202A: Introduction to Topology and Analysis

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THEME 1
METRIC SPACES

My personal view on spaces is that every space I ever work with is either metrizable or is the Zariski topology.

—Evan Chen, [Che22]

1.1 August 24

Good morning everyone. This is my first class of the semester.

1.1.1 Administrative Notes

Here are some housekeeping remarks.

- The webpage for this class is `math.berkeley.edu/ rieffel/202AannF22.html`.
- The midterm date is negotiable. We will have a vote on Friday. The possible dates are Friday 14 October, Monday 17 October, or Wednesday 19 October.
- There will be no vote on the final exam. It is on 15 December at 7PM.
- Homework will be due Fridays by midnight, approximately every week.
- There is no particular text for this course, and any given text covers more than we have time for. That said, we will (very) loosely follow [Lan12], but it is helpful to have a number of different expositions around.
- Please wear a mask during lectures and office hours.

Here is a summary of the course.

- We will spend the next couple of lectures talking about metric spaces.
- We will then spend the first half of the course on general topology. The second half of the course will be on measure and integration.
- Throughout we will see a little on functional analysis.

1.1.2 Metric Spaces

Hopefully we remember something about metric spaces. Here's the definition.

Definition 1.1 (Metric)**.** A *metric* d on a set X is a function $d: X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following rules for any $x, y, z \in X$.

(a) Zero: $d(x, x) = 0$.

(b) Zero: $d(x, y) = 0$ implies $x = y$.

(c) Symmetry: $d(x, y) = d(y, x)$.

(d) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

We call (X, d) a *metric space*.

Remark 1.2. It is occasionally helpful to think about a "reversed" triangle inequality: note $d(x, z) \leq d(x, y) + d(y, z)$ implies $d(x, z) - d(x, y) \leq d(y, z)$. Similarly, $d(x, y) - d(x, z) \leq d(y, z)$, so it follows

$$|d(y, x) - d(x, z)| \leq d(y, z).$$

We will want some "almost" metrics as well. Here are their names.

Definition 1.3 (Semi-metric)**.** A *semi-metric* d on a set X satisfies (a), (c), and (d) of Definition 1.1. We call (X, d) a *semi-metric space*.

Definition 1.4 (Extended metric)**.** An *extended metric* d on a set X is a function $d: X \times X \to \mathbb{R}^{\infty}_{\geq 0}$ satisfying (a)–(d) of Definition 1.1. We call (X, d) an *extended metric space*.

Intuitively, we might want extended metrics if we have points that we never want to be able to get to from other ones.

We can turn spaces with a semi-metric into a space with a metric.

Lemma 1.5. Fix a semi-metric space (X, d), and define the relation $\sim$ on X by $x \sim y$ if and only if $d(x, y) = 0$. Then $\sim$ is an equivalence relation.

Proof. We run these checks by hand. Fix any $x, y, z \in X$.

- Reflexive: $d(x, x) = 0$ means that $x \sim x$.
- Symmetry: if $x \sim y$, then $d(x, y) = 0$, so $d(y, x) = 0$, so $y \sim x$.
- Transitive: if $x \sim y$ and $y \sim z$, then

$$0 \leq d(x, z) \leq d(x, y) + d(y, z) = 0,$$

so $d(x, z) = 0$, so $x \sim z$. ■

As such, given a semi-metric space (X, d), we may look at the set of equivalence classes under $\sim$, which we will denote $X/{\sim}$.[1]

[1] The notation of $/{\sim}$ is intended to make us think of quotients.

Proposition 1.6. Fix a semi-metric space (X,d) and define $\sim$ as in Lemma 1.5. Then d naturally descends to a metric $\widetilde{d}$ on $X/{\sim}$.

Proof. Let $[x]$ denote the equivalence class of $x \in X$ under $\sim$. We claim that the function

$$\widetilde{d}([x],[y]) := d(x,y)$$

is a well-defined metric. We have the following checks; fix any $x,y,z \in X$.

- Well-defined: if $x \sim x'$ and $y \sim y'$, then note that

$$d(x,y) \le d(x,x') + d(x',y) = d(x',y) \le d(x',y') + d(y',y) = d(x',y').$$

  By symmetry, we also have $d(x',y') \le d(x,y)$, so equality follows. So d does descent properly to the quotient $X/{\sim}$.
- Zero: note that $\widetilde{d}([x],[y]) = 0$ if and only if $d(x,y) = 0$ if and only if $x \sim y$ if and only if $[x] = [y]$.
- Symmetry: note that

$$\widetilde{d}([x],[y]) = d(x,y) = d(y,x) = \widetilde{d}([y],[x]).$$

- Triangle inequality: note that

$$\widetilde{d}([x],[z]) = d(x,z) \le d(x,y) + d(y,z) = \widetilde{d}([x],[y]) + \widetilde{d}([y],[z]),$$

  which finishes. ■

Here are some examples of metric spaces.

Example 1.7. Given a connected graph $G = (V,E)$ with a weighting function $w\colon E \to \mathbb{R}_{\ge 0}$, we can build a metric as follows: define the "shortest-path" function $d\colon V \times V \to \mathbb{R}_{\ge 0}$ sending two vertices $v,w \in V$ to the length of the shortest path. If the graph G is not connected, we merely have an extended metric.

Example 1.8 (Euclidean metric)**.** The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$

$$d\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

is a metric.

Observe that it is not completely obvious that Example 1.8 satisfies the triangle inequality, but this will follow from the theory of the next subsections.

1.1.3 Norms on Vector Spaces

Norms provide convenient ways to build metrics.

Definition 1.9 (Norm)**.** Fix a vector space V over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|\colon V \to \mathbb{R}_{\ge 0}$ is a function satisfying the following, for any $r \in \mathbb{R}$ and $v,w \in V$.

(a) Zero: $\|v\| = 0$ if and only if $v = 0$.

(b) Scaling: $\|rv\| = |r| \cdot \|v\|$.

(c) Triangle inequality: $\|v+w\| \le \|v\| + \|w\|$.

Remark 1.10. We can probably work with a more general normed field instead of "merely" $\mathbb{R}$ or $\mathbb{C}$.

And here is our result.

Proposition 1.11. Given a metric space V with a norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$, then the function

$$d(v, w) := \|v - w\|$$

defines a metric on V.

Proof. We run the checks directly. Let $x, y, z \in V$ be points.

- Zero: note that $d(x, y) = 0$ if and only if $\|x - y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$.
- Symmetry: note that

$$d(x, y) = \|x - y\| = |-1| \cdot \|y - x\| = 1 \cdot \|y - x\| = d(y, x).$$

- Triangle inequality: note that

$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z),$$

which finishes the check. ■

Here are the usual examples.

Example 1.12. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then the following are norms on V.

- $\|(x_1, \ldots, x_n)\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.
- $\|(x_1, \ldots, x_n)\|_1 := \sum_{i=1}^n |x_i|$.

Here are some more esotetric examples.

Example 1.13. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then

$$\|(x_1, \ldots, x_n)\|_\infty := \sup\{|x_1|, \ldots, |x_n|\}$$

provides a norm on V.

Example 1.14. Set $V := \mathbb{R}^n$ or $V := \mathbb{C}^n$. Then, given $p \geq 1$,

$$\|(x_1, \ldots, x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

provides a norm on V.

Remark 1.15. Taking the limit as $p \to \infty$ of $\|f\|_p$ gives $\|f\|_\infty$. This justifies the notation.

Remark 1.16. Despite having lots of examples, all of these norms are equivalent in a topological sense.

These normed vector spaces actually allow us to define a metric on any subset.

Proposition 1.17. Given a metric space (X, d) and a subset $Y \subseteq X$, the restriction of d to $Y \times Y$ is a metric.

Proof. All the requirements for d on $Y \times Y$ are satisfied for any points in X, so we are done by doing no work. ■

Example 1.18. Any subset $X \subseteq \mathbb{R}^n$ has an induced metric by restricting the (say) Euclidean metric.

1.1.4 A Hint of L^p Spaces

Here is a more complicated example of a metric.

Example 1.19. Define $V := C([0, 1])$ to be the $\mathbb{R}$-vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0, 1]$. The following are norms.

- $\|f\|_\infty := \sup\{|f(x)| : x \in [0, 1]\}$.
- $\|f\|_1 := \int_0^1 |f(t)|\, dt$.
- $\|f\|_2 := \left(\int_0^1 |f(t)|^2\, dt\right)^{1/2}$.
- More generally, given $p \geq 1$
$$\|f\|_p := \left(\int_0^1 |f(t)|^p\, dt\right)^{1/p}.$$

These integrals are finite because $[0, 1]$ is compact, forcing f to achieve a finite maximum on $[0, 1]$.

Remark 1.20. We can tell the same story for $C(X)$, for any measurable compact space X.

Remark 1.21. Note the analogy of Example 1.19 with Example 1.14. To see this more rigorously, set X to be the finite set $\{1, \ldots, n\}$ so that $C(X) = \mathbb{R}^n$.

We should probably justify the claims of this subsection, so here is our result.

Proposition 1.22. Define $V := C([0, 1])$ to be the vector space of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions on $[0, 1]$. Then, given $p \geq 1$, the function $\|\cdot\|_p : C \to \mathbb{R}_{\geq 0}$ by

$$\|f\| := \left(\int_0^1 |f(t)|^p\, dt\right)^{1/p}$$

is a norm.

Proof. We run the checks directly.

- Zero: if $f = 0$, then of course $\int_0^1 |f(t)|^p\, dt = 0$.
- Zero: suppose that $f \in C([0, 1])$ has $f(t_0) \neq 0$ for any $t_0 \in [0, 1]$; set $y := f(t_0)$. Then $f^{-1}((y/2, 3y/2))$ is a nonempty open subset of X and hence contains a nonempty open interval (a, b) with $a < b$. As such,
$$\int_X |f(t)|^p\, dt \geq \int_a^b |f(t)|^p\, dt \geq \int_a^b |y/2|^p\, dt > 0,$$

so we are done.

- Scaling: given $f \in C([0,1])$ and a scalar r, we have

$$\|rf\| = \left(\int_0^1 |rf(t)|^p\,dt\right)^{1/p} = \left(|r|^p \int_0^1 |f(t)|^p\,dt\right)^{1/p} = |r| \cdot \|f\|.$$

- Triangle inequality: we borrow from [Tao09]. Given $f, g \in C([0,1])$, for psychological reasons we will assume that f and g are nonzero (else this is clear); then $\|f\|, \|g\| \neq 0$, so we may scale everything so that $\|f\| + \|g\| = 1$. In fact, we may again use scaling to find $a, b \in V$ such that

$$f = (1-\theta)a \qquad \text{and} \qquad g = \theta b$$

where $\theta \in (0,1)$ and $\|a\| = \|b\| = 1$. Now, the triangle inequality translates into showing

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p\,dt = \|(1-\theta)a + \theta b\|_p^p \stackrel{?}{\leq} \left(\|(1-\theta)a\|_p + \|\theta b\|_p\right)^p = 1.$$

Well, because $p \geq 1$, the function $t \mapsto t^p$ is convex, so we get to write

$$\int_0^1 |(1-\theta)a(t) + \theta b(t)|^p\,dt \leq (1-\theta)\int_0^1 |a(t)|^p\,dt + \theta \int_0^1 |b(t)|^p\,dt,$$

which is what we wanted.

The above checks complete the proof; note that the proof of the triangle inequality was nontrivial. ■

Remark 1.23. Now, to show Remark 1.21, replace all $\int_0^1$ with $\sum_{i=1}^n$ and adjust all the language accordingly. The point is that "integrating over $[0,1]$" is analogous to "integrating over $\{1,\ldots,n\}$." A more thorough understanding of measure theory will allow us to rigorize this.

Next class we will talk about completeness.

1.2 August 26

Today we're talking about completeness of metric spaces.

1.2.1 Isometries

In mathematics, we are interested in objects not in isolation but as they relate to each other. Namely, we are interested also in the maps between our objects.

The philosophy here comes from category theory, where one is really most interested in the "morphisms" between "objects" instead of the objects themselves. For concreteness, here is a definition of a category.

Definition 1.24 (Category)**.** A *category* $\mathcal{C}$ consists of a class of objects $\operatorname{Ob}\mathcal{C}$ and class of morphisms $\operatorname{Mor}\mathcal{C}$ such that any two objects $A, B \in \operatorname{Ob}\mathcal{C}$ have a morphism class $\operatorname{Mor}(A, B)$. This data satisfy the following properties.

- Composition: given objects $A, B, C \in \operatorname{Ob}\mathcal{C}$, there is a binary composition operation
$$\circ\colon \operatorname{Mor}(B, C) \times \operatorname{Mor}(A, B) \to \operatorname{Mor}(A, C).$$
Explicitly, given $f \in \operatorname{Mor}(A, B)$ and $g \in \operatorname{Mor}(B, C)$, there is a composition $(g \circ f) \in \operatorname{Mor}(A, C)$.
- Given $A \in \operatorname{Ob}\mathcal{C}$, there is an identity morphism $\operatorname{id}_A \in \operatorname{Mor}(A, A)$.
- Identity: any $f \in \operatorname{Mor}(A, B)$ has $f \circ \operatorname{id}_A = f = \operatorname{id}_B \circ f$.
- Associativity: any $f \in \operatorname{Mor}(A, B)$ and $g \in \operatorname{Mor}(B, C)$ and $h \in \operatorname{Mor}(C, D)$ has $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 1.25. There is a category of groups, where the morphisms are group homomorphisms. The identity function gives the identity morphism, and composition of functions gives the required composition.

For completeness, we check that composition is well-defined: given homomorphisms $f\colon A \to B$ and $g\colon B \to C$, we need $(g \circ f)\colon A \to C$ to be a group homomorphism. Well,
$$(g \circ f)(a \cdot a') = g(f(a \cdot a')) = g(f(a) \cdot f(a')) = g(f(a)) \cdot g(f(a')) = (g \circ f)(a) \cdot (g \circ f)(a').$$

In our discussion of metric spaces, there are many possible kinds of morphisms for us to consider. Here is the strongest type.

Definition 1.26 (Isometry)**.** Given metric spaces (X, d_X) and (Y, d_Y), an *isometry* is a function $f\colon X \to Y$ preserving the metric as
$$d_Y(f(x), f(x')) = d_X(x, x').$$

Example 1.27. The 90° rotation $r\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $r(x, y) \mapsto (y, -x)$ is an isometry, where $\mathbb{R}^2$ is given the Euclidean metric. Indeed, any $(x, y), (x', y') \in \mathbb{R}^2$ have
$$\begin{aligned} d\big(r(x, y), r(x', y')\big) &= d\big((y, -x), (y', -x')\big) \\ &= \sqrt{(y - y')^2 + (-x - -x')^2} \\ &= \sqrt{(x - x')^2 + (y - y')^2} \\ &= d\big((x, y), (x', y')\big). \end{aligned}$$

Notation 1.28. Fix two metric spaces (X, d_X) and (Y, d_Y). Given a function $f\colon X \to Y$ with extra structure respecting some aspect of the metric, we might write $f\colon (X, d_X) \to (Y, d_Y)$ to emphasize this.

To show that isometries are valid morphisms, we need to check that the identity function $\operatorname{id}_X\colon X \to X$ is an isometry (which of course it is) and that the composition of two isometries is an isometry. We check this last one in a quick lemma.

Lemma 1.29. Given two isometries $f\colon (X, d_X) \to (Y, d_Y)$ and $g\colon (Y, d_Y) \to (Z, d_Z)$, the composition $g \circ f$ is an isometry.

Proof. Well, any two points $x, x' \in X$ have
$$d_Z(g(f(x)), g(f(x'))) = d_Y(f(x), f(x')) = d_X(x, x'),$$

which is what we wanted. ■

One can restrict further to surjective isometries, where the main point is that (again) the composition of two surjective functions remains surjective. (Note that the identity is of course surjective.) The following is the reason why a surjective isometry is a good notion.

Lemma 1.30. A surjective isometry $f\colon (X, d_X) \to (Y, d_Y)$ is bijective, and its inverse function is also an isometry.

Proof. To see that f is bijective, we only need to know that f is injective. Well, given $x, x' \in X$, note that $f(x) = f(x')$ if and only if $d_Y(f(x), f(x')) = 0$ if and only if $d(x, x') = 0$ if and only if $x = x'$.[2]

Thus, f is indeed bijective; let $g\colon Y \to X$ be its inverse. We now need to show that g is an isometry. Well, given $y, y' \in Y$, we may find $x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$. Then

$$d_X(g(y), g(y')) = d_X((g \circ f)(x), (g \circ f)(x')) = d_X(x, x') \stackrel{*}{=} d_Y(f(x), f(x')) = d_Y(y, y'),$$

where in $\stackrel{*}{=}$ we have used the fact that f is an isometry. ■

Remark 1.31. The above result is somewhat subtle in its importance: the inverse function of a bijection is only an inverse in the category of sets. The above result is saying that this inverse morphism in the category of sets is lifting to an inverse morphism in the category of metric spaces with isometries as morphisms. In general, it is not always true that bijective morphisms are invertible, as we shall soon see.

1.2.2 Lipschitz Continuity

Isometries are somewhat restrictive, so we might weaken this as follows.

Definition 1.32 (Lipschitz continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *Lipschitz continuous* if and only if there is a constant $c \in \mathbb{R}$ such that

$$d_Y(f(x), f(x')) \le c \cdot d_X(x, x').$$

Remark 1.33. Equivalently, we are asking for the ratio

$$\frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

to be uniformly bounded above for all $x \neq x'$. Notably, the inequality is trivially satisfied whenever $x = x'$, or equivalently whenever $d(x, x') = 0$.

Example 1.34. Any isometry $f\colon (X, d_X) \to (Y, d_Y)$ is Lipschitz continuous: indeed, set $c := 1$ so that, for any $x, x' \in X$,

$$d_Y(f(x), f(x')) = d_X(x, x') \le 1 \cdot d_X(x, x').$$

[2] In fact, this argument shows that all isometries are injective. We will shortly see that all actually Lipschitz continuous functions are injective.

Example 1.35. Provide $\mathbb{R}$ and $\mathbb{R}^2$ their usual Euclidean metrics. Then the projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ by $\pi\colon (x,y) \mapsto x$ is Lipschitz continuous: indeed, set $c := 1$ so that, for any $(x,y),(x',y') \in \mathbb{R}^2$, we have

$$d_{\mathbb{R}^2}\big((x,y),(x',y')\big) = \sqrt{(x-x')^2+(y-y')^2} \ge \sqrt{(x-x')^2} = d_{\mathbb{R}}(x,x') = d_{\mathbb{R}}\big(\pi((x,y)),\pi((x',y'))\big).$$

Again, one can see that the identity function $\mathrm{id}_X\colon (X,d_X) \to (X,d_X)$ is Lipschitz continuous (with $c := 1$), and here is our composition check.

Lemma 1.36. If $f\colon (X,d_X) \to (Y,d_Y)$ and $g\colon (Y,d_Y) \to (Z,d_Z)$ are Lipschitz continuous, then the composition $(g \circ f)\colon (X,d_X) \to (Z,d_Z)$ is also Lipschitz continuous.

Proof. We are given constants c and d such that any $x,x' \in X$ and $y,y' \in Y$ have

$$d_Y(f(x),f(x')) \le c \cdot d_X(x,x') \qquad \text{and} \qquad d_Z(g(y),g(y')) \le d \cdot d_Y(y,y').$$

As such, we use the constant cd to witness our Lipschitz continuity: any $x,x' \in X$ have

$$d_Z(g(f(x)),g(f(x'))) \le d \cdot d_Y(f(x),f(x')) \le cd \cdot d_X(x,x'),$$

which is what we wanted. ■

It will be shortly worth our time to talk about the constant c appearing in Definition 1.32.

Lemma 1.37. Fix a Lipschitz continuous function $f\colon (X,d_X) \to (Y,d_Y)$. Then there exists a constant c_f (possibly $-\infty$) such that any real number $c \ge c_f$ is equivalent to the following property: any $x,x' \in X$ have

$$d_Y(f(x),f(x')) \le c \cdot d_X(x,x').$$

Proof. Let S denote the set of all constants c such that any $x,x' \in X$ have

$$d_Y(f(x),f(x')) \le c \cdot d_X(x,x').$$

Equivalently, using Remark 1.33, S is the set of upper-bounds for

$$R := \left\{ \frac{d_Y(f(x),f(x'))}{d_X(x,x')} : x,x' \in X, x \ne x' \right\}.$$

Now, S is nonempty because f is Lipschitz continuity, so we set $c_f := \sup R$ to be the least upper bound for R—observe that $c_f = -\infty$ is permissible when X has one point. It is now pretty clear that $S = [c_f, \infty)$. ■

Note that c_f the property stated in the lemma automatically implies that c_f is the least possible constant and is unique. Being least is immediate (by the backwards direction), and being unique follows from being least. So because we have some uniqueness, we get a definition.

Definition 1.38 (Lipschitz constant)**.** Given a Lipschitz continuous function $f\colon (X,d_X) \to (Y,d_Y)$, the *Lipschitz constant* c_f *for* f is the least real number c such that

$$d_Y(f(x),f(x')) \le c \cdot d_X(x,x').$$

We could, as before, look at surjective Lipschitz continuous functions, but these need not be bijective anymore as shown by Example 1.35. What's worse is that, as warned possible in Remark 1.31, bijective Lipschitz continuous functions need not even have a Lipschitz continuous inverse.

Exercise 1.39. We exhibit a function between metric spaces which is bijective and Lipschitz continuous, but its inverse function is not Lipschitz continuous.

Proof. Set $X := (0,1)$ and $Y := (1,\infty)$, both metric spaces with the Euclidean (subspace) metric, and set $f\colon (0,\infty) \to (0,\infty)$ by $f\colon x \mapsto 1/x$. Notably, $x \in X$ implies $f(x) \in Y$, and $y \in Y$ implies $f(y) \in X$.

- Note $f|_Y$ is bijective with inverse $f|_X$ because $f(f(x)) = f(1/x) = x$ for all $x \in (0,\infty)$.
- Note $f|_Y$ is Lipschitz continuous: set $c := 1$ and note that any $y, y' \in Y$ have

$$|f(y) - f(y')| = \left|\frac{1}{y} - \frac{1}{y'}\right| = \left|\frac{y-y'}{yy'}\right| \le |y - y'|.$$

- But $f|_X$ is not Lipschitz continuous: suppose for contradiction that f_X is Lipschitz continuous, and use Lemma 1.37 to recover the needed constant c_0. Then set $c := \max\{c_0, 4\}$, which must also work as a constant, and set $x := 1/c$ and $x' := 1/(3c)$ so that

$$|f(x) - f(x')| = |c - 3c| = 2c > c \cdot |x - x'|.$$

This is a contradiction, so we are done. ■

Remark 1.40 (Nir)**.** In some sense, the problem here is that the definition of Lipschitz continuity allows $d_Y(f(x), f(x'))$ to be "too small," which permits the inverse function to have distances which blow up.

In light of Exercise 1.39, we introduce a new definition.

Definition 1.41 (Lipschitz isomorphism)**.** Give metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is a *Lipschitz isomorphism* if and only if f is Lipschitz continuous and has an inverse function which is also Lipschitz continuous.

Remark 1.42. A good reason to care about this notion of continuity (and isomorphism) is that all normed $\mathbb{R}$-vector spaces of some finite dimension n are Lipschitz isomorphic.

1.2.3 Fun with Continuity

Here is yet a weaker notion of morphism.

Definition 1.43 (Uniformly continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *uniformly continuous* if and only if every $\varepsilon > 0$ has some $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

for all $x, x' \in X$.

Example 1.44. Any Lipschitz continuous function $f\colon (X, d_X) \to (Y, d_Y)$ is also uniformly continuous: indeed, for any $\varepsilon > 0$, set $\delta := \max\{c_f, 1\}\varepsilon > 0$ (where c_f is the Lipschitz constant) so that

$$d_X(x, x') < \varepsilon \implies d_Y(f(x), f(x')) \le c_f \cdot d(x, x') < \delta.$$

Example 1.45. Give $[0,1]$ the Euclidean (subspace) metric, and set $f\colon [0,1] \to [0,1]$ by $f(x) := \sqrt{x}$.

- Note f is uniformly continuous because it is continuous on a compact set.
- However, f is not Lipschitz continuous: for any constant $c > 0$, set $x = 1/(c+1)^2$ and $x' = 0$ so that
$$\left|\frac{f(x) - f(x')}{x - x'}\right| = \left|\frac{1/(c+1)}{1/(c+1)^2}\right| = |c+1| > c,$$
so Remark 1.33 tells us that we are not Lipschitz continuous.

By rearranging quantifiers, we get another useful (but weaker) notion.

Definition 1.46 (Continuous)**.** Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous at* $x \in X$ if and only if all $\varepsilon > 0$ have some $\delta_x > 0$ such that
$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$
Then f is *continuous* if and only if it is continuous at all $x \in X$.

Example 1.47. All uniformly continuous functions $f\colon (X, d_X) \to (Y, d_Y)$ are continuous. Indeed, at any $x_0 \in X$ with $\varepsilon > 0$, uniform continuity promises $\delta > 0$ so that
$$|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon$$
for all $x, x' \in X$. Setting x' to x_0 recovers continuity.

Example 1.48. Give $\mathbb{R}$ the usual Euclidean metric, and set $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) := x^2$.

- Note $f(x)$ is continuous because it is a polynomial.
- However, $f(x)$ is not uniformly continuous: take $\varepsilon = 1$. Now, for any $\delta > 0$, set $x = 1/\delta$ and $x' = 1/\delta + \delta/2$ so that $|x - x'| < \delta$, but
$$|f(x) - f(x')| = \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2} = 1 + \frac{\delta^2}{4} > \varepsilon.$$

As usual, the identity function is uniformly continuous and continuous (it's an isometry), and these continuities are preserved by composition. We will have a different way to see that continuous functions remain continuous under composition later, so for now we will focus on uniform continuity.

Lemma 1.49. Fix uniformly continuous morphisms $f\colon (X, d_X) \to (Y, d_Y)$ and $g\colon (Y, d_Y) \to (Z, d_Z)$. Then the function $(g \circ f)$ is uniformly continuous.

Proof. For any $\varepsilon > 0$, the uniform continuity of g promises $\delta_g > 0$ such that
$$d_Y(y, y') < \delta_g \implies d_Z(g(y), g(y')) < \varepsilon$$
for any $y, y' \in Y$. Continuing, the uniform continuity of f promises $\delta_f > 0$ such that
$$d_X(x, x') < \delta_X \implies d_Y(f(x), f(x')) < \delta_Y \implies d_Z(g(f(x)), g(f(x'))) < \varepsilon$$
for any $x, x' \in X$, which is what we wanted. ■

Remark 1.50. In some sense, isometries and Lipschitz continuous functions have their definition fundamentally interrelated with the metric. In contrast, the weaker notion of continuity will readily generalize to general topological spaces. Uniform continuity also generalizes to "uniformities," which is a different notion.

1.2.4 Convergence and Completeness

To discuss completeness, we need to talk about convergence.

Definition 1.51 (Converge)**.** Fix a metric space (X, d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n > N \implies d(x_n, x) < \varepsilon.$$

We might write this as "$x_n \to x$ as $n \to \infty$" or "$\lim_{n\to\infty} x_n = x$." In this event, we may say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges*, and its limit is x.

Remark 1.52 (Nir)**.** As a sanity check, the limit of a sequence is unique: if $x_n \to x$ and $x_n \to x'$ as $n \to \infty$, then any $\varepsilon > 0$ can find some large n so that $d(x_n, x), d(x_n, x') < \varepsilon/2$. As such,

$$d(x, x') < d(x_n, x) + d(x_n, x') = \varepsilon$$

for any $\varepsilon > 0$, so $d(x, x') = 0$ and thus $x = x'$ is forced.

We have no reason yet to be convinced that any of our morphisms described previously are good notions, so let's start with continuity.

Lemma 1.53. Fix a continuous function between metric spaces $f : (X, d_X) \to (Y, d_Y)$. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ converges to $f(x) \in Y$.

Proof. For any $\varepsilon > 0$,the continuity of f implies that we can find $\delta_x > 0$ so that

$$d_X(x_n, x) < \delta_x \implies d_Y(f(x_n), f(x)) < \varepsilon$$

for any x_n. But the fact that $x_n \to x$ as $n \to \infty$ means that there is $N > 0$ so that

$$n > N \implies d_X(x_n, x) < \delta_x \implies d_Y(f(x_n), f(x)) < \varepsilon,$$

so indeed, $f(x_n) \to f(x)$ as $n \to \infty$. ■

In fact, the converse also holds.

Lemma 1.54. Fix metric spaces (X, d_X) and (Y, d_Y), and fix a point $x \in X$. Then suppose a function $f : X \to Y$ satisfies that any convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x$ as $n \to \infty$ has $f(x_n) \to f(x)$ as $n \to \infty$. Then f is continuous at x.

Proof. We proceed by contraposition. If f is not continuous at x, then any $n \in \mathbb{N}$ can find x_n such that $d_X(x, x_n) < 1/n$ even though $d_Y(f(x_n), f(x)) \geq 1$. In particular, $x_n \to x$ as $n \to \infty$ (for any ε, choose $N = 1/\varepsilon$), but the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ does not converge to $f(x)$ because no n has $d_Y(f(x), f(x_n)) < 1$. ■

We would like a notion of convergence which only uses data internal to the sequence, and this leads to the following definition.

Definition 1.55 (Cauchy)**.** Fix a metric space (X,d). A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy sequence* if and only if, for any $\varepsilon > 0$, we can find $N > 0$ such that

$$n,m > N \implies d(x_n, x_m) < \varepsilon.$$

It would be rude if continuity was always the best kind of morphism, so this time around preserving Cauchy-ness requires something stronger.

Lemma 1.56. Fix a uniformly continuous function between metric spaces $f\colon (X,d_X) \to (Y,d_Y)$. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is Cauchy, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}} \subseteq Y$ is also Cauchy.

Proof. For any $\varepsilon > 0$, the uniform continuity of f promises $\delta > 0$ so that

$$d_X(x_n, x_m) < \delta \implies d(f(x_n), f(x_m)) < \varepsilon$$

for any x_n, x_m. However, the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy promises N so that

$$n,m > N \implies d_X(x_n,x_m) < \delta \implies d(f(x_n), f(x_m)) < \varepsilon,$$

which is what we wanted. ■

Example 1.57. Continuous functions do not need to preserve Cauchy sequences: $f\colon (0,\infty) \to (0,\infty)$ by $f(x) := 1/x$ is continuous, and the sequence $\{1/n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ is Cauchy (it converges to 0 in $\mathbb{R}$) even though $\{f(1/n)\}_{n\in\mathbb{N}} = \{n\}_{n\in\mathbb{N}}$ certainly does not converge.

Anyway, it is quick to check that convergent sequences are Cauchy.

Lemma 1.58. Fix a metric space (X,d). Then all convergent sequences are Cauchy.

Proof. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$. Then, for any $\varepsilon > 0$, find N so that

$$d(x_n, x) < \varepsilon/2$$

for all $n > N$. Then any $n, m > N$ has

$$d(x_n,x_m) \le d(x_n,x) + d(x_m,x) < \varepsilon,$$

so the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. ■

We in general hope that our Cauchy sequences converge. As such, we have the following definition.

Definition 1.59 (Complete)**.** A metric space (X,d) is *complete* if and only if every Cauchy sequence in X converges to a point in X.

We are sad when a metric space is not complete, so we hope to have a way to make it complete. The most natural way to do this is by using the notion of density.

Definition 1.60 (Dense)**.** Fix a metric space (X,d). Then $S \subseteq X$ is *dense* if and only if, given any $x \in X$ and $\varepsilon > 0$, we may find $x' \in S$ with $d(x,x') < \varepsilon$.

And here is our completion.

Definition 1.61 (Completion)**.** A *completion* of the metric space (X,d) is a metric space $(\overline{X},\overline{d})$ equipped with an isometry $\iota\colon X \to \overline{X}$ such that $(\overline{X},\overline{d})$ is complete and $\operatorname{im}\iota$ is dense in $\overline{X}$.

One can show that any metric space has a completion and that they are all isometric and therefore in some sense the same. We'll do these separately.

1.2.5 Existence of Completions

Let's start with existence.

Theorem 1.62. Any metric space (X, d) has a completion.

Proof. Let $\widetilde{X}$ denote the set of all Cauchy sequences in X. We hope to make $\widetilde{X}$ into our completion, but this requires a little care. To begin, we have the following lemma.

Lemma 1.63. Given a metric space (X, d) with two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, then the sequence

$$\{d(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$$

converges.

Proof. Because $\mathbb{R}$ is a complete metric space, it suffices to show that the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is Cauchy. Well, for any $\varepsilon > 0$, find a sufficiently large N so that

$$n, m > N \implies d(x_n, x_m), d(y_n, y_m) < \varepsilon/2.$$

Then any $n, m > N$ has

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \varepsilon + d(y_m, y_n),$$

and $d(x_m, y_m) < d(x_n, y_n) + \varepsilon$ as well by symmetry. It follows that any $n, m > N$ has

$$\left|d(x_n, y_n) - d(x_m, y_m)\right| < \varepsilon,$$

verifying that our sequence is Cauchy. ■

Remark 1.64. Here is a quick motivational remark for the definition of our metric below: if (X, d) is a metric space with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then we claim $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. Indeed, for any $\varepsilon > 0$, we can find N large enough so that $d(x_n, x), d(y_n, y) < \varepsilon/2$ for any $n > N$. As such,

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n) < d(x, y) + \varepsilon.$$

By symmetry, we get $d(x, y) \le d(x_n, y_n) + \varepsilon$ as well, finishing.

Thus, we define $\widetilde{d} \colon \widetilde{X} \times \widetilde{X} \to \mathbb{R}_{\ge 0}$ by

$$\widetilde{d}\left(\{x_n\}, \{y_n\}\right) := \lim_{n\to\infty} d(x_n, y_n).$$

We claim that $\widetilde{d}$ is a semi-metric on $\widetilde{X}$. We have the following checks; fix Cauchy sequences $\{x_n\}, \{y_n\}, \{z_n\}$.

- Zero: note
$$\widetilde{d}(\{x_n\}, \{x_n\}) = \lim_{n\to\infty} d(x_n, x_n) = 0.$$

- Symmetry: note
$$\widetilde{d}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \widetilde{d}(\{y_n\}, \{x_n\}).$$

- Triangle inequality: note

$$\begin{aligned}\widetilde{d}(\{x_n\},\{y_n\})+\widetilde{d}(\{y_n\},\{z_n\}) &= \lim_{n\to\infty} d(x_n,y_n)+\lim_{n\to\infty} d(y_n,z_n)\\ &= \lim_{n\to\infty}(d(x_n,y_n)+d(y_n,z_n))\\ &\geq \lim_{n\to\infty} d(x_n,z_n)\\ &= \widetilde{d}(x_n,z_n),\end{aligned}$$

  where we have implicitly used a number of limit laws.

So because $\widetilde{d}$ is a semi-metric, Proposition 1.6 tells us that $\widetilde{d}$ will descend naturally to a metric $\overline{d}$ on $\overline{X} := \widetilde{X}/\sim$, where $\{x_n\}\sim\{y_n\}$ if and only if $\widetilde{d}(\{x_n\},\{y_n\})=0$. We will let $[\{x_n\}]$ denote the equivalence class of the Cauchy sequence $\{x_n\}\in\widetilde{X}$ in $\overline{X}$.

We now show that $(\overline{X},\overline{d})$ can be made into a completion for X.

- Given $x\in X$, note that the constant sequence $\{x\}$ is Cauchy (for any $\varepsilon>0$, set $N=0$), so we define $\iota\colon X\to\overline{X}$ by

$$\iota(x) := [\{x\}].$$

  To see that ι is an isometry, note any $x,x'\in X$ have

$$\overline{d}(\iota(x),\iota(x'))=\widetilde{d}(\{x\},\{y\})=\lim_{n\to\infty} d(x,y)=d(x,y).$$

- We show that $\operatorname{im}\iota$ is dense in $\overline{X}$. Indeed, fix some $[\{x_n\}]\in\overline{X}$ and $\varepsilon>0$. Then there is some N so that $n,m>N$ has

$$d(x_n,x_m)<\varepsilon/2.$$

  Fixing a particular n_0 with $n_0>N$, we set $x:=x_{n_0}$ so that

$$\overline{d}([\{x_n\}],\iota(x))=\widetilde{d}(\{x_n\},x_{n_0})=\lim_{n\to\infty} d(x_n,x_{n_0}).$$

  Now, for $n>N$, we have $d(x_n,x_{n_0})<\varepsilon/2$, so we conclude that this limit must be less than ε.

- We show that $(\overline{X},\overline{d})$ is a complete metric space. Fix a Cauchy sequence $\{\overline{x}_k\}$ in $\overline{X}$. To find the Cauchy sequence we are supposed to converge to, we use our density result: for each $k\in\mathbb{N}$, we can find $y_k\in X$ such that $\overline{d}(\overline{x}_k,\iota(y_k))<1/k$.

  We claim that $\{y_k\}$ is Cauchy. Indeed, for any $\varepsilon>0$, we can find N such that $k,\ell>N_0$ has

$$\overline{d}(\overline{x}_k,\overline{x}_\ell)<\varepsilon/3.$$

  Then, setting $N:=\max\{3/\varepsilon,N_0\}$, we note that $k,\ell>N$ has

$$d(y_k,y_\ell)=\overline{d}(\iota(y_k),\iota(d_\ell))\leq\overline{d}(\overline{x}_k,\iota(y_k))+\overline{d}(\overline{x}_\ell,\iota(y_\ell))+\overline{d}(\overline{x}_k,\overline{x}_\ell)<\varepsilon.$$

  Lastly, we claim that $\overline{x}_k\to[\{y_n\}]$ in $\overline{X}$. Indeed, for any $\varepsilon>0$, find some sufficiently large N so that

$$k,\ell>N\implies d(y_k,y_\ell)<\varepsilon/2.$$

  Then $k>\max\{N,2/\varepsilon\}$ has

$$\overline{d}(\overline{x}_k,[\{y_n\}])\leq\overline{d}(\overline{x}_k,\iota(y_k))+\overline{d}([\{y_n\}],\iota(y_k))<\frac{\varepsilon}{2}+\lim_{n\to\infty} d(y_n,y_k).$$

  Because $k>N$, we have $d(y_n,y_k)<\varepsilon/2$ for any $n>N$, so the entire right-hand side must be upper-bounded by ε. This finishes.

The above checks complete the proof. ■

Remark 1.65 (Nir). One might complain that we used the completeness of $\mathbb{R}$ in this proof because one common way to construct the real numbers is as the completion of $\mathbb{Q}$ under the Euclidean metric. To remedy this, one ought to define the equivalence relation on Cauchy sequences more directly, saying that two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of real numbers are equivalent under $\sim$ if and only if

$$\lim_{n\to\infty} d_{\mathbb{R}}(x_n, y_n) = 0.$$

1.2.6 Uniqueness of Completions

We now show that any two completions of a metric space (X,d) are isometric, which is our uniqueness result. Here is the main intermediate result.

Lemma 1.66. Fix a metric space (X,d) and a completion $(\overline{X},\overline{d})$ with its isometry $\iota\colon (X,d)\to(\overline{X},\overline{d})$. Then, for any complete metric space (Y,d') and isometry $\varphi\colon (X,d)\to(Y,d')$, there is a unique isometry $\psi\colon(\overline{X},\overline{d})\to(Y,d')$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \overline{X} \arrow[d, dashed, "\psi"] \\ & Y \end{tikzcd}$$

Proof. We start by showing the uniqueness of ψ. Well, for any $\overline{x}\in\overline{X}$, note that any $n\in\mathbb{N}$ allows us to find $x_n\in X$ with

$$\overline{d}(\overline{x},\iota(x_n)) < 1/n$$

because $\operatorname{im}\iota$ is dense in $\overline{X}$. Now, we notice that $\iota(x_n)\to\overline{x}$ as $n\to\infty$ because any $\varepsilon>0$ can set $N=1/\varepsilon$. As such, we see that Lemma 1.53 applied to any possible $\psi\colon\overline{X}\to Y$ forces

$$\psi(\overline{x}) = \psi\Big(\lim_{n\to\infty}\iota(x_n)\Big) = \lim_{n\to\infty}\psi(\iota(x_n)) = \lim_{n\to\infty}\varphi(x_n).$$

Note that, a priori, we do not know if the sequence $\{\varphi(x_n)\}_{n\in\mathbb{N}}$ converges, but this argument tells us that it must; the limit is unique by Remark 1.52, so $\psi(\overline{x})$ is unique as well.

We now show that ψ exists. As before, any $\overline{x}\in\overline{X}$ can find a sequence $\{x_n\}\subseteq X$ such that $\iota(x_n)\to\overline{x}$ as $n\to\infty$. Thus, we note that $\{\varphi(x_n)\}$ is Cauchy by Lemma 1.56, so the completeness of Y gives it a limit; we set

$$\psi(\overline{x}) \coloneqq \lim_{n\to\infty}\varphi(x_n).$$

We have the following checks on ψ.

- Well-defined: if we have two sequences $\{x_n\}$ and $\{x'_n\}$ such that $\iota(x_n)\to x$ and $\iota(x'_n)\to x$ as $n\to\infty$, we need to show that

  $$\lim_{n\to\infty}\varphi(x_n) = \lim_{n\to\infty}\varphi(x'_n).$$

  For brevity, set y and y' to be the limits of $\{\varphi(x_n)\}$ and $\{\varphi(x'_n)\}$, respectively. Then, for any $\varepsilon>0$, we note that there is a sufficiently large N such that

  $$n>N \implies d_Y(y,\varphi(x_n)), d_Y(y',\varphi(x'_n)) < \varepsilon/4.$$

  Further, we can make N even larger so that

  $$n>N \implies \overline{d}(\overline{x},\iota(x_n)), \overline{d}(\overline{x},\iota(x'_n)) < \varepsilon/4.$$

As such, any $n > N$ has

$$\begin{aligned} d_Y(y,y') &\le d_Y(y,\varphi(x_n)) + d_Y(\varphi(x_n),\varphi(x_n')) + d_Y(y',\varphi(x_n')) \\ &< \varepsilon/4 + d_X(x_n,x_n') + \varepsilon/4 \\ &= \varepsilon/2 + \overline{d}(\iota(x_n),\iota(x_n')) \\ &\le \varepsilon/2 + \overline{d}(\overline{x},\iota(x_n)) + \overline{d}(\overline{x},\iota(x_n')) \\ &< \varepsilon. \end{aligned}$$

It follows $d_Y(y,y') = 0$, so $y = y'$.

- Isometry: given $\overline{x},\overline{x}' \in \overline{X}$, find sequences $\{x_n\}$ and $\{x_n'\}$ in X so that $\iota(x_n) \to \overline{x}$ and $\iota(x_n') \to \overline{x}'$ as $n \to \infty$. Thus,

$$\begin{aligned} d_Y(\psi(\overline{x}),\psi(\overline{x}')) &= d_Y\Big(\lim_{n\to\infty}\varphi(x_n), \lim_{n\to\infty}\varphi(x_n')\Big) \\ &\overset{*}{=} \lim_{n\to\infty} d_Y(\varphi(x_n),\varphi(x_n')) \\ &= \lim_{n\to\infty} d(x_n,x_n') \\ &= \lim_{n\to\infty} \overline{d}(\iota(x_n),\iota(x_n')) \\ &= \overline{d}\Big(\lim_{n\to\infty}\iota(x_n), \lim_{n\to\infty}\iota(x_n')\Big) \\ &\overset{*}{=} \overline{d}(\overline{x},\overline{x}'), \end{aligned}$$

where we have used Remark 1.64 at the $\overset{*}{=}$.

- For any $x \in X$, we see that the (constant) Cauchy sequence $\{\iota(x)\}$ converges to $\iota(x)$, so

$$\psi(\iota(x)) = \lim_{n\to\infty} \varphi(x) = \varphi(x).$$

It follows $\psi \circ \iota = \varphi$.

Thus, we have finished establishing the existence of an isometry $\psi\colon \overline{X} \to Y$ such that $\varphi = \psi \circ \iota$. ■

Remark 1.67. One can also replace all isometries with uniformly continuous functions in the statement.

And here is our uniqueness result.

Theorem 1.68. Fix a metric space (X,d) and two completions $\iota\colon (X,d) \to (\overline{X},\overline{d})$ and $\iota'\colon (X,d) \to (\overline{X}',\overline{d}')$. Then there is a surjective isometry $\psi\colon (\overline{X},\overline{d}) \to (\overline{X}',\overline{d}')$.

Proof. Applying Lemma 1.66 twice, we get isometries $\psi\colon (\overline{X},\overline{d}) \to (\overline{X}',\overline{d}')$ and $\psi'\colon (\overline{X}',\overline{d}') \to (\overline{X},\overline{d})$ making the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \overline{X} \arrow[d, "\psi"] \\ & \overline{X}' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\iota'"] \arrow[dr, "\iota"'] & \overline{X}' \arrow[d, "\psi'"] \\ & \overline{X} \end{tikzcd}$$

In particular, we see that $\psi' \circ \psi$ makes the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\iota"'] & \overline{X} \arrow[d, "\psi'\circ\psi"] \\ & \overline{X} \end{tikzcd}$$

However, using Lemma 1.66 again, this isometry $\psi' \circ \psi$ is unique to make the diagram commute, and we could of course put the isometry $\mathrm{id}_{\overline{X}}$ here if we wanted to. Thus,

$$\psi' \circ \psi = \mathrm{id}_{\overline{X}}.$$

By symmetry, $\psi \circ \psi' = \mathrm{id}_{\overline{X'}}$, so we do see that ψ and ψ' are inverse isometries. This finishes the proof. ■

1.3 August 29

Good morning everyone.

1.3.1 Some Examples

Let's give some more examples of metric spaces. Let's start with spaces of continuous functions.

Definition 1.69. We denote the $\mathbb{R}$-vector space of $\mathbb{C}$-valued continuous function from a topological space X as $C(X)$.

And here are our two examples. The first is of a complete metric space.

Exercise 1.70. Give $V := C([0,1])$ the uniform norm

$$\|f\|_\infty := \sup\{|f(t)| : t \in [0,1]\}.$$

Then V is complete.

Proof. This is merely the statement that a sequence of continuous functions which are uniformly Cauchy will converge uniformly to a continuous function. We will prove this for completeness. Fix a sequence of continuous function $\{f_n\}_{n\in\mathbb{N}}$ which are Cauchy with respect to $\|\cdot\|_\infty$. In other words, for each $\varepsilon > 0$, there exists N_ε so that

$$n, m > N_\varepsilon \implies \|f_n - f_m\|_\infty < \varepsilon,$$

which means that $|f_n(t) - f_m(t)| < \varepsilon$ for all $t \in [0,1]$.

In particular, for any fixed $t \in [0,1]$, the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ (using the same N_ε), so we use the completeness of $\mathbb{R}$ to let this sequence converge to $f(t) \in \mathbb{R}$. We have the following checks.

- To see that $f_n \to f$ as $n \to \infty$ (under our metric), select any $\varepsilon > 0$, and then find N so that

  $$n, m > N \implies \|f_n - f_m\|_\infty < \varepsilon/3.$$

  Further, for any $t \in [0,1]$, we see that we can find a large enough $n_t > N$ so that $|f(t) - f_{n_t}(t)| < \varepsilon/3$. But then $n > N$ has

  $$|f_n(t) - f(t)| \le |f_n(t) - f_{n_t}(t)| + |f_{n_t}(t) - f(t)| < 2\varepsilon/3,$$

  so $\|f - f_n\|_\infty \le 2\varepsilon/3 < \varepsilon$.

- To see that f is continuous, fix $t \in [0,1]$ so that we want to show f is continuous at t. Well, for any $\varepsilon > 0$, find N large enough so that

  $$n, m > N \implies \|f_n - f_m\|_\infty < \varepsilon/4.$$

  Now, select $n_t > N$ large enough so that $|f(t) - f_{n_t}(t)| < \varepsilon/4$, and the continuity of f_{n_t} promises us $\delta > 0$ so that

  $$|t - t'| < \delta \implies |f_{n_t}(t) - f_{n_t}(t')| < \varepsilon/4.$$

In particular, for any t' with $|t-t'|<\delta$, find $n_{t'}>N$ large enough so that $|f(t')-f_{n_{t'}}(t')|<\varepsilon/4$, and then we see

$$|f(t)-f(t')| \le |f(t)-f_{n_t}(t)|+|f_{n_t}(t)-f_{n_t}(t')|+|f_{n_t}(t')-f_{n_{t'}}(t')|+|f_{n_{t'}}(t')-f(t')|<\varepsilon,$$

which is what we wanted. ■

The second example is the same space, but it is no longer complete.

Example 1.71. Fix $p\ge 1$ finite. Give $V:=C([0,1])$ the L^p norm as

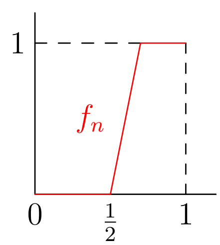
$$\|f\|_p := \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

Then V is not complete.

Proof. For each $n\ge 2$, define f_n as the piecewise continuous function

$$f_n(t) := \begin{cases} 0 & 0\le t\le \frac12, \\ n(t-\frac12) & \frac12 \le t\le \frac12+\frac1n, \\ 1 & \frac12+\frac1n\le t\le 1.\end{cases}$$

Here is the image.



The point is that f_n is trying to converge to a discontinuous function. To help us with the proof here, we pick up the following lemma.

Lemma 1.72. Fix $V:=C([0,1])$ and some finite $p\ge 1$. If we have a convergent sequence $f_n\to f$ as $n\to\infty$ in the $\|\cdot\|_p$ metric, and $f_n(t)=g(t)$ for all sufficiently large n and $t\in U$ for some open $U\subseteq C([0,1])$, then $f|_U(t)=g(t)$.

Proof. Suppose for the sake of contradiction that we have $t_0\in U$ with $f(t_0)\ne g(t_0)$; we show that $\{f_n\}$ does not converge to f. Set $\varepsilon:=|f(t_0)-g(t_0)|$, which is nonzero. The continuity of $f-g$ now promises that there is $\delta>0$ for which

$$|t-t_0|<\delta \implies |(f-g)(t_0)-(f-g)(t)|<\varepsilon/2,$$

so in particular $|(f-g)(t)|\ge\varepsilon/2$. It follows that, for sufficiently large n, we have

$$\|f-f_n\|_p^p = \int_0^1 |f(t)-f_n(t)|^p\,dt \ge \int_U |(f-g)(t)|\,dt \ge \int_{U\cap(t_0-\delta,t_0+\delta)} \frac{\varepsilon}{2}\,dt.$$

Because $U\cap(t_0-\delta,t_0+\delta)$ is open, it has nonzero measure, so this entire right-hand quantity is nonzero, thus violating that $f_n\to f$ as $n\to\infty$. ■

Now suppose for the sake of contradiction that $f_n \to f$ as $n \to \infty$ for some $f \in V$. Then, using $U = (0, 1/2)$, we conclude that $f(t) = 0$ for all $t \in (0, 1/2)$. Similarly, for any n, we set $U_n = (1/2 + 1/n, 1)$, so $f_m|_{U_n}$ returns 1 always for sufficiently large m; this then implies $f(t) = 1$ for any $t \in U_n$ for any n, so $f(t) = 1$ for any $t \in (1/2, 1)$.

However, the sequences $a_n := \frac{1}{2} - \frac{1}{n}$ and $b_n := \frac{1}{2} + \frac{1}{n}$ (for $n \geq 3$) have $a_n \to \frac{1}{2}$ and $b_n \to \frac{1}{2}$ both as $n \to \infty$ while the continuity of f would require

$$0 = \lim_{n \to \infty} f(a_n) = f(1/2) = \lim_{n \to \infty} f(b_n) = 1,$$

which is a contradiction. ■

Remark 1.73. In an attempt to make this metric space complete, we can try to specify which functions we want to look at, which motivates the theory of measure and integration.

Remark 1.74. The $\|\cdot\|_2$ norm on $C(X)$ for some (say) subset $X \subseteq \mathbb{R}$ with finite measure as coming from an inner product

$$\langle f, g \rangle := \int_X f(t)\overline{g(t)}\, dt.$$

When $\|\cdot\|_2$ is complete, we would then get a Hilbert space, which are very nice normed vector spaces, and we'll see more of them in Math 202B.

Remark 1.75 (Nir)**.** In contrast to the finite case, we see that the $\|\cdot\|_\infty$ norm induces a different (metric) topology on $C([0, 1])$ than the $\|\cdot\|_p$ norms with p finite because the former is complete while the latter are not. In fact, all the norms $\|\cdot\|_p$ induce different topologies on $C([0, 1])$.

Part I

Topology

THEME 2
BUILDING TOPOLOGIES

Sets are not doors.

—Munkres

2.1 August 29

We continue lecture by shifting to topology.

2.1.1 Metric Topology

We close our discussion of metric spaces with a taste of topology. Recall the following definition.

Definition 1.46 (Continuous). Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous at $x \in X$* if and only if all $\varepsilon > 0$ have some $\delta_x > 0$ such that

$$d_X(x, x') < \delta_x \implies d_Y(f(x), f(x')) < \varepsilon.$$

Then f is *continuous* if and only if it is continuous at all $x \in X$.

We are going to want to extend this definition to more general topological spaces. To step in that direction, we will want to talk about open sets, so we start with open balls.

Definition 2.1 (Ball). Fix a metric space (X, d). Then the *open ball of radius r centered at $x_0 \in X$* is

$$B(x_0, r) := \{x \in X : d(x, x_0) < r\}.$$

The *closed ball* is $\overline{B(x_0, r)} := \{x \in X : d(x, x_0) \leq r\}$.

We can now restate continuity as follows.

Definition 2.2 (Continuous). Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous* at $x \in X$ if and only if, given any nonempty open ball $B(f(x_0), \varepsilon)$, there exists a nonempty open ball $B(x_0, \delta)$ such that

$$f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon).$$

Namely, we've really only restated our inequalities.

To continue our generalization, we define the pre-image.

Definition 2.3 (Pre-image). Fix a function $f\colon X \to Y$. Then we define the *pre-image* $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

Note that our pre-image notation matches with the notation of an inverse function. In general, no confusion will arise by confusing these two.

As such, let's restate continuity again: observe that $A \subseteq X$ and $B \subseteq Y$ has $f(A) \subseteq B$ if and only if all $a \in A$ have $f(a) \in B$ if and only if all $a \in A$ have $a \in f^{-1}(B)$ if and only if $A \subseteq f^{-1}(B)$.

Definition 2.4 (Continuous). Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is *continuous* at $x \in X$ if and only if, given any nonempty open ball $B(f(x), \varepsilon)$, there exists a nonempty open ball $B(x, \delta)$ such that

$$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)).$$

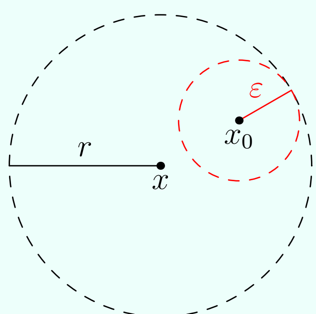
We defined open balls and promised open sets, so now let's define our open sets.

Definition 2.5 (Open set). Fix a metric space (X, d). Then a subset $U \subseteq X$ is *open* if and only if, for each $x \in U$, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. In other words, each point in U has an open ball around it.

Example 2.6. Open balls are open sets. Indeed, given an open ball $B(x, r)$, note that any $x_0 \in B(x, r)$ has $d(x_0, x) < r$, so we take $\varepsilon := r - d(x_0, x)$. To see this works, observe $x' \in B(x_0, \varepsilon)$ will have

$$d(x', x) \le d(x', x_0) + d(x_0, x) < \varepsilon + (r - \varepsilon) = r,$$

so $B(x_0, \varepsilon) \subseteq B(x, r)$ follows. Here is the image for what just happened.



And here is our definition of corresponding definition of continuity.

Lemma 2.7. Given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is continuous at $x \in X$ if and only if, given any open set $U \subseteq Y$ with $f(x) \in U$, there is an open ball $B(x, \delta)$, such that

$$B(x, \delta) \subseteq f^{-1}(U).$$

Proof. Taking f to be continuous, note that we can find $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$ because U is open. Thus, continuity promises $\delta > 0$ such that

$$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)) \subseteq f^{-1}(U).$$

Conversely, if f satisfies the conclusion of the statement, we can take $U = B(f(x), \varepsilon)$ for any $\varepsilon > 0$ by Example 2.6, and the conclusion promises $\delta > 0$ such that

$$B(x, \delta) \subseteq f^{-1}(U) = f^{-1}(B(f(x), \varepsilon)),$$

which is what we wanted. ■

It is cleaner to talk about the entire function being continuous instead of at a point.

Lemma 2.8. Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is continuous if and only if, given any open set $U \subseteq Y$ with $f(x) \in U$, the pre-image $f^{-1}(U)$ is open.

Proof. This is a matter of rearranging our quantifiers correctly. Lemma 2.7 tells us that, for all $x \in X$, all open $U \subseteq Y$ with $f(x) \in U$ has some $\delta > 0$ such that $B(x, \delta) \subseteq U$. Equivalently, for all open $U \subseteq Y$, any $x \in X$ with $x \in f^{-1}(U)$ has some $\delta > 0$ such that $B(x, \delta) \subseteq U$. But by definition of being open, we're just saying that all open $U \subseteq Y$ has $f^{-1}(U)$ also open. ■

So we have the following definition.

Definition 2.9 (Continuous). A function $f : X \to Y$ between metric spaces is *continuous* if and only if, for any open set $U \subseteq Y$, the pre-image $f^{-1}(U)$ is open.

The philosophy here is to try to understand open sets instead of trying to understand the metrics. This is the idea of topology.

2.1.2 Open Sets

Thus, we are motivated to understand open sets. Here are some basic properties.

Proposition 2.10. Fix a metric space (X, d), and let $\mathcal{T}$ be the collection of open sets.

(a) We have $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union

$$\bigcup_{U \in \mathcal{U}} U$$

is open.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \in \mathcal{T}$, we have

$$\bigcap_{i=1}^{n} U_i$$

is open.

Proof. We go in sequence.

(a) To show $X \in \mathcal{T}$, note that any $x \in X$ has $B(x, 1) \subseteq X$ by definition. To show $\varnothing \in \mathcal{T}$, note that any $x \in \varnothing$ has $B(x, 1) \subseteq \varnothing$ because there is no $x \in \varnothing$ at all.

(b) For any $x \in \bigcup_{U \in \mathcal{U}} U$, we have $x \in V$ for some particular $V \in \mathcal{U}$. Then the openness of V tells us we can find $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subseteq V \subseteq \bigcup_{U \in \mathcal{U}} U,$$

which finishes.

(c) Fix x in the common intersection. Then, for any i, we have $x \in U_i$, so we have some $\varepsilon_i > 0$ such that $B(x, \varepsilon_i) \subseteq U$, and so we set

$$\varepsilon := \min_{1 \le i \le n} \varepsilon_i.$$

In particular, $\varepsilon > 0$ because n is finite, and we have

$$B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq U_i$$

for each i, so $B(x, \varepsilon)$ is a subset of our intersection. ■

Remark 2.11. The arbitrary intersection of open sets need not be open: working in $\mathbb{R}$ with the usual metric,

$$\bigcap_{i=1}^{\infty} B(0, 1/n) = \{0\},$$

which is not open. (Namely, no $\varepsilon > 0$ has $B(x, \varepsilon) \subseteq \{0\}$.)

Motivated by Proposition 2.10, we have the following definition.

Definition 2.12 (Topology)**.** Fix a set X. Then a *topology* $\mathcal{T}$ on X is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following.

(a) We have $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, the arbitrary union $\bigcup_{U \in \mathcal{U}} U$ lives in $\mathcal{T}$.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, the intersection $\bigcap_{i=1}^{n} U_i$ lives in $\mathcal{T}$.

We will say that the ordered pair $(X, \mathcal{T})$ is a *topological space*. We say that the sets in $\mathcal{T}$ are *open*.

Example 2.13. By Proposition 2.10, metric spaces with their open sets form a topological space.

Here are some more basic examples.

Definition 2.14 (Discrete topology)**.** Given a set X, the *discrete topology* is the topology $\mathcal{P}(X)$.

Definition 2.15 (Indiscrete topology)**.** Given a set X, the *indiscrete topology* is the topology $\{\varnothing, X\}$.

It is fairly routine to check that the above collections form topologies. In fact, they are closed under both arbitrary union and arbitrary intersection.

Remark 2.16. The discrete topology can be defined by the metric $d \colon X \times X \to \mathbb{R}_{\ge 0}$ by

$$d(x, x') := \begin{cases} 1 & x \ne x', \\ 0 & x = x'. \end{cases}$$

Indeed, for any $x \in X$, we see $B(x, 1/2) = \{x\}$, so any subset $U \subseteq X$ is the open set

$$U = \bigcup_{x \in U} \{x\} = \bigcup_{x \in U} B(x, 1/2).$$

Remark 2.17. If $\#X \geq 2$, the indiscrete topology cannot be given a metric. Indeed, find distinct points $a, b \in X$ and set $r := d(a, b)$, so $a \neq b$ implies $r > 0$. Now, $a \in B(a, r)$, but $b \notin B(a, r)$, so $B(a, r)$ is an open set distinct from both $\varnothing$ and X.

Remark 2.18. One can give topologies a partial order by inclusion. Then the discrete topology is the maximal one (definitionally, any topology is a subset of $\mathcal{P}(X)$), and the indiscrete topology is the minimal one (definitionally, any topology contains $\varnothing$ and X).

And so here is our general definition of continuity.

Definition 2.19 (Continuous)**.** Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Then a function $f \colon X \to Y$ is *continuous* if and only if, for any $U_Y \in \mathcal{T}_Y$, we have $f^{-1}(U_Y) \in \mathcal{T}_X$.

2.2 August 31

It is once again the morning.

2.2.1 Intersections of Topologies

We will want to have lots of topologies to work with. Here is a basic way to build them.

Proposition 2.20. Let X be a set, and pick up some collection of topologies $\{\mathcal{T}_\alpha\}_{\alpha \in \lambda}$. Then the intersection

$$\mathcal{T} := \bigcap_{\alpha \in \lambda} \mathcal{T}_\alpha$$

is also a topology on X.

Proof. This is mostly a matter of writing out the axioms.

(a) Note that $\varnothing, X \in \mathcal{T}_\alpha$ for each α, so $\varnothing, X \in \mathcal{T}$.

(b) Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}$, we have $\mathcal{U} \subseteq \mathcal{T}_\alpha$ for each α, so $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}_\alpha$ for each α, so

$$\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$$

as well.

(c) Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, we have $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}_\alpha$ for each α, so $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha$ for each α, so

$$\bigcap_{i=1}^n U_i \in \mathcal{T}$$

follows. ∎

Corollary 2.21. Fix a set X. Given a collection $\mathcal{S} \subseteq \mathcal{P}(X)$, there is a smallest topology $\mathcal{T}$ containing $\mathcal{S}$.

Proof. Certainly there is some topology containing $\mathcal{S}$, namely the discrete topology $\mathcal{P}(X)$. Thus, we can set our topology to be

$$\mathcal{T}(\mathcal{S}) := \bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S} \\ \mathcal{T} \text{ a topology}}} \mathcal{T},$$

which is a topology (by Proposition 2.20) which contains $\mathcal{S}$ (because each topology in the intersection contains $\mathcal{S}$), and of course any topology $\mathcal{T}$ containing $\mathcal{S}$ will have $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}$. ■

To codify this idea, we have the following idea.

Definition 2.22 (Generated topology)**.** Fix a set X. We say that a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ *generates* its smallest topology $\mathcal{T}$. We will write $\mathcal{T}(\mathcal{S})$ for this topology.

Remark 2.23 (Nir)**.** The topology $\mathcal{T}(\mathcal{S})$ is unique. Indeed, suppose two topologies $\mathcal{T}$ and $\mathcal{T}'$ are minimal topologies containing $\mathcal{S}$. Then $\mathcal{T}\cap\mathcal{T}'$ is also a topology containing $\mathcal{S}$ by Proposition 2.20, but $\mathcal{T}\cap\mathcal{T}' \subseteq \mathcal{T},\mathcal{T}'$ forces $\mathcal{T} = \mathcal{T}\cap\mathcal{T}' = \mathcal{T}'$.

Remark 2.24 (Nir)**.** Given collections $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S}')$. Indeed, we have

$$\mathcal{T}(\mathcal{S}) = \bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S} \\ \mathcal{T} \text{ a topology}}} \mathcal{T} \subseteq \bigcap_{\substack{\mathcal{T}\supseteq\mathcal{S}' \\ \mathcal{T}_0 \text{ a topology}}} \mathcal{T} = \mathcal{T}(\mathcal{S}').$$

Remark 2.25 (Nir)**.** If $\mathcal{T}$ is already a topology on X, then $\mathcal{T}(\mathcal{T}) = \mathcal{T}$. Indeed, of course $\mathcal{T} \subseteq \mathcal{T}(\mathcal{T})$, but then also

$$\mathcal{T}(\mathcal{T}) = \bigcap_{\substack{\mathcal{T}'\supseteq\mathcal{T} \\ \mathcal{T}' \text{ a topology}}} \mathcal{T}' \subseteq \mathcal{T}$$

because $\mathcal{T}$ is a topology containing $\mathcal{T}$.

2.2.2 Sub-bases

On the other side of things, we pick up the following definition.

Definition 2.26 (Sub-base)**.** Let $(X,\mathcal{T})$ be a topological space. A collection $\mathcal{S} \subseteq \mathcal{T}$ is a *sub-base* for $\mathcal{T}$ if and only if the following hold.

(a) $\mathcal{S}$ covers X, in that $X = \bigcup_{U\in\mathcal{S}} U$.

(b) $\mathcal{T}$ is generated by $\mathcal{S}$.

The point is that collections $\mathcal{S}$ are easy to find, so we have therefore found many topologies.

It will be useful to give a more concrete description of the topology generated by a collection $\mathcal{S}$. We start by taking finite intersections.

Lemma 2.27. Fix a set X and a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with $X = \bigcup_{U\in\mathcal{S}} U$. Then set

$$\mathcal{I}^{\mathcal{S}} := \left\{\bigcap_{i=1}^{n} U_i : \{U_i\}_{i=1}^{n} \subseteq \mathcal{S}\right\}.$$

Then $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$ and $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection. Further, the topology generated by $\mathcal{I}^{\mathcal{S}}$ is also the topology generated by $\mathcal{S}$.

Proof. We show the claims in sequence

- That $\{U\} \subseteq \mathcal{S}$ for any $U \in \mathcal{S}$ implies that $U \in \mathcal{I}^{\mathcal{S}}$ for any $U \in \mathcal{S}$, so $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$ follows.
- To show $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection, pick up some finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{I}^{\mathcal{S}}$. Then, for each i, we can find some finite collection $\mathcal{U}_i \subseteq \mathcal{S}$ such that
$$U_i = \bigcap_{V \in \mathcal{U}_i} V.$$
Setting $\mathcal{U} := \bigcup_{i=1}^n \mathcal{U}_i$, we see that $\mathcal{U}$ is finite and that
$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n \bigcap_{V \in \mathcal{U}_i} V = \bigcap_{V \in \mathcal{U}} V$$
must live in $\mathcal{I}^{\mathcal{S}}$.
- Because $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$, Remark 2.24 tells us $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right)$. In the other direction, note that any finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{S}$ also lives in $\mathcal{T}(\mathcal{S})$, so
$$\bigcap_{i=1}^n U_i \in \mathcal{T}(\mathcal{S}).$$
It follows $\mathcal{I}^{\mathcal{S}} \subseteq \mathcal{T}(\mathcal{S})$, so $\mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right) \subseteq \mathcal{T}(\mathcal{T}(\mathcal{S})) = \mathcal{T}(\mathcal{S})$ by Remark 2.25. ■

After taking finite intersections, we take arbitrary unions.

Lemma 2.28. Fix a set X and a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ closed under finite intersection with $\bigcup_{U \in \mathcal{I}} U = X$. Then the collection of (arbitrary) unions of elements in $\mathcal{I}$, denoted
$$\mathcal{T} := \left\{ \bigcup_{U \in \mathcal{U}} U : \mathcal{U} \subseteq \mathcal{I} \right\},$$
is $\mathcal{T}(\mathcal{I})$.

Proof. If $\mathcal{T}'$ is a topology containing $\mathcal{I}$, then note any collection $\mathcal{U} \subseteq \mathcal{I}$ lives in $\mathcal{T}'$, so the arbitrary union
$$\bigcup_{U \in \mathcal{U}} U$$
lives in $\mathcal{T}'$. It follows that $\mathcal{T} \subseteq \mathcal{T}'$, so
$$\mathcal{T} \subseteq \bigcap_{\substack{\mathcal{T}' \supseteq \mathcal{T} \\ \mathcal{T}' \text{ a topology}}} \mathcal{T}' = \mathcal{T}(\mathcal{I}).$$
Thus, it remains to show that $\mathcal{T}$ is in fact a topology, which will imply from $\mathcal{I} \subseteq \mathcal{T}$ that $\mathcal{T}(\mathcal{I}) \subseteq \mathcal{T}(\mathcal{T}) = \mathcal{T}$ by Remark 2.24. Here are our checks.

- Setting $\mathcal{U} = \varnothing \subseteq \mathcal{I}$, we see that $\bigcup_{U \in \mathcal{U}} U = \varnothing$, so $\varnothing \in \mathcal{T}$. Also, by hypothesis, we have
$$X = \bigcup_{U \in \mathcal{I}} U \in \mathcal{T}.$$

- Arbitrary union: let $\mathcal{U} \subseteq \mathcal{T}$ be a subcollection. For any $U \in \mathcal{U}$, we can find a collection $\mathcal{V}_U \subseteq \mathcal{I}$ such that
$$U = \bigcup_{V \in \mathcal{V}_U} V.$$
Now, we set $\mathcal{V}$ to be the union of all the collections of $\mathcal{V}_U$ for each $U \in \mathcal{U}$, which is still contained in $\mathcal{I}$, so that
$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} \bigcup_{V \in \mathcal{V}_U} V = \bigcup_{V \in \mathcal{V}} V \in \mathcal{T}.$$
- Finite intersection: by induction, it suffices to pick up two sets $U, V \in \mathcal{T}$ and show $U \cap V \in \mathcal{T}$. Well, we can find collections $\mathcal{U}, \mathcal{V} \subseteq \mathcal{I}$ such that
$$U = \bigcup_{U' \in \mathcal{U}} U' \qquad \text{and} \qquad V = \bigcup_{V' \in \mathcal{V}} V',$$
from which it follows (by distribution) that
$$U \cap V = \left(\bigcup_{U' \in \mathcal{U}} U' \right) \cap \left(\bigcup_{V' \in \mathcal{V}} V' \right) = \bigcup_{U' \in \mathcal{U}} \left(U' \cap \bigcup_{V' \in \mathcal{V}} V' \right) = \bigcup_{\substack{U' \in \mathcal{U} \\ V' \in \mathcal{V}}} (U' \cap V').$$
Now, $\mathcal{I}$ is closed under finite intersection, so $U' \cap V' \in \mathcal{I}$, so we have witnessed $U \cap V$ as an arbitrary union of elements of $\mathcal{I}$, so $U \cap V \in \mathcal{T}$ follows. ■

Corollary 2.29. Fix a set X and a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ with $X = \bigcup_{U \in \mathcal{S}} U$. Letting $\mathcal{I}^{\mathcal{S}}$ be the collection of finite intersections of $\mathcal{S}$ and then $\mathcal{T}$ be the collection of arbitrary unions of $\mathcal{I}^{\mathcal{S}}$, we have that $\mathcal{T} = \mathcal{T}(\mathcal{S})$.

Proof. By Lemma 2.27, we have $\mathcal{T}(\mathcal{S}) = \mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right)$. Plugging $\mathcal{I}^{\mathcal{S}}$ into Lemma 2.28 (which applies because $\mathcal{I}^{\mathcal{S}}$ is closed under finite intersection and covers X because $\mathcal{S} \subseteq \mathcal{I}^{\mathcal{S}}$), we see that $\mathcal{T}\left(\mathcal{I}^{\mathcal{S}}\right) = \mathcal{T}$, finishing. ■

We quickly point out that the point of discussing sub-bases is that we will be allowed to check continuity on only a sub-base.

Lemma 2.30. Fix a topological space $(X, \mathcal{T}_X)$ and a set Y. Given a function $f \colon X \to Y$, the collection
$$\mathcal{T}(f) := \left\{ U \subseteq Y : f^{-1}(U) \in \mathcal{T}_X \right\}$$
forms a topology on Y.

Proof. Here are our checks.

- Note $f^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$, so $\varnothing \in \mathcal{T}(f)$. Also, $f^{-1}(Y) = X \in \mathcal{T}_X$, so $Y \in \mathcal{T}(f)$.
- Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{T}(f)$, we see that
$$f^{-1}\left(\bigcup_{U \in \mathcal{U}} U \right) = \bigcup_{U \in \mathcal{U}} f^{-1}(U)$$
is a union of elements of $\mathcal{T}_X$ and therefore in $\mathcal{T}_X$. Thus, $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}(f)$.
- Finite intersection: this is identical to the previous check. Given a finite collection $\{U_1, \ldots, U_n\} \in \mathcal{T}(f)$, we see that
$$f^{-1}\left(\bigcap_{i=1}^n U_i \right) = \bigcap_{i=1}^n f^{-1}(U_i)$$
is a finite intersection of elements of $\mathcal{T}_X$ and therefore in $\mathcal{T}_X$. Thus, $\bigcap_{i=1}^n U_i \in \mathcal{T}(f)$. ■

Proposition 2.31. Fix topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and let $\mathcal{S}$ be a sub-base for $\mathcal{T}_Y$. Then a function $f: X \to Y$ is continuous if and only if

$$f^{-1}(U) \in \mathcal{T}_X$$

for all $U \in \mathcal{S}$.

Proof. Certainly if f is continuous then the pre-image of any open set $U \in \mathcal{S} \subseteq \mathcal{T}_Y$ must be open. On the other hand, let $\mathcal{T}(f) \subseteq \mathcal{P}(Y)$ be the collection of subsets U for which $f^{-1}(U) \in \mathcal{T}_X$. This is a topology by Lemma 2.30, and it contains $\mathcal{S}$ by hypothesis, so it follows

$$\mathcal{T}_Y = \mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(f).$$

Thus, $f^{-1}(U) \in \mathcal{T}_X$ for any $U \in \mathcal{T}_Y$, so f is continuous. ■

2.2.3 Bases

Having defined a sub-base, we should be rightly upset that we have not defined a base.

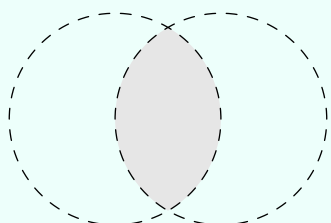
Definition 2.32 (Base). Fix a set X. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *base* (for a topology on X) if and only if the collection of arbitrary unions of $\mathcal{B}$ form a topology on X.

This definition is a little hard to access because we still don't have a good notion of what a topology is.

Example 2.33. Fix a set X. Given any collection $\mathcal{S} \subseteq \mathcal{P}(X)$, the collection of finite intersections $\mathcal{I}^{\mathcal{S}}$ is a base by Lemma 2.28.

However, in general we do not require a base to be closed under finite intersection.

Example 2.34. Fix a metric space (X, d). Then the collection of open balls $\mathcal{B}$ forms a topology by Example 2.13. Notably, the intersection of two open balls need not be an open ball, as follows.



Even though bases are not closed under finite intersection, we do have the following.

Proposition 2.35. Fix a set X and a collection $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a base if and only if

(a) $X = \bigcup_{B \in \mathcal{B}} B$, and

(b) any $B_1, B_2 \in \mathcal{B}$ has some collection $\mathcal{U} \subseteq \mathcal{B}$ such that

$$B_1 \cap B_2 = \bigcup_{B \in \mathcal{U}} B.$$

Proof. In one direction, suppose that $\mathcal{B}$ is a base generating the topology $\mathcal{T}$.

(a) Because $X \in \mathcal{T}$, we see that X is the union of some subcollection $\mathcal{U} \subseteq \mathcal{B}$, so it follows

$$X = \bigcup_{U \in \mathcal{U}} U \subseteq \bigcup_{B \in \mathcal{B}} B \subseteq X.$$

(b) Given $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T}$, we see that $B_1 \cap B_2 \in \mathcal{T}$, so because $\mathcal{T}$ is made of arbitrary unions of $\mathcal{B}$, there is a collection $\mathcal{U} \subseteq \mathcal{B}$ such that

$$B_1 \cap B_2 = \bigcup_{B \in \mathcal{U}} B.$$

We now go in the other direction. Suppose $\mathcal{B}$ satisfies (a) and (b), and define

$$\mathcal{T} := \left\{ \bigcup_{U \in \mathcal{U}} U : \mathcal{U} \subseteq \mathcal{B} \right\}.$$

We now check that $\mathcal{T}$ is a topology.

- Using $\mathcal{U} = \varnothing \subseteq \mathcal{B}$, so we see that $\bigcup_{U \in \mathcal{U}} U = \varnothing$ is in $\mathcal{T}$. Also, by (a), we have

$$X = \bigcup_{B \in \mathcal{B}} B \in \mathcal{T}.$$

- Arbitrary union: this is the same as the check in Lemma 2.28. Given a collection $\mathcal{U} \subseteq \mathcal{T}$, each $U \in \mathcal{U}$ has some collection $\mathcal{V}_U \subseteq \mathcal{B}$ such that $\bigcup_{V \in \mathcal{V}_U} V = U$. Letting $\mathcal{V} \subseteq \mathcal{B}$ be the union of all the $\mathcal{V}_U$, we see

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} \bigcup_{V \in \mathcal{V}_U} V = \bigcup_{V \in \mathcal{V}} V$$

  lives in $\mathcal{T}$.

- Finite intersection: by induction, it suffices to pick up $U_1, U_2 \in \mathcal{T}$ and show $U_1 \cap U_2 \in \mathcal{T}$. Well, find $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$ such that

$$U_1 = \bigcup_{B_1 \in \mathcal{B}_1} B_1 \qquad \text{and} \qquad U_2 = \bigcup_{B_2 \in \mathcal{B}_2} B_2,$$

  which implies

$$U_1 \cap U_2 = \bigcup_{\substack{B_1 \in \mathcal{B}_1 \\ B_2 \in \mathcal{B}_2}} (B_1 \cap B_2).$$

  Now, (b) implies that $B_1 \cap B_2$ for any $B_1, B_2 \in \mathcal{B}$ is a union of elements in $\mathcal{B}$, so $B_1 \cap B_2 \in \mathcal{T}$. Thus, $U_1 \cap U_2$ is the arbitrary union of elements in $\mathcal{T}$, so $U_1 \cap U_2 \in \mathcal{T}$ by the previous check. ■

Remark 2.36 (Nir)**.** Careful readers might realize that we could rearrange the given exposition to show that, given a sub-base $\mathcal{S}$, the collection of finite intersections $\mathcal{I}^{\mathcal{S}}$ is a base instead of going through Lemma 2.28.

Remark 2.37. Of course, any base is also a sub-base. Notably, sub-bases only require that $X = \bigcup_{U \in \mathcal{S}} U$, which must be satisfied for bases.

Example 2.38. Set $X = \mathbb{R}$ with the usual topology $\mathcal{T}$. Then the collection $\mathcal{B}$ of open intervals (a, b) form a base for the usual topology (these are our open balls). In contrast, the collection

$$\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$

forms a sub-base for the usual topology. Namely, certainly $\mathcal{S} \subseteq \mathcal{T}$, and $\mathcal{B} \subseteq \mathcal{T}(\mathcal{S})$ because of the finite intersection $(-\infty, b) \cap (a, \infty) = (a, b)$ for any $a, b \in \mathbb{R}$. Namely, $\mathcal{T} = \mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}(\mathcal{T}(\mathcal{S})) = \mathcal{T}(\mathcal{S})$ follows.

2.2.4 Induced Topologies

We start with the following motivating example.

Example 2.39. Fix a set X, and give it the discrete topology. Then, for any topological space $(Y, \mathcal{T}_Y)$, any function $f\colon X \to Y$ is continuous because the pre-image of any open subset $U_Y \subseteq Y$ is open in X.

In general, we might have some smallish collection of functions which we want to force to be continuous, so we might ask what topology is forced by their continuity.

Definition 2.40 (Induced topology)**.** Fix a set X and a collection of topologies $\{(Y_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$ with some functions $f_\alpha\colon X \to Y_\alpha$ for each $\alpha \in \lambda$. Then

$$\bigcup_{\alpha\in\lambda} \{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha\}$$

is a sub-base for an *induced topology*.

The one thing to check is that X belongs to the arbitrary unions of our collection, which is clear because $X = f_\alpha^{-1}(Y_\alpha)$.

Definition 2.41 (Relative topology)**.** Fix $(Y, \mathcal{T})$ a topological space. Then the *relative topology* for a subset $X \subseteq Y$ is the topology induced by the natural embedding $\iota\colon X \hookrightarrow Y$.

We have the following more concrete description.

Lemma 2.42. Fix $(Y, \mathcal{T}_Y)$ a topological space. Then the relative topology for a subset $X \subseteq Y$ consists of the subsets

$$\{X \cap U : U \in \mathcal{T}_Y\}.$$

Proof. Let $\iota\colon X \hookrightarrow Y$ be the natural embedding. Then we are given the sub-base

$$\mathcal{S} \coloneqq \left\{\iota^{-1}(U) : U \in \mathcal{T}_Y\right\}.$$

Now, $\iota^{-1}(U) = X \cap U$, and then we can check directly that this collection $\mathcal{S}$ gives a topology and finish by Remark 2.25. Here are the checks, which should be completely routine by now.

- Note $\varnothing \in \mathcal{T}_Y$ implies $\varnothing = X \cap \varnothing \in \mathcal{S}$. Also, $Y \in \mathcal{T}_Y$ implies $X = X \cap Y \in \mathcal{S}$.
- Arbitrary union: given a collection $\mathcal{U} \subseteq \mathcal{S}$, for each $U \in \mathcal{U}$ find $U_V \in \mathcal{T}_Y$ such that $U = X \cap U_V$. Then

  $$\bigcup_{U\in\mathcal{U}} = U = \bigcup_{U\in\mathcal{U}} X \cap U_V = X \cap \underbrace{\bigcup_{U\in\mathcal{U}} U_V}_{\in\mathcal{T}_Y}$$

  lives in $\mathcal{S}$.
- Finite intersection: given a finite collection $\{U_1, \ldots, U_n\} \subseteq \mathcal{S}$, find $V_i \in \mathcal{T}_Y$ such that $U_i = X \cap V_i$. Then

  $$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (X \cap V_i) = X \cap \underbrace{\bigcap_{i=1}^n V_i}_{\in\mathcal{T}_Y}$$

  lives in $\mathcal{S}$. ■

2.3 September 2

There are no questions about anything.

2.3.1 Closed Sets

We begin, as always, with a definition.

Definition 2.43 (Closed). Fix a topological space $(X, \mathcal{T})$. A subset $V \subseteq X$ is *closed* if and only if $(X \setminus V) \in \mathcal{T}$.

Here are some basic properties.

Lemma 2.44. Fix a topological space $(X, \mathcal{T})$.

(a) The set $\varnothing$ and X are both closed.

(b) Arbitrary intersection: given a collection of closed sets $\mathcal{V}$, the intersection $\bigcap_{V\in\mathcal{V}} V$ is closed.

(c) Finite union: given a finite collection of closed sets $\{V_1, \ldots, V_n\}$, the union $\bigcup_{i=1}^n V_i$ is closed.

Proof. We proceed in sequence.

(a) Note that $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ are both open so $\varnothing$ and X are closed.

(b) Arbitrary intersection: observe that

$$X \Bigg\backslash \bigcap_{V\in\mathcal{V}} V = \bigcup_{V\in\mathcal{V}} (X \setminus V)$$

is an arbitrary union of open sets and therefore open. Thus, $\bigcap_{V\in\mathcal{V}} V$ is closed.

(c) Finite union: observe that

$$X \Bigg\backslash \bigcup_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (X \setminus V_i)$$

is the finite intersection of open sets and therefore open. Thus, $\bigcup_{i=1}^n V_i$ is closed. ■

Remark 2.45. Observe that both X and $\varnothing$ are both open and closed. This is allowed.

Example 2.46. Fix a metric space (X, d). Then any closed ball $\overline{B(x_0, r)}$ is closed: we need to show

$$U := X \setminus \overline{B(x_0, r)} = \{x \in X : d(x, x_0) > r\}$$

is open. Well, for any $y \in U$, we see $d(y, x_0) > r$, so set $\varepsilon_y := d(y, x_0) - r$, so $y' \in B(y, \varepsilon_y)$ has $d(x_0, y') \geq d(x_0, y) - d(y, y') > r$. Thus, any $y \in U$ has $B(y, \varepsilon_y) \subseteq U$, finishing.

Remark 2.47. In $\mathbb{R}^2$ with the Euclidean metric,

$$\bigcup_{\varepsilon<1}^{\infty} \overline{B(0, \varepsilon)} = \left\{x \in \mathbb{R}^2 : d(0, x) < \varepsilon \text{ for some } \varepsilon < 1\right\} = B(0, 1)$$

is not closed. Indeed, we need to show $U := X \setminus B(0, 1) = \left\{x \in \mathbb{R}^2 : d(0, x) \geq 1\right\}$ is not open. Well, note $(1, 0) \in U$, but any $\varepsilon > 0$ has $(1 - \varepsilon/2, 0) \in B((1, 0), \varepsilon)$ despite $(1 - \varepsilon/2, 0) \notin U$. Thus, U is not open.

Remark 2.48. One can define a topology by defining its closed sets to satisfy the axioms of Lemma 2.44. Then one defines the open sets as the complements of open sets.

In the case of metric spaces, we also have the following characterization of metric spaces.

Lemma 2.49. Fix a metric space (X, d) and $V \subseteq X$. The following are equivalent.

(a) V is closed.

(b) Any sequence $\{x_n\}_{n\in\mathbb{N}}$ in V which converges to a point $x \in X$ actually converges to $x \in V$.

Proof. In one direction, suppose V is closed, and suppose $x_n \to x$ as $n \to \infty$ with $x \notin V$. Then we show that some $n \in \mathbb{N}$ has $x_n \notin V$. Well, $x \in X \setminus V$, and $X \setminus V$ is open, so there is some $\varepsilon > 0$ with

$$B(x, \varepsilon) \subseteq X \setminus V.$$

However, $x_n \to x$ as $n \to \infty$ promises some large n such that $d(x, x_n) < \varepsilon$, implying that $x_n \in X \setminus V$ and so $x_n \notin V$.

In the other direction, suppose V is not closed. Then $X \setminus V$ is not open, so we can find $x \in X \setminus V$ for which there is no $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq X \setminus V$. As such, $x \notin V$ but $B(x, 1/n) \cap V \neq \varnothing$ for all $n \in \mathbb{N}$, so just pick up some

$$x_n \in B(x, 1/n) \cap V$$

for each $n \in \mathbb{N}$. As such, $d(x, x_n) < 1/n$ for all $n \in \mathbb{N}$, so $x_n \to x$ as $n \to \infty$ (take $N = 1/\varepsilon$), and $x_n \in V$ for all $n \in \mathbb{N}$, but the limit x does not live in V. ■

Remark 2.50. The reason we are not generalizing the above lemma to arbitrary topological spaces is because we haven't generalized convergence yet.

Corollary 2.51. Fix a complete metric space (X, d). Then a closed subset $V \subseteq X$ given the restricted metric is also complete.

Proof. Suppose a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in V is Cauchy. Embedding back in X, this sequence is still Cauchy in X, so it has a limit $x \in X$. But Lemma 2.49 then promises $x \in V$, so $\{x_n\}_{n\in\mathbb{N}}$ does in fact have a limit x in V. ■

2.3.2 Closures

Given a general set, we can define the closure as follows.

Definition 2.52 (Closure). Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, we define the *closure* as

$$\overline{S} := \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V.$$

Lemma 2.53. Fix a topological space $(X, \mathcal{T})$. Given a subset $S \subseteq X$, the closure $\overline{S}$ is the unique smallest closed set containing S.

Proof. Note that

$$\overline{S} := \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V$$

is closed as the arbitrary intersection of closed sets, by Lemma 2.44. To see that $\overline{S}$ is a minimal such closed set, note that any closed V containing S must have $\overline{S} \subseteq V$ by definition of $\overline{S}$.

Lastly, to see that $\overline{S}$ is unique, note that if we have two minimal closed sets $\overline{S}_1$ and $\overline{S}_2$ containing S, then note $\overline{S}_1 \cap \overline{S}_2$ are both closed sets containing S by Lemma 2.44, so minimality forces $\overline{S}_1 = \overline{S}_1 \cap \overline{S}_2 = \overline{S}_2$. ■

Example 2.54. If $S \subseteq X$ is closed, then we see

$$S \subseteq \bigcap_{\substack{V \supseteq S \\ V \text{ closed}}} V \subseteq S$$

because S is a closed set containing S. Thus, $S = \overline{S}$.

With the notation, we note that we can move our notion of density from metric spaces to general topology.

Lemma 2.55. Fix a metric space (X, d). Then $S \subseteq X$ is dense if and only if $\overline{S} = X$.

Proof. In one direction, suppose that S is not dense in X, and we show $\overline{S} \subsetneq X$. Well, we are granted $x \in X$ and $\varepsilon > 0$ such that $S \cap B(x, \varepsilon) = \varnothing$, so $S \subseteq X \setminus B(x, \varepsilon)$. However, $X \setminus B(x, \varepsilon)$ is closed, so

$$\overline{S} \subseteq X \setminus B(x, \varepsilon) \subsetneq X,$$

as needed.

In the other direction, suppose $\overline{S} \subsetneq X$, and we show that S is not dense in X. Well, find $x \in X \setminus \overline{S}$. Because $X \setminus \overline{S}$ is open, we may find $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq X \setminus \overline{S}$, implying that

$$B(x, \varepsilon) \cap S \subseteq B(x, \varepsilon) \cap \overline{S} = \varnothing,$$

making S not dense in X. ■

Thus, we can generalize our definition as follows.

Definition 2.56 (Dense)**.** Fix a topological space $(X, \mathcal{T})$. Given subsets $A \subseteq B$, we say A is *dense* in B if and only if $B \subseteq \overline{A}$.

Remark 2.57. We are not requiring that B be closed for the definition of density. For example, $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{Q}$.

2.3.3 The Product Topology

Let's see more examples of induced topologies. We start with the easiest example of the product topology.

Definition 2.58 (Product topology)**.** Fix topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. The *product topology* on $X_1 \times X_2$ is the topology induced by the canonical projection mappings

$$\pi_1 \colon X_1 \times X_2 \to X_1 \qquad \text{and} \qquad \pi_2 \colon X_1 \times X_2 \to X_2.$$

We now give the following more concrete description of the product topology.

Lemma 2.59. Fix topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. The product topology $\mathcal{T}$ on $X := X_1 \times X_2$ has a base given by
$$\mathcal{B} := \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}.$$

Proof. The product topology is the minimal topology making $\pi_1 \colon X_1 \times X_2 \to X_1$ and $\pi_2 \colon X_1 \times X_2 \to X_2$ continuous. Namely, the product topology has a sub-base given by the sets
$$\pi_1^{-1}(U_1) = U_1 \times X_2 \qquad \text{and} \qquad \pi_2^{-1}(U_2) = X_1 \times U_2$$
for any $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$. Using Example 2.33, we let $\mathcal{I}$ denote the finite intersections of these open sets and note $\mathcal{I}$ is a base for our topology.

Now, we finish by claiming $\mathcal{B} = \mathcal{I}$. On one hand, any $U_1 \times U_2 \in \mathcal{B}$ with $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$ can be written as the finite intersection
$$U_1 \times U_2 = (U_1 \times X_2) \cap (X_1 \times U_2) = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \in \mathcal{I}.$$
On the other hand, pick finitely many sets of the form $\pi_1^{-1}(U_1)$ and $\pi_2^{-1}(U_2)$; dividing them into their classes, we can write our finite collection of sets as in $\{U_1^{(i)} \times X_2\}_{i=1}^m$ or $\{X_1 \times U_2^{(j)}\}_{j=1}^n$. Their intersection is
$$\left(\bigcap_{i=1}^m U_1^{(i)} \times X_2\right) \cap \left(\bigcap_{j=1}^n X_1 \times U_2^{(j)}\right) = \underbrace{\left(\bigcap_{i=1}^m U_1^{(i)}\right)}_{U_1 :=} \cap \underbrace{\left(\bigcap_{j=1}^n U_2^{(j)}\right)}_{U_2 :=}.$$
Now, $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ are finite intersection of open sets and therefore open, so our finite intersection takes the form $U_1 \times U_2$ and thus lives in $\mathcal{B}$. ■

Remark 2.60. Later in life we will discuss measurable sets, which are not quite topologies but will have similar ideas in spirit. For example, they will also care deeply about "rectangles."

We can define this more generally.

Definition 2.61 (Product topology)**.** Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$. The *product topology* on $X := \prod_{\alpha \in \lambda} X_\alpha$ is induced by the canonical projection maps
$$\pi_\alpha \colon X \to X_\alpha.$$

Here is our more concrete description.

Lemma 2.62. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$. The product topology on $X := \prod_{\alpha \in \lambda}$ has a base
$$\mathcal{B} := \left\{\prod_{\alpha \in \lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha, U_\alpha = X_\alpha \text{ for all but finitely many } \alpha\right\}.$$

Proof. We are immediately given the sub-base of $\mathcal{S} := \{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{T}_\alpha\}$. Using Example 2.33, we let $\mathcal{I}$ denote the finite intersections of $\mathcal{S}$ so that $\mathcal{I}$ is a base for our product topology.

As before, we finish by claiming $\mathcal{I} = \mathcal{B}$. To stay organized, we proceed in steps.

- We show $\mathcal{B} \subseteq \mathcal{I}$. Namely, for any $\prod_{\alpha \in \lambda} U_\alpha$ in $\mathcal{B}$, we set $\lambda' := \{\alpha : U_\alpha \neq X_\alpha\}$, which we know must be finite. Then
$$\prod_{\alpha \in \lambda} U_\alpha = \bigcap_{\alpha \in \lambda} \pi^{-1}(U_\alpha) = \bigcap_{\alpha \in \lambda'} \pi_\alpha^{-1}(U_\alpha)$$
because $\pi^{-1}(X_\alpha) = X$. The right-hand side is indeed a finite intersection of elements of $\mathcal{S}$ and therefore in $\mathcal{I}$.

- We show $\mathcal{S} \subseteq \mathcal{B}$. For a given β and $U_\beta \in \mathcal{T}_\beta$, set $U_\alpha := X_\alpha$ for each $\alpha \neq \beta$. Then we see that

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in \lambda} U_\alpha$$

is in $\mathcal{B}$ because $U_\alpha = X_\alpha$ for all but a single $\alpha \in \lambda$.

- We show $\mathcal{B}$ is closed under finite intersection. By induction, it suffices to pick up $U, U' \in \mathcal{B}$ and show that $U \cap U' \in \mathcal{B}$. Indeed, write

$$U = \prod_{\alpha \in \lambda} U_\alpha \qquad \text{and} \qquad U' = \prod_{\alpha \in \lambda} U'_\alpha,$$

where $\lambda_0 = \{\alpha : U_\alpha \neq X_\alpha\}$ and $\lambda'_0 = \{\alpha : U'_\alpha \neq X_\alpha\}$ are both finite. Then

$$U \cap U' = \prod_{\alpha \in \lambda} (U_\alpha \cap U'_\alpha),$$

and we have $U_\alpha \cap U'_\alpha = X_\alpha$ whenever $\alpha \notin (\lambda_0 \cup \lambda'_0)$, which is only finitely many exceptions because both λ_0 and λ_0 are finite.

- We show $\mathcal{I} \subseteq \mathcal{B}$. Indeed, $\mathcal{I}$ is made of the finite intersections of $\mathcal{S}$, and we see that $\mathcal{B}$ does indeed contain the finite intersections of $\mathcal{S}$ because $\mathcal{B}$ contains the finite intersections of itself, and $\mathcal{S} \subseteq \mathcal{B}$. ■

Remark 2.63. If λ is finite, then the arguments of Lemma 2.59 generalize to give the cleaner base

$$\left\{\prod_{\alpha \in \lambda} U_\alpha : U_\alpha \in \mathcal{T}_\alpha\right\}.$$

This also follows directly from Lemma 2.62, where we note that the "finitely many exceptions" actually permits all $\alpha \in \lambda$ to be an exception because λ is finite.

Example 2.64. Give $\{0,1\}$ the discrete topology. Then the space $X := \{0,1\}^{\mathbb{N}}$ given the product topology does not have

$$U := \prod_{n \in \mathbb{N}} \{0\}$$

open in X even though $\{0\} \subseteq \{0,1\}$ is always open. To see this, we note U has only a single element. On the other hand, for U to be open, Lemma 2.62 tells us U must contain a basis element B of the form

$$B := \prod_{n \in \mathbb{N}} U_n$$

where $U_n = \{0,1\}$ for all but finitely many n. However, B is infinite as the infinite product of sets containing more than 1 element, so $B \not\subseteq U$.

We quickly remark that the product topology satisfies the following universal property.

Lemma 2.65. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \lambda}$, and give the product $X := \prod_{\alpha \in \lambda} X_\alpha$ the projections $\pi_\alpha \colon X \to X_\alpha$ and the product topology $\mathcal{T}$. Given a topological space $(Y, \mathcal{T}_Y)$ and continuous maps $f_\alpha \colon Y \to X_\alpha$, there is a unique continuous map $f \colon Y \to X$ such that $f_\alpha = \pi_\alpha \circ f$ for each $\alpha \in \lambda$.

Proof. We show uniqueness and existence separately.

- Uniqueness: suppose both f and f' satisfy that $f_\alpha = \pi_\alpha \circ f = \pi_\alpha \circ f'$ for each $\alpha \in \lambda$. Then, for some $y \in Y$, we see that $f(y) = (x_\alpha)_{\alpha\in\lambda}$ and $f'(y) = (x'_\alpha)_{\alpha\in\lambda}$ have

$$x_\beta = (\pi_\beta \circ f)(y) = f_\beta(y) = (\pi_\beta \circ f')(y) = x'_\beta$$

for each $\beta \in \lambda$. So we conclude that $f(y) = f'(y)$ on all inputs. Observe that we have not used continuity anywhere.

- Existence: define $f\colon Y \to X$ by

$$f(y) := (f_\alpha(y))_{\alpha\in\lambda}.$$

We now need to check that f is continuous. By Proposition 2.31, it suffices to check this on the subbase of Lemma 2.62. In particular, pick up some finite $\lambda' \subseteq \lambda$ and set $U_\alpha \in \mathcal{T}_\alpha$ for each $\alpha \in \lambda$ while $U_\alpha = X_\alpha$ for $\alpha \notin \lambda'$. Then our basis element is

$$U := \prod_{\alpha\in\lambda} U_\alpha.$$

In particular,

$$\begin{aligned} f^{-1}(U) &= \{y \in Y : f_\alpha(y) \in U_\alpha \text{ for all } \alpha \in \lambda\} \\ &= \bigcap_{\alpha\in\lambda} f_\alpha^{-1}(U_\alpha) \\ &= \left(\bigcap_{\alpha\in\lambda'} f_\alpha^{-1}(U_\alpha)\right) \cap \left(\bigcap_{\alpha\notin\lambda'} f_\alpha^{-1}(\underbrace{U_\alpha}_{X_\alpha})\right), \end{aligned}$$

which is open because the left term is a finite intersection of open sets and the right term is just Y. ■

Corollary 2.66. Fix a collection of topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Give the product $X := \prod_{\alpha\in\lambda} X_\alpha$ the projections $\pi_\alpha\colon X \to X_\alpha$ and the product topology $\mathcal{T}$. Given a topological space $(Y, \mathcal{T}_Y)$, a function $f\colon Y \to X$ is continuous if and only if the compositions $\pi_\alpha \circ f$ are continuous.

Proof. Certainly if f is continuous, then the continuity of π_α means that each $\pi_\alpha \circ f$ is continuous.

Conversely, set $f_\alpha := \pi_\alpha \circ f$ to be a continuous map $f_\alpha\colon Y \to X_\alpha$. Then Lemma 2.65 promises us a unique continuous map $\widetilde{f}\colon Y \to X$ such that

$$\pi_\alpha \circ \widetilde{f} = f_\alpha = \pi_\alpha \circ f.$$

However, the uniqueness proof of Lemma 2.65 showed that there is in fact one unique map of sets whose projections under π_α are f_α, so we conclude $f = \widetilde{f}$. Thus, f is continuous. ■

2.3.4 Comments on the Dual Space

Given a vector space V with a norm $\|\cdot\|$, we might be interested in the linear functionals on V, but because V is a metric space, we should actually be looking at the continuous linear functional. One can show (in Math 202B) that one has "plenty" of continuous linear functionals. Here is a lemma we will use a few times.

Lemma 2.67. Let $\|\cdot\|$ be a norm on an $\mathbb{R}$-vector space V. Then a linear functional $f\colon V \to \mathbb{R}$ is continuous if and only if there exists a real number $c > 0$ such that

$$|f(v)| \le c\,\|v\| \tag{2.1}$$

for all $v \in V$.

Proof. In one direction, suppose that we can find a real number $c > 0$ satisfying (2.1) for all $v \in V$. To show f is continuous, we use Lemma 1.54: suppose that we have a sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \to v$ as $n \to \infty$. Then, for any $\varepsilon > 0$, find N such that $n > N$ implies

$$\|v - v_n\| < \varepsilon/c$$

so that

$$|f(v) - f(v_n)| \le c\,\|v - v_n\| < \varepsilon.$$

Conversely, suppose that f is continuous. Note that we don't have to worry about $v = 0$ because this gives equality. Now, we can find $\delta > 0$ such that $\|v\| < \delta$ implies $|f(v)| < 1$. It follows that any nonzero $v \in V$ will have

$$\left\|\frac{\delta}{2\,\|v\|}v\right\| < \delta,$$

so we see

$$|f(v)| = \frac{2\,\|v\|}{\delta}\left|f\left(\frac{\delta}{2\,\|v\|}v\right)\right| \le \frac{2}{\delta}\cdot\|v\|\,,$$

so $c := 2/\delta$ will do the trick. ■

Here is an example.

Exercise 2.68. Give $V := C([0,1])$ a p-norm $\|\cdot\|_p$ for some $p \ge 1$ or $p = \infty$. Then $g \in C([0,1])$ defines a continuous linear functional

$$\varphi_g \colon f \mapsto \int_0^1 f(t)g(t)\,dt.$$

Proof. To show φ_g is linear, pick up any $r_1, r_2 \in \mathbb{R}$ and $f_1, f_2 \in V$; then

$$\varphi_g(r_1f_1 + r_2f_2) = \int_0^1 (r_1f_1 + r_2f_2)(t)g(t)\,dt = r_1\int_0^1 f_1(t)g(t)\,dt + r_2\int_0^1 f_2(t)g(t)\,dt = r_1\varphi_g(f_1) + r_2\varphi_g(f_2).$$

Checking continuity is a little more involved. Note $|g|$ is a continuous function on a compact set $[0,1]$ and therefore has a maximum M. We now use Lemma 2.67; we have two cases.

- Suppose $p = \infty$. Then, for any $f \in V$, we see

  $$|\varphi_g(f)| = \left|\int_0^1 f(t)g(t)\,dt\right| \le M\int_0^1 |f(t)|\,dt \le M\,\|f\|_\infty\,,$$

  which finishes by Lemma 2.67.

- Suppose $p \ge 1$ is finite. To begin, we note

  $$|\varphi_g(f)| = \left|\int_0^1 f(t)g(t)\,dt\right| \le M\int_0^1 |f(t)|\,dt.$$

  Now, because the function $x \mapsto x^p$ is convex, we see that

  $$\left(\int_0^1 |f(t)|\,dt\right)^p \le \int_0^1 |f(t)|^p\,dt = \|f\|_p^p\,,$$

  so $|\varphi_g(f)| \le M\,\|f\|_p$. Lemma 2.67 finishes. ■

Even though the linear functionals we found were continuous for all $\|\cdot\|_p$, it is possible to find linear functionals continuous for some of our norms but not others.

Exercise 2.69. Fix $V := C([0,1])$, and select some $t_0 \in [0,1]$. Then

$$\varphi\colon f \mapsto f(t_0)$$

defines a linear functional on V which is continuous for $\|\cdot\|_\infty$ but not for $\|\cdot\|_p$ for any finite $p \geq 1$.

Proof. To see continuity with $\|\cdot\|_\infty$, we note that any $f \in V$ has

$$|\varphi(f)| = |f(t_0)| \leq \|f\|_\infty ,$$

so Lemma 2.67 finishes.

We now show that φ is not continuous for a fixed $\|\cdot\|_p$, where $p \geq 1$ is finite. Using Lemma 2.67, we just have to show that the ratio $|\varphi(v)|/\|v\|_p$ is unbounded for $v \in V$. For this, we define $f_c\colon [0,1] \to \mathbb{R}$ by

$$f(t) := \max\left\{0, c - c^{2p+1}(t-t_0)^2\right\}.$$

The idea here is that f has a sharp bump at t_0. Now, f is a continuous function on $[0,1]$ because it is the composition of continuous functions, so $f \in V$. We can compute

$$\|f\|_p = \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}.$$

Now, $f(t)$ will only be nonzero when $c - c^{2p+1}(t-t_0)^2 \geq 0$, which is equivalent to $t - t_0 \in (-c^{-p}, c^{-p})$, so we bound

$$\|f\|_p^p = \int_0^1 |f(t)|^p\,dt \leq \int_{-c^{-p}}^{c^{-p}} \left(c - c^{2p+1}z^2\right)\,dz \leq 2c^{1-p}.$$

Notably, as $c \to \infty$, we have that $\|f\|_p \leq 2^{1/p} \cdot c^{1/p-1}$ is bounded, but $|\varphi(f)| = c$ grows unbounded. Thus, φ is discontinuous. ■

Remark 2.70. Now, we have exhibited many continuous functions

$$\varphi_g\colon C([0,1]) \to \mathbb{R},$$

so we can ask for the topology on $C([0,1])$ induced by these. It turns out that this induced topology is much weaker than any individual norm topology; this topology is often called the weak topology determined by $C([0,1])$.

Remark 2.71. By the end of the class, we will have a reasonable notion of the dual space of $\|\cdot\|_1$ and $\|\cdot\|_2$. The dual space for $\|\cdot\|_\infty$ will come up in Math 202B.

Remark 2.72. Still working with $C([0,1])$ given a specific norm $\|\cdot\|_p$, one can show that any $g \in C([0,1])$ has some $r_g \in \mathbb{R}$ with

$$\varphi_g(B(0,1)) \subseteq B(0,r_g).$$

It turns out to be helpful to be able to consider the product topology on the (very large) product

$$\prod_{g \in C([0,1])} B(0,r_g).$$

2.4 September 7

It's another day of sun.

2.4.1 Quotient Spaces

Here is a different way to induce a topology, the reverse of the induced topology.

Definition 2.73 (Final topology)**.** Fix a set Y and some topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$. Given functions $f_\alpha \colon X_\alpha \to Y$, we define the *final topology* on Y to be the "strongest" (i.e., with the most open sets) making the f_α continuous.

Remark 2.74. Note that certainly some topology on Y exists making the f_α continuous because we can give Y the indiscrete topology, where $f_\alpha^{-1}(\varnothing) = \varnothing$ and $f_\alpha^{-1}(Y) = X_\alpha$ are open for each $\alpha \in \lambda$.

Here is a more concrete description.

Lemma 2.75. Fix a set Y and some topological spaces $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in\lambda}$, with functions $f_\alpha \colon X_\alpha \to Y$. Then the final topology is

$$\mathcal{T} \coloneqq \bigcap_{\alpha\in\lambda} \left\{S \subseteq Y : f_\alpha^{-1}(S) \in \mathcal{T}_\alpha\right\}.$$

Proof. Certainly each $\left\{S \subseteq Y : f_\alpha^{-1}(S) \in \mathcal{T}_\alpha\right\}$ is a topology by Lemma 2.30, as is their intersection by Proposition 2.20. Thus, $\mathcal{T}$ is a topology.

It remains to show that $\mathcal{T}$ is the strongest topology making each of the f_α continuous. Well, suppose $\mathcal{T}'$ is a topology making each of the f_α continuous. Then, for each $U \in \mathcal{T}'$, we have

$$f_\alpha^{-1}(U) \in \mathcal{T}_\alpha \text{ for each } \alpha \in \lambda,$$

so $U \in \mathcal{T}$ follows. Thus, $\mathcal{T}' \subseteq \mathcal{T}$. ■

We will be primarily interested in the case with just one function.

Remark 2.76. In the case of one function, which is Lemma 2.30, note that we might as well assume that $f \colon X \to Y$ is onto for otherwise we might as well just pass to the relative topology on $\operatorname{im} f$. To be explicit, we see $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open if and only if $f^{-1}(U \cap \operatorname{im} f)$ is open if and only if $U \cap \operatorname{im} f$ is open.

We are now ready to define the quotient space.

Lemma 2.77. Given sets $f \colon X \to Y$, there is an equivalence relation $\sim$ on X with $x \sim x'$ if and only if $f(x) = f(x')$.

Proof. We check the conditions one at a time. Find $x, x', x'' \in X$.

- Reflexive: note $f(x) = f(x)$, so $x \sim x$.
- Symmetric: if $x \sim x'$, then $f(x) = f(x')$, so $f(x') = f(x)$, so $x' \sim x$.
- Transitive: if $x \sim x'$ and $x' \sim x''$, then $f(x) = f(x') = f(x'')$, so $f(x) = f(x'')$, so $x \sim x''$. ■

With an equivalence relation, we may consider the set of equivalence classes $X/{\sim}$.

Remark 2.78. Conversely, given some partition $P \subseteq \mathcal{P}(X)$ of X, we can define $f \colon X \to P$ by $f \colon x \mapsto [x]$, where $[x] \in P$ is the element of P containing x. (Note $[x] \in P$ exists and is well-defined because P is a partition.) The point is that surjective functions give rise to equivalence relations, and equivalence relations give rise to surjective functions.

Anyway, here is our definition.

Definition 2.79 (Quotient topology)**.** Fix an equivalence relation $\sim$ on a set X with a topology $\mathcal{T}$. Then the *quotient topology* on $X/\sim$ is the final topology for the natural projection $X \twoheadrightarrow X/\sim$.

It turns out that we can talk about the quotient space by universal property as well.

Proposition 2.80. Fix an equivalence relation $\sim$ on a set X with a topology $\mathcal{T}$; let $\pi \colon X \twoheadrightarrow (X/\sim)$ be the natural projection. Then, for any continuous map $f \colon X \to Z$ such that any $x \sim x'$ has $f(x) = f(x')$, there is a unique continuous map $\overline{f} \colon (X/\sim) \to Z$ such that

$$f = \overline{f} \circ \pi.$$

Proof. We show uniqueness and existence separately.

- Uniqueness: for any $[x] \in (X/\sim)$, we see that we must have

  $$\overline{f}([x]) = \overline{f}(\pi(x)) = f(x),$$

  so $\overline{f}([x])$ is forced by our other data.

- Existence: for each $[x] \in (X/\sim)$, define $\overline{f}([x]) \coloneqq f(x)$. Note that this is well-defined: if $[x] = [x']$, then $x \sim x'$, so $f(x) = f(x')$ by hypothesis.

  It remains to show that $\overline{f}$ is continuous. Well, for an open set $U \subseteq Z$, we note that

  $$\overline{f}^{-1}(U) = \{[x] : \overline{f}([x]) \in U\} = \{[x] : f(x) \in U\} = \pi\left(f^{-1}(U)\right).$$

  Now, $\pi^{-1}\left(\pi\left(f^{-1}(U)\right)\right) = f^{-1}(U)$ because $x \in \pi^{-1}\left(\pi\left(f^{-1}(U)\right)\right)$ if and only if $\pi(x) \in \pi\left(f^{-1}(U)\right)$, which is equivalent to there being $x' \in f^{-1}(U)$ with $\pi(x) = \pi(x')$, which is equivalent to there being x' with $x \sim x'$ while $f(x) = f(x') \in U$.

  Thus, $\pi^{-1}\left(\pi\left(f^{-1}(U)\right)\right)$ is open, so it follows $\pi\left(f^{-1}(U)\right) \subseteq (X/\sim)$ is open. ■

2.4.2 Homeomorphism

Homeomorphisms are isomorphisms in our category Top. To be technical, here is our definition.

Definition 2.81 (Homeomorphism)**.** A function $f \colon X \to Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a *homeomorphism* if and only if f is continuous and has a continuous inverse. Formally, we require a continuous map $g \colon Y \to X$ such that

$$f \circ g = \mathrm{id}_Y \qquad \text{and} \qquad g \circ f = \mathrm{id}_X.$$

Warning 2.82. It is not enough for f to be continuous and bijective to be a homeomorphism. The hypothesis that the inverse function be continuous is necessary.

Remark 2.83. The definition above does not require that f be bijective, but this follows from f having an inverse.

Here are some examples.

Example 2.84. Fix a nonzero real number a and a real number b. Then the function $\varphi_{a,b} \colon \mathbb{R} \to \mathbb{R}$ by $\varphi_{a,b}(x) \coloneqq ax + b$ is continuous: checking this on the subbase (which is enough by Proposition 2.31), we compute $\varphi_{a,b}^{-1}((c,d)) = ((c-b)/a, (d-b)/a)$. The inverse function is $\varphi_{1/a,-b/a}$—note $\varphi_{1/a,-b/a}(\varphi_{a,b}(x)) = \varphi_{a,b}(\varphi_{1/a,-b/a}(x)) = x$—which is continuous for the same reason, so this function $\varphi_{a,b}$ is a homeomorphism.

Lemma 2.85. Fix a homeomorphism $f\colon (X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$. Further, for any subset $S\subseteq X$, give S and $f(S)$ their respective relative topologies. Then the restriction $f|_S\colon S\to f(S)$ is a homeomorphism.

Proof. For clarity, let $g\colon Y\to X$ be the inverse function for f; note that $g(f(S))=\{g(f(x)):x\in S\}=S$, so $g|_{f(S)}\colon f(S)\to S$. Observe that we still have $g(f(x))=x$ and $f(g(y))$ for each $x\in X$ and $y\in Y$, so $f|_S$ and $g|_S$ are inverse functions by restricting these equations.

It remains to see that f and g are continuous. We will show that f is continuous, and g will follow by symmetry. Well, for an open subset $U\cap f(S)\subseteq f(S)$ (where $U\subseteq X$ is open), we see

$$f|_S^{-1}(U\cap f(S))=\{x\in S: f(x)\in U\cap f(S)\}=S\cap\{x\in X:f(x)\in U\}\cap\{x\in S:f(x)\in f(S)\}=S\cap f^{-1}(U),$$

which is indeed open in the relative topology of S. ■

Example 2.86. Fix real numbers $b>a$. Continuing from Example 2.84, $\varphi_{a,b}\colon\mathbb{R}\to\mathbb{R}$ restricts by Lemma 2.85 to a homeomorphism

$$\varphi_{b-a,a}|_{[0,1]}\colon[0,1]\to[a,b].$$

Namely, $x\in[0,1]$ if and only if $0\le x\le 1$ if and only if $a\le(b-a)x+a\le b$ if and only if $\varphi_{b-a,a}(x)\in[a,b]$.

Example 2.87. Give $\mathbb{R}$ the Euclidean topology, and let $\mathbb{R}_d$ be the real numbers with the discrete topology. Then the identity function $\iota\colon\mathbb{R}_d\to\mathbb{R}$ is continuous because all functions from the discrete topology are continuous. However, ι is its own inverse, and the inverse function

$$\pi\colon\mathbb{R}\to\mathbb{R}_d$$

(which is also the identity on $\mathbb{R}$) is not continuous. For example, $\pi^{-1}(\{0\})=\{0\}$ is not open in $\mathbb{R}$ (by Remark 2.11) even though $\mathbb{R}\setminus\{0\}\subseteq\mathbb{R}_d$ is open.

Here are some more exotic examples.

Exercise 2.88. Give $X:=[0,1]$ the subspace topology, and define the equivalence relation $\sim$ as having equivalence classes $\{0,1\}$ and $\{r\}$ for each $r\in(0,1)$. Then the quotient topology $X/{\sim}$ is homeomorphic to $S^1\subseteq\mathbb{C}$.

Proof. We note that $\sim$ is an equivalence relation because its equivalence classes are a partition. Now, we define the maps

$$\begin{array}{ccc}(X/{\sim}) & \cong & S^1\\ t & \mapsto & e^{2\pi it}\\ \theta/2\pi & \leftarrow & e^{i\theta}\end{array}$$

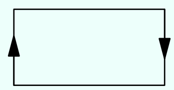
which we can see to be well-defined inverse. Note that $\mathbb{R}\to\mathbb{C}$ by $t\mapsto e^{it}$ is continuous by complex analysis (it's in fact holomorphic). Restricting, we get the continuous map $[0,1]\to S^1$, and then we can see that we can mod out by $0\sim1$ because they both go to the same place (using Proposition 2.80). One can check by hand that the inverse map is continuous, but we won't bother. ■

Remark 2.89 (Nir)**.** Here is a quick way to see that the inverse map is continuous: any continuous bijection $f\colon(X/{\sim})\to S^1$ with $(X/{\sim})$ compact—which is true because X is compact—and S^1 Hausdorff will send closed subsets $V\subseteq(X/{\sim})$ (which are compact) to compact subsets of S^1 (which are closed). Thus, f is a closed map, so its inverse is continuous because f is bijective.

For the next few examples, we won't be very rigorous because we haven't provided good definitions of the relevant spaces.

Example 2.90. Give $X := [0,2] \times [0,1]$ the subspace topology, and define the equivalence relation $\sim$ as requiring $(0,r) \sim (2,r)$ only. Then X is homeomorphic to a circle by gluing its edges. One might draw X as follows.

Example 2.91. Continuing with the drawing style of Example 2.90, we have that



is the Möbius strip.

Remark 2.92. Note that these homeomorphisms do not care for the metric of our spaces. All that matters is the continuity.

Example 2.93. Let X be the unit sphere in $\mathbb{R}^3$ with the subspace topology, and define the equivalence relation on X by equivalence classes $\{v, -v\}$ for each $v \in X$. Then $X/\sim$ turns out to be $\mathbb{RP}^2$, which is hard to visualize.

2.4.3 Group Actions

A space might even have interesting homeomorphisms to itself.

Example 2.94. Fix a real number θ. The circle S^1 in $\mathbb{C}$ (given the subspace topology) has the rotation homeomorphism

$$r_\theta \colon e^{it} \mapsto e^{i(t+\theta)}.$$

Remark 2.95. In general, given a topological space $(X, \mathcal{T})$, we can make the group of homeomorphisms $\operatorname{Aut}(X)$ of homeomorphisms whose operation is composition.

This gives the following definition.

Definition 2.96 (Group action)**.** A *group action* by a group G on a topological space X is a group homomorphism

$$\varphi_\bullet \colon G \to \operatorname{Aut}(X).$$

Example 2.97. The group $\langle\sigma\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on a normed vector space $(V, \|\cdot\|)$ by sending σ^k to

$$\varphi_{\sigma^k} \cdot v := (-1)^k v.$$

Notably, φ_{σ^k} is continuous and its own inverse for any k, so it is a homeomorphism. In fact, we can see directly that $\varphi_{\sigma^k} \circ \varphi_{\sigma^\ell} = \varphi_{\sigma^{k+\ell}}$.

Notably, with a group action comes a partition.

Definition 2.98 (Orbit)**.** Let G act on a topological space X by $\varphi_\bullet : G \to \operatorname{Aut}(X)$. Then the G-*orbit* Gx of a point $x \in x$ is the set

$$Gx \coloneqq \{\varphi_g(x) : g \in G\}.$$

We denote the set of all orbits $\mathcal{O}_x$ be X/G.

Remark 2.99. Note that the map $x \mapsto \mathcal{O}_x$ is a well-defined (surjective) map $X \to X/G$. In particular, we need to know that $x \in \mathcal{O}_{x'}$ implies that $\mathcal{O}_x = \mathcal{O}_{x'}$ so that there is exactly one orbit containing x. Well, $x \in \mathcal{O}_{x'}$ means we can find $g_0 \in G$ such that $x = \varphi_{g_0}(x')$, so

$$\mathcal{O}_x = \{\varphi_g(x) : g \in G\} = \{\varphi_g(\varphi_{g_0}(x')) : g \in G\} = \{\varphi_{gg_0}(x') : g \in G\} \subseteq \mathcal{O}_{x'}.$$

Conversely, we note that $x' = \varphi_{g_0^{-1}}(x)$, so $\mathcal{O}_{x'} \subseteq \mathcal{O}_x$ follows, giving equality.

Thus, the G-orbits partition X, so we can give the set X/G the quotient topology as the final topology of the natural projection $X \twoheadrightarrow X/G$.

Theme 3
Adjectives for Spaces

Rarely is a picture a proof, but I hope a good picture will cement your understanding of why something is true. Seeing is believing.

—Charles C. Pugh, [Pug15]

3.1 September 9

The fun continues. The next problem set is going to be long but only in words, not in what we actually have to prove. We are being told not to be intimidated.

Remark 3.1. We are about to transition from making topologies to coming up with adjectives which will give "lots" of continuous maps to, say, the real numbers. A rigorization of this shall be provided shortly.

3.1.1 Normal Spaces

Last class we briefly mentioned the Hausdorff property.

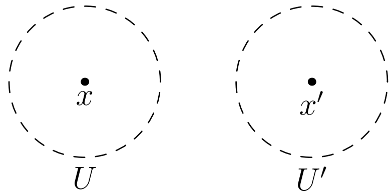
Definition 3.2 (Hausdroff)**.** Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is *Hausdorff* if and only if, for any two distinct points $x, x' \in X$, there are disjoint open sets U and U' such that $x \in U$ and $x' \in U'$.

Example 3.3. A metric space (X, d) is Hausdorff. Indeed, given distinct points $x, x' \in X$, we have $d(x, x') > 0$, so we set $r := \frac{1}{2}d(x, x')$. Then $x \in B(x, r)$ and $x' \in B(x', r)$ (which are open sets by Example 2.6), we see $B(x, r) \cap B(x', r) = \varnothing$. Indeed, if we had $y \in B(x, r) \cap B(x', r)$, then we must have

$$d(x, x') \le d(x, y) + d(x', y) < 2r = d(x, x'),$$

which is a contradiction.

Here is the image



Here is another adjective.

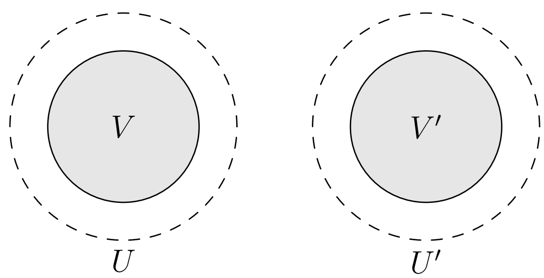
Definition 3.4 (Normal)**.** Fix a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T})$ is *Hausdorff* if and only if, for any two disjoint closed sets $V, V' \subseteq X$, there are disjoint open sets U and U' such that $V \in U$ and $V' \in U'$.

Remark 3.5. Intuitively, Hausdorff is approximately the normal property with singleton sets. In particular, some authors require "Hausdorff" in the definition of a normal space. We will not do this.

Example 3.6. Any set X given the indiscrete topology is normal. The problem here is that the only closed sets $\{\varnothing, X\}$, so the only possible pair of disjoint closed sets are $V_1 := \varnothing$ and $V_2 := \varnothing$, for which the open sets $U_1 := \varnothing$ and $U_2 := \varnothing$ are disjoint and cover these.

Example 3.7. A set X with more than 2 elements is normal, as shown in the previous example, but it is not Hausdorff. Namely, finding distinct points $x_1, x_2 \in X$, the only open subset of X containing x_1 or x_2 is X, so there are no disjoint open subsets U_1 containing x_1 and U_2 containing x_2.

Here is the image.



It is not completely obvious that metric spaces are normal, but we will see that they are.

Here is the main result for today.

Theorem 3.8 (Urysohn's lemma)**.** Fix a topological space $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is normal, then for any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f \colon X \to [0,1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

So the point here is to realize Remark 3.1, where being normal is implying that we have "lots" of continuous functions.

Remark 3.9. Certainly if a topological space $(X, \mathcal{T})$ satisfies the conclusion of Theorem 3.8, then $(X, \mathcal{T})$ is normal. Indeed, for any disjoint closed subsets $V_0, V_1 \subseteq X$, pick up the promised continuous function f. Then

$$V_0 \subseteq f^{-1}((-1/2, 1/2)) \qquad \text{and} \qquad V_1 \subseteq f^{-1}((1/2, 3/2))$$

are disjoint open sets; namely, these are open because f is continuous, and they are disjoint because $f^{-1}((-1/2,1/2)) \cap f^{-1}((1/2,3/2)) = f^{-1}\big((-1/2,1/2) \cap (1/2,3/2)\big) = f^{-1}(\varnothing) = \varnothing$.

3.1.2 Urysohn's Lemma: Metric Spaces

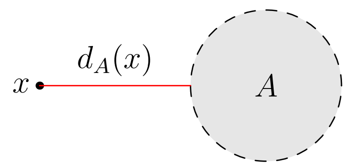
Let's see Theorem 3.8 for metric spaces, which will prove that metric spaces are normal by Remark 3.9. We pick up the following definition.

Definition 3.10. Fix a metric space (X,d). Then we define, for any $x \in X$ and nonempty subset $A \subseteq X$,

$$d_A(x) \coloneqq \inf_{a \in A} d(x,a).$$

Remark 3.11. The infimum here exists because A is nonempty, so the set $\{d(x,a) : a \in A\}$ is nonempty (and bounded below by 0).

The image is that $d_A(x)$ is the distance from x to A.



We have the following continuity check.

Lemma 3.12. Fix a metric space (X,d). Then, for any nonempty subset $A \subseteq X$, the function $d_A \colon X \to \mathbb{R}$ is Lipschitz continuous.

Proof. Fix any $x, y \in X$. Then, for any given $a \in A$, we find that

$$d_A(x) \le d(x,a) \le d(x,y) + d(y,a).$$

Thus, $d_A(x) - d(x,y) \le d(y,a)$ for all $a \in A$, so we conclude that

$$d_A(x) - d(x,y) \le \inf_{a \in A} d(y,a) = d_A(y),$$

so $d_A(x) - d_A(y) \le d(x,y)$. By symmetry, we also have $d_A(y) - d_A(x) \le d(x,y)$, so it follows

$$\left|d_A(x) - d_A(y)\right| \le d(x,y),$$

which is what we need for our Lipschitz continuous. ∎

As a sanity-check that this function behaves like it should, we pick up the following.

Lemma 3.13. Fix a metric space (X,d). Then, for any nonempty subset $A \subseteq X$, we have

$$d_A^{-1}(\{0\}) = \overline{A}.$$

Proof. Certainly $A \subseteq d_A^{-1}(\{0\})$ because $d_A(a) = 0$ for all $a \in A$. (In particular, $d_A(x) \ge 0$ everywhere, and $a \in A$ implies that $d_A(a) \le d(a,a) = 0$.) Because d_A is continuous by Lemma 3.12, we see $d_A^{-1}(\{0\})$ is closed, so containing A forces

$$\overline{A} \subseteq d_A^{-1}(\{0\}).$$

Conversely, suppose that $x \notin X \setminus \overline{A}$, and we show that $d_A(x) > 0$. Indeed, $X \setminus \overline{A}$ is open, so there is some open ball $B(x,\varepsilon)$ with $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq X \setminus \overline{A}$. It follows $B(x,\varepsilon) \cap \overline{A} = \varnothing$, so

$$d(a,x) \ge \varepsilon$$

for all $a \in A$. Thus, $d_A(x) \ge \varepsilon > 0$, so $d_A(x) \ne 0$. ∎

Example 3.14. If $A \subseteq X$ is a dense subset, then $\overline{A} = X$, so $d_A : X \to \mathbb{R}$ is the constantly zero function.

Example 3.15. If $A \subseteq X$ is closed, then $\overline{A} = A$ by Example 2.54, so $d_A^{-1}(\{0\}) = A$. In other words, we have $x \in A$ if and only if $d_A(x) = 0$.

Let's now show Theorem 3.8 for metric spaces.

Proposition 3.16. Fix a metric space (X, d). For any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f : X \to [0,1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

Proof. The point is to use the Lipschitz continuous functions d_{V_0}, d_{V_1}. Then we define

$$f(x) := \frac{d_{V_0}(x)}{d_{V_0}(x) + d_{V_1}(x)}.$$

Note that defining $f : X \to \mathbb{R}$ does not have division-by-zero problems: because $d_{V_0}(x), d_{V_1}(x) \geq 0$, the only way to get zero in the denominator is by $d_{V_0}(x) = d_{V_1}(x) = 0$. However, this forces $x \in V_0 \cap V_1$ by Lemma 3.13 because V_0 and V_1 are closed, but in fact $V_0 \cap V_1 = \varnothing$.

We now run our checks on f.

- Because the quotient of two continuous functions is still continuous, we see that f is continuous.
- Using the fact that $d_A(x) \geq 0$ for any nonempty $A \subseteq X$ and $x \in X$, we find

  $$f(x) = \frac{d_{V_0}(x)}{d_{V_0}(x) + d_{V_1}(x)} \geq 0,$$

  and

  $$f(x) = 1 - \frac{d_{V_1}(x)}{d_{V_0}(x) + d_{V_1}(x)} \leq 1,$$

  so $\operatorname{im} f \subseteq [0,1]$.
- If $x \in V_0$, then $d_{V_0}(x) = 0$, so $f(x) = 0/(0 + d_{V_1}(x)) = 0$. If $x \in V_1$, then $d_{V_1}(x) = 0$, so $f(x) = d_{V_0}(x)/(d_{V_0}(x) + 0) = 1$. ■

And here is our check.

Corollary 3.17. Any metric space (X, d) is normal.

Proof. Plug Proposition 3.16 into Remark 3.9. ■

3.1.3 Urysohn's Lemma: The General Case

We will not prove the general case of Theorem 3.8 today, but we will make some progress. Here is a useful lemma.

Lemma 3.18. Fix a normal topological space $(X, \mathcal{T})$. Given a closed subset $V \subseteq X$ and an open subset $U_0 \subseteq X$ with $V \subseteq U_0$, there is an open set U such that

$$V \subseteq U \subseteq \overline{U} \subseteq U_0.$$

Proof. Because $V \subseteq U_0$, we define $V' := X \setminus U_0$, which is closed because U_0 is open. Further, $V' \subseteq X \setminus U_0 \subseteq X \setminus V$ forces $V \cap V' = \varnothing$. Thus, using the normality of $(X, \mathcal{T})$, we are promised disjoint open sets U and U' such that

$$V \subseteq U \qquad \text{and} \qquad V' \subseteq U'.$$

In particular, we see that

$$U \subseteq X \setminus U'$$

while $X \setminus U'$ is closed by definition. Thus, by definition of the closure, $\overline{U} \subseteq X \setminus U' \subseteq X \setminus V' = U_0$. This finishes the proof. ■

3.2 September 12

There are still no questions.

3.2.1 Urysohn's Lemma: The General Case

We continue the proof from last class.

> **Theorem 3.8** (Urysohn's lemma)**.** Fix a topological space $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is normal, then for any disjoint closed subsets $V_0, V_1 \subseteq X$, there is a continuous function $f \colon X \to [0,1]$ such that $f(V_0) = \{0\}$ and $f(V_1) = \{1\}$.

Proof. To begin, define $U_1 := X \setminus V_1$, which is open because V_1 is closed; notably $V_0 \subseteq U_1$. The idea here is that the points of U_1 will take value at most 1. Now, by Lemma 3.18, we find $U_{1/2}$ with

$$V_0 \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_1.$$

Intuitively, we are going to let f take values at most $1/2$ on $U_{1/2}$. Using Lemma 3.18 again, we can find $U_{1/2}$ with

$$V_0 \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2},$$

and now our function will take values at most $1/4$ on $U_{1/4}$. On the other side, we can use the containment $\overline{U_{1/2}} \subseteq U_1$ in Lemma 3.18 to find $U_{3/4}$ such that

$$\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U_1,$$

and here $U_{3/4}$ our function should take values less than $3/4$.

We can then continue the process for eights and then off to infinity. Let's describe what we have at the end of this inductive process. Set $\Delta := \{k/2^n : 0 < k \le 2^n\}$ to be the set of "dyadic" rationals in $(0,1]$; notably Δ is dense in $[0,1]$.[1] Then each $r \in \Delta$, we get an open set $U_r \subseteq X$. These have the following properties.

- Any $r, s \in \Delta$ with $r < s$ has $\overline{U_r} \subseteq U_s$.
- By construction $U_1 = X \setminus V_1$.
- Also, $V_0 \subseteq U_r$ for all $r \in \Delta$.

We now define

$$f(x) := \begin{cases} 1 & x \in V_1, \\ \inf\{r \in \Delta : x \in U_r\} & x \notin V_1, \end{cases}$$

where $x \notin V_1$ in the second case promises $x \in U_1$ so that the infimum in the second line makes sense. We now run the following checks on f.

[1] The fact we need is that $a, b \in [0,1]$ with $a < b$ have $r \in \Delta$ between them. Well, multiply $b - a$ by a suitably large power of 2 so that $2^n(b-a) > 1$, so there is an integer k in this interval between $2^n a$ and $2^n b$, so $a < k/2^n < b$.

- Note that $\operatorname{im} f(x) \subseteq \overline{\Delta} = [0,1]$.
- By the construction of these open sets, we have $f(x) = 1$ if $x \in V_1$.
- Further, $f(x) < r$ for all $r \in \Delta$ if $x \in V_0$, so $f(x) = 0$ for $x \in V_0$.
- It remains to check that f is continuous. For this, we use Proposition 2.31 to check the continuity on a subbase. Specifically, we use sets of the form $[0,a)$ and $(a,1]$ for $a \in (0,1)$. Indeed, note $[0,a) \cap (b,1] = (a,b)$, so intersections of these can give all open intervals strictly contained $[0,1]$; adding in the "open" intervals $[0,a)$ and $(a,0]$ make all the open intervals in $[0,1]$, which are a basis for our topology.

  We now proceed with our check; fix some $a \in (0,1)$.

  - Note that $x \in X$ has $f(x) < a$ if and only if there is some $r \in \Delta$ such that $f(x) < r < a$ (by density of Δ) if and only if there is some $r \in \Delta$ such that $x \in U_r$ and $r < a$ (by definition of the infimum). As such,
  $$f^{-1}([0,a)) = \bigcup_{r<a} U_r.$$
  - Note that $x \in X$ has $f(x) > a$ if and only if there is an $r, s \in \Delta$ with $f(x) > r > s > a$ (by density). It follows $x \notin U_r$, which contains $\overline{U_s}$, so $x \notin \overline{U_s}$ for some $s \in \Delta$ with $s > a$.
  On the other hand, $x \notin \overline{U_s}$ for some $s \in \Delta$ with $s > a$ implies that $x \notin U_r$ for any $r \in \Delta$ with $r > s > a$, so it follows $f(x) \geq s > a$.
  Thus, $f(x) > a$ if and only if $x \notin \overline{U_s}$ for $s \in \Delta$ with $s > a$, implying
  $$f^{-1}((a,1]) = \bigcup_{s>a} (X \setminus \overline{U_s}).$$

The above checks complete the proof. ■

Remark 3.19. We could not have f output to $\mathbb{Q} \cap [0,1]$ because we used the completeness of $\mathbb{R}$ in the construction of f.

Remark 3.20. It is somewhat noticeable that we have not discussed sequences at all in this class yet, even though they were featured prominently in metric space topology. The reason we have been avoiding them is that we prefer to use open sets and not points to study general topological spaces.

3.2.2 Bounded Functions

We are going to want a little functional analysis before we continue.

Definition 3.21 (Bounded)**.** Fix a metric space (X,d) and a nonempty set A. A subset $A \subseteq X$ is *bounded* if and only if there is an open ball $B(x,r)$ containing A. More generally, a function $f \colon A \to X$ is *bounded* if and only if $\operatorname{im} f \subseteq X$ is bounded, and we let $B(A,X)$ denote the set of all bounded functions $f \colon A \to X$.

We will be particularly interested in the case where X is a normed vector space.

The point of defining bounded functions is that we can provide them with a metric.

Definition 3.22 (Uniform metric)**.** Fix a nonempty set X and a metric space (Y,d). Then the *uniform metric* is the function $d_u \colon B(X,Y)^2 \to \mathbb{R}_{\geq 0}$ defined by
$$d_u(f,g) := \sup\{d(f(x),g(x)) : x \in X\}.$$

Lemma 3.23. Fix a set X and a metric space (Y, d). Then the uniform metric d_u on $B(X, Y)$ is a metric.

Proof. Here are our checks; fix $f, g, h \in B(X, Y)$.

- Well-defined: because f and g bounded, we can find open balls $B(a, r)$ and $B(b, s)$ containing $\operatorname{im} f$ and $\operatorname{im} g$ respectively. It follows that, for any $x \in X$, we have

$$d(f(x), g(x)) \leq d(f(x), a) + d(a, b) + d(b, g(x)) \leq r + d(a, b) + s,$$

so the set $\{d(f(x), g(x)) : x \in X\}$ has an upper bound and hence a supremum.

- Nonnegative: fixing a particular $x \in X$, note $d_u(f, g) \geq d(f(x), g(x)) \geq 0$.
- Zero: note $d_u(f, f)$ is $\sup\{d(f(x), f(x)) : x \in X\} = \sup\{0 : x \in X\} = 0$.
- Zero: note $d_u(f, g) = 0$ implies that $\sup\{d(f(x), g(x)) : x \in X\} = 0$, so $d(f(x), g(x)) \leq 0$ for all $x \in X$, so $d(f(x), g(x)) = 0$ for all $x \in X$, so $f(x) = g(x)$ for all $x \in X$.
- Symmetric: note

$$d_u(f, g) = \sup\{d(f(x), g(x)) : x \in X\} = \sup\{d(g(x), f(x)) : x \in X\} = d_u(g, f).$$

- Triangle inequality: note that

$$d(f(x), h(x)) \leq d(f(x), g(x)) + d(g(x), h(x)) = d_u(f, g) + d_u(g, h)$$

for all $x \in X$, so it follows $d_u(f, h) \leq d_u(f, g) + d_u(g, h)$ by taking the supremum. ■

Here is why we like this metric.

Proposition 3.24. Fix a set X and a complete metric space (Y, d). Then $B(X, Y)$ given the uniform metric is complete.

Proof. Fix a Cauchy sequence $\{f_n\}_{n \in \mathbb{N}}$ in $B(X, Y)$. Namely, for all $\varepsilon > 0$, there exists some N so that

$$n, m > N \implies d(f_n(x), f_m(x)) < \varepsilon$$

for all $x \in X$. In particular, fixing some particular $x \in X$, we see that $\{f_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Y, so the completeness of Y promises some limit $f(x)$.

It remains to check that the data of f assembles to a function $f \in B(X, Y)$. Well, any (fixed) $\varepsilon > 0$ promises an N so that $n, m > N$ forces $d(f_n(x), f_m(x)) < \varepsilon$ for all $x \in X$. Now, fixing some $x \in X$, any $\delta > 0$ has some N' large enough so that $m > N'$ has $d(f_m(x), f(x)) < \delta$, meaning that $n, m > \max\{N, N'\}$ gives

$$d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \varepsilon + \delta$$

for all $\delta > 0$. Thus, fixing some $n > N$, we see $d(f_n(x), f(x)) \leq \varepsilon$ for all $x \in X$.

To finish, we note $f_n \in B(X, Y)$ is bounded, so there is an open ball $B(a, r)$ containing $\operatorname{im} f_n$. Thus, for all $x \in X$,

$$d(a, f(x)) \leq d(a, f_n(x)) + d(f_n(x), f(x)) < r + \varepsilon,$$

so $\operatorname{im} f \subseteq B(a, r + \varepsilon)$. ■

We close with the following result.

Proposition 3.25. Fix a topological space $(X, \mathcal{T})$ and a metric space (Y, d). Let $B_c(X, Y) \subseteq B(X, Y)$ denote the metric subspace of bounded continuous functions $f\colon X \to Y$. Then $B_c(X, Y)$ is a closed subspace of $B(X, Y)$. In particular, if (Y, d) is complete, then $B_c(X, Y)$ is also complete.

Proof. Note that the second claim follows from the first claim by Corollary 2.51; thus, we focus on the first claim. For this, we use Lemma 2.49: fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ of bounded continuous functions such that $f_n \to f$ as $n \to \infty$ where $f\colon X \to Y$ is just some bounded function. We need to show that f is continuous.

Well, fix an open set $U \subseteq Y$ so that we need to show $f^{-1}(U) \subseteq X$ is open. For this, we pick up any element $x \in f^{-1}(U)$, and we find an open neighborhood $U_x \subseteq f^{-1}(U)$ containing x; this will finish because it shows

$$f^{-1}(U) \subseteq \bigcup_{x\in U} U_x \subseteq f^{-1}(U),$$

so $f^{-1}(U)$ is the arbitrary union of open sets.

We now proceed with the proof directly.

1. Because $f(x) \in U$, and U is open, there is some $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$.

2. Because $\{f_n\}_{n\in\mathbb{N}}$ converges to f, there is a sufficiently large N so that $n > N$ has $d(f_n(y), f(y)) < \varepsilon/2$ for all $y \in X$. Fix some $n > N$.

3. Now, for all $y \in f_n^{-1}(B(f(x), \varepsilon/2))$, we see

$$d(f(y), f(x)) \le d(f(y), f_n(y)) + d(f_n(y), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so $f(y) \in U$. As such, we see that $f_n^{-1}(B(f(x), \varepsilon/2))$ is open (because f_n is continuous), it contains x, and it is contained in $f^{-1}(U)$.

The above open neighborhood completes the proof of the first claim. ■

3.3 September 14

The march continues.

3.3.1 The Tietze Extension Theorem

Here is the main result for today.

Theorem 3.26 (Tietze extension)**.** Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to \mathbb{R}$, there exists a continuous function $\widetilde{f}\colon X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$. In fact, if $\operatorname{im} f \subseteq [a, b]$, then we may enforce $\operatorname{im} \widetilde{f} \subseteq [a, b]$ as well.

This property is quite special to $\mathbb{R}$ shared by a few other spaces.

Example 3.27. Take $X := \overline{B(0,1)} \subseteq \mathbb{R}^2$ given the relative topology, and let $A = \partial X$ be the boundary, which is the unit circle. Then the identity function $\operatorname{id}_A\colon A \to A$ does not extend continuously to a function $\widetilde{\operatorname{id}_A}\colon X \to A$. To see this rigorously, take a course in algebraic topology.

Example 3.28. Of course, any set Y given the indiscrete topology will be such that a continuous function $f\colon A \to Y$ can be extended to continuously to a function $\widetilde{f}\colon X \to Y$ because all functions to Y are continuous for free.

Remark 3.29. The condition of $\operatorname{im} f \subseteq [a,b]$ might as well be replaced by $\operatorname{im} f \subseteq [0,1]$ by using the homeomorphism $\mathbb{R} \to \mathbb{R}$ by $x \mapsto (x-a)/(b-b)$ which will send $[a,b]$ to $[0,1]$.

Here is a lemma which will help the proof of Theorem 3.26.

Lemma 3.30. Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to [0,r]$ (where $r > 0$), there exists a continuous function $g\colon X \to [0, r/3]$ such that

$$0 \le f(a) - g(a) \le 2r/3$$

for each $a \in A$.

Proof. Set $B := \{x \in A : f(x) \le r/3\} = f^{-1}([0, r/3])$ and $C := \{x \in A : f(x) \ge 2r/3\} = f^{-1}([(2r/3, r])$. Both $B, C \subseteq A$ and C are closed because they are the pre-image of closed subsets under $f\colon A \to \mathbb{R}$. In fact, by the relative topology, we can write $B = B' \cap A$ where $B' \subseteq X$ is closed. However, B' and A are both closed in X, so $B \subseteq X$ is closed. Similar holds for C.

Thus, so Urysohn's lemma provides a continuous function $g\colon X \to [0,1]$ such that $g|_B = 0$ and $g|_C = 1$. As such, we define $g\colon X \to [0, r/3]$ by

$$g(x) := (r/3) \cdot g(x),$$

which is still continuous because the map $x \mapsto (r/3)x$ is a homeomorphism $[0,1] \to [0, r/3]$ by Example 2.86. We can now see that g satisfies the needed properties. Fix some $a \in A$.

- If $a \in B$, then $g(a) = 0$ while $f(a) \le r/3$, so $0 \le f(a) - g(a) \le r/3$.
- If $a \in C$, then $g(a) = r/3$ while $f(a) \in [2r/3, r]$, so $0 \le f(a) - g(a) \le 2r/3$.
- Lastly, $a \notin B$ and $a \notin C$ means that $r/3 < f(a) < 2r/3$ while $0 \le g(a) \le r/3$, so it follows $0 \le f(a) - g(a) \le 2r/3$ still.

The above checks finish. ■

We now show the following special case of Theorem 3.26.

Proposition 3.31. Fix a normal topological space $(X, \mathcal{T})$, and give some closed subset $A \subseteq X$ the relative topology from X. Given a continuous function $f\colon A \to [0,1]$, there exists a continuous function $\widetilde{f}\colon X \to [0,1]$ such that $\widetilde{f}|_A = f$.

Proof. For brevity, define $\sigma := 2/3$. Taking $r = 1$ in Lemma 3.30, we get a function $g_1\colon X \to [0, 1/3]$ with

$$0 \le f(a) - g_1(a) \le \sigma$$

for all $a \in A$, so define $\widetilde{f}_1 := g_1$. Next applying Lemma 3.30 to $(f - \widetilde{f}_1|_A)\colon A \to [0, \sigma]$ with $r = \sigma$, we get promised a function $g_2\colon X \to [0, \sigma/3]$ with

$$0 \le f(a) - \widetilde{f}_1(a) - g_2(a) \le \sigma^2$$

for any $a \in A$, so define $\widetilde{f}_2 := \widetilde{f}_1 + g_2$.

In general, suppose given a function $\widetilde{f}_n\colon X \to [0,1]$ with

$$0 \le f(a) - \widetilde{f}_n(a) \le \sigma^n$$

for $a \in A$, we can use Lemma 3.30 to $(f - \widetilde{f}_n|_A)\colon A \to [0, \sigma^n]$ to get a function $g_{n+1}\colon X \to [0, \sigma^n/3]$ with

$$0 \le f(a) - \widetilde{f}_n(a) - g_{n+1}(a) \le \sigma^{n+1}$$

for $a \in A$, allowing us to then set $\widetilde{f}_{n+1} := \widetilde{f}_n + g_{n+1}$.

Applying the above process inductively, we get a function

$$\widetilde{f}_n = \sum_{k=1}^{n} g_k$$

going to $[0,1]$ such that $\|g_k\|_\infty \leq \sigma^{k-1}/3$ and $0 \leq f(a) - \widetilde{f}_n(a) \leq (2/3)^n$ for each $a \in A$ and $n \geq 1$. Notably, using the uniform metric d_u, we see that any $n \geq m$ has

$$d_u(\widetilde{f}_n, \widetilde{f}_m) = \sup_{x \in X}\left(\sum_{k=m+1}^{n} g_k(x)\right) \leq \sum_{k=m+1}^{n} \frac{1}{3}\sigma^{k-1} \leq \frac{\sigma^m}{3}\sum_{k=0}^{\infty}\sigma^k = \frac{\sigma^m}{3}\cdot\frac{1}{1-\sigma} = \left(\frac{2}{3}\right)^m,$$

which gets arbitrarily small. Thus, $\{\widetilde{f}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence: for any $\varepsilon > 0$, we can find N with $n > N$ having $(2/3)^n < \varepsilon$, meaning $n, m \geq N$ will have $d_u(\widetilde{f}_n, \widetilde{f}_m) < \varepsilon$. Now, because $[0,1] \subseteq \mathbb{R}$ is a closed subset of a complete metric space and hence complete by Corollary 2.51, the sequence $\{\widetilde{f}_n\}_{n\in\mathbb{N}}$ converges to a continuous function $\widetilde{f}\colon X \to [0,1]$ by Proposition 3.25.

It remains to check that $\widetilde{f}|_A = f$. Well, any $a \in A$ and $n \in \mathbb{N}$ have

$$|f(a) - \widetilde{f}(a)| \leq |f(a) - \widetilde{f}_n(a)| + |\widetilde{f}_n(a) - \widetilde{f}(a)| \leq \left(\frac{2}{3}\right)^n + |\widetilde{f}_n(a) - f(a)|.$$

Because $\widetilde{f}_n \to f$ as $n \to \infty$ under the metric d_u, we see that $|\widetilde{f}_n(a) - f(a)| \to 0$ as $n \to \infty$. Additionally, $(2/3)^n \to 0$ as $n \to \infty$, so the entire right-hand side goes to 0 as $n \to \infty$, meaning that $|f(a) - \widetilde{f}(a)| < \varepsilon$ for all $\varepsilon > 0$. Thus, $f(a) = \widetilde{f}(a)$ for each $a \in A$. ■

3.3.2 The Tietze Extension Theorem: Proof

And here is the proof of the general case of Theorem 3.26.

Proof of Theorem 3.26. Fix a continuous function $f\colon A \to \mathbb{R}$. Note that there is a homeomorphism $\varphi\colon \mathbb{R} \cong (0,1)$, so we name composite

$$A \xrightarrow{f} \mathbb{R} \overset{\varphi}{\cong} (0,1) \subseteq [0,1]$$

g and then extend it to a function $\widetilde{g}\colon X \to [0,1]$. We would like to go back to $(0,1)$ and then back to $\mathbb{R}$, but it is possible for $0, 1 \in \operatorname{im} g$.

Isolating the problem, we set $B_0 := \widetilde{g}^{-1}(\{0\})$ and $B_1 := \widetilde{g}^{-1}(\{1\})$ and note that $A \cap (B_0 \cup B_1) = \varnothing$ because $\widetilde{g}(A) = g(A) \subseteq (0,1)$. Now, by normality of X, we get promised disjoint open subsets $U_1, U_2 \subseteq X$ containing B_0 and B_1. We will finish this proof next class. ■

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