261A: Lie Groups

Nir Elber

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Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1
TOPOLOGICAL BACKGROUND

Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions.

—Ravi Vakil, [Vak17]

1.1 August 28

Today we review differential topology. Here are some logistical notes.

- There will be weekly homeworks, of about 5 problems.
- There will be a final take-home exam.
- This course has a bCourses page.
- We will mostly follow Kirillov's book [Kir08].

1.1.1 Group Objects

The goal of this class is to study symmetries of geometric objects. As such, we are interested in studying (infinite) groups with some extra geometric structure, such as a real manifold or a complex manifold or a scheme structure. Speaking generally, we will have some category $\mathcal{C}$ of geometric objects, equipped with finite products (such as a final object), which allows us to have group objects in $\mathcal{C}$.

Definition 1.1 (group object)**.** Fix a category $\mathcal{C}$ with finite products, such as a final object $*$. A *group object* is the data (G, m, e, i) where $G \in \mathcal{C}$ is an object and $m \colon G \times G \to G$ and $e \colon * \to G$ and $i \colon G \to G$ are morphisms. We require this data to satisfy some associativity, identity, and inverse coherence laws.

For concreteness, we go ahead and write out the coherence diagrams, but they are not so interesting.

- Associative: the following diagram commutes.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}_G \times m} & G \times G \\ {\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array}$$

- Identity: the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}_G \times e"] \arrow[dr, equal] & G \times G \arrow[d, "m"] & G \arrow[l, "e \times \mathrm{id}_G"'] \arrow[dl, equal] \\ & G & \end{tikzcd}$$

- Inverses: the following diagram commutes.

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "i \times \mathrm{id}_G"] \arrow[r] \arrow[dr, "\mathrm{id}_G \times i"'] & * \arrow[r, "e"] & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Example 1.2. In the case where $\mathcal{C} = \mathrm{Set}$, we recover the notion of a group, where G is the set, m is the multiplication law, e is the identity, and i is the inverse.

Example 1.3. Group objects in the category of manifolds will be Lie groups.

1.1.2 Review of Topology

This course requires some topology as a prerequisite, but let's review these notions for concreteness. We refer to [Elb22] for most of these notions.

Definition 1.4 (topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ of a set X and collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of open subsets of X, which we require to satisfy the following axioms.

- $\varnothing, X \in \mathcal{T}$.
- Finite intersection: for $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.
- Arbitrary unions: for a subcollection $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

We will suppress the notation $\mathcal{T}$ from our topological space as much as possible.

Example 1.5. The set $\mathbb{R}$ equipped with its usual (metric) topology is a topological space.

Example 1.6. Given a topological space X and a subset $Z \subseteq X$, we can make Z into a topological space with open subsets given by $U \cap Z$ whenever $U \subseteq Z$ is open.

Definition 1.7 (closed)**.** A subset Z of a topological space X is *closed* if and only if $X \backslash Z$ is open.

One way to describe topologies is via a base.

Definition 1.8 (base)**.** Given a topological space X, a *base* $\mathcal{B} \subseteq \mathcal{P}(X)$ for the topology such that any open subset $U \subseteq X$ is the union of a subcollection of $\mathcal{B}$. Equivalently, for any open subset $U \subseteq X$ and $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example 1.9. The collection of open intervals $(a, b) \subseteq \mathbb{R}$ generates the usual topology. In fact, one can even restrict ourselves to open intervals (a, b) where $a, b \in \mathbb{Q}$, so $\mathbb{R}$ has a countable base.

Our morphisms are continuous maps.

Definition 1.10 (continuous)**.** A function $f \colon X \to Y$ between topological spaces is *continuous* if and only if $f^{-1}(V) \subseteq X$ is open for each open subset $V \subseteq Y$.

Thus, we can define Top as the category of topological spaces equipped with continuous maps as its morphisms. Thinking categorically allows us to make the following definition.

Definition 1.11 (homeomorphism)**.** A *homeomorphism* is an isomorphism in Top. Namely, a function $f \colon X \to Y$ between topological spaces which is continuous and has a continuous inverse.

Remark 1.12. There are continuous bijections which are not homeomorphisms! For example, one can map $[0, 2\pi) \to S^1$ by sending $x \mapsto e^{ix}$, which is a continuous bijection, but the inverse is discontinuous at $1 \in S^1$.

Earlier, we wanted to have finite products in our category. Here is how we take products of pairs.

Definition 1.13 (product topology)**.** Given topological spaces X and Y, we define the topological space $X \times Y$ as having $X \times Y$ as its set and open subsets given by arbitrary unions of sets of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open.

Remark 1.14. Alternatively, we can say that the topology $X \times Y$ has a base given by the "rectangles" $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open. In fact, if $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, respectively, then we can check that the open subsets

$$\{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a base for $X \times Y$.

Remark 1.15. The final object in Top is the singleton space.

Now, group objects in Top are called topological groups, which are interesting in their own right. For example, locally compact topological groups have a good Fourier analysis theory.

Example 1.16. The group $\mathbb{R}$ under addition is a topological group. In fact, $\mathbb{Q}$ under addition is also a topological group, though admittedly a more unpleasant one.

Example 1.17. The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a topological group.

1.1.3 Review of Differential Topology

However, in this course, we will be more interested in manifolds, so let's define these notions. We refer to [Elb24] for (a little) more detail, and we refer to [Lee13] for (much) more detail. To begin, we note arbitrary topological spaces are pretty rough to handle; here are some niceness requirements. The following is a smallness assumption.

Definition 1.18 (separable)**.** A topological space X is *separable* if and only if it has a countable base.

The following says that points can be separated.

Definition 1.19 (Hausdorff)**.** A topological space X is *Hausdorff* if and only if any pair of distinct points $p, q \in X$ have disjoint open neighborhoods.

The following is another smallness assumption, which we will use frequently but not always.

Definition 1.20 (compact)**.** A topological space X is *compact* if and only if any open cover $\mathcal{U}$ (i.e., each $U \in \mathcal{U}$ is open, and $X = \bigcup_{U \in \mathcal{U}} U$) has a finite subcollection which is still an open cover.

We are now ready for our definition.

Definition 1.21 (topological manifold)**.** A *topological manifold of dimension* n is a topological space X satisfying the following.

- X is Hausdorff.
- X is separable.
- Locally Euclidean: X has an open cover $\{U_\alpha\}_{\alpha \in \kappa}$ such that there are open subsets $V_\alpha \subseteq \mathbb{R}^n$ and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$.

Remark 1.22. By passing to open balls, one can require that all the V_α are open balls. By doing a little more yoga with such open balls (noting $B(0,1) \cong \mathbb{R}^n$), one can require that $V_\alpha = \mathbb{R}^n$ always.

Remark 1.23. It turns out that open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ can only be homeomorphic if and only if $n = m$. This implies that the dimension of a connected component of X is well-defined without saying what n is in advance. However, we should say what n is in advance in order to get rid of pathologies like $\mathbb{R} \sqcup \mathbb{R}^2$.

To continue, we must be careful about our choice of U_αs and φ_αs.

Definition 1.24 (chart, atlas, transition function)**.** Fix a topological manifold X of dimension n.

- A *chart* is a pair (U, φ) of an open subset $U \subseteq X$ and homeomorphism φ of U onto an open subset of $\mathbb{R}^n$.
- An *atlas* is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \kappa}$ such that $\{U_\alpha\}_{\alpha \in \kappa}$ is an open cover of X.
- The *transition function* between two charts (U, φ) and (V, ψ) is the composite homeomorphism
$$\varphi(U \cap V) \xleftarrow{\varphi} (U \cap V) \xrightarrow{\psi} \psi(U \cap V).$$
Note that there is also an inverse transition map going in the opposite direction.

Let's see some examples.

Example 1.25. The space $\mathbb{R}^n$ is a topological manifold of dimension n. It has an atlas with the single chart $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.26. The singleton $\{*\}$ is a topological manifold of dimension 0. In fact, $\{*\} = \mathbb{R}^0$.

Example 1.27. The hypersurface $S^n \subseteq \mathbb{R}^{n+1}$ cut out by the equation

$$x_0^2 + \cdots + x_n^2 = 1$$

is a topological manifold of dimension n. It has charts given by stereographic projection out of some choice of north and south poles. Alternatively, it has charts given by the projection maps $\mathrm{pr}_i \colon S^n \to \mathbb{R}^n$ given by deleting the ith coordinate, defined on the open subsets

$$U_i^\pm := \{(x_0, \ldots, x_n) \in \mathbb{R}^n : \pm x_i > 0\}$$

for choice of index i and sign in $\{\pm\}$.

Calculus on our manifolds will come from our transition maps.

Definition 1.28. An atlas $\mathcal{A}$ on a topological manifold X is C^k, real analytic, or complex analytic (if $\dim X$ is even) if and only if the transition maps have the corresponding condition.

1.2 August 30

Today we finish our review of smooth manifolds. Once again, we refer to [Elb24] for a few more details and [Lee13] for many more details.

Notation 1.29. We will use the word *regular* to refer to one of the regularity conditions C^k, smooth, real analytic, or complex analytic. We may abbreviate complex analytic to "complex" when no confusion is possible. We use the field $\mathbb{F}$ to denote the "ground field," which is $\mathbb{C}$ when considering the complex analytic case and $\mathbb{R}$ otherwise.

1.2.1 Smooth Manifolds

We now define a regular manifold.

Definition 1.30 (regular manifold)**.** A *regular manifold* of dimension n is a pair $(M, \mathcal{A})$ of a topological manifold M and a maximal regular atlas $\mathcal{A}$; a chart is called regular if and only if it is in $\mathcal{A}$. We will eventually suppress the $\mathcal{A}$ from our notation as much as possible.

The reason for using a maximal atlas is to ensure that it is more or less unique.

Remark 1.31. Here is perhaps a more "canonical" way to deal with atlas confusion. One can say that two regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if and only if the transition maps between them are also regular; this is the same as saying that $\mathcal{A}_1 \cup \mathcal{A}_2$ is regular. Compatibility forms an equivalence relation, and each equivalence class $[\mathcal{A}]$ has a unique maximal element, which one can explicitly define as

$$\mathcal{A}_{\max} := \{(U, \varphi) : \mathcal{A} \text{ and } (U, \varphi) \text{ are compatible}\}.$$

This explains why it is okay to just work with maximal atlases.

Example 1.32. One can give the topological manifold $\mathbb{R}^2$ many non-equivalent complex structures. For example, one has the usual choice of $\mathbb{R}^2 \cong \mathbb{C}$, but one can also make $\mathbb{R}^2$ homeomorphic to $B(0,1) \subseteq \mathbb{C}$.

Example 1.33. There are "exotic" smooth structures on S^7.

Example 1.34. Given regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, one can form the product manifold $X \times Y$. It should have maximal atlas compatible with the atlas

$$\{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B}\}.$$

1.2.2 Regular Functions

With any class of objects, we should have morphisms.

Definition 1.35. A function $f : X \to Y$ of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is regular if and only if any $p \in X$ has a choice of charts $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $f(U) \subseteq V$ and the composite

$$\varphi(U) \overset{\varphi}{\leftarrow} U \overset{f}{\rightarrow} V \overset{\psi}{\rightarrow} \psi(V)$$

is a regular function between open subsets of Euclidean space.

Remark 1.36. One can replace the single choice of charts above with any choice of charts satisfying $p \in U$ and $f(U) \subseteq V$.

Remark 1.37. Here is another way to state this: for any open $V \subseteq Y$ and smooth function $h : V \to \mathbb{F}$, the composite

$$f^{-1}(U) \overset{f}{\rightarrow} V \overset{h}{\rightarrow} \mathbb{F}$$

succeeds in being smooth (in any local coordinates).

Definition 1.38 (diffeomorphism)**.** A *diffeomorphism* of regular manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is a regular map $f : X \to Y$ with regular inverse.

Remark 1.39. Alternatively, one can say that the charts in $\mathcal{A}$ and the charts in $\mathcal{B}$ are in natural bijection via f. Checking that these notions align is not too hard.

The above definition of regular map is a little rough to handle, so let's break it down into pieces.

Definition 1.40 (local coordinates)**.** Fix a regular manifold $(X, \mathcal{A})$ of dimension n. Then a system of *local coordinates* around some point $p \in X$ is a choice of regular chart $(U, \varphi) \in \mathcal{A}$ for which $\varphi(p) = 0$. From here, our local coordinates $(x_1, \ldots, x_n)$ are the composite of φ with a coordinate projection to the ground field. (In the complex analytic case, we want the ground field to be $\mathbb{C}$; otherwise, the ground field is $\mathbb{R}$.)

Now, we are able to see that a function $f : X \to \mathbb{R}$ is regular if and only if it becomes regular in local coordinates. One can even define regularity with respect to a subset of X.

Regularity allows us to produce lots of manifolds, as follows.

Theorem 1.41. Given regular maps $f_1, \ldots, f_m : X \to \mathbb{F}$, the subset

$$\{p \in \mathbb{F}^n : f_1(p) = \cdots = f_m(p) = 0 \text{ and } \{df_1(p), \ldots, df_n(p)\} \text{ are linearly independent}\}$$

is a manifold of dimension $n - m$.

Sketch. This is more or less by the implicit function theorem; for the $\mathbb{F} = \mathbb{R}$ cases, one can essentially follow [Lee13, Corollary 5.14]. ∎

Example 1.42. The function $\mathbb{R}^{n+1} \to \mathbb{R}$ given by $(x_0, \ldots, x_n) \mapsto x_0^2 + \cdots + x_n^2$ is real analytic and sufficiently regular at the value 1, which establishes that S^n defined in Example 1.27 succeeds at being a real analytic manifold.

Functions to $\mathbb{F}$ have a special place in our hearts, so we take the following notation.

Notation 1.43. Give a regular manifold X and any open subset $U \subseteq X$, we let $\mathcal{O}_X(U)$ denote the set of regular functions $U \to \mathbb{F}$

Remark 1.44. One can check that the data $\mathcal{O}_X$ assembles into a sheaf. Namely, an inclusion $U \subseteq V$ produces restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Remark 1.45. Once we have all of our regular functions out of X, we note that some Yoneda-like philosophy explains that the sheaf of X determines its full regular structure. Here is an explicit statement: given a manifold X and two maximal regular atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ determining sheaves of regular functions $\mathcal{O}_1$ and $\mathcal{O}_2$, having $\mathcal{O}_1 = \mathcal{O}_2$ forces $\mathcal{A}_1 = \mathcal{A}_2$. Indeed, it is enough to show the inclusion $\mathcal{A}_1 \subseteq \mathcal{A}_2$, so suppose (U, φ) is a regular chart in $\mathcal{A}_1$. Then the corresponding local coordinates $(x_1, \ldots, x_n)$ all succeed at being regular for $\mathcal{A}_1$, so they are smooth functions in $\mathcal{O}_1$, so they live in $\mathcal{O}_2$ also, so (U, φ) will succeed at being a regular local diffeomorphism for $\mathcal{A}_2$ and hence be a regular chart.

Sheaf-theoretic notions tell us that we should be interested in germs.

Definition 1.46 (germ)**.** Fix a point p on a regular manifold X. A *germ* of a regular function $f \in \mathcal{O}_X(U)$ (where $p \in U$) is the equivalence class of functions $g \in \mathcal{O}_X(V)$ (for a possibly different open subset V containing p) such that $f|_{U \cap V} = g|_{U \cap V}$. The collection of equivalence classes is denoted $\mathcal{O}_{X,p}$ and is called the stalk at p.

1.3 September 4

Today we hope to finish our review of differential topology.

Convention 1.47. For the remainder of class, our manifolds will be smooth, real analytic, or complex analytic.

1.3.1 Tangent Spaces

Now that we are thinking locally about our functions via germs, we can think locally about our tangent spaces.

Definition 1.48 (derivation)**.** Fix a point p on a regular manifold X. A *derivation* at p is an $\mathbb{F}$-linear map $D\colon \mathcal{O}_{X,p} \to \mathbb{F}$ satisfying the Leibniz rule

$$D(fg) = g(p)D(f) + f(p)D(g).$$

Definition 1.49 (tangent space)**.** Fix a point p on a regular manifold X. Then the *tangent space* T_pX is the $\mathbb{F}$-vector space of all derivations on $\mathcal{O}_{X,p}$.

As with everything in this subject, one desires a local description of the tangent space.

Lemma 1.50. Fix an n-dimensional regular manifold X and a point $p \in X$. Equip p with a chart (U, φ) giving local coordinates $(x_1, \ldots, x_n)$. Then the maps $D_i \colon \mathcal{O}_{X,p} \to \mathbb{F}$ given by

$$D_i \colon [(V, f)] \mapsto \left.\frac{\partial f|_{U \cap V}}{\partial x_i}\right|_p$$

provide a basis for T_pX.

Proof. Checking that this is a derivation follows from the Leibniz rule on the chart. Linear independence of the $D_\bullet$s can also be checked locally by plugging in the germs $[(U, x_i)]$ into any linear dependence.

It remains to check that our derivations span. Well, fix any other derivation D which we want to be in the span of the $D_\bullet$s. By replacing D with $D - \sum_i D(x_i) D_i$, we may assume that $D(x_i) = 0$ for all i. We now want to show that $D = 0$. This amounts to some multivariable calculus. Fix a germ $[(V, f)]$, and shrink U and V enough so that f is defined on U; we want to show $D(f) = 0$. The fundamental theorem of calculus implies

$$f(x_1, \ldots, x_n) = f(0) + \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n)\, dt.$$

However, one can expand out the derivative on the right by the chain rule to see that

$$f(x_1, \ldots, x_n) = f(0) + \sum_{i=1}^n x_i h_i(x_1, \ldots, x_n)$$

for some regular functions $h_1, \ldots, h_n \colon X \to \mathbb{F}$. Applying D, we see that

$$D(f) = \sum_{i=1}^n \underbrace{D(x_i)}_{0} h_i(p) + \underbrace{x_i(p)}_{0} D(h_i) = 0,$$

as required. ■

Tangent spaces have a notion of functoriality.

Definition 1.51. Fix a regular map $F \colon X \to Y$ of regular manifolds. Given $p \in X$, the *differential map* is the linear map $dF_p \colon T_pX \to T_{F(p)}Y$ defined by

$$dF_p(v)(g) := v(g \circ F)$$

for any $v \in T_pX$ and germ $g \in \mathcal{O}_{X,p}$. We may also denote $dF_p(v)$ by F_*v.

One has to check that dF_p is linear (which does not have much to check) and satisfies the Leibniz rule (which is a matter of expansion); we will omit these checks.

Remark 1.52. One also has a chain rule: for regular maps $F \colon X \to Y$ and $G \colon Y \to Z$, one has $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.

1.3.2 Immersions and Submersions

This map at the tangent space is important enough to give us other definitions.

Definition 1.53 (submersion, immersion, embedding)**.** Fix a regular function $F\colon X \to Y$.

- The map F is a *submersion* if and only if dF_p is surjective for all $p \in X$.
- The map F is an *immersion* if and only if dF_p is injective for all $p \in X$.
- The map F is an *embedding* if and only if F is an immersion and a homeomorphism onto its image.

Remark 1.54. One can check that submersions $F\colon X \to Y$ have local sections $Y \to X$. Explicitly, for $Q \in Y$, the fiber $F^{-1}(\{Q\}) \subseteq X$ is a manifold, and if $Q \in \operatorname{im} F$, the fiber has codimension $\dim Y$.

Remark 1.55. If $F\colon X \to Y$ is an embedding, then the image $F(X) \subseteq Y$ inherits a unique manifold structure so that the inclusion $F(X) \subseteq Y$ is smooth.

Example 1.56. The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) := x$ is a submersion.

Example 1.57 (lemniscate)**.** The function $F\colon S^1 \to \mathbb{R}^2$ given by

$$F(\theta) := \left(\frac{\cos\theta}{1+\sin^2\theta}, \frac{\cos\theta\sin\theta}{1+\sin^2\theta}\right)$$

can be checked to be an immersion (namely, $F'(\theta) \neq 0$ always), but it fails to be injective because $F(\pi/4) = F(3\pi/4) = (0, 0)$, so it is not an embedding.

Example 1.58. The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$ is a smooth homeomorphism onto its image, but it is not an immersion.

Example 1.59. For any open subset $U \subseteq X$ of a manifold, the inclusion map $U \to X$ is an embedding. (In fact, it is also a submersion.)

We will want to distinguish between embeddings, notably to get rid of open embeddings.

Definition 1.60 (closed)**.** An embedding $F\colon X \to Y$ of regular manifolds is *closed* if and only if $F(X) \subseteq Y$ is closed.

Example 1.61. Fix a submersion $F\colon X \to Y$. A point $Q \in Y$ gives rise to a fiber $Z := F^{-1}(\{Q\})$, which Remark 1.54 explains is a closed submanifold of X of codimension $\dim Y$. One can check that T_pZ is exactly the kernel of $dF_p\colon T_pX \to T_pY$; see [Lee13, Proposition 5.37].

1.3.3 Lie Groups

We now may stop doing topology.

Definition 1.62 (Lie group)**.** A regular *Lie group* is a group object in the category of regular manifolds. For brevity, we may call (real) smooth Lie groups simply "Lie groups" or "real Lie groups," and we may call complex analytic Lie groups simply "complex Lie groups."

As with any object, we have a notion of morphisms.

Definition 1.63 (homomorphism)**.** A homomorphism of regular Lie groups is a regular map of the underlying manifolds and a homomorphism of the underlying groups; an isomorphism of regular Lie groups is a homomorphism with an inverse which is also a homomorphism.

Remark 1.64. If X is already a regular manifold, and we are equipped with continuous multiplication and inverse maps, to check that X becomes a regular Lie group, it is enough to check that merely the multiplication map is regular. See [Lee13, Exercise 7-3].

Remark 1.65. Hilbert's 5th problem asks when C^0 Lie groups can give rise to real Lie groups, and there is a lot of work in this direction. As such, we will content ourselves to focus on real Lie groups instead of any weaker regularity.

Remark 1.66. Any complex Lie group is also a real Lie group.

Here is a basic check which allows one to translate checks to the identity.

Lemma 1.67. Fix a regular Lie group G. For any $g \in G$, the maps $L_g\colon G \to G$ and $R_g\colon G \to G$ defined by $L_g(x) := gx$ and $R_g(x) := xg$ are regular diffeomorphisms.

Proof. Regularity follows from regularity of multiplication. Our inverses of L_g and R_g are given by $L_{g^{-1}}$ and $R_{g^{-1}}$, which verifies that we have defined regular diffeomorphisms. ■

1.4 September 6

Last time we defined a Lie group. Today and for the rest of the course, we will study them.

1.4.1 Examples of Lie Groups

Here are some examples of Lie groups and isomorphisms.

Example 1.68. For our field $\mathbb{F}$, the $\mathbb{F}$-vector space $\mathbb{F}^n$ is a Lie group over $\mathbb{F}$.

Example 1.69. Any finite (or countably infinite) group given the discrete topology becomes a real and complex Lie group.

Example 1.70. The groups $\mathbb{R}^\times$ and $\mathbb{R}^+$ (under multiplication) are real Lie groups. In fact, one has an isomorphism $\{\pm 1\} \times \mathbb{R}^+ \to \mathbb{R}^\times$ of real Lie groups given by $(\varepsilon, r) \mapsto \varepsilon r$.

Example 1.71. The group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ (under multiplication) is a real Lie group.

Example 1.72. The group $\mathbb{C}^\times$ is a real Lie group. In fact, one has an isomorphism $S^1 \times \mathbb{R}^+ \to \mathbb{C}^\times$ of real Lie groups given again by $(\varepsilon, r) \mapsto \varepsilon r$.

Example 1.73. Over our field $\mathbb{F}$, the set $\mathrm{GL}_n(\mathbb{F})$ of invertible $n \times n$ matrices is a Lie group. Indeed, it is an open subset of $\mathbb{F}^{n^2}$ and thus a manifold, and one can check that the inverse and multiplication maps are rational and hence smooth.

Example 1.74. Consider the collection of matrices

$$\mathrm{SU}_2 := \left\{A \in \mathrm{GL}_2(\mathbb{C}) : \det A = 1 \text{ and } AA^\dagger = 1_2\right\},$$

where $A^\dagger$ is the conjugate transpose. Then SU_2 is an embedded submanifold of $\mathrm{GL}_2(\mathbb{C})$ (cut out by the given equations) and also a subgroup. By writing out $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, one can write out our equations on the coefficients as

$$\begin{cases} ad - bc = 1, \\ a\overline{a} + b\overline{b} = 1, \\ a\overline{c} + b\overline{d} = 0, \\ c\overline{c} + d\overline{d} = 1. \end{cases}$$

In particular, we see that the vector $(a,b) \in \mathbb{C}^2$ is orthogonal to the vector $(\overline{c}, \overline{d})$, so we can solve for this line as providing some $\lambda \in \mathbb{C}$ such that $(\overline{c}, \overline{d}) = \lambda(-b, a)$. But then the determinant condition requires $\lambda = 1$ from $|a|^2 + |b|^2 = 1$. By expanding out $a = w + ix$ and $b = y + iz$, one finds that SU_2 is diffeomorphic to S^3.

The classical groups provide many examples of Lie groups over our field $\mathbb{F}$.

- One has $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_n(\mathbb{F})$, which are subsets of matrices cut out by the conditions $\det A \neq 0$ and $\det A = 1$, respectively.
- Orthogonal: fix a non-degenerate symmetric 2-form Ω on $\mathbb{F}^n$. One can always adjust our basis of $\mathbb{F}^n$ so that Ω is diagonal, and by adjusting our basis by squares, we may assume that Ω has only $+1$ or -1s on the diagonal. If $\mathbb{F} = \mathbb{C}$, we can in fact assume that $\Omega = 1_n$, and then we find that our group is

  $$\mathrm{O}_n(\mathbb{C}) := \{A : A^\intercal A = 1_n\}.$$

  Otherwise, if $\mathbb{F} = \mathbb{R}$, then our adjustment (and rearrangement) of the basis allows us to assume that Ω takes the form $\Omega_{k,n-k} := \operatorname{diag}(+1, \ldots, +1, -1, \ldots, -1)$ with k copies of $+1$ and $n-k$ copies of -1, and we define

  $$\mathrm{O}_{k,n-k}(\mathbb{R}) := \{A : A^\intercal \Omega A = \Omega\}$$

- Special orthogonal: one can add the condition that $\det A = 1$ to all the above orthogonal groups, which makes the special orthogonal groups.
- Symplectic: for $\mathbb{F}^{2n}$, one can fix a non-degenerate symplectic 2-form Ω. It turns out that, up to basis, we find that $\Omega = \left[\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right]$, and we define

  $$\mathrm{Sp}_{2n}(\mathbb{F}) := \{A : A^\intercal \Omega A = 1_{2n}\}.$$

- Unitary: using the non-degenerate Hermitian forms, we can similarly define

  $$\mathrm{U}_n(\mathbb{C}) := \left\{A : A^\dagger A = 1_n\right\}$$

  is a real Lie group. (Conjugation is not complex analytic, so this is not a complex Lie group!)

1.4.2 Connected Components

We will want to focus on connected Lie groups in this class, so we spend a moment describing why one might hope that this is a reasonable reduction. The main point is that it is basically infeasible to classify finite groups, and allowing for disconnected Lie groups forces us to include all these groups in our study by Example 1.69.

Quickly, recall our notions of connectivity; we refer to [Elb22, Appendix A.1] for details.

Definition 1.75 (connected)**.** A topological space X is *disconnected* if and only if there exists disjoint nonempty open subsets $U, V \subseteq X$ covering X. If there exists no such pair of open subsets, then X is *connected*; in other words, the only subsets of X which are both open and closed are $\varnothing$ and X.

Definition 1.76 (connected component)**.** Given a topological space X and a point $p \in X$, the *connected component* of $p \in X$ is the union of all connected subspaces of X containing p.

Remark 1.77. One can check that the connected component is in fact connected and is thus the maximal connected subspace.

Definition 1.78 (path-connected)**.** A topological space X is *path-connected* if and only if any two points $p, q \in X$ have some (continuous) path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Definition 1.79. Given a topological space X and a point $p \in X$, the *path-connected component* of $p \in X$ is the collection of all $q \in X$ for which there is a path connecting p and q.

Remark 1.80. One can check that having a path connecting two points of X is an equivalence relation on the points of X. Then the path-connected components are the equivalence classes for this equivalence relation. From this, one can check that the path-connected components are the maximal path-connected subsets of a topological space.

One has the following lemmas.

Lemma 1.81. Fix a topological space X.

(a) If X is path-connected, then X is connected.

(b) If X is a connected topological n-manifold, then X is path-connected.

Proof. Part (a) is [Elb22, Lemma A.16]. Part (b) is [Elb24, Proposition 1.39]. ■

Lemma 1.82. Fix a continuous surjection $f\colon X \to Y$ of topological spaces. If X is connected, then Y is connected.

Proof. This is [Elb22, Lemma A.8]. ■

Anyway, we are now equipped to return to our discussion of Lie groups.

Lemma 1.83. Fix a Lie group G, and let $G^\circ \subseteq G$ be the connected component of the identity $e \in G$. For any $g \in G$, we see that gG° is the connected component of g.

Proof. Certainly gG° is a connected subset containing g by Lemma 1.82 (note multiplication is continuous), so it is contained in the connected component of g. On the other hand, any connected subset U around g must have $g^{-1}U$ be a connected subset around e, so $g^{-1}U \subseteq G^\circ$, so $U \subseteq gG^\circ$. In particular, the connected component of g is also contained in gG°. ■

Proposition 1.84. Fix a Lie group G, and let $G^\circ \subseteq G$ be the connected component of the identity $e \in G$.

(a) Then G° is a normal subgroup of G.

(b) The quotient $\pi_0(G) := G/G^\circ$ given the quotient topology from the surjection $G \twoheadrightarrow \pi_0(G)$ is a discrete countable group.

Proof. We show the parts independently.

(a) We check this in parts.

- Of course G° is a subgroup: it contains the identity, and the images of the maps $i\colon G^\circ \to G$ and $m\colon G^\circ \times G^\circ \to G$ must land in connected subsets of G containing the identity by Lemma 1.82, so we see that G° is contained
- We now must check that G° is normal. Fix some $g \in G$, and we want to show that $gG^\circ g^{-1} \subseteq G^\circ$. Well, define the map $G^\circ \to G$ by $a \mapsto gag^{-1}$, which we note is continuous because multiplication and inversion are continuous. Lemma 1.82 tells us that the image must be connected, and we see $e \mapsto e$, so the image must actually land in G°.

(b) One knows that $\pi_0(G)$ is a group because G° is normal, and it is discrete because connected components are both closed and open in G, so the corresponding points are closed and open in $\pi_0(G)$. (We are implicitly using Lemma 1.83.) This is countable because a separated topological space must have countably many connected components. ■

Remark 1.85. One can restate the above result as providing a short exact sequence

$$1 \to G^\circ \to G \to \pi_0(G) \to 1$$

of Lie groups. In this way, we can decompose our study of G into connected Lie groups and discrete countable groups. In this course, we will ignore studying discrete countable groups because they are too hard.

1.5 September 9

Today we talk more about subgroups and coverings.

1.5.1 Closed Lie Subgroups

Arbitrary subgroups of Lie groups may not inherit a manifold structure, so we add an adjective to acknowledge this.

Definition 1.86 (closed Lie subgroup)**.** Fix a Lie group G. A *closed Lie subgroup* is a subgroup $H \subseteq G$ which is also an embedded submanifold.

Remark 1.87. On the homework, we will show that closed Lie subgroups are in fact closed subsets of G. It is a difficult theorem (which we will not prove nor use in this class) that being a closed subset and a subgroup implies that it is an embedded submanifold.

Here are some checks on subgroups.

Lemma 1.88. Fix a connected topological group G. Given an open neighborhood U of G of the identity $e \in G$, the group G is generated by U is all of G.

Proof. Let H be the subgroup generated by U. For each $h \in H$, we see that $hU \subseteq H$, which is an open neighborhood (see Lemma 1.67), so $H \subseteq G$ is open. However, we also see that

$$G = \bigsqcup_{[g]} gH,$$

where $[g]$ varies over representatives of cosets. Thus, $G \backslash H$ is again the union of open subsets, so H is also closed, so $G = H$ because G is connected. ■

Lemma 1.89. Fix a homomorphism $f\colon G_1 \to G_2$ of connected Lie groups. If $df_e\colon T_eG_1 \to T_eG_2$ is surjective, then f is surjective.

Proof. By translating around (by Lemma 1.67), we see that f is a submersion. (Explicitly, for each $g \in G_1$, we see that $R_{f(g)} \circ f$ must continue to be a submersion at the identity, but this equals $f \circ R_g$, so f is a submersion at g too.) Because submersions are open [Lee13, Proposition 4.28], we see that f being a submersion means that its image is an open subgroup of G_2, which is all of G_2 by Lemma 1.88. ■

Here is a check to be a closed Lie subgroup.

Lemma 1.90. Fix a regular Lie group G of dimension n. A subgroup $H \subseteq G$ is a closed Lie subgroup of dimension k if and only if there is a single regular chart (U, φ) with $e \in U$ such that

$$U \cap H = \{g \in U : \varphi_{k+1}(g) = \cdots = \varphi_n(g) = 0\}$$

for some.

Proof. We have constructed a slice chart for the identity $e \in H$. We will translate this slice chart around to produce a slice chart for arbitrary $h \in H$, which will complete the proof by [Lee13, Theorem 5.8]. In particular, for any $h \in H$, we know that left translation $L_{h^{-1}}\colon G \to G$ is a diffeomorphism, so the composite

$$hU \overset{L_{h^{-1}}}{\to} U \overset{\varphi}{\to} \varphi(U)$$

continues to be a chart of G with $h \in hU$. Furthermore, we see that $g \in hU$ lives in H if and only if $L_{h^{-1}}g \in U \cap H$, which by hypothesis is equivalent to

$$\varphi_{k+1}\left(L_{h^{-1}}g\right) = \cdots = \varphi_n\left(L_{h^{-1}}g\right) = 0.$$

Thus, we have constructed the desired slice chart. ■

We may want some more flexibility with our subgroups.

Example 1.91. Fix an irrational number $\alpha \in \mathbb{R}$. Then there is a Lie group homomorphism $f\colon \mathbb{R} \to (\mathbb{R}/\mathbb{Z})^2$ defined by $f(t) := (t, \alpha t)$. One can check that $\operatorname{im} f \subseteq (\mathbb{R}/\mathbb{Z})^2$ is a dense subgroup, but it is not closed!

So we have the following definition.

Definition 1.92 (Lie subgroup)**.** Fix a Lie group G. A *Lie subgroup* is a subgroup $H \subseteq G$ which is an immersed submanifold.

1.5.2 Quotient Groups

Along with subgroups, we want to be able to take quotients.

Definition 1.93 (fiber bundle)**.** A *fiber bundle* with fiber F on a smooth manifold X is a surjective continuous map $\pi\colon Y \to X$ such that there is an open cover $\mathcal{U}$ of X and (local) homeomorphisms making the following diagram commute for all $U \in \mathcal{U}$.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & F \times U \arrow[dl, "\mathrm{pr}_2"] \\ & U & \end{tikzcd}$$

Fiber bundles are the correct way to discuss quotients.

Theorem 1.94. Fix a closed Lie subgroup H of a Lie group G.

(a) Then G/H is a manifold of dimension $\dim G - \dim H$ equipped with a quotient map $q\colon G \twoheadrightarrow G/H$.

(b) In fact, q is a fiber bundle with fiber H.

(c) If H is normal in G, then G/H is actually a Lie group (with the usual group structure).

(d) We have $T_e(G/H) \cong T_eG/T_eH$.

Proof. We construct the manifold structure on G/H as follows: for each $g \in G$, we produce a coset $\overline{g} \in G/H$, which we note as $q^{-1}(\overline{g}) = gH$. Now, $gH \subseteq G$ is an embedded submanifold because H is (we are using Lemma 1.67), so one can locally find a submanifold $M \subseteq G$ around g intersecting gH transversally, meaning that

$$T_gG = T_gM \oplus T_g(gH).$$

By shrinking M, we can ensure that the above map continues to be an isomorphism in a neighborhood of g, so the multiplication map $M \times H \to UH$ is a diffeomorphism. Now, MH is an open neighborhood of $g \in G$, and M projects down to an open subset of G/H, so $M \cong q(\overline{M})$ provides our chart.

Now, (a) and (d) follows by inspection of the construction. We see that (b) follows because we built our projection map $G \twoheadrightarrow G/H$ so that it locally looks like $U \times H \twoheadrightarrow \overline{U}$, so we get our fiber bundle. Lastly, (d) follows by the equality $T_gG = T_gM \oplus T_g(gH)$. ■

Remark 1.95. Writing the above out in detail would take several pages; see [Lee13, Theorem 21.10].

Access to quotients permits an isomorphism theorem, which we will prove later when we have talked a bit about Lie algebras.

Theorem 1.96 (Isomorphism)**.** Fix a Lie group homomorphism $f\colon G_1 \to G_2$.

(a) The kernel $\ker f$ is a normal closed Lie subgroup of G_1.

(b) The quotient $G_1/\ker f$ is a Lie subgroup of G_2.

(c) The image $\operatorname{im} f$ is a Lie subgroup of G_2. If $\operatorname{im} f$ is further closed, then $G_1/\ker f \to \operatorname{im} f$ is an isomorphism of Lie subgroups.

1.5.3 Actions

Groups are known by their actions, so let's think about how our actions behave.

Definition 1.97 (action)**.** Fix a Lie group G and regular manifold X. Then a *regular action* of G on X is a regular map $\alpha\colon G \times X \to X$ satisfying the usual constraints, as follows.

(a) Identity: $\alpha(e, x) := x$.

(b) Composition: $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$.

This allows us to define the usual subsets.

Definition 1.98 (orbit, stabilizer)**.** Fix a regular action of a Lie group G on a regular manifold X.

(a) The *orbit* of $x \in X$ is the subspace $Gx := \{gx : g \in G\}$.

(b) The *stabilizer* of $x \in X$ is the subgroup $G_x := \{g \in G : gx = x\}$.

Here are some examples.

Example 1.99. The group $\mathrm{GL}_n(\mathbb{F})$ acts on the vector space $\mathbb{F}^n$.

Example 1.100. The group $\mathrm{SO}_3(\mathbb{R})$ preserves distances in its action on $\mathbb{R}^3$, so its action descends to an action on S^2.

Representations are special kinds of actions.

Definition 1.101 (representation)**.** Fix a Lie group G over $\mathbb{F}$. Then a *representation* of G is the (regular) linear action of G on a finite-dimensional vector space V over $\mathbb{F}$; namely, the map $v \mapsto g \cdot v$ for each $g \in G$ must be a linear map $V \to V$. A *homomorphism* of representations V and W is a linear map $A\colon V \to W$ such that $A(gv) = g(Av)$. These objects and morphisms make the category $\mathrm{Rep}_{\mathbb{F}}(G)$.

Remark 1.102. Equivalently, we may ask for the induced map $G \to \mathrm{GL}(V)$, given by sending $g \in G$ to the map $v \mapsto gv$, to be a Lie group homomorphism.

Remark 1.103. The category $\mathrm{Rep}_{\mathbb{F}}(G)$ comes with many nice operations.

(a) Duals: given a representation $\pi\colon G \to \mathrm{GL}(V)$, we can induce a G-action on $V^* := \mathrm{Hom}(V, \mathbb{F})$ by

$$\big((\pi^* g)(v^*)\big)(v) := v^*\left(g^{-1}v\right).$$

(Here, $\pi^* g$ should be a map $V^* \to V^*$, so it takes a linear functional $v^* \in V^*$ as input and produces the linear functional $(\pi^* g)(v^*)$ as output.)

(b) Tensor products: given representations $\pi\colon G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, we can induce a G-action on $V \otimes W$ by

$$\big((\pi \otimes \pi')(g \otimes g')\big)(v \otimes v') := \pi(g)v \otimes \pi'(g')v'.$$

(c) Hom sets: given representations $\pi\colon\ G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, we can induce a G-action on $\mathrm{Hom}(V, W)$ by

$$(g\varphi)(v) := \pi'(g)\left(\varphi(\pi(g)^{-1}v)\right).$$

(d) Quotients: given representations $\pi\colon G \to \mathrm{GL}(V)$ and $\pi'\colon G \to \mathrm{GL}(V')$, where $V \subseteq V'$ is a G-representation, then we can induce a G-action on V'/V by

$$\pi'(g)(v' + V) := gv' + V.$$

One can check that these operations make $\mathrm{Rep}_{\mathbb{F}}(G)$ into a symmetric monoidal abelian category. Checking that these actually form actions is a matter of writing out the definitions, so we will omit it. (Notably, all of these actions are algebraic combinations of previous actions, so all regularity is inherited.)

Returning to group actions on manifolds, we remark that Theorem 1.96 can be seen as a version of the Orbit–stabilizer theorem.

Theorem 1.104 (Orbit–stabilizer)**.** Fix a regular action of a Lie group G on a regular manifold X. Further, fix $x \in X$.

(a) The orbit Gx is an immersed submanifold of X.

(b) The stabilizer G_x is a closed Lie subgroup of G.

(c) The quotient map $f\colon G/G_x \to X$ given by $g \mapsto gx$ is an injective immersion.

(d) If Gx is an embedded submanifold, then the map f of (c) is a diffeomorphism.

1.6 September 11

Today we talk more about homogeneous spaces.

1.6.1 Homogeneous Spaces

Let's see some applications of Theorem 1.104.

Example 1.105. Suppose a regular Lie group G acts smoothly and transitively on a regular manifold X. For each $x \in X$, we see that $G/G_x \to X$ is a bijective immersion. In particular, Sard's theorem implies that $\dim G/G_x = \dim X$, so we conclude that this map is in fact a bijective local diffeomorphism, which of course is just a diffeomorphism. Thus, Theorem 1.94 tells us that the map $G \to X$ given by $g \mapsto gx$ is a fiber bundle with fiber G_x.

The above situation is so nice that it earns a name.

Definition 1.106 (homogeneous space)**.** Fix a regular Lie group G acting smoothly and transitively on a regular manifold X. If the action of G is transitive, we say that X is a *homogeneous space* of G.

Here are many examples.

Example 1.107. Continuing from Example 1.100, we recall that $\mathrm{SO}_3(\mathbb{R})$ acts on S^2. In fact, one can check that the stabilizer of any $x \in S^3$ is isomorphic to S^1, so Example 1.105 tells us that $\mathrm{SO}_3(\mathbb{R}) \to S^2$ is a fiber bundle with fiber S^1. In general, we find that $\mathrm{SO}_n(\mathbb{R}) \to S^n$ is a fiber bundle with fiber $\mathrm{SO}_{n-1}(\mathbb{R})$.

Example 1.108. The group SU_2 acts on $\mathbb{CP}^1$ by matrix multiplication. We see that the stabilizer of some line in $\mathbb{CP}^1$ consists of the matrices in SU_2 with a nonzero eigenvector on the line. For example, using the computation of Example 1.74, we see that trying to stabilize $[1:0]$ gives rise to the matrices $\left[\begin{smallmatrix} a & -b \\ b & \overline{a} \end{smallmatrix}\right]$ where we require $b = 0$. Thus, we see that our stabilizer is isomorphic to U_1. In particular, our orbits are compact immersed submanifolds of $\mathbb{CP}^1$ of dimension $\dim \mathrm{SU}_2 - \dim \mathrm{U}_1 = \dim \mathbb{CP}^1$, so the action must be transitive in order for orbits to be closed and the correct dimension.

Example 1.109. One can check that SU_n acts on $S^{2n-1} \subseteq \mathbb{C}^n$ with stabilizer isomorphic to SU_{n-1}.

Example 1.110 (flag varieties)**.** Let $\mathcal{F}_n$ be the set of "flags" of $\mathbb{F}^n$, which is an ascending chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{F}^n.$$

Then we see that $\mathrm{GL}_n(\mathbb{F})$ acts on $\mathcal{F}_n$ by matrix multiplication. On the homework, we check that this action is transitive with stabilizer (of the standard flag $\{\operatorname{span}_{\mathbb{F}}(e_1, \ldots, e_i)\}_{i=0}^n$) given by the matrix subgroup $B_n(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ of upper-triangular matrices. Thus, we see that we can realize $\mathcal{F}_n$ as the manifold quotient $\mathrm{GL}_n(\mathbb{F})/B_n(\mathbb{F})$, providing a manifold structure.

Example 1.111 (Grassmannians)**.** Let $\mathrm{Gr}_k(\mathbb{F}^n)$ be the set of vector subspaces $V \subseteq \mathbb{F}^n$ of dimension k. Then we see that $\mathrm{GL}_n(\mathbb{F})$ acts transitively on $\mathrm{Gr}_k(\mathbb{F}^n)$ with stabilizer of $\operatorname{span}(e_1, \ldots, e_k)$ given by matrices of the form

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where $A \in \mathbb{F}^{k \times k}$ and $B \in \mathbb{F}^{k \times (n-k)}$ and $D \in \mathbb{F}^{(n-k)\times(n-k)}$. Thus, we can realize $\mathrm{Gr}_k(\mathbb{F}^n)$ as the manifold quotient of $\mathrm{GL}_n(\mathbb{F})$, providing a manifold structure.

Example 1.112. There are many regular actions of G on itself.

- Regular left: define our action $R_\ell \colon G \times G \to G$ by $(g, x) \mapsto gx$.
- Regular right: define our action $R_r \colon G \times G \to G$ by $(g, x) \mapsto xg^{-1}$.
- Adjoint: define our action $\mathrm{Ad} \colon G \times G \to G$ by $(g, x) \mapsto gxg^{-1}$. (This action is rarely transitive!)

Example 1.113 (adjoint). Fix a regular Lie group G. Note that $\mathrm{Ad}_g(1) = 1$, so we may take the differential to provide a map $(d\,\mathrm{Ad}_g)_e \colon T_eG \to T_eG$, so we get an adjoint representation

$$\begin{aligned} G \times T_eG &\to T_eG \\ (g\ ,\ \ v) &\mapsto (d\,\mathrm{Ad}_g)_e(v) \end{aligned}$$

which we will frequently abuse notation to abbreviate the above as providing some $\mathrm{Ad}_g \in \mathrm{GL}(T_eG)$. Expanding everything in sight into coordinates reveals that this action is smooth; in fact, one can check (again in coordinates) that the map $G \to \mathrm{GL}(T_eG)$ given by $g \mapsto (d\,\mathrm{Ad}_g)_e$ is smooth. Taking the differential of this last map produces a map $T_eG \to \mathrm{End}(T_eG)$, which is sometimes called the adjoint representation of T_eG.

1.6.2 Covering Spaces

It will help to recall some theory around covering spaces. See [Elb23] for (some) more detail about this theory or [Hat01] for (much) more detail.

Definition 1.114 (covering space). A *covering space* is a fibration $p \colon Y \to X$ with discrete fiber S. The *degree* of p equals $\#S$.

In more words, we are asking for each $x \in X$ to have an open neighborhood U such that the restriction $p^{-1}(U) \to U$ is homeomorphic (over U) to $\bigsqcup_{s \in S} p^{-1}(U) \to U$ for some discrete set S.

Remark 1.115. If X is a regular manifold and $\deg p \leqslant |\mathbb{N}|$, then Y is also a regular manifold. Indeed, being a manifold is checked locally, so one can find neighborhoods as in the previous remark to witness the manifold structure.

We are interested in paths in topological spaces, but there are too many. To make this set smaller, we consider it up to homotopy.

Definition 1.116 (homotopy). Fix a topological space X. Two paths $\gamma_0, \gamma_1 \colon [0,1] \to X$ are *homotopic* relative to their endpoints if and only if there is a continuous map $H_\bullet \colon [0,1]^2 \to X$ such that $H_0 = \gamma_0$ and $H_1 = \gamma_1$ and $H_s(0) = \gamma_0(0) = \gamma_1(0)$ and $H_s(1) = \gamma_0(1) = \gamma_1(1)$ for all s. The map H is called a *homotopy*.

Definition 1.117 (simply connected). A topological space X is *simply connected* if and two paths with the same endpoints are homotopic relative to those endpoints.

Example 1.118. One can check that S^1 fails to be simply connected because the path going around the circle is not homotopic to the constant path.

It is important to know that one can lift paths.

Theorem 1.119. Fix a covering space $p \colon Y \to X$. Fix some point $x \in X$ and a path $\gamma \colon [0,1] \to X$ with $\gamma(0) = x$. Then each $\tilde{x} \in p^{-1}(\{x\})$ has a unique path $\tilde{\gamma} \colon [0,1] \to Y$ such that $\tilde{\gamma}(0) = \tilde{x}$ and making the following diagram commute.

$$\begin{tikzcd} {[0,1]} \arrow[r, "\tilde{\gamma}"] \arrow[dr, "\gamma"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

Remark 1.120. One can further check that having two homotopic paths $\gamma_1 \sim \gamma_2$ downstairs produce homotopic paths $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$.

Remark 1.121. More generally, fix a simply connected topological space Z. Then given a map $f: Z \to X$ and a choice of $\widetilde{x} \in p^{-1}(\{x\})$ and $z \in f^{-1}(\{x\})$, there will be a unique lift $\widetilde{f}: Z \to Y$ such that $\widetilde{f}(z) = \widetilde{x}$. In short, given any $z' \in Z$, find a path connecting z and z', send this path into X and then lift it up to Y. Because Z is simply connected (and the above theorem), the choice of path from z to z' does not really matter.

Anyway, we now define our collection of paths.

Definition 1.122 (fundamental group)**.** Fix a point x of a topological space X. Then the set of paths both of whose endpoints are x forms a monoid with operation given by composition (i.e., concatenation). If we take the quotient of this monoid by homotopy classes of paths, then we get a group of path homotopy classes, which we call $\pi_1(X, x)$. This is the *fundamental group*.

Remark 1.123. For any two $x, y \in X$ in the same path-connected component, the path $\alpha: [0,1] \to X$ connecting x to y produces an isomorphism $\pi_1(X, x) \cong \pi_1(X, y)$ by $\gamma \mapsto \alpha \cdot \gamma \cdot \alpha^{-1}$, where $\cdot$ denotes path composition.

Remark 1.124. The above remark allows us to verify that X is simply connected if and only $\pi_1(X, x)$ is trivial for all x. In fact, we only have to check this for one x in each path-connected component.

1.7 September 13

Today we continue our discussion of coverings.

1.7.1 The Universal Cover

There is more or less one covering space which produces all the other ones.

Definition 1.125 (universal cover)**.** Fix a path-connected topological space X. Then a covering space $p: Y \to X$ is the *universal cover* if and only if Y is connected and simply connected.

We now discuss an action of $\pi_1(X, b)$ on covering spaces in order to better understand this universal cover. Fix a covering space $p: Y \to X$ and a basepoint $x \in X$. Then we note that $\pi_1(X, x)$ acts on the fiber $p^{-1}(\{x\})$ as follows: for any $[\gamma] \in \pi_1(X, x)$ and $\widetilde{x} \in p^{-1}(\{x\})$, we define $\widetilde{\gamma}: [0,1] \to Y$ by lifting the path $\gamma: [0,1] \to X$ up to Y so that $\widetilde{\gamma}(0) = \widetilde{x}$; then

$$[\gamma] \cdot \widetilde{x} := \widetilde{\gamma}(1).$$

One can check that this action is well-defined (namely, it does not depend on the representative γ and does provide a group action). Here are some notes.

- If Y is path-connected, then the action is transitive: and $\widetilde{x}, \widetilde{x}' \in p^{-1}(\{x\})$ admit a path $\widetilde{\gamma}: [0,1] \to Y$ with $\widetilde{\gamma}(0) = \widetilde{x}$ and $\widetilde{\gamma}(1) = \widetilde{x}'$, so $\gamma := p \circ \widetilde{\gamma}$ has

$$[\gamma] \cdot \widetilde{x} := \widetilde{x}'$$

  by construction of γ.

- If Y is simply connected, then this action is also free. Indeed, choose two paths $\gamma_1, \gamma_2 : [0,1] \to X$ representing classes in $\pi_1(X,x)$. Now, suppose that $[\gamma_1] \cdot \widetilde{x} = [\gamma_2] \cdot \widetilde{x}$ for each $\widetilde{x} \in p^{-1}(\{x\})$, and we will show that $[\gamma_1] = [\gamma_2]$. Well, choosing lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, the hypothesis implies that they have the same endpoints. Thus, because Y is simply connected, we know $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$. We now see that $\gamma_1 \sim \gamma_2$ by composing the homotopy witnessing $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$ with p.

The conclusion is that $p^{-1}(\{x\})$ is in bijection with $\pi_1(X,x)$ when $p\colon Y \to X$ is the universal cover. Here are some examples.

Example 1.126. One has a covering space $p\colon S^n \to \mathbb{RP}^n$ given by

$$(x_0, \ldots, x_n) \mapsto [x_0 : \ldots : x_n].$$

For $n \geqslant 2$, we know S^n is simply connected, so it will be the universal cover, and we are able to conclude that $\pi_1(\mathbb{RP}^n)$ is isomorphic to a fiber of p, which has two elements, so $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.127. One has a covering space $p\colon \mathbb{R} \to S^1$ given by $p(t) := e^{2\pi i t}$. We can see that $\mathbb{R}$ is simply connected (it's convex), so this is a universal covering. This at least tells us that $\pi_1\left(S^1\right)$ is countable, and one can track through the group law through the above bijections to see that actually $\pi_1\left(S^1\right) \cong \mathbb{Z}$.

Example 1.128. One can show that $\pi_1(\mathbb{C} \backslash \{z_1, \ldots, z_n\})$ is the free group on n generators, basically corresponding to how one goes around each point.

Now, in the context of our Lie groups, we get the following result.

Theorem 1.129. Fix a regular Lie group G, and let $p\colon \widetilde{G} \to G$ be the universal cover.

(a) Then $\widetilde{G}$ has the structure of a regular Lie group.

(b) The projection p is a homomorphism of Lie groups.

(c) The kernel $\ker p \subseteq \widetilde{G}$ is discrete, central, and isomorphic to $\pi_1(G,e)$. In particular, $\pi_1(G,e)$ is commutative.

Proof. Here we go.

(a) Remark 1.115 tells us that $\widetilde{G}$ is a regular manifold, so it really only remains to exhibit the group structure. We will content ourselves with merely describing the group structure. Fix any $\widetilde{e} \in p^{-1}(\{e\})$, which will be our identity.

Now, $\widetilde{G}$ is simply connected, so $\widetilde{G} \times \widetilde{G}$ is also simply connected. Thus, Remark 1.115 explains that the composite

$$\widetilde{G} \times \widetilde{G} \to G \times G \xrightarrow{m} G$$

will lift to a unique map to the universal cover as a map $\widetilde{m}$ making the following diagram commute.

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, "\widetilde{m}"] \arrow[d, "{(p,p)}"'] & \widetilde{G} \arrow[d, "p"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} (\widetilde{e}, \widetilde{e}) \arrow[r, mapsto] \arrow[d, mapsto] & \widetilde{e} \arrow[d, mapsto] \\ (e,e) \arrow[r, mapsto] & e \end{tikzcd}$$

One can construct the inverse map similarly by lifting the map $\widetilde{G} \xrightarrow{p} G \xrightarrow{i} G$ to a map to $\widetilde{G}$ sending $\widetilde{e} \mapsto \widetilde{e}$. Uniqueness of lifting will guarantee that we satisfy the group law.

(b) We see that p is a homomorphism by construction of $\widetilde{m}$ above.

(c) This is on the homework. ■

Example 1.130. Recall that we have the fiber bundle $\mathrm{SO}_n(\mathbb{R}) \to S^{n-1}$ with fiber $\mathrm{SO}_{n-1}(\mathbb{R})$. Thus, the long exact sequence in homotopy groups produces

$$\pi_2\left(S^{n-1}\right) \to \pi_1(\mathrm{SO}_{n-1}(\mathbb{R})) \to \pi_1(\mathrm{SO}_n(\mathbb{R})) \to \pi_1\left(S^{n-1}\right) \to \pi_0\left(\mathrm{SO}_{n-1}(\mathbb{R})\right).$$

Now, for $n \geqslant 4$, one has that $\pi_2\left(S^{n-2}\right) = \pi_1\left(S^{n-1}\right) = 1$, so we have $\pi_1\left(\mathrm{SO}_{n-1}(\mathbb{R})\right) \cong \pi_1\left(\mathrm{SO}_n(\mathbb{R})\right)$. One can check that $\mathrm{SO}_3(\mathbb{R}) \cong \mathbb{RP}^3$, so we see that

$$\pi_1\left(\mathrm{SO}_n(\mathbb{R})\right) \cong \mathbb{Z}/2\mathbb{Z}$$

for $n \geqslant 4$. The universal (double) cover of $\mathrm{SO}_n(\mathbb{R})$ is called Spin_n, and Theorem 1.129 explains that we have a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n \to \mathrm{SO}_n(\mathbb{R}) \to 1.$$

Example 1.131. More concretely, one can show that $\mathrm{SU}_2(\mathbb{C})$ has an action on $\mathbb{R}^3$ preserving distances and orientation, so we get a homomorphism $\mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$. One can check that this map is surjective with kernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In general, Theorem 1.129 explains that we have a short exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$$

for any regular Lie group G, so it does not cost us too much to pass from G to $\widetilde{G}$, allowing us to assume that the Lie groups we study are simply connected. (Note that even though $\pi_1(G)$ is discrete, the short exact sequence does not split: $\widetilde{G}$ succeeds at being connected.)

1.7.2 Vector Fields

Fix a regular manifold X of dimension n. We may be interested in thinking about all our tangent spaces at once.

Definition 1.132 (tangent bundle)**.** Fix a regular manifold X. Then we define the *tangent bundle* as

$$TX := \{(x, v) : v \in T_x X\}.$$

Note that there is a natural projection map $TX \to X$ by $(x, v) \mapsto x$.

Remark 1.133. Locally on a chart (U, φ) of X, we see that φ provides coordinates $(x_1, \ldots, x_n)$ on U, so one has a bijection

$$U \times \mathbb{R}^n \to TU$$

by sending $(x, \partial/\partial x_i) \mapsto \left(\varphi^{-1}(x), d\varphi_x^{-1}(\partial/\partial x_i)\right)$ (In the future, we may conflate $d\varphi_x^{-1}(\partial/\partial x_i)$ with $\partial/\partial x_i$). This provides a chart for TU, and one can check that these charts are smoothly compatible by an explicit computation using the smooth compatibility of charts on X. The point is that $TX \to X$ is a vector bundle of rank n.

Vector bundles are interesting because of their sections.

Definition 1.134 (vector field)**.** Fix a regular manifold X. Then a *vector field* on X is a smooth section $\sigma\colon X \to TX$ of the natural projection map $TX \to X$.

Remark 1.135. Locally on a chart (U, φ) with coordinates $(x_1, \ldots, x_n)$, we see that we can think about a vector field σ locally as

$$\sigma(x) := \sum_{i=1}^{n} \sigma_i(x) \frac{\partial}{\partial x_i}\bigg|_x,$$

where the smoothness of σ enforces the $\sigma_\bullet$s to be smooth. Changing coordinates to (U', φ') with a coordinate expansion $\sigma(x) = \sum_i \sigma_i'(x) \frac{\partial}{\partial x_i'}$, one can change bases using the Jacobian of $\varphi' \circ \varphi^{-1}$ to find that

$$\sigma_i'(x) = \sum_{j=1}^{n} \frac{\partial x_i'}{\partial x_j'} \sigma_j(x).$$

Anyway, the point is that we can define a vector field locally on these coordinates and then going back and checking that we have actually defined something that will glue smoothly up to X.

The reason we care so much about tangent spaces in this class is because they give rise to our Lie algebras, whose representations are somehow our main focus.

Definition 1.136 (Lie algebra)**.** Fix a Lie group G. Then the *Lie algebra* of G is the vector space

$$\mathfrak{g} := T_e G.$$

We may also notate $\mathfrak{g}$ by $\operatorname{Lie}(G)$.

It is somewhat difficult to find structure in this tangent space immediately, so we note that $T_e G$ is isomorphic with another vector space.

Definition 1.137 (invariant vector field)**.** Fix a Lie group G. Then a vector field $\xi \colon G \to TG$ is *left-invariant* if and only if

$$\xi(gx) = dL_g(\xi(x))$$

for any $x, g \in G$. One can define *right-invariant* analogously.

Remark 1.138. We claim that the vector space of left-invariant vector fields is isomorphic to $\mathfrak{g}$. Here are our maps.

- Given a left-invariant vector field ξ, one can produce the tangent vector $\xi(e) \in \mathfrak{g}$.
- Given some $\xi(e) \in \mathfrak{g}$, we define

  $$\xi(g) := dL_g(\xi(e)) \in T_g G.$$

  It is not difficult to check that $\xi \colon G \to TG$ is at least a section of the natural projection $TG \to G$. We omit the check that ξ is smooth because it is somewhat involved.

Remark 1.139. As an aside, we note that the produced left-invariant vector fields parallelizes G after providing a basis of $\mathfrak{g}$; in particular, one has a canonical isomorphism $TG \cong G \times \mathfrak{g}$. One can actually show that TG is a Lie group with Lie group structure given by functoriality of the tangent bundle applied to the group operations of G, and one finds that $TG \cong G \rtimes \mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint action.

Next class we will go back and argue that our classical groups are actually Lie groups and compute their Lie algebras.

1.8 September 16

Today we will talk about Lie algebras of classical groups.

1.8.1 The Exponential Map: The Classical Case

Let's work through our examples by hand. Recall that our classical groups are our subgroups of $\mathrm{GL}_n(\mathbb{F})$ cut out by equations involving $\det$ and preserving a bilinear/sesquilinear form (symmetric, symplectic, or Hermitian).

Example 1.140. We show that $\mathrm{GL}_n(\mathbb{F})$ is a Lie group over $\mathbb{F}$ and compute its Lie algebra.

Proof. Note $\mathrm{GL}_n(\mathbb{F})$ is an open submanifold of $\mathrm{M}_n(\mathbb{F}) \cong \mathbb{F}^{n\times n}$. Matrix multiplication and inversion are rational functions of the coordinates and hence smooth, so $\mathrm{GL}_n(\mathbb{F})$ succeeds at being a Lie group. Lastly, we see that being open implies that our tangent space is

$$T_e \mathrm{GL}_n(\mathbb{F}) \cong T_e \mathrm{M}_n(\mathbb{F}) \cong \mathbb{F}^n,$$

as required. ■

We will postpone the remaining computations until we discuss the exponential. For these computations, we want the exponential map.

Definition 1.141 (exponential)**.** For $X \in \mathfrak{gl}_n(\mathbb{F})$, we define the exponential map $\exp\colon \mathfrak{gl}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{F})$ by

$$\exp(X) := \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Note that $\exp$ is an isomorphism at the identity, so the Inverse function theorem provides a smooth "local" inverse $\log(1_n + X)$ defined in an open neighborhood of 1_n. In fact, one can formally compute that

$$\log(1_n + X) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}.$$

We run a few small checks.

Remark 1.142. Note $\exp(0) = 1$. In fact, one can check that $d\exp_0(A) = A$ for any $A \in \mathfrak{gl}_n(\mathbb{F})$ by taking the derivative term by term.

Remark 1.143. We also see that $\exp\left(AXA^{-1}\right) = A\exp(X)A^{-1}$ and $\exp(X^\intercal) = \exp(X)^\intercal$ and $\exp(X^\dagger) = \exp(X)^\dagger$ by a direct expansion.

What's important about $\exp$ is the following multiplicative property.

Lemma 1.144. Fix $X, Y \in \mathfrak{gl}_n(\mathbb{F})$ which commute. Then

$$\exp(X+Y) = \exp(X)\exp(Y).$$

Proof. We check this in formal power series. Because everything in sight converges, this is safe. The main point is to just expand everything. Indeed,

$$\begin{aligned}\exp(X+Y) &= \sum_{k=0}^{\infty} \frac{(X+Y)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{a+b=k}\binom{k}{a}X^aY^b\right) \\ &= \sum_{a,b=0}^{\infty} \frac{1}{(a+b)!}\cdot\frac{(a+b)!}{a!b!}X^aY^b \\ &= \sum_{a,b=0}^{\infty} \frac{X^a}{a!}\cdot\frac{Y^b}{b!} \\ &= \exp(X)\exp(Y),\end{aligned}$$

as required. ■

Remark 1.145. For fixed X, the previous point implies that the map $\mathbb{F} \to \mathrm{GL}_n(\mathbb{F})$ given by $t \mapsto \exp(tX)$ is a Lie group homomorphism. (Smoothness is automatic by smoothness of $\exp$.) The image of this map is called the "1-parameter subgroup" generated by X.

Remark 1.146. Taking inverses shows that $\log(XY) = \log X + \log Y$ for X and Y close enough to the identity.

Here is another check which is a little more interesting.

Lemma 1.147. Fix $X \in \mathfrak{gl}_n(\mathbb{F})$. Then

$$\det\exp(X) = \exp(\operatorname{tr} X).$$

Proof. The computations do not change if we extend the base field, so we may work over $\mathbb{C}$ everywhere. Thus, we may assume that X is upper-triangular by conjugating (see Remark 1.143) say with diagonal entries $\{d_1, \ldots, d_n\}$. Now, for any $k \geqslant 0$, any X^k continues to be upper-triangular with diagonal entries $\{d_1^k, \ldots, d_n^k\}$. Thus, we see that $\exp(X)$ is upper-triangular with diagonal entries $\{\exp(d_1), \ldots, \exp(d_n)\}$, so

$$\begin{aligned}\det\exp(X) &= \exp(d_1)\cdots\exp(d_n) && (1.1)\\ &= \exp(d_1 + \cdots + d_n) && (1.2)\\ &= \exp(\operatorname{tr} X), && (1.3)\end{aligned}$$

as required. ■

1.8.2 The Classical Groups

For our classical groups, we will show the following result.

Theorem 1.148. For each classical group $G \subseteq \mathrm{GL}_n(\mathbb{F})$, there will exist a vector subspace $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$ (which can be identified with T_eG via the embedding $G \subseteq \mathrm{GL}_n(\mathbb{F})$) and open neighborhoods of the identity $U \subseteq \mathrm{GL}_n(\mathbb{F})$ and $\mathfrak{u} \subseteq \mathfrak{g}$ such that $\exp\colon (U \cap G) \to (\mathfrak{u} \cap \mathfrak{g})$ is a local isomorphism.

Before engaging with the examples, we note the following corollary.

Corollary 1.149. For each classical group G, we see that G is a Lie group with $T_eG = \mathfrak{g}$ and $\dim G = \dim \mathfrak{g}$.

Proof. It suffices to provide a slice chart of the identity for $G \subseteq \mathrm{GL}_n(\mathbb{F})$; we then get slice charts everywhere by translation. Well, ■

Let's now proceed with our examples. We begin with some general remarks.

Lemma 1.150. Let $G \subseteq \mathrm{GL}_n(\mathbb{F})$ be a closed Lie subgroup, and let $SG := \{g \in G : \det g = 1\}$. We show SG is a Lie subgroup and compute $T_1SG \subseteq T_1G$ as

$$T_1SG = \{g \in T_1G : \operatorname{tr} g = 0\}.$$

Proof. Let G act on $\mathbb{F}$ by $\mu\colon G \times \mathbb{F} \to \mathbb{F}$ by $\mu(g,c) := (\det g)c$. Note that μ is a polynomial and hence regular, so this is a regular action upon checking that $\mu(1,c) = c$ and $\mu(g,\mu(h,c)) = \mu(gh,c)$, which hold because $\det$ is a homomorphism.

Now, the stabilizer of $1 \in \mathbb{F}$ consists of the $g \in G$ such that $(\det g) \cdot 1 = 1$, which is equivalent to $\det g = 1$ and hence equivalent to $g \in SG$. Thus, $SG \subseteq G$ is a closed Lie subgroup with

$$T_1SG(\mathbb{F}) = \{v \in T_1G : (d\det)_1(v) = 0\},$$

where $\det\colon G \to \mathbb{F}$ is the determinant map. To compute $(d\det)_1(v)$, we identify $T_1G \subseteq T_1\,\mathrm{GL}_n(\mathbb{F}) = T_1M_n(\mathbb{F}) \cong M_n(\mathbb{F})$; then for any $X \in M_n(\mathbb{F})$, we note that the path $\gamma\colon \mathbb{R} \to M_n(\mathbb{R})$ defined by $\gamma(t) := 1 + tX$ has $\gamma(0) = 1$ and $\gamma'(0) = X$, so

$$(d\det)_1(X) = (d\det)_1(\gamma'(0)) = (\det \circ \gamma)'(0) = \frac{d}{dt}\det(1+tX)\bigg|_{t=0}.$$

Thus, we are interested in the linear terms of the polynomial $\det(1+tX)$. Now, writing X out in coordinates as $X = [X_{ij}]_{1 \leqslant i,j \leqslant n}$ and setting $A_{ij} = 1_{i=j} + tX_{ij}$, we note

$$\det(1+tX) = \det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}.$$

Now, the only way a summand can produce linear terms is if there is at most one non-diagonal entry A_{ij}, which of course forces all entries to be diagonal. Thus,

$$\frac{d}{dt}\det(1+tX)\bigg|_{t=0} = \frac{d}{dt}(1+tX_{11})\cdots(1+tX_{nn})\bigg|_{t=0} \stackrel{*}{=} (X_{11} + \cdots + X_{nn}) = \operatorname{tr} X,$$

where $\stackrel{*}{=}$ holds by an expansion of the terms looking for linear terms. Thus,

$$T_1SG = \{X \in T_1G : \operatorname{tr} X = 0\}.$$

■

Lemma 1.151. Let $J \in M_n(\mathbb{F})$ be some matrix, and let $(-)^*$ denote either of the involutions $(-)^\intercal$ or $(-)^\dagger$. Then one has the subgroup

$$O_J(\mathbb{F}) := \{g \in \mathrm{GL}_n(\mathbb{F}) : g^*Jg = J\}.$$

We claim that $O_J(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ is a closed Lie subgroup (though if $(-)^* = (-)^\dagger$ and $\mathbb{F} = \mathbb{C}$, then $O_J(\mathbb{F})$ is a group over $\mathbb{R}$) and compute that

$$T_1O_J(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^*J + JX = 0\}.$$

Proof. Indeed, let $\mathrm{GL}_n(\mathbb{F})$ act on $M_n(\mathbb{F})$ by $\mu(g, A) := g^*Ag$. This (right!) action is polynomial and hence regular (with the previous parenthetical in mind), and we can check that it is an action because $\mu(1, A) = A$ and $\mu(g, \mu(h, A)) = g^*h^*Ahg = \mu(hg, A)$.

Now, the stabilizer of $J \in M_n(\mathbb{F})$ is precisely $O_J(\mathbb{F})$ by definition, so $O_J(\mathbb{F}) \subseteq \mathrm{GL}_n(\mathbb{F})$ is in fact a closed Lie subgroup. We also go ahead and compute $T_1O_J(\mathbb{F})$. Letting $f\colon \mathrm{GL}_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined by $f(g) := g^*Jg$, we see that

$$T_1O_J(\mathbb{F}) = \ker df_1,$$

so we want to compute df_1. As usual, we identify $T_1G \subseteq T_1\,\mathrm{GL}_n(\mathbb{F}) = T_1M_n(\mathbb{F}) \cong M_n(\mathbb{F})$; then for any $X \in M_n(\mathbb{F})$, we note that the path $\gamma\colon \mathbb{R} \to M_n(\mathbb{R})$ defined by $\gamma(t) := 1 + tX$ has $\gamma(0) = 1$ and $\gamma'(0) = X$, so

$$df_1(X) = df_1(\gamma'(0)) = (f \circ \gamma)'(0) = \frac{d}{dt} f(1 + tX)\bigg|_{t=0}.$$

Thus, we go ahead and compute

$$f(1 + tX) = (1 + tX)^*J(1 + tX) = J + t(X^*J + JX) + t^2X^*JX,$$

so

$$df_1(X) = \frac{d}{dt} f(1 + tX)\bigg|_{t=0} = X^*J + JX.$$

Thus,

$$T_1O_J(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^*J + JX = 0\},$$

as required. ■

We now execute our computations in sequence.

(a) Using the preceding remarks, we see that

$$T_1\,\mathrm{U}_{p,q}(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : X^*B_{p,q} + B_{p,q}X = 0\},$$

where $B_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$ is a diagonal matrix. We now continue as in (c). Set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and then we compute

$$\begin{aligned} X^*B_{p,q} + B_{p,q}X &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^* & -C^* \\ B^* & -D^* \end{bmatrix} + \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &= \begin{bmatrix} A^* + A & B - C^* \\ B^* - C & -D^* - D \end{bmatrix}. \end{aligned}$$

In particular, this will vanish if and only if A and D are skew-Hermitian and $B = C^*$, so

$$\begin{aligned} T_1\,\mathrm{U}_{p,q}(\mathbb{C}) &= \left\{\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^*\right\}, \\ T_1\,\mathrm{U}_n(\mathbb{C}) &= \{A \in M_n(\mathbb{C}) : A = -A^*\}. \end{aligned}$$

Now, the space of $p \times p$ skew-Hermitian matrices A (namely, satisfying $A = -A^*$) is forced to have imaginary diagonal, and then the remaining entries are uniquely determined by their values strictly above the diagonal. Thus, the real dimension of this space is $p + p(p-1) = p^2$. We conclude that

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{U}_{p,q}(\mathbb{C}) &= p^2 + 2pq + q^2 = n^2, \\ \dim_{\mathbb{R}} \mathrm{U}_n(\mathbb{C}) &= n^2. \end{aligned}$$

From here, we address SU by recalling that

$$T_1\,\mathrm{SU}_{p,q}(\mathbb{R}) = \{X \in T_1\,\mathrm{U}_{p,q}(\mathbb{C}) : \operatorname{tr} X\}.$$

In particular,

$$T_1 \operatorname{SU}_{p,q}(\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^*, \operatorname{tr} A + \operatorname{tr} D = 0 \right\},$$
$$T_1 \operatorname{SU}_n(\mathbb{C}) = \{ A \in M_n(\mathbb{C}) : A = -A^*, \operatorname{tr} A = 0 \}.$$

Now, tr continues is real and actually surjects onto $\mathbb{R}$ for $n \geqslant 1$, even for our family of matrices above (for example, for any real number r, the matrix $\operatorname{diag}(r, 0, \ldots, 0)$ has trace r and lives in the above families). Thus, the kernel has dimension one smaller than the total space, giving

$$\dim_{\mathbb{R}} \operatorname{SU}_{p,q}(\mathbb{C}) = n^2 - 1,$$
$$\dim_{\mathbb{R}} \operatorname{SU}_n(\mathbb{C}) = n^2 - 1.$$

Example 1.152. We will show that

$$\operatorname{SL}_n(\mathbb{F}) := \{ A \in \operatorname{GL}_n(\mathbb{F}) : \det A = 1 \}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \operatorname{SL}_n(\mathbb{F}) = n^2 - 1$.

Proof. We use Lemma 1.150. We see that

$$T_1 \operatorname{SL}_n(\mathbb{F}) = \{ X \in M_n(\mathbb{F}) : \operatorname{tr} X = 0 \}.$$

(Note $T_1 \operatorname{GL}_n(\mathbb{F}) = T_1 M_n(\mathbb{F}) \cong M_n(\mathbb{F})$.) As such, we note that $\operatorname{tr}\colon M_n(\mathbb{F}) \to \mathbb{F}$ is surjective (for $n \geqslant 1$), so $\dim_{\mathbb{F}} T_1 \operatorname{SL}_n(\mathbb{F}) = \dim_{\mathbb{F}} \ker \operatorname{tr} = \dim_{\mathbb{F}} M_n(\mathbb{F}) - 1 = n^2 - 1$. ■

We now begin our computations for bilinear forms.

Example 1.153. Let $B := 1_n$ be the standard bilinear form. We will show that

$$\operatorname{O}_n(\mathbb{F}) := \{ A \in \operatorname{GL}_n(\mathbb{F}) : ABA^\intercal B \}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \operatorname{O}_n(\mathbb{F}) = \frac{1}{2} n(n-1)$.

Proof. We use Lemma 1.151. We see that

$$T_1 \operatorname{O}_n(\mathbb{F}) = \{ X \in M_n(\mathbb{F}) : X^\intercal + X = 0 \},$$

which is the space of alternating matrices. Thus, we see that the diagonal of $X \in T_1 \operatorname{O}_n(\mathbb{F})$ vanishes, and the remaining entries are determined by the values strictly above the diagonal, of which there are $\frac{1}{2} n(n-1)$. Thus, $\dim \operatorname{O}_n(\mathbb{F}) = \frac{1}{2} n(n-1)$. ■

Example 1.154. We will show that

$$\operatorname{SO}_n(\mathbb{F}) := \{ A \in \operatorname{O}_n(\mathbb{F}) : \det A = 1 \}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \operatorname{SO}_n(\mathbb{F}) = \frac{1}{2} n(n-1)$.

Proof. Using Lemma 1.150, we see that

$$T_1 \operatorname{SO}_n(\mathbb{F}) = \{ X \in \operatorname{O}_n(\mathbb{F}) : \operatorname{tr} X = 0 \}.$$

However, alternating matrices already have vanishing traces, so $T_1 \operatorname{SO}_n(\mathbb{F})$ is simply the full space of alternating matrices, giving $\dim_{\mathbb{F}} \operatorname{SO}_n(\mathbb{F}) = \frac{1}{2} n(n-1)$. ■

Over $\mathbb{R}$, there are more bilinear forms.

Example 1.155. Let $B_{p,q} := 1_p \oplus 1_q$ where $n = p + q$. We will show that

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : AB_{p,q}A^\intercal B_{p,q}\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \mathrm{O}_{p,q}(\mathbb{R}) = \frac{1}{2}n(n-1)$.

Proof. By Lemma 1.151, we see that

$$T_1 \mathrm{O}_{p,q}(\mathbb{R}) = \{X \in M_n(\mathbb{R}) : X^\intercal B_{p,q} + B_{p,q}X = 0\},$$

where $B_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$ is a diagonal matrix. To compute the dimension of this space of matrices, we set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and then we compute

$$\begin{aligned} X^\intercal B_{p,q} + B_{p,q}X &= \begin{bmatrix} A^\intercal & C^\intercal \\ B^\intercal & D^\intercal \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^\intercal & -C^\intercal \\ B^\intercal & -D^\intercal \end{bmatrix} + \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &= \begin{bmatrix} A^\intercal + A & B - C^\intercal \\ B^\intercal - C & -D^\intercal - D \end{bmatrix}. \end{aligned}$$

In particular, this will vanish if and only if A and D are both alternating, and $B = C^\intercal$, yielding

$$T_1 \mathrm{O}_{p,q}(\mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ B^\intercal & D \end{bmatrix} : A \in M_p(\mathbb{R}) \text{ and } D \in M_q(\mathbb{R}) \text{ are alternating} \right\}.$$

Thus, the dimension of our space is

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{O}_{p,q}(\mathbb{R}) &= \underbrace{\frac{1}{2}p(p-1)}_{A} + \underbrace{pq}_{B=C^\intercal} + \underbrace{\frac{1}{2}q(q-1)}_{D} \\ &= \frac{1}{2}\left(p^2 + 2pq + q^2 - p - q\right) \\ &= \frac{1}{2}(p+q)(p+q-1) \\ &= \frac{1}{2}n(n-1), \end{aligned}$$

where the dimension computations for (the spaces of) A and D are as in. ■

Example 1.156. We will show that

$$\mathrm{SO}_{p,q}(\mathbb{R}) := \{A \in \mathrm{O}_{p,q}(\mathbb{F}) : \det A = 1\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \mathrm{SO}_{p,q}(\mathbb{R}) = \frac{1}{2}n(n-1)$.

Proof. We use Lemma 1.150, we note that

$$T_1 \mathrm{SO}_{p,q}(\mathbb{R}) : \{X \in T_1 \mathrm{O}_{p,q}(\mathbb{R}) : \operatorname{tr} X = 0\},$$

but our description of $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has A and D alternating, so $\operatorname{tr} X = \operatorname{tr} A + \operatorname{tr} D = 0$. Thus, we see $T_1 \mathrm{SO}_{p,q}(\mathbb{R}) = T_1 \mathrm{O}_{p,q}(\mathbb{R})$, so the above description of tangent space and dimension go through. ■

Example 1.157. Let $\Omega_{2n} := \left[\begin{smallmatrix} 0_n & -1_n \\ 1_n & 0_n \end{smallmatrix}\right]$ be the standard symplectic form. We will show that

$$\mathrm{Sp}_{2n}(\mathbb{F}) := \{A \in \mathrm{GL}_n(\mathbb{F}) : A\Omega A^\intercal = A\}$$

is a Lie group over $\mathbb{F}$ and compute its Lie algebra to find $\dim_{\mathbb{F}} \mathrm{Sp}_{2n}(\mathbb{F}) = 2n^2 + n$.

Proof. By Lemma 1.151, we see that

$$T_1 \mathrm{Sp}_{2n}(\mathbb{F}) = \{X \in M_n(\mathbb{F}) : X^\intercal \Omega + \Omega X = 0\},$$

where $\Omega = \begin{bmatrix} & -1_n \\ 1_n & \end{bmatrix}$ is alternating. As usual, we set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and we compute

$$\begin{aligned} X^\intercal \Omega + \Omega X &= \begin{bmatrix} A^\intercal & C^\intercal \\ B^\intercal & D^\intercal \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} + \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} C^\intercal & -A^\intercal \\ D^\intercal & -B^\intercal \end{bmatrix} + \begin{bmatrix} -C & -D \\ A & B \end{bmatrix} \\ &= \begin{bmatrix} C^\intercal - C & -D - A^\intercal \\ A + D^\intercal & B - B^\intercal \end{bmatrix}. \end{aligned}$$

Thus, we see that

$$T_1 \mathrm{Sp}_{2n}(\mathbb{F}) = \left\{ \begin{bmatrix} A & B \\ C & -A^\intercal \end{bmatrix} : A, B, C \in M_n(\mathbb{F}), B = B^\intercal, C = C^\intercal \right\},$$

and our dimension is

$$\dim_{\mathbb{F}} \mathrm{Sp}_{2n}(\mathbb{F}) = \underbrace{n^2}_{A} + \underbrace{\frac{1}{2}n(n+1)}_{B} + \underbrace{\frac{1}{2}n(n+1)}_{C} = 2n^2 + n,$$

where we compute the dimension of space of symmetric matrices exactly analogously to the case of alternating matrices, except now the diagonal is permitted to be nonzero. ■

Lastly, we handle Hermitian forms.

Example 1.158. Let $B_{p,q} := 1_p \oplus 1_q$ where $n = p + q$. We will show that

$$\mathrm{U}_{p,q}(\mathbb{C}) := \left\{A \in \mathrm{GL}_n(\mathbb{F}) : AB_{p,q}A^\dagger B_{p,q}\right\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \mathrm{U}_{p,q}(\mathbb{C}) = n^2$.

Proof. Using Lemma 1.151, we see that

$$T_1 \mathrm{U}_{p,q}(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : X^* B_{p,q} + B_{p,q} X = 0\},$$

where $B_{p,q} = \left[\begin{smallmatrix} 1_p & \\ & -1_q \end{smallmatrix}\right]$ is a diagonal matrix. We now continue as in (c). Set $X := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to have the appropriate dimensions, and then we compute

$$\begin{aligned} X^* B_{p,q} + B_{p,q} X &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^* & -C^* \\ B^* & -D^* \end{bmatrix} + \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &= \begin{bmatrix} A^* + A & B - C^* \\ B^* - C & -D^* - D \end{bmatrix}. \end{aligned}$$

In particular, this will vanish if and only if A and D are skew-Hermitian and $B = C^*$, so

$$T_1 \mathrm{U}_{p,q}(\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^* \right\},$$
$$T_1 \mathrm{U}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = -A^*\}.$$

Now, the space of $p \times p$ skew-Hermitian matrices A (namely, satisfying $A = -A^*$) is forced to have imaginary diagonal, and then the remaining entries are uniquely determined by their values strictly above the diagonal. Thus, the real dimension of this space is $p + p(p-1) = p^2$. We conclude that

$$\dim_{\mathbb{R}} \mathrm{U}_{p,q}(\mathbb{C}) = p^2 + 2pq + q^2 = n^2,$$

as required. ■

Example 1.159. We will show that

$$\mathrm{SU}_{p,q}(\mathbb{C}) := \{A \in \mathrm{SU}_{p,q}(\mathbb{C}) : \det A = 1_n\}$$

is a Lie group over $\mathbb{R}$ and compute its Lie algebra to find $\dim_{\mathbb{R}} \mathrm{SU}_{p,q}(\mathbb{C}) = n^2 - 1$.

Proof. By Lemma 1.150, we see

$$T_1 \mathrm{SU}_{p,q}(\mathbb{R}) = \{X \in T_1 \mathrm{U}_{p,q}(\mathbb{C}) : \operatorname{tr} X\}.$$

In particular,

$$T_1 \mathrm{SU}_{p,q}(\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), A = -A^*, D = -D^*, \operatorname{tr} A + \operatorname{tr} D = 0 \right\},$$
$$T_1 \mathrm{SU}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A = -A^*, \operatorname{tr} A = 0\}.$$

Now, tr continues is real and actually surjects onto $\mathbb{R}$ for $n \geqslant 1$, even for our family of matrices above (for example, for any real number r, the matrix $\operatorname{diag}(r, 0, \ldots, 0)$ has trace r and lives in the above families). Thus, the kernel has dimension one smaller than the total space, giving

$$\dim_{\mathbb{R}} \mathrm{SU}_{p,q}(\mathbb{C}) = n^2 - 1,$$

as required. ■

Theme 2
Passing to Lie Algebras

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

—Emil Artin

2.1 September 18

Today we compute our Lie algebras.

2.1.1 The Exponential Map: The General Case

Given a Lie group G with Lie algebra $\mathfrak{g}$, we would like to define an exponential map $\exp\colon \mathfrak{g} \to G$. Recall that $\exp$ gave rise to our homomorphisms $\gamma\colon \mathbb{R} \to G$ with $\gamma(0) = e$ and $\gamma'(0)$ is specified. This will be our starting point.

Proposition 2.1. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, there exists a unique Lie group homomorphism $\gamma_X : \mathbb{R} \to G$ such that $\gamma_X'(0) = X$.

Proof. We use the theory of integral curves; see [Lee13, Chapter 9]. In particular, we see that we must satisfy $\gamma(s + t) = \gamma(t)\gamma(s)$ for all $s, t \in \mathbb{R}$, which yields

$$\gamma'(t) = \gamma(t)\gamma'(0),$$

where this multiplication really means $dL_{\gamma(t)}(\gamma'(0))$.

Thus, we see that we want to extend $X \in T_eG$ to a left-invariant vector field, and then we let $\gamma\colon \mathbb{R} \to G$ be the integral curve of this vector field satisfying $\gamma(0) = e$. (A priori, γ can only be defined in a neighborhood of the identity, but we can translate around in the group G to get a global solution. See [Lee13, Lemma 9.15] and in particular its corollary [Lee13, Theorem 9.18].) Then

$$\gamma'(t) = X(\gamma(t)) = dL_{\gamma(t)}(X(0)) = dL_{\gamma(t)}(X)$$

for each $t \in \mathbb{R}$.

Thus far we have shown that there is at most one Lie group homomorphism $\gamma_X : \mathbb{R} \to G$ satisfying $\gamma_X'(0) = X$; namely, it will be the above integral curve! It remains to check that the above integral curve actually satisfies $\gamma(t+s) = \gamma(t)\gamma(s)$. Well, for $s \in \mathbb{R}$, we define $\gamma_1(t) = \gamma(t+s)$ and $\gamma_2(t) = \gamma(s)\gamma(t)$. Then we see that γ_1 and γ_2 are both integral curves satisfying the ordinary differential equation

$$\widetilde{\gamma}'(t) = dL_{\widetilde{\gamma}(t)}(\widetilde{\gamma}'(0))$$

with initial condition $\widetilde{\gamma}(0) = \gamma(s)$, so the must be equal, completing the proof. ■

Remark 2.2. Here is one way to conclude without using [Lee13, Theorem 9.18]. The last paragraph of the proof provides a path $\gamma\colon (-\varepsilon, \varepsilon) \to G$ for some $\varepsilon > 0$ satisfying the homomorphism property. But then any $N > 0$ allows us to define $\widetilde{\gamma}\colon (-N, N) \to G$ given by

$$\widetilde{\gamma}(t) := \gamma(t/N)^N.$$

However, we can check that $\widetilde{\gamma}$ satisfies $\widetilde{\gamma}'(t) = dL_{\widetilde{\gamma}(t)}(X)$ with initial condition $\widetilde{\gamma}(0) = e$, so $\widetilde{\gamma}$ extends γ. Thus, we can extend γ to $\bigcup_{N>0}(-N\varepsilon, N\varepsilon) = \mathbb{R}$.

We now define $\exp$ motivated by the classical case.

Definition 2.3 (exponential)**.** Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, define γ_X via Proposition 2.1. Then we define $\exp_G\colon \mathfrak{g} \to G$ by

$$\exp_G(X) := \gamma_X(1).$$

We will omit the subscript from $\exp_G$ as much as possible.

Example 2.4. If $G \subseteq \mathrm{GL}_n(\mathbb{F})$ is classical, we can take $\gamma_X(t) = \exp(tX)$ where $\exp$ is defined as for GL_n. Thus, $\exp(X)$ matches with the above definition.

Example 2.5. Consider the Lie group $\mathbb{R}^n$. Then for each $X \in T_0\mathbb{R}^n$, we identify $T_0\mathbb{R}^n \cong \mathbb{R}^n$ to observe that we can take $\gamma_X(t) := tX$. Thus, $\exp(X) = X$.

Example 2.6. For any G, we can take $\gamma_0(t) := 0$, so $\exp(0) = 1$.

Example 2.7. We can directly compute that

$$(d\exp)_0(X) = \frac{d}{dt}\exp(tX)\bigg|_0 \overset{*}{=} \frac{d}{dt}\gamma_X(t)\bigg|_{t=0} = X.$$

The equality $\exp(tX) \overset{*}{=} \gamma_X(t)$ is explained as follows: we can check that $\gamma_{rX}(t) = \gamma_X(rt)$ for any $r, t \in \mathbb{R}$ by computing the derivative at 0, so $\exp(tX) = \gamma_{tX}(1) = \gamma_X(t)$ follows.

Here are some quick checks.

Proposition 2.8. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then $\exp\colon \mathfrak{g} \to G$ is regular and a local diffeomorphism.

Proof. Note that $\exp$ solves the differential equation given by Example 2.7, for which the theory of integral curves promises that this solution must be regular. Example 2.7 also tells us that $\exp$ is an isomorphism at the identity and hence a local diffeomorphism. ■

We would like to know something like $\exp(A+B) = \exp(A)\exp(B)$ when A and B commute, but one needs to be a little careful in how to state this. Here are some manifestations.

Proposition 2.9. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then

$$\exp((s+t)X) = \exp(sX)\exp(tX)$$

for any $s, t \in \mathbb{R}$ and $X \in \mathfrak{g}$.

Proof. This is a matter of following the definitions around. Let $\gamma_X : \mathbb{R} \to G$ be the one-parameter family for X. Then we see that $\gamma_{rX}(t) = \gamma_X(rt)$ for any $r \in \mathbb{R}$ as explained in Example 2.7, so

$$\exp((s+t)X) = \gamma_{(s+t)X}(1) = \gamma_X(s+t) = \gamma_X(s)\gamma_X(t) = \exp(sX)\exp(tX),$$

as desired. ■

Proposition 2.10. Fix a homomorphism $\varphi\colon G \to H$ of Lie groups. Then

$$\varphi(\exp_G(X)) = \exp_H(d\varphi_0(X))$$

for any $X \in T_eG$.

Proof. This follows from the definition. In particular, we claim that

$$\gamma_{d\varphi_0(X)}(t) \stackrel{?}{=} \varphi(\gamma_X(t)).$$

To see this, note that $t \mapsto \varphi(\gamma_X(t))$ is a Lie group homomorphism $\mathbb{R} \to H$, and we can compute the derivative at 0 to be $d\varphi_0(\gamma_X'(0)) = d\varphi_0(X)$, as required. Plugging in $t = 1$ to the above equation completes the proof. ■

Corollary 2.11. Fix homomorphisms $\varphi_1, \varphi_2\colon G \to H$ of Lie groups. Suppose G is connected. If $d\varphi_1 = d\varphi_2$, then $\varphi_1 = \varphi_2$.

Proof. Using Proposition 2.10, we see that

$$\varphi(\exp(X)) = \exp(d\varphi_0(X))$$

produces the same answer for $\varphi \in \{\varphi_1, \varphi_2\}$. However, $\exp$ is a local diffeomorphism by Proposition 2.8, so we have determined the values of φ_1 and φ_2 on the image of $\exp$, which must contain an open neighborhood of the identity of G. Thus, because G is connected, we see that G is generated by this open neighborhood, so in fact we have fully determined the values of φ_1 and φ_2. ■

Proposition 2.12. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$, and let $\operatorname{Ad}_\bullet\colon G \to \operatorname{GL}(\mathfrak{g})$ be the adjoint representation of Example 1.113. For any $g \in G$ and $X \in \mathfrak{g}$, we have

$$g\exp(X)g^{-1} = \exp(\operatorname{Ad}_g X).$$

Proof. By Proposition 2.10, we see that

$$g\exp(X)g^{-1} = \operatorname{Ad}_g(\exp(X)) = \exp((d\operatorname{Ad}_g)_e X) = \exp(\operatorname{Ad}_g X),$$

where the last equality holds by definition of the adjoint representation. (Yes, the notation is somewhat confusing.) ■

While we are here, we note that there is a logarithm map.

Definition 2.13 (logarithm)**.** Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Because $\exp$ is a local diffeomorphism, there are open neighborhoods $U \subseteq G$ and $\mathfrak{u} \subseteq \mathfrak{g}$ of the identities so that $\log\colon U \to \mathfrak{u}$ is an inverse for $\exp$.

2.1.2 The Commutator

Define the form $\mu\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\mu(X,Y) := \log(\exp(X)\exp(Y)).$$

(Technically, μ is a priori only defined on an open neighborhood of the identity of $\mathfrak{g} \times \mathfrak{g}$.) Expanding out everything into coordinates, we see that μ has a Taylor series expansion as

$$\mu(X,Y) = c + \alpha_1(X) + \alpha_2(Y) + Q_1(X) + Q_2(Y) + \lambda(X,Y) + \cdots,$$

where c is constant, α_1 and α_2 are linear, Q_1 and Q_2 are quadratic, λ is bilinear, and $+\cdots$ denotes cubic and higher-order terms. However, we see that $\mu(X,0) = 0$ and $\mu(0,Y) = 0$ for any $X, Y \in \mathfrak{g}$, so $c = Q_1 = Q_2 = 0$ and $\alpha_1(X) = X$ and $\alpha_2(Y) = Y$. Further, we claim that λ is skew-symmetric: it is enough to show that $\lambda(X,X) = 0$, for which we note that

$$2X = \log(\exp(2X)) = \log(\exp(X)\exp(X)) = \mu(X,X) = X + X + \lambda(X,X) + \cdots,$$

so $\lambda(X,X) = 0$ is forced.

This λ allows us to define the Lie bracket on $\mathfrak{g}$ in a purely group-theoretic way.

Definition 2.14 (Lie bracket)**.** Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then we define the *commutator* as the skew-symmetric form $\frac{1}{2}\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $[-,-]$. In particular, we see that

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \cdots\right), \tag{2.1}$$

where $+\cdots$ denotes higher-order terms (as usual).

Remark 2.15. A priori, the commutator may only be defined on an open neighborhood of the identity of $\mathfrak{g} \times \mathfrak{g}$, so (2.1) only holds (a priori) for sufficiently small X and Y. However, bilinearity allows us to scale our definition of $[-,-]$ from this open neighborhood everywhere.

Example 2.16. We compute the commutator map for GL_n. We see that

$$\exp(X)\exp(Y) = 1 + X + Y + XY + \frac{1}{2}\left(X^2 + Y^2\right) + \cdots, \tag{2.2}$$

$$\exp\left(X + Y + \frac{1}{2}[X+Y] + \cdots\right) = 1 + X + Y + \frac{1}{2}\left(X^2 + Y^2\right) + \frac{1}{2}XY + \frac{1}{2}YX + \frac{1}{2}[X,Y] + \cdots, \tag{2.3}$$

giving $[X,Y] = XY - YX$ subtracting.

To compute the commutator for the classical groups, we need to check some functoriality.

Proposition 2.17. Fix a homomorphism $\varphi\colon G \to H$ of Lie groups. For any $X, Y \in T_eG$, we have

$$d\varphi_0([X,Y]) = [d\varphi_0(X), d\varphi_0(Y)].$$

Proof. We unravel the definitions. Everything in sight is linear, so we may assume that X and Y are sufficiently small, so $d\varphi_0(X)$ and $d\varphi_0(Y)$ are sufficiently small. We now compute

$$\begin{aligned}\exp\left(d\varphi_0(X)+d\varphi_0(Y)+\frac{1}{2}[d\varphi_0(X),d\varphi_0(Y)]\right)&=\exp\left(d\varphi_0(X)\right)\exp\left(d\varphi_0(Y)\right)\\&\overset{*}{=}\varphi(\exp(X))\varphi(\exp(Y))\\&=\varphi(\exp(X)\exp(Y))\\&=\varphi\left(\exp\left(X+Y+\frac{1}{2}[X,Y]+\cdots\right)\right)\\&\overset{*}{=}\exp\left(d\varphi_0(X)+d\varphi_0(Y)+\frac{1}{2}d\varphi_0([X,Y])+\cdots\right),\end{aligned}$$

where we have used Proposition 2.10 at the equalities $\overset{*}{=}$. Because $\exp$ is a diffeomorphism for X and Y sufficiently small, the desired equality follows. ■

Example 2.18. The embedding $\mathrm{SL}_n(\mathbb{F})\to\mathrm{GL}_n(\mathbb{F})$ implies by Proposition 2.17 that the Lie bracket on the Lie algebra $\mathfrak{sl}_n$ can be computed by restricting the commutator Lie bracket on $\mathfrak{gl}_n$ (given by Example 2.16). In particular, we see that $\mathfrak{sl}_n$ is closed under taking commutators, which is not totally obvious a priori! A similar operation permits computation of the Lie bracket of a Lie group G whenever given an embedding $G\subseteq\mathrm{GL}_n$ (such as for the classical groups).

Corollary 2.19. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathrm{Ad}_\bullet\colon G\to\mathrm{GL}(\mathfrak{g})$ denote the adjoint representation. For

$$\mathrm{Ad}_g([X,Y])=[\mathrm{Ad}_g(X),\mathrm{Ad}_g(Y)].$$

Proof. We simply apply Proposition 2.17 to $\mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g})$, which yields

$$(d\mathrm{Ad}_g)_e([X,Y])=[(d\mathrm{Ad}_g)_eX,(d\mathrm{Ad}_g)_eT],$$

which is the original equation after enough abuse of notation. ■

Proposition 2.20. Fix a Lie group G. For sufficiently small $X,Y\in T_eG$, we have

$$\exp(X)\exp(Y)\exp(X)^{-1}\exp(Y)^{-1}=\exp([X,Y]+\cdots).$$

Proof. This is a direct computation. We compute

$$\begin{aligned}\exp(X)\exp(Y)\exp(-X)\exp(-Y)&=\exp\left(X+Y+\frac{1}{2}[X,Y]+\cdots\right)\exp\left(-X-Y+\frac{1}{2}[X,Y]+\cdots\right)\\&=\exp\left([X,Y]+\cdots\right),\end{aligned}$$

where we get some omitted cancellation of lower-order terms in the last equality (and there is a lot of higher-order terms). ■

Corollary 2.21. If G is abelian, then $[X,Y]=0$ for any X and Y.

Proof. It suffices to assume that X and Y are sufficiently small because the conclusion is linear. Now, Proposition 2.20 implies that

$$\exp([X,Y]+\cdots)=0,$$

so because $\exp$ is a diffeomorphism for small enough X and Y, so $[X,Y]=0$ follows. ■

2.2 September 20

Today we continue discussing the Lie bracket.

2.2.1 The Adjoint Action

Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Here is the standard example of a "Lie algebra representation."

Notation 2.22. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Note that the map $(d\mathrm{Ad}_g)_1 \colon G \to \mathrm{GL}(\mathfrak{g})$ is smooth, so we can consider the differential of this map, which we label $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Here are some checks on this map.

Proposition 2.23. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$.

(a) For $X, Y \in \mathfrak{g}$, we have $\mathrm{ad}_X(Y) = [X, Y]$.

(b) For $X \in \mathfrak{g}$, we have $\mathrm{Ad}_{\exp(X)} = \exp(\mathrm{ad}_X)$ as operators $\mathfrak{g} \to \mathfrak{g}$.

Proof. Here we go.

(a) By definition of our differential action, we have

$$(d\mathrm{Ad}_g)_1(Y) = \frac{d}{dt} g \exp(tY) g^{-1}\bigg|_{t=0}$$

for any $g \in G$ and $Y \in \mathfrak{g}$. We would like to compute a derivative of this map with respect to g (at the identity). As such, we plug in $g = \exp(sX)$ to compute

$$\begin{aligned}
\mathrm{ad}_X(Y) &= \frac{d}{ds}(d\mathrm{Ad}_{\exp(sX)})_1(Y)\bigg|_{s=0} \\
&= \frac{d}{ds}\frac{d}{dt}\exp(sX)\exp(tY)\exp(-sX)\bigg|_{t=0}\bigg|_{s=0} \\
&\overset{*}{=} \frac{d}{ds}\frac{d}{dt}\exp(tY + st[X,Y] + \cdots)\bigg|_{t=0}\bigg|_{s=0},
\end{aligned}$$

where in $\overset{*}{=}$ we have used the definition of our bracket. Upon expanding out $\exp$ as a series, we see that the lower-order terms are $1 + tY + st[X, Y] + \cdots$ (everything higher is at least quadratic) for small enough s and t, so the derivative evaluates to $[X, Y]$.

(b) This follows immediately from Proposition 2.10 upon setting $\varphi = (d\mathrm{Ad}_\bullet)_1$. ■

Here is an example computation of what all this adjoint business looks like for GL_n, more directly than appealing to the bracket.

Lemma 2.24. Identify $T\,\mathrm{GL}_n(\mathbb{F})$ with $\mathrm{GL}_n(\mathbb{F}) \times \mathfrak{gl}_n(\mathbb{F})$ via left-invariant vector fields. For $X \in \mathfrak{gl}_n(\mathbb{F})$, we have

$$\begin{cases} dL_g(X) = gX, \\ dR_g(X) = Xg^{-1}, \\ d\,\mathrm{Ad}_g(X) = gXg^{-1}. \end{cases}$$

Proof. Set $G := \mathrm{GL}_n(\mathbb{F})$ and $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{F})$. Note that the adjoint is the composite of L_g and R_g, so the last equality follows from the first two. For the first equality, we are computing the differential of the maps $L_g, R_g \colon G \to G$ at some $h \in G$. Well, L_g and R_g actually extend to perfectly fine linear maps $M_n(\mathbb{F}) \to M_n(\mathbb{F})$, and the differential of any linear map is simply itself upon identifying the tangent spaces of $M_n(\mathbb{F})$ with itself, so we conclude that $dL_g(X) = gX$ and $dR_g(X) = Xg^{-1}$, as required. ■

Lemma 2.25. Fix a homomorphism $\varphi \colon G \to H$ of Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. For any $g \in G$ and $X \in \mathfrak{g}$, we have

$$(d\mathrm{Ad}_{\varphi(g)})_e(d\varphi_1(X)) = d\varphi_1((d\mathrm{Ad}_g)_e(X)).$$

Proof. Simply take the differential (at 1) of the equation $\mathrm{Ad}_{\varphi(g)} \circ \varphi = \varphi \circ \mathrm{Ad}_g$, which is true because φ is a homomorphism. ■

Example 2.26. Given any embedding $G \subseteq \mathrm{GL}_n(\mathbb{F})$ with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$, we can use Lemma 2.25 to compute the adjoint action on $\mathfrak{g}$ by conjugation (via Lemma 2.24)!

Proposition 2.27. Let $(d\mathrm{Ad}_\bullet)_1 \colon \mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}(\mathfrak{gl}_n(\mathbb{F}))$ denote the adjoint representation. Then

$$\mathrm{ad}_X(Y) = XY - YX.$$

Proof. To parse the symbols, we note that $(d(d\mathrm{Ad}_\bullet)_1)_1 \colon \mathfrak{gl}_n(\mathbb{F}) \to \mathrm{End}(\mathfrak{gl}_n(\mathbb{F}))$, so the statement at least makes sense. Now, given $X \in \mathfrak{gl}_n(\mathbb{F})$, define $\gamma \colon \mathbb{F} \to M_n(\mathbb{F})$ by $\gamma(t) := 1 + tX$. Then $\gamma'(0) = X$. As such,

$$(d(d\mathrm{Ad}_\bullet)_1)_1(X) = (d(d\mathrm{Ad}_\bullet)_1)_1(\gamma'(0)) = ((d\mathrm{Ad}_\bullet)_1 \circ \gamma)'(0).$$

In particular, plugging in some $Y \in \mathfrak{gl}_n(\mathbb{F})$, we may use Lemma 2.24 to compute that

$$\begin{aligned}
\frac{d}{dt}((d\mathrm{Ad}_\bullet)_1 \circ \gamma)(t)(Y)\bigg|_{t=0} &= \frac{d}{dt}(d\mathrm{Ad}_{1+tX})_1(Y)\bigg|_{t=0} \\
&= \frac{d}{dt}(1+tX)Y(1+tX)^{-1}\bigg|_{t=0} \\
&= \frac{d}{dt}(1+tX)Y\left(1 - tX + t^2X^2 + \cdots\right)\bigg|_{t=0} \\
&= XY - YX,
\end{aligned}$$

where the series expansion takes t small enough for the series to converge. (For example, one can take t small enough so that all eigenvalues of tX are less than 1.) ■

Example 2.28. Given any embedding $G \subseteq \mathrm{GL}_n(\mathbb{F})$ with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$, we note that the action of G on $\mathfrak{g}$ actually extends to an action of G on $\mathfrak{gl}_n(\mathbb{F})$ (still by conjugation) which happens to stabilize $\mathfrak{g}$. Then the action $G \to \mathrm{GL}(\mathfrak{gl}_n(\mathbb{F}))$ is a restriction of the adjoint action $\mathrm{GL}_n(\mathbb{F}) \to \mathrm{GL}(\mathfrak{gl}_n(\mathbb{F}))$ given by conjugation still, whose differential action $\mathfrak{gl}_n(\mathbb{F}) \to \mathfrak{gl}(\mathfrak{gl}_n(\mathbb{F}))$ we computed above to be given by $\mathrm{ad}_X \colon Y \mapsto XY - YX$. This restricts back to the subspace $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$ (via the inclusion $G \subseteq \mathrm{GL}_n(\mathbb{F})$), where we know that the action must happen to stabilize $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$. The point is that we have computed our adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by the commutator. (Alternatively, one can redo the computation of the above proof.)

2.2.2 Lie Algebras

Here is a standard consequence of this theory.

Proposition 2.29 (Jacobi identity)**.** Fix a Lie group G with Lie algebra $\mathfrak{g}$. Then we have the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Proof. Doing some rearranging with Proposition 2.23 (and the skew-symmetry), we see that this is equivalent to plugging Z into the identity

$$\mathrm{ad}_{[X,Y]} \stackrel{?}{=} \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X.$$

To verify this, we note that the right-hand side is $[\mathrm{ad}_X, \mathrm{ad}_Y]$, where the commutator is taken in $\mathfrak{gl}(\mathfrak{g})$. Thus, we are trying to show that the adjoint preserves a commutator, which we do as follows: recall that $\mathrm{Ad}_\bullet \colon G \to \mathrm{GL}(\mathfrak{g})$ is a morphism of Lie groups, meaning that the differential map ad preserves the commutator by Proposition 2.17. ■

The Jacobi identity is important enough to earn the following definition.

Definition 2.30 (Lie algebra)**.** Fix a field k. Then a *Lie algebra* is a k-vector space $\mathfrak{g}$ equipped with a bilinear form $[-,-] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following.

(a) Skew-symmetric: $[X,X] = 0$ for all $X \in \mathfrak{g}$.

(b) Jacobi identity: for any $X, Y, Z \in \mathfrak{g}$, we have

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

A morphism of Lie algebras is a k-linear morphism preserving the forms.

Definition 2.31 (commutative)**.** A Lie algebra $\mathfrak{g}$ is *commutative* if and only if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Example 2.32. For any k-algebra A, we produce a Lie bracket on A given by

$$[X,Y] := XY - YX.$$

This map is of course linear in both X and Y (because multiplication is k-linear in a k-algebra), and $[X,X] = X^2 - X^2 = 0$. Lastly, to see the Jacobi identity, we expand:

$$\begin{aligned}
[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] &= [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] \\
&= X(YZ - ZY) - (YZ - ZY)X \\
&\quad + Y(ZX - XZ) - (ZX - XZ)Y \\
&\quad + Z(XY - YX) - (XY - YX)Z \\
&= 0.
\end{aligned}$$

For example, one can take A to be $\mathrm{End}_k(V)$ for some k-vector space V; this produces the Lie algebra $\mathfrak{gl}(V)$.

Example 2.33. Given a regular Lie group G, the tangent space at the identity $\mathfrak{g}$ is a Lie algebra according to the above definition.

The above example upgrades into a functor.

Proposition 2.34. Fix a regular Lie group G. For any morphism of Lie groups $\varphi\colon G_1 \to G_2$, the differential $d\varphi_e\colon T_eG_1 \to T_eG_2$ is a (functorial) morphism of Lie algebras. In fact, if G_1 and G_2 is connected, the induced map

$$\mathrm{Hom}_{\mathrm{LieGrp}}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Lie}(k)}(T_eG_1, T_eG_2)$$

is injective. In other words, the functor $G \to T_eG$ from connected Lie groups to Lie algebras is faithful.

Proof. The differential being a homomorphism of Lie algebras follows from Proposition 2.17. Functoriality follows from the corresponding functoriality for differentials of more general smooth maps. The injectivity follows from Corollary 2.11. ■

Remark 2.35. It turns out that the functor above is also full, though we are not in a position to show this yet.

2.2.3 Subalgebras

Lie algebras are interesting enough to study on their own right, but we now note that we have sufficient motivation from Proposition 2.34.

Definition 2.36 (subalgebra, ideal)**.** Fix a Lie algebra $\mathfrak{g}$.

- A *Lie subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace closed under the Lie bracket of $\mathfrak{g}$; note that $\mathfrak{h}$ continues to be a Lie algebra.
- A *Lie ideal* is a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the stronger property that

$$[X, Y] \in \mathfrak{h}$$

  for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

Definition 2.37 (representation)**.** A *representation* of a Lie algebra $\mathfrak{g}$ over a field k is a morphism $\mathfrak{g} \to \mathfrak{gl}(V)$ for some vector space V over k.

Here is how these things relate back to Lie groups.

Proposition 2.38. Fix a regular Lie subgroup H of a regular Lie group G. Let their Lie algebras be $\mathfrak{h}$ and $\mathfrak{g}$, respectively.

(a) Then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.

(b) If H is normal in G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

(c) If G and H are connected, and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then H is normal in G.

Proof. Here we go.

(a) Certainly $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace, so we want to check that $[X, Y] \in \mathfrak{h}$ for $X, Y \in \mathfrak{h}$, where the target bracket is taken in $\mathfrak{g}$. Consider the embedding $\varphi\colon H \to G$ so that $\mathfrak{h} = \operatorname{im} d\varphi_0$. Thus, we use Proposition 2.17 to see that

$$d\varphi_0([X, Y]) = [d\varphi_0(X), d\varphi_0(Y)].$$

Thus, for any $X, Y \in \operatorname{im} d\varphi_0$, we see that $[X, Y] \in \operatorname{im} d\varphi_0$, as required.

(b) For any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, we want to check that $[X, Y] \in \mathfrak{h}$. By Proposition 2.23, we are asking to check that $\mathrm{ad}_X(Y) \in \mathfrak{h}$. Well, for any $g \in G$, we see that $gHg^{-1} \subseteq H$, so the adjoint $\mathrm{Ad}_g \colon G \to G$ restricts to $\mathrm{Ad}_g \colon H \to H$. In particular, by taking the differential, we see that the adjoint $(d\mathrm{Ad}_\bullet)_1 \colon G \to \mathrm{GL}(\mathfrak{g})$ restricts to $(d\mathrm{Ad}_\bullet)_1 \colon G \to \mathrm{GL}(\mathfrak{h})$. (Namely, $(d\mathrm{Ad}_g)_1(Y) \in \mathfrak{h}$ for any $Y \in \mathfrak{h}$.) Taking the differential of this, we see that we get our map $\mathrm{ad}_\bullet \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$, meaning that $\mathrm{ad}_X(Y) \in \mathfrak{h}$ for any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

(c) Recall from Proposition 2.23 that

$$\mathrm{Ad}_{\exp(X)}(Y) = \exp(\mathrm{ad}_X Y).$$

Thus, for any $X \in \mathfrak{g}$, we see that $\mathrm{Ad}_{\exp(X)}$ is an operator $\mathfrak{h} \to \mathfrak{h}$. Thus, for $g \in G$ sufficiently close to the identity, we see that $\mathrm{Ad}_g(Y) \in \mathfrak{h}$ for $Y \in \mathfrak{h}$. Taking the exponential, Proposition 2.12 tells us that $ghg^{-1} \in H$ for $g \in G$ and $h \in H$ both sufficiently close to the identity.

Concretely, we get an open neighborhood U of the identity of G such that $ghg^{-1} \in H$ for any $g \in U$ and $h \in H \cap U$. Now, the subset of G normalizing $U \cap H$ is a subgroup of G containing U, so we see that it must be all of G because G is connected. Then the subset of H normalized by G is again a subgroup of H containing $U \cap H$, so we see that it must be all of H because H is connected. Thus, H is normal in G. ■

Here is some motivation for our definition of ideal.

Lemma 2.39. Fix a morphism $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ of Lie algebras.

(a) The kernel $\ker \varphi \subseteq \mathfrak{g}$ is a Lie ideal.

(b) The image $\operatorname{im} \varphi \subseteq \mathfrak{h}$ is a Lie subalgebra.

Proof. Here we go.

(a) For any $X \in \ker \varphi$ and $Y \in \mathfrak{g}$, we need to check that $[X, Y] \in \ker \varphi$. Well,

$$\varphi([X, Y]) = [\varphi(X), Y] = [0, Y] = 0$$

by the bilinearity of $[-, -]$.

(b) For any $X, Y \in \operatorname{im} \varphi$, we must check that $[X, Y] \in \operatorname{im} \varphi$. Well, find $X_0, Y_0 \in \mathfrak{g}$ such that $X = \varphi(X_0)$ and $Y = \varphi(Y_0)$, and then we see that

$$[X, Y] = [\varphi(X_0), \varphi(Y_0)] = \varphi([X_0, Y_0])$$

is in the image of φ, as required. ■

Remark 2.40. Fix a collection $\{\mathfrak{g}_\alpha\}_{\alpha \in \kappa}$ of Lie ideals of $\mathfrak{g}$. Then we claim that the intersection $\bigcap_{\alpha \in \kappa} \mathfrak{g}_\alpha$ is still a Lie ideal of $\mathfrak{g}$. Indeed, for any $X \in \bigcap_{\alpha \in \kappa} \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}$, we see that $X \in \mathfrak{g}_\alpha$ and hence $[X, Y] \in \mathfrak{g}_\alpha$ for all $\alpha \in \kappa$; thus, $[X, Y] \in \bigcap_{\alpha \in \kappa} \mathfrak{g}_\alpha$.

Lemma 2.41. Fix a Lie ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$. Then the quotient space $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra with bracket given by

$$[X + \mathfrak{h}, Y + \mathfrak{h}]_{\mathfrak{g}/\mathfrak{h}} \coloneqq [X, Y]_{\mathfrak{g}} + \mathfrak{h}.$$

Proof. The main issue is checking that the bracket is well-defined. Well, if $X, Y \in \mathfrak{g}$ and $X', Y' \in \mathfrak{h}$, we must check that

$$[X + X', Y + Y'] + \mathfrak{h} \stackrel{?}{=} [X, Y] + \mathfrak{h},$$

where the bracket is taken in $\mathfrak{g}$. This is a matter of expanding with the bilinearity: note

$$\begin{aligned}[X + X', Y + Y'] &= [X + X', Y] + [X + X', Y'] \\ &= [X, Y] + [X', Y] + [X, Y'] + [X', Y'],\end{aligned}$$

and now we see that the last three terms live in $\mathfrak{h}$ because $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal.

Now, note that we have a canonical surjective linear map $\pi\colon \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ which satisfies

$$\pi([X, Y]) = [\pi(X), \pi(Y)].$$

Thus, the bilinearity, skew-symmetry, and Jacobi identity for $\mathfrak{g}/\mathfrak{h}$ are immediately inherited from the corresponding checks on $\mathfrak{g}$. Rigorously, perhaps one should note that (for example) the Jacobi identity corresponds to showing that some linear functional on $(\mathfrak{g}/\mathfrak{h})^3$ vanishes; however, this linear functional can be checked to vanish on the level of $\mathfrak{g}^3$. ■

Proposition 2.42. Fix a morphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ of Lie algebras. Then the induced quotient map

$$\overline{\varphi}\colon \mathfrak{g}/\ker\varphi \to \operatorname{im}\varphi$$

is an isomorphism.

Proof. Linear algebra implies that $\overline{\varphi}$ is already an isomorphism of vector spaces. Thus, it merely remains to check that $\overline{\varphi}$ is a morphism of Lie algebras. Well, for $X, Y \in \mathfrak{g} =$, we see

$$\begin{aligned}\overline{\varphi}([X + \ker\varphi, Y + \ker\varphi]) &= \overline{\varphi}([X, Y] + \ker\varphi) \\ &= \varphi([X, Y]) + \ker\varphi \\ &= [\varphi(X), \varphi(Y)] + \ker\varphi \\ &= [\overline{\varphi}(X), \overline{\varphi}(Y)],\end{aligned}$$

as required. ■

2.2.4 Lie Algebra of a Vector Field

One can in general provide a Lie algebra of a vector field. Fix a regular vector field $\xi\colon X \to TX$ on a regular manifold X. For any regular function f on an open subset $U \subseteq X$, we may define

$$(\xi f)(x) := \xi_x(f_x),$$

where we recall that $\xi_x \in T_xX$ is some derivation which outputs a number when fed a germ f_x. The point is that ξf is itself a regular function $X \to \mathbb{F}$! We are now able to define a bracket.

Proposition 2.43. Fix a regular manifold X. Given vector fields $\xi, \eta\colon X \to TX$, we define the Lie bracket

$$[\xi, \eta] := \xi\eta - \eta\xi.$$

Then $[-, -]$ is a Lie bracket.

Proof. At each $x \in X$, we have certainly defined a map taking regular functions f on X and outputting an element of $\mathbb{F}$ given by

$$[\xi, \eta]_x(f) := \xi_x(\eta f)_x - \eta_x(\xi f)_x.$$

This is certainly linear in f because ξ and η are. Further, the value of $[\xi, \eta]_x(f)$ only depends on the germ f_x because having $f_x = g_x$ for functions f and g implies $(f - g)_x = 0_x$, and then $\eta(f - g)$ and $\xi(f - g)$ both vanish in a neighborhood of x, so $[\xi, \eta]_x(f - g) = 0$.

It remains to check the product rule. Well, for regular functions f and g and some $y \in X$, we compute

$$(\eta fg)(y) = \eta_y(f_y g_y) = f(y)\eta_y(g_y) + g(y)\eta_y(g_y) = (f \cdot \eta g + g \cdot \eta f)(y),$$

and a similar computation works for ξ. Thus,

$$\begin{aligned}\xi(\eta fg)(x) &= \xi(f\eta g + g\eta f)(x) \\ &= \xi(f\eta g)(x) + \xi(g\eta f)(x) \\ &= f(x)\xi(\eta g)(x) + (\xi f)(x)(\eta g)(x) + g(x)\xi(\eta f)(x) + (\xi g)(x)(\eta f)(x),\end{aligned}$$

and a similar computation holds for $\eta(\xi fg)(x)$. Thus, we see that

$$\begin{aligned}[\xi,\eta]_x(fg) &= f(x)\xi(\eta g)(x) + (\xi f)(x)(\eta g)(x) + g(x)\xi(\eta f)(x) + (\xi g)(x)(\eta f)(x) \\ &\qquad - (f(x)\eta(\xi g)(x) + (\eta f)(x)(\xi g)(x) + g(x)\eta(\xi f)(x) + (\eta g)(x)(\xi f)(x)) \\ &= f(x)[\xi,\eta]_x g + g(x)[\xi,\eta]_x f\end{aligned}$$

after sufficient cancellation and rearranging. ■

Example 2.44. Fix regular functions f and g on some open subset of $U \subseteq \mathbb{R}^m$, and let x_i and x_j be two coordinates. Then we compute

$$\left[f\frac{\partial}{\partial x_i}, g\frac{\partial}{\partial x_j}\right] = f\frac{\partial g}{\partial x_i}\frac{\partial}{\partial x_j} - g\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}.$$

Proof. Fixing some $p \in U$ and regular germ h, we see

$$\begin{aligned}\left[f\frac{\partial}{\partial x_i}, g\frac{\partial}{\partial x_j}\right]_p(h) &= f(p)\frac{\partial}{\partial x_i}g\frac{\partial h}{\partial x_j}\bigg|_p - g(p)\frac{\partial}{\partial x_j}f\frac{\partial h}{\partial x_i}\bigg|_p \\ &= f(p)\frac{\partial g}{\partial x_i}\bigg|_p\frac{\partial h}{\partial x_j}\bigg|_p + f(p)g(p)\frac{\partial h}{\partial x_i\partial x_j}\bigg|_p - g(p)\frac{\partial f}{\partial x_j}\bigg|_p\frac{\partial h}{\partial x_i}\bigg|_p - f(p)g(p)\frac{\partial h}{\partial x_i\partial x_j}\bigg|_p \\ &= f(p)\frac{\partial g}{\partial x_i}\bigg|_p\frac{\partial h}{\partial x_j}\bigg|_p - g(p)\frac{\partial f}{\partial x_j}\bigg|_p\frac{\partial h}{\partial x_i}\bigg|_p,\end{aligned}$$

as required. ■

Remark 2.45. In local coordinates in some chart (U,φ) with $\varphi = (x_1,\ldots,x_m)$ of our regular manifold M, one can write vector fields as

$$\xi = \sum_{i=1}^m a_i\frac{\partial}{\partial x_i} \qquad \text{and} \qquad \eta = \sum_{i=1}^m b_i\frac{\partial}{\partial x_i},$$

where a_i and b_i are regular functions. Then one can expand the bilinearity to see that

$$[\xi,\eta] = \sum_{i,j=1}^m\left(a_j\frac{\partial b_i}{\partial x_j} - b_j\frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}.$$

Indeed, after applying bilinearity, the main point is to compute $\left[f\frac{\partial}{\partial x}, g\frac{\partial}{\partial y}\right]$ for regular functions f and g and coordinates x and y, which we did in the previous example.

Remark 2.46. For example, if ξ and η are tangent to a regular submanifold $N \subseteq M$ of dimension k, then $[\xi,\eta]$ continues to be tangent. One can check this using a local slice chart, where the condition that ξ is tangent to Y is equivalent to having $a_i = 0$ for $i > k$. Combining this with the computation of the previous remark completes the argument.

2.3 September 23

Today we continue talking about vector fields.

2.3.1 Vector Fields on Lie Groups

Let's return to Lie groups.

Lemma 2.47. Fix a regular Lie group G. A vector field ξ on G is left-invariant if and only if

$$\xi(f \circ L_g) = \xi f \circ L_g$$

for any germ f defined in a neighborhood of g.

Proof. We show the two implications separately.

- If ξ is left-invariant, then $\xi_{gh} = (dL_g)_h(\xi_h)$ for any $g, h \in G$. Thus, for any $h \in G$, we see that

  $$\begin{aligned}(\xi f \circ L_g)(h) &= \xi_{gh} f \\ &= ((dL_g)_h \xi_h) f \\ &= \xi_h(f \circ L_g),\end{aligned}$$

  as required.
- Suppose $\xi(f \circ L_g) = \xi f \circ L_g$ for any f. Then plugging in the identity tells us that

  $$\xi_g f = (\xi f \circ L_g)(e) = \xi_e(f \circ L_g) = ((dL_g)_e(\xi_e))(f).$$

  Thus, $\xi_g = (dL_g)_e \xi_e$, as required. ■

Lemma 2.48. Fix a left-invariant vector field ξ on a regular Lie group G. Then for a germ f at a point $g \in G$, one has

$$\xi_g f = \frac{d}{dt} f(g \exp(t\xi_e))\bigg|_{t=0}.$$

Proof. This is more or less the chain rule. For our $g \in G$, Lemma 2.47 tells us that

$$\xi_g f = \xi_e(f \circ L_g).$$

Now, the path $\gamma \colon \mathbb{F} \to G$ given by $\gamma(t) := \exp(t\xi_e)$ has $\gamma'(0) = \xi_e$, so

$$\xi_e(f \circ L_g) = d(f \circ L_g \circ \gamma)'(0) = \frac{d}{dt} f(g \exp(t\xi_e))\bigg|_{t=0},$$

as required. ■

Proposition 2.49. Fix a regular Lie group G with Lie algebra $\mathfrak{g}$. Then the collection of left-invariant vector fields $\operatorname{Vect}^L(G)$ is a Lie subalgebra of $\operatorname{Vect}(G)$ which is isomorphic to $\mathfrak{g}$.

Proof. By Remark 1.138, one certainly has an isomorphism $\operatorname{Vect}^L(G) \to \mathfrak{g}$ given by $\xi \mapsto \xi_e$, with inverse given by $X \mapsto \xi_X$, where ξ_X is the vector field $\xi_X(g) := dL_g(X)$. Now, by Lemma 2.47, ξ is left-invariant if and only if

$$\xi(f \circ L_g) = \xi f \circ L_g$$

for any germ f defined in a neighborhood of g. Thus, we see that $\operatorname{Vect}^L(G)$ is preserved by the commutator of $\operatorname{Vect}(G)$.

It remains to check that our isomorphism with $\mathfrak{g}$ is a morphism of Lie algebras. Fix $X, Y \in \mathfrak{g}$, and we would like to show that $[\xi_X, \xi_Y] = \xi_{[X,Y]}$. It is enough to check this equality after mapping back down to $\mathfrak{g}$, so we want to check that $[\xi_X, \xi_Y]_e = [X, Y]$. This is a direct computation: by Lemma 2.48, any germ f at e has

$$\begin{aligned}
[\xi_X, \xi_Y]_e f &= \frac{d}{dt}\big(\xi_Y f(\exp(tX)) - \xi_X f(\exp(tY))\big)\Big|_{t=0} \\
&= \frac{\partial^2}{\partial s \partial t}\frac{d}{ds}\big(f(\exp(tX)\exp(sY)) - f(\exp(tY)\exp(sX))\big)\Big|_{(s,t)=(0,0)} \\
&= \frac{\partial^2}{\partial s \partial t}\left(f \exp\left(tX + sY + \frac{1}{2}st[X,Y] + \cdots\right) - f \exp\left(tX + sY - \frac{1}{2}st[X,Y] + \cdots\right)\right)\Big|_{(s,t)=(0,0)}.
\end{aligned}$$

Now, one can imagine taking a Taylor series expansion of $f \circ \exp\colon \mathfrak{g} \to \mathbb{R}$ in terms of Z, in which we see that the above derivative will only depend on the st term of the relevant expansion. More precisely, write $(f \circ \exp)(Z) = f(e) + \lambda(Z) + Q(Z) + C(Z)$, where λ is linear, Q is quadratic, and C has vanishing first- and second-order derivatives. Then, after cancellation within λ, we see that

$$\begin{aligned}
[\xi_X, \xi_Y]_e f = &\frac{\partial^2}{\partial s \partial t} st\lambda(st[X,Y])\Big|_{(s,t)=(0,0)} \\
&+ \frac{\partial^2}{\partial s \partial t} Q\left(tX + sY + \frac{1}{2}st[X,Y] + \cdots\right)\Big|_{(s,t)=(0,0)} \\
&+ \frac{\partial^2}{\partial s \partial t} Q\left(tX + sY - \frac{1}{2}st[X,Y] + \cdots\right) + \cdots\Big|_{(s,t)=(0,0)},
\end{aligned}$$

where $+\cdots$ denotes higher-order terms which will not affect the current derivative (for example, containing C). Now, the linear terms inside Q will produce cancelling terms after expansion, so the only term we are left to care about is

$$[\xi_X, \xi_Y]_e f = \lambda([X,Y]) = \frac{d}{dt}(f \circ \exp)(t[X,Y])\Big|_{t=0} = (\xi_{[X,Y]})_e f,$$

as required. ■

2.3.2 Group Actions via Lie Algebras

In general, if G acts on a regular manifold M via the action $a\colon G \times X \to X$, one can define an action of $\mathfrak{g}$ on $\operatorname{Vect}(X)$ by analogy with Lemma 2.48.

Definition 2.50. Fix a left action $a\colon G \times M \to M$ of a regular Lie group G on a regular manifold M. Then we define $a_*\colon \mathfrak{g} \to \operatorname{Vect}(M)$ by

$$(a_* X)_p f := \frac{d}{dt} f(a(\exp(-tX), p))\Big|_{t=0}$$

for any $p \in M$ and germ f at p.

Remark 2.51. Let's explain the sign in the above definition: the action of G on M induces a natural action of G on the regular functions $\mathcal{O}(M)$ by $(g \cdot f)(p) := f\left(g^{-1} \cdot p\right)$. It is this action of G on $\mathcal{O}(M)$ which motivates the above definition.

Remark 2.52. On the homework, we will check that a_* is a morphism of Lie algebras.

We can now prove the Orbit–stabilizer theorem (Theorem 2.53) in the following more precise form.

Theorem 2.53 (Orbit–stabilizer)**.** Fix a left action $a\colon G \times M \to M$ of a regular Lie group G on a regular manifold M. Fix some $p \in M$.

(a) For all $p \in M$, the stabilizer G_p is a closed Lie subgroup with Lie algebra

$$\operatorname{Lie} G_p = \{X \in \mathfrak{g} : (a_* X)_p = 0\}.$$

(b) The induced map $G/G_p \to M$ given by $g \mapsto g \cdot p$ is an injective immersion. In particular, the orbit Go is an immersed submanifold.

(c) If the induced map $G/G_p \to M$ is an embedding, then G/G_p is diffeomorphic to Gp.

Proof. We begin with the proof of (a), which we do in steps.

1. Set
$$\mathfrak{g}_p := \{X \in \mathfrak{g} : (a_* X)_p = 0\}$$
for brevity. We claim that $\mathfrak{g}_p \subseteq \mathfrak{g}$ is a Lie subalgebra. Certainly $X \mapsto (a_* X)_p$ is a linear map $\mathfrak{g} \to \operatorname{Vect}(M) \to T_p M$, so $\mathfrak{g}_p$ is a linear subspace.

   It remains to check that $\mathfrak{g}_p$ is preserved by the bracket. Fix $X, Y \in \mathfrak{g}_p$, and we want to check $[X, Y] \in \mathfrak{g}_p$. Well, because a_* is a homomorphism of Lie algebras, we see
$$a_*[X,Y]_p f = \underbrace{(a_* X)_p}_{0}(a_* Y f) - \underbrace{(a_* Y)_p}_{0}(a_* X f) = 0$$
for any germ f at p. Thus, $a_*[X,Y] = 0$, so $[X,Y] \in \mathfrak{g}_p$.

2. For $X \in \mathfrak{g}_p$, we check that $\exp(X) \in G_p$. Indeed, we claim the two curves $\gamma_1(t) := \exp(-tX) \cdot p$ and $\gamma_2(t) := p$ are both integral curves for $a_* X$ with the same initial condition at 0. This completes the check because it implies that $\exp(X) \cdot p = \gamma_1(-1) = \gamma_2(-1) = p$ by uniqueness of integral curves.

   To prove the claim, we note that γ_2 is constant, so there is nothing to check there. For γ_1, we must check that
$$\gamma_2'(t) \stackrel{?}{=} (a_* X)_{\gamma_2(t)}$$
in $T_{\gamma_2(t)} M$. To check this, we pass through an arbitrary germ f to see that
$$\gamma_2'(t) f = (f \circ \gamma_2)'(t) = \frac{d}{ds} f(\exp(-sX - tX) \cdot p)\bigg|_{s=0},$$
and
$$(a_* X)_{\gamma_2(t)} f = \frac{d}{ds} f(\exp(-tX - sX) \cdot p)\bigg|_{s=0},$$
as required.

3. We attempt to control $\mathfrak{g}/\mathfrak{g}_p$. Choose a complement $\mathfrak{u}$ of $\mathfrak{g}_p \subseteq \mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{u}$. (We do not require that $\mathfrak{u}$ is a Lie subalgebra, despite the font.) Then the map $f\colon \mathfrak{u} \to T_pM$ given by $Z \mapsto (a_*Z)_p$ has kernel $\mathfrak{g}_p \cap \mathfrak{u} = 0$ and hence is injective. Thus, the Implicit function theorem tells us that the map $F\colon \mathfrak{u} \to M$ given by $v \mapsto \exp(-V) \cdot p$ must be an injective immersion for small v because $df_p(V) = dF_p(V)$.

   Instead of using the Implicit function theorem, we can argue using local diffeomorphisms as follows: fix a basis $\{e_1, \ldots, e_k\}$ of $\mathfrak{u}$, and extend the linearly independent set $\{dF_p(e_1), \ldots, df_p(e_k)\} \subseteq T_pM$ to a basis $\{dF_p(e_1), \ldots, dF_p(e_k)\} \sqcup \{e'_{k+1}, \ldots, e'_m\}$. Then define a local map $\widetilde{F}\colon \mathfrak{u} \times \mathbb{F}^{m-k} \to M$ by

$$\widetilde{F}(a_1e_1 + \cdots + a_me_m) = F(a_1e_1 + \cdots + a_ke_k) + a_{k+1}e'_{k+1} + \cdots + a_me'_m,$$

   where the addition on the right-hand side is defined in a local chart of M around p. (Technically, $\widetilde{F}$ is only defined in a neighborhood of $0 \in \mathfrak{u}$.) Then $\widetilde{F}$ is a local diffeomorphism at 0 by construction, so F is an injective immersion in this same neighborhood of 0.

4. We construct a slice chart for $G_p \subseteq G$ at the identity, which will complete the proof (a) by Lemma 1.90. Note that the map $\exp^{\oplus}\colon \mathfrak{g}_p \oplus \mathfrak{u} \to G$ given by $(V, X) \mapsto \exp(V)\exp(X)$ is a local diffeomorphism at 0 (because the differential is simply the identity by checking what happens on each piece $\mathfrak{g}_p$ and $\mathfrak{u}$ separately). Thus, for $g \in G$ sufficiently close to e, we can write g uniquely as in the image of e and thus as $g = \exp(V)\exp(X)$ where $V \in \mathfrak{u}$ and $X \in \mathfrak{g}_p$. Now, we see that $g \in G_p$ if and only if $\exp(V) \in G_p$, which for small enough V is equivalent to $V \in \mathfrak{g}_p$ by the previous step.

   In total, we have constructed a very small open neighborhood $U \subseteq \mathfrak{g}_p \oplus \mathfrak{u}$ of the identity such that $e|_U$ is a diffeomorphism onto its image $\exp^{\oplus}(U) \subseteq G$ and

$$G_p \cap \exp^{\oplus}(U) = \{(V, X) \in \mathfrak{g}_p \oplus \mathfrak{u} : V = 0\},$$

   which is a slice chart.

We now proceed with (b). Let $\overline{\varphi}$ denote the induced map $G/G_p \to M$ given by $\overline{\varphi}(g) := g \cdot p$, which we want to see is an injective immersion. Injectivity follows by definition of G_p: if $\varphi(g_1) = \varphi(g_2)$, then $g_1 \cdot p = g_2 \cdot p$, so $g_1^{-1}g_2 \in G_p$, so $g_1G_p = g_2G_p$. Being an immersion more or less follows from the proof. By translation, it suffices to show that $d\overline{\varphi}_e$ is injective.[1] Well, the Lie algebra of G/G_p is the quotient $\mathfrak{g}/\mathfrak{g}_p$ by Theorem 1.94, which is isomorphic to $\mathfrak{u}$ by construction of $\mathfrak{u}$. But we know that the action map is injective on $\mathfrak{u}$ by the third step above, so we are done.

Lastly, we note that (c) follows immediately from (b) because embeddings are diffeomorphic onto their images by the uniqueness of the smooth structure of embedded submanifolds. ■

Remark 2.54. We also remark that Theorem 1.96 follows quickly from the above result. Indeed, let G act on H via the homomorphism $\varphi\colon G \to H$: $g \cdot h := \varphi(g)h$. Then the stabilizer of any $h \in H$ is given by $\ker\varphi$, proving $\ker\varphi$ is in fact a closed Lie subgroup. Now, passing to $\overline{\varphi}$ as in the above proof shows that $G/\ker\varphi \to \operatorname{im}\varphi$ is an injective immersion.

2.4 September 23

We began class by finishing the proof of Theorem 2.53 and giving an example.

2.4.1 The Orbit–Stabilizer Theorem for Fun and Profit

Let's see an example of Theorem 2.53.

[1] Once $d\overline{\varphi}_e$ is injective, we note that $\overline{\varphi} \circ L_g = L_g \circ \overline{\varphi}$ (where the first L_g is a map $G \to G$ and the second is a map $M \to M$, but both are diffeomorphisms), so $d\overline{\varphi}_g \circ d(L_g)_e = d(L_g)_p \circ d\overline{\varphi}_e$ verifies that $d\overline{\varphi}_g$ is injective.

Example 2.55. Fix a finite-dimensional representation V of a regular Lie group G given by $\rho\colon G \to \mathrm{GL}(V)$. For $v \in V$, its stabilizer G_v has Lie algebra given by

$$\mathfrak{g}_v = \{X \in \mathfrak{g} : (\rho_* X)_v = 0\}.$$

Example 2.56. Fix a finite-dimensional algebra A over a field $\mathbb{F}$. Then we claim that $\mathrm{Aut}_k(A)$ is a closed Lie subgroup of $\mathrm{GL}(A)$, and we claim that

$$\mathrm{Lie}(\mathrm{Aut}\, A) = \mathrm{Der}(A) \subseteq \mathrm{End}(A).$$

Proof. Note that $\varphi \in \mathrm{GL}(A)$ is an automorphism if and only if φ also preserves the multiplication map $\mu\colon A \otimes A \to A$ of A. Now, $\mathrm{GL}(A)$ has a natural action $\rho\colon\ \mathrm{GL}(A) \to \mathrm{GL}(\mathrm{Hom}(A \otimes A, A))$ by

$$(\rho(g)\varphi)(x \otimes y) := g\varphi\left(g^{-1}x \otimes g^{-1}y\right).$$

Precisely speaking, this is the composite of the actions of G on the various pieces by Remark 1.103, so this is in fact a representation of G. Now, $g \in \mathrm{GL}(A)$ preserves the multiplication map μ if and only if

$$g(\mu(a \otimes b)) = \mu(g(a) \otimes g(b))$$

for all $a, b \in A$, which is equivalent to

$$(\rho(g)\mu)(a \otimes b) = g\mu\left(g^{-1}a \otimes g^{-1}b\right) = \mu(a \otimes b)$$

for all $a, b \in A$. Thus, $\mathrm{Aut}(A) \subseteq \mathrm{GL}(A)$ is the stabilizer of $\mu \in \mathrm{Hom}(A \otimes A, A)$ and hence a closed Lie subgroup by Theorem 2.53.

It remains to compute the Lie algebra, which Theorem 2.53 tells us is

$$\mathfrak{gl}(A)_\mu = \{X \in \mathfrak{gl}(A) : (\rho_* X)_\mu = 0\}.$$

Thus, we want to compute $(\rho_* X)_\mu$. Note that $\mathrm{Hom}(A \otimes A, A)$ is some finite-dimensional $\mathbb{F}$-vector space, so for any germ f defined around μ, we may use the chain rule to compute

$$\begin{aligned}(\rho_* X)_\mu f &= \frac{d}{dt} f(\rho(\exp(-tX), \mu))\bigg|_{t=0} \\ &= df_\mu\left(\frac{d}{dt}\rho(\exp(-tX), \mu)\bigg|_{t=0}\right).\end{aligned}$$

Thus, we see that $X \in \mathfrak{gl}(A)_\mu$ if and only if $\frac{d}{dt}\rho(\exp(-tX), \mu)\big|_{t=0} = 0$. Now, linear operators pass through derivatives, and evaluation is a linear operator on $\mathrm{Hom}(A \otimes A, A)$, so it suffices to check when

$$\frac{d}{dt}\rho(\exp(-tX), \mu)(a \otimes b)\big|_{t=0}$$

vanishes, for arbitrary $a, b \in A$. Thus, we compute

$$\begin{aligned}\rho(\exp(-tX), \mu)(a \otimes b) &= \exp(-tX)\mu(\exp(tX)a, \exp(tX)b) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} X^n \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k a \cdot \sum_{\ell=0} \frac{t^\ell}{\ell!} X^\ell b\right) \\ &= \sum_{n,k,\ell=0}^{\infty} \frac{(-1)^n t^{n+k+\ell}}{n!k!\ell!} X^n \left(X^k a \cdot X^\ell b\right),\end{aligned}$$

where we have rearranged the sums with impunity because everything in sight converges absolutely. Furthermore, we can differentiate term-by-term to see that

$$\begin{aligned}\frac{d}{dt}\rho(\exp(-tX),\mu)(a\otimes b)\Big|_{t=0} &= \frac{(-1)^1t^{1+0+0}}{1!0!0!}X^1\left(X^0a\cdot X^0b\right)\\ &\quad+\frac{(-1)^0t^{0+1+0}}{0!1!0!}X^0\left(X^1a\cdot X^0b\right)\\ &\quad+\frac{(-1)^0t^{0+0+1}}{0!0!1!}X^0\left(X^0a\cdot X^1b\right)\\ &= -X(a\cdot b)+Xa\cdot b+a\cdot Xb.\end{aligned}$$

Thus,

$$\operatorname{Lie}(\operatorname{Aut}A)=\{X\in\mathfrak{gl}(A):X(a\cdot b)=Xa\cdot b+a\cdot Xb\},$$

which of course is the set of derivations. ■

Remark 2.57. A close examination of the above proof finds that we only need μ to be an element of $\operatorname{Hom}(A\otimes A,A)$ for the argument to go through. Notably, we may replace (A,μ) above with a Lie algebra $(\mathfrak{g},[-,-])$ to find that $\operatorname{Aut}_{\mathrm{LieAlg}}(\mathfrak{g})\subseteq\operatorname{GL}(\mathfrak{g})$ is a Lie subgroup with Lie algebra given by the derivations

$$\{\varphi\in\mathfrak{gl}(\mathfrak{g}):\varphi([X,Y])=[\varphi(X),Y]+[X,\varphi(Y)]\text{ for all }X,Y\in\mathfrak{g}\}.$$

Remark 2.58. The adjoint map $\operatorname{ad}\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ actually lands in $\operatorname{Der}(\mathfrak{g})$: checking this is tantamount to checking that $X,Y,Z\in\mathfrak{g}$ has

$$\operatorname{ad}_X[Y,Z]\stackrel{?}{=}[\operatorname{ad}_X Y,Z]+[Y,\operatorname{ad}_X Z],$$

which one can check is equivalent to the Jacobi identity of Proposition 2.29.

Remark 2.59. Similarly, the adjoint action $\operatorname{Ad}\colon G\to\operatorname{GL}(\mathfrak{g})$ actually lands in $\operatorname{Aut}_{\mathrm{LieAlg}}(\mathfrak{g})$. Indeed, for $g\in G$ and $X,Y\in\mathfrak{g}$, this amounts to checking that

$$\operatorname{Ad}_g[X,Y]\stackrel{?}{=}[\operatorname{Ad}_g X,\operatorname{Ad}_g Y],$$

which is Corollary 2.19.

Here is another application.

Definition 2.60 (center)**.** Fix a group G. Then the *center* of G is the subset

$$Z(G):=\{z\in G:zg=gz\text{ for all }g\in G\}.$$

Similarly, fix a Lie algebra $\mathfrak{g}$, then the *center* of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}):=\{X\in\mathfrak{g}:[X,Y]=0\text{ for all }Y\in\mathfrak{g}\}.$$

Remark 2.61. We will not bother to check that $Z(G)$ is a subgroup because this is a standard result of group theory. However, in order to do something, let's check that $\mathfrak{z}(\mathfrak{g})$ is a Lie ideal of $\mathfrak{g}$. Note that $\mathfrak{z}(\mathfrak{g})$ is the kernel of the collection of linear maps $X\mapsto[X,Y]$ as $Y\in\mathfrak{g}$ varies, so $\mathfrak{z}(\mathfrak{g})$ is an intersection of Lie ideals (by Lemma 2.39) and hence a Lie ideal by Remark 2.40.

Proposition 2.62. Fix a connected regular Lie group G. Then $Z(G)$ is a closed Lie subgroup with Lie algebra

$$\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Proof. We would like to see that $Z(G)$ is the kernel of the adjoint map $\mathrm{Ad}\colon G \to \mathrm{Aut}\, G$, but it is difficult to make sense of this argument because $\mathrm{Aut}\, G$ is not a manifold.

Instead, we note that $g \in G$ if and only if g commutes with an open neighborhood U of the identity: indeed, commuting with U implies commuting with the subgroup generated by U, but G is connected, so commuting with U is equivalent to commuting with G. Now, we can take U to be some neighborhood of the identity in the image of the local diffeomorphism $\exp\colon \mathfrak{g} \to G$, so $g \in Z(G)$ if and only if

$$g \exp(X) g^{-1} = \exp(X)$$

for all $X \in \mathfrak{g}$ in an open neighborhood of 0. Now, $\mathrm{Ad}_g \exp(X) = \exp(\mathrm{Ad}_g X)$ by Proposition 2.12, so the above equality is equivalent to having $\mathrm{Ad}_g X = X$ for X in a neighborhood of 0.

Thus, Remark 2.54 tells us that $Z(G)$ is the kernel of the representation $\mathrm{Ad}_\bullet \colon G \to \mathrm{GL}(\mathfrak{g})$. We conclude that its Lie algebra is the kernel of the differential of $\mathrm{Ad}_\bullet$, which of course is $\mathrm{ad}_\bullet \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Thus,

$$\mathrm{Lie}\, Z(G) = \{X \in \mathfrak{g} : \mathrm{ad}_X = 0\},$$

but Proposition 2.23 explains that $\mathrm{ad}_X = [X, -]$, so we see that this is simply $\mathfrak{z}(\mathfrak{g})$, as required. ■

Definition 2.63 (adjoint)**.** Fix a connected regular Lie group G. Then the *adjoint group* of G is $G^{\mathrm{ad}} := G/Z(G)$.

Example 2.64. For $G = \mathrm{GL}_n(\mathbb{F})$, one can check that $Z(G)$ is the subgroup $\{cI : c \in \mathbb{F}\}$. The adjoint group is then $\mathrm{PGL}_n(\mathbb{F})$.

2.4.2 The Baker–Campbell–Hausdorff Formula

For completeness, we mention the Baker–Campbell–Hausdorff formula. We will not need this result, so we will not prove it, and the discussion in this subsection will be quite terse. Fix a Lie group G with Lie algebra $\mathfrak{g}$. We would like to understand the group law on G purely in terms of $\mathfrak{g}$. As in our discussion of the commutator, we note that

$$\mu(X, Y) = \log(\exp(X)\exp(Y)),$$

defined in an open neighborhood of $\mathfrak{g}$ can be expanded out as

$$\mu(X, Y) = X + Y + \frac{1}{2}[X, Y] + \mu_3(X, Y) + \mu_4(X, Y) + \cdots = \sum_{n=1}^{\infty} \mu_n(X, Y),$$

where $\mu_n(X, Y)$ consists of the order-n terms in this Taylor expansion. Here is the main result. For example, as above, $\mu_1(X, Y) = X + Y$ and $\mu_2(X, Y) = \frac{1}{2}[X, Y]$.

Theorem 2.65 (Baker–Campbell–Hausdorff)**.** The polynomials μ_n above are independent of G.

One proves this basically by solving differential equations for the μ_n inductively in n.

Example 2.66. One could compute that

$$\mu_3(X, Y) = \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]).$$

2.4.3 The Fundamental Theorems of Lie Theory

To wrap up our transition to Lie algebras, we state the fundamental theorems of Lie theory, which we will mostly not prove.

Theorem 2.67. For a regular Lie group G with Lie algebra $|mfg$, there is a bijection between Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$. This bijection sends $H \subseteq G$ to $\mathfrak{h} := \operatorname{Lie} H$.

Theorem 2.68. Fix a simply connected regular Lie group G with Lie algebra $\mathfrak{g}$. Then for any regular Lie group H with Lie algebra $\mathfrak{h}$, the map

$$\operatorname{Hom}_{\mathrm{LieGrp}}(G, H) \to \operatorname{Hom}_{\mathrm{LieAlg}}(\mathfrak{g}, \mathfrak{h}),$$

given by taking the differential at the identity, is a bijection.

Theorem 2.69. Any finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of some simply connected regular Lie group.

Here is the consequence.

Corollary 2.70. The (full subcategory) of simply connected regular Lie groups is equivalent to the category of finite-dimensional Lie algebras, given by the Lie algebra functor.

Proof. Theorem 2.68 shows that this functor is fully faithful, and Theorem 2.69 shows that this functor is essentially surjective. This completes the proof. ■

We begin with the proof of Theorem 2.67. This requires the theory of distributions.

Definition 2.71 (distribution)**.** Fix a regular manifold M. Then a *k-dimensional distribution* $\mathcal{D}$ on X is a k-dimensional (local) subbundle $\mathcal{D} \subseteq TX$.

Remark 2.72. Locally at a point $p \in M$, we can think about $\mathcal{D}_p$ as being spanned by k linearly independent differentials which spread out over a neighborhood.

Definition 2.73 (integrable)**.** A distribution $\mathcal{D}$ of dimension k on a regular manifold M is *integrable* if and only if each $p \in M$ has a regular chart (U, φ) with local coordinates $\varphi = (x_1, \ldots, x_n)$ such that

$$D|_U = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right\}.$$

Here is a more coordinate-free check for being integrable.

Definition 2.74 (foliation)**.** A distribution $\mathcal{D}$ of dimension k on a regular manifold M is a *foliation* if and only if $p \in M$ has an "integral" immersed submanifold $S_p \subseteq M$, meaning that $T_qS_p = D_q$ for all $q \in S_p$.

Of course, integrable distributions are foliations, but it is a theorem that one can show the converse; see [Lee13, Theorem 19.12].

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List of Definitions