V_\alpha)})(B)$$

  is an isomorphism, so in particular $\varphi_V^\sharp$ is an isomorphism.

  We will go ahead and construct the inverse morphism for $\varphi^\sharp$, for completeness. We may define $\psi_B^\sharp$ as the inverse of $\varphi_B^\sharp$ at all basis sets B. To see that $\psi^\sharp$ assembles to a morphism of sheaves on the base

$\mathcal{B}$, note that any basis elements $B' \subseteq B$ have the left square of

$$\begin{tikzcd} \mathcal{O}_X(B) \arrow[d, "\operatorname{res}_{B,B'}"'] & \varphi_*\mathcal{O}_Y(B) \arrow[l, "\psi_B^\sharp"'] \arrow[d, "\operatorname{res}_{B,B'}"] \\ \mathcal{O}_X(B') & \varphi_*\mathcal{O}_Y(B') \arrow[l, "\psi_{B'}^\sharp"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_X(B) \arrow[r, "\varphi_B^\sharp"] \arrow[d, "\operatorname{res}_{B,B'}"'] & \varphi_*\mathcal{O}_Y(B) \arrow[d, "\operatorname{res}_{B,B'}"] \\ \mathcal{O}_X(B') \arrow[r, "\varphi_{B'}^\sharp"] & \varphi_*\mathcal{O}_Y(B') \end{tikzcd}$$

commutes because it is the same as the square on the right. Thus, $\psi^\sharp$ extends to a full morphism of sheaves $\varphi_*\mathcal{O}_X \to \mathcal{O}_Y$. Additionally, for any basis element $B \in \mathcal{B}$, we see that

$$\varphi_B^\sharp \circ \psi_B^\sharp = \operatorname{id}_{\varphi_*\mathcal{O}_X(B)} = (\operatorname{id}_{\varphi_*\mathcal{O}_Y})_B,$$

so uniqueness of morphisms of sheaves on a base shows that $\varphi^\sharp \circ \psi^\sharp = \operatorname{id}_{\varphi_*\mathcal{O}_X}$. Reversing the roles of $\varphi^\sharp$ with $\psi^\sharp$ and $\varphi_*\mathcal{O}_Y$ with $\mathcal{O}_X$ shows that $\psi^\sharp\colon \varphi^\sharp = \operatorname{id}_{\mathcal{O}_Y}$, so we see that $\psi^\sharp$ is in fact an inverse for $\varphi^\sharp$. This finishes. ■

3.1.4 Diagonal Morphisms

It will turn out that two important classes of morphisms—quasiseparated and separated morphisms—are best thought of in terms of "diagonal morphisms." Thus, we will spend a little time discussing these.

Notation 3.32. Fix schemes X_1 and X_2 over a scheme Y. Given morphisms $\alpha_1\colon Z \to X_1$ and $\alpha_2\colon Z \to X_2$ making the outer square of

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left, "\alpha_1"] \arrow[ddr, bend right, "\alpha_2"'] \\ & X_1 \times_Y X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\ & X_2 \arrow[r] & Y \end{tikzcd}$$

commute, we let (α_1, α_2) denote the induced arrow.

Example 3.33. Given a morphism $\varphi\colon X \to Y$, we note that $\varphi \circ \operatorname{id}_X = \varphi \circ \operatorname{id}_X$, so we have induced a morphism $(\operatorname{id}_X, \operatorname{id}_X)\colon X \to X \times_Y X$. This morphism is called the "diagonal morphism" and is denoted $\Delta\varphi := (\operatorname{id}_X, \operatorname{id}_X)$.

Here is the quickest possible reason to care.

Lemma 3.34. Fix a category $\mathcal{C}$ with fiber products. A morphism $\varphi\colon X \to Y$ is monic if and only if the diagonal morphism $\Delta\varphi\colon X \to X \times_Y X$ is an isomorphism.

Proof. Let $\pi_1, \pi_2\colon X \times_Y X \to X$ be the canonical projections, and set $\Delta := \Delta\varphi$ for brevity. We have two checks.

- Suppose that φ is monic. We claim that

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

is a pullback square. Indeed, for any object Z with morphisms $\alpha, \beta\colon Z \to X$ such that $\varphi \circ \alpha = \varphi \circ \beta$, we see that $\alpha = \beta$ because φ is monic. Now, any morphism $\psi\colon Z \to X$ makes

$$\begin{tikzcd}
Z \arrow[drr, bend left, "\alpha"] \arrow[dr, dashed, "\psi"] \arrow[ddr, bend right, "\beta"'] & & \\
& X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "\varphi"] \\
& X \arrow[r, "\varphi"] & Y
\end{tikzcd}$$

commute if and only if $\alpha = \psi = \beta$, so we see that ψ both exists and is unique.

To finish, it follows by the universal property that the diagonal morphism $\Delta\colon X \to X \times_Y X$ making the diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, equal] \arrow[dr, dashed, "\Delta"] \arrow[ddr, bend right, equal] & & \\
& X \times_Y X \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "\varphi"] \\
& X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

commute must be the canonical isomorphism witnessing $\Delta\colon X \cong X \times_Y X$.

- Suppose that Δ is an isomorphism. Then we note that $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \mathrm{id}_X$ forces π_1 and π_2 to also both be inverses by the uniqueness of the inverse.

  Now, pick up some object Z with morphisms $\alpha, \beta\colon Z \to X$ such that $\varphi \circ \alpha = \varphi \circ \beta$. We need to show that $\alpha = \beta$.

  Well, we are promised a morphism $(\alpha, \beta)\colon Z \to X \times_Y X$ such that $\pi_1 \circ (\alpha, \beta) = \alpha$ and $\pi_2 \circ (\alpha, \beta) = \beta$, so

  $$\alpha = \pi_1 \circ (\alpha, \beta) = \pi_2 \circ \Delta \circ \pi_1 \circ (\alpha, \beta) = \pi_2 \circ (\alpha, \beta) = \beta,$$

  which is what we wanted. ■

Having defined our diagonal morphisms, we now pick up some facts about them. The following pullback square will prove quite helpful.

> **Lemma 3.35** (Magic diagram)**.** Fix morphisms $\alpha_1\colon X_1 \to Y$ and $\alpha_2\colon X_2 \to Y$ and $\iota\colon Y \to Z$. Then the diagram
>
> $$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \times_Z Y_2 \arrow[d] \\
Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$
>
> equipped with the natural maps is a pullback square. Here, we assume that the relevant fiber products exist.

Proof. We proceed by force; we begin by naming our maps. Let $\pi_1\colon X_1 \times_Z X_2 \to X_1$ and $\pi_2\colon X_1 \times_Z X_2 \to X_2$ be the canonical projections. Analogously, we let $\varpi_1\colon X_1 \times_Y X_2 \to X_1$ and $\varpi_2\colon X_1 \times_Y X_2 \to X_2$ be the canonical projections so that the diagram

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr, bend left, "\varpi_1"] \arrow[dr, dashed, "\varpi"] \arrow[ddr, bend right, "\varpi_2"'] & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"'] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "\iota\alpha_1"] \\
& X_2 \arrow[r, "\iota\alpha_2"'] & Z
\end{tikzcd}$$

commutes (indeed, $\alpha_1 \circ \varpi_1 = \alpha_2 \circ \varpi_2$) and thus induces the morphism ϖ making the diagram commute. Namely, $\varpi_1 = \pi_1 \circ \varpi$ and $\varpi_2 = \pi_2 \circ \varpi$. We take a moment to recognize that ϖ is (ϖ_1, ϖ_2).

To induce the map $\Delta\iota\colon Y \to Y \times_Z Y$, we let $\psi_1, \psi_2\colon Y \times_Z Y \to Y$ denote the canonical projections, and we draw the diagram

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\Delta\iota"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& Y \times_Z Y \arrow[r, "\psi_1"'] \arrow[d, "\psi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\iota"'] \\
& Y \arrow[r, "\iota"] & Z
\end{tikzcd}$$

and note that the outer square commutes because we're dealing with identities. This induces our desired diagonal morphism $\Delta\iota$.

Next, we induce the map $(\alpha_1, \alpha_2)\colon X_1 \times_Z X_2 \to Y \times_Z Y$ by noting that the outer "square" of

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, dashed] & X_1 \arrow[dr, "\alpha_1"] & \\
X_2 \arrow[dr, "\alpha_2"'] & Y \times_Z Y \arrow[r, "\psi_1"'] \arrow[d, "\psi_2"] & Y \arrow[d, "\iota"] \\
& Y \arrow[r, "\iota"'] & Z
\end{tikzcd}$$

commutes because $\alpha_1 \circ \pi_1 = \alpha_2 \circ \pi_2$, so we have induced a dashed arrow we name (α_1, α_2). Importantly, $\psi_1 \circ (\alpha_1, \alpha_2) = \alpha_1 \circ \pi_1 = \alpha_2 \circ \pi_2 = \psi_2 \circ (\alpha_1, \alpha_2)$.

Lastly, we induce the map $X_1 \times_Y X_2 \to Y$ just by $\alpha_1 \circ \pi_1 = \alpha_2 \circ \pi_2$. To see that the magic diagram commutes, we note that there is at most morphism $X_1 \times_Y X_2 \to Y \times_Z Y$ which can fill into the dashed arrow of

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\varpi_1"] \arrow[d, "\varpi_2"'] \arrow[dr, dashed] & X_1 \arrow[dr, "\alpha_1"] & \\
X_2 \arrow[dr, "\alpha_2"'] & Y \times_Z Y \arrow[r, "\psi_1"'] \arrow[d, "\psi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\iota"] \\
& Y \arrow[r, "\iota"'] & Z
\end{tikzcd}$$

to make the diagram commute. However, the diagram

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\varpi_1"] \arrow[d, "\varpi_2"'] \arrow[dr] & X_1 \arrow[d, "\alpha_1"'] \arrow[ddrr, "\alpha_1"] & & \\
X_2 \arrow[r, "\alpha_2"] \arrow[ddrr, "\alpha_2"'] & Y \arrow[dr, dashed, "\Delta\iota"] \arrow[drr, equal] \arrow[ddr, equal] & & \\
& & Y \times_Z Y \arrow[r, "\psi_1"'] \arrow[d, "\psi_2"] & Y \\
& & Y &
\end{tikzcd}$$

commutes mostly by construction of $\Delta\iota$, and the diagram

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[dr, "\varpi"] \arrow[drr, bend left, "\varpi_1"] \arrow[ddr, bend right, "\varpi_2"'] & & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, "(\alpha_1,\alpha_2)"] & X_1 \arrow[dr, "\alpha_1"] & \\
& X_2 \arrow[dr, "\alpha_2"'] & Y \times_Z Y \arrow[r, "\psi_1"'] \arrow[d, "\psi_2"] & Y \\
& & Y &
\end{tikzcd}$$

commutes by construction of both ω and (α_1, α_2).

We are now ready to show the universal property. Fix some object W with maps $\varphi_F : W \to X_1 \times_Z Y_2$ and $\varphi_Y : W \to Y$ such that $(\alpha_1, \alpha_2) \circ \varphi_F = \Delta\iota \circ \varphi_Y$. Then we need a unique morphism $\varphi: W \to X_1 \times_Y X_2$ making the diagram

$$\begin{tikzcd}
W \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\varphi_F"] \arrow[ddr, bend right, "\varphi_Y"'] & & \\
& X_1 \times_Y X_2 \arrow[r, "\varpi"'] \arrow[d, "\alpha_1\varpi_1"] & X_1 \times_Z X_2 \arrow[d, "(\alpha_1,\alpha_2)"] \\
& Y \arrow[r, "\Delta\iota"'] & Y \times_Z Y
\end{tikzcd}$$

commute. We show uniqueness and existence separately.

- Uniqueness: given φ, we claim that the diagram

$$\begin{tikzcd}
W \arrow[dr, "\varphi"] \arrow[drr, bend left, "\pi_1\varphi_F"] \arrow[ddr, bend right, "\pi_2\varphi_F"'] & & \\
& X_1 \times_Y X_2 \arrow[r, "\varpi_1"'] \arrow[d, "\varpi_2"] & X_1 \arrow[d, "\alpha_1"] \\
& X_2 \arrow[r, "\alpha_2"'] & Y
\end{tikzcd}$$

  commutes, which will of course uniquely determine φ by our pullback. Well, we see $\varpi_i \circ \varphi = \pi_i \circ \varpi \circ \varphi = \pi_i \circ \varphi_F$ for $i \in \{1, 2\}$, which is what we wanted.

- Existence: as above, we claim that the diagram

$$\begin{tikzcd}
W \arrow[drr, bend left, "\pi_1\varphi_F"] \arrow[ddr, bend right, "\pi_2\varphi_F"'] & & \\
& X_1 \times_Y X_2 \arrow[r, "\varpi_1"'] \arrow[d, "\varpi_2"] & X_1 \arrow[d, "\alpha_1"] \\
& X_2 \arrow[r, "\alpha_2"'] & Y
\end{tikzcd}$$

  commutes, which will induce the desired morphism φ. Indeed, $\alpha_i \circ \pi_i \circ \varphi_F = \psi_i \circ (\alpha_1, \alpha_2) \circ \varphi_F = \psi_i \circ \Delta_Y \circ \varphi_Y = \varphi_Y$ for each $i \in \{1, 2\}$.

  We now run our checks on φ. On one side, we see that $\alpha_1 \circ \varpi_1 \circ \varphi = \alpha_1 \circ \pi_1 \circ \varpi_F$ equals φ_F as computed above. On the other side, we see that

$$\pi_i \circ \varpi \circ \varphi = \varpi_i \circ \varphi = \pi_i \circ \varphi_F,$$

so both $\varpi \circ \varphi$ and φ_F could fill the dashed arrow in the diagram

$$\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, bend left, "\varpi_1\varphi_F"] \arrow[ddr, bend right, "\varpi_2\varphi_F"'] & & \\
& X_1 \times_Z X_2 \arrow[r, "\varpi_1"'] \arrow[d, "\varpi_2"] & X_1 \arrow[d, "\iota\alpha_1"] \\
& X_2 \arrow[r, "\iota\alpha_2"'] & Z
\end{tikzcd}$$

where we know there is space for at most morphism. Thus, $\varpi \circ \varphi = \varphi_F$ follows.

The above checks complete the proof. ■

We now run a few checks on classes of diagonal morphisms.

Notation 3.36. Given a class of morphisms P, we let ΔP denote the class of morphisms π such that $\Delta\pi \in \Delta P$.

Lemma 3.37. Fix a class P of morphisms which is preserved by composition and base change. Then ΔP is preserved by composition.

Proof. Fix morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ in ΔP so that we want to show $\psi \circ \varphi$ is still in ΔP. Namely, we are given that the diagonal morphisms $\Delta\varphi\colon X \to X \times_Y X$ and $\Delta\psi\colon Y \to Y \times_Z Y$ are in P.

Now, Lemma 3.35 promises us the pullback square

$$\begin{tikzcd}
X \times_Y X \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_Z Y \arrow[d] \\
Y \arrow[r, "\Delta\psi"] & Y \times_Z Y
\end{tikzcd}$$

which because P is preserved by base change tells us that the natural map $\delta\colon X \times_Y X \to X \times_Z X$ is in P.

To finish, we note that the diagonal morphism $\Delta(\psi\circ\varphi)\colon X \to X \times_Z X$ is the one induced by $\mathrm{id}_X : X \to X$, but then the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta\varphi"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & & \\
& X \times_Y X \arrow[r] \arrow[d] \arrow[dr, dashed, "\delta"] & X \arrow[dr, equal] & \\
& X \arrow[dr, equal] & X \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "\psi\varphi"'] \\
& & X \arrow[r, "\psi\varphi"] & Z
\end{tikzcd}$$

tells us that the natural composite map $X \overset{\Delta\varphi}{\to} X \times_Y X \overset{\delta}{\to} X \times_Z X$ must also be $\Delta(\psi\circ\varphi)$ by the uniqueness of the definition of $\Delta(\psi\circ\varphi)$ from the bottom-right pullback square. So because $\Delta\varphi$ and δ are both in P, we conclude that their composite $\Delta(\psi\circ\varphi)$ is also in P. Thus, $\psi\circ\varphi$ is in P. ■

Lemma 3.38. Fix a class P of morphisms which is preserved by base change. Then ΔP is preserved by base change.

Proof. Fix a pullback square

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "\varphi'"] & Y' \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

such that $\pi \in \Delta P$. We want to show that $\pi' \in \Delta P$.

Well, we let $\pi_1, \pi_2 \colon X \times_Y X \to X$ and $\pi_1' \colon X' \times_{Y'} X' \to X'$ be the canonical projections so that $\varphi'\pi_1' = \varphi'\pi_2'$, implying $\pi\varphi'\pi_1' = \pi\varphi'\pi_2$, which induces $(\varphi'\pi_1', \varphi'\pi_2') \colon X' \times_{Y'} X' \to X \times_Y X$. The key claim, now, is that the square

$$\begin{tikzcd} X' \arrow[r, "\Delta\pi'"] \arrow[d, "\varphi'"] & X' \times_{Y'} X' \arrow[d, "(\varphi'\pi_1', \varphi'\pi_2')"] \\ X \arrow[r, "\Delta\pi"] & X \times_Y X \end{tikzcd}$$

is a pullback square. This will finish because P is preserved by base change: note $\Delta\pi \in P$ implies $\Delta\pi' \in P$, so $\pi' \in \Delta P$.

Now, at the very least this square commutes because

$$\pi_i \circ (\varphi'\pi_1', \varphi'\pi_2') \circ \Delta\pi' = \varphi' \circ \pi_i' \circ \Delta\pi' = \varphi' = \pi_i \circ \Delta\pi \circ \varphi'$$

for each $i \in \{1, 2\}$, so it follows that $(\varphi'\pi_1', \varphi'\pi_2') \circ \Delta\pi' = \Delta\pi \circ \varphi'$ by the definition of maps into the fiber product.

Continuing, to show that our square is a pullback square, we use Lemma 2.173. Namely, we claim that the outer and right squares of

$$\begin{tikzcd} X' \arrow[r, "\Delta\pi'"] \arrow[d, "\varphi'"] \arrow[rr, bend left, "\pi'"] & X' \times_{Y'} X' \arrow[r, "\pi'\pi_1'"] \arrow[d, "(\varphi'\pi_1', \varphi'\pi_2')"] & Y' \arrow[d, "\varphi"] \\ X \arrow[r, "\Delta\pi"] \arrow[rr, bend right, "\pi"'] & X \times_Y X \arrow[r, "\pi\pi_1"] & Y \end{tikzcd} \tag{3.1}$$

are pullback square. To begin, we note that at least the diagram commutes: namely, the right square has

$$\pi \circ \pi_1 \circ (\varphi'\pi_1', \varphi'\pi_2') = \pi \circ \varphi' \circ \pi_1' = \varphi \circ \pi' \circ \pi_1'.$$

Further, we see that $\pi' \circ \pi_1' \circ \Delta\pi' = \pi'$ and $\pi \circ \pi_1 \circ \Delta\pi = \pi$ by construction of the diagonal morphisms.

Thus, we note that the outer rectangle of (3.1) is in fact a pullback square by hypothesis, so it really only remains to show that the right square of (3.1) is a pullback square. For this, we use Lemma 2.174, writing down

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, "(\pi_1', \pi_2')"] \arrow[d, "\pi'\pi_1'"] \arrow[rr, bend left, "(\varphi'\pi_1', \varphi'\pi_2')"] & X' \times_Y X' \arrow[r, "(\varphi', \varphi')"] \arrow[d, "(\pi'\pi_1', \pi'\pi_2')"] & X \times_Y X \arrow[d, "\pi\pi_1"] \\ Y' \arrow[r, "\Delta\varphi"] \arrow[rr, bend right, "\varphi"'] & Y' \times_Y Y' \arrow[r, "\varphi\varpi_1"] & Y \end{tikzcd}$$

to claim that both squares are pullback squares; here $\varpi_1, \varpi_2 \colon Y \times_X Y \to Y$ are the canonical projections. The left square commutes and is a pullback square by Lemma 3.35. The right square commutes because

$$\pi \circ \pi_1 \circ (\varphi', \varphi') = \pi \circ \varphi' \circ \pi_1' = \varphi \circ \pi' \circ \pi_1' = \varphi \circ \varpi_1 \circ (\pi'\pi_1', \pi'\pi_2').$$

At a high level, the right square is a pullback square because

$$X' \times_Y X' \simeq (Y' \times_Y X) \times_Y (Y' \times_Y X) \simeq (Y' \times_Y Y') \times_Y (X \times_Y X).$$

We can also see this more directly, but I think I personally draw the line at explicitly proving associativity laws. ■

Lemma 3.39. Fix a class P of morphisms which is local on the target and preserved by base change. Then ΔP is local on the target.

Proof. Fix a morphism $\varphi\colon X \to Y$. In one direction, suppose $\varphi \in \Delta P$ and fix some open subset $V \subseteq Y$. Then we note that (2.13) tells us

$$\begin{tikzcd} \varphi^{-1}V \arrow[r, hook] \arrow[d, "\varphi|_{\varphi^{-1}V}"'] & X \arrow[d, "\varphi"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

is a pullback square, so $\varphi|_{\varphi^{-1}V} \in \Delta P$ by Lemma 3.38.

In the other direction, fix an open cover $\{Y_\alpha\}_{\alpha\in\lambda}$ of Y and suppose that the restrictions $\varphi_\alpha\colon \pi^{-1}Y_\alpha \to Y_\alpha$ all live in ΔP; set $X_\alpha := \varphi^{-1}Y_\alpha$ for brevity. Then the pullback squares

$$\begin{tikzcd} X_\alpha \arrow[r, hook, "\jmath_\alpha"] \arrow[d, "\varphi_\alpha"'] & X \arrow[d, "\varphi"] \\ Y_\alpha \arrow[r, hook, "\iota_\alpha"] & Y \end{tikzcd}$$

coming from (2.13) grant us the pullback squares

$$\begin{tikzcd} X_\alpha \arrow[r, hook, "\jmath_\alpha"] \arrow[d, "\Delta\varphi_\alpha"'] & X \arrow[d, "\Delta\varphi"] \\ X_\alpha \times_{Y_\alpha} X_\alpha \arrow[r] & Y \times_X Y \end{tikzcd}$$

which essentially finish the proof. However, $X_\alpha \times_{Y_\alpha} X_\alpha$ is really $\pi_1^{-1}X_\alpha \subseteq X \times_Y X$ by Lemma 2.186, where $\pi_1, \pi_2\colon X \times_Y X \to X$ are the canonical projections. Notably, we see that the open cover Y_α of Y becomes an open cover $\pi_1^{-1}\varphi^{-1}Y_\alpha$ of $X \times_Y X$, so the fact that all the individual restrictions[1] $\Delta\varphi_\alpha\colon X_\alpha \to \pi_1^{-1}X_\alpha$ are in P tells us that $\Delta\varphi$ is also in P because P is local on the target, so $\varphi \in \Delta P$. ■

3.1.5 Quasiseparatedness is Reasonable

It turns out to be very convenient to think about a morphism $\varphi\colon X \to Y$ being quasiseparated in terms of the diagonal morphism $\Delta\colon X \to X \times_Y X$ induced by $\mathrm{id}_X\colon X \to X$.

Lemma 3.40. Fix a morphism $\varphi\colon X \to Y$ of schemes.

(a) π is quasiseparated.

(b) There is an affine open cover $\mathcal{U}$ of Y such that each $\varphi^{-1}(U)$ is quasiseparated for each $U \in \mathcal{U}$.

(c) The diagonal morphism $\Delta\colon X \to X \times_Y X$ is quasicompact.

Proof. As usual, (a) implies (b) by choosing any affine open cover $\mathcal{U}$ of Y, which implies that $\varphi^{-1}(U)$ is quasiseparated for each affine open $U \in \mathcal{U}$ because φ is quasiseparated.

The other implications are harder. Before going further, we set our variables. Let $\pi_1, \pi_2\colon X \times_X Y \to X$ be the canonical inclusions so that $\pi_i \circ \Delta = \mathrm{id}_X$ for $i \in \{1, 2\}$ by definition of Δ.

[1] Namely, the $\Delta\varphi_\alpha$ is a restriction because the pullback square is "actually" being induced by (2.13).

- We show (b) implies (c). Give Y the affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$ such that $\pi^{-1}Y_\alpha$ is quasiseparated for each α. Further, give each $\varphi^{-1}(Y_\alpha)$ an affine open cover $\{U_{\alpha,\beta}\}_{\beta\in\lambda_\alpha}$ and label our diagram as

$$\begin{tikzcd} \pi_1^{-1}U_{\alpha,\beta'}\cap\pi_2^{-1}U_{\alpha,\beta} \arrow[r, hook, "\widetilde{\jmath}_{\alpha,\beta}"] \arrow[d, hook, "\widetilde{\jmath}_{\alpha,\beta'}"] & \pi_1^{-1}U_{\alpha,\beta'} \arrow[r, "\pi_1"] \arrow[d, hook, "\widetilde{\jmath}_{\alpha,\beta'}"] & U_{\alpha,\beta'} \arrow[d, hook, "\jmath_{\alpha,\beta'}"] \\ \pi_2^{-1}U_{\alpha,\beta} \arrow[r, hook, "\widetilde{\jmath}_{\alpha,\beta}"] \arrow[d, "\pi_2"] & (\varphi\pi_1)^{-1}Y_\alpha \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \varphi^{-1}Y_\alpha \arrow[d, "\varphi"] \\ U_{\alpha,\beta} \arrow[r, hook, "\jmath_{\alpha,\beta}"] & \varphi^{-1}Y_\alpha \arrow[r, "\varphi"] & Y_\alpha \end{tikzcd} \tag{3.2}$$

where the key point is that the top-left corner indeed should be $\pi_1^{-1}U_{\alpha,\beta'}\cap\pi_2^{-1}U_{\alpha,\beta}$ as computed in Corollary 2.180.

Now, the bottom-right square of (3.2) is a pullback square by Lemma 2.186, and the remaining square are pullback squares by Lemma 2.178, so repeated applications of Lemma 2.174 tell us that the outer square is a pullback square. In particular, we conclude that $\pi_1^{-1}U_{\alpha,\beta'}\cap\pi_2^{-1}U_{\alpha,\beta}$ is affine using the big pullback square by Lemma 2.176.

We now remember that we are trying to show that Δ is quasicompact. Well, being quasicompact is affine-local on the target by Lemma 3.16, so it suffices to show that $\Delta|_{\Delta^{-1}V}$ is quasicompact as V varies over the various $\pi_1^{-1}U_{\alpha,\beta'}\cap\pi_2^{-1}U_{\alpha,\beta}$. In fact, by Corollary 3.17, it suffices to show that $\Delta^{-1}(V)$ itself is compact for our various V, for which we compute

$$\begin{aligned}\Delta^{-1}(V) &= \left\{x\in X:\Delta(x)\in\pi_1^{-1}U_{\alpha,\beta'}\cap\pi_2^{-1}U_{\alpha,\beta}\right\}\\ &= \{x\in X:\pi_1\Delta(x)\in U_{\alpha,\beta'}\text{ and }\pi_2\Delta(x)\in U_{\alpha,\beta}\}\\ &= U_{\alpha,\beta'}\cap U_{\alpha,\beta},\end{aligned}$$

which is compact because the $U_{\alpha,\beta}$ and $U_{\alpha,\beta'}$ are compact open subsets of the quasiseparated space $\varphi^{-1}Y_\alpha$: indeed, $\varphi^{-1}Y_\alpha$ is quasiseparated because φ is quasiseparated, and Y_α is affine. This finishes.

- We show (c) implies (a); we use Remark 3.12. Fix an affine open subset $V\subseteq Y$ and two affine open subsets $U_1,U_2\subseteq\varphi^{-1}V$; we need to show that $U_1\cap U_2$ is quasicompact. Well, by Lemma 3.35, we are promised a pullback square

$$\begin{tikzcd} V_1\times_X V_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V_1\times_Y V_2 \arrow[d] \\ X \arrow[r, "\Delta"] & X\times_Y X \end{tikzcd}$$

where the bottom morphism is Δ. To compute $V_1\times_X V_2$, we note that the maps $V_1\hookrightarrow X$ and $V_2\hookrightarrow X$ are open embeddings, so $V_1\times_X V_2\simeq V_1\cap V_2$ by Corollary 2.180. With this in mind, we see that the top morphism above is quasicompact because quasicompactness is preserved by base change by Lemma 3.25, so to show $V_1\cap V_2$ is quasicompact, it suffices to show that $V_1\times_Y V_2$ is quasicompact.

Well, noting that $V_1,V_2\subseteq\varphi^{-1}U$, we note that

$$\begin{tikzcd} V_1\times_Y V_2 \arrow[r] \arrow[d] & V_1 \arrow[d, "\varphi"] \\ V_2 \arrow[r, "\varphi"] & U \end{tikzcd}$$

is a pullback square by Lemma 2.186. Thus, $V_1\times_Y V_2$ is affine by Lemma 2.176 because now all of V_1,V_2,U are affine, so $V_1\times_Y V_2$ is quasicompact.

The above implications finish the proof. ■

Here are some quick applications of the above lemma.

Example 3.41. All monomorphisms $\varphi\colon X \to Y$ are quasiseparated: indeed, by Lemma 3.34, the diagonal morphism $\Delta\varphi$ is an isomorphism, which is quasicompact by Example 3.4, which is quasiseparated by Lemma 3.40. For example, open embeddings are quasiseparated by Corollary 2.179.

Corollary 3.42. Fix an affine scheme Y and a scheme morphism $\pi\colon X \to Y$. Then X is quasiseparated if and only if π is quasiseparated.

Proof. If π is quasiseparated, then we note that the affine open subscheme $Y \subseteq Y$ forces $X = \pi^{-1}(Y)$ to be quasiseparated. Conversely, if X is quasiseparated, then we note that the affine open cover $\{Y\}$ of Y has $\pi^{-1}(Y) = X$ quasiseparated, so π is quasiseparated by Lemma 3.40. ■

Example 3.43. Using the fact that $\operatorname{Spec}\mathbb{Z}$ is final in the category of schemes (by Corollary 2.29), we can use Corollary 3.42 to say that a scheme X is quasiseparated if and only if the canonical morphism $X \to \operatorname{Spec}\mathbb{Z}$ is quasiseparated.

Example 3.44. Because affine schemes are quasiseparated by Example 3.10, we conclude by Corollary 3.42 that any morphism of affine schemes is quasiseparated.

Example 3.45. Given a ring A, the canonical projection $\mathbb{P}^n_A \to \operatorname{Spec} A$ is quasicompact. Indeed, by Corollary 3.42, it suffices to show that $\mathbb{P}^n_A$ is quasicompact, which is true because the usual affine open cover of $\mathbb{P}^n_A$ cover $\mathbb{P}^n_A$ by finitely many affine (and therefore quasicompact) open subsets.

Corollary 3.46. The class of quasiseparated morphisms is affine-local on the target.

Proof. In one direction, if $\pi\colon X \to Y$ is quasiseparated, then any affine open cover $\mathcal{U}$ of Y will make $\pi^{-1}U$ quasiseparated for any $U \in \mathcal{U}$ because π is quasiseparated. Thus, π is quasiseparated by Lemma 3.40.

In the other direction, fix a scheme morphism $\pi\colon X \to Y$ and an affine open cover $\{V_\alpha\}_{\alpha\in\lambda}$ such that $\pi|_{\pi^{-1}V_\alpha}\colon \pi^{-1}V_\alpha \to V_\alpha$ is quasiseparated for each V_α. Fixing some V_α, we see $\pi^{-1}V_\alpha$ is affine and makes $\pi^{-1}V_\alpha$ quasiseparated because $\pi|_{\pi^{-1}V_\alpha}$ is integral. Thus, π is integral by Lemma 3.40. ■

Lemma 3.40 lets us turn questions about morphisms being quasiseparated into questions about them being quasicompact. This will more or less automatically prove that being quasiseparated is preserved under composition and base change. Let's see this.

Lemma 3.47. The class of quasiseparated morphisms is preserved by composition.

Proof. Note that the class of quasicompact morphisms is preserved by composition by Corollary 3.22 and preserved by base change by Lemma 3.25. The result now follows from Lemma 3.37 by viewing quasiseparated morphisms as the class of morphisms with quasicompact diagonal, by Lemma 3.40. ■

Lemma 3.48. The class of quasiseparated morphisms is preserved by base change.

Proof. Note that the class of quasicompact morphisms is preserved by base change by Lemma 3.25. The result now follows from Lemma 3.38 by viewing quasiseparated morphisms as the class of morphisms with quasicompact diagonal, by Lemma 3.40. ■

Quasiseparated morphisms also turn out to satisfy a cancellation property.

Lemma 3.49. Fix scheme morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. If the composite $\psi \circ \varphi$ is quasiseparated, then φ is also quasiseparated.

Proof. We use the fact that being quasiseparated is affine-local on the target, as showed in Lemma 3.40. Give Z some affine open cover $\{Z_\alpha\}_{\alpha\in\lambda}$, and then give each $\psi^{-1}Z_\alpha$ an affine open cover $\{Y_{\alpha,\beta}\}_{\beta\in\lambda_\alpha}$. It suffices to show that the restriction $\varphi|_{\varphi^{-1}Y_{\alpha,\beta}}\colon \varphi^{-1}Y_{\alpha,\beta} \to Y_{\alpha,\beta}$ is quasiseparated because being quasiseparated is affine-local on the target.

Thus, fix some $\alpha \in \lambda$ and $\beta \in \lambda_\alpha$. Because $Y_{\alpha,\beta}$ is affine, we want to know that $\varphi^{-1}Y_{\alpha,\beta}$ is quasiseparated by Corollary 3.42. However, $(\psi\circ\varphi)^{-1}Z_\alpha$ is quasiseparated because $\psi\circ\varphi$ is quasiseparated. In particular, the open subset $\varphi^{-1}Y_{\alpha,\beta} \subseteq (\psi\circ\varphi)^{-1}Z_\alpha$ is also quasiseparated by Example 3.9. ■

3.1.6 Affine Morphisms Are Reasonable

Here is our definition.

Definition 3.50 (Affine). A scheme morphism $\pi\colon X \to Y$ if and only if every affine open subset $U \subseteq Y$ has $\pi^{-1}(U)$ affine.

Example 3.51. Let $\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes; we show φ is affine. Well, for any affine open subscheme $\operatorname{Spec} A' \cong U \subseteq \operatorname{Spec} A$, Lemma 2.178 tells us that

$$\begin{tikzcd}
\varphi^{-1}(U) \arrow[r, hook] \arrow[d, "\varphi|_{\varphi^{-1}(U)}"] & \operatorname{Spec} B \arrow[d, "\varphi"] \\
U \arrow[r, hook] & \operatorname{Spec} A
\end{tikzcd}$$

is a pullback square, so $\varphi^{-1}(U) \cong \operatorname{Spec} B \times_{\operatorname{Spec} A} U \cong \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} A'$, which is just $\operatorname{Spec} B \otimes_A A'$ by Lemma 2.176. In particular, $\varphi^{-1}(U)$ is affine.

Remark 3.52. In fact, a scheme morphism $\pi\colon X \to Y$ with Y affine has π is affine if and only if X is affine. Indeed, in one direction, if X is affine, then π is affine by Example 3.51. In the other direction, if π is affine, then the affine open subset $Y \subseteq Y$ tells us that $X = \pi^{-1}Y$ is affine, which finishes.

Remark 3.53. As an example of using our hypotheses, we note that affine morphisms $\pi\colon X \to Y$ are quasicompact and quasiseparated. Indeed, for any affine open $U \subseteq Y$, we see $\pi^{-1}U$ is affine and therefore quasicompact and quasiseparated (quasiseparatedness by Example 3.10).

Remark 3.54. More generally, suppose that P is a class of morphisms which is affine-local on the target and includes morphisms of affine schemes. Then, for any affine morphism $\pi\colon X \to Y$, we see π is in P: give Y any affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$, and then note that the restriction $\pi_\alpha\colon \pi^{-1}Y_\alpha \to Y_\alpha$ is in P because this is a morphism of affine schemes. So because P is affine-local on the target, we conclude π is in P.

Here are the usual sanity checks.

Lemma 3.55. Affine morphisms are preserved by composition.

Proof. Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be affine morphisms, and we want to show $\psi\circ\varphi$ is affine. Well, if $W \subseteq Z$ is an affine open subset, then $\psi^{-1}W \subseteq Y$ is affine, so $(\psi\circ\varphi)^{-1}(W) = \varphi^{-1}\psi^{-1}W \subseteq X$ is also affine. This finishes. ■

It turns out that showing being affine is affine-local on the target is quite tricky. We will slowly build our way up.

Lemma 3.56. Fix an affine morphism $\pi\colon X \to Y$. Then for any open subscheme $V \subseteq Y$, the restriction $\pi|_{\pi^{-1}V}\colon \pi^{-1}V \to V$ is also affine.

Proof. We chase the definitions. Fix an affine open subscheme $V' \subseteq V$. Then $V' \subseteq Y$ is also an affine open subscheme, so $\pi^{-1}V' \subseteq X$ is an affine open subscheme, so $\pi|_{\pi^{-1}V}^{-1}(V') \subseteq \pi^{-1}V'$ is an affine open subscheme. ■

What is hard about being affine-local on the target is the gluing part of Lemma 2.118. We throw the relevant results into some lemmas.

Lemma 3.57. Suppose that the scheme X which has a finite an affine open cover $\{U_i\}_{i=1}^n$ such that each intersection $U_i \cap U_j$ is quasicompact. Then, for any $f \in \mathcal{O}_X(X)$, the natural map $\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$ (induced by restriction and Lemma 2.23) is an isomorphism.

It turns out that the hypothesis on X is equivalent to requiring X to be quasicompact and quasiseparated, but we won't bother showing this.

Proof. We already have our maps of rings as a localization of a restriction map, where the localization is legal because $f \in \mathcal{O}_X(X_f)^\times$ by Lemma 2.23. Thus, it suffices to show that our map $\iota_f\colon \mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$ given by $s/f^n \mapsto (s|_{X_f})(f|_{X_f})^{-n}$ is bijective. We have two checks; set $A := \mathcal{O}_X(X)$ for brevity.

- Injective: we only need quasicompactness. For now, fix an affine open subscheme $\varphi\colon U \cong \operatorname{Spec} \mathcal{O}_X(U)$ setting $B := \mathcal{O}_X(U)$ (using the canonical isomorphism) of X. We are given $a/f^n \in A_f$ such that $a|_{X_f} \cdot (f|_{X_f})^{-n} = 0$, so actually $a|_{X_f} = 0$, so set $a_U := a|_U$ so that $a_U|_{U\cap X_f} = a|_{X_f}|_{X_f \cap U} = 0$ as well.

  Now, we pass through $\varphi^\sharp\colon (\mathcal{O}_X|_U) \to \varphi_*(\mathcal{O}_{\operatorname{Spec} B})$. Because $U \cap X_f = \varphi(D(\varphi_U^\sharp(f|_U)))$, as shown above, we see that the full diagram

$$\begin{tikzcd} \mathcal{O}_X|_U(U) \arrow[r, "\varphi_U^\sharp"] \arrow[d, "\operatorname{res}_{U,U\cap X_f}"] & \mathcal{O}_{\operatorname{Spec} B}(\varphi^{-1}(U)) \arrow[r, equal] \arrow[d, "\operatorname{res}_{\varphi^{-1}(U),\varphi^{-1}(U\cap X_f)}"] & \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \arrow[r, equal] & B \arrow[d] \\ \mathcal{O}_X|_U(U\cap X_f) \arrow[r, "\varphi_{U\cap X_f}^\sharp"'] & \mathcal{O}_{\operatorname{Spec} B}(\varphi^{-1}(U\cap X_f)) \arrow[r, equal] & \mathcal{O}_{\operatorname{Spec} B}(D(\varphi_U^\sharp(f|_U))) \arrow[r, equal] & B_{\varphi_U^\sharp(f|_U)} \end{tikzcd} \tag{3.3}$$

  commutes. In particular, passing a_U with $a_U|_{U\cap X_f} = 0$ through the diagram, we see that $\varphi_U^\sharp(a_U)$ vanishes in $B_{\varphi_U^\sharp(f|_U)}$. Thus, there exists a positive integer n such that

$$0 = \varphi_U^\sharp(f|_U)^n \cdot \varphi_U^\sharp(a_U) = \varphi_U^\sharp\left((f|_U)^n a_U\right) = \varphi_U^\sharp\left((f^n a)|_U\right).$$

  Because $\varphi^\sharp$ is an isomorphism of sheaves (because it is part of an isomorphism of schemes), we see that $\varphi_U^\sharp$ is an isomorphism and is therefore injective, so $(f^n a)|_U = 0$.

  We are now ready to talk about all of X. Because X has an affine open cover, the quasicompactness of X promises a finite affine open cover $\{U_i\}_{i=1}^m$, and the argument above promises positive integers n_i such that

$$(f^{n_i} a)|_{U_i} = 0$$

  for each i. Thus, we set $n := \max\{n_i : 1 \le i \le m\}$ so that

$$(f^n a)|_{U_i} = 0$$

  for all i. However, $\{U_i\}_{i=1}^m$ forms a cover of X, so the identity axiom $\mathcal{O}_X$ forces $f^n a = 0$. This finishes.

- Surjectivity: we now use the fact that the $U_i \cap U_j$ is quasicompact. We proceed in steps. Fix $b \in \mathcal{O}_X(X_f)$.

  1. For now, fix an affine open subscheme $(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ of $(X, \mathcal{O}_X)$. For concreteness, we set $(\varphi, \varphi^\sharp) \colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \cong (U, \mathcal{O}_X|_U)$ to be the isomorphism. Set $b_U := b|_{U \cap X_f}$. Now, $\varphi^\sharp \colon \mathcal{O}_X|_U \to \varphi_* \mathcal{O}_{\operatorname{Spec} B}$ is an isomorphism, so we build the commutative diagram (3.3) again. In particular, we can write $\varphi^\sharp_{U \cap X_f}(b_U) \in B_{\varphi^\sharp(f|_U)}$ as
  $$\varphi^\sharp_{U \cap X_f}(b_U) = \frac{\varphi^\sharp_U(a)}{\varphi^\sharp_U(f|_U)^n}$$
  for some $\varphi^\sharp_U(a) \in B$ (where $a \in \mathcal{O}_X(U)$) and positive integer n. In particular, it follows that
  $$\varphi^\sharp_{U \cap X_f}(a|_{U \cap X_f}) = \varphi^\sharp_{U \cap X_f}(f^n|_{U \cap X_f} \cdot b|_{U \cap X_f}),$$
  so because $\varphi^\sharp$ is an isomorphism, we have $a|_{U \cap X_f} = f^n|_{U \cap X_f} \cdot b|_{U \cap X_f}$. We remark that multiplying a by multiples of f will not change this property.
  2. Returning to X, we give X the promised affine open cover $\{U_i\}_{i=1}^m$. The argument above generates sections $a_i \in \mathcal{O}_X(U_i)$ and positive integers n_i such that
  $$a_i|_{U_i \cap X_f} = f^{n_i}|_{U_i \cap X_f} \cdot b|_{U_i \cap X_f}$$
  for each i. Letting n be the maximum of the finitely many n_i, we can replace a_i with $a_i \cdot f^{n-n_i}|_{U_i}$ (recall multiplying by fs doesn't do anything) and multiply the above equation by f^{n-n_i} so that actually
  $$a_i|_{U_i \cap X_f} = f^n|_{U_i \cap X_f} \cdot b|_{U_i \cap X_f} \tag{3.4}$$
  for each i. We would like to glue these a_i, but the argument is a little technical.
  3. Fix two indices i and j and affine open subset $V \subseteq U_i \cap U_j$ so that we have a ring B equipped with an isomorphism $(\varphi, \varphi^\sharp) \colon \operatorname{Spec} B \cong V$. Now, (3.4) restricted to V tells us that
  $$a_i|_{V \cap X_f} = f^n|_{V \cap X_f} \cdot b|_{V \cap X_f} = a_j|_{V \cap X_f}.$$
  As such, passing $a_i|_V$ and $a_j|_V$ through the commutative diagram (3.3) (where Us are replaced with Vs), we see that
  $$\varphi^\sharp_V(a_i|_V)|_{V \cap X_f} = \varphi^\sharp_V(a_j|_V)|_{V \cap X_f}$$
  as elements in $B_{\varphi^\sharp_V(f|_V)}$, so undoing the localization, there is a positive integer d such that
  $$\varphi^\sharp_V(f|_V)^d \cdot \varphi^\sharp_V(a_i|_V) = \varphi^\sharp_V(f|_V)^d \cdot \varphi^\sharp_V(a_j|_V).$$
  Lastly, undoing $\varphi^\sharp$, we get $(f|_V)^d \cdot a_i|_V = (f|_V)^d \cdot a_j|_V$.
  4. We now return to X. For each pair of indices i and j, we are given that $U_i \cap U_j$ is quasicompact, so provide $U_i \cap U_j$ with a finite affine open cover $\{V_{ijk}\}_{k=1}^{m_{ij}}$. Notably, the above point grants a positive integer d_{ijk} such that
  $$\left(f|_{U_i}^{d_{ijk}} \cdot a_i\right)|_{V_{ijk}} = \left(f|_{U_j}^{d_{ijk}} \cdot a_j\right)|_{V_{ijk}}$$
  for any triple of indices i and j and k. Now, let d be the maximum over all the d_{ijk}, and replace each a_i with $(f|_{U_i})^d \cdot a_i$ and n with $n + d$ (again multiplying by f doesn't do anything) so that we see
  $$a_i|_{V_{ijk}} = a_j|_{V_{ijk}}$$
  for any triple of indices i and j and k. In particular, letting k vary and applying the identity axiom gives $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$, so the gluability axiom gives an $a \in A$ such that $a|_{U_i} = a$.
  In particular, for each U_i, we see
  $$a|_{X_f}|_{U_i \cap X_f} = a_i|_{U_i \cap X_f} = \left(f^n|_{X_f} \cdot b\right)|_{U_i \cap X_f},$$
  so the identity axiom assures us that $a|_{X_f} = f^n|_{X_f} \cdot b$. This finishes. ■

Lemma 3.58. Fix a scheme X. If there exist global sections $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ such that X_{f_i} is affine for each i, then X is affine.

Proof. We combine the previous two lemmas. Set $A := \mathcal{O}_X(X)$. We are given elements $\{f_i\}_{i=1}^r \subseteq A$ such that $(f_1, \ldots, f_r) = A$, and we know that the X_{f_i} are all affine open subschemes of X. The main approach is to show that the morphism

$$\eta_X \colon X \to \operatorname{Spec} A$$

conjured in Lemma 2.25 is an isomorphism; for brevity, set $\varphi := \eta_X$. To stay organized, we will proceed in steps.

1. We check some open covers. Note that the $D(f_i)$ cover $\operatorname{Spec} A$ by Remark 2.119.

   Additionally, we claim that the X_{f_i} form an affine open cover X. Indeed, as computed in Lemma 2.25, we have $\varphi^{-1}(D(f)) = X_f$ for any $f \in A$, so the fact that the $\{D(f_i)\}$ cover $\operatorname{Spec} A$ forces the X_{f_i} to cover X.

2. We claim that $X_{f_i} \cap X_{f_j}$ is quasicompact for each i and j. Indeed, given f_i and f_j and some point $x \in X$, we note that $f_i \notin \mathfrak{m}_{X,x}$ and $f_j \notin \mathfrak{m}_{X,x}$ if and only if $f_i \notin \mathfrak{m}_{X,x}$ so that $x \in X_{f_i}$ and $f_j|_{X_{f_i}} \notin \mathfrak{m}_{X,x}$. Thus, $X_{f_i} \cap X_{f_j} = (X_{f_i})_{f_j}$ is a distinguished open subscheme of X_{f_i} and is therefore quasicompact.

   Explicitly, under the canonical isomorphism $X_{f_i} \cong \operatorname{Spec} \mathcal{O}_X(X_{f_i})$ given by Corollary 2.28, we see that the pre-image of $D(f_j)$ is $(X_{f_i})_{f_j}$ as computed in Lemma 2.25.

3. We now trigger Lemma 3.57 using the affine open cover $\{X_{f_i}\}_{i=1}^n$ so that $A_{f_i} \simeq \mathcal{O}_X(X_{f_i})$ by localizing the restriction map.

   In particular, Corollary 2.28 lets us set $\varphi_i \colon X_{f_i} \to \operatorname{Spec} \mathcal{O}_X(X_{f_i})$ to be the canonical isomorphism. Thus, we see that we have a chain of isomorphisms

   $$X_{f_i} \cong \operatorname{Spec} \mathcal{O}_X(X_{f_i}) \cong \operatorname{Spec} A_{f_i} \cong D(f_i) \subseteq \operatorname{Spec} A. \tag{3.5}$$

   We take a moment to compute this on global sections, which will give a map from $\mathcal{O}_{\operatorname{Spec} A}(D(f_i)) \simeq A_{f_i}$ to $\mathcal{O}_X(X_{f_i})$. Well, this is

   $$\begin{array}{ccccccc} \mathcal{O}_{\operatorname{Spec} A}(D(f_i)) & \cong & \mathcal{O}_{\operatorname{Spec} A_{f_i}}(\operatorname{Spec} A_{f_i}) & \cong & \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X_{f_i})}(\operatorname{Spec} \mathcal{O}_X(X_{f_i})) & \cong & \mathcal{O}_X(X_{f_i}) \\ a/f_i^n & \mapsto & a/f_i^n & \mapsto & a|_{X_{f_i}} \cdot (f_i|_{X_{f_i}})^{-n} & \mapsto & a|_{X_{f_i}} \cdot (f_i|_{X_{f_i}})^{-n} \end{array}$$

   by tracking everything through.

4. Because being an isomorphism is local on the target by Lemma 3.31, we will be done if we show $\varphi|_{X_{f_i}}$ restricted to is the composite of (3.5).

   However, using the adjunction of Theorem 2.26, it suffices to show that the map of (3.5) agrees with $\varphi|_{X_{f_i}}$ on global sections. Well, $\varphi^\sharp_{D(f_i)} \colon \mathcal{O}_{\operatorname{Spec} A}(D(f_i)) \to \mathcal{O}_X(X_{f_i})$ by construction in Lemma 2.25 sends a generic element a/f_i^n to $a|_{X_{f_i}} \cdot (f_i|_{X_{f_i}})^{-n}$, as desired. ■

Lemma 3.59. A morphism $\pi \colon X \to Y$ is affine if and only if we can provide Y with an affine open cover $\mathcal{U}$ such that $\pi^{-1}(U)$ is affine for each $U \in \mathcal{U}$.

Proof. Certainly if π is affine, then any affine open cover $\mathcal{U}$ of Y will have $\pi^{-1}U \subseteq X$ affine for any $U \in \mathcal{U}$.

For the other direction, we use Lemma 2.118. Call an affine open subset $U \subseteq Y$ "acceptable" if and only if $\pi^{-1}U \subseteq X$ is also affine. We now run our checks.

(i) Suppose that U is acceptable and $f \in \mathcal{O}_X(U)$. Then we want U_f to also be acceptable. Well, we recall from Remark 2.24 that

$$\pi^{-1}(U_f) = (\pi^{-1}U)_{\pi^\sharp_U f},$$

which we now see is a distinguished open subscheme of U and therefore an affine scheme.

(ii) Suppose that we have an open subscheme $U \subseteq X$ and sections $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ with $(f_1, \ldots, f_n) = \mathcal{O}_X(U)$ such that U_{f_i} is acceptable for each i. We would like to show that U is acceptable, for which we need to show $\pi^{-1}U$ is affine.

Well, we recall from Remark 2.24 that

$$\pi^{-1}(U_{f_i}) = (\pi^{-1}U)_{\pi^\sharp_U f_i}$$

for each i, so all the $(\pi^{-1}U)_{\pi^\sharp_U f_i}$ are affine. Furthermore, we note $(f_1, \ldots, f_n) = (1)$ implies some $\mathcal{O}_Y(U)$-linear combination of the f_i equals 1, so some $\mathcal{O}_X(U)$-linear combination of the $\pi^\sharp_U f_i$ equals 1. Thus, Lemma 3.58 kicks in and tells us that $\pi^{-1}U$ is affine.

(iii) Lastly, we note that Y has a cover of acceptable affine open subschemes by hypothesis on Y.

The above checks complete the proof by Lemma 2.118. ■

Corollary 3.60. The class of affine morphisms is affine-local on the target.

Proof. One direction of being affine-local on the target is dealt with by Lemma 3.56.

In the other direction, fix a scheme morphism $\pi \colon X \to Y$ and an affine open cover $\{V_\alpha\}_{\alpha\in\lambda}$ such that $\pi|_{\pi^{-1}V_\alpha} \colon \pi^{-1}V_\alpha \to V_\alpha$ is affine for each V_α. Fixing some V_α, we see $\pi^{-1}V_\alpha$ is affine and makes $\pi^{-1}V_\alpha$ affine because $\pi|_{\pi^{-1}V_\alpha}$ is affine. Thus, π is affine by Lemma 3.59. ■

And lastly, here is base change.

Lemma 3.61. The class of affine morphisms is preserved by base change.

Proof. Suppose we have a pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

of schemes such that ψ_Y is affine. We would like to show that π_X is affine. Notably, because being affine is affine-local on the target by Corollary 3.87, we may use Lemma 3.27 to assume that X and S are affine.

However, ψ_Y is affine, so we conclude that $Y = \psi^{-1}Y$ is affine because S is. So because all of X and Y and S are affine, we set $A := \mathcal{O}_X(X)$ and $B := \mathcal{O}_Y(Y)$ and $R := \mathcal{O}_S(S)$ so that Lemma 2.176 tells us

$$X \times_S Y \simeq \operatorname{Spec} A \otimes_R B,$$

so $X \times_S Y$ is in fact affine. It follows that π_X is affine by Remark 3.52 because X and $X \times_S Y = \pi^{-1}X$ are both affine. ■

Next time we move on to talk about finite morphisms, integral morphisms, and morphisms of finite type.

3.2 September 26

The second problem set has been graded. I had a few typos.

3.2.1 Finiteness Conditions

We continue our discussion of finiteness properties. Here is our strongest.

Definition 3.62 (Finite)**.** A scheme morphism $\pi\colon X \to Y$ is *finite* if and only if an affine open subset $U \subseteq Y$ makes $\pi^{-1}(U)$ also affine in such a way that the induced ring morphism $\pi_U^\sharp\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}U)$ on makes $\mathcal{O}_X(\pi^{-1}U)$ a finitely generated $\mathcal{O}_Y(U)$-module.

In particular, being finite includes being affine.

Example 3.63. Suppose that the ring map $f\colon A \to B$ makes B a finitely generated A-module. We will show later that the associated scheme map $\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is finite.

Here is the weakest.

Definition 3.64 (Locally of finite type)**.** A scheme morphism $\pi\colon X \to Y$ is *locally of finite type* if and only if an affine open subset $V \subseteq Y$ with affine open subset $U \subseteq \pi^{-1}(U)$ inducing the ring morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(\pi^{-1}V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(V)$-algebra.

Note that there is a key difference between being finitely generated as a module and an algebra.

Example 3.65. The ring $k[x, y]$ is finitely generated as k-algebra but not as a k-module. We will see shortly that this means the induced map $\operatorname{Spec} k[x, y] \to \operatorname{Spec} k$ is (locally) of finite type but is not finite.

Lastly, here is our medium-strength morphism.

Definition 3.66 (Finite type)**.** A scheme morphism $\pi\colon X \to Y$ is of *finite type* if and only if π is locally of finite type and quasicompact.

Example 3.67. Suppose that $f\colon A \to B$ makes B a finitely generated A-algebra, and let $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced scheme morphism. We will be able to show shortly that π is of finite type: of course, π is affine by Example 3.51 and so quasicompact by Remark 3.53, and π is locally of finite type by Corollary 3.75.

Example 3.68. Suppose that $\pi\colon X \to Y$ is finite; we show π is of finite type. Note π is affine and therefore quasicompact by Remark 3.53. To see that π is locally of finite type, note that for any affine open $U \subseteq Y$, we know $\mathcal{O}_X(\pi^{-1}U)$ is finitely generated as an $\mathbb{O}_Y(U)$-module by $\pi_U^\sharp$, so the same generators convince us that $\mathcal{O}_X(\pi^{-1}U)$ is a finitely generated $\mathcal{O}_Y(U)$-algebra.

Example 3.69. As usual, isomorphisms have all these adjectives. In particular, an isomorphism $\pi\colon X \to Y$ is finite because $\pi_V^\sharp$ is an isomorphism for any affine open subscheme $V \subseteq Y$, so $\mathcal{O}_X(\pi^{-1}V)$ is of course affine and finitely generated as an $\mathcal{O}_Y(V)$-module.

We now slowly run all of our checks.

3.2.2 Locally of Finite Type is Reasonable

We will now run the usual checks on morphisms which are locally of finite type.

Lemma 3.70. Fix a scheme morphism $\pi\colon X \to Y$ locally of finite type. Then, for any open subsets $U \subseteq X$ and $V \subseteq Y$ contained in $\pi(U)$, the restricted map $\pi|_U\colon U \to V$ is still locally of finite type.

Proof. Fix affine open subsets $\operatorname{Spec} A \cong V' \subseteq V \subseteq Y$ and $\operatorname{Spec} B \cong U' \subseteq \pi|_U^{-1}(V') \subseteq X$. Notably, we still see that V' is an affine open subscheme of Y, and U' is an affine open subscheme of X. Thus, because π is locally of finite type, we see that B is a finitely generated A-algebra. ■

The following lemma will find some utility.

Lemma 3.71. Fix a scheme morphism $\pi\colon X \to Y$, where Y is an affine scheme. Suppose that X has an affine open cover $\mathcal{U}$ such that, for each $U \in \mathcal{U}$, we have $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(Y)$-algebra by $\pi^\sharp$. Then each affine open subset of X has this property.

Proof. We use the Affine communication lemma. Say that an affine open subset $U \subseteq X$ is *acceptable* if and only if $\mathcal{O}_X(U)$ is a finitely generated A-algebra by $\pi^\sharp$. Here are our checks; set $A := \mathcal{O}_Y(Y)$.

(i) Suppose U is acceptable and $f \in \mathcal{O}_X(U)$; we show U_f is acceptable. Well, by hypothesis, we may find finitely many generators $b_1, \ldots, b_n$ generating $\mathcal{O}_X(U)$ as an A-algebra by $\pi^\sharp$. However, $\mathcal{O}_X(U_f) \simeq \mathcal{O}_X(U)_f$ is generated over $\mathcal{O}_X(U)$ as an $\mathcal{O}_X(U)$-algebra by the generator $1/f$, so it follows that $\mathcal{O}_X(U_f)$ is generated by $b_1, \ldots, b_n, 1/f$ as an A-algebra. This is what we wanted.

(ii) Suppose an affine open subset $U \subseteq X$ has elements $w_1, \ldots, w_n \in \mathcal{O}_X(U)$ such that $(w_1, \ldots, w_n) = \mathcal{O}_X(U)$, and U_{f_i} is acceptable for each i. We show that U is acceptable.

For brevity, set $C := \mathcal{O}_X(U)$. Translating everything over to commutative algebra, we see that we are given C_{w_k} is a finitely generated A-algebra for each k, and we want to show that C is a finitely generated A-algebra. Well, we can find finitely many elements

$$\left\{c_{k,1}/w_k^{e_{k,1}}, \ldots, c_{k,N_k}/w_k^{e_{k,N_k}}\right\}$$

of C_{w_k} to generate C_{w_k} as an A-algebra. However, we can also generate all the above generators by the elements

$$\{c_{k,1}, \ldots, c_{k,N_k}, 1/w_k\},$$

so we will elect to use these generators instead. By artificially adding in 1s to the end of each S_k without changing the fact we generate, so we may assume that all the S_k have the same size, so set $N := N_k$ to be this uniform length.

We will want a few more elements in our generating set. Note that the $D(w_k)$ cover C by construction, so no prime $\mathfrak{p} \in \operatorname{Spec} C$ contains all the w_k, so $(w_1, \ldots, w_m) = C$, so we can find elements $d_{1,1}, \ldots, d_{m,1} \in C$ such that

$$\sum_{k=1}^m w_k d_{k,1} = 1.$$

Taking this equation to the mMth power, each term has a power of w_k^M for some k by the pigeonhole principle, so we see that we can write

$$\sum_{k=1}^m w_k^M d_{k,M} = 1,$$

where each $d_{k,M}$ is some polynomial in the w_k and the $d_{k,1}$.

We now let our set of generators be $S := \bigcup_{k=1}^m \{c_{k,1}, \ldots, c_{k,N}, w_k, d_{k,1}\}$, which is the finite union of finite sets and therefore finite. We claim that S generates C as an A-algebra. Indeed, pick up some $c \in C$, and we know that in C_{w_k} we may write

$$\frac{c}{1} = p_k\left(c_{k,1}, \ldots, c_{k,N}, 1/w_k\right)$$

for some polynomial $p_k \in A[x_1, \ldots, x_{N+1}]$. Collecting denominators, we can find some M_k such that

$$\frac{c}{1} = \frac{q_k(c_{k,1}, \ldots, c_{k,N}, w_k)}{w_k^{M_k}},$$

where $q_k \in A[x_1, \ldots, x_{N+1}]$. Thus, there is some M_k', M_k'' such that

$$w_k^{M_k'} c = w_k^{M_k''} q_k(c_{k,1}, \ldots, c_{k,N}, w_k).$$

It follows that $w_k^{M_k'} c$ is generated by S. Letting M be the maximum over all the M_k', we see that $w_k^M c$ is still generated by S, so each term of

$$c = \sum_{k=1}^{m} \left(w_k^M c\right) d_{k,M}$$

is still a polynomial with coefficients in A of the terms in S. It follows that c is generated by S. This is what we wanted.

(iii) We have an open cover of acceptable affine open subschemes by hypothesis.

The above checks complete the proof by Lemma 2.118. ■

We now use Lemma 3.71 for fun and profit.

Lemma 3.72. The class of morphisms locally of finite type is preserved by composition.

Proof. Suppose that we have morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ which are locally of finite type. We want to show that the composite $\beta \circ \alpha$ is also locally of finite type.

Fix some affine open subset $\operatorname{Spec} \mathcal{O}_Z(U) \simeq U$ of Z; set $A := \mathcal{O}_Z(U)$. Pulling back to $\psi^{-1}(U)$, given $\psi^{-1}(U)$ an affine open cover $\{V_\alpha\}_{\alpha \in \lambda}$, where $V_\alpha \simeq \operatorname{Spec} \mathcal{O}_Y(V_\alpha)$; set $B_\alpha := \mathcal{O}_Y(V_\alpha)$. We are given that the map $\psi_\alpha^\sharp\colon A \to B_\alpha$ on global sections makes B_α into a finitely-generated A-algebra.

Going further, for any affine open subset $W_{\alpha,\beta} \subseteq \varphi^{-1}(V_\alpha)$ will have $\mathcal{O}_X(W_{\alpha,\beta})$ finitely generated as a B_α-algebra by $\varphi^\sharp$, which is finitely generated as an A-algebra by $\psi^\sharp$, so it follows that each $\mathcal{O}_X(W_{\alpha,\beta})$ is finitely generated as an A-algebra by $(\psi \circ \varphi)^\sharp$.

However, we are now done: because the collection of all affine open subschemes $V_\alpha \subseteq \psi^{-1}(U)$ cover the open subscheme $\psi^{-1}(U)$, the collection of all affine open subschemes $W_{\alpha,\beta} \subseteq \varphi^{-1}(V_\alpha)$ for all α will cover the open subscheme $(\psi \circ \varphi)^{-1}(U)$. But each $W_{\alpha,\beta}$ has $\mathcal{O}_X(W_{\alpha,\beta})$ finitely generated as an A-algebra, so Lemma 3.71 kicks in to tell us that all affine open subschemes $W \subseteq (\psi \circ \varphi)^{-1}(U)$ have $\mathcal{O}_X(W)$ finitely generated as an A-algebra by $(\psi \circ \varphi)^\sharp$. ■

Lemma 3.73. Fix a scheme morphism $\pi\colon X \to Y$ and an affine open cover $\mathcal{V}$ such that any $V \in \mathcal{V}$ and affine open $U \subseteq \pi^{-1}V$ has $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(V)$-algebra by $\pi^\sharp$. Then π is locally of finite type.

Proof. Say that an affine open subscheme $V \subseteq Y$ (with $V \cong \operatorname{Spec} \mathcal{O}_Y(V)$) is "smallish" if and only if any affine open subscheme $U \subseteq \pi^{-1}(V)$ makes $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(V)$-algebra by $\pi^\sharp$. We would like to show that all affine open subscheme of Y are smallish, for which we use Lemma 2.118. Here are our checks.

(i) Fix a smallish affine open subscheme $V \subseteq Y$ and some $f \in \mathcal{O}_Y(V)$. We would like to show that V_f is also smallish. Well, find some affine open subscheme $U \subseteq \pi^{-1}(V_f)$. Then $V_f \subseteq V$ means that $U \subseteq \pi^{-1}(V)$, so because V is smallish, we conclude that $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra by $\pi^\sharp$.

Namely, we have some set finite set of generators $b_1, \ldots, b_n \in \mathcal{O}_X(U)$ generating $\mathcal{O}_X(U)$ as an $\mathcal{O}_Y(V)$-algebra. But then any $b \in \mathcal{O}_X(U)$ can be written as a polynomial in the b_i with coefficients in $\mathcal{O}_Y(V)$, so the same polynomial shows that any $b \in \mathcal{O}_X(U)$ can be written as a polynomial in the b_i with coefficients in $\mathcal{O}_Y(V_f) \simeq \mathcal{O}_Y(V)_f$.

(ii) Fix an affine open subscheme $V \subseteq Y$ and some elements $(f_1, \ldots, f_n) \subseteq \mathcal{O}_Y(V)$ generating $\mathcal{O}_Y(V)$. Given that each V_{f_i} is smallish, we would like to show that V is smallish. For brevity, set $A := \mathcal{O}_Y(V)$.

At this point, we remark that we may essentially restrict π to $\pi|_{\pi^{-1}V}$, though we will not do this.

Anyway, arguing as before, for any affine open subset $W_{i,\beta} \subseteq \varphi^{-1}(V_{f_i})$ will have $\mathcal{O}_X(W_{i,\beta})$ finitely generated as an A_{f_i}-algebra by $\pi^\sharp$. However, $A_{f_i} \simeq A[1/f_i]$ is certainly finitely generated as an A-algebra, so it follows that each $\mathcal{O}_X(W_{i,\beta})$ is finitely generated as an A-algebra by $\pi^\sharp$ as well.

However, we are now done: because the collection of all affine open subschemes $V_{f_i} \subseteq V$ cover V (because $(f_1, \ldots, f_n) = 1$), the collection of all affine open subschemes $W_{i,\beta} \subseteq \varphi^{-1}(V_i)$ for all i will cover the open subscheme $\pi^{-1}(V)$. But each $W_{i,\beta}$ has $\mathcal{O}_X(W_{i,\beta})$ finitely generated as an A-algebra, so Lemma 3.71 kicks in to tell us that all affine open subschemes $W \subseteq \pi^{-1}(U)$ have $\mathcal{O}_X(W)$ finitely generated as an A-algebra by $\pi^\sharp$.

(iii) Lastly, we see that Y has an affine open cover by smallish affine open subschemes.

The above checks complete the proof by Lemma 2.118. ■

Here are the usual applications of a lemma like this.

Corollary 3.74. The class of morphisms locally of finite type is affine local on the target.

Proof. One direction of being affine-local on the target is covered by Lemma 3.70.

In the other direction, fix a scheme morphism $\pi\colon X \to Y$ and an affine open cover $\{V_\alpha\}_{\alpha\in\lambda}$ such that $\pi|_{\pi^{-1}V_\alpha}\colon \pi^{-1}V_\alpha \to V_\alpha$ is locally of finite type for each V_α. Then, fixing some V_α, any affine open subset $U \subseteq \pi^{-1}V_\alpha$ will have $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(V)$-algebra by $\pi^\sharp$ because $\pi|_{\pi^{-1}V_\alpha}$ is locally of finite type. Thus, π is locally of finite type by Lemma 3.73. ■

Corollary 3.75. A morphism $\pi\colon X \to Y$ of affine schemes is locally of finite type if and only if $\mathcal{O}_X(X)$ is a finitely generated $\mathcal{O}_Y(Y)$-algebra by $\pi^\sharp_Y$.

Proof. In one direction, if π is locally of finite type, then the affine open subset $Y \subseteq Y$ forces $\mathcal{O}_X(X)$ to be a finitely generated $\mathcal{O}_Y(Y)$-algebra by $\pi^\sharp_Y$. In the other direction, if $\mathcal{O}_X(X)$ is a finitely generated $\mathcal{O}_Y(Y)$-algebra by $\pi^\sharp_Y$, then the affine open cover $\{Y\}$ of Y tells us that π is locally of finite type by Lemma 3.73. ■

Lemma 3.76. Fix a scheme morphism $\varphi\colon X \to Y$ and an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of X such that the restrictions $\varphi|_{U_\alpha}\colon U_\alpha \to Y$ are all locally of finite type. Then φ is locally of finite type.

Proof. Fix some affine open subscheme $V \subseteq Y$. We need to show that all affine open subschemes $U \subseteq \varphi^{-1}V$ have $\mathcal{O}_X(U)$ a finitely-generated $\mathcal{O}_Y(V)$-algebra by $\varphi^\sharp$.

Well, give each $\varphi^{-1}V \cap U_\alpha$ an affine open cover and then union all of these together into an affine open cover $\mathcal{U}$ of X. Notably, because the restrictions $\varphi|_{U_\alpha}$ are locally of finite type, we see that the restrictions

$$\varphi|_{U_\alpha \cap \varphi^{-1}V}\colon (U_\alpha \cap \varphi^{-1}V) \to V$$

are also locally of finite type by Lemma 3.70. Thus, each $U \in \mathcal{U}$ has $U \subseteq U_\alpha$ for some α, meaning that $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra by $\varphi^\sharp$ because the above restriction is locally of finite type. So Lemma 3.71 finishes. ■

Example 3.77. The canonical projection $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$ is locally of finite type. Indeed, give $\mathbb{P}^n_A$ the usual affine open cover $\{U_i\}_{i=0}^n$, and we see that each $\mathcal{O}_X(U_i) = A[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$ is a finitely-generated A-algebra by $\pi^\sharp$. Thus, $\pi|_{U_i}$ is locally of finite type for each i by Corollary 3.75, so π is locally of finite type by Lemma 3.76.

Corollary 3.78. Open embeddings are locally of finite type.

Proof. Fix an open embedding $\iota\colon X \hookrightarrow Y$, where $X \subseteq Y$ is an open subscheme; notably, $\iota_U^\sharp\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ is just the restriction map $\mathcal{O}_Y(U) \to \mathcal{O}_Y(X \cap U)$. Giving Y an affine open cover $\mathcal{U}$, it suffices by Corollary 3.74 to show that each $\iota|_{\iota^{-1}U}\colon \iota^{-1}U \to U$ is locally of finite type.

Thus, we may rename our variables: set $X := \iota^{-1}U$ and $Y := U$ and $\iota := \iota|_{\iota^{-1}U}$ so that $X \to Y$ is still an open embedding because we just have $\iota^{-1}U = U \cap X$, and the structure sheaves still match because $\mathcal{O}_{\iota^{-1}U} = \mathcal{O}_X|_{\iota^{-1}U} = \mathcal{O}_Y|_X|_{\iota^{-1}U} = \mathcal{O}_Y|_{\iota^{-1}U}$.

In other words, we may assume that Y is affine. But now we may use the distinguished open base $Y_f \simeq D(f) \subseteq \operatorname{Spec} \mathcal{O}_Y(Y)$ (under Corollary 2.28) of an affine scheme to write

$$X = \bigcup_{\alpha\in\lambda} Y_{f_\alpha}$$

for some elements $\{f_\alpha\}_{\alpha\in\lambda} \subseteq \mathcal{O}_Y(Y)$. However, for each α, we see

$$\mathcal{O}_Y(Y_{f_\alpha}) \simeq \mathcal{O}_Y(Y)_{f_\alpha} = \mathcal{O}_Y(Y)[1/f_\alpha]$$

is a finitely generated $\mathcal{O}_Y(Y)$-algebra, given by "restriction" $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(Y_f)$, which is just the localization at f map as coming from Corollary 2.28. Thus, we conclude that ι is locally of finite type by Lemma 3.71. ■

And here is base change.

Lemma 3.79. The class of morphisms locally of finite type is preserved by base change.

Proof. Suppose we have a pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

of schemes such that ψ_Y is locally of finite type. We would like to show that π_X is locally of finite type. Notably, because being locally of finite type is affine-local on the target by Corollary 3.74, we may use Lemma 3.27 to assume that X and S are affine.

Well, fix an affine open subscheme $U \subseteq X$, and we would like to show that all affine open subschemes $V \subseteq \pi_X^{-1}(U)$ have $\mathcal{O}_{X\times_S Y}(V)$ finitely generated as an $\mathcal{O}_X(V)$-algebra by $\pi_X^\sharp$. Using Lemma 3.71, it suffices to just give an open cover of such affine open subschemes V. Well, note that we have built the tower

$$\begin{tikzcd} \pi_X^{-1}U \arrow[r, "\pi_X|_{\pi_X^{-1}U}"] \arrow[d, hook] & U \arrow[d, hook] \\ X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

where once again the bottom square is a pullback square by Lemma 2.186, the top square is a pullback square by Lemma 2.178, so the total rectangle is a pullback square by Lemma 2.174. Thus, we may think about $\pi^{-1}U$ as canonically isomorphic to $U \times_S Y$.

Now, using Lemma 2.183, we can give Y an affine open cover Y_α so that the affine open schemes $U \times_S Y_\alpha$ will manage to cover $U \times_S Y$. As such, Lemma 3.71 tells us that we would like for each $U \times_S Y_\alpha$ to have global sections finitely generated as an $\mathcal{O}_U(U)$-algebra.

But now everything is affine! Set $R := \mathcal{O}_S(S)$ and $A_\alpha := \mathcal{O}_Y(Y_\alpha)$ and $B := \mathcal{O}_U(U)$. Then Lemma 2.176 tells us that

$$U \times_S Y_\alpha \simeq \operatorname{Spec} B \otimes_R A_\alpha.$$

Because A_α is finitely generated as an R-algebra (by $\psi_Y^\sharp$), we may write $A_\alpha \simeq R[x_1, \ldots, x_n]/I$ for some n and some ideal I. Chasing our maps around, we see

$$B \otimes_R A_\alpha \simeq B \otimes_R \frac{R[x_1, \ldots, x_n]}{I} \simeq \frac{B[x_1, \ldots, x_n]}{IB},$$

where the last isomorphism is by $b \otimes r(x_1, \ldots, x_n) \mapsto b \cdot r(x_1, \ldots, x_n)$. Thus, $B \otimes_R A_\alpha$ is a finitely generated B-algebra, so we are done. ■

As a fun aside, we note that morphisms locally of finite type satisfies a cancellation property.

Lemma 3.80. Fix scheme morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. If the composite $\psi \circ \varphi$ is locally of finite type, then φ is also locally of finite type.

Proof. Give Z an affine open cover $\{Z_\alpha\}_{\alpha \in \lambda}$; for each α, give $\psi^{-1}Z_\alpha$ an affine open cover $\{Y_{\alpha\beta}\}_{\beta \in \lambda_\alpha}$. Because being locally of finite type is affine-local on the target by Corollary 3.74, it suffices to show that the restrictions $\varphi_{\alpha\beta}\colon \varphi^{-1}Y_{\alpha\beta} \to Y_{\alpha\beta}$ are locally of finite type.

Well, by Lemma 3.73, it suffices to show that any affine open subscheme $U \subseteq \varphi^{-1}Y_{\alpha\beta}$ has $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(Y_{\alpha\beta})$-algebra by $\operatorname{res}_{\varphi^{-1}Y_{\alpha\beta},U} \circ \varphi_U^\sharp$.

However, because $\psi \circ \varphi$ is locally of finite type, we see that $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Z(Z_\alpha)$-module by $\operatorname{res}_{\varphi^{-1}\psi^{-1}Z_\alpha,U} \circ \varphi^\sharp_{\psi^{-1}Z_\alpha} \circ \psi^\sharp_{Z_\alpha}$, which is also

$$\operatorname{res}_{\varphi^{-1}Y_{\alpha\beta},U} \circ \varphi^\sharp_{Y_{\alpha\beta}} \circ \operatorname{res}_{\psi^{-1}Z_\alpha,Y_{\alpha\beta}} \circ \psi^\sharp_{Z_\alpha}.$$

Thus, passing the generators of $\mathcal{O}_X(U)$ as a $\mathcal{O}_Z(Z_\alpha)$-algebra through $\operatorname{res}_{\psi^{-1}Z_\alpha,Y_{\alpha\beta}} \circ \psi^\sharp_{Z_\alpha}$ to $\mathcal{O}_Y(Y_{\alpha\beta})$ will end up having the same image in $\mathcal{O}_X(U)$ under $\operatorname{res}_{\varphi^{-1}Y_{\alpha\beta},U} \circ \varphi_U^\sharp$, so $\mathcal{O}_X(U)$ is also finitely generated as an $\mathcal{O}_X(Y_{\alpha\beta})$-algebra. ■

3.2.3 Integral Is Reasonable

We are currently building up towards showing that finite morphisms have the usual properties. We could show this directly, but it will be productive to take an intermediate step through integral morphisms. Here is our definition.

Definition 3.81. A scheme morphism $\pi\colon X \to Y$ is *integral* if and only if π is affine and all affine open $U \subseteq Y$ makes $\mathcal{O}_X(\pi^{-1}U)$ an integral extension of $\mathcal{O}_Y(U)$ by $\pi_U^\sharp$.

Example 3.82. If $f\colon A \to B$ is an integral extension of rings, the induced scheme morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is an integral morphism of schemes. We will show this in Corollary 3.88, after we show that being integral is affine-local on the target.

Example 3.83. If $\pi\colon X \to Y$ is a finite morphism, then π is also integral. Indeed, for any affine open subscheme $U \subseteq X$, we see $\pi^{-1}U \subseteq Y$ is affine, and $B := \mathcal{O}_X(\pi^{-1}U)$ is a finitely generated $A := \mathcal{O}_Y(U)$-algebra by $f := \pi_U^\sharp$. Namely, for each $b \in B$, we see that $f(A)[b]$ is finite over A, so b is integral over A, so B is an integral extension of A.

As usual, we go ahead and show that the class of integral morphisms is affine-local on the target, preserved by composition, and preserved by base change.

We begin with composition, which easier this time.

Lemma 3.84. The class of integral morphisms is preserved by composition.

Proof. Fix integral morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. We would like to show that $\psi \circ \varphi$ is integral. Well, fix some affine open subscheme $W \subseteq Z$.

Then $\psi^{-1}W \subseteq Y$ is an affine open subscheme because ψ is affine, and $\psi_W^\sharp\colon \mathcal{O}_Z(W) \to \mathcal{O}_Y(\psi^{-1}W)$ is an integral extension because ψ is integral. Continuing, $\varphi^{-1}\psi^{-1}W \subseteq X$ is an affine open subscheme because φ is affine, and $\varphi^\sharp_{\psi^{-1}W}\colon \mathcal{O}_Y(\psi^{-1}W) \to \mathcal{O}_X(\varphi^{-1}\psi^{-1}W)$ is an integral extension because ψ is integral. It follows that the composite

$$\mathcal{O}_Z(W) \stackrel{\psi_W^\sharp}{\to} \mathcal{O}_Y(\psi^{-1}W) \stackrel{(\psi_*\varphi)_W^\sharp}{\to} \mathcal{O}_X(\varphi^{-1}\psi^{-1}W)$$

is an integral extension of rings.[2] Because the above composite is $(\psi \circ \varphi)_W^\sharp$, we are done. ■

We now move towards to showing that the class of integral morphisms is affine-local on the target.

Lemma 3.85. Fix an integral morphism $\pi\colon X \to Y$ and open subset $V \subseteq Y$. Then $\pi|_{\pi^{-1}V}\colon \pi^{-1}V \to V$ is an integral morphism for each affine open subscheme $U \subseteq Y$.

Proof. Given an affine open subscheme $V' \subseteq V$, we need to show that $\pi^{-1}V' \subseteq \pi^{-1}V$ is affine and has $\mathcal{O}_X(\pi^{-1}V')$ an integral extension of $\mathcal{O}_Y(V')$ by $(\pi|_U)^\sharp_{V'} = \pi^\sharp_{V'}$. Well, note that $V' \subseteq V \subseteq Y$ is an affine open subscheme of Y, so the fact that π is integral tells us that $\mathcal{O}_X(\pi^{-1}V')$ is indeed an integral extension of $\mathcal{O}_Y(V')$ by $\pi^\sharp_{V'}$. ■

Lemma 3.86. Fix a scheme morphism $\pi\colon X \to Y$ and an affine open cover $\mathcal{U}$ of Y such that $\pi^{-1}U$ is affine for each $U \in \mathcal{U}$, and $\pi_U^\sharp\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}U)$ is an integral morphism of rings. Then π is integral.

Proof. We use Lemma 2.118. To begin, we note that the open cover $\mathcal{U}$ tells has $\pi^{-1}U$ affine for each $U \in \mathcal{U}$, so we immediately get to say that π is affine by Corollary 3.60.

Say that an affine open subset $U \subseteq Y$ is "acceptable" if and only if $\mathcal{O}_X(\pi^{-1}U)$ is an integral extension of $\mathcal{O}_Y(U)$ by $\pi_U^\sharp$. We would like to show that all affine open subsets of Y are acceptable, for which we use Lemma 2.118.

(i) Suppose the affine open subscheme $U \subseteq Y$ is acceptable. Then, for some $f \in \mathcal{O}_Y(U)$, we need to show that U_f is also acceptable. Well, we recall from Remark 2.24 that

$$\pi^{-1}(U_f) = \pi^{-1}(U)_{\pi_U^\sharp(f)}.$$

Now, because π is affine, we see that $\pi^{-1}(U)$ is affine, so it follows that $\pi^{-1}(U_f) = \pi^{-1}(U)_{\pi_U^\sharp(f)}$ is also affine and (canonically) isomorphic to $\operatorname{Spec} \mathcal{O}_X(\pi^{-1}U)_{\pi_U^\sharp(f)}$.

[2] This is a commutative algebra fact: if $f\colon A \to B$ and $g\colon B \to C$ are integral extensions, then $g \circ f$ is an integral extension. One sees by finding $c \in C$, giving c a monic polynomial in $B[x]$ with coefficients $b_0, \ldots, b_d$, and now we note that $g(f(A))[g(b_0), \ldots, g(b_d), c]$ is finite over A, so c is integral over A.

Further, we know that $B \coloneqq \mathcal{O}_X(\pi^{-1}U)$ is an integral extension of $A \coloneqq \mathcal{O}_Y(U)$ by $\pi_U^\sharp$. We want to show that $B_{\pi_U^\sharp f}$ is an integral extension of A_f. Well, any $b/(\pi_U^\sharp f)^n$ has $b \in B$ integral over A and therefore satisfying some monic polynomial

$$\sum_{k=0}^{d} \pi_U^\sharp(a_k) b^k = 0$$

with $a_d = 1$ and $\{a_k\}_{k=1}^d \subseteq A$. It follows that $b/(\pi_U^\sharp f)^n$ satisfies

$$\sum_{k=0}^{d} \pi_U^\sharp(a_k)\pi_U^\sharp(f^{nk}) \left(\frac{b}{(\pi_U^\sharp f)^n}\right)^k = 0,$$

so we see that $b/(\pi_U^\sharp f)^n$ satisfied a monic polynomial in A_f after multiplying both sides above by $\pi_U^\sharp(f^{-nd})$.

(ii) Suppose the affine open subscheme $U \subseteq Y$ has sections $f_1, \ldots, f_n \in \mathcal{O}_Y(U)$ generating $\mathcal{O}_Y(U)$ such that each U_{f_i} is acceptable. We would like to show U is acceptable. Note that we already know $\pi^{-1}U$ is affine because π is affine.

For brevity, we set $A \coloneqq \mathcal{O}_Y(U)$ and $B \coloneqq \mathcal{O}_X(\pi^{-1}U)$ and $\iota \coloneqq \pi_U^\sharp$. Because U is affine, we see $\mathcal{O}_Y(U_{f_i}) \simeq A_{f_i}$ for each i, and above we computed that

$$\pi^{-1}(U_{f_i}) = \pi^{-1}(U)_{\iota(f_i)},$$

so $\mathcal{O}_X(\pi^{-1}(U)_{\iota f_i}) \simeq B_{\iota f_i}$. Namely, the localization map $B \to B_{\iota f_i}$ is just a restriction map, so we see that the induced map $A_{f_i} \to B_{\iota f_i}$ is just the localization (i.e., restriction) of $\iota\colon A \to B$.

Thus, because the U_{f_i} are acceptable, we see that the maps $\iota_{f_i}\colon A_{f_i} \to B_{\iota f_i}$ give integral extensions of rings. We want to show that U is acceptable, which translates into showing that $\iota\colon A \to B$ is an integral extension of rings.

Well, pick up some $b \in B$. Fixing some i, we are given that $B_{\iota f_i}$ is an integral A_{f_i}-algebra, so $b/1 \in B_{\iota f_i}$ is the root of some monic polynomial

$$\sum_{k=0}^{d} \iota(a_k)(b/1)^k = 0$$

where $a_d = 1$ and $\{a_k\}_{k=1}^n \subseteq A_{f_i}$. Combining the left-hand side into a single fraction (the coefficient in front of $(b/1)^d$ will be some power of f_i) and then expanding out the equality in A_{f_i} tells us that there is some M_i and elements $a_1', \ldots, a_d' \in A$ with $b_d = 1$ such that

$$f_i^{M_i} \sum_{k=0}^{d} \iota(a_i')b^k = 0.$$

Multiplying both sides by enough f_is, we see

$$\sum_{k=0}^{d} \iota\left(a_i' f_i^{dM_i - kM_i}\right)\left(\iota(f_i)^{M_i} b\right)^k = 0,$$

so $\iota(f_i)^{M_i} b$ satisfies a monic polynomial with coefficients in A and is therefore integral over A.

Now looping through all i, we see that

$$\iota(A)\left[\iota(f_1)^{M_1} b, \ldots, \iota(f_n)^{M_n} b\right] \subseteq B$$

is an extension of A generated by integral elements and therefore integral. To show that b is integral over A, it suffices to show that $b \in \iota(A)\left[\iota(f_1)^{M_1} b, \ldots, \iota(f_n)^{M_n} b\right]$.

Well, we note that $(f_1,\ldots,f_n)=A$ implies that $A=\operatorname{rad}(f_1,\ldots,f_n)=\operatorname{rad}(f_1^{M_1},\ldots,f_n^{M_n})$, so it follows $V((f_1,\ldots,f_n))=V(\operatorname{rad}(f_1,\ldots,f_n))=V((f_1^{M_1},\ldots,f_n^{M_n}))$, so $(f_1^{M_1},\ldots,f_n^{M_n})=A$. So we may find constants $c_1,\ldots,c_n\in A$ such that

$$\sum_{k=1}^n c_k f_k^{M_k}=1.$$

Thus,

$$b=\sum_{k=1}^n \iota(c_k)\iota(f_k)^{M_k}b\in\iota(A)\left[\iota(f_1)^{M_1}b,\ldots,\iota(f_n)^{M_n}b\right].$$

(iii) The assumption on π grants us an acceptable affine open cover.

The above checks allow Lemma 2.118 to kick in, completing the proof. ■

Corollary 3.87. The class of integral morphisms is affine-local on the target.

Proof. One direction of being affine-local on the target is covered by Lemma 3.85.

In the other direction, fix a scheme morphism $\pi\colon X\to Y$ and an affine open cover $\{V_\alpha\}_{\alpha\in\lambda}$ such that $\pi|_{\pi^{-1}V_\alpha}\colon\pi^{-1}V_\alpha\to V_\alpha$ is integral for each V_α. Fixing some V_α, we see $\pi^{-1}V_\alpha$ is affine and makes $\mathcal{O}_X(\pi^{-1}V_\alpha)$ an integral $\mathcal{O}_Y(V_\alpha)$-algebra by $\pi^\sharp$ because $\pi|_{\pi^{-1}V_\alpha}$ is integral. Thus, π is integral by Lemma 3.86. ■

Corollary 3.88. Fix an affine scheme Y and a morphism of schemes $\pi\colon X\to Y$. Then π is integral if and only if X is affine and $\mathcal{O}_X(X)$ is an integral $\mathcal{O}_Y(Y)$ algebra by $\pi_Y^\sharp$.

Proof. If π is integral, then π is affine, so the affine open subscheme $Y\subseteq Y$ must make $\pi^{-1}Y=X$ an affine open subscheme of X, so X is affine. Further, because π is integral, so $\mathcal{O}_X(X)$ must be an integral $\mathcal{O}_Y(Y)$-algebra by $\pi_Y^\sharp$.

Conversely, suppose that X is affine, and $\pi_Y^\sharp\colon\mathcal{O}_Y(Y)\to\mathcal{O}_X(X)$ is an integral extension of rings. Then the affine open cover $\{Y\}$ of Y has $X=\pi^{-1}Y$ affine and makes $\mathcal{O}_X(X)$ an integral $\mathcal{O}_Y(Y)$-algebra by $\pi_Y^\sharp$, so we finish by Lemma 3.86. ■

We are now ready to prove that being integral is preserved by base change.

Lemma 3.89. The class of integral morphisms is preserved by base change.

Proof. Suppose we have a pullback square

$$\begin{tikzcd} X\times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

of schemes such that ψ_Y is integral. We would like to show that π_X is integral. Notably, because being integral is affine-local on the target by Corollary 3.87, we may use Lemma 3.27 to assume that X and S are affine.

However, ψ_Y is integral and therefore affine, so we conclude that Y is affine because Y is. Thus, because now all of X and Y and S are affine, we set $A:=\mathcal{O}_X(X)$ and $B:=\mathcal{O}_Y(Y)$ and $R:=\mathcal{O}_S(S)$ so that we may set $X\times_S Y=\operatorname{Spec}A\otimes_R B$ by Lemma 2.176. Namely, we are now looking at the push-out diagram

$$\begin{tikzcd} R \arrow[r, "\psi_A^\sharp"] \arrow[d, "\psi_B^\sharp"'] & A \arrow[d, "\pi_A^\sharp"] \\ B \arrow[r, "\pi_B^\sharp"] & A\otimes_R B \end{tikzcd}$$

where $\psi_B^\sharp$ is integral; note we are abusing our notation with the sheaf morphisms here. Observe that $\pi_A^\sharp = \operatorname{id}_A \otimes 1$ and $\pi_B^\sharp = 1 \otimes \operatorname{id}_B$.

We now proceed directly. Because X is integral, we may use Corollary 3.88 so that showing π_X is integral is equivalent to showing that $X \times_S Y$ is affine–which we know because $X \times_S Y = \operatorname{Spec} A \otimes_R B$—such that $\pi_B^\sharp \colon B \to A \otimes_R B$ is an integral extension.

Well, $A \otimes_R B$ is generated by finite sums of the form $(a \otimes 1)(1 \otimes b)$, so it suffices to show that each $a \otimes 1$ and $1 \otimes b$ is integral over B. (Namely, the elements of B integral over A is "the integral closure" of B in A and is a subring of B.) On one hand, we see that

$$(1 \otimes b) - \pi_B^\sharp(b) = 0$$

provides a monic polynomial for $1 \otimes b$. On the other hand, for $a \otimes 1$, we note that $\psi_A^\sharp \colon R \to A$ is an integral extension, so a is the root of some monic polynomial

$$\sum_{k=0}^{d} \psi_A^\sharp(r_k) a^k = 0$$

where $r_1, \ldots, r_d \in R$. Thus,

$$\sum_{k=0}^{d} \pi_A^\sharp(\psi_A^\sharp r_k)(a \otimes 1)^k = \left(\sum_{k=0}^{d} \psi_A^\sharp(r_k) r^k \right) \otimes 1 = 0 \otimes 1 = 0,$$

so $a \otimes 1$ is indeed integral over B. This finishes. ■

3.2.4 Reasonability Loose Ends

We now show that morphisms of finite type and finite morphisms form reasonable classes. This will be optimized by the following lemma.

Lemma 3.90. Fix some classes $\{P_\alpha\}_{\alpha \in \lambda}$ of morphisms, and let P be the class of morphisms which live in P_α for each α.

(a) If each P_α is preserved by composition, then P is as well.

(b) If each P_α is preserved by base change, then P is as well.

(c) If each P_α is affine-local on the target, then P is as well.

Proof. We proceed directly.

(a) Suppose $\varphi \colon A \to B$ and $\psi \colon B \to C$ are both in P. Then $\varphi, \psi \in P_\alpha$ for each α, so $\psi \circ \varphi \in P_\alpha$ for each α because P_α is preserved by composition, so $\psi \circ \varphi \in P$.

(b) Suppose $\varphi_X \colon X \to S$ lives in P and therefore in P_α for each α. Given a scheme Y with a morphism $\psi_Y \colon Y \to S$, we note that the canonical projection $\pi_X \colon X \times_S Y \to X$ lives in P_α for each α because P_α is preserved by base change. It follows $\pi_X \in P$.

(c) Suppose $\varphi \colon X \to Y$, and fix an (affine) open cover $\mathcal{U}$ of Y.

In one direction, if $\varphi \colon X \to Y$ is in P, then φ is in P_α for each α, so $\varphi|_{\varphi^{-1}U}$ is in P_α for each α and each $U \in \mathcal{U}$ because P_α is affine-local on the target, so $\varphi|_{\varphi^{-1}U}$ is in P for each $U \in \mathcal{U}$.

In the other direction, if $\varphi|_{\varphi^{-1}U}$ is in P for each $U \in \mathcal{U}$, then $\varphi|_{\varphi^{-1}U}$ is in P_α for each α and each U, so φ is in P_α for each α because P_α is affine-local on the target, so φ is in P. ■

This gives morphisms of finite type immediately.

Corollary 3.91. The class of morphisms of finite type is preserved by composition, is preserved by base change, and is affine-local on the target.

Proof. By definition, a morphism φ is of finite type if and only if φ is quasicompact and locally of finite type.

Thus, Lemma 3.90 tells us that it suffices to show the classes of quasicompact morphisms and of morphisms locally of finite type are both preserved by composition, are preserved by base change, and are affine-local on the target.

- For quasicompact morphisms, these are by Corollary 3.22, Lemma 3.25, and Example 3.21.
- For morphisms locally of finite type, these are by Lemma 3.72, Lemma 3.79, and Corollary 3.74. ■

To extend this approach to finite morphisms, we have the following lemma.

Lemma 3.92. A scheme morphism $\varphi\colon X \to Y$ is finite if and only if φ is both integral and locally of finite type.

Proof. Fix an affine open subscheme $V \subseteq Y$. We need to show that $U := \varphi^{-1}V$ is an affine open subscheme of X and that $\mathcal{O}_X(U)$ is finitely generated as an $\mathcal{O}_Y(V)$-algebra by $\varphi^\sharp$.

Well, φ is integral, so $U = \varphi^{-1}V$ is indeed affine, and in fact $\mathcal{O}_X(U)$ is an integral $\mathcal{O}_Y(V)$-algebra by $\varphi^\sharp_V$. Further, because φ is locally of finite type, the affine open subscheme U of U must have $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(V)$-algebra by $\operatorname{res}_{U,U} \circ \varphi^\sharp_V = \varphi^\sharp_V$.

Now, for brevity, set $A := \mathcal{O}_Y(V)$ and $B := \mathcal{O}_X(U)$ and $f := \varphi^\sharp_V$ so that $f\colon A \to B$ gives

$$B = f(A)[b_1, \ldots, b_n]$$

for some $b_1, \ldots, b_n \in B$ because B is finitely generated over A. However, B is integral over A as well, so each of the b_i are integral over A, so bounded polynomials in the b_i will finitely generate B as an A-module, finishing. More explicitly, each extension of

$$f(A) \subseteq f(A)[b_1] \subseteq f(A)[b_1, b_2] \subseteq \cdots \subseteq f(A)[b_1, \ldots, b_n] = B$$

is finite, so the total extension is finite. ■

Corollary 3.93. The class of finite morphisms is preserved by composition, is preserved by base change, and is affine-local on the target.

Proof. This proof is essentially the same as Corollary 3.91 with some references changed. By Lemma 3.92, a morphism φ is finite if and only if φ is integral and locally of finite type.

Thus, Lemma 3.90 tells us that it suffices to show the classes of integral morphisms and of morphisms locally of finite type are both preserved by composition, are preserved by base change, and are affine-local on the target.

- For integral morphisms, these are by Lemma 3.84, Lemma 3.89, and Corollary 3.87.
- For morphisms locally of finite type, these are by Lemma 3.72, Lemma 3.79, and Corollary 3.74. ■

3.2.5 Fun with Integral Morphisms

Integral morphisms are important mostly because finite morphisms are integral by Example 3.83, but they enjoy some nice properties on their own.

Lemma 3.94. Fix an integral scheme morphism $\pi\colon X \to Y$. Then π is a closed map of topological spaces.

Proof. Give Y an affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$. Quickly, we claim that a subset $S \subseteq Y$ is closed if and only if $S \cap Y_\alpha \subseteq Y_\alpha$ is closed for each α. Indeed, $S\cap Y_\alpha \subseteq Y_\alpha$ is closed if and only if $Y_\alpha \setminus S \subseteq Y_\alpha$ is open for each $\alpha \in \lambda$, which is equivalent to $Y_\alpha \setminus S \subseteq Y$ being open. In this case, we see

$$Y \setminus S = \left(\bigcup_{\alpha\in\lambda} Y_\alpha\right) \setminus S = \bigcup_{\alpha\in\lambda}(Y_\alpha \setminus S)$$

is open, making S is closed. Conversely, if S is closed, then $Y_\alpha \setminus S = Y_\alpha \cap (Y\setminus S)$ is open for each $\alpha\in\lambda$.

The above argument allows us to reduce to the affine case: fix some closed subset $V \subseteq X$, and we want to show that $\pi(V) \subseteq Y$ is closed. It suffices to show that

$$\pi(V) \cap Y_\alpha = \pi(V \cap \pi^{-1}Y_\alpha)\cap Y_\alpha$$

is closed in Y for each $\alpha\in\lambda$. Namely, it suffices to show that $\pi|_{\pi^{-1}Y_\alpha}\colon \pi^{-1}Y_\alpha \to Y_\alpha$ is closed.

Thus, we rename our variables: replace Y with Y_α and X with $\pi^{-1}Y_\alpha$ and π with $\pi|_{\pi^{-1}Y_\alpha}$. Notably, because being integral is affine-local on the target by Corollary 3.87, we see π is still integral. But now, Y is affine, so Corollary 3.88 tells us that X is affine, and $B \coloneqq \mathcal{O}_X(X)$ is an integral $A \coloneqq \mathcal{O}_Y(Y)$-algebra by $f \coloneqq \pi^\sharp_Y$.

At this point, we are essentially asking a commutative algebra. We are given some closed subset $V(I) \subseteq \operatorname{Spec} B$, and we want to know that

$$\pi(V(I)) = \{\pi(\mathfrak{q}) : \mathfrak{q} \in V(I)\} = \left\{f^{-1}\mathfrak{q} : \mathfrak{q} \supseteq I\right\}$$

is a closed subset of $\operatorname{Spec} A$. Well, we claim that this set is equal to $V\left(f^{-1}(I)\right)$. Certainly any $\mathfrak{q} \in \operatorname{Spec} B$ with $\mathfrak{q} \supseteq I$ will have $f^{-1}\mathfrak{q} \supseteq f^{-1}(I)$, so $\pi(V(I)) \supseteq V\left(f^{-1}(I)\right)$.

For the other inclusion, we pick up the following result from commutative algebra.

Lemma 3.95 (Lying over)**.** Fix an integral extension of rings $f\colon A \to B$. Then each prime $\mathfrak{p} \in \operatorname{Spec} A$ has some prime $\mathfrak{q} \in \operatorname{Spec} S$ such that $f^{-1}\mathfrak{q} = \mathfrak{p}$.

Proof. This proof is mildly technical, so we cite [Eis95, §4.4]. (We did cover this result in Math 250B.) ■

To apply the above lemma, we note that $\overline{f}\colon A/f^{-1}(I) \to B/I$ is an integral extension: any element $[b] \in B/I$ is represented by some $b \in B$, which is the root of some monic polynomial in $A[x]$; reducing this polynomial to $A/f^{-1}(I)$ makes $[b]$ the root of some monic polynomial in $A/f^{-1}(I)[x]$.

Thus, for each $\mathfrak{p} \in V\left(f^{-1}(I)\right)$, we see that $\mathfrak{p}$ reduces to a prime $\overline{\mathfrak{p}} \in A/f^{-1}(I)$ by Exercise 1.53, so Lemma 3.95 tells us that there is a prime $\overline{\mathfrak{q}} \in B/I$ such that

$$\overline{f}^{-1}\overline{\mathfrak{q}} = \overline{\mathfrak{p}}.$$

But then Exercise 1.53 again gives us a prime $\mathfrak{q} \in \operatorname{Spec} B$ containing I which reduces down to $\overline{\mathfrak{q}}$. Notably, $a \in \mathfrak{p}$ if and only if $[a] \in \overline{\mathfrak{p}}$, which is equivalent to $[a] \in \overline{f}^{-1}\overline{\mathfrak{q}}$, which is now equivalent to $[f(a)] \in \mathfrak{q}$, which is equivalent to $f(a) \in \mathfrak{q}$, so we do indeed have $\mathfrak{p} = f^{-1}\mathfrak{q}$. It follows $\mathfrak{p} \in \pi(V(I))$. This finishes. ■

Remark 3.96. Later in life, there will be ring maps we care about which are integral but not finite. For example, $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$ is an integral ring map.

Here is another good property of integral morphisms.

Proposition 3.97. Fix an integral scheme morphism $\pi\colon X \to Y$. Then each closed $Z \subseteq X$ has $\pi(Z)$ closed and $\dim Z = \dim \pi(Z)$.

Proof. That $\pi(Z)$ is closed follows from our previous proof. As usual, we reduce to the affine case, where we have an integral extension of rings $\pi^\sharp\colon R \to S$ (where $\pi(Z) = \operatorname{Spec} R$ and $S = \operatorname{Spec} Z$), and we would like to show that $\dim R = \dim S$.

For this, we combine two commutative algebra results.

Lemma 3.98. Fix an integral extension of rings $R \subseteq S$. If $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \subseteq S$ are prime ideals, then their intersections of R are still distinct.

The above result tells us that $\dim Z \le \dim \pi(Z)$. For the other inequality, we need to be able to go up.

Lemma 3.99 (Going up). Fix an integral extension of rings $R \subseteq S$. If there are primes

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq R,$$

with a partial lift of $\mathfrak{q}_i$ such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for $1 \le i \le m$, then we can extend the chain all the way up to n.

The above result gives us $\pi(Z) \le \dim Z$. ■

The point is that integral morphisms are similar to finite ones.

3.2.6 Quasifinite Morphisms

Here's an example of our finiteness conditions doing their job.

Lemma 3.100. Fix a finite scheme morphism $\pi\colon X \to Y$. For any $y \in Y$, the set $\pi^{-1}(\{y\})$ is finite.

Proof. There are arguments avoiding the fiber product, but let's just go ahead and use it. Let $X_y := X \times_Y \{y\}$ so that the topological space of X_y is $\pi^{-1}(\{y\})$. Here is our diagram.

$$\begin{tikzcd} X_y \arrow[r] \arrow[d, "\pi_y"] & X \arrow[d, "\pi"] \\ \{y\} \arrow[r, hook] & Y \end{tikzcd}$$

Namely, the canonical projection $X_y \to \{y\}$ is finite because finite morphisms are preserved by base change by Corollary 3.93. Because X_y is homeomorphic to $\pi^{-1}(\{y\})$ by Lemma 2.197, it suffices to show that X_y is a finite topological space.

Well, $\{y\}$ is the affine scheme $\operatorname{Spec} k(y)$, so because the canonical map $\pi_y\colon X_y \to \{y\}$ is finite, we conclude that X_y is affine, and

$$(\pi_y)^\sharp_{\{y\}}\colon k(y) \to \mathcal{O}_X(X_y)$$

makes $\mathcal{O}_X(X_y)$ finitely generated as a $k(y)$-module. Thus, we set $A := \mathcal{O}_X(X_y)$ and $k := k(y)$ so that A is finitely generated as a k-module, and we want to show $X_y \cong \operatorname{Spec} A$ is finite.

For this, we appeal to commutative algebra: ideals of A must be k-subspaces of A, so because A is a finite-dimensional k-vector space, we conclude that the ideals of A have the descending chain condition. Thus, A is Artinian and therefore has only finitely many prime ideals. ■

Motivated by the above statement, here is yet another finiteness condition.

Definition 3.101 (Quasifinite)**.** A scheme morphism $\pi\colon X \to Y$ is *quasifinite* if and only if π is of finite type and each $y \in Y$ has $\pi^{-1}(\{y\})$ a finite set.

Example 3.102. If $\pi\colon X \to Y$ is a finite morphism of locally Noetherian schemes and $U \subseteq X$ is quasicompact, then the restriction $\pi|_U : U \to Y$ is quasifinite.

Remark 3.103. Being quasifinite is stable under composition, base change, and is affine local on the target. I will not show this, for now, because it is somewhat hard, and I have not been convinced to care about quasifinite morphisms.

Remark 3.104. A morphism is finite if and only if it is quasifinite and integral. We have seen the forward direction: finite morphisms are integral by Example 3.83 and quasifinite by Lemma 3.100. The converse is much harder to prove.

3.2.7 Chevalley's Theorem

We close lecture by stating Chevalley's theorem.

Theorem 3.105 (Chevalley)**.** Fix a scheme morphism $\pi\colon X \to Y$ of finite type. If $C \subseteq X$ is constructible, then $\pi(C)$ is also constructible.

Here is the appropriate definition.

Definition 3.106 (Constructible)**.** Fix a Noetherian topological space X. Then a subset $C \subseteq X$ is constructible if and only if it is the union of subsets of the form $U \subseteq V$ where $U \subseteq X$ is open and $V \subseteq X$ is closed.

There is a different definition on the homework; in particular, the collection of constructible sets

Remark 3.107. It is somewhat important for constructible sets to be living in a Noetherian topological space. The definition must change otherwise.

3.3 September 28

Today we discuss Chevalley's theorem.

3.3.1 Chevalley's Theorem: Comments

Here is the statement.

Theorem 3.108. Fix Noetherian schemes X and Y and a morphism $\pi\colon X \to Y$ of finite type. Then if $C \subseteq X$ is constructible, $\pi(C)$ is also constructible.

We will prove Theorem 3.108 today, but there are analogues when X and Y need not be Noetherian.

Remark 3.109. If we want to allow X to be quasicompact and quasiseparated, we can let constructible subsets be finite unions of sets of the form $U \setminus U'$ where U and U' are quasicompact open sets. Letting $\pi\colon X \to Y$ be of "finite presentation" it will be true that π sends constructible sets to constructible sets.

Here, "finitely presented" means locally finitely presented and quasicompact and quasiseparated. Further, locally finitely presented means that any affine open subset $U = \operatorname{Spec} B \subseteq Y$ with an affine open subset $\operatorname{Spec} A \subseteq \pi^{-1}(U)$ has A a finitely presented B-algebra; i.e., there is a finitely generated ideal $I \subseteq B[x_1, \ldots, x_n]$ such that $A \cong B[x_1, \ldots, x_n]/I$.

Let's see a few examples of Theorem 3.108.

Example 3.110. Take $\pi\colon X \to \mathbb{A}^1_k$ with X connected. Then $\pi(X)$ is connected, so $\pi(X)$ is a single point and so $\pi(X)$ contains an open subset of $\mathbb{A}^1_k$ and is therefore missing only finitely many points.

There are some really nontrivial examples which explain why we want to work with constructible sets in Theorem 3.108.

Example 3.111. Let k be algebraically closed, for psychological reasons. We define $\pi\colon \operatorname{Spec} k[x,y] \to \operatorname{Spec} k[u,v]$ by taking $\mathfrak{m}_{(a,b)} \mapsto \mathfrak{m}_{(a,ab)}$. Namely, this is the scheme morphism coming from the ring homomorphism $k[u,v] \to k[x,y]$ by $u \mapsto x$ and $v \mapsto xy$. In particular, we can see that we map the generic point to generic points, we send a one-dimensional prime $(f(x,y))$ to $f(u,v/u) \cdot u^\bullet$ for some sufficiently large power $u^\bullet$, and in particular we send $(x) \mapsto (u,v)$. Additionally, we have a continuous bijection $\mathbb{A}^2_k \setminus V((x))$ going to $\mathbb{A}^2_k \setminus V((u))$. Thus, $\pi(\mathbb{A}^2_k)$ is $D(u) \cup \{(0,0)\}$, which is very weird.

Remark 3.112. The above example is a "typical" example of a birational map, where away from a closed set we have a continuous bijection, and the remaining closed set gets squeezed down.

Remark 3.113. On a Noetherian scheme, closed points contain all the needed topological data. Namely, continuous maps carry the generic points along for the ride after being told what to do with closed points. Thus, we can reason topologically about schemes like $\mathbb{A}^2_k$ by only paying attention to closed points.

3.3.2 Chevalley's Theorem: Proof

We begin with a few reduction steps to turn this into an affine problem of $\operatorname{Spec} B[x_1, \ldots, x_n] \to \operatorname{Spec} B$. Here are our reduction steps.

1. At any point in the proof, because we are only looking at topological spaces, we can replace X and Y with their reductions: note we have a unique map π_{red} making the diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \\ X_{\text{red}} \arrow[u] \arrow[r, dashed, "\pi_{\text{red}}"] & Y_{\text{red}} \arrow[u] \end{tikzcd}$$

   commute, where the vertical maps are the identity on topological spaces. In particular, π_{red} will still be of finite type, which is something that we can just check on affine open subsets by hand.

2. We replace $\pi(C)$ in general with just $\operatorname{im}\pi$. Given a constructible subset $C \subseteq X$, we want to show $\pi(C)$ is constructible. However, we may write C as a finite union

$$C = \bigcup_{i=1}^n (U_i \cap V_i)$$

of locally closed subsets $U_i \cap V_i$ with

$$\pi(C) = \bigcup_{i=1}^{n} \pi(U_i \cap V_i).$$

In particular, it suffices to show that any given $\pi(U_i \cap V_i)$, so we just replace X with $U_i \cap V_i$, which has a scheme structure as a closed subscheme of an open subscheme. So it suffices to show $\operatorname{im}\pi = \pi(X)$ itself is constructible. Notably, π is still of finite type because we're staying Noetherian (namely, everything is quasicompact).

3. We make X and Y affine. We can write Y as

$$Y = \bigcup_{i=1}^{n} Y_i$$

where Y_i is affine. So we let π_i be the restricted map $\pi^{-1}Y_i \to Y_i$, so we can write

$$\pi(X) = \bigcup_{i=1}^{n} \pi_i\left(\pi^{-1}(Y_i)\right),$$

and it again suffices to just show that each of the π_i are outputting a constructible subset of Y. Notably, everything is Noetherian, so $\pi^{-1}Y_i$ is quasicompact, so we can replace $\pi^{-1}Y_i$ with various restrictions to affine subsets.

4. We are now in the situation where $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of finite type, so we have $A = B[x_1, \ldots, x_n]/I$ where $\pi^\sharp$ is the canonical induced map. We can even force $\operatorname{Spec} A$ and $\operatorname{Spec} B$ to be irreducible because there are only finitely many irreducible components, so we are forcing A and B to be integral domains. Lastly, we will also force B to embed into A, which is equivalent to the map $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} B$ being having dense image; to see this, simply replace $\operatorname{Spec} B$ with $\overline{\pi(\operatorname{Spec} A)}$.

To continue the reduction, we recall Noether normalization.

Theorem 3.114. Fix a field K and R is a finitely generated K-algebra, then we can find $x_1, \ldots, x_n \in R$ such that R is in fact a finitely generated $K[x_1, \ldots, x_n]$-module; i.e., the induced map $K[x_1, \ldots, x_n] \to R$ is a finite ring homomorphism.

To apply Theorem 3.114, we need to upgrade it to go beyond K-algebras.

Lemma 3.115. Fix an integral embedding $B \subseteq A$ such that A is a finitely generated B-algebra. Then there is a nonzero $s \in B$ such that the embedding

$$B_s[x_1, \ldots, x_n] \hookrightarrow A_s$$

is a finite ring homomorphism.

Intuitively, Theorem 3.114 communicates what happens at (0), which is like testing what happens on the "generic fiber." Then the above result "spreads out" the generic fiber to all of $D(s)$.

Proof. We apply Theorem 3.114. Set $K \coloneqq \operatorname{Frac} B$, and we have a canonical embedding $B \hookrightarrow K$. Then we set $S \coloneqq B \setminus \{0\}$ so that $K =\hookrightarrow S^{-1}A$ makes $S^{-1}A$ finitely generated over K.

Thus, applying Theorem 3.114, we can find $x_1, \ldots, x_n \in S^{-1}A$ such that $S^{-1}A$ is finite over $K[x_1, \ldots, x_n]$. We now choose our single element $s \in B$ to be large enough to get the result. To begin, make s divisible by the denominators of the x_i. Further, we know A will be generated by some finitely many y_is over B, but as elements of $S^{-1}A$, they satisfy a monic polynomial with coefficients in $K[x_1, \ldots, x_n]$. Adding the denominators of all these polynomials to s, we see that the y_is are integral over $B_s[x_1, \ldots, x_n]$ and hence contained. It follows that we have generated A, so A_s is indeed finite over $B_s[x_1, \ldots, x_n]$. ■

What?

In total, we currently have a diagram which looks like

$$\begin{tikzcd} \operatorname{Spec} A_s \arrow[r, "\pi_s"] \arrow[d, hook] & \operatorname{Spec} B_s[x_1, \ldots, x_n] \arrow[r] & \operatorname{Spec} B_s \arrow[d, hook] \\ \operatorname{Spec} A \arrow[rr, "\pi"] & & \operatorname{Spec} B \end{tikzcd}$$

where the vertical embeddings are open. We will finish the proof. The point is that we are almost done "locally" by looking at particular open sets.

3.4 September 30

We continue the proof of Theorem 3.108 from last class.

3.4.1 Finishing Chevalley's Theorem

Recall that we had the diagram

$$\begin{tikzcd} \operatorname{Spec} A_s \arrow[r, "\pi_s"] \arrow[d, hook] & \operatorname{Spec} B_s[x_1, \ldots, x_n] \arrow[r] & \operatorname{Spec} B_s \arrow[d, hook] \\ \operatorname{Spec} A \arrow[rr, "\pi"] & & \operatorname{Spec} B \end{tikzcd}$$

from spreading out Noether normalization. Note that π_s is a dominant morphism because $B_s[x_1, \ldots, x_n] \hookrightarrow A_s$ is a dominant morphism. In fact, π_s is a finite morphism by its construction from Noether normalization, so we conclude that $\pi_s(\operatorname{Spec} A_s) = \operatorname{Spec} B_s[x_1, \ldots, x_n]$ by dominance. The map $\operatorname{Spec} B_s[x_1, \ldots, x_n] \to \operatorname{Spec} B_s$ is now certainly surjective, so in total, we see that $\operatorname{im} \pi$ contains the open subset $\operatorname{Spec} B_s \subseteq \operatorname{Spec} B$.

This is actually good enough to finish the proof.

Example 3.116. Set $B = \operatorname{Spec} k[x]$. Then the open subsets of $\operatorname{Spec} k[x]$ are only missing finitely many points and so will stay open!

Motivated by how nice Noetherian schemes are, we have the following lemma.

Lemma 3.117. Fix a Noetherian space Y. A subset $E \subseteq Y$ is constructible if and only if all irreducible closed subsets $Z \subseteq Y$ has either $E \cap Z$ or $Z \setminus E$ containing a nonempty open set.

Proof. This is by Noetherian induction. Namely, if the statement were false, we could make E minimal satisfying the above condition and derive contradiction by looking at the two cases. ∎

Thus, to finish the proof, we use Lemma 3.117. We want to show that $\pi(X)$ is irreducible, so for some irreducible closed subset $Z \subseteq Y$ we use the same reduction steps as above to show that the restricted map

$$\pi \colon \pi^{-1}Z \to Z$$

has, for any open irreducible subset $U \subseteq \overline{\pi(Z)}$ is strictly contained in Z (so that the restricted morphism is not dominant) or $\pi(U)$ contains an open nonempty subset inside Z (because the argument goes through as soon as the restriction is dominant). In the former case, we will get that $Z \setminus \pi(U)$ contains a nonempty open subset, so Lemma 3.117 kicks in to tell us that $\pi(X)$ is constructible.

Remark 3.118. Even though the statement of Theorem 3.108 is non-constructive, one can concretely work through the above proof using a somewhat explicit construction from π_s. In particular, after peeling off the desired open subset B_s, what's left over is approximately speaking some lower-dimensional object, so we can induct downwards.

3.4.2 Closed Embeddings Are Reasonable

It will be helpful for us to collect some facts about closed embeddings before we talk about separated morphisms in a moment. To start off, we're well overdue to show that closed morphisms are monic.

Lemma 3.119. Closed morphisms are monic.

Proof. We show this by hand. Fix a closed morphism $\iota\colon X \to Y$ and two morphisms $\alpha, \beta\colon S \to X$ such that $\iota \circ \alpha = \iota \circ \beta$. We need to show that $\alpha = \beta$, which we show by hand.

- On the level of topological spaces, we note that any $s \in S$ will have $\iota(\alpha(s)) = \iota(\beta(s))$, so $\alpha(s) = \beta(s)$ because ι is a homeomorphism onto its image.
- On the level of sheaves, we are given that $\iota_*\alpha^\sharp \circ \iota^\sharp = \iota_*\beta^\sharp \circ \iota^\sharp$ as sheaf morphisms $\mathcal{O}_Y \to \alpha_*\iota_*\mathcal{O}_S$. Now, fix any $x \in X$, and we will show that $\alpha_x^\sharp = \beta_x^\sharp$ at x, which will finish by Proposition 1.111.

  As an intermediate claim, we show that the composite $\mathcal{O}_{Y,\iota(x)} \to (\iota_*\mathcal{O}_X)_{\iota(x)} \to \mathcal{O}_{X,x}$ is surjective. Well, we know that the map $\iota^\sharp_{\iota(x)}\colon \mathcal{O}_{Y,\iota(x)} \to (\iota_*\mathcal{O}_X)_{\iota(x)}$ is already surjective, so we just need to show that the map $(\iota_*\mathcal{O}_X)_{\iota(x)} \to \mathcal{O}_{X,x}$ is surjective. For this, we pick up some germ $[(U,s)] \in \mathcal{O}_{X,x}$ with U containing x.

  Now, we note that ι is a homeomorphism onto its image, so $\iota(U) \subseteq \operatorname{im}\iota$ is an open subset, so there is some open $V \subseteq Y$ such that $\iota(U) = V \cap \operatorname{im}\iota$, which in particular means $\iota^{-1}V = U$. So we see $[(V,s)] \in (\iota_*\mathcal{O}_X)_{\iota(x)}$ will go to $[(U,s)]$ under the canonical map.

  We now show $\alpha_x^\sharp = \beta_x^\sharp$. For this, we track through the large diagram

$$\begin{tikzcd}
(\mathcal{O}_Y)_{\iota(x)} \arrow[r, two heads] \arrow[dr, two heads] & (\iota_*\mathcal{O}_X)_{\iota(x)} \arrow[r, "(\iota_*\alpha^\sharp)_x"] \arrow[d, two heads] & (\iota_*\alpha_*\mathcal{O}_S)_{\iota(x)} \arrow[d] & {[(V,s)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V,\alpha^\sharp_{\iota^{-1}V}s)]} \arrow[d, mapsto] \\
& \mathcal{O}_{X,x} \arrow[r, "\alpha_x^\sharp"] & (\alpha_*\mathcal{O}_S)_x & {[(\iota^{-1}V,s)]} \arrow[r, mapsto] & {[(\iota^{-1}V,\alpha^\sharp_{\iota^{-1}V}s)]}
\end{tikzcd}$$

  where the left triangle is surjective as shown above, and the right square commutes as shown. Now, for any germ $s \in \mathcal{O}_{X,x}$, we can compute what $\alpha_x^\sharp$ will do to s by pulling s back to a germ in $\mathcal{O}_{Y,\iota(y)}$, then going forward across to $(\iota_*\alpha_*\mathcal{O}_S)_{\iota(x)}$, and lastly going down to $(\alpha_*\mathcal{O}_S)_x$.

  However, all of these steps are independent of $\alpha^\sharp$ in that the composite along the top is the same if we replace $\alpha^\sharp$ with $\beta^\sharp$, and the last downward step only depends on the topological data, which we know from the above aligns. This finishes. ■

Corollary 3.120. Closed morphisms are quasiseparated.

Proof. Monomorphisms are quasiseparated by Example 3.41, so Lemma 3.119 finishes. ■

Here is composition.

Lemma 3.121. Fix a continuous map $f\colon X \to Y$ which is a homeomorphism onto a closed set of Y. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a sheaf morphism which is surjective on stalks, then $f_*\varphi\colon f_*\mathcal{F} \to f_*\mathcal{G}$ is still surjective on stalks.

Proof. Fix some $y \in Y$ so that we want to show $(f_*\varphi)_y$ is surjective. If $y \notin \operatorname{im} f = \overline{\operatorname{im} f}$, then note $(f_*\mathcal{G})_y = 0$ by Remark 1.169, so there is nothing to say.

Otherwise, we have $y \in \operatorname{im} f$, so say $y = f(x)$ for $x \in X$. Starting with a germ $[(U,s)] \in (f_*\mathcal{G})_{f(x)}$, we take this to $[(f^{-1}U,s)] \in \mathcal{G}_x$, where we can use the surjectivity of φ_x to find $[(V,t)] \in \mathcal{F}_x$ with $\varphi_V(t)|_W = s|_W$ for some small W containing x; restricting V enough, we may assume $\varphi_V(t) = s|_W$ and that $V \subseteq f^{-1}U$.

But now we see that the restriction $f|_{f^{-1}U} \colon f^{-1}U \to U$ is a homeomorphism, so $f(V) \subseteq U$ is an open subset of U and hence of Y. At this point we recognize $f^{-1}(f(V)) = V$, so the germ $[(f(V), t)] \in (f_*\mathcal{F})_{f(x)}$ will do the trick. This finishes. ■

Lemma 3.122. The class of closed embeddings is preserved by composition.

Proof. Fix closed embeddings $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$. We need to show that $(\psi \circ \varphi) \colon X \to Z$ is a closed embedding. There are two checks.

- Note that φ is a homeomorphism onto a closed subset $\varphi(X) \subseteq Y$. Further, ψ is a homeomorphism onto a closed subset $\psi(Y) \subseteq Z$, so ψ restricted to $\varphi(X)$ will still be a homeomorphism onto $\psi(\varphi(X))$. Notably, $\varphi(X)$ implies that $\psi(\varphi(X))$ is closed.
- We check that $(\psi \circ \varphi)^\sharp \colon \mathcal{O}_Z \to \psi_*\varphi_*\mathcal{O}_X$ is surjective on stalks. Well, fixing some z, we want to show that the composite
$$\mathcal{O}_{Z,z} \stackrel{\psi^\sharp_z}{\to} (\psi_*\mathcal{O}_Y)_z \stackrel{(\psi_*\varphi^\sharp)_z}{\to} (\psi_*\varphi_*\mathcal{O}_X)_z$$
is surjective. Note that $\psi^\sharp_z$ is already surjective, so it suffices to show that $(\psi_*\varphi^\sharp)_z$ is surjective, which follows from Lemma 3.121. ■

As we should expect, we now move towards being local on the target.

Lemma 3.123. Suppose that $\pi \colon X \to Y$ is a closed embedding. Then, for any open subset $U \subseteq Y$, the restriction $\pi|_{\pi^{-1}U} \colon \pi^{-1}U \to U$ is still a closed embedding.

Proof. Unsurprisingly, there are two checks.

- On the level of topological spaces, we note that the inverse continuous map $\varphi \colon \pi(Y) \to X$ will restrict to an inverse continuous map $\varphi|_U \colon (U \cap \pi(Y)) \to \pi^{-1}(U)$, so we see that $\pi|_{\pi^{-1}U}$ is at least still a homeomorphism onto its image. (Notably, $U \cap \pi(Y) = \pi(\pi^{-1}(U))$.) Additionally, we see that $\operatorname{im} \pi|_{\pi^{-1}U} = U$, so the image is a closed subset of U (in fact, all of U).
- On stalks, we fix some $y \in U$, and we want to show that the map
$$(\pi|_{\pi^{-1}U})^\sharp_y \colon (\mathcal{O}_Y|_U)_y \to \pi_*(\mathcal{O}_X|_{\pi^{-1}U})_y$$
is surjective. Well, by definition of our restriction in Lemma 2.9, $(\pi|_{\pi^{-1}U})^\sharp_V$ just behaves as $\pi^\sharp_V$ for any open $V \subseteq U$, so the definition of the map on stalks will match as $\pi^\sharp_y$, which we already know is surjective. ■

And here is the converse.

Lemma 3.124. Fix a scheme morphism $\pi \colon X \to Y$. Suppose that we have an open cover $\mathcal{U}$ on Y such that $\pi|_{\pi^{-1}U} \colon \pi^{-1}U \to U$ is a closed embedding for each $U \in \mathcal{U}$. Then π is a closed embedding.

Proof. As usual, we have two checks. For brevity, let the open cover by $\{V_\alpha\}_{\alpha\in\lambda}$ and $\pi_\alpha \colon U_\alpha \to V_\alpha$ by the restrictions, where $U_\alpha := \pi^{-1}V_\alpha$.

- On the level of topological spaces, we know that π_α onto its image $\operatorname{im} \pi_\alpha$, and that $\operatorname{im} \pi_\alpha$ is always a closed set. For one, we note that
$$\operatorname{im} \pi_\alpha = \pi\left(\pi^{-1}V_\alpha\right) = \operatorname{im} \pi \cap V_\alpha$$

is a closed subset in V_α for each α. Thus, we conclude that $(Y \setminus \operatorname{im}\pi) \cap V_\alpha = V_\alpha \setminus \operatorname{im}\pi$ is an open subset in V_α and therefore in Y for each α, so

$$Y \setminus \operatorname{im}\pi = \bigcup_{\alpha\in\lambda} (Y \setminus \operatorname{im}\pi) \cap V_\alpha$$

must be an open subset of Y, so $\operatorname{im}\pi$ is closed.

It remains to show that π is a homeomorphism onto $\operatorname{im}\pi$. Well, for each α, we are promised a local inverse continuous function $\varphi_\alpha\colon (\operatorname{im}\pi \cap V_\alpha) \to U_\alpha$. Now, for any $y \in \operatorname{im}\pi$, we find some α such that $y \in V_\alpha$ and define

$$\varphi(y) := \varphi_\alpha(y).$$

Note that these functions glue appropriately: if $\pi(x) \in V_\alpha \cap V_\beta$, then $\varphi_\alpha(\pi(x)) = x = \varphi_\beta(\pi(x))$. Thus, these glue to a continuous function φ by Exercise 1.71. Additionally, we see any $x \in X$ has $f(x) \in V_\alpha$ for some α and therefore

$$\varphi(\pi(x)) = \varphi_\alpha(\pi_\alpha(x)) = x,$$

any $y \in \operatorname{im}\pi$ has $y \in V_\alpha$ for some α and therefore

$$\pi(\varphi(y)) = \pi_\alpha(\varphi_\alpha(y)) = y.$$

Thus, φ is indeed a continuous inverse for π.

- We show that $\pi^\sharp\colon \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is surjective on stalks. Fix any $y \in Y$, and pick up some germ $[(V,f)] \in (\pi_*\mathcal{O}_X)_y$. Now, $y \in V_\alpha$ for some α, so we may restrict the germ to V_α to assume that $V \subseteq V_\alpha$.

  In particular, we thus see that actually $[(V,f)] \in ((\pi_\alpha)_*\mathcal{O}_X|_{U_\alpha})_y$, so the surjectivity of $\pi_\alpha^\sharp$ on stalks tells us that there is some germ $[(V,g)] \in (\mathcal{O}_Y|_{V_\alpha})_y$ (namely, we may assume $g \in \mathcal{O}_Y(V)$ with $U \subseteq V_\alpha$) with

  $$(\pi_\alpha^\sharp)_y([(V,g)]) = [(U, \pi_V^\sharp(g))] = [(V,f)].$$

  This finishes our surjectivity check. ■

Corollary 3.125. The class of closed embeddings is local on the target.

Proof. Combine Lemma 3.123 and Lemma 3.124. ■

We now show that we are affine-local on the target. We will appeal to Proposition 2.147 even though we will shortly give a different proof of this result. There will be no circular logic.

Lemma 3.126. Closed embeddings are finite and in particular affine.

Proof. Fix a closed embedding $\pi\colon X \to Y$; because being finite is affine-local on the target by Corollary 3.93, it suffices to show that $\pi|_{\pi^{-1}U}\colon \pi^{-1}U \to U$ is finite for each affine open $U \subseteq Y$. (Namely, fix any affine open cover of Y.)

Well, note $\pi|_{\pi^{-1}U}$ remains a closed embedding by Lemma 3.123, so we might as well rename Y to U and X to $\pi^{-1}U$ and π to $\pi|_{\pi^{-1}U}$. Then $\pi\colon X \to Y$ is a closed embedding, where Y is affine, and we want to show that π is finite. Further, because isomorphisms are closed embeddings by Example 2.146 and closed embeddings are preserved by composition by Lemma 3.122, we see that we actually have a closed embedding

$$X \to Y \to \operatorname{Spec} A,$$

so we can just assume that Y takes the form $\operatorname{Spec} A$. Thus, by Proposition 2.147, we must have π factor as

$$X \cong \operatorname{Spec} A/I \to \operatorname{Spec} A$$

for some ideal $I \subseteq A$. In particular, we see that $\pi^\sharp$ is surjective on global sections by composing the above morphisms, which means that $\mathcal{O}_X(X)$ is in fact finitely generated as an A-algebra by $\pi^\sharp$ (in fact, with one generator). ■

Lemma 3.127. The class of closed embeddings is preserved by base change.

Proof. Suppose we have a pullback square

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"] & S \end{tikzcd}$$

of schemes such that ψ_Y is a closed embedding. We would like to show that π_X is a closed embedding. Notably, because being a closed embedding is affine-local on the target by Corollary 3.125 (in fact, local on the target), we may use Lemma 3.27 to assume that X and S are affine.

Thus, $\psi_Y : Y \to S$ being a closed embedding forces Y to be affine by Proposition 2.147, where the induced map $\psi_Y^\sharp : \mathcal{O}_S(S) \to \mathcal{O}_Y(Y)$ on global sections is surjective. So for brevity we set $R := \mathcal{O}_S(S)$ and $A := \mathcal{O}_X(X)$ and $B := \mathcal{O}_Y(Y)$. Thus, by Lemma 2.176, we see that

$$X \times_S Y \simeq \operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B \simeq \operatorname{Spec} A \otimes_R B,$$

so we might as well set $X \times_S Y$ to be $\operatorname{Spec} A \otimes_R B$, where the canonical projection $X \times_S Y \to X$ is given by chaining the canonical maps $\operatorname{Spec} A \otimes_R B \to \operatorname{Spec} A$ (from $A \to A \otimes_R B$) with the canonical isomorphism $\operatorname{Spec} A \otimes X$.

Notably, because isomorphisms are closed embeddings by Example 2.146, and closed embeddings are preserved by composition by Lemma 3.122, it suffices to show that the map $\operatorname{Spec} A \otimes_R B \to \operatorname{Spec} A$ is a closed embedding, for which it suffices by Proposition 2.147 to show that the canonical map $A \to A \otimes_R B$ is surjective.

Well, we are given that the canonical map $R \to B$ is surjective. Thus, for any tensor $a \otimes b \in A \otimes_R B$, we can find $r \in R$ which goes to b, so $a \otimes b = ra \otimes 1$, which comes from $ra \in A$ through the inclusion $A \to A \otimes_R B$. Because $A \otimes_R B$ is generated by these elements $a \otimes b$, we are done. ■

3.4.3 Locally Closed Embeddings Are Reasonable

While we're talking about embeddings, we'll go ahead and show that locally closed embeddings have the usual adjectives.

Remark 3.128. Locally closed embeddings are a closed embedding followed by an open embedding, both of which are monic by Lemma 3.119 and Corollary 2.179, respectively. Because the composite of monomorphisms remains monic, we see that locally closed embeddings are monic; for example, they are quasiseparated by Example 3.41.

Lemma 3.129. A locally closed embedding with closed image (topologically!) is a closed embedding.

Proof. Fix a locally closed embedding $\pi : X \to Y$ with closed image. By definition, we can factor π as $\pi = \iota \circ \pi'$, where $\pi' : X \to U$ is a closed embedding, and $\iota : U \to Y$ is an open embedding. We have two checks.

- Topologically, both π' and ι are homeomorphisms onto their image, so their composite is as well: let $\iota_0 : U \to U$ be the continuous inverse of ι, and let $\pi_0' : \iota(X) \to X$ be the continuous inverse of π'.

  Then we note $\iota_0|_{\pi'(X)} : \pi'(X) \to \pi'(X)$ is still a continuous function, and we can check that $\pi_0' \circ \iota_0|_{\pi'(X)}$ is the inverse we're looking for, which is a matter of writing down the equations and remembering $\iota \circ \iota_0$ and $\iota_0 \circ \iota$ and $\pi \circ \pi_0$ and $\pi_0 \circ \pi$ are all identities.

- On the structure sheaf, we need to check that $\pi^\sharp\colon \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is surjective on stalks. By Remark 1.169, we only have to check this on $\overline{\operatorname{im}\pi} = \operatorname{im}\pi$. Well, for some $x \in X$, we pick up some germ $[(V, f)] \in (\pi_*\mathcal{O}_X)_{\pi(x)}$, which means $f \in \mathcal{O}_X(\pi^{-1}V)$.

  Because π factors through U, we see that $\pi(x) \in U$, so we may restrict V to $V \cap U$, meaning we may assume $V \subseteq U$, so actually $[(V, f)] \in (\pi'_*\mathcal{O}_X)_{\pi'(x)}$. However, π' is a closed embedding, so $(\pi')^\sharp\colon \mathcal{O}_Y|_U \to (\pi'_*\mathcal{O}_X)$ is surjective on stalks, so we may find $[(W, g)] \in \mathcal{O}_Y|_U$ such that

$$\pi^\sharp_{f(x)}([(W, g)]) = [(W, \pi^\sharp_W g)] = [(W, (\pi')^\sharp_W g)] = (\pi')^\sharp_{f(x)}([W, g]) = [(V, f)],$$

  which is what we wanted. ■

It turns out that being preserved by composition is the hardest check, so we will do it last. Let's begin with local on the target.

Lemma 3.130. The class of locally closed embeddings is affine-local on the target.

Proof. Fix a scheme morphism $\pi\colon X \to Y$ and an affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$ of Y. In one direction, if π is a locally closed embedding, then we have an open subscheme $U \subseteq Y$ such that π factors through $\pi'\colon X \to U$ as a closed embedding. Now, for any open $V \subseteq Y$, we see that $\pi|_{\pi^{-1}V}$ becomes

$$\pi^{-1}V \overset{\pi'|_{\pi^{-1}V}}{\to} (U \cap V) \hookrightarrow V.$$

Notably, $\pi'|_{\pi^{-1}V} = \pi'|_{(\pi')^{-1}(U\cap V)}$ is a closed embedding by Lemma 3.123, and $U \cap V \hookrightarrow V$ remains an open embedding (it's still an embedding of open subsets, and the structure sheaves continue to match), so we see that $\pi|_{\pi^{-1}V}$ is still a locally closed embedding.

In the other direction, give Y an affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$ such that the restrictions $\pi_\alpha\colon X_\alpha \to Y_\alpha$ are all locally closed embeddings, where $X_\alpha := \pi^{-1}Y_\alpha$. This means that we are promised open subschemes $U_\alpha \subseteq Y_\alpha$ such that each π_α factors as

$$X_\alpha \overset{\pi'_\alpha}{\to} U_\alpha \hookrightarrow Y_\alpha,$$

where π'_α is a closed embedding. Thus, we set

$$U := \bigcup_{\alpha\in\lambda} U_\alpha.$$

Notably, U is covered by the U_α, and $X_\alpha \subseteq (\pi'_\alpha)^{-1}(U_\alpha) \subseteq \pi^{-1}Y_\alpha = X_\alpha$, so in fact π factors through U as π' as

$$X \overset{\pi'}{\to} U \hookrightarrow Y.$$

Further, we see that the restrictions $\pi'|_{\pi^{-1}U_\alpha}\colon X_\alpha \to U_\alpha$ are all closed embeddings, so Lemma 3.124 says that π' is a closed embedding, which finishes. ■

Now, here is base-change.

Lemma 3.131. The class of locally closed embeddings is preserved by base change.

Proof. Fix a closed embedding $\iota\colon X \to U$ and an open embedding $\jmath\colon Z \to U$. Additionally, picking up some auxiliary scheme Z with a map $\varphi\colon Z \to Y$, we would like to show that the induced map $X \times_Y Z \to Z$ in

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "\varphi"] \\ X \arrow[r, "\jmath\circ\iota"] & Y \end{tikzcd} \tag{3.6}$$

is also a locally closed embedding. For this, we expand the diagram into

$$\begin{tikzcd} X \times_Y Z \arrow[r, dashed, "\iota'"] \arrow[d, "\pi"'] & \varphi^{-1}U \arrow[r, "\jmath'"] \arrow[d, "\varphi'"] & Z \arrow[d, "\varphi"] \\ X \arrow[r, "\iota"] & U \arrow[r, "\jmath"] & Y \end{tikzcd}$$

where we note that the map ι' is induced to make the diagram commute because the right square is a pullback square, so we can plug the commutative diagram (3.6) into it. In particular, we have that the projection $X \times_Y Z \to Z$ is $\jmath' \circ \iota'$.

Now, we note that the left square is a pullback square by Lemma 2.174. As such, because ι is a closed embedding, and closed embeddings are preserved by base change by Lemma 3.127, we know that ι' is also a closed embedding. So we have written the map $X \times_Y Z \to Z$ as the composite $\jmath' \circ \iota'$ where ι' is a closed embedding and $\jmath'$ an open embedding, which shows that $\jmath' \circ \iota'$ is a locally closed embedding. ■

We begin with composition.

> **Lemma 3.132.** Suppose that $\jmath\colon X \to V$ is an open embedding, and $\iota\colon V \to Y$ is a closed embedding. Then the composite $(\iota \circ \jmath)\colon X \to Y$ is a locally closed embedding.

Proof. For psychological ease, we identify V topologically with its image in Y, and we identify X topologically with its image in V (and thus in Y) as well. In particular, on the level of topological spaces, X is an open subset of the closed subspace $V \subseteq Y$, so we can find an open subset $U \subseteq Y$ such that $X = U \cap V$. Let $\jmath'\colon U \hookrightarrow Y$ be the corresponding open embedding, and we claim that we can induce a closed embedding $\iota\colon X \to U$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\jmath"] \arrow[d, "\iota'"'] & V \arrow[d, "\iota"] \\ U \arrow[r, "\jmath'"] & Y \end{tikzcd} \tag{3.7}$$

commute; this will finish because it implies that $\iota \circ \jmath = \jmath' \circ \iota'$ is the composite of a closed embedding followed by an open embedding and therefore a locally closed embedding.

At the very least, note that $X = V \cap U$ is a closed subset of U, so we will simply define $\iota'\colon X \to U$ as the identity on topological spaces. In particular, we have that ι' is a homeomorphism from X onto a closed subset of U. We also see that (3.7) now commutes on the level of topological spaces because all the maps are the identity on the level of topological spaces.

It remains to talk about sheaves. In particular, for each open subset $U' \subseteq U$, we must exhibit a map $(\iota')^\sharp_{U'}\colon \mathcal{O}_U(U') \to \iota'_*\mathcal{O}_X(U')$. Well, we note that $\mathcal{O}_U(U') = \mathcal{O}_Y(U)$, and $\iota'_*\mathcal{O}_X(U') = \mathcal{O}_X(X \cap U') = \mathcal{O}_V(X \cap U')$ (note $X \cap U'$ is open in V), so we define our map $(\iota')^\sharp_{U'}$ as either of the diagonal morphisms in

$$\begin{tikzcd} \mathcal{O}_Y(U') \arrow[r, "\iota^\sharp_{U'}"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_V(U' \cap V) \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_Y(U' \cap U) \arrow[r, "\iota^\sharp_{U' \cap U}"] & \mathcal{O}_V(U' \cap X) \end{tikzcd}$$

where the diagram commutes because $\iota^\sharp$ is a sheaf morphism. To see that $(\iota')^\sharp$ assembles into a sheaf map, we pick up open sets $U'' \subseteq U'$ and note that the diagram

$$\begin{tikzcd} \mathcal{O}_Y(U') \arrow[r, "\iota^\sharp_{U'}"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_V(U' \cap V) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & \mathcal{O}_V(U' \cap X) \arrow[r, equal] \arrow[d, "\mathrm{res}"] & \iota'_*\mathcal{O}_X(U') \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_Y(U'') \arrow[r, "\iota^\sharp_{U''}"] & \mathcal{O}_V(U'' \cap V) \arrow[r, "\mathrm{res}"] & \mathcal{O}_V(U'' \cap X) \arrow[r, equal] & \iota'_*(U'') \end{tikzcd}$$

commutes: the left square commutes by naturality of $\iota^\sharp$, and the middle square commutes because everything is just restrictions of the same sheaf $\mathcal{O}_V$.

Further, we see that we assemble into a morphism of locally ringed spaces: for any point $p \in X \subseteq U$, we need to know that the composite

$$\begin{array}{ccccc} \mathcal{O}_{U,p} & \overset{(\iota')^\sharp_p}{\to} & (\iota'_*\mathcal{O}_X)_p & \to & \mathcal{O}_{X,p} \\ [(U',s)] & \mapsto & [(U',(\iota')^\sharp_{U'}(s))] & \mapsto & [(U'\cap X,(\iota')^\sharp_{U'}(s))] \end{array}$$

is a map of local rings. To begin, we note that $(\iota')^\sharp_{U'}(s) = \iota^\sharp_{U'}(s)|_{U'\cap X}$, where the restriction takes place in $\mathcal{O}_V$, so the entire germ is just $\iota^\sharp_{U'}(s)|_p$. Thus, the above map is

$$[U',s] \mapsto \iota^\sharp_{U'}(s)|_p,$$

which we can see is a map of local rings directly because $\iota^\sharp$ is a map of local rings, so any germ on the right side living in $\mathfrak{m}_{U,p}$ is actually living in $\mathfrak{m}_{Y,p}$ and will therefore go to $\mathfrak{m}_{V,p}$ under $\iota^\sharp$. (We are implicitly using the fact that stalks of a restriction sheaf, which is how open embeddings are defined, are identified with stalks of the original sheaf.)

Lastly, we need to show that our diagram commutes. We already know that it commutes on the level of topological spaces. On the level of sheaves, we pick up some open subset $U' \subseteq Y$, and we need the diagram

$$\begin{tikzcd} \mathcal{O}_Y(U') \arrow[r, "\iota^\sharp_{U'}"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_V(V\cap U') \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_{U'}(U\cap U') \arrow[r, "(\iota')^\sharp_{U\cap U'}"] & \mathcal{O}_X(X\cap U') \end{tikzcd}$$

commutes because it is the same as the larger diagram

$$\begin{tikzcd} \mathcal{O}_Y(U') \arrow[r, "\iota^\sharp_{U'}"] \arrow[d, "\mathrm{res}"'] & \mathcal{O}_V(V\cap U') \arrow[d, "\mathrm{res}"'] \arrow[dr, "\mathrm{res}"] & \\ \mathcal{O}_Y(U\cap U') \arrow[r, "\iota^\sharp_{U\cap U'}"] & \mathcal{O}_V(V\cap U\cap U') \arrow[r, "\mathrm{res}"] & \mathcal{O}_V(X\cap U') \end{tikzcd}$$

where now the square commutes by naturality of $\iota^\sharp$, and the triangle commutes because everything is a restriction in $\mathcal{O}_V$. ■

Corollary 3.133. The class of locally closed embeddings is preserved by composition.

Proof. Let our closed embeddings be $X_1 \to X_2$ and $X_2 \to X_3$, where we decompose $X_1 \to X_2$ into a closed embedding $\iota_1 \colon X_1 \to U_1$ and an open embedding $\jmath_1 \colon U_1 \to X_2$, and we do similar for the map $X_2 \to X_3$. Then the composite $X_1 \to X_3$ is just

$$X_1 \overset{\iota_1}{\to} U_1 \overset{\jmath_1}{\to} X_2 \overset{\iota_2}{\to} U_2 \overset{\jmath_2}{\to} X_2.$$

Now, the composite $\iota_2 \circ \jmath_1$ of an open embedding followed by a closed embedding is actually a locally closed embedding as shown in the previous lemma, so we can write it as $\jmath_2' \circ \iota_1'$ where $\jmath_2'$ is a closed embedding and ι_1' is an open embedding.

It follows that the full composite $X_1 \to X_3$ is the open embedding $\iota_1' \circ \iota_1$ (this is still an open embedding because we're still a homeomorphism onto an open set, and the structure sheaves still match) followed by the closed embedding $\jmath_2 \circ \jmath_2'$ (recall closed embeddings are preserved by composition by Lemma 3.122), which in total is a locally closed embedding. ■

3.4.4 Separated Morphisms

We are moving towards defining varieties, for which we need to define separated morphisms. Hartshorne proves some difficult criterion for morphisms to be separated and proper, but many of the corollaries of the criterion can be proven without it, so we will avoid proving the criterion for now.

Anyway, in some sense, a scheme or morphism being separated is the correct analogue of being Hausdorff. However, the statement of the Hausdorff condition requires talking about points and open sets, local to the space, so we will want a more morphism-oriented point of view.

Lemma 3.134. A topological space X is Hausdorff if and only if the diagonal map $\Delta\colon X \to X \times X$ given by $x \mapsto (x, x)$ has closed image.

Proof. We have two implications.

- Suppose X is Hausdorff. We need to show $(X \times X) \setminus \Delta(X)$ is open. Well, for any $(x, x') \in (X \times X) \setminus \Delta(X)$, we see $x \neq x'$, so we are promised disjoint open subsets $U, U' \subseteq X$ containing x and x' respectively. It follows that $(U \times U') \cap \Delta(X) = \varnothing$ (any intersection would witness some $y \in U \cap U'$) and contains (x, x'). Thus, any (x, x') has been given an open neighborhood $U \times U' \subseteq (X \times X) \setminus \Delta(X)$, finishing.
- Suppose $\Delta(X) \subseteq X \times X$ is closed. Then, for any distinct $x, x' \in X$, we see $(x, x') \notin \Delta(X)$. But $(X \times X) \setminus \Delta(X)$ is open, so we can write it out using the distinguished base as
$$(X \times X) \setminus \Delta(X) = \bigcup_{\alpha \in \lambda} (U_\alpha \times U'_\alpha)$$
for open subsets $U_\alpha, U'_\alpha \subseteq X$. So say $(x, x') \in U_\alpha \times U'_\alpha$, so $x \in U_\alpha$ and $x' \in U'_\alpha$. Further, $(U_\alpha \times U'_\alpha) \cap \Delta(X) \neq \varnothing$, so we again see that $U_\alpha \cap U'_\alpha = \varnothing$, so we are done. ■

Thus, we are motivated into the following definition.

Definition 3.135 (Separated)**.** A scheme morphism $\pi\colon X \to Y$ is *separated* if and only if $\Delta\pi\colon X \to X \times_Y X$ is a closed embedding. An S-scheme X is *separated* if and only if the map $X \to S$ is separated; for example, a scheme X is separated if and only if the map $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

Remark 3.136. As usual, the class of separated morphisms is preserved by composition, is preserved by base change, and is affine-local on the target. Here are the checks.

- The class of closed embeddings is preserved by composition by Lemma 3.122 and by base change by Lemma 3.127, so the class of separated morphisms is as well by Lemma 3.37.
- The class of closed embeddings is preserved by base change by Lemma 3.127, so the class of separated morphisms is as well by Lemma 3.38.
- Lastly, the class of closed embeddings is affine-local on the target by Corollary 3.125 and by base change by Lemma 3.127, so the class of separated morphisms is as well by Lemma 3.39.

Remark 3.137. As a sanity check, separated morphisms $\pi\colon X \to Y$ are quasiseparated. Indeed, the diagonal morphism $\Delta\pi\colon X \to X \times_Y X$ is a closed embedding and hence quasicompact by Example 3.3, so we conclude π is quasiseparated by Lemma 3.40.

Example 3.138. Monomorphisms $\pi\colon X \to Y$ are separated: indeed, the diagonal morphism $\Delta\pi$ is an isomorphism by Lemma 3.34, which is a closed embedding by Example 2.146. For example, isomorphisms are separated, open embeddings are separated by Corollary 2.179, closed embeddings are separated by Lemma 3.119, and more generally locally closed embeddings are separated by Remark 3.128.

We're defining separated as having $\Delta\pi$ being a closed embedding, but we might want $\Delta\pi$ to just have closed image from our topological motivation. These turn out to be the same; to see this, we note that the diagonal morphism isn't just any morphism.

Lemma 3.139. Fix a morphism $\pi\colon X \to Y$ of affine schemes. Then the induced diagonal map $\Delta\pi\colon X \to X \times_Y X$ is a closed embedding.

Proof. For brevity, set $\Delta := \Delta\pi$. Because isomorphisms are closed embeddings by Example 2.146, and closed embeddings are preserved by composition by Lemma 3.122, it suffices to show that a morphism of affine schemes $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ makes $\Delta\colon \operatorname{Spec} B \to (\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B)$ a closed embedding.

Well, let $f\colon A \to B$ be the ring map associated to π by Theorem 2.18, and we see we may set

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec} B \otimes_A B$$

by Lemma 2.176, where the canonical projections are given by the ring maps $i_1, i_2\colon B \to B \otimes_A B$ by $\iota_1\colon b \mapsto b \otimes 1$ and $\iota_2\colon b \mapsto 1 \otimes b$. Now, the diagonal map Δ making the diagram

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[dr, dashed, "\Delta"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \operatorname{Spec} B \otimes_A B \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d, "\pi"'] \\
& \operatorname{Spec} B \arrow[r, "\pi"] & \operatorname{Spec} A
\end{tikzcd}$$

commute becomes the diagonal morphism $d\colon B \otimes_A B \to B$ making the diagram

$$\begin{tikzcd}
B & & \\
& B \otimes_A B \arrow[ul, dashed, "d"'] & B \arrow[l, "i_2"] \arrow[ull, equal, bend right] \\
& B \arrow[u, "i_1"'] \arrow[uul, equal, bend left] & A \arrow[l, "f"] \arrow[u, "f"]
\end{tikzcd}$$

commute, which we see must be surjective because $d \circ i_2 = \mathrm{id}_B$. Thus, $\Delta\colon \operatorname{Spec} B \to \operatorname{Spec} B \otimes_A B$ is associated to the surjection $d\colon B \otimes_A B \to B$ and is therefore a closed embedding by Proposition 2.147. ■

Example 3.140. Lemma 3.139 tells us that morphisms of affine schemes are automatically separated. Because the class of separated morphisms is affine-local on the target by Remark 3.136, we see that all affine morphisms are separated by Remark 3.54.

To get to the affine case, we pick up the following lemma, analogous to Lemma 3.134.

Lemma 3.141. Fix a scheme morphism $\varphi\colon X \to Y$, and let $\pi_1, \pi_2\colon X \to X \times_Y X$. Given open subschemes $U_1, U_2 \subseteq X$, we have $U_1 \times_Y U_2 \simeq \pi_1^{-1} U_1 \times_Y \pi_2^{-1} U_2$ canonically, and

$$U_1 \cap U_2 = (\Delta\varphi)^{-1}(U_1 \times_Y U_2),$$

where $U_1 \times_Y U_2$ is realized as canonically embedded into $X \times_Y X$.

Proof. For brevity, set $\Delta := \Delta\varphi$, and let $\pi_1, \pi_2 \colon X \times_Y X \to X$ be the canonical projections. Note the diagram

$$\begin{tikzcd}
\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \arrow[r, hook] \arrow[d, hook] & \pi_1^{-1}(U_1) \arrow[r, "\pi_1"] \arrow[d, hook] & U_1 \arrow[d, hook] \\
\pi_2^{-1}(U_2) \arrow[r, hook] \arrow[d, "\pi_2"] & X \times_Y X \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "\varphi"] \\
U_2 \arrow[r, hook] & X \arrow[r, "\varphi"] & Y
\end{tikzcd}$$

has all squares pullback squares, where all but the bottom-right corner follow from Lemma 2.178. Notably, Thus, we see that

$$U_1 \times_Y U_2 \simeq \underbrace{\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2)}_{U:=},$$

where the canonical projections are given by π_1, π_2. It remains to compute $\Delta^{-1}(U)$. We show our inclusions separately.

- If $x \in U_1 \cap U_2$, then we see $\pi_1\Delta(x) = x \in U_1$ and $\pi_2\Delta(x) \in U_2$, so $\Delta(x) \in \pi_1^{-1}U_1 \cap \pi_2^{-1}U_2$, so $x \in \Delta^{-1}(U)$.
- If $x \in \Delta^{-1}(U)$, then we see $\Delta(x) \in U$, so $x = \pi_1\Delta(x) \in \pi_1(U) \subseteq U_1$, and $x = \pi_2\Delta(x) \in \pi_2(U) \subseteq U_2$, so $x \in U_1 \cap U_2$.

The above inclusions finish the proof. ■

Lemma 3.142. Fix a scheme morphism $\varphi \colon X \to Y$. Then the diagonal morphism $\Delta\varphi \colon X \to X \times_Y X$ is a locally closed embedding.

Proof. To show that we have a locally closed embedding, we need to find the requested intermediate open embedding $U \subseteq X \times_Y X$. The idea is to reduce to the affine case, Lemma 3.139. For brevity, let $\Delta := \Delta\varphi$, let $\pi_1, \pi_2 \colon X \times_Y X \to X$ be the canonical projections, and let $\pi \colon X \times_Y X \to Y$ be $\pi := \varphi \circ \pi_1 = \varphi \circ \pi_2$.

Now, give Y an affine open cover $\{Y_\alpha\}$ and set $X_\alpha := \varphi^{-1}Y_\alpha$ so that Lemma 2.186 and Lemma 2.185 says that the pre-images

$$\pi^{-1}(Y_\alpha) \simeq X_\alpha \times_{Y_\alpha} X_\alpha \simeq X_\alpha \times_Y X_\alpha$$

cover $X \times_Y X$. Continuing, give each $X_\alpha := \pi^{-1}Y_\alpha$ an affine cover $\{X_{\alpha\beta}\}_{\beta\in\lambda_\alpha}$ so that Lemma 3.141 tells us

$$\pi_1^{-1}(X_{\alpha\beta_1}) \cap \pi_2^{-1}(X_{\alpha\beta_2}) \simeq X_{\alpha\beta_1} \times_Y X_{\alpha\beta_2} \simeq X_{\alpha\beta_1} \times_{Y_\alpha} X_{\alpha\beta_2},$$

where the canonical projections are induced by π_1, π_2. Notably, $\varphi(X_{\alpha\beta}) \subseteq Y_\alpha$, so Lemma 2.185 is still applying here.

At this point, we set

$$U := \bigcup_{\alpha\in\lambda} \bigcup_{\beta\in\lambda_\alpha} \big(\underbrace{\pi_1^{-1}(X_{\alpha\beta}) \cap \pi_2^{-1}(X_{\alpha\beta})}_{U_{\alpha\beta}:=}\big).$$

In particular, we see that $U \subseteq X \times_Y X$ is open. Notably, $\Delta^{-1}(U_{\alpha\beta}) = U_{\alpha\beta}$ by Lemma 3.141, so taking the union over all α and β tells us that $\Delta^{-1}(U)$ is open.

Thus, Remark 2.11 tells us that Δ factors as

$$X \overset{\Delta'}{\to} U \hookrightarrow X \times_Y X,$$

where $U \hookrightarrow X \times_Y X$ is a closed embedding. It remains to show that $\Delta' \colon X \to U$ is a closed embedding. Because closed embeddings are affine-local on the target by Corollary 3.125, it suffices to show that the restrictions $\Delta'_{\alpha\beta} \colon \Delta^{-1}U_{\alpha\beta} \to U_{\alpha\beta}$ are all closed embeddings.

To see this, we recall $\Delta^{-1}U_{\alpha\beta} = X_{\alpha\beta}$ from Lemma 3.141. Thus, we are left trying to show that

$$\Delta|_{X_{\alpha\beta}} \colon X_{\alpha\beta} \to U_{\alpha\beta}$$

are closed embeddings. However, $U_{\alpha\beta} \simeq X_{\alpha\beta} \times_{Y_\alpha} X_{\alpha\beta}$, where the canonical projections are given by $\pi_1|_{U_{\alpha\beta}}, \pi_2|_{U_{\alpha\beta}} \colon U_{\alpha\beta} \to X_{\alpha\beta}$, as we said when we constructed U. So the fact that

$$\pi_\bullet|_{U_{\alpha\beta}} \circ \Delta|_{X_{\alpha\beta}} = (\pi_\bullet \circ \Delta)|_{X_{\alpha\beta}} = \mathrm{id}_X|_{X_{\alpha\beta}} = \mathrm{id}_{X_{\alpha\beta}}$$

for each $\pi_\bullet$, using the functoriality of restriction (from Remark 2.43), tells us that $\Delta|_{X_{\alpha\beta}}$ is the diagonal morphism of the morphism of affine schemes $\varphi|_{X_{\alpha\beta}} \colon X_{\alpha\beta} \to Y_\alpha$. So $\Delta|_{X_{\alpha\beta}}$ is a closed embedding by Lemma 3.139. This finishes. ■

Corollary 3.143. A scheme morphism $\pi \colon X \to Y$ is separated if and only if $\Delta\pi(X) \subseteq X \times_Y X$ is closed topologically.

Proof. By Lemma 3.142, $\Delta\pi$ is a locally closed embedding, so $\Delta\pi(X)$ being closed makes $\Delta\pi$ is a closed embedding by Lemma 3.129. ■

As an application, we show that separated morphisms satisfy cancellation, like quasiseparated morphisms (Lemma 3.49).

Lemma 3.144. Fix scheme morphisms $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$. If the composite $\psi \circ \varphi$ is separated, then φ is also separated.

Proof. The key to the proof is to draw the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta\varphi"] \arrow[dr, "\Delta(\psi\varphi)"'] & X \times_Y X \arrow[r] \arrow[d, "\delta"] & Y \arrow[d, "\Delta\psi"] \\ & X \times_Z X \arrow[r] & Y \times_Z Y \end{tikzcd}$$

where the right square is the pullback square of Lemma 3.35. In particular, Lemma 3.142 tells us that δ is a locally closed embedding. Now, $\Delta(\psi \circ \varphi)$ is a closed embedding because $\psi \circ \varphi$ is separated, so $\Delta(\psi \circ \varphi)(X)$ is a closed subset of $X \times_Z X$, so

$$\delta^{-1}(\Delta(\psi \circ \varphi)(X)) = \delta^{-1}(\delta(\Delta\varphi(X))) \subseteq X \times_Y X$$

is also a closed subset by the continuity of δ. However, δ is the composite of morphisms which are homeomorphisms onto their images, so δ is a homeomorphism onto its image (to be formal, use the arguments of Lemma 3.129). Thus,

$$\Delta(\psi \circ \varphi)(X) = \delta(\Delta\varphi(X)) \subseteq \operatorname{im} \delta$$

tells us that $\delta^{-1}(\delta(\Delta\varphi(X))) = \Delta\varphi(X)$ is a closed subset of $X \times_Y X$, so φ is in fact separated by Corollary 3.143. ■

Cancellation allows us to check that being separated is a reasonable notion for S-schemes.

Corollary 3.145. Let S be a separated scheme. Then an S-scheme X is separated over S if and only if X is separated (over $\operatorname{Spec} \mathbb{Z}$).

Proof. Let $\varphi\colon X \to S$ be the promised map. Recall that $\operatorname{Spec}\mathbb{Z}$ is final by Corollary 2.29, so we have unique maps $\tau_S: S \to \operatorname{Spec}\mathbb{Z}$ and $\tau_X: X \to \operatorname{Spec}\mathbb{Z}$ which notably satisfy

$$\tau_X = \tau_S \circ \varphi$$

by the uniqueness of τ_X. Because S is separated, we see τ_S is separated.

Now, if X is separated over S, then φ is separated, so τ_X is separated because separated morphisms are preserved by composition by Remark 3.136; thus, X is separated. Conversely, if X is separated, then τ_X is separated, so φ is separated by Lemma 3.144; thus, X is separated over S. ■

Here is a more extended example.

Exercise 3.146. The affine k-line with doubled origin is not separated over $\operatorname{Spec} k$.

Proof. We will be a little sketchy here because we'll shortly give a more sophisticated proof of this. Glue the affine schemes $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k[y]$ by the isomorphism $\operatorname{Spec} k[x,x^{-1}] \simeq \operatorname{Spec} k[y,y^{-1}]$, which makes the affine k-line with doubled origin X.

We really only have to pay attention to the topological spaces. Note that $X \times_{\operatorname{Spec} k} X$ as a topological space looks like $\mathbb{A}^2_k$ with a doubled y-axis and a doubled x-axis and a quadrupled origin; one can check this through by gluing everything. As such, the diagonal map has image consisting of the line $y = x$, with two of the origins. However, the two origins have closure of all four of the origins, so the total image is not closed. ■

Remark 3.147. However, it is true that our scheme is quasiseparated. Namely, we can see somewhat directly that our pre-images are quasicompact.

3.5 October 3

It's spooky season.

3.5.1 The Cancellation Theorem

We've been speaking a lot about diagonal morphisms and their various classes, so it will be useful to pick up the following piece of general theory.

Theorem 3.148. Fix a class P of morphisms which is preserved by composition and base change. Further, fix objects X, Y, B with morphisms $\alpha\colon X \to B$ and $\beta\colon Y \to B$ and $\varphi\colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & B & \end{tikzcd}$$

commute. If $\alpha \in P$ and $\Delta\beta \in P$, then $\varphi \in P$.

Proof. This is on the homework. ■

Example 3.149. If φ is quasicompact, and β is quasiseparated, then α is quasicompact.

3.5.2 Varieties

We are almost ready to define varieties. We note that projective space is separated.

Proposition 3.150. The canonical map $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.

We will prove this momentarily, but let's explain why we care.

Remark 3.151. By Corollary 3.145, being separated as an A-scheme is equivalent to being separated as a scheme, so Proposition 3.150 is just saying that $\mathbb{P}^n_A$ is a separated scheme.

Example 3.152. Quasiprojective schemes X are those with a locally closed embedding $X \hookrightarrow \mathbb{P}^n_A$. However, locally closed embeddings are separated by Example 3.138, so X is separated as a $\mathbb{P}^n_A$-scheme and therefore separated by Corollary 3.145 and Proposition 3.150.

These notions let us define a variety.

Definition 3.153 (Variety)**.** Fix a field k. A *variety* over k (or "k-variety") is a reduced, separated k-scheme of finite type.

Remark 3.154. By Corollary 3.145, there is no ambiguity calling a k-scheme "separated" because being separated over $\operatorname{Spec} k$ and being separated over $\operatorname{Spec} \mathbb{Z}$ are equivalent.

Example 3.155. Example 3.152 tells us that reduced quasiprojective schemes are varieties. In particular, all our usual affine and projective varieties are indeed varieties.

Before continuing, we pick up the following result.

Lemma 3.156. Fix a scheme morphism $\varphi\colon X \to Y$ where Y is affine. Then the following are equivalent.

(a) φ is separated.

(b) Any affine open subset $U_1, U_2 \subseteq X$ has $U_1 \cap U_2$ affine, and the canonical map

$$\mathcal{O}_X(U_1) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1 \cap U_2)$$

is surjective.

(c) There is an open affine cover $\{U_\alpha\}_{\alpha \in \lambda}$ of X such that all the intersections $U_\alpha \cap U_\beta$ are affine, and the canonical map

$$\mathcal{O}_X(U_\alpha) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(U_\beta) \to \mathcal{O}_X(U_\alpha \cap U_\beta)$$

is surjective.

Proof. Quickly, we note that the canonical maps $\mathcal{O}_X(U_1) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1 \cap U_2)$ are being induced by restriction.

Now, for brevity, set $\Delta\colon X \to X \times_Y X$ to be the diagonal map, and set $A := \mathcal{O}_Y(Y)$. Quickly, note that any two affine open subschemes $U_1, U_2 \subseteq X$ will have

$$U_1 \cap U_2 = \Delta^{-1}(U_1 \times_Y U_2) \simeq \Delta^{-1}\big(\operatorname{Spec}(\mathcal{O}_X(U_1) \times_A \mathcal{O}_X(U_2))\big)$$

by Lemma 3.141 and Lemma 2.176. We will use this fact a few times.

- We show (a) implies (b); because φ is separated, Δ is a closed embedding. Well, pick up two affine open subschemes $U_1, U_2 \subseteq X$. As above, we see
$$U_1 \cap U_2 = \Delta^{-1}(U_1 \times_Y U_2) \simeq \Delta^{-1}\big(\operatorname{Spec}(\mathcal{O}_X(U_1) \times_A \mathcal{O}_X(U_2))\big).$$
Because Δ is a closed embedding, Δ is affine by Lemma 3.126, so $U_1 \cap U_2$ is fact affine.
Thus, because Δ is a closed embedding, we see that its restriction
$$\Delta|_{U_1\cap U_2} \colon U_1 \cap U_2 \to \operatorname{Spec}(\mathcal{O}_X(U_1) \otimes_A \mathcal{O}_X(U_2))$$
is also a closed embedding by Corollary 3.125. Because $U_1 \cap U_2$ is affine, Proposition 2.147 kicks in to tell us that the induced map
$$\mathcal{O}_X(U_1) \otimes_A \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1 \cap U_2) \tag{3.8}$$
is surjective; indeed, we see that this is the right diagram by tracking Δ in the following diagrams passed through Theorem 2.18.
$$\begin{tikzcd} U_1 \cap U_2 \arrow[dr, "\Delta"] \arrow[drr, hook, bend left] \arrow[ddr, hook, bend right] & & \\ & U_1 \times_Y U_2 \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & U_1 \arrow[d, "\varphi"] \\ & U_2 \arrow[r, "\varphi"] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_X(U_1 \cap U_2) & & \\ & \mathcal{O}_X(U_1) \otimes_A \mathcal{O}_X(U_2) \arrow[ul] & \mathcal{O}_X(U_1) \arrow[l] \arrow[ull, bend right, "\text{res}"'] \\ & \mathcal{O}_X(U_2) \arrow[u] \arrow[uul, bend left, "\text{res}"] & A \arrow[l] \arrow[u] \end{tikzcd} \tag{3.9}$$
Notably, the maps $\mathcal{O}_X(U_i) \to \mathcal{O}_X(U_1 \cap U_2)$ are indeed restrictions by how the open embedding $U_1 \cap U_2 \hookrightarrow U_1$ is defined in Example 2.5. This is what we wanted.
- There is nothing to say to show (b) implies (c): just give X any affine open cover.
- We show (c) implies (a). Fix $\{U_\alpha\}_{\alpha\in\lambda}$ as the given affine open cover of X. Again, noting
$$U_\alpha \cap U_\beta = \Delta^{-1}(U_\alpha \times_Y U_\beta) \simeq \Delta^{-1}\big(\operatorname{Spec}(\mathcal{O}_X(U_\alpha) \times_A \mathcal{O}_X(U_\beta))\big),$$
we see that using (3.9) again tells us that $\Delta|_{U_\alpha\cap U_\beta} \colon U_\alpha \cap U_\beta \to \operatorname{Spec}(\mathcal{O}_X(U_\alpha) \times_A \mathcal{O}_X(U_\beta))$ is a morphism of affine schemes associated by Theorem 2.18 to the surjection
$$\mathcal{O}_X(U_\alpha) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(U_\beta) \to \mathcal{O}_X(U_\alpha \cap U_\beta).$$
Thus, $\Delta|_{U_\alpha\cap U_\beta}$ is a closed embedding for any $\alpha, \beta \in \lambda$ by Proposition 2.147, so Δ is a closed embedding by Corollary 3.125. Namely, the $U_1 \times_Y U_2$ cover $X \times_Y X$ by two applications of Corollary 2.181. ■

Remark 3.157. This is intended to be similar to the way we think about quasiseparated morphisms. However, having affine intersections is not good enough to be separated, as we saw with the affine line with doubled origin.

We are now ready to prove Proposition 3.150.

Proof of Proposition 3.150. We choose the usual affine open cover of $\mathbb{P}^n_A$. Namely, set
$$U_i := \operatorname{Spec} \frac{A[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)}$$
for each $i \in \{0, 1, \ldots, n\}$. By the gluing to get $\mathbb{P}^n_A$, our intersections are
$$U_i \cap U_j \simeq \operatorname{Spec} \frac{A[x_{0/i}, \ldots, x_{n/i}, x_{j/i}^{-1}]}{(x_{i/i} - 1)} \simeq \operatorname{Spec} \frac{A[x_{0/j}, \ldots, x_{n/j}, x_{i/j}^{-1}]}{(x_{j/j} - 1)},$$

which are all affine, where the isomorphism on the right is given by $x_{k/i} \mapsto x_{k/j}/x_{i/j}$.

To finish the application of Lemma 3.156, we need to show that the maps

$$\begin{array}{ccccc} \mathcal{O}_X(U_1) & \otimes_A & \mathcal{O}_X(U_2) & \to & \mathcal{O}_X(U_1 \cap U_2) \\ f_1 & \otimes & f_2 & \mapsto & f_1|_{U_1 \cap U_2} \cdot f_2|_{U_1 \cap U_2} \end{array}$$

are surjective. Well, unwrapping our gluing isomorphisms, it suffices to show that the map

$$\begin{array}{ccccc} \frac{A[x_{0/i},\ldots,x_{n/i}]}{(x_{i/i}-1)} & \otimes_A & fracA[x_{0/j},\ldots,x_{n/j}](x_{j/j}-1) & \to & \frac{A[x_{0/i},\ldots,x_{n/i},x_{j/i}^{-1}]}{(x_{i/i}-1)} \\ x_{k/i} & \otimes & x_{\ell/j} & \mapsto & x_{k/i}x_{\ell/i}/x_{j/i} \end{array}$$

which is indeed surjective because $x_{k/i} \otimes 1 \mapsto x_{k/i}$ and $1 \otimes x_{i/j} \mapsto x_{i/i}/x_{j/i} = 1/x_{j/i}$ has hit all the needed generators of $\mathcal{O}_X(U_1 \cap U_2)$, so the total map must be surjective. ■

We close our discussion here by noting that morphisms of k-varieties get some adjectives for free by Theorem 3.148.

Lemma 3.158. A k-morphism of k-varieties is separated and of finite type.

Proof. Fix a k-morphism $\varphi\colon X \to Y$ of k-varieties, and let $\alpha\colon X \to \operatorname{Spec} k$ and $\beta\colon Y \to \operatorname{Spec} k$ be the canonical morphisms.

Because φ is a k-morphism, the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

commutes. Here are our checks.

- Because $\beta = \alpha \circ \varphi$ is separated, Lemma 3.144 tells us that φ is separated.
- Similarly, Lemma 3.80 tells us that φ is locally of finite type because φ is.
- Because α is quasicompact (it's of finite type), and the diagonal morphism $\Delta\beta$ is quasicompact (it's a closed embedding by definition and thus quasicompact by Example 3.3), we see φ is quasicompact by Theorem 3.148. ■

3.5.3 Rational Maps

For this subsection, we let X be a reduced scheme and Y a separated scheme. One can remove this assumption, but it creates problems.

We will use the notation $X \dashrightarrow Y$ to talk about our rational maps before we define it; the reason will be clear from the examples.

Example 3.159. Consider the map $\mathbb{P}^3_{\mathbb{Z}} \dashrightarrow \mathbb{P}^2_{\mathbb{Z}}$ by $[x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : x_2]$. Notably, this is not defined when $x_0 = x_1 = x_2 = 0$, which is just a single point in $\mathbb{P}^3_{\mathbb{Z}}$, so our morphism is technically not well-defined, but that's life.

Example 3.160. Let $X = V(y^2 - x^3 - x)$ be an elliptic curve over k. Then there is a map $X \dashrightarrow \mathbb{A}^1_k$ by $(x, y) \mapsto 1/x$ so that this isn't well-defined at $x = 0$. Notably, there is a way to extend this map to $\mathbb{P}^1_k$, but that is for later.

Example 3.161. Fix an integral scheme X with generic point $\eta \in X$, where $K(X) := \mathcal{O}_{X,\eta}$ is the function field. Now, recall

$$\operatorname{Mor}(X, \mathbb{A}^1_{\mathbb{Z}}) \simeq \operatorname{Hom}(\mathbb{Z}[x], \mathcal{O}_X(X)) \simeq \mathcal{O}_X(X),$$

which embeds into $\mathcal{O}_{X,\eta}$. The point is that $K(X)$ is supposed to be some extra "rational" maps on X to $\mathbb{A}^1_k$.

With enough examples, here is our definition.

Definition 3.162 (Rational)**.** Fix a reduced scheme X and a scheme Y. A *rational map* $f \colon X \dashrightarrow Y$ is an equivalence class of (U, α), where $U \subseteq X$ is a dense open subset, and $\alpha \colon U \to Y$ is a scheme morphism. The equivalence relation is given by $(U, \alpha) \sim (V, \beta)$ if and only if there is an open dense subset $W \subseteq U \cap V$ with $\alpha|_W = \beta|_W$.

Here are some adjectives on our rational maps.

Definition 3.163 (Dominant)**.** Fix a reduced scheme X and a scheme Y. A rational map $f \colon X \dashrightarrow Y$ is *dominant* if and only if each (U, α) representing f has α dominant.

Definition 3.164 (Birational)**.** Fix a reduced scheme X and a scheme Y. A rational map $f \colon X \dashrightarrow Y$ is *birational* if and only if there is another rational map $g \colon Y \dashrightarrow X$ such that $f \circ g$ and $g \circ f$ are equivalent to identities.

Example 3.165. The affine line with doubled origin, $\mathbb{A}^1 \setminus \{0\}$, and $\mathbb{A}^1$ are all birationally equivalent.

Example 3.166. All of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{A}^1 \times \mathbb{A}^1$ and $\mathbb{P}^2$ are all birationally equivalent.

Here is a quick sanity check, which explains why we want our codomain to be separated.

Proposition 3.167. Fix open dense subsets U, U' of a reduced scheme X and a separated scheme Y. Given $\alpha \colon U \to Y$ and $\beta \colon U' \to Y$ which are both representatives of a rational map $f \colon X \dashrightarrow Y$, we have $\alpha|_{U \cap U'} = \beta|_{U \cap U'}$.

In particular, we can glue together α and β and apply transfinite induction induct in order to say that f has a unique representative with largest domain.

Non-Example 3.168. Consider the rational map from the affine line with doubled origin to $\mathbb{A}^1$. Then on each $\operatorname{Spec} k[x]$ for the affine line with doubled origin, there is a map to the affine line, but we can't glue these two maps together to get a full map from the affine line with doubled origin to the affine line.

3.6 October 5

We continue.

3.6.1 Scheme Equalizers

Last time we were in the middle of the following proposition.

Proposition 3.167. Fix open dense subsets U, U' of a reduced scheme X and a separated scheme Y. Given $\alpha\colon U \to Y$ and $\beta\colon U' \to Y$ which are both representatives of a rational map $f\colon X \dashrightarrow Y$, we have $\alpha|_{U\cap U'} = \beta|_{U\cap U'}$.

Before going into the proof, we have the following lemma.

Definition 3.169 (Equalizer)**.** Given objects X and Y with morphisms $\alpha, \beta\colon X \to Y$, we can define the *equalizer* as the limit of the following diagram.

$$X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$

Example 3.170. In the category Set, we have

$$\mathrm{eq}(\alpha, \beta) = \{x \in X : \alpha(x) = \beta(x)\},$$

hence justifying the name "equalizer."

Remark 3.171. It turns out that being an equalizer is equivalent to fitting into the following pullback square.

$$\begin{tikzcd} \mathrm{eq}(\alpha,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ X \arrow[r, "{(\alpha,\beta)}"] & Y \times Y \end{tikzcd}$$

With Remark 3.171, we can define equalizers of schemes.

Lemma 3.172. Fix scheme morphisms $\alpha, \beta\colon X' \to Y$ where Y is separated. Then the canonical map $\mathrm{eq}(\alpha, \beta) \to X'$ is a closed embedding.

Proof. Use the fact that closed embeddings are preserved by base change, combined with the pullback square Remark 3.171. ■

Remark 3.173. One can use the functor of points interpretation to show that the T-points of $\mathrm{eq}(\alpha, \beta)$ are the T-points of X' such that $\alpha_T(x) = \beta_T(x)$.

We are now ready to prove Proposition 3.167.

Proof of Proposition 3.167. The fact that α and β are representatives of the same map promises an open subscheme V of $\mathrm{eq}(\alpha, \beta) \subseteq U \cap U'$ such that $\alpha|_V = \beta|_V$. However, it follows that $\ker(\alpha, \beta) = U \cap U'$ because $U \cap U'$ is reduced (in particular, this gives the uniqueness of the closed subscheme structure, and we have shown that this closed subscheme topologically contains an open dense subset of $U \cap U'$ and therefore must be equal). ■

3.6.2 Graphs

Given a morphism $f\colon X \to Y$, we can try to define its graph. Morally, it should be the morphism induced by the following diagram.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, equal] & & \\
& X \times Y \arrow[r] \arrow[d] & Y \\
& X &
\end{tikzcd}$$

Remark 3.174. If Y is separated, one can use the pullback square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X \times Y \arrow[r] & Y \times Y
\end{tikzcd}$$

to show that the graph morphism is a closed embedding.

There are such properties about the graph morphism that we could prove using the same pullback square.

3.6.3 A Non-reduced Example

Let's quickly talk about why we're adding in the reduced condition. Take $X' := \operatorname{Spec} k[x,y]/\left(x^2, xy\right)$, which is intuitively the y-axis of $\mathbb{A}^2_k$ with some small differential information in the x direction. Also, we set $Y := k[t]$, and can build two scheme morphisms by

$$\begin{array}{rccc} & k[t] & \to & k[x,y]/\left(x^2,xy\right) \\ f_1\colon & t & \mapsto & y \\ f_2\colon & t & \mapsto & x+y \end{array}$$

has $f_1|_{V((x))} = f_2|_{V((x))}$. Now, we can see that $V((x))$ is open and dense in X' (it contains $D(y)$) even though $f_1 \neq f_2$ as scheme morphisms on X'. Actually seeing that $f_1 \neq f_2$ is a little tricky: note

$$\left(x^2, xy\right) = (x) \cap \left(x^2, xy, y^2\right)$$

is a primary decomposition, so we have associated primes (x) and (x,y). Now, even though $D(y)$ is an open dense subset of X', we see that $V((x)) \supseteq D(y)$ has $V((x)) \neq X'$. The problem here is that our open dense subset $D(y)$ of X' has missed a closed embedding from (x,y), which our f_2 could not see.

3.6.4 Birational Maps

Let's continue studying our rational maps.

Proposition 3.175. Fix integral schemes X and Y. Then a rational map $f\colon X \dashrightarrow Y$ is birational if and only if there are open dense subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism $U \cong V$.

Proof?

One can say more for varieties.

Proposition 3.176. Fix a field k and fix two k-schemes X and Y of finite type. Then there is a (natural) bijection

$$\{\text{dominant } f\colon X \dashrightarrow Y\} \simeq \operatorname{Hom}_k(\operatorname{Spec} K(X), \operatorname{Spec} K(Y)) \simeq \operatorname{Hom}_k(K(Y), K(X)).$$

Proof. In one direction, start with a dominant map $f\colon X \to Y$. Note that the generic point $\eta_X \in X$ must go to the generic point $\eta_Y \in Y$. Namely, for any nonempty open subset $V \subseteq Y$, we have

$$f^{-1}(V) \cap U \neq \varnothing$$

for any open $U \subseteq X$, so $\eta_X \in f^{-1}(V)$ is forced. Looping through all open subsets V forces $\eta_X \in f^{-1}(\{\eta_Y\})$. Now, our dominant map is going to give a map $f^\sharp\colon \mathcal{O}_{Y,\eta_Y} \to \mathcal{O}_{X,\eta_X}$, which is exactly the map $K(Y) \to K(X)$ we were looking for.

In the other direction, we need to use the finite type condition. To begin, note that we can replace Y with an open affine subset $\operatorname{Spec} B$, which is still dense; one can do the same for X (to $\operatorname{Spec} A$) with no headaches. Then we see $K(Y) = \operatorname{Frac} B$ and $K(X) = \operatorname{Frac} X$. Now, we are given a map $f^\sharp\colon K(Y) \to K(X)$. Because our scheme is of finite type, we see $B = k[x_1, \ldots, x_n]$ for some finite n, and then we choose some $s \in A$ with enough dominators so that we have an induced map

$$f^\sharp\colon B \to A_s,$$

which induces the required rational map. This map is dominant because it sends the generic point to the generic point. ■

Remark 3.177. In some sense, we are saying that field extensions correspond to dominant morphisms of some corresponding scheme.

We can thus see that X and Y are birationally equivalent if and only if $K(X) = K(Y)$.

Example 3.178. We work the elliptic curve $V\left(Y^2Z - X^3 - Z^3\right) \subseteq \mathbb{P}^2_k$. Then there are two birational maps $X \dashrightarrow \mathbb{P}^1_k$ by $[X : Y : Z] \to X/Z$ and $[X : Y : Z] \to Y/Z$. One can see here that these rational maps in fact extend all the way to give a full scheme morphism to $\mathbb{P}^1_k$.

Example 3.179. We work with $X = \operatorname{Spec} k[x,y]/\left(y^2 - x^3\right)$. Then the normalization is $\widetilde{X} = \operatorname{Spec} k[t]$, where our normalization map is by $\widetilde{X} \to X$ given by $x \mapsto t^2$ and $x \mapsto t^3$. One can check that this map is birational: the fraction fields are $K(X) = k(x)[y]/\left(y^2 - x^3\right)$ and $K(Y) = k(t)$, and the isomorphism we can see fairly directly is just given by $t \mapsto y/x$.

Remark 3.180. Here is how to see that $\widetilde{X}$ is the normalization of X: note $k[t]$ is the integral closure of $k[x,y]/\left(y^2 - x^3\right)$ in $K(X)$, where we see that t is the root of some monic polynomial belonging to y/x. Thus, $k[t]$ is certainly contained in the integral closure. Then one just needs to show that $k[t]$ is integrally closed in $k(t)$, which is true.

Remark 3.181. More generally, the normalization map is birational when our scheme X has finite type over k.

3.6.5 Universally Closed Morphisms

Before we define proper morphisms, we will define the notion of universally closed. Here is a starting definition.

Definition 3.182 (Closed)**.** A scheme morphism $\varphi\colon X \to Y$ is *closed* if and only if it's a closed map of topological spaces.

Example 3.183. Integral morphisms are closed by Lemma 3.94.

Example 3.184. Closed embeddings $\varphi\colon X \to Y$ are closed: note φ is a homeomorphism onto its image $\varphi(X) \subseteq Y$, and $\varphi(X)$ is closed, so any closed subset $V \subseteq X$ has $\varphi(V)$ a closed subset of $\varphi(X)$ and thus closed in Y because $\varphi(X) \subseteq Y$ is closed.

Example 3.185. Isomorphisms are closed because they are homeomorphisms on the level of topological spaces.

The issue here is that closed morphisms are not necessarily preserved by base change. To fix this, we have the following definition.

Definition 3.186 (Universally closed)**.** A scheme morphism $\varphi\colon X \to Y$ is *universally closed* if and only if φ remains a closed under any base change.

Example 3.187. Integral morphisms are universally closed. Indeed, integral morphisms are closed by Lemma 3.94, and integral morphisms are preserved by base change by Lemma 3.89, so any base-change of an integral morphism is still closed.

Example 3.188. Closed embeddings are universally closed. Indeed, closed embeddings are closed by Example 3.184, and closed embeddings are preserved by base change by Lemma 3.127, so any base-change of a closed embedding is still closed.

Let's give the usual checks.

Lemma 3.189. The class of closed morphisms is preserved by composition.

Proof. Fix closed morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. Then any closed subset $C \subseteq X$ makes $\varphi(C) \subseteq Y$ closed and therefore $\psi(\varphi(C)) = (\psi \circ \varphi)(C)$ closed. This finishes. ∎

Lemma 3.190. The class of universally closed morphisms is preserved by composition.

Proof. Fix universally closed morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. We need to show that $\psi\circ\varphi$ is universally closed. Well, fix some scheme morphism $\pi\colon S \to Z$ and build the base-change diagram

$$\begin{tikzcd} X \times_Y (Y \times_Z S) \arrow[r, "\varphi'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \times_Z S \arrow[r, "\psi'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "\pi"] \\ X \arrow[r, "\varphi"] & Y \arrow[r, "\psi"] & Z \end{tikzcd}$$

so that we want to show $\psi' \circ \varphi'$ is a closed embedding: notably, each square is a pullback square, so the total square is a pullback square by Lemma 2.174, so $\psi' \circ \varphi'$ is indeed the base-change of $\psi \circ \varphi$.

Well, ψ' and φ' are closed embeddings because closed embeddings are preserved by base change by Lemma 3.127, so $\psi' \circ \varphi'$ is a closed embedding because closed embeddings are preserved by composition by Lemma 3.122. ∎

Lemma 3.191. The class of universally closed morphisms is preserved by base change.

Proof. Fix a pullback square

$$\begin{array}{ccc} W & \xrightarrow{\pi_Z} & Z \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \psi_Z} \\ X & \xrightarrow{\psi_X} & Y \end{array}$$

such that ψ_X is universally closed, and we want to show that π_Y is also universally closed. Well, pick some morphism $\pi'\colon S \to Z$ and build the pullback square

$$\begin{array}{ccc} W \times_Z S & \xrightarrow{\pi'_Z} & S \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi'} \\ W & \xrightarrow{\pi_Z} & Z \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \psi_Z} \\ X & \xrightarrow{\psi_X} & Y \end{array}$$

so that we want to show π'_Z is also a closed map. Well, the bottom square is a pullback by hypothesis, and the top square is a pullback by construction, so the big rectangle is a pullback by Lemma 2.174. Thus, π'_Z is a closed map because ψ_X is closed. ■

Lemma 3.192. Fix a scheme morphism $\varphi\colon X \to Y$. Given an open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of Y such that the restrictions $\varphi|_{\varphi^{-1}U_\alpha}\colon \varphi^{-1}U_\alpha \to U_\alpha$ are all closed, the morphism φ is also closed.

Proof. Set $\varphi_\alpha := \varphi|_{\varphi^{-1}U_\alpha}$, for brevity. Fix a closed set $C \subseteq X$, and we want to show that $\varphi(C)$ is also closed.

Well, for any α, we know $\varphi_\alpha\colon \varphi^{-1}U_\alpha \to U_\alpha$ is closed, so

$$\varphi_\alpha\left(\varphi^{-1}U_\alpha \cap C\right) = U_\alpha \cap \varphi(C)$$

is also closed in U_α. Thus, $U_\alpha \setminus \varphi(C)$ is open in U_α and therefore open in Y for any α, so

$$\varphi(C) = \bigcup_{\alpha\in\lambda} (U_\alpha \cap \varphi(C))$$

is a union of open subsets and therefore open. This finishes. ■

Lemma 3.193. The class of universally closed morphisms is local on the target.

Proof. Fix a morphism $\varphi\colon X \to Y$. In one direction, suppose φ is universally closed, and we want to show that $\varphi|_{\varphi^{-1}U}$ is universally closed for any open subscheme $U \subseteq Y$. Well, by Lemma 2.178 implies that $\varphi|_{\varphi^{-1}U}$ is a base-change of φ and is therefore universally closed by Lemma 3.191.

In the other direction, give Y an affine open cover $\{Y_\alpha\}_{\alpha\in\lambda}$ such that the restrictions $\varphi_\alpha\colon X_\alpha\colon Y_\alpha$ are all universally closed, where $X_\alpha := \varphi^{-1}Y_\alpha$. We want to show that φ is universally closed. Well, draw a pullback square

$$\begin{array}{ccc} W & \xrightarrow{\varphi'} & Z \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

so that we want to show $\varphi' : W \to Z$ is closed. For this, we define $\delta := \pi \circ \varphi' = \varphi \circ \pi'$, and we define $Z_\alpha := \pi^{-1} Y_\alpha$ and $W_\alpha := \delta^{-1} Y_\alpha$ so that Lemma 2.186 tells us

$$\begin{tikzcd} W_\alpha \arrow[r, "\varphi'|_{W_\alpha}"] \arrow[d, "\pi'|_{W_\alpha}"] & Z_\alpha \arrow[d, "\pi|_{Z_\alpha}"] \\ X_\alpha \arrow[r, "\varphi_\alpha"] & Y_\alpha \end{tikzcd}$$

is also a pullback square. Now, each φ_α is universally closed, so each $\varphi'|_{W_\alpha}$ is closed (it's a base-change), so φ' is also closed by Lemma 3.192. This finishes the proof. ■

3.6.6 Proper Morphisms

We are now ready to define proper morphisms.

Definition 3.194 (Proper)**.** A scheme morphism $f: X \to Y$ is *proper* if and only if f is separated, of finite type, and universally closed.

Here are the standard checks.

Lemma 3.195. The class of proper morphisms is preserved by composition, preserved by base change, and affine-local on the target.

Proof. Being separated has all these adjectives by Remark 3.136, and being finite type has all these adjectives by Corollary 3.91. Lastly, being universally closed is preserved by composition by Lemma 3.190, preserved by composition by Lemma 3.191, and local on the target by Lemma 3.193. So we are done by Lemma 3.90. ■

Example 3.196. Finite morphisms are proper.

- Finite morphisms are affine and therefore separated by Example 3.140.
- They're of finite type by Example 3.68.
- Finite morphisms are integral by Example 3.83 and therefore universally closed by Example 3.187.

For example, closed embeddings are proper by Lemma 3.126, as are isomorphisms by (say) Example 2.146.

3.7 October 7

There may be algebraic groups on the homework. Cool.

3.7.1 Some Proper Facts

We begin by showing that projective schemes are proper.

Lemma 3.197. Fix a ring A. Then the canonical projection $\pi: \mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map.

Proof. We think of $\mathbb{P}^n_A$ as $\operatorname{Proj} A[x_0,\ldots,x_n]$; for brevity, set $R := A[x_0,\ldots,x_n]$.

Fix a closed subset of $\mathbb{P}^n_A$, which we can write down as $V_+(I)$ for some homogeneous ideal $I \subseteq R$. We would like to show that $\pi(V_+(I)) \subseteq \operatorname{Spec} A$ is closed. Suppose that I is generated by the homogeneous elements of positive degree $\{f_\alpha\}_{\alpha\in\lambda}$.

We need to describe the map $\mathbb{P}^n_A \to \operatorname{Spec} A$. Given a prime $\mathfrak{P} \in \operatorname{Proj} R$, we know that $\mathfrak{P}$ does not contain $(x_0,\ldots,x_n)$, so $\mathfrak{P} \in D_+(x_i)$ for some i. From the Proj construction, we have isomorphisms

$$D_+(x_i) \simeq \operatorname{Spec}(R_{x_i})_0 \simeq \operatorname{Spec}\frac{A[x_{0/i},\ldots,x_{n/i}]}{(x_{i/i}-1)},$$

which we now see has an evident canonical projection down to $\operatorname{Spec} A$. Thus, we are taking our prime $\mathfrak{P} \in D_+(x_i)$ over to $(\mathfrak{P}R_{x_i})_0$ down to $\pi(\mathfrak{P}) = A \cap (\mathfrak{P}R_{x_i})_0$ in $\operatorname{Spec} A$.

We thus claim that $\mathfrak{p} \in \pi(V_+(I))$ if and only if the polynomials in I have a common root in $\mathbb{P}^n_{k(\mathfrak{p})}$. This essentially holds by unwinding the map above and looking at constants.

Thus, $\mathfrak{p} \in \pi(V_+(I))$ is equivalent to saying $V_+(\overline{I}) \subseteq \operatorname{Proj} k(\mathfrak{p})[x_0,\ldots,x_n]$ is nonempty; for brevity, set $R^{\mathfrak{p}} := k(\mathfrak{p})[x_0,\ldots,x_n]$. In other words, we are asking for

$$V_+(\overline{I}) \not\subseteq V((x_0,\ldots,x_n)).$$

Well, by Proposition 2.75, this is equivalent to asking for

$$\operatorname{rad}\overline{I} \not\supseteq \operatorname{rad}(x_0,\ldots,x_n) = (x_0,\ldots,x_n).$$

This is equivalent to asking for $x_i \notin \operatorname{rad}\overline{I}$ for each i, which is equivalent to asking for $x_i^N \notin \overline{I}$ for all N for some i. In particular, this implies that

$$(x_0,\ldots,x_n)^N \not\subseteq \overline{I}$$

for each N by looking at the x_i^N term; conversely, if $x_i^N \in \overline{I}$ for some N for each i, then we can find N large enough so that $x_i^N \in \overline{I}$ for each i, so $(x_0,\ldots,x_n)^{Nn} \subseteq \overline{I}$.

Now, the condition we're looking at is equivalent to requiring

$$R^{\mathfrak{p}}_N \not\subseteq \overline{I}$$

for each N. However, now that we see we're focusing on a degree-N part, we see is equivalent to requiring

$$R^{\mathfrak{p}}_N \not\subseteq \sum_{\alpha\in\lambda} \overline{f_\alpha} R^{\mathfrak{p}}_{N-\deg\overline{f_\alpha}},$$

which means we're asking for the map

$$\bigoplus_{\alpha\in\lambda} R^{\mathfrak{p}}_{N-\deg\overline{f_\alpha}} \to R^{\mathfrak{p}}_N$$

by $(r_\alpha)_\alpha \mapsto \sum_\alpha \overline{f_\alpha} r_\alpha$ to not be surjective. Because this is a linear transformation of $k(\mathfrak{p})$-vector spaces, this is equivalent to asking for each of the $\dim R^{\mathfrak{p}}_N \times \dim R^{\mathfrak{p}}_N$-minors of the corresponding matrix to have vanishing determinant. Combining everything over all the primes has carved out a closed subset of $\operatorname{Spec} A$, finishing. ■

Proposition 3.198. Fix a scheme S. Then the canonical projection $\pi\colon \mathbb{P}^n_S \to S$ is proper.

Proof. We proceed in steps.

1. We reduce to the affine case. Because being proper is affine-local on the target by Lemma 3.195, it suffices to show that the restriction $\pi\colon \pi^{-1}U \to U$ is proper for any affine open subscheme $U \subseteq S$, where $U \cong \operatorname{Spec} A$. This allows us to draw the diagram

$$\begin{tikzcd} \mathbb{P}^n_A \arrow[r, dashed] \arrow[d] & \pi^{-1}U \arrow[r, hook] \arrow[d] & \mathbb{P}^n_S \arrow[r] \arrow[d, "\pi"] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec} A \arrow[r] & U \arrow[r, hook] & S \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

where the rightmost square is a pullback by construction of $\mathbb{P}^n_S$, and the middle square is a pullback by Lemma 2.178, so the rightmost two squares make a pullback square by Lemma 2.174.

Thus, because $\operatorname{Spec}\mathbb{Z}$ is final, the fact that we have morphisms $\mathbb{P}^n_A \to \mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_A \to \operatorname{Spec} A \to U$ at all forces a dashed arrow above making the diagram commute, and we note that the total rectangle is a pullback by Exercise 2.198, so the leftmost square is a pullback by Lemma 2.173.

In total, because isomorphisms are preserved by base-change, we see our morphism $\mathbb{P}^n_A \to \pi^{-1}U$ is an isomorphism, so showing that the morphism $\pi^{-1}U \to U$ is proper is equivalent to showing that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. Namely, isomorphisms are proper by Example 3.196, and proper morphisms are preserved by composition by Lemma 3.195.

As such, we relabel our variables so that $S := \operatorname{Spec} A$, and we are looking at the canonical projection $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$.

2. We get rid of our easy adjectives. Notably, this morphism is already separated by Proposition 3.150, locally of finite type by Example 3.77, and quasicompact by Example 3.45. It remains to show that π is universally closed.

3. We describe what being universally closed requires. By taking a base-change morphism $\varphi\colon S \to \operatorname{Spec} A$, we note that we induce the commutative diagram

$$\begin{tikzcd} \mathbb{P}^n_S \arrow[r, dashed] \arrow[d] & \mathbb{P}^n_A \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ S \arrow[r] & \operatorname{Spec} A \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

where the right square is a pullback by Exercise 2.198, the dashed arrow is induced by $\varphi\colon S \to \operatorname{Spec} A$ and the canonical projection $\mathbb{P}^n_S \to \mathbb{P}^n_{\mathbb{Z}}$, the total square is a pullback by construction of $\mathbb{P}^n_{\mathbb{Z}}$, and so the left square is a pullback by Lemma 2.173.

Namely, we want to show that the canonical projection $\mathbb{P}^n_S \to S$ is closed, for any scheme S.

4. We reduce to the affine case again. Being closed is affine-local on the target by Lemma 3.192, is preserved by composition by Lemma 3.189, and contains isomorphisms by Example 3.185. Thus, we can replace the word "proper" with "closed" everywhere in item 1 so that it suffices to show that the canonical projection $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. However, this is true by Lemma 3.197. ■

Approximately speaking, we expect proper to mean compact. Here's an example of this.

Proposition 3.199. Fix an algebraically closed field k and a proper integral k-scheme X. Then, for any affine k-scheme Y, every k-morphism $f\colon X \to Y$ is constant: there is $y \in Y(k)$ such that f factors through y.

Remark 3.200. It follows from Theorem 3.108 that, given a field k and an irreducible scheme X, then the any map $X_k \to \mathbb{A}^1_k$ either has image of a point or we have a full open embedding. Technically, we only need X to be (geometrically) connected for this to be true.

Proof. Because Y is an affine k-scheme, we can embed $Y \hookrightarrow \mathbb{A}_k^I$ for some index set I, but then we can project down to $\mathbb{A}_k^1$ as well; by looking at each individual projection, we will be able to get $X \to Y$ to factor through a point we saw from each projection. As such, we get to reduce to the case where $Y = \mathbb{A}_k^1$ by later pulling everything back through these maps. Now, we have the following diagram.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\text{proper}"'] & & \mathbb{A}_k^1 \arrow[dl, "\text{separated}"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

It follows from that the map $X \to \mathbb{A}_k^1$ is proper by chasing our adjectives around. However, drawing the larger diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathbb{A}_k^1 \arrow[r] & \mathbb{P}_k^1 \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

tells us that the composite $X \to \mathbb{P}_k^1$ is proper, so the image $\pi(X)$ is closed in $\mathbb{P}_k^1$. However, $\mathbb{A}_k^1$ is not closed in $\mathbb{P}_k^1$, so because X is connected (it's integral), we must have $\pi(X)$ to be a point in $\mathbb{P}_k^1$ and thus factor through a point $y \in \mathbb{A}_k^1$. Because X is reduced (it's integral), we see that our morphism f will factor through $X \to \operatorname{Spec} k(y)$.

To see this last claim, we note that we already have a map $X \to \mathbb{A}_k^1$ factoring through y topologically, so because $\operatorname{Spec} k(y)$ is the reduced closed subscheme associated to $\{y\}$, our morphism on the level of sheaves must also be okay. Namely, on the level of sheaves we are looking at a map

$$\mathcal{O}_{\mathbb{A}_k^1,y} \to \mathcal{O}_X(X)$$

where $\mathfrak{m}_y$ is going to $\mathcal{O}_X(X)$. ■

3.7.2 The Valuative Criterion

Here is our result.

Theorem 3.201. Fix a scheme morphism $f\colon X \to Y$ of finite type, where Y is locally Noetherian. Then the following are equivalent.

(i) f is separated/universally closed/proper.

(ii) For any discrete valuation ring A with fraction field K with diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u'"] \arrow[d, "j"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "v"] \arrow[ur, dashed] & Y \end{tikzcd}$$

will induce at most one dashed arrow/at least one dashed arrow/exactly one dashed arrow.

Example 3.202. Checking the condition for $A := \mathbb{Z}_p$ and $Y := \operatorname{Spec} \mathbb{Z}_p$, we are essentially saying that $X(\mathbb{Z}_p) = \mathbb{Q}_p$.

Proof of the easier direction. We begin with the easier direction. To show that (i) implies (ii), we allow A to be any valuation ring. We have two claims; namely, proper follows by combining separated with universally closed.

- Separated: recall that if we have two scheme morphisms $\alpha, \beta \colon X' \to Y'$ where X' is reduced and Y' separated, then agreeing on a dense open subset requires them to be identified. In our case, we look at
$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, shift left, "\widetilde{f}_1"] \arrow[r, shift right, "\widetilde{f}_2"'] \arrow[dr] & X \arrow[dl, "f"] \\ & Y \end{tikzcd}$$
where we see that $\operatorname{Spec} A$ is reduced because it's a valuation ring, and X is already separated. Thus, agreeing at the generic point (which is $\operatorname{Spec} K$) requires $\widetilde{f}_1 = \widetilde{f}_2$. So there is at most one lift.

- Universally closed: to begin, we note that we have a fiber product diagram
$$\begin{tikzcd} \operatorname{Spec} K \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \operatorname{Spec} A \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$
which almost gets us what we want. Note that the left projection is closed because closed embeddings are preserved by base change.

  Notably, a lift $\widetilde{v} \colon \operatorname{Spec} A \to X$ is the same as asking for a map $u' \colon \operatorname{Spec} K \to (\operatorname{Spec} A) \times_Y X$ agreeing with the projections everywhere. Well, choose the correct point $x \in \operatorname{Spec} A \times_Y X$ to be the image of $u'(\operatorname{Spec} K)$, let X' be its Zariski closure, and give it the unique reduced subscheme structure.

  Now, we have a composite
$$X' \to \operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$$
which is a closed morphism in total and so has image closed in $\operatorname{Spec} A$. However, the image must contain $\operatorname{Spec} K$ by construction, so it follows that the composite map is surjective. As such, we find $x' \in X'$ which we will have to the closed point in $\operatorname{Spec} A$, and on the level of sheaves we define
$$A \hookrightarrow \mathcal{O}_{X',x'} \to \mathcal{O}_{X',x} \to K,$$
and this last map is an isomorphism: the ring $\mathcal{O}_{X',x}$ is a field because x is the generic point, and it contains A and lives inside the fraction field of A, so we must have actually $\mathcal{O}_{X',x} \simeq K$.

  In total, we have a map
$$A \hookrightarrow \mathcal{O}_{X',x'} \hookrightarrow K,$$
but because $\mathcal{O}_{X',x'}$ is a local ring with two maps, we conclude that we will also have $A = \mathcal{O}_{X',x'}$. Thus, we have managed to define a scheme morphism from $\operatorname{Spec} A$ to X. ■

The case of curves has a nice application.

Corollary 3.203. In the case of curves over a field k, let Y be a proper k-scheme and X is a normal integral k-variety with dimension 1 (i.e., X is a curve). Then the above result tells us that any rational map $X \dashrightarrow Y$ will extend to a unique morphism of schemes.

Proof. We already understand the uniqueness, so we focus on existence. Notably, for an open dense subset $U \subseteq X$, we know that $X \setminus U$ has dimension 0 and is therefore finite. Now, for any $x \in X \setminus U$, it happens that $\mathcal{O}_{X,x}$ is one-dimensional and normal and therefore a discrete valuation ring, so the above result lets us extend $X \dashrightarrow Y$ up to a morphism from $\mathcal{O}_{X,x}$, but because Y is of finite type, we can actually extend to an open neighborhood around x by bounding our denominators, so inductively repeating this process finishes. ■

Corollary 3.204. There is an equivalence of categories between normal geometric integral projective curves over a field k (equipped with dominant maps) and finitely generated field extensions K/k of transcendence degree 1.

Proof. Rational dominant maps correspond to field extensions, and above we have seen that proper normal maps will automatically extend to full scheme morphisms. In the other direction, a finitely generated K-algebra gives a quasiprojective curve over k, so taking the Zariski closure in a sufficiently large $\mathbb{P}^n_k$ gets an actually projective curve, and lastly taking normalization grants us the actually normal curve. ■

Next class we will finish the proof of the valuative criterion.

Remark 3.205. The reason we care about the harder direction of the valuative criterion is that, in life, a scheme morphism $\pi\colon X \to Y$ will be thinking about X as a moduli space over a base Y, where the valuative criterion has some actually geometric meaning. (Moduli spaces would be a good topic for the term paper.)

3.8 October 10

Today we finish the proof of the valuative criterion.

3.8.1 The Valuative Criterion

Here is our statement, from last class.

Theorem 3.201. Fix a scheme morphism $f\colon X \to Y$ of finite type, where Y is locally Noetherian. Then the following are equivalent.

(i) f is separated/universally closed/proper.

(ii) For any discrete valuation ring A with fraction field K with diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u'"] \arrow[d, "j"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "v"] \arrow[ur, dashed] & Y \end{tikzcd}$$

will induce at most one dashed arrow/at least one dashed arrow/exactly one dashed arrow.

We showed that (i) implies (ii) last class. Today we will sketch that (ii) implies (i), but we are unlikely to actually use the result crucially during the class.

Remark 3.206. This does not mean that the valuative criterion is useless. If we have a variety which we don't know is immediately projective, then the valuative criterion may be quite useful; e.g., we might want to look at moduli spaces.

Proof of the harder direction. We show (ii) implies (i). As before, we have two claims.

- Separated: being separated is affine-local on the target, so we may assume that Y is affine and in particular Noetherian. Because $f\colon X \to Y$ is of finite type, this tells us that X is a Noetherian scheme.

  We want to show that the image of $\Delta_f\colon X \to X \times_Y X$ is closed, but because we are working with Noetherian schemes, we may use Theorem 3.108 to say that $\Delta_f(X)$ is at least constructible, so it suffices to show that $\Delta_f(X)$ is stable under specialization.

Now, for some $z \in \Delta_f(X)$, suppose $z' \in \overline{\{z\}}$, and we want to show $z' \in \Delta_f(X)$. We now claim that there is a discrete valuation ring A with $\operatorname{Spec} A = \{\eta, s\}$ (here, η is generic, and s is closed) and a scheme morphism $u' \colon \operatorname{Spec} A \to X \times_Y X$ with $u'(\eta) = z$ and $u'(s) = z'$. To see this finishes the proof, we draw the diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[drr, dashed, bend left] \arrow[dr, "u'"] \arrow[ddr, dashed, bend right] & & \\
& X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and note that $\pi_1 \circ u'(\eta) = \pi_2 \circ u'(\eta)$ because this is z, so we conclude that $\pi_1 \circ u' = \pi_2 \circ u'$ by checking the uniqueness on (ii). So we see that the morphism $u' \colon \operatorname{Spec} A \to \Delta_f(X)$ must give $z' \in \Delta_f(X)$ by lifting the corresponding specialization from η.

Now, to prove the claim, we appeal to commutative algebra. Well, set $Z \coloneqq X \times_Y X$, and we consider $\mathcal{O}_{Z,z'}/\mathfrak{m}_z$, which comes with it attached an embedding into Z. This is a local integral domain with generic point z and closed point z'. As an aside, if $\mathcal{O}_{Z,z'}/\mathfrak{m}_z$ is one-dimensional, then B we can take a normalization to make B normal, so B is regular (it's a one-dimensional, Noetherian, normal ring), so B is a discrete valuation ring (it's a regular local ring of dimension 1).

So we are mostly interested in trying to force ourselves into a one-dimensional case. One way to finish is by specializing one point at a time to find a one-dimensional subscheme of $\operatorname{Spec} \mathcal{O}_{Z,z'}/\mathfrak{m}_z$, which can be found in [Vak17]. Another way to finish is by blowing up: one can blow up some local question like $k[x,y]_{(x,y)}$ by setting $u = x/y$ and $v = y$ to embed into $k[u,v]_{(v)}$, which turns our closed points into codimension-1 subschemes ("divisors").

- Universally closed: read Hartshorne. ■

Remark 3.207. Without the Noetherian assumption, the claim from the proof of the separated case will hold if we have a general valuation ring instead of just a discrete valuation ring. Indeed, starting from $B \coloneqq \mathcal{O}_{Z,z'}/\mathfrak{m}_z$, we find a maximal local ring A fitting into

$$B \subseteq A \subseteq \operatorname{Frac} B$$

to get a valuation ring by Zorn's lemma, and this will do the trick.

Theme 4
Quasicoherent Sheaves

There was nothing clever to say, so I said something foolish

—Madeline Miller

4.1 October 10

We now shift gears to talk about quasicoherent sheaves.

4.1.1 $\mathcal{O}_X$-modules

Fix a ringed space $(X, \mathcal{O}_X)$. We have the following definition.

Warning 4.1. The term "premodule" is non-standard. It is more common to refer to a presheaf of $\mathcal{O}_X$-modules.

Definition 4.2 ($\mathcal{O}_X$-premodule)**.** Fix a ringed space $(X, \mathcal{O}_X)$. Then an $\mathcal{O}_X$-*premodule* is a presheaf of abelian groups $\mathcal{F}$ on X with a sheaf morphism for scalar multiplication $\cdot\colon \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ making $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module for each open $U \subseteq X$. Namely, given open subsets $V \subseteq U \subseteq X$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\cdot_U"] \arrow[d, "\mathrm{res}_{U,V} \times \mathrm{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\cdot_V"] & \mathcal{F}(V)
\end{tikzcd}$$

Definition 4.3 ($\mathcal{O}_X$-module)**.** Fix a ringed space $(X, \mathcal{O}_X)$. Then an $\mathcal{O}_X$-*module* is an $\mathcal{O}_X$-premodule which is also a sheaf.

Quickly, note that we are in fact allowed to write down $\mathcal{O}_X \times \mathcal{F}$ as a sheaf thanks to Corollary 1.120.

Example 4.4. Given any ringed space $(X, \mathcal{O}_X)$, we may assign $\mathcal{F}(U) := 0$ for any $U \subseteq X$, giving 0 the trivial action.

Remark 4.5. Here's how to think about this definition: for each open $U \subseteq X$, we have a morphism $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$, which is the map endowing $\mathcal{F}(U)$ with $\mathcal{O}_X(U)$-module structure. However, we should remember the various sheaf things floating around, so we make the action respect restriction.

Let's quickly see our definitions in action.

Lemma 4.6. Fix a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-premodule $\mathcal{F}$. For any $p \in X$, we can make $\mathcal{F}_p$ naturally into an $\mathcal{O}_{X,p}$-module.

Proof. Given germs $[(V, r)] \in \mathcal{O}_{X,p}$ and $[(U, f)] \in \mathcal{F}_p$ where $p \in U \cap V$, we simply define

$$[(V, r)] \cdot [(U, f)] := [(U \cap V, r|_{U\cap V} \cdot f|_{U\cap V})].$$

Here are our checks.

- Well-defined: if $[(V, r)] = [(V', r')]$ and $[(U, f)] = [(U', f')]$, then we may find $V'' \subseteq V \cap V'$ and $U'' \subseteq U \cap U'$ such that $r|_{V''} = r'|_{V''}$ and $f|_{U''} = f'|_{U''}$. So we compute

$$\begin{aligned}
[(V, r)] \cdot [(U, f)] &= [(U \cap V, r|_{U\cap V} \cdot f|_{U\cap V})] \\
&\overset{*}{=} [(U'' \cap V'', (r|_{U\cap V} \cdot f|_{U\cap V})|_{U''\cap V''})] \\
&= [(U'' \cap V'', r|_{U''\cap V''} \cdot f|_{U''\cap V''})] \\
&= [(U'' \cap V'', r'|_{U''\cap V''} \cdot f'|_{U''\cap V''})] \\
&\overset{*}{=} [(U'' \cap V'', (r'|_{U'\cap V'} \cdot f'|_{U'\cap V'})|_{U''\cap V''})] \\
&= [(U' \cap V', r'|_{U'\cap V'} \cdot f'|_{U'\cap V'})] \\
&= [(V', r')] \cdot [(U', f')],
\end{aligned}$$

  where the $\overset{*}{=}$ were using the fact that $\mathcal{F}$ is an $\mathcal{O}_X$-premodule.

- Identity: note that $[(X, 1)]$ is the identity of $\mathcal{O}\mathcal{O}_{X,p}$, so we note $[(X, 1)] \cdot [(U, f)] = [(U, f)]$.

- Associativity: given $[(V_1, r_1)], [(V_2, r_2)] \in \mathcal{O}_{X,p}$ and some $[(U, f)] \in \mathcal{F}_p$, we begin by restricting everything to $W := V_1 \cap V_2 \cap U$ (which still contains p) so that the germs we're looking at look like $[(W, r_1), [(W, r_2)] \in \mathcal{O}_{X,p}$ and $[(W, f)] \in \mathcal{F}_p$. Then

$$[(W, r_1)] \cdot ([(W, r_2)] \cdot [(W, f)]) = [(W, r_1)] \cdot [(W, r_2 f)] = [(W, r_1 r_2 f)] = [(W, r_1 r_2)] \cdot [(W, f)].$$

- Distributivity: given $[(V_1, r_1)], [(V_2, r_2)] \in \mathcal{O}_{X,p}$ and $[(U_1, f_1)], [(U_2, f_2)] \in \mathcal{F}_p$, we begin by restricting everything to $W := V_1 \cap V_2 \cap U_1 \cap U_2$ so that the germs we're looking at look like $[(W, r_1)], [(W, r_2)] \in \mathcal{O}_{X,p}$ and $[(W, f_1)], [(W, f_2)] \in \mathcal{F}_p$. Then

$$\begin{aligned}
&([(W, r_1)] + [(W, r_2)]) \cdot ([(W, f_1)] + [(W, f_2)]) \\
&= [(W, r_1 + r_2)] \cdot [(W, f_1 + f_2)] \\
&= [(W, (r_1 + r_2)(f_1 + f_2))] \\
&= [(W, r_1 f_1 + r_1 f_2 + r_2 f_1 + r_2 f_2)] \\
&= [(W, r_1 f_1)] + [(W, r_1 f_2)] + [(W, r_2 f_1)] + [(W, r_2 f_2)] \\
&= [(W, r_1)] \cdot [(W, f_1)] + [(W, r_1)] \cdot [(W, f_2)] + [(W, r_2)] \cdot [(W, f_1)] + [(W, r_2)] \cdot [(W, f_2)].
\end{aligned}$$

The above checks finish. ■

Having defined our objects, we should define our morphisms.

Definition 4.7 (Morphism of $\mathcal{O}_X$-premodules)**.** Fix a ringed space $(X, \mathcal{O}_X)$. As usual, we will define a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-premodules as a morphism of the underlying (pre)sheaves making the diagram following diagram commute.

$$\begin{tikzcd} \mathcal{O}_X \times \mathcal{F} \arrow[r, "\cdot_{\mathcal{F}}"] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X}, \varphi)"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O}_X \times \mathcal{G} \arrow[r, "\cdot_{\mathcal{G}}"] & \mathcal{G} \end{tikzcd}$$

Composition is still just composition of the morphisms of sheaves. As usual, morphism of $\mathcal{O}_X$-modules is just a morphism of the underlying $\mathcal{O}_X$-premodules.

Remark 4.8. Because (pre)sheaf morphisms are determined by their action on open subsets, checking that a (pre)sheaf morphism is a morphism of $\mathcal{O}_X$-(pre)modules amounts to checking that the following diagram commutes for any open subset $U \subseteq X$.

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\cdot_{\mathcal{F}}"] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X(U)}, \varphi_U)"'] & \mathcal{F}(U) \arrow[d, "\varphi_U"] \\ \mathcal{O}_X(U) \times \mathcal{G}(U) \arrow[r, "\cdot_{\mathcal{G}}"] & \mathcal{G}(U) \end{tikzcd}$$

Namely, we require $\varphi_U(rf) = r\varphi_U(f)$ for any $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$.

Let's check that our definitions make sense.

Lemma 4.9. Fix a ringed space $(X, \mathcal{O}_X)$. We have defined a category of $\mathcal{O}_X$-modules, denoted $\mathrm{Mod}_{\mathcal{O}_X}$ as a subcategory of sheaves on X.

Proof. We have to check that the identity is a morphism of $\mathcal{O}_X$-modules, as is the composition of two morphisms of $\mathcal{O}_X$-modules. For the identity, we see that the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\cdot_{\mathcal{F}}"] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X(U)}, (\mathrm{id}_{\mathcal{F}})_U)"'] & \mathcal{F}(U) \arrow[d, "(\mathrm{id}_{\mathcal{F}})_U"] & (r,f) \arrow[r, mapsto] \arrow[d, mapsto] & rf \arrow[d, mapsto] \\ \mathcal{O}_X(U) \times \mathcal{G}(U) \arrow[r, "\cdot_{\mathcal{G}}"] & \mathcal{G}(U) & (r,f) \arrow[r, mapsto] & rf \end{tikzcd}$$

commutes for any $\mathcal{O}_X$-module $\mathcal{F}$, where $\mathrm{id}_{\mathcal{F}}$ has been inherited from Sh_X. For composition, fix morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ of $\mathcal{O}_X$-modules. Then $\psi \circ \varphi$ (defined as the composition in Sh_X) needs to be a morphism of $\mathcal{O}_X$-modules. Well, for any open $U \subseteq X$ and $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$, we see

$$(\psi \circ \varphi)_U(rf) = \psi_U(\varphi_U(rf)) = \psi_U(r\varphi_U(f)) = r\psi_U\varphi_U(f) = r(\psi \circ \varphi)_U(f),$$

which is what we wanted. ■

As usual, we will want a more concrete way to understand isomorphisms.

Lemma 4.10. Fix a ringed space $(X, \mathcal{O}_X)$. A morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism of $\mathcal{O}_X$-modules if and only if it is a morphism of $\mathcal{O}_X$-modules and also a (pre)sheaf isomorphism.

Proof. Certainly an isomorphism of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module homomorphism and also a sheaf isomorphism. Indeed, pick up the inverse morphism $\psi\colon \mathcal{G} \to \mathcal{F}$, and we see $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$ by how identities are defined in $\mathrm{Mod}_{\mathcal{O}_X}$.

Conversely, fix a morphism of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{F}\to\mathcal{G}$ which is a sheaf isomorphism; let $\psi\colon\mathcal{G}\to\mathcal{F}$ be the inverse sheaf morphism. We need to show that ψ is an $\mathcal{O}_X$-module morphism. Well, for any open subset $U\subseteq X$, we note that any $g\in\mathcal{G}(U)$ can be written as $\varphi_U(f)$ for some $f\in\mathcal{F}(U)$, so any $r\in\mathcal{O}_X(U)$ has

$$\psi_U(rg)=\psi_U(r\varphi_U(f))=\psi_U(\varphi_U(rf))=rf=r\psi_U(\varphi_U(f))=r\psi_U(g),$$

which is what we wanted. ■

As expected, we note that taking stalks remains functorial.

Lemma 4.11. Fix a ringed space $(X,\mathcal{O}_X)$ and a morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ of $\mathcal{O}_X$-premodules. Then any $p\in X$ induces a natural map $\varphi_p\colon\mathcal{F}_p\to\mathcal{G}_p$ of $\mathcal{O}_{X,p}$-premodules.

Proof. Note that Proposition 1.111 at least gives us some map $\varphi_p\colon\mathcal{F}_p\to\mathcal{G}_p$ by $\varphi_p([(U,f)])=[(U,\varphi_Uf)]$. To see that this a morphism of $\mathcal{O}_{X,p}$-modules, we note that any $[(V,r)]\in\mathcal{O}_{X,p}$ and $[(U,f)]\in\mathcal{F}_p$ may first everything to $W:=U\cap V$ so that we're looking at the germs $[(W,r)]$ and $[(W,f)]$. Then we compute

$$\begin{aligned}[(W,r)]\cdot\varphi_p([(W,f)])&=[(V,r)]\cdot[(W,\varphi_Wf)]\\&=[(W,r\cdot\varphi_Uf)]\\&=[(W,\varphi_U(rf))]\\&=\varphi_p([(W,rf)])\\&=\varphi_p([(W,r)]\cdot[(W,f)]),\end{aligned}$$

where we have used the fact that φ_U is an $\mathcal{O}_X(U)$-premodule morphism. ■

Remark 4.12. Because we simply defined φ_p the same way it is defined for presheaves, the computation of Remark 1.112 can be directly copied to show that taking stalks is functorial. Namely, the identity and functoriality checks do not change at all.

4.1.2 $\operatorname{Mod}_{\mathcal{O}_X}$ Is Additive

While we're here, we might as well check that our category is additive. This is mostly inherited from Corollary 1.122. Here's our zero.

Lemma 4.13. Fix a ringed space $(X,\mathcal{O}_X)$. Then the zero sheaf $\mathcal{Z}$ on X from Corollary 1.117 is the zero object in the category of $\mathcal{O}_X$-modules.

Proof. To begin, we note that $\mathcal{Z}(U)=0$ is always naturally an $\mathcal{O}_X(U)$-module, and we define our scalar multiplication accordingly. This assembles into an $\mathcal{O}_X$-module by setting open subsets $V\subseteq U\subseteq X$ and noting that

$$\begin{array}{ccccccc}\mathcal{O}_X(U)\times\mathcal{Z}(U)&\xrightarrow{\cdot_U}&\mathcal{Z}(U)&&(r,0)&\longmapsto&0\\ {\scriptstyle\mathrm{res}_{U,V}\times\mathrm{res}_{U,V}}\downarrow&&\downarrow{\scriptstyle\mathrm{res}_{U,V}}&&\downarrow&&\downarrow\\ \mathcal{O}_X(V)\times\mathcal{Z}(V)&\xrightarrow{\cdot_V}&\mathcal{Z}(V)&&(r|_V,0)&\longmapsto&0\end{array}$$

commutes.

It remains to show the universal property. Fix an $\mathcal{O}_X$-module $\mathcal{F}$. We need to show that there are unique morphisms of $\mathcal{O}_X$-modules $\mathcal{F}\to\mathcal{Z}$ and $\mathcal{Z}\to\mathcal{F}$. From Corollary 1.117, we know there is already at most one sheaf morphism in each direction, so there is at most one morphism of $\mathcal{O}_X$-modules.

Thus, we just need to show existence.

- Initial: we know from Corollary 1.117 that the zero maps $\varphi_U\colon \mathcal{Z}(U) \to \mathcal{F}(U)$ assemble into a sheaf map. To see that these assemble into a morphism of $\mathcal{O}_X$-modules, we pick up any $r \in \mathcal{O}_X(U)$ and $0 \in \mathcal{Z}(U)$ and note that
$$\varphi_U(r \cdot 0) = \varphi_U(0) = 0 = r \cdot 0 = r\varphi_U(0).$$
- Terminal: we know from Corollary 1.117 that the zero maps $\psi_U\colon \mathcal{Z}(U) \to \mathcal{F}(U)$ assemble into a sheaf map. To see that these assemble into a morphism of $\mathcal{O}_X$-modules, we pick up any $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$ and note that
$$\psi_U(r \cdot f) = \psi_U(f) = 0 = r \cdot 0 = r\psi_U(f),$$
which is what we wanted. ■

Here's our addition structure.

Lemma 4.14. Fix a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$. Then the set of $\mathcal{O}_X$-morphisms $\mathcal{F} \to \mathcal{G}$ is a subgroup of the sheaf morphisms $\mathcal{F} \to \mathcal{G}$. In particular, $\operatorname{Mor}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an abelian group, where composition distributes over addition.

Proof. We use the subgroup test. There are two checks.

- The subset of $\mathcal{O}_X$-module morphisms is certainly nonempty because we have a morphism of $\mathcal{O}_X$-modules given by the composition of the zero maps $\mathcal{F} \to \mathcal{Z} \to \mathcal{G}$ by Lemma 4.13.
- We need to show that two morphisms $\varphi, \psi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-morphisms have $(\varphi + -\psi)\colon \mathcal{F} \to \mathcal{G}$ a morphism of $\mathcal{O}_X$-modules.

  Well, we pick up an open subset $U \subseteq X$ and some $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$ to check
$$(\varphi + -\psi)_U(rf) = \varphi_U(rf) + (-\psi)_U(rf) = \varphi_U(rf) + -\psi_U(rf) = r(\varphi_U(f) + -\psi_U(f)) = r(\varphi + -\psi)_U(f),$$
  which is what we wanted.

We now note that the last sentence here follows from Lemma 1.113. Indeed, we are abelian because we are a subgroup of an abelian group, and composition still distributes over addition because composition and addition have not changed. ■

Remark 4.15. From Lemma 1.113, we see that our zero morphism $\mathcal{F} \to \mathcal{G}$ is indeed $\mathcal{F} \to \mathcal{Z} \to \mathcal{G}$. Namely, we want the morphism sending everything to 0 automatically.

Lemma 4.16. Fix a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$-modules $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$. The product sheaf $\mathcal{F} := \prod_{\alpha\in\lambda} \mathcal{F}_\alpha$ is naturally an $\mathcal{O}_X$-module.

Proof. We define our scalar multiplication $\cdot\colon \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ at any open subset $U \subseteq X$ by $r \cdot (f_\alpha)_\alpha := (r \cdot f_\alpha)_\alpha$ for any $r \in \mathcal{O}_X(U)$ and $(f_\alpha)_\alpha \in \mathcal{F}(U)$. To see that this makes an $\mathcal{O}_X$-module, we check that the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\cdot_{\mathcal{F}}"] \arrow[d, "\operatorname{res}_{U,V} \times \operatorname{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\cdot_{\mathcal{G}}"] & \mathcal{F}(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
(r, (f_\alpha)_\alpha) \arrow[rr, mapsto] \arrow[d, mapsto] & & (rf_\alpha)_\alpha \arrow[d, mapsto] \\
(r|_V, (f_\alpha)_\alpha|_V) \arrow[r, mapsto] & (r|_V \cdot f_\alpha|_V)_\alpha|_V = (rf_\alpha)_\alpha|_V &
\end{tikzcd}$$

commutes, where in the bottom-right we have used the fact that the $\mathcal{F}_\alpha$ are $\mathcal{O}_X$-modules, meaning $r|_V \cdot f_\alpha|_V = (rf_\alpha)|_V$. ■

Lemma 4.17. Fix a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$-modules $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$, and let $\mathcal{F} := \prod_{\alpha\in\lambda} \mathcal{F}_\alpha$ be the product sheaf. Then $\mathcal{F}$ is also the product in the category of $\mathcal{O}_X$-modules.

Proof. We will use the same projection morphisms $\pi_\alpha \colon \mathcal{F} \to \mathcal{F}_\alpha$ from Corollary 1.120. To check that they are morphisms of $\mathcal{O}_X$-modules, we fix some $\beta \in \lambda$ and note that the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\cdot_{\mathcal{F}}"] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X(U)}, (\pi_\beta)_U)"'] & \mathcal{F}(U) \arrow[d, "(\pi_\beta)_U"] & (r, (f_\alpha)_\alpha) \arrow[r] \arrow[d] & (rf_\alpha)_\alpha \arrow[d] \\
\mathcal{O}_X(U) \times \mathcal{F}_\beta(U) \arrow[r, "\cdot_{\mathcal{F}_\beta}"] & \mathcal{F}_\beta(U) & (r, f_\beta) \arrow[r] & (rf_\beta)
\end{tikzcd}$$

commutes.

We now check the universal property. Fix some $\mathcal{O}_X$-module $\mathcal{G}$ with morphisms $\varphi_\alpha \colon \mathcal{G} \to \mathcal{F}_\alpha$. We need to show that there is a unique morphism $\varphi \colon \mathcal{G} \to \mathcal{F}_\alpha$ of $\mathcal{O}_X$-modules such that $\varphi_\alpha = \pi_\alpha \circ \varphi$ for each α.

Well, we know there is a unique sheaf morphism by Corollary 1.120, so there is certainly at most morphism of $\mathcal{O}_X$-modules. To show that the morphism exists, we set

$$\varphi_U(g) := ((\varphi_\alpha)_U g)_\alpha \in \mathcal{F}(U)$$

for any open $U \subseteq X$ and $g \in \mathcal{G}(U)$. As checked in Corollary 1.120, this assembles into a morphism of our sheaves $\varphi \colon \mathcal{G} \to \mathcal{F}$ such that $\varphi_\alpha = \pi_\alpha \circ \varphi$ for each α, so we just have to check that this is a morphism of $\mathcal{O}_X$-modules. Well, for any open $U \subseteq X$ and $r \in \mathcal{O}_X(U)$ and $g \in \mathcal{G}(U)$, we check

$$\varphi_U(rg) = ((\varphi_\alpha)_U(rg))_\alpha = (r(\varphi_\alpha)_U(g))_\alpha = r((\varphi_\alpha)_U(g))_\alpha = r\varphi_U(g),$$

finishing. ■

Corollary 4.18. Fix a ringed space $(X, \mathcal{O}_X)$. The category of $\mathcal{O}_X$-modules is additive.

Proof. Combine Lemma 4.14, Lemma 4.13, and Lemma 4.17. ■

4.1.3 Kernels for $\mathcal{O}_X$-modules

As usual, kernels are easier than cokernels for sheafification reasons. We will avoid discussing sheafification of $\mathcal{O}_X$-modules for a little while longer.

Proposition 4.19. Fix a ringed space $(X, \mathcal{O}_X)$ and a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. Then the kernel $\ker\varphi$ of φ viewed as a (pre)sheaf is the kernel in the category of $\mathcal{O}_X$-modules.

Proof. Quickly, we recall that

$$(\ker\varphi)(U) := \ker\varphi_U \subseteq \mathcal{F}(U),$$

where the restrictions maps are induced by $\mathcal{F}$.

To make $\ker\varphi$ into an $\mathcal{O}_X$-module, we will give it the action inherited from $\mathcal{F}$: given some open $U \subseteq X$, some $r \in \mathcal{O}_X(U)$, and some $f \in \ker\varphi_U \subseteq \mathcal{F}(U)$, we set $r \cdot f$ to be the $r \cdot f$ from the action of $\mathcal{O}_X(U)$ on $\mathcal{F}(U)$. To see that this is well-defined, we note

$$\varphi_U(r \cdot f) = r \cdot \varphi_U(f) = r \cdot 0 = 0$$

because φ is a morphism of $\mathcal{O}_X$-modules. Thus, $r \cdot f \in \ker \varphi_U$, making our action okay. We also note that $U' \subseteq U \subseteq X$ makes the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times (\ker\varphi)(U) \arrow[r] \arrow[d, "\operatorname{res}_{U,U'} \times \operatorname{res}_{U,U'}"'] & (\ker\varphi)(U) \arrow[d, "\operatorname{res}_{U,U'}"] \\ \mathcal{O}_X(U') \times (\ker\varphi)(U') \arrow[r] & (\ker\varphi)(U') \end{tikzcd} \qquad \begin{tikzcd} (r,f) \arrow[r, mapsto] \arrow[d, mapsto] & r \cdot f \arrow[d, mapsto] \\ (r|_{U'}, f|_{U'}) \arrow[r, mapsto] & (r \cdot f)|_{U'} \end{tikzcd}$$

commute, as inherited from $\mathcal{F}$, so $\ker\varphi$ is indeed an $\mathcal{O}_X$-module.

It remains to show the universal property. Let $\iota\colon \ker\varphi \to \mathcal{F}$ be the canonical morphism; note that ι is an $\mathcal{O}_X$-module morphism because $\iota(rf) = rf = r\iota(f)$ for any $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$ at any open $U \subseteq X$. Now, given some morphism $\psi\colon \mathcal{H} \to \mathcal{F}$ of $\mathcal{O}_X$-modules such that $\varphi \circ \psi = 0$, we need to show that there is a unique map $\overline{\psi}\colon \mathcal{H} \to \ker\varphi$ making the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[dr, dashed, "\overline{\psi}"] \arrow[drr, bend left] \arrow[ddr, bend right, "\psi"'] & & \\ & \ker\varphi \arrow[r] \arrow[d, "\iota"] & \mathcal{Z} \arrow[d] \\ & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd} \tag{4.1}$$

commute. Note that uniqueness follows directly from Lemma 1.125. It remains to show existence. Well, Lemma 1.125 has told us that we may set

$$\overline{\psi}_U(h) := \psi_U(h)$$

for any open $U \subseteq X$ and section $h \in \mathcal{O}_X(U)$ to define a well-defined sheaf morphism $\overline{\psi}\colon \mathcal{H} \to \ker\varphi$ making (4.1) commute as sheaves.

Thus, it roughly remains to show $\overline{\psi}$ defines a morphism of $\mathcal{O}_X$-modules. Well, for any open $U \subseteq X$ and sections $r \in \mathcal{O}_X(U)$ and $h \in \mathcal{H}(U)$, we see

$$\overline{\psi}_U(rh) = \psi_U(rh) = r \cdot \psi_U(h) = r \cdot \overline{\psi}_U(h).$$

So we see that $\overline{\psi}_U$ is a morphism of $\mathcal{O}_X$-modules, and (4.1) now commutes in the category of $\mathcal{O}_X$-modules because composition of morphisms of $\mathcal{O}_X$-modules is the same as the composition of the underlying sheaf morphisms. ■

Corollary 4.20. Fix a ringed space $(X, \mathcal{O}_X)$ and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. The following are equivalent.

(a) φ is monic in the category of sheaves.

(b) φ is monic in the category of $\mathcal{O}_X$-modules.

Proof. By Lemma 1.127, we see that φ is monic in the category of $\mathcal{O}_X$-modules if and only if $\ker\varphi$ vanishes. However, the kernel of φ is the same in the category of sheaves as in the category of $\mathcal{O}_X$-modules by Proposition 4.19, so $\ker\varphi$ vanishing is equivalent to φ being monic in the category of sheaves by Lemma 1.127. ■

4.1.4 Sheafification for $\mathcal{O}_X$-modules

In order to deal with cokernels (and colimits in general), we will want to understand how to use sheafification with $\mathcal{O}_X$-modules. Here are the relevant result.

Lemma 4.21. Fix a ringed space $(X, \mathcal{O}_X)$. Fix a $\mathcal{O}_X$-premodule $\mathcal{F}$.

(a) We can make $\mathcal{F}^{\text{sh}}$ naturally into an $\mathcal{O}_X$-module.

(b) The sheafification map $\text{sh}\colon \mathcal{F} \to \mathcal{F}^{\text{sh}}$ can be made into an $\mathcal{O}_X$-premodule morphism.

Proof. We show the parts in order. Observe that we will have to use the construction of sheafification of Lemma 1.140.

(a) Given an open subset $U \subseteq X$, recall

$$\mathcal{F}^{\text{sh}}(U) := \left\{ (f_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p : (f_p)_{p\in U} \text{ is a compatible system of germs} \right\}$$

with the obvious restriction maps.

To give $\mathcal{F}^{\text{sh}}(U)$ an $\mathcal{O}_X(U)$-module structure, we note that any $p \in U$ can precompose the action map $\mathcal{O}_{X,p} \to \operatorname{Aut}(\mathcal{F}_p)$ with the restriction $\mathcal{O}_X(U) \to \mathcal{O}_{X,p}$ to give an action of $\mathcal{O}_X(U)$ on each stalk $\mathcal{F}_p$ by $r \cdot f_p := r|_p \cdot f_p$. Taking the product of these action morphisms, we get a morphism

$$\mathcal{O}_X(U) \to \prod_{p\in U} \operatorname{Aut}(\mathcal{F}_p) \subseteq \operatorname{Aut}\left(\prod_{p\in U} \mathcal{F}_p\right)$$

(this action is just defining the arbitrary product of some modules), so we have made $\prod_{p\in U} \mathcal{F}_p$ into an $\mathcal{O}_X(U)$-module, where the scalar multiplication is by $r \cdot (f_p)_{p\in U} = (r|_p \cdot f_p)_{p\in U}$.

We now claim that $\mathcal{F}^{\text{sh}}(U) \subseteq \prod_{p\in U} \mathcal{F}_p$ is an $\mathcal{O}_X(U)$-submodule. Certainly we have a subgroup (after all, $\mathcal{F}^{\text{sh}}(U)$ assembles into a sheaf valued in abelian groups), so we just need to be preserved by the $\mathcal{O}_X(U)$-action. Well, suppose $(f_p)_{p\in U}$ is a system of compatible germs so that each $p \in U$ has some open $U_p \subseteq U$ (contained in U after suitable restriction) containing p with a lift $\widetilde{f}_p \in \mathcal{F}(U_p)$ such that $\widetilde{f}_p|_q = f_q$ for each $q \in U$.

Now, for any $r \in \mathcal{O}_X(U)$, the construction of the $\mathcal{O}_X$-action gives

$$(r|_{U_p} \cdot \widetilde{f}_p)|_q = r|_q \cdot \widetilde{f}_p|_q = r|_q \cdot f_q$$

for any $q \in U_p$, for any $p \in U$. Thus, $r|_{U_p} \cdot \widetilde{f}_p \in \mathcal{F}(U_p)$ witnesses that $r \cdot (f_p)_{p\in U} = (r|_p \cdot f_p)_{p\in U}$ is a system of compatible of germs.

Thus, we have made each $\mathcal{F}^{\text{sh}}(U)$ into an $\mathcal{O}_X(U)$-module. To make $\mathcal{F}^{\text{sh}}$ into an $\mathcal{O}_X$-module, we need to check that any $V \subseteq U$ makes the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}^{\text{sh}}(U) \arrow[r] \arrow[d, "\text{res}_{U,V} \times \text{res}_{U,V}"'] & \mathcal{F}^{\text{sh}}(U) \arrow[d, "\text{res}_{U,V}"] & (r, (f_p)_{p\in U}) \arrow[r, mapsto] \arrow[d, mapsto] & (r|_p \cdot f_p)_p \arrow[d, mapsto] \\ \mathcal{O}_X(V) \times \mathcal{F}^{\text{sh}}(V) \arrow[r] & \mathcal{F}^{\text{sh}}(V) & (r|_V, (f_p)_{p\in V}) \arrow[r, mapsto] & (r|_p \cdot f_p)_{p\in V} \end{tikzcd}$$

commute.

(b) We now check that the sheafification map $\text{sh}\colon \mathcal{F} \to \mathcal{F}^{\text{sh}}$ is a morphism of $\mathcal{O}_X$-premodules. Well, Lemma 1.140 already tells us that this map is a sheaf morphism, so we just need to check that we preserve the $\mathcal{O}_X$ structure. For this, we see that any open $U \subseteq X$ makes

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d, "(\text{id}_{\mathcal{O}_X(U)}, \text{sh}_U)"'] & \mathcal{F}(U) \arrow[d, "\text{sh}_U"] & (r, f) \arrow[r, mapsto] \arrow[d, mapsto] & rf \arrow[d, mapsto] \\ \mathcal{O}_X(U) \times \mathcal{F}^{\text{sh}}(U) \arrow[r] & \mathcal{F}^{\text{sh}}(U) & (r, (f|_p)_{p\in U}) \arrow[r, mapsto] & (r|_p \cdot f|_p)_{p\in U} \end{tikzcd}$$

commute. Namely, we have $(rf)|_p = r|_p \cdot f|_p$ in the above computation from the construction of the action of $\mathcal{O}_{X,p}$ on $\mathcal{F}_p$ from Lemma 4.6. ■

Remark 4.22. As above, the sheafification map $\operatorname{sh}\colon \mathcal{F} \to \mathcal{F}^{\operatorname{sh}}$ is a morphism of $\mathcal{O}_X$-premodules, so Lemma 4.11 assures us that $\operatorname{sh}_p\colon \mathcal{F}_p \to \mathcal{F}^{\operatorname{sh}}_p$ is a morphism of $\mathcal{O}_{X,p}$-modules at each $p \in U$. But Proposition 1.141 now assures us that this morphism is an isomorphism on the level of abelian groups, so sh_p is a full isomorphism of $\mathcal{O}_{X,p}$-modules.

Now here is the universal property.

Proposition 4.23. Fix a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-premodule $\mathcal{F}$. Further, let $\operatorname{sh}\colon \mathcal{F} \to \mathcal{F}^{\operatorname{sh}}$ be the sheafification map. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ and $\mathcal{O}_X$-premodule morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, there is a unique $\mathcal{O}_X$-module morphism $\varphi^{\operatorname{sh}}\colon \mathcal{F}^{\operatorname{sh}} \to \mathcal{G}$ such that $\varphi = \varphi^{\operatorname{sh}} \circ \operatorname{sh}$.

Proof. Before doing anything, we note that the statements make sense because $\mathcal{F}^{\operatorname{sh}}$ is an $\mathcal{O}_X$-module by Lemma 4.21.

Because $\mathcal{O}_X$-module morphisms are already sheaf morphisms, Lemma 1.140 tells us that there is at most one sheaf morphism $\varphi^{\operatorname{sh}}\colon \mathcal{F}^{\operatorname{sh}} \to \mathcal{G}$ and thus at most one $\mathcal{O}_X$-module morphism such that $\varphi = \varphi^{\operatorname{sh}} \circ \operatorname{sh}$.

For existence, we note that Lemma 1.140 promises some sheaf morphism $\varphi^{\operatorname{sh}}\colon \mathcal{F}^{\operatorname{sh}} \to \mathcal{G}$ such that $\varphi = \varphi^{\operatorname{sh}} \circ \operatorname{sh}$, which we upgrade into an $\mathcal{O}_X$-module morphism. Indeed, we show in Lemma 1.140 that $(\varphi_p(f_p))_{p\in U}$ is a compatible system of germs for compatible system of germs $(f_p)_{p\in U} \in \mathcal{F}^{\operatorname{sh}}(U)$, which lets us uniquely define $\varphi^{\operatorname{sh}}_U((f_p)_{p\in U})$ by having

$$\varphi^{\operatorname{sh}}_U((f_p)_{p\in U})|_q = \varphi_q(f_q)$$

for any $q \in U$. Namely, we know that this is well-defined and produces a sheaf morphism.

It remains to check that $\varphi^{\operatorname{sh}}$ is in fact a morphism of $\mathcal{O}_X$-modules. For this, we see that any open $U \subseteq X$ and sections $r \in \mathcal{O}_X(U)$ and $(f_p)_{p\in U} \in \mathcal{F}^{\operatorname{sh}}(U)$ will have at q the equalities

$$\left(r \cdot \varphi^{\operatorname{sh}}_U((f_p)_{p\in U})\right)|_q = r|_q \cdot \varphi^{\operatorname{sh}}_U((f_p)_{p\in U})|_q = r|_q \cdot \varphi_q(f_q) \stackrel{*}{=} \varphi_q(r|_q \cdot f_q),$$

where we have used Lemma 4.11 at $\stackrel{*}{=}$. On the other hand, $\varphi^{\operatorname{sh}}_U(r \cdot (f_p)_{p\in U}) = \varphi^{\operatorname{sh}}_U((r|_p \cdot f_p)_{p\in U})$ is defined to be $\varphi_q(r|_q \cdot f_q)$ at q, so we are done. ■

4.1.5 Cokernels for $\mathcal{O}_X$-modules

Now that we have sheafification, we can discuss cokernels. We begin with $\mathcal{O}_X$-premodules.

Lemma 4.24. Fix a ringed space $(X, \mathcal{O}_X)$ and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-premodules. Then the cokernel presheaf $\operatorname{coker}^{\operatorname{pre}} \varphi$ of φ viewed as a presheaf is the cokernel in the category of $\mathcal{O}_X$-premodules.

Proof. As usual, the main point is to give $\operatorname{coker}^{\operatorname{pre}} \varphi$ an $\mathcal{O}_X$-premodule structure; in particular, note that we already have a presheaf by Lemma 1.146. Well, given open $U \subseteq X$, we begin by noting that $\operatorname{im} \varphi_U \subseteq \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-submodule of $\mathcal{G}(U)$: given $\varphi_U(f) \in \operatorname{im} \varphi_U$ and $r \in \mathcal{O}_X(U)$, we see

$$r \cdot \varphi_U(f) = \varphi_U(rf) \in \operatorname{im} \varphi_U.$$

Thus, $\operatorname{im} \varphi_U$ has a $\mathcal{O}_X(U)$-action by restricting from $\mathcal{G}(U)$, so we can give $\operatorname{coker} \varphi_U = \mathcal{G}(U)/\operatorname{im} \varphi_U$ a quotient module action. Explicitly, $r \in \mathcal{O}_X(U)$ and $[g] \in \operatorname{coker} \varphi_U$, we have

$$r \cdot [g] = [rg].$$

Note further that these maps assemble into making an $\mathcal{O}_X$-module by noting that any $V \subseteq U$ makes the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \operatorname{coker} \varphi_U \arrow[r] \arrow[d, "\operatorname{res}_{U,V} \times \operatorname{res}_{U,V}"'] & \operatorname{coker} \varphi_U \arrow[d, "\operatorname{res}_{U,V}"] & (r, [g]) \arrow[r, mapsto] \arrow[d, mapsto] & {[rg]} \arrow[d, mapsto] \\
\mathcal{O}_X(V) \times \operatorname{coker} \varphi_V \arrow[r] & \operatorname{coker} \varphi_V & (r|_V, [g|_V]) \arrow[r, mapsto] & {[(rg)|_V]}
\end{tikzcd}$$

commute. As an additional sanity check, we note that the projection map $\pi\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}} \varphi$ constructed in Lemma 1.146 is a morphism of $\mathcal{O}_X$-premodules: certainly we have a presheaf morphism, and we have a $\mathcal{O}_X$-premodule morphism because any open $U \subseteq X$ makes the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{G}(U) \arrow[r] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X(U)}, \pi_U)"'] & \mathcal{G}(U) \arrow[d, "\pi_U"] & (r, g) \arrow[r, mapsto] \arrow[d, mapsto] & rg \arrow[d, mapsto] \\
\mathcal{O}_X(U) \times \operatorname{coker} \varphi_U \arrow[r] & \operatorname{coker} \varphi_U & (r, [g]) \arrow[r, mapsto] & [rg]
\end{tikzcd}$$

commute.

We now show the universal property. Fix an $\mathcal{O}_X$-premodule $\mathcal{H}$ with a map $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $\psi \circ \varphi = 0$. Then we need a unique map $\overline{\psi}\colon \operatorname{coker}^{\mathrm{pre}} \varphi \to \mathcal{H}$ making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \arrow[ddr, bend left, "\psi"] & \\
\mathcal{Z} \arrow[r] \arrow[drr, bend right] & \operatorname{coker}^{\mathrm{pre}} \varphi \arrow[dr, dashed, "\overline{\psi}"] & \\
& & \mathcal{H}
\end{tikzcd}$$

commute. Note that Lemma 1.146 promises that there is at most one presheaf morphism $\overline{\psi}$ making this diagram commute, so we at least have uniqueness of an $\mathcal{O}_X$-premodule morphism making the diagram commute.

It remains to show existence. Well, Lemma 1.146 does promise that setting

$$\overline{\psi}_U([g]) \coloneqq \psi_U(g)$$

for any $g \in \mathcal{G}(U)$ will assemble into a well-defined presheaf morphism $\overline{\psi}\colon \operatorname{coker}^{\mathrm{pre}} \varphi \to \mathcal{H}$ such that $\psi = \overline{\psi} \circ \pi$. It remains to show that $\overline{\psi}$ is a morphism of $\mathcal{O}_X$-premodules, for which we note that any open $U \subseteq X$ makes the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \operatorname{coker} \varphi_U \arrow[r] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X(U)}, \overline{\psi}_U)"'] & \operatorname{coker} \varphi_U \arrow[d, "\overline{\psi}_U"] & (r, [g]) \arrow[r, mapsto] \arrow[d, mapsto] & {[rg]} \arrow[d, mapsto] \\
\mathcal{O}_X(U) \times \mathcal{H}(U) \arrow[r] & \mathcal{H}(U) & (r, \psi_U(g)) \arrow[r, mapsto] & r\psi_U(g) = \psi_U(rg)
\end{tikzcd}$$

commutes, where we are using the fact that ψ is already a morphism of $\mathcal{O}_X$-premodules. ■

And now we apply sheafification.

> **Proposition 4.25.** Fix a ringed space $(X, \mathcal{O}_X)$ and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. Then the cokernel sheaf $\operatorname{coker} \varphi$ of φ viewed as a presheaf is the cokernel in the category of $\mathcal{O}_X$-modules.

Proof. As in Lemma 1.147, we let $\pi^{\mathrm{pre}}\colon \mathcal{G} \to \operatorname{coker}^{\mathrm{pre}} \varphi$ denote the projection map and $\mathrm{sh}\colon \operatorname{coker}^{\mathrm{pre}} \varphi \to \operatorname{coker} \varphi$ be the sheafification map.

Now, we define $\pi \coloneqq \mathrm{sh} \circ \pi^{\mathrm{pre}}$ to be the canonical projection $\operatorname{coker} \varphi \to \mathcal{G}$. Note that sh and π^{pre} are both morphisms of $\mathcal{O}_X$-premodules by Lemma 4.21 and Lemma 4.24, respectively, so π is also a morphism of $\mathcal{O}_X$-premodules and thus a morphism of $\mathcal{O}_X$-modules because $\mathcal{G}$ and $\operatorname{coker} \varphi$ are both sheaves.

It remains to verify the universal property. Fix an $\mathcal{O}_X$-module $\mathcal{H}$ with a map $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $\psi \circ \varphi$.

Then we need a unique morphism $\overline{\psi}\colon \operatorname{coker}\varphi \to \mathcal{H}$ of $\mathcal{O}_X$-modules making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"'] \arrow[ddr, bend left, "\psi"] & \\
\mathcal{Z} \arrow[r] \arrow[drr, bend right] & \operatorname{coker}\varphi \arrow[dr, dashed, "\overline{\psi}"] & \\
 & & \mathcal{H}
\end{tikzcd}$$

commute. Well, Lemma 1.147 assures us that there is at most one sheaf morphism making this diagram commute, so there is at most one morphism of $\mathcal{O}_X$-modules making this diagram.

For existence, we note that Lemma 4.24 promises a morphism $\overline{\psi}^{\text{pre}}\colon \operatorname{coker}^{\text{pre}}\varphi \to \mathcal{H}$ of $\mathcal{O}_X$-premodules such that

$$\psi = \overline{\psi}^{\text{pre}} \circ \pi^{\text{pre}}.$$

However, $\mathcal{H}$ is a sheaf, so Proposition 4.23 assures us that there is a morphism $\overline{\psi}\colon \operatorname{coker}\varphi \to \mathcal{H}$ of $\mathcal{O}_X$-modules such that $\overline{\psi}^{\text{pre}} = \overline{\psi} \circ \mathrm{sh}$. Thus,

$$\psi = \overline{\psi}^{\text{pre}} \circ \pi^{\text{pre}} = \overline{\psi} \circ \mathrm{sh} \circ \pi^{\text{pre}} = \overline{\psi} \circ \pi,$$

which is what we wanted. ■

Corollary 4.26. Fix a ringed space $(X, \mathcal{O}_X)$ and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. The following are equivalent.

(a) φ is epic in the category of sheaves.

(b) φ is epic in the category of $\mathcal{O}_X$-modules.

Proof. This is similar to Corollary 4.20. By Lemma 1.148, we see that φ is epic in the category of $\mathcal{O}_X$-modules if and only if $\operatorname{coker}\varphi$ vanishes. However, the cokernel of φ is the same in the category of sheaves as in the category of $\mathcal{O}_X$-modules by Proposition 4.25, so $\operatorname{coker}\varphi$ vanishing is equivalent to φ being monic in the category of sheaves by Lemma 1.148. ■

4.1.6 $\operatorname{Mod}_{\mathcal{O}_X}$ Is Abelian

Because we can, we will go ahead and show that $\operatorname{Mod}_{\mathcal{O}_X}$ is abelian. We have two more checks.

Lemma 4.27. Fix a ringed space $(X, \mathcal{O}_X)$ and a monic morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. Then actually $\varphi\colon \mathcal{F} \to \mathcal{G}$ makes $\mathcal{F}$ the kernel of the cokernel $\pi\colon \mathcal{G} \to \operatorname{coker}\varphi$.

Proof. By Lemma 1.151, we know that the map $\varphi\colon \mathcal{F} \to \mathcal{G}$ identifies $\mathcal{F}$ as $\ker\pi$ in the category of sheaves. Now, by Proposition 4.19, we see that $\ker\pi$ in the category of sheaves is the same as in the category of $\mathcal{O}_X$-modules, so we conclude that $\varphi\colon \mathcal{F} \to \mathcal{G}$ remains a kernel in the category of $\mathcal{O}_X$-modules.

To be more explicit, we note that the universal property of the kernel $\ker\pi$ promises a unique sheaf morphism $\overline{\varphi}$ making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, dashed, "\overline{\varphi}"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right] & & \\
 & \ker\pi \arrow[r, "\iota"] \arrow[d] & \mathcal{G} \arrow[d, "\pi"] \\
 & \mathcal{Z} \arrow[r] & \operatorname{coker}\pi
\end{tikzcd}$$

commute. In fact, by Lemma 1.151, we see that $\overline{\varphi}$ is in fact a sheaf isomorphism because $\mathcal{F}$ is also a kernel for π, so we get to apply the usual universal property arguments.

But because φ is a morphism of $\mathcal{O}_X$-modules, we see that $\overline{\varphi}$ is also an $\mathcal{O}_X$-module isomorphism by Lemma 4.10. As such, $\overline{\varphi}\colon \mathcal{F} \cong \ker \pi$ as $\mathcal{O}_X$-modules, which makes $\mathcal{F}$ into a kernel of π in the category of $\mathcal{O}_X$-modules, where the needed inclusion map is $\iota \circ \overline{\varphi} = \varphi$. ■

Lemma 4.28. Fix a ringed space $(X, \mathcal{O}_X)$ and an epic morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. Then actually $\varphi\colon \mathcal{F} \to \mathcal{G}$ makes $\mathcal{G}$ the cokernel of the kernel $\iota\colon \ker \varphi \to \mathcal{F}$.

Proof. This is very similar to the previous proof. By Lemma 1.152, we know that the map $\varphi\colon \mathcal{F} \to \mathcal{G}$ identifies $\mathcal{G}$ as $\operatorname{coker} \iota$ in the category of sheaves. Now, by Proposition 4.25, we see that $\operatorname{coker} \iota$ in the category of sheaves is the same as in the category of $\mathcal{O}_X$-modules, so we conclude that $\varphi\colon \mathcal{F} \to \mathcal{G}$ remains a kernel in the category of $\mathcal{O}_X$-modules.

To be more explicit, we note that the universal property of the kernel $\operatorname{coker} \iota$ promises a unique sheaf morphism $\overline{\varphi}$ making the diagram

$$\begin{tikzcd}
\ker \varphi \arrow[r, "\iota"] \arrow[d] & \mathcal{F} \arrow[d, "\pi"'] \arrow[dd, bend left, "\varphi"] \\
\mathcal{Z} \arrow[r] \arrow[dr, bend right] & \operatorname{coker} \iota \arrow[d, dashed, "\overline{\varphi}"] \\
& \mathcal{G}
\end{tikzcd}$$

commute. In fact, by Lemma 1.151, we see that $\overline{\varphi}$ is in fact a sheaf isomorphism because $\mathcal{G}$ is also a cokernel for ι, so we get to apply the usual universal property arguments.

But because φ is a morphism of $\mathcal{O}_X$-modules, we see that $\overline{\varphi}$ is also an $\mathcal{O}_X$-module isomorphism by Lemma 4.10. As such, $\overline{\varphi}\colon \operatorname{coker} \iota \cong \operatorname{coker} \varphi$ as $\mathcal{O}_X$-modules, which makes $\mathcal{G}$ into a cokernel of ι in the category of $\mathcal{O}_X$-modules, where the needed inclusion map is $\overline{\varphi} \circ \pi = \varphi$. ■

Theorem 4.29. Fix a ringed space $(X, \mathcal{O}_X)$. Then the category of $\mathcal{O}_X$-modules is abelian.

Proof. The category is additive by Corollary 4.18. We have kernels by Proposition 4.19 and cokernels by Proposition 4.25, and they cohere appropriately by Lemma 4.27 and Lemma 4.28. ■

Now that our category is abelian, we can discuss exactness.

Corollary 4.30. Fix a ringed space $(X, \mathcal{O}_X)$. Then a sequence of $\mathcal{O}_X$-modules

$$\mathcal{F} \overset{\alpha}{\to} \mathcal{G} \overset{\beta}{\to} \mathcal{H}$$

is exact at $\mathcal{G}$ in the category of $\mathcal{O}_X$-modules if and only if it is exact at $\mathcal{G}$ in the category of sheaves of abelian groups.

Proof. The complex is exact at $\mathcal{G}$ if and only if the image map $\mathcal{F} \to \operatorname{coker} \ker \alpha$ serves as a kernel of the map $\mathcal{G} \to \mathcal{H}$. However, cokernels and kernels are the exact same in the category of $\mathcal{O}_X$-modules as in the category of sheaves of abelian groups, as discussed in Proposition 4.19 and Proposition 4.25, so these two exactness are equivalent. ■

4.1.7 Direct Sums

We will want to be able to discuss "free $\mathcal{O}_X$-modules" and relatives later on, so for these we must discuss direct sums.

Lemma 4.31. Fix a ringed space $(X, \mathcal{O}_X)$ and some collection $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ of $\mathcal{O}_X$-premodules. Then the data

$$\mathcal{F}(U) := \bigoplus_{\alpha\in\lambda} \mathcal{F}_\alpha(U)$$

assembles into an $\mathcal{O}_X$-premodule. Further, the natural inclusions $\iota_\alpha \colon \mathcal{F}_\alpha \to \mathcal{F}$ are $\mathcal{O}_X$-premodule morphisms.

Proof. Note that each $\mathcal{F}_\alpha(U)$ is an $\mathcal{O}_X(U)$-module, so taking the direct sum of modules tells us that

$$\mathcal{F}(U) := \bigoplus_{\alpha\in\lambda} \mathcal{F}_\alpha(U)$$

remains an $\mathcal{O}_X(U)$-module as a submodule of the product $\prod_{\alpha\in\lambda} \mathcal{F}_\alpha(U)$.

To see that we form a presheaf, fix some open $U \subseteq X$. Then we define our restriction maps for any $(f_\alpha)_{\alpha\in\lambda} \in \mathcal{F}(U)$ and $V \subseteq U$ by

$$(f_\alpha)_{\alpha\in\lambda}|_V := (f_\alpha|_V)_{\alpha\in\lambda}.$$

Here are our checks.

- Well-defined: note that $f_\alpha = 0$ for all but finitely many α, so $f_\alpha|_V$ still vanishes for all but finitely many α, so we do indeed have $(f_\alpha|_V)_{\alpha\in\lambda} \in \mathcal{F}(V)$.
- Identity: if $U = V$, we see $(f_\alpha)_\alpha|_V = (f_\alpha|_V)_\alpha = (f_\alpha)_\alpha$.
- Functoriality: if $W \subseteq V \subseteq U$, then $(f_\alpha)_\alpha|_V|_W = (f_\alpha|_V|_W)_\alpha = (f_\alpha|_W)_\alpha = (f_\alpha)_\alpha|_W$.
- $\mathcal{O}_X$-module: if $r \in \mathcal{O}_X(U)$ and $(f_\alpha)_\alpha \in \mathcal{F}(U)$, then any $V \subseteq U$ will give

$$(r \cdot (f_\alpha)_{\alpha\in\lambda})|_V = (rf_\alpha)_{\alpha\in\lambda}|_V = \big((rf_\alpha)|_V\big)_{\alpha\in\lambda} = \big(r|_V \cdot f_\alpha|_V\big)_{\alpha\in\lambda} = r|_V \cdot (f_\alpha)_{\alpha\in\lambda}|_V.$$

The above checks complete the construction of $\mathcal{F}$.

It remains to discuss the inclusions $\iota_\beta \colon \mathcal{F}_\beta \to \mathcal{F}$ for some fixed $\beta \in \lambda$. Well, for any open $U \subseteq X$, we note that we have a map

$$\mathcal{F}_\beta(U) \to \bigoplus_{\alpha\in\lambda} \mathcal{F}_\alpha(U)$$

induced by the direct sum in the category of $\mathcal{O}_X(U)$-modules as $f \mapsto (1_{\alpha=\beta} f)_{\alpha\in\lambda}$. This is a morphism of presheaves because any open $V \subseteq U$ and $f \in \mathcal{F}_\beta(U)$ give

$$(\iota_\beta)_U(f)|_V = (1_{\alpha=\beta} f)_{\alpha\in\lambda}|_V = (1_{\alpha=\beta} f|_V)_{\alpha\in\lambda} = (\iota_\beta)_V(f|_V).$$

And this is a morphism of $\mathcal{O}_X$-premodules because any $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}_\beta(U)$ will give

$$r\iota_\beta(f) = r \cdot (1_{\alpha=\beta} f)_{\alpha\in\lambda} = (1_{\alpha=\beta} \cdot rf)_{\alpha\in\lambda} = \iota_\beta(rf).$$

This finishes. ■

Because we used sheafification, we will want to understand direct sums at stalks.

Lemma 4.32. Fix a ringed space $(X, \mathcal{O}_X)$ and some collection $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$ of $\mathcal{O}_X$-modules, and let $\mathcal{F}$ be their direct sum. Then there are natural inclusions $\iota_\alpha \colon \mathcal{F}_\alpha \to \mathcal{F}$ yielding the isomorphism of $\mathcal{O}_{X,p}$-modules

$$\bigoplus_{\alpha\in\lambda} \mathcal{F}_{\alpha,p} \to \mathcal{F}_p$$

at each point $p \in X$.

Proof. To begin, we use Lemma 4.31 to define an $\mathcal{O}_X$-premodule by

$$\mathcal{F}^{\text{pre}}(U) \coloneqq \bigoplus_{\alpha\in\lambda} \mathcal{F}_\alpha(U),$$

and we let $\operatorname{sh}\colon \mathcal{F}^{\text{pre}} \to \mathcal{F}$ be the sheafification map. Further, we let $\iota_\alpha^{\text{pre}} \colon \mathcal{F}_\alpha \to \mathcal{F}^{\text{pre}}$ be the inclusions of Lemma 4.31, which lets us set $\iota_\alpha \coloneqq \operatorname{sh} \circ \iota_\alpha^{\text{pre}}$.

Thus, taking stalks, we have induced morphisms $\iota_{\alpha,p} \colon \mathcal{F}_{\alpha,p} \to \mathcal{F}_p$ of $\mathcal{O}_{X,p}$-modules at each p. Namely, $(\iota_\alpha^{\text{pre}})_p$ is a morphism of $\mathcal{O}_{X,p}$-modules by Lemma 4.11, and sh_p is an isomorphism of $\mathcal{O}_{X,p}$-modules by Remark 4.22. So using the direct sum in the category of $\mathcal{O}_{X,p}$-modules, we get a morphism

$$\bigoplus_{\alpha\in\lambda} \mathcal{F}_{\alpha,p} \stackrel{\iota}{\to} \mathcal{F}^{\text{pre}}_{\alpha,p} \stackrel{\operatorname{sh}_p}{\to} \mathcal{F}_p.$$

In particular, it remains to show that the left morphism here is an isomorphism. It's already a morphism of $\mathcal{O}_{X,p}$-modules, so we really just need to show bijectivity.

- Injective: suppose that we have a collection of germs $g_\alpha \coloneqq [(V_\alpha, f_\alpha)] \in \mathcal{F}_{\alpha,p}$ which makes an element of $\bigoplus_{\alpha\in\lambda} \mathcal{F}_{\alpha,p}$ going to 0 under ι. To begin, we note that all but finitely many of the $g_\alpha \in \mathcal{F}_{\alpha,p}$ are allowed to be nonzero, so define
$$V \coloneqq \bigcap_{\substack{\alpha\in\lambda \\ g_\alpha\neq 0}} V_\alpha,$$
which is still contains p and remains open as the finite union of opens. Now, we restrict all the $[(V_\alpha, f_\alpha)]$ to V, where we see that $g_\alpha = 0$ means that $[(V_\alpha, f_\alpha]) = [(V, 0)]$, so we may now just assume that $f_\alpha = 0$ for all but finitely many α.

  In total, we unravel our definitions to give
$$0 = \iota\big((g_\alpha)_{\alpha\in\lambda}\big) = \sum_{\beta\in\lambda} \iota^{\text{pre}}_{\beta,p}([(V, f_\beta)]) = \sum_{\beta\in\lambda} [(V, \iota^{\text{pre}}_{\beta,V} f_\beta)] = [(V, (f_\alpha)_{\alpha\in\lambda})],$$
where these sums are legal because they are (in practice) finite with only finitely many nonzero terms. Thus, we are told that there is some open $W \subseteq V$ containing p such that
$$(f_\alpha)_{\alpha\in\lambda}|_W = 0,$$
which now forces $f_\alpha|_W = 0$ for each α, so $[(V, f_\alpha)] = 0$ for each α, so our original tuple of germs $(g_\alpha)_\alpha$ also vanishes.

- Surjective: Fix a germ $[(V, (f_\alpha)_\alpha)] \in \mathcal{F}_p^{\text{pre}}$. Notably, only finitely many of the f_α are nonzero, so we can expand
$$[(V, (f_\alpha)_\alpha)] = \sum_{\beta\in\lambda} [(V, (1_{\alpha=\beta} f_\alpha)_\alpha)] = \sum_{\beta\in\lambda} [(V, \iota_{\beta,V} f_\beta)] = \sum_{\alpha\in\lambda} \iota_{\beta,p}([(V, f_\beta)]) = \iota\big(([(V, f_\alpha)])_{\alpha\in\lambda}\big),$$
which is the needed element in the pre-image. ■

4.1.8 Tensor Products

The tensor product is an important enough notion for modules that we will also want for our $\mathcal{O}_X$-modules. For example, the tensor product will behave well with modules which are locally $\mathcal{O}_X$.

Lemma 4.33. Fix a ringed space $(X, \mathcal{O}_X)$ and two $\mathcal{O}_X$-premodules $\mathcal{F}$ and $\mathcal{G}$. Then the data of

$$\mathcal{T}(U) \coloneqq \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

assembles into an $\mathcal{O}_X$-premodule.

Proof. To begin, we note that $\mathcal{F}(U)$ and $\mathcal{G}(U)$ are both $\mathcal{O}_X(U)$-modules, so the tensor product at least makes sense and endows $\mathcal{T}(U)$ with an $\mathcal{O}_X(U)$-action. To define our restriction maps, we claim that opens $V \subseteq U$ make the map

$$\begin{array}{ccc} \mathcal{F}(U) \times \mathcal{G}(U) & \to & \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V) \\ (f, g) & \mapsto & (f|_V) \otimes (g|_V) \end{array}$$

$\mathcal{O}_X(U)$-bilinear: indeed, for $f_1, f_2 \in \mathcal{F}(U)$ and $g_1, g_2 \in \mathcal{G}(U)$ and $r_1, r_2, s_1, s_2 \in \mathcal{O}_X(U)$, we compute

$$\begin{aligned} &(r_1 f_1 + r_2 f_2)|_V \otimes (s_1 g_1 + s_2 g_2)|_V \\ &= (r_1|V \cdot f_1|_V + r_2|_V \cdot f_2|_V) \otimes (s_1|_V \cdot g_1|_V + s_2|_V \cdot g_2|_V) \\ &= (r_1 s_1)|_V \cdot (f_1|_V \otimes g_1|_V) + (r_1 s_2)|_V \cdot (f_1|_V \otimes g_2|_V) \\ &\qquad + (r_2 s_1)|_V \cdot (f_2|_V \otimes g_1|_V) + (r_2 s_2)|_V \cdot (f_2|_V \otimes g_2|_V), \end{aligned}$$

where we have used the fact that restriction is a morphism of $\mathcal{O}_X(V)$-modules and a ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Thus, we induce a restriction map $\operatorname{res}_{U,V} \colon \mathcal{T}(U) \to \mathcal{T}(V)$ by extending $f \otimes g \mapsto (f|_V) \otimes (g|_V)$. To see that this assembles into an $\mathcal{O}_X$-premodule, we check that any opens $V \subseteq U$ makes

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{T}(U) \arrow[r] \arrow[d, "{(\operatorname{res}_{U,V}, \operatorname{res}_{U,V})}"'] & \mathcal{T}(U) \arrow[d, "\operatorname{res}_{U,V}"] & (r, f \otimes g) \arrow[r, mapsto] \arrow[d, mapsto] & r(f \otimes g) \arrow[d, mapsto] \\ \mathcal{O}_X(V) \times \mathcal{T}(V) \arrow[r] & \mathcal{T}(V) & (r|_V, f|_V \otimes g|_V) \arrow[r, mapsto] & r|_V(f|_V \otimes g|_V) \end{tikzcd}$$

commute, which follows by extending the computation in the right square additively. ■

So we have the following definition.

Definition 4.34 (Tensor product module)**.** Fix a ringed space $(X, \mathcal{O}_X)$ and some $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$. Then the *tensor product module* $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of the $\mathcal{O}_X$-premodule coming from the data

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

4.1.9 Sheaf Theory for Modules

Let's give some more sheaf-theoretic constructions. As usual, the direct image is good.

Lemma 4.35. Fix a morphism $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module.

Proof. The point is to use the sheaf morphism $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ to define our $\mathcal{O}_Y$-action. Indeed, for an open subset $U \subseteq Y$, we need to define an action of $\mathcal{O}_Y(U)$ on $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$. Well, we have a ring map

$$f_U^\sharp \colon \mathcal{O}_Y(U) \to \mathcal{O}_X\left(f^{-1}U\right),$$

so the action homomorphism $\mathcal{O}_X\left(f^{-1}U\right) \to \operatorname{Aut}\left(\mathcal{F}(f^{-1}U)\right)$ can be pre-composed with $f^\sharp$ to exhibit an action homomorphism $\mathcal{O}_Y(U) \to \operatorname{Aut}(f_*\mathcal{F}(U))$. Explicitly, our action will send $r \in \mathcal{O}_Y(U)$ and $m \in f_*\mathcal{F}(U)$ to $r \cdot m := f_U^\sharp(r) \cdot m$.

It remains to see that this creates an $\mathcal{O}_Y$-module. Well, given open subsets $V \subseteq U \subseteq X$, some $r \in \mathcal{O}_Y(U)$ and some $m \in f_*\mathcal{F}(U)$, we compute

$$\begin{aligned}(r \cdot m)|_V &= (f_U^\sharp(r) \cdot m)|_V \\ &= f_U^\sharp(r)|_V \cdot m|_V \\ &= f_V^\sharp(r) \cdot m|_V,\end{aligned}$$

which is what we wanted. ■

Going the other way requires some care. In particular, we won't want to just look at $f^{-1}\mathcal{F}$ to pull back because it's not clear how to give the sheaf an $\mathcal{O}_X$-module give an $\mathcal{O}_Y$-action.

Definition 4.36 (Direct image module)**.** Fix a morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{O}_X$ remains an $\mathcal{O}_X$-module, so the *direct image module* $f_*\mathcal{F}$ will become an $\mathcal{O}_Y$-module through $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

However, pullback sheaves do change from being "just" $f^{-1}\mathcal{F}$. Nonetheless, let's pick up some facts about f^{-1}.

Lemma 4.37. Fix a ringed space $(X, \mathcal{O}_X)$. Then, for any open $U \subseteq X$, the restriction taking an $\mathcal{O}_X$-module $\mathcal{F}$ to $\mathcal{F}|_U$ defines the actions of a functor $\cdot|_U\colon \operatorname{Mod}_{\mathcal{O}_X} \to \operatorname{Mod}_{\mathcal{O}_X|_U}$ on objects.

Proof. We already have a functor $\operatorname{Sh}_X \to \operatorname{Sh}_U$ by Lemma 1.172, using the inclusion $U \hookrightarrow X$. Thus, we just need to check that all the mappings preserve $\mathcal{O}_X$-structure.

- Given an $\mathcal{O}_X$-module $\mathcal{F}$, we give $\mathcal{F}|_U$ the structure of an $\mathcal{O}_X|_U$-module. Well, for any open $V \subseteq U$, we note that $\mathcal{F}|_U = \mathcal{F}(V)$ already has the structure of an $\mathcal{O}_X(V) = \mathcal{O}_X|_U(V)$-module. To check that this gives $\mathcal{O}_X$-module, we pick up $f \in \mathcal{F}|_U(V)$ and $r \in \mathcal{O}_X(V)$ and note that any $V' \subseteq V$ will have
$$(rf)|_{V'} = r|_{V'} \cdot f|_{V'}.$$
- Given a morphism of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{F} \to \mathcal{G}$, we need to show that $\varphi|_U\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a morphism of $\mathcal{O}_X$-modules. Well, we already have a sheaf morphism. Then for any open $V \subseteq U$ and $f \in \mathcal{F}|_U(V)$, we compute
$$(\varphi|_U)_V(rf) = \varphi_V(rf) = r\varphi_V(f) = r(\varphi|_U)_V(f).$$

We now note that our functoriality checks are in fact included in Lemma 1.172, by our definitions, so we are done. ■

Now here is our pullback module.

Definition 4.38 (Pullback module)**.** Given an $\mathcal{O}_Y$-module $\mathcal{G}$, we see $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. To make this an $\mathcal{O}_X$-module, we would like to use the morphism $f^\flat\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, but this only makes $\mathcal{O}_X$ into an $f^{-1}\mathcal{O}_Y$-module, so we define
$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$
to be the *pullback module* $\mathcal{O}_X$-module.

Remark 4.39. Analogously to Proposition 1.175, we have the adjunction

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \simeq \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

I do not expect to use this neither result nor f^*, so I will not run any checks.

Example 4.40. Given a ring map $\varphi\colon B \to A$, we induce a scheme map $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$. Now, there is a functor from A-modules M to $\mathcal{O}_X$-modules $\widetilde{M}$. Then we see that we can send such a module $\widetilde{M}$ back to a B-module as $f_*(\widetilde{M}) = \widetilde{M}$ by taking global sections and using φ, and we can send a B-module N to $f^*(\widetilde{N}) = \widetilde{N \otimes_B A}$ given the obvious A-module structure.

4.1.10 Sheaves from Modules

To make the above example precise, we need to define $\widetilde{M}$ as an $\mathcal{O}_{\operatorname{Spec} A}$-module.

Lemma 4.41. Fix an affine scheme X, and set $A \coloneqq \mathcal{O}_X(X)$. Given an A-module M, the data given by

$$\widetilde{M}(X_f) \coloneqq S(D(f))^{-1}M \simeq M \otimes_A \mathcal{O}_X(X_f)$$

for $f \in A$ will assemble into a sheaf on the base and thus an $\mathcal{O}_X$-module.

Remark 4.42. The isomorphism $S(D(f))^{-1}A \simeq A_f$ of Remark 1.88 tells us that

$$\begin{array}{ccccccc} \widetilde{M}(X_f) = S(D(f))^{-1}M & \simeq & M \otimes_A \mathcal{O}_X(X_f)^{-1}A & \simeq & M \otimes_A A_f & \simeq & M_f \\ m/g & \mapsto & m \otimes 1/g & \mapsto & m \otimes g^{-1} & \mapsto & g^{-1}m \end{array}$$

so that the naturality of these isomorphisms means that we tend to think of $\widetilde{M}(X_f)$ as just M_f in practice.

Proof. Recall that $S(D(f)) = \{g \in A : V(\{g\}) \subseteq (\operatorname{Spec} A) \setminus D(f)\}$, which are the functions on A which do not vanish on $D(f)$. Thus, $D(f') \subseteq D(f)$ implies $S(D(f)) \subseteq S(D(f'))$, so we induce a natural localization map

$$\operatorname{res}_{D(f),D(f')}\colon S(D(f))^{-1}M \to S(D(f'))^{-1}M.$$

To see that we've made a sheaf on the base, reread the proof of Lemma 1.90 and replace all the numerators from elements of A to elements of M. We won't write this out proof here because it really is just a matter of changing all the variable names.

Thus, we have induced a sheaf on the base, which extends uniquely to a sheaf by Proposition 1.80. It remains to show that $\widetilde{M}$ is an $\mathcal{O}_X$-module. To begin, we need to define our action $\cdot\colon \mathcal{O}_X \times \widetilde{M} \to \widetilde{M}$. By Lemma 1.79, it suffices to exhibit a morphism of sheaves on the base, for which we need to give morphisms

$$\mathcal{O}_X(X_f) \times \widetilde{M}(X_f) \to \widetilde{M}(X_f).$$

Well, we just have to note that $\mathcal{O}_X(X_f)$ has a natural action on $\widetilde{M}(X_f) \simeq M \otimes_A \mathcal{O}_X(X_f)$ by multiplying on the right coordinate. Translating this back over to $S(D(f))^{-1}M$, our natural action takes $a/g \in \mathcal{O}_X(X_f) \simeq S(D(f))^{-1}A$ and $m/h \in \widetilde{M}(D(f))$ and spits out $a/g \cdot m/h = (am)/(gh)$.

To see that we assemble into a morphism of sheaves on the base, we fix distinguished open sets $D(f') \subseteq$

$D(f)$ and note that the diagram

$$\begin{tikzcd}
\mathcal{O}_X(X_f) \times \widetilde{M}(X_f) \arrow[r] \arrow[d] & \widetilde{M}(X_f) \arrow[d] & (a/g, m/h) \arrow[r, mapsto] \arrow[d, mapsto] & (am)/(gh) \arrow[d, mapsto] \\
\mathcal{O}_X(X_f) \times \widetilde{M}(D(f')) \arrow[r] & \widetilde{M}(X_f) & (a/g, m/h) \arrow[r, mapsto] & (am)/(gh)
\end{tikzcd}$$

commutes. So we have induced a scalar multiplication which is a sheaf morphism, so we have made an $\mathcal{O}_X$-module. ■

Let's codify our notation.

Notation 4.43. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Given an A-module M, we define the $\mathcal{O}_X$-module $\widetilde{M}$ from the sheaf on the distinguished base defined by

$$\widetilde{M}(X_f) := S(D(f))^{-1}M,$$

with the usual restrictions. Note we ran our checks in Lemma 4.41.

Example 4.44. From the above example, we can pass A through to see that $\widetilde{A} \simeq \mathcal{O}_X$ is an $\mathcal{O}_X$-module. Indeed, let $\varphi\colon X \cong \operatorname{Spec} A$ be the canonical isomorphism from Corollary 2.28. Then, for each affine open subscheme $X_f \subseteq X$, we see

$$\widetilde{A}(X_f) = S(D(f))^{-1}A = \mathcal{O}_{\operatorname{Spec} A}(D(f)) \overset{\varphi^\sharp_{D(f)}}{\cong} \mathcal{O}_X(X_f)$$

by definition of $\varphi^\sharp$. Notably, this is functorial in X_f because $\varphi^\sharp$ is.

Here is a quick sanity check.

Lemma 4.45. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Let $\varphi\colon X \to \operatorname{Spec} A$ be the canonical isomorphism. Given an A-module M and a point $x \in X$, we have a natural isomorphism $M_{\varphi(x)} \simeq \widetilde{M}_x$.

Proof. We imitate the proof of Lemma 1.102. We begin by exhibiting the natural map $\alpha\colon M_{\varphi(x)} \to \widetilde{M}_x$. Well, if $f \notin \varphi(x)$, then we see $\varphi(x) \in D(f)$, so $x \in X_f$ by construction of φ. Thus, we induce a map

$$\alpha\colon m/f \mapsto \left[\left(X_f, \frac{m}{f}\right)\right].$$

Here are the checks on α.

- Well-defined: if $m/f = m'/f'$ in $M_{\varphi(x)}$ for $f, f' \in \varphi(x)$, then there exists $f'' \notin \varphi(x)$ such that $f''f'm = f''fm'$. Thus, in $\widetilde{M}(X_{f''})$, we see that
$$\frac{m}{f} = \frac{m'}{f'}$$
because $f'' \in S(D(f''))$. Thus, we see that
$$\left[\left(X_f, \frac{m}{f}\right)\right] = \left[\left(X_{f''}, \frac{m}{f}\right)\right] = \left[\left(X_{f''}, \frac{m'}{f'}\right)\right] = \left[\left(X_{f'}, \frac{m'}{f'}\right)\right].$$
- Injective: suppose $[(X_f, m/f)] = [(X_{f'}, m'/f')]$ in $\widetilde{M}_x$. Using the distinguished base, we are promised $X_{f''}$ such that $\frac{m}{f} = \frac{m'}{f'}$ in $\widetilde{M}(X_{f''})$, which means that there is some $g \in S(D(f''))$ such that
$$gf'm = gfm'.$$

Notably, $g \in \varphi(x)$ would imply that $f'' \in \varphi(x)$, which is false, so we must have $g \notin \varphi(x)$. Thus, $\frac{m}{f} = \frac{m'}{f'}$ in $M_{\varphi(x)}$.

- Surjective: using the distinguished base, we can write any germ in $\widetilde{M}_{\varphi(x)}$ in the form $[(X_f, m/g)]$ where $x \notin X_f$ and $g \in S(D(f))$. Notably, $x \notin X_f$, then $\varphi(x) \notin D(f)$ by the construction of φ, so $f \notin \varphi(x)$.

  Further, because $g \in S(D(f))$, then it follows again that $g \notin \varphi(x)$, so $\varphi(x) \in D(g)$, so $x \in X_g$ as usual, and we note that $g \in S(D(f))$ implies that $D(g) \subseteq D(f)$, so $X_g \subseteq X_f$ by construction of φ. The point of all this is that we can restrict m/g to $X_g \subseteq X_f$ as

$$[(X_f, m/g)] = [(X_g, m/g)] = \alpha(m/g),$$

  which is what we wanted. ■

Remark 4.46. The above proof shows that $\widetilde{M}_x$ has the natural structure of an $\mathcal{O}_{X,x}$-module from $M_{\varphi(x)}$.

4.1.11 Geometry Is Opposite Algebra, Again

As in Theorem 2.18, we are going to build towards the following result.

Theorem 4.47. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. The functor from A-modules to $\mathcal{O}_X$-modules taking M to $\widetilde{M}$ is exact and fully faithful. This equivalence also respects $\oplus$ and $\otimes$.

Sketch. Exactness is checked at stalks. Being fully faithful is approximately as hard as Theorem 2.18, where the same machinery approximately works. ■

In particular, just as before, there is an adjunction lying in the background. Before doing anything, let's run the usual checks. Here are our functors.

Lemma 4.48. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. The map sending an A-module M to an $\mathcal{O}_X$-module $\widetilde{M}$ defines the action of a functor $\widetilde{\cdot}\colon \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$ on objects.

Proof. Fix an A-module homomorphism $\varphi\colon M \to N$, and we need to exhibit the corresponding $\mathcal{O}_X$-module morphism $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$. Well, by Lemma 1.79, it suffices to exhibit a morphism on the distinguished base. For this, we define the map $\widetilde{\varphi}$ on some $X_f \subseteq X$ by the diagram

$$\begin{tikzcd}
\widetilde{M}(X_f) \arrow[d, dashed, "\widetilde{\varphi}_{D(f)}"'] & \simeq & M \otimes_A S(D(f))^{-1}A \arrow[d] & m/g \arrow[r, mapsto] \arrow[d, dashed, mapsto] & m \otimes 1/g \arrow[d, mapsto] \\
\widetilde{N}(X_f) & \simeq & N \otimes_A S(D(f))^{-1}A & \varphi(m)/g \arrow[r, mapsto] & \varphi(m) \otimes 1/g
\end{tikzcd}$$

where we note that the map on the right is a homomorphism because it's just $\varphi \otimes_A S(D(f))^{-1}A$. To show that this assembles into a morphism of sheaves on the base, suppose $D(f') \subseteq D(f) \subseteq X$, and we note that the diagram

$$\begin{tikzcd}
\widetilde{M}(X_f) \arrow[r, "\widetilde{\varphi}_{X_f}"] \arrow[d, "\mathrm{res}_{X_f, X_{f'}}"'] & \widetilde{N}(X_f) \arrow[d, "\mathrm{res}_{X_f, X_{f'}}"] & m/g \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(m)/g \arrow[d, mapsto] \\
\widetilde{M}(X_{f'}) \arrow[r, "\widetilde{\varphi}_{X_{f'}}"] & \widetilde{N}(X_{f'}) & m/g \arrow[r, mapsto] & \varphi(m)/g
\end{tikzcd}$$

commutes. Lastly, to see that we have created a morphism of $\mathcal{O}_X$-modules, we check that the diagram

$$\begin{tikzcd} \mathcal{O}_X(X_f) \times \widetilde{M}(X_f) \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{O}_X(X_f)} \times \widetilde{\varphi}_{X_f}"'] & \widetilde{M}(X_f) \arrow[d, "\varphi_{X_f}"] \\ \mathcal{O}_X(X_f) \times \widetilde{N}(X_f) \arrow[r] & \widetilde{N}(X_f) \end{tikzcd} \qquad \begin{tikzcd} (a/g, m/h) \arrow[r, mapsto] \arrow[d, mapsto] & (am)/(gh) \arrow[d, mapsto] \\ (a/g, \varphi(m)/h) \arrow[r, mapsto] & \varphi(am)/(gh) \end{tikzcd}$$

commutes.

We now check the functoriality axioms.

- Identity: we need to show $\widetilde{\mathrm{id}_M} = \mathrm{id}_{\widetilde{M}}$. As usual, Lemma 1.79 tells us that we may check this on the distinguished base. Then, for any $m/g \in \widetilde{M}(X_f)$, we compute

$$(\widetilde{\mathrm{id}_M})_{X_f}(m/g) = \mathrm{id}_M(m)/g = m/g = \mathrm{id}_{\widetilde{M}(X_f)}(m/g) = (\mathrm{id}_{\widetilde{M}})_{\widetilde{M}(X_f)}(m/g).$$

- Functoriality: fix morphism $\varphi\colon M_1 \to M_2$ and $\psi\colon M_2 \to M_3$, and we want to show that $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$. By Lemma 1.79, we may check this on the distinguished base. Well, for any $m/g \in \widetilde{M}_1(X_f)$, we compute

$$\begin{aligned} (\widetilde{\psi \circ \varphi})_{X_f}(m/g) &= (\psi \circ \varphi)(m)/g \\ &= \psi(\varphi(m))/g \\ &= \widetilde{\psi}_{X_f}((\varphi(m))/g) \\ &= (\widetilde{\psi}_{X_f} \circ \widetilde{\varphi}_{X_f})(m/g) \\ &= (\widetilde{\psi} \circ \widetilde{\varphi})_{X_f}(m/g), \end{aligned}$$

which is what wanted. ∎

Lemma 4.49. Fix a ringed space $(X, \mathcal{O}_X)$, and set $A := \mathcal{O}_X(X)$. Then, given an open subset $U \subseteq X$ the map sending an $\mathcal{O}_X$-module $\mathcal{F}$ to $\mathcal{F}(U)$ defines the action of a functor $\mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_X(U)}$ on objects. If $U = X$, we call this functor Γ.

Proof. To define our functor on morphisms, we will just send $\varphi\colon \mathcal{F} \to \mathcal{G}$ to the morphism $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$. Notably, φ_U is a morphism of $\mathcal{O}_X(U)$-modules by definition of an $\mathcal{O}_X$-module homomorphism.

We now check our functoriality axioms.

- Identity: note that $\mathrm{id}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}$ gets sent to the identity map $(\mathrm{id}_{\mathcal{F}})_U\colon \mathcal{F}(U) \to \mathcal{F}(U)$ by definition of this identity map.
- Functoriality: given $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$, we note $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$ by definition of our composition. ∎

Next up we exhibit some natural maps.

Lemma 4.50. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. We exhibit a natural transformation $\varepsilon_\bullet\colon \widetilde{\cdot} \circ \Gamma \Rightarrow \mathrm{id}_{\mathrm{Mod}_{\mathcal{O}_X}}$.

Proof. Fix an $\mathcal{O}_X$-module $\mathcal{F}$ so that we want to exhibit a map $\varepsilon_{\mathcal{F}}\colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$. As usual, Lemma 1.79 lets us define this morphism on the distinguished base. To begin, we acknowledge that we are looking for a map of $\mathcal{O}_X(X_f)$-modules

$$S(D(f))^{-1}\mathcal{F}(X) \to \mathcal{F}(X_f).$$

Well, we will simply define this map by

$$(\varepsilon_{\mathcal{F}})_{X_f}\colon \frac{s}{g} \mapsto \frac{1}{g}(s|_{X_f}),$$

where the right-hand makes sense because $\mathcal{F}(X_f)$ is an $\mathcal{O}_X(X_f)$-module. Here are the checks.

- Well-defined: if $s_1/g_1 = s_2/g_2$, then there is some $g \in S(D(f))$ such that $gg_2s_1 = gg_1s_2$, so the fact $\mathcal{F}$ is an $\mathcal{O}_X$-module tells us
$$gg_2|_{X_f} \cdot s_1|_{X_f} = gg_1|_{X_f} \cdot s_2|_{X_f}$$
which rearranges into $\frac{1}{g_1}(s_1|_{X_f}) = \frac{1}{g_2}(s_2|_{X_f})$.
- Homomorphic: given $a/g, a'/g' \in S(D(f))^{-1}A$ and $m/h, m'/h' \in S(D(f))^{-1}A$, we compute
$$\begin{aligned}(\varepsilon_{\mathcal{F}})_{X_f}\left(\frac{a}{g}\cdot\frac{m}{h}+\frac{a'}{g'}\frac{m'}{h'}\right) &= (\varepsilon_{\mathcal{F}})_{X_f}\left(\frac{g'h'am+gha'm'}{ghg'h'}\right)\\ &= \frac{1}{ghg'h'}(g'h'am|_{X_f}+gha'm'|_{X_f})\\ &= \frac{a}{g}\cdot\frac{1}{h}(m|_{X_f})+\frac{a'}{g'}\cdot\frac{1}{h'}(m'|_{X_f})\\ &= \frac{a}{g}\cdot(\varepsilon_{\mathcal{F}})_{X_f}(m/h)+\frac{a'}{g'}\cdot(\varepsilon_{\mathcal{F}})_{X_f}(m'/h').\end{aligned}$$
- Morphism of sheaves on a base: given $X_{f'} \subseteq X_f$, we see that the diagram
$$\begin{tikzcd} S(D(f))^{-1}\mathcal{F}(X) \arrow[r, "\varepsilon_{\mathcal{F}}"] \arrow[d, "\operatorname{res}_{X_f,X_{f'}}"'] & \mathcal{F}(X_f) \arrow[d, "\operatorname{res}_{X_f,X_{f'}}"] \\ S(D(f'))^{-1}\mathcal{F}(X) \arrow[r, "\varepsilon_{\mathcal{F}}"] & \mathcal{F}(X_{f'}) \end{tikzcd} \qquad \begin{tikzcd} m/g \arrow[r, mapsto] \arrow[d, mapsto] & \frac{1}{g}m|_{X_f} \arrow[d, mapsto] \\ m|_{X_{f'}}/g \arrow[r, mapsto] & \frac{1}{g}m|_{X_{f'}} \end{tikzcd}$$
commutes.

Thus, we have defined a morphism of our sheaves. To define a morphism of $\mathcal{O}_X$-modules, we need to check that the left diagram of
$$\begin{tikzcd} \mathcal{O}_X \times \widetilde{\mathcal{F}(X)} \arrow[r] \arrow[d, "(\mathrm{id}_{\mathcal{O}_X},\varepsilon_{\mathcal{F}})"'] & \widetilde{\mathcal{F}(X)} \arrow[d, "\varepsilon_{\mathcal{F}}"] \\ \mathcal{O}_X \times \mathcal{F} \arrow[r] & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} (a/h, m/g) \arrow[r, mapsto] \arrow[d, mapsto] & (am)/(gh) \arrow[d, mapsto] \\ (a/h, \frac{1}{g}m|_{X_f}) \arrow[r, mapsto] & \frac{1}{gh}(am|_{X_f}) \end{tikzcd}$$
commutes; as usual, we can check the distinguished base by Lemma 1.79, which is what we've done on the right.

We now note that ε is a natural transformation. Fix a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. Then we need the leftmost diagram of
$$\begin{tikzcd} \widetilde{\mathcal{F}(X)} \arrow[r, "\widetilde{\varphi_X}"] \arrow[d, "\varepsilon_{\mathcal{F}}"'] & \widetilde{\mathcal{G}(X)} \arrow[d, "\varepsilon_{\mathcal{G}}"] \\ \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\mathcal{F}(X)}(X_f) \arrow[r, "(\widetilde{\varphi_X})_{X_f}"] \arrow[d, "(\varepsilon_{\mathcal{F}})_{X_f}"'] & \widetilde{\mathcal{G}(X)}(X_f) \arrow[d, "(\varepsilon_{\mathcal{G}})_{X_f}"] \\ \mathcal{F}(X_f) \arrow[r, "\varphi_{X_f}"] & \mathcal{G}(X_f) \end{tikzcd} \qquad \begin{tikzcd} \frac{m}{g} \arrow[r, mapsto] \arrow[d, mapsto] & \frac{\varphi_X(m)}{g} \arrow[d, mapsto] \\ \frac{1}{g}(m|_{X_f}) \arrow[r, mapsto] & \frac{1}{g}\varphi_{X_f}(m|_{X_f}) \end{tikzcd}$$
to commute, for which it suffices by Lemma 1.79 to check on the distinguished base $\{X_f\}_{f\in\mathcal{O}_X(X)}$, which is described on the right two squares. ■

Remark 4.51. If $\mathcal{F} = \widetilde{M}$ for an A-module M, then $\varepsilon_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$. Indeed, by Lemma 1.79, it suffices to check this on the distinguished base. Well, for any $X_f \subseteq X$ and $m/g \in \widetilde{M}(X_f)$, we note
$$(\varepsilon_{\mathcal{F}})_{X_f}(m/g) = \frac{1}{g}(m|_{X_f}) = m/g = (\mathrm{id}_{\mathcal{F}})_{X_f}(m/g).$$

Theorem 4.52. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Then the functors $\widetilde{\cdot}$ and Γ are adjoint. In other words, there is a natural isomorphism

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) & \simeq & \operatorname{Hom}_A(M, \mathcal{F}(X)) \\ \varphi & \mapsto & \varphi_X \\ \varepsilon_{\mathcal{F}} \circ \widetilde{f} & \leftarrow\!\shortmid & f \end{array}$$

for any A-module M and $\mathcal{O}_X$-module $\mathcal{F}$. Here $\varepsilon_\bullet$ is from Lemma 4.50.

Proof. Quickly, note that $\varphi_X : \widetilde{M}(X) \to \mathcal{F}(X)$ is indeed a map $M \to \mathcal{F}(X)$ because $\widetilde{M}(X) = \widetilde{M}(1) = M$.

Now, as in Theorem 2.26, naturality will follow pretty quickly once we have our bijection. We now check that these are inverses; fix an A-module M and $\mathcal{O}_X$-module $\mathcal{F}$.

- Begin with $\varphi \colon \widetilde{M} \to \mathcal{F}$; we need to show that
$$\varphi = \varepsilon_{\mathcal{F}} \circ \widetilde{\varphi_X}$$
as morphisms $\widetilde{M} \to \mathcal{F}$.
By Lemma 1.79, it suffices to show this equality on the distinguished base, so fix some $X_a \subseteq X$, and we want to show
$$\varphi_{X_a} \stackrel{?}{=} (\varepsilon_{\mathcal{F}})_{X_a} \circ (\widetilde{\varphi}_X)_{X_a}.$$
Well, for any $m/b \in \widetilde{M}(X_a) = S(D(a))^{-1}M$, we compute
$$\begin{aligned} \big((\varepsilon_{\mathcal{F}})_{X_a} \circ (\widetilde{\varphi}_X)_{X_a}\big)(m/b) &= (\varepsilon_{\mathcal{F}})_{X_a}(\varphi_X(m)/b) \\ &= \frac{1}{b}(\varphi_X(m)|_{X_a}) \\ &= \frac{1}{b}\varphi_{X_a}(m) \\ &= \varphi_{X_a}(m/b). \end{aligned}$$
- Begin with $f \colon M \to \mathcal{F}(X)$; we need to show that $f = (\varepsilon_{\mathcal{F}} \circ \widetilde{f})_X$ as morphisms $M \to \mathcal{F}(X)$. Well,
$$(\varepsilon_{\mathcal{F}} \circ \widetilde{f})_X = (\varepsilon_{\mathcal{F}})_{X_1} \circ \widetilde{f}_{X_1} = \operatorname{id}_{\mathcal{F}} \circ f = f.$$

And here are our naturality checks.

- Fix an A-module homomorphism $g \colon M \to N$. Then the following diagram commutes.
$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F}) \arrow[d, "-\circ\widetilde{g}"] & \simeq & \operatorname{Hom}_A(N, \mathcal{F}(X)) \arrow[d, "-\circ g"] & \varepsilon_{\mathcal{F}} \circ \widetilde{f} \arrow[d, mapsto] & f \arrow[l, mapsto] \arrow[d, mapsto] \\ \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) & \simeq & \operatorname{Hom}_A(M, \mathcal{F}(X)) & \varepsilon \circ \widetilde{f \circ g} & f \circ g \arrow[l, mapsto] \end{tikzcd}$$
- Fix an $\mathcal{O}_X$-module homomorphism $\psi \circ \mathcal{F} \to \mathcal{G}$. Then the following diagram commutes.
$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \arrow[d, "\psi\circ-"] & \simeq & \operatorname{Hom}_A(M, \mathcal{F}(X)) \arrow[d, "\psi_X\circ-"] & \varphi \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_X \arrow[d, mapsto] \\ \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{G}) & \simeq & \operatorname{Hom}_A(M, \mathcal{G}(X)) & \psi \circ \varphi \arrow[r, mapsto] & (\psi \circ \varphi)_X \end{tikzcd}$$

The above checks finish our check that we have an adjunction. ■

Corollary 4.53. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Then the functor $\widetilde{\cdot}\colon \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$ is fully faithful.

Proof. Given A-modules M and N, we need to show

$$\widetilde{\cdot}\colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$$

is a bijection. The point is to use Theorem 4.52 with $\mathcal{F}$ to set $\widetilde{N}$. Technically, we are told that the map

$$f \mapsto \varepsilon_{\widetilde{N}} \circ \widetilde{f}$$

is a bijection, but Remark 4.51 tells us that $\varepsilon_{\widetilde{N}} = \mathrm{id}_{\widetilde{N}}$, so this bijection really is just $f \mapsto \widetilde{f}$. ∎

Corollary 4.54. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Given an $\mathcal{O}_X$-module $\mathcal{F}$, there exists an A-moodule M such that $\mathcal{F} \cong \widetilde{M}$ if and only if the map $\varepsilon_{\mathcal{F}}\colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$ of Lemma 4.50 is an isomorphism.

Proof. If the map $\varepsilon_{\mathcal{F}}$ is an isomorphism, then take $M := \mathcal{F}(X)$, and the needed isomorphism is just $\varepsilon_{\mathcal{F}}$.

In the other direction, suppose that we have an isomorphism $\varphi\colon \widetilde{M} \cong \mathcal{F}$. Then Theorem 4.52 tells us that

$$\varphi = \varepsilon_{\mathcal{F}} \circ \widetilde{\varphi_X}.$$

Now, φ is an isomorphism, so φ_X is an isomorphism by functoriality of Γ in Lemma 4.49. Thus, $\widetilde{\varphi_X}$ is still an isomorphism by functoriality of $\widetilde{\cdot}$ in Lemma 4.48. Thus, we can write

$$\varepsilon_{\mathcal{F}} = \varphi \circ \widetilde{\varphi_X}^{-1},$$

so $\varepsilon_{\mathcal{F}}$ is the composite of isomorphisms and therefore an isomorphism. ∎

Corollary 4.55. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Given an A-module M and a distinguished open subscheme $X_f \subseteq X$, the morphism $\varepsilon_{\widetilde{M}|_{X_f}}$ is an isomorphism.

Proof. By the functoriality of extending a sheaf on a base from Lemma 1.84, it suffices to show that $\varepsilon_{\widetilde{M}|_{X_f}}$ is an isomorphism on the distinguished base of X_f. Noting that $\mathcal{O}_X(X_f) \simeq A_f$ through the canonical isomorphism $\varphi\colon \operatorname{Spec} A \cong X$, we can write an element of $\mathcal{O}_X(X_f)$ as a/f^n for some $a \in A$ and $n \in \mathbb{N}$, from which we note

$$(X_f)_{a/f^n} = \{\varphi(\mathfrak{p}) \in X : f \notin \mathfrak{p}, a/f^n \notin \mathfrak{p}A_f\} = \{\varphi(\mathfrak{p}) : af \notin \mathfrak{p}\} = X_{af},$$

so our distinguished open subschemes look like X_{af} for $a \in A$.

Thus, we are showing that

$$(\varepsilon_{\widetilde{M}_{X_f}})_{X_{af}}\colon \underbrace{\widetilde{\widetilde{M}(X_f)}(X_{af})}_{S(D(af))^{-1}\widetilde{M}(X_f)} \to \underbrace{\widetilde{M}(X_{af})}_{S(D(af))^{-1}M}$$

is an isomorphism for each $a \in A$; by construction, this morphism sends m/g to $\frac{1}{g}(m/1) = \frac{1}{1}m/g$. Namely, we see that $S(D(af))^{-1}\widetilde{M}(X_f) = S(D(af))^{-1}S(D(f))^{-1}M \simeq S(D(af))^{-1}M$ by sending m/g to $(m/g)/1$ because $D(ag) \supseteq D(f)$, which is this map. ∎

Corollary 4.56. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Given an $\mathcal{O}_X$-module $\mathcal{F}$ and some distinguished open subscheme $X_f \subseteq X$, the following diagram commutes.

$$\begin{tikzcd}
\widetilde{\widetilde{\mathcal{F}(X)}(X_f)} \arrow[r, "\widetilde{\varepsilon_{\mathcal{F}}(X_f)}"] \arrow[d, "\varepsilon_{\widetilde{\mathcal{F}(X)}|_{X_f}}"'] & \widetilde{\mathcal{F}(X_f)} \arrow[d, "\varepsilon_{\mathcal{F}|_{X_f}}"] \\
\widetilde{\mathcal{F}(X)}|_{X_f} \arrow[r, "\varepsilon_{\mathcal{F}}|_{X_f}"] & \mathcal{F}|_{X_f}
\end{tikzcd}$$

Proof. By the functoriality of extending a sheaf on a base from Lemma 1.84, it suffices to check this on the distinguished base. As we saw in the previous corollary, the distinguished base on X_f consists of X_{af} for various $a \in A$.

Thus, for some X_{af}, we check that the square

$$\begin{tikzcd}
\widetilde{\widetilde{\mathcal{F}(X)}(X_f)}(X_{af}) \arrow[r, "\widetilde{\varepsilon_{\mathcal{F}}(X_f)}"] \arrow[d, "\varepsilon_{\widetilde{\mathcal{F}(X)}|_{X_f}}"'] & \widetilde{\mathcal{F}(X_f)}(X_{af}) \arrow[d, "\varepsilon_{\mathcal{F}|_{X_f}}"] \\
\widetilde{\mathcal{F}(X)}(X_{af}) \arrow[r, "\varepsilon_{\mathcal{F}}|_{X_f}"] & \mathcal{F}(X_{af})
\end{tikzcd}
\qquad
\begin{tikzcd}
\frac{b}{(af)^n}\left(\frac{c}{f^m}\cdot x\right) \arrow[r, mapsto] \arrow[d, mapsto] & \frac{b}{(af)^n}\left(\frac{cx}{f^m}\right) \arrow[d, mapsto] \\
\frac{b\left(\frac{c}{f^m}\cdot x\right)}{(af)^n} \arrow[r, mapsto] & \frac{bcx}{(af)^n f^m}
\end{tikzcd}$$

commutes by expanding out all the definitions. At a high level, all of our definitions have "essentially" been identities, so there's not much to worry about here. ■

4.1.12 Extending Geometry Is Opposite Algebra, Again

We are going to want more properties of the functor $\widetilde{\cdot}\colon \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$, where X is an affine scheme and $A = \mathcal{O}_X(X)$. Here is exactness.

Lemma 4.57. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Let $\varphi\colon X \cong \operatorname{Spec} A$ be the canonical isomorphism. For an A-module homomorphism $f\colon M \to N$, the diagram

$$\begin{tikzcd}
M_{\varphi(x)} \arrow[r, "f_{\varphi(x)}"] \arrow[d] & N_{\varphi(x)} \arrow[d] \\
\widetilde{M}_x \arrow[r, "\widetilde{f}_x"] & \widetilde{N}_x
\end{tikzcd}$$

commutes, where the vertical morphisms are the isomorphisms of Lemma 4.45, and $f_{\varphi(x)}$ is the localized map induced map $M_{\varphi(x)} \simeq M \otimes_A A_{\varphi(x)}$.

Proof. We simply compute that

$$\begin{tikzcd}
M_{\varphi(x)} \arrow[r, "f_{\varphi(x)}"] \arrow[d] & N_{\varphi(x)} \arrow[d] \\
\widetilde{M}_x \arrow[r, "\widetilde{f}_x"] & \widetilde{N}_x
\end{tikzcd}
\qquad
\begin{tikzcd}
m/a \arrow[r, mapsto] \arrow[d, mapsto] & f(m)/a \arrow[d, mapsto] \\
{[(X_a, m/a)]} \arrow[r, mapsto] & {[(X_a, f(m)/a)]}
\end{tikzcd}$$

from the construction of $\widetilde{f}$. ■

Proposition 4.58. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. Then the functor $\widetilde{\cdot}\colon \operatorname{Mod}_A \to \operatorname{Mod}_{\mathcal{O}_X}$ of Lemma 4.48 is exact.

Proof. Let $\varphi\colon X \cong \operatorname{Spec} A$ be the canonical isomorphism. Fix an exact sequence of A-modules named

$$0 \to M' \overset{f}{\to} M \overset{g}{\to} M'' \to 0.$$

Because localization is an exact functor on A-modules ([Eis95, Proposition 2.5]), we note that

$$0 \to M'_{\mathfrak{p}} \overset{f_{\mathfrak{p}}}{\to} M_{\mathfrak{p}} \overset{g_{\mathfrak{p}}}{\to} M''_{\mathfrak{p}} \to 0 \tag{4.2}$$

is a short exact sequence for any prime $\mathfrak{p} \in \operatorname{Spec} A$. Thus, for any x, we draw the diagram

$$\begin{tikzcd} 0 \arrow[r] & M'_{\varphi(x)} \arrow[r, "f_{\varphi(x)}"] \arrow[d] & M_{\varphi(x)} \arrow[r, "g_{\varphi(x)}"] \arrow[d] & M''_{\varphi(x)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{M'}_x \arrow[r, "\widetilde{f}_x"] & \widetilde{M}_x \arrow[r, "\widetilde{g}_x"] & \widetilde{M''}_x \arrow[r] & 0 \end{tikzcd}$$

where the vertical morphisms are the isomorphisms of Lemma 4.45. Notably, each square commutes by Lemma 4.57, and the top row is an instance of the exact sequence (4.2), so the bottom row is also exact.

Thus, the sequence

$$0 \to \widetilde{M'} \overset{\widetilde{f}}{\to} \widetilde{M} \overset{\widetilde{g}}{\to} \widetilde{M''} \to 0$$

is exact at the stalk over any $x \in X$, so this is exact by Proposition 1.161. ■

4.1.13 Quasicoherent Sheaves

We are now ready to define quasicoherent sheaves.

Definition 4.59 (Quasicoherent sheaf). Fix a scheme $(X, \mathcal{O}_X)$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* if and only if X has an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ such that $\mathcal{F}|_{U_\alpha} \cong \widetilde{M_\alpha}$ for some $\mathcal{O}_X(U_\alpha)$-module M_α.

As usual, we note that it's annoying that we only have an affine open cover because we would like to have $\mathcal{F}|_U \cong \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M at any affine open subscheme $U \subseteq X$.

Well, here's the appropriate application of the Affine communication lemma.

Lemma 4.60. Fix a quasicoherent sheaf $\mathcal{F}$ on a scheme X. Then, for any affine open subscheme $U \subseteq X$, there is an $\mathcal{O}_X(U)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$.

Proof. We apply Lemma 2.118; call an affine open subscheme $U \subseteq X$ "respectful" if and only if there is an $\mathcal{O}_X(U)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. Because $\mathcal{F}$ is quasicoherent, we are given that there is an open cover of respectful affine open subschemes of X, so it remains to check conditions (i) and (ii).

(i) We are given a respectful affine open subscheme U and some $f \in \mathcal{O}_X(U)$ and would like to show that U_f is respectful. Well, by Corollary 4.54, we are given that $\varepsilon_{\mathcal{F}|_U} : \widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ is an isomorphism of $\mathcal{O}_U$-modules. In particular, by Corollary 4.56, we conclude

$$\begin{tikzcd} \widetilde{\widetilde{\mathcal{F}(U)}(U_f)} \arrow[r, "\widetilde{\varepsilon_{\mathcal{F}|_U}(U_f)}"] \arrow[d, "\varepsilon_{\widetilde{\mathcal{F}(U)}|_{U_f}}"'] & \widetilde{\mathcal{F}(U_f)} \arrow[d, "\varepsilon_{\mathcal{F}|_{U_f}}"] \\ \widetilde{\mathcal{F}(U)}|_{U_f} \arrow[r, "\varepsilon_{\mathcal{F}|_U}|_{U_f}"] & \mathcal{F}|_{U_f} \end{tikzcd}$$

commutes. The left arrow is an isomorphism by Corollary 4.55. The top arrow is an isomorphism by functoriality of taking sections and $\widetilde{\cdot}$ from Lemma 4.48 because $\varepsilon_{\mathcal{F}|_U}$ is an isomorphism and using functoriality of restriction from Lemma 4.37. Lastly, the bottom arrow is an isomorphism functoriality of restriction in Lemma 4.37, so we conclude that $\varepsilon_{\mathcal{F}}|_{U_f}$ is also an isomorphism. We conclude that U_f is respectful.

(ii) We are given an affine open subscheme U and some $\{f_1, \ldots, f_n\}$ so that the U_{f_i} cover U and are respectful, and we need to show that U is respectful. By Corollary 4.54, we see that all the $\varepsilon_{\mathcal{F}|_{U_{f_i}}}$ are isomorphisms, so we use Corollary 4.56 to conclude that the diagram

$$\begin{tikzcd}
\widetilde{\widetilde{\mathcal{F}(U)}(U_f)} \arrow[r, "\widetilde{\varepsilon_{\mathcal{F}|_U}(U_f)}"] \arrow[d, "\varepsilon_{\widetilde{\mathcal{F}(U)}}|_{U_f}"'] & \widetilde{\mathcal{F}(U_f)} \arrow[d, "\varepsilon_{\mathcal{F}|_{U_f}}"] \\
\widetilde{\mathcal{F}(U)}|_{U_f} \arrow[r, "\varepsilon_{\mathcal{F}|_U}|_{U_f}"] & \mathcal{F}|_{U_f}
\end{tikzcd}$$

commutes for each $f \in \{f_1, \ldots, f_n\}$. Notably, now the right arrow is an isomorphism because each U_f is respectful (by Corollary 4.54). The left arrow is an isomorphism by Corollary 4.55. The top arrow is an isomorphism because each U_f is respectful, noting that $\varepsilon_{\mathcal{F}|_U}(U_f) = \varepsilon_{\mathcal{F}|_{U_f}}(U_f)$, so $\varepsilon_{\mathcal{F}|_{U_f}}$ is an isomorphism by Corollary 4.54, so functoriality from Lemma 4.49 and Lemma 4.48 finishes.

Thus, we conclude that the bottom arrow $\varepsilon_{\mathcal{F}|_U}|_{U_f}$ is an isomorphism for each $f \in \{f_1, \ldots, f_n\}$. Noting that the U_{f_i} form a cover of U, we are essentially done.

Indeed, it remains to show that $\varepsilon_{\mathcal{F}|_U}$ is actually an isomorphism of $\mathcal{O}_X$-modules. Certainly this is a morphism of $\mathcal{O}_X$-modules by construction, so Lemma 4.10 says that we just have to check we have an isomorphism of sheaves. Well, by Proposition 1.132, it suffices to check that we are an isomorphism at stalks, for which it suffices to check that we are an isomorphism at stalks as sheaves of just abelian groups.

For this, pick some $p \in U$ and find some i with $p \in U_{f_i}$, and we note the naturality of Lemma 1.174 gives the commutative diagram

$$\begin{tikzcd}
(\mathcal{F}|_U)_p \arrow[r, "(\varepsilon_{\mathcal{F}|_U})_p"] \arrow[d] & (\widetilde{\mathcal{F}(U)})_p \arrow[d] \\
(\mathcal{F}|_{U_{f_i}})_p \arrow[r, "(\varepsilon_{\mathcal{F}}|_{U_{f_i}})_p"] & (\widetilde{\mathcal{F}(U)}|_{U_{f_i}})_p
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(V, s)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[(V, (\varepsilon_{\mathcal{F}|_U})_V s)]} \arrow[d, mapsto] \\
{[(V \cap U_{f_i}, s|_{U_{f_i}})]} \arrow[r, mapsto] & {[(V \cap U_{f_i}, (\varepsilon_{\mathcal{F}|_U})_{V \cap U_{f_i}} s)]}
\end{tikzcd}$$

where we note that the vertical morphisms are isomorphisms. Thus, because all the $(\varepsilon_{\mathcal{F}}|_{U_{f_i}})_p$ are isomorphisms by Proposition 1.132, we conclude that $(\varepsilon_{\mathcal{F}}|_{U_f})_p$ are isomorphisms by Proposition 1.132 again. This finishes. ■

Corollary 4.61. Fix an affine scheme X, and set $A := \mathcal{O}_X(X)$. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if there exists an A-module M such that $\mathcal{F} \cong \widetilde{M}$.

Proof. If $\mathcal{F} \cong \widetilde{M}$, then the affine open cover $\{X\}$ of X shows that $\mathcal{F}$ is quasicoherent. Conversely, if X is quasicoherent, then the fact that X is an affine open subscheme of X implies that $\mathcal{F} = \mathcal{F}|_X \cong \widetilde{M}$ for some A-module M by Lemma 4.60. ■

Corollary 4.62. Fix a scheme X and a quasicoherent sheaf $\mathcal{F}$ of X. For any open subscheme $U \subseteq X$, we see $\mathcal{F}|_U$ is a quasicoherent sheaf on U.

Proof. For any affine open subscheme $V \subseteq U$, we note that V is also an affine open subscheme of X, so there is an $\mathcal{O}_V(V)$-module M such that $\mathcal{F}|_V \cong \widetilde{M}$. So we are done upon choosing any affine open cover for U. ■

4.2 October 12

We continue our discussion of quasicoherent sheaves.

4.2.1 Quasicoherent Sheaves via Modules

It will turn out that many of our sheaf-theoretic constructions will not actually need to think about sheaves very much to define. To get a taste for how this will work, we start with the kernel.

Lemma 4.63. Fix an affine scheme X and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of quasicoherent sheaves. Then $\ker \varphi \simeq \widetilde{\ker \varphi_X}$.

Proof. As usual, by Lemma 1.84, it suffices to exhibit an isomorphism of sheaves on the distinguished base. From Corollary 4.54, let $\varepsilon_{\mathcal{F}}\colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$ and $\varepsilon_{\mathcal{G}}\colon \widetilde{\mathcal{G}(X)} \to \mathcal{G}$ be the canonical isomorphisms.

Now, for any $f \in \mathcal{O}_X(X_f)$, we note that we have the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker \varphi_{X_f} \arrow[r] \arrow[d, dashed] & \mathcal{F}(X_f) \arrow[r, "\varphi_{X_f}"] \arrow[d, "(\varepsilon_{\mathcal{F}})_{X_f}"] & \mathcal{G}(X_f) \arrow[d, "(\varepsilon_{\mathcal{G}})_{X_f}"] \\ 0 \arrow[r] & (\ker \varphi_X)_f \arrow[r] & \mathcal{F}(X)_f \arrow[r, "(\varphi_X)_f"'] & \mathcal{G}(X)_f \end{tikzcd}$$

where the bottom row is exact because localization is an exact functor. Note that the right square commutes by Lemma 4.50, and the two vertical maps are isomorphisms by construction of the εs.

Thus, we see from abstract nonsense that $(\varepsilon_{\mathcal{F}})_{X_f}$ restricts to an isomorphism $\varepsilon_{X_f}\colon \ker \varphi_{X_f} \to (\ker \varphi_X)_f$. To check functoriality on these isomorphisms, just suppose that $X_g \subseteq X_f$ and note that

$$\begin{tikzcd} \ker \varphi_{X_f} \arrow[r, "\varepsilon_{X_f}"] \arrow[d, "\mathrm{res}"'] & (\ker \varphi_X)_f \arrow[d, "\mathrm{res}"] \\ \ker \varphi_{X_g} \arrow[r, "\varepsilon_{X_g}"] & (\ker \varphi_X)_g \end{tikzcd} \qquad \begin{tikzcd} \frac{m}{h} \arrow[r, mapsto] \arrow[d, mapsto] & \frac{1}{h}(m|_{X_f}) \arrow[d, mapsto] \\ \frac{m}{h} \arrow[r, mapsto] & \frac{1}{h}(m|_{X_g}) \end{tikzcd}$$

commutes, identical as in Lemma 4.50. So this isomorphism follows. ■

Corollary 4.64. Fix a scheme X and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of quasicoherent sheaves on X. Then $\ker \varphi$ is a quasicoherent sheaf.

Proof. Fix some affine open cover $\mathcal{U}$ of X, and let U be any affine open subscheme in $\mathcal{U}$. Using Lemma 4.63, we note that the morphism $\varphi|_U\colon \mathcal{F}|_U \to \mathcal{G}|_U$ of quasicoherent sheaves (note $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are quasicoherent by Corollary 4.62) tells us that $\ker(\varphi|_U) \cong \widetilde{K}$ for some $\mathcal{O}_X(U)$-module K. However, for any open $V \subseteq U$ has

$$(\ker \varphi|_U)(V) = \ker(\varphi|_U)_V = \ker \varphi_V = (\ker \varphi)(V) = (\ker \varphi)|_U(V).$$

The restriction maps also match up by construction, so we have actually shown that $(\ker \varphi)|_U \cong \widetilde{K}$. ■

While we're here, we might as well address products.

Lemma 4.65. Fix an affine scheme X and quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. Set $M := \mathcal{F}(X)$ and $N := \mathcal{G}(X)$. Then $\mathcal{F} \times \mathcal{G} \simeq \widetilde{M \times N}$.

Proof. We begin with the standard set-up. By Lemma 1.84, it suffices to exhibit an isomorphism of sheaves on the distinguished base. From Corollary 4.54, let $\varepsilon_\mathcal{F} \colon \widetilde{M} \to \mathcal{F}$ and $\varepsilon_\mathcal{G} \colon \widetilde{N} \to \mathcal{G}$ be the canonical isomorphisms.

Now, for each $f \in \mathcal{O}_X(X)$, we see that we have isomorphisms $(\varepsilon_\mathcal{F})_{X_f} \colon M_f \to \mathcal{F}(X_f)$ and $(\varepsilon_\mathcal{G})_{X_f} \colon N_f \to \mathcal{G}(X_f)$. As such, we note that $(M \times N)_f \simeq M_f \times N_f$ by $\frac{(m,n)}{g} \mapsto \left(\frac{m}{g}, \frac{n}{g}\right)$.[1] Thus, we have an isomorphism

$$\begin{array}{ccccccc} \widetilde{M \times N}(X_f) = (M \times N)_f & \simeq & M_f \times N_f & \simeq & \mathcal{F}(X_f) \times \mathcal{G}(X_f) & = (\mathcal{F} \times \mathcal{G})(X_f) \\ \frac{1}{g}(m,n) & \mapsto & \left(\frac{m}{g}, \frac{n}{g}\right) & \mapsto & \left(\frac{1}{g}(m|_{X_f}), \frac{1}{g}(n|_{X_f})\right) & \end{array}$$

which we name ε_{X_f}. To see that ε_{X_f} assembles into an isomorphism of sheaves on the distinguished base, we note that $X_g \subseteq X_f$ makes

$$\begin{tikzcd} \widetilde{M \times N}(X_f) \arrow[r, "\varepsilon_{X_f}"] \arrow[d, "\mathrm{res}"'] & (\mathcal{F} \times \mathcal{G})(X_f) \arrow[d, "\mathrm{res}"] \\ \widetilde{M \times N}(X_g) \arrow[r, "\varepsilon_{X_g}"] & (\mathcal{F} \times \mathcal{G})(X_g) \end{tikzcd} \qquad \begin{tikzcd} \frac{1}{h}(m,n) \arrow[r, mapsto] \arrow[d, mapsto] & \left(\frac{m}{h}, \frac{n}{h}\right) \arrow[d, mapsto] \\ \frac{1}{h}(m,n) \arrow[r, mapsto] & \left(\frac{m}{h}, \frac{n}{h}\right) \end{tikzcd}$$

commute. This finishes. ■

Corollary 4.66. Fix a scheme X and quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. Then $\mathcal{F} \times \mathcal{G}$ is quasicoherent.

Proof. As before, fix some affine open cover $\mathcal{U}$ of X, and let U be any affine open subscheme in $\mathcal{U}$. We show $(\mathcal{F} \times \mathcal{G})|_U \simeq \widetilde{P}$ for some $\mathcal{O}_X(U)$-module P. Well, we note that any open $V \subseteq U$ gives

$$(\mathcal{F} \times \mathcal{G})|_U(V) = \mathcal{F}(V) \times \mathcal{G}(V) = (\mathcal{F}|_U \times \mathcal{G}|_U)(V),$$

so upon noting that restriction maps also match, we have $(\mathcal{F} \times \mathcal{G})|_U = \mathcal{F}|_U \times \mathcal{G}|_U$. However, $\mathcal{F}|_U \times \mathcal{G}|_U$ is a product of quasicoherent sheaves (by Corollary 4.62) on the affine scheme U, which is quasicoherent by Lemma 4.65. ■

4.2.2 Quasicoherent Sheaves Without Sheafification

Now that we've seen some quasicoherent sheaves with no sheafification, let's add a little sheafification to the mix. We start with a technical lemma.

Lemma 4.67. Fix a topological space X and a sheaf $\mathcal{F}$ and a presheaf $\mathcal{G}$. Given a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ which is an isomorphism on a base $\mathcal{B}$ for the topology on X, we have $\mathcal{F} \cong \mathcal{G}^{\mathrm{sh}}$.

Proof. Let $\mathrm{sh} \colon \mathcal{G} \to \mathcal{G}^{\mathrm{sh}}$. Unsurprisingly, we claim that $\mathrm{sh} \circ \varphi$ is the desired isomorphism. By Proposition 1.132, it suffices to check that this is an isomorphism on stalks. Well, for any $p \in X$, we see that sh_p is already an isomorphism by Proposition 1.141, so we need to show that φ_p is an isomorphism on stalks.

[1] The inverse map is $\left(\frac{m}{g}, \frac{n}{h}\right) \mapsto \frac{(hm,gn)}{gh}$. We won't say more because this is "just" commutative algebra.

Well, using Lemma 1.101, we note that the diagram

$$\begin{array}{ccccc} \mathcal{F}_p & \simeq & \varinjlim \mathcal{F}(B) & [(V,f)] & \longmapsto & [(B,f|_B)] \\ \downarrow{\scriptstyle \varphi_p} & & \downarrow{\scriptstyle \varinjlim \varphi_B} & \downarrow & & \downarrow \\ \mathcal{G}_p & \simeq & \varinjlim \mathcal{G}(B) & [(V,\varphi_V f)] & \longmapsto & [(B,\varphi_B(f|_B))] \end{array}$$

commutes, where $B \in \mathcal{B}$ is some basis element in V containing p. Thus, we note that the horizontal arrows are isomorphisms by Lemma 1.101, and the right arrow is an isomorphism because each φ_B is an isomorphism by hypothesis. Thus, the left arrow is an isomorphism, which is what we wanted. ■

And here is the cokernel.

Lemma 4.68. Fix an affine scheme X and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of quasicoherent sheaves. Then $\widetilde{\operatorname{coker} \varphi_X} \simeq \operatorname{coker} \varphi$.

Proof. By Lemma 4.67, it suffices to show that an exhibit a natural isomorphism $\widetilde{\operatorname{coker} \varphi_X} \simeq \operatorname{coker}^{\text{pre}} \varphi$ of (pre)sheaves on the distinguished base. From Corollary 4.54, let $\varepsilon_{\mathcal{F}}\colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$ and $\varepsilon_{\mathcal{G}}\colon \widetilde{\mathcal{G}(X)} \to \mathcal{G}$ be the canonical isomorphisms.

Now, for any $f \in \mathcal{O}_X(X_f)$, we note that we have the exact sequences

$$\begin{array}{ccccccc} \mathcal{F}(X_f) & \xrightarrow{\varphi_{X_f}} & \mathcal{G}(X_f) & \longrightarrow & \operatorname{coker} \varphi_{Xf} & \longrightarrow & 0 \\ \downarrow{\scriptstyle (\varepsilon_{\mathcal{F}})_{X_f}} & & \downarrow{\scriptstyle (\varepsilon_{\mathcal{F}})_{X_f}} & & \downarrow & & \\ \mathcal{F}(X)_f & \xrightarrow[(\varphi_X)_f]{} & \mathcal{G}(X)_f & \longrightarrow & (\operatorname{coker} \varphi_X)_f & \longrightarrow & 0 \end{array}$$

where the bottom row is exact because localization is an exact functor. In particular, the left square commutes by Lemma 4.50, and the two vertical maps are isomorphisms by construction of the εs.

Thus, abstract nonsense tells us $(\varepsilon_{\mathcal{G}})_{X_f}$ can take quotients to yield an isomorphism $\varepsilon_{X_f}\colon \operatorname{coker} \varphi_{X_f} \to (\operatorname{coker} \varphi_X)_f$. For functoriality, we suppose $X_g \subseteq X_f$ and note that

$$\begin{array}{ccccccc} \operatorname{coker} \varphi_{X_f} & \xrightarrow{\varepsilon_{X_f}} & (\operatorname{coker} \varphi_X)_f & & \left[\frac{m}{h}\right] & \longmapsto & \frac{1}{h}[m|_{X_f}] \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} & & \downarrow & & \downarrow \\ \operatorname{coker} \varphi_{X_g} & \xrightarrow{\varepsilon_{X_g}} & (\operatorname{coker} \varphi_X)_g & & \left[\frac{m}{h}\right] & \longmapsto & \frac{1}{h}[m|_{X_g}] \end{array}$$

commutes, exactly as in Lemma 4.50. Thus, we have exhibited a natural isomorphism $\operatorname{coker}^{\text{pre}} \varphi \simeq \widetilde{\operatorname{coker} \varphi_X}$ of presheaves on the distinguished base. Inverting and noting that being isomorphic to a sheaf on a base forces both of these to be a sheaf on the base finishes by Lemma 4.67. ■

Corollary 4.69. Fix a scheme X and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of quasicoherent sheaves on X. Then $\operatorname{coker} \varphi$ is a quasicoherent sheaf.

Proof. As before, fix some affine open cover $\mathcal{U}$ of X, and let U be any affine open subscheme in $\mathcal{U}$. Restricting preserves being quasicoherent by Corollary 4.62, so we note that $\varphi|_U\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a morphism of quasicoherent sheaves on the affine scheme U, so Lemma 4.68 tells us that $\operatorname{coker}(\varphi|_U) \simeq \widetilde{C}$ for some $\mathcal{O}_U(U)$-module C.

But now we note Lemma 1.182 tells us

$$(\operatorname{coker} \varphi)|_U = (\operatorname{coker}^{\text{pre}} \varphi)^{\text{sh}}|_U \simeq ((\operatorname{coker}^{\text{pre}} \varphi)|_U)^{\text{sh}}.$$

Now, for each open $V \subseteq U$, we note $(\operatorname{coker}^{\mathrm{pre}}\varphi)|_U(V) = \operatorname{coker}\varphi_V$, so $(\operatorname{coker}^{\mathrm{pre}}\varphi)|_U = \operatorname{coker}^{\mathrm{pre}}(\varphi|_U)$ upon noting that the restriction maps also align. So we see that

$$(\operatorname{coker}\varphi)|_U \simeq (\operatorname{coker}^{\mathrm{pre}}(\varphi|_U))^{\mathrm{sh}} = \operatorname{coker}(\varphi|_U) \simeq \widetilde{C}.$$

This finishes. ■

We will also want access to direct sums and tensor products.

Lemma 4.70. Fix an affine scheme X and quasicoherent sheaves $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$. Setting $M_\alpha := \mathcal{F}_\alpha(X)$ and $M := \bigoplus_{\alpha\in\lambda} M_\alpha$, we have

$$\bigoplus_{\alpha\in\lambda}\mathcal{F}_\alpha \simeq \widetilde{M}.$$

Proof. Define $\mathcal{F}$ to be the direct sum "presheaf" so that

$$\mathcal{F}(U) = \bigoplus_{\alpha\in\lambda}\mathcal{F}_\alpha(U)$$

for each open $U \subseteq X$. We would like to show that $\widetilde{M} \simeq \mathcal{F}^{\mathrm{sh}}$, for which we use Lemma 4.67 to note that it suffices to exhibit an isomorphism of (pre)sheaves on the distinguished base. From Corollary 4.54, let $\varepsilon_\alpha \colon \widetilde{M_\alpha} \to \mathcal{F}_\alpha$ denote the canonical isomorphisms. Additionally, set $A := \mathcal{O}_X(X)$.

Now, for any $f \in \mathcal{O}_X(X_f)$, we have the isomorphisms

$$\begin{array}{ccccccc}
\widetilde{M}(X_f) = & M_f & \simeq & \left(\bigoplus_{\alpha\in\lambda} M_\alpha\right)\otimes_A A_f & \overset{*}{\simeq} & \bigoplus_{\alpha\in\lambda}(M_\alpha\otimes_A A_f) & \simeq & \bigoplus_{\alpha\in\lambda}(M_\alpha)_f \\
\frac{1}{g}(m_\alpha)_\alpha \mapsto & & & (m_\alpha)_\alpha\otimes\frac{1}{g} & \mapsto & \left(m_\alpha\otimes\frac{1}{g}\right)_\alpha & \mapsto & \left(\frac{m_\alpha}{g}\right)_\alpha
\end{array}$$

where $\overset{*}{\simeq}$ is by noting that direct sums commute with localization. Explicitly, the inverse is by $(m_\alpha \otimes \frac{a_\alpha}{f^{n_\alpha}}) \mapsto (f^{\max\{n_\alpha\}-n_\alpha}m_\alpha)_\alpha \otimes \frac{1}{f^{\max\{n_\alpha\}}}$, where $\max\{n_\alpha\}$ is a legal thing to write down because only finitely many of the $m_\alpha \otimes \frac{a_\alpha}{f^{n_\alpha}}$ are nonzero, so we may set $n_\alpha = 0$ whenever the term is zero.

We now call the above isomorphism ε_{X_f}. It remains to check funtoriality: suppose $X_g \subseteq X_f$, and we check that

$$\begin{tikzcd}
\widetilde{M}(X_f) \arrow[r, "\varepsilon_{X_f}"] \arrow[d, "\mathrm{res}"'] & \mathcal{F}(X_f) \arrow[d, "\mathrm{res}"] & \frac{1}{h}(m_\alpha)_\alpha \arrow[r, mapsto] \arrow[d, mapsto] & \left(\frac{m_\alpha}{h}\right)_\alpha \arrow[d, mapsto] \\
\widetilde{M}(X_g) \arrow[r, "\varepsilon_{X_g}"] & \mathcal{F}(X_g) & \frac{1}{h}(m_\alpha)_\alpha \arrow[r, mapsto] & \left(\frac{m_\alpha}{h}\right)_\alpha
\end{tikzcd}$$

commutes, which finishes by Lemma 4.67. ■

Corollary 4.71. Fix a scheme X and quasicoherent sheaves $\{\mathcal{F}_\alpha\}_{\alpha\in\lambda}$. Then $\bigoplus_{\alpha\in\lambda}\mathcal{F}_\alpha$ is a quasicoherent sheaf.

Proof. Let $\mathcal{F}$ be the direct sum "presheaf" so that

$$\mathcal{F}(V) := \bigoplus_{\alpha\in\lambda}\mathcal{F}_\alpha(V)$$

for any open $V \subseteq U$ so that we want to show $\mathcal{F}^{\mathrm{sh}}$ is quasicoherent. As usual, fix some affine open cover $\mathcal{U}$ of X, and let U be any affine open subscheme in $\mathcal{U}$.

As is typical, the main point is to show that restriction commutes with direct sums. Indeed, we note that any open $V \subseteq U$ gives

$$\mathcal{F}|_U(V) = \mathcal{F}(V) = \bigoplus_{\alpha\in\lambda}\mathcal{F}_\alpha(V) = \bigoplus_{\alpha\in\lambda}\mathcal{F}_\alpha|_U(V),$$

so $\mathcal{F}|_U$ matches the direct sum presheaf of the $\mathcal{F}_\alpha|_U$ on open sets, and of course the restriction maps match up because they are just term-wise restrictions. Taking sheafification, we see

$$(\mathcal{F}|_U)^{\mathrm{sh}} \simeq \bigoplus_{\alpha\in\lambda} \mathcal{F}_\alpha|_U.$$

The sheaves $\mathcal{F}_\alpha|_U$ are quasicoherent sheaves (by Corollary 4.62) on the affine scheme U, so Lemma 4.70 promises us an $\mathcal{O}_X(U)$-module M such that

$$(\mathcal{F}|_U)^{\mathrm{sh}} \simeq \bigoplus_{\alpha\in\lambda} \mathcal{F}_\alpha|_U \simeq \widetilde{M},$$

which finishes because then $\mathcal{F}^{\mathrm{sh}}|_U \simeq (\mathcal{F}|_U)^{\mathrm{sh}}$ by Lemma 1.182. ■

Lemma 4.72. Fix an affine scheme X and quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$. Setting $M := \mathcal{F}(X)$ and $N := \mathcal{G}(X)$ and $A := \mathcal{O}_X(X)$, we have $\mathcal{F}\otimes\mathcal{G} \simeq \widetilde{M\otimes_A N}$.

Proof. As should be expected by now, we define $\mathcal{H}$ to be the tensor product "presheaf" by

$$\mathcal{H}(U) := \mathcal{F}(U)\otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

for each open $U \subseteq X$. We would like to show $\widetilde{M\otimes N} \simeq \mathcal{H}^{\mathrm{sh}}$, for which we use Lemma 4.67 to note that it suffices to exhibit this isomorphism on the distinguished base. Let $\varepsilon_{\mathcal{F}}\colon \widetilde{M} \to \mathcal{F}$ and $\varepsilon_{\mathcal{G}}\colon \widetilde{N} \to \mathcal{G}$ be the isomorphisms of Corollary 4.54.

Now, for $f \in \mathcal{O}_X(X)$, note that we have isomorphisms

$$\begin{array}{ccccccccc} \widetilde{M\otimes_A N}(X_f) = (M\otimes_A N)_f & \simeq & (M\otimes_A N)\otimes_A A_f & \overset{*}{\simeq} & (M\otimes_A A_f)\otimes_{A_f}(N\otimes_A A_f) & \simeq & M_f\otimes_{A_f} N_f \\ \frac{m\otimes n}{g} & \mapsto & (m\otimes n)\otimes\frac{1}{g} & \mapsto & (m\otimes\frac{1}{1})\otimes(n\otimes\frac{1}{g}) & \mapsto & \frac{m}{1}\otimes\frac{n}{g} \end{array}$$

where $\overset{*}{\simeq}$ is just a property of the tensor product.[2] Label the above isomorphism ε_{X_f}. To finish, we check functoriality: if $X_g \subseteq X_f$, then we check that

$$\begin{tikzcd} \widetilde{M\otimes_A N}(X_f) \arrow[r, "\varepsilon_{X_f}"] \arrow[d, "\mathrm{res}"'] & \mathcal{H}(X_f) \arrow[d, "\mathrm{res}"] \\ \widetilde{M\otimes_A N}(X_g) \arrow[r, "\varepsilon_{X_g}"] & \mathcal{H}(X_g) \end{tikzcd} \qquad \begin{tikzcd} \frac{m\otimes n}{h} \arrow[r, mapsto] \arrow[d, mapsto] & \frac{m}{1}\otimes\frac{n}{h} \arrow[d, mapsto] \\ \frac{m\otimes n}{h} \arrow[r, mapsto] & \frac{m}{1}\otimes\frac{n}{h} \end{tikzcd}$$

commutes, which finishes by Lemma 4.67. ■

Corollary 4.73. Fix a scheme X and quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. Then $\mathcal{F}\otimes\mathcal{G}$ is a quasicoherent sheaf on X.

Proof. Let $\mathcal{H}$ be the tensor product "presheaf" defined by

$$\mathcal{H}(U) := \mathcal{F}(U)\otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

for any open $U \subseteq X$ so that we want to show $\mathcal{H}^{\mathrm{sh}}$ is quasicoherent. As usual, fix some affine open cover $\mathcal{U}$ of X, and let U be any affine open subscheme in $\mathcal{U}$.

[2] Namely, if M and N are R-modules, and S is an R-algebra, then $(M\otimes_R S)\otimes_S(N\otimes_R S) \simeq M\otimes_R(S\otimes_S(S\otimes_R N)) \simeq M\otimes_R(S\otimes_S S)\otimes_R N \simeq M\otimes_R S\otimes_R N$.

As is typical, the main point is to show that restriction commutes with tensor products. Indeed, we note that any open $V \subseteq U$ gives

$$\mathcal{H}|_U(V) = \mathcal{H}(V) = \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V) = \mathcal{F}|_U \otimes_{\mathcal{O}_X|_U(V)} \mathcal{G}|_U(V),$$

so $\mathcal{H}|_U$ matches the tensor product presheaf of $\mathcal{F}|_U$ and $\mathcal{G}|_U$ on open sets, and of course the restriction maps match up because they are just term-wise restrictions. Taking sheafification, we see

$$(\mathcal{H}|_U)^{\text{sh}} \simeq (\mathcal{F}|_U) \otimes (\mathcal{G}|_U).$$

We are now in the home stretch. The sheaves $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are quasicoherent sheaves (by Corollary 4.62) on the affine scheme U, so Lemma 4.72 promises us an $\mathcal{O}_X(U)$-module M such that

$$(\mathcal{H}|_U)^{\text{sh}} \simeq (\mathcal{F}|_U) \otimes (\mathcal{G}|_U) \simeq \widetilde{M},$$

which finishes because now $\mathcal{H}^{\text{sh}}|_U \simeq (\mathcal{H}|_U)^{\text{sh}}$ by Lemma 1.182. ■

4.2.3 The Category of Quasicoherent Sheaves

It turns out that, by breaking down a quasicoherent sheaf to a sheaf on a base, one can fully determine a quasicoherent sheaf based off of the data of what it does on affine open subschemes and the restriction maps coming from distinguished open subschemes. Thus, we have the following.

Lemma 4.74. Fix a scheme X. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if any distinguished open subscheme U_f of an affine open subscheme $U \subseteq X$ has the induced map

$$\mathcal{F}(U)_f \to \mathcal{F}(U_f)$$

coming from restriction an isomorphism.

Proof. Fix any affine open subscheme $U \subseteq X$. We show that $\mathcal{F}|_U \cong \widetilde{M}$ for $M = \mathcal{F}(U)$ if and only if the morphisms

$$\begin{aligned} \operatorname{res}_f \colon \mathcal{F}(U)_f &\to \mathcal{F}(U_f) \\ s/f^n &\mapsto f|_{U_f}^{-n} \cdot s|_{U_f} \end{aligned}$$

are isomorphisms for any $f \in \mathcal{O}_X(U)$. To see that this morphism is a well-defined morphism of $\mathcal{O}_X(U_f)$-modules (here, $\mathcal{F}(U)_f$ has the action of $\mathcal{O}_X(U_f)$ through the isomorphism $\mathcal{O}_X(U)_f \simeq \mathcal{O}_X(U_f)$ by Corollary 2.28), we note that this is the morphism

$$\varepsilon_{U_f} \colon \underbrace{\widetilde{\mathcal{F}(U)}(U_f)}_{\simeq \mathcal{F}(U)_f} \to \mathcal{F}(U_f),$$

from Lemma 4.50; here, $\widetilde{\mathcal{F}(U)}(U_f) \simeq \mathcal{F}(U)_f$ by Remark 4.42. Indeed, tracking ε through, we see that it sends s/g to $g^{-1} \cdot s|_{U_f}$, which is precisely res_f.

We now proceed with the proof. We have two directions.

- Suppose that $\mathcal{F}|_U \cong \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M. Then Corollary 4.54 promises the canonical morphism $\varepsilon \colon \widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ is an isomorphism because U is an affine scheme.

  Thus, by Proposition 1.132 each ε_{U_f} is an isomorphism of abelian groups, and it is already a morphism of $\mathcal{O}_X(U_f)$-morphisms, so in fact ε_{U_f} is an isomorphism of $\mathcal{O}_{U_f}$-modules. This finishes.

- Conversely, suppose that each ε_{U_f} is an isomorphism for each $f \in \mathcal{O}_X(U)$; we show that ε is an isomorphism of $\mathcal{O}_X|_U$-modules. Certainly it is a morphism of $\mathcal{O}_X|_U$-modules by construction of ε, so it remains to show that ε is an isomorphism of sheaves by Lemma 4.10.

  Well, each ε_{U_f} is an isomorphism of sheaves for each $f \in \mathcal{O}_X(U)$, so we assemble into an isomorphism on the distinguished base. Thus, by functoriality of extending a sheaf on the base from Lemma 1.84, we conclude that ε is in total a sheaf isomorphism. This finishes. ■

This test for quasicoherent sheaves is useful, but ultimately have done most of our work with them already. We now pick up some solitary results.

Theorem 4.75. Fix a scheme X. The category of quasicoherent sheaves on X is abelian.

Proof. Most of this inherited directly from Theorem 4.29. Here are the additivity checks.

- Because our category is a full subcategory of the category of $\mathcal{O}_X$-modules, we see that $\operatorname{Mor}(\mathcal{F}, \mathcal{G})$ is already an abelian group with the necessary composition distributions for free.
- We show that we have products; this essentially follows directly from Corollary 4.66 and nothing that we live in a full subcategory. We will show this proof in detail once because it's just a matter of saying all the words out loud.

  Fix quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, and we claim that the quasicoherent sheaf $\mathcal{F} \times \mathcal{G}$ (quasicoherent by Corollary 4.66) with the $\mathcal{O}_X$-module projections $\pi_\mathcal{F} \colon \mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $\pi_\mathcal{G} \colon \mathcal{F} \times \mathcal{G} \to \mathcal{G}$. To check that this data gives the universal property, pick up a quasicoherent sheaf $\mathcal{H}$ with maps $\psi_\mathcal{F} \colon \mathcal{H} \to \mathcal{F}$ and $\psi_\mathcal{G} \colon \mathcal{H} \to \mathcal{G}$, and we need a unique morphism $\psi \colon \mathcal{H} \to \mathcal{F} \times \mathcal{G}$ making the diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\psi_\mathcal{F}"] \arrow[ddr, bend right, "\psi_\mathcal{G}"'] & & \\
& \mathcal{F} \times \mathcal{G} \arrow[r, "\pi_\mathcal{F}"'] \arrow[d, "\pi_\mathcal{G}"] & \mathcal{F} \\
& \mathcal{G} &
\end{tikzcd}$$

  commute. Well such a morphism exists uniquely in the category of $\mathcal{O}_X$-modules, so because we're in a full subcategory, this morphism still exists uniquely in the category of quasicoherent sheaves.
- The zero $\mathcal{O}_X$-module $\mathcal{Z}$ is quasicoherent: for any affine open subscheme $U \subseteq X$, we see that $\mathcal{Z}|_U \simeq \widetilde{0}$ because any $f \in \mathcal{O}_X(U)$ has

$$\mathcal{Z}(X_f) = 0 = 0_f = \widetilde{0}(X_f),$$

  where we are using Lemma 1.84 to check this isomorphism on the distinguished base. Of course, our restriction maps match up because everything is 0, where there is only one morphism to 0 anyway.

  To show the universal property is essentially the same as we just did for products. Namely, for any quasicoherent sheaf $\mathcal{F}$, there are unique $\mathcal{O}_X$-module morphisms $\mathcal{F} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{F}$, and these morphisms will still exist and stay unique in the category of quasicoherent sheaves.

Here are our abelian checks.

- We show kernels and cokernels exist. Now, given a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$, we note that the kernel map $\ker \varphi \to \mathcal{F}$ and cokernel map $\mathcal{G} \to \operatorname{coker} \varphi$ in the category of $\mathcal{O}_X$-modules remains the kernel and cokernel maps in the category of quasicoherent sheaves because these are quasicoherent sheaves by Corollary 4.64 and Corollary 4.69 and because we are a full subcategory with the same zero object.

  Checking this in detail is a matter of saying the universal property out loud to immediately reduce to the category of $\mathcal{O}_X$-modules, as we showed for products.
- Fix an epic morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$. Then the map $\varphi \colon \mathcal{F} \to \mathcal{G}$ is the cokernel of the kernel map $\iota \colon \ker \varphi \to \mathcal{F}$ in the category of $\mathcal{O}_X$-modules by Lemma 4.27.

  However, as usual, saying the universal property out loud tells us that (because $\ker \varphi$ is quasicoherent by Corollary 4.64), $\varphi \colon \mathcal{F} \to \mathcal{G}$ remains the cokernel of the kernel map $\iota \colon \ker \varphi \to \mathcal{F}$ in the category of quasicoherent sheaves.

- Fix a monic morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$. Then the map $\varphi\colon \mathcal{F} \to \mathcal{G}$ is the kernel of the cokernel map $\pi\colon \mathcal{G} \to \operatorname{coker}\varphi$ in the category of $\mathcal{O}_X$-modules by Lemma 4.28.

  However, as usual, saying the universal property out loud tells us that (because $\operatorname{coker}\varphi$ is quasicoherent by Corollary 4.69), $\varphi\colon \mathcal{F} \to \mathcal{G}$ remains the kernel of the cokernel map $\pi\colon \mathcal{G} \to \operatorname{coker}\varphi$ in the category of quasicoherent sheaves.

The above checks complete the proof. ■

As usual, we now discuss exactness.

Corollary 4.76. Fix a scheme X. Then a sequence of quasicoherent sheaves

$$\mathcal{F} \overset{\alpha}{\to} \mathcal{G} \overset{\beta}{\to} \mathcal{H}$$

is exact at $\mathcal{G}$ in the category of quasicoherent if and only if it is exact at $\mathcal{G}$ in the category of sheaves of $\mathcal{O}_X$-modules.

Proof. The complex is exact at $\mathcal{G}$ if and only if the image map $\mathcal{F} \to \operatorname{coker}\ker\alpha$ serves as a kernel of the map $\mathcal{G} \to \mathcal{H}$. However, cokernels and kernels are the exact same in the category of quasicoherent sheaves as in the category of $\mathcal{O}_X$-modules, as discussed in the proof of Theorem 4.75 (due to Corollary 4.64 and Corollary 4.69), so these two exactness are equivalent. ■

4.2.4 Short Exact Sequences for Quasicoherent Sheaves

Let's just start with the main attraction.

Proposition 4.77. Fix an affine scheme X with an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

If $\mathcal{F}'$ is quasicoherent, then

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is also exact.

Proof. Label our short exact sequence as

$$0 \to \mathcal{F}' \overset{\iota}{\to} \mathcal{F} \overset{\pi}{\to} \mathcal{F}'' \to 0.$$

Proposition 1.162 tells us that any $U \subseteq X$ has

$$0 \to \mathcal{F}'(U) \overset{\iota_U}{\to} \mathcal{F}(U) \overset{\pi_U}{\to} \mathcal{F}''(U) \tag{4.3}$$

exact, so we merely have to show that π_X is surjective. As such, pick up some $f'' \in \mathcal{F}''(X)$ which we want to exhibit as being in the image of π_X.

By Lemma 4.74, we note that the map $\mathcal{F}_1(U) \to \mathcal{F}_1(U_f)$ need not be surjective, but it is almost surjective: for any $a \in \mathcal{F}_1(U_f)$, there's going to be some n such that af^n does come from $\mathcal{F}_1(U)$. Now we can repeat the proof from homework that showed exactness from when $\mathcal{F}_1$ is flasque. ■

Corollary 4.78. Fix a scheme X with an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

If $\mathcal{F}'$ is quasicoherent, then

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is also exact for any affine open subscheme $U \subseteq X$.

Proof. Label our short exact sequence as

$$0 \to \mathcal{F}' \overset{\alpha}{\to} \mathcal{F} \overset{\beta}{\to} \mathcal{F}'' \to 0.$$

By Corollary 4.30, it is also exact in the category of sheaves of abelian groups on X. Using Lemma 1.181, we see that

$$0 \to \mathcal{F}'|_U \overset{\alpha|_U}{\to} \mathcal{F}|_U \overset{\beta|_U}{\to} \mathcal{F}''|_U \to 0$$

is also exact in the category of sheaves of abelian groups on U, so unwinding with Corollary 4.30, this is also exact in the category of $\mathcal{O}_X$-modules.

We are now ready to apply Proposition 4.77. Note $\mathcal{F}'|_U$ is still quasicoherent by Corollary 4.62, so Proposition 4.77 tells us that

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is exact, which is what we wanted. ■

Remark 4.79. Corollary 4.78 tells us that (later in life) quasicoherent sheaves are going to have trivial cohomology on affine open subschemes.

As another consequence of our understanding of exactness, we have the following.

Corollary 4.80. Fix a scheme X. Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of $\mathcal{O}_X$-modules, if any two of the terms are quasicoherent, then the third is also quasicoherent.

Proof. As usual, let $\mathcal{U}$ be any affine open cover of X. For any $U \in \mathcal{U}$, we find an $\mathcal{O}_X(U)$-module M such that $\mathcal{F}|_U \simeq \widetilde{M}$.

Label our short exact sequence by

$$0 \to \mathcal{F}' \overset{\alpha}{\to} \mathcal{F} \overset{\beta}{\to} \mathcal{F}'' \to 0.$$

As in Corollary 4.78, we note that Corollary 4.30 tells us that the above is also exact in the category of sheaves of abelian groups on X. Thus, Lemma 1.181 tells us that

$$0 \to \mathcal{F}'|_U \overset{\alpha|_U}{\to} \mathcal{F}|_U \overset{\beta|_U}{\to} \mathcal{F}''|_U \to 0$$

is also exact in the category of sheaves of abelian groups on U and thus in the category of $\mathcal{O}_U$-modules by Corollary 4.30.

Now, set $M' := \mathcal{F}'(U)$ and $M := \mathcal{F}(U)$ and $M'' := \mathcal{F}''(U)$ so that Corollary 4.54 grants us canonical morphisms $\varepsilon' \colon \widetilde{M'} \to \mathcal{F}'$ and $\varepsilon \colon \widetilde{M} \to \mathcal{F}$ and $\varepsilon'' \colon \widetilde{M''} \to \mathcal{F}''$. Quickly, note that

$$0 \to M' \overset{\alpha_U}{\to} M \overset{\beta_U}{\to} M'' \to 0$$

is exact by Proposition 4.77 because $\mathcal{F}'$ is quasicoherent. As such, Proposition 4.58 tells us that

$$0 \to \widetilde{M'} \overset{\widetilde{\alpha_U}}{\to} \widetilde{M} \overset{\widetilde{\beta_U}}{\to} \widetilde{M''} \to 0$$

is exact.

Combining, Lemma 4.50 gives us the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M'} \arrow[r, "\widetilde{\alpha_U}"] \arrow[d, "\varepsilon'"'] & \widetilde{M} \arrow[r, "\widetilde{\beta_U}"] \arrow[d, "\varepsilon"'] & \widetilde{M''} \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & \mathcal{F}'|_U \arrow[r, "\alpha|_U"] & \mathcal{F}|_U \arrow[r, "\beta|_U"] & \mathcal{F}''|_U \arrow[r] & 0
\end{tikzcd}$$

with exact rows, as shown above. But ε' and ε'' are isomorphisms because $\mathcal{F}'$ and $\mathcal{F}''$ are quasicoherent, so the Snake lemma (or alternatively, the Five lemma) tells us that ε must be an isomorphism as well. This finishes. ■

4.2.5 Sheaf Theory for Quasicoherent Sheaves

Here are some quick ways to build quasicoherent sheaves.

Proposition 4.81. Fix a scheme morphism $f\colon X \to Y$.

(a) If $\mathcal{G}$ is a quasicoherent sheaf on Y, then $f^*\mathcal{G}$ is a quasicoherent sheaf on X.

(b) If f is quasicompact and quasiseparated, then any quasicoherent sheaf $\mathcal{F}$ on X gives a quasicoherent sheaf $f_*\mathcal{F}$ on Y.

Proof. Here we go.

(a) Reduce to the affine case, as usual. Namely, the result is local on X, so we may show this on an affine open cover of X. For this, we give Y an affine open cover, pull this back to X, and the affine open cover on X we want is going to come from the affine open covers of the pre-images. Now everything is an affine problem.

(b) As usual, the question is local on Y, so we may assume that Y is affine. Thus, f being quasicompact and quasiseparated forces X to be quasicompact and quasiseparated. In particular, it follows that we can give X a finite affine open cover $\{U_i\}_{i=1}^n$ where each of the intersections $U_i \cap U_j$ have finite affine open covers $\{U_{ijk}\}_{k=1}^{n_{ij}}$.

Now, all the maps $f|_{U_i}$ and $f|_{U_{ij}}$ are morphisms of affine schemes, so $(f|_{U_i})_*(\mathcal{F}|_{U_i})$ will all be quasicoherent sheaves on Y because morphisms between affine schemes are all affine. Thus, we think that we can build $f_*\mathcal{F}$ by hand! Namely, we see that $f_*\mathcal{F} = \mathcal{F}(f^{-1}V)$ fits into the exact sequence

$$0 \to \prod_i \mathcal{F}(f^{-1}V \cap U_i) \to \prod_{i,j,k} \mathcal{F}(f^{-1}V \cap V_{ijk})$$

by using the sheaf condition. Lifting this to the level of sheaves, we see

$$0 \to f_*\mathcal{F} \to \prod_i (f|_{U_i})_*(\mathcal{F}|_{U_i}) \to \prod_{i,j,k}$$

is exact, so the morphism on the left is a kernel of morphisms of quasicoherent sheaves and is therefore a quasicoherent sheaf. ■

4.2.6 Closed Embeddings

Given a sheaf $\mathcal{F}$ on x, recall from long ago that we defined $\operatorname{Supp}\mathcal{F} := \{x \in X : \mathcal{F}_x \neq 0\}$. Now, here is our result.

Definition 4.82 (Ideal sheaf)**.** Fix a scheme X. An *ideal sheaf* $\mathcal{I}$ is an $\mathcal{O}_X$-module which is a subsheaf of $\mathcal{O}_X$.

Proposition 4.83. Fix a scheme X. Then there is an equivalence of categories between closed subschemes of X and quasicoherent ideal sheaves.

Proof. Beginning with a closed embedding $\iota\colon Y \to X$, we get a quasicoherent sheaf by taking $\ker \iota^\sharp$. In the other direction, we take a quasicoherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, and we set $Y := \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ on the level of topological spaces and $f^{-1}(\mathcal{O}_X/\mathcal{I})$ on the level of sheaves.

Here are our checks.

- Recall that closed embeddings are quasicompact and quasiseparated[3], so $f_*\mathcal{O}_Y$ is quasicoherent, so the kernel of $\iota^\sharp$ will also be quasicoherent.
- We check that Y is actually a closed subscheme. Note $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is equal to $\{x \in X : \overline{1}_x \neq 0\}$ and is therefore closed; here $\overline{1}$ refers to the global section coming from $1 \in \mathcal{O}_X(X)$. To check that Y is actually a scheme, we check on the level of affine open subschemes. One can check that this gives a closed embedding by hand.

We omit the checks that this is actually an equivalence of categories. ■

Remark 4.84. The forward direction of the above proposition shows that all closed subschemes look like $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$. Namely, one has to show that quasicoherent ideal sheaves $\mathcal{I}$ of an affine scheme all look like $\widetilde{I}$ where $I \subseteq A$ is an ideal, and the above proof has shown that $\operatorname{Supp}(\widetilde{A}/\widetilde{I}) = V(I)$ and $f^{-1}(\mathcal{O}_{\operatorname{Spec} A}/\mathcal{I})$ has the correct structure sheaf.

4.2.7 Scheme Images

We would like to talk about images of schemes. For example, given a quasicompact and quasiseparated morphism $f\colon X \to Y$, then we see $f_*\mathcal{O}_X$ is quasicoherent on Y, so $I = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is going to be a quasicoherent sheaf on Y, and this will define a closed subscheme of Y as above.

In general, $\ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ need not be quasicoherent, but we can still set $\mathcal{I}$ to be the largest quasicoherent subsheaf, and our image is the largest closed subscheme defined by $\mathcal{I}$. Of course, we should probably talk about what the "largest quasicoherent subsheaf" is, but we can simply define it as the sum of all the quasicoherent subsheaves $\mathcal{I}_Z$ where $Z \hookrightarrow Y$ varies over all the closed embeddings through which $X \to Y$ factors.

We close class with the following definition.

Definition 4.85 (Coherent sheaf)**.** Fix a locally Noetherian scheme X. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if and only if X can be covered by affine open subschemes $U_i \subseteq X$ such that $\mathcal{F}|_{U_\alpha}$ is isomorphic to $\widetilde{M_\alpha}$ where each M_α is a finitely generated $\mathcal{O}_X(U_\alpha)$-module.

Remark 4.86. If we remove the locally Noetherian scheme X, then we're going to want our modules M_i to be finitely presented. In the Noetherian case, these are equivalent.

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Term paper topics will be posted later today. They're going to be a bit vague. Other topics are allowed, but email first. You may talk to other people about the topics, but the papers should be different.

4.3.1 Coherent Sheaves

Things are going to get a little abstract. Here we go.

Last time we discussed coherent sheaves for locally Noetherian schemes. Throughout, $\mathcal{F}$ is an $\mathcal{O}_X$-module. Here are some finiteness conditions.

[3] The fact that closed embeddings are quasicompact is because closed subsets of affine schemes are quasicompact. The fact that closed embeddings are quasiseparated comes because all monomorphisms are quasiseparated.

Definition 4.87 (Finite type)**.** An $\mathcal{O}_X$-module $\mathcal{F}$ is of *finite type* if and only if each $x \in X$ has some open $U \subseteq X$ such that there is some n giving a surjection

$$(\mathcal{O}_X|_U)^n \to \mathcal{F}|_U \to 0.$$

Example 4.88. In the affine case, take $X \simeq \operatorname{Spec} A$ and $\mathcal{F} = \widetilde{M}$, where M is an A-module. Now, $\widetilde{M}$ is of finite type if and only if M is a finitely generated A-module.

Definition 4.89 (Finite presentation)**.** An $\mathcal{O}_X$-module $\mathcal{F}$ is of *finite presentation* if and only if each $x \in X$ has some open $U \subseteq X$ such that there are some n and m giving a surjection

$$(\mathcal{O}_X|_U)^m \to (\mathcal{O}_X|_U)^n \to \mathcal{F}|_U \to 0.$$

Example 4.90. In the affine case, take $X \simeq \operatorname{Spec} A$ and $\mathcal{F} = \widetilde{M}$, where M is an A-module. Now, $\widetilde{M}$ is of finite presentation if and only if M is a finitely presented A-module.

And now, the third has a different name.

Definition 4.91 (Coherent)**.** An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type and any $U \subseteq X$ and any map $(\mathcal{O}_X|_U)^n \to \mathcal{F}$ has finite-type kernel.

Of course, being coherent implies finite presentation implies finite type.

Here are some quick remarks.

Example 4.92. In the affine case, take $X \simeq \operatorname{Spec} A$ and $\mathcal{F} = \widetilde{M}$, where M is an A-module. Now, $\widetilde{M}$ is coherent if and only if M and any map $A^n \to M$ has finitely generated kernel.

Remark 4.93. The above conditions can be checked on any given affine open cover, using Lemma 2.118 as usual.

Remark 4.94. We can see that having finite presentation implies that $\mathcal{F}$ is quasicoherent, by pulling back to force $\mathcal{F}|_U$ to be a module in the sequence.

Remark 4.95. In the locally Noetherian case, everything is equivalent because finitely generated modules are also Noetherian.

Remark 4.96. If $\mathcal{O}_X$ is coherent as an $\mathcal{O}_X$-module, then finite presentation is equivalent to being coherent. Essentially all our time will be spent assuming $\mathcal{O}_X$ is coherent, though there are counterexamples.

Remark 4.97. Fix a morphism $f\colon X \to Y$ of locally Noetherian schemes X, Y.

- If $\mathcal{G}$ is quasicoherent over Y, then $f^*\mathcal{G}$ is also quasicoherent, essentially by checking directly on the affine open subschemes.
- If f is finite, then $\mathcal{F}$ being quasicoherent over X implies that $f_*\mathcal{F}$ is also quasicoherent over Y. Approximately speaking, we need f to be finite because if M is finitely generated over an A-algebra, this does not imply that M is finitely generated over A. One can weaken this in various ways.

4.3.2 Vector Bundles

In analogy to geometry, we will want to talk about modules which are locally free.

Definition 4.98 (Locally free)**.** Fix a scheme $(X, \mathcal{O}_X)$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* if and only if $\mathcal{F} \cong \mathcal{O}_X^\lambda$ for some set λ. Further, $\mathcal{F}$ is *locally free* if and only if there is an open cover $\mathcal{U}$ of X such that $\mathcal{F}|_U$ is a free $\mathcal{O}_U$-module for each $U \in \mathcal{U}$.

Definition 4.99 (Rank)**.** Fix a scheme $(X, \mathcal{O}_X)$ and locally free $\mathcal{O}_X$-module $\mathcal{F}$. Then the *rank* of $\mathcal{F}$ at an open subset U is the size of the index set λ where $\mathcal{F}|_U \cong (\mathcal{O}_U)^\lambda$.

Remark 4.100. If X is connected, then the rank is a constant function. Indeed, we can see directly that it is locally constant.

Remark 4.101. Note that locally free implies $\mathcal{F}$ is quasicoherent, and being locally free of finite type implies that $\mathcal{F}$ is coherent, in the case that $\mathcal{O}_X$ is coherent.

And here is our definition.

Definition 4.102 (Vector bundle)**.** A *vector bundle* over a scheme is a locally free sheaf of finite rank. A *line bundle* is a vector bundle of rank 1. Some authors call line bundles "invertible sheaves."

Remark 4.103. Here is a fact from commutative algebra we will want: a module M over a ring A has $\widetilde{M}$ locally free of finite type as an $\mathcal{O}_{\operatorname{Spec} A}$-module if and only if M is a finitely generated projective A-module.

And now, here are some affine examples.

Example 4.104. Given a field k, set $X := \mathbb{A}_k^1$. Then we note finitely generated projective modules over the principal ideal domain $k[x]$ must all actually be free, so the only vector bundles over X are just $\mathcal{O}_{\mathbb{A}_k^1}^r$, where r is the rank.

Example 4.105. Given a field k, set $X := \mathbb{A}_k^n$. In this case, all line bundles are isomorphic to $\mathcal{O}_{\mathbb{A}_k^n}$! We will prove later when we relate line bundles to divisors, so this result will follow from the fact that $k[x_1, \ldots, x_n]$ is factorial.

Here's an arithmetic example.

Example 4.106. Given a number field K, set $X := \operatorname{Spec} \mathcal{O}_K$. Then fractional ideals of $\mathcal{O}_K$ are finitely generated $\mathcal{O}_K$-submodules of K. The fact that $\mathcal{O}_K$ is a Dedekind domain implies that nonzero fractional ideals $\mathfrak{a}$ are invertible; i.e., there is some $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}_K$, or equivalently, $\mathfrak{a}_\mathfrak{p} \simeq (\mathcal{O}_K)_\mathfrak{p}$ for each maximal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$. The point is that we can spread out from $\mathfrak{p}$ to some small open subset around each $\mathfrak{p}$, so $\widetilde{\mathfrak{a}}$ becomes a locally free sheaf of rank 1, which is an "invertible" sheaf.

Remark 4.107. In the above example, we have $\widetilde{\mathfrak{a}} \otimes \widetilde{\mathfrak{b}} \cong \widetilde{\mathfrak{a} \otimes \mathfrak{b}} \cong \widetilde{\mathfrak{a}\mathfrak{b}}$ for fractional ideals $\mathfrak{a}, \mathfrak{b} \subseteq K$. In general, the tensor product of two line bundles is also a line bundle, which will turn the set of line bundles into an abelian group. The identity is the trivial sheaf, and the inverse will be constructed later; this inverse is what motivates the term "invertible sheaf."

Remark 4.108. In $\mathcal{O}_K$, all invertible modules are isomorphic to some fractional ideal. The point is that there is an isomorphism between $\operatorname{Cl} K$ and the isomorphism classes of line bundles under the group operation $\otimes$. (This isomorphism comes from the fact that principal ideals are the ones giving the trivial line bundle.)

Let's explain why we're calling these vector bundles.

Example 4.109. Take $X := \mathbb{P}^1_k = \operatorname{Proj} k[X_0, X_1]$ so that $U_0 = \operatorname{Spec} k[x_{1/0}]$ and $U_1 = \operatorname{Spec} k[x_{0/1}]$ are our affine open pieces. To build a line bundle $\mathcal{F}$ on X, we note that $\mathcal{F}|_{U_0}$ must be $k[x_{1/0}]$ as discussed above, and we note that $\mathcal{F}|_{U_1}$ must be $k[x_{0/1}]$ for the same reason. We need a way to glue these together to a sheaf on X, so let's glue by

$$\begin{array}{ccc} k[x_{1/0}, x_{1/0}^{-1}] & \simeq & k[x_{0/1}, x_{0/1}^{-1}] \\ f & \mapsto & x_{0/1} f \\ x_{1/0} f & \leftarrow\!\shortmid & f \end{array}$$

which we can check to satisfy the cocycle condition. This gives the sheaf $\mathcal{O}_X(1)$, where the 1 here refers to a twist which we will explain later.

From the above example, we can more or less feel this is very similar to constructing a line bundle: we built something on X which locally looked like the line $\mathbb{A}^1_k$.

Example 4.110. More generally, we can take $X := \mathbb{P}^n_k$ covered by the affine open subschemes

$$U_i := \operatorname{Spec} \frac{k[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)}.$$

A line bundle $\mathcal{F}$ on X must be locally isomorphic to U_i on each U_i, so to glue these together, we pick up some integer m and glue by

$$\begin{array}{ccc} k[x_{0/i}, \ldots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1) & \simeq & k[x_{0/j}, \ldots, x_{n/j}, x_{i/j}^{-1}]/(x_{j/j} - 1) \\ f & \mapsto & x_{i/j}^m f \\ x_{j/i}^m f & \leftarrow\!\shortmid & f \end{array}$$

which we can check to satisfy the cocycle condition. This gives the sheaf $\mathcal{O}_X(m)$.

Next time we will give a Proj construction for quasicoherent sheaves, which will recover the above.

4.4 October 17

Welcome back. Today we talk about line bundles, many times.

4.4.1 Quasicoherent Sheaves from Proj

We take a moment to remember the Proj construction. Fix a $\mathbb{Z}_{\geq 0}$-graded ring S, and recall that we had

$$\operatorname{Proj} S := \{\mathfrak{p} \in \operatorname{Spec} S : \mathfrak{p} \not\supseteq S_+\}.$$

We want to create "graded" quasicoherent sheaves from $\mathbb{Z}$-graded modules.

Definition 4.111 (Graded module)**.** Fix a $\mathbb{Z}$-graded ring S. Then a graded S-module is an S-module $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $S_d \cdot M_e \subseteq M_{d+e}$ for any $d, e \in \mathbb{Z}$.

Of course, $\mathbb{Z}_{\geq 0}$-graded rings are also $\mathbb{Z}$-graded rings, so we have also defined graded modules in the cases we care about. Notably, we want to allow our modules to be $\mathbb{Z}$-graded for twisting reasons.

To define our quasicoherent sheaf $\mathcal{M}$ on S, we imitate the construction from Proj. In other words, we will define

$$\widetilde{M}(D_+(f)) \coloneqq (M_f)_0$$

for any homogeneous $f \in S_+$. The usual restriction maps $\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) \to \mathcal{O}_{\operatorname{Proj} S}(D_+(g))$ whenever $D_+(g) \subseteq D_+(f)$ give the restriction maps $M_f \to M_g$ and hence maps $(M_f)_0 \to (M_g)_0$.

Remark 4.112. One can then check that $\widetilde{M}$ is a quasicoherent sheaf and that $\widetilde{M}_{\mathfrak{p}} = (M_{\mathfrak{p}})_0$.

Exercise 4.113. Take $X \coloneqq \mathbb{P}^n_k$. We show that $\mathcal{O}_X(m)$ is $\widetilde{S(m)}$, where $S(m)$ is the ring S twisted by $m \in \mathbb{Z}$.

Proof. We check what happens on the affine open cover. Namely, set $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$ with $U_i \coloneqq D_+(x_i)$, so we can compute

$$(S(m)_{x_i})_0 = \left\{ \frac{f}{x_i^k} : \deg f = k + m \right\}.$$

Notably, we see $\widetilde{S(m)}|_{U_i} \cong \mathcal{O}_{U_i} = \operatorname{Spec} k[x_0/x_i, \ldots, x_n/x_i]$ by taking some f/x_i^k above to f/x_i^{k+m}. Notably we are being assured that $\widetilde{S(m)}$ is a line bundle here.

Now we check the gluing data. Namely, for some $f/(x_i^k x_j^\ell)$ in $(S(m)_{x_i x_j})$ in $\widetilde{S}(m)(U_i \cap U_j)$, this gets sent to $f/(x_i^{k+m} x_j^\ell)$ when looking in U_i but sent to $f/(x_i^k x_j^{k+\ell})$ over in U_j. Thus, our gluing data must be sending $f/(x_i^{k+m} x_j^\ell)$ over in U_i through to $(x_i/x_j)^m$, which is what we wanted. ■

More generally, we have the following.

Proposition 4.114. Fix a $\mathbb{Z}_{\geq 0}$-graded ring S. If S is generated as an S_0-algebra by elements of S_1, then $\mathcal{O}_{\operatorname{Proj} S}(m) \coloneqq \widetilde{S(m)}$ is a line bundle on $\operatorname{Proj} S$.

Proof. The proof of Exercise 4.113 goes through entirely, where general elements of S_1 will play the role of the elements of x_i. ■

4.4.2 Čech Cohomology

Fix a scheme X and a line bundle $\mathcal{L}$ on X. Give X an open cover $\{U_\alpha\}_{\alpha \in \lambda}$ such that we are provided isomorphisms

$$\varphi_i \colon \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}.$$

In total, we see that we have isomorphisms

$$\begin{aligned} \mathcal{O}_{U_i \cap U_j} &= \mathcal{O}_{U_i}|_{U_j} \\ &\overset{\varphi_i^{-1}}{\cong} \mathcal{L}|_{U_i}|_{U_i \cap U_j} \\ &= \mathcal{L}|_{U_j}|_{U_i \cap U_j} \\ &\overset{\varphi_j}{\cong} \mathcal{O}_{U_j}|_{U_i \cap U_j} \\ &= \mathcal{O}_{U_i \cap U_j}. \end{aligned}$$

Thus, we see that a line bundle is providing some isomorphisms φ_i as well as some isomorphisms $\varphi_{ji} \colon \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$ given by $\varphi_{ij} \coloneqq \varphi_j|_{U_i \cap U_j} \circ \varphi_i^{-1}|_{U_i \cap U_j}$.

Going further, we see that $\alpha_{ji} := \varphi_{ji}(1) \in \mathcal{O}_{U_i \cap U_j}(U_i \cap U_j)^\times$, and in fact the full isomorphism will be determined by where it sends 1 because these are $\mathcal{O}_{U_i\cap U_j}$-modules. Notably, $\alpha_{ij}\alpha_{ji} = 1$ because φ_{ij} and φ_{ji} are inverse. We now draw our complex to make our cohomology: we will build

$$\begin{array}{ccccc} \prod_{\alpha} \mathcal{O}_X^\times(U_\alpha) & \overset{d_1}{\to} & \prod_{\alpha,\beta} \mathcal{O}_X^\times(U_\alpha \cap U_\beta) & \overset{d_2}{\to} & {}_{\alpha,\beta,\gamma}\mathcal{O}_X^\times(U_\alpha \cap U_\beta \cap U_\gamma) \\ (f_\alpha) & \mapsto & (f_\alpha f_\beta^{-1})_{\alpha,\beta} & & \\ & & (f_{\alpha\beta})_{\alpha,\beta} & \mapsto & (f_{\alpha\gamma}^{-1} f_{\alpha\beta} f_{\beta\gamma})_{\alpha,\beta,\gamma} \end{array} \tag{4.4}$$

as our complex; one can in fact check that $d_2 \circ d_1 = 0$, which is essentially by construction. This complex can continue, though we will not write mote.

Remark 4.115. The entire story can be told above for more general line bundles, where multiplicative units (which are just $\mathrm{GL}_1 = (\cdot)^\times$) gets replaced by GL_n. What's difficult here is that GL_n is no longer abelian.

Remark 4.116. Here's how to remember these signs: start with ijk, and number them $0, 1, 2$. Then we are summing $(-1)^0 jk + i(-1)^1 k + ij(-1)^2$, which is the next one.

Remark 4.117. To see that this is the start of the complex we want, we can see that the $\varphi_{ij}(1)$ will need to be in the kernel of d_2, basically by how gluing data works.

Let's see what happens if we are given some different isomorphisms $\varphi_i' \colon \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Then we can chain the isomorphisms

$$\mathcal{O}_{U_i} \overset{\varphi_i^{-1}}{\cong} \mathcal{L}|_{U_i} \overset{\varphi_i'}{\cong} \mathcal{O}_{U_i}.$$

Thus, $\varphi_i' \circ \varphi_i^{-1}$ is just another isomorphism of $\mathcal{O}_{U_i}$ and will again be determined by what it does on the global section 1, so we set $h_i := \varphi_i \circ (\varphi_i')^{-1}(1)$ to track our transition functions.

By unwinding definitions, we see that $h_i\varphi_{ij}'(1) = \varphi_{ij}(1) \cdot h_j$, which means that the "difference" between the isomorphisms φ_i and φ_i' correspond to an element of the image of d_1, namely we are looking at $d_1((h_\alpha)_\alpha)$. Thus, we expect that the first cohomology H^1 of (4.4) is just isomorphisms classes of line bundles! With this motivation, we have the following definition.

Definition 4.118. Given a scheme X, we define

$$\check{H}(X, \mathcal{O}_X^\times) := \varinjlim_{\mathcal{U} \text{ cover of } X} H^1(\mathcal{U}, \mathcal{O}_X^\times),$$

where we are using the complex of (4.4) to define the H^1 on the right.

And here is our result.

Proposition 4.119. Fix a scheme X. Then $\check{H}^1(X, \mathcal{O}_X^\times)$ is in natural bijection with isomorphism classes of line bundles on X.

Proof. Fix two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, endowed with their isomorphisms φ_α^1 and φ_β^2 for some open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of X. Then we can see that

$$(\mathcal{L}_1 \otimes \mathcal{L}_2)|_{U_\alpha} \simeq \mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} \simeq \mathcal{O}_{U_i},$$

so the tensor product is in fact sending line bundles to line bundles.

We now prove the proposition more directly. In one direction, fix some $(f_{\alpha\beta})_{\alpha,\beta\in\lambda} \in \ker d_2$ from (4.4). This gluing data satisfies our cocycle condition (by being in the kernel of d_1), and say $f_{\alpha\alpha} \colon \mathcal{O}_{U_\alpha} \to \mathcal{O}_{U_\alpha}$ tells us that the things we are gluing are in fact locally one-dimensional. In total, we get to glue a sheaf together, which we see is a line bundle.

As another group-theoretic remark, we note that a line bundle $\mathcal{L}$ equipped with its isomorphisms $\varphi_{\alpha\beta}$, gluing the inverse morphisms $\varphi_{\alpha\beta}^{-1}$ together will give us a line bundle $\mathcal{L}^{-1}$, and we can check by our construction of the group law to have $\mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}_X$.

We now return to the proposition. In the other direction, suppose that $\mathcal{L}$ and $\mathcal{L}'$ are line bundles, which have equipped with them their isomorphisms (f_{ij}) and (f'_{ij}), respectively, such that $(f_{ij}) - (f'_{ij}) \in \operatorname{im} d_1$. Notably, by replacing $\mathcal{L}$ and $\mathcal{L}'$ with $\mathcal{L} \otimes (\mathcal{L}')^{-1}$ and $\mathcal{O}_X$ respectively, and using the group law as described above, we may just assume that $(f_{ij}) \in \operatorname{im} d_1$, and we are trying to show that $\mathcal{L}$ is trivial.

Well, we are told $\varphi_i \circ \varphi_j^{-1} = f_{ij} = d_1((h_i))_i = h_i h_j^{-1}$. As such, we may consider the isomorphisms given by

$$h_i \varphi_i^{-1} \colon \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i},$$

and we can check that these isomorphisms glue together (namely, checking the cocycle condition) to get an isomorphism $\mathcal{L} \cong \mathcal{O}_X$, which is what we wanted. ■

Next time we will move on to divisors.

THEME 5
DIVISORS

Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him.

—Felix Klein, [Kle16]

5.1 October 19

Today we discuss divisors somewhat.

5.1.1 Line Bundles

Fix a scheme X.

Notation 5.1. Given a scheme X, we define $\operatorname{Pic} X$ to be the abelian group of isomorphism classes of line bundles, where the group law is given by $\otimes_{\mathcal{O}_X}$.

Remark 5.2. In some cases, $\operatorname{Pic} X$ can further be given a scheme structure.

This abelian group is difficult to study in general, but we will see some examples.

Remark 5.3. Given a scheme morphism $\varphi\colon X \to Y$, a line bundle $\mathcal{L}$ on Y can be pulled back to $\varphi^*\mathcal{L}$, so we induce a group homomorphism $\operatorname{Pic} Y \to \operatorname{Pic} X$.

5.1.2 Divisors

Let's now discuss divisors. Fix a Noetherian integral scheme X.

Definition 5.4 (Weil divisor)**.** A *Weil divisor* of a Noetherian integral scheme X is a (formal) finite $\mathbb{Z}$-linear combination of codimension-1 closed integral subschemes of X. We will often write divisors D as

$$D = \sum_{\substack{Y \subseteq X \\ Y \text{ closed} \\ \operatorname{codim} Y = 1}} n_Y [Y],$$

where the n_Y vanish for all but finitely many Y.

Remark 5.5. The set of Weil divisors has the obvious group structure by addition pointwise.

One can allow non-integral (namely, non-reduced) things to sneak into our divisors by adding in this information to the multiplicity, but we will not do so.

Definition 5.6 (Effective)**.** Fix a Noetherian integral scheme X. A Weil divisor $D = \sum_Y n_Y [Y]$ is *effective* if and only if $n_Y \geq 0$ for each Y.

Definition 5.7 (Support)**.** Fix a Noetherian integral scheme X. The *support* of a Weil divisor $D = \sum_Y n_Y [Y]$ is

$$\operatorname{Supp} D \coloneqq \bigcup_{\substack{Y \subseteq X \\ n_Y \neq 0}} Y.$$

Example 5.8. Fix a number field $K/\mathbb{Q}$ and $X \coloneqq \operatorname{Spec} \mathcal{O}_K$, which we can see is a Noetherian integral scheme. Then Weil divisors are just $\mathbb{Z}$-linear formal combinations of the maximal ideals $[\mathfrak{p}]$. Namely, we are only looking at maximal ideals because we want to have codimension 1.

Example 5.9. Fix a normal quasiprojective curve C over an algebraically closed field k. Then a Weil divisor is again just $\mathbb{Z}$-linear formal combinations of the closed points on C. We are implicitly using the fact that closed points are our codimension-1 closed embeddings.

The fact that a point $x \in X$ might have codimension 1 is saying that $\mathcal{O}_{X,x}$ is a local ring of dimension 1. Notably, we are already regular, namely $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x} = 1$, so the corresponding local rings are all discrete valuation rings. One can even give this valuation explicitly: pick $\pi \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$, and then for any $f \in \operatorname{Frac} \mathcal{O}_{X,x}$, our valuation is the integer n for which $f/\pi^n \in \mathcal{O}_{X,x}^\times$.

These notions give us some divisors.

Definition 5.10 (Principal divisor)**.** Fix a Noetherian integral scheme X whose local rings are regular in codimension-1. Given some $f \in K(X) \setminus \{0\}$, we define the *principal divisor*

$$\operatorname{div}(f) \coloneqq \sum_{Y \subseteq X} v_Y(f)[Y],$$

where v_Y is the valuation of f above at the generic point of Y.

Remark 5.11. If X is normal, then the local rings will be regular.

Remark 5.12. Thus, we have a function div which actually defines a group homomorphism from $K(X)^\times$ to the group of Weil divisors.

5.1.3 Divisors for Line Bundles

We would like to generalize Definition 5.10 to line bundles. More generally, given a line bundle $\mathcal{L}$ over X, some $s \in \Gamma(U, \mathcal{L}) \setminus \{0\}$ for $U \subseteq X$ open and dense is a rational section of $\mathcal{L}$. This exactly generalizes $f \in K(X)^\times$.

This gives the following definition.

Definition 5.13. Fix a line bundle $\mathcal{L}$ on a normal Noetherian integral scheme X. Give X a finite affine open cover $\{U_i\}_{i=1}^n$ such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$ for each i. Then a rational function $s \in \Gamma(U, \mathcal{L})$ will glue together into a divisor $\operatorname{div} s$.

What?

Let's see an example.

Exercise 5.14. Fix $X := \mathbb{P}^1_k$, where k is a field. We run computations on $\mathcal{O}_X$ and $\mathcal{O}_X(1)$.

Proof. Here are some examples. We think of $\mathbb{P}^1_k$ as the one-point compactification of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. Namely, with $\mathbb{P}^1_k = \operatorname{Proj} k[X_0, X_1]$, we have $x = X_1/X_0$. Set $U_0 := \operatorname{Spec} k[X_1/X_0]$ and $U_1 := \operatorname{Spec} k[X_0/X_1]$.

- The rational function $1 \in \Gamma(X, \mathcal{O}_X)$ gives rise to $\operatorname{div} 1 = 0$ because it vanishes nowhere.
- The rational function $X_0/X_1 \in \Gamma(X, \mathcal{O}_X)$ we only need to compute on the open dense subset $\operatorname{Spec} k[x]$, so we get a zero on $[\infty]$ and a pole at $[0]$, so the divisor here is $[\infty] - [0]$.
- The rational function $X_0 \in \Gamma(X, \mathcal{O}_X(1))$ gets twisted over to the generated divisor $[\infty]$. Notably, the number of poles does not match the number of zeroes! Similarly, the rational function $X_1 \in \Gamma(X, \mathcal{O}_X(1))$ goes over to $[0]$. ∎

Definition 5.15 (Weil divisor class group)**.** Fix a Noetherian integral scheme X whose local rings are regular in codimension 1. Then we define the *Weil divisor class group* $\operatorname{Cl} X$ as the Weil divisors modded out by the principal divisors.

Remark 5.16. The divisor map will send pairs $(\mathcal{L}, s)$ of a line bundle and a rational section s on $\mathcal{L}$ to the Weil divisors, using Definition 5.13. Modding out by isomorphism on both sides builds a map $\operatorname{Pic} X \to \operatorname{Cl} X$.

We would like for the morphism of Remark 5.16 to be an isomorphism, but it need not be in general. For this, we need more scheme adjectives.

5.1.4 Normal, Regular, Smooth Schemes

Normal will make the map Remark 5.16 injective, regular will make it an isomorphism, but smoothness is easier to check than regularity.

We start from commutative algebra. Here is regular.

Definition 5.17 (Regular)**.** Fix a Noetherian local ring A with maximal ideal $\mathfrak{m}$. Then A is *regular* if and only if

$$\dim A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2.$$

Definition 5.18 (Regular)**.** A locally Noetherian scheme X is *regular at* $x \in X$ if and only if the stalk $\mathcal{O}_{X,x}$ is regular. The scheme X is *regular* if and only if it is regular everywhere.

Remark 5.19. A scheme X will be regular if and only if it is regular at the closed points.

Remark 5.20. The main point here is that $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ is supposed to be the dimension of the (co)tangent space at a closed point, so we are kind of asking for the tangent space and the actual scheme to have the same dimension. This is analogous to asking for smoothness, as we will see momentarily.

Example 5.21. Set $R := \mathbb{Z}_p[x, y]/(xy - p)$. Then R is regular: its closed point is $\mathfrak{m} := (p, x, y)$, and then we can directly compute

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(p, x, y)}{(xy - p, p^2, x^2, y^2, px, py, xy)} = \frac{(p, x, y)}{(xy, p, x^2, y^2)}$$

is generated by x and y as an $\mathbb{F}_p$-module.

5.2 October 21

Sleep is less important than problem set, I suppose.

5.2.1 Regular Schemes

We pick up a few more commutative algebra facts.

Theorem 5.22. A Noetherian regular local ring A is a unique factorization domain. Every unique factorization domain is normal.

Corollary 5.23. A regular Noetherian scheme is normal.

Proof. Check affine-locally. ■

Let's see some examples.

Example 5.24. Fix a regular Noetherian local ring A so that we get a unique factorization domain. If $\dim A = 0$, then $\mathfrak{m}/\mathfrak{m}^2 = 0$, so $\mathfrak{m}$ is the unique maximal ideal, so A is a field. If $\dim A = 1$, then A is a discrete valuation ring.

Example 5.25. Fix an algebraically closed field k. Then we see $\mathbb{A}_k^n$ is regular. Recall that we only need to check regularity on closed points, which are maximal ideals $\mathfrak{m} := (x_1 - a_1, \ldots, x_n - a_n) \subseteq k[x_1, \ldots, x_n]$. Then we can directly compute our basis as

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = k\langle x_1 - a_1, \ldots, x_n - a_n\rangle.$$

Example 5.26. We check that $\mathbb{A}^n_{\mathbb{Z}}$ is regular. Fix a closed point $\mathfrak{m}_x \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. Note $\mathfrak{m}_x \cap \mathbb{Z} = (p)$ for some prime p, so we can think of $\mathfrak{m}_x/(p)$ has living in $\mathbb{F}_p[x_1, \ldots, x_n]$. Attempting to induct downwards, we next look at

$$\frac{\mathfrak{m}_x}{(p)} \cap \mathbb{F}_p[x_1] = (p_1(x_1))$$

for some irreducible $p_1(x_1)$. Continuing, we can look in

$$\frac{\mathfrak{m}_x}{(p, p_1)} \cap \frac{\mathbb{F}_p[x_1]}{(p_1(x))} = (p_2(x_1, x_2)).$$

After long enough, we get $\mathfrak{m}_x = (p, p_1, \ldots, p_n)$, where $p \in \mathbb{Z}$ and $p_i \in \mathbb{F}_p[x_1, \ldots, x_i]$ are all irreducible. At the end of the day, we can compute $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 \le n+1$ because of our spanning set, but we must have $\dim \mathcal{O}_{X,x} = n + 1 \le \dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2$, so the equality follows.

Non-Example 5.27. Fix $A := k[x, y, z]/(x^2 - yz)$ is not regular. Namely, localizing at (x, y, z) gives a normal but not regular ring.

Remark 5.28. In good enough curves, being normal, regular, and smooth are all the same. This explains why we see a singularity in the previous example despite being normal.

5.2.2 Smooth Schemes

We can kind of feel that being regular is pretty annoying. Here is our definition of smoothness.

Definition 5.29 (Smooth)**.** Fix a k-scheme X. Then X is *smooth of dimension* d if and only if X has an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ such that $U_\alpha \cong \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ has the Jacobian matrix

$$\left[\left.\frac{\partial f_j}{\partial x_i}\right|_x\right]_{ij}$$

has rank $n - d$ at each x.

Remark 5.30. Smooth implies locally of finite type, which we can see from inside the definition.

Remark 5.31. Geometrically, we are saying that we locally look like some $\mathbb{R}^d$. In some sense, this is just using the Implicit function theorem to look locally.

Example 5.32. Fix an elliptic curve $\operatorname{Spec} k[x, y]/\left(y^2 - f(x)\right)$, where $\deg f = 3$ has nonzero discriminant, with $\operatorname{char} k \ne 2$. Now, we can compute our Jacobian is

$$\begin{bmatrix} -f'(x) & 2y \end{bmatrix}$$

so that the rank of the Jacobian vanishes if and only if both $y = 0$ and $f'(x) = 0$. However, this requires $f(x) = 0$ and $f'(x) = 0$, meaning that f has a double root, which is not allowed when f has vanishing discriminant.

Non-Example 5.33. As before, take $\operatorname{Spec} k[x, y, z]/(x^2 - yz)$. Then our Jacobian is

$$\begin{bmatrix} 2x & -z & -y \end{bmatrix}$$

which vanishes at $(x, y, z) = (0, 0, 0)$.

Example 5.34. Given a field k, we see $\mathbb{A}_k^n$ is smooth by doing the check directly. Similarly, $\mathbb{P}_k^n$ is smooth by doing the usual affine open cover.

Here's a check for varieties.

Proposition 5.35. Fix a k-variety X. If X is smooth at a point x, then X is regular at x. If k is perfect (and, say, algebraically closed), the converse also holds.

Proof. As usual, for a regularity check, we only need to check at closed points. Note that our "affine case" $\mathbb{A}_k^n$ is already smooth, so we hope that the Jacobian having the correct rank should save us.

Locally, because X is a k-variety, we may build a map $X \to \mathbb{A}_k^n$ such that $x \mapsto y$ for some closed point $y \in \mathbb{A}_k^n$. (We have implicitly replaced X with an affine open subscheme containing x.) Now, we see that

$$\mathcal{O}_{X,x} \simeq \frac{\mathcal{O}_{\mathbb{A}_k^n,y}}{(f_1, \ldots, f_r)}.$$

The Jacobian having full rank of $n-d$ implies that the elements $\overline{f}_\bullet$ are linearly independent in $\mathfrak{m}_y/\mathfrak{m}_y^2$. (Note that we can filter out from the $f_\bullet$ until we have $r = n - d$ to get the needed linear independence.) Now, we have the following fact.

Lemma 5.36. Fix a regular local ring $(A, \mathfrak{m})$ and some $f \in \mathfrak{m} \setminus \{0\}$. Then A/fA is regular if and only if $f \notin \mathfrak{m}^2$.

Applying the lemma inductively over all our generators implies that $\mathcal{O}_{X,x}$ is regular, using the regularity of $\mathcal{O}_{\mathbb{A}_k^n}$ which we already have.

Conversely, take k algebraically closed. From earlier, we still have the map $\mathcal{O}_{\mathbb{A}_k^n,y} \to \mathcal{O}_{X,x}$. However, applying the lemma again, we see that $\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x} = d$ if and only if the Jacobian has rank d. Indeed, we claim that

$$\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = n - \operatorname{rank} J_x,$$

where J_x is our Jacobian. To see this, simply unwind the above end of the previous direction to go extract the needed $f_\bullet$, using the fact that our field is algebraically closed. ■

Non-Example 5.37. Take $k := \mathbb{F}_p(t)$ for p odd with $X := \operatorname{Spec} k[x, y]/\left(y^2 - x^p + t\right)$. Then $\dim X = 1$, and we can check by hand that X is smooth at all points with $y \neq 0$: our Jacobian is

$$\begin{bmatrix} 0 & 2y \end{bmatrix}$$

which we can see will not be smooth at, say, the point $(y) = (y, x^p - t)$. However, X is actually regular at y. Indeed, we can check by hand that $\mathfrak{m}_{(y)}/\mathfrak{m}_{(y)}^2 \simeq ky$ as k-vector spaces, which is dimension 1.

Let's generalize our definition of smoothness a little.

Definition 5.38 (Smooth)**.** A scheme morphism $f\colon X \to Y$ is *smooth of relative dimension* d if and only if there is an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of X and $\{V_\alpha\}_{\alpha\in\lambda}$ of Y (with $f(U_\alpha) \subseteq V_\alpha$ for each α) such that the induced maps $f|_{U_i}\colon U_i \hookrightarrow V_i$ can be factored through an open embedding

$$U_i \hookrightarrow \operatorname{Spec} \frac{R[x_1, \ldots, x_{d+r}]}{(f_1, \ldots, f_r)} \to \operatorname{Spec} R_i$$

(where $R_i := \mathcal{O}_Y(V_i)$) such that the Jacobian J has rank r at all points in U.

Equivalently, we can ask for

$$\det \left[\frac{\partial f_j}{\partial x_i}\bigg|_x \right]_{1\le i,j\le r}$$

where the x_is have possibly been rearranged.

Remark 5.39. Note that we are implicitly just saying that open embeddings should be smooth.

Example 5.40. The morphism $\mathbb{A}^n_R \to \operatorname{Spec} R$ and $\mathbb{P}^n_R \to \operatorname{Spec} R$ are smooth. One can just check this directly from the definition using the obvious open covers.

Remark 5.41. Smoothness is preserved by composition, base-change, and is affine-local on the target. So the morphisms $\mathbb{A}^n_S \to S$ and $\mathbb{P}^n_S \to S$ are both smooth for general schemes S.

5.3 October 24

We continue.

5.3.1 More on Smooth Schemes

Here is a better way to think about smoothness.

Proposition 5.42. Fix Noetherian schemes X and Y of finite type over a field k. A morphism $f\colon X \to Y$ is smooth if and only if

(i) f is flat, and

(ii) the fiber $X_{\overline{y}}$ is smooth over $\overline{k}$ for any geometric point $\overline{y} \in Y(\overline{k})$.

Proof. Omitted. ∎

Remark 5.43. The point is that we can check smoothness on geometric points, with some small coherence condition on points.

Wait, what does flat mean?

Definition 5.44 (Flat)**.** A scheme morphism $f\colon X \to Y$ is *flat* if and only if each $x \in X$ has $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,f(x)}$-module for any point $x \in X$.

Non-Example 5.45. Note that $X := \operatorname{Spec} k[x,y]/(xy)$ is not flat over $\operatorname{Spec} k[x]$ because the fibers have different dimensions: (x) has one-dimensional fiber while $(x-a)$ has zero-dimensional fiber for any $a \in k^\times$. Alternatively, we can see this algebraically from the fact that $x \in k[x]$ is not a zero-divisor even though $x \in k[x,y]/(xy)$ is.

From Non-Example 5.45, we can kind of feel that dimension-changing is a bad property, and it seems related to breaking flatness. Here is this check.

Theorem 5.46. Fix a morphism $f\colon X \to Y$ of regular schemes. If all fibers have dimension n, then f is flat.

Proof. We omit this proof, but it is very useful. ■

Remark 5.47. One can weaken the hypotheses to make X Cohen–Macaulay, but we will not define this here.

Non-Example 5.48. One can build a morphism between smooth schemes which is not smooth. For example, define $f\colon \mathbb{A}^2_k \to \mathbb{A}^2_k$ to make the fibers squish in some sense; i.e., send $(x,y) \mapsto (xy,y)$.

Nonetheless, we have the following result.

Proposition 5.49. Fix a smooth morphism $f\colon X \to Y$ such that Y is regular and locally Noetherian. Then X is regular.

Proof. The point is to use Nakayama's lemma. As usual, it suffices to check regularity on closed points, so fix a closed point $x \in X$. By continuity, we note that $y := f(x)$ is also a closed point in Y. By induction on $\dim Y$ and using flatness, we get

$$\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}.$$

To see this, note that there is nothing to see in the case where Y is zero-dimensional. Then to upgrade to the one-dimensional case, we note that regularity means that $\mathcal{O}_{Y,y}$ is cutting out by a single equation, so flatness means the cutting-out polynomial is not a zero-divisor and will appropriately lift to $\mathcal{O}_{X_y,x}$.

However, we note X_y is smooth, so we may generate $\mathfrak{m}_{X_y,x}$ by $\dim \mathcal{O}_{X_y,x}$ elements, but the map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_y,x}$ allows us to actually pick these elements from $\mathcal{O}_{X,x}$. On the other hand,

$$\mathcal{O}_{X_y,x} \simeq \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x},$$

but the regularity of Y now forces $\mathfrak{m}_{Y,y}$ to be generated by $\dim \mathcal{O}_{Y,y}$ elements, which then generates $\mathcal{O}_{X,x}$ by the requested number of elements as we see from the previous dimension computation. ■

Example 5.50. If X is smooth over a field k, then X is regular over k.

Example 5.51. Fix a polynomial $f(x)$, and set $X := \operatorname{Spec} \mathbb{Z}[1/N][x,y]/\left(y^2 - f(x)\right)$ for any N such that $p \mid 2 \operatorname{disc} f$ implies $p \mid N$. We computed earlier that X is smooth over $\operatorname{Spec} \mathbb{Z}[1/N]$, so X is also regular.

Remark 5.52. To review, smooth schemes over a regular scheme are regular. Regular schemes have factorial local rings, which implies the scheme is normal. Lastly, a scheme being normal implies being regular in codimension 1.

5.3.2 Back to Divisors

Here is our result.

Theorem 5.53. Fix a Noetherian, integral scheme X which is regular in codimension 1. Now, consider the induced map

$$\operatorname{div}\colon \operatorname{Pic} X \to \operatorname{Cl} X,$$

where $\operatorname{Pic} X$ is of isomorphism classes of line bundles, and $\operatorname{Cl} X$ is the class group of Weil divisors.

(a) If X is normal, then div is injective.

(b) If X is locally factorial, then div is an isomorphism.

Proof of (a). Unsurprisingly, we will proceed one at a time.

1. We show that the map sending pairs $(\mathcal{L}, s)$ (up to isomorphism) to Weil divisors is injective. Suppose that a line bundle $\mathcal{L}$ and a rational section s gives $\operatorname{div}(\mathcal{L}, s) = 0$. This means that s has no poles or zeroes on any closed integral subscheme $Y \subseteq X$ of codimension 1. In particular, reducing to the affine case, picking an affine open subscheme $U \subseteq X$ and some point $x \in U$ of codimension 1, we have

   $$s \in U_x.$$

   However, for a normal Noetherian ring R, we will have

   $$R = \bigcap_{\operatorname{codim} \mathfrak{p} = 1} R_{\mathfrak{p}},$$

   so it follows that we can lift s to $\mathcal{L}(U)$. Gluing over all U, we see that $s \in \mathcal{L}(X)$.

   However, the map $\mathcal{O}_X \to \mathcal{L}$ given by $1 \mapsto s$ is actually an isomorphism because has no zeroes or poles on X. Indeed, s having no zeroes implies that the isomorphism $\mathcal{L}|_U \simeq \mathcal{O}_U$ for each U for which $\mathcal{L}$ is locally trivial, but $\mathcal{O}_X|_U \simeq \mathcal{O}_U$ is an isomorphism and factors through the $1 \mapsto s$ map, so we conclude that $\mathcal{L}$ is in fact trivial.

2. We show that the map $\operatorname{Pic} X \to \operatorname{Cl} X$ is injective. Indeed, we can compute that $\operatorname{div}(\mathcal{L}, s) = \operatorname{div} f$ for some $f \in K(X)$ implies that $\operatorname{div}(\mathcal{L}, f^{-1}s)$, so the above argument forces us to have $\mathcal{L}$ trivialized by $f^{-1}s$. So the divisor here will vanish. ■

To finish the proof of Theorem 5.53, we need to be able to go backward from Weil divisors to line bundles. We do this by introducing Cartier divisors.

Definition 5.54 (Sheaf of total fractions)**.** Fix a scheme X. Then we define the *sheaf of total fractions* $\mathcal{K}_X$ as the sheafification of the presheaf sending each open $U \subseteq X$ to $S_U^{-1}\mathcal{O}_X(U)$, where $S_U \subseteq \mathcal{O}_X(U)$ are the non-zero-divisors.

Definition 5.55 (Cartier divisor)**.** Fix a scheme X. Then a *Cartier divisor* is a global section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$. A *principal Cartier divisor* is a Cartier divisor coming from a global section of $\mathcal{K}_X^\times$,

$$\operatorname{CaCl} X := \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)/\Gamma(X, \mathcal{K}_X^\times).$$

Remark 5.56. Concretely, given an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$, a Cartier divisor of X consists of the data (U_α, f_α) where $f_\alpha \in S_{U_\alpha}^{-1}\mathcal{O}_X(U_\alpha)$ such that $f_\alpha/f_\beta \in \mathcal{O}_{U_\alpha\cap U_\beta}^\times$. This package of data might be called an "effective" Cartier divisor when each f_α lives in $\mathcal{O}_X(U_\alpha)$.

Proposition 5.57. Fix a scheme X. We define an injective group homomorphism $\operatorname{CaCl} X \to \operatorname{Pic} X$ which is an isomorphism when X is integral.

Construction of the map. Fix a Cartier divisor D consisting of the data (U_α, f_α), where $\{U_\alpha\}_{\alpha\in\lambda}$ is some affine open cover. We now define the sheaf $\mathcal{O}_X(D)$ on X such that

$$\mathcal{O}_X(D)|_{U_\alpha} = f_\alpha^{-1}\mathcal{O}_{U_\alpha} \subseteq \mathcal{K}_{U_\alpha}.$$

Note that this is independent of our precise choice of U_α and f_α to represent D by the coherence condition. We now get a line bundle by setting $f_{ij} = f_i|_{U_i\cap U_j} \cdot f_j^{-1}|_{U_i\cap U_j}$, which makes an element of $\check{H}(X, \mathcal{O}_X^\times)$. ■

5.4 October 26

We continue our discussion of divisors.

5.4.1 More Back to Divisors

We continue with the proof of the following result from last class.

Proposition 5.57. Fix a scheme X. We define an injective group homomorphism $\operatorname{CaCl} X \to \operatorname{Pic} X$ which is an isomorphism when X is integral.

Proof. Last class we constructed this map. We now show that the map is injective: we can check that the map $D \mapsto \mathcal{O}_X(D)$ is a group homomorphism by its construction, so it suffices to show that it has trivial kernel.

Well, if we have some $D = \{(U_\alpha, f_\alpha)\}_{\alpha\in\lambda}$ going to 0 in $\operatorname{Pic} X$, then we see that we are promised an isomorphism

$$\mathcal{O}_X \simeq \mathcal{O}_X(D) \subseteq \mathcal{K}_X,$$

so tracking where the global section 1 goes through grants us some $f \in \mathcal{K}_X$ and get that f "generates" $\mathcal{O}_X(D)$. In particular, we see that the global section f has $\operatorname{div}(f^{-1}) = D$ by construction, so $\mathcal{O}_X(D)$ is in fact a principal Cartier divisor.

We now show that we have an isomorphism. Fix a line bundle $\mathcal{L}$. If we have $\mathcal{L} \subseteq \mathcal{K}_X$ already, then we can go pick up its affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ upon which it's locally trivial, and then we can do a spreading argument around each point in X to go find the corresponding Cartier divisor D with $\mathcal{L} = \mathcal{O}_X(D)$.

So we have to provide an embedding $\mathcal{L} \hookrightarrow \mathcal{K}_X$. Well, given an open subscheme $j\colon U \subseteq X$ such that $\mathcal{L}|_U \simeq \mathcal{O}_X|_U$, we get to write

$$\mathcal{L} \subseteq j_*(\mathcal{L}|_U) \simeq j_*(\mathcal{O}_X|_U) \subseteq j_*(\mathcal{K}_U) \simeq \mathcal{K}_X,$$

which is what we wanted. Notably, the last isomorphism holds because X is integral. ■

Remark 5.58. Let's explain this result philosophically. Note that the short exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 1,$$

so a good enough cohomology theory tells us that we should have an exact sequence

$$\check{H}^0(X,\mathcal{K}_X^\times) \to \check{H}^0(X,K_X^\times/\mathcal{O}_X^\times) \to \check{H}^1(X,\mathcal{O}_X^\times) \to \check{H}^1(X,\mathcal{K}_X^\times).$$

By definition, we get $\check{H}^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$, and $\check{H}^1(X,\mathcal{O}_X^\times) \simeq \operatorname{Pic} X$, so we have induced a map

$$\frac{\Gamma(X,\mathcal{K}_X^\times/\mathcal{O}_X^\times)}{\Gamma(X,\mathcal{K}_X^\times)} \hookrightarrow \check{H}^1(X,\mathcal{O}_X^\times) \simeq \operatorname{Pic} X,$$

so our map is indeed injective. One can show that $\check{H}^1(X,\mathcal{K}_X^\times)$ vanishes when X is integral by some computation, which is "why" we get an isomorphism.

Remark 5.59. One can show that effective Cartier divisors form a group isomorphic to pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is a line bundle and $s \in \Gamma(X,\mathcal{L})$ where the map $\mathcal{O}_X \to \mathcal{L}$ given by multiplication by s is injective. When X is reduced, we're essentially requiring that $s \neq 0$.

Do we want s=0?

5.4.2 Adding in Regularity

We now return to the proof of our main result.

Theorem 5.60. Fix a regular Noetherian scheme X. Then we have a natural isomorphism $\operatorname{Pic} X \to \operatorname{Cl} X$.

Proof. All connected components of X we can show to be integral, so we may assume that X is integral. Note that we certainly have a map

$$\operatorname{CaCl} X \to \operatorname{Weil} X$$

which we can see to be injective from the proof of injectivity above. Namely, this is the composite

$$\operatorname{CaCl} X \to \{(\mathcal{L}, s) : s \in \Gamma(X,\mathcal{L})\} \to \operatorname{Weil} X$$

which we can see to be injective.

It remains to go in the other direction. Fix a Weil divisor $D = \sum_{[Y]} n_Y[Y]$; it suffices to just get a Cartier divisor for just $[Y]$. Now, for each $x \in X$, we note that $\mathcal{O}_{X,x}$ is a regular local domain and hence factorial. Now, each closed subscheme Y has associated to it an ideal sheaf $\mathcal{I}_Y \subseteq \mathcal{O}_X$, and so we get an ideal

$$\mathcal{I}_{Y,x} \subseteq \mathcal{O}_{X,x}.$$

Because $\mathcal{O}_{X,x}$ is factorial, all its height-1 primes are principal (!), and $\mathcal{I}_{Y,x}$ has codimension 1, so we conclude that $\mathcal{I}_{Y,x} = (f_x)$ for some f_x by running through all the primes containing it. Because we're looking at germs, we actually have $f_x \in \Gamma(U_x, \mathcal{O}_X)$ for some open neighborhood U_x of x, and it follows from this construction that

$$f_x/f_y \in \Gamma(U_x \cap U_y, \mathcal{O}_X)^\times.$$

In total, we have created a Cartier divisor $\{(U_x, f_x)\}_{x\in X}$, which is what we wanted. ■

Let's see some examples.

Example 5.61. Recall that $\mathbb{A}_k^n$ has all the adjectives, so

$$\operatorname{Pic}(\mathbb{A}_k^n) = \operatorname{Cl}(\mathbb{A}_k^n) = 0.$$

Namely, $\operatorname{Cl}(\mathbb{A}_k^n) = 0$ because $k[x_1, \ldots, x_n]$ is factorial. Indeed, any closed subscheme $Y \subseteq \mathbb{A}_k^n$ of codimension 1 must have $Y = V(f)$ for some $f \in k[x_1, \ldots, x_n]$.

Example 5.62. Given a number field K, we see that $\operatorname{Pic}(\operatorname{Spec} \mathcal{O}_K) = \operatorname{Cl} \mathcal{O}_K$, where $\operatorname{Cl} \mathcal{O}_K$ is the ideal class group.

Exercise 5.63. We show that $\operatorname{Pic} \mathbb{P}_k^n \simeq \mathbb{Z}$ by sending $n \in \mathbb{Z}$ to $\mathcal{O}(n)$.

Proof. To begin, note that this is injective: if $n \neq m$, then by tensoring to ensure that everything is positive, it suffices to do this in the case of $n, m > 0$, but then $\dim \Gamma(X, \mathcal{O}(n)) \neq \dim \Gamma(X, \mathcal{O}(m))$, so $\mathcal{O}(n) \neq \mathcal{O}(m)$.

To continue, we pick up the following result.

Proposition 5.64. Fix a Noetherian integral separated scheme X which is regular in codimension 1. Then any irreducible closed subscheme $Z \subseteq X$ produces the following.

(a) If $\operatorname{codim} Z \geq 2$, then $\operatorname{Cl} X \simeq \operatorname{Cl}(X \setminus Z)$ by taking $Y \mapsto [Y \setminus Z]$.

(b) If $\operatorname{codim} Z = 1$, then

$$\mathbb{Z} \stackrel{1 \mapsto [Z]}{\to} \operatorname{Cl} X \stackrel{[Y] \mapsto [Y \setminus Z]}{\to} \operatorname{Cl}(X \setminus Z) \to 0$$

is exact.

Proof. The map

$$\operatorname{Weil} X \stackrel{[Y] \mapsto [Y \setminus Z]}{\to} \operatorname{Weil}(X \setminus Z)$$

is certainly surjective by taking the Zariski closure to go backwards. If $\operatorname{codim} Z \geq 2$, then we can check that there is nothing in the kernel as well. If $\operatorname{codim} Z = 1$, then we can show that the only way for us to have a Weil divisor which vanishes is for it to come from Z, so there is nothing else to say. ■

Now, to finish the proof, we take $X = \mathbb{P}_k^n$ with some affine open subscheme $Z = \mathbb{A}_k^{n-1}$ so that $X \setminus Z = \mathbb{A}_k^n$. Then the exact sequence above will read as $\mathbb{Z} \twoheadrightarrow \operatorname{Pic} \mathbb{P}_k^n \to 0$ by $n \mapsto \mathcal{O}(n)$, but we already know that this map is injective, so we are done. ■

Remark 5.65. Projective k-varieties do have a reasonable notion of the degree of a divisor. For example, in $\mathbb{P}_k^n$, all closed subschemes of codimension 1 look like $V_+(f)$ for some homogeneous polynomial f, so we define $\deg V_+(f) := \deg f$. Notably, we can check that the isomorphism $\operatorname{Pic} \mathbb{P}_k^n \to \mathbb{Z}$ is given by $\deg \colon [Y] \mapsto \deg Y$.

Remark 5.66. In contrast, degree is not well-behaved for $\mathbb{A}_k^n$ because principal divisors do not necessarily have degree 0, and in particular it would not be helpful to have a map from $\operatorname{Pic} \mathbb{A}_k^n = 0$ anyway.

5.5 October 28

Today we'll focus on computing divisors for curves over a field.

5.5.1 Degrees on Curves

To discuss curves, we should probably define curves.

Definition 5.67 (Curve)**.** Fix a field k. A *curve* is a one-dimensional variety over k.

Let's begin by discussing Cartier divisors.

- Let X be a normal curve. For a Cartier divisor D, we define

$$\deg D \coloneqq \sum_{x \in X} \nu_x(D)[k(x) : k].$$

  Here, $\nu_x(D) = \nu_x(f_i)$, where $D = \{(U_i, f_i)\}$ has $x \in U_i$ for a given i, which makes sense because $\mathcal{O}_{X,x}$ is a discrete valuation ring.

- If X is not normal, then we should pass to its normalization $\widetilde{X}$ with normalization map $\iota \colon \widetilde{X} \to X$. Then we can pull back Cartier divisors D on X to divisors on $\widetilde{X}$ and compute the degree there.

Example 5.68. Take $X = \mathbb{P}^1_{\mathbb{Q}}$ as a $\mathbb{Q}$-curve. Then the closed point $x \coloneqq (x_0^2 - 2x_1^2)$ has $\deg_{\mathbb{Q}}[x] = 2$, where the 2 contribution is coming from $[\mathbb{Q}(x) : \mathbb{Q}] = 2$.

And here is the degree of a Weil divisor.

- For a normal curve X, and a Weil divisor $D = \sum_x n_x[x]$ on X, we again just define

$$\deg D \coloneqq \sum_{x \in X} n_x[k(x) : x].$$

- For general curves, apply the same normalization technique to get the definition in general.

Here is the usual check.

Proposition 5.69. Fix a proper k-curve X. Then $\deg \operatorname{div} f = 0$ for any $f \in K(X)$. Thus, $\deg$ descends to a homomorphism $\operatorname{CaCl} X \to \mathbb{Z}$.

We will prove this shortly.

Non-Example 5.70. With $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$, we note that $\deg \operatorname{div}(t) = \deg[0] = 1$. The point here is that the proper condition on X is necessary.

5.5.2 Pulling Back Divisors

To prove Proposition 5.69, we need to discuss what we meant when we said we're pulling back Cartier divisors.

Remark 5.71. Before continuing, we say out loud some properties of flat morphisms.

- The class of flat morphisms is preserved by composition, is preserved by base change, and is affine-local on the target.
- If $f \colon X \to Y$ is flat, and $y = f(x)$ for some $x \in X$, then $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X_y,x}$.

Fix some flat morphism $f\colon X \to Y$. If $Z \subseteq Y$ is a closed integral subscheme of codimension 1, then we note that $X \times_Y Z$ is a closed subscheme of X. Further, by the flatness, we note that

$$\operatorname{codim}_X(X \times_Y Z) = \operatorname{codim}_Y(Z) = 1$$

provided that $X \times_Y Z$ is nonempty. The point is that this process defines a group homomorphism $\operatorname{Weil}(Y) \to \operatorname{Weil}(X)$ sending some $[Z]$ to the reduced subscheme with topological space $X \times_Y Z \subseteq X$, with some suitably defined multiplicities.

Namely, for an irreducible closed subscheme W of $X \times_Y Z$, we let the multiplicity of $[W]$ be given by choosing an affine open subscheme $U \subseteq W$. Then we take the stalk $\mathcal{O}_W(U)_\eta$, where η is the generic point, is a zero-dimensional ring and hence Artinian and hence has finite length, so we set the multiplicity to the length of $\mathcal{O}_W(U)_\eta$.

Example 5.72. If we have $X \times_Y Z \simeq \operatorname{Spec} k[x]/\left(x^2\right)$, then our multiplicity we can see should be 2, and the corresponding reduction is k.

Example 5.73. If we have $X \times_Y Z \simeq \operatorname{Spec} k[x,y]/\left(x^2\right)$, then our reduction is $k[y]$, and our multiplicity is 2 coming from the length of $k[x,y]/\left(x^2\right)$ at the generic point (y) of $k[y]$.

We can see from the examples that the point of our weird reduction and so on is to keep track of the differential information.

Next we want to pull back Cartier divisors, which we are promised will be less confusing.

Example 5.74. Fix a dominant morphism $f\colon X \to Y$ of integral k-curves. Then a Cartier divisor made of the data $\{(U_i, f_i)\}$ on Y has $f_i \in \mathcal{K}(Y) = K(Y)^\times$. But we see that $f\colon X \to Y$ gives us a map $f^\sharp\colon K(Y) \to K(X)$ because dominance forces f to send the generic point of X to the generic point of Y. As such, we will do the obvious thing and send

$$(U_i, f_i) \mapsto \left(f^{-1}U_i, f^\sharp(f_i)\right)$$

to define our map $f^*\colon \operatorname{CaDiv} Y \to \operatorname{CaDiv} X$.

Generalizing the example, we note that $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is going to induce a morphism $f^\sharp\colon \mathcal{K}_Y \to f_*\mathcal{K}_X$. To make a map on divisors, we want a condition like flatness to preserve dimension and to perhaps send irreducible components to irreducible components. Namely, we are getting a map $f^*\colon \mathcal{K}_Y/\mathcal{O}_Y \to f_*(\mathcal{K}_X^*/\mathcal{O}_X^*)$, which is precisely a map of the Cartier divisors.

Example 5.75. Set $E = V_+\left(ZY^2 = X^3 + aXZ^2 + bZ^3\right) \subseteq \mathbb{P}^2_k$ to be an elliptic curve. Then the map $f\colon E \to \mathbb{P}^1_k$ by $[X:Y:Z] \mapsto [Y:Z]$ (note that E has no points with $Y = Z = 0$) will let us pull back. For example, we can pull back the divisor $[0:1]$ of $\mathbb{P}^1_k$ back to the three roots of $x^3 + ax + b$ as points on our elliptic curve. Similarly, we can pull back $[1:0]$ on $\mathbb{P}^1_k$ to $3[0:1:0]$.

5.6 October 31

Happy Halloween.

5.6.1 Pulling Back on Curves

Let's try to prove that divisors over proper curves have degree zero.

Proposition 5.76. Fix normal integral proper k-curves X and Y and a finite morphism $f\colon X \to Y$. Then any Cartier divisor $D \in \operatorname{CaDiv}(Y)$ have

$$\deg f^*D = [K(X) : K(Y)] \deg D.$$

Proof. Recall that we have a group homomorphism $f^*\colon \operatorname{CaDiv}(Y) \to \operatorname{CaDiv}(X)$, so by linearly extending and passing to Weil divisors, we may consider $D = [y]$, where $y \in Y$ is a closed point. (Namely, codimension-1 integral closed subschemes of Y are just closed points.)

As such, we want to compute $f^*[y]$. As some motivation, we discuss the pull-back. Well, place y in some affine open subscheme $\operatorname{Spec} B \subseteq Y$, and because f is finite, we see that $f^{-1}(\operatorname{Spec} B) \subseteq X$ is affine, so set $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$, where $f^\sharp\colon B \to A$ makes A a finitely generated B-module. Anyway, we approximately have

$$f^*[y] \approx \left[f^{-1}(y)\right],$$

where we now have to count our multiplicities. Namely, our scheme is $\operatorname{Spec} A/\mathfrak{m}_y A$, and localizing for psychological reasons means that we're looking at

$$\operatorname{Spec} \frac{(B \setminus \mathfrak{m}_y)^{-1}A}{\mathfrak{m}_y(B \setminus \mathfrak{m}_y)^{-1}A}.$$

Thinking more explicitly, let's say that $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in X$ are the points in $f^{-1}(\{y\})$, so we see

$$f^*[y] = \sum_{i=1}^n \operatorname{length}_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}/\mathfrak{m}_y A_{\mathfrak{m}_i})[\mathfrak{m}_i],$$

where we have used the decomposition of Artin rings into a product of its localizations. Taking degrees, we want to compute

$$\deg f^*[y] = \sum_{i=1}^n \operatorname{length}_{A_{\mathfrak{m}_i}}(A_{\mathfrak{m}_i}/\mathfrak{m}_y A_{\mathfrak{m}_i}).$$

We now proceed with the proof more directly. Adding together our lengths into a product, we can write

$$\deg f^*[y] = \operatorname{length} \frac{(B \setminus \mathfrak{m}_y)^{-1}A}{\mathfrak{m}_y(B \setminus \mathfrak{m}_y)^{-1}A},$$

which we claim is $[K(X) : K(Y)]$ by Nakayama's lemma. Indeed, set $A' := (B \setminus \mathfrak{m}_y)^{-1}A$, which we see is an integral domain. Now, $K(X) = \operatorname{Frac}(A')$, but we see that A' is a torsion-free and the principal ideal domain $B_{\mathfrak{m}_y}$, so A' must be a free $B_{\mathfrak{m}_y}$-module. Passing to the fraction fields recovers what the rank should be by Nakayama's lemma. ■

It is technically okay in the above proof to work with the non-normal case. We give a few remarks to convince ourselves of this.

Remark 5.77. In general, when X is not normal, given a closed point $x \in X$ and a Cartier divisor $D \in \operatorname{CaDiv}(X)$ represented by the data $\{(U_i, f_i)\}$, we can define the multiplicity of x at D by

$$\operatorname{mult}_x D := \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/f_x),$$

where we are implicitly assuming that $f_x \in \mathcal{O}_{X,x}$ and extending multiplicatively to work with the entire fraction field.

Example 5.78. Fix $X = \operatorname{Spec} k[x,y]/(xy)$, and we compute the multiplicity of (x,y) for the divisor coming from $f = x - y$. Then we see

$$\operatorname{mult}_{(x,y)}(f) = \operatorname{length}\left(\frac{k[x,y]}{(xy, x-y)}\right)_{(x,y)} = \operatorname{length}\left(\frac{k[x]}{x^2}\right)_{(x)} = 2.$$

On the other hand, our normalization is $\widetilde{X} = \operatorname{Spec} k[x] \sqcup \operatorname{Spec} k[y]$, so we can just compute

$$\operatorname{mult}_{(x)}(f) + \operatorname{mult}_{(y)}(f) = 1 + 1 = 2,$$

which matches.

Example 5.79. One could similarly look at the cuspidal cubic $X = \operatorname{Spec} k[x,y]/\left(y^2 = x^3\right)$ and recover a multiplicity of 1 at (x,y).

And here is our result.

Corollary 5.80. Fix a normal integral proper k-curve X. For some rational section $f \in K(X)^\times$, we have

$$\deg \operatorname{div}(f) = 0.$$

Proof. Note we have an induced map $f\colon X \to \mathbb{P}^1_k$: a global section f of X corresponds to a function $X \to \mathbb{A}^1_k$, which can then be extended uniquely to $\mathbb{P}^1_k$. Anyway, we see that $\operatorname{div}(f) = f^*([0] - [\infty])$ by computing the pull-back, which we see has degree 0. ■

Remark 5.81. We will often set $\deg f := [K(X) : K(Y)]$. The reason for this is that we can more or less think about the degree of a polynomial as the degree of the covering it introduces, which is approximately the number of roots (counted with multiplicity), which is the degree of the pull-back introduced in the proposition.

Let's give a few more remarks.

Remark 5.82. Moret–Bailly has a theorem which lets us turn Weil divisors into Cartier divisors (up to a multiple of $\mathbb{Q}$). Indeed, if X is normal equipped with a map $X \to \operatorname{Spec} \mathbb{Z}_p$ of relative dimension 1, then Weil divisors are Cartier divisors up to a multiple of $\mathbb{Q}$.

Remark 5.83. If X is normal, and $D \subseteq X$ is a Cartier divisor (up to a multiple of $\mathbb{Q}$) where we embed D into X as a Weil divisor, as well as some other subscheme $f\colon C \subseteq X$, we can think about our multiplicity of intersections of D and X as f^*D (as a Cartier divisor).

When X is a projective, regular k-scheme, it is still possible to define the degree, but one needs to first choose a line bundle $\mathcal{L}$. For example, taking $X = \mathbb{P}^n_k$ and $Y = V_+(f)$ as some closed subscheme, we define $\deg Y$ as the degree of the divisor on $\mathbb{P}^1_k$ given by $Y \cap \mathbb{P}^1_k$ where $\mathbb{P}^1_k \not\subseteq Y$. Namely, we are defining the degree of Y with its intersections with some line in "general position." But lines are just the intersection of $n-1$ different hyperplanes, and hyperplanes are just divisors of $\mathcal{O}_X(1)$!

Now, thinking more generally, we can think of $\deg_{\mathcal{L}}(Z)$, where $Z \subseteq X$ is some closed subscheme of dimension r, as the degree of the intersections of Z with r total divisors in general position. For example, if $r = \dim X$, then are defining a map $\deg_{\mathcal{L}}\colon \operatorname{Weil} X \to \mathbb{Z}$. We would like this map to be nontrivial, but this requires $\mathcal{L}$ to be an ample line bundle.

5.7 November 2

Today we give some motivational remarks for what is going on.

5.7.1 Homework Exists

Let's talk about the homework a little. On the homework, we may assume that k is algebraically closed.

Exercise 6.9(a) provides a relationship between $\operatorname{Pic} X$ and $\operatorname{Pic} \widetilde{X}$, where X is a proper k-curve and $\pi\colon \widetilde{X} \to X$ is the normalization. In particular, the homework shows that π^* is surjective, and the kernel can be easily described. Let's discuss surjectivity somewhat.

It turns out that understanding push-forwards for finite morphisms will make sense. In some sense, we want to take a finite morphism $f\colon Y_1 \to Y_2$ and then just push directly the closed subset of Y_1 to Y_2 and use the scheme-theoretic image to get the correct multiplicity.

But the main point of our discussion here is the following "projection" formula.

Theorem 5.84. Suppose that $f\colon \widetilde{X} \to X$ is a finite morphism so that the pull-back and push-forward of divisors both make sense. Fix a Weil divisor D on X "away" from the singular points of X. Then $f_*(f^*D) = (\deg f)D$.

In particular, for our normalization map $\pi\colon \widetilde{X} \to X$, we can check that $\deg \pi = 1$, so we get a good section π_* for X. This gives surjectivity, in some sense, and it also tells us that the singular points are the problems here. Computing the kernel of $\pi^*\colon \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$ requires us to keep track of principal divisors, which requires some care.

Remark 5.85. Later in life, we will discuss the Picard group of smooth curves X. It turns out that $\operatorname{Pic} X$ has a connected component with a scheme structure, which is its Jacobian. Then to recover the "generalized Jacobian" for curves, we can use the short exact sequence

$$0 \to \ker \pi^* \to \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \to 0$$

because we now know how $\ker \pi^*$ and $\operatorname{Pic} \widetilde{X}$ behave. We will be able to see some concrete examples on the homework.

On the homework, we will see two different kinds of singularities.

Definition 5.86 (Cusp)**.** A curve X has a *cusp* if it locally looks like $y^2 = x^3$ for some $\mathbb{A}^2_k$.

The issue here is that $(0,0)$ is our singularity, where the point is that the ring

$$\frac{k[x,y]}{(y^2 - x^3)}$$

has normalization $k[t]$ given by the maps $x \mapsto t^2$ and $y \mapsto t^3$. Because we are dealing with cubic curves in $\mathbb{P}^2_k$ on the homework, we may just assume that our cusp looks like $y^2 = x^3$.

Definition 5.87 (Node)**.** A curve X has a *node* if it locally looks like $y^2 = x^2(x+c)$ where $c \neq 0$, for some $\mathbb{A}^2_k$.

Again, $(0,0)$ is our singularity, and the normalization now has two points. Essentially, locally at $(0,0)$, the curves looks like $\operatorname{Spec} k[x,y]/(xy)$.

Remark 5.88. One can show that these are the only possible singularities for a cubic. I think on the homework we will be allowed to just assume that our cubic curves look like this.

As a last remark, we need to define some notation.

Notation 5.89. The group scheme $\mathbb{G}_a$ is isomorphic to $\mathbb{A}^1_k$ as schemes, where the addition is given by the addition in k. The group scheme $\mathbb{G}_m$ is isomorphic to $\mathbb{A}^1_k \setminus \{0\}$ as schemes, where the addition is given by the multiplication in $k^\times$.

5.7.2 Differential Geometry for Algebraic Geometers

Take $k = \mathbb{C}$, and throughout we will let X be a smooth projective k-variety. Smoothness, by definition, grants us some local Implicit function theorem, so X locally looks like some open discs, so we can give X the structure of a complex analytic manifold, which we call X^{an}; notably, X^{an} can be turned into a scheme by using the sheaf of holomorphic functions as our structure sheaf.

Theorem 5.90 (Serre's GAGA)**.** Fix a smooth projective $\mathbb{C}$ variety X. Then $\operatorname{Pic} X \cong \operatorname{Pic} X^{\mathrm{an}}$.

For example, on X^{an}, we have the usual short exact sequence

$$0 \to \mathbb{Z} \stackrel{2\pi i}{\to} \mathcal{O}_{X^{\mathrm{an}}} \stackrel{\exp}{\to} \mathcal{O}_{X^{\mathrm{an}}}^\times \to 0$$

which then gives rise to the exact sequence as follows.

$$\begin{tikzcd} 0 \arrow[r] & H^1(X^{\mathrm{an}}, \mathbb{Z}) \arrow[r] & H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \arrow[r] & H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^\times) \arrow[dll] \\ & H^2(X^{\mathrm{an}}, \mathbb{Z}) \arrow[r] & H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \arrow[r] & \cdots \end{tikzcd}$$

Namely, we see that we have a 0 on the leftmost point because the map $\mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}}^\times$ is surjective even after taking global sections, where we are using the fact that X should be proper.

Now, $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^\times) = \operatorname{Pic} X^{\mathrm{an}}$ by thinking of this cohomology as a line bundle, and $H^2(X^{\mathrm{an}}, \mathbb{Z})$ is the same as the usual singular cohomology. Thus, we have an exact sequence

$$0 \to \frac{H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})}{H^1(X^{\mathrm{an}}, \mathbb{Z})} \to \operatorname{Pic} X^{\mathrm{an}} \to \ker\left(H^2(X^{\mathrm{an}}, \mathbb{Z}) \to H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})\right) \to 0.$$

The right-hand side here is $\mathrm{NS}(X)$, the Néron–Severi group, which (conjecturally) behaves pretty well as a direct object. For example, this is a finitely generated abelian group because $H^2(X^{\mathrm{an}}, \mathbb{Z})$ is a vector space.

Now, Hodge theory says that $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is approximately $\mathbb{C}^g$ where g is half the total dimension of X^{an}, and the denominator $H^1(X^{\mathrm{an}}, \mathbb{Z})$ is a lattice inside it of full rank. Thus, our left-hand object is $\mathbb{C}^g/\mathbb{Z}^{2g}$, which is an abelian variety.

So the point is that $\operatorname{Pic} X^{\mathrm{an}}$ can be put as an extension of an abelian variety by a discrete object, which should roughly be what we expect from the homework.

Remark 5.91. The Lefschetz hyperplane theorem states that a smooth irreducible projective $\mathbb{C}$-variety $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ will have some hyperplane $H \subseteq \mathbb{P}^n_{\mathbb{C}}$ where $Y := X \cap H$ (which is a smooth irreducible $\mathbb{C}$-variety) has the induced map

$$\pi_1(Y) \to \pi_1(X)$$

has a surjection if $\dim Y \geq 2$ and an isomorphism if $\dim Y = 1$. The point here is that we can compute $\pi_1(X)$ by some inductive process by cutting out hyperplanes, gradually dropping the dimension.

Remark 5.92. At the end of the day, we're saying that algebraically trivial line bundles on X can be pulled back to line bundles on X^{an}, where we can use tools of Hodge theory and things.

Theme 6
Ample Line Bundles

6.1 November 4

We now leave line bundles for some time to talk about line bundles.

6.1.1 Building Projective Morphisms

Let's start with a theorem.

Theorem 6.1. Fix an S-scheme X. Then there is a bijection

$$\operatorname{Mor}_S(X, \mathbb{P}^n_S) \simeq \{(\mathcal{L}, s_0, \ldots, s_n) : \mathcal{L}/X \text{ is a line bundle}, s_\bullet \in \Gamma(X, \mathcal{L}) \text{ generating } \mathcal{L}\}/\sim,$$

where the $\sim$ is up to isomorphism.

Wait, what does generating mean?

Definition 6.2 (Generating line bundle)**.** Fix a scheme X and a line bundle $\mathcal{L}$ on X. Then some global sections $S \subseteq \Gamma(X, \mathcal{L})$ *generate* $\mathcal{L}$ if and only if $\{s_x : s \in S\}$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module for each $x \in X$.

We will want a way to relate morphisms to projective space with line bundles, so we pick up an adjective.

Definition 6.3 (Very ample)**.** Fix an affine scheme S and some morphism $f: X \to S$. Then a line bundle $\mathcal{L}$ on X is *very ample relative to* f if there is a locally closed embedding $\iota: X \to \mathbb{P}^n_S$ such that $\mathcal{L} \cong i_*\mathcal{O}_{\mathbb{P}^n_S}(1)$.

For a general scheme S where f is quasicompact, the line bundle $\mathcal{L}$ on X is *very amble relative to* f if and only if S has an affine open cover $\{U_\alpha\}_{\alpha\in\lambda}$ with locally closed embeddings $\iota_\alpha: f^{-1}U_\alpha \hookrightarrow \mathbb{P}^{n_\alpha}_{U_\alpha}$ such that $\mathcal{L}|_{f^{-1}U_\alpha} \cong \iota_\alpha\mathcal{O}_{\mathbb{P}^n_{U_\alpha}}(1)$.

Remark 6.4. We can drop the assumption that f is quasicompact; see `01VR`.

Let's start moving towards a proof of Theorem 6.1.

Example 6.5. Fix a morphism $\iota\colon X \to \mathbb{P}^n_S$ with $\mathcal{L} = \iota_* \mathcal{O}_{\mathbb{P}^n_S}(1)$. Covering S by an affine open cover, we see that $\mathbb{P}^n_S$ is covered by affine pieces $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$, so we can pull back the various x_i along ι_* and glue them together to global sections s_i generating $\mathcal{L}$. In particular, if we make sure that we are always gluing the same letter $x_\bullet$ together, they will assemble properly; more precisely, we can use distinguished open subschemes common to the intersection of any two affine open subschemes of S to do this gluing "correctly."

So here is our proof.

Proof of Theorem 6.1. In one direction, given a morphism $i\colon X \to \mathbb{P}^n_S$, then the arguments of Example 6.5 still carry through.

In the other direction, we are given a line bundle $\mathcal{L}$ on X generated by some global sections $s_0, \ldots, s_n$ of $\mathcal{L}$. We now define $X_j := X_{s_j}$ to be some open subschemes of X, and because the $s_\bullet$ generate $\mathcal{L}$, we must have the X_j covering X; indeed, otherwise all the $(s_\bullet)_x$ vanish simultaneously at a point $x \in X$ and thus do not generate $\mathcal{L}_x$.

We are now ready to build our morphism to $\mathbb{P}^n_S$. Define a morphism $X_j \to \mathbb{A}^n_S$ given by $(s_i/s_j)_{i=0}^n$, which is legal as a tuple in $\mathcal{O}^n_{X_j}$. More precisely, if S is affine, then we are looking at

$$\mathbb{A}^n_{\operatorname{Spec} A} = \operatorname{Spec} \frac{A[X_0, \ldots, X_n]}{(X_j - 1)}$$

where X_i will go to s_i/s_j, and these maps will glue. We can now glue these morphisms to $\mathbb{A}^n_S \subseteq \mathbb{P}^n_S$ together. This can be checked by hand, but we can explain it more deeply. Namely, for some open $U \subseteq X$ with an isomorphism $\varphi\colon \mathcal{L}|_U \simeq \mathcal{O}_U$, the global sections $s_\bullet$ are inducing a map from U to the lines in $\mathbb{A}^{n+1}_S \setminus \{0\}$, which is exactly what $\mathbb{P}^n_S$ is; now these morphisms are more canonical and will glue as such. ■

Remark 6.6. If the $s_\bullet$ do not generate $\mathcal{L}$, then we still get a map from an open subscheme $U \subseteq X$ to $\mathbb{P}^n_S$. Namely, the problem is that the $X_\bullet$ will not fully cover X, but on

$$U := \bigcup_{j=1}^n X_j$$

we will be just fine.

Exercise 6.7. Fix $E := V_+\left(ZY^2 = X^3 + aXZ^2 + XZ^2 + bZ^3\right)$ over an affine base $\operatorname{Spec} A$. We build some line bundles.

Proof. There is a canonical embedding $\iota\colon E \hookrightarrow \mathbb{P}^2_A$. Then we note that the global sections X and Y and Z manage to generate $\iota^*\mathcal{O}(1)$. On the other hand, there is a map $g\colon E \to \mathbb{P}^1_A$ by $[X : Y : Z] \mapsto [Y : Z]$, which builds the line bundle $g^*\mathcal{O}(1)$ generated by the global sections Y and Z.

One can see that these two line bundles are the same because they both come from the same divisor $3[0:1:0]$ either by hand or by measuring the divisor on the $Z = 0$ hyperplane and then pulling back using our theorem for pulling back divisors. The point here is that we are allowed to choose more global sections than we need to generate our line bundle.

As yet another example, consider the line bundle corresponding to the divisor $2[0:1:0]$, whose global sections are the rational functions on E regular on $E \setminus [0:1:0]$ and with a pole of order at most 2 at the point $[0:1:0]$. We can generate this line bundle by the constant function 1 and the global section $s_1 = X/Z$. (Notably, outside $[0:1:0]$, we see that Z vanishes.) The point is that we have extended the map given by

$$[X : Y : Z] \mapsto [X : Z],$$

which doesn't technically make sense as written, but we may extend them as rational maps. ■

Remark 6.8. We said at the start of the class that an elliptic curve is a smooth projective curve with a marked k-point. The reason why we want such a point is so that we can discuss divisors, which then give us line bundles, and then they will give us global sections, which will be the coordinates X, Y, Z that grant us the equation for the elliptic curve!

Example 6.9. Take $X := \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$ with $\mathcal{L} := \mathcal{O}(d)$ for some positive integer d. Now, we define the $s_\bullet$ to be the monomials of degree d in the $x_\bullet$. Then there is a Veronese embedding $\mathbb{P}^n_k \hookrightarrow \mathbb{P}^m_k$.

Example 6.10. Again, take $X := \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$ with $\mathcal{L} := \mathcal{O}(d)$ for some positive integer d. As before, we define the $s_\bullet$ to be the monomials of degree d in the $x_\bullet$. Then there is a Serge embedding $\mathbb{P}^m_k \times \mathbb{P}^n_k \to \mathbb{P}^{mn+m+n}_k$. The global sections here look like $x_i y_j$ for our various letters.

6.2 November 7

The fun continues.

6.2.1 Closed Embeddings to Projective Space

Let's see another example of Theorem 6.1.

Example 6.11. Fix a field k, and we study $\operatorname{Aut} \mathbb{P}^n_k$. Now, Theorem 6.1 tells us that a morphism $\varphi \colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ corresponds to a line bundle $\varphi^*\mathcal{O}(1)$. In particular, this should be a generator of $\operatorname{Pic} \mathbb{P}^n_k \simeq \mathbb{Z}$, so we are either $\mathcal{O}(1)$ or $\mathcal{O}(-1)$. However, we cannot have $\mathcal{O}(-1)$ because $\mathcal{O}(-1)$ has no nontrivial global sections to generate $\varphi^*\mathcal{O}(1)$ at all!

We now see that φ is given by $\mathcal{O}(1)$ and $n+1$ global sections generating $\mathcal{O}(1)$, but we know that $\dim_k \Gamma(\mathbb{P}^n_k, \mathcal{O}(1)) = n+1$ from the homework, so we just go ahead and choose $n+1$ sections to generate $\mathcal{O}(1)$ which will be the same as choosing an invertible matrix in $k^{(n+1)\times(n+1)}$. In total, we see that

$$\operatorname{Aut} \mathbb{P}^n_k \simeq \mathrm{GL}_{n+1}(k)/k^\times.$$

In Theorem 6.1 we constructed morphisms to projective space, but we are really interested in closed embeddings to projective space (e.g., to talk about projective varieties). So we upgrade to the following.

Proposition 6.12. Fix an affine scheme $\operatorname{Spec} A$. Then a morphism $\varphi \colon X \to \mathbb{P}^n_A$ corresponding to the tuple $(\mathcal{L}, s_0, \ldots, s_n)$ is a closed embedding if and only if the following hold.

(i) X_{s_i} is affine.

(ii) For each i, the corresponding ring map $A[y_0, \ldots, y_n] \to \mathcal{O}_X(X_{s_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

Proof. Read Hartshorne. The main point is that closed embeddings are affine and surjective affine-locally, which we can translate over to projective space. ■

Example 6.13. One can use Theorem 6.1 to define a k-variety which is proper but not projective. Approximately speaking, we can glue a hyperplane in one copy of $\mathbb{P}^3$ with a quadratic surface in a different copy of $\mathbb{P}^3$ to make our scheme X, and then do this twice in the reverse direction. This is proper because it's just a gluing of proper schemes, but it's not projective because any closed embedding $\iota \colon X \hookrightarrow \mathbb{P}^N_k$ will induce a line bundle $\iota^*\mathcal{O}(1)$ by going back and forth between our copies of $\mathbb{P}^3$ via some degree maps to track the Picard groups.

Remark 6.14. If you don't understand the above example, that's fine. One can survive without an example.

Remark 6.15. Notably, this example requires dimension to be at least 2 because all proper curves are projective, which we will see when we discuss Riemann–Roch later.

Example 6.16. One can construct a proper k-variety X containing a k-curve $C \subseteq X$ so that the induced map $\operatorname{Pic} X \to \operatorname{Pic} C$ has image in $\operatorname{Pic}^0 C$. This implies that X is not projective: an embedding $\iota\colon X \to \mathbb{P}^n_k$ will have $\mathcal{O}(1)$ pulled back all the way to C having positive degree. Namely, we can intersect some hyperplanes which will retain intersection number, which is our degree.

6.2.2 Ample Line Bundles

Last time we defined very ample line bundles as a pullback of $\mathcal{O}(1)$ from an embedding to projective space. However, this can be too strong, so we have the following restriction.

Definition 6.17 (Ample)**.** Fix a quasicompact quasiseparated scheme X (and finite type over a scheme S). A line bundle $\mathcal{L}$ on X is *ample* if and only if each finitely generated quasicoherent sheaf $\mathcal{F}$ on X has $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ generated by its global sections for sufficiently large n.

Remark 6.18. If X is Noetherian, we are just working with coherent sheaves $\mathcal{F}$.

Example 6.19. If X is affine and Noetherian, all line bundles $\mathcal{L}$ are ample. Namely, coherent sheaves all look like $\widetilde{M}$, so the tensor product $\widetilde{M} \otimes \mathcal{L}^{\otimes n}$ is still a coherent sheaf always and therefore generated by its global sections.

Let's see some more examples.

Theorem 6.20 (Serre)**.** Fix an affine base S and a scheme X over S of finite type. Given a closed embedding $\iota\colon X \hookrightarrow \mathbb{P}^n_S$, then $\mathcal{L} := \iota^*\mathcal{O}(1)$ is an ample line bundle on X. Moreover, $\mathcal{F} \otimes \iota_*\mathcal{O}(1)^{\otimes n}$ is generated by finitely many global sections for sufficiently large n, for each quasicoherent sheaf $\mathcal{F}$ on X.

Proof. We begin by reducing to $X = \mathbb{P}^n_S$, for which we use the following lemma.

Lemma 6.21. Fix everything as above. Fix a closed embedding $\iota\colon X \to Y$ of S-schemes, where Y is a quasicompact and quasiseparated scheme. If $\mathcal{L}/Y$ is ample (with the "moreover" finiteness condition from before), then $i^*\mathcal{L}/X$ is still ample (still with that moreover finiteness condition).

Proof. Fix a quasicoherent sheaf $\mathcal{F}$ on X of finite type. Because ι is a finite morphism, we see that $\iota_*\mathcal{F}$ is a quasicoherent sheaf on Y and still of finite type. (This was an exercise on the homework.) Now, $\mathcal{L}$ is ample on Y, so $i_*\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections, and in fact finitely many by assumption. Thus, we have a surjection

$$\mathcal{O}_Y^m \twoheadrightarrow \iota_*\mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

for sufficiently large n. Now, applying ι^*, which we know is right-exact its adjunction (alternatively, note ι^{-1} is exact, and then the tensor product is right-exact), we get a surjection

$$\mathcal{O}_X^m \twoheadrightarrow \iota^*(\iota_*\mathcal{F}) \to (\iota^*\mathcal{L})^{\otimes n}.$$

However, because ι is affine (it's a closed embedding), we see that we have a surjection $\iota^*\iota_*\mathcal{F} \twoheadrightarrow \mathcal{F}$, so using the right-exactness of the tensor product, we get a surjection

$$\mathcal{O}_X^m \twoheadrightarrow \mathcal{F} \otimes (i^*\mathcal{L})^{\otimes n}.$$

Thus, we are still generated by finitely many global sections, which is what we wanted. ■

Remark 6.22. Technically, we will only need ι to be a quasicompact locally closed embedding. We will make this upgrade later.

The above lemma allows us to reduce to the case of $X = \mathbb{P}^n_S$. So we write $S = \operatorname{Spec} A$ with $X = \operatorname{Proj} A[Y_0, \ldots, Y_n]$, where

$$X = \bigcup_{i=0}^{n} D_+(Y_i).$$

Now, each $D_+(Y_i)$ is affine, so for any quasicoherent $\mathcal{F}$ of finite type, we are granted some finitely many $t'_{ij} \in \mathcal{F}(D_+(Y_i))$ which generate $\mathcal{F}|_{D_+(Y_i)}$. Namely, we are using the fact that when everything is affine, or quasicoherent sheaves are generated by global sections.

To finish the proof, we pick up the following lemma.

Lemma 6.23. Fix a quasicompact and quasiseparated scheme X and a line bundle $\mathcal{L}$ on X. Then any global section $f \in \mathcal{L}(X)$ and quasicoherent sheaf $\mathcal{F}$ on X, all t'in $\mathcal{F}(X_f)$ has some n such that $f^n t'$ is a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

Proof. This proof is identical to our homework problem Exercise II.2.16, which we did on the homework. ■

In total, we see that the lemma allows us to extend each t'_{ij} to some global sections $Y_i^n t'_{ij}$ of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$, where we are still finitely generated everywhere. ■

Remark 6.24. More generally, ι does not have to be a closed embedding: we could set $\mathcal{L}$ to be a very ample line bundle with respect to the projection map $f \colon X \to S$, where $\iota \colon X \hookrightarrow \mathbb{P}^n_S$ is a locally closed embedding with $\mathcal{L} = \iota^*\mathcal{O}(1)$.

Remark 6.25. Again, let X be a proper Noetherian scheme over a Noetherian affine scheme S. Then a line bundle $\mathcal{L}$ is ample over X if and only if $\mathcal{L}^{\otimes n}$ is very ample for some positive n. We will upgrade this statement later. In particular, Riemann–Roch will tell us that line bundles over curves of sufficiently large degree are very ample, so all line bundles over curves are ample.

6.3 November 9

Today we hopefully finish our discussion of ample line bundles.

6.3.1 Projective Spaces

We begin class with an aside on the Proj construction.

Lemma 6.26. Given a graded ring A which is a finitely generated A_0-algebra, we can find a graded ring A' with $A_0 = A'_0$ such that A' is finitely generated over A_0 by elements in A_1 and $\operatorname{Proj} A = \operatorname{Proj} A'$.

Sketch. The idea is to do something like the Veronese embedding. Fix generators $\{a_i\}_{i=1}^n$ be generators of A over A_0, and set $d_i := \deg a_i$. Now, set $d := nd_1d_2\cdots d_n$ and note that the graded ring

$$A^{(d)} := \bigoplus_{m\geq 0} A_{md}$$

is generated by $A_1^{(d)} = A_d$ because we have fit all the generators in there. Then we showed on the homework that $\operatorname{Proj} A^{(d)} \simeq \operatorname{Proj} A$. ■

The point is that working with $\operatorname{Proj} A'$ has a good closed embedding to $\mathbb{P}^n_{A_0}$ for some large enough n by setting $A' = A_0[s_0, \ldots, s_n]$ and using the obvious morphism of graded rings

$$A_0[x_0, \ldots, x_n] \twoheadrightarrow A_0[s_0, \ldots, s_n].$$

Remark 6.27. One can check that Theorem 6.20 works fine for any $\operatorname{Proj} A$ where A is finitely generated over A_0 by finitely many elements in A_1.

6.3.2 A Better Ample

We now return to our discussion of ample line bundles.

Proposition 6.28. Fix a quasicompact and quasiseparated scheme X of finite type over an affine scheme and a line bundle $\mathcal{L}$ on X. Then the following are equivalent.

(a) $\mathcal{L}$ is ample; in other words, for each finitely generated quasicoherent sheaf $\mathcal{F}$ on X has $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ generated by its global sections for sufficiently large n.

(b) The open sets $X_f := \{x \in X : f_x \text{ generates } \mathcal{L}_x^{\otimes n} \simeq \mathcal{O}_{X,x}\}$ as n varies over all positive integers and f varies over the global sections of $\mathcal{L}^{\otimes n}$ form an affine base for the topology of X.

(c) There exists some positive integer d such that there are finitely many global sections $f_\bullet$ of $\mathcal{L}^{\otimes n}$ where the X_{f_i} cover X.

Proof. We first show (a) implies (b). Fix some $x \in X$, and place it in some affine open $U \subseteq X$. We need to find an f for which $x \in X_f \subseteq U$. To use the ample condition, we need to pick up some quasicoherent sheaves of finite type, so here we go.

Lemma 6.29. Fix a quasicompact and quasiseparated scheme X. Then there is a coherent ideal sheaf $\mathcal{I}$ such that $\mathcal{I}$ is a finite type quasicoherent sheaf such that $V(\mathcal{I}) = X \setminus U$.

Proof. In the case that X is Noetherian, then any ideal sheaf $\mathcal{I}$ will do. In general, one can reduce to the affine case because X is quasicompact and quasiseparated, and then the fact that U is quasicompact will let us construct $\mathcal{I}$. ■

We now apply the lemma. By definition of being ample applied to the constructed ideal sheaf $\mathcal{I}$, we know there is some $n > 0$ and a global section f of $\mathcal{I} \otimes \mathcal{L}^{\otimes n}$ such that $f_x \notin \mathfrak{m}_x$. (Namely, $x \in U = X \setminus V(\mathcal{I})$ tells us that $\mathcal{I}_x \not\subseteq \mathfrak{m}_x$, so we can find a germ in $\mathcal{I}_x$ not in $\mathfrak{m}_x$, and the ample condition lets us lift this to a global section.)

It follows $x \in X_f \subseteq U$. Indeed, $x \in X_f$ is straight from the construction, and $X_f \subseteq U$ is because any
What? $y \in X \setminus U$ has all $g \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ with f_y not invertible by the construction of $\mathcal{I}$.

Next we show (b) implies (c). Fix x in an affine open subscheme $U \subseteq X$ with some isomorphism $\varphi\colon \mathcal{L}|_U \simeq \mathcal{O}_U$. Now, (b) promises a global section f of $\mathcal{L}^{\otimes n}$ such that $x \in X_f \subseteq U$. But then

$$\varphi(f|_U) \in \Gamma(U, \mathcal{O}_U)$$

is affine, so $X_f = D(\varphi(f|_U))$ is an affine open subscheme containing x. Then the union of these X_f (over all $x \in X$) covers X, and quasicompactness reduces this to a finite cover.

Lastly, we show (c) implies (a). This requires a little more work because we have to construct a bunch of global sections to generate. Fix our finite affine open cover by X_{f_i}s. Let $\mathcal{F}$ be some quasicoherent sheaf of finite type so that $\mathcal{F}|_{X_{f_i}}$ corresponds to a finitely generated $\mathcal{O}_X(X_{f_i})$-module. This finite generation promises us some sections $t'_{ij} \in \Gamma(X_{f_i}, \mathcal{F})$ generating $\mathcal{F}|_{X_{f_i}}$, but then a result from class grants us some n_0 such that any $n \geq n_0$ has

$$t'_{ij} \otimes f_i^n$$

will extend to some global section $t_{ij} \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes nd})$ where $f_i \in \Gamma(X, \mathcal{L}^{\otimes d})$ by Lemma 6.23. Intuitively, the point is that our line bundle has some prescribed allowed poles in our global section, which disappear upon by multiplying by sufficiently many powers of f. It follows that any $n \geq n_0$ has $\mathcal{F} \otimes \mathcal{L}^{\otimes nd}$ generated by these finitely many sections.

We now need to get $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for any n large enough. Well, we apply the above argument to the finitely many

$$\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes 2}, \ldots, \mathcal{F} \otimes \mathcal{L}^{d-1}$$

and take the maximum of the given ns to upgrade our result. ■

Corollary 6.30. Fix a quasicompact and quasiseparated scheme X. Given a quasicompact locally closed embedding $\iota\colon Z \to X$, if $\mathcal{L}$ is ample on X then $i^*\mathcal{L}$ on Z.

Proof. Given any global section f of $\mathcal{L}^{\otimes n}$, we note that the adjunction of i^* and i_* promises that

$$i^*f \in \Gamma(Z, (i^*\mathcal{L})^{\otimes n}) \simeq \Gamma(X, i_*(i^*\mathcal{L}^{\otimes n})),$$

which has a map from $\Gamma(X, \mathcal{L}^{\otimes n}$. The point is that the X_f form a base for the topology on X, so we find $Z_{i^*f} = X_f \cap Z$ will form a base for the topology on X. ■

Remark 6.31. In particular, taking $\mathbb{P}^n$ in the above corollary tells us that very ample implies ample.

Remark 6.32. It will turn out that, in the case where X is proper, there will be a cohomological criterion for a line bundle to be ample. Such a thing does not exist for being very ample because being very ample is a bit too rigid.

We also note that we have the following result.

Proposition 6.33. Fix an affine base scheme S so that $f\colon X \to S$ is a finite-type morphism making X a quasicompact and quasiseparated scheme. Given a line bundle $\mathcal{L}$ on X, the following are equivalent.

(a) $\mathcal{L}$ is ample.

(b) There is some positive integer n for which $\mathcal{L}^{\otimes n}$ is very ample for f.

(c) Each line bundle $\mathcal{L}'$ on X has some positive integer n_0 such that $\mathcal{L}' \otimes \mathcal{L}^{\otimes n}$ is very ample for f for all $n \geq n_0$.

Proof. We show (b) implies (a). Because $\mathcal{L}^{\otimes n}$ is very ample, we note that Theorem 6.20 can be upgraded with the above results to tell us that $\mathcal{L}^{\otimes n}$ is actually ample. To continue, we have the following check.

Lemma 6.34. Fix a quasicompact and quasiseparated scheme X. Then a line bundle $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ for some positive integer n.

Proof. The forward direction is clear. In the reverse direction, we note that a global section $f \in \Gamma(X, \mathcal{L}^{\otimes d})$ has $X_f = X_{f^n}$ while $f^n \in \Gamma(X, \mathcal{L}^{\otimes nd})$. So if the X_f form a base for our topology, then the X_{f_n} will also form a base for our topology. ■

We show that (a) implies (b). The difficult part here is to construct a projective embedding, so we must construct a map to projective space and then check that it is an embedding. Well, because $\mathcal{L}$ is ample, we are promised finitely many global sections $f_i \in \Gamma(X, \mathcal{L}^{\otimes d})$ such that the X_{f_i} form an affine open cover for X.

Now, let the a'_{ij} be a finite set of generators for $\Gamma(X_{f_i}, \mathcal{O}_X)$ over A and extend them up to global sections a_{ij} extending $a'_{ij} \otimes f_i^{\otimes n}$ for large enough n, as usual. Notably, all these global sections are living in some fixed $\mathcal{L}^{\otimes dn}$.

Now, these generators of $\mathcal{L}^{\otimes nd}$ given by the f_i^n and a_{ij} form a map $X \to \mathbb{P}^n_A$. On each affine piece $X_{f_i} = X_{f_i^n}$, we see that the map looks like

$$A[f_i^n, a_{ij}] \to A$$

is by $a_{ij}/f_i^n \mapsto a'_{ij}$, which is surjective. Thus, we have constructed a closed embedding.

It remains to show that both (a) and (b) imply (c). Well, we know that $\mathcal{L}^{\otimes n}$ is very ample for some n, so we get a locally closed embedding $X \hookrightarrow \mathbb{P}^{N_2}_S$ for some N. Further, using the ample condition, there is some m such that $\mathcal{L}' \otimes \mathcal{L}^{\otimes m}$ is generated by finitely many global sections, which then gives us another locally closed embedding $X \to \mathbb{P}^{N_2}_S$.

To finish, we note that we have a locally closed embedding

$$X \overset{(f_1,f_2)}{\to} \mathbb{P}^{N_1} \times \mathbb{P}^{N_2},$$

which then embeds to $\mathbb{P}^{N_3}$ using the Serge embedding. In total, $i\colon X \hookrightarrow \mathbb{P}^{N_3}$ is still a locally closed embedding. The corresponding line bundle to this locally closed embedding is $i^*\mathcal{O}_{\mathbb{P}^{N_3}}(1)$, which we showed on the homework is the same as $\pi_1^*\mathcal{O}_{\mathbb{P}^{N_1}}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^{N_2}}(2)$, which then pulls back to $\mathcal{L}^{\otimes n} \otimes \mathcal{L}' \otimes \mathcal{L}^{\otimes m} = \mathcal{L}' \otimes \mathcal{L} \otimes (m+n)$. This finishes. ■

The point is that sufficiently large powers make very ample line bundles.

Remark 6.35. There is a notion of being relatively ample, but we will not discuss it in this class.

6.4 November 14

We continue falling. We began class by finishing the proof of Proposition 6.33, which I've edited directly into last class for continuity.

Remark 6.36. Here are some corollaries to Proposition 6.33.

- One can show that $\mathcal{L}$ and $\mathcal{L}'$ being very ample implies that $\mathcal{L} \otimes \mathcal{L}'$ is very ample.
- Similarly, we will show on the homework that $\mathcal{L}$ and $\mathcal{L}'$ are both ample, then $\mathcal{L} \otimes \mathcal{L}'$ is still ample. The point is that being ample means that a sufficiently large power is still ample.
- In fact, if $\mathcal{L}$ is any line bundle on a projective scheme X (so that very ample line bundles exist), then there are very ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^\vee$. The point is to use (c) of Proposition 6.33: given an ample line bundle $\mathcal{M}$, eventually $\mathcal{M}^{\otimes n}$ is very ample, and eventually $\mathcal{L} \otimes \mathcal{M}^{\otimes n}$ is very ample, so we set $\mathcal{L}_1 = \mathcal{L} \otimes \mathcal{M}^{\otimes n}$ and $\mathcal{L}_2 = \mathcal{M}^{\otimes n}$.

Remark 6.37. One can use the above remarks to define heights for projective varieties: of course, we would like to measure heights via a projective embedding, but we would like to keep track of this projective embedding, for which we use the corresponding very ample line bundle.

Anyway, today we're going to focus on blowing up.

6.4.1 Blowing Up

We will have to define blowing up, but we will not really discuss how it helps resolution of singularities. Here is our main example.

Exercise 6.38. Fix an algebraically closed field k, basically so that we can talk about closed points without fear. We blow up $\mathbb{A}^2_k$ at the origin to make $\operatorname{Bl}_{(0,0)}\mathbb{A}^2_k$.

Proof. Set-theoretically, we will have

$$\operatorname{Bl}_{(0,0)}\mathbb{A}^2_k \coloneqq \{(p,[\ell]) \in \mathbb{A}^2_k \times [\ell] : p \in \ell\},$$

where we are thinking about $\mathbb{P}^1_k$ as lines going through the origin. We can see that this defines a closed subset (variety-theoretically) of $\mathbb{A}^2_k \times \mathbb{P}^1_k$, so we give it the reduced subscheme structure.

Another way to think of this construction of $\operatorname{Bl}_{(0,0)}\mathbb{A}^2_k$ is by taking $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ and $\mathbb{P}^1_k = \operatorname{Proj} k[X,Y]$ so that $\operatorname{Bl}_{(0,0)}\mathbb{A}^2_k$ is essentially the closed subscheme carved out by $xY - yX = 0$. We can see this on the usual affine open cover of $\mathbb{P}^1_k$, as follows.

- On $\mathbb{A}^2_k \times D_+(Y)$, we are looking at $\operatorname{Spec} k[x,y,u]/(xu-y)$, where u denotes X/Y.
- On $\mathbb{A}^2_k \times D_+(X)$, we are looking at $\operatorname{Spec} k[x,y,v]/(x-yv)$, where u denotes Y/X.

Let's discuss what's going on here. Note there is a map $\pi\colon \operatorname{Bl}_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2_k$ by sending $(p,[\ell]) \mapsto p$. However, something funny is happening with the fibers.

- At the point we blew up, we see $\pi^{-1}(\{(0,0)\}) = \mathbb{P}^1_k$.
- Otherwise, $\pi|_{\operatorname{Bl}_{(0,0)}\mathbb{A}^2_k \setminus \pi^{-1}(\{(0,0)\})}$ is an isomorphism onto its image, which is $\mathbb{A}^2_k \setminus \{(0,0)\}$.

This first pre-image is so nice only because what we blew up was so nice, but it will be true in general that a blow-up map will be an isomorphism outside what we're blowing up. Here are some other nice properties, which can be checked by hand.

- The map π is proper.
- We see $\operatorname{Bl}_{(0,0)}\mathbb{A}^2_k$ is smooth. One can check this directly from the Jacobian.

Again, the first property is only so nice because we're blowing up is so nice, but the second property will roughly hold in general. ■

Example 6.39. Let's see our blowing up aide in resolution of singularities: define the curve $C \subseteq \mathbb{A}^2_k$ by $\operatorname{Spec} k[x,y]/\left(y^2 - x^3 - x^2\right)$, which is smooth everywhere except a node at the origin. To fix this, we blow up at $(0,0)$, for which we consider

$$\overline{\pi^{-1}(C \setminus \{0\})}_{\text{red}} \subseteq \operatorname{Bl}_{(0,0)} \mathbb{A}^2_k.$$

(In general, we want to consider a scheme-theoretic closure, but the Zariski closure works for now.) Let's see what happens on charts.

- On $D_+(Y)$, we are looking at $\operatorname{Spec} k[x,u]/\left(u^2 - x - 1\right)$ because $y^2 = x^3 + x^2$ implies $x^2u^2 = x^3 + x^2$, which is $u^2 = x + 1$.
- Similarly, on $D_+(X)$, we are looking at $\operatorname{Spec} k[y,v]/\left(1 - v^3y - v^2\right)$.

Notably, because k is algebraically closed, we can see that the above glue together into just $\operatorname{Spec} k[u]$.

The point of the above example is that we were able to recover a normalization map from blowing up.

Now that we're motivated to study blowing up, let's define what it is we're talking about. Our definition will be by universal property.

Definition 6.40 (Blowing up)**.** Fix a closed subscheme Z of a scheme X. Then the *blow-up* of X along Z is a scheme $\operatorname{Bl}_Z X$ equipped with a morphism $\pi\colon \operatorname{Bl}_Z X \to X$ such that $\pi^{-1}(Z)$ (which is $Z \times_X \operatorname{Bl}_Z X$) is an affective Cartier divisor. We also require that $(\operatorname{Bl}_Z X, \pi)$ is final with respect to this data: given any $f\colon W \to X$ such that $f^{-1}(Z) \subseteq W$ is an affective Cartier divisor in W, then there is a unique $g\colon W \to \operatorname{Bl}_Z X$ making the relevant data commute.

Wait, what do we mean when we say that a closed subscheme is an effective Cartier divisor?

Remark 6.41. We'll say that a closed subscheme is an effective Cartier divisor if and only if the corresponding ideal sheaf is a line bundle. One can see this is a sane definition by tracking through what it means for the corresponding Weil divisor to induce an effective Cartier divisor: essentially, we're asking to locally be cut out by a single equation.

We'll show that we blew up $\mathbb{A}^2_k$ at $(0,0)$ properly later.

Here are some more words.

Definition 6.42 (Exceptional divisor)**.** Fix a closed subscheme Z of a scheme X inducing a blow-up map $\pi\colon \operatorname{Bl}_Z X \to X$. We call $\pi^{-1}(Z)$ the *exceptional divisor*.

Definition 6.43 (Total transform)**.** Fix a closed subscheme Z of a scheme X with blow-up $\pi\colon \operatorname{Bl} Z_X \to X$. For a closed subscheme $Y \subseteq X$, we call $\pi^{-1}(Y)$ the *total transform of* Y, and we call $\pi^{-1}(Y \setminus Z)$ the *strict transform of* Y.

Here are some basic properties.

Lemma 6.44. Fix a closed subscheme Z of a scheme X with blow-up $\pi\colon \operatorname{Bl} Z_X \to X$. If $Z \subseteq X$ is an effective Cartier divisor, then $\operatorname{Bl}_Z X = X$.

Proof. One can show directly that the identity satisfies the universal property. ■

Lemma 6.45. Blowing up is functorial: given a closed subscheme Z of a scheme X and a scheme morphism $f\colon X' \to X$, there is a map $\widetilde{f}$ making the following diagram commute.

$$\begin{tikzcd} \operatorname{Bl}_{f^{-1}Z} X' \arrow[r, "\widetilde{f}"] \arrow[d] & \operatorname{Bl}_Z X \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

Proof. Apply the universal property to $f \circ \pi'$. ■

Lemma 6.46. Fix a closed subscheme Z of a scheme X with blow-up $\pi\colon \operatorname{Bl}_Z X \to X$. If $U \subseteq X$ is open, then $\operatorname{Bl}_{Z\cap U} U \simeq \pi^{-1}U \simeq U$.

Proof. Omitted. We will prove this later. ■

The point of the above lemma is that we can construct blow-ups (affine-)locally.

6.5 November 16

Today we continue to blow up. Last class we defined blow-ups by universal property. Today we will actually show that it exists.

6.5.1 Blow-Up Fact Collection

Continuing our fact-collection, we will finish up a proof from last time.

Lemma 6.46. Fix a closed subscheme Z of a scheme X with blow-up $\pi\colon \operatorname{Bl}_Z X \to X$. If $U \subseteq X$ is open, then $\operatorname{Bl}_{Z\cap U} U \simeq \pi^{-1}U \simeq U$.

Proof. The map $U \hookrightarrow X$ grants us a map $\operatorname{Bl}_{Z\cap U} U \to \pi^{-1}U$ because $\pi^{-1}U$ is a fiber product. We would like to construct an inverse for this map. Well, $\pi^{-1}U$ fits into the pullback square

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\pi"] \arrow[d, hook] & U \arrow[d, hook] \\ \operatorname{Bl}_Z X \arrow[r, "\pi"] & X \end{tikzcd}$$

where facts about intersections of locally closed embeddings tell us that the pre-image of $Z \cap U$ in $\pi^{-1}U$ is $\pi^{-1}Z \cap \pi^{-1}U$. However, this is now an effective Cartier divisor in $\pi^{-1}U$, so the universal property grants us a map $\pi^{-1}U \to \operatorname{Bl}_{Z\cap U} U$. Because both of our maps were defined by universal property, it's not hard to check that these maps are inverse. ■

Remark 6.47. Technically, one can weaken the open embedding $U \hookrightarrow X$ to merely be a flat morphism. We haven't defined what a flat morphism is, so we won't say more about this.

Lemma 6.48. Fix a closed subscheme Z of a scheme X with blow-up $\pi\colon \operatorname{Bl}_Z X \to X$. If X is reduced or irreducible, then so is $\operatorname{Bl}_Z X$.

Proof. The point is to reduce to the affine case. Namely, with $X = \operatorname{Spec} A$, we set $U := X \setminus Z$ so that $\pi^{-1}U$ will contain some $D(t)$ for a single $t \in A$, using the distinguished base. However, our closed subscheme is a Cartier divisor and therefore locally generated by a single element, so looking sufficiently locally, Z is $V(t)$
What? for some non-zero-divisor t. It follows that $D(t) = \operatorname{Spec} A_t$ is reduced or irreducible following from A. ■

Lemma 6.49. Fix a closed subscheme Z of an integral scheme X with $U := X \setminus Z$ open and dense. Then the blow-up $\pi\colon \operatorname{Bl}_Z X \to X$ restricts to a map $\pi|_U\colon U \to \pi^{-1}U$ which defines a birational map $X \dashrightarrow \operatorname{Bl}_Z X$.

Proof. This is an isomorphism from Lemma 6.46, so the difficulty here is showing that $\pi^{-1}U$ is open and dense in $\operatorname{Bl}_Z X$. Well, we see that

$$\operatorname{Bl}_Z X \setminus \pi^{-1}U = \pi^{-1}Z$$

is an effective Cartier divisor, so $\pi^{-1}U \subseteq \operatorname{Bl}_Z X$ is fact open and dense scheme-theoretically, which is what we wanted. Note that being birational as a notion makes sense because $\operatorname{Bl}_Z X$ is still integral by Lemma 6.48. ■

Lemma 6.50. Fix a closed subscheme Z of a scheme X with blow-up $\pi\colon \operatorname{Bl}_Z X \to X$. Given a closed subscheme $i\colon Y \subseteq X$, the induced map $\widetilde{i}\colon \operatorname{Bl}_{Y\cap Z} Y \to \operatorname{Bl}_Z X$ is a closed embedding, where the image is the scheme-theoretic image of $\pi^{-1}(Y \setminus Z)$ in $\operatorname{Bl}_Z X$.

Proof. One can use the universal property (there's a hint in Vakil), but we will prove this by construction later. ■

Remark 6.51. The above result is how we compute for closed subschemes. Explicitly, when we wanted to blow up the nodal curve in the affine plane, we were able to blow up the affine plane first and then take the (scheme-theoretic image of the) pre-image of the nodal curve.

6.5.2 How to Blow Up

We now construct the blow-up. Fix a closed subscheme $f\colon Z \subseteq X$ which is an effective Cartier divisor, where $\mathcal{I} := \mathcal{I}_Z = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Z)$. Now, we construct

$$\mathcal{B} := \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots .$$

Note $\mathcal{B}$ is a graded quasicoherent $\mathcal{O}_X$-algebra (where "$\mathcal{O}_X$-algebra" has the obvious definition) generated in degree 1, by construction. We are going to construct $\operatorname{Proj} \mathcal{B}$, as a generalization of our previous Proj over graded algebras.

Well, to do this construction, we unsurprisingly glue locally. Fix an affine open subscheme $U \subseteq X$, and set $A := \mathcal{O}_X(U)$. Then, note

$$\mathcal{B}(U) = \bigoplus_{n\ge 0} \Gamma\left(U, \mathcal{I}^n\right)$$

is a graded A-algebra. So we note that there is a map $\pi\colon \operatorname{Proj} \mathcal{B}(U) \to U$. Further, note that open $V \subseteq U$ induces an isomorphism $\pi^{-1}V \simeq \operatorname{Proj} \mathcal{B}(V)$, so we can glue these affine pieces together to build $\operatorname{Proj} \mathcal{B}$ with a map $\pi\colon \operatorname{Proj} \mathcal{B} \to X$.

Remark 6.52. The above construction will work for any graded quasicoherent $\mathcal{O}_X$-algebra $\mathcal{B}$.

We now use the fact that $\mathcal{B}$ is generated in degree 1. Namely, we define the quasicoherent sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}\mathcal{B}$ by gluing together the various $\widetilde{\mathcal{B}(U)(1)}$ on $\operatorname{Proj}\mathcal{B}(U)$.

Remark 6.53. One can check that $\mathcal{O}(1)$ is a line bundle on $\operatorname{Proj}\mathcal{B}$ when $\mathcal{B}$ is generated in degree 1 and is of finite type.

Remark 6.54. One can check that Proj is compatible with base-change: given a scheme morphism $\varphi\colon X \to X'$, where $\mathcal{B}$ is a graded quasicoherent $\mathcal{O}_X$-module, then the square

$$\begin{tikzcd} \operatorname{Proj}\varphi^*\mathcal{B} \arrow[r] \arrow[d] & X' \arrow[d, "\varphi"] \\ \operatorname{Proj}\mathcal{B} \arrow[r] & X \end{tikzcd}$$

is a pullback square. In fact, we can check that $\pi^*_{\operatorname{Proj}\mathcal{B}}\mathcal{O}_{\operatorname{Proj}\mathcal{B}}(1) = \mathcal{O}_{\operatorname{Proj}\varphi^*\mathcal{B}}(1)$.

Example 6.55. In particular, the above square tells us that $\mathbb{P}^n_X = \mathbb{P}^n_{\mathbb{Z}} \times X$ is a reasonable definition (where X is a scheme). Namely, we are defining

$$\mathbb{P}^n_X = \operatorname{Proj}\mathcal{O}_X[x_0, \ldots, x_n].$$

Proposition 6.56. Fix a closed subscheme Z of X which is an effective Cartier divisor. Then the $\mathcal{O}_X$-algebra $\mathcal{B} \coloneqq \bigoplus_{n \geq 0} \mathcal{I}^n$ from earlier has $\operatorname{Proj}\mathcal{B} \simeq \operatorname{Bl}_Z X$.

Before proving this, let's give an example.

Example 6.57. Take $X = \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ with $Z = (0, \ldots, 0)$ so that $I \coloneqq (x_1, \ldots, x_n)$ gives $\mathcal{I}_Z = \widetilde{I}$. Now, we can compute that

$$\bigoplus_{n \geq 0} \mathcal{I}_Z \simeq k[x_1, \ldots, x_n][X_1, \ldots, X_n]/(x_iX_j - x_jX_i),$$

where the X_i are the elements of $\mathcal{I}_Z$ living in the degree-1 component of our ring. One can check that this aligns with our blow-up of $\mathbb{A}^2_k$ at the origin.

Remark 6.58. In general, if we are blowing up an affine space along $Z \subseteq X$ of dimension Z, then $\pi^{-1}Z$ will be $\mathbb{P}^{n-1-\dim Z}$. For example, if we are blowing up along a line, then we want to mod out by the direction of the line first, which drops the dimension.

We now prove Proposition 6.56.

Proof. We can reduce to the affine case: being an effective Cartier divisor can be checked locally, and showing the universal property will then be something that we can glue together affine-locally because morphisms can be proven to exist and be unique affine-locally. The proof will be finished next class. ∎

Remark 6.59. Hartshorne and Vakil prove this when $\mathcal{I}$ is of finite type because these are associated to having our line bundle being finitely generated, which turns into projective morphisms, which we understand.

Remark 6.60. One can understand morphisms to $\operatorname{Proj} \mathcal{B}$ similar to morphisms to $\mathbb{P}^n_S$, though we will not need this.

BIBLIOGRAPHY

[Wei59] Andre Weil. “Correspondence”. In: *Annals of Mathematics* 69.1 (1959), pp. 247–251.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977.

[Eis95] David Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995. URL: https://books.google.com/books?id=Fm%5C_yPgZBucMC.

[Liu06] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford graduate texts in mathematics. Oxford University Press, 2006. URL: https://books.google.com/books?id=pEGLDAEACAAJ.

[Kid12] Keenan Kidwell. *Is the radical of a homogeneous ideal of a graded ring homogeneous?* Mathematics Stack Exchange. 2012. eprint: https://math.stackexchange.com/q/238203. URL: https://math.stackexchange.com/q/238203.

[Kle16] Felix Klein. *Elementary Mathematics from a Higher Standpoint*. Trans. by Gert Schubring. Vol. II. Springer Berlin, Heidelberg, 2016.

[Vak17] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. 2017. URL: http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf.

[Spa18] Bianca Sparacino. *The Strength in Our Scars*. Thought Catalog Books, 2018.

List of Definitions